Instituto Nacional de Matemática Pura e Aplicada

Bounds for the Riemann zeta-function via Fourier analysis

by

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DISSERTATION

Presented to the Post-graduate Program in Mathematics of the

Instituto de Matemática Pura e Aplicada

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

Rio de Janeiro, 2019

To my family.

Acknowledgments

I would like to express my deepest gratitude to God for allowing me to finish this fantastic journey. I am sure that during each step in these 4 years, He has always been with me.

I would like to thank my thesis advisor Emanuel Carneiro, for all the guidance, support and incessant motivation. He taught me several courses during my stay at IMPA, but the main lesson I learned from him was to learn to trust myself. In addition, I am very grateful for all the support, both academic and financial, that he has provided throughout my graduate studies.

I would like to extend my gratitude to Prof. Kannan Soundararajan (University of Stanford), Prof. Micah Milinovich (University of Mississippi) and Prof. Dimitar Dimitrov (University of São Paulo), for their support, the mathematical discussions and the kind hospitality.

I would also like to thank my committee members: Professors Emanuel Carneiro, Dimitar Dimitrov, Carlos Gustavo Moreira, Felipe Linares, Mikhail Belolipetsky and Micah Milinovich, for their comments and suggestions.

I am very thankful to all my friends from different countries that I made during this time. This fantastic travel was nicer with their company.

I would like to acknowledge the financial support from CNPQ-Brazil (2015-2017) and FAPERJ-Brazil (2017-2019).

I would like to give special thanks to my family for the words of encouragement, the phrases of support and for waiting for me at the end of each year to be together. My parents Andrés Chirre Vega and Celia Lupe Chávez Flores are my motors for everything I do.

I am deeply grateful to my fiancee Jany. Her unconditional love, her words of encouragement and patience were an important part during my studies.

Finally, I would like to thank my friends from the Seventh-day Hispanic Adventist Church from Rio de Janeiro. Their spiritual and moral support was important to feel at home.

Abstract

In this Ph.D. thesis, we establish new bounds for some objects related to the Riemann zeta-function and L-functions, under the Riemann hypothesis, making use of fine tools from analytic number theory, harmonic analysis, and approximation theory. Firstly, we use extremal bandlimited approximations to show bounds for the high moments of the argument of the Riemann zeta-function and for a family of L-functions. Secondly, we use the resonance method of Soundararajan, in the version of Bondarenko and Seip, to obtain large values for the high moments of the argument function. Finally, we improve some estimates related with the distribution of the zeros of the Riemann zeta-function, using the approach of pair correlation of Montgomery and tools from semidefinite programming.

Resumo

Nesta tese de Doutorado estabelecemos novos limites para alguns objetos relacionados à função zeta de Riemann e a uma classe de L-funções, sob a hipótese de Riemann, fazendo uso de ferramentas finas da teoria analítica dos números, análise harmônica e teoria da aproximação. Em primeiro lugar, usamos aproximações extremais de banda limitada para mostrar cotas para os momentos do argumento da função zeta de Riemann e para uma família de L-funções. Em segundo lugar, usamos o método de ressonância de Soundararajan, na versão de Bondarenko e Seip, para obter grandes valores para os momentos da função argumento. Finalmente, melhoramos algumas estimativas relacionadas com a distribuição dos zeros da função zeta de Riemann, usando a abordagem de correlação de pares de Montgomery e ferramentas de programação semidefinida.

Contents

Chapter 1

Introduction

This Ph.D. thesis is focused on the use of different techniques in analytic number theory, harmonic analysis and approximation theory to establish new bounds for some objects related to the Riemann zeta-function and a family of L-functions. The thesis compiles the developments of the following research articles:

[A1] Bounding $S_n(t)$ on the Riemann hypothesis (with E. Carneiro), Mathematical Proceedings of the Cambridge Philosophical Society, vol.164 (2018), 259-283.

[A2] Bandlimited approximations and estimates for the Riemann zeta-function (with E. Carneiro and M. B. Milinovich), to appear in Publicacions Matemàtiques.

[A3] A note on entire L-functions, to appear in Bulletin of the Brazilian Mathematical Society.

[A4] Extreme values for $S_n(\sigma, t)$ near the critical line, to appear in Journal of Number Theory.

[A5] Pair correlation estimates for the zeros of the zeta-function via semidefinite programming (with F. Gonçalves and D. de Laat), preprint, arXiv:1810.08843 (2018).

In Chapter 2 we find new upper and lower bounds for the high moments $S_n(t)$ of the argument of the Riemann zeta-function on the critical line, under the Riemann hypothesis. This extends the work of E. Carneiro, V. Chandee and M. B. Milinovich [16] for the case $n = 0$ and $n = 1$ and substantially improves the previous result of T. Wakasa [91] for the case $n \geqslant 2$. Our method uses special extremal functions of exponential type derived from the Gaussian subordination framework of E. Carneiro, F. Littmann and J. Vaaler [25], and an optimized interpolation argument. This chapter describes the article [A1] which is a joint work with E. Carneiro (IMPA - Brazil).

In Chapter 3 we extend the results of Chapter 2 to the critical strip. In particular, this recovers the results on the critical line and sharpens the error terms in such estimates. New upper and lower bounds for the real part of the logarithmic derivative of the Riemann zeta-function in the critical strip are obtained. This chapter describes the article [A2] which is a joint work with E. Carneiro and M. Milinovich (University of Mississippi - USA).

In Chapter 4 we discuss how to extend the results of the previous chapters to a general family of L-functions in the framework of [56, Chapter 5], under the generalized Riemann hypothesis. This also extends the work of E. Carneiro, V. Chandee and M. Milinovich [17] and the work of E. Carneiro and R. Finder [20]. We also show estimates for the logarithm of L-functions extending a result of E. Carneiro and V. Chandee [14]. This chapter describes the article [A3] and the final part of the article [A1].

In Chapter 5 we obtain new estimates for the extreme values of the argument of the Riemann zeta-function and its high moments near the critical line assuming the Riemann hypothesis. These results extend the work of A. Bondarenko and K. Seip [9]. The main tools are certain convolution formulas and a version of the resonance method. This work can be seen as the counterpart of the estimates in [A2] close to the critical line. In particular we get some omega results for the functions $S_n(t)$. This chapter describes the results of the article [A4].

In Chapter 6 we give improved asymptotic bounds for several quantities related to the zeros of the Riemann zeta-function under H. Montgomery's pair correlation approach [72]. Similar results are obtained for the derivative of the Riemann ξ-function and a family of primitive Dirichlet L-functions. The key idea is to replace the usual bandlimited auxiliary functions by the class of functions used in the linear programming bounds developed by H. Cohn and N. Elkies [32] for the sphere packing problem. The advantage of this framework is that it reduces the problems to certain convex optimization problems that can be solved numerically via semidefinite programming. This chapter describes the results of the article [A5] which is a joint work with F. Gonçalves (Universität Bonn - Germany) and D. de Laat (MIT - USA).

1.1 Notation

Throughout this thesis, we use the classical notation for the usual elements in analytic number theory and harmonic analysis. We consider the following agreements:

1. For $s \in \mathbb{C}$ we write $s = \sigma + it$, where σ and t are real numbers.
2. For $f \in L^1(\mathbb{R})$ we denote by $\widehat{f}$ the Fourier transform of f, defined by
$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x.$$
3. For every sum over zeros the summands should be repeated according to the multiplicity of the zero.

Also, we consider the following:

a) $\mathbb{Z}_{\geqslant 0}$ denotes the set of the integer numbers $\{0, 1, 2, 3, 4, ...\}$.

b) $\Gamma(s)$ denotes the Gamma function.

c) $\Lambda(n)$ denotes the von Mangoldt function defined to be $\log p$ if $n = p^m$ with p a prime number and $m \geqslant 1$ an integer, and zero otherwise.

d) L_k denotes the Laguerre polynomial of degree k with parameter $-1/2$ defined by $L_k(z) = \sum_{j=0}^{n} \binom{n-1/2}{n-j} \frac{(-z)^j}{j!}$.

e) supp (f) denotes the set $\overline{\{x \in \mathrm{Dom}(f) : f(x) \neq 0\}}$.

f) f_+ denotes the function defined by $f_+(x) = \max\{f(x), 0\}$.

g) $f = O(g)$ (or $f \ll g$) means $|f(t)| \leqslant C\,|g(t)|$ for some constant $C > 0$ and for t sufficiently large. In the subscript we indicate the parameters in which such constant C may depend on.

h) $f = o(g)$ means that $\lim_{t \to \infty} f(t)/g(t) = 0$.

i) $f = \Omega_+(g)$ means $f(t) > C\,g(t)$ for some constant $C > 0$ and for some arbitrarily large values of t.

j) $f = \Omega_-(g)$ means $f(t) < -C\,g(t)$ for some constant $C > 0$ and for some arbitrarily large values of t.

k) $f = \Omega_\pm(g)$ means that $f = \Omega_+(g)$ and $f = \Omega_-(g)$.

l) $f = \Omega(g)$ means that $\lim_{t \to \infty} f(t)/g(t) \neq 0$.

Chapter 2

The Riemann zeta-function and bandlimited approximations I

This chapter is comprised of the paper [A1]. Our main goal here is to improve, under the Riemann hypothesis, the known upper and lower bounds for the high moments $\{S_n(t)\}_{n\geqslant 2}$ of the argument of the Riemann zeta-function on the critical line, extending the work of Carneiro, Chandee and Milinovich [16] for $S(t)$ and $S_1(t)$. Our argument relies on the use of certain extremal majorants and minorants of exponential type derived from the Gaussian subordination framework of Carneiro, Littmann and Vaaler [25] and an optimized interpolation argument.

2.1 The Riemann zeta-function

The Riemann zeta-function $\zeta(s)$ is the function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\operatorname{Re} s > 1$. Using the fundamental theorem of arithmetic, one clearly sees the first connection of the Riemann zeta-function with the prime numbers through the relation

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{2.1.1}$$

where the product is over all prime numbers and is absolutely convergent for $\operatorname{Re}(s) > 1$.

In 1859, Riemann [80] showed that $\zeta(s)$ has an analytic continuation to the complex plane. In fact, the Riemann ξ-function defined by

$$\xi(s) = \frac{1}{2}\, s\,(s-1)\, \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right)\, \zeta(s) \tag{2.1.2}$$

is an entire function of order 1 and satisfies the functional equation $\xi(s) = \xi(1-s)$. Riemann also showed a more deeper connection between the behavior of the function $\zeta(s)$ and the distribution of the prime numbers. To be more especific, he showed an explicit formula that expresses the number of primes less than a number x in terms of the zeros of $\zeta(s)$.

It is known that the Riemann zeta-function only has zeros in $\mathrm{Re}\, s < 0$ in each point $s = -2k$ with $k \in \mathbb{N}$. These are called the "trivial zeros" of $\zeta(s)$ and are exactly the poles of the Gamma function that appears in (2.1.2). By the Euler product (2.1.1), the Riemann zeta-function has no zeros in $\mathrm{Re}\,(s) > 1$. Therefore, the "non-trivial zeros" of $\zeta(s)$ lie in the critical strip $0 \leqslant \mathrm{Re}\, s \leqslant 1$. Moreover, using (2.1.2) we see that the non-trivial zeros of $\zeta(s)$ are the zeros of $\xi(s)$. It is also known that $\zeta(s)$ has a countably infinite number of non-trivial zeros and that they are symmetric with respect to the real-axis and the critical line $\mathrm{Re}\, s = \frac{1}{2}$. In the course of his paper [80], Riemann says that he considers it "very likely" that the non-trivial zeros have real part equal to $\frac{1}{2}$, but that he has been unable to prove that this is true. This harmless affirmation is one of the most important open problems in pure mathematics.

Conjecture 2.1 (Riemann hypothesis - 1859). *All non-trivial zeros of $\zeta(s)$ have $\mathrm{Re}\, s = \frac{1}{2}$.*

The experience of Riemann's successors with the Riemann hypothesis has been the same as Riemann's—they also consider its truth "very likely" and they also have been unable to prove it. Hilbert included the problem of proving the Riemann hypothesis in his list [52] of the most important unsolved problems which confronted mathematics in 1900, and the attempt to solve this problem has occupied the best efforts of many of the best mathematicians of the twentieth century. It is now unquestionably the most celebrated problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because it appears tantalizingly vulnerable and because its solution would probably bring to light new techniques of far-reaching importance.[1]

For an overview of the theory of the Riemann zeta-function, we refer the reader to the classic books by Davenport [36], Edwards [37], Ivic [54, 55], Iwaniec and Kowalski [56], Montgomery and Vaughan [74], and Titchmarsh [86] as well as the references contained within these sources.

2.2 Behavior on the critical line: $S_n(t)$

Let $N(t)$ denote the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leqslant t$, counting multiplicities (zeros with ordinate $\gamma = t$ are counted with weight $\frac{1}{2}$). In the study of the distribution of the zeros of $\zeta(s)$, Riemann [80] stated the asymptotic formula for $N(t)$, which was later proved by von Mangoldt [70] in 1895. For $t \geqslant 2$ we have

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right), \tag{2.2.1}$$

[1]H. M. Edwards, *Riemann's zeta-function*, Pag. 6.

where $S(t)$ is defined as follows: If t is not the ordinate of a zero of $\zeta(s)$ we define

$$S(t) = \tfrac{1}{\pi} \arg \zeta\big(\tfrac{1}{2} + it\big),$$

where the argument is obtained by a continuous variation along straight line segments joining the points 2, $2 + it$ and $\frac{1}{2} + it$, with the convention that $\arg \zeta(2) = 0$. If t is the ordinate of a zero of $\zeta(s)$ we define

$$S(t) = \tfrac{1}{2} \lim_{\varepsilon \to 0} \{S(t+\varepsilon) + S(t-\varepsilon)\}\,.$$

The function $S(t)$ has an intrinsic oscillating character and is naturally connected to the distribution of the non-trivial zeros of $\zeta(s)$ via the relation (2.2.1). Useful information on the qualitative and quantitative behavior of $S(t)$ is encoded in its high moments $S_n(t)$. Setting $S_0(t) = S(t)$ we define, for $n \geqslant 1$ and $t > 0$,

$$S_n(t) = \int_0^t S_{n-1}(\tau)\ \mathrm{d}\tau\ + \delta_n\,, \tag{2.2.2}$$

where δ_n are constants given by (see for instance [41, p. 2])

$$\delta_{2k-1} = \frac{(-1)^{k-1}}{\pi} \int_{\frac{1}{2}}^{\infty} \int_{\sigma_{2k-2}}^{\infty} \cdots \int_{\sigma_2}^{\infty} \int_{\sigma_1}^{\infty} \log|\zeta(\sigma_0)|\ \mathrm{d}\sigma_0\ \mathrm{d}\sigma_1\ \ldots\ \mathrm{d}\sigma_{2k-2}$$

for $n = 2k-1$, with $k \geqslant 1$, and

$$\delta_{2k} = (-1)^{k-1} \int_{\frac{1}{2}}^{1} \int_{\sigma_{2k-1}}^{1} \cdots \int_{\sigma_2}^{1} \int_{\sigma_1}^{1} \mathrm{d}\sigma_0\ \mathrm{d}\sigma_1\ \ldots\ \mathrm{d}\sigma_{2k-1} = \frac{(-1)^{k-1}}{(2k)! \cdot 2^{2k}}$$

for $n = 2k$, with $k \geqslant 1$.

Fujii [41] established some interesting formulas between $S_n(t)$ and the non-trivial zeros of $\zeta(s)$. Such formulas allowed him to recast the Riemann hypothesis (RH) as follows:

Theorem 2.2 (Fujii, 2001)**.** *The following statement is equivalent to the Riemann hypothesis: for any integer $n \geqslant 3$, we have $S_n(t) = o(t^{n-2})$, as $t \to \infty$.*

Unconditionally, there are known bounds for the functions $S_n(t)$. For the cases $n = 0$ and $n = 1$ we have the classical bounds $S(t) = O(\log t)$ and $S_1(t) = O(\log t)$ (see for instance [86]). For $n \geqslant 2$, Fujii [41, Theorem 2] established that

$$S_n(t) = O_n\left(\frac{t^{n-1}}{\log t}\right).$$

Under RH, Littlewood [64] (see also Selberg [83]) obtained improved estimates for $S_n(t)$.

In fact, the classical result of Littlewood [64, Theorem 11] states that, under RH,

$$S_n(t) = O\left(\frac{\log t}{(\log\log t)^{n+1}}\right) \tag{2.2.3}$$

for $n \geqslant 0$. The order of magnitude of (2.2.3) has not been improved over the last ninety years, and the efforts have hence been concentrated in optimizing the values of the implicit constants. In the case $n = 0$, the best bound under RH is due to Carneiro, Chandee and Milinovich [16] (see also [17]), who established that

$$|S(t)| \leqslant \left(\frac{1}{4} + o(1)\right)\frac{\log t}{\log\log t}. \tag{2.2.4}$$

This improved upon earlier works of Goldston and Gonek [46], Fujii [42] and Ramachandra and Sankaranarayanan [79], who had obtained (2.2.4) with constants $C = 1/2$, $C = 0.67$ and $C = 1.12$, respectively, replacing the constant $C = 1/4$.

For $n = 1$ the current best bound under RH is also due to Carneiro, Chandee and Milinovich [16], who showed that

$$-\left(\frac{\pi}{24} + o(1)\right)\frac{\log t}{(\log\log t)^2} \;\leqslant\; S_1(t) \;\leqslant\; \left(\frac{\pi}{48} + o(1)\right)\frac{\log t}{(\log\log t)^2}. \tag{2.2.5}$$

This improved upon earlier works of Fujii [43], and Karatsuba and Korolëv [58], who had obtained (2.2.5) with the pair of constants $(C^+, C^-) = (0.32, 0.51)$ and $(C^+, C^-) = (40, 40)$, respectively, replacing the pair $(C^+, C^-) = (\pi/48, \pi/24)$.

For $n \geqslant 2$, under RH, it was recently established by Wakasa [91] that

$$|S_n(t)| \leqslant (W_n + o(1))\frac{\log t}{(\log\log t)^{n+1}}, \tag{2.2.6}$$

with the constant W_n given by

$$W_n = \frac{1}{2\pi n!}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\sum_{j=0}^{n}\frac{n!}{(n-j)!}\left(\frac{1}{e} + \frac{1}{2^{j+1}e^2}\right)\right.$$
$$\left. + \frac{1}{(n+1)}\cdot\frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{1}{n(n+1)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\right\}$$

if n is odd, and

$$W_n = \frac{1}{2\pi n!}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\sum_{j=0}^{n}\frac{n!}{(n-j)!}\left(\frac{1}{e} + \frac{1}{2^{j+1}e^2}\right)\right.$$
$$\left. + \frac{1}{(n+1)}\cdot\frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{\pi}{2}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\right\}$$

if n is even.

2.2.1 Main result

Here we extend the methods of [16] to significantly improve the bound (2.2.6). Our main result is the following.

Theorem 2.3. *Assume the Riemann hypothesis. For $n \geqslant 0$ and t sufficiently large we have*

$$- \left(C_n^- + o(1)\right) \frac{\log t}{(\log\log t)^{n+1}} \leqslant S_n(t) \leqslant \left(C_n^+ + o(1)\right) \frac{\log t}{(\log\log t)^{n+1}}, \tag{2.2.7}$$

where $C_n^\pm$ are positive constants given by:

- *For $n = 0$,*

$$C_0^\pm = \frac{1}{4}.$$

- *For $n = 4k+1$, with $k \in \mathbb{Z}^+$,*

$$C_n^- = \frac{\zeta(n+1)}{\pi \cdot 2^{n+1}} \quad \text{and} \quad C_n^+ = \frac{(1-2^{-n})\,\zeta(n+1)}{\pi \cdot 2^{n+1}}.$$

- *For $n = 4k+3$, with $k \in \mathbb{Z}^+$,*

$$C_n^- = \frac{(1-2^{-n})\,\zeta(n+1)}{\pi \cdot 2^{n+1}} \quad \text{and} \quad C_n^+ = \frac{\zeta(n+1)}{\pi \cdot 2^{n+1}}.$$

- *For $n \geqslant 2$ even,*

$$\begin{aligned} C_n^+ = C_n^- &= \left[\frac{2\left(C_{n+1}^+ + C_{n+1}^-\right) C_{n-1}^+ \, C_{n-1}^-}{C_{n-1}^+ + C_{n-1}^-}\right]^{1/2} \\ &= \frac{\sqrt{2}}{\pi \cdot 2^{n+1}} \left[\frac{\left(1-2^{-n-2}\right)\left(1-2^{-n+1}\right)\zeta(n)\,\zeta(n+2)}{(1-2^{-n})}\right]^{1/2}. \end{aligned}$$

The terms $o(1)$ in (2.2.7) are $O(\log\log\log t/\log\log t)$.[2]

For $n = 0$ and $n = 1$ this is a restatement of the result of Carneiro, Chandee and Milinovich [16]. The novelty here are the cases $n \geqslant 2$. Observe that $C_n^\pm \sim \frac{1}{\pi \cdot 2^{n+1}}$ when n is odd and large and $C_n^\pm \sim \frac{\sqrt{2}}{\pi \cdot 2^{n+1}}$ when n is even and large. We highlight the contrast between these exponentially decaying bounds and the previously known bounds (2.2.6) of Wakasa [91] that verify

$$\lim_{n\to\infty} W_n = \frac{1}{2\pi\left(1-\frac{1}{e}\left(1+\frac{1}{e}\right)\right)} = 0.3203696...$$

[2]We remark that the implicit constants in the $O-$notation in our estimates (as well as in (2.2.3)) are allowed to depend on n.

n	C_n^-	C_n^+	W_n	$W_n \,/\, \max\{C_n^-, C_n^+\}$
2	0.0593564...	0.0593564...	0.6002288...	10.1122762...
3	0.0188406...	0.0215321...	0.3426156...	15.9118250...
4	0.0141490...	0.0141490...	0.3509932...	24.8069103...
5	0.0050598...	0.0049017...	0.3254151...	64.3131985...
6	0.0035192...	0.0035192...	0.3235655...	91.9420229...
7	0.0012387...	0.0012484...	0.3216216...	257.6130647...
8	0.0008792...	0.0008792...	0.3210078...	365.0786196...
9	0.0003111...	0.0003105...	0.3206826...	1030.6078264...
10	0.0002198...	0.0002198...	0.3205263...	1458.2249832...

Table 2.1: Comparison for $2 \leqslant n \leqslant 10$.

Table 2.1 puts in perspective the new bounds of our Theorem 2.3 and the previously known bounds (2.2.6) in the small cases $2 \leqslant n \leqslant 10$. The last column reports the improvement factor.

2.2.2 Strategy outline

Our approach is partly motivated (in the case of n odd) by the ideas of Goldston and Gonek [46], Chandee and Soundararajan [29], and Carneiro, Chandee and Milinovich [16], on the use of the Guinand-Weil explicit formula on special functions with compactly supported Fourier transforms (drawn from [89], [27] and [22, 25], respectively) to bound objects related to the Riemann zeta-function.

The strategy can be broadly divided into the following four main steps:

Step 1: Representation lemma.

The first step is to identify certain particular functions of a real variable naturally connected to the high moments $S_n(t)$. For each $n \geqslant 0$ define a normalized function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

- If $n = 2m$, for $m \in \mathbb{Z}_{\geqslant 0}$, we define

$$f_{2m}(x) = (-1)^m x^{2m} \arctan\left(\frac{1}{x}\right) - \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{2k+1}\, x^{2m-2k-1} - \frac{x}{(2m+1)(1+x^2)}. \quad (2.2.8)$$

- If $n = 2m+1$, for $m \in \mathbb{Z}_{\geqslant 0}$, we define

$$f_{2m+1}(x) = \frac{1}{2m+1}\left[(-1)^{m+1} x^{2m+1} \arctan\left(\frac{1}{x}\right) + \sum_{k=0}^{m} \frac{(-1)^{m-k}}{2k+1} x^{2m-2k}\right]. \quad (2.2.9)$$

We show in Lemma 2.5 below that, under RH, $S_n(t)$ can be expressed in terms of the sum of a translate of f_n over the ordinates of the non-trivial zeros of $\zeta(s)$. From the power series representation (around the origin)

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$$

one can check that $f_{2m}(x) \ll_m |x|^{-3}$ and $f_{2m+1}(x) \ll_m |x|^{-2}$ as $|x| \to \infty$. This rather innocent piece of information is absolutely crucial in our argument.

Step 2: Extremal functions.

Our tool to evaluate sums over the non-trivial zeros of $\zeta(s)$ is the Guinand-Weil explicit formula. However, the functions f_n defined above do not possess the required smoothness to allow a direct evaluation. In fact, we have that f_n is of class $C^{n-1}(\mathbb{R})$ but not higher (the n-th derivative of f_n is discontinuous at $x = 0$). Note also that f_0 is discontinuous at the origin. Then, it will be convenient to replace f_n by one-sided entire approximations of exponential type in a way that minimizes the $L^1(\mathbb{R})-$error. This is the so called *Beurling-Selberg extremal problem* in approximation theory. These special functions have been useful in several classical applications in number theory (see for instance the excellent survey [89] by J. D. Vaaler and some of the references therein) and have recently been used in connection to the theory of the Riemann zeta-function in the works [14, 15, 16, 17, 20, 29, 44, 46]. We shall see that the even functions f_{2m+1}, for $m \in \mathbb{Z}_{\geqslant 0}$, fall under the scope of the Gaussian subordination framework of [25]. This yields the desired existence and qualitative description of the Beurling-Selberg extremal functions in these cases (Lemma 2.8 below) and ultimately leads to the bounds of Theorem 2.3 for n odd. When n is even, our argument is subtler since the functions f_{2m} are odd. The Gaussian subordination framework for odd functions [22] only allows us to solve the Beurling-Selberg problem for a class of functions *with a discontinuity at the origin.* This is the case, for example, with the function $f_0(x) = \arctan(1/x) - x/(1+x^2)$, and this was explored in [16] to show (2.2.4). For $m \geqslant 1$, the functions f_{2m} are all odd and continuous, and the solution of the Beurling-Selberg problem for these functions is quite a delicate issue and currently unknown. We are then forced to take a very different path in this case.

Step 3: Guinand-Weil explicit formula and asymptotic analysis.

In the case of n odd, we bound $S_n(t)$ by applying the Guinand-Weil explicit formula to the Beurling-Selberg majorants and optimizing the size of the support of the Fourier transform. This is possible via a careful asymptotic analysis of all the terms that appear in the explicit formula.

Step 4: Interpolation tools.

Having obtained the desired bounds for all odd n's, we proceed with an interpolation argument to obtain the estimate for the even n's in between, exploring the smoothness of

$S_n(t)$ via the mean value theorem. An optimal choice of the parameters involved in the interpolation argument yields the desired bounds for the even n's.

2.3 Representation lemma I

Our starting point is the following formula motivated by the work of Selberg [81].

Lemma 2.4. *Assume the Riemann hypothesis. For $n \geqslant 0$ and $t > 0$ (t not coinciding with the ordinate of a zero of $\zeta(s)$ when $n = 0$) we have*

$$S_n(t) = -\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{i^n}{n!}\int_{1/2}^{\infty}\left(\sigma - \tfrac{1}{2}\right)^n \frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma\right\}. \tag{2.3.1}$$

Proof. This result is contained in the work of Fujii [41, Lemmas 1 and 2]. We provide here a brief sketch of the proof. Let $R_n(t)$ be the expression on the right-hand side of (2.3.1). The validity of the formula for $n = 0$ is clear. Proceeding by induction, let us assume that the result holds for $n = 0, 1, 2, \ldots, m-1$. Differentiating under the integral sign and using integration by parts one can check that $R_m'(t) = R_{m-1}(t) = S_{m-1}(t)$ (for $m = 1$ we may restrict ourselves to the case when t does not coincide with the ordinate of a zero of $\zeta(s)$). From (2.2.2) it remains to show that $\lim_{t\to 0^+} R_m(t) = \delta_m$ for $m \geqslant 1$. This follows by integrating by parts m times and then taking the limit as $t \to 0^+$. $\square$

The next result establishes the connection between S_n and the functions f_n defined in (2.2.8) - (2.2.9). In the proof of Theorem 2.3 we shall only use the case of n odd, but we state here the representation for n even as well, as a result of independent interest.

Lemma 2.5 (Representation lemma). *For each $n \geqslant 0$ let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as in* (2.2.8) - (2.2.9)*. Assume the Riemann hypothesis. For $t \geqslant 2$ (and t not coinciding with an ordinate of a zero of $\zeta(s)$ in the case $n = 0$) we have:*

(i) If $n = 2m$, for $m \in \mathbb{Z}_{\geqslant 0}$, then

$$S_{2m}(t) = \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma} f_{2m}(t-\gamma) \;+\; O(1). \tag{2.3.2}$$

(ii) If $n = 2m+1$, for $m \in \mathbb{Z}_{\geqslant 0}$, then

$$S_{2m+1}(t) = \frac{(-1)^m}{2\pi(2m+2)!}\log t - \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma} f_{2m+1}(t-\gamma) \;+\; O(1). \tag{2.3.3}$$

The above sums run over the ordinates of the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. We split the proof into two cases: n odd and n even.
Case 1. n odd: Write $n = 2m+1$. It follows from Lemma 2.4 and integration by parts that

$$\begin{aligned} S_{2m+1}(t) &= -\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{i^{2m+1}}{(2m+1)!}\int_{1/2}^{\infty}\left(\sigma-\tfrac{1}{2}\right)^{2m+1}\frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma\right\} \\ &= \frac{(-1)^{m+1}}{\pi(2m+1)!}\,\mathrm{Re}\left\{\int_{1/2}^{\infty}\left(\sigma-\tfrac{1}{2}\right)^{2m+1}\frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma\right\} \\ &= \frac{(-1)^{m}}{\pi(2m)!}\,\mathrm{Re}\left\{\int_{1/2}^{\infty}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log\zeta(\sigma+it)\,\mathrm{d}\sigma\right\} \\ &= \frac{(-1)^{m}}{\pi(2m)!}\left\{\int_{1/2}^{3/2}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log|\zeta(\sigma+it)|\,\mathrm{d}\sigma\right\} + O(1). \end{aligned} \tag{2.3.4}$$

The idea is to replace the integrand by an absolutely convergent sum over the zeros of $\zeta(s)$ and then integrate term-by-term. Using the Hadamard's factorization formula (cf. [36, Chapter 12]) for the Riemann ξ-function defined in (2.1.2), we have

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$, $A \in \mathbb{R}$ and $B = -\sum_{\rho}\mathrm{Re}\,(1/\rho)$. Therefore, assuming the Riemann hypothesis, it follows that

$$\left|\frac{\xi(\sigma+it)}{\xi(\frac{3}{2}+it)}\right| = \prod_{\gamma}\left(\frac{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}{1+(t-\gamma)^2}\right)^{1/2}. \tag{2.3.5}$$

Hence

$$\log|\xi(\sigma+it)| - \log\left|\xi\left(\tfrac{3}{2}+it\right)\right| = \frac{1}{2}\sum_{\gamma}\log\left(\frac{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}{1+(t-\gamma)^2}\right).$$

By Stirling's formula for $\Gamma(s)$ (cf. [36, Chapter 10]) we obtain

$$\log|\zeta(\sigma+it)| = \left(\tfrac{3}{4}-\tfrac{\sigma}{2}\right)\log t - \frac{1}{2}\sum_{\gamma}\log\left(\frac{1+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\right) + O(1), \tag{2.3.6}$$

uniformly for $1/2 \leqslant \sigma \leqslant 3/2$ and $t \geqslant 2$. Inserting (2.3.6) into (2.3.4) yields

$$\begin{aligned} S_{2m+1}(t) &= \frac{(-1)^{m}}{\pi(2m)!}\left(\int_{1/2}^{3/2}\left(\sigma-\tfrac{1}{2}\right)^{2m}\left(\tfrac{3}{4}-\tfrac{\sigma}{2}\right)\,\mathrm{d}\sigma\right)\log t \\ &\quad - \frac{(-1)^{m}}{2\pi(2m)!}\int_{1/2}^{3/2}\sum_{\gamma}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log\left(\frac{1+(t-\gamma)^2}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\right)\,\mathrm{d}\sigma + O(1) \\ &= \frac{(-1)^{m}}{2\pi(2m+2)!}\log t \end{aligned}$$

$$
\begin{aligned}
&- \frac{(-1)^m}{2\pi(2m)!} \sum_\gamma \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \log\left(\frac{1 + (t-\gamma)^2}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} \right) \, \mathrm{d}\sigma + O(1) \\
&= \frac{(-1)^m}{2\pi(2m+2)!} \log t - \frac{(-1)^m}{\pi(2m)!} \sum_\gamma f_{2m+1}(t-\gamma) + O(1), \qquad (2.3.7)
\end{aligned}
$$

where the function f_{2m+1} is (momentarily) defined by

$$
f_{2m+1}(x) = \frac{1}{2} \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \log\left(\frac{1 + x^2}{(\sigma - \frac{1}{2})^2 + x^2} \right) \mathrm{d}\sigma \,, \qquad (2.3.8)
$$

and the interchange between the sum and integral in (2.3.7) is justified by monotone convergence since all the terms involved are nonnegative. Starting from (2.3.8), a change of variables and the use of formula [50, 2.731] yield

$$
\begin{aligned}
f_{2m+1}(x) &= \frac{1}{2} \int_0^1 \sigma^{2m} \log\left(\frac{1+x^2}{\sigma^2 + x^2} \right) \, \mathrm{d}\sigma \\
&= \frac{\log(1+x^2)}{2(2m+1)} - \frac{1}{2} \int_0^1 \sigma^{2m} \log(\sigma^2 + x^2) \, \mathrm{d}\sigma \\
&= \frac{\log(1+x^2)}{2(2m+1)} - \frac{1}{2(2m+1)} \Bigg[\sigma^{2m+1} \log(\sigma^2 + x^2) + (-1)^m 2x^{2m+1} \arctan\left(\frac{\sigma}{x}\right) \\
&\qquad\qquad - 2 \sum_{k=0}^{m} \frac{(-1)^{m-k}}{2k+1} x^{2m-2k} \sigma^{2k+1} \Bigg] \Bigg|_0^1 \\
&= \frac{1}{(2m+1)} \left[(-1)^{m+1} x^{2m+1} \arctan\left(\frac{1}{x}\right) + \sum_{k=0}^{m} \frac{(-1)^{m-k}}{2k+1} x^{2m-2k} \right].
\end{aligned}
$$

This shows that the two definitions (2.2.9) and (2.3.8) agree, which completes the proof in this case.

Case 2. n even: Write $n = 2m$. From Lemma 2.4 it follows that

$$
\begin{aligned}
S_{2m}(t) &= -\frac{1}{\pi} \,\mathrm{Im} \left\{ \frac{i^{2m}}{(2m)!} \int_{1/2}^{\infty} \left(\sigma - \tfrac{1}{2}\right)^{2m} \frac{\zeta'}{\zeta}(\sigma + it) \, \mathrm{d}\sigma \right\} \\
&= \frac{(-1)^{m+1}}{\pi(2m)!} \,\mathrm{Im} \left\{ \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \frac{\zeta'}{\zeta}(\sigma + it) \, \mathrm{d}\sigma \right\} + O(1).
\end{aligned} \qquad (2.3.9)
$$

We again replace the integrand by an absolutely convergent sum over the non-trivial zeros of $\zeta(s)$. Let $s = \sigma + it$. If s is not a zero of $\zeta(s)$, then the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [36, Chapter 12]) and Stirling's formula for $\Gamma'(s)/\Gamma(s)$ (cf. [36, Chapter 10]) imply that

$$
\begin{aligned}
\frac{\zeta'}{\zeta}(s) &= \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1 \right) + O(1) \\
&= \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \log\left(\frac{t}{2} \right) + O(1)
\end{aligned} \qquad (2.3.10)
$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$ and $t \geqslant 2$, where the sum runs over the non-trivial zeros ρ of $\zeta(s)$. Assume that t is not the ordinate of a zero of $\zeta(s)$. Then, from (2.3.9), (2.3.10) and the Riemann hypothesis, it follows that

$$\begin{aligned}
S_{2m}(t) &= \frac{(-1)^{m+1}}{\pi(2m)!} \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \operatorname{Im}\left\{\frac{\zeta'}{\zeta}(\sigma + it)\right\} \mathrm{d}\sigma + O(1) \\
&= \frac{(-1)^{m+1}}{\pi(2m)!} \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \operatorname{Im}\left\{\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}\left(\tfrac{3}{2} + it\right)\right\} \mathrm{d}\sigma + O(1) \\
&= \frac{(-1)^{m}}{\pi(2m)!} \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \sum_{\gamma} \left\{\frac{(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(t-\gamma)}{1 + (t-\gamma)^2}\right\} \mathrm{d}\sigma + O(1) \\
&= \frac{(-1)^{m}}{\pi(2m)!} \sum_{\gamma} \int_{1/2}^{3/2} \left\{\frac{(\sigma - \frac{1}{2})^{2m}(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(\sigma - \frac{1}{2})^{2m}(t-\gamma)}{1 + (t-\gamma)^2}\right\} \mathrm{d}\sigma + O(1) \\
&= \frac{(-1)^{m}}{\pi(2m)!} \sum_{\gamma} \Bigg[\sum_{j=1}^{m} (-1)^{j+1} \frac{(t-\gamma)^{2j-1}}{2m - 2j + 1} + (-1)^m (t-\gamma)^{2m} \arctan\left(\frac{1}{t-\gamma}\right) \\
&\qquad\qquad - \frac{t-\gamma}{(2m+1)(1 + (t-\gamma)^2)}\Bigg] + O(1) \\
&= \frac{(-1)^{m}}{\pi(2m)!} \sum_{\gamma} \Bigg[\sum_{k=0}^{m-1} (-1)^{m-k+1} \frac{(t-\gamma)^{2m-2k-1}}{2k + 1} + (-1)^m (t-\gamma)^{2m} \arctan\left(\frac{1}{t-\gamma}\right) \\
&\qquad\qquad - \frac{t-\gamma}{(2m+1)(1 + (t-\gamma)^2)}\Bigg] + O(1) \\
&= \frac{(-1)^{m}}{\pi(2m)!} \sum_{\gamma} f_{2m}(t-\gamma) + O(1)\,, \qquad (2.3.11)
\end{aligned}$$

where the interchange between the sum and the integral is justified by dominated convergence since $f_{2m}(x) \ll_m |x|^{-3}$ as $|x| \to \infty$. Finally, if $m \geqslant 1$, both sides can be extended continuously when t is the ordinate of a zero of $\zeta(s)$. □

Remark 2.6. *Observe the introduction of a test point $\frac{3}{2} + it$ in a couple of passages in the proof above. This seemingly innocent object is actually quite important in dealing with the convergence issues.*

The sum of $f_{2m+1}(t-\gamma)$ over the non-trivial zeros in (2.3.3) is too complicated to be evaluated directly, mainly due to the fact that f_{2m+1} is only of class $C^{2m}(\mathbb{R})$. The key idea to prove Theorem 2.3 in this case is to replace the function f_{2m+1} in (2.3.3) by an appropriate majorant or minorant of exponential type (thus with a compactly supported Fourier transform by the Paley-Wiener theorem). We then apply the following version of the Guinand-Weil explicit formula which connects the zeros of the zeta-function and the prime powers.

Lemma 2.7 (Guinand-Weil explicit formula)**.** *Let $h(s)$ be analytic in the strip $|\operatorname{Im} s| \leqslant \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\operatorname{Re} s| \to \infty$.*

Let $h(w)$ be real-valued for real w. Then

$$\sum_{\rho} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\widehat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right),$$

where $\rho = \beta + i\gamma$ are the non-trivial zeros of $\zeta(s)$, Γ'/Γ is the logarithmic derivative of the Gamma function, and $\Lambda(n)$ is the von Mangoldt function.

Proof. The proof of this lemma follows from [56, Theorem 5.12]. □

2.4 Extremal bandlimited approximations I

Recall that an entire function $G : \mathbb{C} \to \mathbb{C}$ is said to have exponential type τ if

$$\limsup_{|z|\to\infty} \frac{\log|G(z)|}{|z|} \leqslant \tau.$$

The celebrated Paley-Wiener theorem states that a function $g \in L^2(\mathbb{R})$ has Fourier transform supported in the interval $[-\Delta, \Delta]$ if and only if it is equal almost everywhere to the restriction to $\mathbb{R}$ of an entire function of exponential type $2\pi\Delta$. The term *bandlimited* is commonly used in the applied literature in reference to functions that have compactly supported Fourier transforms.

The problem of finding one-sided approximations of real-valued functions by entire functions of prescribed exponential type, seeking to minimize the $L^1(\mathbb{R})-$error, is a classical problem in approximation theory. This problem has its origins in the works of A. Beurling and A. Selberg, who constructed majorants and minorants of exponential type for the signum function and characteristic functions of intervals, respectively. The survey [89] by J. D. Vaaler is the classical reference on the subject, describing some of the historical milestones of the problem and presenting a number of interesting applications of such special functions to analysis and number theory. In recent years there has been considerable progress both in the constructive aspects and in the range of applications of such extremal bandlimited approximations. For the constructive theory we highlight, for instance, the works [22, 25, 27, 51, 60, 66, 67, 68] in the one-dimensional theory and the works [21, 23, 24, 49, 53] in the multi-dimensional and weighted theory. These allowed new applications in the theory of the Riemann zeta-function and general L-functions, for instance in [14, 15, 16, 17, 18, 19, 20, 29, 31, 44, 46, 71].

The appropriate machinery for our purposes is the *Gaussian subordination framework* of Carneiro, Littmann and Vaaler [25], a method that allows one to solve the Beurling-Selberg extremal problem for a wide class of even functions. In particular, functions $g : \mathbb{R} \to \mathbb{R}$ of the form

$$g(x) = \int_0^\infty e^{-\pi\lambda x^2} \, \mathrm{d}\nu(\lambda), \tag{2.4.1}$$

where ν is a finite nonnegative Borel measure on $(0, \infty)$, fall under the scope of [25]. It turns out that our functions f_{2m+1} defined in (2.2.9) are included in this class. We collect the relevant properties for our purposes in the next lemma. This lemma is the generalization of [16, Lemma 4] that considers the case $m = 0$.

Lemma 2.8 (Extremal functions for f_{2m+1})**.** *Let $m \geqslant 0$ be an integer and let $\Delta \geqslant 1$ be a real parameter. Let f_{2m+1} be the real valued function defined in* (2.2.9)*, i.e.*

$$f_{2m+1}(x) = \frac{1}{2m+1}\left[(-1)^{m+1}x^{2m+1}\arctan\left(\frac{1}{x}\right) + \sum_{k=0}^{m}\frac{(-1)^{m-k}}{2k+1}x^{2m-2k}\right].$$

Then there are unique real entire functions[3] $g^-_{2m+1,\Delta} : \mathbb{C} \to \mathbb{C}$ *and* $g^+_{2m+1,\Delta} : \mathbb{C} \to \mathbb{C}$ *satisfying the following properties:*

(i) For $x \in \mathbb{R}$ we have

$$-\frac{K_{2m+1}}{1+x^2} \leqslant g^-_{2m+1,\Delta}(x) \leqslant f_{2m+1}(x) \leqslant g^+_{2m+1,\Delta}(x) \leqslant \frac{K_{2m+1}}{1+x^2}, \tag{2.4.2}$$

for some positive constant K_{2m+1} independent of Δ. Moreover, for any complex number $z = x + iy$ we have

$$\left|g^\pm_{2m+1,\Delta}(z)\right| \ll_m \frac{\Delta^2}{(1+\Delta|z|)}e^{2\pi\Delta|y|}. \tag{2.4.3}$$

(ii) The Fourier transforms of $g^\pm_{2m+1,\Delta}$, denoted by $\widehat{g}^\pm_{2m+1,\Delta}(\xi)$, are continuous functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\widehat{g}^\pm_{2m+1,\Delta}(\xi) \ll_m 1 \tag{2.4.4}$$

for all $\xi \in [-\Delta, \Delta]$, where the implied constant is independent of Δ.

(iii) The L^1-distances of $g^\pm_{2m+1,\Delta}$ to f_{2m+1} are explicitly given by

$$\int_{-\infty}^{\infty}\left\{f_{2m+1}(x) - g^-_{2m+1,\Delta}(x)\right\} \mathrm{d}x = \frac{1}{\Delta}\int_{1/2}^{3/2}\left(\sigma - \tfrac{1}{2}\right)^{2m}\log\left(\frac{1+e^{-2\pi(\sigma-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\mathrm{d}\sigma \tag{2.4.5}$$

and

$$\int_{-\infty}^{\infty}\left\{g^+_{2m+1,\Delta}(x) - f_{2m+1}(x)\right\} \mathrm{d}x$$

[3] Recall that a *real entire function* is an entire function whose restriction to $\mathbb{R}$ is real-valued.

$$= -\frac{1}{\Delta}\int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \log\left(\frac{1 - e^{-2\pi(\sigma-1/2)\Delta}}{1 - e^{-2\pi\Delta}}\right) \mathrm{d}\sigma. \tag{2.4.6}$$

Proof. For $\Delta \geqslant 1$, we consider the nonnegative Borel measure $\nu_\Delta = \nu_{2m+1,\Delta}$ on $(0,\infty)$ given by

$$\mathrm{d}\nu_\Delta(\lambda) := \int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2} - e^{-\pi\lambda\Delta^2}}{2\lambda}\right) \mathrm{d}\sigma \ \mathrm{d}\lambda\,,$$

and let $F_\Delta = F_{2m+1,\Delta}$ be the function

$$F_\Delta(x) := \int_0^\infty e^{-\pi\lambda x^2} \ \mathrm{d}\nu_\Delta(\lambda).$$

Recall that

$$\frac{1}{2}\log\left(\frac{x^2 + \Delta^2}{x^2 + (\sigma - 1/2)^2\Delta^2}\right) = \int_0^\infty e^{-\pi\lambda x^2}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2} - e^{-\pi\lambda\Delta^2}}{2\lambda}\right) \mathrm{d}\lambda.$$

Multiplying both sides by $(\sigma - 1/2)^{2m}$ and integrating from $\sigma = 1/2$ to $\sigma = 3/2$ yields

$$\begin{aligned}
&\frac{1}{2}\int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \log\left(\frac{x^2 + \Delta^2}{x^2 + (\sigma - 1/2)^2\Delta^2}\right) \mathrm{d}\sigma \\
&\quad = \int_{1/2}^{3/2}\int_0^\infty \left(\sigma - \tfrac{1}{2}\right)^{2m} e^{-\pi\lambda x^2}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2} - e^{-\pi\lambda\Delta^2}}{2\lambda}\right) \mathrm{d}\lambda \ \mathrm{d}\sigma \\
&\quad = \int_0^\infty e^{-\pi\lambda x^2}\int_{1/2}^{3/2} \left(\sigma - \tfrac{1}{2}\right)^{2m} \left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2} - e^{-\pi\lambda\Delta^2}}{2\lambda}\right) \mathrm{d}\sigma \ \mathrm{d}\lambda \\
&\quad = F_\Delta(x),
\end{aligned}$$

where the interchange of the integrals is justified since the terms involved are all nonnegative. It follows from (2.3.8) that

$$f_{2m+1}(x) = F_\Delta(\Delta x). \tag{2.4.7}$$

In particular, this shows that the measure ν_Δ is finite on $(0,\infty)$ since

$$\int_0^\infty \mathrm{d}\nu_\Delta(\lambda) = F_\Delta(0) = f_{2m+1}(0) = \frac{1}{(2m+1)^2}.$$

By [25, Corollary 17], there is a unique extremal minorant $G_\Delta^-(z) = G_{2m+1,\Delta}^-(z)$ and a unique extremal majorant $G_\Delta^+(z) = G_{2m+1,\Delta}^+(z)$ of exponential type 2π for $F_\Delta(x)$, and these functions are given by

$$G_\Delta^-(z) = \left(\frac{\cos\pi z}{\pi}\right)^2 \left\{\sum_{n=-\infty}^{\infty} \frac{F_\Delta\left(n - \frac{1}{2}\right)}{\left(z - n + \frac{1}{2}\right)^2} + \frac{F_\Delta'\left(n - \frac{1}{2}\right)}{\left(z - n + \frac{1}{2}\right)}\right\} \tag{2.4.8}$$

and

$$G_{\Delta}^{+}(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{F_{\Delta}(n)}{(z-n)^2} + \sum_{n\neq 0} \frac{F_{\Delta}'(n)}{(z-n)} \right\}. \tag{2.4.9}$$

Hence, the functions $g_{\Delta}^{-}(z) = g_{2m+1,\Delta}^{-}(z)$ and $g_{\Delta}^{+}(z) = g_{2m+1,\Delta}^{+}(z)$ defined by

$$g_{\Delta}^{-}(z) := G_{\Delta}^{-}(\Delta z) \quad \text{and} \quad g_{\Delta}^{+}(z) := G_{\Delta}^{+}(\Delta z) \tag{2.4.10}$$

are the unique extremal functions of exponential type $2\pi\Delta$ for f_{2m+1}. We claim that these functions verify the conditions of Lemma 2.8.

Part(i) We start by observing that

$$\left|f_{2m+1}(x)\right| \ll_m \frac{1}{1+x^2} \quad \text{and} \quad \left|f_{2m+1}'(x)\right| \ll_m \frac{1}{|x|(1+x^2)}. \tag{2.4.11}$$

This follows from the fact that f_{2m+1} and f_{2m+1}' are bounded functions with power series representations

$$f_{2m+1}(x) = \frac{1}{2m+1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k+2m+1)x^{2k}} \quad \text{and} \quad f_{2m+1}'(x) = \frac{1}{2m+1} \sum_{k=1}^{\infty} \frac{(-1)^{k}(2k)}{(2k+2m+1)x^{2k+1}}$$

for $|x| > 1$. It then follows from (2.4.7) that

$$\left|F_{\Delta}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2+x^2} \quad \text{and} \quad \left|F_{\Delta}'(x)\right| \ll_m \frac{\Delta^2}{|x|(\Delta^2+x^2)}. \tag{2.4.12}$$

Observe that for any complex number z we have

$$\left|\frac{\sin \pi z}{\pi z}\right|^2 \ll \frac{e^{2\pi|\mathrm{Im}\, z|}}{1+|z|^2}. \tag{2.4.13}$$

Expressions (2.4.8) and (2.4.9) can be rewritten as

$$G_{\Delta}^{-}(z) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi(z-n+\frac{1}{2})}{\pi(z-n+\frac{1}{2})}\right)^2 \left\{F_{\Delta}\big(n-\tfrac{1}{2}\big) + (z-n+\tfrac{1}{2})F_{\Delta}'\big(n-\tfrac{1}{2}\big)\right\} \tag{2.4.14}$$

and

$$G_{\Delta}^{+}(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 F_{\Delta}(0) + \sum_{n\neq 0} \left(\frac{\sin \pi(z-n)}{\pi(z-n)}\right)^2 \left\{F_{\Delta}(n) + (z-n)F_{\Delta}'(n)\right\}. \tag{2.4.15}$$

It follows from (2.4.12), (2.4.13), (2.4.14) and (2.4.15) that

$$\left|G_{\Delta}^{\pm}(z)\right| \ll_m \frac{\Delta^2}{1+|z|}\, e^{2\pi|\mathrm{Im}\, z|}$$

and from (2.4.10) this implies (2.4.3).

To bound $G_\Delta^{\pm}$ on the real line, we explore the fact that F_Δ is an even function (and hence F_Δ' is odd) to group the terms conveniently. For the majorant we group the terms n and $-n$ in (2.4.15) to get

$$\begin{aligned} G_\Delta^{+}(x) &= \left(\frac{\sin \pi x}{\pi x}\right)^2 F_\Delta(0) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{\sin^2 \pi(x-n)}{\pi^2(x^2-n^2)^2}\right)\Big\{(2x^2+2n^2)F_\Delta(n) + (x^2-n^2)\,2n\,F_\Delta'(n)\Big\}\,, \end{aligned} \tag{2.4.16}$$

and it follows from (2.4.12) and (2.4.13) that

$$\left|G_\Delta^{+}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2+x^2}. \tag{2.4.17}$$

It may be useful to split the sum in (2.4.16) into the ranges $\{n \leqslant |x|/2\}$, $\{|x|/2 < n \leqslant 2|x|\}$ and $\{2|x| < n\}$ to verify this last claim. The bound

$$\left|G_\Delta^{-}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2+x^2}. \tag{2.4.18}$$

follows in an analogous way, grouping the terms n and $1-n$ (for $n \geqslant 1$) in (2.4.14). From (2.4.10), (2.4.17) and (2.4.18) we arrive at (2.4.2). .

Part (ii) From the inequalities (2.4.2) and (2.4.3), it follows that the functions $g_\Delta^{\pm}$ have exponential type $2\pi\Delta$ and are integrable on $\mathbb{R}$. By the Paley-Wiener theorem, the Fourier transforms $\widehat{g}_\Delta^{\pm}$ are compactly supported on the interval $[-\Delta, \Delta]$. Moreover, using (2.4.2) we obtain

$$\left|\widehat{g}_\Delta^{\pm}(\xi)\right| = \left|\int_{-\infty}^{\infty} g_\Delta^{\pm}(x) e^{-2\pi i x\xi}\,\mathrm{d}x\right| \leqslant \int_{-\infty}^{\infty} \left|g_\Delta^{\pm}(x)\right|\,\mathrm{d}x \leqslant K_{2m+1}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x \ll_m 1.$$

.

Part (iii) From (2.4.7), (2.4.10) and the identities in [25, Section 11, Corollary 17 and Example 3] we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \left\{f_{2m+1}(x) - g_{2m+1,\Delta}^{-}(x)\right\}\mathrm{d}x \\ &= \frac{1}{\Delta}\int_{-\infty}^{\infty} \left\{F_\Delta(x) - G_\Delta^{-}(x)\right\}\mathrm{d}x \\ &= \frac{1}{\Delta}\int_0^{\infty} \left\{\sum_{n\neq 0} (-1)^{n+1}\lambda^{-1/2} e^{-\pi\lambda^{-1}n^2}\right\}\mathrm{d}\nu_\Delta(\lambda) \\ &= \frac{1}{\Delta}\int_0^{\infty}\int_{1/2}^{3/2} \left\{\sum_{n\neq 0} (-1)^{n+1}\lambda^{-1/2} e^{-\pi\lambda^{-1}n^2}\right\}\left(\sigma-\tfrac{1}{2}\right)^{2m}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2} - e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\sigma\,\mathrm{d}\lambda \end{aligned}$$

$$= \frac{1}{\Delta}\int_{1/2}^{3/2}\left(\sigma-\tfrac12\right)^{2m}\int_0^\infty\left\{\sum_{n\neq 0}(-1)^{n+1}\lambda^{-1/2}e^{-\pi\lambda^{-1}n^2}\right\}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\lambda\,\mathrm{d}\sigma$$

$$= \frac{1}{\Delta}\int_{1/2}^{3/2}\left(\sigma-\tfrac12\right)^{2m}\log\left(\frac{1+e^{-2\pi(\sigma-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\mathrm{d}\sigma,$$

where the interchange of integrals is justified since the integrand is nonnegative. In a similar way, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left\{g^{+}_{2m+1,\Delta}(x)-f_{2m+1}(x)\right\}\mathrm{d}x\\
&\quad= \frac{1}{\Delta}\int_{-\infty}^{\infty}\left\{G^{+}_{\Delta}(x)-F_{\Delta}(x)\right\}\mathrm{d}x\\
&\quad= \frac{1}{\Delta}\int_0^\infty\left\{\sum_{n\neq 0}\lambda^{-1/2}e^{-\pi\lambda^{-1}n^2}\right\}\mathrm{d}\nu_\Delta(\lambda)\\
&\quad= \frac{1}{\Delta}\int_0^\infty\int_{1/2}^{3/2}\left\{\sum_{n\neq 0}\lambda^{-1/2}e^{-\pi\lambda^{-1}n^2}\right\}\left(\sigma-\tfrac12\right)^{2m}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\sigma\,\mathrm{d}\lambda\\
&\quad= \frac{1}{\Delta}\int_{1/2}^{3/2}\left(\sigma-\tfrac12\right)^{2m}\int_0^\infty\left\{\sum_{n\neq 0}\lambda^{-1/2}e^{-\pi\lambda^{-1}n^2}\right\}\left(\frac{e^{-\pi\lambda(\sigma-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\lambda\,\mathrm{d}\sigma\\
&\quad= -\frac{1}{\Delta}\int_{1/2}^{3/2}\left(\sigma-\tfrac12\right)^{2m}\log\left(\frac{1-e^{-2\pi(\sigma-1/2)\Delta}}{1-e^{-2\pi\Delta}}\right)\mathrm{d}\sigma.
\end{aligned}$$

This concludes the proof of Lemma 2.8. □

2.5 Proof of Theorem 2.3 in the case of n odd

Let $n = 2m+1$. To simplify notation we disregard one of the subscripts and write $g^{\pm}_{\Delta}(z) := g^{\pm}_{2m+1,\Delta}(z)$. For a fixed $t>0$, we consider the functions $h^{\pm}_{\Delta}(z) := g^{\pm}_{\Delta}(t-z)$. Then $\widehat{h}^{\pm}_{\Delta}(\xi) = \widehat{g}^{\pm}_{\Delta}(-\xi)e^{-2\pi i\xi t}$ and the condition $|h^{\pm}_{\Delta}(s)| \ll (1+|s|)^{-2}$ when $|\mathrm{Re}\, s|\to\infty$ in the strip $|\mathrm{Im}\, s|\leqslant 1$ follows from (2.4.2), (2.4.3) and an application of the Phragmén-Lindelöf principle. We can then apply the Guinand-Weil explicit formula (Lemma 2.7) to get

$$\begin{aligned}
\sum_{\gamma} g^{\pm}_{\Delta}(t-\gamma) = &\left\{g^{\pm}_{\Delta}\left(t-\frac{1}{2i}\right)+g^{\pm}_{\Delta}\left(t+\frac{1}{2i}\right)\right\}-\frac{1}{2\pi}\widehat{g}^{\pm}_{\Delta}(0)\log\pi\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}g^{\pm}_{\Delta}(t-x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac14+\frac{ix}{2}\right)\mathrm{d}x \qquad (2.5.1)\\
&-\frac{1}{2\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\left\{\widehat{g}^{\pm}_{\Delta}\left(-\frac{\log n}{2\pi}\right)e^{-it\log n}+\widehat{g}^{\pm}_{\Delta}\left(\frac{\log n}{2\pi}\right)e^{it\log n}\right\}.
\end{aligned}$$

We now analyze each term on the right-hand side of (2.5.1) separately.

1. *First term*: From (2.4.3) we get

$$\left|g_\Delta^{\pm}\left(t-\frac{1}{2i}\right)+g_\Delta^{\pm}\left(t+\frac{1}{2i}\right)\right| \ll_m \Delta^2\frac{e^{\pi\Delta}}{1+\Delta t}. \tag{2.5.2}$$

2. *Second term*: From (2.4.4) we get

$$\left|\frac{1}{2\pi}\widehat{g}_\Delta^{\pm}(0)\log\pi\right| \ll_m 1. \tag{2.5.3}$$

3. *Third term*: This is the term that requires most of our attention. Using (2.3.8) and [50, 2.733 - Formula 1] we start by observing that

$$\begin{aligned}\int_{-\infty}^{\infty} f_{2m+1}(x)\ \mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \sigma^{2m}\log\left(\frac{1+x^2}{\sigma^2+x^2}\right)\mathrm{d}\sigma\ \mathrm{d}x\\ &= \frac{1}{2}\int_0^1 \sigma^{2m}\int_{-\infty}^{\infty}\log\left(\frac{1+x^2}{\sigma^2+x^2}\right)\mathrm{d}x\ \mathrm{d}\sigma\\ &= \frac{1}{2}\int_0^1 \sigma^{2m}\left[x\log\left(\frac{1+x^2}{\sigma^2+x^2}\right)+2\arctan(x)-2\sigma\arctan\left(\frac{x}{\sigma}\right)\right]\Bigg|_{-\infty}^{\infty}\mathrm{d}\sigma\\ &= \pi\int_0^1\sigma^{2m}(1-\sigma)\ \mathrm{d}\sigma\\ &= \frac{\pi}{(2m+1)(2m+2)}.\end{aligned} \tag{2.5.4}$$

Let us assume without loss of generality that $t \geqslant 10$. Using Stirling's formula for Γ'/Γ (cf. [36, Chapter 10]), together with (2.4.2), (2.4.5), (2.4.6) and (2.5.4), we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta^{\pm}(t-x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)\mathrm{d}x\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta^{\pm}(x)\big(\log t+O(\log(2+|x|))\big)\ \mathrm{d}x\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{f_{2m+1}(x)-\big(f_{2m+1}(x)-g_\Delta^{\pm}(x)\big)\Big\}\big(\log t+O(\log(2+|x|))\big)\ \mathrm{d}x\\ &=\frac{\log t}{2(2m+1)(2m+2)}-\frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log\left(\frac{1\mp e^{-2\pi(\sigma-1/2)\Delta}}{1\mp e^{-2\pi\Delta}}\right)\mathrm{d}\sigma+O(1)\\ &=\frac{\log t}{2(2m+1)(2m+2)}-\frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log\left(1\mp e^{-2\pi(\sigma-1/2)\Delta}\right)\mathrm{d}\sigma\\ &\quad+O\left(e^{-\pi\Delta}\log t\right)+O(1).\end{aligned} \tag{2.5.5}$$

We evaluate this last integral expanding $\log(1\mp x)$ into a power series:

$$\int_{1/2}^{\infty}\left(\sigma-\tfrac{1}{2}\right)^{2m}\log\left(1\mp e^{-2\pi(\sigma-1/2)\Delta}\right)\mathrm{d}\sigma=\int_0^{\infty}\sigma^{2m}\log\left(1\mp e^{-2\pi\sigma\Delta}\right)\mathrm{d}\sigma$$

$$= \int_0^\infty \sigma^{2m} \sum_{k\geqslant 0} \left\{ \mp \frac{e^{-2\pi\sigma\Delta(2k+1)}}{2k+1} - \frac{e^{-2\pi\sigma\Delta(2k+2)}}{2k+2} \right\} \mathrm{d}\sigma$$
$$= \sum_{k\geqslant 0} \int_0^\infty \sigma^{2m} \left\{ \mp \frac{e^{-2\pi\sigma\Delta(2k+1)}}{2k+1} - \frac{e^{-2\pi\sigma\Delta(2k+2)}}{2k+2} \right\} \mathrm{d}\sigma$$
$$= \frac{(2m)!}{(2\pi\Delta)^{2m+1}} \sum_{k\geqslant 0} \left\{ \mp \frac{1}{(2k+1)^{2m+2}} - \frac{1}{(2k+2)^{2m+2}} \right\}.$$

The interchange between integral and sum above is guaranteed by the monotone convergence theorem since all terms involved have the same sign. We have thus arrived at the following two expressions:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta^+(t-x) \,\mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ix}{2}\right) \mathrm{d}x$$
$$= \frac{\log t}{2(2m+1)(2m+2)} + \frac{(2m)!\,\zeta(2m+2)}{(2\pi\Delta)^{2m+2}} \log t + O\left(e^{-\pi\Delta}\log t\right) + O(1) \tag{2.5.6}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta^-(t-x) \,\mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ix}{2}\right) \mathrm{d}x$$
$$= \frac{\log t}{2(2m+1)(2m+2)} - \frac{(2m)!\,\left(1-2^{-2m-1}\right)\zeta(2m+2)}{(2\pi\Delta)^{2m+2}} \log t + O\left(e^{-\pi\Delta}\log t\right) + O(1). \tag{2.5.7}$$

4. *Fourth term*: Recall that the Fourier transforms $\widehat{g}_\Delta^\pm$ are supported on the interval $[-\Delta, \Delta]$. Using (2.4.4), summation by parts and the Prime Number Theorem we obtain

$$\left| \frac{1}{2\pi} \sum_{n\geqslant 2} \frac{\Lambda(n)}{\sqrt{n}} \left\{ \widehat{g}_\Delta^\pm\left(-\frac{\log n}{2\pi}\right) e^{-it\log n} + \widehat{g}_\Delta^\pm\left(\frac{\log n}{2\pi}\right) e^{it\log n} \right\} \right| \ll_m \sum_{n\leqslant e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \ll_m e^{\pi\Delta}. \tag{2.5.8}$$

Final analysis: Finally, recalling that $n = 2m+1$ we consider two cases:

<u>*Case 1: m even.*</u> In this case, by (2.3.3) we have

$$S_{2m+1}(t) = \frac{1}{2\pi(2m+2)!}\log t - \frac{1}{\pi(2m)!} \sum_\gamma f_{2m+1}(t-\gamma) \; + \; O(1).$$

Using (2.4.2) we arrive at

$$\frac{1}{2\pi(2m+2)!}\log t - \frac{1}{\pi(2m)!} \sum_\gamma g^+_{2m+1,\Delta}(t-\gamma) \; + \; O(1)$$
$$\leqslant S_{2m+1}(t)$$
$$\leqslant \frac{1}{2\pi(2m+2)!}\log t - \frac{1}{\pi(2m)!} \sum_\gamma g^-_{2m+1,\Delta}(t-\gamma) \; + \; O(1).$$

From (2.5.1), (2.5.2), (2.5.3), (2.5.6), (2.5.7) and (2.5.8) we find

$$\begin{aligned} -\frac{\zeta(2m+2)}{\pi(2\pi\Delta)^{2m+2}}\log t+O\left(\frac{\Delta^2\, e^{\pi\Delta}}{1+\Delta t}\right)+O\left(e^{-\pi\Delta}\log t\right)+O\left(e^{\pi\Delta}+1\right) \\ \leqslant S_{2m+1}(t) \\ \leqslant \frac{\left(1-2^{-2m-1}\right)\zeta(2m+2)}{\pi(2\pi\Delta)^{2m+2}}\log t+O\left(\frac{\Delta^2\, e^{\pi\Delta}}{1+\Delta t}\right) \\ +O\left(e^{-\pi\Delta}\log t\right)+O\left(e^{\pi\Delta}+1\right). \end{aligned} \tag{2.5.9}$$

Choosing

$$\pi\Delta=\log\log t-(2m+3)\log\log\log t$$

in (2.5.9) we obtain

$$\begin{aligned} -\left(\frac{\zeta(2m+2)}{\pi\cdot 2^{2m+2}}+o(1)\right)\frac{\log t}{(\log\log t)^{2m+2}} &\leqslant S_{2m+1}(t) \\ &\leqslant \left(\frac{\left(1-2^{-2m-1}\right)\zeta(2m+2)}{\pi\cdot 2^{2m+2}}+o(1)\right)\frac{\log t}{(\log\log t)^{2m+2}}, \end{aligned}$$

where the terms $o(1)$ above are $O(\log\log\log t/\log\log t)$.

Case 2: m odd. Using (2.3.3) we get

$$S_{2m+1}(t)=\frac{-1}{2\pi(2m+2)!}\log t+\frac{1}{\pi(2m)!}\sum_{\gamma}f_{2m+1}(t-\gamma)+O(1),$$

and we only need to interchange the roles of g_Δ^+ and g_Δ^- in comparison to the previous case. Similar calculations show that

$$-\left(C_{2m+1}^- + o(1)\right)\frac{\log t}{(\log\log t)^{2m+2}} \leqslant S_{2m+1}(t) \leqslant \left(C_{2m+1}^+ + o(1)\right)\frac{\log t}{(\log\log t)^{2m+2}},$$

where the terms $o(1)$ above are $O(\log\log\log t/\log\log t)$ and

$$C_{2m+1}^-=\frac{\left(1-2^{-2m-1}\right)\zeta(2m+2)}{\pi\cdot 2^{2m+2}} \quad\text{and}\quad C_{2m+1}^+=\frac{\zeta(2m+2)}{\pi\cdot 2^{2m+2}}.$$

This completes the proof of Theorem 2.3 for n odd.

2.6 Proof of Theorem 2.3 in the case of n even

In order to further simplify the notation let us write

$$\ell_n(t):=\frac{\log t}{(\log\log t)^n} \quad\text{and}\quad r_n(t):=\frac{\log t\log\log\log t}{(\log\log t)^n}.$$

Let $n \geqslant 2$ be an even integer (the case $n = 0$ was established in [16]). We have already shown that

$$-C_{n-1}^{-}\ell_n(t) + O(r_{n+1}(t)) \leqslant S_{n-1}(t) \leqslant C_{n-1}^{+}\ell_n(t) + O(r_{n+1}(t)) \tag{2.6.1}$$

and

$$-C_{n+1}^{-}\ell_{n+2}(t) + O(r_{n+3}(t)) \leqslant S_{n+1}(t) \leqslant C_{n+1}^{+}\ell_{n+2}(t) + O(r_{n+3}(t)). \tag{2.6.2}$$

Our goal now is to obtain a similar estimate for $S_n(t)$ that interpolates between (2.6.1) and (2.6.2). We view this as a pure analysis problem and our argument below explores the fact that the function $S_n(t)$, for $n \geqslant 2$, is continuously differentiable.

By the mean value theorem and (2.6.1) we obtain, for $-\sqrt{t} \leqslant h \leqslant \sqrt{t}$,

$$\begin{aligned} S_n(t) - S_n(t-h) &= h\, S_{n-1}(t_h^*) \\ &\leqslant \left(\chi_{h>0}\, |h|\, C_{n-1}^{+} + \chi_{h<0}\, |h|\, C_{n-1}^{-}\right) \ell_n(t_h^*) + |h|\, O(r_{n+1}(t_h^*)) \\ &\leqslant \left(\chi_{h>0}\, |h|\, C_{n-1}^{+} + \chi_{h<0}\, |h|\, C_{n-1}^{-}\right) \ell_n(t) + |h|\, O(r_{n+1}(t))\,, \end{aligned} \tag{2.6.3}$$

where t_h^* is a suitable point in the segment connecting $t-h$ and t, and $\chi_{h>0}$ and $\chi_{h<0}$ are the indicator functions of the sets $\{h \in \mathbb{R};\ h > 0\}$ and $\{h \in \mathbb{R};\ h < 0\}$, respectively.

Let a and b be positive real numbers that shall be properly chosen later (in particular, we will be able to choose them in a way that $a + b = 1$, for instance). Let ν be a real parameter such that $0 < \nu \leqslant \sqrt{t}$. We integrate (2.6.3) with respect to the variable h to get

$$\begin{aligned} S_n(t) &\leqslant \frac{1}{(a+b)\nu} \int_{-a\nu}^{b\nu} S_n(t-h)\ \mathrm{d}h \\ &\quad + \frac{1}{(a+b)\nu} \left[\int_{-a\nu}^{b\nu} \left(\chi_{h>0}\, |h|\, C_{n-1}^{+} + \chi_{h<0}\, |h|\, C_{n-1}^{-}\right)\ \mathrm{d}h\right] \ell_n(t) \\ &\quad + \frac{1}{(a+b)\nu} \left[\int_{-a\nu}^{b\nu} |h|\ \mathrm{d}h\right] O(r_{n+1}(t)) \\ &= \frac{1}{(a+b)\nu}\Big[S_{n+1}(t+a\nu) - S_{n+1}(t-b\nu)\Big] + \left[\frac{b^2C_{n-1}^{+} + a^2C_{n-1}^{-}}{2(a+b)}\right] \nu\, \ell_n(t) \\ &\quad + O(\nu\, r_{n+1}(t)). \end{aligned}$$

We now use (2.6.2) to get

$$\begin{aligned} S_n(t) \leqslant \frac{1}{(a+b)\nu}\Big[&C_{n+1}^{+}\ell_{n+2}(t+a\nu) + C_{n+1}^{-}\ell_{n+2}(t-b\nu) \\ &+ O(r_{n+3}(t+a\nu)) + O(r_{n+3}(t-b\nu))\Big] \\ &+ \left[\frac{b^2C_{n-1}^{+} + a^2C_{n-1}^{-}}{2(a+b)}\right] \nu\, \ell_n(t) + O(\nu\, r_{n+1}(t)) \end{aligned}$$

$$= \left[\frac{C^+_{n+1} + C^-_{n+1}}{(a+b)} \right] \frac{1}{\nu} \; \ell_{n+2}(t) + \left[\frac{b^2 C^+_{n-1} + a^2 C^-_{n-1}}{2(a+b)} \right] \nu \, \ell_n(t)$$

$$+ O\left(\frac{r_{n+3}(t)}{\nu} \right) + O(\nu \, r_{n+1}(t)). \tag{2.6.4}$$

Choosing $\nu = \frac{\alpha}{\log\log t}$ in (2.6.4), where $\alpha > 0$ is a constant to be determined, we find

$$S_n(t) \leqslant \left\{ \left[\frac{C^+_{n+1} + C^-_{n+1}}{(a+b)} \right] \frac{1}{\alpha} + \left[\frac{b^2 C^+_{n-1} + a^2 C^-_{n-1}}{2(a+b)} \right] \alpha \right\} \ell_{n+1}(t) + O(r_{n+2}(t)).$$

We now choose $\alpha > 0$ to minimize the expression in brackets, which corresponds to the choice

$$\alpha = \left[\frac{C^+_{n+1} + C^-_{n+1}}{(a+b)} \right]^{1/2} \left[\frac{b^2 C^+_{n-1} + a^2 C^-_{n-1}}{2(a+b)} \right]^{-1/2}.$$

This leads to the bound

$$S_n(t) \leqslant 2 \left[\frac{\big(C^+_{n+1} + C^-_{n+1}\big)\big(b^2 C^+_{n-1} + a^2 C^-_{n-1}\big)}{2(a+b)^2} \right]^{1/2} \ell_{n+1}(t) + O(r_{n+2}(t)). \tag{2.6.5}$$

We now seek to minimize the right-hand side of (2.6.5) in the variables a and b. It is easy to see that it only depends on the ratio a/b (and hence we can normalize to have $a+b=1$). If we consider $a = bx$ we must minimize the function

$$H(x) = 2 \left[\frac{\big(C^+_{n+1} + C^-_{n+1}\big)\big(C^+_{n-1} + x^2 C^-_{n-1}\big)}{2(x+1)^2} \right]^{1/2}.$$

Note that $C^{\pm}_{n-1} > 0$ and $C^{\pm}_{n+1} > 0$. Such a minimum is obtained when $x = C^+_{n-1}/C^-_{n-1}$, leading to the bound

$$S_n(t) \leqslant \left[\frac{2\big(C^+_{n+1} + C^-_{n+1}\big) \; C^+_{n-1} \; C^-_{n-1}}{C^+_{n-1} + C^-_{n-1}} \right]^{1/2} \ell_{n+1}(t) + O(r_{n+2}(t)).$$

The argument for the lower bound of $S_n(t)$ is entirely symmetric.

This completes the proof of Theorem 2.3.

Chapter 3

The Riemann zeta-function and bandlimited approximations II

This chapter is comprised of the paper [A2]. We provide explicit upper and lower bounds for the argument of the Riemann zeta-function and its high moments in the critical strip under the assumption of the Riemann hypothesis. This extends the bounds of the previous chapter and sharpens the error terms in such estimates. The novelty here is the use of the explicit formulas for the Fourier transforms of the bandlimited approximations that will appear. We also show bounds for the real part of the logarithmic derivative of the Riemann zeta-function in the critical strip.

Although the results in this chapter end up generalizing the results of Chapter 2, we emphasize the fact that this chapter is considerably more technical. For this reason we decided to keep the important case of the critical line in a separate chapter to clarify the true insights and the rightful steps of our method.

3.1 Behavior in the critical strip: $S_n(\sigma, t)$

In this section we extend the definition of the functions $S_n(t)$ in (2.2.2) to the critical strip. Let $\zeta(s)$ denote the Riemann zeta-function and let $\frac{1}{2} \leqslant \sigma \leqslant 1$ be a real number. For $t > 0$ we define

$$S(\sigma, t) = \tfrac{1}{\pi} \arg \zeta\big(\sigma + it\big),$$

where the argument is obtained by a continuous variation along straight line segments joining the points 2, $2 + it$ and $\sigma + it$, assuming that this path has no zeros of $\zeta(s)$, with the convention that $\arg \zeta(2) = 0$. If this path has zeros of $\zeta(s)$ (including the endpoint $\sigma + it$) we set

$$S(\sigma, t) = \tfrac{1}{2} \lim_{\varepsilon \to 0^+} \left\{ S(\sigma, t + \varepsilon) + S(\sigma, t - \varepsilon) \right\}.$$

Similarly to (2.2.2) we define the sequence of high moments $S_n(\sigma,t)$ of $S(\sigma,t)$. Setting $S_0(\sigma,t) = S(\sigma,t)$, for $n \geqslant 1$ and $t > 0$ we define the functions

$$S_n(\sigma,t) = \int_0^t S_{n-1}(\sigma,\tau)\,\mathrm{d}\tau \,+ \delta_{n,\sigma}\,,$$

where $\delta_{n,\sigma}$ is a specific constant depending on σ and n. For $k \in \mathbb{N}$, these constants are given by

$$\delta_{2k-1,\sigma} = \frac{(-1)^{k-1}}{\pi}\int_\sigma^\infty \int_{\sigma_{2k-2}}^\infty \cdots \int_{\sigma_2}^\infty \int_{\sigma_1}^\infty \log|\zeta(\sigma_0)|\,\mathrm{d}\sigma_0\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_{2k-2}$$

for $n = 2k-1$ and by

$$\delta_{2k,\sigma} = (-1)^{k-1}\int_\sigma^1 \int_{\sigma_{2k-1}}^1 \cdots \int_{\sigma_2}^1 \int_{\sigma_1}^1 \mathrm{d}\sigma_0\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_{2k-1} = \frac{(-1)^{k-1}(1-\sigma)^{2k}}{(2k)!}$$

for $n = 2k$. Note that in particular we have that $S_n(\frac{1}{2},t) = S_n(t)$ for $t > 0$.

The main purpose of this chapter is to extend the bounds of Theorem 2.3 to the critical strip in an explicit way. Assuming RH, for $\frac{1}{2} < \sigma < 1$, another function that will play an important role in our study is the derivative[1]

$$S_{-1}(\sigma,t) := S_0'(\sigma,t) = \frac{1}{\pi}\,\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it).$$

3.1.1 Main result

For an integer $n \geqslant 0$ we introduce the function

$$H_n(x) := \sum_{k=0}^\infty \frac{x^k}{(k+1)^n}. \tag{3.1.1}$$

The function $xH_n(x) = Li_n(x)$ is known as *polylogarithm of order n* in the classical terminology of special functions. Note that $H_0(x) = 1/(1-x)$ for $|x| < 1$. Our main result is stated below, in which we regard σ and t as free parameters.

Theorem 3.1. *Assume the Riemann hypothesis and let $n \geqslant -1$. Let $\frac{1}{2} < \sigma < 1$ and $c > 0$ be a given real number. Let $t > 0$ be such that $\log\log t \geqslant 4$. In the range*

$$(1-\sigma)^2\log\log t \geqslant c \tag{3.1.2}$$

we have the uniform bounds:

[1]The derivative is calculated over the variable t, when σ is fixed.

(i) For $n=-1$,

$$-C^{-}_{-1,\sigma}(t)\,(\log t)^{2-2\sigma}+O_c\left(\frac{(\sigma-\frac{1}{2})(\log t)^{2-2\sigma}}{(1-\sigma)^2\log\log t}\right)\leqslant S_{-1,\sigma}(t)=\frac{1}{\pi}\,\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it)$$
$$\leqslant C^{+}_{-1,\sigma}(t)\,(\log t)^{2-2\sigma}+O_c\left(\frac{(\log t)^{2-2\sigma}}{(\sigma-\frac{1}{2})\,(1-\sigma)^2\,(\log\log t)}\right). \tag{3.1.3}$$

(ii) For $n\geqslant 0$,

$$-C^{-}_{n,\sigma}(t)\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{n+1}}+O_{n,c}\left(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)^2\,(\log\log t)^{n+2}}\right)\leqslant S_n(\sigma,t)$$
$$\leqslant C^{+}_{n,\sigma}(t)\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{n+1}}+O_{n,c}\left(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)^2\,(\log\log t)^{n+2}}\right). \tag{3.1.4}$$

Above, $C^{\pm}_{n,\sigma}(t)$ are positive functions given by:

- *For $n\geqslant -1$ odd,*

$$C^{\pm}_{n,\sigma}(t)=\frac{1}{2^{n+1}\pi}\left(H_{n+1}\Big(\pm(-1)^{(n+1)/2}\,(\log t)^{1-2\sigma}\Big)+\frac{2\sigma-1}{\sigma(1-\sigma)}\right). \tag{3.1.5}$$

- *For $n=0$,*

$$C^{\pm}_{0,\sigma}(t)=\Big(2\big(C^{+}_{1,\sigma}(t)+C^{-}_{1,\sigma}(t)\big)\,C^{-}_{-1,\sigma}(t)\Big)^{1/2}. \tag{3.1.6}$$

- *For $n\geqslant 2$ even,*

$$C^{\pm}_{n,\sigma}(t)=\left(\frac{2\big(C^{+}_{n+1,\sigma}(t)+C^{-}_{n+1,\sigma}(t)\big)\,C^{+}_{n-1,\sigma}(t)\,C^{-}_{n-1,\sigma}(t)}{C^{+}_{n-1,\sigma}(t)+C^{-}_{n-1,\sigma}(t)}\right)^{1/2}. \tag{3.1.7}$$

Remark 3.2. *In the course of the proof of Theorem 3.1 we obtain slightly stronger bounds than the ones presented in* (3.1.3) *(see inequalities* (3.5.12) *and* (3.5.15) *below). In the statement of Theorem 3.1 we presented the error terms in* (3.1.3) *and* (3.1.4) *in a convenient way for our interpolation argument in Section 3.6.*

Observe that letting $\sigma\to\frac{1}{2}^{+}$ in our Theorem 3.1 (for $n\geqslant 0$), we obtain a sharpened version of Theorem 2.3 with improved error terms (a factor $\log\log\log t$ has been removed). In particular, we record here the following consequence, a new proof of the best known bound for $S(t)$ under RH (in fact, with a sharpened error term when compared to [16] and [17])[2].

Corollary 3.3. *Assume the Riemann hypothesis. For $t>0$ sufficiently large we have*

$$|S(t)|\leqslant\frac{1}{4}\frac{\log t}{\log\log t}+O\left(\frac{\log t}{(\log\log t)^2}\right).$$

[2]For an explanation of why all these methods lead to the same constant $1/4$ in the bound for $S(t)$, see [17, Section 3].

In order to find bounds for $S(\sigma,t)$ which are stable under the limit $\sigma \to \frac{1}{2}^{+}$ (and hence extend Theorem 2.3), we modified a bit our interpolation method in §3.6.1 to use both bounds for $S_1(\sigma,t)$ and *only the lower bound for* $S_{-1}(\sigma,t)$. Observe that the lower bound for $S_{-1}(\sigma,t)$ in Theorem 3.1 is stable under the limit $\sigma \to \frac{1}{2}^{+}$, whereas the upper bound is not. This is somewhat expected since $S(t)$ has jump discontinuities at the ordinates of the non-trivial zeros of $\zeta(s)$. In our case such blow up comes from the fact that we use a bandlimited majorant for the Poisson kernel and, as $\sigma \to \frac{1}{2}^{+}$, this Poisson kernel converges to a delta function. This lack of stability may be related to the existence of small gaps between ordinates of zeros of $\zeta(s)$. Something similar can be seen in the work of Ki [61] on the distribution of the zeros of $\zeta'(s)$.

If one is interested in bounds as $t \to \infty$ for a *f*ixed σ with $\frac{1}{2} < \sigma < 1$, our Theorem 3.1 yields the following corollary (the bounds below can be made uniform in $\delta > 0$ if we consider $\frac{1}{2} + \delta \leqslant \sigma \leqslant 1 - \delta$.)

Corollary 3.4. *Assume the Riemann hypothesis and let* $n \geqslant -1$. *Let* $\frac{1}{2} < \sigma < 1$ *be a fixed number. Then*

$$|S_n(\sigma,t)| \leqslant \frac{\omega_n}{2^{n+1}\pi}\left(1 + \frac{2\sigma-1}{\sigma(1-\sigma)} + o(1)\right)\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{n+1}},$$

as $t \to \infty$, *where* $\omega_n = 1$ *if* n *is odd and* $\omega_n = \sqrt{2}$ *if* n *is even.*

This plainly follows from (3.1.5) and (3.1.7) for $n \neq 0$. For the case $n = 0$ one would simply perform the full interpolation method as described in §3.6.2 (using the upper and lower bounds for both $S_1(\sigma,t)$ and $S_{-1}(\sigma,t)$) to obtain the optimized constant as in (3.1.7).

Remark 3.5. *The extra factor* $\sqrt{2}$ *in Corollary 3.4 when* n *is even comes from* (3.1.7) *and it is due to our indirect interpolation argument. In principle, if one could directly solve the associated extremal Fourier analysis problem in the case of* n *even, this could lead to a better bound than* (3.1.7). *We note, however, that this is a highly nontrivial problem in approximation theory. See the discussion in §3.1.3 below.*

Finally, notice that we have purposely restricted our range to be strictly inside the critical strip, away from the line $\sigma = 1$. With our methods it is also possible (by means of some additional technical work) to consider the case when the parameter σ is close to 1, obtaining bounds of the sort $S_n(\sigma,t) = O_n(1)$, for $n \geqslant 1$ (with explicit constants). We do not pursue such matters here, feeling that classical methods in the literature are more suitable to treat this range. In fact, bounds for $S_n(1,t)$, for $n \geqslant 1$, are easily obtainable directly from (3.2.1) and the use of Fubini's theorem with the series representation in the region $\{z \in \mathbb{C};\, \mathrm{Re}\, z > 1\}$. These bounds would be equal to our bounds in the cases of n odd, and better in the case of n even, since we use an indirect approach, via interpolation, for these cases. In the particular case of $n = 0$, the known bound $|S(1,t)| \leqslant \frac{1}{\pi}\log\log\log t + O(1)$ (see [74, Corollary 13.16]) is not easily obtainable by our particular interpolation argument.

3.1.2 A result for $\log|\zeta(\frac{1}{2}+it)|$

Using the lower bound for the function $S_{-1}(\sigma,t)$ in Theorem 3.1, we also deduce a new proof of the best known bound, with improved error terms, for $\log|\zeta(\frac{1}{2}+it)|$ under RH (see [29] and [14]).

Corollary 3.6. *Assume the Riemann hypothesis. For $t>0$ sufficiently large we have*

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| \leqslant \frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t}{(\log\log t)^2}\right).$$

Proof. Assuming RH, it follows from [74, Corollary 13.16] that

$$\log|\zeta(\sigma+it)| \leqslant \log\frac{1}{1-\sigma} + O\bigg(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)\log\log t}\bigg)$$

uniformly for $1/2+1/\log\log t \leqslant \sigma \leqslant 1-1/\log\log t$ and $t \geqslant 3$. Therefore, letting $\delta=\delta(t)=\frac{1}{2}+\frac{\log\log\log t}{\log\log t}$, we have

$$\begin{aligned}\log|\zeta(\tfrac{1}{2}+it)| &= -\int_{1/2}^{\delta}\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma + \log|\zeta(\delta+it)| \\ &= -\int_{1/2}^{\delta}\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma + O\left(\frac{\log t}{(\log\log t)^2}\right).\end{aligned}$$

Since the lower bound in (3.1.3) implies that

$$-\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it) \leqslant \frac{(\log t)^{2-2\sigma}}{1+(\log t)^{1-2\sigma}} + O\big((\sigma-\tfrac{1}{2})(\log t)^{2-2\sigma}\big)$$

uniformly for $1/2<\sigma\leqslant\delta$, we see that

$$\log|\zeta(\tfrac{1}{2}+it)| \leqslant \int_{1/2}^{\delta}\left\{\frac{(\log t)^{2-2\sigma}}{1+(\log t)^{1-2\sigma}} + O\big((\sigma-\tfrac{1}{2})(\log t)^{2-2\sigma}\big)\right\}\mathrm{d}\sigma + O\left(\frac{\log t}{(\log\log t)^2}\right).$$

The corollary now follows from the estimates

$$\int_{1/2}^{\delta}\frac{(\log t)^{2-2\sigma}}{1+(\log t)^{1-2\sigma}}\,\mathrm{d}\sigma \;\leqslant\; \int_{1/2}^{1}\frac{(\log t)^{2-2\sigma}}{1+(\log t)^{1-2\sigma}}\,\mathrm{d}\sigma \;=\; \frac{\log 2}{2}\frac{\log t}{\log\log t} - \frac{\log t\log(1+1/\log t)}{2\log\log t}$$

and

$$\int_{1/2}^{\delta}(\sigma-\tfrac{1}{2})(\log t)^{2-2\sigma}\,\mathrm{d}\sigma \ll \frac{\log t}{(\log\log t)^2}.$$

□

3.1.3 Strategy outline

The proof of these results follows the strategy of the previous chapter. It is worth mentioning that here we face severe additional technical challenges in order to fully develop this circle of ideas to reach our desired conclusion.

The strategy is divided into the following four main steps:

Step 1: Representation lemma.

The first step is to identify the functions of a real variable that are naturally connected with the objects to be bounded, in our case the functions $S_n(\sigma, t)$. For each $n \geqslant -1$ and $\frac{1}{2} \leqslant \sigma \leqslant 1$ we define the function $f_{n,\sigma} : \mathbb{R} \to \mathbb{R}$ in the following manner.

- If $n = 2m$, for $m \in \mathbb{Z}_{\geqslant 0}$, we define

$$f_{2m,\sigma}(x) = \int_{\sigma}^{3/2} (\alpha - \sigma)^{2m} \left(\frac{x}{(\alpha - \frac{1}{2})^2 + x^2} - \frac{x}{1 + x^2} \right) \mathrm{d}\alpha. \tag{3.1.8}$$

- If $n = 2m + 1$, for $m \in \mathbb{Z}_{\geqslant 0}$, we define

$$f_{2m+1,\sigma}(x) = \frac{1}{2} \int_{\sigma}^{3/2} (\alpha - \sigma)^{2m} \log \left(\frac{1 + x^2}{(\alpha - \frac{1}{2})^2 + x^2} \right) \mathrm{d}\alpha. \tag{3.1.9}$$

- If $n = -1$, we define

$$f_{-1,\sigma}(x) = \frac{(\sigma - \frac{1}{2})}{(\sigma - \frac{1}{2})^2 + x^2}. \tag{3.1.10}$$

We prove a representation lemma (Lemma 3.8) where we write $S_n(\sigma, t)$, for each $n \geqslant -1$, as a sum of a translate of the function $f_{n,\sigma}$ over the non-trivial zeros of $\zeta(s)$ plus some known terms and a small error.

Step 2: Extremal functions.

As mentioned in the previous chapter, the tool to evaluate sums over the non-trivial zeros of $\zeta(s)$ is the Guinand-Weil explicit formula. However, the functions $f_{n,\sigma}$ defined above do not possess the required smoothness to allow a direct evaluation. In fact, for $\sigma = \frac{1}{2}$ and $n \geqslant 1$, we have that $f_{n,\frac{1}{2}}$ is of class $C^{n-1}(\mathbb{R})$ but not higher (the n-th derivative is discontinuous at the origin). Note also that $f_{0,\frac{1}{2}}$ is discontinuous at the origin and $f_{-1,\frac{1}{2}}$ is identically zero. For $\frac{1}{2} < \sigma$, the functions $f_{n,\sigma}$ are of class $C^{\infty}(\mathbb{R})$ but do not have an analytic extension to the strip $\{z \in \mathbb{C};\ -\frac{1}{2} - \varepsilon < \operatorname{Im} z < \frac{1}{2} + \varepsilon\}$. In fact, the functions $f_{n,\sigma}$ are analytic in the strip $\{z \in \mathbb{C};\ -(\sigma - \frac{1}{2}) < \operatorname{Im} z < (\sigma - \frac{1}{2})\}$ but the n-th derivative of $f_{n,\sigma}$ cannot be extended continuously to the points $\pm(\sigma - \frac{1}{2})i$, for $n \geqslant 0$ (for $n = -1$ the function $f_{-1,\sigma}$ has a pole at $\pm(\sigma - \frac{1}{2})i$).

The idea is then to replace the functions $f_{n,\sigma}$ by suitable bandlimited approximations (real-valued majorants and minorants with compactly supported Fourier transforms) chosen in such a way to minimize the $L^1(\mathbb{R})$−distance. In our case, the situation is markedly different depending upon whether n is even or odd. When $n \geqslant -1$ is odd, the function $f_{n,\sigma}$ is *even*, and the robust Gaussian subordination framework of Carneiro, Littmann and

Vaaler [25] provides the required extremal functions. When n is even, the function $f_{n,\sigma}$ is *odd and continuous* (except in the case $n = 0$ and $\sigma = \frac{1}{2}$, which was considered in [16]). In this general situation, the solution of the Beurling-Selberg extremal problem is unknown. Therefore, we adopt a different approach based on an interpolation argument.

Step 3: Guinand-Weil explicit formula and asymptotic analysis.

In the case of n odd, $n \geqslant -1$, we bound $S_n(\sigma, t)$ by applying the Guinand-Weil explicit formula to the Beurling-Selberg majorants and optimizing the size of the support of the Fourier transform. We do a careful asymptotic analysis of all the terms that appear in the explicit formula. In particular, we highlight that one of the main technical difficulties of this work, when compared to [16, 18], is in the analysis of the sum over primes powers. This term is easily handled in the works [16, 18] when $\sigma = \frac{1}{2}$ but, in the case $\sigma > \frac{1}{2}$ that we treat here, we must perform a much deeper analysis, using the explicit knowledge of the Fourier transform of the majorant function. This refined analysis allows to improve the error term in Theorem 2.3. We collect in the Appendix some of the calculus facts and some of the number theory facts that are needed for this analysis.

Step 4: Interpolation tools.

Having obtained the desired bounds for all odd n's, with $n \geqslant -1$, we proceed (as in the previous chapter) with an interpolation argument to obtain the estimate for the even n's in between, exploring the smoothness of $S_n(\sigma, t)$ via the mean value theorem. In the particular case $n = 0$, we modified a bit our interpolation method to use both bounds for $S_1(\sigma, t)$ and only the lower bound for $S_{-1}(\sigma, t)$.

3.2 Representation lemma II

In this section we collect some useful auxiliary results. Lemmas 3.7 and 3.8 below have appeared in Lemmas 2.4 and 2.5 in the case $\sigma = \frac{1}{2}$. The proofs for general $\frac{1}{2} \leqslant \sigma \leqslant 1$ are essentially analogous. We include here brief versions of these proofs, both for completeness and for the convenience of the reader.

Lemma 3.7. *Assume the Riemann hypothesis.*

(i) For $n \geqslant 0$, $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $t > 0$ (and t not coinciding with the ordinate of a zero of $\zeta(s)$ when $n = 0$ and $\sigma = \frac{1}{2}$), we have

$$S_n(\sigma, t) = -\frac{1}{\pi} \operatorname{Im}\left\{\frac{i^n}{n!} \int_{\sigma}^{\infty} (\alpha - \sigma)^n \, \frac{\zeta'}{\zeta}(\alpha + it) \, \mathrm{d}\alpha\right\}. \tag{3.2.1}$$

(ii) For $n = -1$, $\frac{1}{2} < \sigma \leqslant 1$ and $t > 0$, we have

$$S_{-1}(\sigma, t) := S_0'(\sigma, t) = \frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it). \tag{3.2.2}$$

Proof. For the case $\sigma = \frac{1}{2}$ this is stated in Lemma 2.4 and the proof for (i) in the case general $\frac{1}{2} \leqslant \sigma \leqslant 1$ follows the same outline. Part (ii) just follows from the definition of $S_{-1}(\sigma, t)$. □

We are now in position to state the main result of this section, an expression that connects $S_n(\sigma, t)$ with the functions $f_{n,\sigma}$ defined in (3.1.8), (3.1.9) and (3.1.10). This result is an extension of Lemma 3.8 and the proof follows the same outline. In the proof of Theorem 3.1 we shall only use the case of n odd, but we state here the representation for n even as well, as a result of independent interest.

Lemma 3.8 (Representation lemma)**.** *Assume the Riemann hypothesis. For each $n \geqslant -1$ and $\frac{1}{2} \leqslant \sigma \leqslant 1$ (except $n = -1$ and $\sigma = \frac{1}{2}$), let $f_{n,\sigma} : \mathbb{R} \to \mathbb{R}$ be defined as in* (3.1.8), (3.1.9) *and* (3.1.10)*. For $t \geqslant 2$ (and t not coinciding with an ordinate of a zero of $\zeta(s)$ in the case $n = 0$ and $\sigma = \frac{1}{2}$) the following formulas hold.*

(i) If $n = 2m$, for $m \in \mathbb{Z}_{\geqslant 0}$, then

$$S_{2m}(\sigma, t) = \frac{(-1)^m}{\pi (2m)!} \sum_{\gamma} f_{2m,\sigma}(t - \gamma) \; + \; O_m(1). \tag{3.2.3}$$

(ii) If $n = 2m + 1$, for $m \in \mathbb{Z}_{\geqslant 0}$, then

$$\begin{aligned} S_{2m+1}(\sigma, t) = \frac{(-1)^m}{2\pi (2m+2)!} \left(\tfrac{3}{2} - \sigma\right)^{2m+2} \log t \\ - \frac{(-1)^m}{\pi (2m)!} \sum_{\gamma} f_{2m+1,\sigma}(t - \gamma) + O_m(1). \end{aligned} \tag{3.2.4}$$

(iii) If $n = -1$, then

$$S_{-1}(\sigma, t) = -\frac{1}{2\pi} \log \frac{t}{2\pi} + \frac{1}{\pi} \sum_{\gamma} f_{-1,\sigma}(t - \gamma) + O\left(\frac{1}{t}\right). \tag{3.2.5}$$

The above sums run over the ordinates of the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. We first treat (ii). It follows from Lemma 3.7 and integration by parts that

$$\begin{aligned} S_{2m+1}(\sigma, t) &= -\frac{1}{\pi} \, \mathrm{Im} \left\{ \frac{i^{2m+1}}{(2m+1)!} \int_{\sigma}^{\infty} (\alpha - \sigma)^{2m+1} \, \frac{\zeta'}{\zeta}(\alpha + it) \; \mathrm{d}\alpha \right\} \\ &= \frac{(-1)^{m+1}}{\pi (2m+1)!} \, \mathrm{Re} \left\{ \int_{\sigma}^{\infty} (\alpha - \sigma)^{2m+1} \, \frac{\zeta'}{\zeta}(\alpha + it) \; \mathrm{d}\alpha \right\} \\ &= \frac{(-1)^m}{\pi (2m)!} \, \mathrm{Re} \left\{ \int_{\sigma}^{\infty} (\alpha - \sigma)^{2m} \, \log \zeta(\alpha + it) \; \mathrm{d}\alpha \right\} \\ &= \frac{(-1)^m}{\pi (2m)!} \int_{\sigma}^{3/2} (\alpha - \sigma)^{2m} \, \log |\zeta(\alpha + it)| \; \mathrm{d}\alpha + O_m(1). \end{aligned} \tag{3.2.6}$$

Using the equation (2.3.6) established in the proof of Lemma 2.5 we have

$$\log|\zeta(\alpha+it)| = \left(\tfrac{3}{4}-\tfrac{\alpha}{2}\right)\log t - \frac{1}{2}\sum_{\gamma}\log\left(\frac{1+(t-\gamma)^2}{(\alpha-\frac{1}{2})^2+(t-\gamma)^2}\right) + O(1), \tag{3.2.7}$$

uniformly for $1/2 \leqslant \alpha \leqslant 3/2$ and $t \geqslant 2$. Inserting (3.2.7) into (3.2.6) yields

$$\begin{aligned}
S_{2m+1}(\sigma,t) &= \frac{(-1)^m}{\pi(2m)!}\left(\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\left(\tfrac{3}{4}-\tfrac{\alpha}{2}\right)\,\mathrm{d}\alpha\right)\log t \\
&\quad - \frac{(-1)^m}{2\pi(2m)!}\int_{\sigma}^{3/2}\sum_{\gamma}(\alpha-\sigma)^{2m}\log\left(\frac{1+(t-\gamma)^2}{(\alpha-\frac{1}{2})^2+(t-\gamma)^2}\right)\mathrm{d}\alpha + O_m(1) \\
&= \frac{(-1)^m}{2\pi(2m+2)!}\left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log t \\
&\quad - \frac{(-1)^m}{2\pi(2m)!}\sum_{\gamma}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\log\left(\frac{1+(t-\gamma)^2}{(\alpha-\frac{1}{2})^2+(t-\gamma)^2}\right)\mathrm{d}\alpha + O_m(1) \\
&= \frac{(-1)^m}{2\pi(2m+2)!}\left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log t - \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma}f_{2m+1,\sigma}(t-\gamma) + O_m(1),
\end{aligned}$$

where the interchange between summation and integration can be justified, for instance, by the monotone convergence theorem, since all the terms involved are nonnegative. This concludes the proof of (ii).

We now move to the proof of (iii). Let $s=\sigma+it$ and recall that we are assuming $t \geqslant 2$. From the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [36, Chapter 12]), we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + B + \frac{1}{2}\log\pi - \frac{1}{s-1}, \tag{3.2.8}$$

with $B = -\sum_{\rho}\mathrm{Re}\,(1/\rho)$. Again using Stirling's formula we obtain

$$S_{-1}(\sigma,t) = \frac{1}{\pi}\,\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it) = -\frac{1}{2\pi}\log\frac{t}{2\pi} + \frac{1}{\pi}\sum_{\gamma}f_{-1,\sigma}(t-\gamma) + O\left(\frac{1}{t}\right).$$

This proves (iii).

Finally, the proof of (i) follows along the same lines, starting with (3.2.1), restricting the range of integration to the interval $(\sigma,\frac{3}{2})$, and using the partial fraction decomposition (3.2.8) after adding and subtracting a term $\frac{\zeta'}{\zeta}(\frac{3}{2}+it)$ to balance the equation. The details of the proof are left to the interested reader. □

As mentioned in the previous section, we propose to use the Guinand-Weil explicit formula (Lemma 2.7) to understand the sum of over the non-trivial zeros of $\zeta(s)$ that appear in Lemma 3.8, but the functions $f_{n,\sigma}$ do not possess the required smoothness properties to allow the application of the Guinand-Weil formula. The key idea to prove Theorem 3.1, *in the case of n odd*, is to replace the functions $f_{n,\sigma}$ by appropriate extremal majorants and minorants of exponential type (thus with a compactly supported Fourier transform by the Paley-Wiener theorem). These bandlimited approximations are described in the next section.

3.3 Extremal bandlimited approximations II

As in the previous chapter, we will use the Gaussian subordination framework of Carneiro, Littmann and Vaaler [25] to find our extremal functions. It turns out that our functions $f_{n,\sigma}$ when n is odd, defined in (3.1.9) and (3.1.10), are included in this class, since that these functions can be write in the form (2.4.1). Moreover, it is also crucial for our purposes to have a detailed description of the Fourier transforms of our majorants and minorants in order to analyze the contribution from the primes and prime powers in the explicit formula.

3.3.1 Approximations to the Poisson kernel

We start with the case of the Poisson kernel $f_{-1,\sigma}$. In order to simplify the notation we let $\beta = \sigma - \frac{1}{2}$ and define

$$h_\beta(x) := f_{-1,\sigma}(x) = \frac{\beta}{\beta^2 + x^2}. \tag{3.3.1}$$

The solution of the extremal problem for the Poisson kernel below is of independent interest and may have other applications in analysis and number theory.

Lemma 3.9 (Extremal functions for the Poisson kernel)**.** *Let $\beta > 0$ be a real number and let $\Delta > 0$ be a real parameter. Let $h_\beta : \mathbb{R} \to \mathbb{R}$ be defined as in* (3.3.1)*. Then there is a unique pair of real entire functions $m^-_{\beta,\Delta} : \mathbb{C} \to \mathbb{C}$ and $m^+_{\beta,\Delta} : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) The real entire functions $m^\pm_{\beta,\Delta}$ have exponential type $2\pi\Delta$.

(ii) The inequality

$$m^-_{\beta,\Delta}(x) \leqslant h_\beta(x) \leqslant m^+_{\beta,\Delta}(x)$$

holds pointwise for all $x \in \mathbb{R}$.

(iii) Subject to conditions (i) and (ii), the value of the integral

$$\int_{-\infty}^{\infty} \left\{m^+_{\beta,\Delta}(x) - m^-_{\beta,\Delta}(x)\right\} \mathrm{d}x$$

is minimized.

The functions $m^\pm_{\beta,\Delta}$ are even and verify the following additional properties:

(iv) The L^1-distances of $m^\pm_{\beta,\Delta}$ to h_β are explicitly given by

$$\int_{-\infty}^{\infty} \left\{m^+_{\beta,\Delta}(x) - h_\beta(x)\right\} \mathrm{d}x = \frac{2\pi e^{-2\pi\beta\Delta}}{1 - e^{-2\pi\beta\Delta}} \tag{3.3.2}$$

and

$$\int_{-\infty}^{\infty} \left\{ h_\beta(x) - m^-_{\beta,\Delta}(x) \right\} \mathrm{d}x = \frac{2\pi e^{-2\pi\beta\Delta}}{1+e^{-2\pi\beta\Delta}}. \tag{3.3.3}$$

(v) The Fourier transforms of $m^{\pm}_{\beta,\Delta}$, denoted by $\widehat{m}^{\pm}_{\beta,\Delta}(\xi)$, are even continuous functions supported on the interval $[-\Delta,\Delta]$ given by

$$\widehat{m}^{\pm}_{\beta,\Delta}(\xi) = \pi \left(\frac{e^{2\pi\beta(\Delta-|\xi|)} - e^{-2\pi\beta(\Delta-|\xi|)}}{(e^{\pi\beta\Delta} \mp e^{-\pi\beta\Delta})^2} \right). \tag{3.3.4}$$

(vi) The functions $m^{\pm}_{\beta,\Delta}$ are explicitly given by

$$m^{\pm}_{\beta,\Delta}(z) = \left(\frac{\beta}{\beta^2+z^2} \right) \left(\frac{e^{2\pi\beta\Delta} + e^{-2\pi\beta\Delta} - 2\cos(2\pi\Delta z)}{(e^{\pi\beta\Delta} \mp e^{-\pi\beta\Delta})^2} \right). \tag{3.3.5}$$

In particular, the function $m^-_{\beta,\Delta}$ is nonnegative on $\mathbb{R}$.

(vii) Assume that $0<\beta\leqslant\frac{1}{2}$ and $\Delta\geqslant 1$. For any real number x we have

$$0 < m^-_{\beta,\Delta}(x) \leqslant h_\beta(x) \leqslant m^+_{\beta,\Delta}(x) \ll \frac{1}{\beta(1+x^2)}, \tag{3.3.6}$$

and, for any complex number $z=x+iy$, we have

$$\left| m^+_{\beta,\Delta}(z) \right| \ll \frac{\Delta^2 e^{2\pi\Delta|y|}}{\beta(1+\Delta|z|)} \tag{3.3.7}$$

and

$$\left| m^-_{\beta,\Delta}(z) \right| \ll \frac{\beta\Delta^2 e^{2\pi\Delta|y|}}{1+\Delta|z|}. \tag{3.3.8}$$

Proof. We start by observing that (see (2.4.1))

$$h_\beta(x) = \int_0^\infty e^{-\pi\lambda x^2} \, \mathrm{d}\nu_\beta(\lambda) \,,$$

where ν_β is the finite nonnegative measure given by $\mathrm{d}\nu_\beta(\lambda) = \pi\beta\, e^{-\pi\lambda\beta^2}\, \mathrm{d}\lambda$. Let us define the auxiliary function

$$H_{\beta,\Delta}(x) = h_\beta\left(\frac{x}{\Delta}\right) = \frac{\beta\Delta^2}{\beta^2\Delta^2+x^2} = \int_0^\infty e^{-\pi\lambda x^2}\, \mathrm{d}\nu_{\beta,\Delta}(\lambda) \,,$$

where $\nu_{\beta,\Delta}$ is the finite nonnegative measure given by $\mathrm{d}\nu_{\beta,\Delta}(\lambda) = \pi\beta\Delta^2\, e^{-\pi\lambda\beta^2\Delta^2}\, \mathrm{d}\lambda$.

From [25, Section 11] we know that there is a unique extremal majorant $M^+_{\beta,\Delta}(z)$ of exponential type 2π and a unique extremal minorant $M^-_{\beta,\Delta}(z)$ of exponential type 2π for the real-valued function $H_{\beta,\Delta}$, and these are given by

$$M^{+}_{\beta,\Delta}(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{H_{\beta,\Delta}(n)}{(z-n)^2} + \sum_{n\neq 0} \frac{H'_{\beta,\Delta}(n)}{(z-n)} \right\} \tag{3.3.9}$$

and

$$M^{-}_{\beta,\Delta}(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{H_{\beta,\Delta}\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)^2} + \frac{H'_{\beta,\Delta}\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)} \right\}. \tag{3.3.10}$$

We now set

$$m^{+}_{\beta,\Delta}(z) := M^{+}_{\beta,\Delta}(\Delta z) \quad \text{and} \quad m^{-}_{\beta,\Delta}(z) := M^{-}_{\beta,\Delta}(\Delta z),$$

and a simple change of variables shows that these will be the unique extremal functions of exponential type $2\pi\Delta$ for h_β, as described in (i), (ii) and (iii). From (3.3.9) and (3.3.10) it is clear that $M^{\pm}_{\beta,\Delta}$, and hence $m^{\pm}_{\beta,\Delta}$, are even functions. We now verify the items (iv) - (vii).

Part(iv) Since $M^{\pm}_{\beta,\Delta}$ are entire functions of exponential type 2π whose restrictions to $\mathbb{R}$ belong to $L^1(\mathbb{R})$, a classical result of Plancherel and Pólya [76] (see also [89, Eq. (3.1) and (3.2)]) guarantees that $M^{\pm}_{\beta,\Delta}$ are bounded on the real line and hence belong to $L^2(\mathbb{R})$ as well. Moreover, still by [76], their derivatives $(M^{\pm}_{\beta,\Delta})'$ are also entire functions of exponential type 2π whose restrictions to $\mathbb{R}$ belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. In particular, $M^{\pm}_{\beta,\Delta}$ are integrable and of bounded variation on $\mathbb{R}$, and thus the Poisson summation formula holds pointwise. This can be used to calculate the values of the integrals of $M^{\pm}_{\beta,\Delta}$. Using the fact that $\widehat{M}^{\pm}_{\beta,\Delta}$ are supported in the interval $[-1,1]$ (which follows from the Paley-Wiener theorem) and the fact that $M^{+}_{\beta,\Delta}$ interpolates the values of $H_{\beta,\Delta}$ at $\mathbb{Z}$ (resp. $M^{-}_{\beta,\Delta}$ interpolates the values of $H_{\beta,\Delta}$ at $\mathbb{Z}+\frac{1}{2}$) we find

$$\begin{aligned}\widehat{M}^{+}_{\beta,\Delta}(0) = \sum_{n=-\infty}^{\infty} M^{+}_{\beta,\Delta}(n) = \sum_{n=-\infty}^{\infty} H_{\beta,\Delta}(n) &= \sum_{k=-\infty}^{\infty} \widehat{H}_{\beta,\Delta}(k) \\ &= \sum_{k=-\infty}^{\infty} \pi\Delta e^{-2\pi\beta\Delta|k|} = \pi\Delta\left(\frac{1+e^{-2\pi\beta\Delta}}{1-e^{-2\pi\beta\Delta}}\right)\end{aligned}$$

and

$$\begin{aligned}\widehat{M}^{-}_{\beta,\Delta}(0) = \sum_{n=-\infty}^{\infty} M^{-}_{\beta,\Delta}(n+\tfrac{1}{2}) = \sum_{n=-\infty}^{\infty} H_{\beta,\Delta}(n+\tfrac{1}{2}) &= \sum_{k=-\infty}^{\infty} (-1)^k\, \widehat{H}_{\beta,\Delta}(k) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k\, \pi\Delta e^{-2\pi\beta\Delta|k|} = \pi\Delta\left(\frac{1-e^{-2\pi\beta\Delta}}{1+e^{-2\pi\beta\Delta}}\right).\end{aligned}$$

The relation $\widehat{m}^{\pm}_{\beta,\Delta}(0) = \frac{1}{\Delta}\widehat{M}^{\pm}_{\beta,\Delta}(0)$ and the fact that $\widehat{h}_\beta(0) = \int_{-\infty}^{\infty} h_\beta(x)\,\mathrm{d}x = \pi$ lead us directly to (3.3.2) and (3.3.3). This establishes (iv).

Part (v) We have already noted that the Fourier transforms $\widehat{M}^{\pm}_{\beta,\Delta}$ are continuous functions (since $M^{\pm}_{\beta,\Delta} \in L^1(\mathbb{R})$) supported in the interval $[-1,1]$. From a classical result of Vaaler [89, Theorem 9] one has the explicit expression for the Fourier transform of the majorant, in which we use the fact that $M^{+}_{\beta,\Delta}(n) = H_{\beta,\Delta}(n)$ and $(M^{+}_{\beta,\Delta})'(n) = H'_{\beta,\Delta}(n)$ for all $n \in \mathbb{Z}$,

$$\begin{aligned}\widehat{M}^{+}_{\beta,\Delta}(\xi) &= \sum_{n=-\infty}^{\infty} \left((1-|\xi|)\, M^{+}_{\beta,\Delta}(n) + \frac{1}{2\pi i}\ \mathrm{sgn}(\xi)\, (M^{+}_{\beta,\Delta})'(n) \right) e^{-2\pi i n \xi} \\ &= \sum_{n=-\infty}^{\infty} \left((1-|\xi|)\, H_{\beta,\Delta}(n) + \frac{1}{2\pi i}\ \mathrm{sgn}(\xi)\, H'_{\beta,\Delta}(n) \right) e^{-2\pi i n \xi}\end{aligned} \tag{3.3.11}$$

for $\xi \in [-1,1]$. Using the Poisson summation formula we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H_{\beta,\Delta}(n)\, e^{-2\pi i n \xi} &= \sum_{k=-\infty}^{\infty} \widehat{H}_{\beta,\Delta}(\xi+k) \\ &= \sum_{k=-\infty}^{\infty} \pi\Delta\, e^{-2\pi\beta\Delta|\xi+k|} \\ &= \pi\Delta \left(\frac{e^{-2\pi\beta\Delta|\xi|} + e^{-2\pi\beta\Delta(1-|\xi|)}}{1-e^{-2\pi\beta\Delta}} \right)\end{aligned} \tag{3.3.12}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H'_{\beta,\Delta}(n)\, e^{-2\pi i n \xi} &= \sum_{k=-\infty}^{\infty} \widehat{H'}_{\beta,\Delta}(\xi+k) \\ &= \sum_{k=-\infty}^{\infty} 2\pi i(\xi+k)\widehat{H}_{\beta,\Delta}(\xi+k) \\ &= \sum_{k=-\infty}^{\infty} 2\pi i(\xi+k)\pi\Delta\, e^{-2\pi\beta\Delta|\xi+k|} \\ &= 2\pi^2 i\Delta\, \mathrm{sgn}(\xi) \left(\frac{|\xi|\left(e^{-2\pi\beta\Delta|\xi|} + e^{-2\pi\beta\Delta(1-|\xi|)}\right)}{1-e^{-2\pi\beta\Delta}} - \frac{e^{-2\pi\beta\Delta}\left(e^{2\pi\beta\Delta|\xi|} - e^{-2\pi\beta\Delta|\xi|}\right)}{(1-e^{-2\pi\beta\Delta})^2} \right).\end{aligned} \tag{3.3.13}$$

Plugging (3.3.12) and (3.3.13) into (3.3.11) gives us

$$\widehat{M}^{+}_{\beta,\Delta}(\xi) = \pi\Delta \left(\frac{e^{2\pi\beta\Delta(1-|\xi|)} - e^{-2\pi\beta\Delta(1-|\xi|)}}{(e^{\pi\beta\Delta} - e^{-\pi\beta\Delta})^2} \right),$$

and from the fact that

$$\widehat{m}^{\pm}_{\beta,\Delta}(\xi) = \frac{1}{\Delta}\widehat{M}^{\pm}_{\beta,\Delta}\left(\frac{\xi}{\Delta}\right) \tag{3.3.14}$$

we arrive at (3.3.4) for the majorant.

For the minorant we proceed analogously. From [89, Theorem 9] one has the representation, in which we use the fact that $M^{-}_{\beta,\Delta}(n+\frac{1}{2}) = H_{\beta,\Delta}(n+\frac{1}{2})$ and $(M^{-}_{\beta,\Delta})'(n+\frac{1}{2}) = H'_{\beta,\Delta}(n+\frac{1}{2})$

for all $n \in \mathbb{Z}$,

$$\begin{aligned}\widehat{M}^-_{\beta,\Delta}(\xi) &= \sum_{n=-\infty}^{\infty} \left((1-|\xi|)\, M^-_{\beta,\Delta}(n+\tfrac{1}{2}) + \frac{1}{2\pi i}\,\operatorname{sgn}(\xi)\,(M^-_{\beta,\Delta})'(n+\tfrac{1}{2}) \right) e^{-2\pi i (n+\frac{1}{2})\xi} \\ &= \sum_{n=-\infty}^{\infty} \left((1-|\xi|)\, H_{\beta,\Delta}(n+\tfrac{1}{2}) + \frac{1}{2\pi i}\,\operatorname{sgn}(\xi)\,H'_{\beta,\Delta}(n+\tfrac{1}{2}) \right) e^{-2\pi i (n+\frac{1}{2})\xi} \qquad (3.3.15)\end{aligned}$$

for $\xi \in [-1,1]$. Poisson summation now yields

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H_{\beta,\Delta}(n+\tfrac{1}{2})\, e^{-2\pi i (n+\frac{1}{2})\xi} &= \sum_{k=-\infty}^{\infty} (-1)^k\, \widehat{H}_{\beta,\Delta}(\xi+k) \\ &= \pi\Delta \left(\frac{e^{-2\pi\beta\Delta|\xi|} - e^{-2\pi\beta\Delta(1-|\xi|)}}{1+e^{-2\pi\beta\Delta}} \right)\end{aligned} \qquad (3.3.16)$$

and

$$\sum_{n=-\infty}^{\infty} H'_{\beta,\Delta}(n+\tfrac{1}{2})\, e^{-2\pi i (n+\frac{1}{2})\xi} = \sum_{k=-\infty}^{\infty} 2\pi i\,(\xi+k)\,(-1)^k\, \widehat{H}_{\beta,\Delta}(\xi+k) \qquad (3.3.17)$$

$$= 2\pi^2 i\Delta\,\operatorname{sgn}(\xi) \left(\frac{|\xi|\left(e^{-2\pi\beta\Delta|\xi|} - e^{-2\pi\beta\Delta(1-|\xi|)}\right)}{1+e^{-2\pi\beta\Delta}} + \frac{e^{-2\pi\beta\Delta}\left(e^{2\pi\beta\Delta|\xi|} - e^{-2\pi\beta\Delta|\xi|}\right)}{(1+e^{-2\pi\beta\Delta})^2} \right).$$

Plugging (3.3.16) and (3.3.17) into (3.3.15) gives us

$$\widehat{M}^-_{\beta,\Delta}(\xi) = \pi\Delta \left(\frac{e^{2\pi\beta\Delta(1-|\xi|)} - e^{-2\pi\beta\Delta(1-|\xi|)}}{(e^{\pi\beta\Delta} + e^{-\pi\beta\Delta})^2} \right),$$

and using (3.3.14) we arrive at (3.3.4) for the minorant. This completes the proof of (v).

Part (vi) The proof of (vi) is a direct computation using (v) and Fourier inversion

$$m^{\pm}_{\beta,\Delta}(z) = \int_{-\Delta}^{\Delta} \pi \left(\frac{e^{2\pi\beta(\Delta-|\xi|)} - e^{-2\pi\beta(\Delta-|\xi|)}}{(e^{\pi\beta\Delta} \mp e^{-\pi\beta\Delta})^2} \right) e^{2\pi i \xi z}\, \mathrm{d}\xi.$$

We omit the details of this calculation.

Part (vii) From (3.3.5) it follows directly that $0 < m^-_{\beta,\Delta}(x)$ for all $x \in \mathbb{R}$. We may also write

$$m^+_{\beta,\Delta}(x) = \frac{\beta}{\beta^2+x^2} \left(1 + \frac{4\sin^2(\pi\Delta x)}{(e^{\pi\beta\Delta} - e^{-\pi\beta\Delta})^2} \right). \qquad (3.3.18)$$

We then note that in the range $0 < \beta \leqslant \frac{1}{2}$ and $\Delta \geqslant 1$ the following estimates hold:

$$\frac{\beta}{\beta^2+x^2} \ll \frac{1}{\beta(1+x^2)} \qquad (3.3.19)$$

and

$$\left(\frac{\beta}{\beta^2+x^2}\right)\frac{\sin^2(\pi\Delta x)}{(e^{\pi\beta\Delta}-e^{-\pi\beta\Delta})^2} = \left(\frac{\beta}{\beta^2+x^2}\right)\left(\frac{\sin(\pi\Delta x)}{\Delta x}\right)^2\left(\frac{\Delta x}{\beta\Delta}\right)^2\left(\frac{\beta\Delta}{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}\right)^2$$
$$\ll \left(\frac{\beta}{\beta^2+x^2}\right)\left(\frac{1}{1+\Delta^2x^2}\right)\left(\frac{x}{\beta}\right)^2 \tag{3.3.20}$$
$$\ll \frac{1}{\beta(1+x^2)}.$$

Using (3.3.19) and (3.3.20) in (3.3.18) yields the estimate

$$m^+_{\beta,\Delta}(x) \ll \frac{1}{\beta(1+x^2)}.$$

The idea to analyze the growth in the complex plane is similar. We start by rewriting (3.3.5) as

$$m^{\pm}_{\beta,\Delta}(z) = \frac{4}{\beta}\left(\frac{\sin\pi\Delta(z+i\beta)}{\Delta(z+i\beta)}\right)\left(\frac{\sin\pi\Delta(z-i\beta)}{\Delta(z-i\beta)}\right)\left(\frac{\beta\Delta}{e^{\pi\beta\Delta}\mp e^{-\pi\beta\Delta}}\right)^2 \tag{3.3.21}$$

and then apply the following uniform bounds

$$\left|\frac{\sin w}{w}\right| \ll \frac{e^{|\mathrm{Im}\, w|}}{1+|w|} \tag{3.3.22}$$

and

$$\frac{1}{(1+|w+i\gamma|)}\cdot\frac{1}{(1+|w-i\gamma|)} \ll \frac{1}{1+|w|} \tag{3.3.23}$$

that are valid for any $w \in \mathbb{C}$ and $\gamma > 0$. Using (3.3.22) and (3.3.23) in (3.3.21) we derive that

$$\left|m^{\pm}_{\beta,\Delta}(z)\right| \ll \frac{1}{\beta}\left(\frac{e^{\pi\Delta(|\mathrm{Im}\, z|+\beta)}}{1+\Delta|z+i\beta|}\right)\left(\frac{e^{\pi\Delta(|\mathrm{Im}\, z|+\beta)}}{1+\Delta|z-i\beta|}\right)\left(\frac{\beta\Delta}{e^{\pi\beta\Delta}\mp e^{-\pi\beta\Delta}}\right)^2$$
$$\ll \frac{1}{\beta}\left(\frac{e^{2\pi\Delta|\mathrm{Im}\, z|}}{1+\Delta|z|}\right)\left(\frac{\beta\Delta\, e^{\pi\beta\Delta}}{e^{\pi\beta\Delta}\mp e^{-\pi\beta\Delta}}\right)^2.$$

In the majorant case, we have

$$\left(\frac{\beta\Delta\, e^{\pi\beta\Delta}}{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}\right)^2 \ll 1+(\beta\Delta)^2 \ll \Delta^2\,,$$

and this leads to (3.3.7). In the minorant case we have

$$\left(\frac{\beta\Delta\, e^{\pi\beta\Delta}}{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}\right)^2 \ll (\beta\Delta)^2\,,$$

and this leads to (3.3.8). This concludes the proof of the lemma. □

3.3.2 Approximations to the functions $f_{2m+1,\sigma}$

Our next task is to present the analogue of Lemma 3.9 (i.e. the solution of the Beurling-Selberg extremal problem) for the family of even functions $f_{2m+1,\sigma}$ defined in (3.1.9). This result is an extension of Lemma 2.8, where the case $\sigma = \frac{1}{2}$ was studied. We highlight the explicit description of the Fourier transforms of the extremal bandlimited approximations. This is a slightly technical but extremely important part of this chapter, since these Fourier transforms will play an important role in the evaluation of the sum over prime powers in the explicit formula.

Lemma 3.10 (Extremal functions for $f_{2m+1,\sigma}$). *Let $m \geqslant 0$ be an integer and let $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $\Delta \geqslant 1$ be real parameters. Let $f_{2m+1,\sigma}$ be the real-valued function defined in* (3.1.9), *namely*

$$f_{2m+1,\sigma}(x) = \frac{1}{2}\int_{\sigma}^{3/2} (\alpha - \sigma)^{2m} \log\left(\frac{1+x^2}{(\alpha-\frac{1}{2})^2 + x^2}\right) \mathrm{d}\alpha.$$

Then there is a unique pair of real entire functions $g^{-}_{2m+1,\sigma,\Delta} : \mathbb{C} \to \mathbb{C}$ and $g^{+}_{2m+1,\sigma,\Delta} : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:

(i) The real entire functions $g^{\pm}_{2m+1,\sigma,\Delta}$ have exponential type $2\pi\Delta$.

(ii) The inequality

$$g^{-}_{2m+1,\sigma,\Delta}(x) \leqslant f_{2m+1,\sigma}(x) \leqslant g^{+}_{2m+1,\sigma,\Delta}(x) \tag{3.3.24}$$

holds pointwise for all $x \in \mathbb{R}$.

(iii) Subject to conditions (i) and (ii), the value of the integral

$$\int_{-\infty}^{\infty} \left\{g^{+}_{2m+1,\sigma,\Delta}(x) - g^{-}_{2m+1,\sigma,\Delta}(x)\right\} \mathrm{d}x$$

is minimized.

The functions $g^{\pm}_{2m+1,\sigma,\Delta}$ are even and verify the following additional properties:

(iv) For any real number x we have

$$\left|g^{\pm}_{2m+1,\sigma,\Delta}(x)\right| \ll_m \frac{1}{1+x^2}\,, \tag{3.3.25}$$

and, for any complex number $z = x + iy$, we have

$$\left|g^{\pm}_{2m+1,\sigma,\Delta}(z)\right| \ll_m \frac{\Delta^2 e^{2\pi\Delta|y|}}{(1+\Delta|z|)}, \tag{3.3.26}$$

where the constants implied by the $\ll_m$ notation depend only on m.

(v) The Fourier transforms of $g^{\pm}_{2m+1,\sigma,\Delta}$, denoted by $\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}(\xi)$, are continuous functions supported on the interval $[-\Delta,\Delta]$ and satisfy

$$\left|\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}(\xi)\right| \ll_m 1. \tag{3.3.27}$$

(vi) The L^1-distances of $g^{\pm}_{2m+1,\sigma,\Delta}$ to $f_{2m+1,\sigma}$ are explicitly given by

$$\int_{-\infty}^{\infty}\left\{g^{+}_{2m+1,\sigma,\Delta}(x)-f_{2m+1,\sigma}(x)\right\}\,\mathrm{d}x$$
$$= -\frac{1}{\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\,\log\left(\frac{1-e^{-2\pi(\alpha-1/2)\Delta}}{1-e^{-2\pi\Delta}}\right)\,\mathrm{d}\alpha, \tag{3.3.28}$$

and

$$\int_{-\infty}^{\infty}\left\{f_{2m+1,\sigma}(x)-g^{-}_{2m+1,\sigma,\Delta}(x)\right\}\,\mathrm{d}x$$
$$= \frac{1}{\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\,\log\left(\frac{1+e^{-2\pi(\alpha-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\,\mathrm{d}\alpha. \tag{3.3.29}$$

(vii) At $\xi=0$ we have

$$\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}(0) = \frac{\pi\left(\frac{3}{2}-\sigma\right)^{2m+2}}{(2m+1)(2m+2)}$$
$$-\frac{1}{\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\,\log\left(\frac{1\mp e^{-2\pi(\alpha-1/2)\Delta}}{1\mp e^{-2\pi\Delta}}\right)\,\mathrm{d}\alpha. \tag{3.3.30}$$

(viii) The Fourier transforms $\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}$ are even functions and, for $0<\xi<\Delta$, we have the explicit expressions

$$\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}(\xi) =$$
$$\frac{1}{2}\sum_{k=-\infty}^{\infty}(\pm1)^k\left[\frac{k+1}{|\xi+k\Delta|}\left(\frac{(2m)!\,e^{-2\pi|\xi+k\Delta|(\sigma-1/2)}}{(2\pi|\xi+k\Delta|)^{2m+1}}\right.\right. \tag{3.3.31}$$
$$\left.\left.-\sum_{j=0}^{2m+1}\frac{\gamma_j\,e^{-2\pi|\xi+k\Delta|}}{(2\pi|\xi+k\Delta|)^j}\left(\tfrac{3}{2}-\sigma\right)^{2m+1-j}\right)\right],$$

where $\gamma_j=\frac{(2m)!}{(2m+1-j)!}$, for $0\leqslant j\leqslant 2m+1$.

Proof. Fix $m\geqslant 0$ and $\frac{1}{2}\leqslant\sigma\leqslant 1$. For $\Delta\geqslant 1$ we consider the nonnegative Borel measure $\nu_\Delta=\nu_{2m+1,\sigma,\Delta}$ on $(0,\infty)$ given by

$$\mathrm{d}\nu_\Delta(\lambda) := \int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\left(\frac{e^{-\pi\lambda(\alpha-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\,\mathrm{d}\alpha\,\mathrm{d}\lambda\,,$$

and let $F_\Delta = F_{2m+1,\sigma,\Delta}$ be the function

$$F_\Delta(x) := \int_0^\infty e^{-\pi\lambda x^2}\, \mathrm{d}\nu_\Delta(\lambda).$$

Recall that

$$\frac{1}{2}\log\left(\frac{x^2+\Delta^2}{x^2+(\alpha-\frac{1}{2})^2\Delta^2}\right) = \int_0^\infty e^{-\pi\lambda x^2}\left(\frac{e^{-\pi\lambda(\alpha-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\lambda.$$

Multiplying both sides by $(\alpha-\sigma)^{2m}$ and integrating from $\alpha=\sigma$ to $\alpha=\frac{3}{2}$ yields

$$\begin{aligned}
&\frac{1}{2}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\log\left(\frac{x^2+\Delta^2}{x^2+(\alpha-\frac{1}{2})^2\Delta^2}\right)\mathrm{d}\alpha\\
&\quad=\int_\sigma^{3/2}\int_0^\infty(\alpha-\sigma)^{2m}e^{-\pi\lambda x^2}\left(\frac{e^{-\pi\lambda(\alpha-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\lambda\,\mathrm{d}\alpha\\
&\quad=\int_0^\infty e^{-\pi\lambda x^2}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\frac{e^{-\pi\lambda(\alpha-1/2)^2\Delta^2}-e^{-\pi\lambda\Delta^2}}{2\lambda}\right)\mathrm{d}\alpha\,\mathrm{d}\lambda\\
&\quad=F_\Delta(x),
\end{aligned} \tag{3.3.32}$$

where the interchange of the integrals is justified since the terms involved are all nonnegative. It follows from (3.1.9) that

$$f_{2m+1,\sigma}(x) = F_\Delta(\Delta x). \tag{3.3.33}$$

In particular, this shows that the measure ν_Δ is finite on $(0,\infty)$ since

$$\int_0^\infty \mathrm{d}\nu_\Delta(\lambda) = F_\Delta(0) = f_{2m+1,\sigma}(0).$$

From the Gaussian subordination framework of [25, Section 11], there is a unique extremal majorant $G_\Delta^+(z) = G_{2m+1,\sigma,\Delta}^+(z)$ and a unique extremal minorant $G_\Delta^-(z) = G_{2m+1,\sigma,\Delta}^-(z)$ of exponential type 2π for $F_\Delta(x)$, and these functions are given by

$$G_\Delta^+(z) = \left(\frac{\sin\pi z}{\pi}\right)^2\left\{\sum_{n=-\infty}^{\infty}\frac{F_\Delta(n)}{(z-n)^2}+\sum_{n\neq0}\frac{F_\Delta'(n)}{(z-n)}\right\} \tag{3.3.34}$$

and

$$G_\Delta^-(z) = \left(\frac{\cos\pi z}{\pi}\right)^2\left\{\sum_{n=-\infty}^{\infty}\frac{F_\Delta\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)^2}+\frac{F_\Delta'\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)}\right\}. \tag{3.3.35}$$

Hence, the functions $g_\Delta^+(z) = g_{2m+1,\sigma,\Delta}^+(z)$ and $g_\Delta^-(z) = g_{2m+1,\sigma,\Delta}^-(z)$ defined by

$$g_\Delta^+(z) := G_\Delta^+(\Delta z) \quad\text{and}\quad g_\Delta^-(z) := G_\Delta^-(\Delta z) \tag{3.3.36}$$

are the unique extremal functions of exponential type $2\pi\Delta$ for $f_{2m+1,\sigma}$, as described in (i), (ii) and (iii). From (3.3.34) and (3.3.35) it is clear that $G_{\Delta}^{\pm}$, and hence $g_{\Delta}^{\pm}$, are even functions. We now verify the items (iv) - (viii).

Part (iv) For $\sigma = \frac{1}{2}$, the function $f_{2m+1,\frac{1}{2}} = f_{2m+1}$ (see (2.3.8)) was already used in Lemma 2.8 in connection to bounds for $S_{2m+1}(t)$ in the critical line and is explicitly given by

$$f_{2m+1,\frac{1}{2}}(x) = \frac{1}{(2m+1)}\left[(-1)^{m+1}x^{2m+1}\arctan\left(\frac{1}{x}\right) + \sum_{k=0}^{m}\frac{(-1)^{m-k}}{2k+1}x^{2m-2k}\right].$$

Directly from the definition (3.1.9) we see that

$$0 \leqslant f_{2m+1,\sigma}(x) \leqslant f_{2m+1,\frac{1}{2}}(x) \quad \text{and} \quad 0 \leqslant \left|f'_{2m+1,\sigma}(x)\right| \leqslant \left|f'_{2m+1,\frac{1}{2}}(x)\right| \tag{3.3.37}$$

for all $x \in \mathbb{R}$ and $\frac{1}{2} \leqslant \sigma \leqslant 1$. Therefore, from (2.4.11) and (3.3.37) it follows that

$$\left|f_{2m+1,\sigma}(x)\right| \ll_m \frac{1}{1+x^2} \quad \text{and} \quad \left|f'_{2m+1,\sigma}(x)\right| \ll_m \frac{1}{|x|(1+x^2)}$$

(note that the implicit constants do not depend on σ). It then follows from (3.3.33) that (recall the shorthand notation $F_{\Delta} = F_{2m+1,\sigma,\Delta}$)

$$\left|F_{\Delta}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2 + x^2} \quad \text{and} \quad \left|F'_{\Delta}(x)\right| \ll_m \frac{\Delta^2}{|x|(\Delta^2+x^2)}. \tag{3.3.38}$$

Expressions (3.3.34) and (3.3.35) can be rewritten as

$$G_{\Delta}^{+}(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 F_{\Delta}(0) + \sum_{n\neq 0}\left(\frac{\sin \pi(z-n)}{\pi(z-n)}\right)^2 \Big\{F_{\Delta}(n) + (z-n)F'_{\Delta}(n)\Big\} \tag{3.3.39}$$

and

$$G_{\Delta}^{-}(z) = \sum_{n=-\infty}^{\infty}\left(\frac{\sin \pi(z-n+\frac{1}{2})}{\pi(z-n+\frac{1}{2})}\right)^2 \Big\{F_{\Delta}\big(n-\tfrac{1}{2}\big) + (z-n+\tfrac{1}{2})F'_{\Delta}\big(n-\tfrac{1}{2}\big)\Big\}. \tag{3.3.40}$$

We now use (3.3.38), (3.3.39), (3.3.40) and the bound (2.4.13) to get

$$\left|G_{\Delta}^{\pm}(z)\right| \ll_m \frac{\Delta^2 e^{2\pi|\operatorname{Im} z|}}{1+|z|}.$$

One can break the sums in (3.3.39) and (3.3.40) into the ranges $\{n \leqslant |z|/2\}$, $\{|z|/2 < n \leqslant 2|z|\}$ and $\{2|z| < n\}$ to verify this last claim. From (3.3.36) we arrive at (3.3.26).

To bound the functions $G_{\Delta}^{\pm}$ on the real line, we explore the fact that F_{Δ} is an even function (and hence F'_{Δ} is odd) to group the terms conveniently. For the majorant we group the terms n and $-n$ in (3.3.39) to get

$$G_{\Delta}^{+}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 F_{\Delta}(0) + \sum_{n=1}^{\infty} \left(\frac{\sin^2 \pi(x-n)}{\pi^2(x^2-n^2)^2}\right)\Big\{(2x^2+2n^2)F_{\Delta}(n) + (x^2-n^2)\,2n\,F_{\Delta}'(n)\Big\}\,, \tag{3.3.41}$$

and it follows from (3.3.38) and (2.4.13) that

$$\left|G_{\Delta}^{+}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2+x^2}. \tag{3.3.42}$$

Again, it may be useful to split the sum in (3.3.41) into the ranges $\{n \leqslant |x|/2\}$, $\{|x|/2 < n \leqslant 2|x|\}$ and $\{2|x| < n\}$ to verify this last claim. The bound

$$\left|G_{\Delta}^{-}(x)\right| \ll_m \frac{\Delta^2}{\Delta^2+x^2} \tag{3.3.43}$$

follows in an analogous way, grouping the terms n and $1-n$ (for $n \geqslant 1$) in (3.3.40). From (3.3.36), (3.3.42) and (3.3.43) we arrive at (3.3.25).

Part (v) Since $g_{2m+1,\sigma,\Delta}^{\pm}$ are entire functions of exponential type $2\pi\Delta$ whose restrictions to $\mathbb{R}$ are integrable, it follows from the Paley-Wiener theorem that their Fourier transforms are continuous functions supported on the interval $[-\Delta, \Delta]$. Moreover, from the uniform bounds (3.3.25) we see that

$$\left|\widehat{g}_{2m+1,\sigma,\Delta}^{\pm}(\xi)\right| \leqslant \int_{-\infty}^{\infty} \left|g_{2m+1,\sigma,\Delta}^{\pm}(x)\right| \,\mathrm{d}x \ll_m 1.$$

.

Parts (vi) and (vii) From (3.3.42), (3.3.43), and the fact that the Fourier transforms $\widehat{G}_{\Delta}^{\pm}$ are supported on $[-1,1]$, we may apply the Poisson summation formula pointwise to $G_{\Delta}^{\pm}$. Recalling that G_{Δ}^{+} interpolates the values of F_{Δ} at $\mathbb{Z}$, we use (3.3.32) to derive that

$$\begin{aligned}
\widehat{G}_{\Delta}^{+}(0) &= \sum_{n=-\infty}^{\infty} G_{\Delta}^{+}(n) = \sum_{n=-\infty}^{\infty} F_{\Delta}(n) \\
&= \frac{1}{2}\int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \sum_{n=-\infty}^{\infty} \log\left(\frac{n^2+\Delta^2}{n^2+(\alpha-\frac{1}{2})^2\Delta^2}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \left(2\pi\Delta(\tfrac{3}{2}-\alpha) - 2\log\left(\frac{1-e^{-2\pi(\alpha-1/2)\Delta}}{1-e^{-2\pi\Delta}}\right)\right) \mathrm{d}\alpha \\
&= \frac{\pi\Delta\left(\frac{3}{2}-\sigma\right)^{2m+2}}{(2m+1)(2m+2)} - \int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \log\left(\frac{1-e^{-2\pi(\alpha-1/2)\Delta}}{1-e^{-2\pi\Delta}}\right) \mathrm{d}\alpha.
\end{aligned} \tag{3.3.44}$$

Above we have used the fact that, for $b \geqslant a > 0$ (see, for instance, [14, §4.2.1])

$$\sum_{n=-\infty}^{\infty} \log\left(\frac{n^2+b^2}{n^2+a^2}\right) = 2\pi(b-a) - 2\log\left(\frac{1-e^{-2\pi a}}{1-e^{-2\pi b}}\right).$$

One can prove this directly regarding both sides as a function of the variable b, observing that they agree when $b = a$, and showing that they have the same derivative.

We proceed analogously for the minorant

$$\begin{aligned}\widehat{G}_{\Delta}^{-}(0) &= \sum_{n=-\infty}^{\infty} G_{\Delta}^{-}(n) = \sum_{n=-\infty}^{\infty} F_{\Delta}(n+\tfrac{1}{2}) \\ &= \frac{1}{2}\int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \sum_{n=-\infty}^{\infty} \log\left(\frac{(n+\frac{1}{2})^2+\Delta^2}{(n+\frac{1}{2})^2+(\alpha-\frac{1}{2})^2\Delta^2}\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \left(2\pi\Delta(\tfrac{3}{2}-\alpha) - 2\log\left(\frac{1+e^{-2\pi(\alpha-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\right) \mathrm{d}\alpha \\ &= \frac{\pi\Delta\left(\frac{3}{2}-\sigma\right)^{2m+2}}{(2m+1)(2m+2)} - \int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \log\left(\frac{1+e^{-2\pi(\alpha-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right) \mathrm{d}\alpha,\end{aligned} \tag{3.3.45}$$

now using the fact that, for $b \geqslant a > 0$ (see [14, §4.1.2])

$$\sum_{n=-\infty}^{\infty} \log\left(\frac{(n+\frac{1}{2})^2+b^2}{(n+\frac{1}{2})^2+a^2}\right) = 2\pi(b-a) - 2\log\left(\frac{1+e^{-2\pi a}}{1+e^{-2\pi b}}\right).$$

From (3.3.44), (3.3.45) and the dilation relation

$$\widehat{g}_{\Delta}^{\pm}(\xi) = \frac{1}{\Delta}\widehat{G}_{\Delta}^{\pm}\left(\frac{\xi}{\Delta}\right), \tag{3.3.46}$$

we arrive at (3.3.30). Besides, using the fact that (see, for instance, [50, §2.733 Eq.1])

$$\int_{-\infty}^{\infty} f_{2m+1,\sigma}(x)\ \mathrm{d}x = \frac{\pi\left(\frac{3}{2}-\sigma\right)^{2m+2}}{(2m+1)(2m+2)},$$

we arrive at (3.3.28) and (3.3.29) from (3.3.30).

Part (viii) From relation (3.3.46) it suffices to find the explicit form of $\widehat{G}_{\Delta}^{\pm}(\xi)$ for $-1 \leqslant \xi \leqslant 1$. Since $\widehat{G}_{\Delta}^{\pm}(\xi)$ are even functions, we only need to consider the case $0 < \xi \leqslant 1$ (recall that the values at $\xi = 0$ were computed in the proof of (vii)).
We consider first the majorant. Recall that $G_{\Delta}^{+}(k) = F_{\Delta}(k)$ for all $k \in \mathbb{Z}$ and $(G_{\Delta}^{+})'(k) = F_{\Delta}'(k)$ for all $k \in \mathbb{Z}\backslash\{0\}$. Note also that $(G_{\Delta}^{+})'(0) = 0$, since G_{Δ}^{+} is an even function, and that $F_{\Delta}'(0) = 0$ except in the case $\alpha = \frac{1}{2}$ and $m = 0$, for which F_{Δ} is not differentiable at

$x = 0$. Our starting point is a result of Vaaler [89, Theorem 9] that gives us

$$\begin{aligned}\widehat{G}_\Delta^+(\xi) &= (1-|\xi|)\sum_{k=-\infty}^{\infty} G_\Delta^+(k)\, e^{-2\pi i k\xi} + \frac{1}{2\pi i}\,\operatorname{sgn}(\xi)\sum_{k=-\infty}^{\infty}(G_\Delta^+)'(k)\, e^{-2\pi i k\xi}\\ &= (1-|\xi|)\sum_{k=-\infty}^{\infty} F_\Delta(k)\, e^{-2\pi i k\xi} + \frac{1}{2\pi i}\,\operatorname{sgn}(\xi)\sum_{k\neq 0} F_\Delta'(k)\, e^{-2\pi i k\xi}.\end{aligned} \tag{3.3.47}$$

Using (3.3.32), the first sum in (3.3.47) is given by

$$\begin{aligned}\sum_{k=-\infty}^{\infty} & F_\Delta(k)\, e^{-2\pi i k\xi}\\ &= \sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\log\left(\frac{k^2+\Delta^2}{k^2+(\alpha-\frac{1}{2})^2\Delta^2}\right)\mathrm{d}\alpha\right)e^{-2\pi i k\xi}\\ &= \frac{1}{2}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\sum_{k=-\infty}^{\infty}\log\left(\frac{k^2+\Delta^2}{k^2+(\alpha-\frac{1}{2})^2\Delta^2}\right)e^{-2\pi i k\xi}\right)\mathrm{d}\alpha,\end{aligned} \tag{3.3.48}$$

where the use of Fubini's theorem is justified by the absolute convergence of the sum on the left-hand side (which follows by (3.3.38)). The inner sum in (3.3.48) can be evaluated via Poisson summation applied to the Fourier transform pair

$$h(x) = \log\left(\frac{x^2+b^2}{x^2+a^2}\right) \quad \text{and} \quad \widehat{h}(\xi) = \frac{e^{-2\pi|\xi|a} - e^{-2\pi|\xi|b}}{|\xi|} \tag{3.3.49}$$

for real numbers $b \geqslant a > 0$ (see [14, §4.1.2]). We then arrive at

$$\begin{aligned}\sum_{k=-\infty}^{\infty} & F_\Delta(k)\, e^{-2\pi i k\xi}\\ &= \frac{1}{2}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\sum_{k=-\infty}^{\infty}\frac{e^{-2\pi|\xi+k|(\alpha-1/2)\Delta} - e^{-2\pi|\xi+k|\Delta}}{|\xi+k|}\right)\mathrm{d}\alpha\\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\frac{e^{-2\pi|\xi+k|(\alpha-1/2)\Delta} - e^{-2\pi|\xi+k|\Delta}}{|\xi+k|}\right)\mathrm{d}\alpha.\end{aligned} \tag{3.3.50}$$

We shall use the following indefinite integral [50, §2.321] in our computations

$$\int x^n\, e^{-ax}\,\mathrm{d}x = -e^{-ax}\left(\sum_{\ell=0}^{n}\frac{\ell!\,\binom{n}{\ell}}{a^{\ell+1}}\, x^{n-\ell}\right). \tag{3.3.51}$$

Using (3.3.51) in (3.3.50) we get

$$\sum_{k=-\infty}^{\infty} F_\Delta(k)\, e^{-2\pi i k\xi}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \frac{e^{-2\pi|\xi+k|(\sigma-1/2)\Delta}}{|\xi+k|}\Bigg(\frac{(2m)!}{(2\pi|\xi+k|\Delta)^{2m+1}} - e^{-2\pi|\xi+k|(3/2-\sigma)\Delta}\sum_{\ell=0}^{2m}\frac{\ell!\binom{2m}{\ell}}{(2\pi|\xi+k|\Delta)^{\ell+1}}\big(\tfrac{3}{2}-\sigma\big)^{2m-\ell}\Bigg)$$

$$- \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{e^{-2\pi|\xi+k|\Delta}}{(2m+1)|\xi+k|}\big(\tfrac{3}{2}-\sigma\big)^{2m+1} \tag{3.3.52}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{|\xi+k|}\left[\frac{(2m)!\,e^{-2\pi|\xi+k|(\sigma-1/2)\Delta}}{(2\pi|\xi+k|\Delta)^{2m+1}} - \sum_{j=0}^{2m+1}\frac{\gamma_j\,e^{-2\pi|\xi+k|\Delta}}{(2\pi|\xi+k|\Delta)^{j}}\big(\tfrac{3}{2}-\sigma\big)^{2m+1-j}\right],$$

with $\gamma_j = \frac{(2m)!}{(2m+1-j)!}$, for $0 \leqslant j \leqslant 2m+1$.

We now evaluate the second sum in (3.3.47). Using (3.3.32) we have

$$\sum_{k\neq 0} F'_\Delta(k)\,e^{-2\pi i k\xi}$$

$$= \sum_{k\neq 0}\left(\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\frac{k}{k^2+\Delta^2} - \frac{k}{k^2+(\alpha-\frac{1}{2})^2\Delta^2}\right)\mathrm{d}\alpha\right)e^{-2\pi i k\xi} \tag{3.3.53}$$

$$= \int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\sum_{k=-\infty}^{\infty}\left(\frac{k}{k^2+\Delta^2} - \frac{k}{k^2+(\alpha-\frac{1}{2})^2\Delta^2}\right)e^{-2\pi i k\xi}\right)\mathrm{d}\alpha,$$

where the use of Fubini's theorem is again justified by the absolute convergence of the sum on the left-hand side, which again follows by (3.3.38). The inner sum in (3.3.53) can be evaluated via Poisson summation applied to the Fourier transform pair

$$h(x) = \frac{x}{x^2+a^2} - \frac{x}{x^2+b^2} \quad \text{and} \quad \widehat{h}(\xi) = -\pi i\,\mathrm{sgn}(\xi)\left(e^{-2\pi|\xi|a} - e^{-2\pi|\xi|b}\right) \tag{3.3.54}$$

for real numbers $b \geqslant a > 0$ (see [14, §4.1.2]). We then arrive at the expression

$$\sum_{k\neq 0} F'_\Delta(k)\,e^{-2\pi i k\xi}$$

$$= \pi i\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(\sum_{k=-\infty}^{\infty}\mathrm{sgn}(\xi+k)\left(e^{-2\pi(\alpha-1/2)|\xi+k|\Delta} - e^{-2\pi|\xi+k|\Delta}\right)\right)\mathrm{d}\alpha$$

$$= \pi i\sum_{k=-\infty}^{\infty}\mathrm{sgn}(\xi+k)\int_\sigma^{3/2}(\alpha-\sigma)^{2m}\left(e^{-2\pi(\alpha-1/2)|\xi+k|\Delta} - e^{-2\pi|\xi+k|\Delta}\right)\mathrm{d}\alpha.$$

The latter use of Fubini's theorem can be justified by the absolute convergence of the double integral (one can explicitly sum the exponentials in geometric progressions). In the case $\sigma = \frac{1}{2}$ and $m = 0$ one has to be a bit more careful and group the terms k and $-k-1$, for $k \geqslant 0$, to have convergence. Using (3.3.51) we get

$$\sum_{k\neq 0} F'_\Delta(k)\,e^{-2\pi i k\xi}$$

$$= \pi i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(\xi+k) \Bigg(\frac{(2m)!\, e^{-2\pi|\xi+k|(\sigma-1/2)\Delta}}{(2\pi|\xi+k|\Delta)^{2m+1}} - e^{-2\pi|\xi+k|\Delta} \sum_{\ell=0}^{2m} \frac{\ell! \binom{2m}{\ell}}{(2\pi|\xi+k|\Delta)^{\ell+1}} \left(\tfrac{3}{2}-\sigma\right)^{2m-\ell} \Bigg) \tag{3.3.55}$$

$$- \pi i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(\xi+k) \frac{e^{-2\pi|\xi+k|\Delta}}{(2m+1)} \left(\tfrac{3}{2}-\sigma\right)^{2m+1}$$

$$= \pi i \sum_{k=-\infty}^{\infty} \operatorname{sgn}(\xi+k) \left[\left(\frac{(2m)!\, e^{-2\pi|\xi+k|(\sigma-1/2)\Delta}}{(2\pi|\xi+k|\Delta)^{2m+1}} \right) - \sum_{j=0}^{2m+1} \frac{\gamma_j\, e^{-2\pi|\xi+k|\Delta}}{(2\pi|\xi+k|\Delta)^{j}} \left(\tfrac{3}{2}-\sigma\right)^{2m+1-j} \right],$$

with $\gamma_j = \frac{(2m)!}{(2m+1-j)!}$, for $0 \leqslant j \leqslant 2m+1$.

From (3.3.47), (3.3.52) and (3.3.55) we find, for $0 < \xi \leqslant 1$, that

$$\widehat{G}^{+}_{\Delta}(\xi) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{k+1}{|\xi+k|} \left[\frac{(2m)!\; e^{-2\pi|\xi+k|(\sigma-1/2)\Delta}}{(2\pi|\xi+k|\Delta)^{2m+1}} - \sum_{j=0}^{2m+1} \frac{\gamma_j\, e^{-2\pi|\xi+k|\Delta}}{(2\pi|\xi+k|\Delta)^{j}} \left(\tfrac{3}{2}-\sigma\right)^{2m+1-j} \right].$$

The change of variables (3.3.46) leads us directly to the expression (3.3.31) for the majorant.

The proof for the minorant follows along the same lines, starting with Vaaler's relation [89, Theorem 9] and the fact that $G^{-}_{\Delta}(k+\frac{1}{2}) = F_{\Delta}(k+\frac{1}{2})$ and $(G^{-}_{\Delta})'(k+\frac{1}{2}) = F'_{\Delta}(k+\frac{1}{2})$ for all $k \in \mathbb{Z}$, we have

$$\widehat{G}^{-}_{\Delta}(\xi) = (1-|\xi|) \sum_{k=-\infty}^{\infty} G^{-}_{\Delta}(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi} + \frac{1}{2\pi i} \operatorname{sgn}(\xi) \sum_{k=-\infty}^{\infty} (G^{-}_{\Delta})'(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi}$$

$$= (1-|\xi|) \sum_{k=-\infty}^{\infty} F_{\Delta}(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi} + \frac{1}{2\pi i} \operatorname{sgn}(\xi) \sum_{k=-\infty}^{\infty} F'_{\Delta}(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi}.$$

One now uses Poisson summation with the pairs (3.3.49) and (3.3.54) to derive that

$$\sum_{k=-\infty}^{\infty} F_{\Delta}(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi}$$

$$= \frac{1}{2} \int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \left(\sum_{k=-\infty}^{\infty} \log\left(\frac{(k+\frac{1}{2})^2+\Delta^2}{(k+\frac{1}{2})^2 + (\alpha-\frac{1}{2})^2\Delta^2} \right) e^{-2\pi i (k+\frac{1}{2})\xi} \right) \mathrm{d}\alpha$$

$$= \frac{1}{2} \int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \left(\sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-2\pi|\xi+k|(\alpha-\frac{1}{2})\Delta} - e^{-2\pi|\xi+k|\Delta}}{|\xi+k|} \right) \mathrm{d}\alpha$$

and

$$\sum_{k=-\infty}^{\infty} F'_{\Delta}(k+\tfrac{1}{2})\, e^{-2\pi i (k+\frac{1}{2})\xi}$$

$$= \int_{\sigma}^{3/2} (\alpha-\sigma)^{2m} \left(\sum_{k=-\infty}^{\infty} \left(\frac{(k+\frac{1}{2})}{(k+\frac{1}{2})^2+\Delta^2} - \frac{(k+\frac{1}{2})}{(k+\frac{1}{2})^2+(\alpha-\frac{1}{2})^2\Delta^2} \right) e^{-2\pi i (k+\frac{1}{2})\xi} \right) \mathrm{d}\alpha$$

$$= \pi i \int_{\sigma}^{3/2} (\alpha - \sigma)^{2m} \left(\sum_{k=-\infty}^{\infty} (-1)^k \operatorname{sgn}(\xi + k) \left(e^{-2\pi(\alpha - \frac{1}{2})|\xi+k|\Delta} - e^{-2\pi|\xi+k|\Delta} \right) \right) \, \mathrm{d}\alpha.$$

The remaining computations are analogous to the majorant case. This concludes the proof of the lemma. □

3.4 The sum over prime powers

The idea for our proof of Theorem 3.1, in the case of odd n, is to replace the functions $f_{n,\sigma}$ in our representation lemma (Lemma 3.8) by appropriate majorants and minorants, apply the Guinand-Weil explicit formula (Lemma 2.7), and then asymptotically evaluate the resulting terms. Our majorants and minorants of exponential type $2\pi\Delta$, denoted here by $m_{\Delta}^{\pm}$, are even functions, and hence the resulting sum over prime powers will appear as

$$\frac{1}{\pi} \sum_{n \geqslant 2} \frac{\Lambda(n)}{\sqrt{n}} \, \widehat{m}_{\Delta}^{\pm} \left(\frac{\log n}{2\pi} \right) \cos(t \log n).$$

The purpose of this section is provide a detailed qualitative study of this expression. In order to ease the flow of the proofs below, we collect several auxiliary calculus and number theory facts in two appendices at the end of the thesis.

3.4.1 The case of the Poisson kernel $f_{-1,\sigma}$

Recall that in Lemma 3.9 we denoted the Poisson kernel by $h_\beta(x) := f_{-1,\sigma}(x) = \frac{\beta}{\beta^2 + x^2}$, by introducing the parameter $\beta = \sigma - \frac{1}{2}$.

Lemma 3.11 (Sum over prime powers I)**.** *Assume the Riemann hypothesis. Let* $0 < \beta < \frac{1}{2}$ *and* $\Delta \geqslant 1$*, and let* $m_{\Delta}^{\pm} = m_{\beta,\Delta}^{\pm}$ *be the extremal functions for the Poisson kernel obtained in Lemma 3.9. Then*

$$\begin{aligned} &\frac{1}{\pi} \sum_{n \geqslant 2} \frac{\Lambda(n)}{\sqrt{n}} \, \widehat{m}_{\Delta}^{+} \left(\frac{\log n}{2\pi} \right) \cos(t \log n) \\ &\qquad \geqslant - \frac{2\beta \, e^{(1-2\beta)\pi\Delta} - 2^{\frac{1}{2}-\beta} \left(\frac{1}{2} + \beta\right)^2 + 2^{\frac{1}{2}+\beta} e^{-4\pi\beta\Delta} \left(\frac{1}{2} - \beta\right)^2}{\left(\frac{1}{4} - \beta^2\right)\left(1 - e^{-2\pi\beta\Delta}\right)^2} + O\left(\frac{\Delta^4}{\beta}\right) \end{aligned} \tag{3.4.1}$$

and

$$\begin{aligned} &\frac{1}{\pi} \sum_{n \geqslant 2} \frac{\Lambda(n)}{\sqrt{n}} \, \widehat{m}_{\Delta}^{-} \left(\frac{\log n}{2\pi} \right) \cos(t \log n) \\ &\qquad \leqslant \frac{2\beta \, e^{(1-2\beta)\pi\Delta} - 2^{\frac{1}{2}-\beta} \left(\frac{1}{2} + \beta\right)^2 + 2^{\frac{1}{2}+\beta} e^{-4\pi\beta\Delta} \left(\frac{1}{2} - \beta\right)^2}{\left(\frac{1}{4} - \beta^2\right)\left(1 + e^{-2\pi\beta\Delta}\right)^2} + O\left(\beta\Delta^4\right). \end{aligned} \tag{3.4.2}$$

Proof. Let $x = e^{2\pi\Delta}$ and note that the sums in (3.4.1) and (3.4.2) only run for $2 \leqslant n \leqslant x$.

Using the explicit description for the Fourier transforms $\widehat{m}_\Delta^{\pm}$ given by (3.3.4) we get

$$\begin{aligned} &\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right)\cos(t\log n) \\ &\qquad = \frac{e^{-2\pi\beta\Delta}}{\left(1\mp e^{-2\pi\beta\Delta}\right)^2}\left(\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{1/2}}\left(\frac{e^{2\pi\beta\Delta}}{n^\beta}-\frac{n^\beta}{e^{2\pi\beta\Delta}}\right)\cos(t\log n)\right). \end{aligned} \tag{3.4.3}$$

In the case of the majorant we use that $\cos(t\log n)\geqslant -1$ in (3.4.3), together with Appendix **B.4**, to get

$$\begin{aligned} \frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^{+}\left(\frac{\log n}{2\pi}\right)\cos(t\log n) &\geqslant -\frac{e^{-2\pi\beta\Delta}}{\left(1-e^{-2\pi\beta\Delta}\right)^2}\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{1/2}}\left(\frac{e^{2\pi\beta\Delta}}{n^\beta}-\frac{n^\beta}{e^{2\pi\beta\Delta}}\right) \\ &= -\frac{e^{-2\pi\beta\Delta}}{\left(1-e^{-2\pi\beta\Delta}\right)^2}\left(\frac{2\beta e^{\pi\Delta}-2^{1/2-\beta}e^{2\pi\beta\Delta}\left(\frac12+\beta\right)^2+2^{1/2+\beta}e^{-2\pi\beta\Delta}\left(\frac12-\beta\right)^2}{\frac14-\beta^2}\right. \\ &\qquad\qquad \left. + O\left(\beta\, e^{2\pi\beta\Delta}\Delta^4\right)\right) \\ &= -\frac{2\beta\, e^{(1-2\beta)\pi\Delta}-2^{1/2-\beta}\left(\frac12+\beta\right)^2+2^{1/2+\beta}e^{-4\pi\beta\Delta}\left(\frac12-\beta\right)^2}{\left(\frac14-\beta^2\right)\left(1-e^{-2\pi\beta\Delta}\right)^2}+O\left(\frac{\Delta^4}{\beta}\right), \end{aligned}$$

where we have used the fact

$$\frac{1}{\left(1-e^{-2\pi\beta\Delta}\right)^2}\leqslant\frac{1}{\left(1-e^{-\beta}\right)^2}\ll\frac{1}{\beta^2}.$$

In the case of the minorant we use that $\cos(t\log n)\leqslant 1$ in (3.4.3), together with Appendix **B.4**, to get

$$\begin{aligned} \frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^{-}\left(\frac{\log n}{2\pi}\right)\cos(t\log n) &\leqslant \frac{e^{-2\pi\beta\Delta}}{\left(1+e^{-2\pi\beta\Delta}\right)^2}\sum_{n\leqslant e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{1/2}}\left(\frac{e^{2\pi\beta\Delta}}{n^\beta}-\frac{n^\beta}{e^{2\pi\beta\Delta}}\right) \\ &= \frac{e^{-2\pi\beta\Delta}}{\left(1+e^{-2\pi\beta\Delta}\right)^2}\left(\frac{2\beta e^{\pi\Delta}-2^{1/2-\beta}e^{2\pi\beta\Delta}\left(\frac12+\beta\right)^2+2^{1/2+\beta}e^{-2\pi\beta\Delta}\left(\frac12-\beta\right)^2}{\frac14-\beta^2}\right. \\ &\qquad\qquad \left. + O\left(\beta\, e^{2\pi\beta\Delta}\Delta^4\right)\right) \\ &= \frac{2\beta\, e^{(1-2\beta)\pi\Delta}-2^{1/2-\beta}\left(\frac12+\beta\right)^2+2^{1/2+\beta}e^{-4\pi\beta\Delta}\left(\frac12-\beta\right)^2}{\left(\frac14-\beta^2\right)\left(1+e^{-2\pi\beta\Delta}\right)^2}+O\left(\beta\Delta^4\right). \end{aligned}$$

This proves the lemma. □

3.4.2 The case of $f_{2m+1,\sigma}$, for $m\geqslant 0$

We now consider the sum over prime powers applied to the extremal functions of exponential type $2\pi\Delta$ for the even functions $f_{2m+1,\sigma}$ defined in (3.1.9). The next lemma collects the required bounds for our purposes.

Lemma 3.12 (Sum over prime powers II)**.** *Assume the Riemann hypothesis. Let* $m \geqslant 0$, $\frac{1}{2} \leqslant \sigma < 1$ *and* $\Delta \geqslant 1$. *Let* $g_\Delta^{\pm} = g_{2m+1,\sigma,\Delta}^{\pm}$ *be the extremal functions for* $f_{2m+1,\sigma}$ *obtained in Lemma 3.10, and let* $c > 0$ *be a given real number. In the region*

$$\pi\Delta(1-\sigma)^2 \geqslant c$$

we have

$$\mp\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{g}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right)\cos(t\log n) \leqslant \frac{(2\sigma-1)(2m)!}{\sigma(1-\sigma)}\,\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \tag{3.4.4}$$

Proof. Again we let $x = e^{2\pi\Delta}$ and note that the sum in (3.4.4) only runs for $2 \leqslant n \leqslant x$. Our idea is to explore the formula (3.3.31). First observe that, for $0 < \xi < \Delta$, we have

$$\sum_{k\neq 0}\frac{|k+1|}{|\xi+k\Delta|}\sum_{j=0}^{2m+1}\frac{\gamma_j\,e^{-2\pi|\xi+k\Delta|}}{(2\pi|\xi+k\Delta|)^j}\left(\tfrac{3}{2}-\sigma\right)^{2m+1-j} \ll_m e^{-2\pi\Delta}. \tag{3.4.5}$$

Using (3.3.31), (3.4.5) and the prime number theorem (it suffices to use the weaker estimate $\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}} \ll x^{1/2}$) we find that

$$\begin{aligned}
&\mp\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{g}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right)\cos(t\log n)\\
&\quad = \mp(2m)!\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left(\sum_{k=-\infty}^{\infty}\frac{(\pm 1)^k\,(k+1)\,e^{-|\log nx^k|(\sigma-1/2)}}{|\log nx^k|^{2m+2}}\right)\cos(t\log n)\\
&\qquad \pm\sum_{j=0}^{2m+1}\gamma_j\left(\tfrac{3}{2}-\sigma\right)^{2m+1-j}\,\mathrm{Re}\left(\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{3/2+it}(\log n)^{j+1}}\right) + O_m\big(x^{-1/2}\big)\\
&\quad = \mp(2m)!\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left(\sum_{k=-\infty}^{\infty}\frac{(\pm 1)^k\,(k+1)\,e^{-|\log nx^k|(\sigma-1/2)}}{|\log nx^k|^{2m+2}}\right)\cos(t\log n) + O_m(1).
\end{aligned}$$

It is now convenient to split the inner sum in the ranges $k \geqslant 0$ and $k \leqslant -2$, and regroup them as

$$\begin{aligned}
&\mp\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{g}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right)\cos(t\log n)\\
&= \mp(2m)!\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\sum_{k=0}^{\infty}(\pm 1)^k\left(\frac{k+1}{(\log nx^k)^{2m+2}\,(nx^k)^{\sigma-1/2}}\right.\\
&\qquad\qquad \left. - \frac{k+1}{\left(\log\frac{x^{k+2}}{n}\right)^{2m+2}\left(\frac{x^{k+2}}{n}\right)^{\sigma-1/2}}\right)\cos(t\log n) + O_m(1).
\end{aligned}$$

Using Appendix **B.3** and (7.1.2), we isolate the term $k = 0$ and get

$$\mp\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{g}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right)\cos(t\log n)$$
$$= \mp(2m)!\sum_{n\leqslant x}\left(\frac{\Lambda(n)}{n^{\sigma}(\log n)^{2m+2}} - \frac{\Lambda(n)}{x^{2\sigma-1}\,n^{1-\sigma}(2\log x-\log n)^{2m+2}}\right)\cos(t\log n) \quad (3.4.6)$$
$$+ O_{m,c}\left(\frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}\right).$$

Observe that the terms

$$\frac{\Lambda(n)}{n^{\alpha}(\log n)^{2m+2}} - \frac{\Lambda(n)}{x^{2\sigma-1}\,n^{1-\sigma}(2\log x-\log n)^{2m+2}}$$

are all nonnegative for $n \leqslant x$, and we can get upper bounds in (3.4.6) by just using the trivial inequality

$$-1 \leqslant \cos(t\log n) \leqslant 1. \quad (3.4.7)$$

Estimate (3.4.4) plainly follows from (3.4.6), (3.4.7) and Appendices **B.1** and **B.2**. □

3.5 Proof of Theorem 3.1 in the case of n odd

In this section we prove Theorem 3.1 in the case of odd $n \geqslant -1$.

3.5.1 The case $n = -1$

Here we keep the notation $\beta = \sigma - \frac{1}{2}$, with $0 < \beta < \frac{1}{2}$. To further simplify notation, let $m_\Delta^{\pm} = m_{\beta,\Delta}^{\pm}$ be the extremal functions for the Poisson kernel obtained in Lemma 3.9. From Lemma 3.8 and Lemma 3.9 we have

$$-\frac{1}{2\pi}\log\frac{t}{2\pi} + \frac{1}{\pi}\sum_{\gamma} m_\Delta^{-}(t-\gamma) + O\left(\frac{1}{t}\right)$$
$$\leqslant S_{-1}(\sigma,t) \quad (3.5.1)$$
$$\leqslant -\frac{1}{2\pi}\log\frac{t}{2\pi} + \frac{1}{\pi}\sum_{\gamma} m_\Delta^{+}(t-\gamma) + O\left(\frac{1}{t}\right).$$

For a fixed $t > 0$, we consider the functions $\ell_\Delta^{\pm}(z) := m_\Delta^{\pm}(t-z)$. Then $\widehat{\ell}_\Delta^{\pm}(\xi) = \widehat{m}_\Delta^{\pm}(-\xi)e^{-2\pi i\xi t}$ and the condition $|\ell_\Delta^{\pm}(s)| \ll (1+|s|)^{-2}$ when $|\mathrm{Re}\, s| \to \infty$ in the strip $|\mathrm{Im}\, s| \leqslant 1$ follows from (3.3.6), (3.3.7), (3.3.8) and an application of the Phragmén-Lindelöf principle. Recalling that $\widehat{m}_\Delta^{\pm}$ are even functions, we apply the Guinand-Weil explicit formula (Lemma 2.7) and find that

$$\sum_{\gamma} m_\Delta^{\pm}(t-\gamma) = \left\{ m_\Delta^{\pm}\left(t - \frac{1}{2i}\right) + m_\Delta^{\pm}\left(t + \frac{1}{2i}\right)\right\} - \frac{1}{2\pi}\widehat{m}_\Delta^{\pm}(0)\log\pi$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^\pm(t-x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)\,\mathrm{d}x \qquad (3.5.2)$$
$$-\frac{1}{\pi}\sum_{n\geqslant 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^\pm\left(\frac{\log n}{2\pi}\right)\cos(t\log n).$$

We now proceed with an asymptotic analysis of each term on the right-hand side of (3.5.2).
1. *First term*: From (3.3.7) and (3.3.8) we see that

$$\left|m_\Delta^+\left(t-\frac{1}{2i}\right)+m_\Delta^+\left(t+\frac{1}{2i}\right)\right| \ll \frac{\Delta^2 e^{\pi\Delta}}{\beta(1+\Delta t)} \qquad (3.5.3)$$

and

$$\left|m_\Delta^-\left(t-\frac{1}{2i}\right)+m_\Delta^-\left(t+\frac{1}{2i}\right)\right| \ll \frac{\beta\Delta^2 e^{\pi\Delta}}{1+\Delta t}. \qquad (3.5.4)$$

2. *Second term*: From (3.3.4) it follows that

$$\widehat{m}_\Delta^+(0)=\pi\left(\frac{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}\right)\ll\frac{1}{\beta} \qquad (3.5.5)$$

and

$$\widehat{m}_\Delta^-(0)=\pi\left(\frac{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}\right)\ll\min\{1,\beta\Delta\}. \qquad (3.5.6)$$

3. *Third term*: Recall that the Poisson kernel $h_\beta(x)=\frac{\beta}{\beta^2+x^2}$ defined in (3.3.1) satisfies $\int_{-\infty}^{\infty} h_\beta(x)\,\mathrm{d}x=\pi$. Note also that for $0<\beta\leqslant\frac{1}{2}$ and $|x|\geqslant 1$ we have

$$h_\beta(x)=\frac{\beta}{\beta^2+x^2}\leqslant\frac{1}{1+x^2}. \qquad (3.5.7)$$

Hence, from (3.3.6), we get

$$\begin{aligned}0&\leqslant\int_{-\infty}^{\infty} m_\Delta^-(x)\,\log(2+|x|)\,\mathrm{d}x\\&\leqslant\int_{-\infty}^{\infty} h_\beta(x)\log(2+|x|)\,\mathrm{d}x\\&=\int_{-1}^{1} h_\beta(x)\log(2+|x|)\,\mathrm{d}x+\int_{|x|\geqslant 1} h_\beta(x)\log(2+|x|)\,\mathrm{d}x=O(1).\end{aligned} \qquad (3.5.8)$$

From (3.5.6), (3.5.7), (3.5.8), and Stirling's formula it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)\,\mathrm{d}x&=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(x)\big(\log t+O(\log(2+|x|))\big)\,\mathrm{d}x\\&=\frac{\log t}{2}\left(\frac{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}\right)+O(1).\end{aligned} \qquad (3.5.9)$$

Similarly, using (3.3.6) and (3.5.5), we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{+}(t-x)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)\,\mathrm{d}x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{+}(x)\big(\log t + O(\log(2+|x|))\big)\,\mathrm{d}x \\ &= \frac{\log t}{2}\left(\frac{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}\right) + O\left(\frac{1}{\beta}\right). \qquad (3.5.10)\end{aligned}$$

4. *Fourth term*: This term was treated in Lemma 3.11.

Final analysis (lower bound): Combining the estimates (3.5.1), (3.5.2), (3.5.4), (3.5.6), (3.5.9), and (3.4.2) we derive that

$$\begin{aligned}S_{-1}(\sigma,t) \geqslant -\Bigg[&\frac{\log t}{\pi}\left(\frac{e^{-2\pi\beta\Delta}}{1+e^{-2\pi\beta\Delta}}\right) \\ &+ \frac{2\beta\, e^{(1-2\beta)\pi\Delta} - 2^{1/2-\beta}\big(\frac{1}{2}+\beta\big)^2 + 2^{1/2+\beta}e^{-4\pi\beta\Delta}\big(\frac{1}{2}-\beta\big)^2}{\pi\big(\frac{1}{4}-\beta^2\big)\big(1+e^{-2\pi\beta\Delta}\big)^2}\Bigg] \qquad (3.5.11) \\ &+ O\left(\frac{\beta\Delta^2 e^{\pi\Delta}}{1+\Delta t}\right) + O\big(\min\{1,\beta\Delta\}\big) + O\big(\beta\Delta^4\big).\end{aligned}$$

Note that in deducing (3.5.11), the term $-(1/2\pi)\log t$ in (3.5.1) cancels with part of the leading term in (3.5.9). We now choose $\pi\Delta = \log\log t$ in (3.5.11), which is essentially the optimal choice. Recalling that $\beta = \sigma - \frac{1}{2}$, this choice yields

$$\begin{aligned}S_{-1}(\sigma,t) &\geqslant -\frac{(\log t)^{2-2\sigma}}{\pi}\left(\frac{1}{\big(1+(\log t)^{1-2\sigma}\big)} + \frac{(2\sigma-1)}{\sigma(1-\sigma)\big(1+(\log t)^{1-2\sigma}\big)^2}\right) \\ &\quad + \frac{2^{1-\sigma}\,\sigma^2 - 2^{\sigma}\,(1-\sigma)^2\,(\log t)^{2-4\sigma}}{\pi\sigma(1-\sigma)\big(1+(\log t)^{1-2\sigma}\big)^2} + O\big((\sigma-\tfrac{1}{2})(\log\log t)^4\big). \qquad (3.5.12) \\ &\geqslant -\frac{(\log t)^{2-2\sigma}}{\pi}\left(\frac{1}{\big(1+(\log t)^{1-2\sigma}\big)} + \frac{(2\sigma-1)}{\sigma(1-\sigma)}\right) + O\big((\sigma-\tfrac{1}{2})(\log\log t)^4\big).\end{aligned}$$

In the last inequality we only dismissed nonnegative terms. Note the fact that $2^{1-\sigma}\,\sigma^2 \geqslant 2^{\sigma}\,(1-\sigma)^2$, for $\frac{1}{2} \leqslant \sigma \leqslant 1$. Finally, notice that in the range (3.1.2) we may use (7.1.2) to transform the error term of (3.5.12) into the error term on the left-hand side of (3.1.3).

Final analysis (upper bound): Combining the estimates (3.5.1), (3.5.2), (3.5.3), (3.5.5), (3.5.10), and (3.4.1) we derive that

$$\begin{aligned}S_{-1}(\sigma,t) \leqslant \Bigg[&\frac{\log t}{\pi}\left(\frac{e^{-2\pi\beta\Delta}}{1-e^{-2\pi\beta\Delta}}\right) \\ &+ \frac{2\beta\, e^{(1-2\beta)\pi\Delta} - 2^{1/2-\beta}\big(\frac{1}{2}+\beta\big)^2 + 2^{1/2+\beta}e^{-4\pi\beta\Delta}\big(\frac{1}{2}-\beta\big)^2}{\pi\big(\frac{1}{4}-\beta^2\big)\big(1-e^{-2\pi\beta\Delta}\big)^2}\Bigg] \qquad (3.5.13) \\ &+ O\left(\frac{\Delta^2 e^{\pi\Delta}}{\beta(1+\Delta t)}\right) + O\left(\frac{1}{\beta}\right) + O\left(\frac{\Delta^4}{\beta}\right).\end{aligned}$$

We now choose $\pi\Delta = \log\log t$ in (3.5.13), which again is essentially the optimal choice. Recalling that $\beta = \sigma - \frac{1}{2}$, this yields

$$\begin{aligned} S_{-1}(\sigma,t) \leqslant \frac{(\log t)^{2-2\sigma}}{\pi} & \left(\frac{1}{\big(1-(\log t)^{1-2\sigma}\big)} + \frac{(2\sigma-1)}{\sigma(1-\sigma)\big(1-(\log t)^{1-2\sigma}\big)^2}\right) \\ & - \frac{\big(2^{1-\sigma}\,\sigma^2 - 2^{\sigma}\,(1-\sigma)^2\,(\log t)^{2-4\sigma}\big)}{\pi\sigma(1-\sigma)\big(1-(\log t)^{1-2\sigma}\big)^2} + O\left(\frac{(\log\log t)^4}{\sigma-\frac{1}{2}}\right) \\ \leqslant \frac{(\log t)^{2-2\sigma}}{\pi} & \left(\frac{1}{\big(1-(\log t)^{1-2\sigma}\big)} + \frac{(2\sigma-1)}{\sigma(1-\sigma)\big(1-(\log t)^{1-2\sigma}\big)^2}\right) + O\left(\frac{(\log\log t)^4}{\sigma-\frac{1}{2}}\right), \end{aligned} \tag{3.5.14}$$

where we have just dismissed a nonpositive term in the last inequality. Observe that

$$\left|1 - \frac{1}{\big(1-(\log t)^{1-2\sigma}\big)^2}\right| \ll \frac{(\log t)^{1-2\sigma}}{\big(1-(\log t)^{1-2\sigma}\big)^2} \ll \frac{1}{(\sigma-\frac{1}{2})^2(\log\log t)^2} \ll \frac{1}{(\sigma-\frac{1}{2})^2(\log\log t)}.$$

Therefore we can rewrite (3.5.14) as

$$\begin{aligned} S_{-1}(\sigma,t) \leqslant \frac{(\log t)^{2-2\sigma}}{\pi} & \left(\frac{1}{\big(1-(\log t)^{1-2\sigma}\big)} + \frac{(2\sigma-1)}{\sigma(1-\sigma)}\right) \\ & + O\left(\frac{(\log t)^{2-2\sigma}}{(\sigma-\frac{1}{2})(1-\sigma)\log\log t}\right) + O\left(\frac{(\log\log t)^4}{\sigma-\frac{1}{2}}\right). \end{aligned} \tag{3.5.15}$$

Again, in the range (3.1.2) we may use (7.1.2) to transform the error term of (3.5.15) into the error term on the right-hand side of (3.1.3). This concludes the proof of the theorem in this case.

3.5.2 The case $n \geqslant 1$

Let $n = 2m+1$, with $m \geqslant 0$. For $\frac{1}{2} \leqslant \sigma < 1$ and $\Delta \geqslant 1$, let $g_{\Delta}^{\pm} = g_{2m+1,\sigma,\Delta}^{\pm}$ be the extremal functions for $f_{2m+1,\sigma}$ obtained in Lemma 3.10.

Case 1: m even. In this case, from Lemma 3.8 and Lemma 3.10 we have

$$\begin{aligned} \frac{1}{2\pi(2m+2)!}\left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log t - \frac{1}{\pi(2m)!}\sum_{\gamma} g_{\Delta}^{+}(t-\gamma) + O_m(1) \leqslant S_{2m+1}(\sigma,t) \\ \leqslant \frac{1}{2\pi(2m+2)!}\left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log t - \frac{1}{\pi(2m)!}\sum_{\gamma} g_{\Delta}^{-}(t-\gamma) + O_m(1). \end{aligned} \tag{3.5.16}$$

As observed in the case of the majorants for the Poisson kernel, it follows from (3.3.25), (3.3.26) and the Phragmén-Lindelöf principle that we can then apply the Guinand-Weil explicit formula (Lemma 2.7) to the functions $z \mapsto g_{\Delta}^{\pm}(t-z)$. This yields

$$\sum_{\gamma} g_{\Delta}^{\pm}(t-\gamma) = \left\{ g_{\Delta}^{\pm}\left(t-\frac{1}{2i}\right) + g_{\Delta}^{\pm}\left(t+\frac{1}{2i}\right)\right\} - \frac{1}{2\pi}\widehat{g}_{\Delta}^{\pm}(0)\log\pi$$

$$
\begin{aligned}
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta^{\pm}(t-x) \,\mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ix}{2}\right) \,\mathrm{d}x \\
&- \frac{1}{\pi} \sum_{n \geqslant 2} \frac{\Lambda(n)}{\sqrt{n}} \,\widehat{g}_\Delta^{\pm}\left(\frac{\log n}{2\pi}\right) \cos(t \log n).
\end{aligned} \tag{3.5.17}
$$

We again proceed with an asymptotic analysis of each of the terms in the last expression.
1. *First term*: The estimate (3.3.26) implies that

$$
\left| g_\Delta^{\pm}\left(t - \frac{1}{2i}\right) + g_\Delta^{\pm}\left(t + \frac{1}{2i}\right) \right| \ll_m \frac{\Delta^2 e^{\pi\Delta}}{1 + \Delta t}. \tag{3.5.18}
$$

2. *Second term*: From (3.3.27), it follows that

$$
\left| \widehat{g}_\Delta^{\pm}(0) \right| \ll_m 1. \tag{3.5.19}
$$

3. *Third term*: Using (3.3.25), (3.3.30), and Stirling's formula we find that

$$
\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta^{\pm}(t-x) \,\mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ix}{2}\right) \,\mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_\Delta^{\pm}(x) \big(\log t + O(\log(2+|x|))\big) \,\mathrm{d}x \\
&= \frac{\log t}{2\pi} \left(\frac{\pi\left(\frac{3}{2} - \sigma\right)^{2m+2}}{(2m+1)(2m+2)} - \frac{1}{\Delta} \int_\sigma^{3/2} (\alpha - \sigma)^{2m} \log\left(\frac{1 \mp e^{-2\pi(\alpha - 1/2)\Delta}}{1 \mp e^{-2\pi\Delta}} \right) \mathrm{d}\alpha \right) \\
&\quad + O_m(1).
\end{aligned} \tag{3.5.20}
$$

4. *Fourth term*: This term was treated in Lemma 3.12.

Final analysis (lower bound): We combine the leftmost inequality in (3.5.16) with estimates (3.5.17), (3.5.18), (3.5.19), (3.5.20), and (3.4.4) to get

$$
\begin{aligned}
S_{2m+1}(\sigma, t) &\geqslant \frac{\log t}{(2m)!\, 2\pi^2 \Delta} \int_\sigma^{3/2} (\alpha - \sigma)^{2m} \log\left(\frac{1 - e^{-2\pi(\alpha - 1/2)\Delta}}{1 - e^{-2\pi\Delta}} \right) \mathrm{d}\alpha \\
&\quad - \frac{(2\sigma - 1)}{\pi\sigma(1-\sigma)} \frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_m(1) + O_m\left(\frac{\Delta^2 e^{\pi\Delta}}{1 + \Delta t} \right) \\
&\quad + O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2 \Delta^{2m+3}} \right) \\
&\geqslant \frac{\log t}{(2m)!\, 2\pi^2 \Delta} \int_\sigma^{3/2} (\alpha - \sigma)^{2m} \log\left(1 - e^{-2\pi(\alpha - 1/2)\Delta}\right) \mathrm{d}\alpha \\
&\quad - \frac{(2\sigma - 1)}{\pi\sigma(1-\sigma)} \frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_m(1) + O_m\left(\frac{\Delta^2 e^{\pi\Delta}}{1 + \Delta t} \right) \\
&\quad + O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2 \Delta^{2m+3}} \right).
\end{aligned} \tag{3.5.21}
$$

Observe that

$$\left|\int_{3/2}^{\infty}(\alpha-\sigma)^{2m}\,\log\big(1\pm e^{-2\pi(\alpha-1/2)\Delta}\big)\,\mathrm{d}\alpha\right| \ll \int_{3/2}^{\infty}(\alpha-\tfrac{1}{2})^{2m}\,e^{-2\pi(\alpha-1/2)\Delta}\,\mathrm{d}\alpha$$
$$= \int_{1}^{\infty}\alpha^{2m}e^{-2\alpha\pi\Delta}\,\mathrm{d}\alpha \ll_m \frac{e^{-\pi\Delta}}{\Delta^{2m+2}} \leqslant \frac{e^{(1-2\sigma)\pi\Delta}}{\Delta^{2m+2}}. \tag{3.5.22}$$

We now choose $\pi\Delta=\log\log t$. Using (3.5.22) and (7.1.2) in (3.5.21) leads us to

$$S_{2m+1}(\sigma,t) \geqslant \frac{\log t}{(2m)!\,2\pi^2\Delta}\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\,\log\big(1-e^{-2\pi(\alpha-1/2)\Delta}\big)\,\mathrm{d}\alpha$$
$$-\frac{(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}}+O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \tag{3.5.23}$$

From monotone convergence and (3.3.51) we have

$$\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\,\log\big(1-e^{-2\pi(\alpha-1/2)\Delta}\big)\,\mathrm{d}\alpha = -\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\left(\sum_{k=1}^{\infty}\frac{e^{-2k\pi(\alpha-1/2)\Delta}}{k}\right)\mathrm{d}\alpha$$
$$= -\sum_{k=1}^{\infty}\frac{1}{k}\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\,e^{-2k\pi(\alpha-1/2)\Delta}\,\mathrm{d}\alpha \tag{3.5.24}$$
$$= -\frac{(2m)!}{(2\pi\Delta)^{2m+1}}\sum_{k=1}^{\infty}\frac{e^{-2k\pi(\sigma-1/2)\Delta}}{k^{2m+2}}.$$

Plugging (3.5.24) into (3.5.23) leads us to

$$S_{2m+1}(\sigma,t) \geqslant -\left(\frac{1}{2^{2m+2}\,\pi}\right)\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{2m+2}}\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2m+2}(\log t)^{(2\sigma-1)k}}+\frac{2\sigma-1}{\sigma(1-\sigma)}\right]$$
$$+O_{m,c}\left(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)^2\,(\log\log t)^{2m+3}}\right).$$

Final analysis (upper bound): We combine the rightmost inequality in (3.5.16) with estimates (3.5.17), (3.5.18), (3.5.19), (3.5.20), and (3.4.4) to get

$$S_{2m+1}(\sigma,t) \leqslant \frac{\log t}{(2m)!\,2\pi^2\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\,\log\left(\frac{1+e^{-2\pi(\alpha-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\mathrm{d}\alpha$$
$$+\frac{(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}}+O_m(1)+O_m\left(\frac{\Delta^2e^{\pi\Delta}}{1+\Delta t}\right)$$
$$+O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right)$$
$$\leqslant \frac{\log t}{(2m)!\,2\pi^2\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\,\log\big(1+e^{-2\pi(\alpha-1/2)\Delta}\big)\,\mathrm{d}\alpha$$
$$+\frac{(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}}+O_m(1)+O_m\left(\frac{\Delta^2e^{\pi\Delta}}{1+\Delta t}\right)$$

$$+ O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \tag{3.5.25}$$

We now choose $\pi\Delta = \log\log t$. Using (3.5.22) and (7.1.2) in (3.5.25) leads us to

$$\begin{aligned} S_{2m+1}(\sigma,t) \leqslant \frac{\log t}{(2m)!\,2\pi^2\Delta}\int_\sigma^\infty (\alpha-\sigma)^{2m}\,\log\left(1+e^{-2\pi(\alpha-1/2)\Delta}\right)\,\mathrm{d}\alpha \\ + \frac{(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_{m,c}\left(\frac{e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \end{aligned} \tag{3.5.26}$$

As in (3.5.24), now using dominated convergence, we have

$$\int_\sigma^\infty (\alpha-\sigma)^{2m}\,\log\left(1+e^{-2\pi(\alpha-1/2)\Delta}\right)\,\mathrm{d}\alpha = \frac{(2m)!}{(2\pi\Delta)^{2m+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,e^{-2k\pi(\sigma-1/2)\Delta}}{k^{2m+2}}. \tag{3.5.27}$$

Finally, plugging (3.5.27) into (3.5.26) gives us

$$\begin{aligned} S_{2m+1}(\sigma,t) \leqslant \left(\frac{1}{2^{2m+2}\,\pi}\right)\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{2m+2}}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{2m+2}(\log t)^{(2\sigma-1)k}} + \frac{2\sigma-1}{\sigma(1-\sigma)}\right] \\ + O_{m,c}\left(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)^2\,(\log\log t)^{2m+3}}\right). \end{aligned}$$

<u>*Case 2: m odd.*</u> In the case of m odd, the roles of the majorant g_Δ^+ and minorant g_Δ^- must be interchanged due to the presence of the factor $(-1)^m$ in the representation lemma (3.2.4). The remaining computations are exactly the same as in the case of m even.

This concludes the proof of Theorem 3.1 in the case of odd n.

3.6 Proof of Theorem 3.1 in the case of n even

In this section we prove Theorem 3.1 in the case of even $n \geqslant 0$. Recall that for integer $j \geqslant 0$ we have defined

$$H_j(x) = \sum_{k=0}^{\infty}\frac{x^k}{(k+1)^j},$$

and *for odd* $n \geqslant -1$ we have defined

$$C^{\pm}_{n,\sigma}(t) = \frac{1}{2^{n+1}\pi}\left(H_{n+1}\Big(\pm(-1)^{(n+1)/2}\,(\log t)^{1-2\sigma}\Big) + \frac{2\sigma-1}{\sigma(1-\sigma)}\right). \tag{3.6.1}$$

Throughout this section let us write

$$\ell_{n,\sigma}(t) := \frac{(\log t)^{2-2\sigma}}{(\log\log t)^n} \quad\text{and}\quad r_{n,\sigma}(t) := \frac{(\log t)^{2-2\sigma}}{(1-\sigma)^2(\log\log t)^n}.$$

3.6.1 The case $n = 0$

We now consider $\frac{1}{2} < \sigma < 1$. To treat the case $n = 0$ we proceed with a variant of the method presented in Section 2.6, in which we only use the lower bound for $S_{-1}(\sigma, t)$ since this is stable under the limit $\sigma \to \frac{1}{2}^{+}$.

Let $c > 0$ be a given real number. In the region $(1-\sigma)^2 \geqslant \frac{c/2}{\log\log t}$ we have already shown that

$$-C^{-}_{1,\sigma}(t)\,\ell_{2,\sigma}(t) + O_c(r_{3,\sigma}(t)) \leqslant S_1(\sigma, t) \leqslant C^{+}_{1,\sigma}(t)\,\ell_{2,\sigma}(t) + O_c(r_{3,\sigma}(t)), \tag{3.6.2}$$

and

$$-C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + O_c(r_{1,\sigma}(t)) \leqslant S_{-1}(\sigma, t). \tag{3.6.3}$$

Error terms estimates. Let (σ, t) be such that $(1-\sigma)^2 \geqslant \frac{c}{\log\log t}$. Observe that, in the set $\{(\sigma, \mu);\ t-1 \leqslant \mu \leqslant t+1\}$, estimates (3.6.2) and (3.6.3) apply (note again the use of the constant $c/2$ instead of c in the domains of these estimates). Then, by the mean value theorem and (3.6.3) we obtain, for $0 \leqslant h \leqslant 1$,

$$\begin{aligned} S(\sigma, t) - S(\sigma, t-h) = h\,S_{-1}(\sigma, t_h^*) &\geqslant -h\,C^{-}_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*) + h\,O_c(r_{1,\sigma}(t_h^*)) \\ &= -h\,C^{-}_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*) + h\,O_c(r_{1,\sigma}(t)), \end{aligned} \tag{3.6.4}$$

where t_h^* is a suitable point in the segment connecting $t-h$ and t. From the explicit expression

$$g(t) := C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) = \frac{1}{\pi}\left(\frac{1}{1+(\log t)^{1-2\sigma}} + \frac{2\sigma-1}{\sigma(1-\sigma)}\right)(\log t)^{2-2\sigma}$$

we observe directly that

$$|g'(t)| \ll \frac{1}{t}$$

and hence, by the mean value theorem, that

$$\left|C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) - C^{-}_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*)\right| \ll r_{1,\sigma}(t). \tag{3.6.5}$$

From (3.6.4) and (3.6.5) it follows that

$$S(\sigma, t) - S(\sigma, t-h) \geqslant -h\,C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + h\,O_c(r_{1,\sigma}(t)). \tag{3.6.6}$$

Integrating and optimizing. Let $\nu = \nu_\sigma(t)$ be a real-valued function such that $0 < \nu \leqslant 1$. For a fixed t, we integrate (3.6.6) with respect to the variable h to get

$$\begin{aligned} S(\sigma, t) &\geqslant \frac{1}{\nu}\int_0^\nu S(\sigma, t-h)\,\mathrm{d}h - \frac{1}{\nu}\left(\int_0^\nu h\,\mathrm{d}h\right) C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + \frac{1}{\nu}\left(\int_0^\nu h\,\mathrm{d}h\right) O_c(r_{1,\sigma}(t)) \\ &= \frac{1}{\nu}\big(S_1(\sigma, t) - S_1(\sigma, t-\nu)\big) - \frac{\nu}{2}\,C^{-}_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + O_c(\nu\,r_{1,\sigma}(t)). \end{aligned}$$

From (3.6.2) we then get

$$\begin{aligned} S(\sigma,t) \geqslant \frac{1}{\nu}\Big[&- C^-_{1,\sigma}(t)\,\ell_{2,\sigma}(t) - C^+_{1,\sigma}(t-\nu)\,\ell_{2,\sigma}(t-\nu) + O_c(r_{3,\sigma}(t)) + O_c(r_{3,\sigma}(t-\nu))\Big] \\ &- \frac{\nu}{2}\, C^-_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + O_c(\nu\, r_{1,\sigma}(t)) \\ = -\Big[&C^-_{1,\sigma}(t) + C^+_{1,\sigma}(t)\Big]\frac{1}{\nu}\,\ell_{2,\sigma}(t) - \frac{\nu}{2}\,C^-_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + O_c\left(\frac{r_{3,\sigma}(t)}{\nu}\right) + O_c(\nu\, r_{1,\sigma}(t)), \end{aligned} \tag{3.6.7}$$

where we have used (3.6.16) in the last passage.

We now choose $\nu = \frac{\lambda_\sigma(t)}{\log\log t}$ in (3.6.7), where $\lambda_\sigma(t) > 0$ is a function to be determined. This yields

$$\begin{aligned} S(\sigma,t) \geqslant &-\left[\Big(C^-_{1,\sigma}(t) + C^+_{1,\sigma}(t)\Big)\frac{1}{\lambda_\sigma(t)} + \frac{C^-_{-1,\sigma}(t)}{2}\,\lambda_\sigma(t)\right]\ell_{1,\sigma}(t) \\ &+ O_c\left(\frac{r_{2,\sigma}(t)}{\lambda_\sigma(t)}\right) + O_c(\lambda_\sigma(t)\, r_{2,\sigma}(t)). \end{aligned}$$

Choosing $\lambda_\sigma(t)$ in order to minimize the expression in brackets, we find that

$$\lambda_\sigma(t) = \left(\frac{2\big(C^-_{1,\sigma}(t) + C^+_{1,\sigma}(t)\big)}{C^-_{-1,\sigma}(t)}\right)^{1/2}. \tag{3.6.8}$$

This leads to the bound

$$\begin{aligned} S(\sigma,t) \geqslant &-\Big[2\big(C^-_{1,\sigma}(t) + C^+_{1,\sigma}(t)\big)\,C^-_{-1,\sigma}(t)\Big]^{1/2}\ell_{1,\sigma}(t) \\ &+ O_c\left(\frac{r_{2,\sigma}(t)}{\lambda_\sigma(t)}\right) + O_c(\lambda_\sigma(t)\, r_{2,\sigma}(t)). \end{aligned} \tag{3.6.9}$$

Finally, using the trivial estimates

$$\frac{1}{\pi}\left(\frac{1}{2} + \frac{2\sigma-1}{\sigma(1-\sigma)}\right) \leqslant C^-_{-1,\sigma}(t) \leqslant \frac{1}{\pi}\left(1 + \frac{2\sigma-1}{\sigma(1-\sigma)}\right),$$

$$\frac{1}{4\pi}\left(1 + \frac{2\sigma-1}{\sigma(1-\sigma)}\right) \leqslant C^-_{1,\sigma}(t) \leqslant \frac{1}{4\pi}\left(\zeta(2) + \frac{2\sigma-1}{\sigma(1-\sigma)}\right),$$

and

$$\frac{1}{4\pi}\left(\frac{3}{4} + \frac{2\sigma-1}{\sigma(1-\sigma)}\right) \leqslant C^+_{1,\sigma}(t) \leqslant \frac{1}{4\pi}\left(1 + \frac{2\sigma-1}{\sigma(1-\sigma)}\right),$$

one can show that $\lambda_\sigma(t)$ defined by (3.6.8) verifies the inequalities

$$\frac{1}{2} \leqslant \lambda_\sigma(t) \leqslant 2\,,$$

which shows that indeed $0 < \nu \leqslant 1$ and allows us to write (3.6.9) in our originally intended form of

$$S(\sigma,t) \geqslant -\Big[2\big(C^-_{1,\sigma}(t) + C^+_{1,\sigma}(t)\big)\,C^-_{-1,\sigma}(t)\Big]^{1/2}\ell_{1,\sigma}(t) + O_c(r_{2,\sigma}(t)).$$

The proof of the upper bound for $S_0(\sigma,t)$ follows along the same lines. Instead of (3.6.4), one would start with the following inequality, valid for $0 \leqslant h \leqslant 1$ and $t_h^* \in [t, t+h]$,

$$S(\sigma, t+h) - S(\sigma,t) = h\, S_{-1}(\sigma, t_h^*) \geqslant -h\, C^-_{-1,\sigma}(t_h^*)\, \ell_{0,\sigma}(t_h^*) + h\, O_c(r_{1,\sigma}(t_h^*)).$$

3.6.2 The case $n \geqslant 2$

Let $\frac{1}{2} \leqslant \sigma < 1$. In this subsection we show how to obtain the bounds for $S_n(\sigma,t)$ from the corresponding bounds for $S_{n-1}(\sigma,t)$ and $S_{n+1}(\sigma,t)$. This interpolation argument explores the smoothness of these functions via the mean value theorem in an optimal way. This extends the material that previously appeared in Section 2.6.

Let us consider here the case of $n/2$ *odd*. The case of $n/2$ *even* follows the exact same outline, with the roles of $C^+_{n,\sigma}(t)$ and $C^-_{n,\sigma}(t)$ interchanged.

Let $c > 0$ be a given real number. In the region $(1-\sigma)^2 \geqslant \frac{c/2}{\log\log t}$ we have already established that

$$\begin{aligned} -C^-_{n+1,\sigma}(t)\, \ell_{n+2,\sigma}(t) + O_{n,c}(r_{n+3,\sigma}(t)) &\leqslant S_{n+1}(\sigma,t) \\ &\leqslant C^+_{n+1,\sigma}(t)\, \ell_{n+2,\sigma}(t) + O_{n,c}(r_{n+3,\sigma}(t)), \end{aligned} \tag{3.6.10}$$

and

$$\begin{aligned} -C^-_{n-1,\sigma}(t)\, \ell_{n,\sigma}(t) + O_{n,c}(r_{n+1,\sigma}(t)) &\leqslant S_{n-1}(\sigma,t) \\ &\leqslant C^+_{n-1,\sigma}(t)\, \ell_{n,\sigma}(t) + O_{n,c}(r_{n+1,\sigma}(t)). \end{aligned} \tag{3.6.11}$$

Error term estimates. Let (σ,t) be such that $(1-\sigma)^2 \geqslant \frac{c}{\log\log t}$. Observe that, in the set $\{(\sigma,\mu);\, t-1 \leqslant \mu \leqslant t+1\}$, estimates (3.6.10) and (3.6.11) apply (note the use of $c/2$ instead of c in the domains of these estimates). Then, by the mean value theorem and (3.6.11) we obtain, for $-1 \leqslant h \leqslant 1$,

$$\begin{aligned} S_n(\sigma,t) - S_n(\sigma, t-h) &= h\, S_{n-1}(\sigma, t_h^*) \\ &\leqslant \big(\chi_{h>0}\, |h|\, C^+_{n-1,\sigma}(t_h^*)\, \ell_{n,\sigma}(t_h^*) + \chi_{h<0}\, |h|\, C^-_{n-1,\sigma}(t_h^*)\, \ell_{n,\sigma}(t_h^*)\big) \\ &\quad + |h|\, O_{n,c}(r_{n+1,\sigma}(t_h^*)) \\ &= \big(\chi_{h>0}\, |h|\, C^+_{n-1,\sigma}(t_h^*)\, \ell_{n,\sigma}(t_h^*) + \chi_{h<0}\, |h|\, C^-_{n-1,\sigma}(t_h^*)\, \ell_{n,\sigma}(t_h^*)\big) \\ &\quad + |h|\, O_{n,c}(r_{n+1,\sigma}(t))\,, \end{aligned} \tag{3.6.12}$$

where t_h^* is a suitable point in the segment connecting $t-h$ and t, and $\chi_{h>0}$ and $\chi_{h<0}$ are the indicator functions of the sets $\{h \in \mathbb{R};\, h > 0\}$ and $\{h \in \mathbb{R};\, h < 0\}$, respectively. We would like to change t_h^* by t in the last line of (3.6.12). For all $k \geqslant 0$ let us define

$$f_k(t) = \frac{1}{(\log t)^{(2\sigma-1)k}} \frac{(\log t)^{2-2\sigma}}{(\log\log t)^n} = \frac{(\log t)^{(k+1)(1-2\sigma)+1}}{(\log\log t)^n}.$$

We shall prove that

$$\left|C^{-}_{n-1,\sigma}(t^*_h)\,\ell_{n,\sigma}(t^*_h) - C^{-}_{n-1,\sigma}(t)\,\ell_{n,\sigma}(t)\right| \ll_n r_{n+1,\sigma}(t). \tag{3.6.13}$$

Using the mean value theorem, we have that

$$\begin{aligned}\left|C^{-}_{n-1,\sigma}(t^*_h)\,\ell_{n,\sigma}(t^*_h) - C^{-}_{n-1,\sigma}(t)\,\ell_{n,\sigma}(t)\right| &\ll_n \frac{1}{(1-\sigma)}\sum_{k=0}^{\infty}\frac{1}{(k+1)^n}\left|f_k(t^*_h)-f_k(t)\right| \\ &= \frac{1}{(1-\sigma)}|t^*_h - t|\sum_{k=0}^{\infty}\frac{1}{(k+1)^n}\left|f_k'(t^*_{h,k})\right| \\ &\ll_n \frac{1}{(1-\sigma)}\sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^n\,t^*_{h,k}\,(\log t^*_{h,k})^{(k+1)(2\sigma-1)}(\log\log t^*_{h,k})^n},\end{aligned} \tag{3.6.14}$$

where, for each $k \geqslant 0$, $t^*_{h,k}$ is a point that belongs to the segment connecting t^*_h and t. Observe now that

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^n\,t^*_{h,k}\,(\log t^*_{h,k})^{(k+1)(2\sigma-1)}(\log\log t^*_{h,k})^n} \\ &\qquad\ll_n \sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^n\,t\,(\log t^*_{h,k})^{(k+1)(2\sigma-1)}(\log\log t)^n} \\ &\qquad\ll \frac{1}{t}\left[\sum_{k=0}^{\infty}\frac{2\sigma-1}{(k+1)^{n-1}(\log(t-1))^{(k+1)(2\sigma-1)}}\right]+\frac{1}{t} \\ &\qquad\ll \frac{1}{t} \;\ll\; \ell_{n+1,\alpha}(t).\end{aligned} \tag{3.6.15}$$

From (3.6.14) and (3.6.15), we arrive at (3.6.13). In a similar way we observe that

$$\left|C^{+}_{n-1,\sigma}(t^*_h)\,\ell_{n,\sigma}(t^*_h) - C^{+}_{n-1,\sigma}(t)\,\ell_{n,\sigma}(t)\right| \ll_n r_{n+1,\sigma}(t). \tag{3.6.16}$$

From (3.6.12), (3.6.13), and (3.6.16) we obtain

$$\begin{aligned}S_n(\sigma,t)-S_n(\sigma,t-h) \leqslant{}& \left(\chi_{h>0}\,|h|\,C^{+}_{n-1,\sigma}(t)\,\ell_{n,\sigma}(t)+\chi_{h<0}\,|h|\,C^{-}_{n-1,\sigma}(t)\,\ell_{n,\sigma}(t)\right) \\ &+|h|\,O_{n,c}(r_{n+1,\sigma}(t)).\end{aligned} \tag{3.6.17}$$

Integrating and optimizing. Let $a := a_{n,\sigma}(t)$ and $b := b_{n,\sigma}(t)$ be real-valued functions, that shall be properly chosen later, satisfying $0 \leqslant a, b \leqslant 1$. In particular, we will be able to choose them in a way that $a+b=1$ at the end. Let us just assume for now that $a+b \geqslant 1$ in the following argument. Let $\nu = \nu_{n,\sigma}(t)$ be a real-valued function such that $0 < \nu \leqslant 1$. For a fixed t, we integrate (3.6.17) with respect to the variable h and find that

$$\begin{aligned}S_n(\sigma,t) \leqslant{}& \frac{1}{(a+b)\nu}\int_{-a\nu}^{b\nu} S_n(\sigma,t-h)\,\mathrm{d}h \\ &+\frac{1}{(a+b)\nu}\left[\int_{-a\nu}^{b\nu}\left(\chi_{h>0}\,|h|\,C^{+}_{n-1,\sigma}(t)+\chi_{h<0}\,|h|\,C^{-}_{n-1,\sigma}(t)\right)\mathrm{d}h\right]\ell_{n,\sigma}(t)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(a+b)\nu}\left[\int_{-a\nu}^{b\nu} |h| \,\mathrm{d}h\right] O_{n,c}(r_{n+1,\sigma}(t)) \\
&= \frac{1}{(a+b)\nu}\Big[S_{n+1}(\sigma, t + a\nu) - S_{n+1}(\sigma, t - b\nu)\Big] \\
&+ \left[\frac{b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)}{2(a+b)}\right] \nu\, \ell_{n,\sigma}(t) + O_{n,c}(\nu\, r_{n+1,\sigma}(t)).
\end{aligned}
$$

Using (3.6.10) and the same error term estimates as in (3.6.13) and (3.6.16) we derive that

$$
\begin{aligned}
S_n(\sigma,t) \leqslant{}& \frac{1}{(a+b)\nu}\Big[C^+_{n+1,\sigma}(t+a\nu)\, \ell_{n+2}(\sigma, t+a\nu) + C^-_{n+1,\sigma}(t-b\nu)\, \ell_{n+2}(\sigma, t-b\nu) \\
&+ O_{n,c}(r_{n+3,\sigma}(t+a\nu)) + O_{n,c}(r_{n+3,\sigma}(t-b\nu))\Big] \\
&+ \left[\frac{b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)}{2(a+b)}\right] \nu\, \ell_{n,\sigma}(t) + O_{n,c}(\nu\, r_{n+1,\sigma}(t)) \qquad (3.6.18) \\
={}& \left[\frac{C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)}{(a+b)}\right] \frac{1}{\nu}\; \ell_{n+2,\sigma}(t) + \left[\frac{b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)}{2(a+b)}\right] \nu\, \ell_{n,\sigma}(t) \\
&+ O_{n,c}\left(\frac{r_{n+3,\sigma}(t)}{\nu}\right) + O_{n,c}(\nu\, r_{n+1,\sigma}(t)).
\end{aligned}
$$

Choosing $\nu = \frac{\lambda_{n,\sigma}(t)}{\log\log t}$ in (3.6.18), where $\lambda_{n,\sigma}(t) > 0$ is a function to be determined (recall that we required $0 < \nu \leqslant 1$), we obtain

$$
\begin{aligned}
S_n(\sigma,t) \leqslant{}& \left\{\left[\frac{C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)}{(a+b)}\right] \frac{1}{\lambda_{n,\sigma}(t)} + \left[\frac{b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)}{2(a+b)}\right] \lambda_{n,\sigma}(t)\right\} \ell_{n+1,\sigma}(t) \\
&+ O_{n,c}\left(\frac{r_{n+2,\sigma}(t)}{\lambda_{n,\sigma}(t)}\right) + O_{n,c}(\lambda_{n,\sigma}(t)\, r_{n+2}(t)).
\end{aligned}
$$

We now choose $\lambda_{n,\sigma}(t) > 0$ to minimize the expression in brackets, which corresponds to the choice

$$
\lambda_{n,\sigma}(t) = \left[\frac{C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)}{(a+b)}\right]^{1/2} \left[\frac{b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)}{2(a+b)}\right]^{-1/2}. \qquad (3.6.19)
$$

This leads to the bound

$$
\begin{aligned}
S_n(\sigma,t) \leqslant{}& 2\left[\frac{\big(C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)\big)\big(b^2 C^+_{n-1,\sigma}(t) + a^2 C^-_{n-1,\sigma}(t)\big)}{2(a+b)^2}\right]^{1/2} \ell_{n+1,\sigma}(t) \\
&+ O_{n,c}\left(\frac{r_{n+2,\sigma}(t)}{\lambda_{n,\sigma}(t)}\right) + O_{n,c}\left(\lambda_{n,\sigma}(t)\, r_{n+2,\sigma}(t)\right).
\end{aligned} \qquad (3.6.20)
$$

We seek to minimize the expression in brackets on the right-hand side of (3.6.20) in the variables a and b. It is easy to see that it only depends on the ratio a/b. If we set $a = bx$, we must minimize the function

$$W(x) = 2\left[\frac{\big(C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)\big)\big(C^+_{n-1,\sigma}(t) + x^2 C^-_{n-1,\sigma}(t)\big)}{2(x+1)^2}\right]^{1/2}.$$

Note that $C^{\pm}_{n-1,\sigma}(t) > 0$ and $C^{\pm}_{n+1,\sigma}(t) > 0$. Such a minimum is obtained when

$$x = C^+_{n-1,\sigma}(t)/C^-_{n-1,\sigma}(t), \tag{3.6.21}$$

leading to the bound

$$\begin{aligned} S_n(\sigma,t) \leqslant & \left[\frac{2\big(C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)\big)\; C^+_{n-1,\sigma}(t)\, C^-_{n-1,\sigma}(t)}{C^+_{n-1,\sigma}(t) + C^-_{n-1,\sigma}(t)}\right]^{1/2} \ell_{n+1,\sigma}(t) \\ & + O_{n,c}\left(\frac{r_{n+2,\sigma}(t)}{\lambda_{n,\sigma}(t)}\right) + O_{n,c}\left(\lambda_{n,\sigma}(t)\, r_{n+2,\sigma}(t)\right). \end{aligned} \tag{3.6.22}$$

We may now set $a + b = 1$. From (3.6.21) we then have the exact values of a and b and expression (3.6.19) yields

$$\lambda_{n,\sigma}(t) = \left[\frac{2\big(C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)\big)\big(C^+_{n-1,\sigma}(t) + C^-_{n-1,\sigma}(t)\big)}{C^+_{n-1,\sigma}(t)\, C^-_{n-1,\sigma}(t)}\right]^{1/2}.$$

In the definition of $C^{\pm}_{n-1,\sigma}(t)$ and $C^{\pm}_{n+1,\sigma}(t)$, given by (3.6.1), we now use the bounds (for $j \geqslant 2$)

$$1 \leqslant H_j(x) \leqslant \zeta(j)$$

for $0 < x < 1$, and

$$1 - \frac{1}{2^j} \leqslant H_j(x) \leqslant 1$$

for $-1 < x < 0$. Together with the fact that $n \geqslant 2$, after some computations one arrives at

$$\frac{1}{2} \leqslant \lambda_{n,\sigma}(t) \leqslant 2.$$

Therefore, if $\log\log t \geqslant 4$, we have $\nu = \frac{\lambda_{n,\sigma}(t)}{\log\log t} \leqslant 1$, as we had originally required. Finally, expression (3.6.22) yields

$$S_n(\sigma,t) \leqslant \left[\frac{2\big(C^+_{n+1,\sigma}(t) + C^-_{n+1,\sigma}(t)\big)\; C^+_{n-1,\sigma}(t)\, C^-_{n-1,\sigma}(t)}{C^+_{n-1,\sigma}(t) + C^-_{n-1,\sigma}(t)}\right]^{1/2} \ell_{n+1,\sigma}(t) + O_{n,c}\left(r_{n+2,\sigma}(t)\right),$$

which concludes the proof in this case. The argument for the lower bound of is $S_n(\sigma,t)$ is entirely symmetric. This completes the proof of Theorem 3.1, when $n \geqslant 2$ is even.

This completes the proof of our Theorem 3.1.

Chapter 4

L-functions and bandlimited approximations

This chapter is comprised of the paper [A3]. We exhibit upper and lower bounds with explicit constants for some objects related to L-functions in the critical strip, under the generalized Riemann hypothesis. This is an extension of Theorem 3.1 to a family of entire L-functions. We also include bounds for the logarithm of these functions. In the final part, we briefly present how to extend the previous result to a general class of L-functions (not necessarily entire L-functions), but only in the critical line, extending Theorem 2.3. This is included in the final part of [A1].

4.1 A general family L-functions

In this section we discuss how to extend the results of the previous chapters to a general family of L-functions in the framework of [56, Chapter 5]. Below we adopt the notation

$$\Gamma_{\mathbb{R}}(z) := \pi^{-z/2}\,\Gamma\left(\frac{z}{2}\right),$$

where Γ is the usual Gamma function. We consider a meromorphic function $L(s,\pi)$ on $\mathbb{C}$ which meets the following requirements (for some positive integer d and some $\vartheta \in [0,1]$). The examples include the Dirichlet L-functions $L(s,\chi)$ for primitive characters χ.

(i) There exists a sequence $\{\lambda_\pi(n)\}_{n\geqslant 1}$ of complex numbers ($\lambda_\pi(1) = 1$) such that the series

$$\sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

converges absolutely to $L(s,\pi)$ on $\{s \in \mathbb{C}\,;\, \operatorname{Re} s > 1\}$.

(ii) For each prime number p, there are complex numbers $\alpha_{1,\pi}(p), \alpha_{2,\pi}(p), \ldots, \alpha_{d,\pi}(p)$ such that $|\alpha_{j,\pi}(p)| \leqslant p^{\vartheta}$, where $0 \leqslant \vartheta \leqslant 1$ is independent of p, and

$$L(s,\pi) = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1},$$

with absolute convergence on the half plane $\{s \in \mathbb{C}; \operatorname{Re} s > 1\}$.

(iii) For some positive integer N and some complex numbers $\mu_1, \mu_2, \ldots, \mu_d$ whose real parts are greater than -1 and such that $\{\mu_1, \mu_2, \ldots, \mu_d\} = \{\overline{\mu_1}, \overline{\mu_2}, \ldots, \overline{\mu_d}\}$, we define the function

$$L(s,\pi_\infty) = N^{s/2} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \mu_j),$$

and the completed L-function by

$$\Lambda(s,\pi) := L(s,\pi_\infty) L(s,\pi),$$

which is a meromorphic function of order 1 that has no poles other than 0 and 1. The points 0 and 1 are poles with the same order $r(\pi) \in \{0, 1, \ldots, d\}$[1]. Furthermore, the function $\Lambda(s,\tilde{\pi}) := \overline{\Lambda(\overline{s},\pi)}$ satisfies the functional equation

$$\Lambda(s,\pi) = \kappa\, \Lambda(1-s,\tilde{\pi})$$

for some unitary complex number κ.

Using (ii), the logarithmic derivative of $L(s,\pi)$ has the expression

$$\frac{L'}{L}(s,\pi) = -\sum_p \sum_{j=1}^{d} \frac{\alpha_{j,\pi}(p)}{p^s} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1} \log p\,,$$

where the right-hand side converges absolutely if Re $s > 1$. This shows that the logarithmic derivative of $L(s,\pi)$ has a Dirichlet series

$$\frac{L'}{L}(s,\pi) = -\sum_{n=2}^{\infty} \frac{\Lambda_\pi(n)}{n^s}, \tag{4.1.1}$$

where $\Lambda_\pi(n) = 0$ if n is not a power of prime and $\Lambda_\pi(p^k) = \sum_{j=1}^{d} \alpha_{j,\pi}(p)^k \log p$ if p is prime and k is a positive integer. If follows that

$$\big|\Lambda_\pi(n)\big| \leqslant d\, \Lambda(n)\, n^{\vartheta}. \tag{4.1.2}$$

In what follows we assume the analogous of the Riemann hypothesis to this family of L-functions.

[1]Except for the assumption $r(\pi) \leqslant d$, we are in the same framework as [56, Chapter 5], where many examples may be found.

Conjecture 4.1 (Generalized Riemann hypothesis)**.** $\Lambda(s,\pi) \neq 0$ *if* $\mathrm{Re}\, s \neq \frac{1}{2}$.

4.2 Behavior in the critical strip: $\log|L(\sigma+it,\pi)|$ and $S_n(\sigma,t,\pi)$

For $t > 0$, let $N(t,\pi)$ denote the number of zeros $\rho_\pi = \beta_\pi + i\gamma_\pi$ of $\Lambda(s,\pi)$ which satisfy $0 \leqslant \beta_\pi \leqslant 1$ and $-t \leqslant \gamma_\pi \leqslant t$, counting multiplicities (zeros with ordinate $\gamma_\pi = \pm t$ are counted with weight $\frac{1}{2}$). When t is not an ordinate of a zero of $\Lambda(s,\pi)$, a standard application of the argument principle gives

$$N(t,\pi) = \frac{1}{\pi}\int_{-t}^{t} \mathrm{Re}\,\frac{L'}{L}(\tfrac{1}{2}+iu,\pi_\infty)\,\mathrm{d}u + S(t,\pi) + S(t,\tilde{\pi}) + 2r(\pi) + O(m),$$

where

$$S(t,\pi) = \frac{1}{\pi}\arg L(\tfrac{1}{2}+it,\pi) = -\frac{1}{\pi}\int_{1/2}^{\infty}\frac{L'}{L}(\alpha+it,\pi)\ \mathrm{d}\alpha$$

and the term $O(m)$ corresponds to the contribution of the poles of $L(s,\pi_\infty)$ when $-1 < \mathrm{Re}\,(\mu_j) \leqslant -\frac{1}{2}$. Generically this contribution is equal to $-2\#\{\mu_j : -1 < \mathrm{Re}\,(\mu_j) < -\frac{1}{2}\} - \#\{\mu_j : \mathrm{Re}\,(\mu_j) = -\frac{1}{2}\}$. If t does correspond to an ordinate of a zero of $\Lambda(s,\pi)$, we define

$$S(t,\pi) = \tfrac{1}{2}\lim_{\varepsilon\to 0}\left\{S(t+\varepsilon,\pi) + S(t-\varepsilon,\pi)\right\}.$$

We extend this definition to the critical strip in the following form. Let $n \geqslant 0$ be an integer, $\frac{1}{2} \leqslant \sigma \leqslant 1$ be a real parameter, and $L(s,\pi)$ be an L-function in the above setting. For $t \in \mathbb{R}$ (and t not coinciding with the ordinate of a zero of $L(s,\pi)$ when $n = 0$) we define the *iterates of the argument function* as

$$S_n(\sigma,t,\pi) := -\frac{1}{\pi}\mathrm{Im}\left\{\frac{i^n}{n!}\int_{\sigma}^{\infty}(\alpha-\sigma)^n\,\frac{L'}{L}(\alpha+it,\pi)\ \mathrm{d}\alpha\right\}. \tag{4.2.1}$$

If t is the ordinate of a zero of $L(s,\pi)$ when $n = 0$ we define

$$S_0(\sigma,t,\pi) := \lim_{\varepsilon\to 0}\frac{S_0(\sigma,t+\varepsilon,\pi) + S_0(\sigma,t-\varepsilon,\pi)}{2}.$$

Using the classical notation, we write $S_n(t,\pi) = S_n(\frac{1}{2},t,\pi)$ for $n \geqslant 0$ and $S_0(t,\pi) = S(t,\pi)$. Differentiating under the integral sign and using integration by parts, one can see that $S_n'(\sigma,t,\pi) = S_{n-1}(\sigma,t,\pi)$ for $t \in \mathbb{R}$ (in the case $n = 1$ we may restrict ourselves to the case when t is not the ordinate of a zero of $L(s,\pi)$). We finally define

$$S_{-1}(\sigma,t,\pi) := \frac{1}{\pi}\mathrm{Re}\,\frac{L'}{L}(\sigma+it,\pi),$$

when t is not the ordinate of a zero of $L(s,\pi)$. We can see that $S_{0,\sigma}'(t,\pi) = S_{-1,\sigma}(t,\pi)$.

As in the case of the Riemann zeta-function, the use of extremal functions allows to obtain bounds for some objects related with L-functions. The estimates that we present here are uniform in all parameters, i.e., only will depend of an especial object called *analytic conductor* of $L(s,\pi)$, defined by

$$C(t,\pi) = N\prod_{j=1}^{d}(|it+\mu_j|+3).$$

For instance, Chandee and Soundararajan [29], under the generalized Riemann hypothesis (GRH), showed for $t>0$

$$\log|L(\tfrac{1}{2}+it,\pi)| \leqslant \left((1+2\vartheta)\frac{\log 2}{2}+o(1)\right)\frac{\log C(t,\pi)}{\log\log C(t,\pi)^{3/d}}. \tag{4.2.2}$$

The terms $o(1)$ above are $O(\log\log\log C(t,\pi)^{3/d}/\log\log C(t,\pi)^{3/d})$, where the constant implicit by the O-notation may depend on n but does not depend on d or N. Although they considered explicitly only the case $t=0$, their proof can be adapted to the general case.

For $n=0$ in (4.2.1), Carneiro, Chandee and Milinovich [17], under GRH, showed for $t>0$

$$|S(t,\pi)| \leqslant \left(\frac{1}{4}+\frac{\vartheta}{2}+o(1)\right)\frac{\log C(t,\pi)}{\log\log C(t,\pi)^{3/d}}, \tag{4.2.3}$$

and for $n=1$, Carneiro and Finder [20], under GRH, showed for $t>0$

$$|S_1(t,\pi)| \leqslant \left((1+2\vartheta)^2\frac{\pi}{48}+o(1)\right)\frac{\log C(t,\pi)}{(\log\log C(t,\pi)^{3/d})^2}. \tag{4.2.4}$$

The terms $o(1)$ above are $O(\log\log\log C(t,\pi)^{3/d}/\log\log C(t,\pi)^{3/d})$.

4.2.1 Main result

The main goal here is to extend the above estimates in the critical strip to a family[2] of entire L-functions assuming GRH. We consider an entire function $L(s,\pi)$ on $\mathbb{C}$ which meets the previous requirements and the following additional conditions:

(ii') We restrict ourselves to the case $\vartheta=0$.

(iii') For $1\leqslant j\leqslant d$ we have $\operatorname{Re}\mu_j\geqslant 0$.

(iii") The function $\Lambda(s,\pi)$ is an entire function of order 1 having no zeros in 0 and 1.

To establish the main result for this family of entire L-functions, analogously as Theorem 3.1, we recall the function H_n defined in (3.1.1) as

$$H_n(x) := \sum_{k=0}^{\infty}\frac{x^k}{(k+1)^n}.$$

[2]The examples include the entire Dirichlet L-functions $L(s,\chi)$ for primitive characters χ. Similar families of entire L-functions are studied in [6, 59].

In particular, when $0 < |x| < 1$ we have that

$$\frac{\log(1 \pm x)}{x} = \pm H_1(\mp x). \tag{4.2.5}$$

Theorem 4.2. *Let $L(s,\pi)$ be an entire L-function satisfying the generalized Riemann hypothesis. Let $c > 0$ be a given real number. Then, for $\frac{1}{2} < \sigma < 1$ and $t \in \mathbb{R}$ in the range*

$$(1-\sigma)^2 \log\log C(t,\pi) \geqslant c,$$

we have the following uniform bounds:
(i) For the logarithm,

$$-M_\sigma^-(t)\frac{(\log C(t,\pi))^{2-2\sigma}}{\log\log C(t,\pi)} + O_c\left(\frac{d\,\mu(\sigma)\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2(\log\log C(t,\pi))^2}\right) \leqslant \log|L(\sigma+it,\pi)|$$
$$\leqslant M_\sigma^+(t)\frac{(\log C(t,\pi))^{2-2\sigma}}{\log\log C(t,\pi)} + O_c\left(\frac{d\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2(\log\log C(t,\pi))^2}\right).$$

(ii) For $n \geqslant -1$ an integer,

$$-M_{n,\sigma}^-(t)\frac{(\log C(t,\pi))^{2-2\sigma}}{(\log\log C(t,\pi))^{n+1}} + O_c\left(\frac{d\,\mu_{n,d}^-(\sigma)\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2(\log\log C(t,\pi))^{n+2}}\right) \leqslant S_n(\sigma,t,\pi)$$
$$\leqslant M_{n,\sigma}^+(t)\frac{(\log C(t,\pi))^{2-2\sigma}}{(\log\log C(t,\pi))^{n+1}} + O_c\left(\frac{d\,\mu_{n,d}^+(\sigma)\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2\,(\log\log C(t,\pi))^{n+2}}\right).$$

The functions appearing above are given by:

- *For the logarithm,*

$$M_\sigma^\pm(t) = \frac{1}{2}\left(H_1\Big(\mp(\log C(t,\pi))^{1-2\sigma}\Big) + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\right) \quad \textit{and} \quad \mu(\sigma) = \frac{|\log(\sigma-\frac{1}{2})|}{\sigma-\frac{1}{2}}.$$

- *For $n = -1$,*

$$M_{-1,\sigma}^\pm(t) = \frac{1}{\pi}\left(H_0\Big(\pm(\log C(t,\pi))^{1-2\sigma}\Big) + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\right) \quad \textit{and} \quad \mu_{-1,d}^\pm(\sigma) = (\sigma-\tfrac{1}{2})^{\mp 1}.$$

- *For $n = 0$,*

$$M_{0,\sigma}^\pm(t) = \Big(2\big(M_{1,\sigma}^+(t) + M_{1,\sigma}^-(t)\big)\,M_{-1,\sigma}^-(t)\Big)^{1/2} \quad \textit{and} \quad \mu_{n,d}^\pm(\sigma) = (2\sigma-1)d + 1.$$

- *For $n \geqslant 1$ odd,*

$$M_{n,\sigma}^\pm(t) = \frac{1}{2^{n+1}\pi}\left(H_{n+1}\Big(\pm(-1)^{(n+1)/2}(\log C(t,\pi))^{1-2\sigma}\Big) + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\right),$$

 and $\mu_{n,d}^\pm(\sigma) = 1$.

- *For $n \geqslant 2$ even,*

$$M^{\pm}_{n,\sigma}(t) = \left(\frac{2\big(M^{+}_{n+1,\sigma}(t) + M^{-}_{n+1,\sigma}(t)\big)\, M^{+}_{n-1,\sigma}(t)\, M^{-}_{n-1,\sigma}(t)}{M^{+}_{n-1,\sigma}(t) + M^{-}_{n-1,\sigma}(t)} \right)^{1/2},$$

and $\mu^{\pm}_{n,d}(\sigma) = (2\sigma - 1)d + 1$.

When $\sigma \to \frac{1}{2}^{+}$ in the above theorem we obtain a sharpened version of (4.2.2), (4.2.3) and (4.2.4) for the case of entire L-functions with improved error terms (a factor $\log\log\log C(t,\pi)^{3/d}$ has been removed). Also, we obtain a sharpened version of a similar result for $S_n(t,\pi)$ with $n \geqslant 2$ (see [18, Theorem 6]), as we will see later.

Furthermore, for a fixed $\frac{1}{2} < \sigma < 1$ we obtain bounds as $C(t,\pi) \to \infty$.

Corollary 4.3. *Let $L(s,\pi)$ be an entire L-function satisfying the generalized Riemann hypothesis and let $n \geqslant -1$. Let $\frac{1}{2} < \sigma < 1$ be a fixed number. Then*

$$\log|L(\sigma + it, \pi)| \leqslant \frac{1}{2}\left(1 + o(1) + d\left(\frac{2\sigma - 1}{\sigma(1-\sigma)} + o(1)\right)\right)\frac{(\log C(t,\pi))^{2-2\sigma}}{\log\log C(t,\pi)},$$

and

$$|S_n(\sigma, t, \pi)| \leqslant \frac{\omega_n}{2^{n+1}\pi}\left(1 + o(1) + d\left(\frac{2\sigma - 1}{\sigma(1-\sigma)} + \mu_{d,\sigma}\, o(1)\right)\right)\frac{(\log C(t,\pi))^{2-2\sigma}}{(\log\log C(t,\pi))^{n+1}}$$

as $C(t,\pi) \to \infty$, where $\omega_n = 1$ and $\mu_{d,\sigma} = 1$ if n is odd, and $\omega_n = \sqrt{2}$ and $\mu_{d,\sigma} = (2\sigma-1)d+1$ if n is even.

4.2.2 Strategy ouline

The proof of Theorem 4.2 follows the same circle of ideas used to prove estimates of Theorem 3.1. First, we show the results for $\log|L(\sigma + it, \pi)|$ and $S_n(\sigma, t, \pi)$, when $n \geqslant -1$ is odd. In these cases, we need three ingredients: the representation lemma for our objects, the Guinand-Weil explicit formula for L-functions, and some extremal bandlimited approximations. Later, we show the results for $S_n(\sigma, t, \pi)$, when $n \geqslant 0$ is even, using our argument of interpolation between $S_{n-1}(\sigma, t, \pi)$ and $S_{n+1}(\sigma, t, \pi)$.

4.3 Representation lemma III

Let $m \geqslant 0$ be an integer and $\frac{1}{2} < \sigma \leqslant 1$ be a real number. In this section we consider the function $f_\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$f_\sigma(x) = \log\left(\frac{1 + x^2}{\left(\sigma - \frac{1}{2}\right)^2 + x^2}\right),$$

and the functions $f_{2m+1,\sigma}$ and $f_{-1,\sigma}$ defined in (3.1.9) and (3.1.10) respectively. The following lemma can be considered as an extension of [14, Eq. (2.1)] and Lemma 3.8, where the case of the Riemann zeta-function was studied. The proof for entire L-functions follows the same outline (see [20, Lemma 4]).

Lemma 4.4 (Representation lemma). *Let $L(s,\pi)$ be an entire L-function satisfying the generalized Riemann hypothesis and $m \geqslant 0$ be an integer. Then, for $\frac{1}{2} < \sigma \leqslant 1$ and $t \in \mathbb{R}$ we have*

(i) For the logarithm,

$$\log|L(\sigma+it,\pi)| = \left(\tfrac{3}{4} - \tfrac{\sigma}{2}\right)\log C(t,\pi) - \frac{1}{2}\sum_{\gamma} f_{\sigma}(t-\gamma) + O(d). \tag{4.3.1}$$

(ii) If $n = 2m+1$, for $m \in \mathbb{Z}_{\geqslant 0}$, then

$$\begin{aligned} S_{2m+1}(\sigma,t,\pi) = \frac{(-1)^m}{2\pi(2m+2)!}\,&\left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log C(t,\pi) \\ &- \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma} f_{2m+1,\sigma}(t-\gamma) + O_m(d). \end{aligned} \tag{4.3.2}$$

(iii) If $n = -1$, then

$$S_{-1}(\sigma,t,\pi) = -\frac{1}{2\pi}\log C(t,\pi) + \frac{1}{\pi}\sum_{\gamma} f_{-1,\sigma}(t-\gamma) + O(d). \tag{4.3.3}$$

The sums in (4.3.1), (4.3.2) *and* (4.3.3) *run over all values of γ such that $\Lambda\left(\frac{1}{2}+i\gamma,\pi\right) = 0$, counted with multiplicity.*

Proof. First, we prove (4.3.1). For $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$ we have

$$\begin{aligned} \log\left|\frac{L(\sigma+it,\pi)}{L(\frac{3}{2}+it,\pi)}\right| &= \log\left|\frac{\Lambda(\sigma+it,\pi)}{\Lambda(\frac{3}{2}+it,\pi)}\right| + \log\left|\frac{N^{(3/2+it)/2}}{N^{(\sigma+it)/2}}\right| \\ &\quad + \sum_{j=1}^{d}\log\left|\frac{\Gamma_{\mathbb{R}}\left(\frac{3}{2}+it+\mu_j\right)}{\Gamma_{\mathbb{R}}(\sigma+it+\mu_j)}\right|. \end{aligned} \tag{4.3.4}$$

We treat each term on the right-hand side of (4.3.4). From Hadamard's factorization formula [56, Theorem 5.6 and Eq. (5.29)], the analyticity of $L(s,\pi)$ and the generalized Riemann hypothesis, it follows that

$$\log\left|\frac{\Lambda(\sigma+it,\pi)}{\Lambda\left(\frac{3}{2}+it,\pi\right)}\right| = -\frac{1}{2}\sum_{\gamma}\log\left(\frac{1+(t-\gamma)^2}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}\right), \tag{4.3.5}$$

where the sum runs over all values of γ such that $\Lambda\big(\frac{1}{2}+i\gamma,\pi\big)=0$, counted with multiplicity. A simple computation of the second term show that

$$\log\left|\frac{N^{(3/2+it)/2}}{N^{(\sigma+it)/2}}\right| = \big(\tfrac{3}{4}-\tfrac{\sigma}{2}\big)\log N. \tag{4.3.6}$$

To analyze the third term, we shall use the Stirling's formula in the form

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s) = \frac{1}{2}\log s + O(1), \tag{4.3.7}$$

which is valid for $\operatorname{Re} s \geqslant \frac{1}{2}$. Since $\operatorname{Re}\mu_j \geqslant 0$, we have

$$\operatorname{Re}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\alpha+\mu_j+it) = \frac{1}{2}\log(|\mu_j+it|+3)+O(1) \tag{4.3.8}$$

uniformly in $\frac{1}{2}\leqslant\alpha\leqslant\frac{3}{2}$, so that

$$\begin{aligned}\log\left|\frac{\Gamma_{\mathbb{R}}\big(\frac{3}{2}+it+\mu_j\big)}{\Gamma_{\mathbb{R}}(\sigma+it+\mu_j)}\right| &= \operatorname{Re}\int_{\sigma}^{3/2}(\log\Gamma_{\mathbb{R}}(\alpha+\mu_j+it))'\,\mathrm{d}\alpha\\ &= \int_{\sigma}^{3/2}\operatorname{Re}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\alpha+\mu_j+it)\,\mathrm{d}\alpha\\ &= \big(\tfrac{3}{4}-\tfrac{\sigma}{2}\big)\log(|\mu_j+it|+3)+O(1).\end{aligned} \tag{4.3.9}$$

For the left-hand side of (4.3.4), note that

$$|\log|L(s,\pi)|| \leqslant d\log\zeta(\text{Re } s) \ll \frac{d}{2^{\operatorname{Re}s}} \tag{4.3.10}$$

for any s with Re $s\geqslant\frac{3}{2}$. Then, we get

$$\log|L(\tfrac{3}{2}+it,\pi)| = O(d). \tag{4.3.11}$$

Finally, using (4.3.5), (4.3.6), (4.3.9) and (4.3.11) in (4.3.4) we obtain for $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{2}$ and $t\in\mathbb{R}$ that

$$\log|L(\sigma+it,\pi)| = \big(\tfrac{3}{4}-\tfrac{\sigma}{2}\big)\log C(t,\pi)-\frac{1}{2}\sum_{\gamma}\log\left(\frac{1+(t-\gamma)^2}{\big(\sigma-\frac{1}{2}\big)^2+(t-\gamma)^2}\right)+O(d). \tag{4.3.12}$$

This yields the desired result. In order to prove (4.3.2), we use integration by parts and (4.3.10) to get

$$S_{2m+1}(\sigma,t,\pi) = \frac{(-1)^m}{\pi(2m)!}\left\{\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\log|L(\alpha+it,\pi)|\,\mathrm{d}\alpha\right\}+O_m(d). \tag{4.3.13}$$

Then, inserting (4.3.12) in (4.3.13) and straightforward computations will imply (4.3.2). Finally, we prove (4.3.3). By the partial fraction descomposition of the logarithmic derivative of $L(s,\pi)$ in [56, Theorem 5.6], we have

$$\frac{L'}{L}(\sigma+it,\pi) = \sum_{\rho}\left(\frac{1}{\sigma+it-\rho}+\frac{1}{\rho}\right) + B - \frac{\log N}{2} - \sum_{j=1}^{d}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\sigma+it+\mu_j),$$

where $\operatorname{Re} B = -\operatorname{Re}\sum_{\rho}\rho^{-1}$. Then, taking the real part of this equation, considering that $\rho = \frac{1}{2}+i\gamma$ and using (4.3.8) we obtain (4.3.3) as required. □

As we already know, the sum over the zeros of $\Lambda(s,\pi)$ is complicated to be evaluated directly. One more time, we replace the functions f_σ, $f_{2m+1,\sigma}$ and $f_{-1,\sigma}$ in Lemma (4.4) by an appropriate majorant or minorant of exponential type. We then apply the following version of the Guinand-Weil explicit formula for L-functions. In our setting of entire L-functions we shall use the following version (the proof of the general version can be found in [20, Lemma 5]).

Lemma 4.5. *Let $L(s,\pi)$ be an entire L-function. Let $h(s)$ be analytic in the strip $|\operatorname{Im} s| < \frac{1}{2}+\varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\operatorname{Re} s| \to \infty$. Then*

$$\begin{aligned}\sum_{\rho} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = \frac{\log N}{2\pi}\,\widehat{h}(0) + \frac{1}{\pi}\sum_{j=1}^{d}\int_{-\infty}^{\infty} h(u)\,\operatorname{Re}\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\,\mathrm{d}u \\ -\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\left\{\Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right) + \overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\},\end{aligned}$$

where the sum runs over all zeros ρ of $\Lambda(s,\pi)$ and the coefficients $\Lambda_\pi(n)$ are defined by (4.1.1)*.*

Remark 4.6. *We highlight that for a general L-function, the explicit formula in Lemma 4.11 contains terms that are difficult to estimate in the critical strip, in comparison with the explicit formula for an entire L-function in Lemma 4.5. For this reason, we can not obtain uniform estimates in the critical strip for a general L-function.*

4.4 Extremal bandlimited approximations III

Since that the functions $f_\sigma, f_{2m+1,\sigma}$ and $f_{-1,\sigma}$ do not verify the required smoothness properties to apply the Guinand-Weil formula 4.5, we replace each of these functions by appropriate extremal majorants and minorants. For the extremal functions of $f_{-1,\sigma}$ and $f_{2m+1,\sigma}$ we use the Lemma 3.9 and Lemma 3.10. For the extremal functions of f_σ, the following lemma shows some properties of these functions.

Lemma 4.7 (Extremal functions for f_σ)**.** *Let $\frac{1}{2} < \sigma < 1$ and $\Delta \geqslant 0.02$ be real numbers and let $\Omega(\sigma) = |\log(\sigma-\frac{1}{2})|$. Then there is a pair of real entire functions $g^{\pm}_{\sigma,\Delta} : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For $x \in \mathbb{R}$ we have

$$-\frac{1}{1+x^2} \ll g^-_{\sigma,\Delta}(x) \leqslant f_\sigma(x) \leqslant g^+_{\sigma,\Delta}(x) \ll \frac{\Omega(\sigma)}{(\sigma-\frac{1}{2})^2+x^2}. \tag{4.4.1}$$

Moreover, for any complex number $z = x+iy$ we have

$$\left|g^-_{\sigma,\Delta}(z)\right| \ll \frac{\Delta^2 e^{2\pi\Delta|y|}}{(1+\Delta|z|)}, \tag{4.4.2}$$

and

$$\left|g^+_{\sigma,\Delta}(z)\right| \ll \frac{\Omega(\sigma)\Delta^2 e^{2\pi\Delta|y|}}{(1+\Delta|z|)}. \tag{4.4.3}$$

(ii) The Fourier transforms of $g^\pm_{\sigma,\Delta}$, denoted by $\widehat{g}^\pm_{\sigma,\Delta}$, are even continuous functions supported on the interval $[-\Delta,\Delta]$. For $0<\xi<\Delta$ these are given by

$$\widehat{g}^\pm_{\sigma,\Delta}(\xi) = \sum_{k=-\infty}^{\infty} (\pm 1)^k \frac{(k+1)}{|\xi+k\Delta|}\Big(e^{-2\pi|\xi+k\Delta|(\sigma-1/2)} - e^{-2\pi|\xi+k\Delta|}\Big). \tag{4.4.4}$$

(iii) At $\xi = 0$ we have

$$\widehat{g}^\pm_{\sigma,\Delta}(0) = 2\pi\big(\tfrac{3}{2}-\sigma\big) - \frac{2}{\Delta}\log\left(\frac{1\mp e^{-(2\sigma-1)\pi\Delta}}{1\mp e^{-2\pi\Delta}}\right). \tag{4.4.5}$$

Proof. The proof of this result follows from [13, Lemma 3.2] (see also [14, Lemma 5-8]). □

Remark 4.8. *In the lemmas above mentioned (Lemma 3.9, Lemma 3.10 and Lemma 4.7) we will consider the hypothesis $\Delta \geqslant 0.02$ instead of $\Delta \geqslant 1$. This is possible because in the proof of these results we only used the fact that $1/\Delta \ll 1$.*

4.5 Proof of Theorem 4.2

4.5.1 Proof of Theorem 4.2: the logarithm and the case of n odd

In order to prove Theorem 4.2, we shall first apply the Guinand-Weil explicit formula to the extremal functions and then perform a careful asymptotic analysis of the terms appearing in the process. We highlight that one of the main technical difficulties of our proof, when compared with results in [17, 20, 29], is in the analysis of the sums over prime powers. To obtain the exact asymptotic behavior of such tough terms we shall need explicit formulas for the Fourier transforms of these extremal functions.

Let $m \geqslant 1$ be an integer, and $c > 0$, $\Delta \geqslant 0.02$ and $\frac{1}{2} < \sigma < 1$ be real numbers such that $(1-\sigma)^2\pi\Delta \geqslant c$. Let $t \in \mathbb{R}$, $\beta = \sigma - \frac{1}{2}$ and let $h^\pm_\Delta(s)$ be any of the six extremal functions referred to in Lemmas 3.9, 3.10 and 4.7. As explained in the previous section, we replace each one of the functions $f_{2m+1,\sigma}$, $f_{-1,\sigma}$ and f_σ by its extremal functions in Lemma 4.4. This means that we must bound the sum $h_\Delta^\pm(t-\gamma)$. If we consider the function $h_t(s) := h_\Delta^\pm(t-s)$, then $\widehat{h}_t(\xi) = \widehat{h}_\Delta^\pm(-\xi)e^{-2\pi i\xi t}$. It follows from (3.3.6), (3.3.7), (3.3.8), (3.3.25), (3.3.26), (4.4.1), (4.4.2), (4.4.3) and an application of the Phragmén-Lindelöf principle that $|h_t(s)| \ll (1+|s|)^{-2}$ when $|\mathrm{Re}\, s| \to \infty$ in the strip $|\mathrm{Im}\, s| \leqslant 1$. Therefore, the function $h_t(s)$ satisfies the hypotheses of Lemma 4.11. By the generalized Riemann hypothesis and the fact that $\widehat{h}_\Delta^\pm$ are even functions we obtain

$$\begin{aligned}\sum_\gamma h_\Delta^\pm(t-\gamma) = \frac{\log N}{2\pi}\,\widehat{h}_\Delta^\pm(0) + \frac{1}{\pi}\sum_{j=1}^{d}\int_{-\infty}^{\infty} h_\Delta^\pm(t-u)\,\mathrm{Re}\,\frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}\big(\tfrac12+\mu_j+iu\big)\,\mathrm{d}u \\ -\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{h}_\Delta^\pm\Big(\frac{\log n}{2\pi}\Big)\Big(\Lambda_\pi(n)\,e^{-it\log n} + \overline{\Lambda_\pi(n)}\,e^{it\log n}\Big),\end{aligned} \tag{4.5.1}$$

where the sum runs over all values of γ such that $\Lambda\big(\frac12+i\gamma,\pi\big) = 0$, counted with multiplicity. We now proceed to analyze asymptotically each term on the right-hand side of (4.5.1).

1. *First term*: The first is given by (3.3.4), (3.3.30) and (4.4.5).

2. *Second term*: We first examine the functions $g_{\sigma,\Delta}^\pm$. It follows from (4.4.1), for any $x \neq 0$, that

$$-\frac{1}{x^2} \ll g_{\sigma,\Delta}^-(x) \leqslant f_\sigma(x) \ll \frac{1}{x^2}.$$

Hence, from (4.4.2), we deduce

$$|g_{\sigma,\Delta}^-(x)| \ll \min\Big\{\frac{1}{x^2}, \Delta^2\Big\}.$$

Then, using (4.3.7) and the fact that $\Delta \geqslant 0.02$, we see that

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty} g_{\sigma,\Delta}^-(t-u)\,\mathrm{Re}\,\frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}\big(\tfrac12+\mu_j+iu\big)\,\mathrm{d}u \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{\sigma,\Delta}^-(t-u)\log\big|\tfrac12+\mu_j+iu\big|\,\mathrm{d}u + O(\Delta^2) \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{\sigma,\Delta}^-(u)\big\{\log(|\mu_j+it|+3) + O(\log(|u|+2))\big\}\,\mathrm{d}u + O(\Delta^2) \\ &\quad= \frac{\log(|\mu_j+it|+3)}{2\pi}\,\widehat{g}_{\sigma,\Delta}^-(0) + O(\Delta^2).\end{aligned} \tag{4.5.2}$$

Similarly, the relation

$$|g_{\sigma,\Delta}^+(x)| \ll \Omega(\sigma)\min\Big\{\frac{1}{x^2}, \Delta^2\Big\}$$

implies that

$$\int_{-\infty}^{\infty} g_{\sigma,\Delta}^+(t-u)\,\mathrm{Re}\,\frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}\big(\tfrac12+\mu_j+iu\big)\,\mathrm{d}u = \frac{\log(|\mu_j+it|+3)}{2\pi}\,\widehat{g}_{\sigma,\Delta}^+(0) + O(\Omega(\sigma)\Delta^2). \tag{4.5.3}$$

We next examine the functions $g^{\pm}_{2m+1,\sigma,\Delta}$. Using (3.3.24) and (4.3.7) we obtain

$$\int_{-\infty}^{\infty} g^{\pm}_{2m+1,\sigma,\Delta}(t-u)\,\mathrm{Re}\,\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\,\mathrm{d}u = \frac{\log(|\mu_j+it|+3)}{2\pi}\,\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}(0) + O_m(1). \tag{4.5.4}$$

Finally, we examine the functions $m^{\pm}_{\beta,\Delta}$. If $0<\beta<\frac{1}{2}$ and $|x|\geqslant 1$ then

$$h_\beta(x) = \frac{\beta}{\beta^2+x^2} \leqslant \frac{1}{1+x^2}.$$

Hence we get from (3.3.6) that

$$\begin{aligned} 0 &\leqslant \int_{-\infty}^{\infty} m^{-}_{\beta,\Delta}(x)\,\log(2+|x|)\,\mathrm{d}x \leqslant \int_{-\infty}^{\infty} h_\beta(x)\log(2+|x|)\,\mathrm{d}x \\ &= \int_{-1}^{1} h_\beta(x)\log(2+|x|)\,\mathrm{d}x + \int_{|x|\geqslant 1} h_\beta(x)\log(2+|x|)\,\mathrm{d}x = O(1), \end{aligned}$$

and using (4.3.7) we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty} m^{-}_{\beta,\Delta}(t-u)\,\mathrm{Re}\,\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\,\mathrm{d}u = \frac{\log(|\mu_j+it|+3)}{2\pi}\,\widehat{m}^{-}_{\beta,\Delta}(0) + O(1). \tag{4.5.5}$$

Similarly, (3.3.6) and (4.3.7) imply

$$\frac{1}{\pi}\int_{-\infty}^{\infty} m^{+}_{\beta,\Delta}(t-u)\,\mathrm{Re}\,\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\,\mathrm{d}u = \frac{\log(|\mu_j+it|+3)}{2\pi}\,\widehat{m}^{+}_{\beta,\Delta}(0) + O\left(\frac{1}{\beta}\right). \tag{4.5.6}$$

3. *Third term*: Let $x=e^{2\pi\Delta}$ and note that this term is a sum that only runs for $2\leqslant n\leqslant x$. We start by examining the functions $g^{\pm}_{\sigma,\Delta}$. Observe first that

$$\sum_{k\neq 0} \frac{|k+1|}{|\xi+k\Delta|} e^{-2\pi|\xi+k\Delta|} \ll e^{-2\pi\Delta}, \tag{4.5.7}$$

when $0<\xi<\Delta$. Using (4.1.2) (note that $\vartheta=0$), (4.4.4), (4.5.7) and the prime number theorem we find that

$$\begin{aligned} &\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{g}^{\pm}_{\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n} + \overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\ &\qquad \leqslant 2\,d\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left|\sum_{k=-\infty}^{\infty}(\pm 1)^k\frac{(k+1)}{|\log nx^k|}\left(e^{-|\log nx^k|(\sigma-1/2)} - e^{-|\log nx^k|}\right)\right| \\ &\qquad \leqslant 2\,d\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left|\sum_{k=-\infty}^{\infty}(\pm 1)^k\frac{(k+1)e^{-|\log nx^k|(\sigma-1/2)}}{|\log nx^k|}\right| + O(d). \end{aligned}$$

It is now convenient to split the inner sum in the ranges $k\geqslant 0$ and $k\leqslant -2$, and regroup them as

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{g}^{\pm}_{\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\
&\leqslant 2\,d\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left|\sum_{k=0}^{\infty}(\pm 1)^k\left(\frac{k+1}{(\log nx^k)\,(nx^k)^{\sigma-1/2}}-\frac{k+1}{\left(\log\frac{x^{k+2}}{n}\right)\left(\frac{x^{k+2}}{n}\right)^{\sigma-1/2}}\right)\right|+O(d).
\end{aligned} \tag{4.5.8}$$

For the function $\widehat{g}^{-}_{\sigma,\Delta}$, using Appendices **A.6**, **B.1** and **B.2** in (4.5.8) we obtain that

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{g}^{-}_{\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\
&\qquad\leqslant 2\,d\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left(\frac{1}{n^{\sigma-1/2}\log n}-\frac{n^{\sigma-1/2}}{(2\log x-\log n)x^{2\sigma-1}}\right)+O(d) \qquad (4.5.9)\\
&\qquad=\frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{\pi\Delta}+O_c\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^2}\right).
\end{aligned}$$

For the function $\widehat{g}^{+}_{\sigma,\Delta}$, we isolate the term $k=0$ and using Appendices **B.1**, **B.2** and **B.3** in (4.5.8) we get

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{g}^{+}_{\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\
&\qquad\leqslant\frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{\pi\Delta}+O_c\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(\sigma-\frac{1}{2})(1-\sigma)^2\Delta^2}\right).
\end{aligned} \tag{4.5.10}$$

We next examine the case $g^{\pm}_{2m+1,\sigma,\Delta}$. As we did in the previous case, using (3.3.31), (4.1.2), (4.5.7) and the prime number theorem it follows that

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\
&\leqslant d\,(2m)!\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left|\sum_{k=0}^{\infty}(\pm 1)^k\left(\frac{k+1}{(\log nx^k)^{2m+2}\,(nx^k)^{\sigma-1/2}}-\frac{k+1}{\left(\log\frac{x^{k+2}}{n}\right)^{2m+2}\left(\frac{x^{k+2}}{n}\right)^{\sigma-1/2}}\right)\right| \\
&\quad+O_m(d).
\end{aligned}$$

We isolate the term $k=0$ and using Appendices **B.1**, **B.2** and **B.3** we get

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{g}^{\pm}_{2m+1,\sigma,\Delta}\left(\frac{\log n}{2\pi}\right)\left(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\right)\right| \\
&\qquad\leqslant\frac{d\,(2m)!\,(2\sigma-1)}{\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}}+O_{m,c}\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right)+O_{m,c}(d).
\end{aligned} \tag{4.5.11}$$

We finally examine the case $m^{\pm}_{\beta,\Delta}$. Note that in this case we have $(\frac{1}{2}-\beta)^2\pi\Delta\geqslant c$. Using the fact that $\widehat{m}^{\pm}_{\beta,\Delta}$ are nonnegative (see (3.3.4)), by (4.1.2) and Appendix **B.4** we have that

$$\begin{aligned} &\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{m}^{\pm}_{\beta,\Delta}\left(\frac{\log n}{2\pi}\right)\Big(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\Big)\right| \\ &\qquad\leqslant \frac{d}{(e^{\pi\beta\Delta}\mp e^{-\pi\beta\Delta})^2}\sum_{n\leqslant x}\frac{\Lambda(n)}{\sqrt{n}}\left(\frac{x^\beta}{n^\beta}-\frac{n^\beta}{x^\beta}\right) \\ &\qquad\leqslant \frac{2\,d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{4}-\beta^2)(1\mp e^{-2\pi\beta\Delta})^2}+O_c\left(\frac{d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{2}-\beta)^2\,\Delta\,(1\mp e^{-2\pi\beta\Delta})^2}\right). \end{aligned} \tag{4.5.12}$$

Therefore, for the function $\widehat{m}^{-}_{\beta,\Delta}$ we obtain in (4.5.12) that

$$\begin{aligned} &\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{m}^{-}_{\beta,\Delta}\left(\frac{\log n}{2\pi}\right)\Big(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\Big)\right| \\ &\qquad\leqslant \frac{2d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{4}-\beta^2)(1+e^{-2\pi\beta\Delta})^2}+O_c\left(\frac{d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{2}-\beta)^2\,\Delta}\right). \end{aligned} \tag{4.5.13}$$

As for the function $\widehat{m}^{+}_{\beta,\Delta}$, considering that

$$\frac{1}{\big(1-e^{-2\pi\beta\Delta}\big)^2}\ll\frac{1}{\big(1-e^{-\beta}\big)^2}\ll\frac{1}{\beta^2}.$$

we have

$$\begin{aligned} &\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\,\widehat{m}^{+}_{\beta,\Delta}\left(\frac{\log n}{2\pi}\right)\Big(\Lambda_\pi(n)\,e^{-it\log n}+\overline{\Lambda_\pi(n)}\,e^{it\log n}\Big)\right| \\ &\qquad\leqslant \frac{2\,d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{4}-\beta^2)(1-e^{-2\pi\beta\Delta})^2}+O_c\left(\frac{d\,e^{(1-2\beta)\pi\Delta}}{\beta\,(\frac{1}{2}-\beta)^2\,\Delta}\right). \end{aligned} \tag{4.5.14}$$

Final analysis for $\log|L(\sigma+it,\pi)|$: We first will prove the upper bound. From Lemma 4.4 and (4.4.1) we get

$$\log|L(\sigma+it,\pi)|\leqslant\big(\tfrac{3}{4}-\tfrac{\sigma}{2}\big)\log C(t,\pi)-\frac{1}{2}\sum_{\gamma}g^{-}_{\sigma,\Delta}(t-\gamma)+O(d). \tag{4.5.15}$$

In other hand, using (4.5.2) and (4.5.9) in (4.5.1) we obtain

$$\begin{aligned} \sum_{\gamma}g^{-}_{\sigma,\Delta}(t-\gamma)\geqslant{}&\frac{\log C(t,\pi)}{2\pi}\widehat{g}^{-}_{\sigma,\Delta}(0)-\frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{\pi\Delta} \\ &+O(d\Delta^2)+O_c\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^2}\right). \end{aligned} \tag{4.5.16}$$

Then, combining (4.4.5), (4.5.16) and (7.1.2) in (4.5.15) we get

$$\log|L(\sigma+it,\pi)| \leqslant \frac{1}{2\pi\Delta}\log\left(\frac{1+e^{-(2\sigma-1)\pi\Delta}}{1+e^{-2\pi\Delta}}\right)\log C(t,\pi) + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{2\pi\Delta} + O_c\bigg(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^2}\bigg).$$

Choosing $\pi\Delta = \log\log C(t,\pi)$[3], we have

$$\frac{1}{2\pi\Delta}\log\left(1+e^{-2\pi\Delta}\right)\log C(t,\pi) \ll \frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^2},$$

and the desired result follows from (4.2.5). The proof of the lower bound is similar, combining (4.4.1), (4.4.5), (4.5.1), (4.5.3), (4.5.10), (7.1.2) and (4.3.1).

Final analysis for $S_{-1}(\sigma,t,\pi)$: Let us first prove the lower bound. From Lemma 4.4 and (3.3.6) we have

$$-\frac{1}{2\pi}\log C(t,\pi) + \frac{1}{\pi}\sum_{\gamma} m^{-}_{\beta,\Delta}(t-\gamma) + O(d) \leqslant S_{-1}(\sigma,t,\pi). \tag{4.5.17}$$

Combining (3.3.4), (4.5.1), (4.5.5), (4.5.13) in (4.5.17) we deduce that

$$S_{-1}(\sigma,t,\pi) \geqslant -\frac{\log C(t,\pi)}{\pi}\left(\frac{e^{-2\pi\beta\Delta}}{1+e^{-2\pi\beta\Delta}}\right) - \frac{2\,d\,\beta\,e^{(1-2\beta)\pi\Delta}}{\pi(\frac{1}{4}-\beta^2)\left(1+e^{-2\pi\beta\Delta}\right)^2} + O_c\left(\frac{d\,\beta\,e^{(1-2\beta)\pi\Delta}}{(\frac{1}{2}-\beta)^2\Delta}\right) + O(d).$$

We now choose $\pi\Delta = \log\log C(t,\pi)$. Recalling that $\beta = \sigma - \frac{1}{2}$, by (7.1.2) this choice yields

$$S_{-1}(\sigma,t,\pi) \geqslant -\frac{(\log C(t,\pi))^{2-2\sigma}}{\pi}\left(\frac{1}{\left(1+(\log C(t,\pi))^{1-2\sigma}\right)} + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)\left(1+(\log C(t,\pi))^{1-2\sigma}\right)^2}\right) + O_c\left(\frac{d\,(\sigma-\frac{1}{2})(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2\log\log C(t,\pi)}\right).$$

Observe that this estimate is actually slightly stronger than the one we proposed in Theorem 4.2. For the proof of the upper bound, as before, combining (3.3.4), (3.3.6), (4.5.1), (4.5.6), (4.5.14), (7.1.2) with (4.3.3), and choosing $\pi\Delta = \log\log C(t,\pi)$ we obtain that

$$S_{-1}(\sigma,t,\pi) \leqslant \frac{(\log C(t,\pi))^{2-2\sigma}}{\pi}\left(\frac{1}{\left(1-(\log C(t,\pi))^{1-2\sigma}\right)} + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)\left(1-(\log C(t,\pi))^{1-2\sigma}\right)^2}\right) + O_c\left(\frac{d\,(\log C(t,\pi))^{2-2\sigma}}{(\sigma-\frac{1}{2})(1-\sigma)^2\log\log C(t,\pi)}\right). \tag{4.5.18}$$

[3]Note that we can choose Δ in this form, since that $\log\log C(t,\pi) \geqslant \log\log 3 > 0.09$ and this implies that we need $\Delta \geqslant 0.028...$

Finally, note that if we write $\theta = \log C(t,\pi)$, then $\theta \geqslant \log 3 > 1$, and therefore

$$\left(1 - \frac{1}{\left(1-\theta^{1-2\sigma}\right)^2}\right) \ll \frac{\theta^{1-2\sigma}}{\left(1-\theta^{1-2\sigma}\right)^2} \ll \frac{1}{(\sigma-\frac{1}{2})^2(\log\theta)^2} \ll \frac{1}{(\sigma-\frac{1}{2})^2(\log\theta)}.$$

By applying this bound in (4.5.18), we obtain the desired result.

Final analysis for $S_{2m+1}(\sigma,t,\pi)$: Let us first consider the case where m is even. We will prove the upper bound. From Lemma 4.4 and (3.3.24) we have that

$$\begin{aligned} S_{2m+1}(\sigma,t,\pi) \leqslant \frac{1}{2\pi(2m+2)!} & \left(\tfrac{3}{2}-\sigma\right)^{2m+2}\log C(t,\pi) \\ & - \frac{1}{\pi(2m)!}\sum_{\gamma} g^{-}_{2m+1,\sigma,\Delta}(t-\gamma) + O_m(d). \end{aligned} \tag{4.5.19}$$

Combining (3.3.30), (4.5.1), (4.5.4), (4.5.11) and (7.1.2) in (4.5.19) we get

$$\begin{aligned} S_{2m+1}(\sigma,t,\pi) \leqslant{} & \frac{\log C(t,\pi)}{(2m)!\,2\pi^2\Delta}\int_{\sigma}^{3/2}(\alpha-\sigma)^{2m}\log\left(\frac{1+e^{-2\pi(\alpha-1/2)\Delta}}{1+e^{-2\pi\Delta}}\right)\mathrm{d}\alpha \\ & + \frac{d\,(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_m(d) + O_{m,c}\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \end{aligned} \tag{4.5.20}$$

We now choose $\pi\Delta = \log\log C(t,\pi)$. Using (7.1.2) in (4.5.20) leads us to

$$\begin{aligned} S_{2m+1}(\sigma,t,\pi) \leqslant{} & \frac{\log C(t,\pi)}{(2m)!\,2\pi^2\Delta}\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\,\log\left(1+e^{-2\pi(\alpha-1/2)\Delta}\right)\mathrm{d}\alpha \\ & + \frac{d\,(2\sigma-1)}{\pi\sigma(1-\sigma)}\frac{e^{(2-2\sigma)\pi\Delta}}{(2\pi\Delta)^{2m+2}} + O_{m,c}\left(\frac{d\,e^{(2-2\sigma)\pi\Delta}}{(1-\sigma)^2\Delta^{2m+3}}\right). \end{aligned}$$

Finally, taking into account that

$$\int_{\sigma}^{\infty}(\alpha-\sigma)^{2m}\log\left(1+e^{-2\pi(\alpha-\frac{1}{2})\Delta}\right)\mathrm{d}\alpha = \frac{(2m)!}{(2\pi\Delta)^{2m+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,e^{-2k\pi(\sigma-\frac{1}{2})\Delta}}{k^{2m+2}},$$

we obtain the desired result. The proof of the lower bound is obtained similarly, combining (3.3.24), (3.3.30), (4.5.1), (4.5.4), (4.5.11), (7.1.2) and (4.3.2). When m is odd, the proof is similar, since only the roles of the majorant $g^{+}_{2m+1,\sigma,\Delta}$ and minorant $g^{-}_{2m+1,\sigma,\Delta}$ are interchanged due to the presence of the factor $(-1)^m$ in Lemma 4.4.

4.5.2 Proof of Theorem 4.2: the case of n even

In order to bound the functions $S_{2m}(\sigma,t,\pi)$ when $m \geqslant 0$ is an integer, we follow a different argument to the case of $S_{2m+1}(\sigma,t,\pi)$. Although we can obtain a representation as in Lemma 4.4 (see Lemma 3.8), it is unknown to find extremal majorants and minorants of exponential type for the associated functions in the representation. Therefore, we follow the same outline as in Section 3.6, where similar functions associated with the Riemann zeta-function were studied. Here we present the necessary changes to adapt the proof in 3.6 for our family of entire L-functions. The main change consists in the suitable use of the mean value theorem, since the analytic conductor is not sufficiently smooth.

Since we assume the generalized Riemann hypothesis and $\frac{1}{2} < \sigma < 1$, we have that $S'_{2m+1}(\sigma,t,\pi) = S_{2m}(\sigma,t,\pi)$ and $S'_{2m}(\sigma,t,\pi) = S_{2m-1}(\sigma,t,\pi)$ for all $t \in \mathbb{R}$. For $n \geqslant 0$ we consider the following functions

$$l_{n,\sigma}(t) := \frac{(\log C(t,\pi))^{2-2\sigma}}{(\log\log C(t,\pi))^n} \quad \text{and} \quad r_{n,\sigma}(t) := \frac{d\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2(\log\log C(t,\pi))^n}.$$

Final analysis for $S(\sigma,t,\pi)$: Let $c > 0$ be a given real number. In the range

$$(1-\sigma)^2 \geqslant \frac{c/16}{\log\log C(t,\pi)}$$

we have already shown that

$$-M^-_{1,\sigma}(t)\,\ell_{2,\sigma}(t) + O_c(r_{3,\sigma}(t)) \leqslant S_1(\sigma,t,\pi) \leqslant M^+_{1,\sigma}(t)\,\ell_{2,\sigma}(t) + O_c(r_{3,\sigma}(t)), \tag{4.5.21}$$

and that

$$-M^-_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) + O_c(r_{1,\sigma}(t)) \leqslant S_{-1}(\sigma,t,\pi). \tag{4.5.22}$$

Let (σ,t) be such that $(1-\sigma)^2 \geqslant \frac{c}{\log\log C(t,\pi)}$. By Appendix **A.7** we have that in the set $\{(\sigma,\mu);\, t-25 \leqslant \mu \leqslant t+25\}$, estimates (4.5.21) and (4.5.22) hold. Then, by the mean value theorem and (4.5.22), we obtain for $0 \leqslant h \leqslant 25$,

$$\begin{aligned} S(\sigma,t,\pi) - S(\sigma,t-h,\pi) &= h\,S_{-1}(\sigma,t_h^*,\pi) \\ &\geqslant -h\,M^-_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*) + h\,O_c(r_{1,\sigma}(t_h^*)) \\ &= -h\,M^-_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*) + h\,O_c(r_{1,\sigma}(t)), \end{aligned} \tag{4.5.23}$$

where t_h^* is a suitable point in the segment connecting $t-h$ and t. We claim that

$$|M^-_{-1,\sigma}(t)\,\ell_{0,\sigma}(t) - M^-_{-1,\sigma}(t_h^*)\,\ell_{0,\sigma}(t_h^*)| \ll d\,\mu_{d,\sigma}, \tag{4.5.24}$$

where $\mu_{d,\sigma} = (2\sigma-1)d + 1$. In order to prove this, we define the function

$$g_1(x) = \frac{1}{\pi}\left(\frac{1}{1+x^{1-2\sigma}} + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\right)x^{2-2\sigma}.$$

Note that $|g_1'(x)| \ll \mu_{\mathrm{d}}$ for $x > 1$, and $g_1(\log C(t,\pi)) = M^-_{-1,\sigma}(t)\,\ell_{0,\sigma}(t)$. The mean value theorem applied to the functions g_1 and the logarithm imply that

$$
\begin{aligned}
\big|g_1(\log C(t,\pi)) - g_1(\log C(t_h^*,\pi))\big| &\ll \mu_{d,\sigma} |\log C(t,\pi) - \log C(t_h^*,\pi)| \\
&\leqslant \mu_{d,\sigma} \sum_{j=1}^{d} \big|\log(|\mu_j + it| + 3) - \log(|\mu_j + it_h^*| + 3)\big| \\
&\ll \mu_{d,\sigma} \sum_{j=1}^{d} \big||\mu_j + it| - |\mu_j + it_h^*|\big| \\
&\leqslant \mu_{d,\sigma} \sum_{j=1}^{d} |t - t_h^*| \ll d\, \mu_{d,\sigma}. \qquad (4.5.25)
\end{aligned}
$$

We thus obtain (4.5.24), and using (7.1.2) we have that

$$
\big|M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) - M^-_{-1,\sigma}(t_h^*)\, \ell_{0,\sigma}(t_h^*)\big| \ll \mu_{d,\sigma}\, r_{1,\sigma}(t). \qquad (4.5.26)
$$

From (4.5.23) and (4.5.26) it follows that

$$
S(\sigma,t,\pi) - S(\sigma,t-h,\pi) \geqslant -h\, M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) + h\, O_c(\mu_{d,\sigma}\, r_{1,\sigma}(t)). \qquad (4.5.27)
$$

Let $\nu = \nu_\sigma(t)$ be a real-valued function such that $0 < \nu \leqslant 25$. For a fixed t, we integrate (4.5.27) with respect to the variable h to obtain

$$
\begin{aligned}
S(\sigma,t,\pi) &\geqslant \frac{1}{\nu}\int_0^\nu S(\sigma,t-h,\pi)\, \mathrm{d}h - \frac{1}{\nu}\left(\int_0^\nu h\, \mathrm{d}h\right) M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) \\
&\quad + \frac{1}{\nu}\left(\int_0^\nu h\, \mathrm{d}h\right) O_c(\mu_{d,\sigma}\, r_{1,\sigma}(t)) \\
&= \frac{1}{\nu}\big(S_1(\sigma,t,\pi) - S_1(\sigma,t-\nu,\pi)\big) - \frac{\nu}{2} M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) + O_c(\nu\, \mu_{d,\sigma}\, r_{1,\sigma}(t)).
\end{aligned}
$$

From (4.5.21) we then get

$$
\begin{aligned}
S(\sigma,t,\pi) &\geqslant \frac{1}{\nu}\Big[- M^-_{1,\sigma}(t)\, \ell_{2,\sigma}(t) - M^+_{1,\sigma}(t-\nu)\, \ell_{2,\sigma}(t-\nu) + O_c(r_{3,\sigma}(t)) + O_c(r_{3,\sigma}(t-\nu))\Big] \\
&\quad - \frac{\nu}{2} M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) + O_c(\nu\, \mu_{d,\sigma}\, r_{1,\sigma}(t)) \\
&= -\Big[M^-_{1,\sigma}(t) + M^+_{1,\sigma}(t)\Big] \frac{1}{\nu}\, \ell_{2,\sigma}(t) - \frac{\nu}{2} M^-_{-1,\sigma}(t)\, \ell_{0,\sigma}(t) \\
&\quad + O_c\left(\frac{\mu_{d,\sigma}\, r_{3,\sigma}(t)}{\nu}\right) + O_c(\nu\, \mu_{d,\sigma}\, r_{1,\sigma}(t)), \qquad (4.5.28)
\end{aligned}
$$

where the following was used

$$
\big|M^+_{1,\sigma}(t)\, \ell_{2,\sigma}(t) - M^+_{1,\sigma}(t-\nu)\, \ell_{2,\sigma}(t-\nu)\big| \ll \mu_{d,\sigma}\, r_{3,\sigma}(t). \qquad (4.5.29)
$$

We now prove (4.5.29). For $x > 0$ define

$$g_2(x) = \frac{1}{4\pi}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2 x^{(2\sigma-1)k}} + \frac{d\,(2\sigma-1)}{\sigma(1-\sigma)}\right)\frac{x^{2-2\sigma}}{(\log x)^2}.$$

Note that $M^{+}_{1,\sigma}(t)\,\ell_{2,\sigma}(t) = g_2(\log C(t,\pi))$. For each $k \geqslant 0$ and $x \geqslant \log 3 > 1$ put

$$f_k(x) = \frac{1}{x^{(2\sigma-1)k}}\frac{x^{2-2\sigma}}{(\log x)^2} = \frac{x^{(k+1)(1-2\sigma)+1}}{(\log x)^2}.$$

Then, for $x > y \geqslant \log 3$ using the mean value theorem, we have that

$$\begin{aligned} |g_2(x) - g_2(y)| &\ll \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\big|f_k(x) - f_k(y)\big| + \frac{d\,(2\sigma-1)}{(1-\sigma)}|f_0(x) - f_0(y)| \\ &= |x-y|\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\big|f_k'(\xi_k)\big| + \frac{d\,(2\sigma-1)}{1-\sigma}\big|f_0'(\xi)\big|\right) \\ &\ll |x-y|\left(\sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^2\,\xi_k^{(k+1)(2\sigma-1)}(\log \xi_k)^2} + \frac{d\,(2\sigma-1)}{1-\sigma}\right), \end{aligned} \tag{4.5.30}$$

where $\xi_k, \xi \in]y, x[$ for each $k \geqslant 0$. Observe now that by the mean value theorem

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^2\,\xi_k^{(k+1)(2\sigma-1)}(\log \xi_k)^2} &\leqslant \sum_{k=0}^{\infty}\frac{((k+1)(2\sigma-1)+1)}{(k+1)^2\,y^{(k+1)(2\sigma-1)}(\log y)^2} \\ &\ll \frac{1}{(\log y)^2}\left[\sum_{k=0}^{\infty}\frac{2\sigma-1}{(k+1)y^{(k+1)(2\sigma-1)}} + 1 + \frac{d\,(2\sigma-1)}{1-\sigma}\right] \\ &\leqslant \frac{1}{(\log y)^2}\left[\sum_{k=0}^{\infty}\frac{2\sigma-1}{y^{(k+1)(2\sigma-1)}} + 1 + \frac{d\,(2\sigma-1)}{1-\sigma}\right] \\ &\ll \frac{\mu_{d,\sigma}}{(1-\sigma)(\log y)^2}. \end{aligned}$$

Then, in (4.5.30), by using a similar idea as in (4.5.25), we obtain

$$\begin{aligned} \Big|g_2(\log C(t,\pi)) - g_2(\log C(t-\nu,\pi))\Big| &\ll \frac{\mu_{d,\sigma}}{(1-\sigma)}\frac{|\log C(t,\pi) - \log C(t-\nu,\pi)|}{(\log\log C(t,\pi))^2} \\ &\ll \frac{d\,\mu_{d,\sigma}\,(\log C(t,\pi))^{2-2\sigma}}{(1-\sigma)^2(\log\log C(t,\pi))^3}. \end{aligned}$$

This proves (4.5.29). We now choose $\nu = \frac{\lambda_\sigma(t)}{\log\log C(t,\pi)}$ in (4.5.28), where $\lambda_\sigma(t) > 0$ is a function to be determined. This yields

$$\begin{aligned} S(\sigma,t) \geqslant -&\left[\left(M^{-}_{1,\sigma}(t) + M^{+}_{1,\sigma}(t)\right)\frac{1}{\lambda_\sigma(t)} + \frac{M^{-}_{-1,\sigma}(t)}{2}\,\lambda_\sigma(t)\right]\ell_{1,\sigma}(t) \\ &+ O_c\left(\frac{\mu_{d,\sigma}\,r_{2,\sigma}(t)}{\lambda_\sigma(t)}\right) + O_c(\mu_{d,\sigma}\,\lambda_\sigma(t)\,r_{2,\sigma}(t)). \end{aligned}$$

The optimal $\lambda_\sigma(t)$ minimizing the expression in brackets is

$$\lambda_\sigma(t) = \left(\frac{2\big(M_{1,\sigma}^{-}(t) + M_{1,\sigma}^{+}(t)\big)}{M_{-1,\sigma}^{-}(t)}\right)^{1/2}.$$

and this leads to the bound

$$\begin{aligned} S(\sigma,t) \geqslant -\Big[2\big(M_{1,\sigma}^{-}(t) + M_{1,\sigma}^{+}(t)\big)\, M_{-1,\sigma}^{-}(t)\Big]^{1/2} \ell_{1,\sigma}(t) \\ + O_c\left(\frac{\mu_{d,\sigma}\, r_{2,\sigma}(t)}{\lambda_\sigma(t)}\right) + O_c(\mu_{d,\sigma}\, \lambda_\sigma(t)\, r_{2,\sigma}(t)). \end{aligned} \tag{4.5.31}$$

Finally, using some estimates for $H_n(x)$, one can show that $\frac{1}{2} \leqslant \lambda_\sigma(t) \leqslant 2$, which implies that indeed $0 < \nu \leqslant 25$, and allows us to write (4.5.31) in our originally intended form of

$$S(\sigma,t) \geqslant -\Big[2\big(M_{1,\sigma}^{-}(t) + M_{1,\sigma}^{+}(t)\big)\, M_{-1,\sigma}^{-}(t)\Big]^{\frac{1}{2}} \ell_{1,\sigma}(t) + O_c(\mu_{d,\sigma}\, r_{2,\sigma}(t)).$$

The proof of the upper bound for $S(\sigma,t)$ follows along the same lines.

Final analysis for $S_{2m}(\sigma,t,\pi)$: The proof of this estimates follows the same outline in §3.6.2. The substantial changes in the use of the mean value theorem are similar with (4.5.25) and (4.5.30).

4.6 Behavior on the critical line $S_n(t,\pi)$: general case

In the previous section we established bounds for $S_n(\sigma,t,\pi)$ for a family of entire L-functions defined in §4.2.1. Our purpose here is to extend the case $\sigma = \frac{1}{2}$ to the general family of L-functions (not necessarily entire) defined in Section 4.1. Essentially we want to establish an extension of Theorem 2.3 to the functions $S_n(t,\pi)$ associated to the general family of L-functions.

Theorem 4.9. *For $n \geqslant 0$, let $C_n^{\pm}$ be the constants defined in Theorem 2.3. Let $L(s,\pi)$ be a L-function satisfying the generalized Riemann hypothesis. Then, for all t > 0 we have*

$$\begin{aligned} -\Big((1+2\vartheta)^{n+1} C_n^{-} + o(1)\Big) \frac{\log C(t,\pi)}{(\log\log C(t,\pi)^{3/d})^{n+1}} \leqslant S_n(t,\pi) \\ \leqslant \Big((1+2\vartheta)^{n+1} C_n^{+} + o(1)\Big) \frac{\log C(t,\pi)}{(\log\log C(t,\pi)^{3/d})^{n+1}}. \end{aligned}$$

The terms $o(1)$ above are $O(\log\log\log C(t,\pi)^{3/d}/\log\log C(t,\pi)^{3/d})$, where the constant implicit by the O-notation may depend on n but does not depend on d or N.

The case $n = 0$ of this theorem was established in [17] and the case $n = 1$ was established in [20].

4.6.1 Sketch of the proof

The proof of Theorem 4.9 follows the same circle of ideas used to prove Theorem 2.3. We only give here a brief account of the proof, indicating the changes that need to be made. Notice that we only need to prove Theorem 4.9 for the case n odd, since the case of $n \geqslant 2$ even follows by reproducing the interpolation argument of Section 2.6.

Let f_n be defined by (2.2.8) - (2.2.9) and consider here the dilated functions

$$\widetilde{f}_n(x) = 2^n f_n\left(\tfrac{x}{2}\right). \tag{4.6.1}$$

The following result is the analogue of Lemma 2.5.

Lemma 4.10 (Representation lemma). *Let $L(s,\pi)$ satisfy the generalized Riemann hypothesis. For each $n \geqslant 0$ and $t > 0$ (and t not coinciding with an ordinate of a zero of $L(s,\pi)$ in the case $n = 0$) we have:*

(i) If $n = 2m$, for $m \in \mathbb{Z}^+$, then

$$S_{2m}(t,\pi) = \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma}\widetilde{f}_{2m}(t-\gamma) \; + \; O(d). \tag{4.6.2}$$

(ii) If $n = 2m+1$, for $m \in \mathbb{Z}^+$, then

$$S_{2m+1}(t,\pi) = \frac{(-1)^m\; 2^{2m+1}}{\pi(2m+2)!}\log C(t,\pi) - \frac{(-1)^m}{\pi(2m)!}\sum_{\gamma}\widetilde{f}_{2m+1}(t-\gamma) \; + \; O(d). \tag{4.6.3}$$

The sums in (4.6.2) and (4.6.3) run over all values γ such that $\Lambda(\frac{1}{2}+i\gamma,\pi) = 0$, counted with multiplicity.

Proof. This follows the outline of the proof of Lemma 2.8 and Lemma 4.4, truncating the integrals (2.3.4) and (2.3.9) in the point 5/2 instead of 3/2, and introducing the test point $5/2 + it$ instead of $3/2 + it$ in (2.3.5) and (2.3.11). This is due to the inequality[4]

$$|\log|L(s,\pi)|| \leqslant d\log\zeta(\mathrm{Re}\ s-1) \ll \frac{d}{2^{\mathrm{Re}\, s}} \tag{4.6.4}$$

for any s with Re $s \geqslant \frac{5}{2}$, (4.1.1) and (4.1.2), in order to better deal with the absolute convergence issues, and ultimately causes the replacement of f_n by the dilated version $\widetilde{f}_n$. Full details are given in [17, Section 4.2] for $n = 0$ and in [20, Lemma 4] for $n = 1$. □

The explicit formula for the general family of L-functions takes the following form (compare with Lemma 4.5).

[4]Since now we consider a general L-function, we have that (4.6.4) remains in the range Re $s \geqslant \frac{5}{2}$, while that in the case of a entire L-function it remains in the range Re $s \geqslant \frac{3}{2}$ (see (4.3.10)). For this reason the dilations (4.6.1) appear in the Lemma 4.10.

Lemma 4.11 (Explicit formula for L-functions)**.** *Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \leqslant \frac{1}{2}+\varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then*

$$\begin{aligned}\sum_{\rho} h\left(\frac{\rho-\frac{1}{2}}{i}\right) &= r(\pi)\left\{h\left(\frac{1}{2i}\right)+h\left(-\frac{1}{2i}\right)\right\} + \frac{\log N}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{d}u \\ &+ \frac{1}{\pi}\sum_{j=1}^{d}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\mathrm{d}u \\ &- \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\left\{\Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right)+\overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} \\ &- \sum_{-1<\mathrm{Re}\,\mu_j<-\frac{1}{2}}\left\{h\left(\frac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\frac{\mu_j+\frac{1}{2}}{i}\right)\right\} \\ &- \frac{1}{2}\sum_{\mathrm{Re}\,\mu_j=-\frac{1}{2}}\left\{h\left(\frac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\frac{\mu_j+\frac{1}{2}}{i}\right)\right\},\end{aligned}$$

where the sum runs over all zeros ρ of $\Lambda(s,\pi)$ and the coefficients $\Lambda_\pi(n)$ are defined by (4.1.1).

Conclusion of the proof

For $n = 2m+1$[5] we have the extremal majorants and minorants of exponential type Δ for $\widetilde{f}_{2m+1}$ given by Lemma 2.8 . These are

$$\widetilde{g}^{+}_{2m+1,\Delta}(z) := 2^{2m+1}g^{+}_{2m+1,2\Delta}(z/2) \quad\text{and}\quad \widetilde{g}^{-}_{2m+1,\Delta}(z) := 2^{2m+1}g^{-}_{2m+1,2\Delta}(z/2).$$

We now replace $\widetilde{f}_{2m+1}$ in (4.6.3) and evaluate using the explicit formula. Let us consider, for instance, the upper bound in the case where m is odd. Letting $h(z) := \widetilde{g}^{+}_{2m+1,\Delta}(t-z)$ we have

$$S_{2m+1}(t,\pi) \leqslant -\frac{2^{2m+1}}{\pi(2m+2)!}\log C(t,\pi) + \frac{1}{\pi(2m)!}\sum_{\gamma} h(\gamma) + O(d). \tag{4.6.5}$$

We evaluate $\sum_\gamma h(\gamma)$ from the explicit formula (Lemma 4.11). From Lemma 2.8 we have

$$\begin{aligned}&\left|r(\pi)\left\{h\left(\tfrac{1}{2i}\right)+h\left(-\tfrac{1}{2i}\right)\right\}\right| \\ &+\left|\sum_{-1<\mathrm{Re}\,\mu_j<-\frac{1}{2}}\left\{h\left(\tfrac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\tfrac{\mu_j+\frac{1}{2}}{i}\right)\right\}+\frac{1}{2}\sum_{\mathrm{Re}\,\mu_j=-\frac{1}{2}}\left\{h\left(\tfrac{-\mu_j-\frac{1}{2}}{i}\right)+h\left(\tfrac{\mu_j+\frac{1}{2}}{i}\right)\right\}\right| \\ &\ll_m d\,\Delta^2\,e^{\pi\Delta}.\end{aligned} \tag{4.6.6}$$

[5] We refer the interested reader to [20], where full details are given for the case $n = 1$.

Using Striling's formula in the form

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(z) = \frac{1}{2}\log(2+z) - \frac{1}{z} + O(1),$$

valid for Re $z > -\frac{1}{2}$, we find that

$$\frac{\log N}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{d}u + \frac{1}{\pi}\sum_{j=1}^{d}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\tfrac{1}{2}+\mu_j+iu\right)\,\mathrm{d}u = \frac{\log C(t,\pi)}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{d}u + O(d). \tag{4.6.7}$$

By Lemma 2.8, the Fourier transform $\widehat{h}(\xi)$ is supported on $[-\Delta,\Delta]$ and is uniformly bounded. Also, by (4.1.2)

$$\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\left\{\Lambda_\pi(n)\,\widehat{h}\left(\frac{\log n}{2\pi}\right)+\overline{\Lambda_\pi(n)}\,\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right\} = O\left(d\sum_{n\leqslant e^{2\pi\Delta}}\Lambda(n)n^{\vartheta-1/2}\right) = O\left(d\,e^{(1+2\vartheta)\pi\Delta}\right), \tag{4.6.8}$$

where the last equality follows by the Prime Number Theorem and summation by parts.

From the computations in (2.5.5) and (2.5.6), together with (4.6.5), (4.6.6), (4.6.7) and (4.6.8) we get

$$S_{2m+1}(t,\pi) \leqslant \frac{C^+_{2m+1}}{(\pi\Delta)^{2m+2}}\log C(t,\pi) + O\Big(e^{-2\pi\Delta}\log C(t,\pi)\Big) + O\left(d\,\Delta^2\,e^{(1+2\vartheta)\pi\Delta}\right).$$

for any $t > 0$ and any $\Delta \geqslant 1$. Choosing

$$\pi\Delta = \max\left\{\frac{\log\log C(t,\pi)^{3/d} - (2m+5)\log\log\log C(t,\pi)^{3/d}}{(1+2\vartheta)}\ ,\ \pi\right\}$$

yields the desired result. The lower bound for m odd is analogous, using the minorant $\widetilde{g}^-_{2m+1,\Delta}$. The upper and lower bounds for m even are also analogous, changing the roles of $\widetilde{g}^+_{2m+1,\Delta}$ and $\widetilde{g}^-_{2m+1,\Delta}$.

Chapter 5

The Riemann zeta-function and the resonance method

This chapter is comprised of the paper [A4]. We obtain new estimates for extreme values of the argument of the Riemann zeta-function and its high moments near the critical line assuming the Riemann hypothesis. The proof follows similar ideas from Bondarenko and Seip [9] in the case of $S(t)$ and $S_1(t)$. Our main tools are certain convolution formulas for the functions $S_n(\sigma, t)$ and a new version of the resonance method of Soundararajan given in [9]. In particular, we obtain new omega results for $S_n(t)$.

5.1 Extreme values for $S_n(\sigma, t)$

5.1.1 Behavior in the critical line

The function $S(t)$ has an intrinsic oscillating character and trying to understand its behaviour is a difficult problem up to this date. By Corollary 3.3 we have, under RH,

$$|S(t)| \leqslant \left(\frac{1}{4} + o(1)\right) \frac{\log t}{\log\log t}, \tag{5.1.1}$$

where $o(1) = 1/\log\log t$. The constant $1/4$ and the order of magnitude $\log t/\log\log t$ are the best known up to date. In particular we obtain that

$$\limsup_{t\to\infty} \left| S(t) \frac{\log\log t}{\log t} \right| \leqslant \frac{1}{4}.$$

On the other hand, Montgomery [73, Theorem 2] established the following omega results, under RH,

$$S(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/2}}{(\log\log t)^{1/2}}\right). \tag{5.1.2}$$

This implies that

$$\limsup_{t\to\infty} S(t)\frac{(\log\log t)^{1/2}}{(\log t)^{1/2}} > 0 \quad \text{and} \quad \liminf_{t\to\infty} S(t)\frac{(\log\log t)^{1/2}}{(\log t)^{1/2}} < 0.$$

It is likely that the estimate (5.1.2) is closer to the behavior of the function $S(t)$ than the estimate (5.1.1). In fact, a heuristic argument by Farmer, Gonek and Hughes [38] suggests that $S(t)$ grows as $(\log t \log\log t)^{1/2}$, in the sense that

$$\limsup_{t\to\infty} \frac{S(t)}{(\log\log t)^{1/2}(\log t)^{1/2}} = \frac{1}{\pi\sqrt{2}}.$$

Similarly, for the case $n = 1$, Theorem 3.1 implies that

$$\limsup_{t\to\infty} S_1(t)\frac{(\log\log t)^2}{\log t} \geqslant -\frac{\pi}{24} \quad \text{and} \quad \liminf_{t\to\infty} S_1(t)\frac{(\log\log t)^2}{\log t} \leqslant \frac{\pi}{48}.$$

Also, Tsang [87, Theorem 5] established, under RH,

$$S_1(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/2}}{(\log\log t)^{3/2}}\right), \tag{5.1.3}$$

and this implies that

$$\limsup_{t\to\infty} S_1(t)\frac{(\log\log t)^{3/2}}{(\log t)^{1/2}} > 0 \quad \text{and} \quad \liminf_{t\to\infty} S_1(t)\frac{(\log\log t)^{3/2}}{(\log t)^{1/2}} < 0.$$

For the case $n \geqslant 2$, using the notation in Theorem 3.1, we have

$$\limsup_{t\to\infty} S_n(t)\frac{(\log\log t)^{n+1}}{\log t} \geqslant -C_n^- \quad \text{and} \quad \liminf_{t\to\infty} S_n(t)\frac{(\log\log t)^{n+1}}{\log t} \leqslant C_n^+,$$

but, to the best of our knowledge, there are no known omega results for $S_n(t)$.

Recently, Bondarenko and Seip [9] used their version of the resonance method with a certain convolution formula for $\zeta(s)$ to produce large values of the Riemann zeta-function on the critical line. Besides, using a convolution formula for $\log\zeta(s)$, they obtained similar results for the functions $S(t)$ and $S_1(t)$. They showed the following theorem.

Theorem 5.1 (cf. Bondarenko and Seip [9]). *Assume the Riemann hypothesis. Let $0 \leqslant \beta < 1$ be a fixed real number. Then there exist two positive constants c_0 and c_1 such that, whenever T is large enough,*

$$\max_{T^\beta \leqslant t \leqslant T} |S(t)| \geqslant c_0 \frac{(\log T)^{1/2}(\log\log\log T)^{1/2}}{(\log\log T)^{1/2}}$$

and

$$\max_{T^\beta \leqslant t \leqslant T} S_1(t) \geqslant c_1 \frac{(\log T)^{1/2}(\log\log\log T)^{1/2}}{(\log\log T)^{3/2}}.$$

Theorem 5.1 implies the following omega results for $S(t)$ and $S_1(t)$:

$$S(t) = \Omega\left(\frac{(\log t\ \log\log\log t)^{1/2}}{(\log\log t)^{1/2}}\right) \quad \text{and} \quad S_1(t) = \Omega_+\left(\frac{(\log t\ \log\log\log t)^{1/2}}{(\log\log t)^{3/2}}\right).$$

This result can be compared with the $\Omega_\pm$ results of Montgomery (5.1.2) and the Ω_+ result by Tsang (5.1.3).

5.1.2 Behavior in the critical strip

In Theorem 3.1 we established bounds for $S_n(\sigma, t)$, where $\frac{1}{2} < \sigma < 1$. In particular, for a fixed number $\frac{1}{2} < \sigma < 1$, under RH, we have that

$$S_n(\sigma, t) = O_{n,\sigma}\left(\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{n+1}}\right),$$

for $n \geqslant 0$. On the other hand, under RH, Tsang [87, Theorem 2 and p. 382] states the following lower bound

$$\sup_{t\in[T,2T]} \pm S(\sigma, t) \geqslant c\,\frac{(\log T)^{1/2}}{(\log\log T)^{1/2}}, \tag{5.1.4}$$

for $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + \frac{1}{\log\log T}$, T sufficiently large and some constant $c > 0$. This result shows extreme values for $S(\sigma, t)$ near the critical line. For the critical strip, a result of Montgomery [73] states that, for a fixed $\frac{1}{2} < \sigma < 1$, we have

$$S(\sigma, t) = \Omega_\pm\left((\sigma - \tfrac{1}{2})^2\,\frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}}\right).$$

5.1.3 Main result

The main result of this chapter is to show lower bounds for $S_n(\sigma, t)$ near the critical line, similar to (5.1.4), for $n \geqslant 0$.

Theorem 5.2. *Assume the Riemann hypothesis. Let $0 \leqslant \beta < 1$ be a fixed number. Let $\sigma > 0$ be a real number and $T > 0$ sufficiently large in the range*

$$\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + \frac{1}{\log\log T}.$$

Then there exists a sequence $\{c_n\}_{n\geqslant 0}$ of positive real numbers with the following property.

1. *If $n = 4m + 1$, for $m \in \mathbb{Z}_{\geqslant 0}$:*

$$\max_{T^\beta \leqslant t \leqslant T} S_n(\sigma, t) \geqslant c_n\,\frac{(\log T)^{1-\sigma}(\log\log\log T)^{\sigma}}{(\log\log T)^{\sigma+n}}.$$

2. *In the other cases:*

$$\max_{T^\beta \leqslant t \leqslant T} |S_n(\sigma,t)| \geqslant c_n \frac{(\log T)^{1-\sigma}(\log\log\log T)^\sigma}{(\log\log T)^{\sigma+n}}.$$

Note that when $\sigma = \frac{1}{2}$ and $n = 0$ or 1, we recover Theorem 5.1. Moreover, we obtain the new omega results on the critical line.

Corollary 5.3. *Assume the Riemann hypothesis. Then*

1. *If $n = 4m+1$, for $m \in \mathbb{Z}_{\geqslant 0}$:*

$$S_n(t) = \Omega_+\left(\frac{(\log t\ \log\log\log t)^{1/2}}{(\log\log t)^{n+1/2}}\right). \tag{5.1.5}$$

2. *In the other cases:*

$$S_n(t) = \Omega\left(\frac{(\log t\ \log\log\log t)^{1/2}}{(\log\log t)^{n+1/2}}\right). \tag{5.1.6}$$

Remark 5.4. *It was pointed out to me by M. Milinovich that: for $n \geqslant 3$, Corollary 5.3 holds without the Riemann hypothesis. Assuming RH, Corollary 5.3 follows immediately from Theorem 5.2. If RH fails, an inequality by Fujii [41, Pag. 6] establishes that there is a zero $\beta_0 + i\gamma_0$ of $\zeta(s)$ with $\beta_0 > 1/2$ and $\gamma_0 > 0$ such that*

$$S_n(t) \geqslant A_n\left(\beta_0 - \frac{1}{2}\right)^2 t^{n-2},$$

for $t > 2\gamma_0$, where A_n is a positive constant. This implies (5.1.5) *and* (5.1.6)*.*

5.1.4 Strategy outline

Our approach is motivated by the ideas of Bondarenko and Seip [9] on the use of their version of the resonance method and a convolution formula for $\log\zeta(s)$. Soundararajan [85] introduced the resonance method to produce large values of the Riemann zeta-function on the critical line and large and small central values of L-functions. Also, this method has been the main tool for finding large values for the Riemann zeta-function, L-functions and other objects related to them, in the critical strip (for instance in [1, 2, 3, 7, 8, 9, 11, 63]).

The resonance method. The main goal in the work of Soundararajan [85] is to produce large values of $|\zeta(\frac{1}{2}+it)|$. The idea of the resonance method is to find a Dirichlet polynomial

$$R(t) = \sum_{m \leqslant N} r(m) m^{-it},$$

which "resonates" with $\zeta(\frac{1}{2}+it)$ and picks out its large values. Precisely, we need to compute the smoothed moments

$$M_1(R,T) = \int_{-\infty}^{\infty} |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t, \quad \text{and}$$

$$M_2(R,T) = \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}+it)|R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t,$$

Here Φ denotes a smooth, nonnegative function, compactly supported in $[1,2]$, with $\Phi(t) \leqslant 1$ for all t, and $\Phi(t) = 1$ for $5/4 \leqslant t \leqslant 7/4$. Plainly

$$\max_{T\leqslant t\leqslant 2T} |\zeta(\tfrac{1}{2}+it)| \geqslant \frac{M_2(R,T)}{M_1(R,T)}.$$

When $N \leqslant T^{1-\varepsilon}$ we may evaluate $M_1(R,T)$ and $M_2(R,T)$ easily. These are two quadratic forms in the unknown coefficients $r(n)$, and the problem thus reduces to maximizing the ratio of these quadratic forms. Solving this optimization problem, Soundararajan obtained good lower bounds for $\max_{T\leqslant t\leqslant 2T} |\zeta(\frac{1}{2}+it)|$.

The use of this Dirichlet polynomial is the principal difference between the works of Soundararajan [85], Bondarenko and Seip [9] and the works of Selberg and Tsang, where they used estimates of high moments to detect large values of Dirichlet series. In contrast to the resonance method of Soundararajan [85], Bondarenko and Seip used significantly larger primes, a longer Dirichlet polynomial, and replaced the use of the function $\Phi(t)$ of Soundararajan with the Gaussian function. This replacement produce the change from the interval $[T,2T]$ to $[T^{\beta},T]$, where the function $|\zeta(\frac{1}{2}+it)|$ is maximized.

The strategy of the proof of our results for $S_n(\sigma,t)$ can be broadly divided into the following three main steps:

Step 1: Some results for $S_n(\sigma,t)$.

The first step is to show bounds for $S_n(\sigma,t)$ and for their moments. Bondarenko and Seip only needed to use the Littlewood's estimate (2.2.3) and bounds of Selberg [82] for the moments of $S(t)$ and $S_1(t)$, assuming the Riemann hypothesis. In our case, we will use a weaker version of Theorem 3.1, to estimate the function $S_n(\sigma,t)$ uniformly in the critical strip. As a simple consequence of this result, we will obtain an estimate for its first moment. Finally, we will extend the convolution formula for $\log\zeta(s)$ given in [87, Lemma 5] for the function $S_n(\sigma,t)$. Although we restrict our attention to a region close to the critical line, we will show the bounds for $S_n(\sigma,t)$ in the critical strip, which may be of interest for other applications.

Step 2: The resonator.

The construction of our resonator is similar to that made by Bondarenko and Seip [9, Section 3]. In particular, when $\sigma = \frac{1}{2}$ we obtain the resonator used by them. A deeper analysis in [9, Lemmas 3 and 4] allows us to show these results for a region close to the critical line. This implies that the main relation between the resonator and the convolution formula of $S_n(\sigma, t)$ will follow immediately in the same way as obtained in the case $\sigma = \frac{1}{2}$ [9, Lemma 7].

Step 3: Proof of Theorem 5.2.

We follow the same outline in the proof of [9, Theorem 2]. We will estimate the error terms in the integral that contains the resonator and the convolution formula of $S_n(\sigma, t)$. The main difference in our proof with that of Bondarenko and Seip is in the choice of the sign for a certain Gaussian kernel. This choice will depend on the remainder of n modulo 4. In particular, this allows to obtain Ω_+ results for $S_n(t)$ when $n = 4m + 1$, for $m \in \mathbb{Z}_{\geqslant 0}$, and Ω results in the other cases.

Remark 5.5. *Throughout the following sections, for $n \geqslant 0$ an integer and $\frac{1}{2} \leqslant \sigma \leqslant 1$ a fixed real number, we extend the functions $t \mapsto S_n(\sigma, t)$ to $\mathbb{R}$ in such a way that $S_n(\sigma, t)$ is an odd function when n is even or is an even function when n is odd.*

5.2 Some results for $S_n(\sigma, t)$

The main goal in this section is to show bounds for the functions $S_n(\sigma, t)$ and some convolution formulas of these functions with certain kernels. Throughout this section we let $n \geqslant 0$ be an integer and $0 < \delta \leqslant \frac{1}{2}$ be a real number.

5.2.1 Bounds for $S_n(\sigma, t)$

We will need a weaker version of Theorem 3.1 to bound the functions $S_n(\sigma, t)$.

Theorem 5.6. *Assume the Riemann hypothesis. We have the uniform bound*

$$S_n(\sigma, t) = O_{n,\delta}\left(\frac{(\log t)^{2-2\sigma}}{(\log\log t)^{n+1}}\right)$$

in $\frac{1}{2} \leqslant \sigma \leqslant 1 - \delta < 1$ and $t > 0$ sufficiently large. In particular, we obtain for all $t \in \mathbb{R}$ that

$$S_n(\sigma, t) = O_{n,\delta}(\log(|t| + 2)). \tag{5.2.1}$$

Proof. It is enough to show when $\sigma > \frac{1}{2}$. For t sufficiently large we have that

$$(1 - \sigma)^2 \log\log t \geqslant \delta^2 \log\log t \geqslant 1.$$

Then, by Theorem 3.1 we have

$$\begin{aligned}\big(-C^-_{n,\sigma}(t)+O_{n,\delta}(1)\big)\frac{(\log\log t)^{2-2\sigma}}{(\log\log t)^{n+1}} &\leqslant S_n(\sigma,t)\\ &\leqslant \big(C^+_{n,\sigma}(t)+O_{n,\delta}(1)\big)\frac{(\log\log t)^{2-2\sigma}}{(\log\log t)^{n+1}},\end{aligned} \tag{5.2.2}$$

where $C^{\pm}_{n,\sigma}(t)$ are positive functions. For $n\geqslant 1$ odd, these functions are given by:

$$C^{\pm}_{n,\sigma}(t)=\frac{1}{2^{n+1}\pi}\left(H_{n+1}\Big(\pm(-1)^{(n+1)/2}(\log t)^{1-2\sigma}\Big)+\frac{2\sigma-1}{\sigma(1-\sigma)}\right), \tag{5.2.3}$$

where

$$H_n(x)=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)^n}.$$

Note that when $m\geqslant 2$, we have the bounds $1-2^{-m}\leqslant H_m(x)\leqslant \zeta(m)$, for $|x|\leqslant 1$. Therefore, we obtain in (5.2.3) for $n\geqslant 1$ odd and t sufficiently large

$$a_{n,\delta}\leqslant C^{\pm}_{n,\sigma}(t)\leqslant b_{n,\delta}, \tag{5.2.4}$$

for some positive constants $a_{n,\delta}$ and $b_{n,\delta}$. Using (5.2.2) we obtain the desired result in this case. For $n\geqslant 2$ even, these functions $C^{\pm}_{n,\sigma}(t)$ are given by:

$$C^{\pm}_{n,\sigma}(t)=\left(\frac{2\big(C^+_{n+1,\sigma}(t)+C^-_{n+1,\sigma}(t)\big)\,C^+_{n-1,\sigma}(t)\,C^-_{n-1,\sigma}(t)}{C^+_{n-1,\sigma}(t)+C^-_{n-1,\sigma}(t)}\right)^{1/2}.$$

Since (5.2.4) holds for $C^{\pm}_{n-1,\sigma}(t)$ and $C^{\pm}_{n+1,\sigma}(t)$, we have a similar estimate for $C^{\pm}_{n,\sigma}(t)$, and this implies the desired result in this case. When $n=0$ we have that

$$C^{\pm}_{0,\sigma}(t)=\Big(2\big(C^+_{1,\sigma}(t)+C^-_{1,\sigma}(t)\big)\,C_{-1,\sigma}(t)\Big)^{1/2},$$

where the function $C_{-1,\sigma}(t)$ is defined by

$$C_{-1,\sigma}(t)=\frac{1}{\pi}\left(\frac{1}{1+(\log t)^{1-2\sigma}}+\frac{2\sigma-1}{\sigma(1-\sigma)}\right).$$

Using (5.2.4) and a simple bound for $C_{-1,\sigma}(t)$, we bound $C^{\pm}_{0,\sigma}(t)$ and we conclude. Thefefore, it follows easily that (5.2.1) is valid for $t\geqslant t_0$ where t_0 is sufficiently large, and using the fact that the functions $S_n(\sigma,t)$ are bounded in $[\frac{1}{2},1-\delta]\times[0,t_0]$ we conclude the proof. □

As a simple consequence we have the following estimate

$$\int_0^T |S_n(\sigma,t)|\,\mathrm{d}t=O_{n,\delta}(T\log T), \tag{5.2.5}$$

uniformly in $\frac{1}{2}\leqslant\sigma\leqslant 1-\delta<1$ and $T\geqslant 2$. Although this estimate is weak, it is sufficient for our purposes. For the case $\sigma = \frac{1}{2}$, better estimates are given by Littlewood [65, Theorem 9 and p. 179] for all $n \geqslant 0$.

5.2.2 Convolution formula

Now, we will obtain convolution formulas for the functions $S_n(\sigma, t)$ with certain kernels. The next lemma was introduced by Selberg [82], and was also used by Tsang to study the functions $S(t)$ and $S_1(t)$ [87, 88]. Since we assume the Riemann hypothesis, the factor that contains the zeros outside the critical line disappears.

Lemma 5.7. *Assume the Riemann hypothesis. Suppose that $\frac{1}{2} \leqslant \sigma \leqslant 2$, and let $K(x+iy)$ be an analytic function in the horizontal strip $\sigma - 2 \leqslant y \leqslant 0$ satisfying the growth estimate*

$$V_\sigma(x) := \max_{\sigma-2\leqslant y\leqslant 0} |K(x+iy)| = O\left(\frac{1}{|x|\log^2|x|}\right)$$

when $|x| \to \infty$. Then for every $t \neq 0$, we have

$$\int_{-\infty}^{\infty} \log\zeta(\sigma + i(t+u))K(u)\,\mathrm{d}u = \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^{\sigma+it}\log m}\widehat{K}\left(\frac{\log m}{2\pi}\right) + O\big(V_\sigma(-t)\big). \tag{5.2.6}$$

Proof. See [87, Lemma 5]. □

It is clear that the above lemma gives a convolution formula for the function $S(\sigma, t)$. To obtain a similar formula for the function $S_n(\sigma, t)$ when $n \geqslant 1$, we need an expression that connects the function $S_n(\sigma, t)$ with $\log\zeta(s)$.

Lemma 5.8. *For $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $t \neq 0$ we have*

$$S_n(\sigma, t) = \frac{1}{\pi}\,\mathrm{Im}\left\{\frac{i^n}{(n-1)!}\int_{\sigma}^{\infty} (\alpha-\sigma)^{n-1}\,\log\zeta(\alpha+it)\,\mathrm{d}\alpha\right\}.$$

Proof. This follows from Lemma 3.7 and integration by parts. □

Using this expression we obtain the following convolution formula. This generalizes Tsang's conditional formula in [88] (or [9, Eq. (10)].

Proposition 5.9. *Assume the Riemann hypothesis and the same conditions for the function $K(x+iy)$ as in Lemma 5.7. Suppose further that K is an even real-valued function (or odd real-valued function). Then for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $t \neq 0$, we have*

$$\int_{-\infty}^{\infty} S_n(\sigma, t+s)\,K(s)\,\mathrm{d}r = \frac{1}{\pi}\mathrm{Im}\left\{i^n\sum_{m=2}^{\infty}\frac{\Lambda(m)}{m^{\sigma+it}(\log m)^{n+1}}\,\widehat{K}\left(\frac{\log m}{2\pi}\right)\right\} + O_n\big(V_{1/2}(t) + ||K||_1\big).$$

Proof. For the case $n = 0$, we only need to take imaginary parts in (5.2.6). For $n \geqslant 1$, by Lemma 5.8 we get

$$S_n(\sigma, t) = \frac{1}{\pi} \operatorname{Im} \left\{ \frac{i^n}{(n-1)!} \int_\sigma^2 (\alpha - \sigma)^{n-1} \log \zeta(\alpha + it) \, \mathrm{d}\alpha \right\} + O_n(1).$$

Plugging this in Lemma 5.7 we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} S_n(\sigma, t+s)\, K(s)\, \mathrm{d}s \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Im} \left\{ \frac{i^n}{(n-1)!} \int_\sigma^2 (\alpha - \sigma)^{n-1} \log \zeta(\alpha + i(t+s)) \, \mathrm{d}\alpha \right\} K(s)\, \mathrm{d}s + O_n\big(||K||_1\big) \\
&= \frac{1}{\pi} \operatorname{Im} \left\{ \frac{i^n}{(n-1)!} \int_\sigma^2 (\alpha - \sigma)^{n-1} \left(\int_{-\infty}^{\infty} \log \zeta(\alpha + i(t+s)) K(s)\, \mathrm{d}s \right) \mathrm{d}\alpha \right\} + O_n\big(||K||_1\big) \\
&= \frac{1}{\pi} \operatorname{Im} \left\{ \frac{i^n}{(n-1)!} \int_\sigma^2 (\alpha - \sigma)^{n-1} \left(\sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^{\alpha+it} \log m} \widehat{K}\left(\frac{\log m}{2\pi} \right) \right) \mathrm{d}\alpha \right\} \qquad (5.2.7) \\
&\quad + O_n\big(V_{1/2}(t) + ||K||_1\big) \\
&= \frac{1}{\pi} \operatorname{Im} \left\{ \frac{i^n}{(n-1)!} \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^{it} \log m} \widehat{K}\left(\frac{\log m}{2\pi} \right) \left(\int_\sigma^2 \frac{(\alpha - \sigma)^{n-1}}{m^\alpha} \mathrm{d}\alpha \right) \right\} \\
&\quad + O_n\big(V_{1/2}(t) + ||K||_1\big),
\end{aligned}$$

where the interchange of the integrals is justified by Fubini's theorem, considering the estimates [74, Theorem 13.18, Theorem 13.21]. Using [50, §2.321 Eq.2]) we obtain that

$$\int_\sigma^2 \frac{(\alpha - \sigma)^{n-1}}{m^\alpha} \mathrm{d}\alpha = \frac{\beta_{n-1}}{m^\sigma (\log m)^n} - \frac{1}{m^2} \sum_{k=0}^{n-1} \frac{\beta_k}{(\log m)^{k+1}} (2 - \sigma)^{n-1-k},$$

where $\beta_k = \frac{(n-1)!}{(n-1-k)!}$. This implies that for each $m \geqslant 2$ we get

$$\int_\sigma^2 \frac{(\alpha - \sigma)^{n-1}}{m^\alpha} \mathrm{d}\alpha = \frac{(n-1)!}{m^\sigma (\log m)^n} + O_n\left(\frac{1}{m^{3/2} (\log m)^n} \right).$$

Inserting this in (5.2.7), and considering that $||\widehat{K}||_\infty \leqslant ||K||_1$, we obtain the desired result. □

5.3 The resonator

In this section we will construct the resonator. The construction of our resonator is similar to the resonator developed by Bondarenko and Seip [9, Section 3]. The results presented here are extensions of their results, for a region near the critical line. The resonator is the function of the form $|R(t)|^2$, where

$$R(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it},$$

and $\mathcal{M}'$ is a suitable finite set of integers. Let σ be a positive real number and N be a positive integer sufficiently large, such that

$$\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + \frac{1}{\log \log N}. \tag{5.3.1}$$

Our resonator will depend on σ and N. For simplicity of notation, we write $\log_2 x := \log \log x$ and $\log_3 x := \log \log \log x$. Let P be the set of prime numbers p such that

$$e \log N \log_2 N < p \leqslant \exp\left((\log_2 N)^{1/8}\right) \log N \log_2 N. \tag{5.3.2}$$

We define $f(n)$ to be the multiplicative function supported on the set of square-free numbers such that

$$f(p) := \left(\frac{(\log N)^{1-\sigma} (\log_2 N)^{\sigma}}{(\log_3 N)^{1-\sigma}}\right) \frac{1}{p^{\sigma} (\log p - \log_2 N - \log_3 N)},$$

for $p \in P$ and $f(p) = 0$ otherwise. For each $k \in \left\{1, \cdots, \left[(\log_2 N)^{1/8}\right]\right\}$ we define the following sets:

$$P_k := \left\{p : \text{prime number such that } e^k \log N \log_2 N < p \leqslant e^{k+1} \log N \log_2 N\right\},$$

$$M_k := \left\{n \in \text{supp } (f) : n \text{ has at least } \alpha_k := \frac{3(\log N)^{2-2\sigma}}{k^2 (\log_3 N)^{2-2\sigma}} \text{ prime divisors in } P_k\right\},$$

$$M'_k := \left\{n \in M_k : n \text{ only has prime divisors in } P_k\right\}.$$

Finally, we define the set

$$\mathcal{M} := \text{supp } (f) \setminus \bigcup_{k=1}^{[(\log_2 N)^{1/8}]} M_k.$$

Note that if $m \in \mathcal{M}$ and $d | m$ then $d \in \mathcal{M}$.

Lemma 5.10. *We have that $|\mathcal{M}| \leqslant N$, where $|\mathcal{M}|$ represents the cardinality of $\mathcal{M}$.*

Proof. The proof follows the same outline that [7, Lemma 2]. The main difference is the appearance of the term $(\log_3 N)^{2\sigma - 1}$, which is well estimated, whenever (5.3.1) holds. It allows us to obtain the same estimate for the cardinality of $\mathcal{M}$ as the case $\sigma = \frac{1}{2}$. By [7, Eq. (9)-(10)], we have that

$$\binom{[x]}{[y]} \leqslant \exp\big(y(\log x - \log y) + 2y + \log x\big),$$

for $1 \leqslant y \leqslant x$ and

$$2\binom{m}{n-1} \leqslant \binom{m}{n},$$

for $3n - 1 \leqslant m$. By the prime number theorem, the cardinality of each P_k is at most $e^{k+1} \log N$. Therefore, using the above inequalities and (5.3.1)

$$\begin{aligned}
|\mathcal{M}| &\leqslant \prod_{k=1}^{[(\log_2 N)^{1/8}]} \sum_{j=0}^{[\alpha_k]} \binom{[e^{k+1}\log N]}{j} \leqslant \prod_{k=1}^{[(\log_2 N)^{1/8}]} 2\binom{[e^{k+1}\log N]}{[\alpha_k]} \\
&\leqslant \exp\Bigg(\sum_{k=1}^{[(\log_2 N)^{1/8}]} \frac{3(\log N)^{2-2\sigma}}{(\log_3 N)^{2-2\sigma}} \left(\frac{1}{k} + \frac{3 + 2\log k}{k^2} + \frac{(2\sigma-1)\log_2 N}{k^2} + \frac{(2-2\sigma)\log_4 N}{k^2} \right) \\
&\qquad + 3k + \log_2 N \Bigg) \\
&\leqslant \exp\left(\left(\frac{3}{4} + o(1)\right) (\log N)^{2-2\sigma} (\log_3 N)^{2\sigma - 1} \right) \\
&\leqslant \exp\left(\left(\frac{3}{4} + o(1)\right) (\log N)(\log_3 N)^{2/\log_2 N} \right).
\end{aligned}$$

Then, for N sufficiently large we get that $|\mathcal{M}| \leqslant N$. □

Lemma 5.11. *For all $k = 1, \cdots, [(\log_2 N)^{1/8}]$ we have, as $N \to \infty$*

$$\sum_{p \in P_k} \frac{1}{p^{2\sigma}} = (1 + o(1)) \int_{e^k \log N \log_2 N}^{e^{k+1} \log N \log_2 N} \frac{1}{y^{2\sigma} \log y} \,\mathrm{d}y,$$

where $o(1)$ is independent of k. In particular, we have that

$$(d + o(1)) \frac{1}{(\log_2 N)^{2\sigma}} < \sum_{p \in P_k} \frac{1}{p^{2\sigma}} < (2 + o(1)) \frac{1}{(\log_2 N)^{2\sigma}}, \tag{5.3.3}$$

for some constant $0 < d < 1$.

Proof. Using [74, Theorem 13.1], under the Riemann hypothesis we have

$$\pi(x) = \int_2^x \frac{1}{\log y} \,\mathrm{d}y + O\big(x^{1/2} \log x\big),$$

where $\pi(x)$ is the function that counts the prime numbers not exceeding x. Then, using integration by parts we get

$$\sum_{p\in P_k}\frac{1}{p^{2\sigma}} = \int_{e^k\log N\log_2 N}^{e^{k+1}\log N\log_2 N}\frac{1}{y^{2\sigma}\log y}\,\mathrm{d}y + O\left(\int_{e^k\log N\log_2 N}^{e^{k+1}\log N\log_2 N}\frac{\log y}{y^{2\sigma+1/2}}\,\mathrm{d}y\right)$$
$$= \left(1+O\left(\frac{1}{(\log N)^{1/4}}\right)\right)\int_{e^k\log N\log_2 N}^{e^{k+1}\log N\log_2 N}\frac{1}{y^{2\sigma}\log y}\,\mathrm{d}y.$$

Now we can see that

$$\int_{e^k\log N\log_2 N}^{e^{k+1}\log N\log_2 N}\frac{1}{y^{2\sigma}\log y}\,\mathrm{d}y \leqslant \frac{e^k\log N\log_2 N(e-1)}{(e^k\log N\log_2 N)^{2\sigma}\log\left(e^k\log N\log_2 N\right)} < \frac{2}{(\log_2 N)^{2\sigma}}.$$

On the other hand, we know that $(e^k\log N)^{2\sigma-1} < (\log N)^{4\sigma-2} \leqslant e^4$ for all $1\leqslant k\leqslant [(\log_2 N)^{1/8}]$. Therefore

$$\int_{e^k\log N\log_2 N}^{e^{k+1}\log N\log_2 N}\frac{1}{y^{2\sigma}\log y}\,\mathrm{d}y \geqslant \frac{e^k\log N\log_2 N(e-1)}{(e^{k+1}\log N\log_2 N)^{2\sigma}\log\left(e^{k+1}\log N\log_2 N\right)} > \frac{d}{(\log_2 N)^{2\sigma}},$$

for some constant $0<d<1$. □

The following lemma can be considered as an extension of [9, Lemma 4] to the region (5.3.1).

Lemma 5.12. *We have*

$$\frac{1}{\sum_{l\in\mathbb{N}} f(l)^2}\sum_{n\in\mathcal{M}} f(n)^2\sum_{p|n}\frac{1}{f(p)\,p^{\sigma}} \geqslant c\,\frac{(\log N)^{1-\sigma}(\log_3 N)^{\sigma}}{(\log_2 N)^{\sigma}},$$

for some universal constant $c>0$.

Proof. The proof is similar to [9, Lemma 4]. For each $k\in\left\{1,\cdots,\left[(\log_2 N)^{1/8}\right]\right\}$ we define the following sets:

$$L_k := \left\{n\in \text{supp }(f) : n \text{ has at most } \beta_k := \frac{d\,(\log N)^{2-2\sigma}}{12k^2(\log_3 N)^{2-2\sigma}} \text{ prime divisors in } P_k\right\},$$

where d is the mentioned constant in Lemma 5.11, and

$$L_k' := \left\{n\in L_k : n \text{ only has prime divisors in } P_k\right\}.$$

Finaly, we define the set

$$\mathcal{L} := \mathcal{M}\backslash\bigcup_{k=1}^{[(\log_2 N)^{1/8}]} L_k.$$

Now to prove the lemma, it is enough to show that

$$\frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{n\notin\mathcal{L}} f(n)^2 = o(1), \quad \text{as } N \to \infty. \tag{5.3.4}$$

Indeed, using (5.3.4) and the fact that $\mathcal{L} \subset \mathcal{M}$ we get

$$\begin{aligned}
\frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{n\in\mathcal{M}} f(n)^2 \sum_{p|n} \frac{1}{f(p)\,p^\sigma} &\geqslant \frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{n\in\mathcal{M}} f(n)^2 \min_{n\in\mathcal{L}} \sum_{p|n} \frac{1}{f(p)\,p^\sigma} \\
&\geqslant \big(1 - o(1)\big) \min_{n\in\mathcal{L}} \sum_{p|n} \frac{1}{f(p)\,p^\sigma} \\
&= \big(1 - o(1)\big) \sum_{k=1}^{[(\log_2 N)^{1/8}]} \frac{d\,(\log N)^{2-2\sigma}}{12k^2(\log_3 N)^{2-2\sigma}} \min_{p\in P_k} \frac{1}{f(p)\,p^\sigma} \\
&\geqslant \big(1 - o(1)\big) \sum_{k=1}^{[(\log_2 N)^{1/8}]} \frac{d\,(\log N)^{2-2\sigma}}{12k^2(\log_3 N)^{2-2\sigma}} \left(\frac{k(\log_3 N)^{1-\sigma}}{(\log N)^{1-\sigma}(\log_2 N)^\sigma} \right) \\
&\geqslant c\,\frac{(\log N)^{1-\sigma}(\log_3 N)^\sigma}{(\log_2 N)^\sigma},
\end{aligned}$$

for some constant $c > 0$. Therefore, it remains to prove (5.3.4). Since

$$\mathcal{L} := \text{supp } (f) \setminus \bigcup_{k=1}^{[(\log_2 N)^{1/8}]} \big(M_k \cup L_k\big),$$

it is enough to prove that when $N \to \infty$

$$\frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{k=1}^{[(\log_2 N)^{1/8}]} \sum_{n\in L_k} f(n)^2 = o(1), \tag{5.3.5}$$

and

$$\frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{k=1}^{[(\log_2 N)^{1/8}]} \sum_{n\in M_k} f(n)^2 = o(1). \tag{5.3.6}$$

First we will prove (5.3.5). For each $k \in \big\{1, \cdots, \big[(\log_2 N)^{1/8}\big]\big\}$ and for any $0 < b < 1$ we have

$$\begin{aligned}
\frac{1}{\sum\limits_{l\in\mathbb{N}} f(l)^2} \sum_{n\in L_k} f(n)^2 = \frac{1}{\prod\limits_{p\in P_k} (1 + f(p)^2)} \sum_{n\in L'_k} f(n)^2 &\leqslant b^{-\beta_k} \prod_{p\in P_k} \frac{\big(1 + bf(p)^2\big)}{(1 + f(p)^2)} \\
&\leqslant b^{-\beta_k} \exp\left((b-1) \sum_{p\in P_k} \frac{f(p)^2}{1 + f(p)^2} \right).
\end{aligned} \tag{5.3.7}$$

Since $f(p) \leqslant 1$, using the left-hand side inequality of (5.3.3) we get

$$\begin{aligned}\sum_{p \in P_k} \frac{f(p)^2}{1+f(p)^2} &\geqslant \frac{1}{2} \sum_{p \in P_k} f(p)^2 \\ &= \left(\frac{(\log N)^{2-2\sigma}(\log_2 N)^{2\sigma}}{2(\log_3 N)^{2-2\sigma}} \right) \sum_{p \in P_k} \frac{1}{p^{2\sigma} (\log p - \log_2 N - \log_3 N)^2} \\ &\geqslant \left(\frac{(\log N)^{2-2\sigma}}{8k^2(\log_3 N)^{2-2\sigma}} \right)(d + o(1)).\end{aligned}$$

This implies in (5.3.7) that

$$\frac{1}{\sum_{l \in \mathbb{N}} f(l)^2} \sum_{n \in L_k} f(n)^2 \leqslant \exp\left(\left(\frac{d}{8}(b-1) - \frac{d}{12} \log b + o(1) \right) \frac{(\log N)^{2-2\sigma}}{k^2(\log_3 N)^{2-2\sigma}} \right).$$

Therefore, choosing b close to 1 we obtain $3(b-1) - 2\log b < 0$ and summing over k we obtain (5.3.5). The proof of (5.3.6) is similar. For each $k \in \{1, \cdots, [(\log_2 N)^{1/8}]\}$ and for any $b > 1$ we get

$$\frac{1}{\sum_{l \in \mathbb{N}} f(l)^2} \sum_{n \in M_k} f(n)^2 \leqslant b^{-\alpha_k} \exp\left((b-1) \sum_{p \in P_k} f(p)^2 \right). \tag{5.3.8}$$

Using the right-hand side inequality of (5.3.3) we have

$$\sum_{p \in P_k} f(p)^2 \leqslant \left(\frac{(\log N)^{2-2\sigma}}{k^2(\log_3 N)^{2-2\sigma}} \right)(2 + o(1)).$$

This implies in (5.3.8) that

$$\frac{1}{\sum_{l \in \mathbb{N}} f(l)^2} \sum_{n \in L_k} f(n)^2 \leqslant \exp\left(\left(2(b-1) - 3\log b + o(1) \right) \frac{(\log N)^{2-2\sigma}}{k^2(\log_3 N)^{2-2\sigma}} \right).$$

Finally, choosing b close to 1 we obtain $2(b-1) - 3\log b < 0$ and summing over k we obtain (5.3.6). □

5.3.1 Construction of the resonator

Let $0 \leqslant \beta < 1$ be a fixed number and consider the positive real number $\kappa = (1-\beta)/2$. Note that $\kappa + \beta < 1$. Let σ be a positive real number and T sufficiently large such that

$$\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + \frac{1}{\log\log T}.$$

Then we write $N = [T^\kappa]$. Note that σ and N satisfy the relation (5.3.1). Now, let $\mathcal{J}$ be the set of integers j such that $\left[(1+T^{-1})^j, (1+T^{-1})^{j+1} \right) \bigcap \mathcal{M} \neq \varnothing$, and we define m_j to be the minimum of $\left[(1+T^{-1})^j, (1+T^{-1})^{j+1}\right) \cap \mathcal{M}$ for j in $\mathcal{J}$. Consider the set

$$\mathcal{M}' := \{m_j : j \in \mathcal{J}\}$$

and finally we define

$$r(m_j) := \left(\sum_{n \in \mathcal{M}, (1+T^{-1})^{j-1} \leqslant n \leqslant (1+T^{-1})^{j+2}} f(n)^2 \right)^{1/2},$$

for every $m_j \in \mathcal{M}'$. This defines our Dirichlet polynomial

$$R(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it}.$$

Proposition 5.13. *We have the following properties:*

(i) $|\mathcal{M}'| \leqslant |\mathcal{M}| \leqslant N$.

(ii) $\sum_{m \in \mathcal{M}'} r(m)^2 \leqslant 4 \sum_{l \in \mathcal{M}} f(l)^2$.

(iii) $|R(t)|^2 \leqslant R(0)^2 \ll T^{\kappa} \sum_{l \in \mathcal{M}} f(l)^2$.

Proof. (i) and (ii) follow by the definition of $\mathcal{M}$, $\mathcal{M}'$ and Lemma 5.10. The left-hand side inequality of (iii) is obvious. The right-hand side inequality of (iii) follows by (i), (ii) and the Cauchy-Schwarz inequality. □

5.3.2 Estimates with the resonator

The proofs of the following results are similar to the case $\sigma = \frac{1}{2}$. According to the notation in [9] we write $\Phi(t) = e^{-t^2/2}$. Then $\widehat{\Phi}(t) = \sqrt{2\pi}\,\Phi(2\pi t)$.

Lemma 5.14. *We have*

$$\int_{-\infty}^{\infty} |R(t)|^2\, \Phi\left(\frac{t}{T}\right)\, \mathrm{d}t \ll T \sum_{l \in \mathcal{M}} f(l)^2.$$

Proof. The proof is similar to [9, Lemma 5] and we omit the details. □

Lemma 5.15. *There exists a positive constant $c > 0$ such that if*

$$G(t) := \sum_{m=2}^{\infty} \frac{\Lambda(m)\, a_m}{m^{\sigma+it} \log m}$$

is absolutely convergent and $a_m \geqslant 0$ for every $m \geqslant 2$, then

$$\int_{-\infty}^{\infty} G(t) |R(t)|^2\, \Phi\left(\frac{t}{T}\right)\, \mathrm{d}t \geqslant c\,T\, \frac{(\log T)^{1-\sigma} (\log_3 T)^{\sigma}}{(\log_2 T)^{\sigma}} \left(\min_{p \in P} a_p \right) \sum_{l \in \mathcal{M}} f(l)^2.$$

Proof. The proof follows the same outline of [9, Lemma 7], replacing [9, Lemma 4] by Lemma 5.12. We omit the details. □

5.4 Proof of Theorem 5.2

Assume the Riemann hypothesis. We consider the parameters defined in §5.3.1.

5.4.1 The case n odd

Let $n = 2m + 1$. We consider the entire function

$$K_n(z) = (-1)^m \log_2 T\, \Phi(2\pi \log_2 T\, z),$$

which has Fourier transform

$$\widehat{K_n}(\xi) = \frac{(-1)^{(n-1)/2}}{\sqrt{2\pi}} \Phi\left(\frac{\xi}{\log_2 T}\right) \ll 1. \tag{5.4.1}$$

Firstly we need to estimate the following integral

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_n(\sigma, t+u)\, K_n(u)\, \mathrm{d}u \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t. \tag{5.4.2}$$

This follows by the same computations as in [9, Section 5]. We will divide (5.4.2) into 3 integrals.

1. *First integral*: Using (5.2.1), (5.2.5) and Fubini's theorem we get

$$\begin{aligned}
&\int_{-T^\beta}^{T^\beta} \int_{-\infty}^{\infty} |S_n(\sigma, t+u)\, K_n(u)|\, \mathrm{d}u\ \mathrm{d}t \\
&\quad = \int_{-T^\beta}^{T^\beta} \int_{|u| \leqslant T^\beta} |S_n(\sigma, t+u)\, K_n(u)|\, \mathrm{d}u\ \mathrm{d}t + \int_{-T^\beta}^{T^\beta} \int_{|u| > T^\beta} |S_n(\sigma, t+u)\, K_n(u)|\, \mathrm{d}u\ \mathrm{d}t \\
&\quad \ll_n \int_{-T^\beta}^{T^\beta} \int_{-2T^\beta}^{2T^\beta} |S_n(\sigma, u)\, K_n(u-t)|\, \mathrm{d}u\ \mathrm{d}t + \int_{-T^\beta}^{T^\beta} \int_{|u| > T^\beta} \log(2|u|+2) |K_n(u)|\, \mathrm{d}u\ \mathrm{d}t \\
&\quad \ll_n \int_{-2T^\beta}^{2T^\beta} |S_n(\sigma, u)|\ \mathrm{d}u + T^\beta \ll_n T^\beta \log T.
\end{aligned}$$

Hence, by Proposition 5.13 we obtain

$$\begin{aligned}
\int_{-T^\beta}^{T^\beta} \left(\int_{-\infty}^{\infty} |S_n(\sigma, t+u)\, K_n(u)|\, \mathrm{d}u \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t &\ll_n T^\beta \log T\, R(0)^2 \\
&\ll_n T^{\beta+\kappa} \log T \sum_{l \in \mathcal{M}} f(l)^2.
\end{aligned} \tag{5.4.3}$$

2. *Second integral*: Using the fast decay of $\Phi(t)$, (5.2.1) and Proposition 5.13, it follows that

$$\begin{aligned}
&\int_{|t|>T\log T}\left(\int_{-\infty}^{\infty}|S_n(\sigma,t+u)\,K_n(u)|\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\\
&\ll T^{\kappa}e^{-(\log T)^2/4}\left(\int_{|t|>T\log T}\int_{-\infty}^{\infty}|S_n(\sigma,t+u)\,K_n(u)|\,\mathrm{d}u\,\Phi\left(\frac{t}{2T}\right)\mathrm{d}t\right)\sum_{l\in\mathcal{M}}f(l)^2 \qquad (5.4.4)\\
&=o(1)\sum_{l\in\mathcal{M}}f(l)^2.
\end{aligned}$$

3. *Third integral*:

$$\begin{aligned}
&\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{-\infty}^{\infty}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\\
&=\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{\frac{T^\beta}{2}\leqslant|t+u|\leqslant 2T\log T}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t \qquad (5.4.5)\\
&\quad+\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{\{|u+t|<\frac{T^\beta}{2}\}\cup\{|u+t|>2T\log T\}}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t.
\end{aligned}$$

Now using (5.2.1) and Lemma 5.14, the last integral can be bounded by

$$\begin{aligned}
&\int_{T^\beta\leqslant|t|\leqslant T\log T}\int_{\{|u+t|<\frac{T^\beta}{2}\}\cup\{|u+t|>2T\log T\}}|S_n(\sigma,t+u)\,K_n(u)|\,\mathrm{d}u\,|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\\
&\quad\ll\int_{T^\beta\leqslant|t|\leqslant T\log T}\int_{\{|u|<\frac{T^\beta}{2}\}\cup\{|u|>2T\log T\}}|S_n(\sigma,u)\,K_n(u-t)|\,\mathrm{d}u\,|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\\
&\quad\leqslant\int_{T^\beta\leqslant|t|\leqslant T\log T}\int_{\{|u|<\frac{T^\beta}{2}\}\cup\{|u|>2T\log T\}}\left|S_n(\sigma,u)\,K_n\left(\frac{u}{2}\right)\right|\mathrm{d}u\,|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t \qquad (5.4.6)\\
&\quad\ll_n\int_{T^\beta\leqslant|t|\leqslant T\log T}|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\ll T\sum_{l\in\mathcal{M}}f(l)^2.
\end{aligned}$$

Inserting (5.4.6) in (5.4.5) we obtain that

$$\begin{aligned}
&\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{-\infty}^{\infty}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t\\
&\quad=\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{\frac{T^\beta}{2}\leqslant|t+u|\leqslant 2T\log T}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t \qquad (5.4.7)\\
&\quad+O_n(T)\sum_{l\in\mathcal{M}}f(l)^2.
\end{aligned}$$

Therefore, combining (5.4.3), (5.4.4) and (5.4.7) we have that the integral in (5.4.2) can be written as

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_n(\sigma,\,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t+O_n(T)\sum_{l\in\mathcal{M}}f(l)^2\\
&\quad=\int_{T^\beta\leqslant|t|\leqslant T\log T}\left(\int_{\frac{T^\beta}{2}\leqslant|t+u|\leqslant 2T\log T}S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t. \qquad (5.4.8)
\end{aligned}$$

Final analysis: Finally, recalling that $n = 2m+1$ we consider two cases:

<u>*Case 1: m even.*</u> In this case note that $K_n(u) \geqslant 0$ for all $u \in \mathbb{R}$. Then by Lemma 5.14 and the fact that $S_n(\sigma, t)$ is an even function we obtain in (5.4.8)

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_n(\sigma, t+u)\, K_n(u)\, \mathrm{d}u \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t + O_n(T) \sum_{l \in \mathcal{M}} f(l)^2 \\ \leqslant b\,T \left(\max_{\frac{T^\beta}{2} \leqslant t \leqslant 2T \log T} S_n(\sigma, t) \right) \sum_{l \in \mathcal{M}} f(l)^2,\end{aligned} \tag{5.4.9}$$

for some constant $b > 0$. We define

$$G_n(t) = \sum_{m=2}^{\infty} \frac{\Lambda(m)}{\pi\, m^{\sigma+it} (\log m)^{n+1}} \widehat{K_n}\left(\frac{\log m}{2\pi}\right). \tag{5.4.10}$$

By Proposition 5.9 and (5.4.1) observe that

$$\int_{-\infty}^{\infty} S_n(\sigma, t+u)\, K_n(u)\, \mathrm{d}u = \operatorname{Re} G_n(t) + O_n\big(V_{1/2}(t) + 1\big),$$

for $t \neq 0$. Therefore, the integral on the left-hand side of (5.4.9) takes the form

$$\begin{aligned}&\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_n(\sigma, t+u)\, K_n(u)\, \mathrm{d}u \right) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \\ &= \operatorname{Re} \int_{-\infty}^{\infty} G_n(t) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t + O_n\left(\int_{-\infty}^{\infty} \big(V_{1/2}(t) + 1\big) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \right).\end{aligned} \tag{5.4.11}$$

Using Proposition 5.13, Lemma 5.14 and the definition of $V_{1/2}(t)$ we get

$$\int_{-\infty}^{\infty} \big(V_{1/2}(t) + 1\big) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \ll T \sum_{l \in \mathcal{M}} f(l)^2. \tag{5.4.12}$$

Therefore using (5.4.11) and (5.4.12) we have

$$\begin{aligned}b\,T \left(\max_{\frac{T^\beta}{2} \leqslant t \leqslant 2T \log T} S_n(\sigma, t) \right) \sum_{l \in \mathcal{M}} f(l)^2 \geqslant \operatorname{Re} \int_{-\infty}^{\infty} G_n(t) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t \\ + O_n(T) \sum_{l \in \mathcal{M}} f(l)^2.\end{aligned} \tag{5.4.13}$$

Now using Lemma 5.15 (note that $\widehat{K_n}(t)$ is a positive real function) with

$$a_m = \widehat{K_n}\left(\frac{\log m}{2\pi}\right) \frac{1}{\pi (\log m)^n},$$

for all $m \geqslant 2$ we obtain

$$\operatorname{Re} \int_{-\infty}^{\infty} G_n(t) |R(t)|^2 \Phi\left(\frac{t}{T}\right) \mathrm{d}t$$

$$\geqslant c\,T\,\frac{(\log T)^{1-\sigma}(\log_3 T)^{\sigma}}{(\log_2 T)^{\sigma}}\left(\min_{p\in P}\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\right)\sum_{l\in\mathcal{M}} f(l)^2, \qquad (5.4.14)$$

for some constant $c > 0$. Note that (5.3.2) and (5.4.1) imply

$$\min_{e\log N\log_2 N<p\leqslant \exp\left((\log_2 N)^{1/8}\right)\log N\log_2 N}\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\gg\frac{1}{(\log_2 T)^n}.$$

Inserting this in (5.4.14), we obtain in (5.4.13) that (after simplification)

$$\max_{\frac{T^\beta}{2}\leqslant t\leqslant 2T\log T} S_n(\sigma,t)\geqslant c_n\,\frac{(\log T)^{1-\sigma}(\log_3 T)^{\sigma}}{(\log_2 T)^{\sigma+n}}+O_n(1),$$

for some constant $c_n > 0$. After a trivial adjustment, changing T to $T/2\log T$ and making β slightly smaller, we obtain the restriction $T^\beta \leqslant t \leqslant T$.

<u>*Case 2: m odd.*</u> In this case note that $K_n(u) \leqslant 0$ for all $u \in \mathbb{R}$. Similar to (5.4.9), using the fact that $S_n(t)$ is an even function we find that

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u\right)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t+O_n(T)\sum_{l\in\mathcal{M}}f(l)^2$$
$$\leqslant b\,T\left(\max_{\frac{T^\beta}{2}\leqslant t\leqslant 2T\log T}|S_n(\sigma,t)|\right)\sum_{l\in\mathcal{M}}f(l)^2, \qquad (5.4.15)$$

for some constant $b > 0$. Using the function G_n defined in (5.4.10), by Proposition 5.9 and (5.4.1) we get

$$\int_{-\infty}^{\infty} S_n(\sigma,t+u)\,K_n(u)\,\mathrm{d}u=-\mathrm{Re}\,G_n(t)+O_n\big(V_{1/2}(t)+1\big).$$

A similar analysis as in the previous case shows that, by Lemma 5.15 (note that $-\widehat{K_n}(t)$ is a positive real function)

$$\mathrm{Re}\int_{-\infty}^{\infty}-G_n(t)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t$$
$$\geqslant c\,T\frac{(\log T)^{1-\sigma}(\log_3 T)^{\sigma}}{(\log_2 T)^{\sigma}}\left(\min_{p\in P}-\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\right)\sum_{l\in\mathcal{M}}f(l)^2, \qquad (5.4.16)$$

for some constant $c > 0$. By (5.3.2) and (5.4.1) we have

$$\min_{e\log N\log_2 N<p\leqslant \exp\left((\log_2 N)^{1/8}\right)\log N\log_2 N}-\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\gg\frac{1}{(\log_2 T)^n}.$$

Inserting this in (5.4.16) we obtain in (5.4.15) that (after simplification)

$$\max_{\frac{T^\beta}{2}\leqslant t\leqslant 2T\log T} |S_n(\sigma,t)|\geqslant c_n\,\frac{(\log T)^{1-\sigma}(\log_3 T)^{\sigma}}{(\log_2 T)^{\sigma+n}}+O_n(1),$$

for some constant $c_n > 0$. After the same trivial adjustment of T and β as in the preceding case we obtain the desired result.

5.4.2 The case n even

We consider the entire function

$$K_n(z) = (-1)^{n/2+1}(\log_2 T)^2\, z\, \Phi(2\pi \log_2 T\, z),$$

which has Fourier transform

$$\widehat{K_n}(\xi) = \frac{(-1)^{n/2}\, i}{(2\pi)^{\frac{3}{2}}(\log_2 T)}\xi\, \Phi\left(\frac{\xi}{\log_2 T}\right) \ll 1. \tag{5.4.17}$$

The analysis in this case is similar to the case $n = 2m+1$ with m odd. Using the fact that $S_n(t)$ is an odd function we obtain that (5.4.15) holds. Using the function G_n defined in (5.4.10), by Proposition 5.9 and (5.4.17) note that

$$\int_{-\infty}^{\infty} S_n(\sigma, t+u)\, K_n(u)\, \mathrm{d}u = (-1)^{n/2}\mathrm{Im}\, G_n(t) + O_n\big(V_{1/2}(t) + 1\big).$$

This implies that in (5.4.15) we obtain

$$b\,T\left(\max_{\frac{T^\beta}{2}\leqslant t\leqslant 2T\log T} \big|S_n(\sigma,t)\big|\right)\sum_{l\in\mathcal{M}} f(l)^2 \geqslant \mathrm{Re}\int_{-\infty}^{\infty}(-1)^{n/2+1}\, i\, G_n(t)|R(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t + O_n(T)\sum_{l\in\mathcal{M}} f(l)^2,$$

for some constant $b > 0$. Now, using Lemma 5.15 (note that $i(-1)^{n/2+1}\widehat{K_n}(t)$ is a positive real function for $t \geqslant 0$) it follows that

$$\begin{aligned} T&\left(\max_{\frac{T^\beta}{2}\leqslant t\leqslant 2T\log T} \big|S_n(\sigma,t)\big|\right)\sum_{l\in\mathcal{M}} f(l)^2 \\ &\geqslant c\,T\,\frac{(\log T)^{1-\sigma}(\log_3 T)^\sigma}{(\log_2 T)^\sigma}\left(\min_{p\in P}\mathrm{Im}\left\{(-1)^{n/2}\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\right\}\right)\sum_{l\in\mathcal{M}} f(l)^2, \end{aligned} \tag{5.4.18}$$

for some constant $c > 0$. By (5.3.2) and (5.4.17) we have

$$\min_{e\log N\log_2 N < p\leqslant \exp\left((\log_2 N)^{1/8}\right)\log N\log_2 N}\mathrm{Im}\left\{(-1)^{n/2}\widehat{K_n}\left(\frac{\log p}{2\pi}\right)\frac{1}{(\log p)^n}\right\} \gg \frac{1}{(\log_2 T)^n}.$$

Inserting this in (5.4.18) and doing the same procedure as in the previous cases we obtain the desired result.

Chapter 6

Zeros of the Riemann zeta-function and semidefinite programming

This chapter is comprised of the paper [A5]. We improve the asymptotic bounds for several quantities related to the distribution of the zeros of the Riemann zeta-function (and other functions), under Montgomery's pair correlation approach [72]. The main idea is to replace the usual bandlimited auxiliary functions by the class of functions used in the linear programming bounds developed by Cohn and Elkies [32] for the sphere packing problem. It allows one to relate the considered objects to certain convex optimization problems that can be solved numerically via semidefinite programming.

6.1 The pair correlation of the zeros of the Riemann zeta-function

In 1973, Montgomery [72] made a major contribution to the study of the distribution of the zeros on the critical line: the pair correlation conjecture of the zeros of the Riemann zeta-function. We revisit Montgomery's work in light of the recent techniques in sphere packing, to improve some quantities related to the zeros of the Riemann zeta-function.

The Riemann-von Mangoldt formula (2.2.1), in its weaker form, states that

$$N(T) = (1 + o(1))\frac{T}{2\pi}\log T. \tag{6.1.1}$$

Let

$$N^*(T) := \sum_{0<\gamma\leqslant T} m_\rho,$$

where the sum is over the non-trivial zeros of $\zeta(s)$ counting multiplicities[1] and m_ρ is the multiplicity of ρ. It is clear that $N(T) \leqslant N^*(T)$. On the other hand, in addition to RH, it

[1] We recall that in the sums related to zeros the summands should be repeated according to the multiplicity of the zero. Therefore, the function $N^*(T)$ can also be written as $\sum_{0<\gamma\leqslant T} m_\rho^2$, where the sum runs over the distinct zeros of $\zeta(s)$.

is also conjectured that all zeros of $\zeta(s)$ are simple, and therefore it is *conjectured*[2] that

$$N^*(T) \sim N(T). \tag{6.1.2}$$

One line of research to understand and give evidence for this conjecture is to produce bounds of the form

$$N^*(T) \leqslant \big(C + o(1)\big)N(T), \tag{6.1.3}$$

with $C > 0$ as small as possible, and $T \to \infty$. Under RH, Montgomery [72] was the first to show the constant $C = 1.3333...$. This result was later improved to $C = 1.3275$ by Cheer and Goldston [30]. Assuming GRH, Goldston, Gonek, Özlük and Snyder [47] improved it to $C = 1.3262$.

These results have an important application to estimating the quantity of simple zeros of $\zeta(s)$. Let

$$N_s(T) := \sum_{\substack{0<\gamma\leqslant T\\ m_\rho=1}} 1. \tag{6.1.4}$$

The strong relation between $N^*(T)$ and $N_s(T)$ is due by

$$N_s(T) \geqslant \sum_{0<\gamma\leqslant T} (2 - m_\rho) = 2N(T) - N^*(T). \tag{6.1.5}$$

Under the pair correlation approach the best previous result known is due by Cheer and Goldston [30] showing that at least 67.27% of the zeros are simple. Assuming GRH, Goldston, Gonek, Özlük and Snyder [47] showed that at least 67.38% of the zeros are simple. However, by a different technique, still assuming RH, Bui and Heath-Brown [12] improved the result to 70.37%, which currently is the best.

In order to study the distribution of the spacing between consecutive zeros of $\zeta(s)$, Montgomery [72] also defined the pair correlation function

$$N(T,\beta) := \sum_{\substack{0<\gamma,\gamma'\leqslant T\\ 0<\gamma'-\gamma\leqslant \frac{2\pi\beta}{\log T}}} 1 \tag{6.1.6}$$

and *conjectured* that

$$N(T,\beta) \sim N(T)\int_0^\beta \left\{1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right\} \mathrm{d}x.$$

Note that by (6.1.1) the average gap between zeros is $\frac{2\pi}{\log T}$, hence $N(T,\beta)$ is counting zeros not greater than β times the average gap. To support this conjecture, one wants to produce bounds of the form

$$N(T,\beta) \gg N(T), \tag{6.1.7}$$

[2] It can be seen in the notes of D. A. Goldston [45].

with $\beta > 0$ as small as possible, and $T \to \infty$. Montgomery [72] showed, under RH and (6.1.2) that β can be take as 0.68..., and in [47] it is pointed out that it is not difficult to modify Montgomery's argument to derive the sharper constant $\beta = 0.6695$. This result was improved by Goldston, Gonek, Özlük and Snyder [47] with constant $\beta = 0.6072$. Recently, it was improved to the constant $\beta = 0.6068...$ by Carneiro, Chandee, Littmann and Milinovich [15]. Assuming GRH and (6.1.2), Goldston, Gonek, Özlük and Snyder showed the constant 0.5781....

The direct application of these results is to estimate how small the gaps between consecutive zeros can be related to the total average gap. Ordering the imaginary parts of the zeros of $\zeta(s)$ in the upper half plane $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \cdots$, it is clear that

$$\liminf_{n\to\infty} (\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{2\pi} \leqslant \beta. \tag{6.1.8}$$

Under the pair correlation approach, using the above mentioned constants, we can obtain bounds in (6.1.8). By a different technique, assuming RH, the best result known in (6.1.8) is due to Preobrazhenskiĭ [78], showing the constant 0.5154.

6.1.1 Main results I

Our main goal here is to improve the previous results in (6.1.3) and (6.1.7).

Theorem 6.1. *Assume the Riemann hypothesis. Then, as $T \to \infty$*

$$N^*(T) \leqslant (1.3208 + o(1))N(T).$$

Assume the generalized Riemann hypothesis. Then, as $T \to \infty$

$$N^*(T) \leqslant (1.3155 + o(1))N(T).$$

Using the relation (6.1.5) we obtain the following corollary.

Corollary 6.2. *Assume the Riemann hypothesis. Then, as $T \to \infty$*

$$N_s(T) \geqslant (0.6792 + o(1))N(T).$$

Assume the generalized Riemann hypothesis. Then, as $T \to \infty$

$$N_s(T) \geqslant (0.6845 + o(1))N(T).$$

Using the approach of pair correlation, Corollary 6.2 is the best result (up to date) on the percentage of simple zeros of $\zeta(s)$, but as mentioned previously Bui and Heath-Brown [12] obtained the constant 0.7037 using a different technique. However, we can use Theorem

6.1 and the result of Bui and Heath-Brown to improve the proportion of distinct zeros. Let

$$N_d(T) := \sum_{0<\gamma\leqslant T} \frac{1}{m_\rho}, \tag{6.1.9}$$

be the number of distinct zeros of $\zeta(s)$ with $0 < \gamma \leqslant T$. Using the inequality

$$2N_s(T) \leqslant \sum_{0<\gamma\leqslant T} \frac{(m_\rho - 2)(m_\rho - 3)}{m_\rho} = N^*(T) - 5N(T) + 6N_d(T).$$

in conjunction with the estimate

$$N_s(T) \geqslant (0.7037 + o(1))N(T)$$

and Theorem 6.1, we deduce the following corollary.

Corollary 6.3. *Assume the Riemann hypothesis. Then, as $T \to \infty$*

$$N_d(T) \geqslant (0.8477 + o(1))N(T).$$

Assume the generalized Riemann hypothesis. Then, as $T \to \infty$

$$N_d(T) \geqslant (0.8486 + o(1))N(T).$$

Using the pair correlation approach, the best previous result known is due to Farmer, Gonek and Lee [39] with constant 0.8051. By a different technique, assuming RH, Bui and Heath-Brown [12] improved the constant to 0.8466. To the best of our knowledge, our new bounds are the current best.

We also obtain improved results for Montgomery's pair correlation function.

Theorem 6.4. *Assume the Riemann hypothesis and* (6.1.2)*. Then, for T sufficiently large*

$$N(T, 0.6039) \gg N(T).$$

Assume the generalized Riemann hypothesis and (6.1.2)*. Then, for T sufficiently large*

$$N(T, 0.5769) \gg N(T).$$

As a simple consequence we obtain the best result in (6.1.8), under the pair correlation approach.

Corollary 6.5. *Assume the Riemann hypothesis. Then*

$$\liminf_{n\to\infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_{n+1}}{2\pi} \leqslant 0.6039.$$

Assume the generalized Riemann hypothesis. Then

$$\liminf_{n\to\infty}(\gamma_{n+1}-\gamma_n)\frac{\log\gamma_{n+1}}{2\pi}\leqslant 0.5769.$$

6.2 The pair correlation of the zeros of Dirichlet L-functions

We use the framework established by Chandee, Lee, Liu, and Radziwiłł [28] to improve a result related to the simplicity of the zeros of the primitive Dirichlet L-functions.
Let Φ be a real-valued smooth function supported in the interval $[a,b]$ with $0<a<b<\infty$. Define its Mellin transform by

$$\mathcal{M}\Phi(s)=\int_0^\infty \Phi(x)x^{s-1}\,\mathrm{d}x.$$

For a character χ mod q, let $L(s,\chi)$ be its associated Dirichlet L-function. Under GRH, all non-trivial zeros of $L(s,\chi)$ lie on the critical line $\operatorname{Re}s=1/2$. Let

$$N_\Phi(Q):=\sum_{Q\leqslant q\leqslant 2Q}\frac{W(q/Q)}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \text{primitive}}}\sum_{\gamma_\chi}\big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2,$$

where W is a non-negative smooth function supported in $(1,2)$, and where the last sum is over all non-trivial zeros $\frac{1}{2}+i\gamma_\chi$ of the Dirichlet L-function $L(s,\chi)$. In [28, Lemma 2.1] it is shown that

$$N_\Phi(Q)\sim\frac{A}{2\pi}Q\log Q\int_{-\infty}^{\infty}\big|\mathcal{M}\Phi(ix)\big|^2\,\mathrm{d}x,$$

where

$$A=\mathcal{M}W(1)\prod_{p\ \text{prime}}\left(1-\frac{1}{p^2}-\frac{1}{p^3}\right).$$

Let

$$N_{\Phi,s}(Q)=\sum_{Q\leqslant q\leqslant 2Q}\frac{W(q/Q)}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \text{primitive}}}\sum_{\substack{\gamma_\chi\\ \text{simple}}}\big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2.$$

In addition, we require that $\Phi(x)$ and $\mathcal{M}\Phi(ix)$ are non-negative functions. We note that we can also further relax the conditions on Φ so to include the function given by $\mathcal{M}\Phi(ix)=(\sin x/x)^2$, as was established in [28] and [84].

We want to establish bounds in the form

$$N_{\Phi,s}(Q)\geqslant\big(C+o(1)\big)N_\Phi(Q), \tag{6.2.1}$$

with $C>0$ as small as possible, and $Q\to\infty$. In some sense, (6.2.1) measures (in average) the proportion of simple zeros among all primitive Dirichlet L-functions. Chandee, Lee, Liu, and Radziwiłł [28] showed the constant $C=0.9166...$, assuming GRH. Sono [84] improved the constant to $C=0.9322...$, using similar ideas of the work of Carneiro, Chandee, Littmann and Milinovich [15] for the case of the Riemann zeta-function.

6.2.1 Main result II

The following theorem improves the results above mentioned.

Theorem 6.6. *Assume the generalized Riemann hypothesis. Then, as $Q \to \infty$*

$$N_{\Phi,s}(Q) \geqslant (0.9350 + o(1))N_{\Phi}(Q).$$

Theorem 6.6 shows that at least 93.50% of low-lying zeros of primitive Dirichlet L-functions are simple in a proper sense, under the assumption of the generalized Riemann hypothesis.

6.3 The pair correlation of the zeros of the derivative of the Riemann ξ-function

We can extend our analysis to study the zeros of $\xi'(s)$, using the approach of pair correlation due by Farmer, Gonek and Lee [39]. We recall the definition of the Riemann ξ-function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is known that $\xi'(s)$ has only zeros in the critical strip $0 < \operatorname{Re} s < 1$ and that RH implies that all its zeros satisfy $\operatorname{Re} s = \frac{1}{2}$. Let $N_1(T)$ count the number of zeros $\rho_1 = \beta_1 + i\gamma_1$ of $\xi'(s)$ (with multiplicity) such that $0 < \gamma_1 \leqslant T$. It is also known that

$$N_1(T) = (1 + o(1))\frac{T}{2\pi}\log T.$$

We define the function

$$N_1^*(T) := \sum_{0<\gamma_1\leqslant T} m_{\rho_1},$$

where m_{ρ_1} is the multiplicity of the zero ρ_1. Similarly to the case of the Riemann zeta-function, we want to establish bounds in the form

$$N_1^*(T) \leqslant \big(C + o(1)\big)N_1(T),$$

with $C > 0$ as small as possible, and $T \to \infty$. The previous constant known, assuming RH, is due by Farmer, Gonek and Lee [39], showing the constant $C = 1.1417$. Now, let $N_{1,s}(T)$ be the number of simple zeros of $\xi'(s)$ (similar as (6.1.4)). Using the relation

$$N_{1,s}(T) \geqslant 2N_1(T) - N_1^*(T),$$

we can obtain bounds to the percentage of the simple zeros of $\xi'(s)$. For instance, the result of Farmer, Gonek and Lee [39] implies that more than 85.83% of the zeros of $\xi'(s)$ are simple.

6.3.1 Main result III

We improve the previous result on the percentage of simple zeros of $\xi'(s)$.

Theorem 6.7. *Assume the Riemann hypothesis. Then, as $T \to \infty$*

$$N_1^*(T) \leqslant (1.1175 + o(1))N_1(T).$$

In particular, assuming the Riemann hypothesis we have, as $T \to \infty$

$$N_{1,s}(T) \geqslant (0.8825 + o(1))N_1(T).$$

Also, let $N_{1,d}(T)$ be the number of distinct zeros of $\xi'(s)$ (similar as (6.1.9)). It is clear that the relation

$$N_{1,d}(T) \geqslant \frac{3}{2}N_1(T) - \frac{1}{2}N_1^*(T),$$

can be derived the same way as for $\zeta(s)$. Then, we have the following corollary.

Corollary 6.8. *Assume the Riemann hypothesis. Then, as $T \to \infty$*

$$N_{1,d}(T) \geqslant (0.9412 + o(1))N_1(T).$$

6.4 Strategy oultine

These two problems have been widely studied with several improvements being made over the years. One of the approaches is to use some suitable explicit formula (relating sums with integrals) with an auxiliary function f in some class $\mathcal{A}$ and produce an inequality relating the quantity we are interested to bound with some functional $\mathcal{Q}(f)$ over $\mathcal{A}$. Minimizing (or maximizing) the functional over the class $\mathcal{A}$ would then produce the best-bound one can possibly get with that specific approach. Nowadays, this idea is a standard technique in analytic number theory and has been used in the first chapters of this thesis. Other applications can be see in the following references: Large sieve inequalities [51, 53]; Erdös-Turán inequalities [27, 89]; Hilbert-type inequalities [24, 25, 27, 49, 51, 89]; Tauberian theorems [51]; Bounds in the theory of the Riemann zeta-function and L-functions [14, 15, 16, 17, 18, 19, 20, 29, 31, 44, 46]; Prime gaps [26].

From our point of view, our main contribution connects here. So far the only class $\mathcal{A}$ used for problems (6.1.3) and (6.1.7) was some Paley-Wiener space of bandlimited approximations. We relax the bandlimited condition by requiring only certain sign conditions on the auxiliary function that match exactly with the very same conditions required by the linear programming bounds for the packing problem (see Section 6.5 for a detailed explanation).

This relation is what ultimately inspired and allowed us to perform numerical computations to find good test functions for the functionals we derive in Section 6.5. Furthermore, as far as we know, it is the first time this method is used in the theory of the Riemann zeta-function.

The strategy can be broadly divided into the following two main steps:

Step 1: Derivation of the optimization problems.

The general strategy to study problems (6.1.3) and (6.1.7) is based on Montgomery's function

$$F(\alpha, T) = \frac{1}{N(T)} \sum_{0<\gamma,\gamma' \leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'), \tag{6.4.1}$$

where $\alpha \in \mathbb{R}$, $T \geqslant 2$ and the sum is over pairs of ordinates of zeros (with multiplicity) of $\zeta(s)$ and $w(u) = \frac{4}{4+u^2}$. We use Fourier inversion to obtain

$$\sum_{0<\gamma,\gamma' \leqslant T} g\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma - \gamma') = N(T) \int_{-\infty}^{\infty} \widehat{g}(\alpha) F(\alpha, T)\, \mathrm{d}\alpha, \tag{6.4.2}$$

for suitable functions g, and use some known asymptotic estimate for $F(\alpha, T)$ as $T \to \infty$ (which is proven only under RH or GRH). The main goal here is to note that the inequalities that appear in [28, 30, 39, 47, 72, 84] allows the use of an especial space of functions, denoted by $\mathcal{A}_{LP}$. In particular, these functions are eventually nonpositive and their Fourier transforms are positives. After a series of inequalities, we produce a minimization problem over $\mathcal{A}_{LP}$ for some functional $\mathcal{Z}$.

Step 2: Implementation and numerical issues.

We then approach the problem numerically, using the class of functions $\mathcal{A}_{LP}$ used for the sphere packing problem in [32] and sum-of-squares/semidefinite programming techniques to optimize over these functions, as was done in [62] for the binary sphere packing problem. For the code to generate the semidefinite programs and to perform the post processing we use Julia [4], Nemo [40] and Arb [57].

Although we will only use this framework to study the case of the Riemann zeta-function, $\xi'(s)$ and a certain average of primitive Dirichlet L-functions, the same basic strategy can be, in principle, carried out for other functions where we have a pair correlation approach.

6.5 Derivation of the optimization problems

Let $\mathcal{A}_{LP}$ be the class of even continuous functions $f \in L^1(\mathbb{R})$ satisfying the following conditions:

1. $\widehat{f}(0) = f(0) = 1$;

2. $\widehat{f} \geqslant 0$;

3. f is eventually nonpositive.

By eventually nonpositive we mean that $f(x) \leqslant 0$, for $|x|$ sufficiently large. We then define the last sign change of f by

$$r(f) = \inf\big\{r > 0 : f(x) \leqslant 0 \text{ for } |x| \geqslant r\big\}.$$

It is easy to show that if $f \in \mathcal{A}_{LP}$, then $\widehat{f} \in L^1(\mathbb{R})$.

A remarkable breakthrough in the sphere problem was achieved by Cohn and Elkies in [32], where they showed that if $\Delta(\mathbb{R}^d)$ is the highest sphere packing density in $\mathbb{R}^d$ then

$$\Delta(\mathbb{R}^d) \leqslant \mathcal{Q}(f)$$

for any $f \in \mathcal{A}_{LP}(\mathbb{R}^d)$ (this is the analogous class in higher dimensions defined for radial functions f), where

$$\mathcal{Q}(f) = \frac{\pi^{d/2}}{(d/2)!2^d} r(f)^d.$$

With this approach they generated numerical upper bounds, called linear programming bounds, for the packing density for dimensions up to 36 (nowadays it goes much higher) that improved every single upper bound known at the time and still are the current best. These upper bounds in dimensions 8 and 24 revealed to be extremely close to the lower bounds given by the E_8 root lattice and the Λ_{24} Leech lattice, suggesting that in these special dimensions the linear programming approach could exactly act as the dual problem. This is what inspired Viazovska [90] and Cohn, Kumar, Miller, Radchenko and Viazovska [34], to follow their program and solve the sphere packing problem in dimensions 8 and 24, respectively. What is interesting and surprising to us is that the same space $\mathcal{A}_{LP}$ can be used (but with a functional different than $\mathcal{Q}(f)$) to produce numerical bounds in analytic number theory.

6.5.1 Bounding $N^*(T)$ and $N(T,\beta)$

Ultimately, the functionals we need to define depend on the asymptotic behavior of $F(\alpha,T)$. To analyze the function $N^*(T)$ we define the functionals

$$\mathcal{Z}(f) = r(f) + \frac{2}{r(f)}\int_0^{r(f)} f(x)x\,\mathrm{d}x$$

and

$$\widetilde{\mathcal{Z}}(f) = r(f) + \frac{2}{r(f)}\int_0^{r(f)} f(x)x\,\mathrm{d}x + 3\int_{r(f)}^{3r(f)/2} f(x)\,\mathrm{d}x - \frac{2}{r(f)}\int_{r(f)}^{3r(f)/2} f(x)x\,\mathrm{d}x.$$

Theorem 6.9. *Let $f \in \mathcal{A}_{LP}$. Assuming RH we have, as $T \to \infty$*

$$N^*(T) \leqslant (\mathcal{Z}(f) + o(1))N(T).$$

Assuming GRH, for every fixed small $\delta > 0$ we have, as $T \to \infty$

$$N^*(T) \leqslant (\widetilde{\mathcal{Z}}(f) + O(\delta) + o(1))N(T).$$

Proof. We start assuming only RH. Refining the original work of Montgomery [72], Goldston and Montgomery [48, Lemma 8] stated for the function $F(\alpha, T)$ defined in (6.4.1), that

$$F(\alpha, T) = \big(T^{-2|\alpha|}\log T + |\alpha|\big)(1 + o(1)), \tag{6.5.1}$$

uniformly for $|\alpha| \leqslant 1$. Let $f \in \mathcal{A}_{LP}$ and let $g(x) = \widehat{f}(x/r(f))/r(f)$. We can then use the explicit formula (6.4.2) in conjunction with the asymptotic formula above to obtain

$$\sum_{0<\gamma,\gamma'\leqslant T} g\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg)w(\gamma-\gamma') = N(T)\bigg[\widehat{g}(0) + \int_{-1}^{1}\widehat{g}(\alpha)|\alpha|\,\mathrm{d}\alpha + \int_{|\alpha|>1}\widehat{g}(\alpha)F(\alpha,T)\,\mathrm{d}\alpha + o(1)\bigg],$$

where the $o(1)$ above is justified since $\widehat{g}$ is continuous and $T^{-2|\alpha|}\log T \to \boldsymbol{\delta}_0(\alpha)$ as $T \to \infty$ (in the distributional sense). Moreover, since $F(\alpha, T)$ is non-negative and $\widehat{g}(\alpha) \leqslant 0$ for $|\alpha| \geqslant 1$ we deduce that

$$\begin{aligned}\sum_{0<\gamma,\gamma'\leqslant T} g\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg)w(\gamma-\gamma') &\leqslant N(T)\left[\widehat{g}(0) + 2\int_{0}^{1}\widehat{g}(\alpha)\alpha\,\mathrm{d}\alpha + o(1)\right]\\ &= N(T)\left[\frac{\mathcal{Z}(f)}{r(f)} + o(1)\right].\end{aligned}$$

On the other hand, clearly we have

$$\sum_{0<\gamma,\gamma'\leqslant T} g\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg)w(\gamma-\gamma') \geqslant g(0)\sum_{0<\gamma\leqslant T} m_\rho = \frac{N^*(T)}{r(f)}.$$

Combining these results we show the first inequality in the theorem. Assume now GRH. It is then shown in [47] that for any fixed and sufficiently small $\delta > 0$ we have

$$F(\alpha, T) \geqslant \tfrac{3}{2} - |\alpha| - o(1), \tag{6.5.2}$$

uniformly for $1 \leqslant |\alpha| \leqslant \frac{3}{2} - \delta$, as $T \to \infty$. Using this estimate and the fact that $\widehat{g}(\alpha) \leqslant 0$ for $|\alpha| \geqslant 1$ we obtain

$$\sum_{0<\gamma,\gamma'\leqslant T} \widehat{g}\bigg((\gamma-\gamma')\frac{\log T}{2\pi}\bigg)w(\gamma-\gamma') \leqslant N(T)\left[\widehat{g}(0) + 2\int_{1}^{3/2-\delta}\widehat{g}(\alpha)\left(\tfrac{3}{2}-\alpha\right)\mathrm{d}\alpha + o(1)\right]$$

$$= N(T)\left[\frac{\widetilde{\mathcal{Z}}(f)}{r(f)} + o(1) + O(\delta)\right].$$

Arguing as before we finish the proof. □

To analyze $N(T,\beta)$ we define the function

$$\mathcal{P}(f) = \inf\big\{\lambda > 0 : p_f(\lambda) > 0\big\},$$

where

$$p_f(\lambda) = -1 + \frac{\lambda}{r(f)} + \frac{2r(f)}{\lambda}\int_0^{\lambda/r(f)} \widehat{f}(x)x\,\mathrm{d}x,$$

and the function

$$\widetilde{\mathcal{P}}(f) = \inf\big\{\lambda > 0 : \widetilde{p}_f(\lambda) > 0\big\},$$

where

$$\begin{aligned}\widetilde{p}_f(\lambda) = -1 + \frac{\lambda}{r(f)} + \frac{2r(f)}{\lambda}\int_0^{\lambda/r(f)} \widehat{f}(x)x\,\mathrm{d}x + 3\int_{\lambda/r(f)}^{3\lambda/(2r(f))} \widehat{f}(x)\,\mathrm{d}x \\ - \frac{2r(f)}{\lambda}\int_{\lambda/r(f)}^{3\lambda/(2r(f))} \widehat{f}(x)x\,\mathrm{d}x.\end{aligned}$$

Note that these functions are well defined since p_f and $\widetilde{p}_f$ are of class $C^1(\mathbb{R})$ that assume -1 at $\lambda = 0$, and using the fact that $\widehat{f} \in L^1(\mathbb{R})$ one can show

$$\lim_{\lambda\to\infty} \frac{p_f(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \frac{\widetilde{p}_f(\lambda)}{\lambda} = \frac{1}{r(f)} > 0.$$

Theorem 6.10. *Let $f \in \mathcal{A}_{LP}$ and $\varepsilon > 0$. Assuming RH and* (6.1.2) *we have for T sufficiently large*

$$N(T, \mathcal{P}(f) + \varepsilon) \gg N(T).$$

Assuming GRH we have for T sufficiently large

$$N(T, \widetilde{\mathcal{P}}(f) + \varepsilon) \gg N(T).$$

Proof. In the following we only exhibit the proof assuming RH since under GRH the proof is very similar, and the only extra information needed is in (6.5.2). Let $f \in \mathcal{A}_{LP}$ and $\lambda > 0$. Applying the explicit formula (6.4.2) for $g(x) = f(r(f)x/\lambda)$ in conjunction with (6.5.1) we obtain

$$\sum_{0<\gamma,\gamma'\leqslant T} g\left((\gamma-\gamma')\frac{\log T}{2\pi}\right)w(\gamma-\gamma') = N(T)\int_{-\infty}^{\infty} \widehat{g}(\alpha)F(\alpha,T)\,\mathrm{d}\alpha$$

$$\geqslant N(T)\left[\widehat{g}(0)+2\int_0^1 \widehat{g}(\alpha)\alpha\,\mathrm{d}\alpha+o(1)\right]$$
$$= N(T)\left[1+p_f(\lambda)+o(1)\right].$$

Since $\widehat{f}\geqslant 0$, we have $\|f\|_\infty = f(0) = 1$. Recall now the pair correlation function $N(T,\beta)$ defined in (6.1.6). We have

$$\begin{aligned}
&\sum_{0<\gamma,\gamma'\leqslant T} g\left((\gamma-\gamma')\frac{\log T}{2\pi}\right)w(\gamma-\gamma')\\
&\qquad = N^*(T) + 2\sum_{\substack{0<\gamma,\gamma'\leqslant T\\ 0<\gamma-\gamma'}} f\left((\gamma-\gamma')\frac{r(f)\log T}{2\pi\beta}\right)w(\gamma-\gamma')\\
&\qquad \leqslant N^*(T) + 2\sum_{\substack{0<\gamma,\gamma'\leqslant T\\ 0<\gamma-\gamma'\leqslant\frac{2\pi\beta}{\log T}}} f\left((\gamma-\gamma')\frac{r(f)\log T}{2\pi\beta}\right)w(\gamma-\gamma')\\
&\qquad \leqslant N^*(T) + 2N(T,\beta)\\
&\qquad = (1+o(1))N(T) + 2N(T,\beta),
\end{aligned}$$

where in the last step we have used (6.1.2). Then, we obtain

$$\frac{N(T,\beta)}{N(T)} \geqslant \frac{p_f(\lambda)}{2} + o(1).$$

Noting that $N(T,\beta)$ increases with β, we can then choose β arbitrarily close to $\mathcal{P}(f)$ and obtain the desired result. □

6.5.2 Bounding $N_{\Phi,s}(Q)$

Define the following functional over $\mathcal{A}_{LP}$:

$$\mathcal{L}(f) = \frac{r(f)}{2} + \frac{4}{r(f)}\int_0^{r(f)/2} f(x)x\,\mathrm{d}x + 2\int_{r(f)/2}^{r(f)} f(x)\,\mathrm{d}x.$$

We have the following theorem.

Theorem 6.11. *Let $f\in\mathcal{A}_{LP}$. Assuming GRH, for every fixed small $\delta>0$ we have, as $Q\to\infty$*

$$N_{\Phi,s}(Q) \geqslant (2-\mathcal{L}(f)+O(\delta)+o(1))N_\Phi(Q).$$

Proof. For $Q>1$ and $\alpha\in\mathbb{R}$, we define the pair correlation function F_Φ by

$$F_\Phi(Q^\alpha,W) = \frac{1}{N_\Phi(Q)}\sum_{Q\leqslant q\leqslant 2Q}\frac{W(q/Q)}{\varphi(q)}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \text{primitive}}}\left|\sum_{\gamma_\chi}\mathcal{M}\Phi(i\gamma_\chi)Q^{i\alpha\gamma_\chi}\right|^2.$$

Using the asymptotic large sieve, Chandee, Lee, Liu and Radziwiłł [28] showed the following asymptotic formula under GRH

$$\begin{aligned} &F_\Phi(Q^\alpha, W) \\ &= (1+o(1))\left[1-(1-|\alpha|)_+ + \Phi\big(Q^{-|\alpha|}\big)^2 \log Q \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \big|\mathcal{M}\Phi(it)\big|^2 \,\mathrm{d}t\right)^{-1}\right] \\ &\quad + O\Big(\Phi(Q^{-|\alpha|}) \log^{1/2} Q\Big), \end{aligned} \tag{6.5.3}$$

which holds uniformly for $|\alpha| \leqslant 2-\delta$, as $Q \to \infty$, for any fixed and sufficiently small $\delta > 0$. Let

$$N_\Phi^*(Q) := \sum_{Q \leqslant q \leqslant 2Q} \frac{W(q/Q)}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \text{primitive}}} \sum_{\gamma_\chi} m_{\rho_\chi} \big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2,$$

where m_{ρ_χ} denote the multiplicity of the nontrivial zero $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ of $L(s,\chi)$. Since

$$\sum_{\substack{\gamma_\chi \\ \text{simple}}} \big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2 \geqslant \sum_{\gamma_\chi} (2-m_{\rho_\chi}) \big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2$$

we obtain

$$N_{\Phi,s}(Q) \geqslant 2N_\Phi(Q) - N_\Phi^*(Q). \tag{6.5.4}$$

For any $g \in L^1(\mathbb{R})$ with $\widehat{g} \in L^1(\mathbb{R})$ we have the following explicit formula (Fourier inversion)

$$\begin{aligned} \sum_{Q \leqslant q \leqslant 2Q} \frac{W(q/Q)}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \text{primitive}}} \sum_{\gamma_\chi, \gamma'_\chi} \mathcal{M}\Phi(i\gamma_\chi)\mathcal{M}\Phi(i\gamma'_\chi)\, \widehat{g}\left(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\right) \\ = N_\Phi(Q) \int_{-\infty}^{\infty} g(\alpha) F_\Phi(Q^\alpha, W)\,\mathrm{d}\alpha. \end{aligned}$$

Letting $f \in \mathcal{A}_{LP}$ and $g(x) = f(r(f)x/(2-\delta))$, for any primitive character χ (mod q) we obtain

$$\begin{aligned} &\sum_{\gamma_\chi, \gamma'_\chi} \mathcal{M}\Phi(i\gamma_\chi)\mathcal{M}\Phi(i\gamma'_\chi)\, \widehat{g}\left(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\right) \\ &\qquad = \sum_{\gamma_\chi} m_{\rho_\chi} \big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2 \widehat{g}(0) + \sum_{\gamma_\chi \neq \gamma'_\chi} \mathcal{M}\Phi(i\gamma_\chi)\mathcal{M}\Phi(i\gamma'_\chi)\, \widehat{g}\left(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\right) \\ &\qquad \geqslant \frac{2-\delta}{r(f)} \sum_{\gamma_\chi} m_{\rho_\chi} \big|\mathcal{M}\Phi(i\gamma_\chi)\big|^2. \end{aligned}$$

This implies that

$$\sum_{Q\leqslant q\leqslant 2Q}\frac{W(q/Q)}{\varphi(q)}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \text{primitive}}}\sum_{\gamma_\chi,\gamma'_\chi}\mathcal{M}\Phi(i\gamma_\chi)\mathcal{M}\Phi(i\gamma'_\chi)g\left(\frac{(\gamma_\chi-\gamma'_\chi)\log Q}{2\pi}\right)\geqslant\frac{2-\delta}{r(f)}N^*_\Phi(Q).$$

On the other hand, observing that

$$\Phi\big(Q^{-|\alpha|}\big)^2\log Q\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\big|\mathcal{M}\Phi(it)\big|^2\,\mathrm{d}t\right)\to\boldsymbol{\delta}(\alpha),$$

as $Q\to\infty$ (in the distributional sense) and that

$$(\log Q)^{1/2}\int_{-(2-\delta)}^{2-\delta}g(\alpha)\Phi(Q^{-|\alpha|})\,\mathrm{d}\alpha\leqslant 2\log^{-1/2}Q\int_{Q^{-(2-\delta)}}^{1}\Phi(t)\frac{\mathrm{d}t}{t}=O\big((\log Q)^{-1/2}\big),$$

we can use the asymptotic estimate (6.5.3) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty}g(\alpha)F_\Phi(Q^\alpha,W)\,\mathrm{d}\alpha&\leqslant\int_{-(2-\delta)}^{2-\delta}g(\alpha)F_\Phi(Q^\alpha,W)\,\mathrm{d}\alpha\\&=g(0)+\int_{-(2-\delta)}^{2-\delta}g(\alpha)(1-(1-|\alpha|)_+)\,\mathrm{d}\alpha+O\big((\log Q)^{-1/2}\big)+o(1)\\&=\frac{2\mathcal{L}(f)}{r(f)}+O(\delta)+o(1).\end{aligned}$$

We then conclude that

$$N^*_\Phi(Q)\leqslant N_\Phi(Q)\left(\mathcal{L}(f)+O(\delta)+o(1)\right).$$

Using (6.5.4) we finish the proof. □

6.5.3 Bounding $N_1^*(T)$

Similarly to the case of the Riemann zeta-function, the functionals that we need to define depend on the asymptotic behavior of the function $F_1(\alpha,T)$ defined by

$$F_1(\alpha,T)=N_1(T)^{-1}\sum_{0<\gamma_1,\gamma_1'\leqslant T}T^{i\alpha(\gamma_1-\gamma_1')}w(\gamma_1-\gamma_1'),\tag{6.5.5}$$

where $\alpha\in\mathbb{R}$, $T\geqslant 2$ and the sum is over pairs of ordinates of zeros (with multiplicity) of $\xi'(s)$. To analyze $N_1^*(T)$ we define the following functional

$$\begin{aligned}\mathcal{Z}_1(f)=r(f)&+\frac{2}{r(f)}\int_0^{r(f)}x\,f(x)\,\mathrm{d}x-\frac{8}{r(f)^2}\int_0^{r(f)}x^2\,f(x)\,\mathrm{d}x\\&+\sum_{k=1}^{\infty}\frac{2c_k}{r(f)^{2k+1}}\int_0^{r(f)}x^{2k+1}\,f(x)\,\mathrm{d}x,\end{aligned}$$

where $c_k = 2^{2k+1}\frac{(k-1)!}{(2k)!}$.

Theorem 6.12. *Let $f \in \mathcal{A}_{LP}$. Assuming RH, for every fixed small $\delta > 0$ we have*

$$N_1^*(T) \leqslant (\mathcal{Z}_1(f) + O(\delta) + o(1))N_1(T).$$

Proof. A result similar to (6.5.1) for the function $F_1(\alpha, T)$ defined in (6.5.5) is also known (see [39, Theorem 1.1]), which is the following: for any fixed small $\delta > 0$ we have

$$F_1(\alpha, T) = T^{-2|\alpha|}\log T + |\alpha| - 4|\alpha|^2 + \sum_{k=1}^{\infty} c_k|\alpha|^{2k+1} + o(1)(1 + T^{-2|\alpha|}\log T),$$

uniformly for $|\alpha| \leqslant 1 - \delta$, as $T \to \infty$, where $c_k = 2^{2k+1}\frac{(k-1)!}{(2k)!}$. The proof then follows the same strategy as the proof for $\zeta(s)$ and we leave the details to the reader. □

6.6 Numerically optimizing the bounds

Going back to the sphere packing problem, since we obviously have $\Delta(\mathbb{R}^1) = 1$, this shows $r(f) \geqslant 1$ for all $f \in \mathcal{A}_{LP}$. The last sign change equals 1 for two (suspiciously) well-known functions: the hat function and its Fourier transform

$$H(x) = (1 - |x|)_+ \quad \text{and} \quad \widehat{H}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2,$$

and the Selberg's function with its Fourier transform

$$S(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \frac{1}{(1 - x^2)} \quad \text{and} \quad \widehat{S}(\xi) = \begin{cases} 1 - |\xi| + \frac{\sin(2\pi\xi)}{2\pi} & \text{if } |\xi| \leqslant 1 \\ 0 & \text{if } |\xi| > 1. \end{cases}$$

In particular, we can use these two functions to evaluate the functionals derived in Section 6.5 to obtain bounds, but this does not result in the best possible bounds. To obtain better bounds we use the class of functions used in the linear programming bounds by Cohn and Elkies [32] for sphere packing. That is, we consider the subspace $\mathcal{A}_{LP}(d)$ consisting of the functions $f \in \mathcal{A}_{LP}$ of the form

$$f(x) = p(x)e^{-\pi x^2}, \tag{6.6.1}$$

where p is an even polynomial of degree $2d$.

In [32], optimization over a closely related class of functions is done by specifying the functions by their real roots and optimizing the root locations. For the sphere packing problem this works very well, where in $\mathbb{R}^{24}$ it leads to a density upper bound that is sharp to within a factor $1 + 10^{-51}$ of the optimal configuration [35]. We have also tried this approach for the optimization problems in this chapter, but this did not work very well because the optimal functions seem to have very few real roots, which produces a strange effect in the numerical computations, where the last forced root tends to diverge when you increase the degree of the polynomial[3]. Instead we use sum-of-squares characterizations and semidefinite programming, as was done in [62] for the binary sphere packing problem.

Semidefinite programming is the optimization of a linear functional over the intersection of a cone of positive semidefinite matrices (real symmetric matrices with nonnegative eigenvalues) and an affine space. A semidefinite program is often given in block form, which can be written as

$$\begin{aligned} \text{minimize } \sum_{i=1}^{I} \operatorname{tr}(X_i C_i) : \ & \sum_{i=1}^{I} \operatorname{tr}(X_i A_{i,j}) = b_j \text{ for } j \in [m], \\ & X_1, \ldots, X_I \in \mathbb{R}^{n \times n} \text{ positive semidefinite,} \end{aligned}$$

where $I \in \mathbb{N}$ gives the number of blocks, $\{C_i\} \subseteq \mathbb{R}^{n\times n}$ is the objective, and $\{A_{i,j}\} \subseteq \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^m$ give the linear constraints (for notational simplicity we take all blocks to have the same size). Semidefinite programming is a broad generalization of linear programming (which we recover by setting $n = 1$ in the above formulation), and, as for linear programming, there exist efficient algorithms for solving them. The reason semidefinite programming comes into play here, is that we can model polynomial inequality constraints as sum-of-squares constraints, which in turn can be written as semidefinite constraints; see, e.g., [5].

6.6.1 Proof of Theorems 6.1, 6.6, and 6.7

To obtain the first part of Theorem 6.1 from Theorem 6.9 we need to minimize the functional $\mathcal{Z}$ over the space $\mathcal{A}_{LP}(d)$. We can see this as a bilevel optimization problem, where we optimize over scalars $R \geqslant 1$ in the outer problem, and over functions $f \in \mathcal{A}_{LP}(d)$ satisfying $r(f) = R$ in the inner problem. The outer problem is a simple one dimensional optimization problem for which we use Brent's method [10]. The inner problem can be written as a semidefinite program as we discuss below. The numerical results suggest that the optimal R goes to 1 as $d \to \infty$ (which is itself intriguing and so far we have no explanation), but for fixed d we need to optimize R to obtain a good bound.

A polynomial p that is nonnegative on $[R, \infty)$ can be written as $s_1(x) + (x - R)s_2(x)$, where s_1 and s_2 are sum-of-squares polynomials with $\deg(s_1), \deg(s_2(x)) + 1 \leqslant \deg(p)$; see, e.g., [77]. This shows that functions of the form (6.6.1) that are non-positive on $[R, \infty)$ can be written as

$$f(x) = -\big(s_1(x^2) + (x^2 - R^2)s_2(x^2)\big)e^{-\pi x^2}.$$

Let $v(x)$ be a vector whose entries form a basis of the univariate polynomials of degree at most d. The polynomials s_1 and s_2 are sum-of-squares if and only if they can be written as $s_i(x) = v(x)^T X_i v(x)$ for some positive semidefinite matrices X_i of size $d + 1$. That is, we can parameterize functions of the form (6.6.1) that are non-positive on $[R, \infty)$ by two positive semidefinite matrices X_1 and X_2 of size $d + 1$.

[3] It is worth mentioning that, in a related uncertainty problem, Cohn and Gonçalves [33] discovered the same kind of instability in low dimensions.

The space of functions of the form (6.6.1) is invariant under the Fourier transform. Since a polynomial of degree $2d$ that is nonnegative on $[0, \infty)$ can be written as $s_3(x) + xs_4(x)$, where $s_i(x) = v(x)^T X_i v(x)$ for $i = 3, 4$ are sum-of-squares polynomials of degree $2d$, we have that $\hat{f}$ is of the form

$$\widehat{f}(x) = \big(s_3(x^2) + x^2 s_4(x^2)\big)e^{-\pi x^2}.$$

Let $\mathcal{T}$ be the operator that maps x^{2k} to the function $\frac{k!}{\pi^k} L_k^{-1/2}(\pi x^2)$, where L_k is the Laguerre polynomial of degree k with parameter $-1/2$. Then, for p an even polynomial, we have that $(\mathcal{T}p)(x)e^{-\pi x^2}$ is the Fourier transform of $p(x)e^{-\pi x^2}$. We can now describe the functions of the form (6.6.1) that are non-positive on $[R, \infty)$ and have nonnegative Fourier transform by positive semidefinite matrices $X_1, \ldots, X_4$ of size $d + 1$ whose entries satisfy the linear relations coming from the identity $I(X_1, \ldots, X_4) = 0$, where

$$I(X_1, \ldots, X_4) = \mathcal{T}\big(-s_1(x^2) - (x^2 - R^2)s_2(x^2)\big) - \big(s_3(x^2) + x^2 s_4(x^2)\big).$$

Here $\mathcal{T}(-s_1(x^2) - (x^2 - R^2)s_2(x^2))$ is a polynomial whose coefficients are linear combinations in the entries of X_1 and X_2, and the same for $s_3(x^2) + x^2 s_4(x^2)$ with X_3 and X_4. The linear constraints on the entries of $X_1, \ldots, X_4$ are then obtained by expressing $I(X_1, \ldots, X_4)$ in some polynomial basis and setting the coefficients to zero.

The conditions $f(0) = 1$ and $f(R) = 0$ are linear in the entries of X_1 and X_2, and the condition $\widehat{f}(0) = 1$ is a linear condition on the entries of X_3 and X_4. Finally, the objective $\mathcal{Z}(f)$ is a linear combination in the entries of X_1 and X_2, which can be implemented by using the identity

$$\int x^m e^{-\pi x^2}\, \mathrm{d}x = -\frac{1}{2\pi^{m/2+1/2}}\Gamma\Big(\frac{m+1}{2}, \pi x^2\Big),$$

where Γ is the upper incomplete gamma function. Hence, the problem of minimizing $\mathcal{Z}(f)$ over functions $f \in \mathcal{A}_{LP}(d)$ that satisfy $r(f) = R$ is a semidefinite program.

To obtain the second part of Theorem 6.1 from Theorem 6.9 and to obtain Theorem 6.6 from 6.11 we use the same approach with a different functional. To obtain Theorem 6.7 from Theorem 6.12 we also do the same as above, but now truncate the series in the functional $\mathcal{Z}_1$ at $k = 15$ and add the easy to compute upper bound 10^{-10} on the remainder of the terms.

Implementation and numerical issues

In implementing the above as a semidefinite program we have to make two choices for the polynomial basis that we use: the basis defining the vector $v(x)$, and the basis to enforce the identity $I(X_1, \ldots, X_4) = 0$. This choice of bases is important for the numerical conditioning of the resulting semidefinite program. Following [62] we choose the Laguerre basis $\{L_n^{-1/2}(2\pi x^2)\}$, as this seems natural and performs well in practice (it multiplied by $e^{-\pi x^2}$ is the complete set of even eigenfunctions of the Fourier transform). We solve the semidefinite programs using sdpa-gmp [75], which is a primal-dual interior point solver using high precision floating point arithmetic. For the code to generate the semidefinite programs and to perform the post processing we use Julia [4], Nemo [40], and Arb [57] (where we use Arb for the ball arithmetic used in the verification procedure). For all computations we use $d = 40$. In solving the systems we observe that X_1 can be set to zero everywhere without affecting the bounds, so that $r(f) = R$ holds exactly for the function $f(x) = (R^2 - x^2)v(x^2)^T X_2 v(x^2)e^{-\pi x^2}$ defined by X_2.

The above optimization approach uses floating point arithmetic and a numerical interior point solver. This means the identity $I(0, X_2, X_3, X_4) = 0$ will not be satisfied exactly, and, moreover, because the solver can take infeasible steps the matrices X_2, X_3, and X_4 typically have some eigenvalues that are slightly negative. In practice this leads to incorrect upper bounds if the floating point precision is not high enough in relation to the degree d. Here we explain the procedure we use to obtain bounds that are guaranteed to be correct. This is an adaptation of the method from [62] and [69].

We first solve the above optimization problem numerically to find R and f for which we have a good objective value $v = \mathcal{L}(f)$. Then we solve the semidefinite program again for the same value of R, but now we solve it as a feasibility problem with the additional constraint $\mathcal{L}(f) \leqslant v + 10^{-6}$. The interior point solver will try to give the analytical center of the semidefinite program, so that typically the matrices are all positive definite; that is, the eigenvalues are all strictly positive. Then we use interval arithmetic to check rigorously that X_2, X_3, and X_4 are positive definite, and we compute a rigorous lower bound b on the smallest eigenvalues of X_3 and X_4.

Using interval arithmetic we compute an upper bound B on the largest coefficient of $I(0, X_2, X_3, X_4)$ in the basis given by the $2d + 1$ entries on the diagonal and upper diagonal of the matrix $(R^2 - x^2)v(x^2)v(x^2)^T$. If $b \geqslant (1 + 2d)B$, then it follows tat it is possible to modify the corresponding entries in X_3 and X_4 such that these matrices stay positive definite and such that $I(0, X_2, X_3, X_4) = 0$ holds exactly [69]. This shows that the Fourier transform of the function $f(x) = (R^2 - x^2)v(x^2)^T X_2 v(x^2)e^{-\pi x^2}$ is nonnegative.

We use interval arithmetic to compute $f(0) = R^2 s_2(0)$, $\mathcal{T}((R^2 - x^2)s_2(x^2))(0)$, and $\mathcal{Z}(f)$, $\tilde{\mathcal{Z}}(f)$, $\mathcal{Z}_1(f)$, or $\mathcal{L}(f)$. We can then compute rigorous bounds by observing that, for example, the first part of Theorem 6.1 can be written as follows: Suppose f is a continuous $L^1(\mathbb{R})$ function with $f(x) \leqslant 0$ for $|x| \geqslant R$ and with nonnegative Fourier transform, then

$$N^*(T) \leqslant \left(\frac{f(0)}{\hat{f}(0)} \mathcal{Z}(f) + o(1) \right) N(T).$$

Remark 6.13. *In the link* `https://arxiv.org/abs/1810.08843` *we attach the files 'Z-40.txt', 'tildeZ-40.txt', 'L-40.txt', and 'Z1-40.txt' that contain the value of R on the first line and the matrices X_2, X_3 and X_4 on the next 3 lines (all in 100 decimal floating point values). For convenience it also contains the coefficients of f in the monomial basis on the last line (but these are not used in the verification procedure). We include a script to perform the* *above verification and compute the bounds rigorously, as well as the code for setting up the semidefinite programs, using a custom semidefinite programming specification library.*

Now, we will show these functions (in the monomial basis) that we need to put in Theorems 6.9, 6.11 and 6.12 to prove Theorems 6.1, 6.6 and 6.7. Since that the coefficients of the functions are decimal numbers that have around 100 digits, we will truncate them in the following tables.

The function $f_1(x)$ for the functional $\mathcal{Z}$

In Theorem 6.9, using the function $f_1(x)$ defined by

$$f_1(x) = \sum_{k=0}^{81} a_k \, x^{2k} \, e^{-\pi x^2},$$

to evaluate the functional $\mathcal{Z}$ we obtain the first affirmation of Theorem 6.1. The coefficients a_k are given in the file 'Z-40.txt'. The following table contains the first 11 digits of the coefficients a_k written in the scientific form[4].

k	a_k	k	a_k	k	a_k
0	1.00000000000...e+00	27	- 2.00604252578...e+09	54	6.31600077580...e−13
1	- 5.61930744986...e−01	28	8.88765244247...e+08	55	- 3.58176329682...e−14
2	2.53470012494...e+01	29	- 3.61223860943...e+08	56	1.88492736201...e−15
3	- 5.91540175902...e+02	30	1.34839874801...e+08	57	- 9.19771200837...e−17
4	1.00403659527...e+04	31	- 4.62849221920...e+07	58	4.15752713885...e−18
5	- 1.23558354977...e+05	32	1.46273876941...e+07	59	- 1.73890163846...e−19
6	1.14437313949...e+06	33	- 4.26102745771...e+06	60	6.72107084162...e−21
7	- 8.14064754631...e+06	34	1.14544776797...e+06	61	- 2.39706427262...e−22
8	4.52000281877...e+07	35	- 2.84456407814...e+05	62	7.87512213760...e−24
9	- 1.99244854927...e+08	36	6.53237583584...e+04	63	- 2.37859909049...e−25
10	7.09652171095...e+08	37	- 1.38849606160...e+04	64	6.59016933930...e−27
11	- 2.07779244720...e+09	38	2.73404596754...e+03	65	- 1.67056564436...e−28
12	5.08355055658...e+09	39	- 4.99103491546...e+02	66	3.86307240298...e−30
13	- 1.05537806922...e+10	40	8.45281691947...e+01	67	- 8.12108907968...e−32
14	1.88600790950...e+10	41	- 1.32894661939...e+01	68	1.54588323804...e−33
15	- 2.93980682298...e+10	42	1.94064024091...e+00	69	- 2.65213735790...e−35
16	4.04507506325...e+10	43	- 2.63340360993...e−01	70	4.07836842393...e−37
17	- 4.96450359932...e+10	44	3.32196895016...e−02	71	- 5.58480037924...e−39
18	5.48089027934...e+10	45	- 3.89688519940...e−03	72	6.75672510240...e−41
19	- 5.47806082895...e+10	46	4.25195577287...e−04	73	- 7.15291516397...e−43
20	4.97839424495...e+10	47	- 4.31599240314...e−05	74	6.54672047966...e−45
21	- 4.12453522456...e+10	48	4.07596968487...e−06	75	- 5.10134804489...e−47
22	3.11950633425...e+10	49	- 3.58131005016...e−07	76	3.31644721518...e−49
23	- 2.15534325017...e+10	50	2.92738287922...e−08	77	- 1.74948709650...e−51
24	1.36094861588...e+10	51	- 2.22572598365...e−09	78	7.19154041640...e−54
25	- 7.85672038705...e+09	52	1.57364385611...e−10	79	- 2.16036249384...e−56
26	4.14918386342...e+09	53	- 1.03425977054...e−11	80	4.21713157530...e−59
				81	- 4.01324649596...e−62

[4]We recall that the notation me$\pm n$ means $m \cdot 10^{\pm n}$.

The function $f_2(x)$ for the functional $\widetilde{\mathcal{Z}}$

In Theorem 6.9, using the function $f_2(x)$ defined by

$$f_2(x) = \sum_{k=0}^{81} b_k\, x^{2k}\, e^{-\pi x^2},$$

to evaluate the functional $\widetilde{\mathcal{Z}}$ we obtain the second affirmation of Theorem 6.1. The coefficients b_k are given in the file 'tildeZ-40.txt'. The following table contains the first 11 digits of the coefficients b_k written in the scientific form.

k	b_k	k	b_k	k	b_k
0	1.00000000000...e+00	27	- 5.75974866587...e+07	54	5.40068435403...e−14
1	1.06665168220...e−01	28	2.45777816880...e+07	55	- 3.23922205805...e−15
2	6.81481916247...e+00	29	- 9.80603175046...e+06	56	1.80231560087...e−16
3	- 1.37110374214...e+02	30	3.65016083510...e+06	57	- 9.29503076196...e−18
4	2.26128992850...e+03	31	- 1.26520471124...e+06	58	4.43886712924...e−19
5	- 2.69844166980...e+04	32	4.07758498777...e+05	59	- 1.96064712420...e−20
6	2.38119359029...e+05	33	- 1.22073122870...e+05	60	7.99949934378...e−22
7	- 1.60226943210...e+06	34	3.39300155211...e+04	61	- 3.01029190208...e−23
8	8.41843420767...e+06	35	- 8.75415033780...e+03	62	1.04300993651...e−24
9	- 3.52227724480...e+07	36	2.09667557301...e+03	63	- 3.32082406067...e−26
10	1.19406304232...e+08	37	- 4.66253364582...e+02	64	9.69393663653...e−28
11	- 3.33068477173...e+08	38	9.62960956261...e+01	65	- 2.58777475250...e−29
12	7.74919615522...e+08	39	- 1.84769214729...e+01	66	6.29840798749...e−31
13	- 1.52196020968...e+09	40	3.29476957992...e+00	67	- 1.39289627024...e−32
14	2.55027648723...e+09	41	- 5.46170412293...e−01	68	2.78776155288...e−34
15	- 3.68102336235...e+09	42	8.41897556029...e−02	69	- 5.02590245604...e−36
16	4.61738945493...e+09	43	- 1.20704049913...e−02	70	8.11718658615...e−38
17	- 5.07627927018...e+09	44	1.60990476909...e−03	71	- 1.16677572973...e−39
18	4.93216323989...e+09	45	- 1.99782398529...e−04	72	1.48092961221...e−41
19	- 4.27111829619...e+09	46	2.30692178965...e−05	73	- 1.64382673631...e−43
20	3.32533440042...e+09	47	- 2.47880496990...e−06	74	1.57661969041...e−45
21	- 2.34853701839...e+09	48	2.47841742324...e−07	75	- 1.28668759190...e−47
22	1.51810502093...e+09	49	- 2.30562965247...e−08	76	8.75590310106...e−50
23	- 9.05722190738...e+08	50	1.99534288300...e−09	77	- 4.83207038146...e−52
24	5.02326758272...e+08	51	- 1.60603663997...e−10	78	2.07680004966...e−54
25	- 2.60341739732...e+08	52	1.20189031134...e−11	79	- 6.51937952081...e−57
26	1.26450831957...e+08	53	- 8.35927891328...e−13	80	1.32911223372...e−59
				81	- 1.32027652476...e−62

The function $f_3(x)$ for the functional $\mathcal{L}$

In Theorem 6.11, using the function $f_3(x)$ defined by

$$f_3(x) = \sum_{k=0}^{81} c_k \, x^{2k} \, e^{-\pi x^2},$$

to evaluate the functional $\mathcal{L}$ we obtain Theorem 6.6. The coefficients c_k are given in the file 'L-40.txt'. The following table contains the first 11 digits of the coefficients c_k written in the scientific form.

k	c_k	k	c_k	k	c_k
0	1.00000000000...e+00	27	- 1.62713819169...e+09	54	1.67488448322...e−13
1	- 1.47953929665...e−01	28	6.91432416117...e+08	55	- 9.21143270000...e−15
2	7.46561646903...e+00	29	- 2.69019242288...e+08	56	4.70533150477...e−16
3	- 1.41693776067...e+02	30	9.60033249385...e+07	57	- 2.23052537067...e−17
4	2.82315629755...e+03	31	- 3.14758048606...e+07	58	9.80276873641...e−19
5	- 4.16844775995...e+04	32	9.49576219383...e+06	59	- 3.98948768923...e−20
6	4.42792036366...e+05	33	- 2.63983078885...e+06	60	1.50154512964...e−21
7	- 3.48174087486...e+06	34	6.77177665034...e+05	61	- 5.21859627614...e−23
8	2.09053967925...e+07	35	- 1.60491853060...e+05	62	1.67189309621...e−24
9	- 9.86082020920...e+07	36	3.51826021315...e+04	63	- 4.92764518062...e−26
10	3.74601385112...e+08	37	- 7.14144888016...e+03	64	1.33308619818...e−27
11	- 1.17092463108...e+09	38	1.34352537567...e+03	65	- 3.30165720252...e−29
12	3.06697777952...e+09	39	- 2.34468342999...e+02	66	7.46377080670...e−31
13	- 6.83589677473...e+09	40	3.79874157642...e+01	67	- 1.53473749557...e−32
14	1.31328377541...e+10	41	- 5.71758267322...e+00	68	2.85900983404...e−34
15	- 2.19780303480...e+10	42	7.99953183571...e−01	69	- 4.80249057795...e−36
16	3.23153514097...e+10	43	- 1.04092919473...e−01	70	7.23417344271...e−38
17	- 4.20343793764...e+10	44	1.26029040153...e−02	71	- 9.70802347115...e−40
18	4.86364413100...e+10	45	- 1.42024956485...e−03	72	1.15148315210...e−41
19	- 5.02808494687...e+10	46	1.49011074771...e−04	73	- 1.19555396941...e−43
20	4.66133336021...e+10	47	- 1.45583045301...e−05	74	1.07357113943...e−45
21	- 3.88715083841...e+10	48	1.32459400923...e−06	75	- 8.21029998894...e−48
22	2.92384944296...e+10	49	- 1.12237508525...e−07	76	5.24023483503...e−50
23	- 1.98868177565...e+10	50	8.85607265152...e−09	77	- 2.71468483935...e−52
24	1.22595905682...e+10	51	- 6.50604105163...e−10	78	1.09617620394...e−54
25	- 6.86513734151...e+09	52	4.44884923783...e−11	79	- 3.23552971770...e−57
26	3.49949228901...e+09	53	- 2.83056293932...e−12	80	6.20724710063...e−60
				81	- 5.80679295834...e−63

The function $f_4(x)$ for the functional $\mathcal{Z}_1$

In Theorem 6.12, using the function $f_4(x)$ defined by

$$f_4(x) = \sum_{k=0}^{81} d_k\, x^{2k}\, e^{-\pi x^2},$$

to evaluate the functional $\mathcal{Z}_1$ we obtain Theorem 6.7. The coefficients d_k are given in the file 'Z1-40.txt'. The following table contains the first 11 digits of the coefficients d_k written in the scientific form.

k	d_k	k	d_k	k	d_k
0	9.99999999999...e−01	27	- 1.21739898850...e+09	54	7.33828073532...e−13
1	3.42888040970...e−01	28	5.47472801084...e+08	55	- 4.27557647019...e−14
2	8.62434074947...e+00	29	- 2.26364482361...e+08	56	2.31181372249...e−15
3	- 1.92714557575...e+02	30	8.61069449683...e+07	57	- 1.15907009029...e−16
4	3.95349282450...e+03	31	- 3.01579160626...e+07	58	5.38325088226...e−18
5	- 6.05529323704...e+04	32	9.73414002879...e+06	59	- 2.31349758399...e−19
6	6.55351358594...e+05	33	- 2.89835879345...e+06	60	9.18795709960...e−21
7	- 5.15583483128...e+06	34	7.96885439807...e+05	61	- 3.36702063429...e−22
8	3.05343039445...e+07	35	- 2.02512227067...e+05	62	1.13659437771...e−23
9	- 1.40321299164...e+08	36	4.76126626898...e+04	63	- 3.52732505443...e−25
10	5.13245942674...e+08	37	- 1.03654777780...e+04	64	1.00412614146...e−26
11	- 1.52644075646...e+09	38	2.09123707416...e+03	65	- 2.61524107399...e−28
12	3.75986830304...e+09	39	- 3.91277860362...e+02	66	6.21332653301...e−30
13	- 7.79552991505...e+09	40	6.79398761236...e+01	67	- 1.34194006086...e−31
14	1.38061105251...e+10	41	- 1.09541608687...e+01	68	2.62424977373...e−33
15	- 2.11711001129...e+10	42	1.64086820335...e+00	69	- 4.62502476396...e−35
16	2.84693484648...e+10	43	- 2.28455587633...e−01	70	7.30586885340...e−37
17	- 3.39711275245...e+10	44	2.95749901834...e−02	71	- 1.02762854379...e−38
18	3.63574085438...e+10	45	- 3.56099424334...e−03	72	1.27697251610...e−40
19	- 3.52220170924...e+10	46	3.98876506824...e−04	73	- 1.38840744239...e−42
20	3.11106477178...e+10	47	- 4.15709505586...e−05	74	1.30501681974...e−44
21	- 2.51798139126...e+10	48	4.03140031534...e−06	75	- 1.04424840952...e−46
22	1.87282572851...e+10	49	- 3.63774216039...e−07	76	6.97083689452...e−49
23	- 1.28161059657...e+10	50	3.05406002297...e−08	77	- 3.77555102113...e−51
24	8.06981195014...e+09	51	- 2.38514645395...e−09	78	1.59335518452...e−53
25	- 4.67342328757...e+09	52	1.73231592460...e−10	79	- 4.91360551692...e−56
26	2.48807657650...e+09	53	- 1.16964668909...e−11	80	9.84542414954...e−59
				81	- 9.61648178295...e−62

6.6.2 Proof of Theorem 6.4

To obtain the first part of Theorem 6.4 from Theorem 6.10 we need to minimize the function $\mathcal{P}$ over the space $\mathcal{A}_{LP}$. We can formulate this as a bilevel optimization problem in which we optimize over $R \geqslant 1$ in the outer problem. In the inner problem we perform a binary search over Λ to find the smallest Λ for which there exists a function $f \in \mathcal{A}_{LP}(d)$ that satisfies $f(R) = 0$, $f(x) \leqslant 0$ for $|x| \geqslant R$, and $p_f(\Lambda) \geqslant 0$.

To get a bound whose correctness we can verify rigorously we replace the constraints $f(0) = 1$, $\widehat{f}(0) = 1$, and $p_f(\Lambda) \geqslant 0$ by $f(0) = 1-10^{-10}$, $\widehat{f}(0) = 1+10^{-10}$, and $p_f(\Lambda) \geqslant 10^{-10}$. We then use the above optimization approach to find good values for R and Λ. We then add 10^{-6} to Λ and solve the feasibility problem again to get the strictly feasible matrices X_2, X_3, and X_4. By performing the same procedure as in 6.6.1 we can verify that the Fourier transform of the function f defined by X_2 is nonnegative everywhere, and using interval arithmetic we can check that the inequalities $f(0) \leqslant 1$, $\widehat{f}(0) \geqslant 1$, and $p_f(\Lambda) > 0$ all hold. Note that this verification procedure does not actually check that Λ is equal to or even close to $\mathcal{P}(f)$, but the proof of Theorem 6.10 also works if we replace $\mathcal{P}(f)$ by any Λ for which $p_f(\Lambda)$ is strictly positive. To obtain the second part of the theorem, we do the same except that we replace p_f by $\widetilde{p}_f$.

Remark 6.14. *In the link `https://arxiv.org/abs/1810.08843` we attach the files 'P-40.txt', 'tildeP-40.txt', that have the same layout as the files mentioned in 6.6.1, with an additional line containing the value of* Λ. *We again include the code to perform the verification and to produce the files.*

Now, we will show these functions (in the monomial basis) that we need to put in Theorem 6.10 to prove Theorem 6.4. Since that the coefficients of the functions are decimal numbers that have around 100 digits, we will truncate them in the following tables.

The function $f_5(x)$ for the functional $\mathcal{P}$

In Theorem 6.10, using the function $f_5(x)$ defined by

$$f_5(x) = \sum_{k=0}^{81} h_k \, x^{2k} \, e^{-\pi x^2},$$

to evaluate the functional $\mathcal{P}$ we obtain the first affirmation of Theorem 6.4. The coefficients h_k are given in the file 'P-40.txt'. The following table contains the first 11 digits of the coefficients h_k written in the scientific form.

k	h_k	k	h_k	k	h_k
0	9.99999999899...e−01	27	- 2.86914456546...e+08	54	6.51474548305...e−14
1	7.94132965649...e−01	28	1.25801976387...e+08	55	- 3.66543897877...e−15
2	4.58924844700...e+00	29	- 5.04928972482...e+07	56	1.91375946657...e−16
3	- 5.46240567761...e+01	30	1.85867378275...e+07	57	- 9.26387211404...e−18
4	4.74094778540...e+02	31	- 6.28582298290...e+06	58	4.15326060530...e−19
5	- 3.63771002122...e+03	32	1.95614912652...e+06	59	- 1.72248358970...e−20
6	2.64054225852...e+04	33	- 5.60996740119...e+05	60	6.59919313741...e−22
7	- 1.78884628040...e+05	34	1.48464174938...e+05	61	- 2.33190384609...e−23
8	1.06308836764...e+06	35	- 3.63011354411...e+04	62	7.58632526338...e−25
9	- 5.31859854914...e+06	36	8.20999417529...e+03	63	- 2.26756514046...e−26
10	2.21249623889...e+07	37	- 1.71922020984...e+03	64	6.21262812276...e−28
11	- 7.67460850361...e+07	38	3.33645434583...e+02	65	- 1.55600621464...e−29
12	2.23846189810...e+08	39	- 6.00567342312...e+01	66	3.55164221297...e−31
13	- 5.54293209856...e+08	40	1.00341178482...e+01	67	- 7.36185981921...e−33
14	1.17615184025...e+09	41	- 1.55710260797...e+00	68	1.38007193483...e−34
15	- 2.15678057442...e+09	42	2.24552464415...e−01	69	- 2.32858377264...e−36
16	3.44398100572...e+09	43	- 3.01082908649...e−02	70	3.51652660973...e−38
17	- 4.82117558548...e+09	44	3.75483417992...e−03	71	- 4.72135477088...e−40
18	5.95224307789...e+09	45	- 4.35676664728...e−04	72	5.59066088338...e−42
19	- 6.51565440493...e+09	46	4.70438640957...e−05	73	- 5.78159915473...e−44
20	6.35411193002...e+09	47	- 4.72788183876...e−06	74	5.15855052456...e−46
21	- 5.54409593739...e+09	48	4.42260311100...e−07	75	- 3.90982168592...e−48
22	4.34474248411...e+09	49	- 3.85054353274...e−08	76	2.46642462092...e−50
23	- 3.06885201539...e+09	50	3.11993063978...e−09	77	- 1.25923320536...e−52
24	1.95997402875...e+09	51	- 2.35208223876...e−10	78	4.99594272182...e−55
25	- 1.13514178605...e+09	52	1.64933364060...e−11	79	- 1.44425015692...e−57
26	5.97765655014...e+08	53	- 1.07530724996...e−12	80	2.70454837045...e−60
				81	- 2.46093963203...e−63

The function $f_6(x)$ for the functional $\widetilde{\mathcal{P}}(f)$

In Theorem 6.10, using the function $f_6(x)$ defined by

$$f_6(x) = \sum_{k=0}^{81} j_k\, x^{2k}\, e^{-\pi x^2},$$

to evaluate the functional $\widetilde{\mathcal{P}}(f)$ we obtain the second affirmation of Theorem 6.4. The coefficients j_k are given in the file 'tildeP-40.txt'. The following table contains the first 11 digits of the coefficients j_k written in the scientific form.

k	j_k	k	j_k	k	j_k
0	9.99999999899...e−01	27	- 1.27065723098...e+09	54	3.30671952407...e−14
1	7.46321420919...e−01	28	5.28134626882...e+08	55	- 1.58145385656...e−15
2	1.34437583052...e+01	29	- 2.01523901671...e+08	56	6.95427402084...e−17
3	- 4.49802718765...e+02	30	7.06577362276...e+07	57	- 2.81297135448...e−18
4	9.03498933773...e+03	31	- 2.27824912283...e+07	58	1.04880280566...e−19
5	- 1.20474288673...e+05	32	6.76068228575...e+06	59	- 3.62221467987...e−21
6	1.13367854426...e+06	33	- 1.84779880130...e+06	60	1.16964211898...e−22
7	- 7.92467843245...e+06	34	4.65488953673...e+05	61	- 3.58400774583...e−24
8	4.28672860773...e+07	35	- 1.08157748267...e+05	62	1.06214317301...e−25
9	- 1.85234402075...e+08	36	2.31946034755...e+04	63	- 3.09706067285...e−27
10	6.55035956018...e+08	37	- 4.59373339347...e+03	64	8.93850975692...e−29
11	- 1.93050352411...e+09	38	8.40697629773...e+02	65	- 2.52746385116...e−30
12	4.80811884789...e+09	39	- 1.42242137740...e+02	66	6.85049985153...e−32
13	- 1.02318487175...e+10	40	2.22597522485...e+01	67	- 1.73720864386...e−33
14	1.87754961403...e+10	41	- 3.22308549919...e+00	68	4.03802844482...e−35
15	- 2.99498151229...e+10	42	4.31918748556...e−01	69	- 8.47147306837...e−37
16	4.18380327494...e+10	43	- 5.35785142539...e−02	70	1.58545800407...e−38
17	- 5.15364408403...e+10	44	6.15282598216...e−03	71	- 2.62161285132...e−40
18	5.63399311209...e+10	45	- 6.54101602948...e−04	72	3.79509447524...e−42
19	- 5.49863231365...e+10	46	6.43642772940...e−05	73	- 4.76227643830...e−44
20	4.81684610156...e+10	47	- 5.86090089095...e−06	74	5.11962404543...e−46
21	- 3.80540307750...e+10	48	4.93673398982...e−07	75	- 4.64566920746...e−48
22	2.72231101464...e+10	49	- 3.84459071508...e−08	76	3.48919510158...e−50
23	- 1.76950624351...e+10	50	2.76641133343...e−09	77	- 2.11093203805...e−52
24	1.04796869377...e+10	51	- 1.83782601851...e−10	78	9.88537409836...e−55
25	- 5.66748772113...e+09	52	1.12623139508...e−11	79	- 3.36246819076...e−57
26	2.80379107554...e+09	53	- 6.36013436082...e−13	80	7.39090043478...e−60
				81	- 7.88001814579...e−63

Chapter 7

Appendices

7.1 Prelude

Throughout these appendices we encounter the following setting in multiple situations: let $c > 0$ be a given real number and $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 3$ be such that

$$(1-\sigma)^2 \log x \geqslant c. \tag{7.1.1}$$

Let us note that, if $0 \leqslant \theta_1, \theta_2$ are real numbers, it follows from (7.1.1) that

$$(1-\sigma)^{\theta_1} (\log x)^{\theta_2} \ll_{c,\theta_1,\theta_2} x^{1-\sigma}. \tag{7.1.2}$$

In fact, if $\theta_1 > \theta_2$ we simply observe that

$$(1-\sigma)^{\theta_1} (\log x)^{\theta_2} \leqslant (1-\sigma)^{\theta_2} (\log x)^{\theta_2} \ll_{\theta_2} x^{1-\sigma}.$$

On the other hand, if $0 \leqslant \theta_1 \leqslant \theta_2$, we let $\ell = \theta_2 - \theta_1 \geqslant 0$ and $\eta = \theta_1 + 2\ell = \theta_2 + \ell$ to obtain

$$(1-\sigma)^{\theta_1} (\log x)^{\theta_2} \ll_{c,\theta_1,\theta_2} (1-\sigma)^{\theta_1} (\log x)^{\theta_2} \big((1-\sigma)^2 \log x\big)^{\ell} = \big((1-\sigma) \log x\big)^{\eta} \ll_{\eta} x^{1-\sigma}.$$

We now proceed with the calculus facts required for our analysis.

7.2 Appendix A: Calculus facts

A.1 *Let $c > 0$ be a given real number and $m \geqslant 0$ be a given integer. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ such that $(1-\sigma)^2 \log x \geqslant c$, we have*

$$\int_2^x \frac{1}{t^{\sigma} (\log t)^{2m+2}} \, \mathrm{d}t = \frac{x^{1-\sigma}}{(1-\sigma)(\log x)^{2m+2}} + O_{m,c}\left(\frac{x^{1-\sigma}}{(1-\sigma)^2 (\log x)^{2m+3}} \right).$$

Proof. Using integration by parts we get

$$\begin{aligned}\int_2^x \frac{1}{t^\sigma(\log t)^{2m+2}}\,\mathrm{d}t =&\frac{x^{1-\sigma}}{(1-\sigma)(\log x)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(\log 2)^{2m+2}}\\ &+\frac{(2m+2)}{(1-\sigma)}\int_2^x \frac{1}{t^\sigma\,(\log t)^{2m+3}}\,\mathrm{d}t.\end{aligned} \tag{7.2.1}$$

From (7.1.2) we have

$$\frac{2^{1-\sigma}}{(1-\sigma)(\log 2)^{2m+2}} \ll_m \frac{1}{(1-\sigma)} \ll_{m,c} \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}, \tag{7.2.2}$$

and

$$\begin{aligned}\int_2^x \frac{1}{t^\sigma\,(\log t)^{2m+3}}\,\mathrm{d}t &= \int_2^{x^{2/3}} \frac{1}{t^\sigma(\log t)^{2m+3}}\,\mathrm{d}t + \int_{x^{2/3}}^x \frac{1}{t^\sigma(\log t)^{2m+3}}\,\mathrm{d}t\\ &\leqslant \frac{1}{(\log 2)^{2m+3}}\int_2^{x^{2/3}} \frac{1}{t^\sigma}\,\mathrm{d}t + \frac{1}{(\log(x^{2/3}))^{2m+3}}\int_{x^{2/3}}^x \frac{1}{t^\sigma}\,\mathrm{d}t\\ &\ll_m \frac{x^{\frac{2}{3}(1-\sigma)}}{(1-\sigma)} + \frac{x^{1-\sigma}}{(1-\sigma)(\log x)^{2m+3}}\\ &\ll_{m,c} \frac{x^{1-\sigma}}{(1-\sigma)(\log x)^{2m+3}}.\end{aligned} \tag{7.2.3}$$

The desired inequality follows by combining (7.2.1), (7.2.2), and (7.2.3). □

A.2 *Let $c > 0$ be a given real number and $m \geqslant 0$ and $k \geqslant 1$ be given integers. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ such that $(1-\sigma)^2 \log x \geqslant c$, we have*

$$\begin{aligned}\int_2^x \frac{1}{t^\sigma(k\log x + \log t)^{2m+2}}\,\mathrm{d}t = &\frac{x^{1-\sigma}}{(1-\sigma)((k+1)\log x)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(k\log x + \log 2)^{2m+2}}\\ &+ O_{m,c}\left(\frac{x^{1-\sigma}}{(1-\sigma)^2((k+1)\log x)^{2m+3}}\right).\end{aligned}$$

Proof. Using the change of variables $y = x^k t$ and **A.1** we obtain

$$\begin{aligned}\int_2^x \frac{1}{t^\sigma(k\log x + \log t)^{2m+2}}\,\mathrm{d}t &= x^{-k+k\sigma}\int_{2x^k}^{x^{k+1}} \frac{1}{y^\sigma(\log y)^{2m+2}}\,\mathrm{d}y\\ &= x^{-k+k\sigma}\left[\int_2^{x^{k+1}} \frac{1}{y^\sigma(\log y)^{2m+2}}\,\mathrm{d}y - \int_2^{2x^k} \frac{1}{y^\sigma(\log y)^{2m+2}}\,\mathrm{d}y\right]\\ &= x^{-k+k\sigma}\Bigg[\frac{(x^{k+1})^{1-\sigma}}{(1-\sigma)(\log x^{(k+1)})^{2m+2}} + O_{m,c}\left(\frac{(x^{k+1})^{1-\sigma}}{(1-\sigma)^2(\log x^{(k+1)})^{2m+3}}\right)\\ &\qquad - \frac{(2x^k)^{1-\sigma}}{(1-\sigma)(\log(2x^k))^{2m+2}} + O_{m,c}\left(\frac{(2x^k)^{1-\sigma}}{(1-\sigma)^2(\log(2x^k))^{2m+3}}\right)\Bigg]\\ &= \frac{x^{1-\sigma}}{(1-\sigma)((k+1)\log x)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(k\log x + \log 2)^{2m+2}}\end{aligned}$$

$$+ O_{m,c}\bigg(\frac{x^{1-\sigma}}{(1-\sigma)^2((k+1)\log x)^{2m+3}}\bigg) + O_{m,c}\bigg(\frac{1}{(1-\sigma)^2(k\log x+\log 2)^{2m+3}}\bigg).$$

Since

$$\frac{1}{(1-\sigma)^2(k\log x+\log 2)^{2m+3}} \leqslant \frac{2^{2m+3}}{(1-\sigma)^2((k+1)\log x)^{2m+3}} \ll_m \frac{x^{1-\sigma}}{(1-\sigma)^2((k+1)\log x)^{2m+3}},$$

we obtain the desired result. □

A.3 *Let $m \geqslant 0$ and $k \geqslant 0$ be given integers. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ we have*

$$\int_2^x \frac{1}{t^{1-\sigma}((k+2)\log x - \log t)^{2m+2}}\,\mathrm{d}t = \frac{x^\sigma}{\sigma((k+1)\log x)^{2m+2}} - \frac{2^\sigma}{\sigma((k+2)\log x - \log 2)^{2m+2}} + O_m\bigg(\frac{x^\sigma}{((k+1)\log x)^{2m+3}}\bigg).$$

Proof. Let $y = \frac{x^{k+2}}{t}$. The integral becomes

$$x^{(k+2)\sigma}\int_{x^{k+1}}^{x^{k+2}/2} \frac{1}{y^{1+\sigma}(\log y)^{2m+2}}\,\mathrm{d}y = \frac{x^\sigma}{\sigma((k+1)\log x)^{2m+2}} - \frac{2^\sigma}{\sigma((k+2)\log x - \log 2)^{2m+2}} - \frac{(2m+2)x^{(k+2)\sigma}}{\sigma}\int_{x^{k+1}}^{x^{k+2}/2}\frac{1}{y^{1+\sigma}(\log y)^{2m+3}}\,\mathrm{d}y\,,$$

where we have used integration by parts. Finally, the result follows from the fact that

$$\int_{x^{k+1}}^{x^{k+2}/2} \frac{1}{y^{1+\sigma}(\log y)^{2m+3}}\,\mathrm{d}y \ll \frac{1}{((k+1)\log x)^{2m+3}}\int_{x^{k+1}}^{x^{k+2}/2}\frac{1}{y^{1+\sigma}}\,\mathrm{d}y \ll \frac{1}{x^{(k+1)\sigma}((k+1)\log x)^{2m+3}}.$$

□

A.4 *For $\frac{1}{2} < \sigma < 1$ and $x \geqslant 2$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{\left(x^{\sigma-1/2}\right)^k} \leqslant \frac{1}{(\sigma-\frac{1}{2})\log x}.$$

Proof. Using the mean value theorem we have that

$$\sum_{k=1}^{\infty} \frac{1}{\left(x^{\sigma-1/2}\right)^k} = \frac{1}{x^{\sigma-1/2}-1} = \frac{1}{(\sigma-\frac{1}{2})x^\xi\log x} \leqslant \frac{1}{(\sigma-\frac{1}{2})\log x},$$

where ξ is a point in the interval $(0, \sigma-\frac{1}{2})$. □

A.5 *Let $m \geqslant 0$ be a given integer. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ we have*

$$\sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \left| \frac{2^{\sigma}}{x^{2\sigma-1}\left((k+2)\log x - \log 2\right)^{2m+2}} - \frac{2^{1-\sigma}}{(k\log x + \log 2)^{2m+2}} \right| \ll_m \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}},$$

and

$$\sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \left| \frac{2^{\sigma}}{x^{2\sigma-1}\sigma((k+2)\log x - \log 2)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(k\log x + \log 2)^{2m+2}} \right| \ll_m \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.$$

Proof. Using the mean value theorem for the functions $y \mapsto y^{2m+2}$ and $y \mapsto 2^{\sigma-y}x^y$ we obtain, for $k \geqslant 1$, that

$$\begin{aligned}
&\left| \frac{2^{\sigma}}{x^{2\sigma-1}((k+2)\log x - \log 2)^{2m+2}} - \frac{2^{1-\sigma}}{(k\log x + \log 2)^{2m+2}} \right| \\
&\leqslant \left| \frac{2^{\sigma}}{x^{2\sigma-1}} \left(\frac{1}{((k+2)\log x - \log 2)^{2m+2}} - \frac{1}{(k\log x + \log 2)^{2m+2}} \right) \right| \\
&\quad + \left| \frac{1}{(k\log x + \log 2)^{2m+2}} \left(\frac{2^{\sigma}}{x^{2\sigma-1}} - 2^{1-\sigma} \right) \right| \\
&= \frac{2^{\sigma}}{x^{2\sigma-1}} \left(\frac{((k+2)\log x - \log 2)^{2m+2} - (k\log x + \log 2)^{2m+2}}{((k+2)\log x - \log 2)^{2m+2}(k\log x + \log 2)^{2m+2}} \right) \\
&\quad + \frac{1}{(k\log x + \log 2)^{2m+2}} \left(\frac{2^{1-\sigma}x^{2\sigma-1} - 2^{\sigma}}{x^{2\sigma-1}} \right) \\
&\leqslant \frac{2^{\sigma}}{x^{2\sigma-1}} \left(\frac{2(2m+2)(\log x - \log 2)((k+2)\log x - \log 2)^{2m+1}}{((k+2)\log x - \log 2)^{2m+2}(k\log x + \log 2)^{2m+2}} \right) \\
&\quad + \frac{(2\sigma-1)2^{1-\sigma}(\log x - \log 2)}{(k\log x + \log 2)^{2m+2}} \\
&\ll_m \frac{1}{x^{2\sigma-1}(k+1)^{2m+3}(\log x)^{2m+2}} + \frac{(2\sigma-1)}{(k+1)^{2m+2}(\log x)^{2m+1}}.
\end{aligned}$$

Therefore, summing over all $k \geqslant 1$ and using **A.4**, we arrive at

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \left| \frac{2^{\sigma}}{x^{2\sigma-1}((k+2)\log x - \log 2)^{2m+2}} - \frac{2^{1-\sigma}}{(k\log x + \log 2)^{2m+2}} \right| \\
&\ll_m \sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \left(\frac{1}{x^{2\sigma-1}(k+1)^{2m+3}(\log x)^{2m+2}} + \frac{(2\sigma-1)}{(k+1)^{2m+2}(\log x)^{2m+1}} \right) \\
&\leqslant \frac{1}{x^{2\sigma-1}(\log x)^{2m+2}} \sum_{k=1}^{\infty} \frac{1}{\left(x^{\sigma--1/2}\right)^k (k+1)^{2m+2}} + \frac{2\sigma-1}{(\log x)^{2m+1}} \sum_{k=1}^{\infty} \frac{1}{\left(x^{\sigma-1/2}\right)^k} \\
&\ll \frac{1}{(\log x)^{2m+2}} \ll \frac{x^{1-\sigma}}{(1-\sigma)(\log x)^{2m+3}} \ll \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}},
\end{aligned}$$

which establishes our first proposed estimate. To prove the second, we use the first one and **A.4** as follows

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left|\frac{2^{\sigma}}{x^{2\sigma-1}\,\sigma((k+2)\log x-\log 2)^{2m+2}}-\frac{2^{1-\sigma}}{(1-\sigma)(k\log x+\log 2)^{2m+2}}\right| \\
&\quad\leqslant \frac{1}{\sigma}\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left|\frac{2^{\sigma}}{x^{2\sigma-1}((k+2)\log x-\log 2)^{2m+2}}-\frac{2^{1-\sigma}}{(k\log x+\log 2)^{2m+2}}\right| \\
&\qquad+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left|\frac{2^{1-\sigma}}{\sigma(k\log x+\log 2)^{2m+2}}-\frac{2^{1-\sigma}}{(1-\sigma)(k\log x+\log 2)^{2m+2}}\right| \\
&\quad\ll_m \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}+\frac{2\sigma-1}{\sigma(1-\sigma)}\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left(\frac{2^{1-\sigma}}{(k\log x+\log 2)^{2m+2}}\right) \\
&\quad\ll \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}+\frac{2\sigma-1}{\sigma(1-\sigma)(\log x)^{2m+2}}\sum_{k=1}^{\infty}\frac{1}{\left(x^{\sigma-1/2}\right)^k} \\
&\quad\ll \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}+\frac{1}{(1-\sigma)(\log x)^{2m+3}} \\
&\quad\ll \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.
\end{aligned}$$

□

A.6 *For $\frac{1}{2}<\sigma<1$ and $2\leqslant n\leqslant x$ we have*

$$0\leqslant\sum_{k=0}^{\infty}(-1)^k\left(\frac{k+1}{(\log nx^k)\,(nx^k)^{\sigma-1/2}}-\frac{k+1}{\left(\log\frac{x^{k+2}}{n}\right)\left(\frac{x^{k+2}}{n}\right)^{\sigma-1/2}}\right)$$
$$\leqslant\frac{1}{n^{\sigma-1/2}\log n}-\frac{n^{\sigma-1/2}}{(2\log x-\log n)x^{2\sigma-1}}.$$

Proof. See [14, Eq. (2.14), (2.16) and Lemma 6]. □

A.7 *Let z,w be complex numbers such that $|w|\leqslant 25$. Then*

$$(\log(|z+w|+3))^{16}\geqslant\log(|z|+3).$$

Proof. If $|z|>25$, then

$$(\log(|z+w|+3))^{16}\geqslant\log(|z|-|w|+3)(\log 3)^{15}>4\log(|z|-22)\geqslant\log(|z|+3),$$

since $(\log 3)^{15}>4$. On the other hand, if $|z|\leqslant 25$

$$(\log(|z+w|+3))^{16}\geqslant(\log 3)^{16}>4>\log(28)\geqslant\log(|z|+3).$$

□

7.3 Appendix B: Number theory facts

Recall that, under the Riemann hypothesis, the prime number theorem takes the form ([74, Section 13.1])

$$\sum_{n\leqslant x} \Lambda(n) = x + O\big(x^{1/2}(\log x)^2\big). \tag{7.3.1}$$

In what follows we shall use in integration by parts in multiple occasions. Let $\varepsilon > 0$ be a small number and $f : \Omega \to \mathbb{R}$, where $\Omega = \{(x, y) \in \mathbb{R}^2;\ 2 \leqslant x < \infty\ ;\ 1 \leqslant y \leqslant x + 2\varepsilon\}$, be a function such that $y \mapsto f(x, y)$ is continuously differentiable in $(1, x + \varepsilon)$, for all $x \in [2, \infty)$. Using (7.3.1) we obtain

$$\begin{aligned}\sum_{n\leqslant x} \Lambda(n) f(x, n) = \int_2^x f(x, y)\ \mathrm{d}y + 2f(x, 2) + O\big(x^{1/2}(\log x)^2\,|f(x, x)|\big)\\ + O\bigg(\int_2^x y^{1/2}(\log y)^2 \left|\frac{\partial}{\partial y} f(x, y)\right|\ \mathrm{d}y\bigg).\end{aligned} \tag{7.3.2}$$

We now proceed with the number theory facts required for our analysis. We assume the Riemann hypothesis in all the statements below.

B.1 *Let $c > 0$ be a given real number and $m \geqslant 0$ be an integer or $m = -\frac{1}{2}$. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ such that $(1 - \sigma)^2 \log x \geqslant c$, we have*

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n^\sigma (\log n)^{2m+2}} = \frac{x^{1-\sigma}}{(1 - \sigma)(\log x)^{2m+2}} + O_{m,c}\left(\frac{x^{1-\sigma}}{(1 - \sigma)^2(\log x)^{2m+3}}\right).$$

Proof. We will prove the case $m \geqslant 0$ be an integer. The case $m = -\frac{1}{2}$ need refinement in the calculus but it follows the same idea. Using (7.3.2), together with **A.1** and (7.1.2), we obtain

$$\begin{aligned}\sum_{n\leqslant x} \frac{\Lambda(n)}{n^\sigma (\log n)^{2m+2}} &= \int_2^x \frac{1}{y^\sigma (\log y)^{2m+2}}\,\mathrm{d}y + \frac{2^{1-\sigma}}{(\log 2)^{2m+2}} + O\bigg(\frac{x^{1/2}(\log x)^2}{x^\sigma (\log x)^{2m+2}}\bigg)\\ &\quad + O\left(\int_2^x y^{1/2}(\log y)^2 \left|\frac{\partial}{\partial y}\left[\frac{1}{y^\sigma (\log y)^{2m+2}}\right]\right|\ \mathrm{d}y\right)\\ &= \frac{x^{1-\sigma}}{(1 - \sigma)(\log x)^{2m+2}} + O_{m,c}\left(\frac{x^{1-\sigma}}{(1 - \sigma)^2(\log x)^{2m+3}}\right)\\ &\quad + O_m\left(\int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y\right).\end{aligned}$$

We now analyze the last term. From (7.1.2) we have

$$\int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y \leqslant \int_2^x \frac{1}{y}\ \mathrm{d}y \leqslant \log x \ll_{m,c} \frac{x^{1-\sigma}}{(1 - \sigma)^2(\log x)^{2m+3}}\,, \tag{7.3.3}$$

and this concludes the proof. □

B.2 *Let $c > 0$ be a given real number and $m \geqslant 0$ be an integer or $m = -\frac{1}{2}$. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ such that $(1-\sigma)^2 \log x \geqslant c$, we have*

$$\frac{1}{x^{2\sigma-1}} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1-\sigma}(2\log x - \log n)^{2m+2}} = \frac{x^{1-\sigma}}{\sigma(\log x)^{2m+2}} + O_{m,c}\left(\frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}\right).$$

Proof. We will prove the case $m \geqslant 0$ be an integer. Using (7.3.2) together with **A.3**, we have

$$\begin{aligned}
&\frac{1}{x^{2\sigma-1}} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1-\sigma}(2\log x - \log n)^{2m+2}} \\
&= \frac{1}{x^{2\sigma-1}} \int_2^x \frac{1}{y^{1-\sigma}(2\log x - \log y)^{2m+2}}\,\mathrm{d}y + \frac{2^\sigma}{x^{2\sigma-1}\,(2\log x - \log 2)^{2m+2}} \\
&\quad + O\left(\frac{1}{x^{\sigma-1/2}(\log x)^{2m}}\right) + O\left(\frac{1}{x^{2\sigma-1}} \int_2^x y^{1/2}(\log y)^2 \left|\frac{\partial}{\partial y}\left[\frac{1}{y^{1-\sigma}(2\log x - \log y)^{2m+2}}\right]\right| \mathrm{d}y\right) \\
&= \frac{x^{1-\sigma}}{\sigma(\log x)^{2m+2}} + O_m\left(\frac{x^{1-\sigma}}{(\log x)^{2m+3}}\right) + O(1) \\
&\quad + O\left(\frac{1}{x^{2\sigma-1}} \int_2^x y^{1/2}(\log y)^2 \left|\frac{\partial}{\partial y}\left[\frac{1}{y^{1-\sigma}(2\log x - \log y)^{2m+2}}\right]\right| \mathrm{d}y\right). \qquad (7.3.4)
\end{aligned}$$

We further analyze the last term

$$\begin{aligned}
\frac{1}{x^{2\sigma-1}} \int_2^x y^{1/2}(\log y)^2 &\left|\frac{\partial}{\partial y}\left[\frac{1}{y^{1-\sigma}(2\log x - \log y)^{2m+2}}\right]\right| \mathrm{d}y \\
&\ll_m \int_2^x \frac{(\log y)^2}{x^{2\sigma-1}\,y^{3/2-\sigma}(2\log x - \log y)^{2m+2}}\,\mathrm{d}y \\
&\leqslant \int_2^x \frac{(\log y)^2}{y^{2\sigma-1}\,y^{3/2-\sigma}(2\log x - \log y)^{2m+2}}\,\mathrm{d}y \\
&\leqslant \int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y.
\end{aligned}$$

Therefore, using (7.1.2) and (7.3.3) in (7.3.4) we obtain the result. □

B.3 *Let $c > 0$ be a given real number and $m \geqslant 0$ be a given integer. For $\frac{1}{2} \leqslant \sigma < 1$ and $x \geqslant 2$ such that $(1-\sigma)^2 \log x \geqslant c$, we have*

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} &\left| \sum_{n \leqslant x} \Lambda(n) \left(\frac{1}{n^\sigma(k\log x + \log n)^{2m+2}} - \frac{1}{x^{2\sigma-1}\,n^{1-\sigma}((k+2)\log x - \log n)^{2m+2}} \right) \right| \\
&\ll_{m,c} \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.
\end{aligned}$$

Besides, when $\frac{1}{2} < \sigma < 1$, we have

$$\sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \left| \sum_{n \leqslant x} \Lambda(n) \left(\frac{1}{n^\sigma(k\log x + \log n)} - \frac{1}{x^{2\sigma-1}\,n^{1-\sigma}((k+2)\log x - \log n)} \right) \right|$$

$$\ll_{m,c} \frac{x^{1-\sigma}}{(\sigma-\frac{1}{2})(1-\sigma)^2(\log x)^2}.$$

Proof. We will prove the first result. The second result follows the same outline. Using (7.3.2), **A.2** and **A.3** we have, for any $k \geqslant 1$,

$$\begin{aligned}
&\sum_{n\leqslant x} \Lambda(n)\left(\frac{1}{n^{\sigma}(k\log x+\log n)^{2m+2}} - \frac{1}{x^{2\sigma-1}\, n^{1-\sigma}((k+2)\log x-\log n)^{2m+2}}\right)\\
&= \int_2^x \left(\frac{1}{y^{\sigma}(k\log x+\log y)^{2m+2}} - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x-\log y)^{2m+2}}\right) \mathrm{d}y\\
&\quad + 2\left(\frac{1}{2^{\sigma}(k\log x+\log 2)^{2m+2}} - \frac{1}{x^{2\sigma-1}\, 2^{1-\sigma}((k+2)\log x-\log 2)^{2m+2}}\right)\\
&\quad + O\left(\int_2^x y^{1/2}(\log y)^2\left|\frac{\partial}{\partial y}\left[\frac{1}{y^{\sigma}(k\log x+\log y)^{2m+2}}\right.\right.\right.\\
&\qquad\qquad \left.\left.\left. - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x-\log y)^{2m+2}}\right]\right| \mathrm{d}y\right)\\
&= \frac{2\sigma-1}{\sigma(1-\sigma)}\frac{x^{1-\sigma}}{((k+1)\log x)^{2m+2}}\\
&\quad + \left(\frac{2^{\sigma}}{x^{2\sigma-1}\,\sigma((k+2)\log x-\log 2)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(k\log x+\log 2)^{2m+2}}\right)\\
&\quad + \left(\frac{2^{1-\sigma}}{(k\log x+\log 2)^{2m+2}} - \frac{2^{\sigma}}{x^{2\sigma-1}\,((k+2)\log x-\log 2)^{2m+2}}\right)\\
&\quad + O_{m,c}\left(\frac{x^{1-\sigma}}{(1-\sigma)^2((k+1)\log x)^{2m+3}}\right)\\
&\quad + O\left(\int_2^x y^{1/2}(\log y)^2\left|\frac{\partial}{\partial y}\left[\frac{1}{y^{\sigma}(k\log x+\log y)^{2m+2}}\right.\right.\right.\\
&\qquad\qquad \left.\left.\left. - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x-\log y)^{2m+2}}\right]\right| \mathrm{d}y\right). \qquad (7.3.5)
\end{aligned}$$

We now sum over $k \geqslant 1$ and analyze each term that appears in (7.3.5).

1. *First term*: Using **A.4** we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left(\frac{2\sigma-1}{\sigma(1-\sigma)}\frac{x^{1-\sigma}}{((k+1)\log x)^{2m+2}}\right) &\leqslant \frac{2\sigma-1}{\sigma(1-\sigma)}\frac{x^{1-\sigma}}{(\log x)^{2m+2}}\sum_{k=1}^{\infty}\frac{1}{\left(x^{\sigma-1/2}\right)^k}\\
&\ll \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.
\end{aligned}$$

2. *Second and third terms*: Using **A.5** we obtain

$$\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left|\frac{2^{\sigma}}{x^{2\sigma-1}\,\sigma((k+2)\log x-\log 2)^{2m+2}} - \frac{2^{1-\sigma}}{(1-\sigma)(k\log x+\log 2)^{2m+2}}\right|$$

$$+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\left|\frac{2^{1-\sigma}}{(k\log x+\log 2)^{2m+2}}-\frac{2^{\sigma}}{x^{2\sigma-1}((k+2)\log x-\log 2)^{2m+2}}\right|$$

$$\ll_m \frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.$$

3. *Fourth term*:

$$\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\frac{x^{1-\sigma}}{(1-\sigma)^2((k+1)\log x)^{2m+3}}\ll\frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^{2m+3}}.$$

4. *Fifth term*: Using **A.4** again we have

$$\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left|\frac{\partial}{\partial y}\left[\frac{1}{y^{\sigma}(k\log x+\log y)^{2m+2}}\right.\right.$$
$$\left.\left.-\frac{1}{x^{2\sigma-1}\,y^{1-\sigma}((k+2)\log x-\log y)^{2m+2}}\right]\right|\mathrm{d}y$$

$$=\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left|\frac{2m+2}{y^{1+\sigma}(k\log x+\log y)^{2m+3}}+\frac{\sigma}{y^{1+\sigma}(k\log x+\log y)^{2m+2}}\right.$$
$$\left.+\frac{1}{x^{2\sigma-1}}\left(\frac{2m+2}{y^{2-\sigma}((k+2)\log x-\log y)^{2m+3}}-\frac{1-\sigma}{y^{2-\sigma}((k+2)\log x-\log y)^{2m+2}}\right)\right|\mathrm{d}y$$

$$\leqslant\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{2m+2}{y^{1+\sigma}(k\log x+\log y)^{2m+3}}\right.$$
$$\left.+\frac{2m+2}{x^{2\sigma-1}\,y^{2-\sigma}((k+2)\log x-\log y)^{2m+3}}\right)\mathrm{d}y$$

$$+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{\sigma}{y^{1+\sigma}(k\log x+\log y)^{2m+2}}\right.$$
$$\left.-\frac{1-\sigma}{x^{2\sigma-1}\,y^{2-\sigma}((k+2)\log x-\log y)^{2m+2}}\right)\mathrm{d}y$$

$$\leqslant\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{4m+4}{y^{1+\sigma}(k\log x+\log y)^{2m+3}}\right)\mathrm{d}y$$
$$+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{2\sigma-1}{y^{1+\sigma}(k\log x+\log y)^{2m+2}}\right)\mathrm{d}y$$
$$+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{1-\sigma}{y^{1+\sigma}(k\log x+\log y)^{2m+2}}\right.$$
$$\left.-\frac{1-\sigma}{x^{2\sigma-1}\,y^{2-\sigma}((k+2)\log x-\log y)^{2m+2}}\right)\mathrm{d}y$$

$$\leqslant\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{4m+4}{y^{1+\sigma}((k+1)\log y)^{2m+3}}\right)\mathrm{d}y$$
$$+\sum_{k=1}^{\infty}\frac{k+1}{\left(x^{\sigma-1/2}\right)^k}\int_2^x y^{1/2}(\log y)^2\left(\frac{2\sigma-1}{y^{1+\sigma}((k+1)\log y)^{2m+2}}\right)\mathrm{d}y$$

$$
\begin{aligned}
&+ (1-\sigma) \sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \int_2^x \frac{(\log y)^2}{y^{1/2}} \bigg(\frac{1}{y^{\sigma}(k\log x + \log y)^{2m+2}} \\
&\qquad\qquad - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x - \log y)^{2m+2}} \bigg) \,\mathrm{d}y \\
&\ll_m \int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y + (2\sigma-1)\left(\int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y \right) \sum_{k=1}^{\infty} \frac{1}{\left(x^{\sigma-1/2}\right)^k} \\
&\quad + \sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \int_2^x \bigg(\frac{1}{y^{\sigma}(k\log x + \log y)^{2m+2}} \\
&\qquad\qquad - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x - \log y)^{2m+2}} \bigg) \,\mathrm{d}y \\
&\ll \int_2^x \frac{1}{y^{\sigma+1/2}}\,\mathrm{d}y + \sum_{k=1}^{\infty} \frac{k+1}{\left(x^{\sigma-1/2}\right)^k} \int_2^x \bigg(\frac{1}{y^{\sigma}(k\log x + \log y)^{2m+2}} \\
&\qquad\qquad - \frac{1}{x^{2\sigma-1}\, y^{1-\sigma}((k+2)\log x - \log y)^{2m+2}} \bigg) \,\mathrm{d}y.
\end{aligned} \tag{7.3.6}
$$

We can see that the last sum already appeared in our analysis, in the first, second and fourth terms treated above. Therefore, an application of (7.3.3) in (7.3.6) concludes the proof. □

B.4 *For* $0 \leqslant \beta < \frac{1}{2}$ *and* $x \geqslant 2$*, we have*

$$
\begin{aligned}
\sum_{n\leqslant x} \frac{\Lambda(n)}{n^{1/2}} \left(\frac{x^{\beta}}{n^{\beta}} - \frac{n^{\beta}}{x^{\beta}} \right) &= \frac{2\beta x^{1/2} - 2^{1/2-\beta} x^{\beta} \left(\frac{1}{2}+\beta\right)^2 + 2^{1/2+\beta} x^{-\beta} \left(\frac{1}{2}-\beta\right)^2}{\frac{1}{4}-\beta^2} \\
&\quad + O\left(\beta\, x^{\beta}\, (\log x)^4 \right).
\end{aligned}
$$

Besides, for $0 < \beta < \frac{1}{2}$ *and* $x \geqslant 2$ *such that* $(\frac{1}{2}-\beta)^2 \log x \geqslant c$, we have

$$
\sum_{n\leqslant x} \frac{\Lambda(n)}{n^{1/2}} \left(\frac{x^{\beta}}{n^{\beta}} - \frac{n^{\beta}}{x^{\beta}} \right) = \frac{2\beta x^{1/2}}{\frac{1}{4}-\beta^2} + O_c\left(\frac{\beta x^{1/2}}{(\frac{1}{2}-\beta)^2 \log x} \right).
$$

Proof. We will prove the first result. The second result follows using the mean value theorem. Using (7.3.2) we have that

$$
\begin{aligned}
\sum_{n\leqslant x} \frac{\Lambda(n)}{n^{1/2}} \left(\frac{x^{\beta}}{n^{\beta}} - \frac{n^{\beta}}{x^{\beta}} \right) &= \int_2^x \left(\frac{x^{\beta}}{y^{\beta+1/2}} - \frac{x^{-\beta}}{y^{1/2-\beta}} \right) \mathrm{d}y + 2^{1/2-\beta} x^{\beta} - 2^{1/2+\beta} x^{-\beta} \\
&\quad + O\left(\int_2^x \left| \frac{-(\frac{1}{2}+\beta) x^{\beta}}{y^{\beta+3/2}} - \frac{(\beta-\frac{1}{2}) x^{-\beta}}{y^{3/2-\beta}} \right| y^{1/2} (\log y)^2 \,\mathrm{d}y \right) \\
&= \frac{x^{1/2}}{\frac{1}{2}-\beta} - \frac{2^{1/2-\beta} x^{\beta}}{\frac{1}{2}-\beta} - \frac{x^{1/2}}{\frac{1}{2}+\beta} + \frac{2^{\beta+1/2} x^{-\beta}}{\frac{1}{2}+\beta} \\
&\quad + 2^{1/2-\beta} x^{\beta} - 2^{1/2+\beta} x^{-\beta} \\
&\quad + O\left(\int_2^x \left(\frac{(\frac{1}{2}+\beta) x^{\beta}}{y^{1+\beta}} - \frac{(\frac{1}{2}-\beta) x^{-\beta}}{y^{1-\beta}} \right) (\log y)^2 \,\mathrm{d}y \right).
\end{aligned} \tag{7.3.7}
$$

Using the mean value theorem for the function $t \mapsto (\frac{1}{2}+t)x^t$ we find

$$\begin{aligned}\int_2^x \left(\frac{(\frac{1}{2}+\beta)x^\beta}{y^{1+\beta}} - \frac{(\frac{1}{2}-\beta)x^{-\beta}}{y^{1-\beta}} \right) (\log y)^2 \,\mathrm{d}y &\leqslant \int_2^x \left(\frac{(\frac{1}{2}+\beta)x^\beta}{y} - \frac{(\frac{1}{2}-\beta)x^{-\beta}}{y} \right) (\log y)^2 \,\mathrm{d}y \\ &\ll \Big[\big(\tfrac{1}{2}+\beta\big)x^\beta - \big(\tfrac{1}{2}-\beta\big)x^{-\beta} \Big] (\log x)^3 \\ &\ll \beta x^\beta (\log x)^4. \end{aligned} \tag{7.3.8}$$

The desired estimate follows from (7.3.7) and (7.3.8). □

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