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Instituto de Matemática
Pura e Aplicada

Topics on Riemannian groupoids

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Rio de Janeiro
May 8, 2019.

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Abstract

This thesis develops the theory of Riemannian geometry on singular spaces (differentiable stacks) through the study of Riemannian groupoids and their Morita-invariant properties. With this aim, we consider generalized curves on Riemannian groupoids and define their normal length, relating it to the natural notion of distance in the orbit space. We introduce the notion of geodesic on Riemannian groupoids, verifying that it makes sense on the underlying Riemannian stack. We establish several foundational results, such as the existence and uniqueness of geodesics, a stacky Gauss Lemma, and a stacky Hopf-Rinow theorem. Using the stacky Hopf-Rinow theorem, we investigate the relations between invariant linearization for Lie groupoids and the existence of complete metrics.

Keywords: geodesics, linearization, stacks.

Resumo

Esta tese desenvolve a teoria da geometria Riemanniana em espaços singulares (stacks diferenciáveis) através do estudo de grupóides Riemannianos e suas propriedades Morita-invariantes. Com este objetivo, consideramos curvas generalizadas em grupóides Riemannianos e definimos seu comprimento normal, relacionando-o com a noção natural de distância no espaço de órbitas. Introduzimos a noção de geodésica em grupóides Riemannianos, verificando que esta noção descende para uma noção de geodésica nos stacks Riemannianos. Estabelecemos resultados fundamentais, como a existência e a unicidade de geodésicas, versões do Lema de Gauss e do teorema de Hopf-Rinow para stacks. Utilizando o teorema de Hopf-Rinow, investigamos as relações entre linearização invariante para grupóides de Lie e a existência de métricas completas.

Palavras-chave: geodésicas, linearização, stacks.

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Ariano Suassuna. Pesquisa Fapesp, 2008.

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Introduction

This thesis develops the theory of Riemannian geometry on singular spaces through the study of Riemannian groupoids and their Morita-invariant properties. In what follows we describe the main contributions of this thesis, based on our work [22, 23].

Lie groupoids constitute a unified framework to deal with group actions, foliations and fibrations, among other constructions, often providing new insights to classic geometric questions and results. Lie groupoids have been present in many areas, including Poisson geometry [36, 55, 56], Lie theory [17], Riemannian geometry [40, 59], noncommutative geometry [15] and foliations [32]. Our main interest in Lie groupoids in this thesis lies on their role as models for singular spaces, as we explain next.

The study of singular spaces arising in moduli problems in algebraic geometry led to the notion of *stack*, introduced by Grothendieck in [30]. Stacks can be formally defined as sheaf-like objects, which one describes locally and glues imposing coherence conditions. In the smooth setting, stacks have manifolds and orbifolds as special cases, providing a framework in which one can do differential geometry on singular spaces. For some stacks, known as *differentiable* or *geometric*, one can avoid the original abstract definition and define them using Lie groupoids instead.

Recall that a *Lie groupoid* is roughly defined by two manifolds G and M, called the manifolds of *arrows* and *objects*, respectively, and a partial multiplication on G for which all elements are invertible and M is the set of units. The manifold G encodes a natural equivalence relation on M (two objects are equivalent if they are connected by an arrow), and its equivalence classes are called *orbits*. Intuitively, one thinks of a Lie groupoid G as a model, or atlas, for the quotient orbit space, denoted by M/G. Just as different atlases can represent the same differentiable structure on a manifold, different Lie groupoids can represent the same "differentiable stack". The extension of the notion of equivalence of atlases to the context of Lie groupoids is called *Morita equivalence*, which codifies the "transverse" geometry of the groupoid. In this way, one can define *differentiable stacks* as Morita classes of Lie groupoids. This perspective has proven fruitful [8, 25, 53] and is fundamental in our work.

The notion of *Morita equivalence* build on the definition of *Morita map* (or weak equivalence), i.e., a groupoid map that preserves "the normal structure" around the orbits and the topological orbit space. Then we formally add an inverse for each Morita map. This process of adding formal inverses is known in category theory as localization. The Morita maps satisfy some properties that ensure that their localization can be constructed in a particularly simple way using fractions of maps. For Lie groupoids G and H, a fraction $\psi/\phi : G \dashrightarrow H$ is a pair of maps $\psi : \widetilde{G} \to H$ and $\phi : \widetilde{G} \to G$ from another Lie groupoid $\widetilde{G}$ with ϕ being a Morita map. Two groupoids G and H are Morita equivalent if there exists a fraction where both maps are Morita maps. In this way, Morita maps become isomorphisms at the level of differentiable stacks. More generally, we think of fractions $\psi/\phi : G \dashrightarrow H$ as representing *generalized maps* between Lie groupoids, and as models for maps between the corresponding stacks.

Although Riemannian geometry is vastly developed for smooth manifolds, it is much less studied on spaces which display singularities. This leads us to consider Riemannian geometry on Lie groupoids. The notion of metric on Lie groupoids, suitably compatible with the groupoid structure, was introduced in [24], extending several previous attempts to establish such a concept [28, 33, 40]. The exponential map of these metrics on groupoids is the key tool to prove a linearization theorem for Riemannian groupoids [24], which provides a simpler proof and a stronger version of the Weinstein-Zung linearization theorem for proper Lie groupoids [58, 62].

In [25], a type of Morita invariance for metrics on Lie groupoids is established, yielding a notion of metric on underlying stacks that extends the well-known definitions of Riemannian manifolds, orbifolds, and more. Our goal is to develop foundational aspects of such "Riemannians stacks", such as the study of geodesics and completeness properties.

Contributions

If two Lie groupoids are Morita equivalent, their underlying orbit spaces are homeomorphic. For Riemannian groupoids, the orbit spaces carry more information, for instance, the (pseudo-)distance discussed in [47]. In this work, we present a geometric interpretation of this distance in terms of the length of "generalized curves', which are our model for curves on stacks.

As part of our definition of length for generalized curves, we show that the norm of the orthogonal component of a curve varies continuously. We do it in the general setup of singular Riemannian foliations.

Proposition 1. ([23]) Let $(M, \eta, \mathcal{F})$ be a singular Riemannian foliated manifold. If $a : I \to M$ is a smooth curve, then $\left\| a'(t)^{\perp} \right\|$ is continuous on t.

We conjecture in the case where the singular foliation comes from a proper Lie groupoid that the norm varies smoothly. The normal length of a generalized curve is defined using the orthogonal component of the curves. This notion of length gives a geometric meaning for the distance in the orbit spaces as follows:

Theorem 1. ([23]) Given $(G \rightrightarrows M, \eta)$ a Riemannian groupoid, and given $x, y \in M$, the quotient pseudo-distance $d_N(\bar{x}, \bar{y})$ is the infimum of lengths of generalized curves connecting the points:

$$d_N(\bar{x}, \bar{y}) = \inf\{\ell_N(\alpha) : \bar{x}, \bar{y} \in \operatorname{im}(\bar{\alpha})\}.$$

Since a Morita equivalence between Riemannian stacks yields a correspondence between classes of generalized curves which preserve the normal lengths, we conclude that the (pseudo-)distance on the orbit space is a well-defined object associated with the Riemannian stack.

We then introduce geodesics for metrics on stacks, extending previous definitions of geodesics on orbifolds and orbit spaces of isometric actions [41, 31, 52]. We establish the existence and uniqueness of geodesics and other foundational results such as Gauss Lemma and Hopf-Rinow theorem.

In Riemannian geometry, the Gauss's lemma asserts that any sufficiently small geodesic sphere centered at a point is perpendicular to every geodesic through the point, and moreover, in radial directions, the exponential map is an isometry [14, p.69]. We prove a version of this second property for stacks: our stacky Gauss Lemma says that locally geodesics are the "shortest" paths to connect a given point to other points.

Proposition 2. ([23]) Let $G \rightrightarrows M$ be a proper Riemannian groupoid and $x \in M$. Then there exists $\epsilon > 0$ such that

$$d_N(\bar{x}, \overline{\exp}_{\bar{x}}([v])) = \|v\|, \qquad \forall\, v \in B^N_\epsilon \subset N_x M.$$

Geodesics on a Riemannian manifold are characterized as the curves that locally minimize distances. It turns out that this property of locally minimizing distances does not hold for geodesics on stacks, so we reformulate it in terms of "local rays". The Gauss Lemma shows that geodesics are "local rays", and we also show the converse that local rays are geodesics.

Theorem 2. Let $(G \rightrightarrows M, \eta)$ be a proper Riemannian groupoid. A curve fraction $a/\phi_{\mathcal{U}} : I \dashrightarrow G$ is a local ray if only if it is a geodesic fraction.

So, being a geodesic depends on the normal distance, which is a Riemannian Morita invariant. We use this to show that our definition of the geodesic is, in fact, well-defined for Riemannian stacks.

On a Riemannian manifold the assumptions of completeness as metric space and geodesic completeness are equivalent; this is the content of the Hopf-Rinow theorem [14, p.146]. We show that the same holds for Riemannian stacks: *the geodesics on a stack are defined for all time if and only if the underlying orbit space is complete as a metric space.* As simple corollaries, we conclude that stacks with compact orbit space are complete and that every stack admits a complete metric.

Theorem 3. ([23]) The stack $([M/G], [\eta])$ is geodesically complete if and only if $(M/G, d_N)$ is a complete metric space.

The Hopf-Rinow-Cohn-Vossen theorem states that a locally compact length space is complete if only if curves that locally minimize distances can be extended [11, Thm. 2.5.28]. Using the Riemannian stack structure of the locally compact length space $(M/G, d_N)$ we can see our theorem as an improvement of the Hopf-Rinow-Cohn-Vossen theorem since our geodesics do not minimize distances locally.

Finally we discuss the interplay of metrics on Lie groupoids and linearization problems. The following results are part of the project [22]. For a Lie groupoid $G \rightrightarrows M$ and an orbit $\mathcal{O} \subset M$, let $G\mathcal{O}$ be given by the restriction of G to $\mathcal{O}$. We have the linear model $NG\mathcal{O} \rightrightarrows N\mathcal{O}$, where $NG\mathcal{O}$ (resp. $N\mathcal{O}$) is the normal to $G\mathcal{O}$ in G (resp. $\mathcal{O}$ in G). The linearization problem asks if there exists a groupoid isomorphism from a neighborhood of $G\mathcal{O} \rightrightarrows \mathcal{O}$ in $G \rightrightarrows M$ to a neighborhood of $G\mathcal{O} \rightrightarrows \mathcal{O}$ in $NG\mathcal{O} \rightrightarrows N\mathcal{O}$. This problem has been studied by many authors [19, 24, 58, 62]. A linearization is called *invariant* when the neighborhoods of $\mathcal{O}$ in M and in $N\mathcal{O}$ can be taken to be saturated.

The invariant linearization for s-proper groupoids [19, 24, 58, 62] covers classical results, such as Ehresmann's fibration theorem [27], Reeb's local stability theorem for foliations [48], and linearization of compact group actions [44]. Inspired by the linearization theorem for Riemannian groupoids presented in [24] we replace the s-properness condition by the existence of complete metrics, showing that this implies invariant linearization.

Proposition 3. If a proper groupoid admits a groupoid metric such that the metric on the units is complete, then this is an invariantly linearizable groupoid.

The linearization for s-proper groupoids does not imply the Tube theorem for Lie groups proper actions, as observed in [19]. In this direction, we use the above result to set the Tube theorem for proper actions into the groupoid linearization perspective as proposed in [24, Remark 5.15]. We do this building a complete invariant metric for proper Lie group actions (cf. [35]). This construction is independent of the classic tube theorem.

In [21] it is shown that locally trivial fiber bundles admit complete fibered metrics; this fixes previous attempts to show the existence of complete fibered metric in locally trivial fiber bundle [29, 38, 60]. Based on this proof we show the following for invariantly linearizable groupoids:

Proposition 4. If a proper Lie groupoid is invariantly linearizable, then there exists a complete transversely invariant metric on its units.

This result was stated for proper Lie groupoids in [47, Prop 3.14], but the proof relies on an argument about re-scaling transversely invariant metrics that only works for groupoids with compact orbits (Remark 4.4.2).

We believe that the above proposition can be improved to the existence of a groupoid metric complete on the units. This improvement could be used together with 3 to produce the following characterization:

Conjecture:*A proper groupoid is invariantly linearizable if and only if it admits a groupoid metric complete on the units.*

Chapter 1

Lie groupoids and stacks

In this chapter, we give a short review of the definitions and properties of Lie groupoids, Morita equivalence, and how they provide an approach to differentiable stacks. This chapter will proceed as follows:

- In Section 1.1 we set our notation and recall the basic definitions and properties of Lie groupoids and their maps.
- In Section 1.2 we proceed to define Morita equivalence of Lie groupoids, leading us to differentiable stacks. We also describe some Morita invariants, which can be viewed as objects associated to stacks, including coarse orbit space, normal representation, coarse tangent space, and generalized maps.

1.1 Lie groupoids

Groupoids were first defined by Brandt in the 1920s. In differential geometry, Lie groupoids were introduced by Ehresmann in the 1950s. The development of Lie theory for groupoids was initiated by Pradines. The surveys [10, 57] contain a good exposition of the history of groupoids in differential geometry and other areas.

Lie groupoids constitute a unified framework to deal with group actions, foliations and fibrations, among other constructions, often providing new insights to classic geometric questions and results. Most of the content of this section comes from [16, 20, 42].

Definition

From a categorical viewpoint, groupoids are small categories in which every morphism is an isomorphism. A **Lie groupoid** $G \rightrightarrows M$ is formed by a pair of manifolds G, M, two surjective submersions $s, t : G \to M$ called source and target maps, and a smooth associative multiplication $m : G^{(2)} \to G$ admitting units $u : M \to G$ and inverses $i : G \to G$. The notation $G^{(2)}$ means the pairs (g, h) in $G \times G$ with $t(h) = s(g)$. Because s and t are submersions $G^{(2)}$ is a submanifold of $G \times G$. The associative, unit and inverse conditions in the definition are explicitly described as follows:

(associativity) if $t(hf) = s(g)$ or $s(gh) = t(f)$, then $g(hf) = (gh)f$;

(unity) if $t(g) = y$ and $s(g) = x$, then $e_y g = g$ and $g e_x = g$;

(inverse) if $g \in G$, then $gg^{-1} = e_{s(g^{-1})}$ and $g^{-1}g = e_{s(g)}$.

We adopt the following notations: $m(g, h) = gh$, $u(x) = e_x$ and $i(g) = g^{-1}$. Unless otherwise stated, all our manifolds are second countable and Hausdorff. An exception to this convention is the total space of a Lie groupoid G which is allowed to be non-Hausdorff. When necessary we reserve the term **Hausdorff groupoid** to stress the fact that G is Hausdorff. A groupoid $G \rightrightarrows M$ is called a **proper** groupoid if the map $s \times t : G \to M \times M$ is a proper map. If G, M have the same dimension, we call $G \rightrightarrows M$ an **étale groupoid**.

Properties and notation

The points of M are called **objects**; we call the elements of G **arrows**, and the elements of $G^{(2)}$ are the **composable arrows**. We denote an arrow g in G by $y \xleftarrow{g} x$, where $x = s(g)$, $y = t(g)$ are its **source** and **target**. For a point x in M, the submanifolds $G(-, x) = s^{-1}(x)$ and $G(x, -) = t^{-1}(x)$ of G are called the **source fiber** and the **target fiber** of x respectively, or s-fiber and t-fiber.

The set $G_x := s^{-1}(x) \cap t^{-1}(x)$, together with the restriction of the groupoid multiplication, is a group called the **isotropy group** at x. Showing that the map $t : G(-, x) \to M$ has constant rank we conclude that the fibers $G(y, x) = t^{-1}(x) \cap s^{-1}(x)$ are embedded submanifolds (see [42, Thm. 5.4]). In particular, G_x becomes a Lie group with the restricted multiplication.

The groupoid structure defines an equivalence relation on M such that two points x and y are related if there is an arrow $g \in G$ with $y \xleftarrow{g} x$. The equivalence classes are called the **orbits**. For a point $x \in M$ its orbit is denoted by $\mathcal{O}_x$, noticing that $\mathcal{O}_x = t(s^{-1}(x))$. The isotropy group G_x acts (on the right) freely and properly on $G(-,x)$. The map $t : G(-,x) \to M$ is G_x invariant and its fibers coincide with the orbits of $G(-,x) \curvearrowleft G_x$, so we can identify the quotient $G(-,x)/G_x$ with the orbit $\mathcal{O}_x \subset M$, and regard it as an immersed submanifold. The **coarse orbit space** M/G is the space of orbits with the quotient topology.

The connected components of the orbits are the leaves of a singular foliation $\mathcal{F}_M$ on M, called the **characteristic foliation** (see [6]). Also, there are singular foliations $\mathcal{F}_G = s^*\mathcal{F}_M$ and $\mathcal{F}_{G^{(2)}} = m^*\mathcal{F}_G$ on G and $G^{(2)}$, respectively. Recall that, by a **singular foliation** in the sense of Stefan-Sussmann (cf.[50, 51]) on M, we mean a partition $\{L_x\}$ of M into immersed submanifolds such that for each point $x \in M$, there exists a local chart ϕ on M around x with the following properties:

- ϕ is a diffeomorphism $\phi : U \to V \times W$, where V and W are neighborhoods of the origin in the euclidean space;
- $\phi(x) = (0, 0)$;
- if $L \in \mathcal{F}$, then $\phi(L \cap U) = U \times W_L$ where $W_L = \{w \in W : (0, w) \in \phi(L)\}$.

A chart (U, ϕ) which fulfills the above conditions is called a **foliated chart** around x.

The **normal space** at a point of $M, G, G^{(2)}$ is the normal space to the leaf of respectively $\mathcal{F}_M, \mathcal{F}_G, \mathcal{F}_{G^{(2)}}$ at this point. For an orbit $\mathcal{O} \subset M$, denote $G_{\mathcal{O}} = s^{-1}(\mathcal{O})$ and $G^{(2)}_{\mathcal{O}} = m^{-1}(G_{\mathcal{O}})$. Thus, the normal spaces will be denoted as $N_x\mathcal{O} = T_xM/T_x\mathcal{O}$, $N_gG_{\mathcal{O}} = T_gG/T_gG_{\mathcal{O}}$ and $N_gG^{(2)}_{\mathcal{O}} = T_{(g,h)}G^{(2)}/T_{(g,h)}G^{(2)}_{\mathcal{O}}$.

The **normal representation** of the isotropy group G_x over $N_x\mathcal{O} = T_xM/T_x\mathcal{O}$ is given by $g \cdot [v] = [d_gt(w)]$, where w is such that $d_gs(w) = v$. The normal representation is an invariant of the orbit, in the sense that if x and y are in the same orbit, then the normal representations $G_x \curvearrowright N_x\mathcal{O}$ and $G_y \curvearrowright N_y\mathcal{O}$ are isomorphic. Indeed, fix an arrow $y \xleftarrow{g} x$. Since $T_gG_{\mathcal{O}} = ds^{-1}(T_x\mathcal{O}) = dt^{-1}(T_y\mathcal{O})$, the following two vertical maps are isomorphisms:

$$\begin{array}{ccccc} & & N_gG\mathcal{O} & & \\ & \overline{dt}\swarrow & & \searrow\overline{ds} & \\ N_y\mathcal{O} & \xleftarrow{\quad\quad g\quad\quad} & & & N_x\mathcal{O}, \end{array} \tag{1.1}$$

thus the isomorphism $g = \overline{dt} \circ \overline{ds}^{-1}$ together with the left multiplication $g : G_x \to G_y$ give an isomorphism of the normal representations.

Examples

Example 1.1.1. A **Lie group** G seen as Lie groupoid is a groupoid which has just one object $G \rightrightarrows \{*\}$. Thus, the source and target map are trivial, the multiplication and inversion are given by the Lie group structure.

Example 1.1.2. The **unit groupoid** associated to manifold M is the groupoid $M \rightrightarrows M$ which has only unit arrows, where the five structural maps are the identity id_M.

Example 1.1.3. Let M be a manifold. Given a **equivalence** relation $R \subset M \times M$ such that $\pi_1 : R \to M$ and $\pi_2 : R \to M$, the projections on the first and second factors are submersions, then $R \rightrightarrows M$ is a Lie groupoid with trivial isotropy and the orbits are the same as the equivalence classes.

Example 1.1.4. A surjective submersion $p : M \to B$ gives rise to a **submersion groupoid**, where $G = M \times_B M$, source and target are the projections, and the multiplication is $(z, y) \cdot (y, x) = (z, x)$. The orbits are the fibers of p, the isotropies are trivial, and the coarse orbit space identifies with B.

The submersion groupoid arising from the identity id_M is the unit groupoid, the one arising from the projection $M \to \{*\}$ is the **pair groupoid** $M \times M \rightrightarrows M$, and the one arising from the inclusions $\coprod U_i \to M$ of an open cover is the **Cech groupoid**

$$G_{\mathcal{U}} = \left(\coprod U_j \cap U_i \rightrightarrows \coprod U_i\right).$$

Example 1.1.5. A smooth action $K \curvearrowright M$ of a Lie group on a manifold gives rise to an **action groupoid** $K \ltimes M \rightrightarrows M$, with $s(k, x) = x$, $t(k, x) = k \cdot x$ and $(l, y) \cdot (k, x) = (lk, x)$. Notice that orbits and isotropy groups of $K \ltimes M$ coincide with the usual notions of orbits and isotropy groups of the action.

Example 1.1.6. Let G be a Lie group and $p : P \to B$ a G-principal bundle. We can recover $p : P \to B$ as the quotient projection $P \to P/G$ of a free and proper action $G \curvearrowright P$. The **gauge groupoid** $\mathrm{Gauge(P)} \rightrightarrows B$ is the quotient of the pair groupoid $P \times P \rightrightarrows P$ by the action of G. Clearly gauge groupoids are **transitive**, in the sense that they have only one orbit. Conversely, if we start with a transitive groupoid $G \rightrightarrows M$ we recover it as the gauge groupoid associated to the G_x-principal bundle $t : G(-, x) \to \mathcal{O}_x = M$, for any $x \in M$.

Example 1.1.7. If $E \to M$ is a vector bundle, there is an associated **general linear groupoid**, denoted by $GL(E)$, which is similar to the general linear group associated to a vector space. The objects of $GL(E)$ are the points of M and the arrows between two points x and y in M consist of linear isomorphisms $E_y \leftarrow E_x$. We can recover $GL(E) \rightrightarrows M$ as the gauge groupoid associated to the frame bundle $FE \to M$ of $E \to M$. If the vector bundle $E \to M$ carries a metric we can talk about the orthonormal linear groupoid $O(E) \rightrightarrows M$ in the same fashion.

Example 1.1.8. Let be M a smooth manifold. The arrows of the **fundamental groupoid** $\Pi_1(M) \rightrightarrows M$ consist of homotopy classes of paths with fixed end points. The source and target maps $s, t : \Pi_1(M) \to M$ are defined by $s([\gamma]) = [\gamma(0)]$ and $t([\gamma]) = [\gamma(1)]$. The multiplication of homotopy classes is given by the concatenation of representative paths. Thinking of $\widetilde{M} \to M$ the universal covering as a $\pi_1(M)$ principal bundle, the fundamental groupoid is the gauge groupoid associated to it.

For $x \in M$, we recover the **fundamental group** $\pi_1(M, x)$ exactly as the isotropy group $\Pi(M)_x$. The fundamental groupoid does not require the choice of a point to be defined, so it still makes sense for non-path-connected spaces. Thus, the groupoid approach is a way to glue all these pieces of data into a single object.

Example 1.1.9. A regular foliation $\mathcal{F}$ on a manifold M gives rise to a **monodromy groupoid** $\Pi_1(\mathcal{F}) \rightrightarrows M$, whose arrows are the leafwise homotopy classes of paths. Its orbits are exactly the leaves of $\mathcal{F}$ and the isotropy groups are their fundamental groups. Each arrow $[\gamma] \in \Pi_1(\mathcal{F})$ induces the germ of a transverse diffeomorphism, the holonomy of the path, and the quotient of $\Pi_1(\mathcal{F}) \rightrightarrows M$ by holonomy classes is still a Lie groupoid, the **holonomy groupoid** $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ (see [42]). The characteristic foliations of the monodromy and holonomy groupoids are the original foliation.

Generalizing the previous example, we can think of Lie groupoids as a way to present singular foliations and to perform differential and Riemannian geometry in their leaf spaces. In contrast with regular foliations, it may be unclear how to represent a given singular foliation by a Lie groupoid. An approach to holonomy groupoids of singular foliations (which are not necessarily Lie) can be found in [6]. For singular Riemannian foliations, see [4] for progress in identifying underlying Lie groupoids.

Groupoid maps

A **groupoid map** (or just a map) $\phi : (H \rightrightarrows N) \to (G \rightrightarrows M)$ is a pair of smooth maps $\phi^{(1)} : G \to H$ and $\phi^{(0)} : M \to N$ such that $s \circ \phi^{(1)} = \phi^{(0)} \circ s$, $t \circ \phi^{(1)} = \phi^{(0)} \circ t$, and $\phi^{(1)}(gh) = \phi^{(1)}(g)\phi^{(1)}(h)$ for all $(g,h) \in G^{(2)}$. This induces a map $\phi^{(2)} : G^{(2)} \to H^{(2)}$ on the composable arrows. We will denote by ϕ both $\phi^{(0)}$, $\phi^{(1)}$ and $\phi^{(2)}$ when there is no risk of confusion. If ϕ is invertible we call it a groupoid **isomorphism**.

The condition that the maps commute with the source and target implies that groupoid maps send orbits to orbits. Given a groupoid map $\phi : G \to H$, we denote by $\overline{\phi} : M/G \to H/N$ the continuous map induced by ϕ.

A groupoid map yields Lie group morphisms between the isotropy groups $\phi_x : H_x \to G_{\phi(x)}$, and since it must send orbits to orbits, it also yields linear maps $\overline{d_x\phi} : N_x\mathcal{O} \to N_{\phi(x)}\mathcal{O}$. This gives rise to a morphism $\phi_x : G_x \curvearrowright N_x\mathcal{O} \to H_{\phi(x)} \curvearrowright N_{\phi(x)}\mathcal{O}$ between the normal representations.

When we see groupoids as categories, a groupoid map is by definition a smooth functor, so it makes sense to talk about natural isomorphism of groupoid maps. A **natural isomorphism** between two groupoid maps $\phi \overset{\gamma}{\Rightarrow} \psi$ is a smooth map $\gamma : N \to G$ with $s \circ \gamma = \phi$ and $t \circ \gamma = \psi$ and satisfying $\psi(g)\gamma_x = \gamma_y\phi(g)$ for all $y \overset{g}{\leftarrow} x$. If ϕ, ψ are isomorphic then $\bar{\phi} = \bar{\psi} : N/H \to M/G$ and the maps $\phi_x, \overline{d_x\phi}$ are related to $\psi_x, \overline{d_x\psi}$ by conjugation by γ_x.

Examples of maps

Example 1.1.10. Lie group homomorphisms and smooth maps between manifolds are standard examples of Lie groupoid maps.

Example 1.1.11. Let $K \curvearrowright M$, $L \curvearrowright N$ be Lie group actions on manifolds. If $\varphi : M \to N$ is a smooth action and $\lambda : K \to L$ a Lie group homomorphism, then $\phi : K \ltimes M \to L \ltimes N$ given by $\phi(k,x) = (\lambda(k), \varphi(x))$ is a Lie groupoid map. If the groups are discrete and the manifolds are connected, then any groupoid map $K \ltimes M \to L \ltimes N$ has this form (see [13, Lem. 3.2]).

Example 1.1.12. The fundamental groupoid can be thought of as a functor from smooth manifolds to Lie groupoids, in the same fashion as the fundamental group. A smooth map $f : M \to N$ defines a groupoid map

$$f_* : \Pi_1(M) \to \Pi_1(N) \quad \text{by} \quad f_*([\gamma]) = [f \circ \gamma].$$

This construction extends to foliated manifolds: given a foliated map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$, every leafwise path in M is sent to a leafwise path in M', and this defines a groupoid map

$$f_* : \Pi_1(\mathcal{F}) \to \Pi_1(\mathcal{F}').$$

Example 1.1.13. The holonomy of paths inside leaves is constant in the homotopy classes, so it can be viewed as a groupoid map $\Pi_1(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F})$ which covers the identity on M.

Example 1.1.14. Let $(E \to M, \nabla)$ be a vector bundle with a flat connection. The flat connection ∇ defines a parallel transport along the curves in M which depends only on the curve homotopy class, so each homotopy class $[\gamma]$ gives rise to a linear isomorphism $E_{\gamma(1)} \leftarrow E_{\gamma(0)}$. Hence, the parallel transport defines a groupoid map

$$\Pi_1(M) \to GL(E).$$

Actions and representations

Let $G \rightrightarrows M$ be a Lie groupoid and $\mu : P \to M$ a smooth map. A (left) **Lie groupoid action** of G over P with moment map μ is a map

$$\theta : G \times_M P \to P$$

such that the following action identities are satisfied:

i) $\mu(gp) = t$;

ii) $\theta(g, \theta(h, p)) = \theta(gh, p)$, for all $(g, h) \in G^{(2)}$ and $(h, p) \in G \times_M P$;

iii) $\theta(e_{\mu(p)}, p) = p$, for all $p \in P$.

An action θ realizes the arrows of the groupoid $G \rightrightarrows M$ as symmetries of the fibers of the moment map, which means that to each arrow $y \xleftarrow{g} x$ we have a diffeomorphism $\theta_g : P_x \to P_y$. Associated to the action we can construct the action groupoid $G \times_M P \rightrightarrows P$, where the source is the projection, the target is the action, the multiplication is the induced by those on G.

If $\mu : P \to M$ is a vector bundle and each θ_g is a linear isomorphism, then we call it a **representation**.

Example 1.1.15. Parallel transport of flat connections are simple examples of groupoid representations.

Example 1.1.16. Let $G \rightrightarrows M$ be a Lie groupoid and consider an orbit $\mathcal{O} \subset M$. Given $x \in M$, we have defined the normal representation $G_x \curvearrowright N_x\mathcal{O}$. Note that this is encoded in the groupoid representation of the restriction groupoid $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$ over the normal bundle $N\mathcal{O} \to \mathcal{O}$, where for each arrow $y \overset{g}{\leftarrow} x$ in $G_{\mathcal{O}}$ the isomorphism $g : N_x\mathcal{O} \to N_y\mathcal{O}$ is the one built from the diagram (1.1). We will denote the action groupoid from $G\mathcal{O} \curvearrowright N\mathcal{O}$ by $G\mathcal{O} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O}$.

1.2 Differentiable stacks

The stack concept is a generalization of the notion of "space". These objects permit the study of the singular behavior of geometric and algebraic structures, such as those present in the theory of moduli spaces. The references used as a base to differentiable stacks in this chapter are [8, 12, 20].

We use the point of view that differentiable stacks can be represented by Morita equivalence classes of Lie groupoids, avoiding the technicalities from the categorical point of view. For more details about the equivalence between groupoids modulo Morita equivalence and differentiable stacks, see [8, Sec. 2] and [54, Thm. 1.3.27].

Morita maps

A Lie groupoid map $\phi : (H \rightrightarrows N) \to (G \rightrightarrows M)$ is a **Morita map** if the following conditions hold:

- $\bar{\phi} : N/H \to M/G$ is an homeomorphism;
- $\phi_x : H_x \curvearrowright N_x\mathcal{O} \to G_{\phi(x)} \curvearrowright N_{\phi(y)}\mathcal{O}$ is an isomorphism of representations for all $x \in M$.

For instance, if N/H and M/G are smooth manifolds the map $\bar{\phi}$ is smooth and ϕ be Morita is exactly the condition to $\bar{\phi}$ be a diffeomorphism.

The above definition is a more geometric formulation for Morita maps provided in [20, Thm. 4.3.1]. The classical approach to Morita maps is in terms of equivalence functors, re-writing the fully faithful and essentially surjective conditions in the smooth setup. Recall that a Lie groupoid map $\phi : H \to G$ is **fully faithful** if the following diagram is a fiber product

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ {\scriptstyle t\times s}\downarrow & & \downarrow{\scriptstyle t\times s} \\ N \times N & \xrightarrow{\phi\times\phi} & M \times M, \end{array}$$

and it is **essentially surjective** if the map $t\mathrm{pr}_1 : G \times_M N \to M$ given by $(y \xleftarrow{g} \phi(x), x) \mapsto y$ is a surjective submersion; it is a Morita map if it is both essentially surjective and fully faithful.

Remark 1.2.1. [20, Cor. 4.3.2] It directly follows from the definition that:

- the composition of two Morita maps is again a Morita map;
- if two maps are isomorphic and one of them is Morita, then so is the other.

Examples of Morita maps

Example 1.2.2. A Morita map between Lie groups is the same as a Lie group isomorphism.

Example 1.2.3. Let $\{(U_i, \phi_i)\}$ be an atlas for a smooth manifold M. The inclusion maps $U_i \to M$ give rise to a groupoid map

$$G_{\mathcal{U}} = \left(\coprod U_j \cap U_i \rightrightarrows \coprod U_i\right) \xrightarrow{\phi} (M \rightrightarrows M)$$

between the Cech groupoid and the unit groupoid. Clearly,

$$\phi_x : \{e_x\} \curvearrowright T_x U_i \to \{e_x\} \curvearrowright T_x M$$

is an isomorphism of representations and the map between the quotients is a homeomorphism. Hence, ϕ is a Morita map.

Example 1.2.4. Let $G \rightrightarrows M$ be a transitive groupoid. Fix a point $x \in M$. The inclusion $(G_x \rightrightarrows \{x\}) \to (G \rightrightarrows M)$ is Morita map, since the orbit spaces have just one orbit and $G_x \curvearrowright \{0\} \to G_x \curvearrowright \{0\}$ is clearly an isomorphism of trivial representations.

Example 1.2.5. Let $\mathcal{O}$ be an orbit of a Lie groupoid $G \rightrightarrows M$. For any x in $\mathcal{O}$ the inclusion:

$$(G_x \ltimes N_x\mathcal{O} \rightrightarrows N_x\mathcal{O}) \to (G\mathcal{O} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O})$$

is a Morita map between the normal representation $G_x \curvearrowright N_x\mathcal{O}$ and the normal representation groupoid $G\mathcal{O} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O}$. To see that, we check that the inclusion is fully faithful and essentially surjective. The fiber product

$$(G\mathcal{O} \ltimes N\mathcal{O}) \times_{N_x\mathcal{O} \times N_x\mathcal{O}} (N\mathcal{O} \times N\mathcal{O})$$

is giving by the points in $G\mathcal{O} \ltimes N\mathcal{O} \times N\mathcal{O} \times N\mathcal{O}$ of the form $(x \xleftarrow{g} x, v, v, v)$ with v in $N_x\mathcal{O}$. So,

$$G_x \ltimes N_x\mathcal{O} \cong (G\mathcal{O} \ltimes N\mathcal{O}) \times_{N_x\mathcal{O}\times N_x\mathcal{O}} (N\mathcal{O} \times N\mathcal{O})$$

showing that the inclusion is fully faithful. The fiber product $G\mathcal{O} \times N\mathcal{O} \times_{N\mathcal{O}} N_x\mathcal{O}$ is formed by the points of the form $(y \xleftarrow{g} x, v, v)$ with v in $N_x\mathcal{O}$, so the map $t \circ \mathrm{pr}_1 : G\mathcal{O} \times N\mathcal{O} \times_{N\mathcal{O}} N_x\mathcal{O} \to N\mathcal{O}$ is a surjective submersion.

Example 1.2.6. Let $(M, \mathcal{F})$ be a foliated manifold. Recall that a complete transversal is an immersed (not necessarily connected) submanifold $W \subset M$ with dimension equal to the codimension of F, which is transversal to the leaves of $\mathcal{F}$ and intersects any leaf in at least one point. For instance, one can take W to be the union of a countable disjoint family of (local) transversal sections. We can obtain an étale groupoid by restricting the holonomy groupoid $\mathrm{Hol}(\mathcal{F})$ to a complete transversal W. The inclusion of $\mathrm{Hol}(\mathcal{F})_W$ into $\mathrm{Hol}(\mathcal{F})$ induces a map

$$\pi : W/\mathrm{Hol}(\mathcal{F})_W \to M/\mathcal{F},$$

which is a bijection by the definition of W, and it is an open map because the saturation of a transversal section is an open set, thus π is a homeomorphism. The normal representation

$$\mathrm{Hol}(\mathcal{F})_x \curvearrowright N_x\mathcal{F}$$

represents the linear holonomy at x in M, i.e., the first jet of germs of diffeomorphisms on transversal sections. From this we see that $\mathrm{Hol}(\mathcal{F})_x \curvearrowright N_x\mathcal{F}$ is isomorphic to $(\mathrm{Hol}(\mathcal{F})_W)_x \curvearrowright T_xW$. Hence, the inclusion $\mathrm{Hol}(\mathcal{F})_W$ into $\mathrm{Hol}(\mathcal{F})$ is a Morita map. A groupoid which is equivalent to a étale groupoid is known as **foliation groupoid**.

Morita equivalence

We would like to think of Morita maps as a notion of isomorphism, but such maps are not necessarily invertible. A way to circumvent this problem is by using "fractions" to formally invert them.

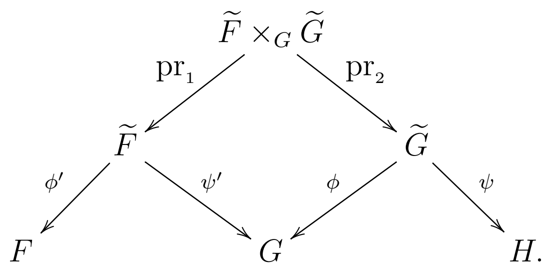
Given $G \rightrightarrows M$ and $H \rightrightarrows N$ Lie groupoids, a **fraction** $\psi/\phi : G \dashrightarrow H$ is given by a pair of maps

$$\begin{array}{ccccc} & & \widetilde{G} \rightrightarrows \widetilde{M} & & \\ & \overset{\phi}{\swarrow} & & \overset{\psi}{\searrow} & \\ G \rightrightarrows M & & & & H \rightrightarrows N \end{array}$$

where ϕ is a Morita map.

Two Lie groupoids $G \rightrightarrows M$ and $H \rightrightarrows N$ are **Morita equivalent** if there is a fraction $\psi/\phi : H \dashrightarrow G$, where both $\phi : \widetilde{G} \to H$ and $\psi : \widetilde{G} \to G$ are Morita maps.

Morita equivalence is, in fact, an equivalence relation. Reflexivity and symmetry follow from the definition. For transitivity, we need to use homotopy fiber products. More precisely, given Lie groupoids $F \rightrightarrows L$, $G \rightrightarrows M$, $H \rightrightarrows N$ and Morita equivalences $\psi'/\phi' : F \dashrightarrow G$ and $\psi/\phi : G \dashrightarrow H$, we consider the groupoid homotopy fiber product (see [20, 42]):



The fraction $(\phi' \circ \mathrm{pr}_1)/(\psi \circ \mathrm{pr}_2)$ is then a Morita equivalence between $F \rightrightarrows L$ and $H \rightrightarrows N$.

Stacks

A **differentiable stack** is a Lie groupoid up to Morita equivalence. We write $[M/G]$ for the differentiable stack presented by the Lie groupoid $G \rightrightarrows M$.

Differentiable stacks are enhanced topological spaces on which it makes sense to perform differential geometry. For a Lie groupoid $G \rightrightarrows M$, the normal representation $G_x \curvearrowright N_x\mathcal{O}$ is a model for the tangent space of $[M/G]$ at $\bar{x} \in M/G$, see [26] for more details about the tangent stack $[TM/TG]$. We define the **coarse tangent space** $T_{\bar{x}}[M/G]$ at $\bar{x}$ as the coarse orbit space of the normal representation $G_x \ltimes N_x\mathcal{O} \rightrightarrows N_x\mathcal{O}$. This is well-defined, in the sense that two points in the same orbit have isomorphic normal representations and Morita equivalences preserve the normal representations.

A stack $[M/G]$ is **separated** if it is presented by a proper Lie groupoid $G \rightrightarrows M$. This notion is well-defined, since properness is a property preserved by Morita maps (see [20, Prop. 5.1.3]). Since proper groupoids are linearizable (see Thm 3.2.1), separated stacks can be locally recovered from their normal representations, see Corollary 3.2.2.

Examples

Example 1.2.7. Smooth manifolds can be identified with separated differentiable stacks that have no isotropy. In fact, a Lie groupoid $G \rightrightarrows M$ is Morita equivalent to the unit groupoid of a manifold $N \rightrightarrows N$ if and only if $G \rightrightarrows M$ is a submersion groupoid arising from some submersion $M \to N$. This identification is functorial.

Example 1.2.8. Orbifolds were first introduced as Hausdorff spaces locally modeled by the orbit space of a finite group acting effectively on Euclidean space (see [49]). In order to define suborbifolds and orbifold maps, it is convenient not to force the local actions to be effective. From a modern perspective, we can define orbifolds as separated differentiable stacks with finite isotropy groups, i.e., stacks that can be presented by proper étale groupoids. Details on the correspondence between the classic and new approaches can be found in [1, 39, 42].

Example 1.2.9. Given K a Lie group, the differentiable stack $[*/K]$ associated to the groupoid $K \rightrightarrows *$ is called the *classifying stack* for K. This is a finite-dimensional stacky model for the usual infinite-dimensional classifying space BK; in fact, they are equivalent from a homotopy-theoretic point of view.

Example 1.2.10. Given $K \curvearrowright M$ a Lie group acting on a manifold, the differentiable stack $[M/K]$ arising from the action groupoid encodes the equivariant geometry of the action. When the action is free and proper this is just the quotient manifold. In general, $[M/K]$ can be used to compute the equivariant cohomology, see [7, p. 263]. If K is discrete and $H^1(M) = 0$, then $[M/K]$ captures all the "dynamical data" up to conjugation [13].

Example 1.2.11. The leaf space $M/\mathcal{F}$ of a foliated manifold $(M, \mathcal{F})$ can be realized as a stack $[M/\mathcal{F}]$ presented by the holonomy groupoid

$$\mathrm{Hol}(\mathcal{F}) \rightrightarrows M.$$

We can simplify this representative to a étale one, by restricting the holonomy groupoid to a complete transversal $W \subset M$ of the foliation, namely

$$\mathrm{Hol}(\mathcal{F})|_W \rightrightarrows W,$$

see Example 1.2.6). The bisections of $\mathrm{Hol}(\mathcal{F})_W$ give rise to a pseudo-group of diffeomorphisms on W. Different choices of transversals give Morita equivalent étale groupoids, and Morita equivalent étale groupoids produce equivalent pseudo-groups. Seeing $M/\mathcal{F}$ as a stack we recover the approach of its differential geometry by pseudo-groups. For more details about the pseudo-groups and its relation with foliation groupoids see [43, Appendix D] and [42].

Stacky maps

When representing stacks by Lie groupoids, isomorphisms of stacks correspond to Morita equivalences. More general maps between stacks are described as follows. Two fractions ψ_1/ϕ_1, ψ_2/ϕ_2 are **equivalent** if there are Morita maps α_1, α_2, such that $\psi_1 \circ \alpha_1$ is isomorphic to $\psi_2 \circ \alpha_2$, and $\phi_1 \circ \alpha_1$ is isomorphic to $\phi_2 \circ \alpha_2$. This can be visualized in the following diagram:

$$
\begin{array}{ccccc}
 & & \widetilde{G}_1 \rightrightarrows \widetilde{M}_1 & & \\
 & \swarrow^{\phi_1}_{\sim} & \wr\uparrow \alpha_1 & \searrow^{\psi_1} & \\
G \rightrightarrows M & \Downarrow & \widetilde{G}_3 \rightrightarrows \widetilde{M}_3 & \Downarrow & H \rightrightarrows N \\
 & \nwarrow^{\sim}_{\phi_2} & \wr\downarrow \alpha_2 & \nearrow_{\psi_2} & \\
 & & \widetilde{G}_2 \rightrightarrows \widetilde{M}_2. & &
\end{array}
$$

A class of fraction between two groupoids is called a **generalized map** between them.

A **stacky map** $[\phi/\psi] : [N/H] \to [M/G]$ of differentiable stacks is the same as a generalized map for us. The identity is id/id and composition can be defined by using homotopy fiber products of Lie groupoids (see [20, 42]). This easily follows from the definition of Morita maps that a fraction ψ/ϕ is invertible if and only if ψ is Morita as well.

Maps of stacks admit a **cocycle description**. Given $G \rightrightarrows M$ a Lie groupoid and $\mathcal{U} = \{U_i\}$ an open cover of M, a new groupoid can be defined,

$$G_{\mathcal{U}} = \left(\coprod_{j,i} G(U_j, U_i) \rightrightarrows \coprod_i U_i \right)$$

with arrows $(y, j) \xleftarrow{(g,j,i)} (x, i)$ for $x \in U_i$, $y \in U_j$ and $y \xleftarrow{g} x$ is an arrow in G. The composition is set by $(h, k, j)(g, j, i) = (hg, k, i)$. The obvious projection

$\phi_{\mathcal{U}} : G_{\mathcal{U}} \to G$ is Morita, and it can be proven (see [20, Prop. 4.5.4]) that every map of stacks can be expressed as a cocycle fraction $\psi/\phi_{\mathcal{U}}$ for some open cover $\mathcal{U}$ of N.

Examples of stacky maps

Example 1.2.12. We have seen in Example 1.2.7 that there is a natural map from the Cech groupoid of an atlas to the manifold, so generalized maps for manifolds can be interpreted as smooth maps given by charts.

Example 1.2.13. Let M be a manifold and G a Lie group. We saw that generalized maps admit a cocycle description. Then a generalized map $f : [M/M] \to [G/\{*\}]$ can be seen as a fraction $\psi/\phi_{\mathcal{U}}$, where ψ is a groupoid map from the Cech groupoid $\coprod_{j,i} U_j \cap U_i \rightrightarrows \coprod_i U_i$ to G. But a map between the Cech groupoid and a Lie group is exactly a cocycle $\psi_{ji} : U_j \cap U_i \to G$. Hence, generalized maps between manifolds and Lie groups are the same as isomorphism classes of principal bundles.

Example 1.2.14. Let $p : M \to B$ be a submersion. The submersion groupoid $M \times_B M \rightrightarrows M$ is an atlas for B thought of as a stack. If $\gamma : I \to B$ is a smooth curve, there is an open covering $\{U_i\}$ of I such that in U_i we can lift $\gamma|_{U_i}$ to M along p, say to a curve $\gamma_i : U_i \to M$. If $x \in U_j \cap U_i$, then $p(\gamma_j(x)) = p(\gamma_i(x))$, so the local lifts define a groupoid map

$$\widetilde{\gamma} : \left(\coprod_{j,i} U_j \cap U_i \rightrightarrows \coprod_i U_i \right) \to (M \times_B M \rightrightarrows M).$$

Thus generalized maps from the open interval unit groupoid $I \rightrightarrows I$ to $M \times_B M \rightrightarrows M$ are exactly the smooth curves on B.

Coarse differential and coarse map

The stacky maps have well-defined maps at the level of coarse quotient space and normal representation. The Morita maps preserve the normal representation and the coarse quotient space, and isomorphic groupoid maps have the same induced maps on the quotient space and on the normal representations.

Let $f : [M/G] \to [N/H]$ be a stacky map, and ψ/ϕ a fraction presenting f. We define the **coarse map** of f as

$$\overline{f} := \overline{\psi} \circ \overline{\phi}^{-1} : M/G \to N/H,$$

and the **coarse differential** of f at $\bar{x} \in M/G$ by

$$\overline{d_{\bar{x}} f} := \overline{d_z \psi} \circ \overline{d_x \phi}^{-1} : T_{\bar{x}}[M/G] \to T_{\bar{y}}[N/H],$$

where $\psi(z) = x$ and $\phi(z) = y$. Because the maps $\overline{\phi}$, $\overline{\psi}$, $\overline{d_x \phi}$, $\overline{d_z \psi}$ are invariant under isomorphisms of groupoid maps, then we conclude that the definitions of $\overline{f}$ and $\overline{d_{\bar{x}} f}$ do not depend on the fraction which represents f.

Remark 1.2.15. Given M a manifold and K a Lie group, stacky maps

$$M \to [*/K]$$

into the classifying stack $BK = [*/K]$ are in 1-1 correspondence with isomorphism classes of principal K-bundles over M. When representing stacky maps as fractions $\psi/\phi_{\mathcal{U}}$ this gives the usual characterization of principal bundles by cocycles. This correspondence between maps and bundles also makes sense when replacing both the base manifold and the structure group by arbitrary Lie groupoids. The resulting approach to stacky maps via groupoid bundles is followed by several authors [12, 20, 34, 45, 61].

Chapter 2

Riemannian groupoids and Stacks

In this chapter, we give a short review of the definitions and properties of Riemannian groupoids, Morita fibrations, and how to define a notion of metric on stacks. We introduce the length of a stacky curve and explore its relations with the natural notion distance on the coarse orbit space. The chapter is organized as follows:

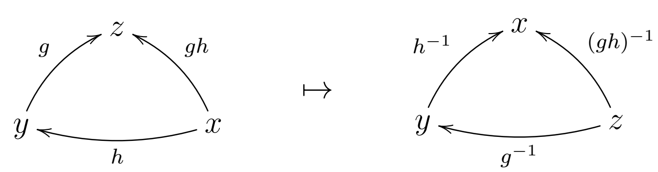
- In Section 2.1 we set our notation and recall the basic definitions and properties of Riemannian groupoids.
- In Section 2.2 we review the pullback and pushforward of groupoid metrics along Morita fibrations, leading us to Riemannian stacks. We also study some Morita invariants such as the pseudo-distance on the coarse orbit space and normal representations by isometries.
- In Section 2.3 we define the normal length of a stacky curve. As part of this definition, we show that the speed of a stacky curve depends continuously on the curve parameter. We recover the normal pseudo-distance between two points as the infimum over all the normal length of stacky curves connecting the points, and conclude that the coarse orbit space together with the normal distance is a Riemannian stack invariant.

2.1 Riemannian groupoids

The notion of metric on Lie groupoids, suitably compatible with the multiplication, was introduced in [24], extending several previous attempts to establish such a concept [28, 33, 40]. It is shown that many families of Lie groupoids admit such metrics, including the important class of proper Lie groupoids. The exponential map of these metrics is the key tool to prove a linearization theorem for Riemannian groupoids in [24], which provides a simpler proof and a stronger version of the Weinstein-Zung linearization theorem for proper Lie groupoids [19, 58, 62].

Definition

Given a Lie groupoid $G \rightrightarrows M$, the space of pairs of composable arrows $G^{(2)} = G \times_M G$ can be identified with the space of commutative triangles whose vertices are points of M and the edges are the arrows of G. So $G^{(2)}$ carries a natural S_3-action by permuting the vertices. For instance, in the diagram below, the permutation (13)(2) acts as follows:



(2.1)

In order to define groupoid metrics, we need a preliminary notion of metrics fibered along submersions. Let (M, η) be a Riemannian manifold and $p : M \to B$ a submersion. For x in M denote

$$T_xM/\ker dp_x \quad \text{by} \quad N_xM, \quad \text{and} \quad p^{-1}(p(x)) \quad \text{by} \quad F_x.$$

Using the metric η we identify N_xM with $T_xF^\perp$. Given $x, y \in M$ belonging to the same fiber $p^{-1}(b)$, we have isomorphisms:

$$N_xM \xrightarrow{dp} T_bB \xleftarrow{dp} N_yM.$$

We call the resulting composition

$$\tau_{yx} = (dp_y)^{-1} \circ (dp_x)$$

by **normal transportation** from x to y. If for all pairs x, y with $p(x) = p(y)$ the map

$$\tau_{yx} : (N_xM, \eta_x) \to (N_yM, \eta_y)$$

is an isometry, we say that η is **fibered** with respect to η. When η is p-fibered we can endow B with a **pushforward metric** $p_*\eta$, this is the unique metric on B which makes $N_xM \xrightarrow{dp} T_{p(x)}B$ an isometry.

A submersion $p : (M, \eta^M \to (B, \eta^B)$ between Riemannian manifolds is called a **Riemannian submersion** if η^M is p-fibered and $p_* \eta^M = \eta^B$.

Definition 2.1.1. ([24, 25]) A **groupoid metric** on $G \rightrightarrows M$ is a metric η on $G^{(2)}$ that is S_3-invariant and is fibered to the multiplication $m : G^{(2)} \to G$. We say that two groupoid metrics η_1, η_2 on a Lie groupoid $G \rightrightarrows M$ are **equivalent** if for every point $x \in M$ the induced inner products on $N_x\mathcal{O}$ coincide.

Remark 2.1.2. Groupoids can be thought of as simplicial manifolds by means of their nerves, whose k-simplices are chains of k composable arrows and whose face and degeneracy maps are induced by the multiplication and unit maps. There is a more general notion of metric for simplicial manifolds, see [24]; in this context, a groupoid metric, as defined above, is the same as a 2-metric, which is the original terminology in [24].

Properties

A groupoid metric η induces a metric $\eta^{(1)}$ on G satisfying that the maps $m, \pi_1, \pi_2 : G^{(2)} \to G$ are Riemannian submersions. This because η is m-fibered and the isometries of S_3 acting in $G^{(2)}$ interchange the maps m, π_1, π_2. The groupoid metric η on $G \rightrightarrows M$ also induces a metric $\eta^{(0)}$ on M, by showing that η is $s \circ m$-fibered, see [24, Prop 3.16].

Note that there is a permutation σ in S_3 such that $i \circ m = m \circ \sigma$ (see Diag. 2.1), then the inversion map i preserves the metric $\eta^{(1)}$. The units $u(M) \subset G$ are formed by connected components of the fixed points of the inversion map. Since the inversion map is an isometry, $u(M)$ is a totally geodesic submanifold.

Given an arrow $y \xleftarrow{g} x$, using the metrics $\eta^{(1)}$ and $\eta^{(0)}$ we identify $N_g G_{\mathcal{O}} \cong T_g G_{\mathcal{O}}^{\perp}$ and $N_x\mathcal{O} \cong T_x\mathcal{O}^{\perp}$. The resulting isomorphism $g : N_x\mathcal{O} \to N_y\mathcal{O}$ from the following diagram,

$$\begin{array}{ccc} & N_g G\mathcal{O} & \\ {\scriptstyle \overline{dt}} \swarrow & & \searrow {\scriptstyle \overline{ds}} \\ N_y\mathcal{O} & \xleftarrow{\quad g \quad} & N_x\mathcal{O}, \end{array}$$

is an isometry of vector spaces, since s and t are Riemannian submersions. The argument above gives more, namely that the normal representation $G_x \curvearrowright N_x\mathcal{O}$ is by isometries.

A **singular Riemannian foliation** $\mathcal{F}$ on a Riemannian manifold (M, η) is a singular foliation such that every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets. The characteristic foliation of a Riemannian groupoid is a singular Riemannian foliation (see [24, Prop 3.8]). To briefly justify this claim, note first that the foliations on G given by the source and target fibers are Riemannian, since they are fibers of Riemannian submersions. For every $g \in G$ we have $T_gG_{\mathcal{O}}^{\perp} = (\ker ds_g)^{\perp} \cap (\ker dt_g)^{\perp}$, so the foliation $\mathcal{F}_G$ on G is also Riemannian. If γ is a geodesic in M orthogonal to $\mathcal{F}_M = s_*(\mathcal{F}_G)$ at some point, we can locally lift γ along s to a geodesic which is orthogonal to $\mathcal{F}_G$ and conclude that γ is orthogonal to $\mathcal{F}_M$.

We say that a geodesic in G (or M) is **orthogonal** if this is orthogonal to $\mathcal{F}_G$ ($\mathcal{F}_M$) in each time.

The orthogonal geodesics satisfy a multiplicative property [24], as we will see. Let w and v be normal vectors in N_gG and N_hG satisfying $ds(w) = dt(v)$. Denote by $\alpha_{dm(w,v)}$ the geodesic in G with initial conditions gh and $dm(w, v)$. Consider in $G^{(2)}$ the geodesic $\alpha_{(w,v)}(t)$ with initial conditions (g, h) and (w, v). This is perpendicular to the fibers of m, π_1 and π_2 at $t = 0$. Since $m, \pi_1, \pi_2 : G^2 \to G$ are both Riemannian submersions, $\alpha_{(w,v)}(t)$ stays perpendicular to those fibers. We conclude that $m(\alpha_{(w,v)}(t))$, $\pi_1(\alpha_{(w,v)}(t))$ and $\pi_2(\alpha_{(w,v)}(t))$ are both geodesics in G with initial conditions $dm(w, v)$, w and v respectively. Hence, the following holds:

$$\alpha_{(w,v)}(t) = (\alpha_w(t), \alpha_v(t)), \tag{2.2}$$
$$\alpha_{dm(w,v)}(t) = m(\alpha_w(t), \alpha_v(t)). \tag{2.3}$$

In general, if $d : X \times X \to \mathbb{R}$ is a distance function and $R \subset X \times X$ is an equivalence relation, the naive function $d' : X/R \times X/R \to \mathbb{R}$, $d'(\bar{x}, \bar{y}) = \inf_{a\in\bar{x}, b\in\bar{y}} d(a, b)$ fails to be a distance, for $d'(\bar{x}, \bar{y})$ may vanish even when $\bar{x} \neq \bar{y}$, and the triangle inequality may not hold. This second issue can be fixed by considering discrete chains and defining a **quotient pseudo-**

distance $\bar{d} : X/R \times X/R \to \mathbb{R}$ as

$$\bar{d}(\bar{x}, \bar{y}) = \inf \left\{ \sum_{i=1}^{n} d(x_i, y_i) : n \in \mathbb{N}, x_1 \sim x, y_i \sim x_{i+1}, y_n \sim y \right\} \tag{2.4}$$

The previous construction defines the **normal pseudo-distance** d_N on the coarse orbit space M/G of a Riemannian groupoid $(G \rightrightarrows M, \eta)$. This d_N was studied in [47], in the more general setting of a Lie groupoid $G \rightrightarrows M$ equipped with a metric on the units that is transversely invariant. If the groupoid $G \rightrightarrows M$ is proper, then the pseudo-distance d_N is in fact a distance [47, Thm 4.1].

The following theorem provides a large family of examples of Lie groupoids which inherit a groupoid metric; this result can be thought of as a generalization of the fact that Hausdorff paracompact manifolds admit a Riemannian metric.

Theorem 2.1.3. [24, Thm.4.13] Any Hausdorff proper Lie groupoid $G \rightrightarrows M$ admits a Riemannian structure.

Examples

Example 2.1.4. Groupoid metrics on the unit groupoid $M \rightrightarrows M$ are the same thing as metrics on M. The normal space at each poitn x in M is T_xM, so two metrics in $M \rightrightarrows M$ are equivalent if only if they are equal.

Example 2.1.5. Let $p : M \to B$ be a submersion and $M \times_B M \rightrightarrows M$ be the associated submersion groupoid. Denote the projections of $M \times_B M \times_B M$ to M by p_1, p_2, p_3, and p_B the projection to B. Given metrics η^M and η^B such that p is a Riemannian submersion, the following expression defines a groupoid metric on the submersion groupoid:

$$\eta = p_1^*\eta^M + p_2^*\eta^M + p_3^*\eta^M - 2p_N^*\eta^B,$$

where $\eta^{(0)} = \eta$. This is a standard fiber product construction for Riemannian submersions. Conversely, if we start with a groupoid metric η on $M \times_B M \rightrightarrows M$ then the metric $\eta^{(0)}$ is p-fibered and makes it into a Riemannian submersion. Two groupoid metrics η, η' on $M \times_B M$ are equivalent if only if they induce the same metric on B, i.e., $p_*\eta^{(0)} = p_*\eta'^{(0)}$.

Example 2.1.6. Given $G \curvearrowright (M, \eta^M)$ an isometric action of a Lie group, a metric η can be built on the action groupoid $G \ltimes M \rightrightarrows M$ through the following recipe:

i) The submersion groupoid of $s : G \times M \to M$ is isomorphic to $G \times G \times M \rightrightarrows G \times M$, the product of the pair groupoid $G \times G \rightrightarrows G$ with the unit groupoid $M \rightrightarrows M$.

ii) The group G acts on $G \times M$, $G \times G \times M$ and $G \times G \times G \times M$ by $k \cdot (g, x) = (gk^{-1}, kx)$, $k \cdot (g, h, x) = (gk^{-1}, hk^{-1}, kx)$, $k \cdot (f, g, h, x) = (fk^{-1}, gk^{-1}, hk^{-1}, kx)$. The groupoid structure is G-invariant. Since the actions are free and proper we take the quotient groupoid $(G \times G \times M \rightrightarrows G \times M)/G$, which is isomorphic to $G \ltimes M \rightrightarrows M$. For instance, at the level of composable arrows the quotient map is $\pi : G \times G \times G \times M \to G \ltimes M^{(2)}$, $\pi(f, g, h, x) = (fg^{-1}, gh^{-1}, hx)$.

(iii) The product metric on $G \times G \times G \times M$ is a groupoid metric and G-invariant. Therefore, the quotient metric η on $(G \times G \times G \times M)/G = G \ltimes M^{(2)}$ is a groupoid metric.

Given x a point in M, the kernel of dt at (e, x) is equal to

$$\ker dt|_{(e,x)} = \{(\xi, -X_\xi(x)) : \xi \in T_eG\},$$

where X_ξ is the fundamental vector field associated to ξ. Let $\{\xi_i\}$ be an ortonormal basis for T_eG, then a straightforward calculus shows that

$$\ker dt|_{(e,x)}{}^{\perp} = \left\{ \left(\sum_i \eta^M_x \left(X_{\xi_i}(x), v\right) \xi_i, v \right) : v \in T_xM \right\}.$$

Remember that $t_\#(\eta^G \times \eta^M) = \eta^{(0)}$, so we can lift T_xM to $\ker dt|_{(e,x)}{}^{\perp}$ and obtain $\eta^{(0)}$ at x. In fact, we have the following expression

$$\eta^{(0)}_x(v, w) = \eta^M_x(v, w) + \sum_i \eta^M_x(X_{\xi_i}(x), v)\, \eta^M_x(X_{\xi_i}(x), w).$$

for all v, w in T_xM. If v in T_xM is such that $\eta^M_x(v, u) = 0$ for all u in $T_x\mathcal{O}$, then $\eta^{(0)}_x(v, w) = \eta^M_x(v, w)$ for all w in T_xM. We conclude that the metrics $\eta^{(0)}$ and η^M have the same orthogonal complement to the orbits. Moreover, they agree on the normal directions to the orbits, i.e., they are equivalent.

The above recipe is a "baby" example of the gauge trick developed in [24] to prove Theorem 2.1.3. If G is compact and we consider a bi-invariant metric on it, then the metric $\eta^{(0)}$ on M is also known as Cheeger deformation of η^M, see [2, Sec. 6.1].

Example 2.1.7. An irrational flow $\mathbb{R} \curvearrowright T^2$ is an isometric action with respect to the standard metric on T^2. So, we can give to $\mathbb{R} \ltimes T^2 \rightrightarrows T^2$ a groupoid metric. Because $T^2/\mathbb{R}$ is non-Hausdorff the pseudo-distance d_N can not be a distance.

Example 2.1.8. Let $G \rightrightarrows M$ be an étale Lie groupoid with a Riemannian structure. In this case, the source and target maps are local isometries. Thus the bisections on $G \rightrightarrows M$ for a isometry pseudo-group. Conversely, if we start with an isometry pseudo-group on a Riemannian manifold M, then the germs of this pseudo-group form an étale groupoid (see [42]). We can pullback the metric from M to the germ groupoid, turning this into a Riemannian groupoid.

Example 2.1.9. Let $\mathcal{F}$ be a foliation on M. A groupoid metric η on its holonomy groupoid induces a metric on M that makes $\mathcal{F}$ into a Riemannian foliation, as we previously observed.

In the other direction, let $\mathcal{F}$ be a Riemannian foliation for a metric η^M. In the following, we use the fact that for Riemannian foliations the holonomy and the linear holonomy coincide. The holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ acts by isometries in the transversal sections. The germs of isometries are completely determined by their differentials. So, we have an injective groupoid map

$$(\mathrm{Hol}(\mathcal{F}) \rightrightarrows M) \to (O(N\mathcal{F}) \rightrightarrows M).$$

Because the gauge groupoid $O(N\mathcal{F}) \rightrightarrows M$ is Hausdorff, then $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ is Hausdorff.

By [24, Prop.3.11] the metric η^M can be lift to a metric $\eta^{\mathrm{Hol}(\mathcal{F})}$ on $\mathrm{Hol}(\mathcal{F})$ which satisfies $t_\#\eta^{\mathrm{Hol}(\mathcal{F})} = \eta^M$ and $s_\#\eta^{\mathrm{Hol}(\mathcal{F})} = \eta^M$. From the fact that the map

$$s : (\mathrm{Hol}(F), \eta^{\mathrm{Hol}(\mathcal{F})}) \to (M, \eta^M)$$

is a Riemannian submersion, we can build a groupoid metric $\tilde{\eta}$ on the submersion groupoid

$$\mathrm{Hol}(F) \times_s \mathrm{Hol}(\mathcal{F}) \rightrightarrows \mathrm{Hol}(\mathcal{F})$$

with $\tilde{\eta}^{(0)} = \eta^{\mathrm{Hol}(\mathcal{F})}$. Supposing that the holonomy groupoid is proper, then Theorem 2.1.3 proceeds by averaging the metric $\tilde{\eta}$, producing a metric $\underline{\tilde{\eta}}$ which is fibered with respect to the fibration

$$\begin{array}{ccccc}
\mathrm{Hol}(\mathcal{F}) \times_s \mathrm{Hol}(\mathcal{F}) \times_s \mathrm{Hol}(\mathcal{F}) & \Rrightarrow & \mathrm{Hol}(\mathcal{F}) \times_s \mathrm{Hol}(\mathcal{F}) & \rightrightarrows & \mathrm{Hol}(\mathcal{F}) \\
\downarrow {\scriptstyle (h,g,f)\mapsto(hg^{-1},gf^{-1})} & & \downarrow {\scriptstyle (h,g)\mapsto(hg^{-1})} & & \downarrow {\scriptstyle (y\xleftarrow{g}x)\mapsto y} \\
\mathrm{Hol}(\mathcal{F})^{(2)} & \Rrightarrow & \mathrm{Hol}(\mathcal{F}) & \rightrightarrows & M
\end{array}$$

So, $\underline{\tilde{\eta}}$ descends to a groupoid metric η on the holonomy groupoid. The facts that $t_\#\tilde{\eta}^{(0)} = \eta^M$ and $t_\#\underline{\tilde{\eta}}^{(0)} = t_\#\tilde{\eta}^{(0)}$ implies $\eta^{(0)} = \eta^M$, see [24, Prop.4.12]. Thus, we conclude that given a Riemannian foliation with proper holonomy groupoid, then there exists a groupoid metric whose the metric on the objects is equal to the first one.

2.2 Metrics on stacks

In [25], a type of Morita invariance for metrics is established, yielding a notion of metric on stacks that extends the well-known definitions of Riemannian manifolds and orbifolds.

Metrics and Morita fibrations

A **Riemannian submersion** $\phi : (\widetilde{G} \rightrightarrows \widetilde{M}, \widetilde{\eta}) \to (G \rightrightarrows M, \eta)$ between Riemannian groupoids is a groupoid map for which the induced map $\phi^{(2)} : \widetilde{G}^{(2)} \to G^{(2)}$ is a Riemannian submersion in the usual sense. This implies that the maps $\phi^{(1)} : \widetilde{G} \to G$ and $\phi^{(0)} : \widetilde{M} \to M$ are Riemannian submersions.

A **Morita fibration** is a special case of fibration, where $\phi : (\widetilde{G} \rightrightarrows \widetilde{M}) \to (G \rightrightarrows M)$ is a Morita map and $\phi^{(0)} : \widetilde{M} \to M$ is surjective submersion, see [25] for a discussion about fibrations. If ϕ is also a Riemannian submersion we say that it is a **Riemannian Morita map**.

Riemannian Morita maps preserve normal representations together with their inner products. To see that, let $\phi : (\widetilde{G}, \widetilde{\eta}) \to (G, \eta)$ be a Riemannian Morita map. Given $y \in M$, the fiber $F = \phi^{-1}(y)$ is included in an orbit of $\widetilde{G}$. Now note that $N_x F \xrightarrow{d\phi} T_x M$ is an isometry that takes $N_x \widetilde{\mathcal{O}}$ to $N_y \mathcal{O}$.

Similar to the pullback of Riemannian metrics along submersions from the choice of a connection, it is possible to pullback groupoid metrics along Morita fibrations in the same fashion.

Proposition 2.2.1 (Prop. 6.3.1, [25])**.** If $\phi : \widetilde{G} \to G$ is a Morita fibration and η a metric on G, then there exists a metric $\widetilde{\eta}$ on $\widetilde{G}$ that makes ϕ into a Riemannian submersion.

Given a submersion $M \to B$, we know that metrics in M are not projectable to B in general. Similarly, the pushforward of a groupoid metric along Morita fibrations is not always possible. But, as we will see now, it is possible by an averaging process to obtain an equivalent metric which is "projectable" along the Morita fibration.

The kernel K of a fibration $\phi : \widetilde{G} \to G$ consists of the arrows in $\widetilde{G}$ that are mapped into identities. We can think of $G \rightrightarrows M$ as a quotient of $\widetilde{G} \rightrightarrows \widetilde{M}$ by $K \rightrightarrows \widetilde{M}$ (see [25, Prop 6.1.2]), for instance $G^{(2)}$ is the quotient of the action

$K^3 \curvearrowright \widetilde{G}^{(2)}$,

$$(k_1, k_2, k_3) \cdot (g, h) = (k_3 g k_2^{-1}, k_2 h k_1^{-1}).$$

Given a metric $\widetilde{\eta}$ on $\widetilde{G}$, we can use this action to produce a "cotangent average metric" $\underline{\widetilde{\eta}}$ (see [25, Def. 3.1.2]) on $\widetilde{G}$, which depends on choices of a Haar system and a connection on K^3. (Recall that a connection is a vector bundle map $\sigma : s^*TM \to TG$ such that $ds \circ \sigma = \mathrm{id}_{s^*TM}$ and $\sigma|_M = du$.)

Proposition 2.2.2 (Prop. 6.3.2, [25])**.** If $\phi : \widetilde{G} \to G$ is a Morita fibration with kernel K and $\widetilde{\eta}$ is a groupoid metric on $\widetilde{G}$, then the cotangent average metric $\underline{\widetilde{\eta}}$ is a groupoid metric equivalent to η that is projectable to G, making ϕ Riemannian.

It follows that, up to equivalence, metrics can be pulled back and pushed forward along Morita fibrations, and since every Morita equivalence can be realized as a fraction of Morita fibrations (see [25]), metrics up to equivalence are instrinsically associated with the underlying stack.

Theorem 2.2.3 (Thm. 6.3.3,[25])**.** A Morita equivalence yields a 1-1 correspondence between equivalence classes of groupoid metrics. In particular, if two Lie groupoids are Morita equivalent and one admits a groupoid metric, then so does the other.

Riemannian stacks

A **stacky metric** $[\eta]$ on $[M/G]$ is defined as the equivalence class of a metric on the groupoid $G \rightrightarrows M$. This notion of metric generalizes the usual notions of metrics for manifolds and orbifolds, and allows us to perform Riemannian geometry on more general differentiable stacks.

Examples

Example 2.2.4. If $M \times_B M \rightrightarrows M$ is the submersion groupoid arising from $p : M \to B$, then a metric η on $M \times_B M \rightrightarrows M$ induces metrics η^M, η^B on M, B making p a Riemannian submersion, see Example 2.1.5. Two groupoid metrics η, η' on $M \times_B M \rightrightarrows M$ are equivalent if and only if $\eta^B = \eta'^B$. Therefore, stacky metrics on $[M/M \times_B M]$ are the same as Riemannian metrics on B.

Example 2.2.5. Let (M, η^M) be a Riemannian manifold, and $G \curvearrowright M$ a proper isometric action. We can proceed as in the Example 2.1.6 and build a groupoid metric η making sense of the orbit space M/G as Riemannian stack. The quotient M/G has a natural distance induced by the distance in M (see 2.4), because the orbits are closed we have

$$d_{M/G}(\bar{x}, \bar{y}) = \inf\{d(x, gy) \,:\, g \in G\}.$$

Assuming that η^M is complete the distance can be also recovered as the infimum of the lengths of all curves connecting the orbits $\bar{x}$ to $\bar{y}$ (cf. [41, Sec. 3]), we recover this in Theorem 2.3.10 by improving the notion of length. Modifications of the metric η^M to equivalent metrics shrinking the metric in the orbits directions and unchanging in the perpendicular directions have been used as approximations (in the Gromov-Hausdorff sense) for the metric space M/G, see [2, Sec. 6.1]. Hence, Riemannian stacks offer a "differential geometry" approach to the quotient space.

Example 2.2.6. If $G \rightrightarrows M$ is an étale groupoid, the normal spaces are the same as the tangent spaces. Then two metrics are equivalent if and only if they are equal. If $G \rightrightarrows M$ is proper and étale, then $[M/G]$ is an orbifold, and (equivalence classes of) groupoid metrics agree with the orbifold metrics as classically defined (see e.g. [31])

Example 2.2.7. The leaf space of a Riemannian foliation $(M, \eta, \mathcal{F})$ is in many cases an orbifold, for instance, if M is compact and the leaves are closed. This has guided the efforts to approach the transversal geometry of Riemannian foliations via pseudo-groups of local isometries [3]. Pseudo-groups of local isometries give rise to étale Riemannian groupoids, and two pseudo-groups of local isometries are equivalent if they have étale Riemannian groupoids representing the same Riemannian stack (cf. [43, Appendix D]).

2.3 Lenght of stacky curves

We will see now that the distance on the coarse orbit space of a Riemannian groupoid (see Sec. 2.1) can be obtained by measuring distances in the orbit space M/G by the length of stacky curves. From now on, let us fix a Riemannian groupoid $(G \rightrightarrows M, \eta)$, and consider its stack $([M/G], [\eta])$.

Curves

Let I be a real interval, viewed as a stack represented by the unit groupoid $I \rightrightarrows I$. A **stacky curve** is defined as a stacky map $\alpha : I \to [M/G]$.

Since stacky curves are particular cases of stacky maps, they also admit a cocycle description. Namely, a stacky curve $\alpha : I \to [M/G]$ can be presented by a covering $\mathcal{U} = \{U_i\}$ of I, and maps $a_i : U_i \to M$, $a_{ji} : U_j \cap U_i \to G$ with $a_j(x) \xleftarrow{a_{ji}(x)} a_i(x)$ for all x in $U_i \cap U_j$, and $a_{kj}(x)a_{ji}(x) = a_{ki}(x)$ for all x in $U_k \cap U_j \cap U_i$. This gives rise to fraction

$$(I \rightrightarrows I) \underset{\sim}{\xleftarrow{\phi_{\mathcal{U}}}} (\coprod U_{ji} \rightrightarrows \coprod U_i) \xrightarrow{a} (G \rightrightarrows M).$$

We call $a/\phi_{\mathcal{U}}$ a **good fraction** if $\mathcal{U} = \{U_i\}$ is indexed by a subset of $\mathbb{Z}$ in such way that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \neq 1$. Since the interval I has topological dimension 1, we can use refinement arguments to show that any curve can be presented by a good fraction. Two good fractions define the same curve if, after restricting to a common refinement, they are isomorphic.

A good fraction $a/\phi_{\mathcal{U}}$ for a stacky curve should be compared with the notion of G**-path (Hafliger paths)** [31, 2.3], [42, 3.3]. They are defined as a sequence alternating continuous paths $x_k \overset{\gamma_k}{\rightsquigarrow} y_k$ on M and arrows $y_k \xrightarrow{g_k} x_{k+1}$ in G. Given a good fraction $a/\phi_{\mathcal{U}}$ for a stacky curve we can build a G-path by splitting the interval, choosing $t_k \in U_{k+1,k}$, and setting $\gamma_k = a_k|_{[t_k,t_{k+1}]}$ and $g_k = a_{k+1,k}(t_k)$. Conversely, a G-path on which every γ_k is smooth gives rise to a good cocycle by first extending g_k to a smooth curve $\tilde{g}_k : (t_k - \epsilon, t_k + \epsilon) \to G$, $\tilde{g}_k(t_k) = g_k$, and then modifying γ_k and γ_{k+1} near t_k so as to agree with $s(\tilde{g}_k)$ and $t(\tilde{g}_k)$. Even though these operations depend on choices, they are well-defined up to equivalence classes of fractions and small *deformations of G-paths*. The advantages of our fractions is that they fit the general theory of stacky maps, without the need of an ad-hoc definition, and they allow us to make sense of smoothness.

Examples

Example 2.3.1. If $M \times_B M \rightrightarrows M$ is the submersion groupoid associated to the submersion $p : M \to B$. A good fraction $a/\phi_{\mathcal{U}}$ for a stacky curve is the same as a curve on B with local lifts a_i to M. In this case the maps $a_{ji} : I \to M \times_B M$ are given by $(a_j(t), a_i(t))$. Two good fractions are equivalent if they induce the same curve on B.

Example 2.3.2. Let $G \rightrightarrows M$ be a proper étale groupoid and $O = [M/G]$ its underlying orbit orbifold. A smooth curve $\alpha : I \to O$ is classicaly defined as a continuous curve $a : I \to |O| = M/G$ on the coarse orbit space that can locally be lifted to a smooth curve $a_i : I_i \to U_i$ on an orbifold chart, see [42, 2.4]. A stacky curve $\alpha : I \to [M/G]$ induces a curve in this classic sense, for if a/ϕ_U is a cocycle representing it, then the segments a_i serve as local lifts into an orbifold chart. But the classic notion of curve has not a clear interpretation in terms of stacks. For instance, the smooth curves

$$a_\pm : \mathbb{R} \to \mathbb{R}^2, \qquad a_\pm(t) = \begin{cases} (t, e^{1/t}) & t \le 0 \\ (t, \pm e^{-1/t}) & t \ge 0 \end{cases}$$

define the same curve on the quotient $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflections around x-axis. But since they are not related by the action of the group, a_+ and a_- present different stacky curves.

Example 2.3.3. If $G \curvearrowright M$ is an action and $G \ltimes M \rightrightarrows M$ the resulting action groupoid, we can represent a stacky curve $\alpha : I \to [M/G]$ as good fraction given by curves $a_i : U_i \to M$ and $g_{i+1,i} : U_{i+1,i} \to G$ such that $g_{i+1,i}(t)a_i(t) = a_{i+1}(t)$ for $t \in U_{i+1,i}$. The collection $\{g_{ji}\}$ defines a G-cocycle over the covering $\{U_i\}$ of the interval, and since every principal G-bundle over I is trivial, we can integrate the cocycle, gaining a global representative $a : I \to M$ for any stacky curve α.

Example 2.3.4. Recall that a codimension q regular foliation $\mathcal{F}$ on a manifold M can be described by a family of submersions $f_i : V_i \to \mathbb{R}^q$, where $\{V_i\}$ is an open covering for M, such that for every $x \in V_{ji}$ there exists a germ of diffeomorphism $h_{ji}(x) \in \mathrm{Diff}_0(\mathbb{R}^q)$ satisfying $f_j = h_{ji}f_i$ on some neighborhood of x (see e.g. [9]). If we use the holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ to make sense of the leaf space $M/\mathcal{F}$ as a differentiable stack, and we fix the defining submersions $f_i : V_i \to \mathbb{R}^q$, then a stacky curve $\alpha : I \to [M/\mathcal{F}]$ always admits a good fraction with $a_i : U_i \to V_i$ and $a_{ji} = h_{ji}$. The relevant information of each segment a_i is that of the composition $b_i = f_i a_i : U_i \to \mathbb{R}^q$. Thus a stacky curve on the leaf space is the same as a family of curves $b_i : U_i \to \mathbb{R}^q$ that are connected by the defining cocycle h_{ji}, compare this with the definition of curves for pseudo-groups [43, App. D, Def. 1.9].

Velocity and speed

Given $\alpha : I \to [M/G]$ a stacky curve and $t_0 \in I$, the **velocity** $\alpha'(t_0)$ is defined in terms of the coarse differential as $[\overline{d_{t_0}\alpha}(\partial_t)] \in T_{\mathcal{O}}[M/G]$, where $\mathcal{O} = \overline{\alpha}(t_0)$ (see Sec. 1.2). If $a/\phi_{\mathcal{U}}$ is a fraction presenting α, and t_0 in U_k, then we can present $\alpha'(t_0)$ as $[a_k'(t_0)]$ in $[N_{a_k(t_0)}\mathcal{O}/G_{a_k(t_0)}] \cong T_{\mathcal{O}}[M/G]$.

Since the normal representations are by isometries, the **speed** of α at time t_0 can be defined as $\|\alpha'(t_0)\| := \|a_k'(t_0)_N\|$, where $a_k'(t_0)_N$ is the normal component of $a_k'(t_0)$ in the decomposition $T_xM = T_x\mathcal{O} \perp T_x\mathcal{O}^\perp$ $(x = a_k(t_0))$.

Because equivalent metrics yield the same speed, this only depends on the stacky metric $[\eta]$ on $[M/G]$. In the next subsection we will prove the following technical fact:

Proposition 2.3.5. The speed $\|\alpha'(t)\|$ of a stacky curve $\alpha : I \to [M/G]$ varies continuously on $t \in I$.

Knowing that the speed of a stacky curve varies continuously, it makes sense to define the **normal length** of a curve.

Definition 2.3.6. Let $\alpha : I \to ([M/G], [\eta])$ be a stacky curve into a Riemannian stack. The length of α with respect to $[\eta]$ is defined as

$$\ell_N(\alpha) = \int_I \|\alpha'(t)\| \, dt.$$

In general, for a good fraction $\alpha = a/\phi_{\mathcal{U}}$, we have

$$\begin{aligned} \ell_N(\alpha) &= \sum_i \int_{U_i} \|\alpha'(t)\| \, dt - \int_{U_i \cap U_{i-1}} \|\alpha'(t)\| \, dt \\ &= \sum_i \int_{U_i} \|a_i'(t)_N\| \, dt - \int_{U_i \cap U_{i-1}} \left\|a_{i-1}'(t)_N\right\| dt. \end{aligned}$$

Continuity of speed and technical results

In this subsection we will show Proposition 2.3.5. Working locally, we can assume that $\alpha = a/1 : (I \rightrightarrows I) \to (G \rightrightarrows M)$ is given by a groupoid morphism, and moreover, that $a(I)$ is completely included within a foliated chart for the corresponding singular Riemannian foliation.

Given $\mathcal{F}$ a singular foliation on M and $x \in M$. If $\mathcal{F}$ is regular around x then we can find a chart for which $\phi^{-1}(\mathbb{R}^p \times y)$ are plaques of $\mathcal{F}$, and if η is a metric on M that makes $\mathcal{F}$ a Riemannian foliation, the projection $U \to \mathbb{R}^q$ becomes a Riemannian submersion. For a singular Riemannian foliation, we have seen that they have foliated charts (Sec.1.1), now the dimensions of the leaves may vary. So, it is rather unclear the existence of foliated charts for which the second projection is a Riemannian submersion.

As seen in Sec. 2.1: if $\pi : M \to B$ is a submersion, for x, x' in M there is an isomorphism $\tau_{x',x} : N_xM \to N_x'M$. Where $N_xM = T_xM/\ker_x d\pi$ and we identify it with $\ker d_x\pi^{\perp}$. We refer to the isomorphism $\tau_{x,x'}$ as normal transportation.

Lemma 2.3.7. Given (M,η) a Riemannian manifold, $\pi : M \to B$ a surjective submersion, $x_0 \in M$, and $\epsilon > 0$, there exists an open $U \subset M$ around x_0 such that

$$\big|\, |\langle \tau_{x',x} v, \tau_{x',x} w\rangle| - |\langle v, w\rangle| \,\big| \leq \epsilon \, \|v\| \, \|w\|$$

for all $x, x' \in U$, $\pi(x) = \pi(x')$ and $v, w \in N_x M$.

Proof. If either $v = 0$ or $w = 0$ then the inequality clearly holds. Then we may assume that $v, w \neq 0$. We will assume $\|v\| = \|w\| = 1$. Denote by $\pi_1 : M \times_B M \to M$ the first projection and let $S(NM) \to M$ be the sphere bundle of the normal bundle to the fibers. Consider the bundle $S(NM) \times_M S(NM) \to M$ and its pullback $\pi_1^*(S(NM) \times_M S(NM))$. The function

$$f : \pi_1^*(S(NM) \times_M S(NM)) \to \mathbb{R}, \quad f(v, w, (x, x')) = |\langle \tau_{x,x'} v, \tau_{x,x'} w\rangle| - |\langle v, w\rangle|$$

is continuous and equal to 0 on the fiber over (x_0, x_0). Then $W = f^{-1}((-\epsilon, \epsilon))$ is an open containing the fiber over (x_0, x_0). Since the projection

$$\pi_1^*(S(NM) \times_M S(NM)) \to M \times_B M$$

is proper, there must exist a basic open $U \times U$ around (x_0, x_0) such that its preimage is contained in W. The result follows. □

In the statement of the previous lemma, if π were a Riemannian submersion, then the left-hand side of the inequality would vanish. We will use the previous lemma to compare the speed of our curve $a : I \to U \subset M$ with that of the orthogonal lift of its projection along the chart second projection $\pi : U \to \mathbb{R}^q$. The next lemma will deal with the orthogonal lift, which we know that at time t_0 is orthogonal to the foliation.

Lemma 2.3.8. Let $(M, \eta, \mathcal{F})$ be a singular Riemannian foliated manifold. Let $a : I \to M$ be a smooth curve, such that $a'(0) \neq 0$ is orthogonal to $\mathcal{F}$ and $\epsilon > 0$. Then the angle between $a'(t)$ and $\mathcal{F}$ is greater than $\pi/2 - \epsilon$ near 0.

Proof. Denote $x_0 = a(0)$. Let P be a compact neighborhood of x_0 on the leaf, such that the normal bundle $N\mathcal{F}|_P$ can be trivialized, namely $N\mathcal{F}|_P \cong P \times \mathbb{R}^q$. Denote by B_δ the set of vectors in $N\mathcal{F}|_P$ whose norm is smaller than δ. For δ sufficiently small the map $\exp : P \times B_\delta \to M$ is a foliated chart around x_0 (see [43, p.192]). We denote by R the vector field in $V = \exp(P \times B_\delta)$ given by the radial vector field $(0, \frac{\partial}{\partial r})$ in $P \times B_\delta$.

The radial vector field R is always orthogonal to $\mathcal{F}$. In order to show that the angle between $a'(t)$ and $\mathcal{F}$ is close to $\pi/2$, it is enough to show that the angle between R and a is close to 0, or equivalently, that its cosine is close to 1. We will proceed by looking at the product $P \times B_\delta$ with the pullback metric and pullback foliation from M. If we call $v = a'(0)$ and write $a(t) = tv + O(t^2)$ in the foliated chart, then $a'(t) = v + O(t)$. For the metric we have the following expansion around $(x_0, 0)$: for all $X, Y \in \mathfrak{X}(P \times B_\delta)$,

$$\eta\left(X(x,v), Y(x,v)\right)_{(x,v)} = \langle X(x,v), Y(x,v)\rangle + O(|x-x_0|^2 + |v|^2).$$

We can conclude that

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{\eta\left(a'(t), R_{a(t)}\right)}{\|a'(t)\|_{a(t)}\left\|R_{a(t)}\right\|_{a(t)}}\right)^2 &= \lim_{t\to 0}\frac{\left\langle a'(t), R_{a(t)}\right\rangle^2 + O(t^3)}{\|a'(t)\|^2\left\|R_{a(t)}\right\|^2 + O(t^4)} \\
&= \lim_{t\to 0}\frac{\left\langle a'(t), a(t)\right\rangle^2 + O(t^3)}{\|a'(t)\|^2\|a(t)\|^2 + O(t^4)} \\
&= \lim_{t\to 0}\frac{t^2\left\langle v, v\right\rangle^2 + O(t^3)}{t^2\left\langle v, v\right\rangle^2 + O(t^4)} = 1. \quad \square
\end{aligned}$$

Proposition 2.3.5 immediately follows from the next result on singular Riemannian foliations. Given $v \in T_xM$, we denote by v_N the normal component of v in the orthogonal decomposition $T_xM = T_x\mathcal{F} \oplus T_x\mathcal{F}^\perp$.

Proposition 2.3.9. Let $(M, \eta, \mathcal{F})$ be a singular Riemannian foliated manifold. If $a : I \to M$ is a smooth curve, then $\|a'(t)_N\|$ is continuous on t.

Proof. Fix $t_0 \in I$. We can suppose without loss of generality that $t_0 = 0$ and denote $x_0 = a(0)$. We assume that we are within a foliated chart $\pi : U \times V \to V$ around x_0. On each point $x \in U \times V$ we have $T_xU \subset T_xF \subset T_xM$. Let S_x be the orthogonal complement of T_xU inside the foliation. Then we have the orthogonal decomposition

$$T_xM = T_xU \oplus S_x \oplus T_x\mathcal{F}^\perp.$$

For a vector $v \in T_xM$ we write $v = v_1 + v_2 + v_3$, relative to this decomposition. Given x, x' on the same fiber of π, the transportation $\tau_{x',x} : S_x \oplus N_x\mathcal{F} \to S_{x'} \oplus N_{x'}\mathcal{F}$ preserves the foliation, i.e. $\tau_{x',x}(S_x) = S_{x'}$.

Observe that $\|a'(t)_T\|^2 = \|a'(t)_1\|^2 + \|a'(t)_2\|^2$, and the first term is clearly smooth, so we need to show that the second one is also continuous. This is enough to show that near 0 the contribution of the second term is arbitrarily small.

Let b be the horizontal lift through x_0 of the projection of $\pi \circ a$ and write $x = a(t)$ and $x' = b(t)$. Note that $a'(t)_2$ is the orthogonal projection of $\tau_{x,x'}b'(t)$ over S_x. Fix $\{w_i\}$ an orthonormal basis of $N_x\mathcal{F}$. From Lemma 2.3.7, for every $\epsilon > 0$ there is a neighborhood of 0 where the following estimate holds:

$$\begin{aligned}
\|a'(t)_2\| &= \left\| \sum_i \langle \tau_{x,x'}b'(t), w_i\rangle\, w_i \right\| \\
&\leq \sum_i |\langle \tau_{x,x'}b'(t), w_i\rangle| \\
&\leq \sum_i |\langle \underline{\,'\,}(t), \tau_{x',x}w_i\rangle| + \epsilon\, \|b'(t)\|\, \|\tau_{x',x}w_i\| \\
&= \sum_i |\langle b'(t)_2, \tau_{x',x}w_i\rangle| + \epsilon\, \|b'(t)\|\, \|\tau_{x',x}w_i\| \\
&\leq \sum_i \|b'(t)_2\|\, \|\tau_{x',x}w_i\| + \epsilon\, \|b'(t)\|\, \|\tau_{x',x}w_i\| \\
&\leq (\|b'(t)_2\| + \epsilon\, \|b'(t)\|) \sum_i \|\tau_{x',x}w_i\| \\
&\leq (\|b'(t)_2\| + \epsilon\, \|b'(t)\|)(1+\epsilon)^{\frac{1}{2}} \dim S_x \\
&\leq \left(\frac{\|b'(t)_2\|}{\|b'(t)\|} + \epsilon \right) \|b'(t)\|\, (1+\epsilon) \dim M.
\end{aligned}$$

Consequently,

$$\|a'(t)_2\|^2 \leq \left(\frac{\|b'(t)_2\|}{\|b'(t)\|} + \epsilon \right)^2 \left(\|b'(t)\|\, (1+\epsilon) \dim M\right)^2.$$

This leads us to

$$0 \leq \lim_{t\to 0} \|a'(t)_2\|^2 \leq c(1+\epsilon)^2 \lim_{t\to 0} \left(\frac{\|b'(t)_2\|}{\|b'(t)\|} + \epsilon \right)^2,$$

where $c = (\|b'(0)\| \dim M)^2$. By Lemma 2.3.8 we have

$$\begin{aligned}
0 &\leq \lim_{t\to 0} \frac{\|b'(t)_2\|^2}{\|b'(t)\|^2} \\
&= \lim_{t\to 0} 1 - \frac{\|b'(t)\|^2 - \|b'(t)_2\|^2}{\|b'(t)\|^2} \\
&= 1 - \lim_{t\to 0} \frac{\|b'(t)_3\|^2}{\|b'(t)\|^2} \\
&\leq 1 - \lim_{t\to 0} \left(\frac{\eta\left(b'(t), R_{b(t)}\right)_t}{\|b'(t)\|_t \left\|R_{b(t)}\right\|_t} \right)^2 \\
&= 0.
\end{aligned}$$

It implies that for all $\epsilon > 0$ the following holds:

$$0 \leq \lim_{t\to 0} \|a'(t)_2\|^2 \leq \epsilon(1+\epsilon)^2 c.$$

We conclude that

$$\lim_{t\to 0} \|a'(t)_2\|^2 = 0.$$

□

Normal pseudo-distance

We have seen that a groupoid metric η on a groupoid $G \rightrightarrows M$ induce a pseudo-distance on M/G defined by

$$\bar{d}_N(\bar{x},\bar{y}) = \inf\left\{\sum_{i=1}^{n} d(x_i,y_i) : y \xleftarrow{g_{n+1}} y_n, \cdots, x_{i+1} \xleftarrow{g_{i+1}} y_i, \cdots, x_1 \xleftarrow{g_1} x, n \in \mathbb{N}\right\}.$$

Our first main theorem shows the quotient pseudo-distance d_N can be measured with stacky curves.

Theorem 2.3.10. Given $(G \rightrightarrows M, \eta)$ a Riemannian groupoid, and given $x, y \in M$, the quotient pseudo-distance $d_N(\bar{x},\bar{y})$ is the infimum of lengths of generalized curves connecting the points:

$$d_N(\bar{x},\bar{y}) = \inf\{\ell_N(\alpha) : \bar{x},\bar{y} \in \mathrm{im}(\bar{\alpha})\}.$$

Proof. Fix x, y in M. We will show that for any $\epsilon > 0$ there is a stacky curve $\alpha : I \to [M/G]$ connecting $\bar{x},\bar{y} \in M/G$ such that $\ell_N(\alpha) < d_N(x,y)+\epsilon$. By the definition of d_N, we know there are points $x_1, y_1, \ldots, x_n, y_n$ such that $y \xleftarrow{g_{n+1}} y_n, \cdots, x_{i+1} \xleftarrow{g_{i+1}} y_i, \cdots, x_1 \xleftarrow{g_1} x$, and

$$\sum_{i=1}^{n} d(x_i,y_i) < d(\bar{x},\bar{y}) + \epsilon/2.$$

For each i we will choose a curve $a_i : I \to M$ connecting x_i and y_i and use the curves a_i to define a fraction

$$(\coprod_i U_{i+1,i} \rightrightarrows \coprod_i U_i) \to (G \rightrightarrows M).$$

In order to get a well-defined stacky map, we pick the a_i inductively. After picking a_i, let t_i be the point in $U_i \cap U_{i+1}$ such that $a_i(t_i) = y_i$. Consider a $a_{i+1,i} : J_i \to G$ a local lift of a_i along s such that $a_{i+1,i}(t_i) = g_{i+1}$. We can pick the next curve a_{i+1} to agree with $t \circ a_{i+1,i}$ in a small neighborhood of t_i. Since the size of the small neighborhood is arbitrary, we can pick the a_i's satisfying $\ell(a_i) \leq d(x_i, y_i) + \frac{\epsilon}{2n}$. The resulting sequences a_i, $a_{i+1,i}$ give rise to a stacky curve $\alpha : I \to [M/G]$ of length smaller than $d_N(\bar{x}, \bar{y}) + \epsilon$:

$$\ell_N(\alpha) \leq \sum_{i=1}^{n} \ell_N(a_i) \leq \sum_{i=1}^{n} \ell(a_i) \leq \sum_{i=1}^{n} \big(d(x_i, y_i) + \epsilon/2n\big) \leq d_N(\bar{x}, \bar{y}) + \epsilon.$$

Let us now show that, given $\alpha : I \to [M/G]$ a stacky curve connecting $\bar{x}$ and $\bar{y}$, its normal length must be greater or equal than $d_N(\bar{x}, \bar{y})$. By the additivity of the integral, and by the triangle inequality for d_N, we can subdivide I into small intervals and show the inequality for each little curve. This allows us to work locally. Thus, without loss of generality, we can assume that the original α is defined by single curve $a : [0, 1] \to M$.

Given $t_0 \in I$, write $x_0 = a(t_0)$. We will work locally, modifying the curve a around t_0. Let $\pi : U \times V \to V$ be a foliated chart around x_0. Then we build a new curve a, which is just the horizontal lift along π of the projection of a. We claim that:

$$\|b'(t)\| \leq (1 + \epsilon) \|b'_3(t)\| \leq (1 + \epsilon) \|\tau(a'_3(t))\| \leq (1 + \epsilon)^2 \|a'_3(t)\| ,$$

in small neighborhood of t_0. The first inequality holds because of Lemma 2.3.8, the second one is because the transportation preserves the foliation and therefore $a'_2(t)$ does not contribute to $b'_3(t)$, the third one is because of Lemma 2.3.7. It follows that for every $s \in [0, 1]$, there is an open interval $(s - \delta_s, s + \delta_s) \subset [0, 1]$ such that:

$$d_N(\bar{a}(s - \delta_s), \bar{a}(s + \delta_s)) \leq d(b(s - \delta_s), b(s + \delta_s)) \leq \int_{s-\delta_s}^{s+\delta_s} \|b'(t)\| \, dt \leq$$
$$\leq (1 + \epsilon)^2 \int_{s-\delta_s}^{s+\delta_s} \|a'_3(t)\| \, dt = (1 + \epsilon)^2 \ell_N(a \,|_{[s-\delta_s, s+\delta_s]}).$$

By compactness we can subdivide the interval $[0, 1]$ into finitely many intervals $[s_i, s_{i+1}]$, and get

$$d_N(\bar{x}, \bar{y}) \leq \sum_{i=1}^{n} d_N(\bar{a}(s_i), \bar{a}(s_{i+1})) \leq (1+\epsilon)^2 \sum_{i=1}^{n} \ell_N(a \,|_{[s_i, s_{i+1}]}) = (1+\epsilon)^2 \ell_N(a).$$

Since ϵ is arbitrary, we have shown that $d_N(\bar{x}, \bar{y}) \leq \ell_N(a)$, and the proof is complete. □

We close this section with some immediate corollaries of our characterization of d_N by using stacky curves.

Corollary 2.3.11. Equivalent metrics on the same Lie groupoid $G \rightrightarrows M$ yield the same pseudo-distance on the coarse orbit space M/G.

Proof. It follows from the previous theorem and the fact that, with respect to equivalent metrics, normal vectors have the same norm and consequently generalized curves have the same normal length. □

Corollary 2.3.12. If $\phi : \widetilde{G} \to G$ is a Riemannian Morita map, then the map $\bar{\phi} : \widetilde{M}/\widetilde{G} \to M/G$ between the orbit spaces preserves distances.

Proof. We have a 1-1 correspondence between curves in both stacks. Since the distance can be measured by curves, and since a curve α on $[\widetilde{M}/\widetilde{G}]$ has the same speed as $\phi \circ \alpha$, we conclude that they have the same length and that $\bar{\phi}$ is distance-preserving. □

Corollary 2.3.13. The pseudo-distance d_N on M/G is a Riemannian Morita invariant; it depends only on the underlying Riemannian stack $([M/G], [\eta])$.

Remark 2.3.14. Theorem 2.3.10 says that the quotient length structure can be recovered by the normal length structure on the generalized curves on $[M/G]$. If d_N is a distance, then $(M/G, d_N)$ clearly inherits a quotient length structure from (M, η^M) (see [11, pp.63]).

Chapter 3

Geodesics on Riemannian stacks

In this chapter, we introduce the definition of geodesic for metrics on stacks. We establish several foundational results, such as the existence and uniqueness of geodesics, and a stacky Gauss Lemma. Our main result is a stacky version of the classical Hopf-Rinow theorem. This chapter will proceed as follows:

- In Section 3.1 we introduce the preliminary notion of geodesic fractions and provide a uniqueness result for them.
- In Section 3.2 we discuss how Riemannian metrics can be used to provide local models for stacks, and we show a stacky version of the Gauss lemma.
- In Section 3.3 we review the classic notion of distance minimizer curve and how our geodesic fractions minimize distances. We recall what are rays, then we show that "local rays" are equivalent to geodesic fractions.
- In Section 3.4 we introduce our definition of geodesics on stacks from our preliminary notion of geodesic fractions. We check that this is well-defined using the equialence between "local rays" and geodesic fractions.
- In Section 3.5 we state and prove our extension of the Hopf-Rinow theorem to Riemannian stacks, and present some consequences.

3.1 Geodesic fractions

In order to define geodesics on stacks, we need a preliminary notion. We will introduce the notion of geodesic fractions and discuss existence and uniqueness for them. In the remainder of this section, we will fix a Riemannian groupoid $(G \rightrightarrows M, \eta)$.

Definition 3.1.1. A **geodesic fraction** is a fraction $a/\phi_{\mathcal{U}} : I_{\mathcal{U}} \dashrightarrow G$, such that the maps $a_{ij} : U_i \cap U_j \to G$ are orthogonal geodesics to $\mathcal{F}_G$.

Since s and t are Riemannian submersions, the curves $a_i : U_i \to M$ are geodesics orthogonal to $\mathcal{F}_M$. We will use the term "geodesic" instead of "geodesic fraction" when there is no risk of confusion.

The same argument based on refinement applied to curves (see Sec.2.3) shows that a geodesic fraction is always equivalent to a good geodesic fraction. From now on we make the assumption that all geodesic fractions are good fractions.

The **local existence** of geodesics through a point with a given velocity is a consequence of the local existence for geodesics on manifolds. For each point $\bar{x} \in M/G$ and each coarse vector $[v] \in T_{\bar{x}}[M/G]$, there exists a geodesic a on M which satisfies $\overline{a(0)} = \bar{x}$ and $[a'(0)] = [v]$. This induces a geodesic fraction $a/1 : (I \rightrightarrows I) \dashrightarrow (G \rightrightarrows M)$ with the desired initial data.

Lemma 3.1.2. A curve fraction $a/\psi_{\mathcal{U}} : I \dashrightarrow G$ is equivalent to a geodesic fraction if and only if for every $t \in I$ there is a smaller interval $t \in J \subset I$ such that $(a/\psi_U)|_J$ is equivalent to a geodesic fraction.

Proof. A restriction of a geodesic is clearly a geodesic. Conversely, let $\alpha : I \to [M/G]$ be a curve that is locally a geodesic. Then we can split I into a sequence of intervals $[t_r, t_{r+1}]$ in such a way that $\alpha|_{I^r}$ is a stacky geodesic, $[t_r, t_{r+1}] \subset I^r$. By definition, we can represent $\alpha|_{I^r}$ by a finite good fraction of orthogonal geodesics (a^r_{ji}, U^r_i). The goal is to cook up a good fraction for the whole a out of this local ones. Clearly we can assume that $U^{r-1}_j \cap U^r_i = \emptyset$ except for $j = j_{\max}$ maximal and $i = i_{\min}$ minimal, by shrinking the overlapping around t_r. Choose g^r connecting $a^{r-1}_{j_{\max}}(t_{r-1})$ and $a^r_{i_{\min}}(t_r)$, and let $\gamma^r : J^r \to G$ be the orthogonal geodesic satisfying $\gamma^r(t_r) = g^r$ and projecting onto $a^{r-1}_{i_{\max}}$ via the source map. Then we get a good geodesic fraction for α by merging those of the local restrictions $\alpha|_{I^r}$ and connecting them with the γ^r's. □

Remark 3.1.3. A geodesic fraction $a/\phi_{\mathcal{U}}$ should be compared with the notion of G-geodesic defined for étale groupoids present in [31, 2.3]. They are defined as a sequence alternating geodesics $x_k \overset{\gamma_k}{\rightsquigarrow} y_k$ on M and arrows $y_k \xrightarrow{g_k} x_{k+1}$ in G satisfying $g_k\gamma_k'(t_{k+1}) = \gamma_{k+1}'(t_{k+1})$. We can address non-étale groupoids just requiring that the geodesics γ_k being orthogonal. Given a good geodesic fraction $a/\phi_{\mathcal{U}}$ we can build a G-geodesic by splitting the interval, choosing $t_k \in U_{k+1,k}$, and setting $\gamma_k = a_k|_{[t_k,t_{k+1}]}$ and $g_k = a_{k+1,k}(t_k)$. Conversely, a G-geodesic gives rise to a good geodesic fraction by first extending g_k to a geodesic $a_{k+1,k} : (t_k - \epsilon, t_k + \epsilon) \to G$, with $a_{k+1,k}(t_k) = g_k$ and $s_* a'_{k+1,k}(t_k) = \gamma_k'(t_k)$, and setting $a_k = \gamma_k$.

Examples

Example 3.1.4. In Riemannian submersions, the geodesics on the base are recovered from the horizontal geodesics in the total space. Let $p : M \to B$ be a Riemannian submersion. Given a curve $a : I \to B$, it is a geodesic if only if it lifts locally to horizontal geodesics on M. We can re-write all these local lifts as a groupoid map $\widetilde{a} : \coprod_{j,i} U_i \cap U_j \to M \times_B M$. Therefore, a is a geodesic on B if only if $\widetilde{a}/\phi_{\mathcal{U}} : I \dashrightarrow M \times_B M$ is a geodesic.

Example 3.1.5. If $G \curvearrowright (M, \eta^M)$ is an isometric action, we have seen in the Examples 2.1.6, 2.2.5 how it is fit in the theory of Riemannian stacks. The groupoid metric η built from η^M induces a metric $\eta^{(1)}$ on $G \times M$. By construction, $\eta^{(1)}$ is such that the following are Riemannian submersions:

$$\begin{array}{ccc} (G \times G \times M, \eta^G \times \eta^G \times \eta^M) & \xrightarrow{\pi_2} & (G \times M, \eta^G \times \eta^M) \\ \downarrow{\scriptstyle (h,g,x)\mapsto(hg^{-1},gx)} & & \downarrow{\scriptstyle (g,x)\mapsto gx} \\ (G \times M, \eta^{(1)}) & \xrightarrow{s} & (M, \eta^M). \end{array}$$

A geodesic on $(G \times M, \eta^{(1)})$ which is orthogonal to the submanifolds $G \times \mathcal{O}_x$ has the form $(k, a(t))$, with k in G constant and $a(t)$ a geodesic of (M, η^M) which is orthogonal to the orbits. Thus, a geodesic fraction on $(G \ltimes M \rightrightarrows M, \eta)$ is a collection of geodesics $\{a_i : U_i \to (M, \eta^M)\}$ orthogonal to the orbits and elements $\{g_{ji}\}$ of G, such that $g_{ji}a_i(t) = a_j(t)$. But, choosing a good covering for I, and using that the elements $g_{i+1,i}$ induce global isometries on (M, η^M), we show that the geodesic fraction is equivalent to single geodesic $a : I \to M$ orthogonal to the orbits (cf. [41]).

Example 3.1.6. Let $(G \rightrightarrows M, \eta)$ be a proper étale Riemannian groupoid and $O = [M/G]$ its orbit Riemannian orbifold. An orbifold geodesic $\alpha : I \to O$ is classicaly defined as a continuous curve $\alpha : I \to M/G$ having the property that for any $t \in I$ there exists a subinterval $I_i \subset I$ containing t and an orbifold chart (U_i, G_i, ϕ) around $\alpha(t)$ such that the restriction $\alpha|_{I_i}$ lifts to a smooth geodesic $a_i : I_i \to U_i$. This definition for orbifolds matches our approach, the local lifts define geodesics $a_i : I_i \to M$, and it can always be extended at the level of arrows, in a unique way up to equivalence of geodesic fractions. The definition of geodesics for orbifolds on [31] are covered by the equivalence discussed in the Remark 3.1.3.

Example 3.1.7. Recall that a Riemannian foliated manifold $(M, \mathcal{F}, \eta)$ can be described by a family of Riemannian submersions $f_i : (V_i, \widetilde{\eta}_i) \to (\mathbb{R}^q, \eta_i)$, where $\{V_i\}$ is an open covering for M, such that for every $x \in V_{ji}$ there exists a germ of isometry $h_{ji}(x) \in \mathrm{Diff}_0(\mathbb{R}^q)$ satisfying $f_j = h_{ji} f_i$ on some neighborhood of x. Using the holonomy groupoid $(\mathrm{Hol}(\mathcal{F}) \rightrightarrows M, \eta)$ to make sense of the leaf space $M/\mathcal{F}$ as a Riemannian stack, and fixing the defining Riemannian submersions $f_i : (V_i, \widetilde{\eta}_i \to (\mathbb{R}^q, \eta_i)$, then we can always assume that a geodesic fraction satisfies $a_i : U_i \to V_i$ and $a_j = h_{ji} a_i$. The relevant information of each segment a_i is that of the composition $b_i = f_i a_i : U_i \to \mathbb{R}^q$. Thus a geodesic fraction on the leaf space is the same as a family of geodesics $b_i : U_i \to (\mathbb{R}^q, \eta_i)$ that are connected by the defining cocycle h_{ji}, compare this with the definition of geodesics in [5].

Uniqueness

The uniqueness here is understood as the uniqueness of the class of fractions. Let $a/\psi_{\mathcal{U}} : I \dashrightarrow G$ and $b/\phi_{\mathcal{V}} : I \dashrightarrow G$ be geodesic fractions. If there is $t_0 \in I$ such that $\overline{a/\psi_{\mathcal{U}}}(t_0) = \overline{b/\phi_{\mathcal{V}}}(t_0)$ and $(a/\phi_{\mathcal{U}})'(t_0) = (b/\psi_{\mathcal{V}})'(t_0)$, we plan to show that $a/\psi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$ are equivalent fractions.

As we will now see, geodesic fractions satisfy **local uniqueness** for given initial data. Given two orthogonal geodesics $a, b : I \to M$ and $b(0) \overset{g}{\leftarrow} a(0)$ an arrow with $g \cdot a'(0) = b'(0)$, then there is locally a natural isomorphism $\gamma : J \to G$ between a and b. Take $\gamma : J \to G$ as the geodesic of G with initial conditions $\gamma(0) = g$ and $\gamma'(0) = v$, where v is the horizontal lift of $a'(0)$ along ds. The geodesic γ is orthogonal to s-fibers and t-fibers, and this implies that $s \circ \gamma$ and $t \circ \gamma$ are geodesics. From the uniqueness of geodesics on M we have $s \circ \gamma = a$ and $t \circ \gamma = b$ in their intervals of definition. The first point to get the global uniqueness is extend to the natural isomorphism γ to the whole interval I.

Lemma 3.1.8. (cf. [25, Lem. 4.1.2]) Let $G \rightrightarrows M$ be a proper Riemannian groupoid. Take $a, b : I \to M$ orthogonal geodesics on M, and $b(0) \overset{g}{\leftarrow} a(0)$

an arrow with $g \cdot a'(0) = b'(0)$. Then the orthogonal geodesic $\gamma : J \subset I \to G$ is extended to I.

Proof. Let $(p, q) \subset I$ be the maximal interval where γ is defined, and without loss of generality suppose that $\gamma'(t)$ is unitary. Consider a sequence $t_n \nearrow b$. Then the sequence $\{(\gamma(t_n), \gamma'(t_n))\}$ is contained in the sphere bundle of TG over $G(a([p, q]), b([p, q]))$, that we will denote by K. By properness of G the set K is compact. Hence, there exists a convergent subsequence $(\gamma(t_n), \gamma'(t_n)) \to (g_0, v_0)$. There are $\epsilon > 0$ and W a neighborhood of (g_0, v_0) in TG, such that every geodesic flow line starting in W is defined at least in the interval $(-\epsilon, \epsilon)$. For n sufficiently large, we have $(\gamma(t_n), \gamma'(t_n)) \in W$ and $q - t_n < \epsilon$. Thus, we can extend γ to the interval $(p, t_n + \epsilon)$. But $q < t_n + \epsilon$, which leads to a contradiction. $\square$

Example 3.1.9. If $\mathcal{F}$ is the foliation in $\mathbb{R}^2 \setminus \{0\}$ given by the vertical lines, then its holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows \mathbb{R}^2 \setminus \{0\}$ is not proper. Consider the orthogonal geodesics $a(t) = (t, 1)$ and $b(t) = (t, -1)$ in $\mathbb{R}^2 \setminus \{0\}$. There is a natural isomorphism $\gamma : (0, +\infty) \to \mathrm{Hol}(\mathcal{F})$ between a and b in $(0, +\infty)$, but it can not be extended to 0 once $a(0)$ and $b(0)$ are not in the same leaf. The lack of uniqueness in this example can be thought of a consequence that the orbit space of a non-proper groupoid might not be Hausdorff.

The following proposition is about global uniqueness of geodesics with given initial conditions in proper groupoids.

Proposition 3.1.10. Let $G \rightrightarrows M$ be a proper Riemannian Lie groupoid. Suppose that $a/\psi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$ are geodesic fractions. If there is $t_0 \in I$ such that $\overline{a}(t_0) = \overline{b}(t_0)$ and $a'(t_0) = b'(t_0)$, then $a/\psi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$ are equivalent fractions.

Proof. Choosing a good refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$, then $a/\psi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$ are equivalent to $a/\psi_{\mathcal{W}}$ and $b/\phi_{\mathcal{W}}$ respectively.

For simplicity, we assume $t_0 \in W_0$. Lemma 3.1.8 says that there is a natural isomorphism $\gamma_0 : W_0 \to G$ between a_0 and b_0. As next step, choose a point t_1 in $W_0 \cap W_1$. So, we can compose the arrows

$$\begin{array}{ccc} b_1(t_1) & \dashleftarrow & a_1(t_1) \\ {\scriptstyle b_{10}(t_1)}\uparrow & & \downarrow{\scriptstyle a_{01}(t_1)} \\ b_0(t_1) & \xleftarrow[\gamma_0(t_1)]{} & a_0(t_1) \end{array}$$

to obtain an arrow $b_1(x_0) \overset{g_1}{\longleftarrow} a_1(x_0)$ which satisfies $g_1 \cdot a_1'(x_0) = beta_1'(x_0)$. Applying the lemma again, we get a natural isomorphism $\gamma_1 : W_1 \to G$ between a_1 and b_1. The equation $b_{10}(t) = \gamma_1(t)a_{10}(t)\gamma_0(t)^{-1}$ holds for all $y \in W_0 \cap W_1$, because the product and inverse of orthogonal geodesics is again a orthogonal geodesic (see Sec. 2.1) and both sides have the same initial conditions at t_1.

We inductively construct natural isomorphisms γ_i between a_i and b_i satisfying $b_{i+1,i}(t) = \gamma_{i+1}(t)a_{i+1,i}(t)\gamma_i(t)^{-1}$ for all $t \in W_i \cap W_{i+1}$. The map $\gamma : \sqcup W_i \to G$ induced by the γ_i's is a natural isomorphism between a and b. Clearly, the equations $s \circ \gamma = a$ and $t \circ \gamma = b$ hold, because they hold for each γ_i. We shall check that $b_{ij}(t) = \gamma_i(t)a_{ij}(t)\gamma_j(t)^{-1}$. Since $U_i \cap U_j = \emptyset$ if $|j - i| \neq 1$, we only need check that $b_{i+1,i}(x) = \gamma_{i+1}(x)a_{i+1,i}(x)\gamma_i(x)^{-1}$, which is true by construction. □

Corollary 3.1.11. Let $G \rightrightarrows M$ be a Riemannian proper groupoid. If $a/\psi_{\mathcal{U}} : I \dashrightarrow G$ and $b/\phi_{\mathcal{V}} : J \dashrightarrow G$ are geodesic fractions such there exists $t_0 \in I \cap J$ satisfying $a(t_0) = \beta(t_0)$ and $a'(t_0) = b'(t_0)$, then there exists a geodesic fraction $c/\phi_{\mathcal{W}} : I \cup J \dashrightarrow G$ that extends $a/\psi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$.

Remark 3.1.12. If the metric on G is complete then the hypothesis about properness can be dropped in all the results above, i.e., global uniqueness holds when the metric on G is complete.

3.2 Normal neighbourhoods

The **linear model** of G around $\mathcal{O}$ is the groupoid formed by the normal bundle $NG\mathcal{O} \rightrightarrows N\mathcal{O}$, whose objects and arrows are given by $N\mathcal{O} = TM|_{\mathcal{O}}/T\mathcal{O}$ and $NG\mathcal{O} = TG|_{G\mathcal{O}}/TG\mathcal{O}$, and structure maps are induced by differentiating those of G. The action groupoid from the normal representation $G_{\mathcal{O}} \curvearrowright N\mathcal{O}$ is isomorphic to the linear model $NG_{\mathcal{O}} \rightrightarrows N\mathcal{O}$ (see [20, Sec 3.4]), so we also refer to it as normal representation.

Denote by $D_{\mathcal{O}} \subset N\mathcal{O}$ the domain of the exponential map in the normal directions to $\mathcal{O}$ in M, the same for $D_{G\mathcal{O}} \subset NG\mathcal{O}$. If G is a proper groupoid Lemma 3.1.8 shows that $D_{G\mathcal{O}} = ds^{-1}(D_{\mathcal{O}}) \cap dt^{-1}(D_{\mathcal{O}})$. By the multiplicity property of the orthogonal geodesics (see Eq. 2.3) the exponential map

$$\exp : (D_{G\mathcal{O}} \rightrightarrows D_{\mathcal{O}}) \to (G \rightrightarrows M) \tag{3.1}$$

is a groupoid map. This is a crucial fact in the proof of the following theorem.

Theorem 3.2.1 ([24]). Let $(G \rightrightarrows M, \eta)$ be a proper Riemannian groupoid. Given $\mathcal{O} \subset M$ an orbit, there is a neighborhood $V \subset D_{\mathcal{O}}$ of $\mathcal{O}$ such that the map

$$\exp : (NG\mathcal{O}_V \rightrightarrows V) \to (G \rightrightarrows M)$$

is an isomorphism onto its image.

The above theorem is just a re-statement of the linearization result presented in [24], we will review linearizations in Sec. 4.1. From this statement, and thinking of $NG\mathcal{O} \rightrightarrows N\mathcal{O}$ as a model for $T_{\mathcal{O}}[M/G]$ (see Ex. 1.2.5), we realize that the theorem shows the existence of "normal neighborhoods" for Riemannian stacks. This leads us to formulate a version of the normal neighborhood theorem in terms of the normal representations. This will illustrate the fact that separated stacks are modeled in terms of the normal representations.

By the uniqueness of geodesic fractions, given $[v] \in T_{\mathcal{O}}[M/G]$ there is a unique class of geodesic fraction α, with $\overline{\alpha}_{[v]}(0) = \mathcal{O}$ and $\alpha'_{[v]}(0) = [v]$. The set $D_{\mathcal{O}}/D_{G\mathcal{O}} \subset T_{\mathcal{O}}[M/G]$ is precisely the set of directions $[v] \in T_{\mathcal{O}}[M/G]$ such that $\alpha_{[v]}$ is defined at least in the interval $[0,1]$. The **coarse exponential map**

$$\overline{\exp}_{\mathcal{O}} : D_{\mathcal{O}}/D_{G\mathcal{O}} \to M/G$$

is defined by

$$\overline{\exp}_{\mathcal{O}}([v]) = \alpha_{[v]}(1).$$

The coarse exponential map is the induced map by the exponential map on the normal directions (Eq. 3.1).

Let us introduce some notations and definitions. For a $S \subset M$, we call by its **saturation** the set $t(s^{-1}(S))$. Denote by B^N_ϵ the set of vectors in N_xM with norm smaller than ϵ. The normal representations locally model $[M/G]$ in the following sense:

Corollary 3.2.2. Let $G \rightrightarrows M$ be a proper Riemannian groupoid and $x \in M$. Then there exists $\epsilon > 0$ such that the saturation U of $\exp(B^N_\epsilon)$ is open and

$$\exp : (G_x \ltimes B^N_\epsilon \rightrightarrows B^N_\epsilon) \to (G_U \rightrightarrows U)$$

is a Morita map.

Gauss lemma

In Riemannian geometry, the Gauss's lemma asserts that any sufficiently small geodesic sphere centered at a point is perpendicular to every geodesic through the point, and moreover, in radial directions, the exponential map is an isometry. Next, we present a version of this second property for stacks.

Proposition 3.2.3 (Gauss Lemma)**.** Let $G \rightrightarrows M$ be a proper Riemannian groupoid and $x \in M$. Then there exists $\epsilon > 0$ such that

$$d_N(\bar{x}, \overline{\exp}_{\bar{x}}([v])) = \|v\|, \qquad \forall\, v \in B_\epsilon^N \subset N_x M.$$

Proof. From Corollary 3.2.2, there is ϵ sufficiently small such that $\exp : G_x \ltimes B_\epsilon^N(x) \to G$ is a Morita map onto the saturation of its image, denoted by $G_U \rightrightarrows U$. We can transport the groupoid metric class on $G_U \rightrightarrows U$ to $G_x \ltimes B_\epsilon^N(x) \rightrightarrows B_\epsilon^N(x)$, since they are Morita equivalent. The spaces $(B_\epsilon^N(x)/G_x, d_N)$ and $(U/G_U, d_N)$ are isometric by Corollary 2.3.13. We have reduced the problem to the case where x is a fixed point.

Consider an orbit $\mathcal{O} \subset B_\epsilon(x)$. Since $\mathcal{O}$ is closed, the distance as subsets of M between x and $\mathcal{O}$ is realized, i.e., there exists $y \in \mathcal{O}$ such that $d(x, \mathcal{O}) = d(x, y)$. Let $v \in B_\epsilon(x)$ be a vector such that $y = \exp(v)$. By the classic Gauss lemma, we have that $d(x, y) = \|v\|$.

We want to show that $d_N(\bar{x}, \mathcal{O}) = d(x, \mathcal{O})$. Given $y' = \exp(v')$ another point in the same orbit $\mathcal{O}$, we have that $d(x, y') = \|v\|$, and that there exists an arrow $y' \xleftarrow{g} y$. There is $(g_0, v_0) \in G_x \ltimes B_\epsilon(x)$ such that $\exp(g_0, v_0) = g$. Since exp commutes with source and target, we have $\exp(v) = s(g) = s(\exp((g_0, v_0))) = \exp(s(g_0, v_0)) = \exp(v_0)$, thus $v = v_0$, and the same argument shows that $v' = g_0 \cdot v$. We conclude that $d(x, \mathcal{O}) = \|v\| = \|v'\| = d(x, y')$ for all $y \in \mathcal{O}$. Hence, the orbits are contained in the geodesic spheres around x.

From the definitions we have $d_N(\bar{x}, \bar{y}) \leq d(x, \mathcal{O}_y)$. Suppose that there is a chain $c = \{a_1, b_1, \ldots, a_n, b_n\}$ connecting $\bar{x}$ to $\bar{y}$ with $\sum_{i=1}^n d(a_i, b_i) < d(x, \mathcal{O}_y)$. Because the orbit $\mathcal{O}_y$ is equidistant to x and by triangle inequality, the following inequality holds:

$$d(x, \mathcal{O}_y) = d(a_1, b_n) \leq \sum_{i=1}^n d(a_i, b_i) < d(x, \mathcal{O}_y).$$

Changing y by $\overline{\exp}_{\bar{x}}([v])$, we conclude that

$$d_N(\bar{x}, \overline{\exp}_{\bar{x}}([v]) = d(x, \mathcal{O}_{\exp_x(v)}) = \|v\|.$$

□

For x in M and $\epsilon > 0$ as in the lemma above, we call the set $B_\epsilon(\bar{x}) = \overline{\exp}\left(B_\epsilon^N/G_x\right)$ by **the normal ball** and the set $S_\epsilon(\bar{x}) = \overline{\exp}\left(S_\epsilon^N/G_x\right)$ by **the geodesic sphere**. From their definitions both are compact sets.

3.3 Distance-minimizing curves

In Riemannian geometry, geodesics are locally distance-minimizing curves, but geodesics fractions may not minimize distances, see Example 3.3.1. We will explain in this section how our geodesic fractions are related to the normal distance, i.e., that geodesic fractions minimize distances in special way.

A curve fraction $a/\phi_{\mathcal{U}} : I \dashrightarrow G$ is a **distance minimizer** if

$$d_N(\overline{a}(s), \overline{a}(t)) = \ell_N\left(a/\phi_{\mathcal{U}}|_{[s,t]}\right)$$

for all $s, t \in I$. We say that $a/\phi_{\mathcal{U}}$ is a **local distance minimizer** if for all $t \in I$ there is an open interval containing t such that $a/\phi_{\mathcal{U}}|_J$ minimizes distances.

Example 3.3.1. Let $\mathbb{Z}_2 \ltimes \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be the action groupoid given by the reflection around the x-axis, with the Euclidean metric as groupoid metric. Consider the straight line $a : \mathbb{R} \to \mathbb{R}^2$, $a(t) = (t, -t)$; so, $a/1$ is a geodesic fraction. The normal length $\ell_N(a/1)$ is equal to the length $\ell(a)$, thus $\ell_N(a/1|_{[s,t]}) = |t - s|$. Therefore, there is no open interval J containing 0 such that $a/1|_J$ is distance minimizer.

We say that a curve fraction $a/\phi_{\mathcal{U}} : I \dashrightarrow G$ is a **local ray**, if for any $t_0 \in I$, there is an open interval $J \subset I$ containing t_0 such that

$$d_N(\overline{a}(t_0), \overline{a}(t)) = \ell_N(a/\phi_{\mathcal{U}}|_{[t_0,t]})$$

for all $t \in J$. Since this is defined by the normal length, being a local ray is a property of the fraction class. From the Gauss lemma, geodesics are clearly local rays, and the converse is covered by the next proposition.

Theorem 3.3.2. Let $(G \rightrightarrows M, \eta)$ be a proper Riemannian groupoid. If a curve fraction $a/\phi_{\mathcal{U}} : I \dashrightarrow G$ is a local ray, then it is equivalent to a geodesic fraction.

Proof. Suppose that a is a local ray around every $t \in I$. Fix t_0 in I. By restricting to a neighborhood of t_0 we can assume that a is a groupoid curve $a : I \to G$. Call $x = a(t_0)$. We will restrict to the local model around x in a similar fashion as done in the proof of Gauss lemma 3.2.3. For ϵ sufficiently small Corollary 3.2.2 says that

$$\exp : (G_x \ltimes B^N_\epsilon \rightrightarrows B^N_\epsilon) \to (G_U \rightrightarrows U)$$

is a Morita map, where U is the saturation of $\exp(B^N_\epsilon)$. We pullback the metric η to $G_x \ltimes B^N_\epsilon \rightrightarrows B^N_\epsilon$ along the exponential map. So, we can locally lift the curve a to an isomorphic one $\widetilde{a} : J \to B^N_\epsilon$ through the origin in B^N_ϵ. Then for t near t_0 we have that

$$\ell_N(a|_{[t_0,t]}) = d_N(\overline{a}(t_0), \overline{a}(t)) = d_N(\overline{\widetilde{a}}(t_0), \overline{\widetilde{a}}(t)) = d(x, \overline{\widetilde{a}}(t))$$

here the first identity is because a is a local ray at t_0, the second because a Riemannian Morita map preserves distances (Cor. 2.3.13) and the last one is because of Gauss lemma (3.2.3). It follows that $\widetilde{a}$ is a local ray at t_0 in the manifold sense, and therefore $\widetilde{a}$ is an orthogonal geodesic. This proves that the stacky curve a is locally a geodesic, and therefore a geodesic by Lemma 3.1.2. □

3.4 Stacky geodesics

In Riemannian geometry, the concept of geodesic is the generalization of straight lines in Euclidian space. We will define geodesics for Riemannian stacks and present basic properties. Our definition extends previous definitions of geodesics on orbifolds and orbit spaces of isometric actions [41, 31].

Definition 3.4.1. A **geodesic** on a Riemannian stack $([M/G], [\eta])$ is a curve $\alpha : I \to [M/G]$ that can be represented by a geodesic fraction.

We need to check that the above definition is, in fact, a Riemannian stack object. We will show that this definition is invariant by the equivalence of metrics and Riemannian Morita fibrations. The fact that geodesic fractions do not depend on the metric in the class comes from the characterization of the geodesic fractions as minimizing distances curves in the Theorem 3.3.2.

Proposition 3.4.2. Let $G \rightrightarrows M$ be a proper groupoid. Consider η_1, η_2 equivalent groupoid metrics on G. A fraction $a/\phi_{\mathcal{U}} : I \dashrightarrow G$ is equivalent to a geodesic fraction of η_1 if and only if it is equivalent to a geodesic fraction of η_2.

Proof. A curve being a local ray depends only on the normal distance d_N and the normal length. We have seen that d_N and ℓ_N are invariant in the metric classes. So, if $a/\phi_{\mathcal{U}}$ is equivalent to a geodesic fraction of η_1, then it is a local ray for both η_1 and η_2. By Proposition 3.3.2 we conclude that $a/\phi_{\mathcal{U}}$ is equivalent to a geodesic fraction of η_2. □

Since ϕ is a Morita map, if $\widetilde{a}/\phi_{\widetilde{\mathcal{U}}} : I \dashrightarrow \widetilde{G}$ and $\widetilde{b}/\phi_{\widetilde{\mathcal{V}}} : I \dashrightarrow \widetilde{G}$ are equivalent fractions, then $(\phi \circ \widetilde{a})/\phi_{\widetilde{\mathcal{U}}}$ and $(\phi \circ b)/\phi_{\widetilde{\mathcal{V}}}$ are equivalent. The same holds for lifts, i.e., if $\widetilde{a}/\phi_{\widetilde{\mathcal{U}}}$ and $\widetilde{b}/\phi_{\widetilde{\mathcal{V}}}$ are lifts of equivalent curve fractions $a/\phi_{\mathcal{U}}$ and $b/\phi_{\mathcal{V}}$ in G, then they are equivalent. Note that we only need to show that ϕ sends geodesic fractions to geodesic fractions, and it is possible to lift geodesic fractions (up to equivalence class) to geodesic fractions along Riemannian Morita fibrations. The next proposition states precisely what we need.

Proposition 3.4.3. Let $\phi : (\widetilde{G}, \widetilde{\eta}) \to (G, \eta)$ be a Riemannian Morita fibration. The following holds:

i) ϕ projects geodesic fractions to geodesic fractions;

ii) for each geodesic fraction $a/\phi_{\mathcal{U}}$ in G, there is an equivalent geodesic fraction $a/\phi_{\mathcal{W}}$ in G, which admits a lift to a geodesic fraction $\widetilde{a}/\phi_{\mathcal{W}}$ in $\widetilde{G}$.

Proof. i) Let $\widetilde{a}/\phi_{\widetilde{\mathcal{U}}} : I \dashrightarrow \widetilde{G}$ be a geodesic fraction. The maps $\widetilde{a}_{ij} : U_i \cap U_j \to \widetilde{G}$ are orthogonal geodesics, so the maps $\phi \circ \widetilde{a}_{ij} : U_i \cap U_j \to G$ are also orthogonal geodesics by the observation that Riemannian Morita maps send the normal spaces of $\widetilde{G}$ isometrically into the normal spaces of G.

ii) Because $\phi^{(1)} : \widetilde{G} \to G$ is a Riemannian submersion which sends the normal spaces in $\widetilde{G}$ to the normal spaces in G, we can locally lift orthogonal geodesics on G to orthogonal geodesics on $\widetilde{G}$. Let $\alpha/\phi_{\mathcal{U}}$ be a geodesic fraction in G. Consider $\mathcal{W} = \{W_k\}$ a refinement of $\mathcal{U}$ which is a good covering, and such that the geodesics $\alpha_{kl} : W_k \cap W_l \to G$ have global lifts $\widetilde{a}_{kl} : W_k \cap W_l \to \widetilde{G}$. Then the geodesic fraction $\widetilde{a}/\phi_{\mathcal{W}}$ is a lift of the geodesic fraction $a/\phi_{\mathcal{W}}$, which is equivalent to $a/\phi_{\mathcal{U}}$.

□

3.5 Hopf-Rinow theorem

One of the foundational results of Riemannian geometry is the Hopf-Rinow theorem. This theorem states that completeness as metric space for a Riemannian manifold is equivalent to the geodesic completeness condition (i.e. all geodesics are defined for all time). We present here a stacky version of this theorem, whose statement is basically the same, but the proof contains significant differences. For instance, the geodesics on stacks are not local distance minimizers (see 3.3.1).

We start with an elementary comment and lemma about concatenation of stacky curves, which we will use in the proof of the Hopf-Rinow's theorem. So far, we have avoided the use of "piecewise smooth" stacky curves, but we will briefly cover the cases of interest.

Given two stacky curves $\alpha, \beta : I \to [M/G]$ through the same point, i.e. $\overline{\alpha}(0) = \overline{\beta}(0) = \bar{x}$, it is possible to combine them into a continuous stacky curve ξ, called **concatenation**, defined by

$$\xi(t) = \begin{cases} \alpha(t) & t \leq 0 \\ \beta(t) & t \geq 0. \end{cases}$$

The resulting curve is not necessarily a smooth stacky curve. But the velocity $\xi'(t)$ is defined everywhere except for $t = 0$, so it still makes sense to compute its normal length $\ell_N(\xi)$.

If a stacky curve α is presented by a fraction $\widetilde{\alpha}_{ij}/\phi_U$ where the pieces $\widetilde{\alpha}_i$ are curves orthogonal to the characteristic foliation $\mathcal{F}_M$, then we say that α is an **orthogonal curve**.

Lemma 3.5.1. If $\alpha, \beta : (-\epsilon, \epsilon) \to [M/G]$ are orthogonal curves through the same point such that their concatenation $\xi = \alpha * \beta$ is a distance minimizer, then ξ is actually a stacky curve, which is moreover geodesic.

Proof. Because of the local nature of the problem, we can suppose that both are curves in M. Orthogonal curves are preserved by horizontal lifts and projections along s and t. We can suppose that α and β are orthogonal curves on M which start at the same point x in M. Since $\xi'(t)$ is orthogonal where it is defined, the following equality holds: $\ell_N(\xi|_{[a,b]}) = \ell(\xi|_{[a,b]})$. Therefore

$$d(\xi(a), \xi(b)) \geq d_N\left(\overline{\xi}(a), \overline{\xi}(b)\right) = \ell(\xi|_{[a,b]}).$$

Thus ξ is a curve in M that minimizes distance, and by classic differential geometry, ξ has to be a geodesic on M. □

We say that a Riemannian stack $([M/G],[\eta])$ is **geodesically complete** if any geodesic $\alpha : I \dashrightarrow G$ can be extended to a geodesic $\alpha : \mathbb{R} \dashrightarrow G$.

Theorem 3.5.2 (Hopf-Rinow)**.** The stack $([M/G],[\eta])$ is geodesically complete if and only if $(M/G, d_N)$ is a complete metric space.

Proof. Suppose first that $(M/G, d_N)$ is complete. Let $\alpha : I \to [M/G]$ be a geodesic with $\|\alpha'(t)\| = 1$, and let $I = (a,b) \subset \mathbb{R}$ be the maximal interval where α is defined. Suppose that $b < \infty$, and consider t_n an increasing sequence converging to b. Thus $d_N(\overline{\alpha}(t_n), \overline{\alpha}(t_m)) \leq |t_n - t_m|$, and therefore

$$\lim_{n\to\infty} \overline{\alpha}(t_n) = \bar{x}.$$

Fix $x \in \pi^{-1}(\bar{x})$. There exists $\epsilon > 0$ and a neighborhood $U \subset M$ of x where every unitary geodesic is defined at least for time $(-\epsilon, \epsilon)$. Since $\pi(U)$ is an open neighborhood of $\bar{x}$, there is n_0 sufficiently large such that $\overline{\alpha}(t_{n_0})$ is in $\pi(U)$ and $b - t_{n_0} < \epsilon$. Choosing $z \in U$ such that $\pi(z) = \overline{\alpha}(t_{n_0})$, and choosing $v \in T_zM$ a normal vector representing $\alpha'(t_{n_0})$, we extend α to the interval $(a, t_{n_0} + \epsilon)$. This contradicts the maximality of (a,b).

We assume now that the stack $[M/G]$ is geodesically complete. The argument mimics the standard proof of Hopf-Rinow for manifolds. Given $\bar{x}, \bar{y}$ points in M/G, we will show that there exists a minimizing-distance geodesic α connecting them. Fix x in the orbit $\bar{x}$, consider the normal $B_\epsilon(\bar{x})$ given by Gauss lemma 3.2.3 applied to x. Since $S_\epsilon(\bar{x})$ is compact, there is a direction $[v]$ with $\|[v]\| = 1$ such that

$$d_N(\bar{x}, \overline{\exp}(\epsilon[v])) = \inf\{d_N(\bar{y}, \overline{\exp}([w])) : w \in S_\epsilon(\bar{x})\}.$$

Denote by α the geodesic through $\bar{x}$ with velocity $[v]$. We are going to show that $\overline{\alpha}(r) = \bar{y}$, where $r = d_N(\bar{x}, \bar{y})$. Consider the times at which α minimizes distances from $\bar{x}$ to $\bar{y}$:

$$A := \{s \in [0,r] \,|\, d_N(\bar{y}, \alpha(s)) = d_N(\bar{x}, \bar{y}) - s\}.$$

Note that $0 \in A$ and that A is closed by continuity. Let b be the supremum of A. Suppose that $b < r$, otherwise we are done. By applying the stacky Gauss lemma 3.2.3, and reasoning as before, we know that there exists a direction $[w]$ in $T_{\overline{\alpha}(b)}[M/G]$ defining a geodesic β starting at $\overline{\alpha}(b)$ satisfying

$$d_N(\overline{\beta}(s), \bar{y}) = d_N(\overline{\alpha}(b), \bar{y}) - s = r - b - s.$$

By the triangle inequality,

$$d_N\left(\bar{x}, \overline{\beta}(s)\right) \geq d_N(\bar{x}, \bar{y}) - d_N(\overline{\beta}(s), \bar{y}) = r - (r - b - s) = b + s.$$

The concatenation ξ of α and β minimizes distances by Lemma 3.5.1. By the global uniqueness of geodesics, we have $\xi = \alpha$, thus α minimizes distances even after r, contradicting the maximality.

Finally, since any limited set A of M/G is contained in a normal ball $B_R(\bar{x})$ for some $R \in \mathbb{R}$, its closure is compact, and in particular, any Cauchy sequence must be convergent. □

Corollary 3.5.3. If the coarse orbit space M/G is compact, then $([M/G], [\eta])$ is geodesically complete.

Corollary 3.5.4. If $(G \rightrightarrows M, \eta)$ is a proper Riemannian groupoid, then there is a smooth function on $G^{(2)}$ such that $([M/G], [f\eta])$ is geodesically complete.

Proof. For each point $\bar{x} \in M/G$ set the value

$$b(\bar{x}) = \sup\{r : \overline{B_r(\bar{x})} \text{ is compact}\}.$$

If b reaches infinity at some point, the triangle inequality implies that every bounded set in M/G is compact, and from Hopf-Rinow $[\eta]$ is complete. We assume that $b(\bar{x}) < \infty$ for all points in M. The function b satisfies

$$|b(\bar{x}) - b(\bar{y})| \leq d_N(\bar{x}, \bar{y}).$$

The Gauss lemma shows that $b(\bar{x}) > 0$ for all $\bar{x} \in M/G$. Hence, $1/b(\bar{x})$ is a continuous function.

Using a partition of unity we can construct a smooth function f_0 on M such that $h(x) \geq 1/b(\bar{x}) > 0$. Choosing a normalized Haar system $\{\mu^x\}_{x \in M}$ we can take the average $\underline{h}$ which is an invariant smooth function (see [18, p.51]), and satisfies $\underline{h}(x) \geq 1/b(\bar{x})$. Define f to be the smooth function on the composable arrows giving by $f = \pi_1^* s^* \underline{h}$. Because $\underline{h}$ is constant along the orbits, f is invariant under the S_3 action on $G^{(2)}$ and consequently constant along the fibers of π_2 and m. Thus, $\widetilde{\eta} = f^2\eta$ is a groupoid metric on $G \rightrightarrows M$ by the properties of f.

The sets in M/G with diameter smaller than $1/3$ with respect to $[\widetilde{\eta}]$ are relatively compact. Fix $\bar{x} \in M/G$. Denote by $\widetilde{B}_{1/3}(\bar{x})$ the ball for the distance of $[\widetilde{\eta}]$. Let $\bar{y}$ be a point in $\widetilde{B}_{1/3}(\bar{x})$, and suppose that $b(\bar{x}) \leq d_N(\bar{x}, \bar{y})$. Then $b(\bar{x}) \leq \ell_N(\alpha)$ for any generalized curve α connecting $\bar{x}$ to $\bar{y}$. We give a lower bound for the $[\widetilde{\eta}]$-length of α in terms of its $[\eta]$-length:

$$\begin{aligned} \widetilde{\ell}_N(\alpha) &= \int_I \overline{f} \circ \alpha(t) \, \|\alpha'(t)\| \, dt \\ &= \overline{f}(\alpha(t_0)) \ell_N(\alpha) \\ &> \frac{\ell_N(\alpha)}{b(\alpha(t_0))}, \end{aligned}$$

where the second equality is ensured by the mean value theorem for definite integrals of continuous functions, since $\overline{f} : M/G \to \mathbb{R}$ and the speed $\|\alpha'(t)\|$ are continuous functions. Because $b(\alpha(t_0)) \leq b(\bar{x}) + d_N(\bar{x}, \alpha(t_0))$, we have $b(\alpha(t_0)) \leq b(\bar{x}) + \ell_N(\alpha)$. Hence,

$$\widetilde{\ell}_N(\alpha) \quad > \quad \frac{\ell_N(\alpha)}{b(\bar{x}) + \ell_N(\alpha)} \geq \frac{L}{2L + L} = \frac{1}{3},$$

and we conclude that $\widetilde{d}_N(\bar{x}, \bar{y}) \geq 1/3$, a contradiction with the fact that $\bar{y} \in \widetilde{B}_{1/3}(\bar{x})$. Thus $\widetilde{B}_{1/3}(\bar{x}) \subset B_{b(\bar{x})/2}(\bar{x})$ which is compact.

Because closed sets with diameter smaller than 1/3 are compact, we can ensure that any Cauchy sequence is convergent. Therefore, by the Hopf-Rinow theorem $[f^2\eta]$ is complete. □

Chapter 4

Linearization versus completeness

In this chapter, we deal with the relations between the linearization of groupoids and the existence of complete metrics. This chapter will be organized as follows:

- In Section 4.1 we recall linearization and invariant linearization for groupoids and some related results.
- In Section 4.2 we review the characterization of locally trivial fiber bundles as the submersions that admit a complete and fibered metric and relate this with a possible characterization of the invariantly linearizable groupoids.
- In Section 4.3 we show that the existence of complete groupoid metrics implies the existence of invariant linearization. As a consequence, we deduce the Tube theorem for proper Lie group actions from the groupoid linearization perspective.
- In Section 4.4 we build a transversely invariant metric from the hypothesis of the existence of invariant linearizations, and we discuss how we expect to improve it into a characterization of the invariant linearizable groupoids.

4.1 Review of linearization

The linearization of a Lie groupoid around an orbit consists of establishing an isomorphism between the linear model (see Sec. 3.2) and the original groupoid in suitable neighborhoods. There are some possibilities in the choice of these neighborhoods, as we will see. Let $G \rightrightarrows M$ be a Lie groupoid, and let $\mathcal{O} \subset M$ be an orbit. We say that:

- G is **weakly linearizable** around $\mathcal{O}$ if there is a groupoid isomorphism

  $$(\widetilde{V} \rightrightarrows V) \overset{\phi}{\cong} (\widetilde{U} \rightrightarrows U),$$

  where $\widetilde{V} \rightrightarrows V$ and $\widetilde{U} \rightrightarrows U$ are neighborhoods of $G\mathcal{O} \rightrightarrows \mathcal{O}$ in $NG\mathcal{O} \rightrightarrows N\mathcal{O}$ and $G \rightrightarrows M$, respectively, and ϕ is the identity on $G\mathcal{O} \rightrightarrows \mathcal{O}$.

- G is **linearizable** around the orbit $\mathcal{O}$ if $\widetilde{V} = NG\mathcal{O}_V$ and $\widetilde{U} = G_U$.

- G is **invariantly linearizable** at $\mathcal{O}$ if it is linearizable, and the neighborhoods U and V can be taken saturated.

We call the pair $(\widetilde{V} \rightrightarrows V, \phi)$ a linearization around $\mathcal{O}$; we will not mention ϕ unless it is necessary. If $G \rightrightarrows M$ is invariantly linearizable around any orbit, we call it an **invariantly linearizable groupoid**.

The multiplicative property of the geodesics on a Riemannian groupoid (see Eq. 2.3) shows that the exponential map in the normal directions of an orbit is a groupoid map. This provides the following linearization theorem:

Theorem 4.1.1 ([24], Thm.5.11)**.** Let $(G \rightrightarrows M, \eta)$ be a Riemannian groupoid, and let $\mathcal{O} \subset M$ be an orbit. Then the exponential map defines a weak linearization of G around $\mathcal{O}$.

Linearization results for proper groupoids are present in [19, 58, 62]. In [24] the authors deduce from the above theorem the main result on linearization of proper groupoids:

Corollary 4.1.2 (cf. Thm.3.2.1)**.** If $G \rightrightarrows M$ is a proper groupoid and $\mathcal{O} \subset M$ is an orbit, then G is linearizable at $\mathcal{O}$.

Invariant linearization of s-proper groupoids covers a large number of related classical results: Ehresmann's fibration theorem [27], Reeb's local stability theorem for foliations [48], and linearization of compact group actions [44].

Corollary 4.1.3. If $G \rightrightarrows M$ is a Lie groupoid whose source map is proper and $\mathcal{O} \subset M$ is an orbit, then G has an invariant linearization around $\mathcal{O}$.

Recall that the existence of invariant linearizations for action groupoids which come from proper Lie group actions is covered by the Tube theorem for proper Lie group actions [46]. The above corollary does not cover this case since the existence of invariant linearization holds without requiring s-properness. We will see how Theorem 4.1.1 can be used to provide invariant linearizations in this case (see Corollary 4.3.5).

4.2 Invariant linearization: from submersions to groupoids

We state in this section how we plan to deal with a characterization of the invariantly linearizable groupoids. A characterization of invariantly linearizable groupoid has been discussed in many works [19, 20, 24]. There is a particular case where this is well understood: locally trivial fiber bundles. In this case, a submersion being a locally trivial fiber bundle is equivalent to its submersion groupoid being invariantly linearizable.

The locally trivial fiber bundles are characterized by the existence of complete fiber metrics. This result has been claimed by several authors [29, 38, 60], but the proofs there always relied on the false assumption that fibered metrics are closed under convex combinations [24, Ex. 2.3]. Then [21] provides a correct proof of the existence of complete fibered metrics:

Theorem 4.2.1. Given $p : M \to B$ a submersion, then p is locally trivial if only if there is a p-fibered and complete metric on M.

The locally trivializations are obtained from the exponential map of the complete metric, observing that the orthogonal bundle to fibers is trivial and using the following proposition:

Proposition 4.2.2. ([24, Prop.5.9]) Let $p : (M, \eta^M) \to (B, \eta^B)$ be a Riemannian submersion with η^M a complete metric. If $S \subset B$ is an embedded submanifold and $\tilde{S} = p^{-1}(S)$, then for any open subsets $\tilde{S} \subset \tilde{U} \subset N\tilde{S}$ and $S \subset U \subset NS$ such that $dp(\tilde{U}) \subset U$, the following square commutes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\exp} & M \\ {\scriptstyle dp}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\exp} & B \end{array}$$

Moreover, if $\exp : U \to B$ is an embedding then $\exp : \tilde{U} \to M$ is also an embedding.

The original statement in [24] does not require completeness on the metric η^M, but this simplifies notations and covers our interests. This proposition together with Theorem 4.1.1 reduce the invariantly linearization problem to the problem of finding saturated tubular neighborhoods. We deduce the existence of saturated tubular neighborhoods from the existence of groupoid metrics which are complete on the units, see Proposition 4.3.1.

The strategy used in [21] to show the existence of a complete fibered metric has three fundamental steps:

1° choose a complete metric η^B on the base;

2° choose a complete metric η^F on the fiber;

3° gluing the product metric $\eta^B \times \eta^F$ on the local trivializations by a smart partition of unit ("tube trick").

This process produces a p-fibered and complete metric η^M on M. The metric η^M can be extended to a groupoid metric on the submersion groupoid $M \times_B \rightrightarrows M$, see Example 2.1.5. We conclude that: if the submersion groupoid $M \times_B M \rightrightarrows M$ is invariantly linearizable, then it admits a groupoid metric that is complete on the units.

Fibered metrics on submersions are the same as transversely invariant metrics on their submersion groupoids. Recall that if $G \rightrightarrows M$ is a Lie groupoid then a **transversely invariant** metric on M is a metric such that the normal representation acts by isometries. We use the strategy of [21] to build complete transversely invariant metrics on invariantly linearizable groupoids, see 4.4.1. As in the case of submersions, we expect to extend our result to

If $G \rightrightarrows M$ is an invariantly linearizable proper Lie groupoid, then there exists a groupoid metric η such that the metric η^M is complete.

This statement together with Proposition 4.3.1 would lead to a characterization for the invariantly linearizable groupoids:

A proper groupoid is invariantly linearizable if and only if it admits a groupoid metric complete on the units.

4.3 Completeness implies invariant linearization

In Riemannian geometry the compactness of a Riemannian manifold can be replaced by completeness in many circumstances. In this spirit, we will substitute the s-properness condition for the existence of invariant linearizations by a completeness condition. We will show that the orbits have a positive injective radius, i.e., the normal exponential is an embedding for vectors with norm smaller than a fixed number. This combined with the proof of Theorem 4.1.1 provides the invariant linearizations.

Proposition 4.3.1. If $(G \rightrightarrows M, \eta)$ is a proper Riemannian groupoid such that η^M is complete and $\mathcal{O}$ is an orbit, then $G \rightrightarrows M$ has an invariant linearization around $\mathcal{O}$.

Proof. Since $G \rightrightarrows M$ is proper and the metric η^M is complete the orthogonal geodesics on G are defined for all time by Lemma 3.1.8. Let $V \subset N\mathcal{O}$ be a neighborhood around $\mathcal{O}$. Because η is a groupoid metric the exponential map

$$\exp : (NG\mathcal{O}_V \rightrightarrows V) \to (G \rightrightarrows M)$$

is a groupoid map. The fact that the source maps is a Riemannian submersion and $ds(NG\mathcal{O}_V) \subset V$ says that: if the map $\exp : V \to M$ is an embedding, then $\exp : NG\mathcal{O} \to G$ is an embedding (see [24, Prop. 5.9]). We will conclude the theorem if we provide a saturated neighborhood $V \subset N\mathcal{O}$ of $\mathcal{O}$ such that $\exp : V \to M$ is an embedding, and $\exp(V)$ is saturated. The next lemmas will cover these points. □

Let S be a closed saturated manifold in $G \rightrightarrows M$. We will denote $V_r := \{v \in NS : \|v\| < r\}$. For a point x_0 in M, we will see that the saturation of $V_r|_{x_0}$ is V_r. Suppose $(x_0, gv) \xleftarrow{(g,v)} (y, v)$ is an arrow in NG_S with (x_0, gv) in V_r. The norm of normal vectors are preserved by the normal representation, so (y, v) is in V_r. We conclude that V_r is saturated for all r.

Lemma 4.3.2. Let $(G \rightrightarrows M, \eta^M)$ be a proper Riemannian groupoid with the metric η^M being complete. If $S \subset M$ is a saturated submanifold, then for all $r > 0$ the image of the map $\exp : V_r \to M$ is a saturated set.

Proof. Suppose that $\mathcal{O}$ is an orbit that intersects $\exp(V_r)$. Let p be a point in $\mathcal{O}$. Take an arrow $p \xleftarrow{g} q$ with $q \in \mathcal{O} \cap \exp(V_r)$. So, there is an orthogonal geodesic β connecting S to q with $\|\beta'(0)\| < r$. The completeness on the orthogonal geodesics in G allows us to lift β along s to a orthogonal geodesic starting in g. This geodesic projects along t to an orthogonal geodesic connecting p to S with length smaller than r, so p is in $\exp(V_r)$. $\square$

Lemma 4.3.3. Let $(G \rightrightarrows M, \eta^M)$ be a proper Riemannian groupoid with the metric η^M being complete. If $\mathcal{O}$ is an orbit of G, then there is $\epsilon > 0$ such that $\exp : V_\epsilon \to M$ is regular.

Proof. Fix a point $x_0 \in \mathcal{O}$. Take U a relatively compact neighborhood of x_0 in $\mathcal{O}$. By the compactness of $\overline{U}$ there is $\epsilon > 0$ such that $\exp : V_\epsilon|_U \to M$ is an embedding.

To show that $\exp : V_\epsilon \to M$ is regular, let (y, w) be a point in V_ϵ. Take an arrow $(x_0, w_0) \xleftarrow{(g,\widetilde{w})} (y, w)$ in $NG\mathcal{O}_{V_\epsilon}$. Since the target map is a Riemannian submersion, we have the following map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker d_{(g,\widetilde{w})}(dt) & \longrightarrow & T_{(g,\widetilde{w})}NG\mathcal{O} & \xrightarrow{d(dt)} & T_{(x_0,w_0)}N\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d\exp} & & \downarrow & & \\ 0 & \longrightarrow & \ker d_{\exp(g,\widetilde{w})}t & \longrightarrow & T_{\exp(g,\widetilde{w})}G & \xrightarrow{dt} & T_{\exp(x_0,w_0)}M & \longrightarrow & 0 \end{array}$$

The first arrow is a surjection between the tangent spaces of the fibers of dp and p, and the last arrow is an isomorphism by hypothesis. This implies that the middle arrow is an isomorphism. As before, we have the following diagram for the source map:

$$\begin{array}{ccc} T_{(g,\widetilde{w})}NG\mathcal{O} & \xrightarrow{d(ds)} & T_{(y,w)}N\mathcal{O} \\ \downarrow{\scriptstyle d\exp} & & \downarrow{\scriptstyle d\exp} \\ T_{\exp(g,\widetilde{w})}G & \xrightarrow{ds} & T_{\exp(y,w)}M. \end{array}$$

We conclude that $d_{(y,w)}\exp : T_{(y,w)}N\mathcal{O} \to T_{\exp(y,w)}$ is an isomorphism. $\square$

Lemma 4.3.4. Let $(G \rightrightarrows M, \eta^M)$ be a proper Riemannian groupoid with the metric η^M being complete. If $\mathcal{O}$ is an orbit, then there is $\epsilon > 0$ such that $\exp : V_\epsilon \to M$ is injective.

Proof. Fix a point $x_0 \in \mathcal{O}$. Take U a relatively compact neighborhood of x_0 in $\mathcal{O}$. By the compactness of $\overline{U}$ there is $\delta > 0$ such that $\exp : V_r|_U \to M$ is a diffeomorphism for all $r < \delta$. Take $\epsilon < \delta$ sufficiently small such that the intersection of $\mathcal{O}$ with the normal neighborhood $B_{2\epsilon}(x_0)$ is inside of U. Thus, $\exp : V_\epsilon|_{\mathcal{O}\cap B_{2\epsilon}(x_0)} \to M$ is injective.

Let (x, v) and (y, w) be different points in V_ϵ with $\exp(x, v) = \exp(y, w)$. Denote $a(t) = \exp(x, tv)$ and $b(t) = \exp(y, tw)$. So, a and b are orthogonal geodesics. Let $x_0 \xleftarrow{g} x$ be an arrow connecting x to x_0. Denote by γ the lift of a along s through g; the lift is global because the orthogonal geodesics in G are complete. Denote by a_1 the projection of γ along t. We also lift the orthogonal geodesic $\beta(1-t)$ along s through the arrow $a_0(1) \xleftarrow{\gamma(1)} a(1)$ to a orthogonal geodesic in $\xi(t)$ in G. Consider $b_0(t)$ the geodesic $(t \circ \xi)(t)$.

Observe that $a_0(0) = x_0$ and $a_0(1) = b_0(1)$. Denote by $y' = b_0(0)$, $v' = a_0'(0)$, and $w' = b_0'(0)$. We have $\exp(x_0, v') = \exp(y', w')$. By our construction the following holds:

$$d(x_0, y') \leq d(x_0, a_0(1)) + d(b_0(1), y') < 2\epsilon.$$

This implies that y' is in $B_{2\epsilon}(x_0)$, which is a contradiction with the fact that $\exp : V_\epsilon|_{\mathcal{O}\cap B_{2\epsilon}(x_0)} \to M$ is injective. Therefore $\exp : V_\epsilon \to M$ is injective. □

Application to proper Lie group actions

We can use Proposition 4.3.1 to deduce the Tube theorem for proper Lie groups actions (cf. [46]). We give a proof for the existence of complete invariant metrics for proper actions (cf. [35]), then we construct a groupoid metric as in Example 2.1.6 which is complete on the units.

Corollary 4.3.5. If $G \curvearrowright M$ is a proper Lie group action, then $G \ltimes M \rightrightarrows M$ is invariantly linearizable.

Proof. By Example 2.1.6, we can build a groupoid metric on $G \ltimes M$ from a right-invariant metric on G and a G-invariant metric on M. The metric on M induced by the groupoid metric on $G \ltimes M$ is the quotient of the product metric on $G \times M$ by the diagonal action. Since right-invariant metrics are complete because they are homogeneous (see [37, p. 175]), we conclude the corollary if we show that there exist complete invariant metrics. This is the content of the next lemma. □

Lemma 4.3.6. If $\theta : G \curvearrowright M$ is a proper Lie group action, there is a complete invariant metric on M.

Proof. Fix μ a left-invariant volume form on G. There exists a cut-off function c for the action $\theta : G \curvearrowright M$, i.e., a smooth function with the following properties:

- the saturation of $\operatorname{supp} c$ is equal to M;
- $(G \times \operatorname{supp} c) \cap \theta^{-1}(K)$ is compact if K is compact;
- $\int_G c(gx)\mu = 1$ for all x in M.

See [18, Prop. 11.6] for the existence of cut-off functions. So, we can take the average of a metric η on M:

$$\underline{\eta}_x(v, w) = \int_G \eta_{gx}(gv, gw)\, c(gx)\mu.$$

Thus $\underline{\eta}$ is a G-invariant metric on M. The same argument used in Corollary 3.5.4 implies that there is a smooth and G-invariant function $f : M \to \mathbb{R}$ such that $f\underline{\eta}$ induces a complete distance on M/G. So, we can assume η invariant and such that $(M/G, d_N)$ is complete.

We use the homogeneity along the orbits to conclude the proof. Suppose $a : (a, b) \to M$ is an unitary geodesic in its maximal interval of definition. Because d_N is complete, and $d_N(\overline{a}(t), \overline{a}(s)) \leq |t - s|$ the limit $\lim_{t\to b} a(t)$ exits. So, denote by $\mathcal{O}$ the orbit which is the $\lim_{t\to b} \overline{a}(t)$. Fix x in $\mathcal{O}$. There exists $\epsilon > 0$ and a neighborhood $U \subset M$ of x where every unitary geodesic is defined at least for time $(-\epsilon, \epsilon)$. The orbit is inside of the open $G \cdot U$. Thus there is $t_0 > b - \epsilon$ such that $a(t_0)$ is in $g \cdot U$ for some g in G. Because $g : U \to g \cdot U$ is an isometry we can extend a to $(a, t_0 + \epsilon)$, and we get a contradiction.

□

Remark 4.3.7. The proof above is independent of the Tube theorem. The only fact in the proof that could be dependent on Tube theorem relies on the existence of cut-off functions, but it is an independent property of the proper groupoids as we can see in [18, Prop. 11.6].

4.4 Invariant linearization implies completeness

We present in this section the groupoid versions of the three steps of the strategy [21] to built complete fibered metrics, which lead us to the following result:

Proposition 4.4.1. If $G \rightrightarrows M$ is an invariantly linearizable proper groupoid, then there is a transversely invariant complete metric $\widetilde{\eta}$ on M.

Remark 4.4.2. The submersion groupoids are ever proper, and we have seen that the existence of complete transversely invariant metrics implies locally triviality of the submersion. So, any submersion which is not locally trivial is a counterexample for the completeness stated in the result [47, Prop 3.14].

Step 1: complete metric on the base

We will assume M to be connected. Let η be a groupoid metric on $G \rightrightarrows M$ such that $([M/G],[\eta])$ is complete; the existence of such metric is ensured by Corollary 3.5.4. We use the following lemma to split the proof between the cases where all the orbits are compact or all the orbits are non-compact.

Lemma 4.4.3. If $G \rightrightarrows M$ is an invariantly linearizable proper groupoid and M is a connected manifold, then all orbits of G are either compact or non-compact.

Proof. It is enough to prove that being compact or being non-compact are open conditions. Let $\mathcal{O}$ be an orbit of $G \rightrightarrows M$. Take $NGO_V \rightrightarrows V$ an invariant linearization around $\mathcal{O}$. Fix $x \in \mathcal{O}$. The intersection $\mathcal{O}' \cap N_x\mathcal{O}$ of an orbit $\mathcal{O}'$ wiht $N_x\mathcal{O}$ is an orbit $\mathcal{O}_l$ of the normal representation $G_x \curvearrowright N_x$. Since G_x is compact, the orbit $\mathcal{O}_l$ is compact. So, the orbit $\mathcal{O}'$ is the quotient $\mathcal{O}_l \times P_x/G_x$. We conclude that $\mathcal{O}'$ is a bundle over $\mathcal{O}$ with compact fibers. Hence, $\mathcal{O}'$ is compact if only if $\mathcal{O}$ is compact. □

If the orbits are compact, then we apply the following lemma:

Lemma 4.4.4. Let $(G \rightrightarrows M, \eta)$ be a Riemannian groupoid with all orbits being compact. If $([M/G],[\eta])$ is complete, then η^M is complete.

Proof. Let $a:(p,q) \to M$ be a geodesic in its maximal definition interval. The completeness of $[M/G]$ implies the existence of the limit point $\lim_{t\to q} \pi \circ a(t) = \mathcal{O}$. Because $G \rightrightarrows M$ is invariantly linearizable and $\mathcal{O}$ is compact, there is an invariant neighborhood V of $\mathcal{O}$ with closure $\overline{V}$ compact. This implies that the set $a^{-1}((q-\delta,q))$ is included in a compact. Thus a is extendable, which is a contradiction. Hence, a is defined for all time. □

If the orbits are non-compact we will locally modify the metric η to obtain an equivalent metric which is complete in the orbits directions. Then we obtain a complete transversely invariant metric $\widetilde{\eta}$ on M by gluing the modified metrics with a "tube trick".

Step 2: complete metric on the orbits

The next lemma re-interprets the conditions that we need in the tube trick and provides an analog of the "choice of a complete metric on the fiber".

Lemma 4.4.5. Let $O \subset M$ be an orbit of G. Take V a saturated neighborhood of $\mathcal{O}$ in $N\mathcal{O}$ such that $\pi(V)$ is relatively compact in M/G. Given η a groupoid metric in $NG\mathcal{O}_V$, there exists a groupoid metric $\hat{\eta}$ in $NG\mathcal{O}_V$ equivalent to η with the following property:

If $C \subsetneq V$ is a closed and limited set with respect to $\hat{\eta}$, then it is a compact.

Proof. Fix x in $\mathcal{O}$. Denote $N = N_x\mathcal{O} \cap V$, $H = G_x$ and $P = G(-,x)$. We have the following pair of Morita fibrations:

$$\begin{array}{ccc} & (H \ltimes N \rightrightarrows N) \times (P \times P \rightrightarrows P) & \\ \swarrow^{\psi} & & \searrow^{\phi} \\ H \ltimes N \rightrightarrows N & & NG\mathcal{O}_V \rightrightarrows N\mathcal{O}. \end{array}$$

We pullback the metric η along ϕ to a metric $\widetilde{\eta}$ on $(H \ltimes N) \times (P \times P)$ (see Prop. 2.2.1). Denote by $\underline{\widetilde{\eta}}$ the cotangent average of $\widetilde{\eta}$ with respect to the kernel of ψ (see Prop. 2.2.2). So, there is a groupoid metric $\check{\eta}$ on $H \ltimes N$ giving by the projection of $\underline{\widetilde{\eta}}$ along ψ.

Observe that $K = \ker \phi = \{(h, v, ha^{-1}, a) : (h, v, a) \in H \times N \times P\}$, and that the fiber product

$$K^3 \times_{(N \times P)^3} (H^2 \times N \times P^3)$$

is isomorphic to $H^3 \times H^2 \times N \times P^3$. So, the groupoid action $K^3 \curvearrowright H^2 \times N \times P^3$ is reduced to a group action $H^3 \curvearrowright H^2 \times N \times P^3$,

$$(k_3, k_2, k_1) \cdot (h_2, h_1, v, a_3, a_2, a_1) = (k_3 h_2 k_2^{-1}, k_2 h_1 k_1^{-1}, k_1 v, a_3 k_3^{-1}, a_2 k_2^{-1}, a_2 k_2^{-1}).$$

Note that we can split the action above into two actions $H^3 \curvearrowright P^3$ and $H^3 \curvearrowright H^2 \times N$. Since H^3 is compact we can take the cotangent average of $\check{\eta}$ with respect to the action $H^3 \curvearrowright H^2 \times N$, which is a H^3-invariant groupoid metric for $H \ltimes N$. Denote by $\underline{\check{\eta}}$ the cotangent average. Consider on P^3 a metric η^{P^3} given by the product of a H-invariant complete metric η^P on P. The existence of such invariant complete metric is guaranteed by Lemma 4.3.6. Thus, the product metric $\underline{\check{\eta}} \times \eta^{P^3}$ is H^3-invariant. This metric is projected along ϕ to a groupoid metric $\hat{\eta}$ in $NG\mathcal{O}_V$. We will see that $\hat{\eta}$ satisfies the desired properties.

We will use here the fact that pullback and pushforward along Morita fibrations preserve the metric classes. The metrics $\underline{\check{\eta}} \times \eta^{P^3}$ and $\underline{\widetilde{\eta}}$ have equivalent projections $\underline{\check{\eta}}$ and $\check{\eta}$ on $H \ltimes N$, so they are equivalent. Recall that, $\widetilde{\eta}$ is equivalent to its cotangent average $\underline{\widetilde{\eta}}$. The metrics $\hat{\eta}$ and η are projections of equivalent metrics $\underline{\check{\eta}} \times \eta^{P^3}$ and $\widetilde{\eta}$ respectively, so they are equivalent.

Let $C \subset V$ be a closed and limited set with respect to $\hat{\eta}$. To check that C is compact, we will use the following facts:

- Denote by $p : V \to \mathcal{O}$ the map induced by the projection $N\mathcal{O} \to \mathcal{O}$. We will see that $\hat{\eta}^{(0)}$ is a p-fibered metric. The map

$$\pi_P : (N \times P, \underline{\check{\eta}}^{(0)} \times \eta^P) \to (P, \eta^P)$$

  is a Riemannian submersion. Because these metrics are H-invariant, the quotient submersion $p : (V, \hat{\eta}^{(0)}) \to (\mathcal{O}, \eta^{\mathcal{O}})$ is a Riemannian submersion. Observe that the completeness of η^P implies the completeness of $\eta^{\mathcal{O}}$.

- The map $p : \overline{V} \to \mathcal{O}$ is proper. Note that the set $\overline{N} = \overline{V} \cap N_x\mathcal{O}$ is an invariant set for the normal representation $G_x \curvearrowright N_x\mathcal{O}$. Because $\pi(V)$ is relatively compact in M/G we deduce that $\overline{N}$ is compact. So, p is proper because it is the quotient of the proper map $\pi_P : \overline{N} \times P \to P$ by a compact group.

So, $p(C)$ is closed and $\operatorname{diam} p(C) \leq \operatorname{diam} U < \infty$. Since the metric $\eta^{\mathcal{O}}$ is complete, we have that $p(C)$ is compact. Therefore, C is a compact set. □

Step 3: tube trick

Let $G \rightrightarrows M$ be an invariantly linearizable groupoid with a Riemannian metric η. Take $\{NGO_{W_i} \rightrightarrows W_i\}$ a family of invariant linearizations for $G \rightrightarrows M$, such that there are saturated opens V_i with $\overline{V_i} \subset W_i$ and $\{V_i\}$ is a locally finite covering for M, and $\pi(W_i)$ are relatively compact in M/G. Consider in each NGO_{W_i} the metric η_i induced by the restriction of η. Apply Lemma 4.4.5 to each metric η_i obtaining an equivalent metric $\hat{\eta}_i$ with the property that limited closed sets in W_i are compact.

Fix $f : M \to [0, +\infty)$ a smooth proper function. We claim that, for each pair $(i, n) \in \mathbb{N}$ there exists a value $l(i, n)$ in N such that:

$$d_{\hat{\eta}_i}\left(\overline{V}_i \cap f^{-1}(n), \overline{V}_i \cap f^{-1}(n + l(i, n))\right) > 1.$$

If there is no such $l(i, n)$, then in the following inclusion,

$$\overline{V_i} \subset f^{-1}([0, n]) \cup f^{-1}([n, +\infty)),$$

the set on the right side is limited. Thus, $\overline{V_i}$ is limited. Hence, by Lemma 4.4.5 the set $\overline{V_i}$ is compact, which is a contradiction with the fact that all orbits are non-compact.

We refer to the sets $T_i(n) = \overline{V_i} \cap f^{-1}([n, n + l(i, n)])$ as tubes *tubes*. Next, we will inductively build a family of tubes such that there are infinitely many tubes over each V_i, and they are disjoint. Starting with the tube $T_1(1)$ over V_1, then by induction we build a tube $T_i(n_i^1)$ over V_i, where n_i^1 is greater than $n_{i-1}^1 + l(i-1, n_{i-1}^1)$. After that, we will construct a second tube $T_1(n_1^2)$ over V_1 with $n_1^1 + l(1, n_1^1) < n_1^2$, and by induction a tube $T_i(n_i^2)$ over V_i with

$$\max\{\, n_{i-1}^2 + l(i-1, n_{i-1}^2)\, ,\, n_i^1 \,\} < n_i^2.$$

This process will end up providing a family of tubes $\{T_i(n_i^j)\}$ with infinitely many tubes over each open set V_i. By construction the intersection of any pair of tubes is empty. Denote

$$\widetilde{T}_i = \bigcup_j T_i(n_i^j).$$

End of the proof

Proof Proposition 4.4.1. Let be η a groupoid metric on $G \rightrightarrows M$ such that $([M/G], [\eta])$ is complete; the existence of such metric is ensured by Corollary 3.5.4. Suppose the orbits are all non-compact, otherwise use Lemma 4.4.4. Let $\{NGO_{W_i} \rightrightarrows W_i\}$ be a family of invariant linearizations, and the sets $\{\widetilde{T}_i\}$ build in the previous subsection. Set $\{\lambda_i\}_i$ a partition of unity subordinated to $U_i = W_i \setminus \bigcup_{i \neq k} \widetilde{T}_k$. Define the metric

$$\tilde{\eta} = \left(\sum_i \lambda_i \left(\hat{\eta}_i^{(0)}\right)^* \right)^* .$$

We will check that $\tilde{\eta}$ is equivalent to η. Let α and β be covectors in the annihilator $T_p\mathcal{O}^\circ$. We have the following equation:

$$\tilde{\eta}(\alpha, \beta) = \sum_i \lambda_i \left(\hat{\eta}_i^{(0)}\right)^* (\alpha, \beta) = \sum_i \lambda_i \left(\eta^{(0)}\right)^* (\alpha, \beta) = \left(\eta^{(0)}\right)^* (\alpha, \beta).$$

The second equality holds because $\hat{\eta}_i$ and η_i are equivalent.

Let $a : [p, q) \to M$ be a unit-speed geodesic. The projection $\pi \circ a$ is a Lipschitz map, and since $(M/G, d)$ is a complete metric space, there exists the limit $\mathcal{O}_{x_0} = \lim_n \pi \circ a(q - \frac{1}{n}) = \lim_{t \to q} \pi \circ \alpha(t)$. The orbit $\mathcal{O}_{x_0}$ is contained in a open V_i, so $\alpha(q - \delta, q) \subset V_i$ for δ closes to 0. If $a(q - \delta, q)$ is included in some compact $K \subset M$ then the geodesic can be extended. If there is no such K, then $a(q - \delta, b)$ must cross infinitely many tubes over V_i on finite time. Since $\tilde{\eta}$ and $\hat{\eta}_i$ agree over the tubes, a will have at least length 1 to cross each of them, which leads to a contradiction.

□

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