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Stationary States of Exclusion Processes With Complex Boundary Conditions and Metastability of Markov Chains With Valleys of Different Depths

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Abstract

The thesis deals with two different topics about the scaling limit of Markov Processes. In the first part of the thesis, we deduce the hydrostatic limit of three types of boundary driven exclusion processes with non-reversible boundary dynamics. In the second part, we study metastability of continuous time finite state Markov chains. For a sequence of Markov chains with certain assumptions on the jump rates, we present a recursive procedure which permits to determine all valleys with different depths from shallow to deep, and the corresponding time scale.

Keywords: Hydrostatic Limit, Boundary Driven Exclusion Processes, Metastability of Markov Processes. Slow Variables

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Contents

CHAPTER 1 INTRODUCTION

The thesis is composed of two research works[1][2]. The first one concerns with the hydrostatic limit of boundary driven exclusion process, which is the content of Chapter 2. Chapter 3 is based on the second paper, which is devoted to study the metastability of finite state Markov processes.

Hydrostatic Limit. The object that we study in Chapter 2 is the simple exclusion process with complex open boundaries. Let us start with the model with simple boundary condition. Let $\Lambda_N = \{1, 2, \cdots, N-1\}$ and $\Omega_N = \{0,1\}^{\Lambda_N}$. Consider the one-dimensional symmetric simple exclusion process $\{\eta^N(t) : t \geq 0\}$ on a finite lattice Λ_N with open boundaries. Particles jumps to nearest neighbors performing simple symmetric random walks with the exclusion rule: a jump is suppressed if site is already occupied. At the left boundary, particles are created with rate α and annihilated with rate $1-\alpha$. On the right boundary this is done with rates β and $1-\beta$. The infinitesimal generator of this Markov process acting on a function $f : \Omega_N \to \mathbb{R}$ is given by:

$$(Lf)(\eta) := (T_l f)(\eta) + (T_r f)(\eta) + \sum_{j=1}^{N-2} [f(\sigma^{j,j+1}\eta) - f(\eta)] \tag{1.0.1}$$

where

$$(T_l f)(\eta) = \{\alpha(1-\eta_1) + (1-\alpha)\eta_1\}[f(\sigma^1\eta) - f(\eta)] \tag{1.0.2}$$

$$(T_r f)(\eta) = \{\beta(1-\eta_{N-1}) + (1-\beta)\eta_{N-1}\}[f(\sigma^{N-1}\eta) - f(\eta)] \tag{1.0.3}$$

In the above equations, $\sigma^{j,j+1}\eta$ is the the configuration obtained from η by exchanging the occupation variables η_j, η_{j+1}, and $\sigma^j\eta$ stands for the configuration obtained from η by flipping the occupation variables η_j.

Our goal is to describe the stationary state of the Markov process with generator L given in (1.0.1) for general boundary dynamics. For the simplest case where T_l and T_r are given by (1.0.2) and (1.0.3), the stationary state is well understood. If $\alpha = \beta$, the Markov chain $\{\eta^N(t) : t \geq 0\}$ is reversible with respect to the product measure with marginal density α. If $\alpha \neq \beta$, even though the stationary measure μ_N cannot be written explicitly, it is well approximated by a product measure in the following sense: for any continuous function $G : [0,1] \to \mathbb{R}$,

$$\lim_{N\to\infty} E_{\mu_N}\Big[\Big|\frac{1}{N}\sum_{k=1}^{N-1} G(k/N)\,[\eta_k - \bar{u}(k/N)]\Big|\Big] = 0\,,$$

where $\bar{u}$ is the unique solution of the linear equation

$$\begin{cases} 0 = \Delta u\,, \\ u(0) = \alpha\,, \quad u(1) = \beta\,. \end{cases} \tag{1.0.4}$$

This is the so-called hydrostatic limit of the particle system. The result above can be proved either using the Derrida's formula for the stationary state as a product of matrices, or by first deriving the hydrodynamic limit though the entropy method introduced by Guo, Papanicolaou and Varadhan[20] and then apply the technique adopted by Farfan, Landim and Mourragui in [21].

A question raised by H.Spohn concerns exclusion processes with complex boundary conditions. If creation and annihilation rates at the boundary depend locally on the configuration, what is the hydrodynamic limit, or the hydrostatic limit? More precisely, the generator of the boundary dynamics T_l is given by:

$$(T_l f)\eta \;=\; c_l(\eta_1, \eta_2, \cdots, \eta_p)\,[f(\sigma^1\eta) \;-\; f(\eta)] \tag{1.0.5}$$

where $p \geq 1$ is a fixed integer, and c is non-negative function: $c : \Omega_{p+1} \to \mathbb{R}_{\geq 0}$. A more general setting is that particles can be created and annihilated at all p left most sites, with rates being a local function defined as c_1. Note that the entropy method is no more available in the general case, since the stationary state cannot be well approximated by a product measure. The matrix method of Derrida is also difficult to apply, because the number of equations for matrix to satisfy is 2^{p+1} and hence it is almost impossible to find a solution(matrix) in simple form even for $p = 2$.

We provide partial answers to this question. We investigate the hydrostatic limit of three classes of boundary driven exclusion processes whose boundary dynamics do not satisfy a detailed balance condition. The first one consists of all boundary dynamics whose generator does not increase the degree of functions of degree 1 and 2. The second class includes all dynamics whose interaction with the reservoirs depends weakly on the configuration. Finally, the third class comprises all exclusion processes whose boundary dynamics is speeded up. Using duality techniques, we prove a law of large numbers for the empirical measure under the stationary state for these three types of interaction with the reservoirs.

Chapter 2 is organized as follows. In section 2.1 we introduce three types of models and state the main results. In sections 2.2 and 2.3 we prove the hydrostatic limit of the first type, proofs of the hydrostatic limit for the second and third type are carried out in section 2.4 and section 2.5 respectively.

Metastability of Finite State Markov Chains. Metastability of continuous-time Markov chains has attracted a lot of attention over the last several decades. Cassandro et al. proposed in [17] the so called pathwise approach to derive the metastablility behavior of continuous-time Markov chains. In [18][19], Bovier et al. introduced a new approach to prove the metastable behavior of continuous-time Markov chains, known as the potential theoretic approach.

Recently, Beltrán and Landim introduced a martingale approach to derive the metastable behavior of continuous-time Markov chains, particularly convenient in the presence of several valleys with the same depth [9, 11, 12]. In the context of finite state Markov chains [10], it permits to identify the slow variables and to reduce the model and the state space.

More precisely, consider a sequence of continuous-time, irreducible Markov chains $\{\eta_t^N, t \geq 0\}$ with a finit-state space E. Given a partition $\mathscr{E}_1, \dots, \mathscr{E}_\mathfrak{n}, \Delta$ of the set E, let $\mathscr{E} = \cup_{1\leq x\leq \mathfrak{n}} \mathscr{E}_x$ and define a projection $\phi_\mathscr{E} : E \to \{0, 1, \dots, \mathfrak{n}\}$ by

$$\phi_\mathscr{E}(\eta) \;=\; \sum_{x=1}^{\mathfrak{n}} x\, \mathbf{1}\{\eta \in \mathscr{E}_x\} \;.$$

In general, $X_N(t) = \phi_\mathscr{E}(\eta_t^N)$ is not Markovian, except some trivial cases. We are interested in finding a proper projection $\phi_\mathscr{E}(\eta)$ such that in a certain time scale X_N converges to some Markov chain under a suitable topology. Therefore we give the following definition: $\phi_\mathscr{E}$ is a slow variable if there exists a time-scale θ_N for which the dynamics of $X_N(t\theta_N)$ is asymptotically Markovian and $X_N(t\theta_N)$ spends a negligible period of time on the set Δ.

The sets $\mathscr{E}_1, \dots, \mathscr{E}_\mathfrak{n}$ are called valleys and Δ is called negligible set. In the time-scale θ_N the chain remains a negligible amount of time in the set Δ and performs transitions between distinct valleys at a time which is asymptotically exponential. We say that the chain η_t^N exhibits a metastable behavior among the valleys $\mathscr{E}_1, \dots, \mathscr{E}_\mathfrak{n}$ in the time-scale θ_N whenever we prove the existence of a slow variable.

The procedure presented above allows to reduce a complicated Markov chain to a simpler one with smaller state space, while keeping the essential features of the dynamics at the same time. For the reduced Markov chain, we can perform the procedure again to obtain an even simpler one. The procedure may continue until the number of sets in the partition is reduced to 2. In this case, the reduced Markov chain has one absorbing point and one transient point. In a certain time-scale, it remains for an exponential time on a subset of the state space after which it jumps to another set where it remains forever.

In Chapter 3 we present a recursive procedure which permits to determine all slow variables of the chain. It provides a sequence of time-scales $\theta_N^1, \dots, \theta_N^\mathfrak{p}$ and of partitions $\{\mathscr{E}_1^j, \dots, \mathscr{E}_{\mathfrak{n}_j}^j, \Delta_j\}$, $1 \leq j \leq \mathfrak{p}$, of the set E with the following properties.

- The time-scales are increasing: $\lim_{N\to\infty} \theta_N^j/\theta_N^{j+1} = 0$ for $1 \le j < \mathfrak{p}$. This relation is represented as $\theta_N^j \ll \theta_N^{j+1}$.

- The partitions are coarser. Each set of the $(j+1)$-th partition is obtained as a union of sets in the j-th partition. Thus $\mathfrak{n}_{j+1} < \mathfrak{n}_j$ and for each a in $\{1, \dots, \mathfrak{n}_{j+1}\}$, $\mathscr{E}_a^{j+1} = \cup_{x\in A} \mathscr{E}_x^j$ for some subset A of $\{1, \dots, \mathfrak{n}_j\}$.

- The sets Δ_j, which separates the valleys, increase: $\Delta_j \subset \Delta_{j+1}$. Actually, $\Delta_{j+1} = \Delta_j \cup_{x\in B} \mathscr{E}_x^j$ for some subset B of $\{1, \dots, \mathfrak{n}_j\}$.

- The projection $\Psi_N^j(\eta) = \sum_{1\le x\le \mathfrak{n}_j} x\, \mathbf{1}\{\eta \in \mathscr{E}_x^j\} + N\, \mathbf{1}\{\eta \in \Delta_j\}$ is a slow variable which evolves in the time-scale θ_N^j.

Our proof is based on a multiscale analysis. We only make a minimal assumption (see assumption (3.1.6)) on the jump rates, which can be easily checked to be satisfied or not. We do not need the assumption of reversibility.

Chapter 3 is organized as follows. In section 3.1 we state the main results. In sections3.2−3.4 we introduce the tools needed to prove these results, which is carried out in 3.5−3.7.

CHAPTER 2 STATIONARY STATES OF BOUNDARY DRIVEN EXCLUSION PROCESSES WITH NONREVERSIBLE BOUNDARY DYNAMICS

2.1 Notation and Results

Consider the symmetric, simple exclusion process on $\Lambda_N = \{1, \dots, N-1\}$ with reflecting boundary conditions. This is the Markov process on $\Omega_N = \{0,1\}^{\Lambda_N}$ whose generator, denoted by $L_{b,N}$, is given by

$$(L_{b,N} f)(\eta) \;=\; \sum_{k=1}^{N-2} \{ f(\sigma^{k,k+1}\eta) - f(\eta) \} \; . \tag{2.1.1}$$

In this formula and below, the configurations of Ω_N are represented by the Greek letters η, ξ, so that $\eta_k = 1$ if site $k \in \Lambda_N$ is occupied for the configuration η and $\eta_k = 0$ otherwise. The symbol $\sigma^{k,k+1}\eta$ represents the configuration obtained from η by exchanging the occupation variables η_k, η_{k+1}:

$$(\sigma^{k,k+1}\eta)_j = \begin{cases} \eta_{k+1} & \text{if } j = k \\ \eta_k & \text{if } j = k+1 \\ \eta_j & \text{if } j \in \Lambda_N \setminus \{k, k+1\} \; . \end{cases}$$

This dynamics is put in contact at both ends with non-conservative dynamics. On the right, it is coupled to a reservoir at density $\beta \in (0,1)$. This interaction is represented by the generator $L_{r,N}$ given by

$$(L_{r,N} f)(\eta) = \{\beta(1-\eta_{N-1}) + (1-\beta)\eta_{N-1}\} \{ f(\sigma^{N-1}\eta) - f(\eta) \} \; , \tag{2.1.2}$$

where $\sigma^k\eta$, $k \in \Lambda_N$, is the configuration obtained from η by flipping the occupation variable η_k,

$$(\sigma^k\eta)_j = \begin{cases} 1 - \eta_k & \text{if } j = k \\ \eta_j & \text{if } j \in \Lambda_N \setminus \{k\} \; . \end{cases}$$

On the left, the system is coupled with different non-conservative dynamics. The purpose of this paper is to investigate the stationary state induced by these different interactions.

2.1.1 Boundary dynamics which do not increase degrees

The first left boundary dynamics we consider are those which keep the degree of functions of degree 1 and 2: those whose generator, denoted by $L_{l,N}$, are such that for all $j \neq k$,

$$\begin{aligned} L_{l,N}\,\eta_j \;&=\; a^j \;+\; \sum_{\ell} a^j_\ell\,\eta_\ell\;, \\ L_{l,N}\,\eta_j\,\eta_k \;&=\; b^{j,k} \;+\; \sum_{\ell} b^{j,k}_\ell\,\eta_\ell \;+\; \sum_{\ell,m} b^{j,k}_{\ell,m}\,\eta_\ell\,\eta_m \end{aligned} \tag{2.1.3}$$

for some coefficients a^j, a^j_ℓ, $b^{j,k}$, $b^{j,k}_\ell$, $b^{j,k}_{\ell,m}$.

Fix $p \geq 0$, and let $\Lambda^*_p = \{-p, \dots, 0\}$, $\Omega^*_p = \{0,1\}^{\Lambda^*_p}$. Consider the generators of Markov chains on Ω^*_p given by

$$(L_R f)(\eta) \;=\; \sum_{j\in\Lambda^*_p} r_j\,\big[\,\alpha_j\,(1-\eta_j) \;+\; \eta_j\,(1-\alpha_j)\,\big]\,\{\,f(\sigma^j\eta) - f(\eta)\,\}\;,$$

$$(L_C f)(\eta) \;=\; \sum_{j\in\Lambda^*_p}\sum_{k\in\Lambda^*_p} c_{j,k}\,\big[\,\eta_k\,(1-\eta_j) \;+\; \eta_j\,(1-\eta_k)\,\big]\,\{\,f(\sigma^j\eta) - f(\eta)\,\}\;,$$

$$(L_A f)(\eta) \;=\; \sum_{j\in\Lambda^*_p}\sum_{k\in\Lambda^*_p} a_{j,k}\,\big[\,\eta_k\,\eta_j \;+\; (1-\eta_j)\,(1-\eta_k)\,\big]\,\{\,f(\sigma^j\eta) - f(\eta)\,\}\;.$$

In these formulae and below, r_j, $c_{j,k}$ and $a_{j,k}$ are non-negative constants, $0 \leq \alpha_j \leq 1$, and $c_{j,j} = a_{j,j} = 0$ for $j \in \Lambda^*_p$.

The generator L_R models the contact of the system at site j with an infinite reservoir at density α_j. At rate $r_j \geq 0$, a particle, resp. a hole, is placed at site j with probability α_j, resp. $1-\alpha_j$. The generator L_C models a replication mechanism, at rate $c_{j,k} \geq 0$, site j copies the value of site k. The generator L_A acts in a similar way. At rate $a_{j,k} \geq 0$, site j copies the inverse value of site k. We add to these dynamics a stirring evolution which exchange the occupation variables at nearest-neighbor sites:

$$(L_S f)(\eta) \;=\; \sum_{j=-p}^{-1} \{f(\sigma^{j,j+1}\eta) - f(\eta)\}\;.$$

The evolution at the left boundary we consider consists in the superposition of the four dynamics introduced above. The generator, denoted by L_l, is thus given by

$$L_l \;=\; L_S \;+\; L_R \;+\; L_C \;+\; L_A\;.$$

Denote by L_G the generator of a general Glauber dynamics on Ω_p^*:

$$(L_G f)(\eta) \;=\; \sum_{k=-p}^{0} c_k(\eta)\, \{f(\sigma^k\eta) - f(\eta)\}\;, \tag{2.1.4}$$

where c_k are non-negative jump rates which depend on the entire configuration $(\eta_{-p},\dots,\eta_0)$. We prove in Lemma 2.2.2 that any Markov chain on Ω_p^* whose generator L_D is given by $L_D = L_S + L_G$ and which fulfills conditions (2.1.3) can be written as $L_S + L_R + L_C + L_A$ [we show that there are non-negative parameters r_j, $c_{j,k}$, $a_{j,k}$ such that $L_G = L_R + L_C + L_A$]. Therefore, by examining the Markov chain whose left boundary condition is characterized by the generator L_l we are considering the most general evolution in which a stirring dynamics is superposed with a spin flip dynamics which fulfills condition (2.1.3).

We prove in Lemma 2.2.3 that the Markov chain induced by the generator L_l has a unique stationary state if

$$\sum_{j\in\Lambda_p^*} r_j \;+\; \sum_{j\in\Lambda_p^*}\sum_{k\in\Lambda_p^*} a_{j,k} \;>\; 0\;. \tag{2.1.5}$$

Assume that this condition is in force. Denote by μ the unique stationary state, and let

$$\rho(k) \;=\; E_\mu[\eta_k]\;, \quad k\in\Lambda_p^*\;, \tag{2.1.6}$$

be the mean density at site k under the measure μ. Clearly, $0\le\rho(k)\le 1$ for all $k\in\Lambda_p^*$. Since $E_\mu[L_l\eta_j]=0$, a straightforward computation yields that

$$0 \;=\; r_j\,[\alpha_j - \rho(j)] \;+\; (\mathscr{C}\rho)(j) \;+\; (\mathscr{A}\rho)(j) \;+\; (\mathscr{T}\rho)(j)\;, \quad j\in\Lambda_p^*\;, \tag{2.1.7}$$

where

$$(\mathscr{C}\rho)(j) \;=\; \sum_{k\in\Lambda_p^*} c_{j,k}\,[\rho(k)-\rho(j)]\;, \quad (\mathscr{A}\rho)(j) \;=\; \sum_{k\in\Lambda_p^*} a_{j,k}\,[1-\rho(k)-\rho(j)]\;,$$

$$(\mathscr{T}\rho)(j) \;=\; \begin{cases} \rho(-p+1)-\rho(-p) & \text{if } j=-p\;,\\ \rho(-1)-\rho(0) & \text{if } j=0\;,\\ \rho(j+1)+\rho(j-1)-2\rho(j) & \text{otherwise.}\end{cases}$$

We prove in Lemma 2.2.4 that (2.1.7) has a unique solution if condition (2.1.5) is in force.

Let $\Lambda_{N,p}=\{-p,\dots,N-1\}$. Consider the boundary driven, symmetric, simple exclusion process on $\Omega_{N,p} = \{0,1\}^{\Lambda_{N,p}}$ whose generator, denoted by L_N, is given by

$$L_N \;=\; L_l \,+\, L_{0,1} \,+\, L_{b,N} \,+\, L_{r,N}\;, \tag{2.1.8}$$

where $L_{0,1}$ represent a stirring dynamics between sites 0 and 1:

$$(L_{0,1}f)(\eta) \;=\; f(\sigma^{0,1}\eta) - f(\eta)\;.$$

There is a little abuse of notation in the previous formulae because the generators are not defined on the space $\Omega_{N,p}$ but on smaller spaces. We believe, however, that the meaning is clear.

Due to the right boundary reservoir and the stirring dynamics, the process is irreducible. Denote by μ_N the unique stationary state, and let

$$\rho_N(k) \;=\; E_{\mu_N}[\eta_k]\;, \quad k \in \Lambda_{N,p}\;, \tag{2.1.9}$$

be the mean density at site k under the stationary state. Of course, $0 \le \rho_N(k) \le 1$ for all $k \in \Lambda_{N,p}$, $N \ge 1$. We prove in Lemma 2.2.5 that under condition (2.1.5) there exists a finite constant C_0, independent of N, such that

$$\big|\,\rho_N(k) \,-\, \rho(k)\,\big| \;\le\; C_0/N\;, \quad \text{for all} \;\; -p \,\le\, k \,\le\, 0\;,$$

where ρ is the unique solution of (2.1.7).

The first main result of this article establishes a law of large numbers for the empirical measure under the stationary state μ_N.

Theorem 2.1.1. *Assume that* $\sum_{j\in\Lambda_p^*} r_j > 0$. *Then, for any continuous function* $G : [0,1] \to \mathbb{R}$,

$$\lim_{N\to\infty} E_{\mu_N}\Big[\,\Big|\frac{1}{N}\sum_{k=1}^{N-1} G(k/N)\,[\eta_k - \bar{u}(k/N)]\Big|\,\Big] = 0\;,$$

where $\bar{u}$ *is the unique solution of the linear equation*

$$\begin{cases} 0 = \Delta u\,, \\ u(0) = \rho(0)\,, \;\; u(1) = \beta\,. \end{cases} \tag{2.1.10}$$

We refer to Section 2.2 for the notation used in the next remark.

Remark 2.1.2. *We believe that Theorem 2.1.1 remains in force if* $\sum_{j\in\Lambda_p^*} r_j \;=\; 0$ *and* $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. *This assertion is further discussed in Remark 2.3.5.*

Remark 2.1.3. *The case* $\sum_{j\in\Lambda_p^*} r_j + \sum_{j,k\in\Lambda_p^*} a_{j,k} = 0$ *provides an example in which at the left boundary sites behave as a voter model and acquire the value of one of their neighbors. One can generalize this model and consider an exclusion process in which, at the left boundary, the first site takes the value of the majority in a fixed interval* $\{2,\dots,2p\}$*, the left boundary generator being given by*

$$(L_l f)(\eta) \;=\; f(M\eta) - f(\eta)\;,$$

where $(M\eta)_k = \eta_k$ *for* $k \ge 2$*, and* $(M\eta)_1 = \mathbf{1}\{\sum_{2\le j\le 2p}\eta_j \ge p\}$*. In this case it is conceivable that the system alternates between two states, one in which the left density is close to* 1 *and one in which it is close to* 0*.*

The proof of Theorem 2.1.1 is presented in Sections 2.2 and 2.3. It relies on duality computations. As the boundary conditions do not increase the degrees of a function, the equations obtained from the identities $E_{\mu_N}[L_N\eta_j] = 0$, $E_{\mu_N}[L_N\eta_j\eta_k] = 0$ can be expressed in terms of the density and of the correlation functions.

2.1.2 Small perturbations of flipping dynamics

We examine in this subsection a model in which the rate at which the leftmost occupation variable is flipped depends locally on the configuration. Consider the generator

$$L_N \;=\; L_l \;+\; L_{b,N} \;+\; L_{r,N}\;, \tag{2.1.11}$$

where $L_{b,N}$ and $L_{r,N}$ were defined in (2.1.1), (2.1.2). The left boundary generator is given by

$$(L_l f)(\eta) \;=\; c(\eta_1,\dots,\eta_p)\,[f(\sigma^1\eta) - f(\eta)]\;.$$

for some non-negative function $c:\{0,1\}^p \to \mathbb{R}_+$.

Let

$$A \;=\; \min_{\xi\in\Omega_p} c(0,\xi)\;,\quad B \;=\; \min_{\xi\in\Omega_p} c(1,\xi) \tag{2.1.12}$$

be the minimal creation and annihilation rates, and denote by

$$\lambda(0,\xi) \;:=\; c(0,\xi) \;-\; A\;,\quad \lambda(1,\eta) \;:=\; c(1,\xi) \;-\; B$$

the marginal rates. We allow ourselves below a little abuse of notation by considering λ as a function defined on Ω_N and which depends on the first p coordinates, instead of a function defined on Ω_{p+1}. With this notation the left boundary generator can be written as

$$(L_l f)(\eta) \;=\; \big[A + (1-\eta_1)\,\lambda(\eta)\big]\,[f(T^1\eta) - f(\eta)] \;+\; \big[B + \eta_1\,\lambda(\eta)\big]\,[f(T^0\eta) - f(\eta)]\;,$$

where for $a = 0$, 1,

$$(T^a\eta)_k \;=\; \begin{cases} a & \text{if } k=1, \\ \eta_k & \text{otherwise.} \end{cases}$$

The Markov chain with generator L_N has a unique stationary state because it is irreducible due to the stirring dynamics and the right boundary condition. Denote by μ_N the unique stationary state of the generator L_N, and by E_{μ_N} the corresponding expectation. Let $\rho_N(k) = E_{\mu_N}[\eta_k]$, $k \in \Lambda_N$.

Theorem 2.1.4. *Suppose that*

$$(p-1)\sum_{\xi\in\Omega_p}\{\lambda(0,\xi) + \lambda(1,\xi)\} \;<\; A \,+\, B\;. \tag{2.1.13}$$

Then, the limit

$$\alpha := \lim_{N\to\infty} \rho_N(1)$$

exists, and it does not depend on the boundary conditions at $N-1$*. Moreover, for any continuous function* $G : [0,1] \to \mathbb{R}$*,*

$$\lim_{N\to\infty} E_{\mu_N}\Big[\,\Big|\frac{1}{N}\sum_{k=1}^{N-1} G(k/N)\,[\eta_k - \bar{u}(k/N)]\Big|\,\Big] = 0\;,$$

where $\bar{u}$ *is the unique solution of the linear equation* (2.1.10) *with* $\rho(0) = \alpha$*.*

Remark 2.1.5. *There is not a simple closed formula for the left density* α*. By coupling, it is proven that the sequence* $\rho_N(1)$ *is Cauchy and has therefore a limit. The density* $\rho_N(1)$ *can be expressed in terms of the dual process, a stirring dynamics with creation and annihilation at the boundary.*

Remark 2.1.6. *A similar result holds for boundary driven exclusion processes in which particles are created at sites* $1 \le k \le q$ *with rates depending on the configuration through the first p sites, provided the rates depend weakly [in the sense* (2.1.13)*] on the configuration.*

Remark 2.1.7. *One can weaken slightly condition* (2.1.13)*. For* $\zeta \in \{0,1\}^q$*,* $0 \le q \le p-1$*, let* $A(\zeta) = \min_\xi c(\zeta,\xi)$*, where the minimum is carried over all configurations*

$\xi \in \{0,1\}^{p-q}$. For $a = 0,\ 1$, and $\zeta \in \cup_{0 \le q \le p-1} \{0,1\}^q$, let $R(\zeta, a) = A(\zeta, a) - A(\zeta) \ge 0$ be the marginal rate. The same proof shows that the assertion of Theorem 2.1.4 holds if

$$\sum_{q=2}^{p} (q-1) \sum_{\zeta \in \{0,1\}^q} R(\zeta) \;<\; A \,+\, B \;.$$

Remark 2.1.8. *In [3], Erignoux proves that the empirical measure evolves in time as the solution of the heat equation with the corresponding boundary conditions.*

The proof of Theorem 2.1.4 is presented in Section 2.4. It is based on a duality argument which consists in studying the process reversed in time. We show that under the conditions of Theorem 2.1.4, to determine the value of the occupation variable η_1 at time 0, we only need to know from the past the behavior of the process in a finite space-time window.

2.1.3 Speeded-up boundary condition

Recall the notation introduced in Subsection 2.1.1. Fix $p > 1$ and consider an irreducible continuous-time Markov chain on Ω_p^*, $p > 0$. Denote by L_l the generator of this process, and by μ the unique stationary state. Let

$$\rho(k) \;=\; E_\mu[\eta_k] \;, \quad k \in \Lambda_p^* \;, \tag{2.1.14}$$

be the mean density at site k under the measure μ. Clearly, $0 < \rho(k) < 1$ for all $k \in \Lambda_p^*$. The density cannot be 0 or 1 because every configuration has a strictly positive weight under the stationary measure.

Fix a sequence $\ell_N \to \infty$, and consider the boundary driven, symmetric, simple exclusion process on $\Omega_{N,p}$ whose generator, denoted by L_N, is given by

$$L_N \;=\; \ell_N \, L_l \,+\, L_{0,1} \,+\, L_{b,N} \,+\, L_{r,N} \;,$$

where $L_{0,1}$ represent a stirring dynamics between sites 0 and 1, introduced below (2.1.8). Note that the left boundary dynamics has been speeded-up by ℓ_N.

Due to the right boundary reservoir and the stirring dynamics, the process is irreducible. Denote by μ_N the unique stationary state, and let

$$\rho_N(k) \;=\; E_{\mu_N}[\eta_k] \;, \quad k \in \Lambda_{N,p} \;,$$

be the mean density at site k under the stationary state.

Theorem 2.1.9. *There exists a finite constant C_0, independent of N, such that $|\rho_N(0) - \rho(0)| \le C_0/\sqrt{\ell_N}$. Moreover, for any continuous function $G : [0,1] \to \mathbb{R}$,*

$$\lim_{N\to\infty} E_{\mu_N}\Big[\Big|\frac{1}{N}\sum_{k=1}^{N-1} G(k/N)\,[\eta_k - \bar{u}(k/N)]\Big|\Big] = 0 ,$$

where $\bar{u}$ is the unique solution of the linear equation (2.1.10)*.*

Remark 2.1.10. *The proof of this theorem is based on duality computations, and does not requires one and two-blocks estimates. There is an alternative proof relying on an estimate of the entropy production along the lines presented in [4, Proposition 2], [7, Proposition 3.3]. This proof applies to gradient and non-gradient models [6], but it requires ℓ_N to grow at least as N.*

The proof of Theorem 2.1.9 is presented in Section 2.5. As the boundary condition has been speeded-up, each time the occupation variables η_0, η_1 are exchanged, the distribution of the variable η_0 is close to its stationary distribution with respect to the left-boundary dynamics.

2.2 Proof of Theorem 2.1.1: one point functions

We prove in this section that the density of particles under the stationary state μ_N is close to the solution of the linear parabolic equation (2.1.10). We first show that the left boundary dynamics we consider is indeed the most general one which does not increase the degree of functions of degree 1 and 2.

For $A \subset \Lambda_p^*$, let $\Psi_A : \Omega_p^* \to \mathbb{R}$ be given by $\Psi_A(\eta) = \prod_{k\in A} \eta_k$. Clearly, any function $f : \Omega_p^* \to \mathbb{R}$ can be written as a linear combination of the functions Ψ_A. A function f is said to be a monomial of order n if it can be written as a linear combination of functions Ψ_A where $|A| = n$ for all A. It is said to be a polynomial of order n if it can be written as a sum of monomials of order $m \le n$.

Recall the definition of the generator L_G given in (2.1.4). Fix $-p \le k \le 0$, and write the jump rate c_k as

$$c_k \;=\; \sum_{A\subset\Lambda_p^*} R_{k,A}\,\Psi_A ,$$

where the sum is carried over all subsets A of Λ_p^*.

Lemma 2.2.1. *The functions $L_G\Psi_{\{j\}}$, resp. $L_G\Psi_{\{j,k\}}$, $-p \le j \ne k \le 0$, are polynomials of order 1, resp. of order 2, if and only if there exists constants $R_{l,\varnothing}$, $R_{l,\{m\}}$, l, $m \in \Lambda_p^*$*

such that

$$c_j(\eta) \;=\; R_{j,\varnothing} \;+\; R_{j,\{j\}}\,\eta_j \;+\; \sum_{k:k\neq j} R_{j,\{k\}}\,\eta_k\,(1-2\eta_j)\;. \tag{2.2.1}$$

Proof. Fix $j \in \Lambda_p^*$. A straightforward computation shows that

$$L_G\Psi_{\{j\}} \;=\; \sum_{A\not\ni j} R_{j,A}\,\Psi_A \;-\; \sum_{A\not\ni j}(2R_{j,A}+R_{j,A\cup\{j\}})\,\Psi_{A\cup\{j\}}\;.$$

Hence, $L_G\Psi_{\{j\}}$ is a polynomial of order 1 if and only if $R_{j,B} = R_{j,B\cup\{j\}} = 0$ for all $B \subset \Lambda_p^*$ such that $|B| \geq 2$, $j \notin B$. This proves that $L_G\Psi_{\{j\}}$ is a polynomial of order 1 if and only if condition (2.2.1) holds.

If the rates are given by (2.2.1), for all $j \neq k \in \Lambda_p^*$,

$$(L_G\Psi_{\{j\}})(\eta) \;=\; R_{j,\varnothing}\,(1-2\eta_j) \;-\; R_{j,\{j\}}\,\eta_j \;+\; \sum_{\ell:\ell\neq j} R_{j,\{\ell\}}\,\eta_\ell\;,$$

and

$$\begin{aligned}(L_G\Psi_{\{j,k\}})(\eta) \;=\;& R_{j,\varnothing}\,(1-2\eta_j)\,\eta_k \;+\; R_{k,\varnothing}\,(1-2\eta_k)\,\eta_j \;-\; \big(R_{j,\{j\}}+R_{k,\{k\}}\big)\,\eta_j\eta_k \\ &+\; \sum_{\ell:\ell\neq j,k} R_{j,\{\ell\}}\,\eta_k\,\eta_\ell \;+\; \sum_{\ell:\ell\neq j,k} R_{k,\{\ell\}}\,\eta_j\,\eta_\ell\;,\end{aligned}$$

which is a polynomial of degree 2. This proves the lemma. ∎

Note: Observe that at this point we do not make any assertion about the sign of the constants $R_{j,\varnothing}$, $R_{j,\{k\}}$.

The next result states that a generator L_G whose rates satisfy condition (2.2.1) can be written as $L_R + L_C + L_A$. Denote by $\mathbb{P}_j$, resp. $\mathbb{N}_j$, $-p \leq j \leq 0$, the subset of points $k \in \Lambda_p^* \setminus \{j\}$, such that $R_{j,\{k\}} \geq 0$, resp. $R_{j,\{k\}} < 0$.

Lemma 2.2.2. *The rates $c_j(\eta)$ given by* (2.2.1) *are non-negative if and only if*

$$\begin{aligned} p_j \;&:=\; R_{j,\varnothing} \;+\; R_{j,\{j\}} \;-\; \sum_{k\in\mathbb{P}_j} R_{j,\{k\}} \;\geq\; 0\;, \\ q_j \;&:=\; R_{j,\varnothing} \;+\; \sum_{k\in\mathbb{N}_j} R_{j,\{k\}} \;\geq\; 0\;. \end{aligned}$$

In this case, there exist non-negative rates r_j, $c_{j,k}$, $a_{j,k}$ and densities $\alpha_j \in [0,1]$, $k \neq j \in$

Λ_p^*, *such that for all* $j \in \Lambda_p^*$, $\eta \in \Omega_p^*$,

$$c_j(\eta) \;=\; r_j \left[\alpha_j \,(1-\eta_j) \,+\, (1-\alpha_j)\,\eta_j \right] \,+\, \sum_{k\in\Lambda_p^*} c_{j,k} \left[\eta_j \,(1-\eta_k) \,+\, \eta_k\,(1-\eta_j) \right] \,, \\ +\, \sum_{k\in\Lambda_p^*} a_{j,k} \left[\eta_j \,\eta_k \,+\, (1-\eta_k)\,(1-\eta_j) \right] \,.$$

Proof. The first assertion of the lemma is elementary and left to the reader. For $j \neq k \in \Lambda_p^*$, define

$$c_{j,k} \;=\; R_{j,\{k\}}\, \mathbf{1}\{k \in \mathbb{P}_j\} \;\geq\; 0, \quad a_{j,k} \;=\; -\,R_{j,\{k\}}\, \mathbf{1}\{k \in \mathbb{N}_j\} \;\geq\; 0 \,, \\ r_j \;:=\; p_j \,+\, q_j \;\geq\; 0 \,, \quad \alpha_j \;:=\; \frac{q_j}{p_j+q_j}\, \mathbf{1}\{r_j \neq 0\} \;\in\; [0,1] \,.$$

It is elementary to check that the second assertion of the lemma holds with these definitions. ∎

Lemma 2.2.3. *The Markov chain induced by the generator* L_l *has a unique stationary state if* $\sum_{j\in\Lambda_p^*} r_j + \sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. *In contrast, if* $\sum_{j\in\Lambda_p^*} r_j + \sum_{j,k\in\Lambda_p^*} a_{j,k} = 0$ *and* $\sum_{j,k\in\Lambda_p^*} c_{j,k} > 0$, *then the Markov chain induced by the generator* L_l *has exactly two stationary states which are the Dirac measures concentrated on the configurations with all sites occupied or all sites empty.*

Proof. Assume first that $\sum_{j\in\Lambda_p^*} r_j > 0$. Let $j \in \Lambda_p^*$ such that $r_j > 0$. If $\alpha_j > 0$, the configuration in which all sites are occupied can be reached from any configuration by moving with the stirring dynamics each empty site to j, and then filling it up with the reservoir. This proves that under this condition there exists a unique stationary state concentrated on the configurations which can be attained from the configuration in which all sites are occupied. Analogously, if $\alpha_j = 0$, the configuration in which all sites are empty can be reached from any configuration.

Suppose that $\sum_{j\in\Lambda_p^*} r_j = 0$ and $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. We claim that from any configuration we can reach any configuration whose total number of occupied sites is comprised between 1 and $|\Lambda_p^*| - 1 = p$. Since the stirring dynamics can move particles and holes around, we have only to show that it is possible to increase, resp. decrease, the number of particles up to $|\Lambda_p^*| - 1$, resp. 1.

Let $k \neq j \in \Lambda_p^*$ such that $a_{j,k} > 0$. To increase the number of particles up to $|\Lambda_p^*| - 1$, move the two empty sites to j and k, and create a particle at site j. Similarly one can decrease the number of particles up to 1. This proves that under the previous assumptions there exists a unique stationary state concentrated on the set of configurations whose total number of particles is comprised between 1 and $|\Lambda_p^*| - 1$.

Assume that $\sum_{j\in\Lambda_p^*} r_j = 0$, $\sum_{j,k\in\Lambda_p^*} a_{j,k} = 0$ and $\sum_{j,k\in\Lambda_p^*} c_{j,k} > 0$. In this case, the configuration with all sites occupied and the one with all sites empty are absorbing states. Let $k \neq j \in \Lambda_p^*$ such that $c_{j,k} > 0$. If there is at least one particle, to increase the number of particles, move the empty site to j, the occupied site to k, and create a particle at site j. Similarly, we can decrease the number of particle if there is at least one empty site. This proves that in this case the set of stationary states is a pair formed by the configurations with all sites occupied and the one with all sites empty. ∎

Lemma 2.2.4. *Suppose that* $\sum_{j\in\Lambda_p^*} r_j + \sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. *Then, there exists a unique solution to* (2.1.7).

Proof. Equation (2.1.6) provides a solution and guarantees existence. We turn to uniqueness. Suppose first that $\sum_{j\in\Lambda_p^*} r_j > 0$ and $\sum_{j,k\in\Lambda_p^*} a_{j,k} = 0$. In this case, the operator $\mathscr{A}$ vanishes. Consider two solution $\rho^{(1)}$, $\rho^{(2)}$, and denote their difference by γ. The difference satisfies the linear equation

$$0 \;=\; -\, r_j\, \gamma(j) \;+\; (\mathscr{C}\gamma)(j) \;+\; (\mathscr{T}\gamma)(j)\;, \quad j \in \Lambda_p^*\;.$$

Let π be the unique stationary state of the random walk on Λ_p^* whose generator is $\mathscr{C} + \mathscr{T}$. Multiply both sides of the equation by $\gamma(j)\,\pi(j)$ and sum over j to obtain that

$$0 \;=\; -\sum_{j\in\Lambda_p^*} r_j\, \gamma(j)^2 \pi(j) \;+\; \langle (\mathscr{C} + \mathscr{T})\gamma\,,\, \gamma\rangle\;,$$

where $\langle f, g\rangle$ represents the scalar product in $L^2(\pi)$. As all terms on the right-hand side are negative, the identity $\langle (\mathscr{C} + \mathscr{T})\gamma\,,\, \gamma\rangle = 0$ yields that γ is constant. Since, by hypothesis, $\sum_j r_j > 0$, $\gamma \equiv 0$, which proves the lemma.

Suppose next that $\sum_{j\in\Lambda_p^*} r_j > 0$ and $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. Define the rates $t_{j,k} \geq 0$, $j \neq k \in \Lambda_p^*$, so that

$$(\mathscr{T} f)(j) \;=\; \sum_{k:k\neq j} t_{j,k}\, [f(k) - f(j)]\;, \quad j \in \Lambda_p^*\;.$$

Let $\Lambda_p^{\rm ext} = \{-1, 1\} \times \Lambda_p^*$. Points in $\Lambda_p^{\rm ext}$ are represented by the symbol (σ, k), $\sigma = \pm 1$, $-p \leq k \leq 0$. We extend the definition of a function $f : \Lambda_p^* \to \mathbb{R}$ to $\Lambda_p^{\rm ext}$ by setting $f(1, k) = f(k)$, $f(-1, k) = 1 - f(k)$, $k \in \Lambda_p^*$. This new function is represented by $\widehat{f} : \Lambda_p^{\rm ext} \to \mathbb{R}$.

With this notation we may rewrite equation (2.1.7) as

$$0 \;=\; r_{(1,j)}\,[\alpha_{(1,j)} - \widehat{\rho}(1,j)] + (\widehat{\mathscr{C}}\,\widehat{\rho})\,(1,j) \;+\; (\widehat{\mathscr{A}}\,\widehat{\rho})\,(1,j) \;+\; (\widehat{\mathscr{T}}\,\widehat{\rho})\,(1,j)\;,\;\; j\in\Lambda_p^*\;, \tag{2.2.2}$$

where, $r_{(1,j)} = r_j$, $\alpha_{(1,j)} = \alpha_j$,

$$(\widehat{\mathscr{A}}\,\widehat{\rho})\,(1,j) \;=\; \sum_{k\in\Lambda_p^*} a_{j,k}\,[\,\widehat{\rho}(-1,k) - \widehat{\rho}(1,j)\,]\;,$$

and $\widehat{\mathscr{C}}$, $\widehat{\mathscr{T}}$ are the generators of the Markov chains on $\Lambda_p^{\rm ext}$ characterized by the rates $\widehat{c}$, $\widehat{t}$ given by

$$\begin{aligned} \widehat{c}\,[\,(\pm 1,j)\,,\,(\pm 1,k)\,] \;&=\; c_{j,k}\;, \quad \widehat{c}\,[\,(\pm 1,j)\,,\,(\mp 1,k)\,] \;=\; 0\;, \\ \widehat{t}\,[\,(\pm 1,j)\,,\,(\pm 1,k)\,] \;&=\; t_{j,k}\;, \quad \widehat{t}\,[\,(\pm 1,j)\,,\,(\mp 1,k)\,] \;=\; 0\;. \end{aligned}$$

Multiply equation (2.1.7) by -1 to rewrite it as

$$0 \;=\; r_{(-1,j)}\,[\alpha_{(-1,j)} - \widehat{\rho}(-1,j)] + (\widehat{\mathscr{C}}\,\widehat{\rho})(-1,j) \;+\; (\widehat{\mathscr{A}}\,\widehat{\rho})(-1,j) \;+\; (\widehat{\mathscr{T}}\,\widehat{\rho})(-1,j) \tag{2.2.3}$$

for any $j\in\Lambda_p^*$, where $r_{(-1,j)} = r_j$, $\alpha_{(-1,j)} = 1-\alpha_j$, and

$$(\widehat{\mathscr{A}}\,\widehat{\rho})(-1,j) \;=\; \sum_{k\in\Lambda_p^*} a_{j,k}\,[\,\widehat{\rho}(1,k) - \widehat{\rho}(-1,j)\,]\;.$$

Since the operator $\widehat{\mathscr{C}} + \widehat{\mathscr{A}} + \widehat{\mathscr{T}}$ defines an irreducible random walk on $\Lambda_p^{\rm ext}$, we may proceed as in the first part of the proof to conclude that there exists a unique solution of (2.1.7).

Finally, suppose that $\sum_{j\in\Lambda_p^*} r_j = 0$ and $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. Let ρ be a solution to (2.1.7). Then, its extension $\widehat{\rho}$ is a solution to (2.2.2), (2.2.3). The argument presented in the first part of the proof yields that any solution of these equations is constant. Since $\widehat{\rho}(1,k) = \rho(k) = 1-\widehat{\rho}(-1,k)$, we conclude that this constant must be $1/2$. This proves that in the case where $\sum_{j\in\Lambda_p^*} r_j = 0$, $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$, the unique solution to (2.1.7) is constant equal to $1/2$. ∎

Recall from (2.1.9) the definition of ρ_N.

Lemma 2.2.5. *Suppose that* $\sum_{j\in\Lambda_p^*} r_j + \sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. *Then, for* $0\le k < N$,

$$\rho_N(k) \;=\; \frac{k}{N}\,\beta \;+\; \frac{N-k}{N}\,\rho_N(0)\;. \tag{2.2.4}$$

Moreover, there exists a finite constant C_0, independent of N, such that

$$\big|\,\rho_N(k) \,-\, \rho(k)\,\big| \;\le\; C_0/N\;, \quad -p \,\le\, k \,\le\, 0\;,$$

where ρ is the unique solution of (2.1.7).

Proof. Fix $1 \le k < N$. As μ_N is the stationary state, $E_{\mu_N}[L_N\,\eta_k] = 0$. Recall that $\rho_N(k) = E_{\mu_N}[\eta_k]$. Note that $\rho_N(N) = \beta$, $(\Delta_N \rho_N)(k) := \rho_N(k-1)+\rho_N(k+1)-2\rho_N(k) = 0$. In particular, ρ_N solves the discrete difference equation

$$(\Delta_N \rho_N)(k) \;=\; 0\;,\;\; 1 \le k \,<\, N\;, \quad \rho_N(N) \;=\; \beta\;, \quad \rho_N(0) \;=\; \rho_N(0)\;,$$

whose unique solution is given by (2.2.4). This proves the first assertion of the lemma.

We turn to the second statement. It is clear that $\rho_N(j)$ fulfills (2.1.7) for $-p \le j < 0$. For $j = 0$ the equation is different due to the stirring dynamics between 0 and 1 induced by the generator $L_{0,1}$. We have that

$$0 \;=\; r_0\,[\alpha_0 - \rho_N(0)] \,+\, (\mathscr{C}\rho_N)(0) \;+\; (\mathscr{A}\rho_N)(0) \;+\; (\Delta_N\rho_N)(0)\;.$$

By (2.2.4), we may replace $\rho_N(1)$ by $[1-(1/N)]\,\rho_N(0)+(1/N)\beta$, and the previous equation becomes

$$0 \;=\; r_0\,[\alpha_0 - \rho_N(0)] + (\mathscr{C}\rho_N)(0) \;+\; (\mathscr{A}\rho_N)(0) \;+\; (\mathscr{T}\rho_N)(0) \;+\; \frac{1}{N}\,\big[\,\beta - \rho_N(0)\,\big]\;. \quad (2.2.5)$$

This equation corresponds to (2.1.7) with $r_0' = r_0 + (1/N)$ and $\alpha_0' = (\alpha_0\,r_0 + \beta/N)/[r_0 + (1/N)]$.

By Lemma 2.2.4, equation (2.1.7) for $j \neq 0$ and (2.2.5) for $j = 0$ has a unique solution. Let $\gamma_N = \rho_N - \rho$, where ρ is the solution of (2.1.7). γ_N satisfies

$$0 \;=\; \frac{1}{N}\,[\beta - \rho_N(0)]\,\delta_{0,j} \;-\; r_j\,\gamma_N(j) \,+\, (\mathscr{C}\gamma_N)(j) \;+\; (\mathscr{A}\gamma_N)(j) \;+\; (\mathscr{T}\gamma_N)(j)\;,$$

where $\delta_{0,j}$ is equal to 1 if $j = 0$ and is equal to 0 otherwise.

We complete the proof in the case $\mathscr{A} = 0$. The other cases can be handled by increasing the space, as in the proof of Lemma 2.2.4. Denote by π the stationary state of the generator $\mathscr{C} + \mathscr{T}$. Multiply both sides of the previous equation by $\pi(j)\gamma_N(j)$ and sum over j to obtain that

$$\sum_{j\in\Lambda_p^*} r_j\,\gamma_N(j)^2\pi(j) \;+\; \langle -\,(\mathscr{C} + \mathscr{T})\,\gamma_N\,,\,\gamma_N\rangle \;=\; \theta_N\,\gamma_N(0)\,\pi(0)\;,$$

where $\theta_N = (1/N)\,[\beta - \rho_N(0)]$. Let $k \in \Lambda_p^*$ such that $r_k > 0$. Such k exists by assumption. Rewrite $\gamma_N(0)$ as $\sum_{k<j\le 0}[\gamma_N(j) - \gamma_N(j-1)] + \gamma_N(k)$ and use Young's inequality to obtain that there exists a finite constant C_0, depending only on p, π and on the rates $c_{j,k}$, r_j such that

$$\theta_N\,\gamma_N(0)\,\pi(0) \;\le\; (1/2)\,r_k\,\gamma_N(k)^2\,\pi(k) \;+\; (1/2)\,\langle -(\mathscr{C}+\mathscr{T})\,\gamma_N\,,\,\gamma_N\rangle \;+\; C_0\,\theta_N^2\;.$$

Here and throughout the article, the value of the constant C_0 may change from line to line. The two previous displayed equations and the fact that $|\beta - \rho_N(0)| \le 1$ yield that

$$\sum_{j\in\Lambda_p^*} r_j\,\gamma_N(j)^2\pi(j) \;+\; \langle -(\mathscr{C}+\mathscr{T})\,\gamma_N\,,\,\gamma_N\rangle \;\le\; \frac{C_0}{N^2}\;\cdot$$

In particular, $\gamma_N(k)^2 \le C_0/N^2$ and $[\gamma_N(j+1) - \gamma_N(j)]^2 \le C_0/N^2$ for $-p \le j < 0$. This completes the proof of the lemma. ∎

2.3 Proof of Theorem 2.1.1: two point functions.

We examine in this section the two-point correlation function under the stationary state μ_N. Denote by $\mathbb{D}_N$ the discrete simplex defined by

$$\mathbb{D}_N \;=\; \{(j,k) : -p \le j < k \le N-1\} \quad \text{and set} \quad \Xi_N = \{-1,1\}\times\mathbb{D}_N\;.$$

Let

$$\bar\eta_m \;=\; 1-\eta_m\;,\quad \bar\rho_N(m) \;=\; 1-\rho_N(m)\;,\quad m\in\Lambda_{N,p}\;,$$

and define the two-point correlation function $\varphi_N(\sigma,j,k)$, $(\sigma,j,k)\in\Xi_N$, by

$$\begin{aligned} \varphi_N(1,j,k) \;&=\; E_{\mu_N}\big[\,\{\eta_j - \rho_N(j)\}\,\{\eta_k - \rho_N(k)\}\,\big]\;,\\ \varphi_N(-1,j,k) \;&=\; E_{\mu_N}\big[\,\{\bar\eta_j - \bar\rho_N(j)\}\,\{\eta_k - \rho_N(k)\}\,\big]\;. \end{aligned} \tag{2.3.1}$$

Note that $\varphi_N(-1,j,k) = -\,\varphi_N(1,j,k)$. The identity $E_{\mu_N}[L_N\{\eta_j - \rho_N(j)\}\,\{\eta_k - \rho_N(k)\}] = 0$ provides a set of equations for φ_N. Their exact form requires some notation.

Denote by $\mathscr{L}_N^{\rm rw}$ the generator of the symmetric, nearest-neighbor random walk on $\mathbb{D}_N$.

This generator is defined by the next two sets of equations. If $k-j>1$,

$$(\mathscr{L}_N^{\mathrm{rw}}\phi)(j,k) \;=\; \begin{cases} (\boldsymbol{\Delta}\phi)(j,k) & \text{if } j>-p,\ k<N-1, \\ (\boldsymbol{\nabla}_1^+\phi)(-p,k)+(\boldsymbol{\Delta}_2\phi)(-p,k) & \text{if } j=-p,\ k<N-1, \\ (\boldsymbol{\Delta}_1\phi)(j,N-1)+(\boldsymbol{\nabla}_2^-\phi)(j,N-1) & \text{if } j>-p,\ k=N-1, \\ (\boldsymbol{\nabla}_1^+\phi)(-p,N-1)+(\boldsymbol{\nabla}_2^-\phi)(-p,N-1) & \text{if } j=-p,\ k=N-1, \end{cases}$$

while for $-p<k<N-2$,

$$\begin{aligned} (\mathscr{L}_N^{\mathrm{rw}}\phi)(k,k+1) \;&=\; (\boldsymbol{\nabla}_1^-\phi)(k,k+1) \;+\; (\boldsymbol{\nabla}_2^+\phi)(k,k+1)\;, \\ (\mathscr{L}_N^{\mathrm{rw}}\phi)(-p,-p+1) \;&=\; (\boldsymbol{\nabla}_2^+\phi)(-p,-p+1)\;, \\ (\mathscr{L}_N^{\mathrm{rw}}\phi)(N-2,N-1) \;&=\; (\boldsymbol{\nabla}_1^-\phi)(N-2,N-1)\;. \end{aligned}$$

In these formulae, $\boldsymbol{\nabla}_i^\pm$, resp. $\boldsymbol{\Delta}_i$, represents the discrete gradients, resp. Laplacians, given by

$$\begin{aligned} &(\boldsymbol{\nabla}_1^\pm\phi)(j,k) \;=\; \phi(j\pm1,k)-\phi(j,k)\;, \quad (\boldsymbol{\nabla}_2^\pm\phi)(j,k) \;=\; \phi(j,k\pm1)-\phi(j,k)\;, \\ &(\boldsymbol{\Delta}_1\phi)(j,k)=\phi(j-1,k)+\phi(j+1,k)-2\phi(j,k)\;, \\ &(\boldsymbol{\Delta}_2\phi)(j,k)=\phi(j,k-1)+\phi(j,k+1)-2\phi(j,k)\;, \\ &(\boldsymbol{\Delta}\phi)(j,k)=(\boldsymbol{\Delta}_1\phi)(j,k)+(\boldsymbol{\Delta}_2\phi)(j,k)\;. \end{aligned}$$

Let L_N^{ex} be the generator given by $L_N^{\mathrm{ex}}=L_S+L_{0,1}+L_{b,N}$. A straightforward computation yields that for $(j,k)\in\mathbb{D}_N$,

$$E_{\mu_N}\big[\, L_N^{\mathrm{ex}}\,\{\eta_j-\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}\,\big] \;=\; (\mathscr{L}_N^{\mathrm{rw}}\varphi_N)(1,j,k) \;+\; F_N(1,j,k)\;,$$

where it is understood that the generator $\mathscr{L}_N^{\mathrm{rw}}$ acts on the last two coordinates keeping the first one fixed, and

$$F_N(\sigma,j,k) \;=\; -\,\sigma\,[\rho_N(j+1)-\rho_N(j)]^2\,\mathbf{1}\{k=j+1\}\;. \tag{2.3.2}$$

Similarly,

$$E_{\mu_N}\big[\, L_N^{\mathrm{ex}}\,\{\bar\eta_j-\bar\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}\,\big] \;=\; (\mathscr{L}_N^{\mathrm{rw}}\varphi_N)(-1,j,k) \;+\; F_N(-1,j,k)\;.$$

For the next generators, we do not repeat the computation of the action of the generator on the product $\{\bar\eta_j-\bar\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}$ because it can be inferred from the action on $\{\eta_j-\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}$.

We turn to the remaining generators. Extend the definition of the rates r_j, $c_{j,k}$ and $a_{j,k}$ to $\Lambda_{N,p}$ by setting

$$r_j \;=\; c_{j,k} \;=\; a_{j,k} \;=\; 0 \quad \text{if } j \not\in \Lambda_p^* \text{ or } k \not\in \Lambda_p^* \;.$$

To present simple expressions for the equations satisfied by the two-point correlation function, we add cemetery points to the state space Ξ_N. Let $\overline{\Xi}_N = \Xi_N \cup \partial\, \Xi_N$, where

$$\begin{aligned} \partial\, \Xi_N \;&=\; \big\{(\sigma, k) : \sigma = \pm 1\,,\, -p \le k < N\big\} \,\cup\, \big\{(\sigma, k, k) : \sigma = \pm 1\,,\, -p \le k \le 0\big\} \\ &\cup\, \big\{(\sigma, k, N) : \sigma = \pm 1\,,\, -p \le k < N-1\big\} \end{aligned} \tag{2.3.3}$$

is the set of absorbing points.

A straightforward computation yields that for $(j,k) \in \mathbb{D}_N$,

$$E_{\mu_N}\big[\, L_R\, \{\eta_j - \rho_N(j)\}\, \{\eta_k - \rho_N(k)\}\,\big] \;=\; (\mathscr{L}_R^\dagger\, \varphi_N)(1,j,k)\;,$$

where

$$(\mathscr{L}_R^\dagger\, \phi)(\sigma,j,k) \;=\; r_j\, [\varphi_N(\sigma,k) - \varphi_N(\sigma,j,k)] \,+\, r_k\, [\varphi_N(\sigma,j) - \varphi_N(\sigma,j,k)]$$

provided we set

$$\varphi_N(\sigma, m) \;=\; b_N(\sigma,m) \;:=\; 0\;, \quad -p \;\le\; m \;<\; N\;,\;\; \sigma \;=\; \pm 1\;. \tag{2.3.4}$$

Similarly, an elementary computation yields that for $(j,k) \in \mathbb{D}_N$,

$$E_{\mu_N}\big[\, L_{r,N}\, \{\eta_j - \rho_N(j)\}\, \{\eta_k - \rho_N(k)\}\,\big] \;=\; (\mathscr{L}_{r,N}^\dagger\, \varphi_N)(1,j,k)\;,$$

where

$$(\mathscr{L}_{r,N}^\dagger\, \varphi_N)(\sigma,j,k) \;=\; \mathbf{1}\{k = N-1\}\, [\varphi_N(\sigma,j,N) - \varphi_N(\sigma,j,k)]\;,$$

provided we set

$$\varphi_N(\sigma, m, N) \;=\; b_N(\sigma,m,N) \;:=\; 0\;, \quad -p \;\le\; m \;\le\; N-2\;,\;\; \sigma \;=\; \pm 1\;. \tag{2.3.5}$$

We turn to the generator L_C. An elementary computation yields that for $(j,k) \in \mathbb{D}_N$,

$$E_{\mu_N}\big[\, L_C\, \{\eta_j - \rho_N(j)\}\, \{\eta_k - \rho_N(k)\}\,\big] \;=\; (\mathscr{L}_C^\dagger\, \varphi_N)(1,j,k)\;,$$

where

$$(\mathscr{L}_C^\dagger \phi)(\sigma,j,k) \;=\; \sum_{m:m\neq j} c_{j,m}\{\phi(\sigma,m,k)-\phi(\sigma,j,k)\} \;+\; \sum_{m:m\neq k} c_{k,m}\{\phi(\sigma,j,m)-\phi(\sigma,j,k)\}\;,$$

provided we set

$$\varphi_N(\sigma,m,m) \;=\; b_N(\sigma,m,m) \;:=\; \sigma\,\rho_N(m)\,[1-\rho_N(m)]\;, \quad -p\le m\le 0\;. \tag{2.3.6}$$

Finally, we claim that for $(j,k)\in\mathbb{D}_N$,

$$E_{\mu_N}\big[\,L_A\,\{\eta_j-\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}\,\big] \;=\; (\mathscr{L}_A^\dagger\,\varphi_N)(1,j,k)\;,$$

where

$$(\mathscr{L}_A^\dagger \phi)(\sigma,j,k) \;=\; \sum_{m:m\neq j} a_{j,m}\{\phi(-\sigma,m,k)-\phi(\sigma,j,k)\} + \sum_{m:m\neq k} a_{k,m}\{\phi(-\sigma,j,m)-\phi(\sigma,j,k)\}\;,$$

and $\varphi_N(\sigma,k,k)$ is given by (2.3.6). Hence, the generator $\mathscr{L}_A^\dagger$ acts exactly as $\mathscr{L}_C^\dagger$, but it flips the value of the first coordinate. Note that it is the only generator which changes the value of the first coordinate.

Let $\mathscr{L}_N^\dagger$ be the generator on $\overline{\Xi}_N$ given by

$$\mathscr{L}_N^\dagger \;=\; \mathscr{L}_N^{\rm rw} \,+\, \mathscr{L}_R^\dagger \,+\, \mathscr{L}_{r,N}^\dagger \,+\, \mathscr{L}_C^\dagger \,+\, \mathscr{L}_A^\dagger\;.$$

If $\sum_j \sum_{j,k} a_{j,k} = 0$, the generator $\mathscr{L}_A^\dagger$ vanishes, the first coordinate is kept constant by the dynamics and we do not need to introduce the variable σ. Note that the points in $\partial\,\Xi_N$ are absorbing points.

As $E_{\mu_N}[L_N\{\eta_j-\rho_N(j)\}\,\{\eta_k-\rho_N(k)\}]=0$, the previous computations yield that the two-point correlation function φ_N introduced in (2.3.1) solves

$$\begin{cases} (\mathscr{L}_N^\dagger\psi_N)(\sigma,j,k)+F_N(\sigma,j,k)=0\;, & (\sigma,j,k)\in\Xi_N\;,\\ \psi_N(\sigma,j,k) \;=\; b_N(\sigma,j,k)\;, & (\sigma,j,k)\in\partial\,\Xi_N\;, \end{cases} \tag{2.3.7}$$

where F_N and b_N are the functions defined in (2.3.2), (2.3.4), (2.3.5), (2.3.6).

As $\mathscr{L}_N^\dagger$ is a generator, (2.3.7) admits a unique solution [on the set $\{(1,j,k):(j,k)\in\mathbb{D}_N\}$ if $\mathscr{L}_A^\dagger$ vanishes]. This solution can be represented in terms of the Markov chain induced by the generator $\mathscr{L}_N^\dagger$.

Denote by $\varphi_N^{(1)}$, resp. $\varphi_N^{(2)}$, the solution of (2.3.7) with $b_N=0$, resp. $F_N=0$. It is clear that $\varphi_N = \varphi_N^{(1)} + \varphi_N^{(2)}$. Denote by $X_N(t)$ the continuous-time Markov chain on $\overline{\Xi}_N$ associated to the generator $\mathscr{L}_N^{\dagger}$. Let $\boldsymbol{P}_{(\sigma,j,k)}$ be the distribution of the chain X_N starting from (σ, j, k). Expectation with respect to $\boldsymbol{P}_{(\sigma,j,k)}$ is represented by $\boldsymbol{E}_{(\sigma,j,k)}$.

Let H_N be the hitting time of the boundary $\partial\,\Xi_N$:

$$H_N \;=\; \inf\big\{t \ge 0 : X_N(t) \in \partial\,\Xi_N\,\big\}\;.$$

It is well known (cf. [5, Theorem 6.5.1] in the continuous case) that

$$\varphi_N^{(1)}(\sigma,j,k) \;=\; \boldsymbol{E}_{(\sigma,j,k)}\Big[\int_0^{H_N} F_N(X_N(s))\,ds\Big]\;.$$

It is also well known that

$$\varphi_N^{(2)}(\sigma,j,k) \;=\; \boldsymbol{E}_{(\sigma,j,k)}\big[b_N(X_N(H_N))\big]\;.$$

To estimate $\varphi_N^{(1)}$ and $\varphi_N^{(2)}$ we need to show that the process $X_N(t)$ attains the boundary $\partial\,\Xi_N$ at the set $\{(\sigma,k,k) : \sigma = \pm 1\,,\, -p \le k \le 0\}$ with small probability. This is the content of the next two lemmata.

For a subset A of $\overline{\Xi}_N$, denote by $H(A)$, resp. $H^+(A)$, the hitting time of the set A, resp. the return time to the set A:

$$H(A) \;=\; \inf\big\{t \ge 0 : X_N(t) \in A\big\}\;, \quad H^+(A) \;=\; \inf\big\{t \ge \tau_1 : X_N(t) \in A\big\}\;,$$

where τ_1 represents the time of the first jump: $\tau_1 = \inf\{s > 0 : X_N(s) \ne X_N(0)\}$.

The next lemma, illustrated in Figure 1, translates to the present model the fact that starting from $(1,0)$ the two-dimensional, nearest-neighbor, symmetric random walk hits the line $\{(0,k) : k \in \mathbb{Z}\}$ at a distance n or more from the origin with a probability less than C/n.

Let $\widehat{\boldsymbol{Q}}_{(l,m)}$ be the law of such a random walk evolving on $\mathbb{Z}^2$ starting from (l,m). Denote by $B_r(l,m)$ the ball of radius $r > 0$ and center $(l,m) \in \mathbb{Z}^2$, and by $\mathbb{L}$ the segment $\{(\sigma,0,a) : \sigma = \pm 1\,,\, 1 \le a < N\}$. Represent the coordinates of $X_N(t)$ by $(\sigma_N(t), X_N^1(t), X_N^2(t))$.

Lemma 2.3.1. *Let $p' = p+1$. There exists a finite constant C_0 such that for all n,*

$$\max_{\sigma=\pm 1}\max_{l,m} \boldsymbol{P}_{(\sigma,l,m)}\big[\,H(\mathbb{L}) = \infty \text{ or } X_N^2(H(\mathbb{L})) \le m - p'n\,\big] \;\le\; \frac{C_0}{n}\;,$$

where the maximum is carried over all pairs (l,m) such that $1 \le l \le p'$, $\{(a,b) \in B_{p'n}(0,m) : a \ge 0\} \subset \mathbb{D}_N^0 = \{(a,b) \in \mathbb{D}_N : a \ge 0\}$.

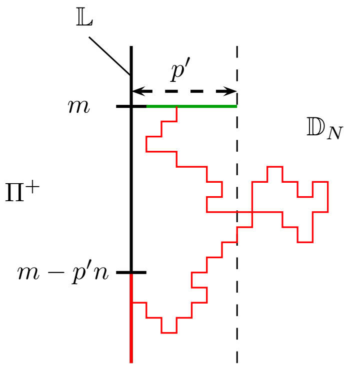

Figure 1: Lemma 2.3.1 states that a random walk (red trajectory) started from the green segment has a probability at most of order $1/n$ of hitting $\mathbb{L}$ in the red half-line.

Proof. Let $\mathbb{L}_r = \{(0,l) : -r \le l \le r\}$. By [8, Proposition 2.4.5], there exists a finite constant C_0 such that for all $n \ge 1$,

$$\widehat{\boldsymbol{Q}}_{(1,0)}\big[H(B_n(0,0)^{\complement}) < H(\mathbb{L}_n)\big] \;\le\; \frac{C_0}{n}\,\cdot$$

Let $\mathbb{L}_r(l,m) = \{(\sigma,l,a) : \sigma = \pm 1\,,\, m-r \le a \le m+r\}$. By the previous displayed equation, if $\mathbb{L}_n(l,m)$ is contained in $\mathbb{D}_N^0$,

$$\boldsymbol{P}_{(\sigma,l+1,m)}\big[\,H(B_n(l,m)^{\complement}) < H(\mathbb{L}_n(l,m))\,\big] \;\le\; \frac{C_0}{n}\,\cdot$$

Iterating this estimate i times yields that

$$\boldsymbol{P}_{(\sigma,l+i,m)}\big[\,H(B_{in}(l,m)^{\complement}) < H(\mathbb{L}_{in}(l,m))\,\big] \;\le\; \frac{C_0\, i}{n}$$

provided all sets appearing in this formula are contained in $\mathbb{D}_N^0$. The assertion of the lemma follows from this estimate and the following observation:

$$\{\,H(\mathbb{L}) = \infty \text{ or } X_N^2(H(\mathbb{L})) \le m - p'n\,\} \subseteq \{\,H(B_{p'n}(0,m)^{\complement}) < H(\mathbb{L}_{p'n}(0,m))\,\}.$$

∎

The next lemma presents the main estimate needed in the proof of the bounds of the two-point correlation functions. Recall from (2.3.3) that we denote by (σ,k), (σ,k,N) some cemetery points. Let

$$\Sigma = \{(\sigma,l,0) : \sigma = \pm 1\,,\, -p \le l < 0\}\,,$$
$$\partial_N \;=\; \big\{(\sigma,k) : \sigma = \pm 1\,,\, -p \le k < N\big\} \,\cup\, \big\{(\sigma,k,N) : \sigma = \pm 1\,,\, -p \le k < N-1\big\}\,.$$

Lemma 2.3.2. *For all $\delta > 0$,*

$$\lim_{N\to\infty} \max_{\substack{(j,k)\in\mathbb{D}_N \\ j>\delta N}} \boldsymbol{P}_{(1,j,k)}\big[H(\Sigma) < H(\partial_N)\big] \;=\; 0\;.$$

Proof. Fix $\delta > 0$ and $(j, k) \in \mathbb{D}_N$ such that $j > \delta N$. Let

$$\partial_N^0 \;=\; \{(\sigma, 0, m) : \sigma = \pm 1\,,\, 0 < m < N\} \,\cup\, \big\{(\sigma, k, N) : \sigma = \pm 1\,,\, -p \le k < N-1\big\}\;,$$

and set $\tau = H(\partial_N^0)$. Clearly, $\tau < H(\Sigma)$. Hence, by the strong Markov property, the probability appearing in the statement of the lemma is equal to

$$\boldsymbol{E}_{(1,j,k)}\Big[\boldsymbol{P}_{X_N(\tau)}\big[H(\Sigma) < H(\partial_N)\big]\Big]\;. \tag{2.3.8}$$

Up to time τ, the process X_N evolves as a symmetric random walk on $\mathbb{D}_N$

Let ℓ_N be a sequence such that $\ell_N \ll N$. We claim that for all $\delta > 0$,

$$\lim_{N\to\infty} \max_{(l,m)} \boldsymbol{P}_{(1,l,m)}\big[\, X_N^2(\tau) \le \ell_N \,\big] \;=\; 0\;, \tag{2.3.9}$$

where the maximum is carried out over all pairs $(l, m) \in \mathbb{D}_N$ such that $l > \delta N$. The proof of this statement relies on the explicit form of the harmonic function for a 2-dimensional Brownian motion.

Up to time τ, the process $Y_N(t) = (X_N^1(t), X_N^2(t))$ evolves on the set $\triangle_N = \{(a, b) : 0 \le a < b \le N\}$. Let $\square_N = \{0, \dots, N-1\} \times \{1, \dots, N\}$. Denote by $Z_N(t) = (Z_N^1(t), Z_N^2(t))$ the random walk on $\square_N$ which jumps from a point to any of its neighbors at rate 1. Let $\Phi_N : \square_N \to \triangle_N$ the projection defined by $\Phi_N(a, b) = (a, b)$ if $(a, b) \in \triangle_N$, and $\Phi_N(a, b) = (b-1, a+1)$ otherwise. The process $\Phi_N(Z_N(t))$ does not evolve as $Y_N(t)$ because the jumps of $\Phi_N(Z_N(t))$ on the diagonal $\{(d, d+1) : 0 \le d < N\}$ are speeded-up by 2, but the sequence of sites visited by both processes has the same law. Therefore,

$$\boldsymbol{P}_{(1,l,m)}\big[\, X_N^2(\tau) \le \ell_N \,\big] \;=\; \boldsymbol{Q}_{(l,m)}\big[\, Z_N(\widehat{\tau}) \in \angle_N \big]\;,$$

where $\boldsymbol{Q}_{(l,m)}$ represents the law of the process Z_N starting from (l, m), $\widehat{\tau}$ the hitting time of the boundary of $\square_N$ and $\angle_N$ the set $\{(0, a) : 1 \le a \le \ell_N\} \cup \{(b, 1) : 0 \le b \le \ell_N - 1\}$.

Denote by $B(r) \subset \mathbb{R}^2$, $r > 0$, the ball of radius r centered at the origin. In the event $\{Z_N(\widehat{\tau}) \in \angle_N\}$, the process Z_N hits the ball of radius ℓ_N centered at the origin before reaching the ball of radius $2N$ centered at the origin: $\{Z_N(\widehat{\tau}) \in \angle_N\} \subset \{H(B(\ell_N)) <$

$H(B(2N))\}$, so that

$$\boldsymbol{Q}_{(l,m)}\big[\, Z_N(\widehat{\tau}) \in \angle_N \big] \;\le\; \widehat{\boldsymbol{Q}}_{(l,m)}\big[\, H(B(\ell_N)) < H(B(2N))\,\big]\;.$$

By [8, Exercice 1.6.8], this later quantity is bounded by

$$\frac{\log 2N - \log |(l,m)| + C\ell_N^{-1}}{\log 2N - \log \ell_N}$$

for some finite constant independent of N. This proves (2.3.9) because $|(l,m)| \ge \delta N$ and $\ell_N \ll N$.

We return to (2.3.8). If $X_N(\tau) \in \partial_N$, the probability vanishes. We may therefore insert inside the expectation the indicator of the set $X_N(\tau) \not\in \partial_N$ It is also clear that $\sigma_N(t)$ does not change before time τ. Hence, by (2.3.9), (2.3.8) is bounded by

$$\begin{aligned} &\boldsymbol{E}_{(1,j,k)}\Big[\mathbf{1}\{X_N(\tau)\in \mathbb{L}^+(\ell_N)\}\, \boldsymbol{P}_{X_N(\tau)}\big[H(\Sigma) < H(\partial_N)\big]\Big] \;+\; o_N(1) \\ &\quad \le\; \max_{m\ge \ell_N} \boldsymbol{P}_{(1,0,m)}\big[H(\Sigma) < H(\partial_N)\big] \;+\; o_N(1)\;, \end{aligned}$$

where $\mathbb{L}^+(r) = \{(\sigma,0,l) : \sigma = \pm 1\,,\, l \ge r\}$, $o_N(1)$ converges to 0 as $N \to \infty$, uniformly over all $(j,k) \in \mathbb{D}_N$, $j > \delta N$, and ℓ_N is a sequence such that $\ell_N \ll N$. Hence, up to this point, we proved that

$$\max_{\substack{(j,k)\in\mathbb{D}_N\\ j>\delta N}} \boldsymbol{P}_{(1,j,k)}\big[H(\Sigma) < H(\partial_N)\big] \;\le\; \max_{m\ge\ell_N} \boldsymbol{P}_{(1,0,m)}\big[H(\Sigma) < H(\partial_N)\big] \;+\; o_N(1)\;, \quad (2.3.10)$$

where $o_N(1)$ converges to 0 as $N \to \infty$, and ℓ_N is a sequence such that $\ell_N \ll N$.

It remains to estimate the probability appearing in the previous formula. If $m > p'$, starting from $(1,0,m)$, in p' jumps the process $X_N(t)$ can not hit Σ. Hence, if $\tau(k)$ stands for the time of the k-th jump, by the strong Markov property,

$$\begin{aligned} \boldsymbol{P}_{(1,0,m)}\big[H(\Sigma) < H(\partial_N)\big] \;&=\; \boldsymbol{P}_{(1,0,m)}\big[H(\partial_N) > \tau(p')\,,\, H(\Sigma) < H(\partial_N)\big] \\ &=\; \boldsymbol{E}_{(1,0,m)}\Big[\,\mathbf{1}\{H(\partial_N) > \tau(p')\}\, \boldsymbol{P}_{X_N(\tau(p'))}\big[H(\Sigma) < H(\partial_N)\big]\Big]\;. \end{aligned}$$

Let $\varrho = \boldsymbol{P}_{(1,0,m)}[H(\partial_N) > \tau(p')] = \boldsymbol{P}_{(-1,0,m)}[H(\partial_N) > \tau(p')]$. Note that this quantity does not depend on m in the set $\{(\sigma,0,b) : \sigma = \pm 1\,,\, b > p'\}$. Moreover, as $\sum_j r_j > 0$, $\varrho < 1$. With this notation, the previous expression is less than or equal to

$$\varrho \max_{\sigma=\pm 1}\, \max_{a,b}\, \boldsymbol{P}_{(\sigma,a,b)}\big[H(\Sigma) < H(\partial_N)\big]\;,$$

where the maximum is carried over all (a, b) which can be attained in p' jumps from $(0, m)$. This set is contained in the set $\{(c, d) : -p \le c \le p' , \ m - p' \le d \le m + p'\}$.

Recall the definition of the set $\mathbb{L}$ introduced just before the statement of Lemma 2.3.1. If $a \ge 1$, the process $X_N(t)$ hits the set $\mathbb{L}$ before the set Σ. Hence, by Lemma 2.3.1, if q_N is an increasing sequence to be defined later, by the strong Markov property, for $1 \le a \le p'$, $b \gg q_N$,

$$\begin{aligned} &\boldsymbol{P}_{(\sigma,a,b)}\big[H(\Sigma) < H(\partial_N)\big] \\ &\quad \le \frac{C_0}{q_N} + \boldsymbol{P}_{(\sigma,a,b)}\big[X_N^2(H(\mathbb{L})) \ge b - p'q_N \ , \ H(\Sigma) < H(\partial_N)\big] \\ &\quad \le \frac{C_0}{q_N} + \max_{b' \ge b - p'q_N} \boldsymbol{P}_{(\sigma,0,b')}\big[H(\Sigma) < H(\partial_N)\big] \ . \end{aligned}$$

On the other hand, if $a \le -1$, let $\mathbb{C}_d = \{(\sigma, c, d) : \sigma = \pm 1 \, , \, -p \le c < 0\}$. In this case, starting from (a, b), in p' jumps the process $X_N(t)$ may hit the set $\mathbb{L}$. Hence, by the strong Markov property, for $a < 0$, $b > np'$, $\boldsymbol{P}_{(\sigma,a,b)}\big[H(\mathbb{C}_{b-np'}) < H(\mathbb{L}) \wedge H(\partial_N)\big] \le \varrho_1^n$ for some $\varrho_1 < 1$. Therefore, by the strong Markov property, for $a < 0$ and $b \gg q_N$,

$$\begin{aligned} &\boldsymbol{P}_{(\sigma,a,b)}\big[H(\Sigma) < H(\partial_N)\big] \\ &\quad \le \boldsymbol{P}_{(\sigma,a,b)}\big[H(\mathbb{L}) \wedge H(\partial_N) < H(\mathbb{C}_{b-p'q_N}) \, , \, H(\Sigma) < H(\partial_N)\big] + \varrho_1^{q_N} \\ &\quad \le \max_{\sigma'=\pm 1} \max_{b' \ge b - p'q_N} \boldsymbol{P}_{(\sigma',0,b')}\big[H(\Sigma) < H(\partial_N)\big] + \varrho_1^{q_N} \ . \end{aligned}$$

Let

$$T_N(b) \ = \ \max_{\sigma=\pm 1} \max_{c \ge b} \boldsymbol{P}_{(\sigma,0,c)}\big[H(\Sigma) < H(\partial_N)\big] \ .$$

Note that the first term appearing on the right-hand side of (2.3.10) is $T_N(\ell_N)$ because the probability does not depend on the value of σ. By the previous arguments, there exists a finite constant C_0 such that for all $b \gg q_N$,

$$T_N(b) \ \le \ \varrho \Big\{ T_N(b - p'q_N) \ + \ \frac{C_0}{q_N} \Big\}$$

because $\varrho_1^q \le 1/q$ for all q large enough. Iterating this inequality r_N times, we get that for all $b \gg q_N r_N$,

$$T_N(b) \ \le \ \frac{C_0}{q_N}\{\varrho + \cdots + \varrho^{r_N}\} \ + \ \varrho^{r_N} \ \le \ \frac{\varrho}{1-\varrho} \frac{C_0}{q_N} \ + \ \varrho^{r_N} \ .$$

In view of (2.3.10) and of the previous estimate, to complete the proof of the lemma, it remains to choose sequences q_N, r_N such that $q_N \to \infty$, $r_N \to \infty$, $r_N \, q_N \ll \ell_N$. ∎

Lemma 2.3.3. *Assume that* $\sum_j r_j > 0$. *Then, for every* $\delta > 0$,

$$\lim_{N\to\infty} \max_{\substack{(j,k)\in\mathbb{D}_N \\ j>\delta N}} \big|\, \varphi^{(1)}_N(1,j,k) \,\big| \;=\; 0\;.$$

Proof. Fix $(j,k) \in \mathbb{D}_N$ such that $0 < j < k$. Denote by D_N the diagonal, $D_N = \{(\sigma, l, l+1) : \sigma = \pm 1\,,\, -p \le l < N-1\}$, and by $D_{N,p}$ its restriction to Λ^*_p, $D_{N,p} = \{(\sigma, l, l+1) : \sigma = \pm 1\,,\, -p \le l \le 0\}$. By Lemma 2.2.5, there exists a finite constant C_0 such that for all $(l,m) \in \mathbb{D}_N$,

$$|F_N(\sigma, l.m)| \;\le\; \frac{C_0}{N^2}\, \mathbf{1}\{D_N \setminus D_{N,p}\}(\sigma,l,m) \;+\; C_0\, \mathbf{1}\{D_{N,p}\}(\sigma,l,m)\;.$$

Therefore, recalling that H_N was defined as the hitting time of the boundary $\partial\Xi_N$,

$$\begin{aligned} \big|\, \varphi^{(1)}_N(1,j,k) \,\big| \;&\le\; \frac{C_0}{N^2}\, \boldsymbol{E}_{(1,j,k)}\Big[\int_0^{H_N} \mathbf{1}\{D_N \setminus D_{N,p}\}(X_N(s))\, ds\Big] \\ &\quad +\; C_0\, \boldsymbol{E}_{(1,j,k)}\Big[\int_0^{H_N} \mathbf{1}\{D_{N,p}\}(X_N(s))\, ds\Big]\;. \end{aligned} \tag{2.3.11}$$

We claim that there exists a finite constant C_0 such that

$$\max_{\sigma=\pm 1} \max_{\substack{(j,k)\in\mathbb{D}_N \\ 0<j<k}} \boldsymbol{E}_{(\sigma,j,k)}\Big[\int_0^{H_N} \mathbf{1}\{D_N \setminus D_{N,p}\}(X_N(s))\, ds\Big] \;\le\; C_0\, N\;. \tag{2.3.12}$$

To bound this expectation, let $R_N = \{(\sigma, 0, m) : \sigma = \pm 1\,,\, 2 \le m \le N-1\}$, and denote by G_N the hitting time of the set $R_N \cup \partial\,\Xi_N$. Note that starting from (j,k), $0 < j < k$, only the component $\big\{(\sigma, l, N) : -p \le l < N-1\big\}$ of the set $\partial\,\Xi_N$ can be attained before the set R_N. Moreover, before G_N the process $X_N(t)$ behaves as a symmetric random walk.

Rewrite the expectation in (2.3.12) as

$$\boldsymbol{E}_{(\sigma,j,k)}\Big[\int_0^{G_N} \mathbf{1}\{D_N \setminus D_{N,p}\}(X_N(s))\, ds\Big] \;+\; \boldsymbol{E}_{(\sigma,j,k)}\Big[\int_{G_N}^{H_N} \mathbf{1}\{D_N \setminus D_{N,p}\}(X_N(s))\, ds\Big]\;. \tag{2.3.13}$$

Since before time G_N the process $X_N(t)$ evolves as a symmetric random walk, the first expectation can be computed. It is equal to $j(N-k)/(N-1) \le C_0\, N$. By the strong Markov property, the second expectation is bounded above by

$$\max_{2\le m<N} \boldsymbol{E}_{(\sigma,0,m)}\Big[\int_0^{H_N} \mathbf{1}\{D_N \setminus D_{N,p}\}(X_N(s))\, ds\Big]\;.$$

Denote by Υ_N the previous expression and by G^+_N the return time to $R_N \cup \partial\,\Xi_N$. By the strong Markov property, the previous expectation is bounded above by

$$\boldsymbol{E}_{(\sigma,0,m)}\Big[\int_0^{G_N^+} \mathbf{1}\{D_N\setminus D_{N,p}\}(X_N(s))\,ds\Big] \;+\; \Upsilon_N \max_{0\le m'<N-1} \boldsymbol{P}_{(\sigma,0,m')}\big[\,G_N^+ < H_N\,\big]\;.$$

The first term vanishes unless the first jump of $X_N(s)$ is to $(\sigma,1,m)$. Suppose that this happens. Starting from $(\sigma,1,m)$, up to time G_N^+, $X_N(s)$ behaves as a symmetric random walk. Hence, by explicit formula for the first term in (2.3.13), the expectation is equal to $(N-m)/(N-1)\le 1$. Hence,

$$\Upsilon_N \;\le\; \max_{0\le m'<N-1} \frac{1}{\boldsymbol{P}_{(\sigma,0,m')}\big[\,H_N < G_N^+\,\big]}\;.$$

As $\sum_j r_j>0$, $P_{(\sigma,0,m')}[\,H_N < G_N^+\,]$ is bounded below by the probability that the process jumps to a site (σ,l,m') such that $r_l>0$ and then hits the set $\partial\,\Xi_N$. Hence, there exists a positive constant c_0 such that $P_{(\sigma,0,m')}[\,H_N < G_N^+\,]\ge c_0$ for all $2\le m'\le N-1$. This proves that $\Upsilon_N\le C_0$. Assertion (2.3.12) follows from this bound and the estimate for the first term in (2.3.13).

We turn to the second term in (2.3.11). Recall the notation introduced just before Lemma 2.3.2. Since the integrand vanishes before hitting the set $D_{N,p}$ and since the set Σ is attained before $D_{N,p}$, for $j>\delta N$

$$\begin{aligned}&\boldsymbol{E}_{(1,j,k)}\Big[\int_0^{H_N} \mathbf{1}\{D_{N,p}\}(X_N(s))\,ds\Big]\\ &\quad=\;\boldsymbol{E}_{(1,j,k)}\Big[\mathbf{1}\{H(\Sigma)<H(\partial_N)\}\int_{H(D_{N,p})}^{H_N} \mathbf{1}\{D_{N,p}\}(X_N(s))\,ds\Big]\;.\end{aligned}$$

Applying the strong Markov property twice, we bound this expression by

$$\boldsymbol{P}_{(1,j,k)}\big[\,H(\Sigma)<H(\partial_N)\,\big]\;\max_{(\sigma,a,b)\in D_{N,p}} \boldsymbol{E}_{(\sigma,a,b)}\Big[\int_0^{H_N} \mathbf{1}\{D_{N,p}\}(X_N(s))\,ds\Big]\;.$$

By Lemma 2.3.2 the first term vanishes as $N\to\infty$, uniformly over $(j,k)\in\mathbb{D}_N$, $j>\delta N$.

It remains to show that there exists a finite constant C_0 such that

$$\max_{(\sigma,j,k)\in D_{N,p}} \boldsymbol{E}_{(\sigma,j,k)}\Big[\int_0^{H_N} \mathbf{1}\{D_{N,p}\}(X_N(s))\,ds\Big]\;\le\; C_0\;. \tag{2.3.14}$$

Denote this expression by Υ_N, and by J_N^+ the return time to $D_{N,p}$. For $(\sigma,j,k)\in D_{N,p}$, the previous expectation is less than or equal to

$$C_0 \;+\; \Upsilon_N \, \boldsymbol{P}_{(\sigma,j,k)}\big[\, J_N^+ < H_N \,\big] \;.$$

As in the first part of the proof, since $\sum_j r_j > 0$, the process hits $\partial\,\Xi_N$ before returning to $D_{N,p}$ with a probability bounded below by a strictly positive constant independent of N: $\min_{(\sigma,j,k)\in D_{N,p}} \boldsymbol{P}_{(\sigma,j,k)}[\, H_N < J_N^+ \,] \ge c_0 > 0$. Therefore, $\Upsilon_N \le C_0$. This completes the proof of assertion (2.3.14) and the one of the lemma. ∎

Lemma 2.3.4. *Assume that $\sum_j r_j > 0$. Then, for every $\delta > 0$,*

$$\lim_{N\to\infty} \max_{\substack{(j,k)\in\mathbb{D}_N \\ j>\delta N}} \big|\, \varphi^{(2)}_N(1,j,k) \,\big| \;=\; 0 \;.$$

Proof. Fix $\delta > 0$ and $(j,k) \in \mathbb{D}_N$ such that $j > \delta N$. Recall the notation introduced just before Lemma 2.3.2. In view of the definition of b_N, given in (2.3.4), (2.3.5), (2.3.6),

$$|\varphi^{(2)}_N(1,j,k)| \;\le\; \boldsymbol{P}_{(1,j,k)}\big[\, H(\Sigma) < H(\partial_N) \,\big] \;.$$

The assertion of the lemma follows from Lemma 2.3.2. ∎

Proof of Theorem 2.1.1. The proof is straightforward. It is enough to prove the result for continuous functions with compact support in $(0,1)$. Fix such a function G and let $\delta > 0$ such that the support of G is contained in $[\delta, 1-\delta]$. By Schwarz inequality and by (2.3.1), the square of the expectation appearing in the statement of the theorem is bounded above by

$$C(G)\,\Big(\frac{1}{N}\sum_{k=1}^{N-1} \big|\, \rho_N(k) - \bar u(k/N) \,\big|\Big)^2 \;+\; \frac{C(G)}{N^2} \sum_{j,k=1}^{N-1} G(j/N)\, G(k/N)\, \varphi_N(1,j,k) \;,$$

where φ_N has been introduced in (2.3.1) and $C(G)$ a finite constant which depends only on G. By Lemmata 2.2.5, 2.3.3 and 2.3.4 this expression vanishes as $N \to \infty$. ∎

Remark 2.3.5. *Assume that $\sum_{j\in\Lambda_p^*} r_j = 0$ and $\sum_{j,k\in\Lambda_p^*} a_{j,k} > 0$. The proof that the correlations vanish, presented in Lemmata 2.3.3 and 2.3.4, requires a new argument based on the following observation. Under the conditions of this remark, the boundary $\partial\,\Xi_N$ of the set Ξ_N is reduced to the set*

$$\big\{(\sigma,k,k) : \sigma = \pm 1 \,,\, -p \le k \le 0\big\} \,\cup\, \big\{(\sigma,k,N) : \sigma = \pm 1 \,,\, -p \le k < N-1\big\} \;.$$

To prove that the correlations vanish, one has to show that by the time the process $X_N(t)$ hits the set $\{(\sigma, k, k) : \sigma = \pm 1\,,\ -p \le k \le 0\}$ its coordinate σ has equilibrated and takes the value ± 1 with probability close to $1/2$.

2.4 Proof of Theorem 2.1.4

The proof of Theorem 2.1.4 is based on a graphical construction of the dynamics through independent Poisson point processes.

Recall the definition of the rates A, B introduced in (2.1.12), that $\Omega_p = \{0,1\}^{\{1,\dots,p-1\}}$, and that $\lambda(0,\xi) = c(0,\xi) - A$, $\lambda(1,\xi) = c(1,\xi) - B$, $\xi \in \Omega_p$. Further, recall that we assume

$$(p-1) \sum_{\xi \in \Omega_p} \{\, \lambda(0,\xi) + \lambda(1,\xi) \,\} \;<\; A + B\,.$$

The left boundary generator can be rewritten as

$$\begin{aligned}(L_l f)(\eta) \;&=\; A\,[f(T^1\eta) - f(\eta)] \;+\; B\,[f(T^0\eta) - f(\eta)] \\ &+ \sum_{a=0}^{1} \sum_{\xi \in \Omega_p} \lambda(a,\xi)\, \mathbf{1}\{\Pi_p \eta = (a,\xi)\}\, [f(T^{1-a}\eta) - f(\eta)]\,,\end{aligned}$$

provided $\Pi_p : \Omega_N \to \Omega_p^\star := \{0,1\}^{\{1,\dots,p\}}$ represents the projection on the first p coordinates: $(\Pi_p \eta)_k = \eta_k$, $1 \le k \le p$. Similarly, the right boundary generator can be expressed as

$$(L_{r,N} f)(\eta) \;=\; \beta\,[f(S^1\eta) - f(\eta)] \;+\; (1-\beta)\,[f(S^0\eta) - f(\eta)]\,,$$

where

$$(S^a\eta)_k \;=\; \begin{cases} a & \text{if } k = N-1, \\ \eta_k & \text{otherwise.} \end{cases}$$

2.4.1 Graphical construction

Let $P := 2^{p-1} = |\Omega_p|$. We present in this subsection a graphical construction of the dynamics based on $N + 2P + 2$ independent Poisson point processes defined on $\mathbb{R}_+$.

- $(N-2)$ processes $\mathfrak{N}_{i,i+1}(t)$, $1 \le i \le N-2$, with rate 1.
- 2 processes $\mathfrak{N}^{+,l}(t)$, $\mathfrak{N}^{-,l}(t)$ with rates A, B, respectively, representing creation and annihilation of particles at site 1, regardless of the boundary condition.

– $2P$ processes $\mathfrak{N}_{(a,\xi)}(t)$, $a = 0, 1$, $\xi \in \Omega_p$, with rates $\lambda(a, \xi)$ to take into account the influence of the boundary in the creation and annihilation of particles at site 1.

– 2 processes $\mathfrak{N}^{+,r}(t)$, $\mathfrak{N}^{-,r}(t)$, with respective rates β and $1 - \beta$, to trigger creation and annihilation of particles at site $N - 1$.

Place arrows and daggers on $\{1, \dots, N - 1\} \times \mathbb{R}$ as follows. Whenever the process $\mathfrak{N}_{i,i+1}(t)$ jumps, place a two-sided arrow over the edge $(i, i+1)$ at the time of the jump to indicate that at this time the occupation variables η_i, η_{i+1} are exchanged. Analogously, each time the process $\mathfrak{N}_{(a,\xi)}(t)$ jumps, place a dagger labeled (a, ξ) over the vertex 1. Each time $\mathfrak{N}^{\pm,l}(t)$ jumps, place a dagger labeled $\pm$ over the vertex 1. Finally, each time $\mathfrak{N}^{\pm,r}(t)$ jumps, place a dagger labeled $\pm$ over the vertex $N - 1$.

Fix a configuration $\zeta \in \Omega_N$ and a time $t_0 \in \mathbb{R}$. Define a path $\eta(t)$, $t \geq t_0$, based on the configuration ζ and on the arrows and daggers as follows. By independence, we may exclude the event that two of those processes jump simultaneously. Let $\tau_1 > t_0$ be the first time a mark (arrow or dagger) is found after time t_0. Set $\eta(t) = \zeta$ for any $t \in [t_0, \tau_1)$. If the first mark is an arrow labeled $(i, i + 1)$, set $\eta(\tau_1) = \sigma^{i,i+1}\eta(\tau_1-)$. If the mark is a dagger labeled (a, ξ), set $\eta(\tau_1) = T^a\eta(\tau_1-)$ if $\Pi_p\eta(\tau_1-) = (a, \xi)$. Otherwise, let $\eta(\tau_1) = \eta(\tau_1-)$. Finally, if the mark is a dagger on site 1, resp. $N - 1$, labeled $\pm$, set $\eta(\tau_1) = T^{[1\pm1]/2}\eta(\tau_1-)$, resp. $\eta(\tau_1) = S^{[1\pm1]/2}\eta(\tau_1-)$.

At this point, the path η is defined on the segment $[t_0, \tau_1]$. By repeating the previous construction on each time-interval between two consecutive jumps of the Poisson point processes, we produce a trajectory $(\eta(t) : t \geq t_0)$. We leave the reader to check that $\eta(t)$ evolves as a continuous-time Markov chain, started from ζ, whose generator is the operator L_N introduced in (2.1.11).

2.4.2 Dual Process

To determine whether site 1 is occupied or not at time $t = 0$ we have to examine the evolution backward in time. This investigation, called the revealment process, evolves as follows.

Let mark mean an arrow or a dagger. To know the value of $\eta_1(0)$ we have to examine the past evolution. Denote by $\tau_1 < 0$ the time of the last mark involving site 1 before $t = 0$. By the graphical construction, the value of η_1 does not change in the time interval $[\tau_1, 0]$.

Suppose that the mark at time τ_1 is an arrow between 1 and 2. In order to determine if site 1 is occupied at time 0 we need to know if site 2 is occupied at time τ_1-. The arrows are thus acting as a stirring dynamics in the revealment process. Each time an arrow is found, the site whose value has to be determined changes.

If the mark at time τ_1 is a dagger labeled $+$ at site 1, $\eta_1(0) = \eta_1(\tau_1) = 1$, and we do not need to proceed further. Analogously, daggers labeled $-$ or $+$ at sites 1, $N-1$ reveal the value of the occupation variables at these sites at the time the mark appears. Hence, these marks act an annihilation mechanism.

Suppose that the mark at time τ_1 is a dagger labeled (a, ξ). To determine whether site 1 is occupied at time 0 we need to know the values of $\eta_1(\tau_1-), \dots, \eta_p(\tau_1-)$. Indeed, if $\Pi_p \eta(\tau_1-) = (a, \xi)$, $\eta_1(0) = \eta_1(\tau_1) = 1 - a$, otherwise, $\eta_1(0) = \eta_1(\tau_1) = \eta_1(\tau_1-)$. Hence, marks labeled (a, ξ) act as branching events in the revealment process.

It follows from this informal description that to determine the value at time 0 of site 1, we may be forced to find the values of the occupation variables of a larger subset $\mathscr{A}$ of Λ_N at a certain time $t < 0$.

Suppose that we need to determine the values of the occupation variables of the set $\mathscr{A} \subset \Lambda_N$ at time $t < 0$. Let $\tau < t$ be the first [backward in time] mark of one of the Poisson processes: there is a mark at time τ and there are no marks in the time interval $(\tau, t]$. Suppose that the mark at time τ is

(a) an arrow between i and $i + 1$;

(b) a dagger labeled $\pm$ at site 1;

(c) a dagger labeled $\pm$ at site $N - 1$;

(d) a dagger labeled (a, ξ) at site 1.

Then, to determine the values of the occupation variables in the set $\mathscr{A}$ at time τ (and thus at time t), we need to find the values of the occupation variables in the set

(a) $\sigma^{i,i+1}\mathscr{A}$, defined below in (2.4.1);

(b) $\mathscr{A} \setminus \{1\}$;

(c) $\mathscr{A} \setminus \{N - 1\}$;

(d) $\mathscr{A} \cup \{1, \dots, p\}$ if $1 \in \mathscr{A}$, and $\mathscr{A}$ otherwise

at time $\tau-$. Since independent Poisson processes run backward in time are still independent Poisson processes, this evolution corresponds to a Markov process taking values in Ξ_N, the set of subsets of Λ_N, whose generator $\mathfrak{L}_N$ is given by

$$\mathfrak{L}_N \;=\; \mathfrak{L}_l \;+\; \mathfrak{L}_{0,N} \;+\; \mathfrak{L}_{r,N} \;,$$

where

$$(\mathfrak{L}_{0,N} f)(\mathscr{A}) \;=\; \sum_{i=1}^{N-2} [f(\sigma^{i,i+1}\mathscr{A}) - f(\mathscr{A})] \;;$$

$$\begin{aligned}(\mathfrak{L}_l f)(\mathscr{A}) \;&=\; (A+B)\,\mathbf{1}\{1\in\mathscr{A}\}\;(f(\mathscr{A}\setminus\{1\}) - f(\mathscr{A})) \\ &+\; \sum_{\xi\in\Omega_p} \lambda(\xi)\,\mathbf{1}\{1\in\mathscr{A}\}\;(f(\mathscr{A}\cup\{1,\dots,p\}) - f(\mathscr{A}))\;;\end{aligned}$$

$$(\mathfrak{L}_{r,N} f)(\mathscr{A}) = f(\mathscr{A}\setminus\{N-1\}) - f(\mathscr{A})\;.$$

In these formulae, $\lambda(\xi) = \lambda(0,\xi) + \lambda(1,\xi)$, and

$$\sigma^{i,i+1}\mathscr{A} = \begin{cases} \mathscr{A}\cup\{i+1\}\setminus\{i\} & \text{if } i\in\mathscr{A}, i+1\notin\mathscr{A} \\ \mathscr{A}\cup\{i\}\setminus\{i+1\} & \text{if } i\notin\mathscr{A}, i+1\in\mathscr{A} \\ \mathscr{A} & \text{otherwise}\;. \end{cases} \tag{2.4.1}$$

Denote by $\mathscr{A}(s)$ the Ξ_N-valued process whose generator is $\mathfrak{L}_N$ and which starts from $\{1\}$. If $\mathscr{A}(s)$ hits the empty set at some time $T>0$ due to the annihilations, this means that we can reconstruct the value of site 1 at time 0 only from the Poisson point processes in the time interval $[-T,0]$, and with no information on the configuration at time $-T$, $\eta(-T)$.

On the other hand, it should be verisimilar that if the number of daggers labeled $\pm$ is much larger that the number of daggers labeled (a,ξ), that is, if the rates $\lambda(a,\xi)$ are much smaller than $A+B$, the process $\mathscr{A}(s)$ should attain the empty set. The next lemma shows that this is indeed the case.

Let

$$T \;=\; \inf\{s>0 : \mathscr{A}(s) = \varnothing\}\;.$$

It is clear that for any $s>0$, the value of $\eta_1(0)$ can be recovered from the configuration $\eta(-s)$ and from the Poisson marks in the interval $[-s,0]$. The next lemma asserts that $\eta_1(0)$ can be obtained only from the Poisson marks in the interval $[-T,0]$.

Lemma 2.4.1. *Assume that $T<\infty$. The value of $\eta_1(0)$ can be recovered from the marks in the time interval $[-T,0]$ of the $N+2(P+1)$ Poisson point processes $\mathfrak{N}$ introduced in the beginning of this section.*

Proof. Let $\Xi'_N = \{0,1,u\}^{\Lambda_N}$, where u stands for unknown. Denote by ζ the configurations of Ξ'_N. We first construct, from the marks of the Poisson point processes $\mathfrak{N}(t)$ on $[-T,0]$, a Ξ'_N-valued evolution $\zeta(s)$ on the time interval $[(-T)-,0]$ in which the set $B(s) = \{k\in$

$\Lambda_N : \zeta_k(s) \neq u\}$ represents the sites whose occupation variables can be determined by the Poisson point processes only.

Let $\zeta_k([-T]-) = u$ for all $k \in \Lambda_N$. By definition of the evolution of $\mathscr{A}(s)$, T corresponds to a mark of one of the Poisson point processes $\mathfrak{N}^{\pm,l}$, $\mathfrak{N}^{\pm,r}$. We define $\zeta(-T)$ as follows. If it is a mark from $\mathfrak{N}^{\pm,l}$ we set $\zeta_1(-T) = [1 \pm 1]/2$ and $\zeta_k(-T) = u$ for $k \neq 1$. Analogously, if it is a mark from $\mathfrak{N}^{\pm,r}$ we set $\zeta_{N-1}(-T) = [1 \pm 1]/2$ and $\zeta_k(-T) = u$ for $k \neq N-1$.

Denote by $-T = \tau_0 < \tau_1 < \cdots < \tau_M < 0 < \tau_{M+1}$ the successive times at which a dagger of type $\pm$ occurs at site 1 or $N-1$. If τ_j corresponds to a mark from $\mathfrak{N}^{\pm,l}$ we set $\zeta_1(\tau_j) = [1 \pm 1]/2$ and we leave the other values unchanged. We proceed analogously if τ_j corresponds to a mark from $\mathfrak{N}^{\pm,r}$. There are (almost surely) a finite number of such times because $T < \infty$ by assumption.

In the intervals (τ_j, τ_{j+1}), holes, particles and unknowns exchange their positions according to the marks of $\mathfrak{N}_{i,i+1}(t)$. Each time σ a dagger of type $\lambda(a,\xi)$ is found, if $(\zeta_1(\sigma-), \ldots, \zeta_p(\sigma-)) = (a,\xi)$, we update the configuration accordingly. Otherwise, we leave the configuration unchanged. This completes the description of the evolution of the process $\zeta(s)$.

We claim that

$$B(s) \supset \mathscr{A}([-s]-) \quad \text{for all} \quad -T \leq s \leq 0\,. \tag{2.4.2}$$

The left limit $(-s)-$ in $\mathscr{A}([-s]-)$ appears because by convention the processes $\zeta(s)$ and $\mathscr{A}(s)$ are both right-continuous and the latter one is run backwards in time.

We prove this claim by recurrence. By construction, $B([-T]-) = \mathscr{A}(T) = \varnothing$ and $B(-T) = \mathscr{A}(T-) = \{1\}$ or $\{N-1\}$, depending on the mark occurring for $\mathscr{A}$ at time T. It is clear that if $B(\tau-) \supset \mathscr{A}(-\tau)$, where $\tau \in [-T, 0)$ is an arrow of type $\mathfrak{N}_{i,i+1}$ or a mark of type $\mathfrak{N}^{\pm,l}$, $\mathfrak{N}^{\pm,r}$, then $B(\tau) \supset \mathscr{A}([-\tau]-)$. Observe that the inclusion may be strict. For example, if $\tau \in [-T, 0)$ is a mark of type $\mathfrak{N}^{+,l}$ and $\mathscr{A}([-\tau]-)$ does not contain 1. This mark permits to determine the value of site 1 at time τ, so that $B(\tau) \ni 1$ but $\mathscr{A}([-\tau]-) \not\ni 1$.

Similarly, suppose that $B(\tau-) \supset \mathscr{A}(-\tau)$ and that $\tau \in (-T, 0)$ is a mark of type $\mathfrak{N}_{(a,\xi)}$. If 1 belongs to $\mathscr{A}([-\tau]-)$, then $\mathscr{A}(-\tau)$ contains $\{1, \ldots, p\}$ and so does $B(\tau-)$ because $B(\tau-) \supset \mathscr{A}(-\tau)$. Hence, all information to update site 1 is available at time $\tau-$ and $1 \in B(\tau) = B(\tau-)$. Since $\mathscr{A}([-\tau]-)$ is contained in $\mathscr{A}(-\tau)$ [it can be strictly contained because some points $m \in \{2, \ldots, p\}$ may not belong to $\mathscr{A}([-\tau]-)$], $B(\tau) \supset \mathscr{A}([-\tau]-)$.

On the other hand, if 1 does not belong to $\mathscr{A}([-\tau]-)$, then $\mathscr{A}([-\tau]-) = \mathscr{A}(-\tau)$, while

$B(\tau) \supset B(\tau-)$. [This relation may be strict because it might happen that $1 \notin B(\tau-)$ and there might be enough information to determine the value of site 1 at time τ.] Thus $B(\tau) \supset B(\tau-) \supset \mathscr{A}(-\tau) = \mathscr{A}([-\tau]-)$. This proves claim (2.4.2).

Since $\mathscr{A}(0) = \mathscr{A}(0-) = \{1\}$, by (2.4.2), $B(0) \ni 1$, which proves the lemma. ∎

Denote by $\mathbb{Q}_N$ the probability measure on $D(\mathbb{R}_+, \Xi_N)$ induced by the process $\mathscr{A}(s)$ starting from $\{1\}$. Expectation with respect to $\mathbb{Q}_N$ is represented by $\mathbb{Q}_N$ as well.

Denote by $C(s)$ the total number of particles created up to time s. The next lemma provides a bound for the total number of particles created up to the absorbing time T.

Lemma 2.4.2. *Let* $\lambda = \sum_{\xi\in\Omega_p}\{\lambda(0,\xi) + \lambda(1,\xi)\}$. *Then,*

$$\mathbb{Q}_N\left[C(T)\right] \;\le\; \frac{(p-1)\lambda}{A+B-(p-1)\lambda}\;.$$

Proof. Let $X(t)$ be a continuous-time random walk on $\mathbb{Z}$ which jumps from k to $k-1$, resp. $k+p-1$, at rate $A+B$, resp. λ. Suppose that $X(0)=1$, and let T_0 be the first time the random walk hits the origin. As $X(t\wedge T_0) + [A+B-(p-1)\,\lambda]\,(t\wedge T_0)$ is an integrable, mean-1 martingale,

$$[A+B-(p-1)\,\lambda]\,E\big[\,t\wedge T_0\,\big] \;=\; 1 \;-\; E\big[\,X(t\wedge T_0)\,\big] \;\le\; 1\;.$$

Letting $t\to\infty$ we conclude that $E[T_0] \le 1/(A+B-(p-1)\lambda)$.

Let $R(s)$ be the total number of jumps to the right of the random walk X up to time s. R is a Poisson process of rate λ so that $R(s)-\lambda\,s$ is a martingale. Hence, $E[R(s\wedge T_0)] = \lambda\,E[s\wedge T_0]$. Letting $s\to\infty$, we obtain that

$$E[R(T_0)] \;=\; \lambda\,E[T_0] \;\le\; \frac{\lambda}{A+B-(p-1)\lambda}\;.$$

Consider the process $\mathscr{A}(s)$ associated to the generator $\mathfrak{L}_N$. Denote the cardinality of a set $B\in\Xi_N$ by $|B|$. $|\mathscr{A}(s)|$ only changes when the set $\mathscr{A}(s)$ contains 1 or $N-1$. The Poisson daggers at $N-1$ may only decrease the cardinality of the set. When $\mathscr{A}(s)$ contains 1, Poisson daggers of type $\pm$ appear at site 1 at rate $A+B$ and they decrease the cardinality of $\mathscr{A}(s)$ by 1. Analogously, the other daggers appear at site 1 at rate λ and increase the cardinality by at most $p-1$. This shows that we may couple $|\mathscr{A}(s)|$ with the random walk $X(s)$ in such a way that $|\mathscr{A}(s)|\le X(s)$ and that $C(s)\le(p-1)R(s)$ for all $0\le s\le T_0$. The assertion of the lemma follows from the bound obtained in the first part of the proof. ∎

As the total number of particles created in the process $\mathscr{A}(s)$ has finite expectation, and since these particles are killed at rate $A+B$ when they reach site 1, the life-span T_0 of $\mathscr{A}(s)$ can not be large and the set of sites ever visited by a particle in $\mathscr{A}(s)$ can not be large. This is the content of the next two lemmata.

Lemma 2.4.3. *For any sequence $\ell_N \to \infty$,*

$$\lim_{N\to\infty} \mathbb{Q}_N\big[T > N\,\ell_N \big] \;=\; 0\;.$$

Proof. Fix a sequence $\ell_N \to \infty$, let $m_N = \sqrt{\ell_N}$, and write

$$\mathbb{Q}_N\big[T > N\,\ell_N \big] \;\le\; \mathbb{Q}_N\big[T > N\,\ell_N \,,\, C(T) \le m_N \big] \;+\; \mathbb{Q}_N\big[C(T) > m_N \big]\;.$$

By the Markov inequality and Lemma 2.4.2, the second term at the right-hand side vanishes as $N \to \infty$.

Denote by T_1 the lifespan of the particle initially at 1, and by T_k, $2 \le k \le C(T)$, the lifespan of the k-th particle created in the process $\mathscr{A}(s)$. By lifespan, we mean the difference $\tau_k - \sigma_k$, where σ_k, resp. τ_k, represents the time the k-th particle has been created, resp. annihilated. Clearly,

$$T \;\le\; \sum_{k=1}^{C(T)} T_k\;.$$

Set $T_k = 0$ for $k > C(T)$. The first term on the right-hand side of the penultimate formula is bounded above by

$$\mathbb{Q}_N\Big[\sum_{k=1}^{m_N} T_k > N\,\ell_N \Big] \;\le\; \frac{m_N}{N\,\ell_N} \sup_{k\ge 1} \mathbb{Q}_N[\,T_k\,]\;.$$

It remains to show that there exists a finite constant C_0 such that for all $k \ge 1$,

$$\mathbb{Q}_N[\,T_k\,] \;\le\; C_0\,N\;. \tag{2.4.3}$$

Particles are created at one of the first p sites. After being created, they perform a symmetric random walk at rate 1 on Λ_N. Each time a particle hits site 1, resp. $N-1$, it is destroyed at rate $A+B$, resp. 1. We overestimate the lifespan by ignoring the annihilation at the right boundary.

Consider a particle performing a rate 1 random walk on Λ_N with reflection at the boundary $N-1$ and annihilated at rate $A+B$ at site 1. Denote by $\boldsymbol{P}_k$ the distribution of this random walk started from site k, and by $\boldsymbol{E}_k$ the corresponding expectation. Let T_Y be the time this particle is killed at site 1, and Y_t, $t \le T$ its position at time t. By the strong Markov property, $\boldsymbol{E}_k[T_Y]$ increases with k. Hence,

$$\mathbb{Q}_N[\,T_k\,] \;\le\; \boldsymbol{E}_p[\,T_Y\,]\,.$$

Divide the lifespan T_Y in excursions away from 1. To keep notation simple, assume that the random walk Y keeps evolving after being killed. Denote by $\{t_j : j \ge 1\}$ the successive hitting times of site 1: $t_0 = 0$, and for $i \ge 1$,

$$t_i \;=\; \inf\big\{t > t_{i-1} : Y(t) = 1 \text{ and } Y(t-) \neq 1\big\}\,.$$

Denote by u_i, $i \ge 1$, the time the random walk $Y(t)$ leaves site 1 after t_i:

$$u_i \;=\; \inf\big\{t > t_i : Y(t) \neq 1\big\}\,,$$

and set $u_0 = 0$. Let $\sigma_i = u_i - t_i$, resp. $s_i = t_i - u_{i-1}$, be duration of the i-th sojourn at 1, resp. the duration of the i-th excursion away from 1.

Denote by A_k the event "the particle is annihilated during its k-th sojourn at site 1". With this notation we have that

$$T_Y \;\le\; (s_1 + \sigma_1) \;+\; \sum_{i\ge 2} \mathbf{1}\{A_1^c \cap \cdots \cap A_{i-1}^c\}\,(s_i \;+\; \sigma_i)\,.$$

By the strong Markov property at time u_{i-1},

$$\boldsymbol{E}_p\Big[\,\mathbf{1}\{A_1^c \cap \cdots \cap A_{i-1}^c\}\,(s_i \;+\; \sigma_i)\Big] \;=\; \boldsymbol{P}_p\Big[\,A_1^c \cap \cdots \cap A_{i-1}^c\Big]\,\boldsymbol{E}_2[\,s_1 + \sigma_1\,]\,.$$

Since the particle is annihilated at rate $A + B$ and leaves site 1 at rate 1, each time it hits site 1 it is killed during its sojourn at 1 with probability $(A+B)/(A+B+1)$. Thus, by the strong Markov property, the probability on the right hand side of the previous displayed equation is equal to α^{i-1}, where $\alpha = 1/(A+B+1)$, so that

$$\boldsymbol{E}_p[\,T_Y\,] \;\le\; \boldsymbol{E}_p[\,s_1 + \sigma_1\,] \;+\; \frac{1}{A+B}\,\boldsymbol{E}_2[\,s_1 + \sigma_1\,]\,.$$

On the one hand, for any $k \in \Lambda_N$, $\boldsymbol{E}_k[\,\sigma_1\,] = 1$, On the other hand, $\boldsymbol{E}_2[\,s_1\,] \le \boldsymbol{E}_p[\,s_1\,]$. Since the random walk is reflected at $N-1$, by solving the elliptic difference equation satisfied by $f(k) = \boldsymbol{E}_k[\,s_1\,]$, we obtain that $\boldsymbol{E}_p[\,s_1\,] \le C_0 N$ for some finite constant C_0 independent of N. This completes the proof (2.4.3) and the one of the lemma. ∎

The proof of the previous lemma shows that each new particle performs only a finite number of excursions, where by excursion we mean the trajectory between the time the particle leaves site 1 and the time it returns to 1. In each excursion the particle visits only a finite number of sites. This arguments yields that during its lifespan the process $\mathscr{A}(s)$ does not visit many sites. This is the content of the next result.

Lemma 2.4.4. *For any sequence ℓ_N such that $\ell_N \to \infty$, $\ell_N \le N-1$,*

$$\lim_{N\to\infty} \mathbb{Q}_N\big[\mathscr{A}(s) \ni \ell_N \text{ for some } s\ge 0\big] \;=\; 0\,.$$

Proof. Fix a sequence ℓ_N satisfying the assumptions of the lemma. Denote by $X_k(s)$ the position at time s of the k-th particle created. Before its creation and after its annihilation we set the position of the particle to be 0. The probability appearing in the statement of the lemma can be rewritten as

$$\mathbb{Q}_N\Big[\bigcup_{l=1}^{C(T)} \big\{X_l(s)=\ell_N \text{ for some } s\ge 0\big\}\Big]\,.$$

Let $m_N = \sqrt{\ell_N}$. The previous expression is bounded by

$$\mathbb{Q}_N\Big[\bigcup_{l=1}^{C(T)} \big\{X_l(s)=\ell_N \text{ for some } s\ge 0\big\}\,,\, C(T)\le m_N\Big] \;+\; \frac{1}{m_N}\,\mathbb{Q}_N[C(T)]\,.$$

By Lemma 2.4.2, the second term vanishes as $N\to\infty$. Set $X_l(s)=0$ for any $l>C(T)$, $s\ge 0$. With this notation, we can replace $C(T)$ by m_N in the union, to bound the first term in the previous equation by

$$\sum_{l=1}^{m_N} \mathbb{Q}_N\big[\,X_l(s)=\ell_N \text{ for some } s\ge 0\big]\,.$$

It remains to show that there exists a finite constant C_0 such that for all $l\ge 1$,

$$\mathbb{Q}_N\big[\,X_l(s)=\ell_N \text{ for some } s\ge 0\big] \;\le\; \frac{C_0}{\ell_N}\,\cdot \tag{2.4.4}$$

To derive (2.4.4), recall the notation introduced in the proof of the previous lemma. Clearly, for any $l\ge 1$,

$$\mathbb{Q}_N\big[\,X_l(s)=\ell_N \text{ for some } s\ge 0\big] \;\le\; \boldsymbol{P}_p\big[\,Y(s)=\ell_N \text{ for some } s\le T_Y\,\big]\,.$$

Note that this is not an identity because the l-th particle may have been created at a site

$k < p$.

Denote by U_k the event that the particle Y visits the site ℓ_N in the time interval $[u_{k-1}, t_k]$. Hence,

$$\big\{Y^j(s) = \ell_N \text{ for some } s \ge 0\big\} \;\subset\; U_1 \cup \bigcup_{i\ge 2} \Big(A_1^c \cap \cdots \cap A_{i-1}^c \cap U_i \Big) \;.$$

By the strong Markov property applied at time u_{i-1},

$$\boldsymbol{P}_p\big[\, Y^j(s) = \ell_N \text{ for some } s \ge 0 \,\big] \;\le\; \boldsymbol{P}_p\big[\, U_1 \,\big] \;+\; \sum_{i\ge 2} \boldsymbol{P}_p\big[\, A_1^c \cap \cdots \cap A_{i-1}^c \,\big]\, \boldsymbol{P}_2\big[\, U_1 \,\big] \;.$$

If $Y(0) = k$, the event U_1 corresponds to the event that a symmetric random walk starting from k hits ℓ_N before it attains 1, so that $\boldsymbol{P}_k[U_1] = [k-1]/[\ell_N - 1]$. Since the particle is annihilated with probability $(A+B)/(1+A+B)$ in each of its sojourn at site 1, by the strong Markov property, the previous sum is equal to

$$\frac{p-1}{\ell_N - 1} \;+\; \frac{1}{A+B}\, \frac{1}{\ell_N - 1} \;.$$

This proves assertion (2.4.4). ∎

We have now all elements to show that the sequence $\rho_N(1)$ converges.

Proposition 2.4.5. *Suppose that conditions* (2.1.13) *are in force. The limit*

$$\alpha := \lim_{N\to\infty} \rho_N(1)$$

exists, and it does not depend on the boundary conditions at $N-1$.

Proof. The proof of this proposition is based on coupling a system evolving on Λ_N with a system evolving on Λ_M, $1 < N < M$ by using the same Poisson point processes to construct both evolutions.

Let $\{\mathfrak{N}^{\pm,r,b}(t) : t \in \mathbb{R}\}$, $b = 1, 2$, be independent Poisson point processes, where $\mathfrak{N}^{+,r,b}$ has rate β and $\mathfrak{N}^{-,r,b}$ rate $1-\beta$. Use the Poisson point processes $\mathfrak{N}_{i,i+1}(t)$, $1 \le i < N-1$, $\mathfrak{N}^{\pm,l}(t)$, $\mathfrak{N}_{(a,\xi)}(t)$, $\mathfrak{N}^{\pm,r,1}(t)$, $t \in \mathbb{R}$, to construct trajectories of a Markov chain $\eta^N(t)$ whose generator is L_N introduced in (2.1.11). Similarly, use the Poisson point processes $\mathfrak{N}_{i,i+1}(t)$, $1 \le i < M-1$, $\mathfrak{N}^{\pm,l}(t)$, $\mathfrak{N}_{(a,\xi)}(t)$, $\mathfrak{N}^{\pm,r,2}(t)$ to construct trajectories of a Markov chain $\eta^M(t)$ whose generator is L_M. Note that on the left boundary and on Λ_N the same Poisson processes are used to construct both chains.

Denote by $\mathscr{A}_N(t)$, $\mathscr{A}_M(t)$, $t \geq 0$, the dual processes evolving according to the Poisson marks described at the beginning of subsection 2.4.2 with initial condition $\mathscr{A}_N(0) = \mathscr{A}_M(0) = \{1\}$. By construction, $\mathscr{A}_N(t) = \mathscr{A}_M(t)$ for all $t \geq 0$ if $N-1 \notin \mathscr{A}_N(t)$ for all $t \geq 0$. Hence, since the value of $\eta^N(0)$ can be recovered from the trajectory $\{\mathscr{A}_N(t) : t \geq 0\}$,

$$\{\eta^N(0) \neq \eta^M(0)\} \subset \{\mathscr{A}_N(t) \ni N-1 \text{ for some } t \geq 0\} . \tag{2.4.5}$$

Denote by $\widehat{\mathbb{P}}_{N,M}$ the probability measure associated to the Poisson processes $\mathfrak{N}_{i,i+1}(t)$, $1 \leq i < M-1$, $\mathfrak{N}^{\pm,l}(t)$, $\mathfrak{N}_{(a,\xi)}(t)$, $\mathfrak{N}^{\pm,r,a}(t)$. Expectation with respect to $\widehat{\mathbb{P}}_{N,M}$ is represented by $\widehat{\mathbb{E}}_{N,M}$. With this notation, $\rho_N(1) = E_{\mu_N}[\eta_1] = \widehat{\mathbb{E}}_{N,M}[\eta_1^N(0)]$. Hence,

$$\big|\, \rho_N(1) - \rho_M(1) \,\big| \;\leq\; \widehat{\mathbb{E}}_{N,M}\big[\, \big|\, \eta_1^N(0) - \eta_1^M(0) \,\big|\, \big] .$$

By (2.4.5), this expression is less than or equal to

$$\widehat{\mathbb{P}}_{N,M}\big[\, \mathscr{A}_N(t) \ni N-1 \text{ for some } t \geq 0 \big] \;=\; \mathbb{Q}_N\big[\, \mathscr{A}(t) \ni N-1 \text{ for some } t \geq 0 \big] .$$

By Lemma 2.4.4 the right-hand side vanishes as $N \to \infty$. This shows that the sequence $\rho_N(1)$ is Cauchy and therefore converges.

Since the argument relies on the fact that the dual process $\mathscr{A}_N(t)$ reaches $N-1$ with a vanishing probability, the same proof works if the process $\eta^M(t)$ is defined with any other dynamics at the right boundary, e.g., reflecting boundary condition. ∎

In the next result we derive an explicit expression for the density $\rho_N(k)$ in terms of β and $\rho_N(1)$.

Lemma 2.4.6. *For all $k \in \Lambda_N$,*

$$\rho_N(k) \;=\; \frac{N-k}{N-1}\, \rho_N(1) \;+\; \frac{k-1}{N-1}\, \beta .$$

Proof. Recall that we denote by Δ_N the discrete Laplacian: $(\Delta_N f)(k) = f(k-1) + f(k+1) - 2f(k)$. Since μ_N is the stationary state, $E_{\mu_N}[L_N f] = 0$ for all function $f : \Omega_N \to \mathbb{R}$. Replacing f by η_k, $2 \leq k \leq N-1$, we obtain that

$$(\Delta_N \rho_N)(k) \;=\; 0 \quad \text{for} \quad 2 \leq k \leq N-1 ,$$

provided we define $\rho_N(N)$ as β. The assertion of the lemma follows from these equations. ∎

Fix $k \in \Lambda_N \setminus \{1\}$, and place a second particle at site k at time 0. This particle moves according to the stirring dynamics in Λ_N until it reaches site 1, when it is annihilated. This later specification is not very important in the argument below, any other convention for the evolution of the particle after the time it hits 1 is fine. Denote by $Z^k(s)$ the position of the extra particle at time s and by $d(A, j)$, $A \subset \Lambda_N$, $j \in \Lambda_N$, the distance between $\{j\}$ and A. The next lemma asserts that the process $\mathscr{A}(s)$ is extincted before the random walk $Z^k(s)$ gets near to $\mathscr{A}(s)$ if $k \geq \sqrt{N}$.

Lemma 2.4.7. *Let ℓ_N be a sequence such that $\ell_N \to \infty$, $\ell_N \sqrt{N} \leq N - 1$. Then,*

$$\lim_{N\to\infty} \max_{\ell_N \sqrt{N} \leq k < N} \mathbb{Q}_N \big[d(\mathscr{A}(s), Z^k(s)) = 1 \textit{ for some } s \geq 0 \big] \;=\; 0 \,.$$

Proof. Recall that we denote by T the extinction time of the process $\mathscr{A}(s)$. The probability appearing in the lemma is bounded above by

$$\mathbb{Q}_N \big[\mathscr{A}(s) \ni \ell_N \sqrt{N}/3 \text{ for some } s \geq 0 \big] \;+\; \mathbb{Q}_N \big[\sup_{s \leq T} |Z^k(s) - Z^k(0)| \geq \ell_N \sqrt{N}/3 \big] \,.$$

By Lemma 2.4.4, the first term vanishes as $N \to \infty$. Let m_N be a sequence such that $m_N \to \infty$, $m_N/\ell_N^2 \to 0$. By Lemma 2.4.3, the second term is bounded by

$$\mathbb{Q}_N \big[\sup_{s \leq N m_N} |Z^k(s) - Z^k(0)| \geq \ell_N \sqrt{N}/3 \big] \;+\; o_N(1) \,,$$

where $o_N(1) \to 0$ as $N \to \infty$. Since Z^k evolves as a symmetric, nearest-neighbor random walk and $m_N/\ell_N^2 \to 0$, the first term vanishes as $N \to \infty$. ∎

To prove a law of large numbers for the empirical measure under the stationary state, we examine the correlations under the stationary state. For j, $k \in \Lambda_N$, $j < k$, let

$$\rho_N(k) \;=\; E_{\mu_N}[\,\eta_k\,] \,, \quad \varphi_N(j,k) \;=\; E_{\mu_N}[\,\eta_j\, \eta_k\,] \;-\; \rho_N(j)\, \rho_N(k) \,. \tag{2.4.6}$$

Lemma 2.4.8. *Let ℓ_N be a sequence such that $\ell_N \to \infty$, $\ell_N \sqrt{N} \leq N - 1$. Then,*

$$\lim_{N\to\infty} \max_{\ell_N \sqrt{N} \leq k < N} \big| \varphi_N(1,k) \big| \;=\; 0 \,.$$

Proof. The probability $\rho_N(k) = \mu_N(\eta_k = 1)$, $k \in \Lambda_N$, can be computed by running the process $\mathscr{A}(s)$ starting from $\mathscr{A}(0) = \{k\}$ until it is extincted, exactly as we estimated $\rho_N(1)$. Similarly, to compute $E_{\mu_N}[\eta_1\, \eta_k]$, we run a process $\mathscr{A}(s)$ starting from $\mathscr{A}(0) = \{1, k\}$. In this case, denote by $\mathscr{A}_1(s)$, $\mathscr{A}_2(s)$ the sets at time s formed by all descendants of 1, k, respectively. Note that $\mathscr{A}_1(s)$ and $\mathscr{A}_2(s)$ may have a non-empty intersection. For instance, if a particle in $\mathscr{A}_1(s)$ branches and a site $k \le p$ is occupied by a particle in $\mathscr{A}_2(s)$.

To compare $E_{\mu_N}[\eta_1\, \eta_k]$ with $E_{\mu_N}[\eta_1]\, E_{\mu_N}[\eta_k]$, we couple a process $\mathscr{A}(s)$ starting from $\{1, k\}$ with two independent processes $\hat{\mathscr{A}_1}(s)$, $\hat{\mathscr{A}_2}(s)$, starting from $\{1\}$, $\{k\}$, respectively. We say that the coupling is successful if $\mathscr{A}_i(s) = \hat{\mathscr{A}_i}(s)$, $i = 1$, 2, for all $s \ge 0$. In this case, the value of the occupation variables η_1, η_k coincide for both processes.

Until $d(\mathscr{A}_1(s), \mathscr{A}_2(s)) = 1$, it is possible to couple $\mathscr{A}(s)$ and $\hat{\mathscr{A}}(s)$ in such a way that $\mathscr{A}_i(s) = \hat{\mathscr{A}_i}(s)$, $i = 1$, 2. Hence, by Lemma 2.4.7, since $k \ge \ell_N \sqrt{N}$, the coupling is successful with a probability which converges to 1 as $N \to \infty$. ∎

Lemma 2.4.9. *For every $\delta > 0$,*

$$\lim_{N\to\infty} \max_{\delta N \le j < k < N} \big| \varphi_N(j,k) \big| \;=\; 0\;.$$

The proof of this lemma is similar to the one Lemmata 2.3.3, 2.3.4. As the arguments are exactly the same, we just present the main steps. Denote by $\widehat{\mathbb{D}}_N$ the discrete simplex defined by

$$\widehat{\mathbb{D}}_N \;=\; \{(j,k) : 2 \le j < k \le N-1\}\;,$$

and by $\partial\, \widehat{\mathbb{D}}_N$ its boundary: $\partial\, \widehat{\mathbb{D}}_N = \{(1,k) : 3 \le k \le N-1\} \cup \{(j,N) : 2 \le j \le N-2\}$. Note that the points $(1,k)$ belong to the boundary and not to the set.

Denote by $\mathscr{L}_N$ the generator of the symmetric, nearest-neighbor random walk on $\widehat{\mathbb{D}}_N$ with absorption at the boundary: For $(j,k) \in \widehat{\mathbb{D}}_N$,

$$\begin{aligned}
(\mathscr{L}_N \phi)(j,k) \;&=\; (\boldsymbol{\Delta}\phi)(j,k)\;, \quad \text{for} \quad k-j>1\;, \\
(\mathscr{L}_N \phi)(k,k+1) \;&=\; (\boldsymbol{\nabla}_1^- \phi)(k,k+1) \;+\; (\boldsymbol{\nabla}_2^+ \phi)(k,k+1) \quad \text{for} \quad 1<k<N-2\;.
\end{aligned}$$

In these formulae, $\boldsymbol{\nabla}_i^{\pm}$, resp. $\boldsymbol{\Delta}$, represent the discrete gradients, resp. Laplacians, introduced below equation (2.3.1).

As $E_{\mu_N}[L_N\{\eta_j - \rho_N(j)\}\, \{\eta_k - \rho_N(k)\}] = 0$, straightforward computations yield that the two-point correlation function φ_N introduced in (2.4.6) is the unique solution of

$$\begin{cases} (\mathscr{L}_N \psi_N)(j,k) + F_N(j,k) = 0\;, \;\; (j,k) \in \widehat{\mathbb{D}}_N\;, \\ \psi_N(j,k) \;=\; b_N(j,k)\;, \;\; (j,k) \in \partial\, \widehat{\mathbb{D}}_N\;, \end{cases} \tag{2.4.7}$$

where $F_N : \widehat{\mathbb{D}}_N \to \mathbb{R}$ and $b_N : \partial\, \widehat{\mathbb{D}}_N \to \mathbb{R}$ are given by

$$F_N(j,k) \;=\; -\,[\rho_N(j+1) - \rho_N(j)]^2\, \mathbf{1}\{k = j+1\}\;, \quad b_N(j,k) \;=\; \varphi_N(j,k)\, \mathbf{1}\{j=1\}\;.$$

Denote by $\varphi_N^{(1)}$, resp. $\varphi_N^{(2)}$, the solution of (2.4.7) with $b_N = 0$, resp. $F_N = 0$. It is clear that $\varphi_N = \varphi_N^{(1)} + \varphi_N^{(2)}$. Let $X_N(t) = (X_N^1(t), X_N^2(t))$ be the continuous-time Markov chain on $\widehat{\mathbb{D}}_N \cup \partial\, \widehat{\mathbb{D}}_N$ associated to the generator $\mathscr{L}_N$. Let $\boldsymbol{P}_{(j,k)}$ be the distribution of the chain X_N starting from (j,k). Expectation with respect to $\boldsymbol{P}_{(j,k)}$ is represented by $\boldsymbol{E}_{(j,k)}$.

Proof of Lemma 2.4.9. The piece $\varphi_N^{(1)}$ of the covariance has an explicit expression. In view of Lemma 2.4.6, for $1 \le j < k \le N$,

$$\varphi_N^{(1)}(j,k) \;=\; -\,\frac{[\beta - \rho_N(1)]^2}{(N-1)^2}\,\frac{(j-1)\,(N-k)}{N-2} \;\le\; \frac{C_0}{N}$$

for some finite constant C_0, independent of N. The piece $\varphi_N^{(2)}$ requires a more careful analysis.

Let H_N be the hitting time of the boundary $\partial\, \widehat{\mathbb{D}}_N$:

$$H_N \;=\; \inf\big\{ t \ge 0 : X_N(t) \in \partial\, \widehat{\mathbb{D}}_N \big\}\;.$$

We have that

$$\varphi_N^{(2)}(j,k) \;=\; \boldsymbol{E}_{(j,k)}\big[b_N(X_N(H_N)) \big] \;=\; \boldsymbol{E}_{(j,k)}\big[\varphi_N(X_N(H_N))\, \mathbf{1}\{ X_N^1(H_N) = 1\} \big]\;.$$

Let k_N be a sequence such that $k_N \ll N$. By (2.3.9), for all $\delta > 0$,

$$\lim_{N\to\infty} \max_{\delta N \le l < m < N} \boldsymbol{P}_{(l,m)}\big[\, X_N^2(H_N) \le k_N \,\big] \;=\; 0\;.$$

Therefore, setting $k_N = \ell_N \sqrt{N}$, where $1 \ll \ell_N \ll \sqrt{N}$, by Lemma 2.4.8,

$$\lim_{N\to\infty} \max_{\substack{(j,k)\in \widehat{\mathbb{D}}_N \\ j > \delta N}} \big|\, \varphi_N^{(2)}(j,k) \,\big| \;\le\; \lim_{N\to\infty} \max_{\ell_N \sqrt{N} \le k < N} \big| \varphi_N(1,k) \big| \;=\; 0\;.$$

This proves the lemma. ∎

Proof of Theorem 2.1.4. The first assertion of the theorem has been proved in Lemma 2.4.6. The proof of the second one is identical to the proof of Theorem 2.1.1. ∎

2.5 Speeded-up boundary conditions

Recall that we denote by μ, resp. μ_N, the stationary state of the Markov chain on Ω_p^*, resp. $\Omega_{N,p}$. Fix a smooth profile $u : [0,1] \to (0,1)$ such that $u(0) = \rho(0)$, $u(1) = \beta$, and let $\nu_{N,p}$ be the product measure defined by

$$\nu_{N,p}(\xi,\eta) \;=\; \mu(\xi)\, \nu_u^N(\eta)\;, \quad \xi\in\Omega_p^*\,,\, \eta\in\Omega_N\;,$$

where ν_u^N is the product measure on Ω_N with marginals given by $\nu_u^N\{\eta_k = 1\} = u(k/N)$.

Denote by f_N the density of μ_N with respect to $\nu_{N,p}$, and by $F_N : \Omega_p^* \to \mathbb{R}_+$ the density given by

$$F_N(\xi) \;=\; \int_{\Omega_N} f_N(\xi,\eta)\, \nu_u^N(d\eta)\;.$$

Lemma 2.5.1. *There exists a finite constant C_0 such that*

$$\big|\, \rho_N(0) - \rho(0)\, \big| \;\le\; C_0/\sqrt{\ell_N}$$

for all $N \ge 1$.

Proof. Fix a function $g : \Omega_p^* \to \mathbb{R}$. As μ_N is the stationary state, and since $L_N g = \ell_N L_l g + L_{0,1} g$

$$0 \;=\; E_{\mu_N}\big[L_N g\big] \;=\; E_{\mu_N}\big[\ell_N\, L_l g + L_{0,1} g\big]\;,$$

so that $|\, E_{\mu_N}[L_l g]\,| \le 2\|g\|_\infty/\ell_N$. Since

$$E_{\mu_N}[L_l g] \;=\; \int_{\Omega_{N,p}} (L_l g)(\xi)\, f_N(\xi,\eta)\, \nu_{N,p}(d\xi, d\eta) \;=\; \int_{\Omega_p^*} (L_l g)(\xi)\, F_N(\xi)\, \mu(d\xi)\;,$$

for every $g : \Omega_p^* \to \mathbb{R}$,

$$\Big|\int_{\Omega_p^*} g(\xi)\, (L_l^* F_N)(\xi)\, \mu(d\xi)\Big| \;\le\; 2\|g\|_\infty/\ell_N\;,$$

where L_l^* represents the adjoint of L_l in $L^2(\mu)$. Since μ is the stationary state, L_l^* is the generator of a irreducible Markov chain on Ω_p^*. It follows from the previous identity that

$$\int_{\Omega_p^*} \big|\, (L_l^* F_N)(\xi)\, \big|\, \mu(d\xi) \;\le\; C_0/\ell_N$$

for some finite constant C_0. Hence, since $\mu(\xi) > 0$ for all $\xi\in\Omega_p^*$, $\|L_l^* F_N\|_\infty \le C_0/\ell_N$. In particular,

$$-\int_{\Omega_p^*} F_N(\xi)\, (L_l^* F_N)(\xi)\, \mu(d\xi) \;\le\; (C_0/\ell_N) \int_{\Omega_p^*} F_N(\xi)\, \mu(d\xi) \;\le\; C_0/\ell_N\;.$$

Note that the expression on the left hand side is the Dirichlet form. Hence, by its explicit expression, $\max_{\xi,\xi'}[F_N(\xi') - F_N(\xi)]^2 \le C_0/\ell_N$, where the maximum is carried over all configuration pairs ξ, ξ' such that $R(\xi,\xi') + R(\xi',\xi) > 0$, R being the jump rate. In particular, as the chain is irreducible,

$$\| F_N - 1 \|_\infty \;=\; \Big\| F_N - \int_{\Omega_p^*} F_N(\xi)\, \mu(d\xi) \Big\|_\infty \;\le\; C_0/\sqrt{\ell_N}\;.$$

We are now in a position to prove the lemma. One just needs to observe that

$$\big|\, \rho_N(0) - \rho(0) \,\big| \;=\; \Big| E_{\mu_N}[\eta_0] - E_\mu[\eta_0] \Big| \;=\; \Big| \int_{\Omega_p^*} \xi_0 F_N(\xi)\, \mu(d\xi) - \int_{\Omega_p^*} \xi_0\, \mu(d\xi) \Big|\;,$$

and that this expression is bounded by $\| F_N - 1 \|_\infty$. ∎

Let

$$\varphi_N(j,k) \;=\; E_{\mu_N}[\, \eta_j\, \eta_k \,] \;-\; \rho_N(j)\, \rho_N(k)\;, \quad j\,,\, k \,\in\, \Lambda_{N,p}\,,\; j<k\;.$$

Lemma 2.5.2. *There exists a finite constant C_0 such that $|\varphi_N(0,k)| \le C_0/\sqrt{\ell_N}$ for all $2 \le k < N$.*

Proof. The argument is similar to the one of the previous lemma. Fix $0 < k < N$, and denote by $G_N = G_N^{(k)} : \Omega_p^* \to \mathbb{R}_+$ the non-negative function given by

$$G_N(\xi) \;=\; \int_{\Omega_N} \eta_k\, f_N(\xi,\eta)\, \nu_u^N(d\eta)\;.$$

With this notation,

$$E_{\mu_N}[\, \eta_0\, \eta_k \,] \;=\; \int_{\Omega_p^*} \xi_0\, G_N(\xi)\, \mu(d\xi)\;. \tag{2.5.1}$$

Fix $g : \Omega_p^* \to \mathbb{R}$ and $k \ge 2$. As $k \ge 2$, $L_N(g\,\eta_k) = \eta_k\, L_N g + g L_N \eta_k$. Thus, since μ_N is the stationary state,

$$0 \;=\; E_{\mu_N}\big[\, L_N(g\,\eta_k) \big] \;=\; \int_{\Omega_{N,p}} (\ell_N\, L_l + L_{0,1})\, g\, \eta_k\, f_N\, d\nu_{N,p} \;+\; E_{\mu_N}\big[\, g\, L_N \eta_k \,\big]\;.$$

By definition of G_N and since $|L_N \eta_k| \le 2$, $|L_{0,1}\, g| \le 2\|g\|_\infty$,

$$\Big| \int_{\Omega_p^*} (L_l\, g)(\xi)\, G_N(\xi)\, \mu(d\xi) \Big| \;\le\; (4/\ell_N)\, \|g\|_\infty\;.$$

The argument presented in the proof of the previous lemma yields that

$$\Big\| G_N - \int_{\Omega_p^*} G_N(\xi)\, \mu(d\xi) \Big\|_\infty \;\le\; C_0/\sqrt{\ell_N}\;.$$

Therefore,

$$\Big| \int_{\Omega_p^*} \xi_0 \Big\{ G_N(\xi) - \int_{\Omega_p^*} G_N(\xi')\, \mu(d\xi') \Big\}\, \mu(d\xi) \Big| \;\le\; C_0/\sqrt{\ell_N}\ .$$

By definition of G_N and by (2.5.1), the expression inside the absolute value is equal to

$$E_{\mu_N}[\,\eta_0\, \eta_k\,] \;-\; \rho(0)\, \rho_N(k)\ .$$

The assertion of the lemma follows from the penultimate displayed equation and from Lemma 2.5.1. ∎

Proof of Theorem 2.1.9. The first assertion of the theorem is the content of Lemma 2.5.1. The proof of Lemma 2.4.9 [with $\widehat{\mathbb{D}}_N$ defined as $\widehat{\mathbb{D}}_N \;=\; \{(j,k) : 1 \le j < k \le N-1\}$] yields that for every $\delta > 0$,

$$\lim_{N\to\infty} \max_{\delta N \le j < k < N} \big|\, \varphi_N(j,k)\, \big| \;=\; 0\ .$$

A Schwarz inequality, as in the proof of Theorem 2.1.1, completes the argument because $\rho_N(k) = (k/N)\, \beta + [1 - (k/N)]\, \rho_N(0)$, $1 \le k \le N$. ∎

3.1 Notation and main results

Consider a finite set E. The elements of E are called configurations and are denoted by the Greek letters η, ξ, ζ. Consider a sequence of continuous-time, E-valued, irreducible Markov chains $\{\eta^N_t : t \geq 0\}$. Denote the jump rates of η^N_t by $R_N(\eta,\xi)$, and by μ_N the unique invariant probability measure.

Denote by $D(\mathbb{R}_+, E)$ the space of right-continuous functions $x : \mathbb{R}_+ \to E$ with left-limits endowed with the Skorohod topology, and by $\mathbb{P}_\eta = \mathbb{P}^N_\eta$, $\eta \in E$, the probability measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov chain η^N_t starting from η. Expectation with respect to $\mathbb{P}_\eta$ is represented by $\mathbb{E}_\eta$.

Denote by $H_{\mathscr{A}}$, $H^+_{\mathscr{A}}$, $\mathscr{A} \subset E$, the hitting time and the time of the first return to $\mathscr{A}$:

$$H_{\mathscr{A}} \;=\; \inf\big\{t > 0 : \eta^N_t \in \mathscr{A}\big\}\;, \quad H^+_{\mathscr{A}} \;=\; \inf\big\{t > \tau_1 : \eta^N_t \in \mathscr{A}\big\}\;, \tag{3.1.1}$$

where τ_1 represents the time of the first jump of the chain η^N_t: $\tau_1 = \inf\{t > 0 : \eta^N_t \neq \eta^N_0\}$.

Denote by $\lambda_N(\eta)$, $\eta \in E$, the holding rates of the Markov chain η^N_t and by $p_N(\eta,\xi)$, η, $\xi \in E$, the jump probabilities, so that $R_N(\eta,\xi) = \lambda_N(\eta) p_N(\eta,\xi)$. For two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E, denote by $\mathrm{cap}_N(\mathscr{A},\mathscr{B})$ the capacity between $\mathscr{A}$ and $\mathscr{B}$:

$$\mathrm{cap}_N(\mathscr{A},\mathscr{B}) \;=\; \sum_{\eta\in\mathscr{A}} \mu_N(\eta)\,\lambda_N(\eta)\,\mathbb{P}_\eta[H_{\mathscr{B}} < H^+_{\mathscr{A}}]\;. \tag{3.1.2}$$

Consider a partition $\mathscr{E}_1, \dots, \mathscr{E}_{\mathfrak{n}}$, Δ of the set E, which does not depend on the parameter N and such that $\mathfrak{n} \geq 2$. Fix two sequences of positive real numbers α_N, θ_N such that $\alpha_N \ll \theta_N$, where this notation stands for $\lim_{N\to\infty} \alpha_N/\theta_N = 0$.

Now let us briefly recall the trace processs of a continuous-time Markov chain. Given a continuous-time Markov chain $\{\eta_t\}_{t\geq 0}$ with state space E and jump rate $R(\cdot,\cdot)$. Fix a subset $F \subset E$. The trace process of $\{\eta_t\}_{t\geq 0}$ on F can be thought as the stochastic process whose trajectory is obtained from the trajectory of $\{\eta_t\}_{t\geq 0}$, by deleting the part of the on $E \setminus F$ and gluing the rest following the time order. Denote by $\{\eta^F_t\}_{t\geq 0}$ the trace process. It can be proved that the process $\{\eta^F_t\}_{t\geq 0}$ is actually a Markov process, with jump rate $R^F(\cdot,\cdot)$ given by: for any $\eta \neq \xi$, $\eta, \xi \in F$,

$$R^F(\eta,\xi) \;=\; R(\eta,\xi) + \sum_{\zeta\in F^c} R(\eta,\zeta)\mathbb{P}_\zeta[T_F = T_\xi].$$

where T_A is the first hitting time of set A:

$$T_A := \inf\{t \geq 0 : \eta_t \in A\}.$$

For the rigorous definition and more properties of the trace process, we refer to Section 6 of [9].

Let $\mathscr{E} = \cup_{x \in S} \mathscr{E}_x$, where $S = \{1, \dots, \mathfrak{n}\}$. Denote by $\{\eta^{\mathscr{E}}_t : t \geq 0\}$ the trace of $\{\eta^N_t : t \geq 0\}$ on $\mathscr{E}$, and by $R^{\mathscr{E}}_N : \mathscr{E} \times \mathscr{E} \to \mathbb{R}_+$ the jump rates of the trace process $\eta^{\mathscr{E}}_t$. Denote by $r^{\mathscr{E}}_N(\mathscr{E}_x, \mathscr{E}_y)$ the mean rate at which the trace process jumps from $\mathscr{E}_x$ to $\mathscr{E}_y$:

$$r^{\mathscr{E}}_N(\mathscr{E}_x, \mathscr{E}_y) \;=\; \frac{1}{\mu_N(\mathscr{E}_x)} \sum_{\eta \in \mathscr{E}_x} \mu_N(\eta) \sum_{\xi \in \mathscr{E}_y} R^{\mathscr{E}}_N(\eta, \xi)\;. \tag{3.1.3}$$

Assume that for every $x \neq y \in S$,

$$\begin{aligned} r_{\mathscr{E}}(x,y) \;&:=\; \lim_{N\to\infty} \theta_N\, r^{\mathscr{E}}_N(\mathscr{E}_x, \mathscr{E}_y) \;\in\; \mathbb{R}_+\;, \\ \text{and that} \quad & \sum_{x\in S}\sum_{y\neq x} r_{\mathscr{E}}(x,y) > 0\;. \end{aligned} \tag{\textbf{H1}}$$

The symbol := in the first line of the previous displayed equation means that the limit exists, that it is denoted by $r_{\mathscr{E}}(x, y)$, and that it belongs to $\mathbb{R}_+$. This convention is used throughout the article.

Assume that for every $x \in S$ for which $\mathscr{E}_x$ is not a singleton and for all $\eta \neq \xi \in \mathscr{E}_x$,

$$\liminf_{N\to\infty} \alpha_N\, \frac{\mathrm{cap}_N(\eta, \xi)}{\mu_N(\mathscr{E}_x)} \;>\; 0\;. \tag{\textbf{H2}}$$

Note that the liminf in the above equation is not necessary to be finite.

Finally, assume that in the time scale θ_N the chain remains a negligible amount of time outside the set $\mathscr{E}$: For every $t > 0$,

$$\lim_{N\to\infty} \max_{\eta\in E} \mathbb{E}_\eta\Big[\int_0^t \mathbf{1}\{\eta^N_{s\theta_N} \in \Delta\}\, ds\Big] \;=\; 0\;. \tag{\textbf{H3}}$$

Denote by $\Psi_N : E \to \{1, \dots, \mathfrak{n}, N\}$ the projection defined by $\Psi_N(\eta) = x$ if $\eta \in \mathscr{E}_x$, $\Psi_N(\eta) = N$, otherwise:

$$\Psi_N(\eta) \;=\; \sum_{x\in S} x\, \mathbf{1}\{\eta \in \mathscr{E}_x\} \;+\; N\, \mathbf{1}\{\eta \in \Delta\}\;.$$

Recall from [16] the definition of the soft topology.

Theorem 3.1.1. *Assume that conditions* (H1)–(H3) *are in force. Fix $x \in S$ and a configuration $\eta \in \mathscr{E}_x$. Starting from η, the speeded-up, hidden Markov chain $\boldsymbol{X}_N(t) = \Psi_N\big(\eta^N(\theta_N t)\big)$ converges in the soft topology to the continuous-time Markov chain $X_{\mathscr{E}}(t)$ on $\{1,\dots,\mathfrak{n}\}$ whose jump rates are given by $r_{\mathscr{E}}(x,y)$ and which starts from x.*

This theorem is a straightforward consequence of known results of [9],[11],[16]. We state it here in sake of completeness and because all the analysis of the metastable behavior of η^N_t relies on it.

Remark 3.1.2. *Theorem 3.1.1 states that in the time scale θ_N, if we just keep track of the set $\mathscr{E}_x$ where η^N_t is and not of the specific location of the chain, we observe an evolution on the set S close to the one of a continuous-time Markov chain which jumps from x to y at rate $r_{\mathscr{E}}(x,y)$.*

Remark 3.1.3. *The function Ψ_N represents a slow variable of the chain. Indeed, we will see below that the sequence α_N^{-1} stands for the order of magnitude of the jump rates of the chain. Theorem 3.1.1 states that on the time scale θ_N, which is much longer than α_N, the variable $\Psi_N(\eta^N_t)$ evolves as a Markov chain. In other words, under conditions (H1)–(H3), one still observes a Markovian dynamics after a contraction of the configuration space through the projection Ψ_N. Theorem 3.1.1 provides therefore a mechanism of reducing the degrees of freedom of the system, keeping the essential features of the dynamics, as the ergodic properties.*

Remark 3.1.4. *It also follows from assumptions (H1)–(H3) that the exit time from a set $\mathscr{E}_x$ is asymptotically exponential. More precisely, let $\breve{\mathscr{E}}_x$, $x \in S$, be the union of all set $\mathscr{E}_y$ except $\mathscr{E}_x$:*

$$\breve{\mathscr{E}}_x \;=\; \bigcup_{y\neq x} \mathscr{E}_y\;. \tag{3.1.4}$$

For every $x \in S$ and $\eta \in \mathscr{E}_x$, under $\mathbb{P}_\eta$ the distribution of $H_{\breve{\mathscr{E}}_x}/\theta_N$ converges to an exponential distribution.

Remark 3.1.5. *Under the assumptions (H1)–(H3), the sets $\mathscr{E}_x$ are cycles in the sense of [22]. More precisely, for every $x \in S$ for which $\mathscr{E}_x$ is a not a singleton, and for all $\eta \neq \xi \in \mathscr{E}_x$,*

$$\lim_{N\to\infty} \mathbb{P}_\eta\big[H_\xi < H_{\breve{\mathscr{E}}_x}\big] \;=\; 1\;.$$

This means that starting from $\eta \in \mathscr{E}_x$, the chain visits all configurations in $\mathscr{E}_x$ before hitting the set $\breve{\mathscr{E}}_x$.

3.1.1 The main assumption

We present in this subsection the main and unique hypothesis made on the sequence of Markov chains η_t^N. Fix two configurations $\eta \neq \xi \in E$. We assume that the jump rate from η to ξ is either constant equal to 0 or is always strictly positive:

$$R_N(\eta,\xi) \;=\; 0 \;\text{ for all } N \ge 1 \;\text{ or }\; R_N(\eta,\xi) \;>\; 0 \;\text{ for all } N \ge 1\;.$$

This assumption permits to define the set of ordered bonds of E, denoted by $\mathbb{B}$, as the set of ordered pairs (η,ξ) such that $R_N(\eta,\xi) > 0$:

$$\mathbb{B} \;=\; \big\{(\eta,\xi) \in E \times E : \eta \neq \xi\,,\, R_N(\eta,\xi) > 0\big\}\;.$$

Note that the set $\mathbb{B}$ does not depend on N.

Our analysis of the metastable behavior of the sequence of Markov chain η_t^N relies on the assumption that the set of ordered bonds can be divided into equivalent classes in such a way that the all jump rates in the same equivalent class are of the same order, while the ratio between two jump rates in different classes either vanish in the limit or tend to $+\infty$. Some terminology is necessary to make this notion precise.

Ordered sequences: Consider a set of sequences $(a_N^r : N \ge 1)$ of nonnegative real numbers indexed by some finite set $r \in \mathfrak{R}$. The set of sequences $(a_N^r : N \ge 1)$ is said to be *ordered* if for all $r \neq s \in \mathfrak{R}$ the sequence a_N^r/a_N^s converges to either some finite constant $C \ge 0$ or ∞ as $N \uparrow \infty$.

In the examples below the set $\mathfrak{R}$ will be the set of configurations E or the set of bonds $\mathbb{B}$. Let $\mathbb{Z}_+ = \{0,1,2,\dots\}$, and let $\mathfrak{A}_m$, $m \ge 1$, be the set of functions $k : \mathbb{B} \to \mathbb{Z}_+$ such that $\sum_{(\eta,\xi)\in\mathbb{B}} k(\eta,\xi) = m$.

Assumption 3.1.6. *We assume that for every $m \ge 1$ the set of sequences*

$$\Big\{ \prod_{(\eta,\xi)\in\mathbb{B}} R_N(\eta,\xi)^{k(\eta,\xi)} : N \ge 1\Big\}\,, \quad k \in \mathfrak{A}_m$$

is ordered.

We assume from now on that the sequence of Markov chains η_t^N fulfills Assumption 3.1.6. In particular, the sequences $\{R_N(\eta,\xi) : N \ge 1\}$, $(\eta,\xi) \in \mathbb{B}$, are ordered. The example we have in mind are zero-temperature limits of non-reversible dynamics in a finite state space.

3.1.2 The shallowest valleys, the fastest slow variable

We identify in this subsection the shortest time-scale at which a metastable behavior is observed, we introduce the shallowest valleys, and we prove that these valleys form a partition which fulfills conditions (H1)–(H3).

We first identify the valleys. Let

$$\frac{1}{\alpha_N} \;=\; \sum_{\eta\in E}\sum_{\xi:\xi\neq\eta} R_N(\eta,\xi)\;.$$

We could also have defined α_N^{-1} as $\max\{R_N(\eta,\xi) : (\eta,\xi)\in\mathbb{B}\}$. By Assumption 3.1.6, there exists a function $R(\cdot,\cdot)$ such that, for every $\eta\neq\xi\in E$, $\alpha_N R_N(\eta,\xi)\to R(\eta,\xi)\in[0,1]$. Let $\lambda(\eta)=\sum_{\xi\neq\eta}R(\eta,\xi)\in\mathbb{R}_+$, and denote by E_0 the subset of points of E such that $\lambda(\eta)>0$. For all $\eta\in E_0$ let $p(\eta,\xi)=R(\eta,\xi)/\lambda(\eta)$. It is clear that for all η, ζ in E, $\xi\in E_0$,

$$\lim_{N\to\infty}\alpha_N\,\lambda_N(\eta)\;=\lambda(\eta)\;,\quad \lim_{N\to\infty}p_N(\xi,\zeta)\;=\;p(\xi,\zeta)\;. \tag{3.1.5}$$

Denote by $X_R(t)$ the E-valued Markov chain whose jump rates are given by $R(\eta,\xi)$. Not that this Markov chain might not be irreducible. However, by definition of α_N, there is at least one bond $(\eta,\xi)\in\mathbb{B}$ such that $R(\eta,\xi)>0$.

Denote by $\mathscr{E}_1,\mathscr{E}_2,\dots,\mathscr{E}_{\mathfrak{n}}$ the recurrent classes of the Markov chain $X_R(t)$, and by Δ the set of transient points, so that $\{\mathscr{E}_1,\dots,\mathscr{E}_{\mathfrak{n}},\Delta\}$ forms a partition of E:

$$E\;=\;\mathscr{E}\sqcup\Delta\;,\quad \mathscr{E}\;=\;\mathscr{E}_1\sqcup\cdots\sqcup\mathscr{E}_{\mathfrak{n}}\;. \tag{3.1.6}$$

Here and below we use the notation $\mathscr{A}\sqcup\mathscr{B}$ to represent the union of two disjoint sets $\mathscr{A}$, $\mathscr{B}$: $\mathscr{A}\sqcup\mathscr{B}=\mathscr{A}\cup\mathscr{B}$, and $\mathscr{A}\cap\mathscr{B}=\varnothing$.

Note that the sets $\mathscr{E}_x$, $x\in S=\{1,\dots,\mathfrak{n}\}$, do not depend on N. If $\mathfrak{n}=1$, the chain does not possess valleys. This is the case, for instance, if the rates $R_N(x,y)$ are independent of N. Assume, therefore, and up to the end of this subsection, that $\mathfrak{n}\geq 2$.

Let θ_N be defined by

$$\frac{1}{\theta_N}\;=\;\sum_{x\in S}\frac{\mathrm{cap}_N(\mathscr{E}_x,\breve{\mathscr{E}}_x)}{\mu_N(\mathscr{E}_x)}\;. \tag{3.1.7}$$

Theorem 3.1.7. *The partition $\mathscr{E}_1,\dots,\mathscr{E}_{\mathfrak{n}},\Delta$ and the time scales α_N, θ_N fulfill the conditions* (H1)–(H3). *Moreover, For every $x\in S$ and every $\eta\in\mathscr{E}_x$, there exists $m_x(\eta)\in(0,1]$ such that*

$$\lim_{N\to\infty}\frac{\mu_N(\eta)}{\mu_N(\mathscr{E}_x)}\;=\;m_x(\eta)\;. \tag{H0}$$

Remark 3.1.8. *The jump rates $r_{\mathscr{E}}(x, y)$ which appear in condition (H1) are introduced in Lemma 3.6.1. It follows from Theorems 3.1.1 and 3.1.7 that in the time-scale θ_N the chain η_t^N evolves among the sets $\mathscr{E}_x$, $x \in S$, as a Markov chain which jumps from x to y at rate $r_{\mathscr{E}}(x, y)$.*

In the next three remarks we present some outcomes of Theorem 3.1.1 and 3.1.7 on the evolution of the chain η_t^N in a time-scale longer than θ_N. These remarks anticipate the recursive procedure of the next subsection.

Remark 3.1.9. *The jump rates $r_{\mathscr{E}}(x, y)$ define a Markov chain on S, represented by $X_{\mathscr{E}}(t)$. Denote by T the set of transient points of this chain and assume that $T \neq \varnothing$. It follows from Theorem 3.1.1 that in the time-scale θ_N, starting from a set $\mathscr{E}_x$, $x \in T$, the chain η_t^N leaves the set $\mathscr{E}_x$ at an asymptotically exponential time, and never returns to $\mathscr{E}_x$ after a finite number of visits to this set. In particular, if we observe the chain η_t^N in a longer time-scale than θ_N, starting from $\mathscr{E}_x$ the chain remains only a negligible amount of time at $\mathscr{E}_x$.*

Remark 3.1.10. *Denote by A the set of absorbing points of $X_{\mathscr{E}}(t)$, and assume that $A \neq \varnothing$. In this case, in the time-scale θ_N, starting from a set $\mathscr{E}_x$, $x \in A$, the chain η_t^N never leaves the set $\mathscr{E}_x$. To observe a non-trivial behavior starting from this set one has to consider longer-time scales.*

Remark 3.1.11. *Finally, denote by $\mathscr{C}_1, \dots, \mathscr{C}_{\mathfrak{p}}$ the equivalent classes of $X_{\mathscr{E}}(t)$. Suppose that there is a class, say $\mathscr{C}_1$, of recurrent points which is not a singleton. In this case, starting from a set $\mathscr{E}_x$, $x \in \mathscr{C}_1$, in the time-scale θ_N, the chain η_t^N leaves the set $\mathscr{E}_x$ at an asymptotically exponential time, and returns to $\mathscr{E}_x$ infinitely many times.*

Suppose now that there are at least two classes, say $\mathscr{C}_1$ and $\mathscr{C}_2$, of recurrent points. This means that in the time-scale θ_N, starting from a set $\mathscr{E}_x$, $x \in \mathscr{C}_1$, the process never visits a set $\mathscr{E}_y$ for $y \in \mathscr{C}_2$. For this to occur one has to observe the chain η_t^N in a longer time-scale.

Denote by $R_1, \dots, R_{\mathfrak{m}}$ the recurrent classes of $X_{\mathscr{E}}(t)$. In the next subsection, we derive a new time-scale at which one observes jumps from sets of the form $\mathscr{F}_a = \cup_{x \in R_a} \mathscr{E}_x$ to sets of the form $\mathscr{F}_b = \cup_{x \in R_b} \mathscr{E}_x$.

3.1.3 All deep valleys and slow variables

We obtained in the previous subsection two time-scales α_N, θ_N, and a partition $\mathscr{E}_1, \dots, \mathscr{E}_{\mathfrak{n}}, \Delta$ of the state space E which satisfy conditions (H0)–(H3). We present in this subsection a recursive procedure. Starting from two time-scales β_N^-, β_N, and a partition $\mathscr{F}_1, \dots, \mathscr{F}_{\mathfrak{p}}, \Delta_{\mathscr{F}}$

of the state space E satisfying the assumptions (H0)–(H3) and such that $\mathfrak{p} \geq 2$, it provides a longer time-scale β_N^+ and a coarser partition $\mathscr{G}_1, \ldots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}$ which fulfills conditions (H0)–(H3) with respect to the sequences β_N, β_N^+.

Consider a partition $\mathscr{F}_1, \ldots, \mathscr{F}_{\mathfrak{p}}$, $\Delta_{\mathscr{F}}$ of the set E and two sequences β_N^-, β_N such that $\beta_N^-/\beta_N \to 0$. Assume that $\mathfrak{p} \geq 2$ and that the partition and the sequences β_N^-, β_N satisfy conditions (H0)–(H3). Denote by $r_{\mathscr{F}}(x,y)$ the jump rates appearing in assumption (H1).

The coarser partition. Let $P = \{1, \ldots, \mathfrak{p}\}$ and let $X_{\mathscr{F}}(t)$ be the P-valued Markov chain whose jumps rates are given by $r_{\mathscr{F}}(x,y)$.

Denote by $G_1, G_2, \ldots, G_{\mathfrak{q}}$ the recurrent classes of the chain $X_{\mathscr{F}}(t)$, and by $G_{\mathfrak{q}+1}$ the set of transient points. The sets $G_1, \ldots, G_{\mathfrak{q}+1}$ form a partition of P. We claim that $\mathfrak{q} < \mathfrak{p}$. Fix $x \in P$ such that $\sum_{y \neq x} r_{\mathscr{F}}(x,y) > 0$, whose existence is guaranteed by hypothesis (H1). Suppose that the point x is transient. In this case the number of recurrent classes must be smaller than $\mathfrak{p}$. If, on the other hand, x is recurrent, the recurrent class which contains x must have at least two elements, and the number of recurrent classes must be smaller than $\mathfrak{p}$.

Let $Q = \{1, \ldots, \mathfrak{q}\}$,

$$\mathscr{G}_a \;=\; \bigcup_{x \in G_a} \mathscr{F}_x \;, \quad \Delta_* \;=\; \bigcup_{x \in G_{\mathfrak{q}+1}} \mathscr{F}_x \;, \quad \Delta_{\mathscr{G}} = \Delta_{\mathscr{F}} \cup \Delta_* \;, \quad a \in Q \;. \tag{3.1.8}$$

Since, by (3.1.6), $\{\mathscr{F}_1, \ldots, \mathscr{F}_{\mathfrak{p}}, \Delta_{\mathscr{F}}\}$ forms a partition of E, $\{\mathscr{G}_1, \ldots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$ also forms a partition of E:

$$E \;=\; \mathscr{G} \sqcup \Delta_{\mathscr{G}} \;, \quad \mathscr{G} \;=\; \mathscr{G}_1 \sqcup \cdots \sqcup \mathscr{G}_{\mathfrak{q}} \;. \tag{3.1.9}$$

The longer time-scale. For $a \in Q = \{1, \ldots, \mathfrak{q}\}$, let $\breve{\mathscr{G}}_a$ be the union of all leaves except $\mathscr{G}_a$:

$$\breve{\mathscr{G}}_a \;=\; \bigcup_{b \neq a} \mathscr{G}_b \;.$$

Assume that $\mathfrak{q} > 1$, and let β_N^+ be given by

$$\frac{1}{\beta_N^+} \;=\; \sum_{a \in Q} \frac{\mathrm{cap}_N(\mathscr{G}_a, \breve{\mathscr{G}}_a)}{\mu_N(\mathscr{G}_a)} \;. \tag{3.1.10}$$

Theorem 3.1.12. *The partition $\mathscr{G}_1, \ldots, \mathscr{G}_q$, $\Delta_{\mathscr{G}}$ and the time scales (β_N, β_N^+) satisfy conditions* (H0)–(H3)*.*

Remark 3.1.13. *It follows from Theorems 3.1.1 and 3.1.12 that the chain η_t^N exhibits a metastable behavior in the time-scale β_N^+ if $\mathfrak{q} > 1$. We refer to Remarks 3.1.2, 3.1.3, 3.1.4 and 3.1.5.*

Remark 3.1.14. *As* $\mathfrak{q} < \mathfrak{p}$ *and as we need* $\mathfrak{p}$ *to be greater than or equal to* 2 *to apply the iterative procedure, this recursive algorithm ends after a finite number of steps.*

If $\mathfrak{q} = 1$, β_N *is the longest time-scale at which a metastable behavior is observed. In this time-scale, the chain* η_t^N *jumps among the sets* $\mathscr{F}_x$ *as does the chain* $X_{\mathscr{F}}(t)$ *until it reaches the set* $\mathscr{G}_1 = \cup_{x \in G_1} \mathscr{F}_x$. *Once in this set, it remains there forever jumping among the sets* $\mathscr{F}_x$, $x \in G_1$, *as the Markov chain* $X_{\mathscr{F}}(t)$, *which restricted to* G_1 *is an irreducible Markov chain.*

The successive valleys: Observe that the valleys $\mathscr{G}_a$ were obtained as the recurrent classes of the Markov chain $X_{\mathscr{F}}(t)$: $\mathscr{G}_a = \cup_{x \in G_a} \mathscr{F}_x$, where G_a is a recurrent class of $X_{\mathscr{F}}(t)$. In particular, at any time-scale the valleys are formed by unions of the valleys obtained in the first step of the recursive argument, which were denoted by $\mathscr{E}_x$ in the previous subsection. Moreover, by (H0), each configuration in $\mathscr{G}_a$ has measure of the same order.

Conclusion: We presented an iterative method which provides a finite sequence of time-scales and of partitions of the set E satisfying conditions (H0)-(H3). At each step, the time scales become longer and the partitions coarser. By Theorem 3.1.1, to each pair of time-scale and partition corresponds a metastable behavior of the chain η_t^N. This recursive algorithm provides all time-scales at which a metastable behavior of the chain η_t^N is observed, and all slow variables which keep a Markovian dynamics.

3.2 What do we learn from Assumption 3.1.6?

We prove in this section that the jump rates of the trace processes satisfy Assumption 3.1.6, and that some sequences, such as the one formed by the measures of the configurations, are ordered.

Assertion 3.2.A. *Let* F *be a proper subset of* E *and denote by* $R_N^F(\eta, \xi)$, $\eta \neq \xi \in F$, *the jump rates of the trace of* η_t^N *on* F. *The jump rates* $R_N^F(\eta, \xi)$ *satisfy Assumption 3.1.6.*

Proof. We prove this assertion by removing one by one the elements of $E \setminus F$. Assume that $F = E \setminus \{\zeta\}$ for some $\zeta \in E$. By Corollary 6.2 in [9] and by the equation following the proof of this corollary, for $\eta \neq \xi \in F$, $R_N^F(\eta, \xi) = R_N(\eta, \xi) + R_N(\eta, \zeta) p_N(\zeta, \xi)$. Hence,

$$R_N^F(\eta, \xi) \;=\; \frac{\sum_{w \in E} R_N(\eta, \xi) R_N(\zeta, w) + R_N(\eta, \zeta) R_N(\zeta, \xi)}{\sum_{w \in E} R_N(\zeta, w)} \,. \tag{3.2.1}$$

It is easy to check from this identity that Assumption 3.1.6 holds for the jump rates R_N^F. It remains to proceed recursively to complete the proof. ∎

Lemma 3.2.1. *The sequences* $\{\mu_N(\eta) : N \geq 1\}$, $\eta \in E$, *are ordered.*

Proof. Fix $\eta \neq \xi \in E$ and let $F = \{\eta, \xi\}$. By [9, Proposition 6.3], the stationary state of the trace of η_t^N on F, denoted by μ_N^F, is given by $\mu_N^F(\eta) = \mu_N(\eta)/\mu_N(F)$. As μ_N^F is the invariant probability measure, $\mu_N^F(\eta)R_N^F(\eta,\xi) = \mu_N^F(\xi)R_N^F(\xi,\eta)$. Therefore, $\mu_N(\eta)/\mu_N(\xi) = \mu_N^F(\eta)/\mu_N^F(\xi) = R_N^F(\xi,\eta)/R_N^F(\eta,\xi)$. By Assertion 3.2.A, the sequences $\{R_N^F(a,b) : N \geq 1\}$, $a \neq b \in \{\eta, \xi\}$ are ordered. This completes the proof of the lemma. ∎

The previous lemma permits to divide the configurations of E into equivalent classes by declaring η equivalent to η', $\eta \sim \eta'$, if $\mu_N(\eta)/\mu_N(\eta')$ converges to a real number belonging to $(0, \infty)$.

Assertion 3.2.B. *Let* F *be a proper subset of* E. *For every bond* $(\eta', \xi') \in \mathbb{B}$ *and every* $m \geq 1$ *the set of sequences*

$$\Big\{ \prod_{(\eta,\xi)\in\mathbb{B}} R_N^F(\eta,\xi)^{k(\eta,\xi)} R_N(\eta',\xi') : N \geq 1 \Big\}\,, \quad k \in \mathfrak{A}_m$$

is ordered.

Proof. We proceed as in the proof of Assertion 3.2.A, by removing one by one the elements of $E \setminus F$. Fix $\zeta \in E \setminus F$. It follows from (3.2.1) and from Assumption 3.1.6 that the claim of the assertion holds for $F' = E \setminus \{\zeta\}$.

Fix $\zeta' \in E \setminus F$, $\zeta' \neq \zeta$. By using formula (3.2.1), to express the rates $R^{E\setminus\{\zeta,\zeta'\}}$ in terms of the rates $R^{E\setminus\{\zeta\}}$, and the statement of this assertion for $F' = E \setminus \{\zeta\}$ we prove that this assertion also holds for $F' = E \setminus \{\zeta, \zeta'\}$. Iterating this algorithm we complete the proof of the assertion. ∎

Denote by $c_N(\eta,\xi) = \mu_N(\eta)R_N(\eta,\xi)$, $(\eta,\xi) \in \mathbb{B}$, the (generally asymmetric) conductances.

Lemma 3.2.2. *The conductances* $\{c_N(\eta,\xi) : N \geq 1\}$, $(\eta,\xi) \in \mathbb{B}$, *are ordered.*

Proof. Consider two bonds (η,ξ), (η',ξ') in $\mathbb{B}$. As in the proof of Lemma 3.2.1, we may express the ratio of the conductances as

$$\frac{c_N(\eta,\xi)}{c_N(\eta',\xi')} = \frac{\mu_N(\eta)R_N(\eta,\xi)}{\mu_N(\eta')R_N(\eta',\xi')} = \frac{R_N^F(\eta',\eta)R_N(\eta,\xi)}{R_N^F(\eta,\eta')R_N(\eta',\xi')}\,,$$

where $F = \{\eta, \eta')$. It remains to recall the statement of assertion 3.2.B to complete the proof of the lemma. ∎

Denote by $\mathbb{B}^s$ the symmetrization of the set $\mathbb{B}$, that is, the set of bonds (η, ξ) such that (η, ξ) or (ξ, η) belongs to $\mathbb{B}$:

$$\mathbb{B}^s \;=\; \big\{(\eta,\xi)\in E\times E : \eta\neq\xi\,,\, (\eta,\xi)\in\mathbb{B} \;\text{ or }\; (\xi,\eta)\in\mathbb{B}\big\}\;.$$

Denote by $c^s_N(\eta, \xi)$, $(\eta, \xi) \in \mathbb{B}^s$, the symmetric part of the conductance:

$$c^s_N(\eta,\xi) \;=\; \frac{1}{2}\big\{c_N(\eta,\xi)+c_N(\xi,\eta)\big\}\;. \tag{3.2.2}$$

Next result is a straightforward consequence of the previous lemma.

Corollary 3.2.3. *The symmetric conductances* $\{c^s_N(\eta, \xi) : N \geq 1\}$, $(\eta, \xi) \in \mathbb{B}^s$, *are ordered.*

As in Lemma 3.2.1, the previous corollary permits to divide the set $\mathbb{B}^s$ into equivalent classes by declaring (η, ξ) equivalent to (η', ξ'), $(\eta, \xi) \sim (\eta', \xi')$, if $c^s_N(\eta, \xi)/c^s_N(\eta', \xi')$ converges to a constant in $(0, \infty)$.

It is possible to deduce from Assumption 3.1.6 that many other sequences are ordered. We do not present these results here as we do not use them below.

3.3 Cycles, sector condition and capacities

We prove in this section that the generator of a Markov chain on a finite set can be decomposed as the sum of cycle generators and that it satisfies a sector condition. This last bound permits to estimate the capacity between two sets by the capacity between the same sets for the reversible process.

Throughout this section, E is a fixed finite set and $\mathcal{L}$ represents the generator of an E-valued, continuous-time Markov chain. We adopt all notation introduced in Section 3.1, removing the index N since the chain is fixed. We start with some definitions.

In a finite set, the decomposition of a generator into cycle generators is very simple. The problem for infinite sets is much more delicate. We refer to [13] for a discussion of the question.

Cycle: A cycle is a sequence of distinct configurations $(\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n = \eta_0)$ whose initial and final configuration coincide: $\eta_i \neq \eta_j \in E$, $i \neq j \in \{0, \dots, n-1\}$. The number n is called the length of the cycle.

Cycle generator: A generator $\mathcal{L}$ of an E-valued Markov chain, whose jump rates are denoted by $R(\eta, \xi)$, is said to be a cycle generator associated to the cycle $\mathfrak{c} = (\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n =$

η_0) if there exists reals $r_i > 0$, $0 \le i < n$, such that

$$R(\eta,\xi) \;=\; \begin{cases} r_i & \text{if } \eta = \eta_i \text{ and } \xi = \eta_{i+1} \text{ for some } 0 \le i < n\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

We denote this cycle generator by $\mathcal{L}_{\mathfrak{c}}$. Note that

$$(\mathcal{L}_{\mathfrak{c}} f)(\eta) \;=\; \sum_{i=0}^{n-1} \mathbf{1}\{\eta = \eta_i\}\, r_i\, [f(\eta_{i+1}) - f(\eta_i)]\ .$$

Sector condition: A generator $\mathcal{L}$ of an E-valued, irreducible Markov chain, whose unique invariant probability measure is denoted by μ, is said to satisfy a sector condition if there exists a constant $C_0 < \infty$ such that for all functions f, $g : E \to \mathbb{R}$,

$$\langle \mathcal{L} f, g\rangle_\mu^2 \;\le\; C_0 \langle (-\mathcal{L} f), f\rangle_\mu \, \langle (-\mathcal{L} g), g\rangle_\mu\ .$$

In this formula, $\langle f, g\rangle_\mu$ represents the scalar product in $L^2(\mu)$:

$$\langle f, g\rangle_\mu \;=\; \sum_{\eta\in E} f(\eta)\, g(\eta)\, \mu(\eta)\ .$$

We claim that every cycle generator satisfies a sector condition and that every generator $\mathcal{L}$ of an E-valued Markov chain, stationary with respect to a probability measure μ, can be decomposed as the sum of cycle generators which are stationary with respect to μ.

Assertion 3.3.A. *Consider a cycle $\mathfrak{c} = (\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n = \eta_0)$ of length $n \ge 2$ and let $\mathcal{L}$ be a cycle generator associated to $\mathfrak{c}$. Denote the jump rates of $\mathcal{L}$ by $R(\eta_i, \eta_{i+1})$. A measure μ is stationary for $\mathcal{L}$ if and only if*

$$\mu(\eta_i)\, R(\eta_i, \eta_{i+1}) \;\; \textit{is constant}\ . \tag{3.3.1}$$

The proof of the previous assertion is elementary and left to the reader. The proof of the next one can be found in [15, Lemma 5.5.8].

Assertion 3.3.B. *Let $\mathcal{L}$ be a cycle generator associated to a cycle $\mathfrak{c}$ of length n. Then, $\mathcal{L}$ satisfies a sector condition with constant $2n$: For all f, $g : E \to \mathbb{R}$,*

$$\langle \mathcal{L} f, g\rangle_\mu^2 \;\le\; 2n\, \langle (-\mathcal{L} f), f\rangle_\mu \, \langle (-\mathcal{L} g), g\rangle_\mu\ .$$

Lemma 3.3.1. *Let $\mathcal{L}$ be a generator of an E-valued, irreducible Markov chain. Denote by μ the unique invariant probability measure. Then, there exists cycles $\mathfrak{c}_1, \dots, \mathfrak{c}_p$ such that*

$$\mathcal{L} \;=\; \sum_{j=1}^{p} \mathcal{L}_{\mathfrak{c}_j} \;,$$

where $\mathcal{L}_{\mathfrak{c}_j}$ are cycle generators associated to $\mathfrak{c}_j$ which are stationary with respect to μ.

Proof. The proof consists in eliminating successively all 2-cycles (cycles of length 2), then all 3-cycles and so on up to the $|E|$-cycle if there is one left. Denote by $R(\eta, \xi)$ the jump rates of the generator $\mathcal{L}$ and by $\mathbb{C}_2$ the set of all 2-cycles (η, ξ, η) such that $R(\eta, \xi) R(\xi, \eta) > 0$. Note that the cycle (η, ξ, η) coincide with the cycle (ξ, η, ξ).

Fix a cycle $\mathfrak{c} = (\eta, \xi, \eta) \in \mathbb{C}_2$. Let $\bar{c}(\eta, \xi) = \min\{\mu(\eta) R(\eta, \xi), \mu(\xi) R(\xi, \eta)\}$ be the minimal conductance of the edge (η, ξ), and let $R_{\mathfrak{c}}(\eta, \xi)$ be the jump rates given by $R_{\mathfrak{c}}(\eta, \xi) = \bar{c}(\eta, \xi)/\mu(\eta)$, $R_{\mathfrak{c}}(\xi, \eta) = \bar{c}(\eta, \xi)/\mu(\xi)$. Observe that $R_{\mathfrak{c}}(\zeta, \zeta') \le R(\zeta, \zeta')$ for all (ζ, ζ'), and that $R_{\mathfrak{c}}(\xi, \eta) = R(\xi, \eta)$ or $R_{\mathfrak{c}}(\eta, \xi) = R(\eta, \xi)$.

Denote by $\mathcal{L}_{\mathfrak{c}}$ the generator associated the the jump rates $R_{\mathfrak{c}}$. Since $\mu(\eta) R_{\mathfrak{c}}(\eta, \xi) = \bar{c}(\eta, \xi) = \mu(\xi) R_{\mathfrak{c}}(\xi, \eta)$, by (3.3.1), μ is a stationary state for $\mathcal{L}_{\mathfrak{c}}$ (actually, reversble). Let $\mathcal{L}_1 = \mathcal{L} - \mathcal{L}_{\mathfrak{c}}$ so that

$$\mathcal{L} \;=\; \mathcal{L}_1 \;+\; \mathcal{L}_{\mathfrak{c}} \;.$$

As $R_{\mathfrak{c}}(\zeta, \zeta') \le R(\zeta, \zeta')$, $\mathcal{L}_1$ is the generator of a Markov chain. Since both $\mathcal{L}$ and $\mathcal{L}_{\mathfrak{c}}$ are stationary for μ, so is $\mathcal{L}_1$. Finally, if we draw an arrow from ζ to ζ' if the jump rate from ζ to ζ' is strictly positive, the number of arrows for the generator $\mathcal{L}_1$ is equal to the number of arrows for the generator $\mathcal{L}$ minus 1 or 2. This procedure has therefore strictly decreased the number of arrows of $\mathcal{L}$.

We may repeat the previous algorithm to $\mathcal{L}_1$ to remove from $\mathcal{L}$ all 2-cycles (η, ξ, η) such that $R(\eta, \xi) R(\xi, \eta) > 0$. Once this has been accomplished, we may remove all 3-cycles $(\eta_0, \eta_1, \eta_2, \eta_3 = \eta_0)$ such that $\prod_{0 \le i < 3} R(\eta_i, \eta_{i+1}) > 0$. At each step at least one arrow is removed from the generator which implies that after a finite number of steps all 3-cycles are removed.

Once all k-cycles have been removed, $2 \le k < |E|$, we have obtained a decomposition of $\mathcal{L}$ as

$$\mathcal{L} \;=\; \sum_{k=2}^{|E|-1} \mathcal{L}_k \;+\; \hat{\mathcal{L}} \;,$$

where $\mathcal{L}_k$ is the sum of k-cycle generators and is stationary with respect to μ, and $\hat{\mathcal{L}}$ is a generator, stationary with respect to μ, and with no k-cycles, $2 \le k < |E|$. If $\hat{\mathcal{L}}$ has an arrow, as it is stationary with respect to μ and has no k-cycles, $\hat{\mathcal{L}}$ must be an $|E|$-cycle generator, providing the decomposition stated in the lemma. ∎

Remark 3.3.2. *Observe that a generator $\mathcal{L}$ is reversible with respect to μ if and only if it has a decomposition in* 2*-cycles. Given a measure μ on a finite state space, for example the Gibbs measure associated to a Hamiltonian at a fixed temperature, by introducing k-cycles satisfying* (3.3.1) *it is possible to define non-reversible dynamics which are stationary with respect to μ. The previous lemma asserts that this is the only way to define such dynamics.*

Corollary 3.3.3. *The generator $\mathcal{L}$ satisfies a sector condition with constant bounded by* $2|E|$*: For all f, $g : E \to \mathbb{R}$,*

$$\langle \mathcal{L}f, g\rangle_\mu^2 \;\le\; 2|E|\, \langle (-\mathcal{L}f), f\rangle_\mu \, \langle (-\mathcal{L}g), g\rangle_\mu \, .$$

Proof. Fix f and $g : E \to \mathbb{R}$. By Lemma 3.3.1,

$$\langle \mathcal{L}f, g\rangle_\mu^2 \;=\; \Big(\sum_{j=1}^{p} \langle \mathcal{L}_{\mathfrak{c}_j} f, g\rangle_\mu \Big)^2 \, ,$$

where $\mathcal{L}_{\mathfrak{c}_j}$ is a cycle generator, stationary with respect to μ, associated to the cycle $\mathfrak{c}_j$. By Assertion 3.3.B and by Schwarz inequality, since all cycles have length at most $|E|$, the previous sum is bounded by

$$2|E| \sum_{j=1}^{p} \langle (-\mathcal{L}_{\mathfrak{c}_j} f), f\rangle_\mu \sum_{k=1}^{p} \langle (-\mathcal{L}_{\mathfrak{c}_k} g), g\rangle_\mu \;=\; 2|E|\, \langle (-\mathcal{L}f), f\rangle_\mu \, \langle (-\mathcal{L}g), g\rangle_\mu \, ,$$

as claimed ∎

Denote by $R^s(\eta,\xi)$ the symmetric part of the jump rates $R^s(\eta,\xi)$:

$$R^s(\eta,\xi) \;=\; \frac{1}{2}\Big\{ R(\eta,\xi) \;+\; \frac{\mu(\xi)}{\mu(\eta)}\, R(\xi,\eta) \Big\} \, . \tag{3.3.2}$$

Denote by η^s_t the E-valued Markov chain whose jump rates are given by R^s. The chain η^s_t is called the reversible chain.

For two disjoint subsets A, B of E, denote by $\mathrm{cap}(A,B)$ (resp. $\mathrm{cap}^s(A,B)$) the capacity between A and B (for the reversible chain). Next result follows from Corollary 3.3.3 and Lemmas 2.5 and 2.6 in [14]

Corollary 3.3.4. *Fix two disjoint subsets A, B of E. Then,*

$$\mathrm{cap}^s(A,B) \;\le\; \mathrm{cap}(A,B) \;\le\; 2|E|\, \mathrm{cap}^s(A,B) \, .$$

We conclude the section with an identity and an inequality which will be used several times in this article. Let A and B be two disjoint subsets of E. By definition of the capacity

$$\operatorname{cap}(A,B) \;=\; \sum_{\eta\in A}\mu(\eta)\,\lambda(\eta)\,\mathbb{P}_\eta\big[H_B < H_A^+\big] \;=\; \sum_{\eta\in A}\mu(\eta)\,\lambda(\eta)\sum_{\xi\in B}\mathbb{P}_\eta\big[H_\xi = H_{A\cup B}^+\big]\;.$$

Therefore, if we denote by $R^{A\cup B}(\eta,\xi)$, $\eta\neq\xi\in A\cup B$, the jump rates of the trace of the chain η_t on the set $A\cup B$, by [9, Proposition 6.1],

$$\operatorname{cap}(A,B) \;=\; \sum_{\eta\in A}\mu(\eta)\sum_{\xi\in B}R^{A\cup B}(\eta,\xi)\;. \tag{3.3.3}$$

Let A be a non-empty subset of E and denote by $R^A(\eta,\xi)$ the jump rates of the trace of η_t on A. We claim that for all $\eta\neq\xi\in A$,

$$\mu(\eta)\,R^A(\eta,\xi) \;\le\; \operatorname{cap}(\eta,\xi)\;. \tag{3.3.4}$$

Denote by $\lambda^A(\zeta)$ the holding rates of the trace process on A and by $p^A(\zeta,\zeta')$ the jump probabilities. By definition,

$$R^A(\eta,\xi) \;=\; \lambda^A(\eta)\,p^A(\eta,\xi) \;=\; \lambda^A(\eta)\,\mathbb{P}_\eta[H_\xi = H_A^+] \;\le\; \lambda^A(\eta)\,\mathbb{P}_\eta[H_\xi < H_\eta^+]\;.$$

Multiplying both sides of this inequality by $\mu_A(\eta) = \mu(\eta)/\mu(A)$, by definition of the capacity we obtain that

$$\mu_A(\eta)\,R^A(\eta,\xi) \;\le\; \operatorname{cap}_A(\eta,\xi)\;,$$

where $\operatorname{cap}_A(\eta,\xi)$ stands for the capacity with respect to the trace process on A. To complete the proof of (3.3.4), it remains to recall formula (A.10) in [11].

3.4 Reversible chains and capacities

We present in this section some estimates for the capacity of reversible, finite state Markov chains obtained in in [10]. There are useful below since if proved in Corollary 3.3.4 that the capacity between two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E is of the same order as the capacity with respect to the reversible chain.

Recall from (3.2.2) that we denote by $c^s_N(\eta,\xi)$ the symmetric conductance of the bond (η,ξ). Fix two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E. A self-avoiding path γ from $\mathscr{A}$ to $\mathscr{B}$ is a sequence of configurations $(\eta_0,\eta_1,\dots,\eta_n)$ such that $\eta_0\in\mathscr{A}$, $\eta_n\in\mathscr{B}$, $\eta_i\neq\eta_j$, $i\neq j$, $c^s_N(\eta_i,\eta_{i+1})>0$, $0\le i<n$. Denote by $\Gamma_{\mathscr{A},\mathscr{B}}$ the set of self-avoiding paths from $\mathscr{A}$ to $\mathscr{B}$

and let

$$\boldsymbol{c}^s_N(\gamma) \;=\; \min_{0\le i<n} c^s_N(\eta_i,\eta_{i+1})\;, \quad \boldsymbol{c}^s_N(\mathscr{A},\mathscr{B}) \;=\; \max_{\gamma\in\Gamma_{\mathscr{A},\mathscr{B}}} \boldsymbol{c}^s_N(\gamma)\;. \tag{3.4.1}$$

For two configurations η, ξ, we represent $\boldsymbol{c}^s_N(\{\eta\},\{\xi\})$ by $\boldsymbol{c}^s_N(\eta,\xi)$. Note that $\boldsymbol{c}^s_N(\eta,\xi) \le c^s_N(\eta,\xi)$, with possibly a strict inequality.

Fix two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E and a configuration $\eta\notin \mathscr{A}\cup\mathscr{B}$. We claim that

$$\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B}) \;\ge\; \min\{\boldsymbol{c}^s_N(\mathscr{A},\eta)\,,\, \boldsymbol{c}^s_N(\eta,\mathscr{B})\}\;. \tag{3.4.2}$$

Indeed, there exist a self-avoiding path γ_1 from $\mathscr{A}$ to η, and a self-avoiding path γ_2 from η to $\mathscr{B}$ such that $\boldsymbol{c}^s_N(\mathscr{A},\eta)=\boldsymbol{c}^s_N(\gamma_1)$, $\boldsymbol{c}^s_N(\eta,\mathscr{B})=\boldsymbol{c}^s_N(\gamma_2)$. Juxtaposing the paths γ_1 and γ_2, we obtain a path γ from $\mathscr{A}$ to $\mathscr{B}$. Of course, the path γ may not be self-avoiding, may return to $\mathscr{A}$ before reaching $\mathscr{B}$, or may reach $\mathscr{B}$ before hitting η. In any case, we may obtain from γ a subpath $\hat\gamma$ which is self-avoiding and which connects $\mathscr{A}$ to $\mathscr{B}$. Subpath in the sense that all bonds (η_i,η_{i+1}) which appear in $\hat\gamma$ also appear in γ. In particular,

$$\boldsymbol{c}^s_N(\hat\gamma) \;\ge\; \boldsymbol{c}^s_N(\gamma) \;=\; \min\{\boldsymbol{c}^s_N(\gamma_1)\,,\,\boldsymbol{c}^s_N(\gamma_2)\} \;=\; \min\{\boldsymbol{c}^s_N(\mathscr{A},\eta)\,,\,\boldsymbol{c}^s_N(\eta,\mathscr{B})\}\;.$$

To complete the proof of claim (3.4.2), it remains to observe that $\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B})\ge \boldsymbol{c}^s_N(\hat\gamma)$.

Fix two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E and configurations $\eta_i\notin\mathscr{A}\cup\mathscr{B}$, $1\le i\le n$, such that $\eta_i\neq\eta_j$, $i\neq j$. Iterating inequality (3.4.2) we obtain that

$$\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B}) \;\ge\; \min\{\boldsymbol{c}^s_N(\mathscr{A},\eta_1)\,,\,\boldsymbol{c}^s_N(\eta_1,\eta_2)\,,\,\dots\,,\,\boldsymbol{c}^s_N(\eta_{n-1},\eta_n)\,,\,\boldsymbol{c}^s_N(\eta_n,\mathscr{B})\}\;. \tag{3.4.3}$$

We conclude this section relating the symmetric capacity between two sets $\mathscr{A}$, $\mathscr{B}$ of E to the symmetric conductances $\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B})$. By Corollary 3.2.3, the sequences of symmetric conductances $\{c^s_N(\eta,\xi) : N\ge 1\}$, $(\eta,\xi)\in\mathbb{B}^s$, are ordered. It follows from this fact and from the proof of Lemmas 4.1 in [10] that there exists constants $0<c_0<C_0<\infty$ such that

$$c_0 \;<\; \liminf_{N\to\infty}\frac{\mathrm{cap}^s_N(\mathscr{A},\mathscr{B})}{\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B})} \;\le\; \limsup_{N\to\infty}\frac{\mathrm{cap}^s_N(\mathscr{A},\mathscr{B})}{\boldsymbol{c}^s_N(\mathscr{A},\mathscr{B})} \;\le\; C_0\;. \tag{3.4.4}$$

3.5 Proof of Theorem 3.1.1

In view of Theorem 5.1 in [16], Theorem 3.1.1 follows from from condition (H3) and from Propositions 3.5.1 below. Denote by $\psi_{\mathscr{E}}:\mathscr{E}\to\{1,\dots,\mathfrak{n}\}$ the projection defined by $\psi_{\mathscr{E}}(\eta)=x$ if $\eta\in\mathscr{E}_x$:

$$\psi_{\mathscr{E}}(\eta) \;=\; \sum_{x\in S} x\,\mathbf{1}\{\eta\in\mathscr{E}_x\}\;.$$

Proposition 3.5.1. *Fix $x \in S$ and a configuration $\eta \in \mathscr{E}_x$. Starting from η, the speeded-up, hidden Markov chain $X_N(t) = \psi_{\mathscr{E}}\big(\eta^{\mathscr{E}}(\theta_N t)\big)$ converges in the Skorohod topology to the continuous-time Markov chain $X_{\mathscr{E}}(t)$, introduced in Theorem 3.1.1, which starts from x.*

Lemma 3.5.2. *For every $x \in S$ for which $\mathscr{E}_x$ is not a singleton and for all $\eta \neq \xi \in \mathscr{E}_x$,*

$$\lim_{N\to\infty} \frac{\mathrm{cap}_N(\mathscr{E}_x, \breve{\mathscr{E}}_x)}{\mathrm{cap}_N(\eta,\xi)} \;=\; 0\ .$$

Proof. Fix $x \in S$. By (3.3.3), applied to $A = \mathscr{E}_x$, $B = \breve{\mathscr{E}}_x$, and by assumption (H1),

$$\lim_{N\to\infty} \theta_N \frac{\mathrm{cap}_N(\mathscr{E}_x, \breve{\mathscr{E}}_x)}{\mu_N(\mathscr{E}_x)} \;=\; \sum_{y\neq x} r_{\mathscr{E}}(x,y) \;\in\; \mathbb{R}_+\ .$$

The claim of the lemma follows from this equation, from assumption (H2) and from the fact that $\alpha_N/\theta_N \to 0$. ∎

Proof of Proposition 3.5.1. In view of Theorem 2.1 in [11], the claim of the proposition follows from condition (H1), and from Lemma 3.5.2. ∎

3.6 Proof of Theorem 3.1.7

The proof of Theorem 3.1.7 is divided in several steps.

1. The measure of the metastable sets. We start proving that condition (H0) is in force. Recall from Section 3.1 that we denote by $X_R(t)$ the E-valued chain which jumps from η to ξ at rate $R(\eta,\xi)$. Denote by $\mathscr{C}_1,\dots,\mathscr{C}_{\mathfrak{m}}$ the equivalent classes of the chain $X_R(t)$.

Assertion 3.6.A. *For all $1 \le j \le \mathfrak{m}$, and for all $\eta \neq \xi \in \mathscr{C}_j$, there exists $m(\eta,\xi) \in (0,\infty)$ such that*

$$\lim_{N\to\infty} \frac{\mu_N(\eta)}{\mu_N(\xi)} \;=\; m(\eta,\xi)\ .$$

Proof. Fix $1 \le j \le \mathfrak{m}$ and $\eta \neq \xi \in \mathscr{C}_j$. By assumption, there exists a path $(\eta = \eta_0,\dots,\eta_n = \xi)$ such that $R(\eta_i,\eta_{i+1}) > 0$ for $0 \le i < n$. On the other hand, since μ_N is an invariant probability measure,

$$\begin{aligned}\lambda_N(\xi)\,\mu_N(\xi) \;&=\; \sum_{\zeta_0,\zeta_1,\dots,\zeta_{n-1}\in E} \mu_N(\zeta_0)\,\lambda_N(\zeta_0)\,p_N(\zeta_0,\zeta_1)\cdots p_N(\zeta_{n-1},\xi)\\ &\ge\; \mu_N(\eta_0)\,\lambda_N(\eta_0)\,p_N(\eta_0,\eta_1)\cdots p_N(\eta_{n-1},\xi)\ .\end{aligned}$$

Therefore,

$$\frac{\mu_N(\xi)}{\mu_N(\eta)} \;\geq\; \frac{\lambda_N(\eta)}{\lambda_N(\xi)}\, p_N(\eta,\eta_1)\cdots p_N(\eta_{n-1},\xi)\;.$$

Since $R(\eta_i,\eta_{i+1}) > 0$ for $0 \leq i < n$, by (3.1.5), $p_N(\eta_i,\eta_{i+1})$ converges to $p(\eta_i,\eta_{i+1}) > 0$. For the same reason, $\alpha_N\lambda_N(\eta)$ converges to $\lambda(\eta) \in (0,\infty)$. Finally, as ξ and η belong to the same equivalent class, there exists a path from ξ to η with similar properties to the one from η to ξ, so that $\alpha_N\lambda_N(\xi)$ converges to $\lambda(\xi) \in (0,\infty)$. In conclusion,

$$\liminf_{N\to\infty} \frac{\mu_N(\xi)}{\mu_N(\eta)} \;>\; 0\;.$$

Replacing η by ξ we obtain that $\liminf \mu_N(\eta)/\mu_N(\xi) > 0$. Since by Lemma 3.2.1 the sequences $\{\mu_N(\zeta) : N \geq 1\}$, $\zeta \in E$, are ordered, $\mu_N(\eta)/\mu_N(\xi)$ must converge to some value in $(0,\infty)$. ∎

By the previous assertion for every $x \in S$ and $\eta \in \mathscr{E}_x$,

$$m_x(\eta) \;:=\; \lim_{N\to\infty} \frac{\mu_N(\eta)}{\mu_N(\mathscr{E}_x)} \;\in\; (0,1]\;, \tag{3.6.1}$$

where we adopted the convention established in condition (H1) of Section 3.1.

2. The time-scale. In this subsection, we introduce a time-scale γ_N, we prove that it is much longer than α_N and that it is of the same order of θ_N. In particular the requirement $\alpha_N/\theta_N \to 0$ is in force.

Denote by $\{\eta^{\mathscr{E}}_t : t \geq 0\}$ the trace of η^N_t on the set $\mathscr{E}$, and by $R^{\mathscr{E}}_N : \mathscr{E}\times\mathscr{E} \to \mathbb{R}_+$ the jump rates of $\eta^{\mathscr{E}}_t$. Let

$$\frac{1}{\gamma_N} \;=\; \sum_{x\in S}\sum_{\eta\in\mathscr{E}_x}\sum_{\xi\in\breve{\mathscr{E}}_x} R^{\mathscr{E}}_N(\eta,\xi)\;, \tag{3.6.2}$$

where $\breve{\mathscr{E}}_x$ has been introduced in (3.1.4). The sequence γ_N represents the time needed to reach the set $\breve{\mathscr{E}}_x$ starting from $\mathscr{E}_x$ for some $x \in S$. This time scale might be longer for other sets $\mathscr{E}_y$, $y \neq x$, but it is of the order γ_N at least for one $x \in S$. We could as well have defined γ_N as $\max_{x\in S}\max_{\eta\in\mathscr{E}_x}\max_{\xi\in\breve{\mathscr{E}}_x} R^{\mathscr{E}}_N(\eta,\xi)$.

Assertion 3.6.B. *The time scale γ_N is much longer than the time-scale α_N:*

$$\lim_{N\to\infty}\frac{\alpha_N}{\gamma_N} \;=\; 0\;.$$

Proof. We have to show that $\alpha_N R^{\mathscr{E}}_N(\eta,\xi)$ converges to 0 as $N\uparrow\infty$, for all $\eta\in\mathscr{E}_x$, $\xi\in\mathscr{E}_y$, $x \neq y \in S$. Fix $x\neq y\in S$, $\eta\in\mathscr{E}_x$, $\xi\in\mathscr{E}_y$. Since $\mathscr{E}_x$ is a recurrent class, $R(\eta,\zeta) = 0$ for all

$\zeta \notin \mathscr{E}_x$. On the other hand, by [9, Proposition 6.1] and by the strong Markov property,

$$R^{\mathscr{E}}_N(\eta,\xi) \;=\; \lambda_N(\eta)\,\mathbb{P}_\eta[H_\xi = H^+_{\mathscr{E}}] \;=\; R_N(\eta,\xi) \;+\; \sum_{\zeta\notin\mathscr{E}} R_N(\eta,\zeta)\,\mathbb{P}_\zeta[H_\xi = H_{\mathscr{E}}]\;.$$

Since $R(\eta,\zeta) = 0$ for all $\zeta \notin \mathscr{E}_x$, it follows from the previous identity and from the definition of $R(\eta,\zeta)$ that $\alpha_N R^{\mathscr{E}}_N(\eta,\xi) \to 0$, as claimed. ∎

By Assertion 3.2.A, for all $x \in S$, $\eta \in \mathscr{E}_x$, $\xi \in \breve{\mathscr{E}}_x$, with the convention adopted in condition (H1) of Section 3.1,

$$r_{\mathscr{E}}(\eta,\xi) \;:=\; \lim_{N\to\infty} \gamma_N\, R^{\mathscr{E}}_N(\eta,\xi) \;\in\; [0,1]\;. \tag{3.6.3}$$

Assertion 3.6.C. *For all $x \in S$,*

$$\ell_x \;:=\; \lim_{N\to\infty} \gamma_N \,\frac{\mathrm{cap}_N(\mathscr{E}_x,\breve{\mathscr{E}}_x)}{\mu_N(\mathscr{E}_x)} \;\in\; \mathbb{R}_+\;. \quad \textit{Moreover}\,,\;\; \ell \;=\; \sum_{x\in S}\ell_x \;>\; 0\;.$$

Proof. By (3.3.3), applied to $A = \mathscr{E}_x$, $B = \breve{\mathscr{E}}_x$, by (3.6.1) and by (3.6.3),

$$\lim_{N\to\infty} \gamma_N \,\frac{\mathrm{cap}_N(\mathscr{E}_x,\breve{\mathscr{E}}_x)}{\mu_N(\mathscr{E}_x)} \;=\; \sum_{\eta\in\mathscr{E}_x} m_x(\eta) \sum_{\xi\in\breve{\mathscr{E}}_x} r_{\mathscr{E}}(\eta,\xi) \;\in\; \mathbb{R}_+\;,$$

which completes the proof of the first claim of the assertion.

By (3.6.2) and by definition of $r_{\mathscr{E}}(\eta,\xi)$,

$$\sum_{x\in S}\sum_{\eta\in\mathscr{E}_x}\sum_{\xi\in\breve{\mathscr{E}}_x} r_{\mathscr{E}}(\eta,\xi) \;=\; 1\;,$$

so that

$$\ell \;=\; \sum_{x\in S}\ell_x \;=\; \sum_{x\in S}\sum_{\eta\in\mathscr{E}_x} m_x(\eta) \sum_{\xi\in\breve{\mathscr{E}}_x} r_{\mathscr{E}}(\eta,\xi) \;\ge\; \min_{x\in S}\min_{\eta\in\mathscr{E}_x} m_x(\eta) \;>\; 0\;,$$

which is the second claim of the assertion. ∎

It follows from Assertion 3.6.C that the time-scale γ_N is of the same order of θ_N in the sense that γ_N/θ_N converges as $N\uparrow\infty$:

$$\lim_{N\to\infty}\frac{\gamma_N}{\theta_N} \;=\; \ell \;\in\; (0,\infty)\;. \tag{3.6.4}$$

3. The average jump rate, condition (H1). Denote by $r_N(\mathscr{E}_x, \mathscr{E}_y)$ the mean rate at which the trace process jumps from $\mathscr{E}_x$ to $\mathscr{E}_y$:

$$r_N(\mathscr{E}_x, \mathscr{E}_y) \;=\; \frac{1}{\mu_N(\mathscr{E}_x)} \sum_{\eta\in\mathscr{E}_x} \mu_N(\eta) \sum_{\xi\in\mathscr{E}_y} R_N^{\mathscr{E}}(\eta,\xi) \;. \tag{3.6.5}$$

Next lemma follows from (3.6.1), (3.6.3) and (3.6.4).

Lemma 3.6.1. *For every $x \neq y \in S$,*

$$r_{\mathscr{E}}(x,y) \;:=\; \lim_{N\to\infty} \theta_N\, r_N(\mathscr{E}_x, \mathscr{E}_y) \;=\; \frac{1}{\ell} \sum_{\eta\in\mathscr{E}_x} m_x(\eta) \sum_{\xi\in\mathscr{E}_y} r_{\mathscr{E}}(\eta,\xi) \;\in\; \mathbb{R}_+$$

4. Inside the metastable sets, condition (H2). Next assertion shows that condition (H2) is in force.

Assertion 3.6.D. *For every $x \in S$ for which $\mathscr{E}_x$ is not a singleton and for all $\eta \neq \xi \in \mathscr{E}_x$, there exist constants $0 < c_0 < C_0 < \infty$ such that*

$$c_0 \;\le\; \liminf_{N\to\infty} \alpha_N \frac{\mathrm{cap}_N(\eta,\xi)}{\mu_N(\mathscr{E}_x)} \;\le\; \limsup_{N\to\infty} \alpha_N \frac{\mathrm{cap}_N(\eta,\xi)}{\mu_N(\mathscr{E}_x)} \;\le\; C_0 \;.$$

Proof. Fix $x \in S$ for which $\mathscr{E}_x$ is not a singleton, and $\eta \neq \xi \in \mathscr{E}_x$. On the one hand, by definition of the capacity

$$\alpha_N \frac{\mathrm{cap}_N(\eta,\xi)}{\mu_N(\mathscr{E}_x)} \;\le\; \frac{\mu_N(\eta)}{\mu_N(\mathscr{E}_x)}\, \alpha_N\, \lambda_N(\eta) \;.$$

By (3.1.5) and (3.6.1), the right hand side converges to $\lambda(\eta) m_x(\eta) < \infty$, which proves one of the inequalities.

On the other hand, as $\mathscr{E}_x$ is an equivalent class which is not a singleton, $\lambda(\zeta) > 0$ for all $\zeta \in \mathscr{E}_x$, or, in other words, $\mathscr{E}_x \subset E_0$. Since $\eta \sim \xi$, there exists a path $(\eta = \eta_0, \dots, \eta_n = \xi)$ such that $R(\eta_i, \eta_{i+1}) > 0$ for $0 \le i < n$. Since,

$$\mathbb{P}_\eta\big[H_\xi < H_\eta^+\big] \;\ge\; p_N(\eta,\eta_1) \cdots p_N(\eta_{n-1},\xi) \;,$$

in view of the formula (3.1.2) for the capacity, we have that

$$\alpha_N \frac{\mathrm{cap}_N(\eta,\xi)}{\mu_N(\mathscr{E}_x)} \;\ge\; \frac{\mu_N(\eta)}{\mu_N(\mathscr{E}_x)}\, \alpha_N\, \lambda_N(\eta)\, p_N(\eta,\eta_1) \cdots p_N(\eta_{n-1},\xi) \;.$$

The right hand side converges to $m_x(\eta)\lambda(\eta)p(\eta,\eta_1)\cdots p(\eta_{n-1},\xi) > 0$, which completes the proof of the assertion. ∎

5. Condition (H3) holds. To complete the proof of Theorem 3.1.7 it remains to show that the chain η^N_t spends a negligible amount of time on the set Δ in the time scale θ_N.

Lemma 3.6.2. *For every $t > 0$,*

$$\lim_{N\to\infty} \max_{\eta\in E} \mathbb{E}_\eta\Big[\int_0^t \mathbf{1}\{\eta^N_{s\theta_N} \in \Delta\}\, ds\Big] \;=\; 0\;.$$

Proof. Since $\alpha_N/\theta_N \to 0$, a change of variables in the time integral and the Markov property show that for every $\eta \in E$, for every $T > 0$ and for every N large enough,

$$\mathbb{E}_\eta\Big[\int_0^t \mathbf{1}\{\eta^N_{s\theta_N} \in \Delta\}\, ds\Big] \;\le\; \frac{2t}{T} \max_{\xi\in E} \mathbb{E}_\xi\Big[\int_0^T \mathbf{1}\{\eta^N_{s\alpha_N} \in \Delta\}\, ds\Big]\;.$$

Note that the process on the right hand side is speeded up by α_N instead of θ_N.

We estimate the expression on the right hand side of the previous formula. We may, of course, restrict the maximum to Δ. Let T_1 be the first time the chain η^N_t hits $\mathscr{E}$ and let T_2 be the time it takes for the process to return to Δ after T_1:

$$T_1 \;=\; H_{\mathscr{E}}\;, \quad T_2 \;=\; \inf\big\{s > 0 : \eta^N_{T_1+s} \in \Delta\big\}\;.$$

Fix $\eta \in \Delta$ and note that

$$\begin{aligned} &\mathbb{E}_\eta\Big[\frac{1}{T}\int_0^T \mathbf{1}\{\eta^N_{s\alpha_N} \in \Delta\}\, ds\Big] \\ &\qquad \le\; \mathbb{P}_\eta\big[T_1 > t_0\alpha_N\big] \;+\; \mathbb{P}_\eta\big[T_2 < T\alpha_N\big] \;+\; \frac{t_0}{T} \end{aligned} \tag{3.6.6}$$

for all $t_0 > 0$ because the time average is bounded by 1 and because on the set $\{T_1 \le t_0\alpha_N\}\cap\{T_2 \ge T\alpha_N\}$ the time average is bounded by t_0/T. By Assertion 3.6.E below, the first term on the right hand side vanishes as $N\uparrow\infty$ and then $t_0\uparrow\infty$. On the other hand, by the strong Markov property, the second term is bounded by $\max_{\xi\in\mathscr{E}} \mathbb{P}_\xi[H_\Delta \le T\alpha_N]$. By definition of the set $\mathscr{E}$, for every $\eta \in \mathscr{E}$ and every $\xi \in \Delta$, $\alpha_N R_N(\eta,\xi) \to 0$ as $N\uparrow\infty$. This shows that for every $T > 0$ the second term on the right hand side of (3.6.6) vanishes as $N\uparrow\infty$, which completes the proof of the lemma. ∎

Assertion 3.6.E. *For every $\eta \in \Delta$,*

$$\lim_{t\to\infty} \limsup_{N\to\infty} \mathbb{P}_\eta\big[H_{\mathscr{E}} \ge t\alpha_N\big] \;=\; 0\;.$$

Proof. Recall that we denote by $X_R(t)$ the continuous-time Markov chain on E which jumps from η to ξ at rate $R(\eta,\xi) = \lim_N \alpha_N R_N(\eta,\xi)$. Note that the set $\mathscr{E}$ consists of recurrent points for the chain $X_R(t)$, while points in Δ are transient. Since the jump rates converge, the chain $\eta^N_{t\alpha_N}$ converges in the Skorohod topology to $X_R(t)$. Therefore, for all $t>0$, $\eta\in\Delta$,

$$\limsup_{N\to\infty} \mathbb{P}_\eta\big[H_{\mathscr{E}} \ge t\,\alpha_N\big] \;\le\; \mathbf{P}_\eta\big[H_{\mathscr{E}} \ge t\big]\;,$$

where $\mathbf{P}_\eta$ stands for the law of the chain $X_R(t)$ starting from η. Since the set of recurrent points for $X_R(t)$ is equal to $\mathscr{E}=\Delta^c$, the previous probability vanishes as $t\uparrow\infty$. ∎

We conclude this section with an observation concerning the capacities of the metastable sets $\mathscr{E}_x$.

Assertion 3.6.F. *The sequences* $\{\mathrm{cap}_N(\mathscr{E}_x,\breve{\mathscr{E}}_x)/\mu_N(\mathscr{E}_x) : N\ge 1\}$, $x\in S$, *are ordered.*

Proof. Fix $x\in S$. By (3.3.3) applied to $A=\mathscr{E}_x$, $B=\breve{\mathscr{E}}_x$,

$$\mathrm{cap}_N(\mathscr{E}_x,\breve{\mathscr{E}}_x) \;=\; \sum_{\eta\in\mathscr{E}_x}\mu_N(\eta)\sum_{\xi\in\breve{\mathscr{E}}_x} R^{\mathscr{E}}_N(\eta,\xi)\;.$$

The claim of the assertion follows from this identity, from Assertion 3.2.A and from (3.6.1). ∎

3.7 Proof of Theorem 3.1.12

Theorem 3.1.12 is proved in several steps.

1. The measure of configurations in $\mathscr{G}_a$. We assumed in (H0) that all configurations in a set $\mathscr{F}_x$ have measure of the same order. We prove below in Assertion 3.7.A that a similar property holds for the sets $\mathscr{G}_a$.

Let

$$\lambda^{\mathscr{F}}_N(\mathscr{F}_x) \;=\; \sum_{y:y\neq x} r^{\mathscr{F}}_N(\mathscr{F}_x,\mathscr{F}_y)\;,\quad p^{\mathscr{F}}_N(\mathscr{F}_x,\mathscr{F}_y) \;=\; \frac{r^{\mathscr{F}}_N(\mathscr{F}_x,\mathscr{F}_y)}{\lambda^{\mathscr{F}}_N(\mathscr{F}_x)} \;\text{ if }\; \lambda^{\mathscr{F}}_N(\mathscr{F}_x)>0\;.$$

Denote by P_0 the subset of points in P such that $\lambda_{\mathscr{F}}(x)=\sum_{y\neq x} r_{\mathscr{F}}(x,y)>0$. For all $x\in P_0$ let $p_{\mathscr{F}}(x,y)=r_{\mathscr{F}}(x,y)/\lambda_{\mathscr{F}}(x)$. It follows from assumption (H1) that for all x, z in P, $y\in P_0$,

$$\lim_{N\to\infty}\beta_N\,\lambda^{\mathscr{F}}_N(\mathscr{F}_x) \;=\lambda_{\mathscr{F}}(x)\;,\quad \lim_{N\to\infty} p^{\mathscr{F}}_N(\mathscr{F}_y,\mathscr{F}_z) \;=\; p_{\mathscr{F}}(y,z)\;. \tag{3.7.1}$$

Recall that $X_{\mathscr{F}}(t)$ is the P-valued Markov chain which jumps from x to y at rate $r_{\mathscr{F}}(x,y)$. Denote by C_a, $a\in P_1=\{1,\dots,\mathfrak{q}_1\}$, the equivalent classes of the Markov chain

$X_{\mathscr{F}}(t)$, and let $\mathscr{C}_a = \cup_{x\in C_a}\mathscr{F}_x$. All configurations in a set $\mathscr{C}_a$ have probability of the same order.

Assertion 3.7.A. *For all equivalent classes C_a, $a \in P_1$, and for all $\eta \neq \xi \in \mathscr{C}_a$, there exists $m(\eta,\xi) \in (0,\infty)$ such that*

$$\lim_{N\to\infty} \frac{\mu_N(\eta)}{\mu_N(\xi)} \;=\; m(\eta,\xi)\;.$$

Proof. The argument is very close to the one of Assertion 3.6.A Denote by $\bar X_N(t)$ the chain $\eta^{\mathscr{F}}(t)$ in which each set $\mathscr{F}_x$ has been collapsed to a point. We refer to the Section 3 of [14] for a precise definition of the collapsed chain and for the proof of the results used below.

The chain $\bar X_N(t)$ takes value in the set P, its jump rate from x to y, denoted by $\bar r_N(x,y)$, is equal to $r_N^{\mathscr{F}}(\mathscr{F}_x,\mathscr{F}_y)$ introduced in (3.1.3), and its unique invariant probability measure, denoted by $\bar\mu_N(x)$, is given by $\bar\mu_N(x) = \mu_N(\mathscr{F}_x)/\mu_N(\mathscr{F})$.

Fix an equivalent class C_a and $\eta \neq \xi \in \mathscr{C}_a$. If η and ξ belong to the same set $\mathscr{F}_x$, the claim follows from Assumption (H0). Suppose that $\eta \in \mathscr{F}_x$, $\xi \in \mathscr{F}_y$ for some $x \neq y \in C_a$. By assumption, there exists a path $(x = x_0,\dots,x_n = y)$ such that $r_{\mathscr{F}}(x_i,x_{i+1}) > 0$ for $0 \le i < n$.

Denote by $\bar\lambda_N(x)$, $x \in P$, the holding rates of the collapsed chain $\bar X_N(t)$, and by $\bar p_N(x,y)$, $x \neq y \in P$, the jump probabilities. Since $\bar\mu_N$ is the invariant probability measure for the collapsed chain,

$$\begin{aligned}\bar\lambda_N(y)\,\bar\mu_N(y) \;&=\; \sum_{z_0,z_1,\dots,z_{n-1}\in P} \bar\mu_N(z_0)\,\bar\lambda_N(z_0)\,\bar p_N(z_0,z_1)\cdots\bar p_N(z_{n-1},y)\\ &\ge\; \bar\mu_N(x_0)\,\bar\lambda_N(x_0)\,\bar p_N(x_0,x_1)\cdots\bar p_N(x_{n-1},y)\;.\end{aligned}$$

Therefore,

$$\frac{\bar\mu_N(y)}{\bar\mu_N(x)} \;\ge\; \frac{\bar\lambda_N(x)}{\bar\lambda_N(y)}\,\bar p_N(x,x_1)\cdots\bar p_N(x_{n-1},y)\;.$$

Since $r_{\mathscr{F}}(x_i,x_{i+1}) > 0$ for $0 \le i < n$, by (3.7.1), $\bar p_N(x_i,x_{i+1})$ converges to $p_{\mathscr{F}}(x_i,x_{i+1}) > 0$. For the same reason, $\beta_N\bar\lambda_N(x) = \beta_N\lambda_N^{\mathscr{F}}(\mathscr{F}_x)$ converges to $\lambda_{\mathscr{F}}(x) \in (0,\infty)$. As y and x share the same properties, inverting their role we obtain that $\beta_N\bar\lambda_N(y)$ converges to $\lambda_{\mathscr{F}}(y) \in (0,\infty)$. In conclusion,

$$\liminf_{N\to\infty} \frac{\bar\mu_N(x)}{\bar\mu_N(y)} \;>\; 0\;.$$

Replacing x by y we obtain that $\liminf \bar\mu_N(y)/\bar\mu_N(x) > 0$. By [14], $\bar\mu_N(z) = \mu_N(\mathscr{F}_z)$,

$z \in P$. To complete the proof it remains to recall the statement of Lemma 3.2.1 and Assumption (H0). ∎

By the previous assertion for every $a \in Q$ and $\eta \in \mathscr{G}_a$,

$$m^*_a(\eta) \;:=\; \lim_{N\to\infty} \frac{\mu_N(\eta)}{\mu_N(\mathscr{G}_a)} \;\in\; (0,1]\;. \tag{3.7.2}$$

Thus, assumption (H0) holds for the partition $\{\mathscr{G}_1, \dots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$.

2. The time scale. We prove in this subsection that the time-scale β_N^+ introduced in (3.1.10) is much longer than β_N.

Assertion 3.7.B. *We have that*

$$\lim_{N\to\infty} \frac{\beta_N}{\beta_N^+} \;=\; 0\;.$$

Proof. We have to show that

$$\lim_{N\to\infty} \beta_N \frac{\mathrm{cap}_N(\mathscr{G}_a, \breve{\mathscr{G}}_a)}{\mu_N(\mathscr{G}_a)} \;=\; 0$$

for each $a \in Q$. Fix $a \in Q$ and recall from (3.1.8) the definition of the set $\mathscr{G}_a$. Since G_a is recurrent class for the chain $X_{\mathscr{F}}(t)$, $r_{\mathscr{F}}(x,y) = 0$ for all $x \in G_a$, $y \in P \setminus G_a$. By definition of the capacity,

$$\begin{aligned}\frac{\mathrm{cap}_N(\mathscr{G}_a, \breve{\mathscr{G}}_a)}{\mu_N(\mathscr{G}_a)} \;&=\; \sum_{\eta\in\mathscr{G}_a} \frac{\mu_N(\eta)}{\mu_N(\mathscr{G}_a)} \lambda_N(\eta)\, \mathbb{P}_\eta\big[H_{\breve{\mathscr{G}}_a} < H^+_{\mathscr{G}_a}\big] \\ &\le\; \sum_{\eta\in\mathscr{G}_a} \frac{\mu_N(\eta)}{\mu_N(\mathscr{G}_a)} \lambda_N(\eta)\, \mathbb{P}_\eta\big[H_{\mathscr{F}\setminus\mathscr{G}_a} < H^+_{\mathscr{G}_a}\big]\;.\end{aligned}$$

By [9, Proposition 6.1], this sum is equal to

$$\sum_{\eta\in\mathscr{G}_a} \frac{\mu_N(\eta)}{\mu_N(\mathscr{G}_a)} \sum_{\xi\in\mathscr{F}\setminus\mathscr{G}_a} R_N^{\mathscr{F}}(\eta,\xi) \;=\; \sum_{x\in G_a} \frac{\mu_N(\mathscr{F}_x)}{\mu_N(\mathscr{G}_a)} \sum_{y\in P\setminus G_a} r_N^{\mathscr{F}}(x,y)\;.$$

Since $r_{\mathscr{F}}(x,y) = 0$ for all $x \in G_a$, $y \in P \setminus G_a$, by assumption (H1) the previous sum multiplied by β_N converges to 0 as $N \uparrow \infty$. ∎

3. Condition (H1) is fulfilled by the partition $\{\mathscr{G}_1, \dots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$. We first obtain an alternative formula for the time-scale β_N^+. The arguments and the ideas are very similar to the ones presented in the previous section. Let

$$\frac{1}{\gamma_N} \;=\; \sum_{a\in Q}\sum_{\eta\in\mathscr{G}_a}\sum_{\xi\in\breve{\mathscr{G}}_a} R^{\mathscr{G}}_N(\eta,\xi)\;.$$

By Assertion 3.2.A, for all $a \in Q$, $\eta \in \mathscr{G}_a$, $\xi \in \breve{\mathscr{G}}_a$, with the convention adopted in condition (H1) of Section 3.1,

$$r_{\mathscr{G}}(\eta,\xi) \;:=\; \lim_{N\to\infty} \gamma_N\, R^{\mathscr{G}}_N(\eta,\xi) \;\in\; [0,1]\;. \tag{3.7.3}$$

Assertion 3.7.C. *For all $a \in Q$,*

$$\hat\lambda_{\mathscr{G}}(a) \;:=\; \lim_{N\to\infty} \gamma_N\, \frac{\mathrm{cap}_N(\mathscr{G}_a,\breve{\mathscr{G}}_a)}{\mu_N(\mathscr{G}_a)} \;\in\; \mathbb{R}_+\;. \quad \textit{Moreover}\,, \;\; \hat\lambda_{\mathscr{G}} \;=\; \sum_{a\in Q}\hat\lambda_{\mathscr{G}}(a) \;>\; 0\;.$$

Proof. Fix $a \in Q$. By (3.3.3), applied to $A = \mathscr{G}_a$, $B = \breve{\mathscr{G}}_a$, by (3.7.2) and by (3.7.3),

$$\lim_{N\to\infty} \gamma_N\, \frac{\mathrm{cap}_N(\mathscr{G}_a,\breve{\mathscr{G}}_a)}{\mu_N(\mathscr{G}_a)} \;=\; \sum_{\eta\in\mathscr{G}_a} m^*_a(\eta) \sum_{\xi\in\breve{\mathscr{G}}_a} r_{\mathscr{G}}(\eta,\xi) \;\in\; \mathbb{R}_+\;,$$

which completes the proof of the first claim of the assertion.

By definition of γ_N and by definition of $r_{\mathscr{G}}(\eta,\xi)$,

$$\sum_{a\in Q}\sum_{\eta\in\mathscr{G}_a}\sum_{\xi\in\breve{\mathscr{G}}_a} r_{\mathscr{G}}(\eta,\xi) \;=\; 1\;,$$

so that

$$\hat\lambda_{\mathscr{G}} \;=\; \sum_{a\in Q}\hat\lambda_{\mathscr{G}}(a) \;=\; \sum_{a\in Q}\sum_{\eta\in\mathscr{G}_a} m^*_a(\eta) \sum_{\xi\in\breve{\mathscr{G}}_a} r_{\mathscr{G}}(\eta,\xi) \;\ge\; \min_{a\in Q}\min_{\eta\in\mathscr{G}_a} m^*_a(\eta) \;>\; 0\;,$$

which is the second claim of the assertion. ∎

It follows from the previous assertion that the time-scale γ_N is of the same order of β^+_N:

$$\lim_{N\to\infty} \frac{\gamma_N}{\beta^+_N} \;=\; \hat\lambda_{\mathscr{G}} \;\in (0,\infty)\;. \tag{3.7.4}$$

Denote by $r^{\mathscr{G}}_N(\mathscr{G}_a,\mathscr{G}_b)$ the mean rate at which the trace process jumps from $\mathscr{G}_a$ to $\mathscr{G}_b$:

$$r^{\mathscr{G}}_N(\mathscr{G}_a,\mathscr{G}_b) \;:=\; \frac{1}{\mu_N(\mathscr{G}_a)} \sum_{\eta\in\mathscr{G}_a}\mu_N(\eta)\sum_{\xi\in\mathscr{G}_b} R^{\mathscr{G}}_N(\eta,\xi)\;. \tag{3.7.5}$$

Lemma 3.7.1. *For every $a \neq b \in Q$,*

$$r_{\mathscr{G}}(a,b) \;:=\; \lim_{N\to\infty} \beta_N^{+}\, r_N^{\mathscr{G}}(\mathscr{G}_a, \mathscr{G}_b) \;=\; \frac{1}{\hat{\lambda}_{\mathscr{G}}} \sum_{\eta\in\mathscr{G}_a} m_a^*(\eta) \sum_{\xi\in\mathscr{G}_b} r_{\mathscr{G}}(\eta,\xi) \;\in\; \mathbb{R}_+$$

Moreover,

$$\sum_{a\in Q}\sum_{b:b\neq a} r_{\mathscr{G}}(a,b) \;=\; 1\ .$$

Proof. The first claim of this lemma follows from (3.7.2), (3.7.3) and (3.7.4). On the other hand, by the explicit formula for $r_{\mathscr{G}}(a,b)$ and by the formula for $\hat{\lambda}_{\mathscr{G}}(a)$ obtained in the previous assertion,

$$\sum_{a\in Q}\sum_{b:b\neq a} r_{\mathscr{G}}(a,b) \;=\; \frac{1}{\hat{\lambda}_{\mathscr{G}}} \sum_{a\in Q}\sum_{\eta\in\mathscr{G}_a} m_a^*(\eta) \sum_{b:b\neq a}\sum_{\xi\in\mathscr{G}_b} r_{\mathscr{G}}(\eta,\xi) \;=\; \frac{1}{\hat{\lambda}_{\mathscr{G}}} \sum_{a\in Q} \hat{\lambda}_{\mathscr{G}}(a)\ .$$

This expression is equal to 1 by definition of $\hat{\lambda}_{\mathscr{G}}$. ∎

4. Condition (H2) is fulfilled by the partition $\{\mathscr{G}_1, \dots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$. The proof of condition (H2) is based on the next assertion.

Assertion 3.7.D. *For every $a \in Q$ for which $\mathscr{G}_a$ is not a singleton and for all $\eta \neq \xi \in \mathscr{G}_a$,*

$$\liminf_{N\to\infty} \beta_N \frac{\mathrm{cap}_N(\eta,\xi)}{\mu_N(\mathscr{G}_a)} \;>\; 0\ .$$

Proof. Throughout this proof c_0 represents a positive real number independent of N and which may change from line to line. Fix $a \in Q$ for which $\mathscr{G}_a$ is not a singleton, and $\eta \neq \xi \in \mathscr{G}_a$. By definition, $\mathscr{G}_a = \cup_{x\in G_a} \mathscr{F}_x$. If η and ξ belongs to the same $\mathscr{F}_x$, the result follows from assumption (H2) and from Assertion 3.7.A.

Fix $\eta \in \mathscr{F}_x$ and $\xi \in \mathscr{F}_y$ for some $x \neq y$, $\mathscr{F}_x \cup \mathscr{F}_y \subset \mathscr{G}_a$. Recall that we denote by $\mathrm{cap}_N^s(\mathscr{A},\mathscr{B})$ the capacity between two disjoint subsets $\mathscr{A}$, $\mathscr{B}$ of E with respect to the reversible chain introduced in (3.3.2).

Since G_a is a recurrent class for the chain $X_{\mathscr{F}}(t)$, there exists a sequence ($x = x_0, x_1, \dots, x_n = y$) such that $r_{\mathscr{F}}(x_i, x_{i+1}) > 0$ for $0 \le i < n$. in view of assumptions (H0) and (H1), there exist $\xi_i \in \mathscr{F}_{x_i}$, $\eta_{i+1} \in \mathscr{F}_{x_{i+1}}$ such that $\beta_N R_N^{\mathscr{F}}(\xi_i, \eta_{i+1}) \ge c_0$. Therefore, by Corollary 3.3.4 and (3.3.4),

$$\beta_N\, \mathrm{cap}_N^s(\xi_i, \eta_{i+1}) \;\ge\; \frac{\beta_N}{2|E|}\, \mathrm{cap}_N(\xi_i, \eta_{i+1}) \;\ge\; c_0\, \mu_N(\xi_i)\ , \tag{3.7.6}$$

so that, by (3.4.4), $\beta_N\, \boldsymbol{c}_N^s(\xi_i, \eta_{i+1}) \ge c_0\, \mu_N(\xi_i)$.

Since the configuration η and ξ_0 belongs to the same set $\mathscr{F}_x$, by assumption (H2), $\beta_N^- \text{cap}_N(\eta, \xi_0)/\mu_N(\mathscr{F}_x) \ge c_0$. A similar assertion holds for the pair of configurations η_i, ξ_i, $1 \le i < n$, and for the pair η_n, ξ. Hence, if we set $\eta_0 = \eta$, $\xi_n = \xi$, by Corollary 3.3.4 and (3.4.4), we have that

$$\beta_N^- \boldsymbol{c}_N^s(\eta_i, \xi_i) \;\ge\; c_0 \mu_N(\mathscr{F}_{x_i}) \,.$$

By (3.7.2), we may replace $\mu_N(\mathscr{F}_{x_i})$ by $\mu_N(\mathscr{G}_a)$ in the previous inequality, and $\mu_N(\xi_i)$ by $\mu_N(\mathscr{G}_a)$ in (3.7.6). By (3.4.3),

$$\boldsymbol{c}_N^s(\eta, \xi) \;\ge\; \min_{0 \le i < n} \min\big\{\boldsymbol{c}_N^s(\eta_i, \xi_i), \boldsymbol{c}_N^s(\xi_i, \eta_{i+1}), \boldsymbol{c}_N^s(\eta_n, \xi_n)\big\} \,.$$

Since $\beta_N^- \ll \beta_N$, it follows from the previous estimates that $\beta_N \boldsymbol{c}_N^s(\eta, \xi) \ge c_0 \mu_N(\mathscr{G}_a)$. To complete the proof, it remains to recall that, by Corollary 3.3.4 and (3.4.4), $\text{cap}_N(\eta, \xi) \ge \text{cap}_N^s(\eta, \xi) \ge c_0 \boldsymbol{c}_N^s(\eta, \xi)$. ∎

5. Condition (H3) is fulfilled by the partition $\{\mathscr{G}_1, \dots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$. Lemma 3.7.2 shows that it is enough to prove condition (H3) for the trace process $\eta^{\mathscr{F}}(t)$.

Lemma 3.7.2. *Assume that*

$$\lim_{N\to\infty} \max_{\eta \in \mathcal{F}} \mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N^+} \in \Delta_*\}\, ds \Big] \;=\; 0 \,,$$

where $\Delta_* = \cup_{x \in G_{\mathfrak{q}+1}} \mathscr{F}_x$ *has been introduced in* (3.1.8)*. Then,*

$$\lim_{N\to\infty} \max_{\eta \in E} \mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta^{N}_{s\beta_N^+} \in \Delta_{\mathscr{G}}\}\, ds \Big] \;=\; 0 \,.$$

Proof. Fix $\eta \in E$. Since $\Delta_{\mathscr{G}} = \Delta_* \cup \Delta_{\mathscr{F}}$,

$$\begin{aligned} &\mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta_{s\beta_N^+} \in \Delta_{\mathscr{F}} \cup \Delta_*\}\, ds \Big] \\ &\qquad \le\; \mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta_{s\beta_N^+} \in \Delta_{\mathscr{F}}\}\, ds \Big] \;+\; \max_{\xi \in \mathscr{F}} \mathbb{E}_\xi \Big[\int_0^t \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N^+} \in \Delta_*\}\, ds \Big] \,. \end{aligned}$$

The second term vanishes as $N \uparrow \infty$ by assumption. The first one is bounded by

$$\frac{\beta_N}{\beta_N^+} \sum_{n=0}^{[\beta_N^+/\beta_N]} \mathbb{E}_\eta \Big[\int_{nt}^{(n+1)t} \mathbf{1}\{\eta_{s\beta_N} \in \Delta_{\mathscr{F}}\}\, ds \Big] \,,$$

where $[r]$ stands for the integer part of r. By the Markov property, this expression is bounded above by

$$2 \max_{\xi\in E} \mathbb{E}_\xi \Big[\int_0^t \mathbf{1}\{\eta_{s\beta_N} \in \Delta_{\mathscr{F}}\}\, ds \Big] \;,$$

which vanishes as $N \uparrow \infty$ by assumption (H3). ∎

To prove that condition (H3) is fulfilled by the partition $\{\mathscr{G}_1, \dots, \mathscr{G}_{\mathfrak{q}}, \Delta_{\mathscr{G}}\}$ it remains to show that the assumption of the previous lemma is in force. The proof of this claim relies on the next assertion. Denote by $\mathbb{P}^{\mathscr{F}}_\eta$ the probability measure on $D(\mathbb{R}_+, \mathscr{F})$ induced by the trace chain $\eta^{\mathscr{F}}_t$ starting from η.

Assertion 3.7.E. *For every $\eta \in \Delta_*$,*

$$\lim_{t\to\infty} \limsup_{N\to\infty} \mathbb{P}^{\mathscr{F}}_\eta \big[H_{\mathscr{G}} \ge t\beta_N \big] \;=\; 0 \;.$$

Proof. Fix $\eta \in \mathscr{F}_x \subset \Delta_*$. Since the partition $\mathscr{F}_1, \dots, \mathscr{F}_{\mathfrak{p}}$, $\Delta_{\mathscr{F}}$ satisfy the conditions (H1)–(H3), by Proposition 3.5.1, starting from η the process $X_N(t) = \psi_{\mathscr{F}}(\eta^{\mathscr{F}}_{t\beta_N})$ converges in the Skorohod topology to the Markov chain $X_{\mathscr{F}}(t)$ on $P = \{1, \dots, \mathfrak{p}\}$ which starts from x and which jumps from y to z at rate $r_{\mathscr{F}}(y,z)$. Therefore,

$$\limsup_{N\to\infty} \mathbb{P}^{\mathscr{F}}_\eta \big[H_{\mathscr{G}} \ge t\,\beta_N \big] \;\le\; \mathbf{P}_x \big[H_R \ge t \big] \;,$$

where $\mathbf{P}_x$ represents the distribution of the chain $X_{\mathscr{F}}(t)$ starting from x and $R = \cup_{1\le a\le \mathfrak{q}} G_a$. Since R corresponds to the set of recurrent points of the chain $X_{\mathscr{F}}(t)$, the previous expression vanishes as $t \uparrow \infty$. ∎

Lemma 3.7.3. *For all $t > 0$,*

$$\lim_{N\to\infty} \max_{\eta\in\mathscr{F}} \mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N^+} \in \Delta_*\}\, ds \Big] \;=\; 0 \;.$$

Proof. Since $\beta_N/\beta_N^+ \to 0$, a change of variables in the time integral, similar to the one performed in the proof of Lemma 3.7.2, and the Markov property show that for every $\eta \in \mathscr{F}$, every $T > 0$ and every N large enough,

$$\mathbb{E}_\eta \Big[\int_0^t \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N^+} \in \Delta_*\}\, ds \Big] \;\le\; \frac{2t}{T} \max_{\xi\in\mathscr{F}} \mathbb{E}_\xi \Big[\int_0^T \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N} \in \Delta_*\}\, ds \Big] \;.$$

Note that the process on the right hand side is speeded up by β_N instead of β_N^+.

We estimate the expression on the right hand side of the previous formula. We may, of course, restrict the maximum to Δ_*. Let T_1 be the first time the trace process $\eta^{\mathscr{F}}_t$ hits

$\mathscr{G}$ and let T_2 be the time it takes for the process to return to Δ_* after T_1:

$$T_1 \;=\; H_{\mathscr{G}}\;, \quad T_2 \;=\; \inf\big\{s>0 : \eta^{\mathscr{F}}_{T_1+s}\in\Delta_*\big\}\;.$$

Fix $\eta\in\Delta_*$ and note that

$$\begin{aligned}&\mathbb{E}_\eta\Big[\frac{1}{T}\int_0^T \mathbf{1}\{\eta^{\mathscr{F}}_{s\beta_N}\in\Delta_*\}\,ds\Big]\\ &\qquad \le\; \mathbb{P}^{\mathscr{F}}_\eta\big[T_1>t_0\beta_N\big] \;+\; \mathbb{P}^{\mathscr{F}}_\eta\big[T_2\le T\beta_N\big] \;+\; \frac{t_0}{T}\end{aligned}$$

for all $t_0>0$. By Assertion 3.7.E, the first term on the right hand side vanishes as $N\uparrow\infty$ and then $t_0\uparrow\infty$. On the other hand, by the strong Markov property, the second term is bounded by $\max_{\xi\in\mathscr{G}}\mathbb{P}^{\mathscr{F}}_\xi[H_{\Delta_*}\le T\beta_N]$. Since, by Proposition 3.5.1, the process $\psi_{\mathscr{F}}(\eta^{\mathscr{F}}_{t\beta_N})$ converges in the Skorohod topology to the Markov chain $X_{\mathscr{F}}(t)$,

$$\limsup_{N\to\infty}\max_{\xi\in\mathscr{G}}\mathbb{P}^{\mathscr{F}}_\xi[H_{\Delta_*}\le T\beta_N] \;\le\; \max_{1\le a\le \mathfrak{q}}\max_{x\in G_a}\mathbf{P}_x[H_{G_{\mathfrak{q}+1}}\le T]\;,$$

where, as in the proof of the previous assertion, $\mathbf{P}_x$ represents the distribution of the chain $X_{\mathscr{F}}(t)$ starting from x. Since the sets G_a are recurrent classes for the chain $X_{\mathscr{F}}(t)$, $r_{\mathscr{F}}(x,y)=0$ for all $x\in\cup_{1\le a\le\mathfrak{q}}G_a$, $y\in G_{\mathfrak{q}+1}$. Therefore, the previous probability is equal to 0 for all $T>0$, which completes the proof of the lemma. ∎

CHAPTER A SOFT TOPOLOGY

For any positive integer $m \geq 1$, let $S_m = \{1, \cdots, m\}$ and $S_\delta = \mathbb{N} \cup \delta$, where $\delta = \infty$. Endow the space S_δ with the metric d given by $d(k,j) = |k^{-1} - j^{-1}|$.

Definition A.0.1 (Soft left-limit)**.** *A measurable function $x : [0,T] \to S_\delta$ is said to have a soft left-limit at $t \in (0,T]$ if one of the following two alternatives holds:*

1. *The trajectory x has a left-limit at t, denoted by $x(t)$;*
2. *The set of cluster points of $x(s)$, $s \uparrow t$, is a pair formed by δ and a point in $\mathbb{N}$, denoted by $x(t\ominus)$.*

A soft right-limit at $t \in [0,T)$ is defined analogously. In this case, the right-limit, when it exists, is denoted by $x(t+)$, and the cluster point of the sequence $x(s), s \downarrow t$, which belongs to S when the second alternative is in force is denoted by $x(t\oplus)$.

Definition A.0.2 (Soft right-continuous)**.** *A trajectory $x : [0,T] \to S_\delta$ which has a soft right-limit at t is said to be soft right-continuous at t if one of the following three alternatives holds*

1. *$x(t+)$ exists and is equal to δ;*
2. *$x(t+)$ exists, belongs to S, and $x(t+) = x(t)$;*
3. *$x(t\oplus)$ exists and $x(t\oplus) = x(t)$.*

A trajectory $x : [0,T] \to S_\delta$ which is soft right-continuous at every point $t \in [0,T]$ is said to be soft right-continuous.

Definition A.0.3. *Let $\mathbb{E}([0,T], S_\delta)$ be the space of soft right-continuous trajectories $x : [0,T] \to S_\delta$ with soft left-limits.*

For a trajectory $x \in \mathbb{E}([0,T], S_\delta)$, let $\delta^x_\infty(t)$ be the time of the last visit to S:

$$\sigma^x_\infty(t) := \sup\{s \leq t : x(s) \in \mathbb{N}\},$$

with the convention that $\sigma^x_\infty(t) = 0$ if $x(s) = \delta$ for $0 \leq s \leq t$. When there is no ambiguity and it is clear to which trajectory we refer to, we denote $\sigma^x_\infty(t)$ by $\sigma_\infty(t)$.

Let $\mathcal{R}_\infty$ be the trajectory which records the last site visited in $\mathbb{N}$: $(\mathcal{R}_\infty x)(t) = 1$ if $x(s) = \delta$ for all $0 \leq s \leq t$, and

$$(\mathcal{R}_\infty x)(t) = \begin{cases} x(\sigma_\infty(t)) & \text{if } x(\sigma_\infty(t)) \in \mathbb{N}, \\ x(\sigma_\infty(t)-) & \text{if } x(\sigma_\infty(t)) \notin \mathbb{N} \text{ and if } x(\sigma_\infty(t)) \text{ exists} \\ x(\sigma_\infty(t)\ominus) & \text{otherwise.} \end{cases}$$

if there exists $0 \leq s \leq t$ such that $x(s) \in \mathbb{N}$. As for the operator $\mathcal{R}_m$, the convention that $(R_\infty x)(0) = 1$ if $x(0) = \delta$ corresponds in assuming that the trajectory is defined for $t < 0$ and that $x(t) = 1$ for $t < 0$.

Definition A.0.4. *Denote by $E([0,T], S_\delta)$ the set of trajectories in $\mathbb{E}([0,T], S_\delta)$ such that $x(0) \in \mathbb{N}$ and which fulfill the following condition: If $x(t) = \delta$ for some $t \in (0,T]$, then $\sigma_\infty(t) > 0$ and $x(\sigma_\infty(t)) = x(\sigma_\infty(t)-) = \infty$.*

Now we define a metric on the space $E([0,T], S_\delta)$. For two trajectories $x, y \in \mathbb{E}([0,T], S_\infty)$, let

$$\mathbf{d}(x,y) = \sum_{m \geq 1} \frac{1}{2^m} d_m(x,y), \text{where } d_m(x,y) = d_S(\mathcal{R}_m x, \mathcal{R}_m y).$$

In the last fomula, the metric $d_S(x,y)$ is the metric corresponding to the Skorohod topology. To make sense of the metric $d_S(x,y)$ in the definition,we state here the following result without giving a proof: For any trajectory x in $\mathbb{E}([0,T], S_\delta)$, and for each $m \geq 1$, $\mathcal{R}_m x$ is a trajectory in $D([0,T], S_m)$. It turns out that $\mathbf{d}(\cdot,\cdot)$ is not a metric on $\mathbb{E}([0,T], S_\infty)$. However it can be proved that $\mathbf{d}(\cdot,\cdot)$ is actually a metric on a smaller space $E([0,T], S_\delta)$. Moreover, The space $E([0,T], S_\delta)$ endowed with the metric $\mathbf{d}(x,y)$ is complete and separable. We call **soft topology** the topology in $E([0,T], S_\delta)$ induced by the metric $\mathbf{d}$.

The soft topology is a weaker topology compared to the Skorohod topology. In fact, fix $m \geq 1$ and consider a sequence x_n in $D([0,T], S_m)$ converging to x in the Skorohod topology. Then, x_n converges to x in $E([0,T], S_\infty)$. Another interesting result about the relation between the soft topology and Skorohod topology is as follows:

Theorem A.0.5. *A sequence of probability measures P_n on $E([0,T], S_\delta)$ converges weakly in the soft topology to a measure P if and only if for each $m \geq 1$ the sequence of probability measures $P_n \circ \mathcal{R}_m^{-1}$ defined on $D([0,T], S_m)$ converges weakly to $P_n \circ \mathcal{R}_m^{-1}$ with respect to the Skorohod topology.*

Bibliography

[1] C. Erignoux, C. Landim, T. Xu. Stationary states of boundary driven exclusion processes with nonreversible boundary dynamics, Journal of Statistical Physics, **171**, Issue 4, 599-631,2018

[2] C. Landim. ; T. Xu. Metastability of finite state Markov chains: a recursive procedure to identify slow variables for model reduction. ALEA-Latin American Journal of Probability and Mathematical Statistics, **13**, 725-751, 2016.

[3] C. Erignoux: Hydrodynamic limit of boundary exclusion processes with nonreversible boundary dynamics. Journal of Statistical Physics, **172**, Issue 5,1327-1357,2018

[4] G. Eyink, J. Lebowitz, H. Spohn: Hydrodynamics of stationary non-equilibrium states for some stochastic lattice gas models. Comm. Math. Phys. **132** ,253–283 ,1990.

[5] A. Friedman: *Stochastic differential equations and applications.* Volume 1, Academic Press, New York, 1975.

[6] C. Kipnis, C. Landim and S. Olla: Macroscopic properties of a stationary nonequilibrium distribution for a nongradient interacting particle system, Ann. Inst. H. Poincaré, Prob. et Stat. **31**, 191-221, 1995.

[7] C. Landim, S. Olla, S. Volchan: Driven tracer particle in one-dimensional symmetric simple exclusion. Comm. Math. Phys. **192**, 287–307, 1998.

[8] G. F. Lawler: *Intersections of Random Walks.* Modern Birkhäuser Classics, Birkhäuser Basel, 1991.

[9] J. Beltrán, C. Landim: Tunneling and metastability of continuous time Markov chains. J. Stat. Phys. **140**, 1065-1114, 2010.

[10] J. Beltrán, C. Landim: Metastability of reversible finite state Markov processes. Stoch. Proc. Appl. **121**, 1633-1677, 2011.

[11] J. Beltrán, C. Landim: Tunneling and metastability of continuous time Markov chains II. J. Stat. Phys. **149**, 598-618, 2012.

[12] J. Beltrán, C. Landim: A Martingale approach to metastability, Probab. Th. Rel. Fields. **161**, 267-307, 2015.

[13] D. Gabrielli, C. Valente: Which random walks are cyclic? ALEA, Lat. Am. J. Probab. Math. Stat. **9**, 231-267, 2012

[14] A. Gaudillière, C. Landim; A Dirichlet principle for non reversible Markov chains and some recurrence theorems. Probab. Theory Related Fields **158**, 55–89 (2014)

[15] T. Komorowski, C. Landim and S. Olla; *Fluctuations in Markov Processes, Time Symmetry and Martingale Approximation*, Grundlheren der mathematischen Wissenschaften **345**, Springer-Verlag, Berlin, New York, 2012.

[16] C. Landim; A topology for limits of Markov chains. Stoch. Proc. Appl. **125**, 1058-1098, 2014

[17] M. Cassandro, A. Galves, E. Olivieri, M. E. Vares. Metastable behavior of stochastic dynamics: A pathwise approach. J. Stat. Phys. **35**, 603-634, 1984

[18] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein. Metastability in stochastic dynamics of disordered mean field models. Probab. Theory Relat. Fields **119**, 99-161, 2001

[19] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein. Metastability and low lying spectra in reversible Markov chains. Commun. Math. Phys. **228**, 219-255, 2002.

[20] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. Comm. Math. Phys. , **118**, 31-59, 1988.

[21] J. Farfan, C. Landim, M. Mourragui. Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. Stochastic Process. Appl.121, 725-758 ,2011.

[22] M. I. Freidlin and A. D. Wentzell. Random perturbations of dynamical systems, volume 260 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, second edition (1998). ISBN 0-387-98362-7.