Genuine infinitesimal bendings of Euclidean submanifolds

Miguel Ibieta Jimenez

Advisor: Marcos Dajczer

Instituto de Matemática Pura e Aplicada-IMPA

April 2019

A mis padres y a mis hermanos....

Acknowledgements

First of all I would like to express my eternal gratitude to my parents and siblings. This work certainly would not have been possible without their love and constant support.

I am truly grateful to my advisor, Marcos Dajczer, for being always available to answer my questions and mainly for his patience during my time at IMPA. I also thank him for his advises and the amusing conversations that we held.

I want to thank Sergio Chion, Luis Florit, Felippe Guimarães, Walcy Santos and Sergio Silva, for accepting to be members of the jury, and also for reading and commenting the present work.

To all my friends at IMPA I give special thanks. Certainly my experience in Rio would not have been the same without them. I will not name all of them in order to avoid unintentional omissions.

In these last four years I was lucky to live with amazing people. First I mention Sheila, Julia and Wagner, with whom I lived most of my time in Rio, and finally Rosi and Marcelo. To all of you many thanks.

I thank the whole community from IMPA for offering me a good environment to develop this work.

Finally, I thank the CNPq for the financial support.

Abstract

A notion of bending of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is associated to smooth variations of f by immersions that are isometric up to the first order. More precisely, an infinitesimal bending of f is the variational vector field associated to such variation.

A very basic question in submanifold theory is whether a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with low codimension admits, locally or globally, a genuine infinitesimal bending. That is, if f admits an infinitesimal bending that is not determined by an infinitesimal bending of a submanifold of larger dimension that contains $f(M)$. We show that a strong necessary local condition to admit such a bending is the submanifold to be ruled and we give a lower bound to the dimension of the rulings. In the global case, we describe the situation for infinitesimal bendings of compact submanifolds with dimension at least five in codimension two.

In the codimension one case, a local description of the non-flat infinitesimally bendable Euclidean hyeprsurfaces was recently given by Dajczer and Vlachos. From their classification, it follows that this class is much larger than the class of isometrically bendable ones. In this work we also prove that a complete Euclidean hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, having no open subset where f is totally geodesic or a cylinder over an unbounded hypersurface of $\mathbb{R}^4$, is infinitesimally bendable only along ruled strips. In particular, if the hypersurface is simply connected, this implies that any infinitesimal bending of f is the variational vector field of an isometric bending, in contrast with the local case.

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Chapter 1

Introduction

Given an isometric immersion of a Riemannian manifold M^n into Euclidean space $\mathbb{R}^{n+p}$, a natural question is whether there exist, locally or globally, other isometric immersions, apart from compositions with isometries of the ambient space. If such immersions exist they are called isometric deformations of the given immersion. In the absence of isometric deformations the submanifold is said to be *rigid*.

An *isometric bending* of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is a smooth variation $\mathcal{F}\colon I \times M^n \to \mathbb{R}^{n+p}$ of f for an interval $0 \in I \subset \mathbb{R}$ such that $f_t = \mathcal{F}(t,\cdot)\colon M^n \to \mathbb{R}^{n+p}$ is an isometric immersion for any $t \in I$. In other words, the immersions f_t induce the same metric on M^n. An easy way to construct an isometric bending is to compose the immersion with a one-parameter family of isometries of $\mathbb{R}^{n+p}$. In this case we say that the isometric bending is *trivial*. If an isometric immersion admits a non-trivial isometric bending we say it is *isometrically bendable*.

The study of the isometric deformability of hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, goes back to the first part of the last century. In the local case, the classical Beez-Killing theorem [1, 19] states that a hypersurface having at least three nonzero principal curvatures at any point is rigid. On the other hand, hypersurfaces whose second fundamental form has rank at most one are flat, and hence locally highly deformable. In the case where the second fundamental form has rank two, the local classification of isometrically bendable hypersurfaces is due to Sbrana [23] in 1909 and Cartan [3] in 1916. A modern presentation of their parametric classifications, as well as further results, can be found in [7] or [12]. In the global case, Sacksteder [21] proved that any compact hypersurface in $\mathbb{R}^{n+1}$ is isometrically unbendable. Dajczer and Gromoll [8] extended Sacksteder's result by showing that a complete hypersurface in $\mathbb{R}^{n+1}$ is isometrically bendable only along ruled strips, provided that it does not contain an open subset that is a cylinder over an unbounded hypersurface in $\mathbb{R}^4$. The precise definitions of cylinder and ruled strip are given in Chapter 2.

When considering submanifolds in codimension larger that one, one has to take into account deformations that are induced by deformations of submanifolds of larger dimension. Dajczer and Florit [5] introduced the concept of *genuine rigidity* of submanifolds in order to deal with that kind of deformations. In the global case, Dajczer and Gromoll [9] proved that along connected components of an open dense subset an isometrically deformable compact Euclidean submanifold of dimension at least five and codimension two is either isometrically rigid or is contained in a deformable hypersurface with possible singularities and any isometric deformation of the former is given by an isometric deformation of the latter. This result was extended by Florit and Guimarães [17] to other low codimensions. The necessity to admit singularities was justified in [16]. Also in [17] Florit and Guimarães allowed singularities in the study of the genuine rigidity of submanifolds and some of their techniques are essential in the present work.

This thesis deals with the classical concept of infinitesimal bending of a submanifold. It can be seen as the infinitesimal analogue of an isometric bending and refers to smooth variations that preserve lengths “up to the first order". Let $\mathcal{F}\colon I\times M^n\to\mathbb{R}^{n+p}$ be an isometric bending of f, and let g_t denote the metric induced by f_t, where f_t is as above. In this case we have that the induced metrics satisfy $g_t'=0$. An *infinitesimal bending* of an isometric immersion $f\colon M^n\to\mathbb{R}^{n+p}$ is the variational vector field associated to a variation of $f=f_0$ by immersions f_t whose induced metrics satisfy $g_t'(0)=0$. Let $\mathfrak{X}(M)$ denote the set of tangent vector fields of M^n and $\Gamma(E)$ the sections of a bundle E over M^n. Observe that the variational vector field $\tau\in\Gamma(f^*T\mathbb{R}^{n+p})$ of a variation $\mathcal{F}$ as above satisfies

$$\langle f_*X,\tau_*X\rangle=0$$

for any $X\in\mathfrak{X}(M)$. On the other hand, given an isometric immersion $f\colon M^n\to\mathbb{R}^{n+p}$ and a vector field $\tau\in\Gamma(f^*T\mathbb{R}^{n+p})$ satisfying the previous equation, we have that the variation $\mathcal{F}(t,x)=f(x)+t\tau(x)$ satisfies

$$\|f_{t*}X\|^2=\|f_*X\|^2+t^2\|\tau_*X\|^2$$

for all $X\in\mathfrak{X}(M)$.

Clearly an isometric bending determines an infinitesimal bending. Hence, those infinitesimal bendings given by trivial isometric bendings are also said to be *trivial*. The problem of the existence of non-trivial infinitesimal bendings of an isometric immersion, gives rise to another notion of rigidity. Namely, an isometric immersion is said to be *infinitesimally rigid* if it only admits trivial infinitesimal bendings. Otherwise, we say that the submanifold is *infinitesimally bendable*. Infinitesimal bendings of surfaces in $\mathbb{R}^3$ are treated, for instance, in the books by Bianchi [2] and Eisenhart [14] published in 1903 and 1909 respectively. A modern account is presented by Spivak [25].

The "infinitesimal" analogue of the Beez-Killing theorem states that a hypersurface with at least three nonzero principal curvatures at every point is infinitesimally rigid, this result appears in the book by Cesàro [4] published in 1896. Dajczer and Rodríguez [11] proved that well known algebraic conditions on the second fundamental form of the immersion that give isometric rigidity also yield infinitesimal rigidity. In codimension larger than one these conditions are given in terms of the type number or the s-nullities of the immersion.

At the beginning of the 20^{th} century infinitesimal bendings of Euclidean hypersurfaces were considered by Sbrana [22] in 1908, followed by Schouten [24] in 1928. After the pioneering work of Sbrana, a complete parametric local classification of the non-flat infinitesimally bendable hypersurfaces was given by Dajczer and Vlachos [13]. In particular, they showed that this class is much larger than the class of isometrically bendable ones. In the global case, infinitesimal bendings of compact hypersurfaces were considered in [11], where it is shown that a compact hypersurface is infinitesimally rigid provided it does not contain open totally geodesic subsets. In this work we study infinitesimal bendings of complete hypersurfaces, which was a case still to be understood.

The following fact has to be taken in mind when trying to understand the geometry of the infinitesimally bendable submanifolds in codimension greater than one. An infinitesimal bending of an isometric immersion $F\colon \tilde{M}^{n+l} \to \mathbb{R}^{n+p}$, $0 < l < p$, induces an infinitesimal bending of $f = F|_M$ where $M^n \subset \tilde{M}^{n+l}$ is an embedded submanifold. Roughly speaking, we want to know what are the necessary conditions for a submanifold to admit an infinitesimal bending that is not a restriction as above. We point out that, as in [17], we allow singularities in our approach to genuine infinitesimal rigidity. We see that a strong necessary local condition for a submanifold to admit such a genuine infinitesimal bending is to be ruled, and we give a lower bound to the dimension of the rulings.

The outline of this thesis is as follows. In Chapter 2 we list some basic facts about isometric immersions. We pay special attention to the relative nullity distribution and its properties. We also state some results concerning the "splitting tensor" associated to the relative nullity, which has an important role in the last chapter. Some results concerning singular extensions of submanifolds are stated. We finish that chapter with results about flat bilinear forms, introduced by Moore, that have shown to be useful in the study of rigidity of isometric immersions. In the present work we show that they are also useful when dealing with infinitesimal bendings.

In Chapter 3 we give some basic facts about infinitesimal bendings. More specifically, we are interested in the properties of a tensor associated to an infinitesimal bending. In the case of surfaces in $\mathbb{R}^3$ this tensor corresponds to the associated rotation field; see [25].

Next in Chapter 4 we prove results of local nature. As stated above, we show that a genuinely infinitesimally bendable submanifold has to be ruled. For that we use convenient flat bilinear forms defined in terms of the associated tensor mentioned above. We also see that similar results hold when we change the ambient space for space forms of non-zero curvature.

The last chapter is devoted to results of global nature. First we deal with infinitesimal bendings of compact submanifolds with dimension at least five in codimension two. Then, inspired in the result by Dajczer and Gromoll [8], we describe the complete Euclidean hypersurfaces that admit non-trivial infinitesimal bendings. We show that a complete hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, that has no open subset where f is either totally geodesic or a cylinder over an unbounded hypersurface of $\mathbb{R}^4$, is infinitesimally bendable only along ruled strips. The main idea for the proofs of the previous results is to transport information along geodesics in the relative nullity. This technique has been widely used, for instance in [8], [9] and [17].

The results of this thesis are contained in [10] and [18].

Chapter 2

Preliminaries

We begin by giving the background material that is needed in the following chapters. All of this material is covered in full detail in [12], where the reader can find detailed proofs as well as other results on these subjects. We assume that the reader is familiar with the basic concepts of Riemannian geometry, some of which are listed below.

Let (M^n, g) be a Riemannian manifold and let ∇ be its Levi-Civita connection. Then, the curvature tensor of (M^n, g) is defined in terms of the Levi-Civita connection as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $\mathfrak{X}(M)$ denotes the set of smooth tangent vector fields of M^n. The Ricci tensor $\text{Ric} : TM \times TM \to C^\infty(M)$ is given by

$$\text{Ric }(X,Y) = tr(Z \to R(Z,X)Y),$$

and the Ricci curvature in the direction of the unit vector $X \in TM$ is defined by

$$\text{Ric }(X) = \frac{1}{n-1}\text{Ric }(X,X),$$

where $X, Y, Z \in \mathfrak{X}(M)$.

The Riemannian metric g determines a distance function on M^n. The distance between two points $x, y \in M$ is defined as the infimum of the arc lengths among the paths that join x and y. A Riemannian manifold is said to be complete if it is complete as a metric space. Equivalently, by the Hopf-Rinow theorem, M^n is complete if every geodesic $\gamma(t)$ is defined for any $t \in \mathbb{R}$. Clearly any compact Riemannian manifold is complete.

Given an immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 1$, of a Riemannian manifold (M^n, g) we say that f is an *isometric immersion* if

$$g(X,Y) = \langle f_*X, f_*Y \rangle$$

for any $X,Y \in \mathfrak{X}(M)$, where $\langle , \rangle$ is the Euclidean metric in $\mathbb{R}^{n+p}$. As stated in the introduction, given a vector bundle E over M^n, the set of smooth sections of E is denoted by $\Gamma(E)$. For an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, the vector bundle $f^*T\mathbb{R}^{n+p}$ splits orthogonally as

$$f^*T\mathbb{R}^{n+p} = f_*TM \oplus N_fM$$

where N_fM is called the *normal bundle* of f. Hence, according to that decomposition any section $Z \in \Gamma(f^*T\mathbb{R}^{n+p})$ decomposes as the sum of its tangent and normal components, $Z = Z_{TM} + Z_{N_fM}$. The *codimension* of f is the dimension of $N_fM(x)$, which in this case is denoted by "p".

The Euclidean connection, denoted by $\tilde{\nabla}$, induces a connection on $f^*T\mathbb{R}^{n+p}$ which we also write as $\tilde{\nabla}$. Let $X,Y \in \mathfrak{X}(M)$ be tangent vector fields, then the tangent component of $\tilde{\nabla}_X f_*Y$ coincides with $f_*\nabla_X Y$ and its normal component

$$\alpha(X,Y) = (\tilde{\nabla}_X f_*Y)_{N_fM},$$

is called the *second fundamental form* of f. It follows that the second fundamental form of f is a symmetric tensor with values in N_fM. Hence, we can decompose $\tilde{\nabla}_X f_*Y$ as

$$\tilde{\nabla}_X f_*Y = f_*\nabla_X Y + \alpha(X,Y),$$

which is known as the Gauss formula.

An isometric immersion is said to be totally geodesic if its second fundamental form vanishes. We recall that the totally geodesic submanifolds of $\mathbb{R}^{n+p}$ are open subsets of affine subspaces.

Given $\xi \in \Gamma(N_fM)$, the *shape operator* $A_\xi : TM \to TM$ with respect to ξ is defined by

$$\langle A_\xi X, Y \rangle = \langle \alpha(X,Y), \xi \rangle,$$

and it follows that A_ξ is a symmetric endomorphism of TM. Also induced by the Euclidean connection we have the *normal connection* in the normal bundle. It is given by $\nabla^\perp_X \xi = (\tilde{\nabla}_X \xi)_{N_fM}$, where $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$. Then, from the relation between the

Euclidean connection and the Levi-Civita connection of M we have the Weingarten formula:

$$\tilde{\nabla}_X \xi = -f_* A_\xi X + \nabla^\perp_X \xi.$$

Comparing the curvature tensors of $\mathbb{R}^{n+p}$ and M^n we obtain the fundamental equations of an isometric immersion. These are the Gauss, Codazzi and Ricci equations, exposed next in that order:

$$\begin{aligned}\langle R(X,Y)Z,W\rangle =&\langle \alpha(X,W),\alpha(Y,Z)\rangle - \langle \alpha(X,Z),\alpha(Y,W)\rangle,\\ (\nabla^\perp_X \alpha)(Y,Z) =&(\nabla^\perp_Y \alpha)(X,Z),\\ \langle R^\perp(X,Y)\xi,\eta\rangle =&\langle [A_\xi,A_\eta]X,Y\rangle,\end{aligned}$$

where $X,Y,Z,W \in \mathfrak{X}(M)$, $R^\perp$ denotes the normal curvature tensor and $\xi,\eta \in \Gamma(N_fM)$. The Codazzi equation can also be written in terms of the shape operators as follows:

$$(\nabla_X A_\xi) - A_{\nabla^\perp_X \xi} Y = (\nabla_Y A_\xi)X - A_{\nabla^\perp_Y \xi} X.$$

The first normal space of $f\colon M^n \to \mathbb{R}^{n+p}$ at $x \in M^n$ is

$$N_1(x) = \operatorname{span}\{\alpha(X,Y) : X,Y \in T_xM\}.$$

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is called 1-*regular* if the first normal spaces $N_1(x)$ have constant dimension $k \le p$ on M^n and thus form a subbundle N_1 of rank k of the normal bundle.

2.1 The relative nullity distribution

An important distribution associated to an isometric immersion is the relative nullity distribution. Given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, the *relative nullity* subspace $\Delta(x)$ of f at $x \in M^n$ is the kernel of the second fundamental form $\alpha\colon T_xM \times T_xM \to N_fM(x)$, that is,

$$\Delta(x) = \{X \in T_xM : \alpha(X,Y) = 0 \text{ for all } Y \in T_xM\}.$$

The dimension $\nu(x)$ of $\Delta(x)$ is called the *index of relative* nullity at $x \in M^n$. We have that the index of relative nullity is upper semicontinuous, in particular, the subset of M^n where ν attains its minimum value is open. Moreover, on open subsets where ν is constant Δ is a smooth distribution.

A distribution $D \subset TM$ is totally geodesic if for any $X, Y \in \Gamma(D)$ we have $\nabla_X Y \in \Gamma(D)$. It follows from the Codazzi equation that, on open subsets where ν is constant, Δ is totally geodesic and hence integrable. Thus, Δ determines a foliation whose leaves are mapped by f to open subsets of totally geodesic submanifolds of $\mathbb{R}^{n+p}$.

Let $D \subset TM$ be a totally geodesic distribution of M^n. Decompose the tangent bundle orthogonally by $TM = D \oplus D^\perp$, according to this decomposition we write

$$X = X_D + X_{D^\perp}$$

for any $X \in \mathfrak{X}(M)$.

The splitting tensor $C\colon \Gamma(D) \times \Gamma(D^\perp) \to \Gamma(D^\perp)$ of D is defined by

$$C(S,X) = C_S X = -(\nabla_X S)_{D^\perp}$$

where $S \in \Gamma(D)$ and $X \in \Gamma(D^\perp)$. Observe that if $T \in D(x)$ then it determines an endomorphim $C_T\colon D^\perp(x) \to D^\perp(x)$. Also notice that the distribution $D^\perp$ is integrable if and only if C_T is self-adjoint for every $T \in \Gamma(D)$. Moreover, $D^\perp$ is totally geodesic if and only if C vanishes.

Next we state some of the properties of the splitting tensor of Δ, in fact, we state them for a slightly more general case. Let D be a totally geodesic distribution such that $D(x) \subset \Delta(x)$ for any $x \in M^n$ and call C its splitting tensor. Since D is totally geodesic, then the Gauss equation gives

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S}.$$

for any $S, T \in \Gamma(D)$. In particular, if γ is a unit speed geodesic contained in a leaf of D then

$$\frac{D}{dt} C_{\gamma'} = C_{\gamma'}^2. \tag{2.1}$$

Let ∇^h denote the connection induced on $D^\perp$, then the Gauss equation also implies the following:

Lemma 1. *We have that*

$$(\nabla_X^h C_T)Y - (\nabla_Y^h C_T)X = C_{(\nabla_X T)_D} Y - C_{(\nabla_Y T)_D} X$$

for any $X, Y \in \Gamma(D^\perp)$ and $T \in \Gamma(D)$.

Proof. Since

$$(\nabla_X^h C_T)Y = \nabla_X^h C_T Y - C_T \nabla_X^h Y,$$

the proof follows from the Gauss equation and the fact that D is totally geodesic. □

The Codazzi equation gives that

$$\nabla_T A_\xi = A_\xi C_T + A_{\nabla^\perp_T \xi}$$

for any $T \in \Gamma(D)$ and $\xi \in \Gamma(N_f M)$. In particular the tensor $A_\xi C_T$ is symmetric and therefore

$$\nabla_T A_\xi = C'_T A_\xi + A_{\nabla^\perp_T \xi}, \tag{2.2}$$

where C'_T denotes the transpose of C_T.

The following proposition provides a way to transport information along geodesics contained in leaves of the relative nullity.

Proposition 2. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion such that $\nu > 0$ is constant on an open subset $U \subset M^n$. If $\gamma\colon [0,b] \to M^n$ is a unit speed geodesic such that $\gamma([0,b))$ is contained in a leaf of Δ in U, then $\Delta(\gamma(b)) = \mathcal{P}_0^b(\Delta(\gamma(0)))$ where $\mathcal{P}_0^t$ is the parallel transport along γ from $\gamma(0)$ to $\gamma(t)$. In particular, we have $\nu(\gamma(b)) = \nu(\gamma(0))$. Moreover, the tensor $C_{\gamma'}$ extends smoothly to $\gamma(b)$ and (2.2) holds on $[0,b]$.*

The proof of the previous proposition is similar to that of Proposition 33 in Chapter 5. For more details on these facts we refer to the first section of Chapter 7 in [12].

An isometric immersion $G\colon M^n \times \mathbb{R}^k \to \mathbb{R}^{n+p} \times \mathbb{R}^k$ of the Riemannian product $M^n \times \mathbb{R}^k$ is called a *k-cylinder* (cylinder) over the isometric immersion $g\colon M^n \to \mathbb{R}^{n+p}$, if it factors as

$$G = g \times Id\colon M^n \times \mathbb{R}^k \to \mathbb{R}^{n+p} \times \mathbb{R}^k$$

where $Id\colon \mathbb{R}^k \to \mathbb{R}^k$ is the identity. Observe that if G is as above, at any point $(x,y) \in M^n \times \mathbb{R}^k$, we have that $\{x\} \times \mathbb{R}^k$ lies in the leaf of the relative nullity foliation passing through (x,y).

For an isometric immersion f as above, if $D \subset \Delta$ is a smooth totally geodesic distribution, of rank $0 < d \leq \nu$, such that its splitting tensor C is identically zero, then we have that the manifold is locally a Riemannian product and the immersion is locally a piece of a k-cylinder (see for instance Proposition 7.4 in [12]).

When the leaves of the relative nullity of an isometric immersion are complete we have the following.

Lemma 3. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion. Assume that $U \subset M^n$ is an open subset where $\nu(x) = \nu_0$ is constant and the relative nullity leaves are complete. Then, for any $x_0 \in U$ and $T_0 \in \Delta(x_0)$ the only possible real eigenvalue of C_{T_0} is zero. Moreover, if $\gamma(s)$*

is a geodesic through x_0 *tangent to* T_0 *then*

$$C_{\gamma'(s)} = \mathcal{P}_0^s C_{T_0}(Id - sC_{T_0})^{-1}(\mathcal{P}_0^s)^{-1} \tag{2.3}$$

where $\mathcal{P}_0^s$ *is the parallel transport along* γ *from* x_0*. In particular,* $\ker C_{\gamma'}$ *is parallel along the geodesic* γ*.*

Proof. See Lemma 1.8 in [8] or Proposition 13.8 in [12]. □

The following result is useful when characterizing hypersurfaces of constant rank 2 whose relative nullity leaves are complete. Here the rank of a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is the dimension of the orthogonal complement of Δ, that is $\dim \Delta^\perp = n - \nu$.

Lemma 4. *Let* $f\colon M^n \to \mathbb{R}^{n+1}$ *be an isometric immersion. If* $U \subset M^n$ *is an open subset where* f *has constant rank* 2 *and the leaves of the relative nullity are complete, then the codimension of*

$$C_0 = \{T \in \Delta : C_T = 0\}$$

is at most one. Moreover, if $\dim C_0^\perp = 1$ *and* C_T *is invertible for* $T \in \Gamma(C_0^\perp)$*, then* $f|_U$ *is a cylinder over a hypersurface* $g\colon L^3 \to \mathbb{R}^4$ *that carries a one-dimensional relative nullity distribution with complete leaves.*

Proof. See Lemmas 1.9 and 1.10 in [8] or Corollaries 13.9 and 13.10 in [12]. □

2.2 Ruled hypersurfaces

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is said to be *r-ruled* if there exists an r-dimensional smooth totally geodesic tangent distribution whose leaves (rulings) are mapped diffeomorphically by f to open subsets of affine subspaces of $\mathbb{R}^{n+p}$. Notice that a special class of ruled submanifolds are the ones with a relative nullity foliation.

In the case of codimension $p = 1$, we say that a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is *ruled* if it is $(n-1)$-ruled. For a hypersurface with boundary we say it is ruled if, in addition to the previous condition, the rulings are tangent to the boundary. A connected component of the subset of M^n where the rulings are all complete is called a *ruled strip*.

It follows that a ruled hypersurface has rank at most two, and if it is not totally geodesic the leaves of relative nullity are contained in the rulings.

Next we give a local parametrization for a ruled hypersurface in terms of a curve orthogonal to the rulings.

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a ruled hypersurface and let $c\colon I \to M^n$ be a unit speed curve orthogonal to the rulings. The rulings form an affine vector bundle over $\tilde{c} = f \circ c$ in $\mathbb{R}^{n+1}$. Then let $T_i(s)$, $1 \le i \le n-1$, be orthonormal tangent fields on the corresponding bundle along c which are parallel with respect to the induced connection. Set $f_* c' = \tilde{T}_0$, $\tilde{T}_i = f_* T_i$ and let N be a unit vector field along c normal to f. We have

$$\begin{cases} \tilde{\nabla}_{\partial/\partial s}\tilde{T}_0 = -\sum_{i=1}^{n-1}\varphi_i\tilde{T}_i + \theta N \\ \tilde{\nabla}_{\partial/\partial s}\tilde{T}_i = \varphi_i\tilde{T}_0 + \beta_i N, \end{cases}$$

where $\theta = \langle AT_0, T_0\rangle$, $\varphi_i = \langle \nabla_{T_0} T_i, T_0\rangle$ and $\beta_i = \langle AT_i, T_0\rangle$.

We parametrize a neighborhood of $\tilde{c}$ in $f(M)$ by means of $\tilde{f}\colon W \subset I \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$ given by

$$\tilde{f}(s, u_1, \ldots, u_{n-1}) = \tilde{c}(s) + \sum_{i=1}^{n-1} u_i \tilde{T}_i(s). \tag{2.4}$$

We have at $(s, u_1, \ldots, u_{n-1})$ that

$$\tilde{f}_* \partial/\partial s = (1 + \sum_i u_i \varphi_i)\tilde{T}_0 + \sum_i u_i \beta_i N.$$

Therefore, the map $\tilde{f}$ has maximal rank if and only if

$$|\tilde{f}_* \partial/\partial s|^2 = (1 + \sum_i u_i \varphi_i)^2 + (\sum_i u_i \beta_i)^2 \neq 0.$$

Note that the directions for which $\sum_i u_i \beta_i = 0$ are in the relative nullity of f at $c(s)$.

The following result can be found in [8] for submanifolds of arbitrary codimension. But for the convenience of the reader we give a proof in the hypersurface case.

Lemma 5. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion and let $U \subset M^n$ be an open subset where f has constant rank 2. Assume that $f|_U$ is ruled and has complete relative nullity leaves. Suppose that $\delta\colon [0,a] \to M^n$ is a unit speed geodesic orthogonal to Δ such that $\delta([0,a)) \subset U$ is contained on a ruling. Then the rank of f at $\delta(a)$ is 2. Moreover, every point in U has a neighborhood V such that $f|_V$ extends to a ruled strip of constant rank 2.*

Proof. Let $W \subset I \times \mathbb{R}^{n-1}$ be an open subset where the parametrization (2.4) is defined and write $W_s = W \cap (\{s\} \times \mathbb{R}^{n-1})$. Assume that the geodesic δ is contained on the ruling determined by $\tilde{f}|_{W_s}$ and has T_{n-1} as its tangent vector field. Notice that $\tilde{f}(s, 0, \ldots, 0, r)$ is a parametrization of δ. Since $\beta_{n-1}(s) \neq 0$ the map $\tilde{f}$ has maximal rank along δ and at $\delta(r)$

we have

$$\tilde{f}_*(\partial/\partial s) = (1 + r\varphi_{n-1})\tilde{T}_0 + r\beta_{n-1}N.$$

Let $\tilde{N}(\delta(r)) = \alpha(r)\tilde{T}_0 + N$ be a vector field normal to f along δ (not necessarily unitary). Then

$$0 = \langle \tilde{f}_*(\partial/\partial s), \alpha\tilde{T}_0 + N\rangle = \alpha(1 + r\varphi_{n-1}) + r\beta_{n-1}.$$

Taking $r \in (0,a]$ we see that $1 + r\varphi_{n-1} \neq 0$, then

$$\alpha(r) = -\frac{r\beta_{n-1}}{(1 + r\varphi_{n-1})}.$$

Therefore, we have that

$$\begin{aligned}\langle \tilde{\nabla}_{\delta'(r)}\tilde{f}_*(\partial/\partial s), \tilde{N}(\delta(r))\rangle &= \langle \varphi_{n-1}\tilde{T}_0 + \beta_{n-1}N, \alpha(r)\tilde{T}_0 + N\rangle \\ &= \beta_{n-1}\left(\frac{1}{1 + r\varphi_{n-1}}\right)\end{aligned}$$

which does not vanish. Thus the rank of $\tilde{f}$ at $\delta(a)$ is 2 and hence the same holds for f.

It remains to prove that $f|_U$ extends locally to a ruled strip. Fix $x \in U$ and let $V \subset U$ be a neighborhood of x parametrized by (2.4). Extend $\tilde{f}$ to $I \times \mathbb{R}^{n-1}$ with the same expression. We claim that this extension defines a ruled strip of constant rank 2. We first prove that $\tilde{f}$ has no singular points. As seen previously, $\tilde{f}$ is singular at points where

$$(1 + \sum_i u_i\varphi_i)^2 + (\sum_i u_i\beta_i)^2 = 0.$$

Then, it suffices to prove that $\sum_i u_i\varphi_i = 0$ for any $T = \sum_i u_iT_i(s) \in \Delta(c(s))$.

Given $T \in \Delta(c(s))$, we have that

$$\sum_i u_i\varphi_i = \langle \nabla_{T_0}T, T_0\rangle = -\langle C_TT_0, T_0\rangle.$$

If the splitting tensor vanishes there is nothing to prove. Otherwise, if X is a unit vector field on V tangent to a ruling and orthogonal to the relative nullity, it follows from the completeness of the relative nullity leaves of f that $C_TX = 0$ for any $T \in \Gamma(\Delta)$. Finally, since the only real eigenvalue of C_T is zero by Lemma 3, then $\langle C_TT_0, T_0\rangle = 0$, and thus $\tilde{f}$ has no singular points.

It follows from Proposition 2 that the open subset where $\tilde{f}$ has rank two is a union of complete relative nullity leaves. From the previous discussion we have that the rank of $\tilde{f}$ along any ruling is two, and the claim follows. □

2.3 Extensions

When dealing with an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ in codimension greater that one, we are interested in the genuine aspects of the submanifold, by this we mean the aspects that are not inherited from another submanifold $\tilde{M}^{n+l} \subset \mathbb{R}^{n+p}$, $0 < l < p$, such that $f(M) \subset \tilde{M}^{n+l}$. By this reason we discuss next some results about extensions of submanifolds.

A smooth map $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, from a differentiable manifold into Euclidean space is said to be a *singular extension* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ if there exists an embedding $j\colon M^n \to \tilde{M}^{n+\ell}$, $0 < \ell < p$, such that F is an immersion along $\tilde{M}^{n+\ell} \setminus j(M)$ and $f = F \circ j$. Observe that we allow the map F to have singularities along $j(M)$. The necessity to admit the existence of such singularities in the above definition was already well established in the study of genuine isometric deformations in both the local and global situation in [9] and [17].

An easy way to build a singular extension of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, is to extend it along lines, more precisely we can define $F\colon I \times M^n \to \mathbb{R}^{n+p}$ by

$$F(t,x) = f(x) + t\lambda(x),$$

where $\lambda(x) \in \Gamma(f^*T\mathbb{R}^{n+p})$ is a suitable nowhere vanishing vector field. Notice that F is not an immersion at $t = 0$ whenever λ is tangent to f.

Next we recall a result due to Florit and Guimarães [17] that is a key ingredient in the proofs of the results presented in this work.

Proposition 6. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and let D be a smooth tangent distribution of dimension $d > 0$. Assume that there is no open subset $U \subset M^n$ and $Z \in \Gamma(D|_U)$ such that the map $F\colon U \times I \to \mathbb{R}^{n+p}$ given by*

$$F(x,t) = f(x) + tf_*Z(x)$$

is a singular extension of f on some open neighborhood of $U \times \{0\}$. Then f is d-ruled along every connected component of an open dense subset of M^n.

Proof. See Proposition 12.42 in [12] or Proposition 13 in [17]. □

Remark 7. Notice that the distribution D above is not assumed to be totally geodesic. From the proof of the result it follows that the rulings are determined as follows: At any $x \in M^n$ there is an open neighborhood U of the origin in $D(x)$ such that $f_*(x)U \subset f(M)$.

If $\lambda \in \Gamma(f^*T\mathbb{R}^{n+p})$ is nowhere tangent to f, then the map $F(t,x) = f(x) + t\lambda$ is always an extension. Next we sketch a way to find such a section λ such that the lines defined by the parameter t lie in the relative nullity of F.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion. Suppose that there is a smooth line normal subbundle $R \subset N_fM$, such that the tangent subspaces

$$D(x) = \mathcal{N}(\alpha_R)(x) = \{Y \in TM(x)\colon \langle \alpha(Y,X), \eta\rangle = 0 \;\; \forall X \in TM(x), \eta \in R(x)\}$$

have constant dimension $n-k$ and therefore form a smooth tangent subbubdle D. Assume further that R is parallel along D with respect to the normal connection (hence in $\mathbb{R}^{n+2}$).

Consider the orthogonal splittings

$$TM = D \oplus E, \;\; N_fM = P \oplus R$$

and define at each $x \in M^n$

$$\Gamma(x) = \text{span}\{(\tilde{\nabla}_X \eta)_{f_*E\oplus P}\colon X \in E(x), \eta \in R(x)\}.$$

It follows from our assumptions that Γ is a smooth rank k subbundle of $f_*E \oplus P$. Define Λ by the orthogonal decomposition $f_*E \oplus P = \Gamma \oplus \Lambda$ and let $\lambda \in \Gamma(\Lambda)$ be a nowhere vanishing section of Λ. Then we have the following:

Proposition 8. *The map $F(t,x) = f(x) + t\lambda(x)$, where $t \in (-\varepsilon, \varepsilon)$ for some fixed $\varepsilon > 0$, is a hypersurface whose second fundamental form has rank k. Moreover ∂_t lies in the relative nullity distribution of F.*

Proof. See Proposition 4 in [9] or Proposition 12.4 in [12] for a more general result. □

2.4 Flat bilinear forms

Flat bilinear forms were introduced by Moore [20] after the pioneering work of E. Cartan to deal with rigidity questions on isometric immersions in space forms. As this work shows, they are also useful when studying infinitesimal bendings of submanifolds.

Let V and U be finite dimensional real vector spaces and let $W^{p,q}$ be a real vector space of dimension $p+q$ endowed with an indefinite inner product of type (p,q). A bilinear form $\mathcal{B}\colon V \times U \to W^{p,q}$ is said to be *flat* if

$$\langle \mathcal{B}(X,Z), \mathcal{B}(Y,W)\rangle - \langle \mathcal{B}(X,W), \mathcal{B}(Y,Z)\rangle = 0$$

for all $X,Y \in V$ and $W,Z \in U$. An element $X \in V$ is called a (left) *regular element* of $\mathcal{B}$ if

$$\dim \mathcal{B}_X(U) = \max\{\dim \mathcal{B}_Y(U) \colon Y \in V\}$$

where $\mathcal{B}_X(Y) = \mathcal{B}(X,Y)$ for any $Y \in U$. The set $RE(\mathcal{B})$ of regular elements of $\mathcal{B}$ is open dense in V (see Proposition 4.4 in [12]).

The following basic fact was given in [20].

Lemma 9. *Let $\mathcal{B} \colon V \times U \to W$ be a flat bilinear form. If $Y \in RE(\mathcal{B})$ then*

$$\mathcal{B}(X, \ker \mathcal{B}_Y) \subset \mathcal{B}_Y(U) \cap \mathcal{B}_Y(U)^{\perp}$$

for any $X \in V$.

The next is a fundamental result in the theory of symmetric flat bilinear forms. It turns out to be false for $p \geq 6$ as shown in [6].

Lemma 10. *Let $\mathcal{B} \colon V^n \times V^n \to W^{p,q}$, $p \leq 5$ and $p+q < n$, be a symmetric flat bilinear form and set*

$$\mathcal{N}(\mathcal{B}) = \{X \in V : \mathcal{B}(X,Y) = 0 \text{ for all } Y \in V\}.$$

If $\dim \mathcal{N}(\mathcal{B}) \leq n-p-q-1$ then there is an orthogonal decomposition

$$W^{p,q} = W_1^{\ell,\ell} \oplus W_2^{p-\ell,q-\ell},\ 1 \leq \ell \leq p,$$

such that the W_j-components $\mathcal{B}_j$ of $\mathcal{B}$ satisfy:

(i) $\mathcal{B}_1$ *is nonzero and satisfies*

$$\langle \mathcal{B}_1(X,Y), \mathcal{B}_1(Z,W) \rangle = 0$$

for all $X,Y,Z,W \in V$.

(ii) $\mathcal{B}_2$ *is flat and* $\dim \mathcal{N}(\mathcal{B}_2) \geq n-p-q+2\ell$.

Proof. See Lemma 4.22 in [12] or [5]. □

Chapter 3

Infinitesimal bendings

Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion, an *isometric bending* of f is a one-parameter variation of f by isometric immersions, this is, a smooth map

$$F\colon (-\varepsilon,\varepsilon)\times M^n \to \mathbb{R}^{n+p}$$

such that $F(0,\cdot) = f(\cdot)$ and $F(t,\cdot)$ is an isometric immersion for each $t \in (-\varepsilon,\varepsilon)$. In other words, the metrics g_t induced by the immersions $f_t = F(t,\cdot)$ satisfy $g_t = g_0$ for any $t \in (-\varepsilon,\varepsilon)$.

A way to construct an isometric bending is to compose f with a smooth one-parameter family of rigid motions in $\mathbb{R}^{n+p}$, that is,

$$F(t,x) = C(t)f(x) + v(t)$$

where $C(t)$ is an orthogonal transformation of $\mathbb{R}^{n+p}$ and $v(t) \in \mathbb{R}^{n+p}$ for each $t \in (-\varepsilon,\varepsilon)$. An isometric bending of f given by the expression above is said to be *trivial*. If f admits a non-trivial isometric bending then it is called *isometrically bendable*. Otherwise, f is said to be *isometrically unbendable*.

Let F be an isometric bending of f and let $\tau(x) = F_*\partial/\partial t|_{t=0}(x) \in \Gamma(f^*T\mathbb{R}^{n+p})$ be the associated variational field. Then τ satisfies

$$\langle \tilde\nabla_X \tau, f_*Y\rangle + \langle \tilde\nabla_Y \tau, f_*X\rangle = 0 \tag{3.1}$$

for any tangent vector fields $X,Y \in \mathfrak{X}(M)$, where $\tilde\nabla$ denotes the Euclidean connection. The classical concept of an infinitesimal bending of a submanifold is the infinitesimal analogue of an isometric bending and refers to smooth variations that preserve lengths “up to the first order", that is, the metrics g_t induced by f_t satisfy $g_t'(0) = 0$. Hence, we say that a section

τ of $f^*T\mathbb{R}^{n+p}$ is an *infinitesimal bending* of $f\colon M^n \to \mathbb{R}^{n+p}$ if (3.1) holds. Given a smooth variation whose variation vector field τ is an infinitesimal bending, by keeping only the terms of first order of the variation we obtain the smooth variation $\mathcal{F}\colon \mathbb{R}\times M^n \to \mathbb{R}^{n+p}$ with variational vector field τ defined as $f_t = f + t\tau$. Then (3.1) gives

$$\|f_{t*}X\|^2 = \|f_*X\|^2 + t^2\|\tilde{\nabla}_X\tau\|^2$$

for any $X \in TM$.

Of course, we always have the *trivial infinitesimal bendings* obtained as variational vector fields of trivial isometric bendings. In other words, they are locally the restriction to the submanifold of a Killing vector field of $\mathbb{R}^{n+p}$. More precisely, a trivial infinitesimal bending τ has the form

$$\tau(x) = \mathcal{D}f(x) + w$$

where $\mathcal{D}$ is a skew-symmetric linear endomorphism of $\mathbb{R}^{n+p}$ and $w \in \mathbb{R}^{n+p}$.

An isometric immersion is said to be *infinitesimally rigid* if it only admits trivial infinitesimal bendings. Otherwise, we say that the submanifold is *infinitesimally bendable*. Next we state the infinitesimal analogue of the Beez-Killing theorem, for a more general result we refer to Theorem 2 in [11] or Theorem 14.4 in [12].

Proposition 11. *A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ whose second fundamental form has rank at least 3 at every point is infinitesimally rigid.*

Hence, for a hypersurface to infinitesimally bendable it is a necessary condition to have at most two nonzero principal curvatures at any point. After the work of Sbrana [22] in 1908 a complete parametric local classification of the non-flat infinitesimally bendable hypersurfaces was given by Dajczer and Vlachos [13]. In particular, they showed that this class is much larger than the class of isometrically bendable ones.

When trying to understand the geometry of the infinitesimally bendable submanifolds in codimension larger than one the following fact has to be taken into consideration. If $\tilde{\tau}$ is an infinitesimal bending of an isometric immersion $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, and $j\colon M^n \to \tilde{M}^{n+\ell}$ is an embedding, then $\tau = \tilde{\tau}|_{j(M)}$ is an infinitesimal bending of $f = F \circ j\colon M^n \to \mathbb{R}^{n+p}$. This basic observation motivates the following definition.

We say that an infinitesimal bending τ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ *extends in the singular sense* if there is a singular extension $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ of f and a smooth map $\tilde{\tau}\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ such that $\tilde{\tau}$ is an infinitesimal bending of $F_{\tilde{M}\setminus j(M)}$ and $\tau = \tilde{\tau}|_{j(M)}$.

An infinitesimal bending τ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, is called a *genuine infinitesimal bending* if τ does not extend in the singular sense when restricted to any open subset of M^n. If f admits such a bending we say that it is *genuinely infinitesimally bendable*. On the other hand, we say that $f\colon M^n \to \mathbb{R}^{n+p}$ is *genuinely infinitesimally rigid* if given any infinitesimal bending τ of f there is an open dense subset of M^n such that τ restricted to any connected component extends in the singular sense. As one expects, trivial infinitesimal bendings are never genuine (see example 16 (1) below). If $f(M) \subset \mathbb{R}^{n+\ell} \subset \mathbb{R}^{n+p}$, $\ell < p$, and $e \in \mathbb{R}^{n+p}$ is orthogonal to $\mathbb{R}^{n+\ell}$ then $\tau = \phi e$ for $\phi \in C^\infty(M)$ is another example of an infinitesimal bending that is not genuine.

3.1 The associated tensor

Let τ be an infinitesimal bending of a isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ and let $L \in \Gamma(\mathrm{Hom}(TM, f^*T\mathbb{R}^{n+p}))$ be the tensor defined by

$$LX = \tilde{\nabla}_X \tau,$$

where $\tilde{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^{n+p}$. Hence (3.1) can be written as

$$\langle LX, f_*Y\rangle + \langle LY, f_*X\rangle = 0 \tag{3.2}$$

for any $X,Y \in \mathfrak{X}(M)$.

Let $\mathcal{B}\colon TM \times TM \to f^*T\mathbb{R}^{n+p}$ the symmetric tensor defined as

$$\mathcal{B}(X,Y) = (\tilde{\nabla}_X L)Y$$

for any $X,Y \in \mathfrak{X}(M)$. Taking tangent and normal components we have

$$\mathcal{B}(X,Y) = f_*\mathcal{Y}(X,Y) + \beta(X,Y) \tag{3.3}$$

where $\mathcal{Y}\colon TM \times TM \to TM$ and $\beta\colon TM \times TM \to N_fM$ are also symmetric tensors.

Proposition 12. *The tensor L satisfies*

$$(\tilde{\nabla}^2 L)(X,Y)(Z) - (\tilde{\nabla}^2 L)(Y,X)(Z) = -LR(X,Y)Z \tag{3.4}$$

for any $X,Y,Z \in \mathfrak{X}(M)$.

Proof. Since

$$(\tilde{\nabla}^2 L)(X,Y)(Z) = \tilde{\nabla}_X(\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z, \tag{3.5}$$

the proof follows from the definition of the curvature tensor. □

Let $f_t\colon I\times M^n\to\mathbb{R}^{n+p}$ be a smooth variation of f having τ as variational vector field . Then the metrics g_t induced by f_t satisfy

$$\partial/\partial t|_{t=0}g_t(X,Y)=0 \tag{3.6}$$

for any $X,Y\in\mathfrak{X}(M)$. Hence, the Levi-Civita connections and curvature tensors of g_t verify

$$\partial/\partial t|_{t=0}\nabla^t_XY=0 \tag{3.7}$$

and

$$\partial/\partial t|_{t=0}g_t(R^t(X,Y)Z,W)=0 \tag{3.8}$$

for any $X,Y,Z,W\in\mathfrak{X}(M)$. Taking the derivative with respect to t at $t=0$ of the Gauss formula for f_t, namely, of

$$\tilde\nabla_X f_{t*}Y=f_{t*}\nabla^t_XY+\alpha^t(X,Y),$$

where every term is seen as a vector in $\mathbb{R}^{n+p}$, we obtain

$$\mathcal{B}(X,Y)=\partial/\partial t|_{t=0}\alpha^t(X,Y). \tag{3.9}$$

Proposition 13. *The tensor* $\mathcal{Y}\colon TM\times TM\to TM$ *satisfies*

$$\langle\alpha(X,Y),LZ\rangle+\langle\mathcal{Y}(X,Y),Z\rangle=0 \tag{3.10}$$

for any $X,Y,Z\in\mathfrak{X}(M)$.

Proof. Given $\eta(t)\in\Gamma(N_{f_t}M)$, let $\mathcal{Y}_\eta$ be the tangent vector field given by

$$f_*\mathcal{Y}_\eta=(\partial/\partial t|_{t=0}\eta(t))_{f_*TM}.$$

The derivative of $\langle f_{t*}Z,\eta(t)\rangle=0$ with respect to t at $t=0$ yields

$$\langle\eta,LZ\rangle+\langle\mathcal{Y}_\eta,Z\rangle=0$$

where $Z\in\mathfrak{X}(M)$ and $\eta=\eta(0)$. In particular,

$$\langle\alpha(X,Y),LZ\rangle+\langle\mathcal{Y}_{\alpha(X,Y)},Z\rangle=0$$

for any $X,Y,Z \in \mathfrak{X}(M)$. On the other hand, we obtain from (3.9) that

$$\mathcal{Y}_{\alpha(X,Y)} = \mathcal{Y}(X,Y)$$

for any $X,Y \in \mathfrak{X}(M)$. □

Proposition 14. *The tensor $\beta\colon TM \times TM \to N_fM$ satisfies*

$$\begin{aligned}\langle\beta(X,W),\alpha(Y,Z)\rangle + \langle\alpha(X,W),\beta(Y,Z)\rangle \\ = \langle\beta(X,Z),\alpha(Y,W)\rangle + \langle\alpha(X,Z),\beta(Y,W)\rangle\end{aligned} \tag{3.11}$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$.

Proof. Take the derivative with respect to t at $t=0$ of the Gauss equations for f^t, that is,

$$g_t(R^t(X,Y)Z,W) = g_t(\alpha^t(X,W),\alpha^t(Y,Z)) - g_t(\alpha^t(X,Z),\alpha^t(Y,W))$$

and use (3.6), (3.8) and (3.9). □

Lemma 15. *The tensor β satisfies*

$$\begin{aligned}(\nabla^{\perp}_X\beta)(Y,Z) - (\nabla^{\perp}_Y\beta)(X,Z) \\ = \alpha(Y,\mathcal{Y}(X,Z)) - \alpha(X,\mathcal{Y}(Y,Z)) - (LR(X,Y)Z)_{N_fM}\end{aligned} \tag{3.12}$$

for any $X,Y,Z \in \mathfrak{X}(M)$.

Proof. We have from the definition of β that

$$\nabla^{\perp}_X\beta(Y,Z) = (\tilde{\nabla}_X(\tilde{\nabla}_Y L)Z - \tilde{\nabla}_X\mathcal{Y}(Y,Z))_{N_fM}.$$

Moreover,

$$(\nabla^{\perp}_X\beta)(Y,Z) = (\tilde{\nabla}^2 L)(X,Y)(Z)_{N_fM} - \alpha(X,\mathcal{Y}(Y,Z))$$

and

$$(\nabla^{\perp}_Y\beta)(X,Z) = (\tilde{\nabla}^2 L)(Y,X)(Z)_{N_fM} - \alpha(Y,\mathcal{Y}(X,Z)),$$

and (3.12) follows from (3.4). □

We discuss next the simplest examples of infinitesimal bendings.

Examples 16. (1) Let τ be a trivial infinitesimal bending of $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, then

$$\tau = \mathcal{D}f(x) + w$$

where $\mathcal{D}$ is a skew-symmetric linear transformation of $\mathbb{R}^{n+p}$ and $w \in \mathbb{R}^{n+p}$. Take $\lambda \in \Gamma(f^*T\mathbb{R}^{n+p})$ such that $F\colon \tilde{M}^{n+1} = M^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+p}$, given by

$$F(x,t) = f(x) + t\lambda(x),$$

is a singular extension of f. Then τ extends in the singular sense since $\tilde{\tau}(x,t) = \tau + t\mathcal{D}\lambda$ is a (trivial) infinitesimal bending of F on the open subset where F is an immersion.

(2) Given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$. If $Z \in \mathfrak{X}(M)$ is a Killing field and $\delta \in \Gamma(N_1^\perp)$ is a smooth normal vector field, then $\tau = f_*Z + \delta$ is an infinitesimal bending of f.

3.1.1 The hypersurface case

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion, N a unitary vector field normal to f and $\mathcal{Y}_N$ be the tangent vector field such that $\langle LX, N\rangle + \langle \mathcal{Y}_N, X\rangle = 0$ for any $X \in \mathfrak{X}(M)$. Hence, we can write (3.3) as

$$(\tilde{\nabla}_X L)Y = \langle BX, Y\rangle N + \langle AX, Y\rangle f_*\mathcal{Y}_N, \tag{3.13}$$

where $B\colon TM \to TM$ is given by $\langle BX, Y\rangle = \langle \beta(X,Y), N\rangle$, and A is the second fundamental form of f. In this case (3.11) becomes

$$BX \wedge AY - BY \wedge AX = 0, \tag{3.14}$$

where $X, Y \in \mathfrak{X}(M)$ and $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$.

Proposition 17. *The tensor B is a Codazzi tensor, that is*

$$(\nabla_X B)Y = (\nabla_Y B)X, \tag{3.15}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. Since in this case the codimension is $p = 1$, we have from the Gauss equation that the right side term on (3.12) vanishes, and hence equation (3.15) follows. □

The following result is essential in the last part of the final chapter, it follows from Theorem 13 in [13] where existence and uniqueness of infinitesimal bendings is discussed in terms of the tensor B. However, for the convenience of the reader, we include a proof.

Proposition 18. *An infinitesimal bending τ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+1}$ is trivial if and only if $B \equiv 0$.*

Proof. If τ is a trivial infinitesimal bending, it follows from (3.13) that

$$\langle BX,Y\rangle = \langle(\tilde{\nabla}_X L)Y,N\rangle = \langle \mathcal{D}(\tilde{\nabla}_X f_*)Y,N\rangle = \langle AX,Y\rangle\langle \mathcal{D}N,N\rangle = 0$$

for all $X,Y \in \mathfrak{X}(M)$. Hence we have that B is identically 0.

Conversely, assume that τ is an infinitesimal bending of f for which B vanishes. For each point $x \in M$ define the skew-symmetric endomorphisms $\mathcal{D}(x)\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\mathcal{D}(x)f_*X = L(x)X$$

for $X \in T_xM$ and

$$\mathcal{D}(x)N(x) = f_*\mathcal{Y}_N(x).$$

Since B vanishes, it follows from (3.13) that

$$\begin{aligned}\tilde{\nabla}_X \mathcal{D}Y - \mathcal{D}\tilde{\nabla}_X Y =& (\tilde{\nabla}_X L)Y - \langle AX,Y\rangle f_*\mathcal{Y}_N \\ =& 0.\end{aligned}$$

Taking the derivative of $\langle f_*\mathcal{Y}_N,N\rangle = 0$ and $\langle LY,N\rangle + \langle \mathcal{Y}_N,Y\rangle = 0$ in the direction of $X \in \mathfrak{X}(M)$, we obtain that

$$\tilde{\nabla}_X f_*\mathcal{Y}_N + f_*BX + LAX = 0$$

for any $X \in \mathfrak{X}(M)$. Hence, we have that

$$\begin{aligned}\tilde{\nabla}_X \mathcal{D}N - \mathcal{D}\tilde{\nabla}_X N =& \tilde{\nabla}_X f_*\mathcal{Y}_N + \mathcal{D}f_*AX \\ =& 0.\end{aligned}$$

Therefore, the family of endomorphisms $\mathcal{D}(x)$ is in fact constant $\mathcal{D}(x) = \mathcal{D}$. That and the definition of L imply that $\tau(x) - \mathcal{D}f(x)$ is a constant map and thus τ is trivial. □

For more details on infinitesimal bendings of hypersurfaces and the properties of the tensor B we refer to [13].

Chapter 4

Local results

In this chapter, it is shown that a strong necessary local condition for a submanifold to be genuinely infinitesimally bendable is that the submanifold has to be ruled. Our first local result is the following:

Theorem 19. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be a genuinely infinitesimally bendable isometric immersion. Then f is r-ruled with $r \geq n-2p$ along connected components of an open dense subset of M^n.*

The following is an immediate consequence of the above result.

Corollary 20. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be an isometric immersion. If M^n has positive Ricci curvature then f is genuinely infinitesimally rigid.*

In the case of low codimension and with a substantial additional effort, we obtain our second local result:

Theorem 21. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p$, be a genuinely infinitesimally bendable isometric immersion. If $2 \leq p \leq 5$, then one of the following holds along any connected component, say U, of an open dense subset of M^n:*

(i) $f|_U$ is ν-ruled by leaves of relative nullity with $\nu \geq n-2p$.

(ii) $f|_U$ has $\nu < n-2p$ at any point and is r-ruled with $r \geq n-2p+3$.

In the above result if $p=2$ we are always in case (i) since a $(n-1)$-ruled submanifold in that codimension has $\nu \geq n-3$ at any point.

4.1 The first local result

Theorem 19 is a corollary of the following result.

Theorem 22. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p$, be an isometric immersion and let τ be an infinitesimal bending of f. Then along each connected component of an open and dense subset either τ extends in the singular sense or f is r-ruled with $r \geq n-2p$.*

We first show that associated to any infinitesimal bending there is a flat bilinear form.

Lemma 23. *Let τ be an infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$. Then the bilinear form $\theta\colon TM \times TM \to N_fM \oplus N_fM$ defined at any point of M^n by*

$$\theta(X,Y) = (\alpha(X,Y) + \beta(X,Y), \alpha(X,Y) - \beta(X,Y)) \tag{4.1}$$

is flat with respect to the inner product in $N_fM \oplus N_fM$ given by

$$\langle\langle(\xi_1,\eta_1),(\xi_2,\eta_2)\rangle\rangle_{N_fM\oplus N_fM} = \langle\xi_1,\xi_2\rangle_{N_fM} - \langle\eta_1,\eta_2\rangle_{N_fM}.$$

Proof. A straightforward computation shows that

$$\begin{aligned}\frac{1}{2}(\langle\langle\theta(X,Z),\theta(Y,W)\rangle\rangle - \langle\langle\theta(X,W),\theta(Y,Z)\rangle\rangle) &= \langle\beta(X,Z),\alpha(Y,W)\rangle \\ + \langle\alpha(X,Z),\beta(Y,W)\rangle - \langle\beta(X,W),\alpha(Y,Z)\rangle &- \langle\alpha(X,W),\beta(Y,Z)\rangle,\end{aligned}$$

and the proof follows from (3.11). □

Under the assumption of regularity of the first normal bundle we have the following statement equivalent to Lemma 23.

Lemma 24. *Assume that f is 1-regular and let $\beta_1\colon TM \times TM \to N_1$ be the N_1-component of β. Then the bilinear form $\hat{\theta}\colon TM \times TM \to N_1 \oplus N_1$ defined at any point by*

$$\hat{\theta}(X,Y) = (\alpha(X,Y) + \beta_1(X,Y), \alpha(X,Y) - \beta_1(X,Y)) \tag{4.2}$$

is flat with respect to the inner product induced on $N_1 \oplus N_1$.

Proof of Theorem 22: Let τ be an infinitesimal bending of f. With the use of (3.2) and (3.10) we easily obtain

$$\langle f_*X + \tilde{\nabla}_X Y, LX + \tilde{\nabla}_X LY\rangle = \langle\alpha(X,Y),\beta(X,Y)\rangle \tag{4.3}$$

for any $X,Y \in \mathfrak{X}(M)$.

By Lemma 23, we have at any point of M^n that the symmetric tensor θ is flat. Given $Y \in RE(\theta)$, denote $D = \ker \theta_Y$ where $\theta_Y(X) = \theta(Y,X)$. Notice that $Z \in D$ just means that $\alpha(Y,Z) = 0 = \beta(Y,Z)$.

Let $U \subset M^n$ be an open subset where $Y \in \mathfrak{X}(U)$ satisfies $Y \in RE(\theta)$ and D has dimension d at any point. Lemma 9 gives

$$\langle\langle \theta(X,Z), \theta(X,Z) \rangle\rangle = 0$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$. Equivalently, the right hand side of (4.3) vanishes and thus

$$\langle f_*X + \tilde{\nabla}_X Z, LX + \tilde{\nabla}_X LZ \rangle = 0 \tag{4.4}$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$.

Assume that there is a nowhere vanishing $Z \in \Gamma(D)$ defined on an open subset V of U such that $F\colon V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+p}$ given by

$$F(x,t) = f(x) + t f_*Z(x)$$

is a singular extension of $f|_V$. Then the map $\tilde{\tau}\colon V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+p}$ given by

$$\tilde{\tau}(x,t) = \tau(x) + tLZ(x)$$

is an infinitesimal bending that extends $\tau|_V$ in the singular sense. In fact, we have

$$\langle F_*\partial_t, \tilde{\nabla}_{\partial_t} \tilde{\tau} \rangle = \langle f_*Z, LZ \rangle = 0,$$

$$\langle \tilde{\nabla}_{\partial_t} \tilde{\tau}, F_*X \rangle + \langle \tilde{\nabla}_X \tilde{\tau}, F_*\partial_t \rangle = \langle LZ, f_*X + t\tilde{\nabla}_X Z \rangle + \langle LX + t\tilde{\nabla}_X LZ, f_*Z \rangle = 0$$

and

$$\langle F_*X, \tilde{\nabla}_X \tilde{\tau} \rangle = \langle f_*X + t\tilde{\nabla}_X Z, LX + t\tilde{\nabla}_X LZ \rangle = 0$$

where the last equality follows from (4.4).

Let $W \subset U$ be an open subset such that a $Z \in \Gamma(D)$ as above does not exist along any open subset of W. By Proposition 6 the immersion is d-ruled along any connected component of an open dense subset of W. Moreover, we have $d = \dim D = n - \dim Im(\theta_Y) \geq n - 2p$. $\square$

Remark 25. In Theorem 19 if f is 1-regular with $\dim N_1 = q < p$ we obtain the better estimate $r \geq n - 2q$ since the proof still works using Lemma 24 instead of Lemma 23.

4.2 The second local result

Let $F\colon \tilde{M}^{n+1} \to \mathbb{R}^{n+p}$ be an isometric immersion and let $\tilde{\tau}$ be an infinitesimal bending of F. Given an isometric embedding $j\colon M^n \to \tilde{M}^{n+1}$ consider the composition of isometric immersions $f = F \circ j\colon M^n \to \mathbb{R}^{n+p}$. Then $\tau = \tilde{\tau}|_{j(M)}$ is an infinitesimal bending of f. It is easy to see that

$$\mathcal{B}(X,Y) = \tilde{\mathcal{B}}(X,Y) + \langle \tilde{\nabla}_X Y, F_*\eta\rangle \tilde{L}\eta$$

for $\eta \in \Gamma(N_jM)$ of unit length and $X,Y \in \mathfrak{X}(M)$. Then (3.10) gives

$$\langle \beta(X,Y), F_*\eta\rangle + \langle \alpha^f(X,Y), \tilde{L}\eta\rangle = 0$$

for any $X,Y \in \mathfrak{X}(M)$. We will see that satisfying a condition of this type may guarantee that an infinitesimal bending is not genuine. In fact, this was already proved by Florit [15] in a special case.

We say that an infinitesimal bending of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, satisfies the *condition* $(*)$ if there is $\eta \in \Gamma(N_fM)$ nowhere vanishing and $\xi \in \Gamma(R)$, where R is determined by the orthogonal splitting $N_fM = P \oplus R$ and $P = \operatorname{span}\{\eta\}$, such that

$$B_\eta + A_\xi = 0 \tag{4.5}$$

where $B_\eta = \langle \beta, \eta\rangle$. We choose η of unit length for simplicity. Thus, that (4.5) holds means

$$\langle \beta(X,Y), \eta\rangle + \langle \alpha(X,Y), \xi\rangle = 0 \tag{4.6}$$

for any $X,Y \in \mathfrak{X}(M)$.

The following result is of independent interest since it does not require the codimension to satisfy $p \leq 5$ as is the case in Theorem 21.

Theorem 26. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let τ be a genuine infinitesimal bending of f that satisfies the condition $(*)$. Then f is r-ruled with $r \geq n-2p+3$ on connected components of an open dense subset of M^n.*

We extend L to the tensor $\bar{L} \in \Gamma(\operatorname{Hom}(TM \oplus P, f^*T\mathbb{R}^{n+p})$ defining

$$\bar{L}\eta = f_*Y + \xi$$

where $Y \in \mathfrak{X}(M)$ is given by

$$\langle f_*Y, f_*X\rangle + \langle LX, \eta\rangle = 0$$

for any $X \in \mathfrak{X}(M)$. Then $\bar{L}$ satisfies

$$\langle \bar{L}X, \eta\rangle + \langle f_*X, \bar{L}\eta\rangle = 0$$

for any $X \in \mathfrak{X}(M)$.

Given $\lambda \in \Gamma(f_*TU \oplus P)$ nowhere vanishing where U is an open subset of M^n, we define the map $F\colon U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+p}$ by

$$F(x,t) = f(x) + t\lambda(x). \tag{4.7}$$

Notice that F is not an immersion at least for $t = 0$ at points where λ is tangent to U. Let $\tilde{\tau}\colon U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+p}$ be the map given by

$$\tilde{\tau}(x,t) = \tau(x) + t\bar{L}\lambda(x). \tag{4.8}$$

We have

$$\langle F_*\partial_t, \tilde{\nabla}_{\partial_t}\tilde{\tau}\rangle = 0.$$

Moreover, since $\langle \bar{L}\lambda, \lambda\rangle = 0$ we obtain

$$\langle \tilde{\nabla}_{\partial_t}\tilde{\tau}, F_*X\rangle + \langle \tilde{\nabla}_X\tilde{\tau}, F_*\partial_t\rangle = \langle \bar{L}\lambda, f_*X\rangle + \langle LX, \lambda\rangle + tX\langle \bar{L}\lambda, \lambda\rangle = 0 \tag{4.9}$$

for any $X \in \mathfrak{X}(M)$ and $t \in (-\varepsilon, \varepsilon)$. Thus $\tilde{\tau}$ is an infinitesimal bending of F on the open subset $\tilde{U}$ of $U \times (-\varepsilon, \varepsilon)$ where F is an immersion if and only if

$$\langle F_*X, \tilde{\nabla}_X\tilde{\tau}\rangle = 0,$$

or equivalently, if

$$\langle f_*X + t\tilde{\nabla}_X\lambda, LX + t\tilde{\nabla}_X\bar{L}\lambda\rangle = 0$$

for any $X \in \mathfrak{X}(M)$.

In the sequel we consider F restricted to $\tilde{U}$. By the above, in order to have that $\tilde{\tau}$ is an infinitesimal bending of F the strategy is to make use of the condition $(*)$ to construct a subbundle $D \subset f_*TM \oplus P$ such that

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = 0$$

for any $X \in \mathfrak{X}(M)$ and any $\lambda \in \Gamma(D)$.

Lemma 27. *Assume that the condition* $(*)$ *holds. Then*

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = \langle(\tilde{\nabla}_X\lambda)_R, (\tilde{\nabla}_X\bar{L})\lambda\rangle, \tag{4.10}$$

where $X \in \mathfrak{X}(M)$, $\lambda \in \Gamma(f_*TM \oplus P)$ *and*

$$(\tilde{\nabla}_X\bar{L})\lambda = \tilde{\nabla}_X\bar{L}\lambda - \bar{L}\nabla'_X\lambda,$$

being ∇' *the connection induced on* $f_*TM \oplus P$.

Proof. Set $\lambda = f_*Z + \phi\eta$ where $Z \in \mathfrak{X}(M)$ and $\phi \in C^\infty(M)$. Then

$$\begin{aligned}
&\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = \langle f_*(\tilde{\nabla}_X\lambda)_{TM} + (\tilde{\nabla}_X\lambda)_P + (\tilde{\nabla}_X\lambda)_R, \tilde{\nabla}_X\bar{L}\lambda\rangle \\
&\quad + \langle\tilde{\nabla}_X\lambda, LX\rangle + \langle f_*X, \tilde{\nabla}_X\bar{L}\lambda\rangle \\
&= \langle f_*(\tilde{\nabla}_X\lambda)_{TM}, (\tilde{\nabla}_XL)Z + L\nabla_XZ + X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle \\
&\quad + (\langle A_\eta X, Z\rangle + X(\phi))\langle\eta, (\tilde{\nabla}_XL)Z + L\nabla_XZ + X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle \\
&\quad + \langle(\tilde{\nabla}_X\lambda)_R, \tilde{\nabla}_X\bar{L}\lambda\rangle + \langle\tilde{\nabla}_X\lambda, LX\rangle + \langle f_*X, \tilde{\nabla}_X\bar{L}\lambda\rangle
\end{aligned} \tag{4.11}$$

for any $X \in \mathfrak{X}(M)$. Using (3.10) and (4.6) we obtain

$$\begin{aligned}
&\langle f_*(\tilde{\nabla}_X\lambda)_{TM}, (\tilde{\nabla}_XL)Z + L\nabla_XZ\rangle \\
&\qquad = -\langle L(\tilde{\nabla}_X\lambda)_{TM}, \alpha(X,Z)\rangle - \phi\langle f_*A_\eta X, L\nabla_XZ\rangle
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&\langle f_*(\tilde{\nabla}_X\lambda)_{TM}, X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle = \phi\langle(\tilde{\nabla}_X\lambda)_{TM}, \nabla_XY\rangle \\
&\qquad - X(\phi)\langle L(\tilde{\nabla}_X\lambda)_{TM}, \eta\rangle - \phi\langle\alpha(X, (\tilde{\nabla}_X\lambda)_{TM}), \xi\rangle
\end{aligned} \tag{4.13}$$

where for the first term in the right hand side of (4.13) we have

$$\begin{aligned}
\langle(\tilde{\nabla}_X\lambda)_{TM}, \nabla_XY\rangle &= X\langle(\tilde{\nabla}_X\lambda)_{TM}, Y\rangle - \langle\nabla_X(\tilde{\nabla}_X\lambda)_{TM}, Y\rangle \\
&= -X\langle L(\tilde{\nabla}_X\lambda)_{TM}), \eta\rangle + \langle L\nabla_X(\tilde{\nabla}_X\lambda)_{TM}, \eta\rangle \\
&= -\langle(\tilde{\nabla}_XL)(\tilde{\nabla}_X\lambda)_{TM}, \eta\rangle - \langle L(\tilde{\nabla}_X\lambda)_{TM}, \tilde{\nabla}_X\eta\rangle \\
&= \langle\alpha(X, (\tilde{\nabla}_X\lambda)_{TM}), \xi\rangle - \langle L(\tilde{\nabla}_X\lambda)_{TM}, \tilde{\nabla}_X\eta\rangle.
\end{aligned} \tag{4.14}$$

Moreover,

$$\langle \eta, (\tilde{\nabla}_X L)Z + L\nabla_X Z\rangle = -\langle \alpha(X,Z),\xi\rangle + \langle \eta, L\nabla_X Z\rangle, \tag{4.15}$$

$$\begin{aligned}\langle \eta, X(\phi)\bar{L}\eta + \phi\tilde{\nabla}_X\bar{L}\eta\rangle &= -\phi\langle\tilde{\nabla}_X\eta,\bar{L}\eta\rangle\\ &= -\phi\langle LA_\eta X,\eta\rangle - \phi\langle\nabla_X^\perp\eta,\xi\rangle\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&\langle\tilde{\nabla}_X\lambda, LX\rangle + \langle f_*X,\tilde{\nabla}_X\bar{L}\lambda\rangle = -\langle\tilde{\nabla}_X X,\bar{L}\lambda\rangle - \langle\lambda,\tilde{\nabla}_X LX\rangle\\ &= -\langle f_*\nabla_X X,\bar{L}\lambda\rangle - \langle\alpha(X,X),\bar{L}\lambda\rangle - \langle\lambda, L\nabla_X X\rangle - \langle\lambda,(\tilde{\nabla}_X L)X\rangle = 0.\end{aligned} \tag{4.17}$$

Now, a straightforward computation replacing (4.12) through (4.17) in (4.11) yields

$$\begin{aligned}&\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda\rangle = \langle(\tilde{\nabla}_X\lambda)_R,\tilde{\nabla}_X\bar{L}\lambda\rangle - \langle L(\tilde{\nabla}_X\lambda)_{TM},\alpha(X,Z)_R\rangle\\ &-\phi\langle L(\tilde{\nabla}_X\lambda)_{TM},\nabla_X^\perp\eta\rangle - \langle\alpha(X,Z),\bar{L}(\tilde{\nabla}_X\lambda)_P\rangle - \phi\langle\nabla_X^\perp\eta,\bar{L}(\tilde{\nabla}_X\lambda)_P\rangle\\ &= \langle(\tilde{\nabla}_X\lambda)_R,(\tilde{\nabla}_X\bar{L})\lambda\rangle,\end{aligned}$$

which concludes the proof. □

In view of (4.10) the next step is to construct a subbundle $D\subset f_*TM\oplus P$ such that

$$\langle(\tilde{\nabla}_X\lambda)_R,(\tilde{\nabla}_X\bar{L})\lambda\rangle = 0 \tag{4.18}$$

for any $X\in\mathfrak{X}(M)$ and $\lambda\in\Gamma(D)$.

Lemma 28. *Assume that the condition* $(*)$ *holds. Then the bilinear form* $\varphi\colon TM\times f_*TM\oplus P\to R\oplus R$ *defined by*

$$\varphi(X,\lambda) = ((\tilde{\nabla}_X\lambda)_R + ((\tilde{\nabla}_X\bar{L})\lambda)_R, (\tilde{\nabla}_X\lambda)_R - ((\tilde{\nabla}_X\bar{L})\lambda)_R).$$

is flat with respect to the indefinite inner product given by

$$\langle\langle(\xi_1,\mu_1),(\xi_2,\mu_2)\rangle\rangle_{R\oplus R} = \langle\xi_1,\xi_2\rangle_R - \langle\mu_1,\mu_2\rangle_R.$$

Proof. We need to show that

$$\Theta = \langle\langle\varphi(X,\lambda),\varphi(Y,\delta)\rangle\rangle - \langle\langle\varphi(X,\delta),\varphi(Y,\lambda)\rangle\rangle = 0$$

for any $X,Y\in\mathfrak{X}(M)$ and $\lambda,\delta\in f_*TM\oplus P$. We have

$$\begin{aligned}\frac{1}{2}\Theta =&\langle(\tilde\nabla_X\lambda)_R,((\tilde\nabla_Y\bar L)\delta)_R\rangle+\langle(\tilde\nabla_Y\delta)_R,((\tilde\nabla_X\bar L)\lambda)_R\rangle\\ &-\langle(\tilde\nabla_X\delta)_R,((\tilde\nabla_Y\bar L)\lambda)_R\rangle-\langle(\tilde\nabla_Y\lambda)_R,((\tilde\nabla_X\bar L)\delta)_R\rangle.\end{aligned}$$

Clearly $\Theta=0$ if $\lambda,\delta\in\Gamma(P)$. If $\lambda,\delta\in\mathfrak{X}(M)$, we have

$$\begin{aligned}\frac{1}{2}\Theta =&\langle\alpha(X,\lambda)_R,((\tilde\nabla_Y\bar L)\delta)_R\rangle+\langle\alpha(Y,\delta)_R,((\tilde\nabla_X\bar L)\lambda)_R\rangle\\ &-\langle\alpha(X,\delta)_R,((\tilde\nabla_Y\bar L)\lambda)_R\rangle-\langle\alpha(Y,\lambda)_R,((\tilde\nabla_X\bar L)\delta)_R\rangle\\ =&\langle\alpha(X,\lambda)_R,((\tilde\nabla_Y L)\delta)_R\rangle-\langle A_\eta Y,\delta\rangle\langle\alpha(X,\lambda)_R,\bar L\eta\rangle\\ &+\langle\alpha(Y,\delta)_R,((\tilde\nabla_X L)\lambda)_R\rangle-\langle A_\eta X,\lambda\rangle\langle\alpha(Y,\delta)_R,\bar L\eta\rangle\\ &-\langle\alpha(X,\delta)_R,((\tilde\nabla_Y L)\lambda)_R\rangle+\langle A_\eta Y,\lambda\rangle\langle\alpha(X,\delta)_R,\bar L\eta\rangle\\ &-\langle\alpha(Y,\lambda)_R,((\tilde\nabla_X L)\delta)_R\rangle+\langle A_\eta X,\delta\rangle\langle\alpha(Y,\lambda)_R,\bar L\eta\rangle.\end{aligned}$$

Using first (4.6) and then (3.11) we obtain

$$\begin{aligned}\frac{1}{2}\Theta =&\langle\alpha(X,\lambda),\beta(Y,\delta)\rangle+\langle\alpha(Y,\delta),\beta(X,\lambda)\rangle\\ &-\langle\alpha(X,\delta),\beta(Y,\lambda)\rangle-\langle\alpha(Y,\lambda),\beta(X,\delta)\rangle=0.\end{aligned}$$

Finally, we consider the case $\lambda=\eta$ and $\delta=Z\in\mathfrak{X}(M)$. Then

$$\begin{aligned}\frac{1}{2}\Theta =&\langle\nabla^\perp_X\eta,((\tilde\nabla_Y L)Z)_R\rangle-\langle A_\eta Y,Z\rangle\langle\nabla^\perp_X\eta,\bar L\eta\rangle+\langle\alpha(Y,Z)_R,((\tilde\nabla_X\bar L)\eta)_R\rangle\\ &-\langle\nabla^\perp_Y\eta,((\tilde\nabla_X L)Z)_R\rangle+\langle A_\eta X,Z\rangle\langle\nabla^\perp_Y\eta,\bar L\eta\rangle-\langle\alpha(X,Z)_R,((\tilde\nabla_Y\bar L)\eta)_R\rangle.\end{aligned}$$

Since

$$\begin{aligned}\langle\nabla^\perp_X\eta,\bar L\eta\rangle =&\langle\tilde\nabla_X\eta,\bar L\eta\rangle+\langle A_\eta X,\bar L\eta\rangle=-\langle\eta,\tilde\nabla_X\bar L\eta\rangle-\langle LA_\eta X,\eta\rangle\\ =&-\langle\eta,(\tilde\nabla_X\bar L)\eta\rangle\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{2}\Theta =&\langle\nabla^\perp_X\eta,(\tilde\nabla_Y L)Z\rangle-\langle\nabla^\perp_Y\eta,(\tilde\nabla_X L)Z\rangle\\ &+\langle\alpha(Y,Z),(\tilde\nabla_X\bar L)\eta\rangle-\langle\alpha(X,Z),(\tilde\nabla_Y\bar L)\eta\rangle.\end{aligned}$$

For the first term using (3.10) and (4.6) we obtain

$$\begin{aligned}
\langle\nabla^{\perp}_{X}\eta,(\tilde\nabla_Y L)Z\rangle =& X\langle\eta,(\tilde\nabla_Y L)Z\rangle-\langle\eta,\tilde\nabla_X(\tilde\nabla_Y L)Z\rangle+\langle f_*A_\eta X,(\tilde\nabla_Y L)Z\rangle\\
=& -X\langle\alpha(Y,Z),\bar L\eta\rangle-\langle\alpha(Y,Z),LA_\eta X\rangle\\
&-\langle\eta,(\tilde\nabla^2 L)(X,Y)(Z)+(\tilde\nabla_{\nabla_X Y}L)Z+(\tilde\nabla_Y L)\nabla_X Z\rangle\\
=& -\langle(\nabla^{\perp}_X\alpha)(Y,Z)+\alpha(\nabla_X Y,Z)+\alpha(Y,\nabla_X Z),\bar L\eta\rangle\\
&-\langle\eta,(\tilde\nabla^2 L)(X,Y)(Z)+(\tilde\nabla_{\nabla_X Y}L)Z+(\tilde\nabla_Y L)\nabla_X Z\rangle\\
&-\langle\alpha(Y,Z),\tilde\nabla_X\bar L\eta\rangle-\langle\alpha(Y,Z),LA_\eta X\rangle+\langle A_{\alpha(Y,Z)}X,\bar L\eta\rangle\\
=& -\langle(\nabla^{\perp}_X\alpha)(Y,Z),\bar L\eta\rangle-\langle\eta,(\tilde\nabla^2 L)(X,Y)(Z)\rangle\\
&-\langle\alpha(Y,Z),(\tilde\nabla_X\bar L)\eta\rangle-\langle LA_{\alpha(Y,Z)}X,\eta\rangle.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
\langle\nabla^{\perp}_{Y}\eta,(\tilde\nabla_X L)Z\rangle =& -\langle(\nabla^{\perp}_Y\alpha)(X,Z),\bar L\eta\rangle-\langle\eta,(\tilde\nabla^2 L)(Y,X)(Z)\rangle\\
&-\langle\alpha(X,Z),(\tilde\nabla_Y\bar L)\eta\rangle-\langle LA_{\alpha(X,Z)}Y,\eta\rangle.
\end{aligned}$$

From (3.4) and the Codazzi equation

$$(\nabla^{\perp}_X\alpha)(Y,Z)=(\nabla^{\perp}_Y\alpha)(X,Z)$$

we obtain

$$\frac{1}{2}\Theta=\langle L(R(X,Y)Z-A_{\alpha(Y,Z)}X+A_{\alpha(X,Z)}Y),\eta\rangle.$$

Then $\Theta=0$ from the Gauss equation, as we wished. □

Proof of Theorem 26: By Lemma 28 the bilinear form φ is flat. Let $U\subset M^n$ be an open subset where there is $Y\in\mathfrak{X}(U)$ such that $Y\in RE(\varphi)$ and $D=\ker\varphi_Y$ has dimension d at any point. Then Lemma 9 gives

$$\langle\langle\varphi(X,\lambda),\varphi(X,\lambda)\rangle\rangle=0$$

for any $X\in\mathfrak{X}(U)$ and $\lambda\in\Gamma(D)$. Notice that this implies that (4.18) holds for any $\lambda\in\Gamma(D)$. Then $D\subset f_*TM$ since otherwise there is $\lambda\in\Gamma(D)$ but $\lambda\notin TM$ such that $\tau|_U$ extends in the singular sense via (4.7) and (4.8), and this is a contradiction. Hence D is a tangent distribution and we conclude from Proposition 6 that $f|_U$ is d-ruled on connected components of an open dense subset of M^n. Moreover, the dimension of the rulings is bounded from below by $n+1-\dim Im(\varphi_Y)\geq n-2p+3$. □

Proof of Theorem 21: We work on the open dense subset of M^n where f is 1-regular on any connected component. Consider an open subset of a connected component where the index of relative nullity is $\nu \leq n-2p-1$ at any point. Lemma 10 applies and thus the flat bilinear form $\hat{\theta}$ in (4.2) decomposes at any point as $\hat{\theta} = \theta_1 + \theta_2$ where θ_1 is as in part (i) of that result. Hence, on any open subset where the dimension of $\mathcal{S}(\theta_1) = \mathcal{S}(\hat{\theta}) \cap \mathcal{S}(\hat{\theta})^\perp$ is constant there are smooth local unit vector fields $\zeta_1, \zeta_2 \in N_1$ such that $(\zeta_1, \zeta_2) \in \mathcal{S}(\theta_1)$. Equivalently,

$$\langle \beta(X,Y), \zeta_1 + \zeta_2 \rangle + \langle \alpha(X,Y), \zeta_1 - \zeta_2 \rangle = 0 \tag{4.19}$$

for any $X,Y \in \mathfrak{X}(M)$. We have $\zeta_1 + \zeta_2 \neq 0$ since otherwise $\zeta_1 - \zeta_2 \in N_1^\perp$. Hence τ satisfies the condition $(*)$ and the proof follows from Theorem 26. □

4.3 Nonflat ambient spaces

Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion where $\mathbb{Q}_c^{n+p}$ denotes either the sphere $\mathbb{S}_c^{n+p}$ or the hyperbolic space $\mathbb{H}_c^{n+p}$ of sectional curvature $c \neq 0$. Then we say that $\tau \in \Gamma(f^*T\mathbb{Q}_c^{n+p})$ is an infinitesimal bending of f if it satisfies (3.1) with respect to the connection in $\mathbb{Q}_c^{n+p}$. And that f is *r-ruled* means that there is an r-dimensional smooth totally geodesic distribution whose leaves are mapped by f to open subsets of totally geodesic submanifolds of $\mathbb{Q}_c^{n+p}$.

In the sequel, we also denote by f the composition of the immersion with the umbilical inclusion of $\mathbb{Q}_c^{n+p}$ into $\mathbb{O}^{n+p+1}$, where $\mathbb{O}^{n+p+1}$ stands for either Euclidean or Lorentzian space, depending on whether $c > 0$ or $c < 0$, respectively.

Let τ be an infinitesimal bending of f and let $f_t\colon M^n \to \mathbb{Q}_c^{n+p}$, $t \in I$, be a smooth variation of $f_0 = f$ having τ as variational vector field. In this case we still have that (3.6), (3.7) and (3.8) hold. And also as before, associated to τ we have the tensors

$$LX = \tilde{\nabla}_X \tau.$$

and

$$\mathcal{B}(X,Y) = (\tilde{\nabla}_X L)Y$$

for any $X,Y \in \mathfrak{X}(M)$, where $\tilde{\nabla}$ denotes the connection in $\mathbb{Q}_c^{n+p}$. Now

$$\mathcal{B}(X,Y) = \mathcal{Y}(X,Y) + \beta(X,Y) + c\langle f_*Y, \tau\rangle f_*X - c\langle X,Y\rangle \tau$$

where the tensors $\mathcal{Y}\colon TM\times TM\to f_*TM$ and $\beta\colon TM\times TM\to N_fM$ are the tangent and normal component of $\partial/\partial t|_{t=0}\alpha^t$ respectively, and α^t denotes the second fundamental form of f_t as a submanifold in $\mathbb{Q}_c^{n+p}$. In particular, we have that (3.11) holds.

In this case, an infinitesimal bending of f is said to satisfy the *condition* $(*)$ if there is $\eta\in\Gamma(N_fM)$ of unit length and $\xi\in\Gamma(R)$, where R is determined by the orthogonal splitting $N_fM=P\oplus R$ and $P=\operatorname{span}\{\eta\}$, such that

$$B_\eta+A_\xi+c\langle\tau,\eta\rangle I=0$$

where $B_\eta=\langle\beta,\eta\rangle$.

The cone over an isometric immersion $f\colon M^n\to\mathbb{Q}_c^{n+p}$ is defined by

$$\begin{aligned}\hat f\colon \hat M^{n+1}=(0,\infty)\times M^n &\to \mathbb{O}^{n+p+1}\\ (s,x)&\mapsto sf(x).\end{aligned}$$

Notice that ∂_s lies in the relative nullity of $\hat f$ and that $N_{\hat f}\hat M$ is the parallel transport of N_fM along the lines parametrized by s. Observe that if $c<0$, then the cone over f is a Lorentzian submanifold of $\mathbb{L}^{n+p+1}$ and hence $N_{\hat f}\hat M$ has positive definite metric.

If τ is an infinitesimal bending of f, it is easy to see that $\hat\tau(s,x)=s\tau(x)$ is an infinitesimal bending of $\hat f$ in $\mathbb{O}^{n+p+1}$, that is, $\hat\tau$ is a vector field that satisfies (3.1) with respect to the connection in $\mathbb{O}^{n+p+1}$. Moreover, if τ satisfies the condition $(*)$ then $\hat\tau$ satisfies the condition $(*)$ for a flat ambient space.

Let $\hat f$ be the cone over an immersion f in $\mathbb{Q}_c^{n+p}$. Notice that the parameter s defines lines parallel to the position vector. Thus, if the map $\hat f+t\lambda$, is a singular extension of $\hat f$ for some vector field λ then the intersection of its image with $\mathbb{Q}_c^{n+p}$ determines a singular extension of f.

Assume that τ is a genuine infinitesimal bending of f. We claim that $\hat f$ and $\hat\tau$ cannot admit extensions of the form

$$\hat F(t,s,x)=\hat f(s,x)+t\lambda(s,x)\ \text{ and }\ \hat\tau'(t,s,x)=\hat\tau(s,x)+t\bar L\lambda(s,x)$$

as in the proofs of Theorems 19 and 26. Notice that

$$\begin{aligned}\langle\hat F(t,s,x),\hat\tau'(t,s,x)\rangle&=\langle\hat f(s,x)+t\lambda(s,x),\hat\tau(s,x)+t\bar L\lambda(s,x)\rangle\\&=st\langle f(x),\bar L\lambda\rangle+st\langle\lambda,\tau\rangle\\&=0,\end{aligned}$$

where in the last step we used that $\hat{L}\partial_s = \tau(x)$, that is, that $\hat{\tau}'$ is orthogonal to the position vector. From that observation we have that if $\hat{F}$ determines a singular extension of f then τ extends in the singular sense. But that was ruled out by our assumption on τ and proves the claim. In this sense, if τ is genuine we obtain that $\hat{f}$ has to be ruled as in Theorems 19 and 26. Finally, observe that being $\hat{f}$ the cone over f, then these rulings determine rulings of f. Therefore we have the following:

Fact 29. *Theorems 19, 21 and 26 hold if the ambient space is replaced by* $\mathbb{Q}_c^{n+p}$.

Chapter 5

Global results

This chapter is devoted to results of global nature. In order to prove those results, we first show some properties of the kernel Δ^* of the tensor θ. As we see next, on open subsets where Δ^* has constant dimension it determines a totally geodesic distribution contained in the relative nullity. Our strategy is to transport information along geodesics contained in leaves of that distribution. This allows us to describe the situation for infinitesimal bendings of compact submanifolds in codimension 2, more precisely we prove the following result:

Theorem 30. *Let $f\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 5$, be an isometric immersion of a compact Riemannian manifold with no open flat subset and let τ be an infinitesimal bending of f. Then one of the following holds along any connected component, say U, of an open dense subset of M^n:*

(i) The infinitesimal bending $\tau|_U$ extends in the singular sense.

(ii) There is an orthogonal splitting $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus span\{e\}$ such that $f(U) \subset \mathbb{R}^{n+1}$ and $\tau|_U$ is a sum of infinitesimal bendings $\tau|_U = \tau_1 + \tau_2$ where $\tau_1 \in \mathbb{R}^{n+1}$, $\tau_2 = \phi e$ for $\phi \in C^\infty(U)$ and both extend in the singular sense.

Inspired by the classification of the isometrically bendable complete Euclidean hypersurfaces by Dajczer and Gromoll in [8], we finish this chapter proving the following result:

Theorem 31. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be an isometric immersion of a complete Riemannian manifold. Assume that there is no open subset of M^n where f is either totally geodesic or a cylinder over a hypersurface in $\mathbb{R}^4$ with complete one-dimensional leaves of relative nullity. Then f admits non-trivial infinitesimal bendings only along ruled strips.*

We point out that the existence of complete nonruled isometrically bendable hypersurfaces of constant rank two in $\mathbb{R}^4$, that are not surface-like, is an open problem [8]. By surface-like we mean a hypersurface that is a cylinder over either a surface in $\mathbb{R}^3$ or a cone over a surface in an umbilical submanifold of $\mathbb{R}^4$.

It follows from the proof of Theorem 30 that the assumption on the open flat subset can be replaced by the weaker hypothesis that there is no open subset of M^n where the index of relative nullity satisfies $\nu \geq n-1$. That and the hypothesis of the absence of open totally geodesic subsets in Theorem 31 can be justified by the fact any section $\delta \in \Gamma(N_1^\perp)$ is an infinitesimal bending of f, as was observed in Examples 16 (2). Moreover, we will see that cases (i) and (ii) in Theorem 30 are not disjoint.

The following two results, of independent interest, are essential in the proofs of the theorems mentioned above. In the sequel τ is an infinitesimal bending of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$.

Let β be the symmetric tensor defined by (3.3) and let θ be the flat bilinear form given by (4.1). Calling $\nu^*(x) = \dim \Delta^*(x)$ at $x \in M^n$ where

$$\Delta^*(x) = \mathcal{N}(\theta)(x) = \Delta \cap \mathcal{N}(\beta)(x),$$

we have the following:

Proposition 32. *On any open subset of M^n where ν^* is constant the distribution Δ^* is totally geodesic and its leaves are mapped by f onto open subsets of affine subspaces of $\mathbb{R}^{n+p}$.*

Proof. From (3.10) we have $\Delta \subset \mathcal{N}(\mathcal{Y})$. Then (3.12) and the Gauss equation give

$$(\nabla_X^\perp \beta)(Z,Y) = (\nabla_Z^\perp \beta)(X,Y) = 0$$

for any $X,Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Let $\nabla^* = (\nabla^\perp, \nabla^\perp)$ be the compatible connection in $N_fM \oplus N_fM$. Hence

$$0 = (\nabla_X^* \theta)(Z,Y) = \theta(Z, \nabla_X Y)$$

for any $X,Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Thus $\Delta^* \subset \Delta$ is totally geodesic. $\square$

On an open subset of M^n where $\nu^* > 0$ is constant consider the orthogonal splitting $TM = \Delta^* \oplus E$ and let $C\colon \Gamma(\Delta^*) \times \Gamma(E) \to \Gamma(E)$ be the corresponding splitting tensor.

The next result provides a way to transport information along geodesics contained in leaves of Δ^*. This technique has been widely used, for instance in [8], [9] and [17].

Proposition 33. *Let $\nu^* > 0$ be constant on an open subset $U \subset M^n$. If $\gamma\colon [0,b] \to M^n$ is a unit speed geodesic such that $\gamma([0,b))$ is contained in a leaf of Δ^* in U, then $\Delta^*(\gamma(b)) = \mathcal{P}_0^b(\Delta^*(\gamma(0)))$ where $\mathcal{P}_0^t$ is the parallel transport along γ from $\gamma(0)$ to $\gamma(t)$. In particular, we have $\nu^*(\gamma(b)) = \nu^*(\gamma(0))$ and the tensor $C_{\gamma'}$ extends smoothly to $[0,b]$.*

Proof. We mimic the proof of Lemma 27 in [17]. Let the tensor $J\colon E \to E$ be the solution in $[0,b)$ of
$$\frac{D}{dt}J + C_{\gamma'} \circ J = 0$$
with initial condition $J(0) = Id$. It follows from (2.1) that J satisfies $\frac{D^2}{dt}J = 0$, and hence it extends smoothly to $\mathcal{P}_0^b(E(0))$ in $\gamma(b)$. Let Y and Z be parallel vector fields along γ such that $Y(t) \in E(t)$ for each $t \in [0,b)$. Since γ' is in Δ^*, it follows from (3.12) that

$$(\nabla^*_{\gamma'}\theta)(JY,Z) = (\nabla^*_{JY}\theta)(\gamma',Z).$$

This and the definition of J imply that $\theta(JY,Z)$ is parallel along γ. In particular J is invertible in $[0,b]$. By continuity $\mathcal{P}_0^b(\Delta^*(\gamma(0))) \subset \Delta^*(\gamma(b))$, and since $Z(0)$ is arbitrary, then $\mathcal{P}_0^b(\Delta^*(\gamma(0))) = \Delta^*(\gamma(b))$. Finally we extend the splitting tensor $C_{\gamma'} = -DJ/dt \circ J^{-1}$. □

The following basic result is used in the proofs of both theorems in this chapter. In the following $M_n(\mathbb{R})$ denotes the set of $n \times n$ real matrices.

Lemma 34. *Let $U\colon [0,b] \to M_n(\mathbb{R})$ be a solution of the ordinary differential equation*

$$U'(s) = T(s)U(s),$$

where $T\colon [0,b] \to M_n(\mathbb{R})$ is continuous. Then the rank of $U(s)$ is constant on $[0,b]$.

Proof. Let $v \in \mathbb{R}^n$ and define $v(s)$ as $v(s) = U(s)v$ for $s \in [0,b]$. Observe that $v(s)$ satisfies the differential equation:
$$v'(s) = U'(s)v = T(s)v(s)$$
on $[0,b]$. Therefore, if $v(s_0) = 0$ for some $s_0 \in [0,b]$ we necessarily have that $v(s) = 0$ for any $s \in [0,b]$. From that we conclude that the dimension of the kernel of $U(s)$ is constant on $[0,b]$. □

5.1 The compact case

Lemma 35. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \leq 5$ and $n > 2p$ be an isometric immersion of a compact Riemannian manifold and let τ be an infinitesimal bending of f. Then, at any $x \in M^n$ there is a pair of vectors $\zeta_1, \zeta_2 \in N_fM(x)$ of unit length such that $(\zeta_1,\zeta_2) \in (\mathcal{S}(\theta))^{\perp}(x)$ where*

$$\mathcal{S}(\theta)(x) = span\{\theta(X,Y) : X,Y \in T_xM\}.$$

Moreover, on any connected component of an open dense subset of M^n the pair ζ_1, ζ_2 at $x \in M^n$ extend to smooth vector fields ζ_1 and ζ_2 parallel along Δ^* that satisfy the same conditions.

Proof. We claim that the subset of points U of M^n where there is no such a pair, that is, the metric induced on $(\mathcal{S}(\theta))^\perp$ is positive or negative definite, is empty. It is not difficult to see that U is open. From Lemma 10 we have $\nu^* > 0$ in U. Let $V \subset U$ be the open subset where $\nu^* = \nu_0^*$ is minimal. Take $x_0 \in V$ and a unit speed geodesic γ in M^n contained in a maximal leaf of Δ^* with $\gamma(0) = x_0$. Since M^n is compact, there is $b > 0$ such that $\gamma([0,b)) \subset V$ and $\gamma(b) \notin V$. Proposition 33 gives $\nu^*(\gamma(b)) = \nu_0^*$ which implies $\gamma(b) \notin U$. Hence, there are unit vectors $\zeta_1, \zeta_2 \in N_f M(\gamma(b))$ such that $(\zeta_1, \zeta_2) \in (\mathcal{S}(\theta))^\perp(\gamma(b))$.

Let $\zeta_i(t)$ be the parallel transport along γ of ζ_i, $i = 1, 2$. Then

$$\langle\langle \theta(X,Y), (\zeta_1, \zeta_2)\rangle\rangle = \langle (A_{\zeta_1 - \zeta_2} + B_{\zeta_1 + \zeta_2})X, Y\rangle.$$

It follows from (3.10) and (3.12) that

$$(\nabla_T^* \theta)(X,Y) = (\nabla_X^* \theta)(T,Y) \tag{5.1}$$

where $T \in \Gamma(\Delta^*)$ extends γ' and $X, Y \in \mathfrak{X}(M)$. Along γ this gives

$$\frac{D}{dt}\mathcal{C}_{\zeta_1,\zeta_2} = \mathcal{C}_{\zeta_1,\zeta_2} C_{\gamma'} = C'_{\gamma'} \mathcal{C}_{\zeta_1,\zeta_2}$$

where $\mathcal{C}_{\zeta_1,\zeta_2} = A_{\zeta_1 - \zeta_2} + B_{\zeta_1 + \zeta_2}$ and $C'_{\gamma'}$ denotes the transpose of $C_{\gamma'}$. Moreover, by Proposition 33 this ODE holds on $[0,b]$. Given that $\mathcal{C}_{\zeta_1,\zeta_2}(\gamma(b)) = 0$, then it follows from Lemma 34 that $\mathcal{C}_{\zeta_1,\zeta_2}$ vanishes along γ. This is a contradiction and proves the claim.

We have from (5.1) that

$$(\nabla_T^* \theta)(X,Y) = -\theta(\nabla_X T, Y) \in \Gamma(\mathcal{S}(\theta))$$

for any $T \in \Gamma(\Delta^*)$ and $X, Y \in \mathfrak{X}(M)$. Thus $\mathcal{S}(\theta)$ is parallel along the leafs of Δ^*. Let U_0 be a connected component of the open dense subset of M^n where the dimension of Δ^*, $\mathcal{S}(\theta), \mathcal{S}(\theta) \cap \mathcal{S}(\theta)^\perp$ and the index of the metric induced on $\mathcal{S}(\theta)^\perp \times \mathcal{S}(\theta)^\perp$ are all constant. Hence on U_0 the vector fields ζ_1, ζ_2 can be taken parallel along the leafs of Δ^*. □

We are now in conditions to prove Theorem 30.

Proof of Theorem 30: We assume that there is no open subset of M^n where the index of relative nullity satisfies $\nu \geq n-1$. By Lemma 35, on connected components of an open dense subset of M^n there are $\zeta_1, \zeta_2 \in \Gamma(N_f M)$ with $\|\zeta_1\| = \|\zeta_2\| = 1$ parallel along the leaves of Δ^* and such that

$$\langle\langle \theta(X,Y), (\zeta_1, \zeta_2) \rangle\rangle = 0$$

for any $X,Y \in \mathfrak{X}(M)$. It follows from (4.1) that (4.19) holds on connected components of an open dense subset of M^n. Let $U \subset M^n$ be an open subset where ζ_1, ζ_2 are smooth and $\zeta_1 + \zeta_2 \neq 0$. Thus $\tau|_U$ satisfies the condition $(*)$. Let $\tilde{V} \subset U$ be an open subset where τ is a genuine infinitesimal bending. By Theorem 26 we have that f is $(n-1)$-ruled on each connected component V of an open dense subset of $\tilde{V}$. Since our goal is to show that V is empty we assume otherwise.

Proposition 6 and the proof of Theorem 26 yield that the rulings on V are determined by the tangent subbundle $D = \ker \varphi_Y$ where φ was given in Lemma 28 and $Y \in RE(\varphi)$. Also from that proof $\dim Im(\varphi_Y) = 2$ and therefore $Im(\varphi_Y) = R \oplus R$ where $N_f M = P \oplus R$ as in Lemma 28. Lemma 9 gives

$$\varphi_X(D) \subset Im(\varphi_Y) \cap Im(\varphi_Y)^{\perp} = \{0\}$$

for any $X \in \mathfrak{X}(M)$, that is, $D = \mathcal{N}(\varphi)$. In particular, from the definition of φ it follows that $D \subset \mathcal{N}(\alpha_R)$. Hence, by dimension reasons either $\mathcal{N}(\alpha_R) = TM$ or $D = \mathcal{N}(\alpha_R)$. Next we contemplate both possibilities.

Let $V_1 \subset V$ be an open subset where $\mathcal{N}(\alpha_R) = TM$ holds, that is, $N_1 = P$. Thus N_1 is parallel relative to the normal connection since, otherwise, the Codazzi equation gives $\nu = n-1$, and that has been ruled out. Hence $f|_{V_1}$ reduces codimension, that is, $f(V_1)$ is contained in an affine hyperplane $\mathbb{R}^{n+1}$. Decompose $\tau = \tau_1 + \tau_2$ where τ_1 and τ_2 are tangent and normal to $\mathbb{R}^{n+1}$, respectively. It follows that τ_1 is an infinitesimal bending of $f|_{V_1}$ in $\mathbb{R}^{n+1}$. Since τ satisfies the condition $(*)$ then Proposition 18 gives that τ_1 is trivial, that is, the restriction of a Killing vector field of $\mathbb{R}^{n+1}$ to $f(V_1)$. Extending τ_2 as a vector field normal to $\mathbb{R}^{n+1}$ it follows that $\tau|_{V_1}$ extends in the singular sense and this is a contradiction.

Let $V_2 \subset V$ be an open subset where $D = \mathcal{N}(\alpha_R)$. By assumption $D \neq \Delta$. Let $\hat{D}$ be the distribution tangent to the rulings in a neighborhood V_2' of $x_0 \in V_2$. From Proposition 6 we have $D(x_0) = \hat{D}(x_0)$. Let $W \subset V_2'$ be an open subset where $D \neq \hat{D}$, that is, where D is not totally geodesic. Then there are two transversal $(n-1)$-dimensional rulings passing through any point $y \in W$. It follows easily that $N_1 = P$ on W. As above we obtain that $\tau|_W$ extends in the singular sense, leading to a contradiction. Let $V_3 \subset V_2$ be the interior of the subset where D is totally geodesic. On V_3 the Codazzi equation gives

$$\nabla^{\perp}_X \alpha(Z,Y) \in \Gamma(P)$$

for all $X,Y \in \Gamma(D)$ and $Z \in \mathfrak{X}(M)$. Thus R is parallel along D relative to the normal connection. We have from Proposition 8 that f admits a singular extension

$$F(x,t) = f(x) + t\lambda(x)$$

for $\lambda \in \Gamma(f_*TM \oplus P)$ as a flat hypersurface. Moreover, F has R as normal bundle and ∂_t belongs to the relative nullity distribution. Then $(\tilde{\nabla}_X \lambda)_R = 0$ for any $X \in \mathfrak{X}(V_3)$. Hence (4.18) is satisfied and thus $\tau|_{V_3}$ extends in the singular sense. This is a contradiction which shows that V is empty, and hence also is $\tilde{V}$.

It remains to consider the existence of an open subset $U' \subset M^n$ where ζ_1, ζ_2 are smooth and $\zeta_1 + \zeta_2 = 0$. It follows from (4.19) that $\zeta_1 - \zeta_2 \perp N_1$. Once more, we obtain that $f(U') \subset \mathbb{R}^{n+1}$. Thus, we have an orthogonal decomposition of $\tau|_{U'}$ as in part (ii) of the statement and τ_1, τ_2 extend in the singular sense as follows:

(i) $\bar{\tau}_1(x,t) = \tau_1(x)$ to $F\colon U \times \mathbb{R} \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x) + te$.

(ii) For instance locally as $\bar{\tau}_2(x,t) = \tau_2(x)$ to $F\colon U \times I \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x) + tN$ being N is a unit normal field to $f|_U$ in $\mathbb{R}^{n+1}$. □

Remarks 36. In case (ii) of Theorem 30 if τ_1 is trivial then τ_1 and τ_2 extend in the same direction, and hence τ also does. Thus we are also in case (i).

Notice that for $p = 2$, as part of the proof we have shown that an infinitesimal bending of a submanifold without flat points as in in part (ii) of Theorem 21 cannot be genuine.

5.2 The complete case

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion, and let τ be an infitesimal bending of f. If B is the symmetric tensor associated to τ, we have in this case that $\Delta^* = \Delta \cap \ker B$. Observe that (3.15) implies that

$$\nabla_T B = BC_T = C_T' B \tag{5.2}$$

for any $T \in \Gamma(\Delta^*)$, where C is the splitting tensor of Δ^*.

If f is such that rank $A \geq 2$, we have from (3.14) that rank $B \leq 2$ and $\Delta \subset \ker B$, in particular $\Delta^* = \Delta$. The following result allows us to transport information along geodesics in rulings that are not necessarily relative nullity.

Lemma 37. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be a ruled hypersurface of constant rank 2 with complete relative nullity leaves. Assume that the splitting tensor of the relative nullity does not vanish* *on any open subset. If τ is an infinitesimal bending of f, then its associated symmetric tensor B satisfies*

$$B|_{\Delta^\perp} = \begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix} \tag{5.3}$$

with respect to a local orthonormal basis $\{Y,X\}$ of $\Delta^\perp$ such that Y is orthogonal to the rulings. Moreover, the smooth function θ verifies

$$X(\theta) = \langle \nabla_Y Y, X\rangle \theta. \tag{5.4}$$

Proof. On the open dense subset where $C \neq 0$, let $T \in \Gamma(C_0^\perp)$ be unitary. Locally take $X,Y \in \Gamma(\Delta^\perp)$ orthonormal such that Y is orthogonal to the rulings. We have seen in the proof of Lemma 5 that $X \in \Gamma(\ker C_T)$. Moreover, Lemma 3 implies that $C_T = \mu J$ for some smooth function μ, where $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ is defined by $JX = 0$ and $JY = X$.

We denote the restrictions of A and B to $\Delta^\perp$ by the same letters and let $D \in \Gamma(\mathrm{End}(\Delta^\perp))$ be given by $D = A^{-1}B$. From (2.2) and (5.2) we have

$$ADC_T = C_T' AD = AC_T D.$$

Hence $A[D,C_T] = 0$, and thus D commutes with J. This gives $D = \phi_1 Id + \phi_2 J$ and

$$B = \phi_1 A + \phi_2 AJ.$$

Since the immersion is ruled, then A has the form

$$A = \begin{bmatrix} \lambda & \nu \\ \nu & 0 \end{bmatrix}.$$

We easily have from (3.14) that $\phi_1 = 0$, and therefore B has the form (5.3). Finally (5.4) follows from (3.15). □

Remark 38. From the above and Theorem 13 in [13], the set of infinitesimal bendings of a ruled hypersurface satisfying the assumptions of the preceding result, is in one to one correspondence with the set of smooth functions on an interval.

The following result is essential in the proof of Theorem 31.

Lemma 39. *Let $\nu^* > 0$ be constant on an open subset $U \subset M^n$. If $\gamma\colon [0,b] \to M^n$ is a unit speed geodesic such that $\gamma([0,b))$ is contained in a leaf of Δ^* in U, then (5.2) holds on $[0,b]$.*

Proof. It follows immediately from Lemma 33. □

Proof of Theorem 31: Let τ be a non-trivial infinitesimal bending of f and let B be its associated symmetric tensor. We consider the subsets of M^n defined by

$$M_i = \{x \in M^n : \operatorname{rank} A(x) \geq i\}.$$

Observe that $M_2 \neq \emptyset$, because otherwise we would have from Lemma 3 that f would be a cylinder over a curve which is ruled out by our assumptions (this also follows from a well known result due to Hartman). It follows from Proposition 11 and Proposition 18 that $B|_{M_3} = 0$. Let $V \subset W_2 = M_2 \setminus \bar{M}_3$ be the open subset of M^n defined by

$$V = \{x \in W_2 : B(x) \neq 0\}.$$

We claim that the leaves of relative nullity in V are complete. Otherwise, there is a geodesic $\gamma\colon [0,b] \to M^n$ contained in a leaf of the relative nullity such that $\gamma([0,b)) \subset V$ and $\gamma(b) \notin V$. From Lemma 39 we have that B satisfies

$$\nabla_{\gamma'(s)} B = C'_{\gamma'(s)} B \tag{5.5}$$

on $[0,b]$ with $B(b) = 0$, where $C'_{\gamma'}$ denotes the transpose of $C_{\gamma'}$. Take a parallel orthonormal basis of $\Delta^\perp$ along γ and regard (5.5) as a differential equation of matrices. Since $B(b) = 0$, Lemma 34 implies that B vanishes along γ. This is a contradiction, and proves the claim.

We show next that $f|_V$ is ruled using arguments from the proof of Proposition 2.1 in [8]. By Lemma 4 the codimension of C_0 is at most one. The assumption that $f(M)$ does not contain a cylinder gives that the subset

$$V_0 = \{x \in V : C(x) = 0\}$$

has empty interior. Let $T \in \Gamma(\Delta)$ be a local unit vector field on the open subset $V_1 = V \setminus V_0$ spanning the orthogonal complement of C_0. Using again Lemma 4 it follows that rank $C_T = 1$. Moreover, we have from (2.3) that V_1 and V_0 are both union of complete relative nullity leaves.

We claim that the smooth distribution $\Delta \oplus \ker C_T$ on V_1 is totally geodesic. If $\ker C_T$ is locally spanned by a unit vector field X, then $(\nabla_X T)_{\Delta^\perp} = 0$. From Lemma 3 we have that $\nabla_T X = 0$. Since Δ is totally geodesic, then $\Delta \oplus \ker C_T$ is integrable. It remains to show that $\langle \nabla_X X, Y \rangle = 0$ where $Y \in \Gamma(\Delta^\perp)$ is a unit vector field orthogonal to X. Since the only real eigenvalue of C_T is zero, then $C_T Y = \mu X$ for a smooth non vanishing function μ. Lemma 1 yields

$$(\nabla^h_X C_T)Y = (\nabla^h_Y C_T)X,$$

which is equivalent to

$$X(\mu) = \langle \nabla_Y Y, X \rangle \mu \tag{5.6}$$

and

$$\mu \langle \nabla_X X, Y \rangle = 0.$$

The last equation proves the claim.

Since C_T is nilpotent, we have that $\ker C_T' = ImC_T'$. Since $C_T' A = AC_T$ we have $C_T' AX = 0$, and then

$$\langle AX, X \rangle = 0.$$

Thus the leaves of $\Delta \oplus \ker C_T$ are totally geodesic submanifolds of $\mathbb{R}^{n+1}$, that is, $f|_{V_1}$ is ruled.

Next we prove that the rulings contained in V_1 are complete. Recall that the leaves of relative nullity in V_1 are complete. Assume otherwise that there is an incomplete ruling in V_1. Thus, there is a geodesic $\delta\colon [0,a] \to M^n$ in the direction of X such that $\delta(a) \notin V_1$. We have from Lemma 5 that the rank of f at $\delta(a)$ is 2. Moreover, from the second statement on that lemma, it follows that (5.6) extends to $\delta(a)$ where $Y \in \Gamma(\Delta^\perp)$ is as before. Since μ is not zero along δ we have that $\delta(a) \notin V_0$, and hence $\delta(a) \notin V$. On the other hand, Lemma 37 yields that B has the form (5.3) with respect to $\{Y, X\}$ and that $\theta \in C^\infty(M)$ verifies (5.4). Using again Lemma 5 we see that (5.4) extends smoothly to $[0,a]$ with $X = \delta'$. But then B has to vanish along δ, and that is a contradiction.

Let S be a connected component of V_1 and let $x \in \partial \bar{S}$ together with a sequence $x_j \in S$ be such that $x_j \to x$. Let L_j be the affine subspace of $\mathbb{R}^{n+1}$ determined by the ruling through $f(x_j)$. Since the rulings are complete, there is an affine subspace L through $f(x)$ which is the limit of the sequence determined by L_j. In fact, suppose that there are two subsequences L_j' and L_j'' converging to different subspaces L' and L'' that intersect at $f(x)$. Then, in a neighborhood of x different subspaces L_j' and L_j'' would intersect, and this is a contradiction. Clearly $L \subset f(\partial \bar{S})$, and thus $f|_{\bar{S}}$ is a ruled strip.

Notice that if two ruled strips have common boundary then their union is also a ruled strip. Take $x \in V_0$. Since V_1 is dense in V, then $f(x) \in L \subset f(M)$ where L is an affine $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$ that is the limit of a sequence of rulings. Suppose that there exist two sequences of rulings L_j' and L_j'' converging to affine subspaces $L' \neq L''$ that intersect at $f(x)$. Then L_j' intersects L'' in a hyperplane for large values of j. Fixing j large enough, the same holds for any ruling in a neighborhood of rulings of L_j'.

Let Z' and Z'' be vector fields tangent to L_j' and L'', respectively, and let R be a vector field tangent to $L'' \cap L_j'$. Since $\tilde{\nabla}_R Z'$ and $\tilde{\nabla}_R Z''$ have no normal components, it follows that $L'' \cap L_j'$ is a complete relative nullity leaf. The same holds for the nearby rulings. In a neighborhood of $y \in L'' \cap L_j'$, as before take unit vector fields $T \in \Gamma(C_0^\perp)$, $X \in \Gamma(\ker C_T)$

and Y such that $C_T Y = \mu X$ with $\mu \neq 0$. Let γ be the unit speed geodesic of M^n such that $f \circ \gamma$ lies in L'', $f(\gamma(0)) = y$ and is orthogonal to Δ. Then $\gamma' = aX + bY$ with $b \neq 0$. Hence $\langle C_T \gamma', \gamma' \rangle = 0$ is equivalent to $ab\mu = 0$. This yields $a = 0$, and thus $\gamma' = Y$ is orthogonal to X. Since $f_* \nabla_{\gamma'} T$ is tangent to L'' and $f_* X$ is orthogonal to L'', then $C_T Y = 0$, and this is a contradiction. Therefore, we have seen that any sequence of points in V_1 converging to x, determines the same affine subspace L as the limit of the correspondent rulings. Moreover, we have shown that L does not intersect $f(V_1)$.

We have proved that there exists an open neighborhood U of x such that $f|_U$ is ruled and has complete relative nullity leaves. Using Lemma 37 as above, we obtain that the affine subspace L is contained in $f(V_0)$. Hence, every connected component of V defines a ruled strip.

To conclude the proof of the theorem it remains to show that $B = 0$ on the open subset $W_1 = M_1 \setminus \bar{M}_2$, that is, that B vanishes outside ruled strips. It follows from (3.14) that $B(\Delta) \subset ImA$, hence rank $B \leq 2$ on W_1. Let V' be the open subset of W_1 defined as

$$V' = \{x \in W_1 : B(x) \neq 0\},$$

then $\Delta^* = \ker B$ on V'.

We claim that V' is empty. Suppose otherwise. Let $V'' \subset V'$ be the open subset where ν^* attains its minimum in V', say ν_0^*. We see next that the leaves of Δ^* are complete on V''. On the contrary, suppose that there is a geodesic $\gamma\colon [0,b] \to M^n$ such that $\gamma([0,b)) \subset V'$ is contained on a leaf of Δ^* and that $\gamma(B) \notin V''$. By lemmas 2 and 33, we know that $\nu(\gamma(b)) = n-1$ and $\nu^*(\gamma(b)) = \nu_0^*$. Then $\gamma(b) \in \bar{M}_2$ and $B(\gamma(b)) \neq 0$. Take a neighborhood of $\gamma(b)$ where $B \neq 0$. Since $\gamma(b) \in \bar{M}_2$, there is a sequence $x_k \in V$ such that $x_k \to \gamma(b)$. Recall that each connected component of V defines a ruled strip. Let L_k be the affine subspace of $\mathbb{R}^{n+1}$ given by the ruling through $f(x_k)$. As before, there is an affine subspace $L \subset f(\bar{M}_2)$ of dimension $n-1$ which is the limit of the sequence L_k and contains $f(\gamma(b))$. Since $A\gamma'(b) = 0$ and the geodesic $f \circ \gamma$ is transversal to L, we have that $A(\gamma(b)) = 0$, and that is a contradiction. Hence Δ^* has complete leaves in V''.

The leaves of the relative nullity foliation cannot be complete on any open subset of W_1. This follows easily from Lemma 3 and the assumptions on f. Hence we necessarily have that $\nu_0^* = n-2$.

Take local orthonormal vector fields X and Y in V' orthogonal to $\ker B$ such that X is an eigenfield of A. Then A and B have the expressions

$$A|_{\ker B^\perp} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad B|_{\ker B^\perp} = \begin{bmatrix} \mu & \rho \\ \rho & 0 \end{bmatrix}$$

with respect to the frame $\{X,Y\}$ and $\lambda \neq 0 \neq \rho$.

Given $T \in \Gamma(\Delta)$ let c_T be defined by $C_T X = c_T X$. Since X is parallel along the relative nullity leaves, we have

$$(\nabla_X B)Y = (X(\rho) + c_Y \mu)X + 2c_Y \rho Y + \rho (\nabla_X X)_{\ker B}$$

and

$$(\nabla_Y B)X = Y(\mu)X + Y(\rho)Y + \rho \nabla_Y Y.$$

In particular (3.15) yields

$$Y(\rho) = 2c_Y \rho. \tag{5.7}$$

Let $W_1' \subset W_1$ be the dense subset where the the relative nullity leaves are not complete. Take a point $x \in V'' \cap W_1'$. Since the leaf of the relative nullity foliation through x is not complete, there is a geodesic $\delta\colon [0,b] \to M^n$ contained in that leaf tangent to Y such that $\delta([0,b)) \subset V''$ and $\delta(b) \notin V''$. By the same transversality argument as above we see that $\rho(\delta(b)) = 0$. It follows from (5.7) that $\rho = 0$ along δ, and that is a contradiction proving the claim that V' is empty.

□

Corollary 40. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold M^n satisfying the hypothesis of Theorem 10. If τ is a non-trivial infinitesimal bending of f, then τ is the variational field of an isometric bending.*

Proof. Let B be the symmetric tensor associated to the infinitesimal bending τ. It is easy to see using (5.3) and (5.4) that the symmetric tensors $A + tB$, $t \in \mathbb{R}$, satisfy the Gauss and Codazzi equations. Then, they give rise to an isometric bending of f having τ as its variational field. □

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