On the cauchy problem associated to the Brinkman flow in $\mathbb{R}^3_+$.

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Abstract

In this work we continue our study of the Cauchy problem associated to the Brinkman equations (see (2)-(3) below) which model fluid flow in certain types of porous media. Here we will consider the flow in the upper half-space

$$\mathbb{R}^3_+ = \left\{ (x, y, z) \in \mathbb{R}^3 \,|z \geqslant 0 \right\}, \tag{1}$$

under the assumption that the plane $z = 0$ is impenetrable to the fluid. This means that we will have to introduce boundary conditions that must be attached to the Brinkman equations. We study local and global well-posedness in appropriate Sobolev spaces introduced below, using Kato's theory for quasilinear equations, parabolic regularization and a comparison principle for the solutions of the problem.

1 Introduction.

In this article we continue our study of the Brinkman equations (see ([I-IA]), ([I-IAM]) and the references therein). This time we will consider the system

$$\begin{aligned} \partial_t \rho + div\,(\rho v) &= F\,(t, \rho)\,, \\ (1 - \Delta)\, v &= -\nabla P\,(\rho)\,, \\ (\rho\,(0)\,, v\,(0)) &= (\rho_0, v_0)\,, \end{aligned} \tag{2}$$

$$\rho = \rho\,(t, x, y, z)\,, v = v\,(t, x, y, z)\,, \tag{3}$$

in $\mathbb{R}^3_+$. We assume that the horizontal plane, that is, $z = 0$, to be *impenetrable to the fluid*. Thus, we must impose a boundary condition at $z = 0$, compatible with this assumption. This means that the fluid flow must be zero in the (downwards) z direction at $z = 0$ for all $(x, y, 0)\,, (x, y) \in \mathbb{R}^2$.

Now, if S is a C^1 surface (say), and $\mathbf{n}$ a continuous unit normal to S. This orients the surface and determines the sign of the fluid flow, which is defined by the component of the velocity in the the direction of $\mathbf{n}$, that is, $v \bullet \mathbf{n}$. where $\bullet$

denotes the usual inner product in Euclidian spaces. Since we expect $(1-\Delta)$ to be invertible[1], we must have

$$v = -(1-\Delta)^{-1} \nabla P(\rho). \tag{4}$$

Let S be the plane $z=0$. We choose

$$\mathbf{n} = (0,0,-1). \tag{5}$$

In view of the condition at $z=0$,we have

$$v \bullet \mathbf{n} = -(1-\Delta)^{-1} \nabla P(\rho) \bullet \mathbf{n} = 0. \tag{6}$$

So,

$$\frac{\partial}{\partial z} P(\rho) = P'(\rho) \frac{\partial \rho}{\partial z} = 0. \tag{7}$$

If $P'(\rho) \neq 0$ for all $\rho \neq 0$ (as we will assume later) it follows that we must have

$$\frac{\partial \rho}{\partial z} = 0 \text{ at } z = 0. \tag{8}$$

Thus ρ must satisfy a Neumann boundary condition at $z=0$.

This paper is organized as follows. In Section 2 we define the operator $-\Delta$, mentioned above, and the Sobolev spaces associated to it. In Section 3 we establish local well-posedness for the Cauchy problem in question. Section 4 deals with the comparison principle for the solutions of the problem, which in turn is used in Section 5 to establish global results.

2 Distributions and Sobolev Spaces.

Let $\mathfrak{S}\left(\mathbb{R}^3_+\right) = \mathcal{S}\left(\mathbb{R}^2 \times [0,\infty)\right)$ [2] denote the set of all C^∞ functions $f : \mathbb{R}^3_+ \longrightarrow \mathbb{C}$ such that

$$\|f\|_{\alpha,\beta} = \sup_{\mathbb{R}^3_+} \left| w^\alpha D^\beta f(w) \right| < \infty. \tag{9}$$

where α, β, are (tridimensional) multi-indexes, $w = (x,y,z) \in \mathbb{R}^3_+$, $D = \frac{1}{i}\nabla$ (see [C-C1] Chapter 1 page 8, [1] Chapter 7 page 323 and [S] Chapter One page 2). Moreover the derivatives with respect to z at $z=0$, are taken from above.

This defines a countable collection of seminorms in $\mathfrak{S}\left(\mathbb{R}^3_+\right)$, which turns this vector space into a Frèchet space (see [R-R1]). Let $\mathfrak{S}'\left(\mathbb{R}^3_+\right)$ denote the topological dual of $\mathfrak{S}\left(\mathbb{R}^3_+\right)$ that is $f \in \mathfrak{S}'\left(\mathbb{R}^3_+\right)$ if and only if $f : \mathfrak{S}\left(\mathbb{R}^3_+\right) \longrightarrow \mathbb{C}$

[1] Whatever Δ means. This will be explained along the article. See also the remark at the end of Section 2.

[2] See [S] Chapter Two page 33.

is linear and is continuous in the following sense[3], for any convergent net $f_\lambda \in \Lambda$ we have

$$f_\lambda \xrightarrow{\Lambda} f \Longleftrightarrow f_\lambda(\varphi) \xrightarrow{\mathbb{C}} f(\varphi) \ \ \forall\, \varphi \in \mathfrak{S}\left(\mathbb{R}^3_+\right). \tag{10}$$

Now, let $\mathfrak{L}^2\left(\mathbb{R}^3_+\right) = L^2\left(\mathbb{R}^2 \times [0,\infty)\right)$. It is not difficult to show that

$$\mathfrak{S}\left(\mathbb{R}^3_+\right) \hookrightarrow \mathfrak{L}^2\left(\mathbb{R}^3_+\right) \hookrightarrow \mathfrak{S}'\left(\mathbb{R}^3_+\right), \tag{11}$$

where the symbol $\hookrightarrow$, in the remainder of this article, will always mean *that the inclusion is continuous and dense with respect to the relevant topologies involved.* Next consider the following operator

$$\mathfrak{D}\left(\widetilde{\Delta}\right) = \left\{\varphi \in \mathfrak{S}\left(\mathbb{R}^3_+\right) \middle| \widetilde{\partial}_z \varphi(x,y,0) = 0\right\}, \tag{12}$$
$$-\widetilde{\Delta}\varphi(x,y,z) = \left(\partial_x^2 + \partial_y^2 + \widetilde{\partial}_z^2\right)\varphi(x,y,z).$$

However it is necessary to explain what the z derivative means. Define $\widetilde{d}_z^2$ by the equations,

$$\mathfrak{D}\left(\widetilde{d}_z^2\right) = \{(f \in \mathcal{S}([0,\infty))\,|f'(0) = 0\} \tag{13}$$
$$-\widetilde{d}_z^2 f = \frac{d^2 f}{dz^2},\ \ f \in \mathfrak{D}\left(\widetilde{d}_z^2\right),$$

where the derivative at zero is taken from above. Using the Fourier Cosine transform and its Inversion formula (see [CH], Section 54),

$$(\mathfrak{F}_c f)(\alpha) = \int_0^\infty f(x)\cos(\alpha x)\,dx,\ \ x,\ \ \alpha \in [0,\infty), \tag{14}$$
$$\left(\mathfrak{F}_c^{-1} g\right)(x) = \frac{2}{\pi}\int_0^\infty g(\alpha)\cos(\alpha x)\,d\alpha,$$

and the fact that

$$(\mathfrak{F}_c f'')(\alpha) = -\alpha^2(\mathfrak{F}_c f)(\alpha) - f'(0), \tag{15}$$

it is easy to see that $\widetilde{d}_z^2$ is essentially self- adjoint. Let d_z^2 denote its unique self adjoint extension. Next, if $(x,y) \in \mathbb{R}^2$ is fixed and $\varphi \in \mathfrak{S}\left(\mathbb{R}^3_+\right)$ then $\psi(z) = \varphi(x,y,z) \in S([0,\infty))$ so we may define

$$\widetilde{\partial}_z^2 \varphi(x,y,z) = \widetilde{d}_z^2 \varphi(x,y,z) = d_z^2 \varphi(x,y,z). \tag{16}$$

Once again, it is easy to show that $\left(-\widetilde{\Delta}\right)$ is essentially self adjoint. We will denote its unique self adjoint extension by $(-\Delta)$. Now, it is necessary to introduce a Fourier transform associated to the operator $(-\Delta)$. This can be done noting that

$$\Theta(x,y,z) = \exp(ix\xi)\exp(iy\eta)\cos(\alpha z), \tag{17}$$

[3] Which is general, because nets define the topology of a space. See([R-R1]).

satisfies,

$$(-\Delta)\,\Theta(x,y,z) = \left(\xi^2+\eta^2+\alpha^2\right)\Theta(x,y,z)\,. \tag{18}$$

So if $\varphi \in \mathfrak{L}^1\left(\mathbb{R}^3_+\right)$, we define

$$\begin{aligned}\widehat{\varphi}(\xi,\eta,\alpha) &= (\mathfrak{F}\varphi)(\xi,\eta,\alpha) \\ &= \left(\frac{1}{2\pi}\right)\int_{\mathbb{R}^3_+}\varphi(x,y,z)\,\overline{\Theta(x,y,z)}dxdydz.\end{aligned} \tag{19}$$

Employing the usual methods, ([R-R2], [I-I1] , [1]), we can extend this operator as an unitary map from $\mathfrak{L}^2\left(\mathbb{R}^3_+\right)$ into itself. Its inverse is given by

$$\begin{aligned}\overset{\vee}{\omega}(x,y,z) &= \left(\mathfrak{F}^{-1}\omega\right)(x,y,z) \\ &= \left(\frac{1}{\pi}\right)^2\int_{\mathbb{R}^3_+}\omega(\xi,\eta,\alpha)\,\Theta(x,y,z)\,d\xi d\eta d\alpha.\end{aligned} \tag{20}$$

The usual methods employed to extend the transform Fourier in $\mathbb{R}^n$ can be used in this case to define $\mathfrak{F}$ in $\mathfrak{L}^2\left(\mathbb{R}^3_+\right)$ and $\mathfrak{S}'\left(\mathbb{R}^3_+\right)$. Note that

$$-\Delta f = \mathfrak{F}^{-1}\Phi\mathfrak{F}f \tag{21}$$

where Φ denotes (with a little abuse of notation) the maximal operator of multiplication by

$$\Phi(\xi,\eta,\alpha) = \left(\xi^2+\eta^2+\alpha^2\right) \tag{22}$$

in $\mathfrak{L}^2\left(\mathbb{R}^3_+\right)$. It deserves remark that the Fourier transform $\mathfrak{F}$ is a topological isomorphism from $\mathfrak{S}\left(\mathbb{R}^3_+\right)$ into itself, so that by the usual duality argument, it has the same property in $\mathfrak{S}'\left(\mathbb{R}^3_+\right)$. Moreover, it is a unitary operator in $\mathfrak{L}^2\left(\mathbb{R}^3_+\right)$ [4] We are now in position to introduce the resolvent $z \longrightarrow R(z)$ of $(-\Delta)$ and the Sobolev spaces associated to it. To begin with, it is not difficult to see that

$$\sum(-\Delta) = [0,\infty)\,, \tag{23}$$

and that the function $z \longrightarrow R(z)$ defined by,'

$$\begin{aligned}R(z)\,f &= (-\Delta - z)^{-1} f, \\ &= \mathfrak{F}^{-1}\left(\xi^2+\eta^2+\alpha^2-z\right)^{-1}\mathfrak{F}f, \\ &z \in \mathbb{C}\backslash[0,\infty)\,,\ f \in \mathfrak{L}\left(\mathbb{R}^3_+\right),\end{aligned} \tag{24}$$

satisfies,

$$\begin{aligned}R(z)(-\Delta - z)\,f &= f \quad \forall\, f \in \mathfrak{D}(-\Delta)\,, \\ (-\Delta - z)\,R(z)\,g &= g\ \forall\ g \in \mathfrak{L}^2\left(\mathbb{R}^3_+\right).\end{aligned} \tag{25}$$

[4] Note that $\mathfrak{F}_c$, the Fourier cossine transfor is an unitary operator in $\mathfrak{L}^2(\mathbb{R}_+)\,,\ \mathbb{R}_+ = [0,\infty)$. See ([C-C2]).

Next, let $s \in \mathbb{R}$ and denote by $\mathfrak{H}^s\left(\mathbb{R}_+^3\right)$ the Sobolev space of order s, that is,

$$\mathfrak{H}^s\left(\mathbb{R}_+^3\right) = \left\{ f \in \mathfrak{S}'\left(\mathbb{R}_+^3\right) \middle| (1-\Delta)^{\frac{s}{2}} f \in \mathfrak{L}^2\left(\mathbb{R}_+^3\right) \right\}. \tag{26}$$

These spaces have the same properties as the Sobolev spaces in $\mathbb{R}^n$, that is,

SB1 $\mathfrak{H}^s\left(\mathbb{R}_+^3\right)$ are Hilbert spaces when endowed with the inner product

$$(f\,|g)_s = \left((1-\Delta)^{\frac{s}{2}} f \middle| (1-\Delta)^{\frac{s}{2}} g\right),\ \forall f,g \in \mathfrak{H}^s\left(\mathbb{R}_+^3\right). \tag{27}$$

SB2 If $s \geqslant \ell$ then $\mathfrak{H}^s\left(\mathbb{R}_+^3\right) \hookrightarrow \mathfrak{H}^\ell\left(\mathbb{R}_+^3\right)$ for all $s, \ell \in \mathbb{R}$.

SB3 (Sobolev´s Lemma.) Let $s > \frac{3}{2}$. Then $\mathfrak{H}^s\left(\mathbb{R}_+^3\right) \hookrightarrow C_0\left(\mathbb{R}_+^3\right)$ where $C_0\left(\mathbb{R}_+^3\right)$ denotes the set of all continuous functions that tend to zero at infinity.

SB4 Let $f, g \in \mathfrak{H}^s\left(\mathbb{R}_+^3\right)$, $s > \frac{3}{2}$. Then the pointwise product [5] $fg \in \mathfrak{H}^s\left(\mathbb{R}_+^3\right)$ and

$$\|fg\|_s \leq C\,\|f\|_s\,\|g\|_s \tag{28}$$

where $C > 0$ is a constant. Note that this turns $\mathfrak{H}^s\left(\mathbb{R}_+^3\right)$ into a Banach algebra.

The proofs of these properties are exactly the same as the corresponding ones in the case of $\mathbb{R}^n$, and will be omitted. The interested reader can consult ([C-C1]), ([R-R2]), ([1]), for example.

Remark 1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The Dirichlet Laplacian and the Neumann Laplacian, denoted Δ_D^Ω and Δ_N^Ω are the unique self-adjoint operators associated with the quadratic form*

$$q(f,g) = \int_\Omega \nabla f \bullet \overline{\nabla g}\,dx \tag{29}$$

with domains $C_0^\infty(\Omega)$ and $H^1(\Omega)$ where ∇ denotes the distributional gradient. (See [R-R3].) This is a very elegant, but rather abstract definition. In many applications one must find the self- adjoint operator in order to deal with actual computations. The Laplacian defined above is the Neumann Laplacian corresponding to $\Omega = \mathbb{R}_+^3 \backslash \{(x,y,z)\,|z=0\}$, that is, the interior of $\mathbb{R}_+^3$.

3 Local Well-posedness

We begin reminding the reader of our definition of well-posedness. The Cauchy problem

$$\begin{aligned} \partial_t u &= G(t,u) \in X, \\ u(0) &= u_0 \in Y, \end{aligned} \tag{30}$$

[5] The product is well defined in view of **SB3**.

$Y \hookrightarrow X$, $t \in [0, T_0]$, $G : [0, T_0] \times Y \longrightarrow X$ is (at least continuous[6]) is said to be *locally well posed* if there exists a $T \in (0, T_0]$ and a function $u : [0, T] \longrightarrow Y$ such that $u(0) = u_0$ and satisfies the differential equation with respect to the norm of X,

$$\lim_{h \longrightarrow 0} \left\| \frac{u(t+h) - u(t)}{h} - G(t, u(t)) \right\|_X = 0 \tag{31}$$

Moreover, the solution must depend continuously on the initial data (and on any other relevant parameters occurring in the equation), in appropriate topologies. In what follows we will consider only the initial data. In that case what we mean is: assume that $u_0^{(j)} \in Y$, $j = 1, 2, 3, ..., \infty$, let $u^{(j)}$ be the corresponding solutions. Suppose that

$$\lim_{j \to \infty} \left\| u_0^{(j)} - u_0^{(\infty)} \right\|_Y = 0. \tag{32}$$

Then, for all $T' \in (0, T)$ we have,

$$\lim_{j \to \infty} \sup_{t \in [0, T']} \left\| u^{(j)}(t) - u^{(\infty)}(t) \right\|_Y = 0. \tag{33}$$

If any of these properties fail, we say that the problem is *ill-posed*[7]. In case G is defined for all $t \in \mathbb{R}$ and the preceding properties are valid for all $T > 0$, we say that the problem is *globally well-posed*.

Using the definitions and notations of the previous section we can solve for v as indicated in (4) and inserting this formula into the first equation of (2), we obtain the Cauchy problem

$$\begin{aligned} \partial_t \rho &= \operatorname{div}\left(\rho (1 - \Delta)^{-1} \Delta P(\rho) \right) + F(t, \rho) \\ \rho(0) &= \rho_0. \end{aligned} \tag{34}$$

Moreover the compatibility condition

$$v_0 = -(1 - \Delta) P(\rho_0) \tag{35}$$

must be satisfied. Note also that the boundary conditions are inserted in the definition of the operators appearing in (34).

Now, there are several ways to solve (34). We mention our favorites, namely

- Kato´s Theory of Quasilinear Equations
- Parabolic Regularization.

[6] In fact some kind of Lipichtz condition must be introduced since Peano's Theorem for ODE's does not hold in infinite dimensions.

[7] It deserves mention that there are examples that show that any of these properties may fail. Morever note that the definition we adopted above includes the notion of *permanence*, that is, the solution "lives" in the same space to which the initial condition belongs. There are strinking examples where this does not hold (see [I-I4] and the references therein).

3.1 Application of Kato´s Theory

A very large class of relevant evolution equations can be written in *quasilinear form*, that is,

$$\begin{cases} \partial_t u + A(t,u)\, u = F(t,u) \in X, \\ u(0) = \phi \in Y. \end{cases} \tag{36}$$

Here X and Y are Banach spaces, as before with $Y \hookrightarrow X$ and $A(t,u)$ is bounded from Y into X (for fixed t) and is the (negative) generator of a C^0 semigroup for each $(t,u) \in [0.T] \times W$, W open in Y. In its most general formulation, X and Y may be non-reflexive ([K-K1])[8]. Since we will deal exclusively with reflexive spaces, we restrict ourselves to a simpler version, which can be found in ([K-K2]). (See also ([I-I1].) The essential assumption of the theory is the existence of an isomorphism S from Y onto X such that

$$SA(t,u)\, S^{-1} = A(t,u) + B(t,u) \tag{37}$$

where $B(t,u) \in \mathcal{B}(X)$, with the strict domain relation implied by the equation. This is, in fact, a condition on the commutator $[S, A(t,u)]$ because (37)[9] can be rewritten as

$$[S, A(t,u)]\, S^{-1} = B(t,u)\,. \tag{38}$$

There are also lesser requirements, involving Lipschtz conditions on the operators in question. For example, $A(t,u)$ must satisfy

$$\|A(t,w) - A(t,\widetilde{w})\|_{\mathcal{B}(Y,X)} \leq \mu \|w - \widetilde{w}\|_X\,,\ \ \mu > 0,\ \text{constant} \tag{39}$$

for all pairs $(t,w)\,, (t,\widetilde{w})$ in $[0.T] \times W$. Both $B(t,u)$ and $F(t,u)$ must satisfy similar conditions. Once these assumptions are satisfied, Kato tells you that (36) is locally well-posed.

Now we must write the integrodifferential equation (34) in quasilinear form. Consider the linear operator:

$$f \longmapsto A(\rho)\, f = -\operatorname{div}\left(f\,(1-\Delta)^{-1}\, \nabla P(\rho)\right). \tag{40}$$

Thus the equation in (34) can be written in the form presented in (36). Next we choose our function spaces. Due to certain technical estimates needed to control the commutator mentioned above, we take $Y = \mathfrak{H}^s\left(\mathbb{R}^3_+\right)$, $s > 5/2$, $X = \mathfrak{L}^2\left(\mathbb{R}^3_+\right)$ and W an arbitrary open ball centered at zero in Y.

Now assume

- P maps $\mathfrak{H}^s(\mathbb{R})$ into itself, $P(0) = 0$ and is Lipschitz in the following sense:

$$\|P(\rho) - P(\widetilde{\rho})\|_s \leq L_s\left(\|\rho\|_s\,, \|\widetilde{\rho}\|_s\right) \|\rho - \widetilde{\rho}\|_s \tag{41}$$

[8] This result is very important because it can be used to show that, as in the linear case, continous dependence follows from existence and uniqueness. See ([K-K1]).

[9] A condition on a commutator is to be expeted.is to be expected. See ([I-I1]).

- $F : [0, T_0] \times \mathfrak{H}^s(\mathbb{R}) \longrightarrow \mathfrak{H}^s(\mathbb{R})$, $F(t, 0) = 0$ and satisfies the following Lipschitz condition:

$$\|F(t,\rho) - F(t,\widetilde{\rho})\|_s \leq M_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s. \tag{42}$$

  where $L_s, M_s : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and monotone non-decreasing functions with respect to each of its arguments.

If T is a linear operator and belongs to the class $G(X, 1, 0)$, that is, if $(-T)$ generates a contraction semigroup, we say that T is *maximally accretive (or m-accretive)*. If $T \in G(X, 1, \beta)$, that is, T generates a semigroup $U(t)$ such that $\|U(t)\|_{B(X)} \leq Me^{-t\beta}$, T is said to be *quasi- maximally accretive (or quasi m-accretive)*. Since X is a Hilbert space, it suffices to prove, in our case, that $T = A(\rho)$ is maximally accretive in X. (See [K-K3],[P] and [R-R1]).

$$\big\langle A(\rho)f, f\big\rangle \geq -\beta\|f\|^2, \forall f \in D(A(\rho)) = Y; \rho \in W \subset Y \tag{43}$$

Let

$$\Theta(\rho) = (1-\Delta)^{-1}\nabla P(\rho), \tag{44}$$

. Integrating by parts and applying Sobolev Lemma, we obtain

$$\begin{aligned}
\big\langle A(\rho)f, f\big\rangle &= \big\langle -\operatorname{div}(f\,\Theta(\rho)), f\big\rangle = -\sum_{i=1}^{3}\int f\partial_{x_i}(f\,\Theta_i(\rho))\,dx \\
&= \sum_{i=1}^{3}\int f\partial_{x_i}f\,\Theta_i(\rho)\,dx = \frac{1}{2}\sum_{i=1}^{3}\int \partial_{x_i}(f^2)\Theta_i(\rho)\,dx \\
&= -\frac{1}{2}\sum_{i=1}^{3}\int f^2\partial_{x_i}\Theta_i(\rho)\,dx = -\frac{1}{2}\int(\operatorname{div}\vec{\Theta}(\rho))f^2\,dx \\
&\geq -\underbrace{\frac{\|\operatorname{div}\Theta(\rho)\|_{L^\infty}}{2}}_{\beta}\|f\|^2
\end{aligned} \tag{45}$$

$$R(A(\rho)+\lambda) = X = \mathfrak{L}^2(\mathbb{R}^3_+), \forall\lambda > \beta$$

The fact that $A(\rho)$ is a closed operator combined with the inequality (45) shows that $(A(\rho) + \lambda)$ has closed range for all $\lambda > \beta$
Thus it suffices to show that $(A(\rho) + \lambda)$ has dense range for $\lambda > \beta$. For this, is sufficient to prove that $R(A(\rho) + \lambda)^\perp = \{0\}$, because $A(\rho)$ is a linear operator.

Let $g \in \mathfrak{L}^2(\mathbb{R}^3_+)$ satisfy,

$$\big\langle(A(\rho)+\lambda)f, g\big\rangle = 0, \forall f \in D(A(\rho)) = \mathfrak{H}^s(\mathbb{R}^3_+). \tag{46}$$

Integrating by parts, yields

$$\begin{aligned}
\big\langle(A(\rho)+\lambda)f, g\big\rangle = 0 &\Rightarrow \big\langle A(\rho)f, g\big\rangle + \big\langle\lambda f, g\big\rangle = 0 \\
&\Rightarrow \big\langle f, \nabla g\,\Theta(\rho)\big\rangle + \big\langle\lambda f, g\big\rangle = 0 \\
&\Rightarrow \big\langle f, \nabla g\,\Theta(\rho) + \lambda g\big\rangle = 0, \forall f \in D(A(\rho)) = \mathfrak{H}^s(\mathbb{R}^n) \\
&\Rightarrow \nabla g\,\Theta(\rho) + \lambda g = 0
\end{aligned} \tag{47}$$

Therefore, multiplying by g, integrating by parts, and using (45) we have:

$$\begin{aligned} g\nabla g\,\Theta(\rho)+\lambda g^2=0 &\Rightarrow \frac{1}{2}\int \nabla(g^2)\,\Theta(\rho)\,dx+\lambda\|g\|^2=0 \\ &\Rightarrow \underbrace{-\frac{1}{2}\int g^2\mathrm{div}\;\Theta(\rho)\,dx}_{=\left\langle A(\rho)g,g\right\rangle}+\lambda\|g\|^2=0 \\ &\Rightarrow \left\langle A(\rho)g,g\right\rangle+\lambda\|g\|^2=0 \qquad (48)\\ &\Rightarrow 0\geq -\beta\|g\|^2+\lambda\|g\|^2=(\lambda-\beta)\|g\|^2 \\ &\Rightarrow g=0 \end{aligned}$$

Finally, we choose the isomorphism $S:\mathfrak{D}\left(S\right)=\mathfrak{h}^s\left(\mathbb{R}^3_+\right)\longrightarrow\mathfrak{L}^2\left(\mathbb{R}^3_+\right)$ to be

$$S=\left(1-\Delta\right)^{s/2}. \qquad (49)$$

Then the proof of (37) is exactly the same of the corresponding fact in $\mathbb{R}^n$ (see [I-IAM]).

In view of these remarks, Kato's quasilinear theory implies the following result.

Theorem 2 *The Cauchy problem (34) is locally well posed in* $\mathfrak{H}^s\left(\mathbb{R}^3_+\right)$ *in the sense described at the beginning of this section for all* $s>5/2$.

3.2 Parabolic Regularization.

It is easy to see that if we integrate (34) with respect to time we obtain

$$\underbrace{\rho\left(t\right)}_{\mathfrak{H}^s\left(\mathbb{R}^3_+\right)}=\underbrace{\phi}_{\mathfrak{H}^s\left(\mathbb{R}^3_+\right)}+\underbrace{\int_0^t \mathrm{div}\left(\rho\left(1-\Delta\right)^{-1}\nabla P\left(\rho\right)\right)\left(t'\right)dt'}_{\mathfrak{H}^{s-1}\left(\mathbb{R}^3_+\right)}. \qquad (50)$$

so we cannot apply Banach's Fixed Point Theorem and Gronwall's inequality to establish local well posedness. However, we can introduce an artificial viscosity $\mu>0$ to obtain the regularized Cauchy Problem

$$\left\{\begin{array}{c}\partial_t\rho_\mu=\mathrm{div}\left(\rho_\mu\left(1-\Delta\right)^{-1}\nabla P\left(\rho_\mu\right)\right)+\mu\Delta\rho_\mu \\ \rho_\mu\left(0\right)=\rho_0.\end{array}\right. \qquad (51)$$

which is equivalent to the integral equation

$$\rho_\mu\left(t\right)=U_\mu\left(t\right)\rho_0+\int_0^t U_\mu\left(t-t'\right)\left[A\left(\rho_\mu\left(t'\right)\right)\rho_\mu\left(t'\right)\right]dt', \qquad (52)$$

where $U_\mu(t)$ is the infinitely smoothing C^0 semigroup

$$U_\mu(t) f = \exp(\mu t\Delta) f = \mathfrak{F}^{-1} e^{-\mu t(\xi^2+\eta^2+\alpha^2)} \mathfrak{F} f. \tag{53}$$

Then we can show that (see [I-IA] and [I-IAM])

Theorem 3 *Assume that $\mu > 0$ and that P satisfy (41) for all (fixed) $s > 3/2$. Then (52) is locally well-posed in $\mathfrak{H}^s(\mathbb{R}^3_+)$. Moreover, if $(0, T_\mu]$ is an interval of existence, then $\rho_\mu \in C((0, T_\mu]; \mathfrak{H}^\infty(\mathbb{R}^3_+))$, where $\mathfrak{H}^\infty(\mathbb{R}^3_+) = \bigcap_{s\in\mathbb{R}^3_+} \mathfrak{H}^s(\mathbb{R}^3_+)$ provided with its natural Frechet space topology.*

It should be noted that the proof (even in $\mathbb{R}^3$) relies heavily on the inequality

$$\|U_\mu(t)\phi\|_{r+\lambda} \leq K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \|\phi\|_r \tag{54}$$

where $K_\lambda > 0$ depends only on λ and holds for all $\phi \in \mathfrak{H}^r(\mathbb{R}^3_+)$, $r \in \mathbb{R}$, $\lambda \geq 0$, and μ, $t > 0$.(See [1], [I-I3], [I-IA] and [I-IAM] for example.) An easy bootstrapping argument combining (52) and (54), (with λ fixed in the interval $(1, 2)$) shows that the RHS of (52) is locally integrable near $t = 0$). This implies the last statement of Theorem 2.

Next, the, usual limiting process involved in the method of parabolic regularization (see [1] and [I-I3]) we are able to show existence and uniqueness of solutions in $AC([0,T]; \mathfrak{H}^{s-1}(\mathbb{R}^3_+)) \cap L^\infty([0,T]; \mathfrak{H}^s(\mathbb{R}^3_+))$. Due to technical reasons (lack of invariance under certain changes of variables, see [I-I1], [I-I3] and [K-K3]), so far we were unable to prove that the solution we obtained in this way actually belongs to $C([0,T]; \mathfrak{H}^s(\mathbb{R}^3_+)) \cap C^1([0,T]; \mathfrak{H}^{s-1}(\mathbb{R}^3_+))$, $s > 3/2$ as we would have liked. However, combining what we already have, with the results in Theorem 2, proved using Kato´s theory when $s > 5/2$, we see that the solutions must coincide, due to uniqueness, if $s > 5/2$.

4 Comparison principle

To simplify the notation we will write

$$\mathcal{B}f = R(-1)f = \mathfrak{F}^{-1}\left(\xi^2 + \eta^2 + \alpha^2 + 1\right)^{-1} \mathfrak{F}f, \quad f \in \mathfrak{L}^2(\mathbb{R}^3_+).$$

In order to state our results, we define of the fractional power spaces associated with Neumann Laplacian $-\Delta$. Following the arguments found in [W] and [KF]. For $\alpha > 0$ and $f \in L^2(\mathbb{R}^3_+)$, define[10]

[10] Of course we could also have used the Fourier transform defined above to introduce these operators.

$$R^\alpha(-1)\, f = (1-\Delta)^{-\alpha}\ f = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-t}\, e^{t\,\Delta} f\ dt.$$

Then $(1-\Delta)^{-\alpha}$ is a bounded, one-to-one operator on $\mathfrak{L}^2(\mathbb{R}^3_+)$. We let $\mathfrak{J}^\alpha = (1-\Delta)^\alpha$ be the inverse of $(1-\Delta)^{-\alpha}$. For $s > 0$, the Hilbert space $\mathfrak{H}^s(\mathbb{R}^3_+)$ is the range of $(1-\Delta)^{-s/2}$ with the inner product

$$\langle f,\, g\rangle_{\mathfrak{H}^s} = \int_{L^2(\mathbb{R}^3_+)} \mathfrak{J}^{s/2} f\, \overline{\mathfrak{J}^{s/2} g}\ dx. \tag{55}$$

Consider the initial value problem (34) with $F(t,\rho) = 0$[11], $P(\rho) = \rho^2 k, k = 1, 2, 3.....$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho\, \mathbf{v}) = 0,\ x \in \mathbb{R}^3_+,\ t \in (0, T_0] \\ \mathbf{v} = -\mathcal{B}\,\nabla \rho^{2k} = -\vec{\Theta}(\rho^{2k}) \\ (\rho(0),\, \mathbf{v}(0)) = (\rho_0,\, \mathbf{v}_0) \end{cases} \tag{56}$$

Theorem 4 ***(Comparison Principle)***. *Let (ρ, v) and (η, w) be solutions of (34) with $P(\rho) = \rho^{2k}, P(\eta) = \eta^{2k}, k = 1, 2, 3...$[12]; and initial values (ρ_0, v_0) and (η_0, w_0) respectively. Then*

$$0 \le \eta_0(x) \le \rho_0(x)\ \ in\ \ \Omega \Rightarrow 0 \le \eta(x,t) \le \rho(x,t)\ \ in\ \ \Omega \times [0, T_0] \tag{57}$$

Proof. In this proof, we use the same idea employed by Alarcon, Iorio and Del Sol in the study of Brinkman flow in $\mathbb{R}^n$ ([I-IAM]). Let

$$R(t,y) = \rho(\phi(t,y),t); S(t,y) = \eta(\psi(t,y),t) \tag{58}$$

and

$$Q(t,y) = R(t,y) - S(t,y) \tag{59}$$

where $\phi(t,y)$ and $\psi(t,y)$ satisfy the following ordinary differential equations,

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t,y) = \mathbf{v}(\phi(t,y),t) & \phi(t,y) = \big(\phi_1(t,y), \phi_2(t,y), \phi_3(t,y)\big) \\ \phi(0,y) = y & v_i = -\partial_{x_i} \mathcal{B}(\rho^{2k}) \end{cases}, \tag{60}$$

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(t,y) = \mathbf{w}(\psi(t,y),t) & \psi(t,y) = \big(\psi_1(t,y), \psi_2(t,y), \psi_3(t,y)\big) \\ \psi(0,y) = y & w_i = -\partial_{x_i} \mathcal{B}(\eta^{2k}) \end{cases}, \tag{61}$$

[11] For simplicity's sake. It is not very difficult to include the external force in the result.
[12] A motivation for this choice can be found in [I-IA],

Now,

$$\begin{cases} \dfrac{dR}{dt} = -R\,\mathrm{div}\,\mathbf{v} & \dfrac{dS}{dt} = -S\,\mathrm{div}\,\mathbf{w} \\ R(0,y) = \rho_0(y) & S(0,y) = \eta_0(y) \end{cases} \tag{62}$$

Solving (62), we obtain:

$$R(t) = R(0)\exp\big[-\int_0^t \mathrm{div}\, v(\phi(s,y),s)\,ds\big] \overbrace{\Rightarrow}^{\rho_0(y)\geq 0} R(t) \geq 0 \tag{63}$$

Analogously, we have that:

$$S(t) = S(0)\exp\big[-\int_0^t \mathrm{div}\, w(\psi(s,y),s)ds\big] \overbrace{\Rightarrow}^{\eta_0(y)\geq 0} S(t) \geq 0 \tag{64}$$

On the other hand, differentiating $Q(t)$:

$$\begin{aligned} \frac{dQ}{dt} &= \frac{dR}{dt} - \frac{dS}{dt} = (-\mathrm{div}\, v)R(t) + (\mathrm{div}\,\mathbf{w})S(t) \\ &= -\rho\,\mathrm{div}\, v + \eta\,\mathrm{div}\, w \\ &= -(\rho-\eta)\mathrm{div}\, v + \eta(\mathrm{div}\, w - \mathrm{div}\, v) \\ &= -Q(t)(\mathrm{div}\,\mathbf{v}) + S(t)(\mathrm{div}\, w - \mathrm{div}\, v) \end{aligned} \tag{65}$$

where

$$\mathrm{div}\, v = \rho^{2k} - \mathcal{B}(\rho^{2k}), \quad \mathrm{div}\, w = \eta^{2k} - \mathcal{B}(\eta^{2k}) \tag{66}$$

Substituting (66) in (65), we obtain a new ordinary differential equation for $Q(t)$,

$$\begin{cases} \dfrac{dQ}{dt} = -\big[\mathrm{div}\,\mathbf{v} + S(t)P(R(t),S(t))\big]Q(t) + B(t,Q(t)) \\ Q(0) = \rho_0(y) - \eta_0(y) \end{cases} \tag{67}$$

with

$$P(R(t),S(t)) = P(\rho,\eta) = \sum_{i=0}^{2k-1} \rho^{2k-1-i}\eta^i \tag{68}$$

and

$$B(t,Q(t)) = S(t)(1-\Delta)^{-1}\Big[Q(t)P(R(t),S(t))\Big]. \tag{69}$$

Thus,

$$Q(t) = U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q(s))\,ds \tag{70}$$

where

$$U(t,s) = \exp\big[-\int_s^t [\mathrm{div}\,(v(\phi(\tau,y),\tau)) + S(\tau)P(R(\tau),S(\tau))]d\tau\big]. \tag{71}$$

In view of conditions for ρ_0 and η_0, we have that $R(t)\geq 0$ and $S(t)\geq 0$.

Consider the sequence

$$Q_n(t) = \begin{cases} Q_n(t) = U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q_{n-1}(s))ds, & \text{se } n = 1,2,....; \\ \rho_0(y) - \eta_0(y), & \text{se } n = 0. \end{cases}$$

If $Q(0) \geq 0$, then $Q_n(t) \geq 0$, for all n. Therefore,

$$Q(t) = \rho\big(\phi(t,y),t\big) - \eta\big(\psi(t,y),t\big) = \lim_{n\to\infty} Q_n(t) \geq 0 \tag{72}$$

To complete the proof we need to show the functions $y \in \Omega \to \phi(t,y) \in \Omega$ and $y \in \Omega \to \psi(t,y) \in \Omega$ are onto. To do this, we analyze in detail the map $y \in \Omega \to \phi(t,y) \in \Omega$.

The Neumann boundary condition $\rho_z = 0$ for $z = 0$ and Brinkman's condition $v = -\nabla\mathcal{B}(\rho^{2k})$[13], implies that

$$v_3((x_1,\, x_2,\, 0),\, t) = 0,\ \forall (x_1,\, x_2) \in \mathbb{R}^2 \text{ and } t \in [0,\, T], \tag{73}$$

then $\phi_3(t,\, (x_1,\, x_2,\, 0) = 0$ for all $(x_1,\, x_2) \in \mathbb{R}^2$ and $t \in [0,\, T]$.

We will show that Ω it is invariant under the flow $\phi(t,\, y)$, that is,

$$\phi\,(\Omega) \subset \Omega. \tag{74}$$

By (73), the plane $\Pi = \big\{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \colon x_3 = 0\big\}$ is invariant under the flow $\phi(t,\, y)$, i.e. $\phi(\Pi) \subset \Pi$. Next we show that (74) holds. To this end, it is enough to verify that

$$\phi(t,\, x_1,\, x_2,\, x_3) \in \Omega,\ \forall (x_1,\, x_2) \in \mathbb{R}^2,\ x_3 > 0. \tag{75}$$

If (75) does not hold, there is a $w = (w_1,\, w_2,\, w_3)$ with $w_3 > 0$ and $0 < t_1 < t_2 \leq T$ such that $\phi(t_1,\, w) \in \Pi$ and $\phi(t_2,\, w) \notin \Omega$. But (73) implies that

$$\phi(t_2,\, w) = \phi(t_1,\, w) + \int_{t_1}^{t_2} v(\phi(s,\, w),\, s)\, ds, \tag{76}$$

so that $\phi_3(t_2,\, w) = 0$. This contradiction proves (74). From (60), integrating from de 0 to t, we get :

$$\phi_i(t) - y_i = \int_0^t v_i(\phi(\tau,y),\tau)\, d\tau;\ i = 1,2,3, \tag{77}$$

so that

$$|\phi_i(t) - y_i| \leq \int_0^t |v_i(\phi(\tau,y),\tau)|\, d\tau \leq a_i(\|\rho_0\|_s,t)\ , i = 1,2,3,\ s > \frac{5}{2}, \tag{78}$$

$$y_i - a_i(\|\rho_0\|_s,t) \leq \phi_i(t,y) \leq y_i + a_i(\|\rho_0\|_s,t),\ \forall y = (y_1,\, y_2,\, y_3) \in \Omega. \tag{79}$$

[13] See (4).

Let $(z_1,\, z_2,\, z_3) \in \mathbb{R}^3_+$. Taking $y_i^{(1)} << 0$ for $i = 1,\, 2$; $y_i^{(2)} >> 0$ for $i = 1,\, 2,\, 3$, such that $z_i \in (y_i^{(1)}, y_i^{(2)})$ for $i = 1,\, 2$ and $0 < z_3 < y_3^{(2)}$ we have:

$$y_i^{(1)} + a_i(\|\rho_0\|_s, t) < z_i < y_i^{(2)} - a_i(\|\rho_0\|_s, t) \tag{80}$$

and

$$0 < z_3 < \phi_i(t,\, y_3^{(2)}). \tag{81}$$

Therefore

$$\phi_i(t, y_i^{(1)}) < z_i < \phi_i(t, y_i^{(2)}), \text{ for } i = 1,\, 2,\, 3. \tag{82}$$

Applying the Intermediate Value Theorem to ϕ_i implies that there exists $y_i \in (y_i^{(1)}, y_i^{(2)})$ satisfying $\phi_i(t, y_i) = z_i$. For $z_3 = 0$ the proof is analogous, since the plane $x_3 = 0$ is invariant by the flow $\phi(t,\, y)$, a consequence of (73). ■

5 Global results in $\mathfrak{H}^s(\mathbb{R}^3_+)$, $s > 5/2$

In this section we obtain the global $\mathfrak{H}^s$-estimate for the solution of the Brinkman Flow equation. This will be a consequence of global-well posedness. of the regularized problem.
First, we introduce the following estimates

Lemma 5 *If $s > 0$, then*

$$\Big\| \sum_{k=1}^{n} \big[\partial_{x_k} \mathfrak{J}^s(g \partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} \mathfrak{J}^s g)\big] \Big\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leq c \Big(\|\mathfrak{J}^2 f\|_{L^\infty(\mathbb{R}^3_+)} \|\mathfrak{J}^s g\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} + \|\mathfrak{J}^{s+2} f\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \|g\|_{L^\infty(\mathbb{R}^3_+)} \Big) \tag{83}$$

Proof. The proof of this Lemma is similar to that of Lemma X1 in [K-K4], is based on the following result due to R. R. Coifman and Y. Meyer (Lemma A.1.2). See Lemma A.1.3 in ([M]). ■

Lemma 6 *If $s > 0$, then $\mathfrak{H}^s(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+)$ is a Banach Algebra. Moreover*

$$\|fg\|_s \leq c(\|f\|_{L^\infty(\mathbb{R}^3_+)} \|g\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} + \|f\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \|g\|_{L^\infty(\mathbb{R}^3_+)}) \tag{84}$$

Proof. See [K-K4]. ■

Lemma 7 *Let $f \in X^s(\mathbb{R}^3_+)$, $s > \frac{5}{2}$, $k = 1, 2, \ldots$. Then*

$$\|f^{2k}\|_s \lesssim \|f\|^{2k-1}_{L^\infty(\mathbb{R}^3_+)} \|f\|_s,$$

where $A \lesssim B$ means that exist a constant $c > 0$ such that $A \leq c\, B$.

Now, we are ready to establish the following result.

Theorem 8 ***(Global Solution)****. Let $s > 5/2$, $P(\rho) = \rho^{2k}$, $F \equiv 0$ and $\rho_0 \in \mathfrak{H}^s(\mathbb{R}^3_+)$ with $0 \le \rho_0(x) \le 1$ in $\mathbb{R}^3_+$. Then (56) is globally well-posed in the sense described in Section 3 and satisfies $0 \le \rho(x,t) \le 1$, $\forall t \ge 0$.*

Proof. The Comparison Principle implies that $0 \le \rho(x,t) \le 1$. Using the regularized initial value problem, with the simplified notations $\rho_\mu(t) \equiv \tilde\rho$, $v_\mu(t) \equiv v$.

$$\begin{cases} \partial_t \tilde\rho - \mu\, \Delta_N\, \tilde\rho + \text{div}\, [\tilde\rho\, \mathbf{v}] = 0 \\ \mathbf{v} = -\mathcal{B}\, \nabla\, \tilde\rho^{2k}\ . \\ (\tilde\rho(0), \mathbf{v}(0)) = (\tilde\rho_0, \mathbf{v}_0) \end{cases} \tag{85}$$

Applying $\mathfrak{J}^s$ to regularized equation:

$$\frac{d}{dt}(\mathfrak{J}^s \tilde\rho) - \mu(\mathfrak{J}^s \Delta_N\, \tilde\rho) + \mathfrak{J}^s \text{div}\, (\tilde\rho\, v) = 0. \tag{86}$$

Multiplying (86) by $\mathfrak{J}^s\tilde\rho$ and integrating over $\mathbb{R}^3_+$ we get,

$$\frac{1}{2}\frac{d}{dt}\int (\mathfrak{J}^s\tilde\rho)^2\, dx = \mu \int (\mathfrak{J}^s\tilde\rho)\mathfrak{J}^s(\Delta_N\, \tilde\rho)\, dx - \int (\mathfrak{J}^s\tilde\rho)(\mathfrak{J}^s \text{div}\, (\tilde\rho\, v))\, dx, \tag{87}$$

so that

$$\frac{1}{2}\frac{d}{dt}\int (\mathfrak{J}^s\tilde\rho)^2\, dx = \underbrace{\mu\int (\mathfrak{J}^s\tilde\rho)\Delta_N\, (\mathfrak{J}^s\tilde\rho)\, dx}_{\le 0} - \sum_{i=1}^{3}\int (\mathfrak{J}^s\tilde\rho)\partial_{x_i}\mathfrak{J}^s(\tilde\rho\, v_i)\, dx. \tag{88}$$

Using the commutator $[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho = \partial_{x_i}\mathfrak{J}^s(\tilde\rho\, v_i) - v_i\partial_{x_i}J^s\tilde\rho$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int (J^s\tilde\rho)^2\, dx \le -\sum_{i=1}^{3}\int (\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\, dx - \sum_{i=1}^{3}\int (\mathfrak{J}^s\tilde\rho)v_i\partial_{x_i}\mathfrak{J}^s\tilde\rho\, dx. \tag{89}$$

Integration by parts in (89) yields,

$$\frac{1}{2}\frac{d}{dt}\int (\mathfrak{J}^s\tilde\rho)^2\, dx \le -\sum_{i=1}^{3}\int (\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\, dx + \frac{1}{2}\int (\mathfrak{J}^s\tilde\rho)^2 \text{div}\, v\, dx. \tag{90}$$

Taking (66) into (90) we obtain,

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int (\mathfrak{J}^s\tilde\rho)^2\, dx \le -\textstyle\sum_{i=1}^{3}\int (\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\, dx + \tfrac{1}{2}\textstyle\int (\mathfrak{J}^s\tilde\rho)^2\tilde\rho^{2k}\, dx \\ &-\tfrac{1}{2}\textstyle\int (\mathfrak{J}^s\tilde\rho)^2\mathfrak{J}^{-1/2}(\tilde\rho^{2k})\, dx. \end{aligned} \tag{91}$$

From the second equation in (85) we have $v_i = -\partial_{x_i}B_i\left(\tilde\rho^{2k}\right)$. Substituting it in (91)

$$\begin{aligned} &\tfrac{d}{dt}\textstyle\int (\mathfrak{J}^s\tilde\rho)^2\, dx \le \textstyle\int (\mathfrak{J}^s\tilde\rho)^2\tilde\rho^{2k}\, dx - \overbrace{\int (\mathfrak{J}^s\tilde\rho)^2\mathfrak{J}^{-1/2}\tilde\rho^{2k}\, dx}^{\ge 0} \\ &+2\textstyle\int (\mathfrak{J}^s\tilde\rho)\big(\sum_{i=1}^{3}[\partial_{x_i}\mathfrak{J}^s, \partial_{x_i}B_i\left(\tilde\rho^{2k}\right)]\tilde\rho\big)\, dx. \end{aligned} \tag{92}$$

Noting that the third term in (92) is non negative, and applying Cauchy Schwartz inequality in the fourth term we get,

$$\frac{d}{dt}\int(\mathfrak{J}^s\tilde{\rho})^2\,dx \leq \|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\int(\mathfrak{J}^s\tilde{\rho})^2\,dx + 2\,\|\mathfrak{J}^s\tilde{\rho}\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\Big\|\sum_{i=1}^{3}\left[\partial_{x_i}\mathfrak{J}^s,\partial_{x_i}\mathfrak{J}^{-1/2}\tilde{\rho}^{2k}\right]\tilde{\rho}\Big\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \quad (93)$$

Using Lemma 6.1 in (93), with $f=\mathfrak{J}^{-1/2}\tilde{\rho}^{2k}$ and $g=\tilde{\rho}$, we obtain:

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \leq \|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\|\tilde{\rho}\|_s^2 + 2c\|\tilde{\rho}\|_s\left[\|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\|\tilde{\rho}\|_s + \|\tilde{\rho}^{2k}\|_s\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)}\right] \quad (94)$$

Applying Lemma 6.3 in (94):

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \lesssim \|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)}^{2k}\|\tilde{\rho}\|_s^2 \quad (95)$$

Now, we need to estimate $\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)}$. Applying the Comparison Principle for ρ together with Sobolev's Lemma we have

$$\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\tilde{\rho}-\rho\|_{L^\infty(\mathbb{R}^3_+)} + \|\rho\|_{L^\infty(\mathbb{R}^3_+)} \lesssim 1 + \|\tilde{\rho}-\rho\|_s \quad (96)$$

Since $\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s|$ we proceed as follows: in the analysis of the weak convergence of sequence ρ_μ (see the proof of Theorem 4.2) we obtained

$$\begin{aligned}|\langle\rho_\mu(t)-\rho_\eta(t),\varphi\rangle_s| &\leq \|\rho_\mu(t)-\rho_\eta(t)\|_s\|\varphi-\varphi_\epsilon\|_s + \|\rho_\mu(t)-\rho_\eta(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\,\|\varphi_\epsilon\|_{2s} \quad (97)\\ &\leq 2M\epsilon + \|\rho_\mu(t)-\rho_\eta(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\,\|\varphi_\epsilon\|_{2s}\end{aligned}$$

Taking the limit as $\eta\to 0$ in (97), it follows that,

$$|\langle\rho_\mu(t)-\rho(t),\varphi\rangle_s| \leq 2M\epsilon + \|\rho_\mu(t)-\rho(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\,\|\varphi_\epsilon\|_{2s} \quad (98)$$

Noting that $\|\rho_\mu(t)-\rho_\nu(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leq 2M\sqrt{n\tilde{T}_s|\mu-\nu|}e^{\tilde{T}_s L_0(M,M)}$ (see Theorem 4.2) and taking the limit as $\nu\to 0$,it follows that,

$$\|\rho_\mu(t)-\rho(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leq 2M\sqrt{n\,\tilde{T}_s\,\mu}\,e^{\tilde{T}_s L_0(M,M)} = \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu} \quad (99)$$

Substituting (99) in (98) and noting that $\|\varphi_\epsilon\|_{2s}\leq\epsilon^{-s}\|\varphi\|_s$ with φ_ϵ constructed as in [I-I5, Lemma 2.6, pg 900], yields

$$|\langle\rho_\mu(t)-\rho(t),\varphi\rangle_s| \leq 2M\epsilon + \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\|\varphi\|_s \quad (100)$$

Then

$$\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s| \leq 2M\epsilon + \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s} \quad (101)$$

and

$$\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)} \lesssim 1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}, \forall \epsilon > 0 \tag{102}$$

Since $r(\tau) = \tau^{2k}$ is a non-decreasing function, it follows that:

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \lesssim r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s})\|\tilde{\rho}\|_s^2 \tag{103}$$

Integrating from 0 to t in (103)

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 + r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}) \int_0^t \|\tilde{\rho}(\tau)\|_s^2 \, d\tau \tag{104}$$

Applying Gronwall 's inequality to (104), we obtain a priori-estimate in $\mathfrak{H}^s(\mathbb{R}^3_+);\ s > 5/2$

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 \ e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s},\ \forall \tilde{T}_s > 0, \forall \epsilon > 0 \tag{105}$$

Finally, applying [Y, Theo. 1, pg 120] in (105) we obtain the final estimate

$$\begin{aligned} \|\rho(t)\|_s^2 &\leq \liminf_{\mu \to 0} \|\rho_\mu(t)\|_s^2 \\ &\leq \liminf_{\mu \to 0} \|\rho_0\|_s^2 \ e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\ &= \lim_{\mu \to 0} \|\rho_0\|_s^2 \ e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\ &= \|\rho_0\|_s^2 \ e^{r(1+2M\epsilon)\tilde{T}_s}\ \forall \epsilon > 0 \end{aligned} \tag{106}$$

Therefore, taking the limit as ϵ tends to zero, follows the final estimate

$$\|\rho(t)\|_s^2 \leq \|\rho_0\|_s^2 \ e^{\tilde{T}_s},\ \forall t \in [0, \tilde{T}_s], \tag{107}$$

and the proof is complete. ■

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