Super-diffusive Limit of an Exclusion Process and Convergence From Coalescence to the Kingman's Coalescent

Enrique Idael Chavez Sarmiento
Advisor: Claudio Landim

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"For Sanni the point of life is like happiness. To be with people that make you feel fulfilled, and to have a good time. For me it is all about performance. The thing is– anybody can be happy and cozy. Nothing good happens in the world by being happy and cozy. You know, and nobody achieves anything great because they are happy and cozy."

—Alex Honnold

For my mother, all the good
and bad things you gave me
made this possible.

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Abstract

This thesis has three parts. In the first part we consider a one-dimensional, weakly asymmetric, boundary driven exclusion process on a one-dimensional discrete interval, in a super-diffusive time scale. In this part we derive an equation which describes the evolution of the density of the particles until certain point. In the second part we consider a process associated to the simple random walk in a discrete torus. We place a particle at each site of the torus and let them evolve as independent, nearest-neighbor, symmetric, continuous-time random walks. Each time two particles meet, they coalesce into one. We prove that, in a convenient scale of time, the sequence of total number of particles of these processes, when the size of the torus grows, converges to the total number of partitions in Kingman's coalescent. Finally, in the third part of this thesis, we consider a process similar to the previous one. We consider a finite number of i.i.d. irreducible and transitive Markov chains in continuous time, over a finite state space. Each time two chains meet, they stay together. This mechanism induces a process in the set of partitions of the first natural numbers. Starting from the invariant measure, we find conditions under which a sequence of these processes, in an appropriate scale of time, converges to the Kingman's coalescent that starts with finite equivalence classes. In particular, we prove this convergence in the reversible case under a condition that involves the relaxation time.

Key words: Super-diffusive limit, Kingman's coalescent, martingale approach, statistical mechanics.

Contents

Chapter 1

Introduction

This PhD thesis is divided in three parts, each of them contained in one of the following chapters, namely Chapters 2, 3 and 4. Though the last two chapters are related, each part of this thesis can be read independently.

1.1 Super-diffusive Limit of Exclusion Processes

In Chapter 2 we consider a weakly asymmetric, boundary driven exclusion process in $\{0, 1, \dots, N\}$, in a superdiffusive time scale $N^2\epsilon_N^{-1}$, with a fix density at the boundaries. We describe the evolution of the density of the particles up to order ϵ_N. Here $\epsilon_N \to 0$ as N goes to infinity.

More specifically, we examine the correction to the hydrodynamic equation of this process. Assume that $\epsilon_N^4 N \to \infty$, and that the exclusion process starts from a local equilibrium state associated to the density profile $\bar{\rho}_{\lambda(0),E(0)} + \epsilon_N v_0$. Then, for all $t > 0$, the system remains close, in the scale ϵ_N^{-1}, to a local equilibrium state whose density profile is given by $\bar{\rho}_{\lambda(t),E(t)} + \epsilon_N v_t$, where v_t is the solution of the following elliptic equation

$$\begin{cases} \partial_t \bar{\rho}_{\lambda(t),E(t)} = \partial_x^2 \big(D(\bar{\rho}_{\lambda(t),E(t)}) v \big) - \partial_x \big(\chi'(\bar{\rho}_{\lambda(t),E(t)}) E(t) v \big) \ , \\ v(0) = v(1) = 0 \ . \end{cases} \tag{1.1.1}$$

More precisely, for every cylinder function Ψ, and every continuous function $H : [0, 1] \to \mathbb{R}$, if η_t^N represents the state at time t of the speeded-up exclusion process,

$$\frac{1}{\epsilon_N N} \sum_{j=1}^{N-1} H(j/N) \big\{ \tau_j \Psi(\eta_t^N) - E_{\bar{\rho}_{\lambda(t),E(t)} + \epsilon_N v_t}[\Psi] \big\} \to 0 \ .$$

In this formula, $\{\tau_j : j \in \mathbb{Z}\}$ represents the group of translations and E_γ the expectation with respect to the local equilibrium state associated to the density profile γ.

The proof of the main results, Theorems 2.1.4 and 2.2.1, follows the strategy proposed by [14,31], which consists in estimating the relative entropy of the state of the process with respect to the local equilibrium state whose density profile solves equation

$$\begin{cases} \partial_t \rho = \epsilon^{-1}\{\partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E(t))\} \;, \\ f'(\rho(t,0)) = \lambda_0(t) \;,\;\; f'(\rho(t,1)) = \lambda_1(t) \;, \\ \rho(0,\cdot) = \bar\rho_{\lambda(0),E(0)}(\cdot) + \epsilon v_0(\cdot) \;, \end{cases} \tag{1.1.2}$$

with $\epsilon = \epsilon_N$.

If $H_N(t)$ represents this latter relative entropy, the main result asserts that for all $t > 0$,

$$\frac{1}{N\epsilon_N^2} H_N(t) \;\to\; 0 \;.$$

1.1.1 Context and Related Work

A theory of thermodynamic transformations for nonequilibrium stationary states has been proposed recently [4,5] in the framework of the Macroscopic Fluctuation Theory [6,7]. It defined the renormalized work performed by a transformation between two nonequilibrium stationary states in driven diffusive systems, and it proved a Clausius inequality which postulates that the renormalized work is always larger than the variation of the equilibrium free energy between the final and the initial nonequilibrium states.

In quasi-static transformations, transformations in which the variations of the environment are very slow, the renormalized work coincides asymptotically with the variation of the equilibrium free energy. More precisely, fix a transformation $u(t)$, $t \geq 0$, between two nonequilibrium stationary states, and denote by $W^{\text{ren}}(u)$ the renormalized work performed by u. Let u_ϵ be the transformation u slowed down by a parameter $\epsilon > 0$, $u_\epsilon(t) = u(t\epsilon)$. Then, $\lim_{\epsilon\to 0} W^{\text{ren}}(u_\epsilon) = \Delta F$, where ΔF represents the variation of the equilibrium free energy between the final and the initial nonequilibrium states. Note that the asymptotic identity is attained independently of the transformation u chosen.

Let us mention that the theory of thermodynamic transformations between nonequilibrium states, and the analysis of quasi-static transformations has been extended to the framework of stochastic perturbations of microscopic Hamiltonian dynamics in contact with heat baths in [27–29].

To select, among the slow transformations between two nonequilibrium stationary states, the one which minimizes the renormalized work we have to examine the first order term in the expansion in ϵ of the renormalized work. This question has been addressed in [8], where it was shown that for slow transformations between two equilibrium states the first order correction of the renormalized work is minimized by transformations whose intermediate states are equilibrium states, and where a partial differential equation which describes the evolution of the optimal transformation has been derived.

A time-change permits to convert a slow transformation in an ordinary transformation whose differential operator is multiplied by ϵ^{-1}. This observation brings us to the question of the correction to the hydrodynamic equation of boundary driven interacting particle systems.

Consider a symmetric, one-dimensional dynamics in contact with reservoirs and in the presence of an external field. At the macroscopic level the system is described by a local density $\rho(t,x)$, $x \in [0,1]$, which evolves according to the driven diffusive equation

$$\begin{cases} \partial_t \rho = \partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E) \\ f'(\rho(t,a)) = \lambda_a(t) \quad \text{for} \quad a \;=\; 0\,,\,1\;, \end{cases} \tag{1.1.3}$$

where D is the diffusivity, χ the mobility, $E(t,x)$ an external field, $\lambda_0(t)$, $\lambda_1(t)$ time-dependent chemical potentials, which fix the density at the boundaries, and f the equilibrium free energy density.

For a fixed external field $E(x)$ and a chemical potential $\lambda = (\lambda_0, \lambda_1)$, denote by $\bar{\rho}_{\lambda,E}$ the solution of the elliptic equation

$$\begin{cases} \partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E) \ = \ 0 \, , \\ f'(\rho(a)) = \lambda_a \quad \text{for} \quad a \ = \ 0 \, , 1 \, . \end{cases} \tag{1.1.4}$$

Consider the driven diffusive equation (1.1.3) speeded up by ϵ^{-1}. Fix a transformation $(\lambda(t), E(t))$, $\epsilon > 0$, and a bounded profile $v_0 : [0,1] \to \mathbb{R}$. Denote by $\rho_\epsilon(t)$ the solution of (1.1.2). A formal expansion in ϵ yields that, for $t > 0$, $u_\epsilon(t) = \epsilon^{-1}[\rho_\epsilon(t) - \bar{\rho}_{\lambda(t),E(t)}]$ converges to $v(t)$, the solution of the elliptic equation (1.1.1). Note that the limit v_t does not depend on the initial condition v_0.

The main results of Chapter 2 state a similar result for a microscopic dynamics speeded-up super-diffusively. Consider a one-dimensional, weakly asymmetric, exclusion process evolving on $\{1, \dots, N-1\}$, and in contact with reservoirs at the boundaries. Assume that the density of each reservoir evolves smoothly in the macroscopic time-scale, and that the dynamics is speeded-up by $N^2\epsilon_N^{-1}$, where $\epsilon_N \to 0$ as $N \uparrow \infty$. De Masi and Olla [12] proved that starting from any initial distribution, at all macroscopic time $t > 0$ the system converges to a local equilibrium state whose density profile is given by the solution of the elliptic equation (1.1.4) with chemical potential $\lambda(t)$.

The results presented here have a similarity to the correction to the hydrodynamic equation, examined in [14, 22] in the asymmetric case in dimension $d \geq 3$ and in [15] in the symmetric case.

1.2 From Coalescence to Kingman's Coalescent

The main results of Chapters 3 and 4 are related to the Kingman's coalescent, a process that we present in the following subsection. Loosely speaking, we shall see that some projections of this process emerge as the limit of certain sequences of processes, which in turn are projections of some sequences of Markov processes. These last Markov processes are coalescing process associated to a Markov chain.

1.2.1 The Kingman's Coalescent

In the early eighties, Kingman [17] presented a Markov process over the set of partitions or equivalence classes, of $\mathbb{N} := \{1, 2, \dots\}$. The dynamic of this process can be described as follows. No matter in which partition we start, at any time $t > 0$ we are in a state with finite number of equivalence classes. Now, assume that we are in a partition π compound of the equivalence classes $A_1, \dots, A_n$. Staying here, we chose uniformly at random a pair of different equivalence classes, say A_i and A_j, and in an exponential time of mean $\binom{n}{2}^{-1}$ we move to the partition obtained from π by coalescing A_i with A_j. We call this partition $(i,j)[\pi]$. Then we repeat the process with $(i,j)[\pi]$; and continue in this way until we get the partition made by the entire set of naturals numbers. We stay in this last state forever.

1.2.2 Coalescence in the Discrete Torus

In Chapter 3 we focus our attention into the coalescence in the discrete torus. Fix $d \geq 2$, and denote by $\mathbb{T}_N^d = \{0, \dots, N-1\}^d$ the discrete, d-dimensional torus with N^d points. Consider independent, nearest-neighbor, symmetric, continuous-time coalescing random walks evolving on $\mathbb{T}_N^d$. This dynamics can be informally described as follows. Place a particle at each point of $\mathbb{T}_N^d$. Each particle evolves, independently from the others, as a continuous-time random walk which jumps from x to $x \pm e_j$ with probability $1/2d$, where the summation is taken modulo N and $\{e_1, \ldots, e_d\}$ stands for the canonical basis of $\mathbb{R}^d$. Whenever a particle jumps to a site occupied by another particle, the two particles coalesce into one.

Let C_N be the first time the set of particles is reduced to a singleton, and let $s_N = N^d$ in dimension $d \geq 3$, $s_N = N^2 \log N$ in dimension 2. Cox [11] proved that C_N / s_N converges in distribution to a random variable τ which can be expressed as

$$\tau \;=\; \sum_{k \geq 2} T_k \;, \tag{1.2.1}$$

where $(T_k)_{k\geq 2}$ is a sequence of independent, exponential random variables whose expectations are given by

$$E[T_n] \;=\; \frac{2}{n\,(n-1)}\;, \quad \text{for } n \geq 2\;.$$

This is related with the Kingman's coalescent. Consider $(\mathscr{N}_t)_{t\geq 0}$, the process that records the number of equivalence classes in the Kingman's coalescent that starts from a partition with infinite equivalence classes. This process is a pure death process on $\mathbb{N} \cup \{\infty\}$, starting at ∞, finite at any positive time, and jumping from k to $k-1$ at rate $k(k-1)/2$. A path of $(\mathscr{N}_t)_{t\geq 0}$ can be sampled as follows. Recall the definition of the random variables $(T_n)_{n\geq 2}$, and set $T_1 = \infty$. Note that, since $E[\tau] < \infty$, with probability one $\sum_{n=2}^{\infty} T_n < \infty$ and so

$$\Big[\sum_{n=k+1}^{\infty} T_n, \sum_{n=k}^{\infty} T_n\Big), \quad k \in \mathbb{N}\;,$$

turns to be a partition of $(0, \infty)$. Set $\mathscr{N}_0 = \infty$ and, for every $t > 0$ and $k \geq 1$, define

$$\mathscr{N}_t = k \iff \sum_{n=k+1}^{\infty} T_n \leq t < \sum_{n=k}^{\infty} T_n\;. \tag{1.2.2}$$

Notice that this process is not continuous at $t = 0$ unless every neighborhood of $\infty \in \mathbb{N} \cup \{\infty\}$ has finite complement.

We shall use an alternative description of this process, more suitable to our purposes. Consider the bijection

$$\begin{array}{ccc} \{1, 2, \ldots, \infty\} & \to & S := \{1, 1/2, 1/3, \ldots, 0\} \\ x & \mapsto & 1/x\;, \end{array}$$

taking ∞ to 0, and endow S with the standard differential structure inherited by the real line. The first result of Chapter 3 characterizes the law of

$$\mathscr{X}_t = 1/\mathscr{N}_t\;, \quad t \geq 0\;, \quad (\text{where } 1/\infty = 0) \tag{1.2.3}$$

as the unique solution of a martingale problem.

The second main result of that chapter asserts that in an appropriate time-scale the process which records the [inverse of the] total number of particles on the torus at a given time converges in the Skorokhod topology to $\mathscr{X}_t$.

1.2.3 Coalescence Associated to a Markov Chain

In Chapter 4 we study the coalescence of a fixed number of particles, associated to a Markov chain. To describe our approach, suppose that ξ is an irreducible and transitive Markov chain. Fix $n \geq 2$, take $\boldsymbol{\xi} := (\xi^1, \ldots, \xi^n)$, a vector of i.i.d. copies of ξ, and consider the following dynamic. We start with $\boldsymbol{\eta} := (\eta^1, \ldots, \eta^n)$ being a copy of $\boldsymbol{\xi}$ and let the time pass. When two coordinates meet, they stay together and follow the motion of the one with the smaller label. This dynamic induces a process in the set of partitions of $\{1, 2, \ldots, n\}$. To make it clearer, suppose that at some point the particles with labels $\{2, 8, 9\}$ are evolving together, as well as the particles corresponding to $\{7, 3, 5\}$. If they meet each other, before any other particle outside $\{\eta^2, \eta^8, \eta^9\} \cup \{\eta^7, \eta^3, \eta^5\}$, then all the particles corresponding to $\{2, 8, 9, 7, 3, 5\}$ stay together, all of them following the motion of η^2. Now suppose that we have $(\xi^N)_N$, a sequence of irreducible and transitive Markov chains that generates the sequence of processes $(X^N)_N$, previously described, over the partitions of $\{1, 2, \ldots, n\}$. We define θ_N as the mean meeting time of two particles starting from the invariant measure. Then we prove that, starting from the invariant measure, the sequence $(X^N_{t\theta_N}, t \geq 0)_N$ converges in distribution, considering the Skorokhod topology, to the Kingman's coalescent, starting from a partition with n equivalence classes, if the following conditions are fulfilled:

1. With probability converging to one, there is no coalescence in a scale of time smaller than $(\theta_N)_N$.
2. The coupling $\boldsymbol{\xi}^N := (\xi^1, \ldots, \xi^n)$ exhibits a local ergodic behavior, i.e. there exists a scale of time $(\alpha_N)_N$ smaller than $(\theta_N)_N$ such that
$$\mathbb{E}^N_{\boldsymbol{m}^n}\Big[\Big|\frac{1}{\alpha_N}\int_0^{\alpha_N} f(\boldsymbol{\xi}^N_s)\, ds\Big|\Big] \xrightarrow{N\to\infty} 0\,,$$
for all functions f with $\boldsymbol{m}^n$-mean zero. Here $\boldsymbol{m}^n$ denotes the invariant measure of $\boldsymbol{\xi}^N$.
3. The times when two particles meet, starting from the invariant probability measure, are uniformly bounded in $L^{1+\varepsilon}$, for some $\varepsilon > 0$.

Condition 1 ensures that the partitions are meta-stable states of $(X^N)_N$, after rescaling it. This also allow us to proof the tightness of $(X^N_{t\theta_N}, t \geq 0)_N$. Condition 2 permit us to make convenient averages in the $(\alpha_N)_N$ scale of time, whereas condition 3 is technical, among other things, it help us to interchange some limits with expectations.

To prove our result we follow the same general strategy we used in Chapter 3. After obtaining the tightness, we use a replacement condition to show that all the limit processes of $(X^N_{t\theta_N}, t \geq 0)_N$ solve a martingale problem with unique solution. An abstraction of this step is described in Subsection 4.2.1. In our proof, we reduce the replacement condition to the study of the first coalescence of any $m \leq n$ particles starting from the invariant measure. In regard to this, we proof in Section 4.3 that, under conditions 1, 2 and 3 for $n = 2$, the time of the first coalescence of m particles starting from the invariant measure behaves asymptotically as an exponential time of mean $\binom{m}{2}^{-1}$. We also treat the reversible case. In this case we study the following condition introduced by Aldous [1]:

$$\lim_{N\to\infty} \frac{\gamma_N}{\theta_N} = 0\,, \tag{1.2.4}$$

where γ_N denotes the relaxation time of each ξ^N. This condition is satisfied for all the chains studied in [26] in the reversible case. In particular the discrete torus satisfies it. We show that (1.2.4) implies conditions 1, 2 and 3 for every $n \in \mathbb{N}$. Therefore, in this case, we prove our main result provided that (1.2.4) holds.

1.2.4 Context and Related Work

The results obtained in Chapters 3 and 4 sharpen and extend previous ones. For the discrete torus, in dimension 2, Zähle et al. [32] proved the convergence of the one-dimensional distribution of $\mathscr{N}_t$ provided the particles are initially spread out. In dimension $d \geq 3$, Limic and Sturm [24] proved the convergence of $\mathscr{N}_t$, excluding a neighborhood of $t = 0$. In $\mathbb{T}^d_N$, by adopting the view point of a martingale problem, we are able to handle the convergence in the Skorokhod topology for all times and to avoid assuming that particles are initially spread out.

Since Cox' article [11], the asymptotic behavior of the coalescence time C_N has been the subject of several papers. Consider a connected graph $\mathbb{G}_N$ with N vertices. If $\mathbb{G}_N$ is the complete graph, the distribution of C_N can be computed exactly and the process which records the total number of particles is Markovian. This example is called the mean-field model, and one expects that, under some mixing conditions on the random walk on the graph $\mathbb{G}_N$, the asymptotic behavior of the coalescence time C_N resembles the one of the mean field model.

Denote by h_N the expected hitting time of a vertex starting from the stationary distribution, and by θ_N the expected meeting time of two independent random walks over $\mathbb{G}_N$, both starting from the stationarity state. Aldous and Fill [2, Chapter 14] conjectured in Open Problem 12 that under some mixing conditions $E[C_N]$ is of the same order of h_N, as in the mean-field case.

Durrett [13] proved mean field behavior in a small world random graph and Cooper, Frieze and Radzik [10, Theorem 8] in random d-regular graphs. Oliveira [25, 26] showed that under some reasonable mixing conditions C_N/θ_N converges to τ, the random time introduced in (1.2.1), in transitive, reversible, irreducible Markov chains. A direct consequence of our study in Chapter 4 shows that in the reversible case, under the weaker condition (1.2.4) related with the relaxation time, starting with n particles in the invariant measure, C_N/θ_N converges to

$$\sum_{k=2}^{n} T_k ,$$

where n is an arbitrary, but fixed, natural number. This put in evidence that, under condition (1.2.4), in general settings, our understanding of $\mathscr{N}_t$ starting from ∞, is not enough to get results like the one we obtain for the discrete torus.

To the best of our knowledge, the results we present in chapter 4 are the first ones that show, under abstract conditions, the convergence of processes related to the coalescence of Markov chains. As we mentioned before, in the reversible case these conditions are implied by (1.2.4), which is weaker than the mixing hypothesis assumed in previous works.

Chapter 2

A Correction to the Hydrodynamic Limit of Boundary Driven Exclusion Processes in a Super-diffusive Time Scale*

Abstract

We consider a one-dimensional, weakly asymmetric, boundary driven exclusion process on the interval $[0, N] \cap \mathbb{Z}$ in the super-diffusive time scale $N^2 \epsilon_N^{-1}$, where $1 \ll \epsilon_N^{-1} \ll N^{1/4}$. We assume that the external field and the chemical potentials, which fix the density at the boundaries, evolve smoothly in the macroscopic time scale. We derive an equation which describes the evolution of the density up to the order ϵ_N.

2.1 Notation and Main Results

2.1.1 The Model

We examine a one-dimensional weakly asymmetric exclusion process in contact with reservoirs. Fix $\Lambda = (0, 1)$, and let $\Lambda_N = \{1, \ldots, N-1\}$, $N \geq 1$, be a discretization of Λ, i.e. the microscopic point $j \in \Lambda_N$ represents the macroscopic location $j/N \in \Lambda$. Particles evolve on Λ_N under an exclusion rule which allows at most one particle per site. The state space is denoted by $\Sigma_N = \{0, 1\}^{\Lambda_N}$, and the configurations are represented by the Greek letters η, ξ so that $\eta(j) = 1$ if site $j \in \Lambda_N$ is occupied for the configuration η, and $\eta(j) = 0$ otherwise.

*Joint work with Claudio Landim

Let A_0 be a finite subset of $\mathbb{Z}$ which contains the set $\{0,1\}$. Consider a strictly positive function $c : [0,1]^{\mathbb{Z}} \to \mathbb{R}_+$ which does not depend on the variables $\eta(0)$ and $\eta(1)$ and whose support is contained in A_0:

$$c(\eta) \;=\; c_\varnothing \;+\; \sum_{\substack{A\subset A_0\\ A\cap\{0,1\}=\varnothing}} c_A \prod_{k\in A} \eta(k)\;,$$

where c_A are coefficients which may be negative. In the case where $A_0 = \{0,1\}$, $c(\eta)$ is constant equal to $c_\varnothing$.

Denote by $\{\tau_k : k \in \mathbb{Z}\}$ the group of translations in $\{0,1\}^{\mathbb{Z}}$ so that $\tau_k\eta$ is the configuration defined by $(\tau_k\eta)(j) = \eta(k+j)$, k, $j \in \mathbb{Z}$. The action is extended to cylinder functions $\Psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, in the usual way: $(\tau_k\Psi)(\eta) = \Psi(\tau_k\eta)$.

We assume throughout this chapter that the jump rate c satisfies the *gradient* condition: There exist $m \geq 1$, cylinder functions $h_1, \ldots, h_m$, and finite-range, signed measures $\mu_1, \ldots, \mu_m$ on $\mathbb{Z}$ with vanishing total mass such that

$$[\eta(0) - \eta(1)]\, c(\eta) \;=\; \sum_{a=1}^{m} \sum_{j\in\mathbb{Z}} \mu_a(j)\, (\tau_{-j}h_a)(\eta)\;. \tag{2.1.1}$$

This decomposition is clearly not unique. In the case $c(\eta) = 1 + \eta(-1) + \eta(2)$, one may take $m = 3$, $h_1(\eta) = \eta(-1)\eta(0)$, $h_2(\eta) = \eta(0)\eta(2)$, $h_3(\eta) = \eta(0)$, $\mu_1(0) = 1 = -\mu_1(2)$, $\mu_2(0) = 1 = -\mu_2(-1)$, $\mu_3(0) = 1 = -\mu_3(-1)$.

Fix a chemical potential $\lambda : \partial\Lambda \to \mathbb{R}$, where $\partial\Lambda$ represents the boundary of Λ. In one dimension, λ is simply a pair (λ_0, λ_1). Let $\alpha = (\alpha_0,\, \alpha_1)$ be the density of particles associated to the chemical potential λ:

$$\alpha_0 \;=\; \frac{e^{\lambda_0}}{1+e^{\lambda_0}}\;, \quad \alpha_1 \;=\; \frac{e^{\lambda_1}}{1+e^{\lambda_1}}\;.$$

Let $\tau_j^{N,\lambda} : \Sigma_N \to \{\alpha_0, \alpha_1, 0, 1\}^{\mathbb{Z}}$, $N \geq 1$, $j \in \mathbb{Z}$, be the operators defined by

$$(\tau_j^{N,\lambda}\eta)(k) \;=\; \eta(k+j) \;\text{ if } k+j \in \Lambda_N\;, \quad (\tau_j^{N,\lambda}\eta)(k) \;=\; \begin{cases} \alpha_0 & \text{if } k+j \leq 0,\\ \alpha_1 & \text{if } k+j \geq N, \end{cases}$$

for $k \in \mathbb{Z}$. As before the action of the operator $\tau_j^{N,\lambda}$ can be extended to functions defined on Σ_N. For $N \geq 1$, $1 \leq j < N-1$, let the functions $c_{j,j+1}^{N,\lambda} : \Sigma_N \to \mathbb{R}_+$ be given by

$$c_{j,j+1}^{N,\lambda} \;=\; \tau_j^{N,\lambda} c\;,$$

so that $c_{j,j+1}^{N,\lambda}(\eta) = c(\tau_j^{N,\lambda}\eta)$. Note that $c_{0,1}^{N,\lambda}$ is usually not equal to c. It follows from (2.1.1) that for $N \geq 1$, $1 \leq j < N-1$,

$$w_{j,j+1}^{N,\lambda} \;:=\; [\eta(j) - \eta(j+1)]\, c_{j,j+1}^{N,\lambda}(\eta) \;=\; \sum_{a=1}^{m} \sum_{k\in\mathbb{Z}} \mu_a(k)\, (\tau_{j-k}^{N,\lambda} h_a)(\eta)\;. \tag{2.1.2}$$

We are now in a position to define the jump rates of the boundary driven exclusion process. Fix a smooth *external field* $E : [0,1] \to \mathbb{R}$, and let

$$\begin{aligned} c^{N,\lambda,E}_{0,1}(\eta) \;&=\; r^{\lambda}_{0,1}(\eta)\, e^{(1/2N)\,E(0)\,[1-2\eta(1)]}\, c^{N,\lambda}_{0,1}(\eta)\;, \\ c^{N,\lambda,E}_{j,j+1}(\eta) \;&=\; e^{(1/2N)\,E(j/N)\,[\eta(j)-\eta(j+1)]}\, c^{N,\lambda}_{j,j+1}(\eta)\;, \quad 1 \le j \le N-2\;, \\ c^{N,\lambda,E}_{N-1,N}(\eta) \;&=\; r^{\lambda}_{N-1,N}(\eta)\, e^{(-1/2N)\,E(1)\,[1-2\eta(N-1)]}\, c^{N,\lambda}_{N-1,N}(\eta)\;, \end{aligned}$$

where,

$$\begin{aligned} r^{\lambda}_{0,1}(\eta) \;&=\; \alpha_0[1-\eta(1)] \;+\; \eta(1)[1-\alpha_0]\;, \\ r^{\lambda}_{N-1,N}(\eta) \;&=\; \alpha_1[1-\eta(N-1)] \;+\; \eta(N-1)[1-\alpha_1]\;. \end{aligned}$$

Denote by $L^{\lambda,E}_N = L_N$ the generator whose action on functions $f : \Sigma_N \to \mathbb{R}$ is given by

$$(L_N f)(\eta) \;=\; \sum_{j=0}^{N-1} c^{N,\lambda,E}_{j,j+1}(\eta)\,\{f(\sigma^{j,j+1}\eta) - f(\eta)\}\;. \tag{2.1.3}$$

In this formula, the configuration $\sigma^{j,j+1}\eta$, $1 \le j \le N-2$, represents the configuration obtained from η by exchanging the occupation variables $\eta(j)$, $\eta(j+1)$,

$$(\sigma^{j,j+1}\eta)(k) = \begin{cases} \eta(j+1), & k = j\;, \\ \eta(j), & k = j+1\;, \\ \eta(k), & k \ne j, j+1\;, \end{cases}$$

while $\sigma^{0,1}\eta$, $\sigma^{N-1,N}\eta$ represent the configuration obtained from η by flipping the occupation variables $\eta(1)$, $\eta(N-1)$, respectively:

$$(\sigma^{j}\eta)(k) = \begin{cases} \eta(k), & k \ne j\;, \\ 1-\eta(k), & k = j\;, \end{cases} \qquad\qquad j \in \{1, N-1\}\;,$$

where $\sigma^1\eta$ represents $\sigma^{0,1}\eta$, and $\sigma^{N-1}\eta$ represents $\sigma^{N-1,N}\eta$.

2.1.2 Transformations

The dynamics introduced in the previous subsection is a finite-state, irreducible, continuous-time Markov chain. It has therefore a unique stationary state, denoted by $\nu^N_{\lambda,E}$. If the external field $E(x)$ vanishes and the chemical potentials coincide, $\lambda_0 = \lambda_1 = \lambda$, this stationary state is the Bernoulli product measure with density $\rho = e^{\lambda}/(1+e^{\lambda})$.

For a given continuous density profile $\gamma : [0,1] \to [0,1]$, denote by $\nu^N_{\gamma(\cdot)}$ the product measure on Σ_N with marginals given by

$$\nu^N_{\gamma(\cdot)}\{\eta(j) = 1\} \;=\; \gamma(j/N)\;, \quad j \in \Lambda_N\;. \tag{2.1.4}$$

Similarly, for $0 \le \theta \le 1$, ν_θ, stands for the Bernoulli product on $\{0,1\}^{\mathbb{Z}}$ with density θ:

$$\nu_\theta\{\eta(j) = 1\} \;=\; \theta\;, \quad j \in \mathbb{Z}\;.$$

To describe the macroscopic evolution of the density, denote the *diffusivity* by D :

$[0,1] \to \mathbb{R}_+$, and the *mobility* by $\chi : [0,1] \to \mathbb{R}_+$:

$$D(\theta) \;=\; E_{\nu_\theta}[c(\eta)]\;, \quad \chi(\theta) \;=\; \frac{1}{2}\, E_{\nu_\theta}\big[\, [\eta(1)-\eta(0)]^2 c(\eta)\big] \;=\; \theta(1-\theta)\, E_{\nu_\theta}[c(\eta)]\;. \tag{2.1.5}$$

The transport coefficients D and χ are related through the local Einstein relation

$$D(\theta) \;=\; \chi(\theta) f''(\theta)\;, \tag{2.1.6}$$

where $f : [0,1] \to \mathbb{R}$ the equilibrium free energy:

$$f(\theta) \;=\; \theta \log\theta \;+\; [1-\theta]\log(1-\theta)\;.$$

Let $A = [0,1]$ or $\mathbb{R}_+$. Denote by $C^r(A)$, $r \geq 0$, the set of functions $F : A \to \mathbb{R}$ which are $[r]$-times differentiable, where $[r]$ stands for the integer part of r, and whose $[r]$-th derivative is Hölder continuous with exponent $r-[r]$, and by $C^r_0([0,1])$ the set of functions in $C^r([0,1])$ which vanish at the boundary. If r is an integer, we require the $[r]$-th derivative to be continuous. Similarly, $C^{r,s}(\mathbb{R}_+ \times [0,1])$, r, $s \geq 0$, represents the set of functions $F : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ which are $[r]$-times differentiable in the time variable, $[s]$-times differentiable in the space variable and whose $[r]$-th (resp. $[s]$-th) time (resp. space) derivative is Hölder continuous with exponent $r-[r]$ (resp. $s-[s]$). As before, if r or s is an integer, we require the corresponding derivative to be continuous.

Assume that $\lambda_a : \mathbb{R}_+ \to \mathbb{R}$, $a = 1, 2$, are functions in $C^1(\mathbb{R}_+)$ and that $E : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is a function in $C^{1,2}(\mathbb{R}_+ \times [0,1])$. Fix a density profile $\gamma : [0,1] \to (0,1)$ in $C^2([0,1])$ and assume that there exists a function ψ in $C^{1+\beta/2,2+\beta}(\mathbb{R}_+ \times [0,1])$, for some $\beta > 0$, such that $f'(\psi(t,a)) = \lambda_a(t)$ for $a = 0, 1$, $\psi(0,x) = \gamma(x)$ for $x \in [0,1]$, and such that

$$\partial_t \psi \;=\; \partial_x(D(\psi)\partial_x\psi) - \partial_x(\chi(\psi)E(t)) \quad \text{at } (t,x) = (0,0) \text{ and } (t,x) = (0,1).$$

Denote by $\rho(t,\cdot)$ the unique classical solution of the parabolic equation

$$\begin{cases} \partial_t\rho = \partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E(t))\;, \\ f'(\rho(t,0)) = \lambda_0(t)\;, \;\; f'(\rho(t,1)) = \lambda_1(t)\;, \\ \rho(0,\cdot) = \gamma(\cdot)\;. \end{cases} \tag{2.1.7}$$

We refer to Theorem 6.1 of Chapter V in [20] for the existence and the uniqueness of classical solutions of equation (2.1.7).

Denote by $\mathscr{M}_N = \mathscr{M}(\Sigma_N)$ the set of probability measures on Σ_N endowed with the weak topology. For two probability measures μ, π in $\mathscr{M}_N$, let $H_N(\mu|\pi)$ be the relative entropy of μ with respect to π:

$$H_N(\mu|\pi) \;=\; \sup_f \Big\{ \int f\, d\mu \;-\; \log \int e^f\, d\pi \Big\}\;,$$

where the supremum is carried over all functions $f : \Sigma_N \to \mathbb{R}$. It is well known [18] that the relative entropy has an explicit expression:

$$H_N(\mu|\pi) \;=\; \begin{cases} \displaystyle\int \log\frac{d\mu}{d\pi}\, d\mu & \text{if } \mu \ll \pi, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1.8}$$

Denote by $L_N(t)$, $t \geq 0$, the generator L_N introduced in (2.1.3) in which the pair (E,λ) is replaced by $(E(t),\lambda(t))$, and by $\{S^N_t : t \geq 0\}$ the semigroup associated to the generators

$N^2 L_N(t)$: $(d/dt)S_t^N = N^2 L_N(t) S_t^N$. Note that time has been speeded-up diffusively since the generator has been multiplied by N^2.

Theorem 2.1.1. *Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures, $\mu_N \in \mathscr{M}_N$, such that*

$$\lim_{N\to\infty} \frac{1}{N} H_N(\mu_N | \nu^N_{\gamma(\cdot)}) \;=\; 0\,.$$

Then, for every $t > 0$,

$$\lim_{N\to\infty} \frac{1}{N} H_N(\mu_N S_t^N | \nu^N_{\rho(t,\cdot)}) \;=\; 0\,.$$

Corollary 2.1.2. *Under the assumptions of Theorem 2.1.1, for every $t \geq 0$, every continuous function $H : [0,1] \to \mathbb{R}$, and every cylinder function $\Psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$,*

$$\lim_{N\to\infty} E_{\mu_N S_t^N} \Big[\Big| \frac{1}{N} \sum_{k=1}^{N-1} H(k/N)\, (\tau_k^{N,\lambda(t)} \Psi)(\eta) - \int_0^1 H(x)\, E_{\nu_{\rho(t,x)}}[\Psi]\, dx \Big| \Big] \;=\; 0\,.$$

2.1.3 Quasi-static Transformations

Fix $\nu > 0$, a function λ in $C^1(\mathbb{R}_+)$, and let $\alpha : \mathbb{R}_+ \to (0,1)$ be given by

$$\alpha(t) \;=\; f'(\lambda(t))\,, \tag{2.1.9}$$

Fix a function $v_0 = v_0^\nu$ in $C_0^2([0,1])$, and assume that there exists a function ψ in $C^{1+\beta/2,2+\beta}(\mathbb{R}_+ \times [0,1])$, for some $\beta > 0$, such that $\psi(t,0) = \psi(t,1) = \alpha(t)$, $t \geq 0$, $\psi(0,x) = \alpha(0) + \nu^{-1} v_0(x)$, $x \in [0,1]$, and

$$\partial_t \psi \;=\; \nu\, \partial_x (D(\psi) \partial_x \psi) \quad \text{at } (t,x) = (0,0) \text{ and } (t,x) = (0,1).$$

This means that we assume that

$$\alpha'(a) \;=\; \partial_x \Big\{ D\big(\alpha(0) + \nu^{-1} v_0(a)\big)\, \partial_x v_0(a) \Big\} \quad \text{for } a = 0, 1\,.$$

Denote by $\rho(t,x) = \rho_\nu(t,x)$ the unique classical solution of the initial–boundary value problem

$$\begin{cases} \partial_t \rho = \nu\, \partial_x (D(\rho) \partial_x \rho)\,, \\ \rho(t,0) = \rho(t,1) = \alpha(t)\,, \\ \rho(0,x) = \alpha(0) + \nu^{-1} v_0\,. \end{cases}$$

Let $u_\nu : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be the function given by

$$u_\nu(t,x) \;=\; \nu\, \{\rho_\nu(t,x) - \alpha(t)\}\,,$$

and, for each $t \geq 0$, let $v_t : [0,1] \to \mathbb{R}$ be the unique solution of the linear elliptic equation

$$\begin{cases} \partial_x (D(\alpha(t)) \partial_x v_t) = \alpha'(t)\,, \\ v_t(0) = v_t(1) = 0\,. \end{cases} \tag{2.1.10}$$

Proposition 2.1.3. *Assume that λ belongs to $C^2(\mathbb{R}_+)$ and that v_0 belongs to $C_0^4([0,1])$. Then, for each $t \geq 0$,*

$$\lim_{\nu\to\infty} \int_0^1 [u_\nu(t,x) - v_t(x)]^2\, dx \;=\; 0\,.$$

One can strengthen the topology in which the convergence occurs, but we do not seek optimal conditions here.

Inspired by the previous result, consider a function λ in $C^1(\mathbb{R}_+)$, and let $\alpha : \mathbb{R}_+ \to (0,1)$ be given by (2.1.9). Fix a sequence ϵ_N which vanishes as $N \to \infty$ and a function $\gamma = \gamma_N$ in $C_0^2([0,1])$. Assume that there exists a function ψ in $C^{1+\beta/2,2+\beta}(\mathbb{R}_+ \times [0,1])$, for some $\beta > 0$, such that $\psi(t,0) = \psi(t,1) = \alpha(t)$, $t \geq 0$, $\psi(0,x) = \alpha(0) + \epsilon_N \gamma(x)$, $x \in [0,1]$, such that

$$\alpha'(a) \;=\; \partial_x \Big\{ D\big(\alpha(a) + \epsilon_N \gamma(a)\big)\, \partial_x \gamma(a) \Big\} \quad \text{for } a = 0, 1\,.$$

Denote by $\rho_N(t,x)$ the solution of

$$\begin{cases} \partial_t \rho = \epsilon_N^{-1} \partial_x (D(\rho) \partial_x \rho)\,, \\ \rho(t,0) = \rho(t,1) = \alpha(t)\,, \\ \rho(0,x) = \alpha(0) + \epsilon_N \gamma(x)\,. \end{cases} \tag{2.1.11}$$

Denote by $\mathscr{L}_N(t)$ the generator L_N introduced in (2.1.3) with $E = 0$ and $\lambda_0 = \lambda_1 = \lambda(t)$. Let $\{T_t^N : t \geq 0\}$ be the semigroup associated to the generator $\epsilon_N^{-1} N^2 \mathscr{L}_N(t)$. Note that time has been speed-up by $\epsilon_N^{-1} N^2$.

Theorem 2.1.4. *Assume that $\epsilon_N^4 N \to \infty$, that λ belongs to $C^2(\mathbb{R}_+)$, and that γ belongs to $C_0^4([0,1])$. Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures, $\mu_N \in \mathscr{M}_N$, such that*

$$\lim_{N\to\infty} \frac{1}{N\epsilon_N^2}\, H_N(\mu_N | \nu^N_{\rho_N(0,\cdot)}) \;=\; 0\,. \tag{2.1.12}$$

Then, for every $t > 0$,

$$\lim_{N\to\infty} \frac{1}{N\epsilon_N^2}\, H_N(\mu_N T_t^N | \nu^N_{\rho_N(t,\cdot)}) \;=\; 0\,.$$

Corollary 2.1.5. *Under the assumptions of Theorem 2.1.4, for every $t \geq 0$, every continuous function $H : [0,1] \to \mathbb{R}$, and every cylinder function $\Psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$,*

$$\lim_{N\to\infty} E_{\mu_N T_t^N} \Big[\frac{1}{\epsilon_N} \Big| \frac{1}{N} \sum_{k=1}^{N-1} H(k/N)\, (\tau_k^{N,\lambda(t)} \Psi)(\eta) - \int_0^1 H(x)\, E_{\nu_{v_N(t,x)}}[\Psi]\, dx \Big| \Big] \;=\; 0\,,$$

where $v_N(t,x) = \alpha(t) + \epsilon_N v(t,x)$, $v(t,x)$ being the unique classical solution of the elliptic equation (2.1.10).

2.2 Proof of the Main Results

We present in Theorem 2.2.1 below a general statement from which one can easily deduce Theorems 2.1.1 and 2.1.4. For a fixed chemical potential $\lambda = (\lambda_0, \lambda_1)$ and a continuous external field $E : [0,1] \to \mathbb{R}$, denote by $\bar\rho_{\lambda,E} : [0,1] \to \mathbb{R}$ the solution of the elliptic equation

$$\begin{cases} \partial_x (D(\rho) \partial_x \rho) - \partial_x (\chi(\rho) E) \;=\; 0\,, \\ f'(\rho(0)) = \lambda_0\,, \;\; f'(\rho(1)) = \lambda_1\,, \end{cases} \tag{2.2.1}$$

Fix sequences $\{\epsilon_N : N \geq 1\}$, $\{\ell_N : N \geq 1\}$ such that $\ell_N \to \infty$, $\epsilon_N \to 0$. Consider a time-dependent external field E in $C^{1,2}(\mathbb{R}_+ \times [0,1])$ and a time-dependent chemical potential $\lambda(t) = (\lambda_0(t), \lambda_1(t))$ such that λ_0, $\lambda_1 \in C^1(\mathbb{R}_+)$. Fix a density profile $\gamma = \gamma_N$ in $C^2([0,1])$ and assume that there exists a function ψ in $C^{1+\beta/2,2+\beta}(\mathbb{R}_+ \times [0,1])$, $\beta > 0$, such that $f'(\psi(t,a)) = \lambda_a(t)$ for $a = 0, 1$, $\psi(0,x) = \bar{\rho}_{\lambda(0),E(0)}(x) + \epsilon_N \gamma(x)$ for $x \in [0,1]$, and such that

$$\partial_t \psi = \ell_N \Big\{ \partial_x (D(\psi)\partial_x \psi) - \partial_x(\chi(\psi)E(t)) \Big\} \quad \text{at } (t,x) = (0,0) \text{ and } (t,x) = (0,1).$$

Denote by $\rho_N(t,\cdot)$ the unique weak solution of the parabolic equation

$$\begin{cases} \partial_t \rho = \ell_N \Big\{ \partial_x (D(\rho)\partial_x \rho) - \partial_x(\chi(\rho)E) \Big\} , \\ f'(\rho(t,0)) = \lambda_0(t) \ , \ \ f'(\rho(t,1)) = \lambda_1(t) \ , \\ \rho(0,x) = \bar{\rho}_{\lambda(0),E(0)} + \epsilon_N \gamma(x) \ . \end{cases} \tag{2.2.2}$$

In Theorem 2.2.1 the following conditions on the solution of equation (2.2.2) are needed: For every $T > 0$, there exists $0 < \delta < 1$ such that

$$\delta \ \le \ \rho_N(t,x) \ \le \ 1 - \delta \quad \text{for all} \quad 0 \le x \le 1 \ , \ \ 0 \le t \le T \ , \ \ N \ge 1 \ . \tag{2.2.3}$$

To explain the second condition, observe that we may rewrite the PDE (2.2.2) as

$$\partial_t \rho = \ell_N \partial_x \big\{ \chi(\rho) \big[\partial_x f'(\rho) - E \big] \big\}$$

because $\chi(\rho) f''(\rho) = D(\rho)$ by Einstein relation (2.1.6). Let

$$F_N(t,x) \ = \ \partial_x f'(\rho_N(t,x)) \ - \ E(t,x)$$

We assume that for every $T > 0$, there exists a finite constant C_0 such that for all $N \ge 1$, $0 \le t \le T$,

$$\|F_N(t)\|_\infty \ \le \ \frac{C_0}{\ell_N} \ , \quad \|\partial_x F_N(t)\|_\infty \ \le \ \frac{C_0}{\ell_N} \ . \tag{2.2.4}$$

Note that for this condition to be fulfilled at $t = 0$, we need $\ell_N \epsilon_N$ to be bounded:

$$\ell_N \, \epsilon_N \ \le \ C_0 \tag{2.2.5}$$

for some finite constant C_0.

Consider two non-decreasing sequences K_N, J_N. We write

$$K_N \ll J_N \text{ if } K_N / J_N \to 0 \text{ as } N \to \infty.$$

Recall that we denote by $L_N(t)$ the generator L_N introduced in (2.1.3) with $E(t)$, $\lambda(t)$ in place of E, λ, respectively. Let $\{\mathfrak{S}_t^N : t \ge 0\}$ be the semigroup associated to the generators $\{\ell_N N^2 L_N(s) : s \ge 0\}$: $(d/dt)\mathfrak{S}_t^N = \ell_N N^2 \mathfrak{S}_t^N L_N(t)$.

Theorem 2.2.1. *Consider a continuous external field $E(t,x)$ and a continuous chemical potential $\lambda(t) = (\lambda_0(t), \lambda_1(t))$. Assume that γ belongs to $C_0^2([0,1])$, that conditions (2.2.3), (2.2.4), (2.2.5) hold, and that $\epsilon_N^{-4} \ll N$. Let $\{\mu_N : N \ge 1\}$ be a sequence of probability measures, $\mu_N \in \mathscr{M}_N$, such that*

$$\lim_{N \to \infty} \frac{1}{N \epsilon_N^2} \, H_N(\mu_N | \nu^N_{\rho_N(0,\cdot)}) \ = \ 0 \ . \tag{2.2.6}$$

Then, for every $t > 0$,

$$\lim_{N \to \infty} \frac{1}{N \epsilon_N^2} \, H_N(\mu_N \mathfrak{S}_t^N | \nu^N_{\rho_N(t,\cdot)}) \ = \ 0 \ .$$

The proof of Theorem 2.2.1 is divided in several steps. Fix a density $\theta \in (0,1)$, and denote by $\nu_\theta = \nu_\theta^N$ the product measure on Σ_N with density θ:

$$\nu_\theta(\eta) \;=\; \frac{1}{Z_N(\theta)}\,\exp\Big\{\beta\sum_{j=1}^{N-1}\eta_j\Big\}\;, \tag{2.2.7}$$

where $Z_N(\theta)$ is the partition function which turns ν_θ into a probability measure, and

$$\beta \;:=\; f'(\theta) \;=\; \log\frac{\theta}{1-\theta}\;. \tag{2.2.8}$$

We use the same notation ν_θ to represent the the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with density θ.

Let $L^2(\nu_\theta)$ be the space of functions $f:\Sigma_N\to\mathbb{R}$ endowed with the scalar product

$$\langle f\,,\,g\rangle_{\nu_\theta} \;=\; \int f(\eta)\,g(\eta)\,\nu_\theta(d\eta)\;.$$

Denote by $L_N^* = L_N^{\lambda,E,*}$ the adjoint in $L^2(\nu_\theta)$ of the generator L_N introduced in (2.1.3). A simple computation shows that for all $f:\Sigma_N\to\mathbb{R}$,

$$(L_N^* f)(\eta) \;=\; (L_{0,1}^* f)(\eta) \;+\; \sum_{j=1}^{N-2}(L_{j,j+1}^* f)(\eta) \;+\; (L_{N-1,N}^* f)(\eta)\;, \tag{2.2.9}$$

where, for $1\le j\le N-2$,

$$\begin{aligned}
(L_{N-1,N}^* f)(\eta) \;&=\; c_{N-1,N}^{N,\lambda,E}(\sigma^{N-1,N}\eta)\,e^{\beta(1-2\eta_{N-1})}\,f(\sigma^{N-1,N}\eta) \;-\; c_{N-1,N}^{N,\lambda,E}(\eta)\,f(\eta)\;,\\
(L_{j,j+1}^* f)(\eta) \;&=\; c_{j,j+1}^{N,\lambda,E}(\sigma^{j,j+1}\eta)\,f(\sigma^{j,j+1}\eta) \;-\; c_{j,j+1}^{N,\lambda,E}(\eta)\,f(\eta)\;,\\
(L_{0,1}^* f)(\eta) \;&=\; c_{0,1}^{N,\lambda,E}(\sigma^{0,1}\eta)\,e^{\beta(1-2\eta_1)}\,f(\sigma^{0,1}\eta) \;-\; c_{0,1}^{N,\lambda,E}(\eta)\,f(\eta)\;.
\end{aligned}$$

In this formula, β is the chemical potential associated to the density θ, which has been introduced in (2.2.8). It follows from the previous formula that the adjoint of $L_N(t)$ in $L^2(\nu_\theta)$, denoted by $L_N^*(t)$, is given by (2.2.9) with E and λ replaced by $E(t)$ and $\lambda(t)$.

Proof of Theorem 2.2.1. Fix sequences ℓ_N, ϵ_N satisfying the assumptions of the theorem, and let γ be a function in $C_0^2([0,1])$. Denote by $\rho(t,x)=\rho_N(t,x)$ the solution of (2.2.2). Consider a sequence of probability measures $\{\mu_N : N\ge 1\}$, $\mu_N\in\mathscr{M}_N$, satisfying (2.2.6). Let $\alpha(t)=(\alpha_0(t),\,\alpha_1(t))$ be the density of particles associated to the chemical potential $\lambda(t)$:

$$\alpha_0(t) \;=\; \frac{e^{\lambda_0(t)}}{1+e^{\lambda_0(t)}}\;,\quad \alpha_1(t) \;=\; \frac{e^{\lambda_1(t)}}{1+e^{\lambda_1(t)}}\;. \tag{2.2.10}$$

Recall that $\{\mathfrak{S}_t^N : t\ge 0\}$ represents the semigroup associated to the generator $N^2\ell_N L_N(t)$, and let

$$f_t \;=\; \frac{d\mu_N\mathfrak{S}_t^N}{d\nu_\theta}\;,\quad \psi_t \;=\; \frac{d\nu^N_{\rho_N(t,\cdot)}}{d\nu_\theta}\;. \tag{2.2.11}$$

A simple computation yields

$$\psi_t(\eta) \;=\; \frac{Z_N(\theta)}{Z_N(\rho(t))}\,\exp\Big\{\sum_{j=1}^{N-1}\eta_j[f'(\rho(t,j/N))-f'(\theta)]\Big\}\;, \tag{2.2.12}$$

where $\rho(t,x)=\rho_N(t,x)$ is the solution of equation (2.2.2), $Z_N(\theta)$, f have been introduced in (2.2.7), (2.1.6), respectively, and $Z_N(\rho(t))$ is the normalizing constant given by

$$Z_N(\rho(t)) \;=\; \exp\Big\{ -\sum_{j=1}^{N-1} \log[1-\rho(t,j/N)] \Big\} \,.$$

With this notation, in view of (2.1.8),

$$H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t,\cdot)}) \;=\; \int f_t \log \frac{f_t}{\psi_t}\, d\nu_\theta \,.$$

Moreover, a usual computation shows that the density f_t solves the Kolmogorov forward equation

$$\frac{d}{dt} f_t \;=\; N^2 \ell_N L_N^*(t) f_t \,. \tag{2.2.13}$$

The proof of Theorem 2.2.1 is divided in three steps.

Step 1: Entropy production. A computation, similar to the one presented in the proof of Lemma 1.4 in [18, Chapter 6], yields that

$$\frac{d}{dt} H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t,\cdot)}) \;\le\; \int \frac{N^2 \ell_N L_N^*(t)\psi_t - \partial_t \psi_t}{\psi_t}\, f_t\, d\nu_\theta \,. \tag{2.2.14}$$

Let h and $g : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ be the cylinder functions given by

$$h(\xi) \;=\; \sum_{a=1}^{m} m_a\, h_a(\xi)\,, \quad g(\xi) \;=\; \frac{1}{2}\, [\xi_1 - \xi_0]^2\, c(\xi)\,, \tag{2.2.15}$$

where $m_a = \sum_k k \mu_a(k)$. Recall the definition of the operators $\tau_j^{N,\lambda}$, $N \ge 1$, $1 \le j < N-1$, introduced just above (2.1.2), and let $\tau_j^N(t) = \tau_j^{N,\lambda(t)}$. A long, but straightforward, computation which uses the identity (2.1.2), yields that

$$\frac{N^2 \ell_N L_N^*(t)\psi_t - \partial_t \psi_t}{\psi_t} \;=\; \ell_N \{ I_1 \,+\, I_2 \,+\, I_3 \} \,+\, O_N(\ell_N)\,,$$

where $O_N(\ell_N)$ represents an error absolutely bounded by $C_0 \ell_N$, C_0 being a finite constant independent of N, and where

$$\begin{aligned} I_1 \;=\; & \sum_{j=1}^{N-2} G_1(t,j/N)\, (\tau_j^N(t)\, h)(\eta) \;+\; \sum_{j=1}^{N-2} G_2(t,j/N)\, (\tau_j^N(t)\, g)(\eta) \\ & -\; \sum_{j=1}^{N-1} \frac{(\partial_t \rho)}{\chi(\rho)}(t,j/N)\, [\eta(j) - \rho(t,j/N)]\,, \end{aligned}$$

$$\begin{aligned} I_2 \;=\; & N\, H_-(t) \sum_{a=1}^{m} \sum_{k\in\mathbb{Z}} \mu_a(k) \sum_{j=1-k}^{0} (\tau_j^N(t)\, h_a)(\eta) \\ & -\; N\, H_+(t) \sum_{a=1}^{m} \sum_{k\in\mathbb{Z}} \mu_a(k) \sum_{j=N-1-k}^{N-2} (\tau_j^N(t)\, h_a)(\eta)\,, \end{aligned}$$

$$I_3 \;=\; N\, H_+(t)\, (\tau_{N-1}^N(t)\, c)(\eta)\, [\eta_{N-1} - \alpha_1(t)] \;-\; N\, H_-(t)\, (\tau_0^N(t)\, c)(\eta)\, [\eta_1 - \alpha_0(t)]\,.$$

In these formulas,

$$
\begin{aligned}
G_1(t,x) &= \partial_x\{\partial_x f'(\rho(t,x)) - E(t,x)\}\,, \\
G_2(t,x) &= \partial_x f'(\rho(t,x))\{\partial_x f'(\rho(t,x)) - E(t,x)\}\,, \\
H_-(t) &= \partial_x f'(\rho(t,0)) - E(t,0)\,, \quad H_+(t) = \partial_x f'(\rho(t,1)) - E(t,1)\,,
\end{aligned}
$$

and $\alpha_0(t)$, $\alpha_1(t)$ are the densities at the boundary, defined in (2.2.10). Note that $G_1 = \partial_x F_N$, $G_2 = F_N^2 + EF_N$, $H_-(t) = F_N(t,0)$ and $H_+(t) = F_N(t,1)$. In particular, by (2.2.4), there exists a finite constant C_0 such that for all $N \geq 1$, $0 \leq t \leq T$,

$$
\|G_1(t)\|_\infty \;\leq\; \frac{C_0}{\ell_N}\,, \quad \|G_2(t)\|_\infty \;\leq\; \frac{C_0}{\ell_N}\,, \quad \|H_\pm(t)\|_\infty \;\leq\; \frac{C_0}{\ell_N}\,. \tag{2.2.16}
$$

For a cylinder function $\Psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, let $\hat\Psi : [0,1] \to \mathbb{R}$ be the polynomial given by

$$
\hat\Psi(\theta) \;=\; E_{\nu_\theta}[\Psi]\,, \tag{2.2.17}
$$

where, we recall, ν_θ is the Bernoulli product measure with density θ. By (2.2.15), (2.1.5) and the equality (2.5.2) we shall prove in section 2.5,

$$
\hat h'(\theta) \;=\; D(\theta)\,, \quad \hat g(\theta) \;=\; \chi(\theta)\,. \tag{2.2.18}
$$

We claim that, in the first line of I_1, the replacement of the cylinder functions $\tau_j^N(t)h$, $\tau_j^N(t)g$ by $\tau_j^N(t)h - \hat h(\rho(t,j/N))$, $\tau_j^N(t)g - \hat g(\rho(t,j/N))$, respectively, produces an error absolutely bounded by a finite constant independent of N. Similarly, the replace-cement in the two lines of I_2 of the cylinder functions $\tau_j^N(t)h_a$, $j \sim 0$, $\tau_k^N(t)h_a$, $k \sim N$, by $\tau_j^N(t)h_a - \hat h_a(\alpha_0(t))$, $\tau_k^N(t)h_a - \hat h_a(\alpha_1(t))$ produces an error of the same order.

Indeed, denote by J_1 (resp. J_2) the first line of I_1 (resp. the two lines of I_2) with the cylinder functions $\tau_j^N(t)h$, $\tau_j^N(t)g$ (resp. $\tau_j^N(t)h_a$, $j \sim 0$, $\tau_k^N(t)h_a$, $k \sim N$) replaced by $\hat h(\rho(t,j/N))$, $\hat g(\rho(t,j/N))$ (resp. $\hat h_a(\alpha_0(t))$, $\hat h_a(\alpha_1(t))$). In the expression of J_2, observe that $\sum_k k\mu_a(k) = m_a$. For any Lipschitz-continuous function $G : [0,1] \to \mathbb{R}$, and for any non-negative integers p, q,

$$
\sum_{j=p}^{N-q} G(j/N) \;=\; N\int_0^1 G(x)\,dx \;+\; O_N(1)\,,
$$

where $O_N(1)$ represents an error absolutely bounded by a finite constant independent of N. It follows from this estimate, from an integration by parts, and from the identities (2.1.6), (2.2.18) that $J_1 + J_2$ is absolutely bounded by a finite constant independent of N, proving the claim.

An elementary computation gives that

$$
\frac{(\partial_t\rho)}{\chi(\rho)} \;=\; \{\partial_x^2 f'(\rho) - \partial_x E\}\,D(\rho) \;+\; \{[\partial_x f'(\rho)]^2 - E\,\partial_x f'(\rho)\}\,\chi'(\rho)\,.
$$

In conclusion, in view of (2.2.18), up to this point, we have shown that

$$
\frac{N^2\ell_N L_N^*(t)\psi_t - \partial_t\psi_t}{\psi_t} \;=\; \ell_N\{\hat I_1 \;+\; \hat I_2 \;+\; I_3\} \;+\; O(\ell_N)\,, \tag{2.2.19}
$$

where

$$\hat{I}_1(t,\eta) \;=\; \sum_{j=1}^{N-2} G_1(t,j/N)\, V_N(h;t,j,\eta) \;+\; \sum_{j=1}^{N-2} G_2(t,j/N)\, V_N(g;t,j,\eta)\;,$$

$$\begin{aligned}\hat{I}_2(t,\eta) \;&=\; N\, H_-(t) \sum_{a=1}^{m} \sum_{k\in\mathbb{Z}} \mu_a(k) \sum_{j=1-k}^{0} \big\{ (\tau_j^N(t) h_a)(\eta) - \hat{h}_a(\alpha_0(t)) \big\} \\ &-\; N\, H_+(t) \sum_{a=1}^{m} \sum_{k\in\mathbb{Z}} \mu_a(k) \sum_{j=N-1-k}^{N-2} \big\{ (\tau_j^N(t) h_a)(\eta) - \hat{h}_a(\alpha_1(t)) \big\}\;,\end{aligned}$$

and, for a cylinder function $\varphi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$,

$$V_N(\varphi;t,j,\eta) \;=\; (\tau_j^N(t)\varphi)(\eta) - \hat{\varphi}(\rho(t,j/N)) - \hat{\varphi}'(\rho(t,j/N))[\eta_j - \rho(t,j/N)]\;.$$

Step 2: A mesoscopic entropy estimate. Denote by $D(\mathbb{R}_+,\Sigma_N)$ the space of right-continuous trajectories $x : \mathbb{R}_+ \to \Sigma_N$ with left limits. For each probability measure μ in $\mathscr{M}_N$, denote by $\mathbb{P}_\mu^N$ the probability measure on $D(\mathbb{R}_+,\Sigma_N)$ induced by the Markov chain with generator $\ell_N N^2 L_N(t)$ starting from the distribution μ. Expectation with respect to $\mathbb{P}_\mu^N$ is represented by $\mathbb{E}_\mu^N$.

Recall that $\epsilon_N^{-4} \ll N$. Let $M_N = \epsilon_N^{-2}$, and fix a sequence $\{K_N : N \ge 1\}$ such that $M_N \ll K_N$, $M_N\, K_N \ll N$. Let

$$\tilde{I}_{1,N}(t,\eta) \;=\; \sum_{j=K_N+1}^{N-K_N-1} G_1(t,j/N)\, V_N(\hat{h};t,j,\eta) \;+\; \sum_{j=1}^{N-1} G_2(t,j/N)\, V_N(\hat{g};t,j,\eta)\;,$$

where, for a smooth function $\hat{\varphi} : [0,1] \to \mathbb{R}$,

$$V_N(\hat{\varphi};t,j,\eta) \;=\; \hat{\varphi}(\eta^{K_N}(j)) - \hat{\varphi}(\rho(t,j/N)) - \hat{\varphi}'(\rho(t,j/N))[\eta^{K_N}(j) - \rho(t,j/N)]\;.$$

Note that in the definition of $\hat{I}_1(t,\eta)$ the sum is carried over $1 \le j \le N-1$, while in the definition of $\tilde{I}_1(t,\eta)$ it is carried over $K_N + 1 \le j \le N - K_N - 1$. In view of (2.2.16), this produces an error of order K_N/ℓ_N in the difference between $\hat{I}_1(t,\eta)$ and $\tilde{I}_1(t,\eta)$.

By (2.2.16) and Lemma 2.3.2, since $M_N K_N \ll N$,

$$\lim_{N\to\infty} \frac{M_N \ell_N}{N}\, \mathbb{E}_{\mu_N}\Big[\int_0^t \{ \hat{I}_{1,N}(s,\eta_s) - \tilde{I}_{1,N}(s,\eta_s) \}\, ds \Big] \;=\; 0\;,$$

and

$$\lim_{N\to\infty} \frac{M_N \ell_N}{N}\, \mathbb{E}_{\mu_N}\Big[\int_0^t \hat{I}_2(s,\eta_s)\, ds \Big] \;=\; 0\;, \quad \lim_{N\to\infty} \frac{M_N \ell_N}{N}\, \mathbb{E}_{\mu_N}\Big[\int_0^t I_3(s,\eta_s)\, ds \Big] \;=\; 0\;.$$

On the other hand, by definition of M_N, by (2.2.5) and by the assumption on ϵ_N, $M_N \ell_N = \epsilon_N^{-2} \ell_N \le C_0 \epsilon_N^{-3} \ll N$. Therefore, in view of (2.2.14), (2.2.19),

$$\begin{aligned}\frac{1}{N\varepsilon_N^2}\, H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t,\cdot)}) \;&\le\; \frac{1}{N\varepsilon_N^2}\, H(\mu_N | \nu_{\rho_N(0,\cdot)}) \\ &+\; \frac{M_N \ell_N}{N}\, \mathbb{E}_{\mu_N}\Big[\int_0^t \tilde{I}_{1,N}(s,\eta_s)\, ds \Big] \;+\; R_N\;,\end{aligned}$$

where R_N vanishes as $N \to \infty$.

Step 3: A large deviations estimate. A Taylor expansion up to the second order shows that $V_N(\hat{\varphi};t,j,\eta)$ is absolutely bounded by $C_0[\eta^{K_N}(j)-\rho(t,j/N)]^2$. The second term on the right hand side of the previous equation is thus bounded above by

$$C_0\,\mathbb{E}_{\mu_N}\Big[\int_0^t \frac{M_N\ell_N}{N}\sum_{j=K_N+1}^{N-K_N-1} J(s,j/N)\,[\eta_s^{K_N}(j)-\rho(s,j/N)]^2\,ds\Big]\,,$$

where $J(t,x)=|G_1(t,x)|+|G_2(t,x)|$. Since $M_N=\epsilon_N^{-2}$, by the entropy inequality, the previous expression is less than or equal to

$$\frac{C_0}{AN\varepsilon_N^2}\int_0^t H(\mu_N\mathfrak{S}_s^N|\nu_{\rho_N(s,\cdot)})\,ds$$
$$+\int_0^t\frac{\varepsilon_N^{-2}}{AN}\log E_{\nu_{\rho_N(s,\cdot)}}\Big[\exp\Big\{A\ell_N\sum_{j=K_N+1}^{N-K_N-1}J(s,j/N)\,[\eta^{K_N}(j)-\rho(s,j/N)]^2\Big\}\Big]\,ds$$

for every $A>0$. By Hölder's inequality and since $\nu_{\rho_N(s,\cdot)}$ is a product measure, the second term of the last sum is less than or equal to

$$\int_0^t\frac{\varepsilon_N^{-2}}{ANK_N}\sum_{j=K_N+1}^{N-K_N-1}\log E_{\nu_{\rho_N(s,\cdot)}}\Big[\exp\Big\{A\ell_N J(s,j/N)\,K_N\,[\eta^{K_N}(j)-\rho(s,j/N)]^2\Big\}\Big]\,ds$$

By (2.2.3) and (2.2.16), $\ell_N J(s,j/N)\le C_0$ and $\delta\le\rho_N(s,x)\le 1-\delta$ for some $\delta>0$. Therefore, since $\nu_{\rho_N(s,\cdot)}$ is the product measure with density $\rho_N(s,\cdot)$, there exists A_0 such that for

$$E_{\nu_{\rho_N(s,\cdot)}}\Big[\exp\Big\{AC_0K_N\,[\eta^{K_N}(j)-\rho(s,j/N)]^2\Big\}\Big]\;\le\;C_0'$$

for all $0<A\le A_0$. The previous integral is therefore less than or equal to $C_0\varepsilon_N^{-2}/AK_N\ll 1$. This proves that there exists a finite constant C_0 such that

$$\frac{1}{N\varepsilon_N^2}H(\mu_N\mathfrak{S}_t^N|\nu_{\rho_N(t,\cdot)})\;\le\;\frac{1}{N\varepsilon_N^2}H(\mu_N|\nu_{\rho_N(0,\cdot)})$$
$$+\;C_0\int_0^t\frac{1}{N\varepsilon_N^2}H(\mu_N\mathfrak{S}_s^N|\nu_{\rho_N(s,\cdot)})\,ds\;+\;R_N\,,$$

where R_N vanishes as $N\to\infty$. To conclude the proof of Theorem 2.2.1 it remains to apply Gronwall inequality. □

Proof of Theorem 2.1.1. Set $\epsilon_N=\ell_N=1$. (2.2.3). Conditions (2.2.4) and (2.2.5) are trivially satisfied. The assertion of Theorem 2.1.1 follows therefore from Theorem 2.2.1. □

Proof of Theorem 2.1.4. Assume that the external field vanishes: $E(t,x)=0$ and that the left and right chemical potentials are equal, $\lambda_0(t)=\lambda_1(t)$, $t\ge 0$. In this case the solution of the elliptic equation (2.2.1) $\bar\rho_{\lambda,0}$ is constant in space, $\bar\rho_{\lambda,0}(x)=\alpha$, where $\alpha=f'(\lambda)$.

Condition (2.2.3) for N large enough follows from Proposition 2.4.1. Condition (2.2.4), which can be read as conditions on $\partial_x\rho_N$ and $\partial_x^2\rho_N$, follows from Propositions 2.4.1 and 2.4.2. □

Proofs of Corollaries 2.1.2 and 2.1.5. The proofs are analogous to the one of Corollary 1.3, Chapter 6 in [18], provided we replace in the statement of Corollary 2.1.5 $\nu_{v_N(t,x)}$ by $\nu_{\rho_N(t,x)}$. However, since Ψ is a cylinder function,

$$\frac{1}{\epsilon_N}\Big|\int_0^1 H(x)\{\,E_{\nu_{v_N(t,x)}}[\Psi]-E_{\nu_{\rho_N(t,x)}}[\Psi]\}\,dx\Big|\;\le\;\frac{C_0}{\epsilon_N}\int_0^1\big|v_N(t,x)-\rho_N(t,x)\big|\,dx\,.$$

By definition of v_N, the right hand side is equal to

$$C_0 \int_0^1 \big| u_N(t,x) - v(t,x) \big| \, dx \ ,$$

where $u_N(t) = \epsilon_N^{-1}[\rho_N(t) - \alpha(t)]$. It remains to recall the statement of Proposition 2.1.3. □

2.3 Entropy Estimates

We adopt in this section the notation and the set-up introduced in the previous one. Recall from (2.2.7) that $\theta \in (0,1)$ is a fixed parameter and that ν_θ is the product measure on Σ_N with density θ. It is not difficult to show that there exists a finite constant $C_0 = C_0(\theta)$ such that

$$\sup_\mu H(\mu|\nu_\theta) \ \le \ C_0 N \ ,$$

where the supremum is carried over all probability measures μ on Σ_N.

Fix a smooth function $\widehat{\lambda} : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $\widehat{\lambda}(t,a) = \lambda_a(t)$, $t \ge 0$, $a = 0, 1$. Let $\alpha(t,x) = e^{\widehat{\lambda}(t,x)}/[1+e^{\widehat{\lambda}(t,x)}]$, and denote by $\nu^N_{\alpha(t)}$, $t \ge 0$, the product measure on Σ_N associated to the density $\alpha(t,x)$:

$$\nu^N_{\alpha(t)}(\eta) \ = \ \frac{1}{\hat{Z}_N(t)} \, \exp\Big\{ \sum_{j=1}^{N-1} \eta_j \, f'(\alpha(t,j/N)) \Big\} \ ,$$

where $\hat{Z}_N(t)$ is the normalizing constant given by

$$\hat{Z}_N(t) \ = \ \exp\Big\{ - \sum_{j=1}^{N-1} \log[1-\alpha(t,j/N)] \Big\} \ .$$

Note that $\alpha(t,x)$ takes values in $(0,1)$. In particular, the quantities introduced above are well defined.

Recall that $\{\mathfrak{S}^N_t : t \ge 0\}$ represents the semigroup associated to the generator $N^2\ell_N L_N(t)$, $(d/dt)\mathfrak{S}^N_t = N^2\ell_N L_N(t)\mathfrak{S}^N_t$, and that $H_N(\mu|\pi)$ stands for the relative entropy of μ with respect to π. Denote by $D^{\alpha(t)}_N(\cdot)$, $t \ge 0$, the functional which acts on functions $h : \Sigma_N \to \mathbb{R}$, as

$$\begin{aligned} D^{\alpha(t)}_N(h) \ &= \ \sum_{j=1}^{N-2} \int c^{N,\lambda(t)}_{j,j+1}(\eta) \, [h(\sigma^{j,j+1}\eta) - h(\eta)]^2 \, d\nu^N_{\alpha(t)}(\eta) \\ &+ \ \int c^{N,\lambda(t)}_{0,1}(\eta) \, r^{\lambda(t)}_{0,1}(\eta) \, [h(\sigma^{0,1}\eta) - h(\eta)]^2 \, d\nu^N_{\alpha(t)}(\eta) \\ &+ \ \int c^{N,\lambda(t)}_{N-1,N}(\eta) \, r^{\lambda(t)}_{N-1,N}(\eta) \, [h(\sigma^{N-1,N}\eta) - h(\eta)]^2 \, d\nu^N_{\alpha(t)}(\eta) \ , \end{aligned}$$

Lemma 2.3.1. *Fix a sequence $\{\mu_N : N \ge 1\}$ of probability measures, $\mu_N \in \mathscr{M}_N$. For every $T > 0$, there exists a finite constant C_0, depending only on $E(t)$, $\alpha(t)$, $0 \le t \le T$, such that for all $0 \le t \le T$,*

$$H_N(\mu_N \mathfrak{S}^N_t | \nu^N_{\alpha(t)}) \ \le \ -\frac{N^2\,\ell_N}{2} \int_0^t D^{\alpha(s)}_N(\sqrt{g_s}) \, ds \ + \ C_0 \, N \, \ell_N \ ,$$

where $g_t = g^N_t = d\mu_N \mathfrak{S}^N_t / d\nu^N_{\alpha(t)}$.

Proof. In this proof, C_0 represents a finite constant which may depend only on θ, $E(t)$, $\alpha(t)$, $0 \le t \le T$, but not on N.

Fix a sequence $\{\mu_N : N \ge 1\}$ of probability measures, $\mu_N \in \mathscr{M}_N$. Recall the definition of $f_t = f_t^N$, introduced in (2.2.11), and let ϕ_t, $t \ge 0$, be given by

$$\phi_t \;=\; \frac{d\nu^N_{\alpha(t)}}{d\nu_\theta} \quad \text{so that} \quad g_t \;=\; \frac{f_t}{\phi_t}\;.$$

By definition,

$$H_N(t) \;:=\; H(\mu_N \mathfrak{S}^N_t | \nu^N_{\alpha(t)}) \;=\; \int f_t \log \frac{f_t}{\phi_t}\, d\nu_\theta\;,$$

so that

$$\frac{d}{dt} H_N(t) \;=\; N^2\, \ell_N \int f_t\, L_N(t) \log \frac{f_t}{\phi_t}\, d\nu_\theta \;-\; \int f_t\, \partial_t \log \phi_t\, d\nu_\theta\;.$$

The second term on the right hand side is clearly bounded by $C_0 N$. On the other hand, since $a \log b/a \le 2\sqrt{a}(\sqrt{b} - \sqrt{a})$, for all a, $b > 0$, the first term on the right hand side is less than or equal to

$$2N^2\, \ell_N \int h_t\, L_N(t)\, h_t\, d\nu^N_{\alpha(t)}\;,$$

where $h_t = \sqrt{g_t} = \sqrt{f_t/\phi_t}$.

Recall the definition of the generator $L_N(t)$ introduced in (2.1.3). Denote by $L^o_N(t)$ the piece of $L_N(t)$ which corresponds to the sum for j in the range $1 \le j \le N-2$, and denote by $L^b_N(t)$ the remaining two terms. A change of variables $\eta' = \sigma^{j,j+1}\eta$, $1 \le j \le N-2$, yields

$$2 \int h_t\, L^o_N(t)\, h_t\, d\nu^N_{\alpha(t)} \;=\; -\sum_{j=1}^{N-2} \int c^{N,\lambda(t)}_{j,j+1}(\eta)\, [h_t(\sigma^{j,j+1}\eta) - h_t(\eta)]^2\, d\nu^N_{\alpha(t)}(\eta) \;+\; R_N\;,$$

where R_N is a remainder absolutely bounded by

$$\frac{C_0}{N} \sum_{j=1}^{N-2} \int c^{N,\lambda(t)}_{j,j+1}(\eta)\, [h_t(\sigma^{j,j+1}\eta) + h_t(\eta)]\, |h_t(\sigma^{j,j+1}\eta) - h_t(\eta)|\, d\nu^N_{\alpha(t)}(\eta)$$

for some finite constant C_0. By Young's inequality, and since $g_t = h_t^2$ is a density with respect to $\nu^N_{\alpha(t)}$, the previous expression is bounded by the sum of a term which can be absorbed by the first term on the right hand side of the penultimate displayed equation with a term bounded by C_0/N, that is,

$$2 \int h_t\, L^o_N(t)\, h_t\, d\nu^N_{\alpha(t)} \;\le\; -\frac{1}{2} \sum_{j=1}^{N-2} \int c^{N,\lambda(t)}_{j,j+1}(\eta)\, [h_t(\sigma^{j,j+1}\eta) - h_t(\eta)]^2\, d\nu^N_{\alpha(t)}(\eta) \;+\; \frac{C_0}{N}\;.$$

Since $\widetilde{\lambda}(t)$ is equal to $\lambda(t)$ at the boundary of the interval $[0,1]$

$$\frac{\nu^N_{\alpha(t)}(\sigma^{0,1}\eta)}{\nu^N_{\alpha(t)}(\eta)}\, \frac{r^{\lambda(t)}_{0,1}(\sigma^{0,1}\eta)}{r^{\lambda(t)}_{0,1}(\eta)} \;=\; 1 \;+\; R_N\;,$$

where R_N is absolutely bounded by C_0/N. In view of this identity, and with a similar computation to the one presented for the interior piece of the generator, we conclude that

$$2\int h_t\, L_N^b(t)\, h_t\, d\nu^N_{\alpha(t)} \;\le\; -\frac{1}{2}\int c_{0,1}^{N,\lambda(t)}(\eta)\, r_{0,1}^{\lambda(t)}(\eta)\, [h_t(\sigma^{0,1}\eta) - h_t(\eta)]^2\, d\nu^N_{\alpha(t)}(\eta)$$
$$-\;\frac{1}{2}\int c_{N-1,N}^{N,\lambda(t)}(\eta)\, r_{N-1,N}^{\lambda(t)}(\eta)\, [h_t(\sigma^{N-1,N}\eta) - h_t(\eta)]^2\, d\nu^N_{\alpha(t)}(\eta) \;+\; \frac{C_0}{N^2}\,.$$

It follows from the previous estimates that

$$H_N(t) \;-\; H_N(0) \;\le\; -\frac{N^2\,\ell_N}{2}\int_0^t D_N(s,\sqrt{g_s})\, ds \;+\; C_0\, N\, \ell_N\,,$$

which concludes the proof of the lemma since $H_N(0) \le C_0 N$, as observed at the beginning of this section. □

For a positive integer k, denote by $\eta^k(j)$ the density of particles in an interval of length $2k+1$ centered at j:

$$\eta^k(j) \;=\; \frac{1}{2k+1}\sum_{i\in I_k(j)\cap\Lambda_N} \eta(i)\,,$$

where $I_k(j) = \{j-k,\dots,j+k\}$.

Recall the definition of the polynomial $\hat{h} : [0,1] \to \mathbb{R}$ given in (2.2.17), where h is a cylinder function, and recall the definition of the probability measures $\mathbb{P}^N_\mu$ introduced at the beginning of Step 2 in the previous section.

Lemma 2.3.2. *Let $G_N : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ (resp. $H_N : \mathbb{R}_+ \to \mathbb{R}$), $N \ge 1$, be a sequence of functions in $C^{0,1}(\mathbb{R}_+ \times [0,1])$ (resp. $C(\mathbb{R}_+)$) such that for all $T > 0$,*

$$\sup_{N\ge1}\sup_{0\le t\le T} \|G_N(t)\|_\infty \;<\; \infty\,, \quad \sup_{N\ge1}\sup_{0\le t\le T} \|H_N(t)\|_\infty \;<\; \infty\,.$$

Let $h : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ be a cylinder function. Fix a sequence $\{\mu_N : N \ge 1\}$ of probability measures, $\mu_N \in \mathscr{M}_N$. Consider two sequences $M_N \uparrow \infty$ and $K_N \uparrow \infty$ such that $M_N \ll K_N$, $M_N K_N \ll N$. Then, for every $T > 0$,

$$\lim_{N\to\infty} M_N\, \mathbb{E}_{\mu_N}\Big[\int_0^T \frac{1}{N}\sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)\big\{(\tau_j h)(\eta_s) - \hat{h}(\eta_s^{K_N}(j))\big\}\, ds\Big] \;=\; 0\,,$$

and

$$\lim_{N\to\infty} M_N \mathbb{E}_{\mu_N}\Big[\int_0^T H_N(s)\big\{\tau_0^{N,\lambda(s)} h(\eta_s) - \hat{h}(\alpha(s,0))\big\}\, ds\Big] \;=\; 0\,,$$
$$\lim_{N\to\infty} M_N \mathbb{E}_{\mu_N}\Big[\int_0^T H_N(s)\big\{\tau_N^{N,\lambda(s)} h(\eta_s) - \hat{h}(\alpha(s,1))\big\}\, ds\Big] \;=\; 0\,.$$

Proof. Fix $T > 0$ and $0 \le t \le T$. Every cylinder function can be written as a linear combination of the functions $\Psi_A = \prod_{j\in A}\eta_j$, A a finite subset of $\mathbb{Z}$. It is therefore enough to prove the lemma for such functions. We present the details for $h = \Psi_{\{0,1\}}$, it will be clear that the arguments apply to all cases.

Fix a sequence of continuous function $G_N : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ satisfying the assumptions of the lemma and note that $\hat{h}(\theta) = \theta^2$ in the case where $h = \Psi_{\{0,1\}}$. It follows from the assumptions of the lemma and from a summation by parts that

$$\frac{1}{N}\sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)\Big\{\eta(j)\eta(j+1) - \frac{1}{2K_N+1}\sum_{k=-K_N}^{K_N}\eta(j+k)\eta(j+k+1)\Big\}$$

in the time interval $[0,T]$ is absolutely bounded by a term of order K_N/N. On the other hand, we may write the difference $(2K_N+1)^{-1}\sum_{|k|\le K_N}\eta(j+k)\eta(j+k+1) - \hat h(\eta^{K_N}(j))$ as

$$\frac{1}{(2K_N+1)^2}\sum_{k,\ell}\eta(j+k)\,[\eta(j+k+1)-\eta(j+\ell)] \;+\; O\Big(\frac{1}{K_N}\Big)\;,$$

where the sum is carried over all k, ℓ such that $|k|\le K_N$, $|\ell|\le K_N$, $k\neq\ell$. The error term takes into account the diagonal terms $k=\ell$. Denote by $V_{j,K_N}(\eta)$ the first term of the previous formula.

In view of the former estimates, the first expectation appearing in the statement of the lemma is equal to

$$\mathbb{E}_{\mu_N}\Big[\int_0^T \frac{1}{N}\sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)\,V_{j,K_N}(\eta_s)\,ds\Big] \;+\; R_N\;,$$

where R_N is a remainder absolutely bounded by $C_0\{(K_N/N)+(1/K_N)\}$. Here and below, C_0 is a finite constant which does not depend on N, and which may change from line to line.

Recall the definition of the density g_s, introduced at the beginning of the proof of Lemma 2.3.1. The first term of the previous formula is equal to

$$\int_0^T ds\int\frac{1}{N}\sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)\,V_{j,K_N}(\eta)\,g_s(\eta)\,\nu^N_{\alpha(s)}(d\eta)\;. \tag{2.3.1}$$

Recall the definition of $V_{j,K_N}(\eta)$ and represent the previous integral, denoted by I, as $(1/2)I+(1/2)I$. In one of the halves, perform the change of variables $\eta'=\sigma^{j+k+1,j+\ell}\eta$ to rewrite the previous expression as

$$\begin{aligned}&\frac{1}{2}\,\frac{1}{(2K_N+1)^2}\sum_{k,\ell}\int_0^T ds\int\frac{1}{N}\sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)\,\eta(j+k)\;\times\\ &\quad\times\;[\eta(j+k+1)-\eta(j+\ell)]\{g_s(\eta)-g_s(\sigma^{j+k+1,j+\ell}\eta)\}\,\nu^N_{\alpha(s)}(d\eta)\;+\;R_N\;.\end{aligned}$$

In this formula, R_N is a remainder which appears from the change of measures $\nu^N_{\alpha(s)}(\sigma^{j+k+1,j+\ell}\eta)/\nu^N_{\alpha(s)}(\eta)$, and which is bounded by C_0K_N/N. Rewrite $g_s(\eta)-g_s(\eta')$ as $[\sqrt{g_s(\eta)}-\sqrt{g_s(\eta')}]\,[\sqrt{g_s(\eta)}+\sqrt{g_s(\eta')}]$ and apply Young's inequality to estimate the previous expression by

$$\begin{aligned}&\frac{1}{4A}\,\frac{1}{\widetilde K_N^2}\sum_{k,\ell}\int_0^T ds\int\frac{1}{N}\sum_j G_N(s,j/N)^2\Big\{\sqrt{g_s(\eta)}+\sqrt{g_s(\sigma^{j+k,j+\ell}\eta)}\Big\}^2\,\nu^N_{\alpha(s)}(d\eta)\\ &+\;\frac{A}{4}\,\frac{1}{\widetilde K_N^2}\sum_{k,\ell}\int_0^T ds\int\frac{1}{N}\sum_j\Big\{\sqrt{g_s(\eta)}-\sqrt{g_s(\sigma^{j+k,j+\ell}\eta)}\Big\}^2\,\nu^N_{\alpha(s)}(d\eta)\end{aligned}$$

for every $A>0$. In this formula, $\widetilde K_N=2K_N+1$. Since g_s is a density with respect to $\nu^N_{\alpha(s)}$, the first term of the previous expression is bounded by

$$\frac{C_0}{A} \int_0^T ds\, \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s,j/N)^2 \;\le\; \frac{C_0}{A}\;.$$

On the other hand, by the path lemma, explained in pages 94-95 of [18] and in details below equation (3.7) in [21], the second term of the previous formula is bounded above by

$$\frac{C_0 A K_N^2}{N} \int_0^T ds\, D_N^{\alpha(s)}(\sqrt{g_s})\;.$$

Recall that in the path lemma, a change of variables $\eta' = \sigma^{j,j+1}\sigma^{j+1,j+2}\cdots\sigma^{k-1,k}\eta$ is performed. Usually, the Jacobian of this change of variables is equal to 1 because the reference measure is a homogeneous product measure. In the present case, where the measure $\nu^N_{\alpha(s)}$ is a local equilibrium, the Jacobian is equal to $\exp\{h(\eta)\}$, where h is uniformly bounded by K_N/N. By Lemma 2.3.1, the previous displayed equation is less than or equal to $C_0 A (K_N/N)^2$. Optimizing over A, we conclude that (2.3.1) is bounded by $C_0 K_N/N$.

To compete the proof of the first assertion of the lemma it remains to recollect all the previous estimates and to recall the assumptions on the sequences M_N and K_N.

We turn to the second assertion. Here again, we present the proof for the left boundary in the case where $h(\eta) = \eta_1\eta_2$. Note that, by definition of $\tau_0^{N,\lambda}$, the case $h(\eta) = \eta_0\eta_1$ reduces to the case $h(\eta) = \eta_1$.

By definition of g_s, the expectation appearing in the statement of the lemma is equal to

$$\int_0^T ds\, H_N(s) \int \{\eta_1\eta_2 - \alpha(s,0)^2\}\, g_s(\eta)\, \nu^N_{\alpha(s)}(d\eta)\;.$$

Fix s and write the difference $E_{\nu^N_{\alpha(s)}}[\eta_1\eta_2 g_s] - \alpha(s,0)^2$ as

$$\begin{aligned}&\Big\{E_{\nu^N_{\alpha(s)}}[\eta_1\eta_2 g_s] \;-\; E_{\nu^N_{\alpha(s)}}[\eta_2 g_s]\,\alpha(s,0)\Big\}\\ &\qquad +\;\Big\{E_{\nu^N_{\alpha(s)}}[\eta_2 g_s]\,\alpha(s,0) \;-\; E_{\nu^N_{\alpha(s)}}[g_s]\,\alpha(s,0)^2\Big\}\;.\end{aligned} \tag{2.3.2}$$

Since $1 = \eta_1 + (1-\eta_1)$, the first term inside braces can be written as

$$[1-\alpha(s,0)]\, E_{\nu^N_{\alpha(s)}}[\eta_1\eta_2 g_s] \;-\; \alpha(s,0)\, E_{\nu^N_{\alpha(s)}}[(1-\eta_1)\eta_2 g_s]\;.$$

Performing a change of variables $\eta' = \sigma^{0,1}\eta$ in the first expectation, this difference becomes

$$\alpha(s,1/N)\,\frac{(1-\alpha(s,0))}{1-\alpha(s,1/N)}\, E_{\nu^N_{\alpha(s)}}\big[(1-\eta_1)\eta_2 g_s(\sigma^{0,1}\eta)\big] \;-\; \alpha(s,0)\, E_{\nu^N_{\alpha(s)}}\big[(1-\eta_1)\eta_2 g_s\big]\;,$$

Since $|\alpha(s,1/N) - \alpha(s,0)| \le C_0/N$, the previous expression is equal to

$$\alpha(s,0)\, E_{\nu^N_{\alpha(s)}}\Big[(1-\eta_1)\eta_2\{g_s(\sigma^{0,1}\eta) \;-\; g_s(\eta)\}\Big] \;+\; R_N\;,$$

where R_N is a remainder bounded by C_0/N in view of the assumptions on the sequence H_N. At this point, we may repeat the arguments presented in the first part of the proof to bound the first term by $C_0\{D_N^{\alpha(s)}(\sqrt{g_s})\}^{1/2}$, whose time integral, in view of Lemma 2.3.1, is bounded by $C_0 N^{-1/2}$. A similar argument permits to estimate the second term in

(2.3.2). This completes the proof of the lemma. □

2.4 The Hydrodynamic Equation

We prove in this section Proposition 2.1.3 and some estimates, stated below in Propositions 2.4.1 and 2.4.2, on the solution of equation (2.4.1). Recall the definition of the spaces $C^k([0,1])$ and $C^k_0([0,1])$, $k \ge 1$, introduced just below (2.1.6). Denote by $\|f\|_p$, $p \ge 1$, the L^p-norm of a function $f : [0,1] \to \mathbb{R}$,

$$\|f\|_p^p \;=\; \int_0^1 |f(x)|^p \, dx \; .$$

Fix $\nu > 0$, a smooth function $\alpha : \mathbb{R}_+ \to (0,1)$, and an initial condition ρ_0 in $C^4([0,1])$ such that $\rho_0(0) = \rho_0(1) = \alpha(0)$. Denote by $\rho(t,x) = \rho_\nu(t,x)$ the solution of the initial–boundary value problem

$$\begin{cases} \partial_t \rho = \nu \, \partial_x (D(\rho) \partial_x \rho) \; , \\ \rho(t,0) = \rho(t,1) = \alpha(t) \; , \\ \rho(0,x) = \rho_0 \; . \end{cases} \tag{2.4.1}$$

Proposition 2.4.1. *For every $t_0 \ge 0$, there exists $\nu_0 < \infty$, such that for all $\nu \ge \nu_0$, there exist positive constants $0 < b < B < \infty$, depending only on D, $\alpha(t)$, $0 \le t \le t_0$, such that for all $0 \le t \le t_0$,*

$$\|\rho(t) - \alpha(t)\|_\infty^2 \;\le\; B \, e^{-b\nu t} \|\partial_x \rho_0\|_2^2 \;+\; \frac{B}{\nu^2} \; ,$$

$$\|(\partial_x \rho)(t)\|_\infty^2 \;\le\; B \, e^{-b\nu t} \Big\{ \|\partial_x^2 \rho_0\|_2^2 \;+\; \|\partial_x \rho_0\|_4^4 \;+\; \frac{1}{\nu} \, \|\partial_x \rho_0\|_2^2 \Big\} \;+\; \frac{B}{\nu^2} \; .$$

In this proposition, $e^{-b\nu t}$ corresponds to the speed of convergence to equilibrium of the solution of (2.4.1) in the case where the boundary condition $\alpha(t)$ does not change in time, while $1/\nu^2$ stands for the relaxation time due to the evolution of the boundary conditions.

Proposition 2.4.2. *Assume that $\rho_0 = \alpha(0) + \varepsilon v_0$, where v_0 belongs to $C^4_0([0,1])$. For every $t_0 \ge 0$, there exists $\varepsilon_0 > 0$ and $\nu_0 < \infty$, depending on D, v_0, $\alpha(t)$, $0 \le t \le t_0$, such that for all $\epsilon < \epsilon_0$, $\nu \ge \nu_0$, there exist positive constants $B < \infty$, such that for all $0 \le t \le t_0$,*

$$\|(\partial_x^2 \rho)(t)\|_\infty^2 \;\le\; B \, \Big\{ \varepsilon^2 + \frac{1}{\nu^4} \Big\} \; .$$

The proof of these propositions is divided in a sequence of assertions. The Poincaré's inequality plays a fundamental role in the argument. It states that there exists a finite constant K_1 such that for every $C^1([0,1])$ function f which vanishes at some point $x \in [0,1]$,

$$\int_0^1 f(x)^2 \, dx \;\le\; K_1 \int_0^1 [f'(x)]^2 \, dx \; .$$

Throughout this subsection, c_0, C_0 represent small and large constants which depend only on K_1 and D.

Let

$$\beta_1(t) \;=\; \sup_{0 \le s \le t} |\alpha'(s)| \; , \quad \beta_2(t) \;=\; \sup_{0 \le s \le t} |\alpha''(s)| \; . \tag{2.4.2}$$

Assertion A. *There exist positive constants* $0 < c_0 < C_0 < \infty$ *such that for all* $t \ge 0$,

$$\int_0^1 [\rho(t) - \alpha(t)]^2\, dx \;\le e^{-c_0 \nu t} \int_0^1 [\rho(0) - \alpha(0)]^2\, dx \;+\; \frac{C_0}{\nu^2}\, \beta_1(t)^2 \big(1 - e^{-c_0 \nu t}\big)\;.$$

Proof. The proof follows classical arguments. Since $\rho(t) = \alpha(t)$ at the boundary, an integration by parts and the fact that $\alpha(t)$ is space independent yield that

$$\frac{1}{2}\,\frac{d}{dt} \int_0^1 [\rho(t) - \alpha(t)]^2\, dx \;=\; -\,\nu \int_0^1 D(\rho(t))\, (\partial_x \rho(t))^2\, dx \;-\; \alpha'(t) \int_0^1 [\rho(t) - \alpha(t)]\, dx\;.$$

Since the diffusivity is bounded below by a strictly positive constant, in the first term we may replace $D(\rho(t))$ by c_0 and the identity by an inequality. By Poincaré's inequality, the integral of $-(\partial_x \rho(t))^2$ is bounded by the integral of $-K_1^{-1}[\rho(t) - \alpha(t)]^2$. The second term on the right hand side can be estimated by Young's inequality. One of the terms is absorbed by what remained of the first term. The other one is $(C_0/\nu)\alpha'(t)^2$.

Up to this point we have shown that

$$\frac{1}{2}\,\frac{d}{dt} \int_0^1 [\rho(t) - \alpha(t)]^2\, dx \;\le\; -\,c_0\, \nu \int_0^1 [\rho(t) - \alpha(t)]^2\, dx \;+\; \frac{C_0}{\nu}\, \alpha'(t)^2\;.$$

To complete the proof, it remains to apply Gronwall inequality. □

Let $d : [0,1] \to \mathbb{R}$ be a primitive of D, $d' = D$, and let

$$c_1 \;=\; \inf_{0 \le \alpha \le 1} D(\alpha)\;, \quad C_1 \;=\; \|(\log D)'\|_\infty\;. \tag{2.4.3}$$

Assertion B. *Assume that* $2K_1 C_1 \beta_1(t_0) < c_1 \nu$ *for some* $t_0 > 0$. *Then, there exists a positive constants* $C_0 < \infty$ *such that for all* $0 \le t \le t_0$,

$$\int_0^1 [\partial_x d(\rho(t))]^2\, dx \;\le\; e^{-a_\nu\, t} \int_0^1 [\partial_x d(\rho(0))]^2\, dx \;+\; \frac{C_0}{\nu} \int_0^t e^{-a_\nu (t-s)} \beta_1(s)^2\, ds\;,$$

where $a_\nu = (c_1/K_1)\nu - 2C_1 \beta_1(t_0)$.

Proof. The proof is similar to the previous one. Adding and subtracting $\alpha'(t)$ we have that

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt} \int_0^1 [\partial_x d(\rho(t))]^2\, dx \;&=\; \int_0^1 \partial_x d(\rho(t))\, \partial_x \Big\{ D(\rho(t))\, \big[\nu\, \partial_x^2 d(\rho(t)) - \alpha'(t)\big] \Big\}\, dx \\ &\quad +\; \alpha'(t) \int_0^1 \partial_x d(\rho(t))\, \partial_x D(\rho(t))\, dx\;.\end{aligned}$$

Since $\alpha(t) = \rho(t,0) = \rho(t,1)$, $\alpha'(t) = \nu\, \partial_x^2 d(\rho(t,0)) = \nu\, \partial_x^2 d(\rho(t,1))$. In particular, we may integrate by parts the first term on the right hand side. This operation yields a negative term and one involving $\alpha'(t)$. This latter expression can be estimated through Young's inequality. The first piece is absorbed into the negative term and the second piece is bounded by $(C_0/\nu)\alpha'(t)^2$. Hence,

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt} \int_0^1 [\partial_x d(\rho(t))]^2\, dx \;&\le\; -\frac{c_1 \nu}{2} \int_0^1 [\partial_x^2 d(\rho(t))]^2\, dx \;+\; \frac{C_0}{\nu}\, \alpha'(t)^2 \\ &\quad +\; C_1\, |\alpha'(t)| \int_0^1 [\partial_x d(\rho(t))]^2\, dx\;.\end{aligned}$$

Since $\int_0^1 \partial_x d(\rho(t)) dx = 0$, applying Poincaré inequality to the first term on the right hand side, we obtain that the last expression is bounded above by

$$-\Big[\frac{c_1\,\nu}{2K_1} - C_1\beta_1(t)\Big]\int_0^1 [\partial_x d(\rho(t))]^2\,dx \;+\; \frac{C_0}{\nu}\,\beta_1(t)^2\;.$$

To complete the proof, it remains to replace $\beta_1(t)$ by $\beta_1(t_0)$ in the term inside brackets, getting an expression which is positive by assumption, and to apply Gronwall inequality. □

Lemma 2.4.3. *Assume that $c_1\nu > 2K_1C_1\beta_1(t_0)$ for some $t_0 > 0$. Then, there exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $0 \le t \le t_0$,*

$$\|\rho(t) - \alpha(t)\|_\infty^2 \;\le\; C_0 e^{-c_0\nu\, t} e^{C_0\beta(t_0)t}\int_0^1 [\partial_x d(\rho(0))]^2\,dx \;+\; \frac{1}{\nu^2}\frac{C_0\beta_1(t)^2}{1-(A_1\beta_1(t_0)/\nu)}\;,$$

where $A_1 = 2K_1C_1/c_1$.

Proof. Assume that $2K_1C_1\beta_1(t_0) < c_1\nu$ for some $t_0 > 0$ and fix $0 < t \le t_0$. Since $\alpha(t) = \rho(t,0)$, by Schwarz inequality there exists a finite constant C_0 such that for every $x \in [0,1]$,

$$|\rho(t,x) - \alpha(t)|^2 \;\le\; C_0\int_0^1 [\partial_x\rho(t)]^2\,dx \;\le\; C_0\int_0^1 [\partial_x d(\rho(t))]^2\,dx\;.$$

To complete the proof, it remains to recall Assertion B and to estimate the term $\beta_1(s)^2$ appearing in the time integral by $\beta_1(t)^2$. □

Let F_n, $G_n : \mathbb{R}_+ \to \mathbb{R}$, $n \ge 1$, be given by

$$F_n(t) \;=\; \int_0^1 [\partial_x d(\rho(t))]^{2n}\,dx\;, \quad G_n(t) \;=\; \int_0^1 [\partial_x^2 d(\rho(t))]^2\,[\partial_x d(\rho(t))]^{2n}\,dx\;. \tag{2.4.4}$$

Assertion C. *For all $n \ge 2$, there exist positive constants $0 < c_0 < C_0 < \infty$, $b_0 > 0$, such that for all $0 < b < b_0$, $t \ge 0$,*

$$\begin{aligned} &F_n(t) \;+\; c_0\,n^2\,\nu\int_0^t G_{n-1}(s)\,e^{-b\nu(t-s)}\,ds \\ &\qquad\le\; e^{-b\nu t}\,F_n(0) \;+\; C_0\frac{n^2}{\nu}\,\beta_1(t)^2\int_0^t F_{n-1}(s)\,e^{-b\nu(t-s)}\,ds\;. \end{aligned}$$

Proof. Since $\alpha'(t) = \partial_t\rho(t,1) = \nu\partial_x^2 d(\rho(t,1))$, adding and subtracting $\alpha'(t)$, and then integrating by parts yield that

$$\begin{aligned} F_n'(t) \;&=\; -2n(2n-1)\,\nu\int_0^1 D(\rho(t))\,[\partial_x d(\rho(t))]^{2n-2}\,[\partial_x^2 d(\rho(t))]^2\,dx \\ &+\; 2n(2n-1)\,\alpha'(t)\int_0^1 D(\rho(t))\,[\partial_x d(\rho(t))]^{2n-2}\,\partial_x^2 d(\rho(t))\,dx \\ &+\; 2n\,\alpha'(t)\int_0^1 [\partial_x d(\rho(t))]^{2n-1}\,\partial_x D(\rho(t))\,dx\;. \end{aligned}$$

Apply Young's inequality to the second term on the right hand side to bound it by the sum of two terms. The first one can be absorbed by the first term on the right hand side, and the second one is bounded by $C_0(n^2/\nu)\alpha'(t)^2F_{n-1}(t)$. In the last term on the right hand side, replace $\partial_x D(\rho(t))$ by $\partial_x[D(\rho(t)) - D(\alpha(t))]$ and integrate by parts to obtain that it is equal to

$$-2n(2n-1)\,\alpha'(t)\int_0^1 [\partial_x d(\rho(t))]^{2(n-1)}\,\partial_x^2 d(\rho(t))\,[D(\rho(t)) - D(\alpha(t))]\,dx\;.$$

Apply Young's inequality to bound this expression by the sum of two terms. The first one can be absorbed by the first term on the penultimate formula, while the second one is less than or equal to $C_0 n^2 \alpha'(t)^2 \nu^{-1} F_{n-1}(t)$. Therefore,

$$F_n'(t) \;\le\; -c_0\, n^2\, \nu\, G_{n-1}(t) \;+\; C_0\, n^2\, \frac{\alpha'(t)^2}{\nu} F_{n-1}(t)\;.$$

Let $f(x) = [\partial_x d(\rho(t,x))]^n$. Since $\int_0^1 \partial_x d(\rho(t,x))\,dx = 0$, there exists $x_0 \in [0,1]$ such that $\partial_x d(\rho(t,x_0)) = 0$, so that $f(x_0) = 0$. We may therefore apply Poincaré's inequality to $[\partial_x d(\rho(t,x))]^n$ to obtain that

$$F_n(t) \;=\; \int f(x)^2\,dx \;\le\; K_1 \int f'(x)^2\,dx \;=\; K_1 n^2 G_{n-1}(t)\;.$$

It follows from the previous estimates that

$$F_n'(t) \;\le\; -b_0\,\nu\, F_n(t) \;-\; c_0\, n^2\, \nu\, G_{n-1}(t) \;+\; C_0\, n^2\, \frac{\alpha'(t)^2}{\nu} F_{n-1}(t)$$

for some $b_0 > 0$. The same inequality remains in force for any $0 < b < b_0$. It remains to apply Gronwall inequality to complete the proof. □

Iterating the inequality appearing in the previous assertion without the term G_{n-1} yields

Assertion D. *For all $n \ge 2$, there exist positive constants $0 < c_0 < C_0 < \infty$, $b_0 > 0$, such that for all $0 < b < b_0$, $t \ge 0$,*

$$F_n(t) \;+\; c_0 n^2 \nu \int_0^t e^{-b\nu(t-s)} G_{n-1}(s)\,ds \;\le\; r_n(t)\;,$$

where $r_n(t) = r_n(t,b,C_0)$ is given by

$$\begin{aligned} r_n(t) \;&=\; C_0 \sum_{k=2}^{n} F_k(0) \Big(\frac{t[n\beta_1(t)]^2}{\nu}\Big)^{n-k} \frac{e^{-b\nu t}}{(n-k)!} \\ &+\; C_0 \Big(\frac{[n\beta_1(t)]^2}{\nu}\Big)^{n-1} \int_0^t e^{-b\nu(t-s)}\, \frac{(t-s)^{n-2}}{(n-2)!}\, F_1(s)\,ds\;. \end{aligned}$$

Let $w : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be given by

$$w(t,x) \;=\; [\partial_x^2 d(\rho_t)](x) \;-\; \frac{1}{\nu}\,\alpha'(t)\;. \tag{2.4.5}$$

Assertion E. *There exist positive constants $0 < c_0 < C_0 < \infty$ such that*

$$\int_0^1 w(t)^2\,dx \;\le\; e^{-c_0\nu t} \int_0^1 w(0)^2\,dx \;+\; \frac{C_0}{\nu^3} \int_0^t \alpha''(s)^2\,ds + r_2(t)\;,$$

where the remainder r_2 has been introduced in Assertion D.

Proof. The proof is similar to the one of Assertion A. We first show that

$$\frac{d}{dt}\int_0^1 w(t)^2\,dx \;\le\; -\,\nu \int_0^1 D(\rho_t)\,[\partial_x^3 d(\rho_t)]^2\,dx \;+\; \frac{1}{A\nu^3}\,\alpha''(t)^2 \\ +\;\nu\int_0^1 \frac{1}{D(\rho_t)}\,[\partial_x D(\rho_t)]^2\,[\partial_x^2 d(\rho_t)]^2\,dx \;+\; A\,\nu\int_0^1 w(t)^2\,dx$$

for any $A > 0$. As $w(t)$ vanishes at $x = 0$, apply Poincaré's to this functions to get that

$$\int_0^1 w(t)^2\,dx \;\le\; K_1 \int_0^1 [\partial_x^3 d(\rho_t)]^2\,dx\;.$$

Hence, choosing A small enough yields

$$\frac{d}{dt}\int_0^1 w(t)^2\,dx \;\le\; -\,c_0\nu\int_0^1 w(t)^2\,dx \;+\; \frac{C_0}{\nu^3}\,\alpha''(t)^2 \;+\; C_0\nu G_1(t)\;,$$

where the function G_1 has been introduced in (2.4.4). We may replace the constant c_0 by one which is smaller than the constant b_0 appearing in the statement of Assertion D. By Gronwall inequality,

$$\int_0^1 w(t)^2\,dx \;\le\; e^{-c_0\nu t}\int_0^1 w(0)^2\,dx \;+\; C_0\int_0^t e^{-c_0\nu(t-s)}\Big\{\frac{\alpha''(s)^2}{\nu^3} \;+\; \nu G_1(s)\Big\}\,ds\;.$$

To complete the proof of the assertion, it remains to recall the statement of Assertion D. □

Lemma 2.4.4. *Assume that $c_1\nu > 2K_1C_1\beta_1(t_0)$ for some $t_0 > 0$. Then, there exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $0 \le t \le t_0$,*

$$\|\partial_x d(\rho(t))\|_\infty^2 \;\le\; C_0 e^{-c_0\nu t}\Big\{\int_0^1 w(0)^2\,dx \;+\; F_2(0) \;+\; \frac{1}{\nu}e^{C_0\beta_1(t_0)}\,t\,F_1(0)\Big\} \\ +\;\frac{C_0}{\nu^2}\Big\{\alpha'(t)^2 \;+\; \frac{1}{\nu}\int_0^t \alpha''(s)^2\,ds \;+\; \frac{1}{\nu^2}e^{C_0\beta_1(t_0)[1+t]}\Big\}\;.$$

Proof. Assume that $2K_1C_1\beta_1(t_0) < c_1\nu$ for some $t_0 > 0$ and fix $0 < t \le t_0$. Since $\int_0^1(\partial_x d(\rho(t))\,dx = 0$, subtracting this integral and applying Schwarz inequality, we get that for all $x \in [0,1]$,

$$[\partial_x d(\rho(t,x))]^2 \;\le\; C_0\int_0^1 [\partial_x^2 d(\rho(t,y))]^2 dy\;.$$

Adding and subtracting $\alpha'(t)/\nu$, by Young's inequality, the previous expression is less than or equal to

$$C_0\int_0^1 w(t)^2 dy \;+\; C_0\frac{\alpha'(t)^2}{\nu^2}\;.$$

By Assertion E, the first term of the previous expression is less than or equal to

$$C_0 e^{-c_0\nu t}\int_0^1 w(0)^2\,dx \;+\; \frac{C_0}{\nu^3}\int_0^t \alpha''(s)^2\,ds \;+\; r_2(t)\;.$$

By its definition and by Assertion B,

$$r_2(t) \;\le\; C_0 e^{-c_0\nu t}\Big\{F_2(0) \;+\; \frac{1}{\nu}e^{C_0\beta_1(t_0)}\,t\,F_1(0)\Big\} \;+\; \frac{C_0}{\nu^4}e^{C_0\beta_1(t_0)[1+t]}\;.$$

To complete the proof it remains to recollect all previous estimates. □

Let

$$q(t) \;=\; \|\partial_x d(\rho_t)\|_\infty\;, \quad Q(t) \;=\; \sup_{0\le s\le t} q(s)\;, \quad t\ge 0\;. \tag{2.4.6}$$

Assertion F. *Suppose that $Q(t_0)(1+n^2K_1) < c_1$ for some $t_0 > 0$. Then, there exists a positive constant $C_0 < \infty$ such that for all $0 \le t \le t_0$,*

$$\int_0^1 w(t)^{2n}\, dx \;\le\; e^{-a_n\nu t}\int_0^1 w(0)^{2n}\, dx \;+\; \int_0^t e^{-a_n\nu(t-s)} H_{n-1}(s)\, ds\;,$$

where $a_n = [c_1 - Q(t_0)(1+n^2K_1)]/K_1$ and

$$H_{n-1}(s) \;=\; C_0\Big\{n^2 q(s)^2\,\frac{\alpha'(s)^2}{\nu} \;+\; \frac{\alpha''(s)^2}{\nu^3}\Big\}\int_0^1 w(s)^{2(n-1)}\, dx\;.$$

Proof. Since $w(t)$ vanishes at the boundary, an integration by parts yields that the time derivative of $\int_0^1 w(t)^{2n}\,dx$ is equal to

$$\begin{aligned}
&-2n(2n-1)\,\nu\int_0^1 w(t)^{2(n-1)} D(\rho_t)\,[\partial_x^3 d(\rho_t)]^2\, dx \qquad (2.4.7)\\
&-\;2n(2n-1)\,\nu\int_0^1 w(t)^{2(n-1)}\partial_x D(\rho_t)\, w(t)\,\partial_x^3 d(\rho_t)\, dx\\
&-\;2n(2n-1)\alpha'(t)\int_0^1 w(t)^{2(n-1)}\partial_x D(\rho_t)\,\partial_x^3 d(\rho_t)\, dx \;-\; \frac{2n\alpha''(t)}{\nu}\int_0^1 w(t)^{2n-1}\, dx\;.
\end{aligned}$$

In this formula, in the second line, we added and subtracted $(1/\nu)\alpha'(t)$ to recover $w(t)$ from $\partial_x^2 d(\rho_t)$.

Recall the definition of $q(t)$, introduced in (2.4.6), and the one of the constants c_1, C_1, defined in (2.4.3). Estimating $\partial_x D(\rho_t)$ by $C_1 q(t)$, and applying Young inequality to the last three terms of the previous displayed equation, we obtain that the time derivative of $\int_0^1 w(t)^{2n}\,dx$ is bounded by

$$\begin{aligned}
&-2n\,(2n-1)\,\nu\,\Big\{c_1 - \frac{q(t)}{2} - \frac{1}{A}\Big\}\int_0^1 w(t)^{2(n-1)}\,[\partial_x^3 d(\rho_t)]^2\, dx\\
&\quad+\;2n(2n-1)\,\nu\,\Big\{\frac{q(t)}{2} + \frac{1}{A}\Big\}\int_0^1 w(t)^{2n}\, dx\\
&\quad+\;C_0\Big\{n^2 A q(t)^2\,\frac{\alpha'(t)^2}{\nu} \;+\; \frac{\alpha''(t)^2}{\nu^3}\Big\}\int_0^1 w(t)^{2(n-1)}\, dx\;,
\end{aligned}$$

for every $A > 0$.

Let $f(x) = w(t,x)^n$. Since f vanishes at the boundary and since $f'(x) = n w(t,x)^{n-1}\partial_x^3 d(\rho_t)$, by Poincaré's inequality,

$$\int_0^1 w(t)^{2n}\, dx \;=\; \int_0^1 f^2\, dx \;\le\; K_1\int_0^1 [f']^2\, dx \;=\; K_1 n^2\int_0^1 w(t)^{2(n-1)}\,[\partial_x^3 d(\rho_t)]^2\, dx\;.$$

Set $A = 2(1+n^2K_1)/c_1$. With this choice, by assumption, $c_1 - q(t)/2 - 1/A > 0$. In particular, the sum of the first two lines of the penultimate displayed equation is less than or equal to

$$-\frac{2n(2n-1)\,\nu}{2n^2K_1}\,\Big\{c_1 - q(t)(1+n^2K_1)\Big\}\int_0^1 w(t)^{2n}\, dx\;.$$

Since $2n(2n-1)/2n^2 \geq 1$, to complete the proof it remains to apply Gronwall inequality. □

Assertion G. *Assume that* $2(1+K_1)C_1Q(t_0) < c_1$ *for some* $t_0 > 0$*. Then, there exist positive constants* $0 < c_0 < C_0 < \infty$ *such that for all* $0 \leq t \leq t_0$*,*

$$\int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \;\leq e^{-a\nu t} \int_0^1 [\partial_x^3 d(\rho_0)]^2 \, dx \;+\; \int_0^t e^{-a\nu(t-s)} H(s) \, ds \,,$$

where $a = [c_1 - 2(1+K_1)C_1Q(t_0)]/2K_1$*, and*

$$H(s) \;=\; C_0 \, \nu \, q(s)^2 \int_0^1 [\partial_x^2 d(\rho_s)]^2 \, dx \;+\; C_0 \, \nu \int_0^1 [\partial_x^2 d(\rho_s)]^4 \, dx \;+\; \frac{C_0}{\nu^3} [\alpha''(s)]^2 \,.$$

Proof. The proof is similar to the one of Assertion B. Fix $t_0 > 0$ satisfying the hypothesis of the lemma and consider some $t < t_0$. Since $\rho(t,1) = \alpha_t$, $\alpha''(t) = \nu^2 \partial_x^2 [D(\rho(t))\partial_x^2 d(\rho(t))]$ at $x = 0, 1$. Adding and subtracting $\nu^{-1}\alpha''(t)$, and integrating by parts, we have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \;&=\; -\,\nu \int_0^1 \partial_x^4 d(\rho_t) \, \partial_x^2 \{ D(\rho_t) \, \partial_x^2 d(\rho_t) \} \, dx \\ &\quad +\; \nu^{-1} \alpha''(t) \int_0^1 \partial_x^4 d(\rho_t) \, dx \,. \end{aligned} \tag{2.4.8}$$

Let $D_1(\alpha) = (\log D)'(\alpha)$, $D_2(\alpha) = (\log D)''(\alpha)/D(\alpha)$. Expand $\partial_x^2 \{ D(\rho_t) \, \partial_x^2 d(\rho_t) \}$, and observe that $\partial_x^2 D(\rho_t) \;=\; D_1(\rho_t) \partial_x^2 d(\rho_t) + D_2(\rho_t) [\partial_x d(\rho_t)]^2$ to write the first term on the right hand side of the previous formula as

$$\begin{aligned} &-\,\nu \int_0^1 D(\rho_t) \, [\partial_x^4 d(\rho_t)]^2 \, dx \;-\; 2\nu \int_0^1 \partial_x D(\rho_t) \, \partial_x^3 d(\rho_t) \, \partial_x^4 d(\rho_t) \, dx \\ &-\,\nu \int_0^1 D_1(\rho_t) \, [\partial_x^2 d(\rho_t)]^2 \, \partial_x^4 d(\rho_t) \, dx \;-\; \nu \int_0^1 D_2(\rho_t) \, [\partial_x d(\rho_t)]^2 \partial_x^2 d(\rho_t) \, \partial_x^4 d(\rho_t) \, dx \,. \end{aligned}$$

Recall the definition of $q(t)$ introduced in (2.4.6), and recall that $c_1 = \inf_{0 \leq \alpha \leq 1} D(\alpha)$, $C_1 = \|D_1\|_\infty$. Apply Young's inequality to the last three terms and to the last term in (2.4.8) to obtain that the left hand side of (2.4.8) is less than or equal to

$$\begin{aligned} &-\,\nu \, [c_1 - \frac{2}{A} - C_1 q(t)] \int_0^1 [\partial_x^4 d(\rho_t)]^2 \, dx \;+\; C_1 \, \nu \, q(t) \int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \\ &\quad +\; A \, C_0 \, \nu \, q(t)^2 \int_0^1 [\partial_x^2 d(\rho_t)]^2 \, dx \;+\; A \, C_0 \, \nu \int_0^1 [\partial_x^2 d(\rho_t)]^4 \, dx \;+\; \frac{A}{\nu^3} [\alpha''(t)]^2 \end{aligned} \tag{2.4.9}$$

for all $A > 0$.

Since $\partial_x^2 d(\rho(t,1)) = \partial_x^2 d(\rho(t,0))$, $\int_0^1 \partial_x^3 d(\rho_t) \, dx = 0$. Therefore, by Poincaré's inequality,

$$\int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \;\leq\; K_1 \int_0^1 [\partial_x^4 d(\rho_t)]^2 \, dx \,.$$

Set $A = 4/c_1$. Since, by hypothesis, $2C_1 q(t) \leq 2C_1 Q(t_0) < c_1$, the first line of (2.4.9) is bounded by

$$-\,\frac{\nu}{2K_1} \, [c_1 - 2C_1 \, [1 + K_1] \, q(t)] \int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \;\leq\; -a \, \nu \int_0^1 [\partial_x^3 d(\rho_t)]^2 \, dx \,,$$

where a has been introduced in the statement of the assertion. To complete the proof, it remains to apply Gronwall inequality. □

Proof of Proposition 2.4.1. The claims are straightforward consequences of Lemmas 2.4.3 and 2.4.4. We turn to the third assertion. □

Proof of Proposition 2.4.2. Since $\partial_x^2 d(\rho)(t,1) - (1/\nu)\alpha'(t)$ vanish as $x = 0$, by Schwarz inequality, for any $x_0 \in [0,1]$,

$$[\partial_x^2 d(\rho_t)(x_0) - \alpha'(t)]^2 \;\le\; \int_0^1 [\partial_x^3 d(\rho_t)(x)]^2 \,.$$

Fix $t_0 > 0$. By Proposition 2.4.1, $Q(t_0)^2 \le C_0\delta^2$, where $\delta^2 = \varepsilon^2 + \nu^{-2}$. Therefore, there exist $\varepsilon_0 > 0$ and $\nu_0 < \infty$ with the property that the hypothesis of Assertion G is in force for all $\varepsilon < \varepsilon_0$, $\nu > \nu_0$. In particular, the previous expression is bounded by

$$C_0\varepsilon^2 \;+\; \int_0^t e^{-a\nu(t-s)} H(s)\, ds\;,$$

where H has been introduced in the statement of Assertion G. By Proposition 2.4.1, which permits to estimate $q(s)^2$, by adding and subtracting $\alpha'(s)$ to $\partial_x^2 d(\rho_s)$, which permits to recover the function $w(s)$ introduced in (2.4.5), the second term of the previous equation is less than or equal to

$$C_0\Big\{\frac{1}{\nu^4} + \frac{\varepsilon^2}{\nu^2}\Big\} \;+\; C_0\,\nu \int_0^t ds\, e^{-a\nu(t-s)} \int_0^1 \{\delta^2 w(s)^2 + [\partial_x^2 d(\rho_s)]^4\}\, dx\,.$$

By Assertion E, this sum is bounded by

$$C_0\Big\{\frac{1}{\nu^4} + \varepsilon^4\Big\} \;+\; C_0\,\nu \int_0^t ds\, e^{-a\nu(t-s)}\Big\{\delta^2 r_2(s) + \int_0^1 [\partial_x^2 d(\rho_s)]^4\, dx\Big\}\,,$$

By Assertion B, $r_2(s) \le C_0\delta^2$. We may thus remove r_2 from the previous formula. By Young inequality, by Assertion F and by Proposition 2.4.1, the second term without $r_2(s)$ is less than or equal to

$$C_0\Big\{\frac{1}{\nu^4} + \varepsilon^4\Big\} \;+\; C_0\delta^2 \int_0^t ds\, e^{-a\nu(t-s)} \int_0^s dr\, e^{-a'\nu(s-r)} \int_0^1 w(r)^2\, dx\,.$$

By Assertions E and B, the previous expression is less than or equal to $C_0\delta^4$. This concludes the proof of the proposition. □

Proof of Proposition 2.1.3. By Proposition 2.4.1, for every $t_0 > 0$, there exists $\nu_0 < \infty$ and $B < \infty$, where B depends on $\alpha(s)$, $0 \le s \le t_0$, and on the initial condition v_0, such that for all $0 \le t \le t_0$

$$\|u_\nu(t)\|_\infty^2 \;\le\; B\,, \qquad \int_0^1 [\partial_x u_\nu(t)]^2\, dx \;\le\; B \tag{2.4.10}$$

Fix $t_0 > 0$ and $0 \le t < t_0$. By definition, $u_\nu(t,0) = u_\nu(t,1) = 0$, and

$$\partial_t u_\nu \;=\; \nu\Big\{\partial_x\big[D(\alpha_t + \varepsilon u_\nu)\partial_x u_\nu\big] - \alpha'(t)\Big\} \;=\; \nu\,\partial_x\Big\{D(\alpha_t + \varepsilon u_\nu)\partial_x u_\nu - D(\alpha_t)\partial_x v_t\Big\}\,.$$

where $\varepsilon = \nu^{-1}$.

Therefore, for every $t \geq 0$, an integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [u_\nu(t) - v_t]^2\, dx \;=\; \int_0^1 [u_\nu(t) - v_t]\, \partial_t v_t\, dx \\ -\, \nu \int_0^1 \partial_x [u_\nu(t) - v_t]\, \big\{ D(\alpha_t + \varepsilon u_\nu)\partial_x u_\nu - D(\alpha_t)\partial_x v_t \big\}\, dx\;.$$

The second term is less than or equal to

$$-\,\nu \int_0^1 D(\alpha_t)\, [\partial_x u_\nu(t) - \partial_x v_t]^2\, dx \;+\; C_0 \int_0^1 |\partial_x u_\nu(t) - \partial_x v_t|\, |u_\nu|\, |\partial_x u_\nu|\, dx \\ \leq\; -c_0 \nu \int_0^1 [\partial_x u_\nu(t) - \partial_x v_t]^2\, dx \;+\; \frac{C_0}{\nu}\int_0^1 u_\nu(t)^2 [\partial_x u_\nu(t)]^2\, dx\;.$$

By (2.4.10), the second term is bounded by B/ν. Therefore, by Young's inequality and by Poincaré's inequality,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [u_\nu(t) - v_t]^2\, dx \;\leq\; -c_0 \nu \int_0^1 [u_\nu(t) - v_t]^2\, dx \;+\; \frac{C_0}{\nu}\int_0^1 (\partial_t v_t)^2\, dx + \frac{B}{\nu}\;.$$

To conclude the proof, it remains to apply Gronwall inequality. □

2.5 The Diffusion Coefficient

We provide in this section a formula for the diffusion coefficient.

Fix a cylinder function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, and recall from (2.2.17) that $\hat{f} : [0,1] \to \mathbb{R}$ represents the polynomial defined by

$$\hat{f}(\theta) \;=\; E_{\nu_\theta}[f(\xi)]\;.$$

Write $E_{\nu_{\theta+h}}[f(\xi)]$ as $E_{\nu_\theta}[f(\xi) N_h]$, where N_h is the Radon-Nikodym derivative of $\nu_{\theta+h}$, restricted to the support of f, with respect to ν_θ, to get that

$$\hat{f}'(\theta) \;=\; \frac{1}{\mathfrak{c}(\theta)} \sum_{k\in\mathbb{Z}} \langle f ; \eta(k)\rangle_\theta\;, \tag{2.5.1}$$

where $\langle f;g\rangle_\theta$ represents the covariance between two cylinder functions f, g in $L^2(\nu_\theta)$: $\langle f;g\rangle_\theta = E_{\nu_\theta}[fg] - E_{\nu_\theta}[f]E_{\nu_\theta}[g]$, and $\mathfrak{c}(\theta)$ the static compressibility, given by $\mathfrak{c}(\theta) = \theta(1-\theta)$.

Recall the definitions of the cylinder function h, introduced in (2.2.15). We claim that

$$\hat{h}'(\theta) \;=\; D(\theta)\;. \tag{2.5.2}$$

Indeed, since the cylinder function $c(\eta)$ does not depend on $\eta(0)$ and $\eta(1)$,

$$\mathfrak{c}(\theta)\, D(\theta) \;=\; -\sum_{k\in\mathbb{Z}} k\, \big\langle [\eta(0) - \eta(1)] c(\eta)\, ;\, \eta(k) \big\rangle_\theta\;.$$

Note that all terms in this sum vanish but the one with $k = 1$, and that the sum over k is finite because c is a cylinder function. By (2.1.1) and by a change of variables, the right hand side is equal to

$$-\sum_{k\in\mathbb{Z}} k \sum_{a=1}^{m} \sum_{j\in\mathbb{Z}} \mu_a(j) \left\langle \tau_{-j} h_a \,;\, \eta(k) \right\rangle_\theta \;=\; -\sum_{a=1}^{m} \sum_{k,j\in\mathbb{Z}} k\, \mu_a(j) \left\langle \tau_{-(j+k)} h_a \,;\, \eta(0) \right\rangle_\theta .$$

Note that sum over j is finite because μ_a has finite support. By definition of m_a, and since the total mass of μ_a vanishes, $\sum_j \mu_a(j) = 0$, performing the change of variables $k' = j + k$ last term becomes

$$\sum_{a=1}^{m} m_a \sum_{k,\in\mathbb{Z}} \left\langle \tau_{-k} h_a \,;\, \eta(0) \right\rangle_\theta \;=\; \sum_{a=1}^{m} m_a \sum_{k,\in\mathbb{Z}} \left\langle h_a \,;\, \eta(k) \right\rangle_\theta \;=\; \mathfrak{c}(\theta) \sum_{a=1}^{m} m_a \hat{h}_a'(\theta) \,,$$

where the last identity follows from (2.5.1). This last expression is equal to $\mathfrak{c}(\theta)\hat{h}'(\theta)$, which concludes the proof of (2.5.2).

Chapter 3

From Coalescing Random Walks on a Torus to Kingman's Coalescent*

Abstract

Let $\mathbb{T}_N^d$, $d \geq 2$, be the discrete d-dimensional torus with N^d points. Place a particle at each site of $\mathbb{T}_N^d$ and let them evolve as independent, nearest-neighbor, symmetric, continuous-time random walks. Each time two particles meet, they coalesce into one. Denote by C_N the first time the set of particles is reduced to a singleton. Cox [11] proved the existence of a time-scale θ_N for which C_N/θ_N converges to the sum of independent exponential random variables. Denote by Z_t^N the total number of particles at time t. We prove that the sequence of Markov chains $(Z_{t\theta_N}^N)_{t\geq 0}$ converges to the total number of partitions in Kingman's coalescent.

3.1 Notation and Results

Denote by p the probability measure on $\mathbb{Z}^d$ given by

$$p(x) \;=\; \frac{1}{2d} \text{ if } x \in \{\pm e_1, \dots, \pm e_d\}\,, \quad \text{and } p(x) = 0 \text{ otherwise}\,. \tag{3.1.1}$$

Let E_N be the family of nonempty subsets of $\mathbb{T}_N^d$. The coalescing random walks introduced in the previous section is the E_N-valued, continuous-time Markov chain, represented by $\{A_N(t) : t \geq 0\}$, whose generator L_N is given by

$$(L_N f)(A) \;=\; \sum_{x\in A}\sum_{y\notin A} p(y-x)\{f(A_{x,y}) - f(A)\} \;+\; \sum_{x\in A}\sum_{y\in A} p(y-x)\{f(A_x) - f(A)\}\,, \tag{3.1.2}$$

*Joint work with Claudio Landim and Johel Beltrán where $A_{x,y}$ (resp. A_x) is the set obtained from A by replacing the point x by y (resp. removing the element x):

$$A_{x,y} \;=\; [A \setminus \{x\}] \cup \{y\}\,, \quad A_x \;=\; A \setminus \{x\}\,.$$

3.1.1 Kingman's Coalescent

Recall from subsection 1.2.2 the definition of the process $(\mathscr{N}_t)_{t\geq 0}$ associated to the Kingman's coalescent, and the definition of the set S. Denote by $D(\mathbb{R}_+, S)$ the space of S-valued, right-continuous trajectories with left-limits, endowed with the Skorokhod topology. The respective coordinate maps are denoted by

$$X_t : D(\mathbb{R}_+, S) \to S\,, \quad t \geq 0\,.$$

Consider the canonical filtration

$$\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)\,, \quad t \geq 0\,.$$

It is known that $\mathcal{G}_\infty := \sigma(X_t : t \geq 0)$ coincides with the corresponding Borel σ-field on $D(\mathbb{R}_+, S)$. Let $C^1(S)$ be the set of functions $f : S \to \mathbb{R}$ of class C^1, that is $f \in C^1(S)$ is the restriction to S of a continuously differentiable function defined on a neighborhood of S. For each $f \in C^1(S)$ define $\mathscr{L} f : S \to \mathbb{R}$ as

$$(\mathscr{L} f)(y) := \begin{cases} \binom{n}{2} \left\{ f\left(\frac{1}{n-1}\right) - f\left(\frac{1}{n}\right) \right\}, & \text{if } y = \frac{1}{n} \text{ and } n \geq 2\,, \\ 0\,, & \text{if } y = 1\,, \\ (1/2) f'(0)\,, & \text{if } y = 0\,. \end{cases} \tag{3.1.3}$$

The following proposition guarantees existence and uniqueness for the $\big(C^1(S), \mathscr{L}\big)$-martingale problem and that $(\mathscr{X}_t)_{t\geq 0}$, defined in (1.2.3), provides the unique solution starting at $0 \in S$.

Proposition 3.1.1. *For each $x \in S$, there exists a unique solution for the $\big(C^1(S), \mathscr{L}\big)$-martingale problem starting at x. That is, there exists a unique probability measure $\mathscr{P}_x$ on the measurable space $\big(D(\mathbb{R}_+, S), \mathcal{G}_\infty\big)$ such that $\mathscr{P}_x[X_0 = x] = 1$ and, for every $f \in C^1(S)$,*

$$f(X_t) - \int_0^t (\mathscr{L} f)(X_s)\, ds\,, \quad t \geq 0\,, \tag{3.1.4}$$

is a $\mathscr{P}_x$-martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$. Moreover, $\mathscr{P}_0$ coincides with the law of $(\mathscr{X}_t)_{t\geq 0}$.

3.1.2 Main Result

Recall that E_N stands for the set of nonempty subsets of $\mathbb{T}^d_N$. Consider the partition of E_N according to the number of elements:

$$E_N = \bigcup_{n\in\mathbb{N}} \mathscr{E}^n_N\,, \quad \text{where} \quad \mathscr{E}^n_N := \{A \subset \mathbb{T}^d_N : |A| = n\}\,,\ n \in \mathbb{N}\,, \tag{3.1.5}$$

and $|A|$ stands for the number of elements of A. Let $\Psi_N : E_N \to S$ be the projection corresponding to partition (3.1.5)

$$\Psi_N(A) = 1/|A|\,, \quad A \in E_N\,.$$

For each $A \in E_N$, let $\mathbf{P}^N_A$ denote a probability measure under which the process $\big(A_N(t)\big)_{t\geq 0}$ corresponds to a coalescing random walk on $\mathbb{T}^d_N$ starting at A, i.e. a Markov chain with state space E_N and generator L_N (defined in (3.1.2)) such that $\mathbf{P}^N_A[A_N(0) = A] = 1$. When $A = \mathbb{T}^d_N$, we denote $\mathbf{P}^N_A$ simply by $\mathbf{P}^N$. Expectation with respect to $\mathbf{P}^N_A$, $\mathbf{P}^N$ is represented by $\mathbf{E}^N_A$, $\mathbf{E}^N$, respectively.

Consider two independent random walks $(x^N_t)_{t\geq 0}$ and $(y^N_t)_{t\geq 0}$ on $\mathbb{T}^d_N$, both with jump probability given by $p(\cdot)$, starting at the uniform distribution. Let θ_N be the expected meeting time:

$$\theta_N \ := \ E\big[\ \min\{t\geq 0 : x^N_t = y^N_t\}\ \big]\ .$$

Since $x^N_t - y^N_t$ evolves as a random walk speeded-up by 2, θ_N represents the expectation of the hitting time of the origin for a simple symmetric random walk speeded-up by 2 which starts from the stationary state. By [3, Proposition 6.10], we may express this expectation in terms of capacities. Sharp bounds for the capacity then provide an asymptotic formula for θ_N.

Consider a continuous-time, random walk $(x_t)_{t\geq 0}$ on $\mathbb{Z}^d$ with jump probabilities given by (3.1.1) and which starts from the origin. Assume that $d \geq 3$, and denote by τ_1 the time of the first jump, $\tau_1 = \inf\{t\geq 0 : x_t \neq 0\}$, and by H^+ the return time to the origin: $H^+ = \inf\{t\geq \tau_1 : x_t = 0\}$. Let v_d be the escape probability: $v_d = P[H^+ = \infty]$. By the argument presented in the previous paragraph, by [16, Corollary 6.8] in dimension $d \geq 3$, and by [16, Corollary 6.12] in dimension 2,

$$\begin{aligned} \lim_{N\to\infty} \frac{\theta_N}{N^d} \ &= \ \frac{1}{2\,v_d} \quad \text{in dimension } d\geq 3\ , \\ \lim_{N\to\infty} \frac{\theta_N}{N^2\log N} \ &= \ \frac{1}{\pi} \quad \text{in dimension } d = 2\ . \end{aligned} \tag{3.1.6}$$

The factor 2 in the denominator appears because the process has been speeded-up by 2. In particular, in $d = 2$, $1/\pi$ should be understood as $(1/2)(2/\pi)$.

Consider the rescaled reduced process

$$\mathbb{X}_N(t) \ = \ \Psi_N(A_N(\theta_N t))\ , \quad t\geq 0\ . \tag{3.1.7}$$

Notice that $\mathbb{X}_N(t)$ is not a Markov chain, but only a hidden Markov chain. Denote by $\mathscr{P}^N$ the probability law on $\big(D(\mathbb{R}_+, S), \mathcal{G}_\infty\big)$ induced by the reduced process $\big(\mathbb{X}_N(t)\big)_{t\geq 0}$ under $\mathbf{P}^N$ (i.e. starting from all vertices in $\mathbb{T}^d_N$ occupied). The main result of this chapter reads as follows

Theorem 3.1.2. *For every $d \geq 2$, the sequence of measures $\mathscr{P}^N$ converges to $\mathscr{P}_0$.*

It follows from Theorem 3.1.2 that, under $\mathbf{P}^N$,

$$\big(\mathbb{X}_N(t)\big)_{t\geq 0} \ \xrightarrow{\text{Law}} \ (\mathscr{X}_t)_{t\geq 0}\ , \quad \text{for } d\geq 2.$$

The scaling limit for the coalescing times obtained in [11] immediately follows from these results.

Remark 3.1.3. *The proofs apply to the case in which the jump probability $p(\cdot)$ is symmetric and has finite range. It also applies if the initial condition $\mathbb{T}^d_N$ is replaced by a finite set $A = \{x_1, \dots, x_n\}$ whose points are scattered: $\|x_i - x_j\| \geq a_N$ for $1 \leq i \neq j \leq n$, where a_N is the sequence introduced in* (3.2.3).

3.1.3 Sketch of the Proof

The proof of Theorem 3.1.2 is divided in two steps. We first show that the sequence $(\mathscr{P}^N)$ is tight, and then we guarantee uniqueness of limit points by proving that every limit point solves the $(C^1(S), \mathscr{L})$-martingale problem.

For the later step, consider a smooth function $f : \mathbb{R} \to \mathbb{R}$, and denote by $M_N(t)$ the martingale given by

$$f(\mathbb{X}_N(t)) - f(\mathbb{X}_N(0)) - \int_0^t \theta_N \, (L_N f)\big(\Psi_N(A_N(s\theta_N))\big) \, ds \, .$$

Since

$$(L_N f)(\Psi_N(A)) = R(A) \Big\{ f\Big(\frac{x}{1-x}\Big) - f(x) \Big\} \, ,$$

where $x = \Psi_N(A)$, and $R(A)$ is the jump rate given by

$$R(A) = \sum_{x \in A} \sum_{y \in A \setminus \{x\}} p(y - x) \, , \tag{3.1.8}$$

the martingale $M_N(t)$ can be written as

$$f(\mathbb{X}_N(t)) - f(\mathbb{X}_N(0)) - \theta_N \int_0^t R(A_N(s\theta_N)) \Big\{ f\Big(\frac{\mathbb{X}_N(s)}{1 - \mathbb{X}_N(s)}\Big) - f(\mathbb{X}_N(s)) \Big\} \, ds \, .$$

If the martingale $M_N(t)$ were expressed in terms of the process $\mathbb{X}_N$, that is if $\theta_N \, R(A_N(s\theta_N)) = r(\mathbb{X}_N(s))$, we could pass to the limit and argue that

$$f(\mathbb{X}(t)) - f(\mathbb{X}(0)) - \int_0^t r(\mathbb{X}(s)) \Big\{ f\Big(\frac{\mathbb{X}(s)}{1 - \mathbb{X}(s)}\Big) - f(\mathbb{X}(s)) \Big\} \, ds \, . \tag{3.1.9}$$

is a martingale for every limit point $\mathscr{P}^*$ of the sequence $\mathscr{P}^N$. This result together with the uniqueness of solutions of the martingale problem (3.1.9) on $\big(D(\mathbb{R}_+, S), \mathcal{G}_\infty\big)$ would yield the uniqueness of limit points.

The previous argument evidences that the main point of the proof consists in "closing" the martingale $M_N(t)$ in terms of the reduced process $\mathbb{X}_N(s)$, that is, that the major difficulty lies in the proof of the existence of a function $r : S \to \mathbb{R}$ such that

$$\int_0^t \Big\{ \theta_N \, R(A_N(s\theta_N)) - r(\mathbb{X}_N(s)) \Big\} g(\mathbb{X}_N(s)) \, ds \longrightarrow 0$$

for all smooth functions $g : \mathbb{R} \to \mathbb{R}$. This is the so-called "replacement lemma" or the "local ergodic theorem". One has to replace a function $\theta_N R(A)$ which does not vanish only in a tiny portion of the state space (in the present context for subsets of $(\mathbb{T}_N^d)^n$ which contain at least two neighborhing points) and which is very large (here of order θ_N) when it does not vanish, by a function of order 1 in the entire space.

The statement of the local ergodic theorem requires some notation. Denote by $D(\mathbb{R}_+, E_N)$ the right-continuous trajectories $\omega : \mathbb{R}_+ \to E_N$ wich have left-limits. Let

$$\boldsymbol{r}\Big(\frac{1}{n}\Big) := \lambda(n) := \binom{n}{2} \, , \quad n \geq 2 \, . \tag{3.1.10}$$

Proposition 3.1.4. *Let $F : \mathbb{N} \to \mathbb{R}$ be a function which eventually vanishes: there exists $k_0 \geq 1$ such that $F(k) = 0$ for all $k \geq k_0$. Let $t_0 > 0$ and let $(B^N : D(\mathbb{R}_+, E_N) \to \mathbb{R}; N \geq 1)$ be a sequence of uniformly bounded functions, with each B^N measurable with respect to $\sigma\big(A_N(s\theta_N)$:*

$0 \le s \le t_0\big)$. *Then, for every* $t > t_0$,

$$\lim_{N\to\infty} \mathbf{E}^N \Big[B^N \int_{t_0}^{t} \{\theta_N R(A_N(s\theta_N)) - \mathfrak{n}_{s\theta_N}\} \, F(|A_N(s\theta_N)|) \, ds \Big] \;=\; 0 \,,$$

where $\mathfrak{n}_s = \boldsymbol{\lambda}(|A_N(s)|)$.

This chapter is organized as follows. In Section 3.2, we present the results on coalescing random walks needed in the proof of Proposition 3.1.4, which is presented in the following section. In Section 3.4, we prove Theorem 3.1.2 and, in Section 3.5, Proposition 3.1.1.

3.2 Coalescing Random Walks on $\mathbb{T}_N^d$

We present in this section some results on coalescing randoms walks obtained by Cox [11]: Propositions 3.2.1, 3.2.5 and 3.2.6. We start with some notation.

Throughout this section, P_x^N represents the distribution of a $\mathbb{T}_N^d$-valued random walk, *speeded-up by* 2, whose jump probability is $p(\cdot)$, introduced in (3.1.1), and initial position is x. Denote by $p_t(x,y) = P_x^N[x(t) = y]$ the transition probabilities of this process and by π_N its stationary state, which is the uniform measure on $\mathbb{T}_N^d$.

The first result, Proposition (4.1) in [11], provides a bound on the expectation of the number of particles still present at time t. Let

$$g_N(t) \;=\; \begin{cases} N^2\, t^{-1} \log(1+t) & d = 2\,, \\ N^d/t & d \ge 3\,. \end{cases}$$

Proposition 3.2.1. *There exists a finite constant* c_d *such that*

$$\mathbf{E}^N[\,|A_N(t)|\,] \;\le\; c_d \, \max\{1, g_N(t)\}$$

for all $t > 0$, $N \ge 1$.

Recall from (3.1.5) that we denote by $\mathscr{E}_N^n$ the subsets of $\mathbb{T}_N^d$ with n elements. Denote by τ_j, $j \ge 1$, the time when the process $A_N(t)$ is reduced to a set of j elements:

$$\tau_j \;=\; \inf\big\{t \ge 0 : |A_N(t)| = j\big\} \;=\; \inf\big\{t \ge 0 : A_N(t) \in \mathscr{E}_N^j\big\}\,. \tag{3.2.1}$$

Lemma 3.2.2. *There exists a finite constant* C_0 *such that for all* $j \ge 2$,

$$\max_{A \in \mathscr{E}_N^j} \frac{1}{\theta_N} \, \mathbf{E}_A^N\big[\tau_{j-1}\big] \;\le\; C_0\,.$$

Proof. Fix two points x, y in A and denote by $\tau_{x,y}$ the first time these particles meet: $\tau_{x,y} = \inf\{t > 0 : x(t) = y(t)\}$. Since $\tau_{j-1} \le \tau_{x,y}$, and since the difference $x(t) - y(t)$ evolves as a random walk speeded-up by 2, the expectation appearing in the statement of the lemma is bounded by

$$\max_{x \in \mathbb{T}_N} \frac{1}{\theta_N} \, E_x^N\big[H_0\big]\,,$$

where H_0 represents the hitting time of the origin. By [23, Proposition 10.13], this quantity is bounded by a finite constant independent of N. □

It follows from the previous result that for every $j \ge 2$,

$$\lim_{M\to\infty} \limsup_{N\to\infty} \max_{A\in\mathscr{E}_N^j} \mathbf{P}_A^N \big[\tau_{j-1} \;\ge\; M\,\theta_N\big] \;=\; 0\,. \tag{3.2.2}$$

Denote by $\|\mu-\nu\|_{\rm TV}$ the total variation distance between two probability measures, μ, ν, defined on a countable state space E:

$$\|\mu-\nu\|_{\rm TV} \;=\; \frac{1}{2}\sum_{a\in E} |\,\mu(a)-\nu(a)\,|\,.$$

Hereafter, the symbol $\alpha_N \ll \beta_N$, for two non-decreasing sequences α_N, β_N, means that $\alpha_N/\beta_N \to 0$. Denote by a_N an increasing sequence such that $1 \ll a_N \ll N$. In dimension 2, assume further that $N/\sqrt{\log N} \ll a_N$. Denote by $\mathfrak{G}_N(n,a_N)$ the scattered subsets of E_N. These are the sets $A=\{y_1,\dots,y_n\}$ in $\mathscr{E}_N^n$ such that

$$\min_{i\neq j} |y_i-y_j| \;\ge\; a_N\,. \tag{3.2.3}$$

Lemma 3.2.3. *For every $n\ge 2$, $t>0$,*

$$\lim_{N\to\infty}\max_{A\in\mathscr{E}_N^n} \mathbf{P}_A^N\Big[A_N(t\theta_N)\notin \mathscr{E}_N^1 \cup \bigcup_{k=2}^{n} \mathfrak{G}_N(k,a_N)\Big] \;=\; 0\,.$$

Proof. Since n is finite and since the difference of two random walks evolves as a random walk speeded-up by 2, this assertion follows from the claim that for every $t>0$

$$\begin{aligned}
&\lim_{N\to\infty}\max_{x\in\mathbb{T}_N^d} \mathbf{P}^N_{\{0,x\}}\big[A_N(t\theta_N)\notin \mathscr{E}_N^1\cup\mathfrak{G}_N(2,a_N)\big] \\
&\quad =\; \lim_{N\to\infty}\max_{x\in\mathbb{T}_N^d} P_x^N\big[H_0>t\theta_N\,,\,|x(t\theta_N)|\le a_N\big] \;=\; 0\,.
\end{aligned}$$

By the Markov property, the previous probability is bounded by

$$E_x^N\Big[P^N_{x(t\theta_N/2)}\big[\,|x(t\theta_N/2)|\le a_N\big]\Big]\,.$$

Recall from the beginning of this section that π_N represents the stationary state of the random walk on $\mathbb{T}_N^d$. The previous expectation is less than or equal to

$$P^N_{\pi_N}\big[|x(t\theta_N/2)|\le a_N\big] \;+\; 2\,\|\pi_N(\cdot)-p_{t\theta_N/2}(x,\cdot)\|_{\rm TV}\,,$$

where $p_t(x,y)$ represents the transition probabilities of a random walk evolving on $\mathbb{T}_N^d$ speeded-up by 2. The first term is bounded by $C_0(a_N/N)^d \to 0$, while the second one vanishes because $\theta_N \gg t^N_{\rm mix}$. □

Corollary 3.2.4. *For every $t>0$,*

$$\lim_{N\to\infty}\mathbf{P}^N\Big[A_N(t\theta_N)\notin\mathscr{E}_N^1\cup\bigcup_{k=2}^{N^d}\mathfrak{G}_N(k,a_N)\Big] \;=\; 0\,.$$

Proof. Fix $t>0$, and let $\mathscr{H}_s = \big\{A_N(s\theta_N)\notin\mathscr{E}_N^1\cup\bigcup_{k=2}^{N^d}\mathfrak{G}_N(k,a_N)\big\}$, $s>0$. Clearly, for every $M > 0$,

$$\mathbf{P}^N[\mathscr{H}_t] \;\le\; \mathbf{P}^N\big[\,|A_N(t\theta_N/2)| \le M\,,\,\mathscr{H}_t\,\big] \;+\; \mathbf{P}^N\big[\,|A_N(t\theta_N/2)| > M\,\big]\;.$$

By Proposition 3.2.1, the second term is bounded by $C(d,t)/M$, where $C(d,t)$ is a constant depending only on d and t. Hence, by the Markov property,

$$\mathbf{P}^N[\mathscr{H}_t] \;\le\; \max_{2\le k\le M}\,\max_{A\in\mathscr{E}_N^k} \mathbf{P}_A^N[\mathscr{H}_{t/2}] \;+\; \frac{C(d,t)}{M}\;.$$

By Lemma 3.2.3, the first term on the right-hand side vanishes as $N\to\infty$ for every $M\ge 2$. This proves the corollary. □

Proposition 3.2.5. *For every* $2\le j<k$*,*

$$\lim_{N\to\infty}\,\max_{A\in\mathfrak{G}_N(k,a_N)} \mathbf{P}_A^N\big[A_N(\tau_j)\notin\mathfrak{G}_N(j,a_N)\big] \;=\; 0\;.$$

Proof. Fix $2\le j<k$. By (3.2.2), it is enough to prove that for all $M>0$,

$$\lim_{N\to\infty}\,\max_{A\in\mathfrak{G}_N(k,a_N)} \mathbf{P}_A^N\big[A_N(\tau_j)\notin\mathfrak{G}_N(j,a_N)\,,\,\tau_j\le M\,\theta_N\big] \;=\; 0\;.$$

This is exactly assertions (3.7) and (3.8) in [11]. □

Denote by π_N^n, $n\ge 2$, the uniform measure on $\mathscr{E}_N^n$. Recall the definition of $\boldsymbol{\lambda}(\cdot)$ given in (3.1.10). Next proposition is a weak version of [11, Theorem 5].

Proposition 3.2.6. *For all* $j\ge 2$*,*

$$\lim_{N\to\infty}\mathbf{P}^N_{\pi_N^j}\big[\,\tau_{j-1}\ge t\,\theta_N\,\big] \;=\; e^{-\lambda(j)\,t}\;.$$

It follows from the previous result that for every $n\ge 1$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbf{P}^N\big[\tau_n\le\delta\theta_N\big] \;=\; 0\;. \tag{3.2.4}$$

Indeed, fix $n\ge 1$ and consider a set $A\in\mathscr{E}_N^{n+1}$. Since $A\subset\mathbb{T}_N^d$, $\mathbf{P}^N[\tau_n\le\delta\theta_N]\le\mathbf{P}_A^N[\tau_n\le\delta\theta_N]$. Averaging over A with respect to π_N^{n+1} we obtain that $\mathbf{P}^N[\tau_n\le\delta\theta_N]\le\mathbf{P}^N_{\pi_N^{n+1}}[\tau_n\le\delta\theta_N]$. By Proposition 3.2.6, the previous quantity vanishes as $N\to\infty$ and then $\delta\to 0$.

Denote by γ_N a sequence much larger than the mixing time and much smaller than the hitting time:

$$t^N_{\rm mix} \;\ll\; \gamma_N \;\ll\; \theta_N\;. \tag{3.2.5}$$

Let $(\ell_N : n\ge 1)$ be a sequence such that $1\ll\ell_N\ll N$. In dimension 2, we assume that $N^\alpha\ll\ell_N\ll N$ for all $0<\alpha<1$, so that

$$\lim_{N\to\infty}\frac{\ell_N}{N} \;=\; 0\,,\quad \lim_{N\to\infty}\frac{\log\ell_N}{\log N} \;=\; 1\;. \tag{3.2.6}$$

Note that in dimension 2 the conditions imposed on ℓ_N are weaker than the ones assumed on a_N in [11, Theorem 4].

Lemma 3.2.7. *For every* $n\ge 2$*,*

$$\lim_{N\to\infty}\,\max_{A\in\mathfrak{G}_N(n,\ell_N)} \mathbf{P}_A^N\big[\tau_{n-1}\le\gamma_N\big] \;=\; 0\;.$$

Proof. The probability is bounded by

$$\binom{n}{2} \max_{\|x\|\geq \ell_N} P^N_x[H_0 \leq \gamma_N]\,,$$

where, recall, H_0 stands for the hitting time of the origin. Since $\gamma_N \ll \theta_N$, by equation (6.18) in [16], this expression vanishes in the limit. □

In the next lemma we compare the dynamics $A_N(t)$ with the one of independent random walks. Fix $n \geq 2$, and denote by $(\mathbf{x}^n_N(t))_{t\geq 0}$, the evolution of n independent random walks on $\mathbb{T}^d_N$ with jump probabilities $p(\cdot)$ given by (3.1.1). The stationary state of this dynamics, denoted by $\pi^{\otimes n}_N$, is the product measure on $[\mathbb{T}^d_N]^n$ in which each component is the measure π_N.

Denote by $p^{(n)}_t(\mathbf{x},\mathbf{y})$ the transition probabilities of $\mathbf{x}^n_N(t)$, and by $t^{N,n}_{\text{mix}}$ the corresponding mixing time. Since the dynamics amounts to the evolution of a random walk on $\mathbb{T}^{nd}_N$, there exist constants $0 < c(d,n) < C(d,n) < \infty$ such that $c(d,n)N^2 \leq t^{N,n}_{\text{mix}} \leq C(d,n)N^2$ (cf. [23, Section 5.3 and 7.4]).

Denote by $x_j(t) \in \mathbb{T}^d_N$ the j-th coordinate of $\mathbf{x}^n_N(t)$, $1 \leq j \leq n$. Up to time τ_{n-1} the process $A_N(t)$ evolves as $\{\mathbf{x}^n_N(t)\} := \{x_1(t),\dots,x_n(t)\}$. More precisely, fix $A = \{a_1,\dots,a_n\} \in \mathscr{E}^n_N$, and let

$$\mathscr{E}^{\leq n}_N \;:=\; \bigcup_{k=1}^{n} \mathscr{E}^k_N\,.$$

There exists a probability measure on $D\big(\mathbb{R}_+, \mathscr{E}^{\leq n}_N \times (\mathbb{T}^d_N)^n\big)$, denoted by $\widehat{\mathbf{P}}^N_A$, which fulfills the following conditions. The distribution of the first, resp. second, coordinate corresponds to the distribution induced by $A_N(t)$, resp. $\mathbf{x}^n_N(t)$. Furthermore, $A_N(0) = A$, $\mathbf{x}^n_N(0) = (a_1,\dots,a_n)$, and $A_N(t) = \{\mathbf{x}^n_N(t)\}$ for all $0 \leq t \leq \tau_{n-1}$, $\widehat{\mathbf{P}}^N_A$ almost surely.

Lemma 3.2.8. *Fix $n \geq 2$. Let $F_N : \mathscr{E}^{\leq n}_N \to \mathbb{R}$ be a sequence of uniformly bounded functions, $\|F\| := \sup_{N\geq 1} \max_{A\in\mathscr{E}^{\leq n}_N} |F_N(A)| < \infty$, and let $(\beta_N)_{N\geq 1}$ be a non-negative sequence. Then, for every $A = \{a_1,\dots a_n\} \in \mathscr{E}^n_N$,*

$$\begin{aligned}
&\mathbf{E}^N_A\Big[F_N(A_N(\beta_N))\,\mathbb{1}\{\tau_{n-1} > \beta_N\}\Big] \;-\; E_{\pi^n_N}[F_N] \\
&\quad= \;-\,\widehat{\mathbf{E}}^N_A\Big[F_N(\{\mathbf{x}^n_N(\beta_N)\})\,\mathbb{1}\big\{\tau_{n-1} \leq \beta_N\,,\,\{\mathbf{x}^n_N(\beta_N)\} \in \mathscr{E}^n_N\,\big\}\Big] \;+\; R_N\,,
\end{aligned}$$

where

$$|R_N| \;\leq\; \|F\|\,\Big\{2\,\|p^{(n)}_{\beta_N}(\mathbf{a},\cdot) - \pi^{\otimes n}_N(\cdot)\|_{\text{TV}} \;+\; c_N\Big\}$$

and $\lim_{N\to\infty} c_N = 0$.

Proof. Fix $A = \{a_1,\dots a_n\} \in \mathscr{E}^n_N$. We may rewrite the expectation appearing in the statement of the lemma as

$$\widehat{\mathbf{E}}^N_A\Big[F_N(A_N(\beta_N))\,\mathbb{1}\{\tau_{n-1} > \beta_N\}\Big]\,.$$

Since $A_N(t) = \{\mathbf{x}^n_N(t)\}$ in the time interval $[0,\tau_{n-1}]$, we may replace in the previous equation $A_N(\beta_N)$ by $\{\mathbf{x}^n_N(\beta_N)\}$ and then add the indicator function of the set $\{\mathbf{x}^n_N(\beta_N)\} \in \mathscr{E}^n_N$. After these replacements, the previous expression becomes

$$\begin{aligned}
&\widehat{\mathbf{E}}^N_A\Big[F_N(\{\mathbf{x}^n_N(\beta_N)\})\,\mathbb{1}\{\,\{\mathbf{x}^n_N(\beta_N)\} \in \mathscr{E}^n_N\,\}\Big] \\
&\quad-\;\widehat{\mathbf{E}}^N_A\Big[F_N(\{\mathbf{x}^n_N(\beta_N)\})\,\mathbb{1}\big\{\tau_{n-1} \leq \beta_N\,,\,\{\mathbf{x}^n_N(\beta_N)\} \in \mathscr{E}^n_N\,\big\}\Big]\,.
\end{aligned}$$

We estimate the first term. Recall that we denote by $p_t^{(n)}(\mathbf{x},\mathbf{y})$ the transition probabilities of $\mathbf{x}_N^n(t)$. With this notation, we may write this term as

$$\sum_{\mathbf{x}\in[\mathbb{T}_N^d]^n} F_N(\{\mathbf{x}\})\,\mathbb{1}\{\,\{\mathbf{x}\}\in\mathscr{E}_N^n\,\}\,\pi_N^{\otimes n}(\mathbf{x}) \;+\; R_N^{(1)}\;,$$

where

$$\big|\,R_N^{(1)}\,\big| \;\le\; 2\,\|F\|\,\|p^{(n)}_{\beta_N}(\mathbf{a},\cdot)-\pi_N^{\otimes n}(\cdot)\|_{\rm TV}\;.$$

and $\mathbf{a}=(a_1,\dots,a_n)$.

To bound the first term of the penultimate formula, recall that we denote by π_N^n the uniform measure on $\mathscr{E}_N^n$. Let

$$R^{(2)}_{N,n} \;:=\; \sum_{A\in\mathscr{E}_N^n}\Big|\,\pi_N^n(A)\;-\;\sum_{\mathbf{x}\in[\mathbb{T}_N^d]^n}\mathbb{1}\{\,\{\mathbf{x}\}=A\}\,\pi_N^{\otimes n}(\mathbf{x})\,\Big|\;. \tag{3.2.7}$$

An elementary computation shows that $\lim_{N\to\infty}R^{(2)}_{N,n}=0$ for every $n\ge 2$. The assertion of the lemma follows from the previous estimates. □

The next lemma is a consequence of [11, Theorem 5] in dimension $d\ge 3$. In dimension 2 is a slight generalization since our assumptions on ℓ_N are weaker. Recall (3.2.6).

Lemma 3.2.9. *Let ℓ_N be a sequence satisfying the conditions introduced in* (3.2.6). *Then, for all $t>0$,*

$$\lim_{N\to\infty}\max_{A\in\mathfrak{G}(n,\ell_N)}\Big|\,\mathbf{P}_A^N\big[\tau_{n-1}\ge t\theta_N\big]\;-\;e^{-\lambda(n)\,t}\,\Big|\;=\;0\;.$$

Proof. We present the proof in dimension $d=2$. The one in higher dimension is analogous. Fix a set $A=\{a_1,\dots,a_n\}$ in $\mathfrak{G}_N(n,\ell_N)$ and a sequence $1\ll t_N\ll\log N$. Recall from the previous lemma the definition of the measure $\widehat{\mathbf{P}}_A^N$. Since the first coordinate evolves as $A_N(t)$,

$$\mathbf{P}_A^N\big[\tau_{n-1}\ge t\theta_N\big] \;=\; \widehat{\mathbf{P}}_A^N\big[\tau_{n-1}\ge t\theta_N\big]\;.$$

By the Markov property and Lemma 3.2.7,

$$\widehat{\mathbf{P}}_A^N\big[\tau_{n-1}\ge t\theta_N\big] \;=\; \widehat{\mathbf{E}}_A^N\Big[\widehat{\mathbf{P}}^N_{A_N(\gamma_N)}\big[\tau_{n-1}\ge t\theta_N-\gamma_N\big]\,\mathbb{1}\{\tau_{n-1}>\gamma_N\}\Big]\;+\;o_N(1)\;,$$

where $\gamma_N=t_NN^2$.

We apply Lemma 3.2.8 with $\beta_N=\gamma_N$ to estimate the right-hand side. Let $F_N:\mathscr{E}_N^{\le n}\to\mathbb{R}$ be the function defined by

$$F_N(A) \;=\; \mathbf{P}_A^N\big[\tau_{n-1}\ge t\theta_N-\gamma_N\big]\;,\quad A\,\in\,\mathscr{E}_N^n\;,$$

and $F_N(A)=0$ for $A\notin\mathscr{E}_N^n$. By Lemma 3.2.8, the right hand side of the penultimate formula is equal to

$$\mathbf{P}^N_{\pi_N^n}\big[\tau_{n-1}\ge t\theta_N-\gamma_N\big]\;+\;\boldsymbol{R}_N\;,$$

where

$$\big|\boldsymbol{R}_N\big| \;\le\; \widehat{\mathbf{P}}_A^N\big[\tau_{n-1}\le\gamma_N\big]\;+\;2\,\|p^{(n)}_{\gamma_N}(\mathbf{a},\cdot)-\pi_N^{\otimes n}(\cdot)\|_{\rm TV}\;+\;c_N\;,$$

with $\lim_{N\to\infty}c_N=0$.

Each term of the previous expression is negligible. In the first one, we may replace $\widehat{\mathbf{P}}_A^N$ by $\mathbf{P}_A^N$, and apply Lemma 3.2.7 to conclude that this expression vanishes as $N\to\infty$. The second one also vanishes in the limit because $\gamma_N\gg t^N_{\rm mix}$ and $t^{N,n}_{\rm mix}$ is of the same order of

$t^N_{\rm mix}$. To complete the proof of the lemma, as $\gamma_N \ll \theta_N$, it remains to apply Proposition 3.2.6. □

Recall the properties of the sequence a_N introduced in (3.2.3). By the previous result, for all $k > j \geq 2$,

$$\lim_{N\to\infty} \max_{A\in\mathfrak{G}_N(k,a_N)} \Big| \mathbf{P}^N_A\big[\, \tau_{j-1} - \tau_j \ge t\,\theta_N \,\big] \,-\, e^{-\lambda(j)\,t} \Big| \;=\; 0\,. \tag{3.2.8}$$

Indeed, by Proposition 3.2.5, we may intersect the event appearing inside the probability with the set $\{A_N(\tau_j) \in \mathfrak{G}_N(j,a_N)\}$. Then, applying the strong Markov property at time τ_j we reduce assertion (4.5.4) to Lemma 3.2.9.

The next result together with the previous lemma entails the convergence of $\mathbf{E}^N_{A_N}[\tau_{n-1}/\theta_N]$ to $\lambda(n)^{-1}$ for any sequence $A_N \in \mathfrak{G}(n,\ell_N)$.

Lemma 3.2.10. *For every $n \ge 2$, $m \ge 1$, there exists a finite constant $C(n,m)$ such that for all $N \ge 1$,*

$$\max_{A\in\mathscr{E}^n_N} \mathbf{E}^N_A\big[\,(\tau_{n-1}/\theta_N)^m\,\big] \;\le\; C(n,m)\,.$$

Proof. By the Markov property, for all $k \ge 1$,

$$\max_{A\in\mathscr{E}^n_N} \mathbf{P}^N_A[\tau_{n-1}/\theta_N \ge k] \;\le\; \Big(\max_{A\in\mathscr{E}^n_N} \mathbf{P}^N_A[\tau_{n-1}/\theta_N \ge 1]\Big)^k\,.$$

We claim that

$$\max_{A\in\mathscr{E}^n_N} \mathbf{P}^N_A[\tau_{n-1} \ge \theta_N] \;\le\; \mathbf{P}^N_{\pi^n_N}[\tau_{n-1} \ge \theta_N/2] \;+\; \delta_N\,. \tag{3.2.9}$$

where $\delta_N \to 0$. Indeed, fix $A = \{a_1,\dots a_n\} \in \mathscr{E}^n_N$, and apply the Markov property to obtain that

$$\mathbf{P}^N_A[\tau_{n-1} \ge \theta_N] \;=\; \mathbf{E}^N_A\Big[\mathbf{P}^N_{A_N(\theta_N/2)}[\tau_{n-1} \ge \theta_N/2]\,\mathbb{1}\{\tau_{n-1} \ge \theta_N/2\}\Big]\,.$$

Let $F_N : \mathscr{E}^{\le n}_N \to \mathbb{R}$ be the function defined by

$$F_N(A) \;=\; \mathbf{P}^N_A\big[\tau_{n-1} \ge \theta_N/2\big]\,,\quad A\,\in\,\mathscr{E}^n_N\,,$$

and $F_N(A) = 0$ for $A \notin \mathscr{E}^n_N$. Since F_N is non-negative, by Lemma 3.2.8, the right-hand side of the penultimate formula is bounded above by

$$\mathbf{P}^N_{\pi^n_N}[\tau_{n-1} \ge \theta_N/2] \;+\; 2\,\|p^{(n)}_{\theta_N/2}(\mathbf{a},\cdot) - \pi^n_N(\cdot)\|_{\rm TV} \;+\; c_N\,,$$

where $\mathbf{a} = (a_1,\dots a_n)$. Assertion (3.2.9) follows from the facts that $\theta_N \gg t^N_{\rm mix}$ and that $t^{N,n}_{\rm mix}$ is of the same order of $t^N_{\rm mix}$.

By Proposition 3.2.6, under the measure $\mathbf{P}^N_{\pi^n_N}$, τ_{n-1}/θ_N converges weakly to an exponential random variable of parameter $\lambda(n)$. Thus, the right-hand side of (3.2.9) converges to $e^{-\lambda(n)/2} < 1$. Therefore, there exists $\delta < 1$ such that for all $N \ge 1$,

$$\max_{A\in\mathscr{E}^n_N} \mathbf{P}^N_A[\tau_{n-1}/\theta_N \ge k] \;\le\; \delta^k\,.$$

This proves the lemma. □

Corollary 3.2.11. *For every $n \geq 2$,*

$$\lim_{N\to\infty} \max_{A\in\mathfrak{G}_N(n,\ell_N)} \Big| \frac{1}{\theta_N} \mathbf{E}^N_A[\tau_{n-1}] - \frac{1}{\lambda(n)} \Big| = 0 .$$

Proof. Fix a sequence $A_N \in \mathfrak{G}_N(n,\ell_N)$, $N \geq 1$. The convergence in law of the sequence τ_{n-1}/θ_N under the measure $\mathbf{P}^N_{A_N}$ to an exponential random variable of parameter $\lambda(n)$ follows from Lemma 3.2.9. By the previous lemma the sequence τ_{n-1}/θ_N is uniformly integrable. □

Recall that we denote by $(e_1,\dots,e_d)$ the canonical basis of $\mathbb{R}^d$.

Lemma 3.2.12. *Assume that $d \geq 3$ and $n \geq 2$. Fix a sequence of sets $A_N \in \mathscr{E}^n_N$ such that $A_N = \{x_N, x_N \pm e_j\} \cup B_N$, where $B_N \cup \{x_N\}$ belongs to $\mathfrak{G}_N(n-1,\ell_N)$. For all $t > 0$,*

$$\lim_{N\to\infty} \mathbf{P}^N_{A_N}[\tau_{n-1} \geq t\theta_N] = v_d\, e^{-\lambda(n)t} . \tag{3.2.10}$$

Proof. Denote by $x(t)$, $y(t)$ the position at time t of the particle initially at x_N, $x_N \pm e_j$, respectively. Let D_r, $r \geq 0$, be the first time the distance between these particles attains r: $D_r = \inf\{t > 0 : \|x(t) - y(t)\| = r\}$, and let $H = D_0 \wedge D_{\ell_N}$. As $\ell_N \ll N$, an elementary computation shows that

$$\lim_{N\to\infty} \mathbf{P}^N_{A_N}[H > N^2] = 0 .$$

We may therefore insert the set $\{H \leq N^2\}$ in the probability appearing in equation (3.2.10). On the event $\{H \leq N^2\}$, when $tN^{d-2} > 1$, we have that $\{D_0 < D_{\ell_N}\} \cap \{\tau_{n-1} \geq t\theta_N\} = \varnothing$. Note that here we used that $d \geq 3$. Hence,

$$\mathbf{P}^N_{A_N}[\tau_{n-1} \geq t\theta_N] = \mathbf{P}^N_{A_N}\Big[H \leq N^2\,,\, D_0 > D_{\ell_N}\,,\, \tau_{n-1} \geq t\theta_N\Big] + o_N(1)\,,$$

where $o_N(1) \to 0$ as $N \to \infty$.

By the Markov property, the probability on the right hand side is equal to

$$\mathbf{E}^N_{A_N}\Big[\mathbb{1}\{H \leq N^2\,,\, D_0 > D_{\ell_N}\,,\, \tau_{n-1} \geq N^2\}\, \mathbf{P}^N_{A_N(N^2)}[\tau_{n-1} \geq t\theta_N - N^2]\Big] .$$

On the event $\{\tau_{n-1} \geq N^2\}$, we may replace the distribution of $A_N(N^2)$ by the one of the position at time N^2 of n independent random walks starting from A_N. After this replacement, we may insert in the expectation the indicator of the set $\{A_N(N^2) \in \mathfrak{G}_N(n,\ell_N)\}$ because the probability of the complement vanishes as $N \to \infty$ [indeed, whatever the initial position of a random walk, its probability to be a distance ℓ_N from the origin at time N^2 vanishes]. After this insertion, we write the previous expectation as

$$e^{-\lambda(n)t}\, \mathbf{P}^N_{A_N}\Big[H \leq N^2\,,\, D_0 > D_{\ell_N}\,,\, A_N(N^2) \in \mathfrak{G}_N(n,\ell_N)\,,\, \tau_{n-1} \geq N^2\Big] + R_N\,,$$

where the absolutely value of R_N is bounded by

$$\max_{A\in\mathfrak{G}_N(n,\ell_N)} \Big|\mathbf{P}^N_A[\tau_{n-1} \geq t\theta_N - N^2] - e^{-\lambda(n)t}\Big| .$$

By Lemma 3.2.9, this expression vanishes as $N \to \infty$. Hence, up to this point we proved that the probability appearing in (3.2.10) is equal to

$$e^{-\lambda(n)t}\, \mathbf{P}^N_{A_N}\Big[H \leq N^2\,,\, D_0 > D_{\ell_N}\,,\, A_N(N^2) \in \mathfrak{G}_N(n,\ell_N)\,,\, \tau_{n-1} \geq N^2\Big] + o_N(1) .$$

On the set $\{H \le N^2, D_0 > D_{\ell_N}, \tau_{n-1} \le N^2\}$ two particles which were at distance at least ℓ_N met in a time interval of length bounded by N^2. Indeed, the time τ_{n-1} may correspond to the coalescence of two particles on the set B_N or one particle in the set B_N and one in the set $\{x_N, x_N \pm e_j\}$. In both cases, these particles were initially at distance at least ℓ_N from each other. The time τ_{n-1} may also correspond to the coalescence of the particles initially at x_N, $x_N \pm e_j$. In this case, at time $H \le N^2 \wedge D_0$ these particles were at distance ℓ_N.

By Lemma 3.2.7 with $n = 2$, the probability that two particles which are at distance ℓ_N meet before time N^2 vanish as $N \to \infty$. We may therefore remove from the previous probability the event $\{\tau_{n-1} \ge N^2\}$. We may also remove, as explained above in the proof, the events $\{H \le N^2\}$ and $\{A_N(N^2) \in \mathfrak{G}_N(n, \ell_N)\}$, so that

$$\mathbf{P}^N_{A_N}[\tau_{n-1} \ge t\theta_N] \;=\; e^{-\lambda(n)t}\, \mathbf{P}^N_{A_N}[D_0 > D_{\ell_N}] \;+\; o_N(1)\;.$$

As $N \to \infty$, this latter probability converges to the escape probability, denoted by v_d, which proves the lemma. □

The next result follows from the previous lemma and from the uniform integrability provided by Lemma 3.2.10.

Corollary 3.2.13. *Assume that $d \ge 3$ and $n \ge 2$. Fix a sequence of sets $A_N \in \mathscr{E}^n_N$ such that $A_N = \{x_N, x_N \pm e_j\} \cup B_N$, where $B_N \cup \{x_N\}$ belongs to $\mathfrak{G}_N(n-1, \ell_N)$. Then,*

$$\lim_{N\to\infty} \frac{1}{\theta_N}\, \mathbf{E}^N_{A_N}[\tau_{n-1}] \;=\; \frac{v_d}{\lambda(n)}\;\cdot$$

By (3.1.6), the previous limit can be written as

$$\lim_{N\to\infty} \frac{1}{N^d}\, \mathbf{E}^N_{A_N}[\tau_{n-1}] \;=\; \frac{1}{2\,\lambda(n)}\;\cdot \tag{3.2.11}$$

We turn to the 2-dimensional case.

Lemma 3.2.14. *Assume that $d = 2$ and $n \ge 2$. Fix a sequence of sets $A_N \in \mathscr{E}^n_N$ such that $A_N = \{z_N, z_N \pm e_j\} \cup B_N$, where $B_N \cup \{z_N\}$ belongs to $\mathfrak{G}_N(n-1, \ell_N)$. Then,*

$$\lim_{N\to\infty} \frac{1}{N^2}\, \mathbf{E}^N_{A_N}[\tau_{n-1}] \;=\; \frac{1}{2\,\lambda(n)}\;\cdot$$

Proof. Fix a sequence of sets A_N satisfying the hypotheses of the lemma. Enumerate the points of $A_N = \{x_1, \dots, x_n\}$ in such a way that $x_1 = z_N$, $x_2 = z_N \pm e_j$. Denote by $x_i(t)$ the position at time t of the random walks initially at x_i.

Let $(\ell_N : N \ge 1)$, $(m_N : N \ge 1)$ be the sequences $\ell_N = N/(\log N)^4$, $m_N = N/\log N$. Notice that both sequences fulfill the conditions above (3.2.6). Let $T_{1,2}$ be the first time the difference $x_1(t) - x_2(t)$ reaches the distance ℓ_N, $T_{1,2} = \inf\{t > 0 : \|x_1(t) - x_2(t)\| \ge \ell_N\}$, and denote by T_i, $1 \le i \le n$, the first time the particle x_i reaches a distance m_N from its original position: $T_i = \inf\{t > 0 : \|x_i(t) - x_i(0)\| \ge m_N\}$. The proof of the lemma relies on the estimates (3.2.12), (3.2.13) and (3.2.14).

Since the difference $x_1(t) - x_2(t)$ evolves as a random walk speeded-up by 2,

$$\mathbf{E}^N_{A_N}[T_{1,2}] \;=\; E^N_{e_1}[\bar{D}_{\ell_N}]\;,$$

where $\bar{D}_{\ell_N}$ is the first time the particle reaches a distance ℓ_N from the origin, and $P^N_{e_1}$ represents the distribution of a symmetric, nearest-neighbor random walk speeded-up by

2, starting from e_1. Denote by $B(x,r)$ the ball centered at x of radius r. By equation (6.5) in [16] and a simple estimate of the capacity between 0 and $B(0,\ell_N)^c$, $E^N_{e_1}[\bar{D}_{\ell_N}] \le C_0 \ell_N^2$ for some constant C_0 independent of N. Hence,

$$\lim_{N\to\infty} \frac{1}{N^2}\, \mathbf{E}^N_{A_N}\big[T_{1,2}\big] \;=\; 0\,. \tag{3.2.12}$$

For every $1\le i\le n$, and every sequence $(S_N)_{N\ge 1}$ of non-negative numbers,

$$\mathbf{P}^N_{A_N}[T_i \le S_N] \;=\; \widehat{P}^N_0[\bar{D}_{m_N} \le S_N] \;=\; \widehat{P}^N_0\big[\sup_{t\le S_N} \|x(t)\| \ge m_N\big]\,,$$

where $\widehat{P}^N_0$ stands for the distribution of a nearest-neighbor, symmetric, random walk starting from the origin. The difference with respect to P^N_0 is that the random walk is not speeded-up by 2 under $\widehat{P}^N_0$. An elementary random walk estimation yields that the right hand side multiplied by $\log N$ vanishes as $N\to\infty$ if we choose $S_N = N^2/(\log N)^4$. Hence, wit this definition for S_N, for all $1\le i\le n$,

$$\lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}[T_i \le S_N] \;=\; 0\,.$$

In contrast,

$$\mathbf{P}^N_{A_N}[T_{1,2} \ge S_N] \;=\; P^N_{e_1}[\bar{D}_{\ell_N} \ge S_N] \;=\; P^N_{e_1}\big[\sup_{t\le S_N} \|x(t)\| \le \ell_N\big]\,.$$

Another elementary random walk estimation yields that the right hand side multiplied by $\log N$ vanishes for the same choice of the sequence S_N. Hence,

$$\lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}[T_{1,2} \ge S_N] \;=\; 0\,.$$

It follows from the last two estimates that

$$\lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}\big[T_{1,2} \ge \min_i T_i\big] \;=\; 0\,. \tag{3.2.13}$$

Denote by $\tau_{i,j}$, $1\le i\ne j\le n$, the first time the particles x_i, x_j meet, $\tau_{i,j} = \inf\{t>0 : x_i(t) = x_j(t)\}$. The arguments used to derive (3.2.13) show that for all pairs $\{i,j\}\ne\{1,2\}$,

$$\lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}\big[T_{1,2} \ge \tau_{i,j}\big] \;=\; 0\,. \tag{3.2.14}$$

We are now in a position to prove the lemma. By the strong Markov property,

$$\begin{aligned}
\mathbf{E}^N_{A_N}[\tau_{n-1}] \;&=\; \mathbf{E}^N_{A_N}\Big[\, \big[T_{1,2} + \tau_{n-1}\circ\vartheta_{T_{1,2}}\big]\, \mathbb{1}\{T_{1,2} < \tau_{n-1}\}\Big] \;+\; \mathbf{E}^N_{A_N}\Big[\tau_{n-1}\, \mathbb{1}\{\tau_{n-1} < T_{1,2}\}\Big] \\
&=\; \mathbf{E}^N_{A_N}\Big[\mathbf{E}^N_{A_N(T_{1,2})}\big[\tau_{n-1}\big]\, \mathbb{1}\{T_{1,2} < \tau_{n-1}\}\Big] \;+\; \mathbf{E}^N_{A_N}\Big[\tau_{n-1}\, \mathbb{1}\{\tau_{n-1} < T_{1,2}\}\Big]\,.
\end{aligned}$$

The second term is bounded by $\mathbf{E}^N_{A_N}[T_{1,2}]$. By (3.2.12), this expectation divided by N^2 vanishes as $N\to\infty$. On the other hand,

$$\begin{aligned}
&\frac{1}{N^2}\, \mathbf{E}^N_{A_N}\Big[\mathbf{E}^N_{A_N(T_{1,2})}\big[\tau_{n-1}\big]\, \mathbb{1}\{T_{1,2} \ge \min_i T_i\,,\, T_{1,2} < \tau_{n-1}\}\Big] \\
&\quad\le\; \sup_{A\in\mathscr{E}^n_N} \frac{1}{(\log N)\, N^2}\, \mathbf{E}^N_A\big[\tau_{n-1}\big]\, (\log N) \mathbf{P}^N_{A_N}\big[T_{1,2} \ge \min_i T_i\big]\,.
\end{aligned}$$

This expression vanishes as $N \to \infty$ because, by Lemma 3.2.10, the first term is uniformly bounded and, by (3.2.13), the second term tends to 0.

Up to this point, we proved that

$$\lim_{N\to\infty} \frac{1}{N^2} \mathbf{E}^N_{A_N}[\tau_{n-1}] \;=\; \lim_{N\to\infty} \frac{1}{N^2}\, \mathbf{E}^N_{A_N}\Big[\mathbf{E}^N_{A_N(T_{1,2})}[\tau_{n-1}]\, \mathbb{1}\{T_{1,2} < \min_i\{\tau_{n-1}, T_i\}\}\Big]\;.$$

On the set $\{T_{1,2} < \min_i T_i\}$, $A_N(T_{1,2})$ belongs to $\mathfrak{G}_N(n, \ell_N)$. Hence, by Corollary 3.2.11 and by (3.1.6),

$$\frac{1}{N^2}\, \mathbf{E}^N_{A_N(T_{1,2})}[\tau_{n-1}] \;=\; (\log N)\, \frac{\pi^{-1}}{\boldsymbol{\lambda}(n)}\, [1 + o_N(1)]\;,$$

so that

$$\lim_{N\to\infty} \frac{1}{N^2} \mathbf{E}^N_{A_N}[\tau_{n-1}] \;=\; \frac{\pi^{-1}}{\boldsymbol{\lambda}(n)} \lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}\big[T_{1,2} < \min_i\{\tau_{n-1}, T_i\}\big]\;.$$

By (3.2.13), in the previous expression we may remove the indicator of the set $\{T_{1,2} < \min_i T_i\}$. By (3.2.14), we may also exclude the sets $\{\tau_{i,j} \le T_{1,2}\}$ for $\{i,j\} \neq \{1,2\}$. Hence, the previous expression is equal to

$$\frac{\pi^{-1}}{\boldsymbol{\lambda}(n)} \lim_{N\to\infty} (\log N)\, \mathbf{P}^N_{A_N}\big[T_{1,2} < \tau_{1,2}\big] \;=\; \frac{\pi^{-1}}{\boldsymbol{\lambda}(n)} \lim_{N\to\infty} (\log N)\, P^N_{e_1}\big[\bar{D}_{\ell_N} < H_0\big]\;,$$

where H_0 represents the hitting time of the origin. By [16, Lemma 6.10], the previous expression is equal to $1/[2\boldsymbol{\lambda}(n)]$, which completes the proof of the lemma. □

Recall the definition of the jump rate R introduced in (3.1.8).

Lemma 3.2.15. *For every $n \ge 2$,*

$$\lim_{N\to\infty} \sum_{A \in \mathscr{E}^n_N} \pi^n_N(A)\, \mathbf{E}^N_A[\tau_{n-1}]\, R(A) \;=\; 1\;.$$

Proof. Since $R(A) = 0$ unless A contains two nearest-neighbor points, for all sets A such that $R(A) > 0$, $\mathbf{E}^N_A[\tau_{n-1}] \le E_{e_1}[H_0]$, where H_0 represents the hitting time of the origin. By [23, Proposition 10.13], this latter expectation is bounded by $C_0 N^d$.

Since $R(A)$ is uniformly bounded, $\mathbf{E}^N_A[\tau_{n-1}] \le C_0 N^d$, and $\pi^n_N(A) = 1/\binom{N^d}{n}$, we may restrict the sum appearing in the statement of the lemma to sets $A = \{x, x \pm e_j\} \cup B$, where $B \cup \{x\} \in \mathfrak{G}_N(n-1, \ell_N)$. The number of such sets A is $(n-1)d\binom{N^d}{n-1}[1 + o_N(1)]$. For them $R(A) = 1/d$, and, by (3.2.11) and Lemma 3.2.14,

$$\mathbf{E}^N_A[\tau_{n-1}] \;=\; \frac{N^d}{2\boldsymbol{\lambda}(n)}[1 + o_N(1)]\;.$$

Hence, the sum alluded to above is equal to

$$[1 + o_N(1)]\,(n-1)\,d\, \frac{\binom{N^d}{n-1}}{\binom{N^d}{n}}\, \frac{N^d}{2\,\boldsymbol{\lambda}(n)}\, \frac{1}{d} \;=\; [1 + o_N(1)]\, \frac{n(n-1)}{2\,\boldsymbol{\lambda}(n)}\;.$$

The result follows from the definition of $\boldsymbol{\lambda}(n)$ given in (3.1.10). □

3.3 Local Ergodicity

We prove in this section Proposition 3.1.4. It states that we may replace the time integral of a function $f(A_N(s))$ by the time integral of a function $F(|A_N(s)|)$. The proof is divided in a sequence of lemmata.

Lemma 3.3.1. *For every $n \geq 2$, there exists a finite constant $C(n)$ such that*

$$\max_{A\in\mathscr{E}_N^n} \mathbf{E}_A^N\Big[\int_0^\infty R(A_N(s))\,ds\Big] \;\le\; C(n)\;.$$

Proof. Since $R(B)=0$ if $|B|=1$,

$$\int_0^\infty R(A_N(s))\,ds \;=\; \int_0^{\tau_1} R(A_N(s))\,ds\;.$$

It is therefore enough to prove that for each $n\geq 2$, there exists a finite constant $C(n)$ such that

$$\max_{A\in\mathscr{E}_N^n} \mathbf{E}_A^N\Big[\int_0^{\tau_{n-1}} R(A_N(s))\,ds\Big] \;\le\; C(n)\;.$$

Fix $n\geq 2$ and a set $A=\{x_1,\dots,x_n\}$ in $\mathscr{E}_N^n$. Denote by $x_i(s)$ the position at time s of the particle x_i and by $\tau_{i,j}$ the collision time of particles i and j: $\tau_{i,j}=\inf\{t>0 : x_i(t)=x_j(t)\}$. As

$$\int_0^{\tau_{n-1}} R(A_N(s))\,ds \;\le\; \sum_{i\neq j}\int_0^{\tau_{i,j}} \mathbb{1}\{|x_i(s)-x_j(s)|=1\}\,ds\;,$$

it is enough to estimate

$$\mathbf{E}_{\{x_i,x_j\}}^N\Big[\int_0^{\tau_{i,j}} \mathbb{1}\{|x_i(s)-x_j(s)|=1\}\,ds\Big]\;.$$

As the difference evolves as a random walk speeded-up by 2, it is enough to bound, for $x\in\mathbb{T}_N^d$,

$$E_x^N\Big[\int_0^{H_0} \mathbb{1}\{x(s)=e_1\}\,ds\Big]\;,$$

where H_0 stands for the hitting time of the origin. This integral represents the time spent at e_1 before hitting the origin. In particular, it is bounded by a geometric sum of independent exponential random variables, which completes the proof of the lemma. □

Remark 3.3.2. *It follows from last lemma and the strong Markov property at time τ_n that there exists a finite constant $C(n)$ such that*

$$\max_{A\in E_N} \mathbf{E}_A^N\Big[\int_0^\infty R\big(A_N(s)\big)\,\mathbb{1}\{|A_N(s)|\le n\}\Big] \;\le\; C(n)\;.$$

Recall the definition of the sequence a_N introduced in (3.2.3), and that π_N^n represents the uniform measure in $\mathscr{E}_N^n$.

Lemma 3.3.3. *For every $n\geq 2$,*

$$\lim_{N\to\infty}\max_{A\in\mathfrak{G}_N(n,a_N)}\Big|\,\mathbf{E}_A^N\Big[\int_0^{\tau_{n-1}} R(A_N(s))\,ds\Big] \;-\; \sum_{B\in\mathscr{E}_N^n}\pi_N^n(B)\,\mathbf{E}_B^N[\tau_{n-1}]\,R(B)\Big| \;=\; 0\;.$$

Proof. The goal is to replace the initial condition A by the pseudo-invariant measure π_N^n and then to apply Lemma 3.3.5. To carry out this strategy, we remove from the time integral an interval large enough for the process to relax and small enough not to interfere with the overall value of the time integral.

Fix a set A in $\mathfrak{G}_N(n, a_N)$, enumerate its elements, $A = \{x_1, \dots, x_n\}$, and denote by $x_i(t)$ the position at time t of the particle initially at x_i. Let D_1 be the first time two particles are at distance 1 from each other: $D_1 = \inf\{t \geq 0 : \|x_i(t) - x_j(t)\| = 1 \text{ for some } i \neq j\}$. Note that $R(A_N(s)) = 0$ for $s \leq D_1$ and that $D_1 \leq \tau_{n-1}$.

Let γ_N be the sequence introduced in (3.2.5). We claim that

$$\lim_{N\to\infty} \max_{A\in\mathfrak{G}_N(n,a_N)} \mathbf{E}^N_A\Big[\mathbb{1}\{\tau_{n-1} \leq \gamma_N\} \int_0^{\tau_{n-1}} R(A_N(s))\, ds\Big] \;=\; 0\,.$$

Indeed, as $R(A_N(s)) = 0$ for $s < D_1$ and $D_1 \leq \tau_{n-1}$, we may replace the lower limit in the integral by D_1 and include in the indicator the condition $D_1 \leq \gamma_N$ to bound the previous expectation by

$$\mathbf{E}^N_A\Big[\mathbb{1}\{D_1 \leq \gamma_N\} \int_{D_1}^{\tau_{n-1}} R(A_N(s))\, ds\Big]\,. \tag{3.3.1}$$

By the strong Markov property, this expression is bounded by

$$\mathbf{P}^N_A\big[D_1 \leq \gamma_N\big] \max_{B\in\mathscr{E}^n_N} \mathbf{E}^N_B\Big[\int_0^{\tau_{n-1}} R(A_N(s))\, ds\Big]\,.$$

By Lemma 3.3.1 the above expectation is bounded, and by equation (6.18) in [16] the probability vanishes as $N \to \infty$ uniformly in $A \in \mathfrak{G}_N(n, a_N)$. Note that in dimension $d \geq 3$, by equation (6.6) in [16], the result (6.18) holds for any sequence l_N such that $1 \ll l_N \ll N$. This proves the claim.

Denote by $\vartheta_s : D(\mathbb{R}_+, E_N) \to D(\mathbb{R}_+, E_N)$, $s \geq 0$, the time translation operators such that $(\vartheta_s\omega)(t) = \omega(t+s)$ for all $t \geq 0$. It follows from the previous assertion that we may introduce the indicator of the set $\{\gamma_N < \tau_{n-1}\}$ in the expectation appearing in the statement of the lemma. After the inclusion in the expectation of the indicator of the set $\{\gamma_N < \tau_{n-1}\}$, in the upper limit of the integral rewrite τ_{n-1} as $\gamma_N + \tau_{n-1}\circ\vartheta_{\gamma_N}$ and apply the Markov property to get that the expectation is equal to

$$\begin{aligned} &\mathbf{E}^N_A\Big[\mathbb{1}\{\gamma_N < \tau_{n-1}\} \int_0^{\gamma_N} R(A_N(s))\, ds\Big] \\ &\quad + \mathbf{E}^N_A\Big[\mathbb{1}\{\gamma_N < \tau_{n-1}\}\, \mathbf{E}^N_{A_N(\gamma_N)}\Big[\int_0^{\tau_{n-1}} R(A_N(s))\, ds\Big]\Big]\,. \end{aligned} \tag{3.3.2}$$

We claim that the first term vanishes as $N \to \infty$, uniformly in $A \in \mathfrak{G}_N(n, a_N)$. Recall the definition of the hitting time D_1. If $\gamma_N \leq D_1$, the expression inside the expectation vanishes because $R(A_N(s)) = 0$ for $s \leq D_1$. We may therefore assume that $D_1 \leq \gamma_N$. We may also replace the lower limit of the integral by D_1 and the upper limit by τ_{n-1} to find out that the first term in (3.3.2) is bounded by (3.3.1). Since the expectation in (3.3.1) vanishes as $N \to \infty$, uniformly in $A \in \mathfrak{G}_N(n, a_N)$, the claim is proved.

It remains to examine the second expectation in (3.3.2). To apply Lemma 3.2.8, let $F : \mathscr{E}^{\leq n}_N \to \mathbb{R}$ be the function given by

$$F(B) \;=\; \mathbf{E}^N_B\Big[\int_0^{\tau_{n-1}} R(A_N(s))\, ds\Big]\,, \quad B \in \mathscr{E}^n_N\,, \tag{3.3.3}$$

$F(B) = 0$ for $B \notin \mathscr{E}^n_N$. By Lemma 3.3.1, F is uniformly bounded, $\|F\| \leq C(n)$, and therefore fulfills the condition of Lemma 3.2.8. Hence, by this result, the second term in (3.3.2) can be written as

$$\mathbf{E}^N_{\pi^n_N}\Big[\int_0^{\tau_{n-1}} R(A_N(s))\,ds\Big] \;+\; R_N\;,$$

where the absolute value of the remainder R_N is bounded by

$$C(n)\,\Big\{\,\mathbf{P}^N_A[\tau_{n-1}\le\gamma_N] \;+\; 2\,\|p^{(n)}_{\gamma_N}(\mathbf{a},\cdot)-\pi^{\otimes n}_N(\cdot)\|_{\rm TV} \;+\; c_N\Big\}\;.$$

In this formula, $\mathbf{a}=(a_1,\dots,a_n)$, a_j are the elements of A and c_N a constant which vanishes as $N\to\infty$. Since $\gamma_N\gg t^N_{\rm mix}$, the second term inside braces vanishes as $N\to\infty$, uniformly in $A\in\mathscr{E}^n_N$. By Lemma 3.2.7, the first term inside braces vanishes as $N\to\infty$, uniformly in $A\in\mathfrak{G}_N(n,a_N)$. To complete the proof of the lemma, it remains to apply Corollary 3.3.6. □

Lemma 3.3.4. *Let $F:\mathbb{N}\to\mathbb{R}$ be a function which eventually vanishes: there exists $k_0\ge 0$ such that $F(k)=0$ for all $k>k_0$. For all $t>0$, $n>1$,*

$$\lim_{N\to\infty}\max_{A\in\mathfrak{G}_N(n,a_N)}\Big|\,\mathbf{E}^N_A\Big[\int_0^{t\theta_N}\{R(A_N(s))-\theta_N^{-1}\,\mathfrak{n}_s\}\,F(|A_N(s)|)\,ds\Big]\,\Big| \;=\; 0\;.$$

Proof. Fix $n\ge 2$ and A in $\mathfrak{G}_N(n,a_N)$. Since $R(A')$, $\boldsymbol{\lambda}(|A'|)$ vanish for $|A'|=1$, if $k_0\le 1$ there is nothing to prove. Assume, therefore, that $k_0\ge 2$. Since $F(k)=0$ for $k>k_0$, we may start the integral from τ_{n_0}, where $n_0=n\wedge k_0$. If $t\theta_N\le\tau_{n_0}$, the integral vanishes. We may therefore insert inside the expectation the indicator function of the set $\{t\theta_N>\tau_{n_0}\}$, which can be written as the disjoint union of the sets $\{\tau_j<t\theta_N\wedge\tau_1\le\tau_{j-1}\}$, $2\le j\le n_0$. Hence, the time-integral appearing in the statement of the lemma can be written as

$$\begin{aligned}&\sum_{j=2}^{n_0}\mathbb{1}\{\tau_j<t\theta_N\wedge\tau_1\le\tau_{j-1}\}\int_{\tau_{n_0}}^{\tau_{j-1}}\widehat{R}(A_N(s))\,F(|A_N(s)|)\,ds\\ &\quad-\;\sum_{j=2}^{n_0}\mathbb{1}\{\tau_j<t\theta_N\wedge\tau_1\le\tau_{j-1}\}\int_{t\theta_N}^{\tau_{j-1}}\widehat{R}(A_N(s))\,F(|A_N(s)|)\,ds\;,\end{aligned}\tag{3.3.4}$$

where $\widehat{R}(A)=R(A)-\theta_N^{-1}\,\boldsymbol{\lambda}(|A|)$.

We consider each term separately. Write the integral appearing in the first line as a sum of integrals on the intervals $[\tau_i,\tau_{i-1})$ and sum by parts to obtain that the first expression is equal to

$$\sum_{i=2}^{n_0}\mathbb{1}\{\tau_i<t\theta_N\wedge\tau_1\le\tau_1\}\,F(i)\int_{\tau_i}^{\tau_{i-1}}\widehat{R}(A_N(s))\,ds\;,$$

where we used the fact that F is constant in the time interval $[\tau_i,\tau_{i-1})$. Remove from the indicator the condition $\{t\theta_N\wedge\tau_1\le\tau_1\}$, which is always satisfied, and replace $\{\tau_i<t\theta_N\wedge\tau_1\}$ by $\{\tau_i<t\theta_N\}$. Fix $2\le i\le n$, disregard the constant $F(i)$, and consider the expectation with respect to $\mathbf{P}^N_A$:

$$\mathbf{E}^N_A\Big[\mathbb{1}\{\tau_i<t\theta_N\}\int_{\tau_i}^{\tau_{i-1}}\widehat{R}(A_N(s))\,ds\Big]\;.\tag{3.3.5}$$

We claim that

$$\lim_{N\to\infty}\mathbf{E}^N_A\Big[\mathbb{1}\{A_N(\tau_i)\not\in\mathfrak{G}_N(i,a_N)\}\int_{\tau_i}^{\tau_{i-1}}\{R(A_N(s))-\theta_N^{-1}\mathfrak{n}_s\}\,ds\Big]\;=\;0\;.$$

Indeed, by the strong Markov property, the absolute value of the previous expectation is less than or equal to

$$\mathbf{P}^N_A\big[A_N(\tau_i)\not\in \mathfrak{G}_N(i,a_N)\big]\ \max_{B\in\mathscr{E}^i_N}\Big\{\mathbf{E}^N_B\Big[\int_0^{\tau_{i-1}} R(A_N(s))\,ds\Big]\ +\ \boldsymbol{\lambda}(i)\,\theta_N^{-1}\mathbf{E}^N_B\big[\tau_{i-1}\big]\Big\}\ .$$

By Lemmata 3.2.10 and 3.3.1, the maximum is bounded. On the other hand, since A belongs to $\mathfrak{G}_N(n,a_N)$, by Proposition 3.2.5, the probability vanishes as $N\to\infty$, which proves the claim.

We may therefore insert in (3.3.5) the indicator of the set $\{A_N(\tau_i)\in\mathfrak{G}_N(i,a_N)\}$. By the strong Markov property, this expectation is equal to

$$\mathbf{E}^N_A\Big[\mathbf{1}\{\tau_i<t\theta_N\,,\,A_N(\tau_i)\in\mathfrak{G}_N(i,a_N)\}\,\mathbf{E}^N_{A_N(\tau_i)}\Big[\int_0^{\tau_{i-1}}\widehat{R}(A_N(s))\,ds\Big]\Big]\ .$$

By Lemmata 3.3.3 and 3.2.15,

$$\lim_{N\to\infty}\mathbf{E}^N_B\Big[\int_0^{\tau_{i-1}} R(A_N(s))\,ds\Big]\ =\ 1$$

uniformly for $B\in\mathfrak{G}_N(i,a_N)$. By Corollary 3.2.11, as $N\to\infty$, $\boldsymbol{\lambda}(i)\,\mathbf{E}^N_B[\tau_{i-1}/\theta_N]$ converges to 1 uniformly for $B\in\mathfrak{G}_N(i,a_N)$.

It remains to examine the second expression in (3.3.4). The argument is similar to the one presented above. Fix $2\le j\le n_0$ and take the expectation with respect to $\mathbf{P}^N_A$ for $A\in\mathfrak{G}_N(n,a_N)$. Since $\tau_1\ge\tau_j$, we may remove τ_1 from the indicator. For $j=2$ the set becomes $\{\tau_2<t\theta_N\}$, while for $2<j\le n_0$ it is given by $\{\tau_j<t\theta_N\le\tau_{j-1}\}$. In the first case, to uniform the notation, we insert the condition $t\theta_N\le\tau_1$. This is possible because the integral vanishes if this bound is not fulfilled.

We claim that

$$\lim_{N\to\infty}\mathbf{E}^N_A\Big[\mathbf{1}\{\mathscr{G}_N\}\int_{t\theta_N}^{\tau_{j-1}}\big\{R(A_N(s))-\theta_N^{-1}\mathfrak{n}_s\big\}\,ds\Big]\ =\ 0\,,$$

where $\mathscr{G}_N$ is the set $\{\tau_j<t\theta_N\le\tau_{j-1}\,,\,A_N(t\theta_N)\not\in\mathfrak{G}_N(j,a_N)\}$. The proof of this claim is identical to the one produced below (3.3.5). Observe that on the set $\{\tau_{j-1}\ge t\theta_N\}$ we may write τ_{j-1} as $t\theta_N+\tau_{j-1}\circ\vartheta_{t\theta_N}$. Apply the Markov property at time $t\theta_N$, estimate the conditional expectation by the supremum over all sets in $\mathscr{E}^j_N$, and apply Lemmata 3.2.10 and 3.3.1, and Lemma 3.2.3 (instead of Proposition 3.2.5).

After inserting in the expectation the indicator of the set $\{A_N(t\theta_N)\in\mathfrak{G}_N(j,a_N)\}$, applying the Markov property at time $t\theta_N$, the expectation becomes

$$\mathbf{E}^N_A\Big[\mathbf{1}\{\mathscr{M}_N\}\,\mathbf{E}^N_{A_N(t\theta_N)}\Big[\int_0^{\tau_{j-1}}\widehat{R}(A_N(s))\,ds\Big]\Big]\ ,$$

where $\mathscr{M}_N=\{\tau_j<t\theta_N\le\tau_{j-1}\,,\,A_N(t\theta_N)\in\mathfrak{G}_N(j,a_N)\}$. By the first part of the proof, this expression vanishes as $N\to\infty$. □

Proof of Proposition 3.1.4. Fix $\varepsilon>0$. In view of Proposition 3.2.1, choose $M\in\mathbb{N}$ such that $\mathbf{P}^N[\,|A_N(t_0\theta_N)|>M]\le\varepsilon$. Let $W(A)=\{\theta_N\,R(A)-\boldsymbol{\lambda}(|A|)\}\,F(|A|)$. There exists a finite constant $C(F,B,t)$ such that

$$\Big|\,\mathbf{E}^N\Big[B^N\mathbf{1}\{|A_N(t_0\theta_N)|>M\}\int_{t_0}^t W\big(A_N(s\theta_N)\big)\,ds\Big]\,\Big|\ \le\ C(F,B,t)\,\varepsilon\,. \qquad (3.3.6)$$

To prove this assertion, apply the Markov property to write the expectation appearing in the left-hand side as

$$\mathbf{E}^N\Big[B^N\mathbf{1}\{|A_N(t_0\theta_N)|>M\}\,\mathbf{E}^N_{A(t_0\theta_N)}\Big[\int_0^{t-t_0}W\big(A_N(s\theta_N)\big)\,ds\Big]\Big]\,.$$

We claim that the absolute value of the expectation with respect to $\mathbf{P}^N_{A(t_0\theta_N)}$ is bounded by a constant depending on F and t. On the one hand, the function $\lambda(|A|)\,F(|A|)$ is bounded because $F(k)=0$ for all k large enough. On the other hand, since F vanishes outside a finite subset of $\mathbb{N}$, by Remark 3.3.2, the expectation of the time integral of $\theta_N\,R(A_N(s\theta_N))\,F(A_N(s\theta_N))$ is bounded. This proves the claim.

It follows from this claim that the absolute value of the expectation appearing in the last displayed equation is bounded by

$$C(F,B,t)\,\mathbf{P}^N\big[\,|A_N(t_0\theta_N)|>M\,\big]\;,$$

Assertion (3.3.6) follows from the choice of M.

A similar argument, using Corollary 3.2.4 instead of Proposition 3.2.1, proves that for all N sufficiently large

$$\Big|\,\mathbf{E}^N\Big[B^N\mathbf{1}\Big\{A_N(t_0\theta_N)\notin\mathscr{E}^1_N\cup\bigcup_{k=2}^{N^d}\mathfrak{G}_N(k,a_N)\Big\}\times\int_{t_0}^{t}W\big(A_N(s\theta_N)\big)\,ds\Big]\,\Big|\;\le\;C(F,B,t)\,\varepsilon\,.$$

It follows from the previous two estimates that we may restrict our attention to the expectation

$$\mathbf{E}^N\Big[B^N\mathbf{1}\{\mathscr{M}_N(M,t_0)\}\int_{t_0}^{t}W\big(A_N(s\theta_N)\big)\,ds\Big]\,,$$

where $\mathscr{M}_N(M,t_0)=\big\{|A_N(t_0\theta_N)|\le M\,,\,A_N(t_0\theta_N)\in\mathscr{E}^1_N\cup\bigcup_{k=2}^{M}\mathfrak{G}_N(k,a_N)\big\}$. Applying the Markov property at time $t_0\theta_N$ yields that the absolute value of the previous expectation is bounded by

$$C(B)\max_{A\in\mathscr{E}^1_N\cup\bigcup_{k=2}^{M}\mathfrak{G}_N(k,a_N)}\Big|\mathbf{E}^N_A\Big[\int_0^{t-t_0}W\big(A_N(s\theta_N)\big)\,ds\Big]\Big|\,,$$

where the constant $C(B)$ is an upper bound for $(|B^N|:N\in\mathbb{N})$. This expression vanishes as $N\to\infty$ by Lemma 3.3.4, which completes the proof of the proposition. □

3.3.1 Equilibrium Expectation of Hitting Times

We conclude this section with a result on the equilibrium expectation of hitting times. Let X_t be a reversible, irreducible, continuous-time Markov chain on a finite set E. Denote by π the unique stationary state and by H_B, $B\subset E$, the hitting time of the set B: $H_B=\inf\{t\ge 0: X_t\in B\}$. Denote by $\mathbb{P}_x$ the distribution of the Markov chain X_t starting from x. Expectation with respect to $\mathbb{P}_x$ is represented by $\mathbb{E}_x$. As usual, for a probability measure μ on E, $\mathbb{P}_\mu=\sum_{x\in E}\mu(x)\,\mathbb{P}_x$.

Lemma 3.3.5. *For all subsets B of E, and all functions $f:E\to\mathbb{R}$,*

$$\mathbb{E}_\pi\Big[\int_0^{H_B}f(X_s)\,ds\Big]\;=\;\sum_{x\in E}\pi(x)\,f(x)\,\mathbb{E}_x[H_B]\,.$$

Proof. Denote by $(Y_k)_{k\ge 0}$ the skeleton of the chain X_t. This is the discrete-time Markov chain which keeps track of the sequence of elements of E visited by the process. Denote by $\lambda(x)$, $x \in E$, the holding time at x. Representing the process X_t in terms of the chain Y_k and independent, mean-one, exponential random variables (cf. Section 6 of [3]), the expectation appearing in the statement of the lemma can be written as

$$\mathbb{E}_\pi\Big[\sum_{k=0}^{h_B-1}\frac{f(Y_k)}{\lambda(Y_k)}\Big] \;=\; \sum_{k\ge 0}\sum_{x\notin B}\sum_{y\notin B}\pi(x)\,\frac{f(y)}{\lambda(y)}\,\mathbb{P}_x\big[Y_k=y\,,\,h_B>k\big]\;,$$

where h_B stands for the hitting time of the set B by the Markov chain Y_k: $h_B=\min\{j\ge 0 : Y_j\in B\}$. By reversibility, the previous expression is equal to

$$\sum_{k\ge 0}\sum_{x\notin B}\sum_{y\notin B}\pi(y)\,\frac{f(y)}{\lambda(x)}\,\mathbb{P}_y\big[Y_k=x\,,\,h_B>k\big] \;=\; \sum_{y\in E}\pi(y)\,f(y)\,\mathbb{E}_y\Big[\sum_{k=0}^{h_B-1}\frac{1}{\lambda(Y_k)}\Big]\,.$$

The last expectation is equal to $\mathbb{E}_y[H_B]$, which completes the proof of the lemma. □

Corollary 3.3.6. *For every $n\ge 2$,*

$$\lim_{N\to\infty}\Big|\,\mathbf{E}^N_{\pi^n_N}\Big[\int_0^{\tau_{n-1}}R(A_N(s))\,ds\Big]\;-\;\sum_{B\in\mathscr{E}^n_N}\pi^n_N(B)\,\mathbf{E}^N_B[\tau_{n-1}]\,R(B)\,\Big|\;=\;0\,.$$

Proof. Let $F:\mathscr{E}^{\le n}_N\to\mathbb{R}$ be the function given by (3.3.3), and recall that it is uniformly bounded. The expectation appearing in the statement of the lemma is equal to $E_{\pi^n_N}[F]$. By (3.2.7) and since F vanishes on $\mathscr{E}^m_N$, $m<n$, and is uniformly bounded, this expectation is equal to $E_{\pi^{\otimes n}_N}[F(\{\mathbf{x}\})]+c_N$, where $\lim_N c_N=0$.

By definition of F,

$$E_{\pi^{\otimes n}_N}[F(\{\mathbf{x}\})] \;=\; \sum_{\mathbf{x}\in[\mathbb{T}^d_N]^n}\pi^{\otimes n}_N(\mathbf{x})\,\mathbf{1}\{\{\mathbf{x}\}\in\mathscr{E}^n_N\}\,\mathbf{E}^N_{\{\mathbf{x}\}}\Big[\int_0^{\tau_{n-1}}R(A_N(s))\,ds\Big]\,.$$

Up to time τ_{n-1} the evolution of $A_N(s)$ corresponds to the evolution of n independent particles. We may thus replace $A_N(s)$ by $\{\mathbf{x}^n_N(s)\}$ inside the expectation, where τ_{n-1} represents in this context the first time two particles meet. The previous sum is thus equal to

$$\sum_{\mathbf{x}\in[\mathbb{T}^d_N]^n}\pi^{\otimes n}_N(\mathbf{x})\,\mathbf{1}\{\{\mathbf{x}\}\in\mathscr{E}^n_N\}\,\widetilde{\mathbf{E}}^N_{\mathbf{x}}\Big[\int_0^{\tau_{n-1}}R(\{\mathbf{x}^n_N(s)\})\,ds\Big]\,,$$

where $\widetilde{\mathbf{P}}^N_{\mathbf{x}}$ represents the distribution of $\mathbf{x}^n_N$ starting from $\mathbf{x}$.

Since $\tau_{n-1}=0$ if the process $\mathbf{x}^n_N(s)$ starts from a configuration $\mathbf{x}$ such that $\{\mathbf{x}\}\notin\mathscr{E}^n_N$, we may remove the indicator in the previous sum. As the process is reversible and $\pi^{\otimes n}_N$ is its unique stationary state, by Lemma 3.3.5, the sum is equal to

$$\sum_{\mathbf{x}\in[\mathbb{T}^d_N]^n}\pi^{\otimes n}_N(\mathbf{x})\,\widetilde{\mathbf{E}}^N_{\mathbf{x}}[\tau_{n-1}]\,R(\{\mathbf{x}\})\,.$$

As $\tau_{n-1}=0$ if the process $\mathbf{x}^n_N(s)$ starts from a configuration $\mathbf{x}$ such that $\{\mathbf{x}\}\notin\mathscr{E}^n_N$, we may restrict the sum to configurations $\mathbf{x}$ such that $\{\mathbf{x}\}\in\mathscr{E}^n_N$. For such a configuration,

$\widetilde{\mathbf{E}}^N_{\mathbf{x}}[\tau_{n-1}] = \mathbf{E}^N_{\{\mathbf{x}\}}[\tau_{n-1}]$. Hence, the last sum is equal to

$$\sum_{\mathbf{x}\in[\mathbb{T}^d_N]^n} \pi_N^{\otimes n}(\mathbf{x})\,\mathbf{E}^N_{\{\mathbf{x}\}}[\tau_{n-1}]\,R(\{\mathbf{x}\}) \;=\; \sum_{A\in\mathscr{E}^n_N} \mathbf{E}^N_A[\tau_{n-1}]\,R(A) \sum_{\{\mathbf{x}\}=A} \pi_N^{\otimes n}(\mathbf{x})\,,$$

where the last sum is performed over all configuration $\mathbf{x} \in [\mathbb{T}^d_N]^n$ such that $\{\mathbf{x}\} = A$. Comparing $\sum_{\{\mathbf{x}\}=A} \pi_N^{\otimes n}(\mathbf{x})$ with $\pi^n_N(A)$ yields that the previous sum is equal to

$$(1 + O(N^{-d}))\sum_{A\in\mathscr{E}^n_N} \mathbf{E}^N_A[\tau_{n-1}]\,R(A)\,\pi^n_N(A)\,,$$

where $O(N^{-d})$ is a sequence of numbers whose absolute value is bounded by $C_0 N^{-d}$ for some finite constant C_0. By Lemma 3.2.15, the sum converges to 1. In particular, the term $O(N^{-d})$ times the sum is negligible. This completes the proof of the corollary. □

3.4 Proof of Theorem 3.1.2

The proof of Theorem 3.1.2 is divided in two steps. We show in Lemma 3.4.3 that the sequence $(\mathscr{P}^N)_N$ is tight, and in Lemma 3.4.1 that all limit points solve the $(C^1(S), \mathscr{L})$-martingale problem introduced in Proposition 3.1.1.

Denote by $\mathbb{P}^N_A$, $A \in E_N$, the probability measure on $D(\mathbb{R}_+, E_N)$ induced by the Markov chain $A_N(t)$ *speeded-up by* θ_N starting from A. When $A = \mathbb{T}^d_N$, we denote $\mathbb{P}^N_A$ simply by $\mathbb{P}^N$. Expectation with respect to $\mathbb{P}^N_A$, $\mathbb{P}^N$ are represented by $\mathbb{E}^N_A$ and $\mathbb{E}^N$, respectively. Note that

$$\mathscr{P}^N \;=\; \mathbb{P}^N \circ \widehat{\Psi}_N^{-1}\,, \tag{3.4.1}$$

where $\widehat{\Psi}_N : D(\mathbb{R}_+, E_N) \to D(\mathbb{R}_+, S)$ is given by $[\widehat{\Psi}_N(\omega)](t) = \Psi_N(\omega(t))$.

In the next lemmata, expectation with respect to $\mathscr{P}^N$, $\mathscr{P}$ are represented by $E_{\mathscr{P}^N}$, $E_{\mathscr{P}}$, respectively.

Lemma 3.4.1. *Let $\mathscr{P}$ be a limit point of the sequence $(\mathscr{P}^N)_N$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function in C^1 which is constant in a neighborhood of the origin: there exists $\delta > 0$ such that $f(x) = f(0)$ for $x \le \delta$. Then, under $\mathscr{P}$, the process defined by (3.1.4) is a martingale.*

Proof. Assume without loss of generality that $(\mathscr{P}^N)_N$ converges to $\mathscr{P}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in C^1 which is constant in a neighborhood of the origin. Denote by $M_N(t)$ the $\mathbb{P}^N$-martingale given by

$$f(\mathbb{X}_N(t)) \,-\, f(\mathbb{X}_N(0)) \,-\, \int_0^t \theta_N\,(L_N f)(\Psi_N(A_N(s\theta_N)))\,ds\,,$$

where $\mathbb{X}_N(t) = \Psi_N(A_N(t\theta_N))$. Since

$$(L_N f)(\Psi_N(A)) \;=\; R(A)\Big\{f\Big(\frac{x}{1-x}\Big) - f(x)\Big\}\,,$$

where $x = \Psi_N(A)$, and $R(A)$ is the jump rate introduced in (3.1.8), the martingale $M_N(t)$ can be written as

$$f(\mathbb{X}_N(t)) \,-\, f(\mathbb{X}_N(0)) \,-\, \theta_N \int_0^t R(A_N(s\theta_N))\Big\{f\Big(\frac{\mathbb{X}_N(s)}{1-\mathbb{X}_N(s)}\Big) - f(\mathbb{X}_N(s))\Big\}\,ds\,.$$

Fix $0 \le t_0$, $k \ge 1$, $0 \le s_1 < \cdots < s_k \le t_0$, a bounded function $G : \mathbb{R}^k \to \mathbb{R}$, and let

$B^N = G(\mathbb{X}_N(s_1), \ldots, \mathbb{X}_N(s_k))$. Since M_N is a martingale, for every $t_0 \leq t$,

$$\mathbb{E}^N\Big[B^N\left\{M_N(t) - M_N(t_0)\right\}\Big] \;=\; 0\,.$$

By Proposition 3.1.4, in the integral part of the martingale we may replace the rate $\theta_N R(A_N(s\theta_N))$ by $\boldsymbol{\lambda}(|A_N(s\theta_N)|) = \boldsymbol{r}(\mathbb{X}_N(s))$ to obtain that

$$\lim_{N\to\infty} \mathbb{E}^N\Big[B^N\left\{\widehat{M}_N(t) - \widehat{M}_N(t_0)\right\}\Big] \;=\; 0\,, \tag{3.4.2}$$

where

$$\widehat{M}_N(t) \;=\; f(\mathbb{X}_N(t)) \;-\; f(\mathbb{X}_N(0)) \;-\; \int_0^t \boldsymbol{r}(\mathbb{X}_N(s))\left\{f\Big(\frac{\mathbb{X}_N(s)}{1-\mathbb{X}_N(s)}\Big) - f(\mathbb{X}_N(s))\right\} ds\,.$$

Notice that the process $\widehat{M}_N(t)$ is expressed as a function of $\mathbb{X}_N$. Therefore, in view of (3.4.1), we may replace in (3.4.2) the probability $\mathbb{P}^N$ by $\mathscr{P}^N$ and write

$$\lim_{N\to\infty} E_{\mathscr{P}^N}\Big[B^N\left\{\widehat{M}_N(t) - \widehat{M}_N(t_0)\right\}\Big] \;=\; 0\,,$$

Since, by assumption, $(\mathscr{P}_N)_N$ converges to $\mathscr{P}$,

$$E_{\mathscr{P}}\Big[B^N\left\{\widehat{M}_N(t) - \widehat{M}_N(t_0)\right\}\Big] \;=\; 0\,.$$

This shows that (3.1.4) is a martingale under $\mathscr{P}$ and completes the proof of the lemma. □

We turn to the tightness of $(\mathscr{P}^N)_N$. Remember that for $w \in D(\mathbb{R}_+, S)$, the *modified modulus of continuity* is defined as

$$\tilde{\omega}(w,t,\delta) \;:=\; \inf_{\Delta}\max_{k}\sup_{t_k\leq r,s<t_{k+1}} \|w(s)-w(r)\|\,,\quad t>0\,,\quad \delta>0\,,$$

where the infimum extends over all partitions $\Delta = \{0 = t_0 < t_1 < \cdots < t_\ell < t\}$ such that $t_{k+1} - t_k \geq \delta$ for $k = 1, \ldots, \ell - 1$. It is well known (see for instance [19, Theorem 4.8.1]) that the tightness follows from

1. for any $t \in \mathbb{R}_+$, the sequence $\big(\mathbb{X}_N(t)\big)_N$ is tight in S; and
2. for all $\varepsilon > 0$, $t > 0$,
$$\lim_{\delta\to 0}\sup_N \mathscr{P}^N[\tilde{\omega}(\mathbb{X}_N,t,\delta) > \varepsilon] \;=\; 0\,. \tag{3.4.3}$$

Since $\mathbb{X}_N(t) \in S$ for all $t \in \mathbb{R}_+$ and S is compact, condition (1) holds immediately thanks to Prohorov's criterion. Denote by σ_j, $j \geq 1$, the hitting time of $1/j$: $\sigma_j = \inf\{t \geq 0 : \mathbb{X}(t) = 1/j\}$.

Lemma 3.4.2. *Condition* (2) *follows from*

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathscr{P}^N\big[\sigma_{j-1} - \sigma_j \leq \delta\big] \;=\; 0\,,\quad \forall\, j \geq 2\,. \tag{3.4.4}$$

Proof. Assume that (3.4.4) holds, fix $\varepsilon > 0$, $t > 0$, $\eta > 0$ and choose $n \in \mathbb{N}$ such that $1/n \leq \varepsilon$. By Proposition 3.2.1 and by the Markov inequality

$$\mathscr{P}^N[\mathbb{X}_N(t) \leq 1/n] = \;\mathbf{P}^N[\,|A_N(t\theta_N)| \geq n\,] \;\leq\; \frac{\mathbf{E}^N[\,|A_N(t\theta_N)|\,]}{n} \;\leq\; \frac{C(t,d)}{n}\,,$$

where $C(t,d)$ is a positive constant depending only on t and d. Then, increasing n if necessary, we can assume that

$$\mathscr{P}^N[\sigma_n < t] \; > \; 1 - \eta/3 \, .$$

Our assumption implies that there are $\delta_0 > 0$ and $M \in \mathbb{N}$ such that

$$\mathscr{P}^N[\sigma_{j-1} - \sigma_j \geq \delta_0, \text{ for all } j \in \{2,3,\dots,n\}] \; > \; 1 - \eta/3, \quad \forall N > M.$$

Let $m := \min\{j \geq 1 : \sigma_j < t\}$. On the set $\{\sigma_n < t\}$, define the random partition $\Delta := \{0 = t_0 < t_1 = \sigma_n < \cdots < t_\ell = \sigma_m < t\}$. Since $\mathbb{X}_N(r)$ is constant in the intervals $[\sigma_j, \sigma_{j-1})$, using this partition we deduce that

$$\tilde{\omega}(\mathbb{X}_N, t, \delta) \leq 1/n \leq \varepsilon \, , \quad \forall \, \delta < \delta_0 \, , \quad N > M \, ,$$

on the event

$$\{\sigma_n < t\} \cap \{\sigma_{j-1} - \sigma_j \geq \delta_0, \text{ for all } j \in \{2,3,\dots,n\}\} \, ,$$

that has probability at least $1 - 2\eta/3$. Hence

$$\sup_{N>M} \mathscr{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] < 2\eta/3 \, , \quad \forall \delta < \delta_0 \, .$$

On the other hand, it is clear that there is $\delta_1 > 0$ such that

$$\mathscr{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] \; < \; \eta/3 \, , \quad N \leq M \, , \quad \forall \delta < \delta_1 .$$

Therefore

$$\sup_N \mathscr{P}^N[\tilde{\omega}(\mathbb{X}_N, t, \delta) > \varepsilon] < \eta \, , \quad \forall \, \delta < \min\{\delta_0, \delta_1\} \, ,$$

which completes the proof, since $\eta > 0$ was arbitrary. □

We complete the proof of the tightness in the next lemma.

Lemma 3.4.3. *The sequence of measures* $(\mathscr{P}^N)_N$ *is tight.*

Proof. By Lemma 3.4.2 it is enough to show (3.4.4). In terms of the measure $\mathbf{P}^N$, the probability appearing in (3.4.4) can be rewritten as

$$\mathbf{P}^N\big[\tau_{j-1} - \tau_j \leq \delta \, \theta_N\big] \, .$$

Fix $\epsilon > 0$ and $M > j$. In view of (3.2.4), choose $\alpha > 0$ small enough for $\mathbf{P}^N[\tau_M \leq 3\alpha\theta_N] \leq \epsilon$ for all N sufficiently large. By Proposition 3.2.1, choose $K \geq M$ such that $\mathbf{P}^N[\,|A_N(\alpha\theta_N)| \geq K] \leq \epsilon$ for all N sufficiently large. Hence, the probability appearing in (3.4.4) is less than or equal to

$$\mathbf{P}^N\Big[\,|A_N(\alpha\theta_N)| \leq K \, , \, \tau_M \geq 3\alpha\theta_N \, , \, \tau_{j-1} - \tau_j \leq \delta\theta_N\Big] \; + \; 2\epsilon \, .$$

By Lemma 3.2.3, this expression is less than or equal to

$$\mathbf{P}^N\Big[\,|A_N(\alpha\theta_N)| \leq K \, , \, A_N(2\alpha\theta_N) \in \mathfrak{G}_N \, , \, \tau_M \geq 3\alpha\theta_N \, , \, \tau_{j-1} - \tau_j \leq \delta\theta_N\Big] \; + \; 3\epsilon \, .$$

By the Markov property, this sum is bounded by

$$\max_{M \leq n \leq K} \, \max_{A \in \mathfrak{G}_N(n, a_N)} \mathbf{P}^N_A\big[\,\tau_{j-1} - \tau_j \leq \delta\theta_N\big] \; + \; 3\epsilon \, .$$

By Propositions 3.2.5, 3.2.6 and the strong Markov property at time τ_j, the first term of the previous expression vanishes as $N \uparrow \infty$ and $\delta \to 0$. □

3.5 Uniqueness

In order to state the uniqueness result as it has been used in Section 3.4 we need to introduce the subset $\mathscr{D}_0 \subseteq C^1(S)$ of functions $f : S \to \mathbb{R}$ which are constant on a neighborhood of zero: $f \in \mathscr{D}_0$ if and only if for some $k(f) \in \mathbb{N}$ we have

$$f(0) = f(1/k)\,, \quad \forall k > k(f)\,.$$

We shall say that a probability measure $\mathscr{P}$ on the measurable space $(D(\mathbb{R}_+, S), \mathcal{G}_\infty)$ is a solution of the $(\mathscr{D}_0, \mathscr{L})$ (resp. $(C^1(S), \mathscr{L})$)-martingale problem if

$$M_t^f \;:=\; f(X_t) - \int_0^t (\mathscr{L} f)(X_s)\, ds\,, \quad t \geq 0 \tag{3.5.1}$$

is a $\mathscr{P}$-martingale for every $f \in \mathscr{D}_0$ (resp. $f \in C^1(S)$). In addition, we say that $\mathscr{P}$ is starting at $x \in S$ whenever $\mathscr{P}\{X_0 = x\} = 1$.

3.5.1 Uniqueness on $S \setminus \{0\}$

For each $k \in \mathbb{N}$, let $\mathscr{P}_{1/k}$ be the law on $(D(\mathbb{R}_+, S), \mathcal{G}_\infty)$ of a Markov process on S starting at $1/k$ and with transition rates

$$q\Big(\frac{1}{n}, \frac{1}{n-1}\Big) \;=\; \binom{n}{2}\,, \quad \text{for } 2 \leq n \leq k$$

and zero elsewhere. By Dinkyn's martingales, the process

$$f(X_t) - \int_0^t (\mathscr{L}^k f)(X_s)\, ds\,, \quad t \geq 0 \tag{3.5.2}$$

is a $\mathscr{P}_{1/k}$-martingale, for all $f : S \to \mathbb{R}$, where

$$\mathscr{L}^k f(x) \;:=\; \begin{cases} \binom{n}{2} \Big\{ f\left(\frac{1}{n-1}\right) - f\left(\frac{1}{n}\right) \Big\}\,, & \text{if } x = \frac{1}{n} \in [\frac{1}{k}, \frac{1}{2}]\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

In particular, $\mathscr{P}_{1/k}$ is a solution of the $(\mathscr{D}_0, \mathscr{L}^k)$-martingale problem. Moreover, uniqueness for this problem can be obtained by standard methods so that

Remark 3.5.1. *For each $k \in \mathbb{N}$, $\mathscr{P}_{1/k}$ is the unique solution of the $(\mathscr{D}_0, \mathscr{L}^k)$-martingale problem starting at $1/k$.*

Since $\mathscr{P}_{1/k}\{X_t \geq 1/k\,,\ \forall\, t \geq 0\} = 1$ and

$$\mathscr{L}^k f(x) \;=\; \mathscr{L} f(x)\,, \quad \text{for all } x \geq 1/k \tag{3.5.3}$$

we may then replace $\mathscr{L}^k$ by $\mathscr{L}$ in (3.5.2). Therefore,

Remark 3.5.2. *For each $x \in S \setminus \{0\}$, $\mathscr{P}_x$ is a solution of the $(C^1(S), \mathscr{L})$, and so, also the $(\mathscr{D}_0, \mathscr{L})$-martingale problem.*

We now prove that, for all $x \in S \setminus \{0\}$, $\mathscr{P}_x$ is actually the unique solution for both martingale problems when starting at x. Of course, it is enough to prove this assertion for $(\mathscr{D}_0, \mathscr{L})$. In virtue of Remark 3.5.1, it suffices to prove that under any such solution $X_t \geq 1/k$, $\forall\, t \geq 0$ almost surely.

Lemma 3.5.3. *For each $x \in S \setminus \{0\}$, $\mathscr{P}_x$ is the unique solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem starting at $x \in S$.*

Proof. Fix some $x = 1/k$ and let $\mathscr{P}$ be a probability satisfying the assumption. Consider the $(\mathcal{G}_t)$-stopping time

$$\tau \ := \ \min\{t \geq 0 : X_t < 1/k\} \, .$$

Since

$$\int_0^{t\wedge\tau} \mathscr{L} f(X_s) ds \ = \ \int_0^t \mathscr{L}^k f(X_{s\wedge\tau}) ds \, , \quad \forall t \geq 0 \, ,$$

then

$$f(X_{t\wedge\tau}) - \int_0^t \mathscr{L}^k f(X_{s\wedge\tau}) ds \, , \quad t \geq 0$$

is a $\mathscr{P}$-martingale, for any $f \in \mathscr{D}_0$. Equivalently, if $X^\tau : D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)$ denotes the measurable map defined by

$$X_t \circ X^\tau \ = \ X_{t\wedge\tau} \, , \quad \forall t \geq 0$$

then the law of X^τ under $\mathscr{P}$, denoted by $\mathscr{P} \circ (X^\tau)^{-1}$, turns out to be a solution of the $(\mathscr{D}_0, \mathscr{L}^k)$-martingale problem. By Remark 3.5.1 we conclude that

$$\mathscr{P} \circ (X^\tau)^{-1} \ = \ \mathscr{P}_{1/k} \, , \tag{3.5.4}$$

which in turn implies that

$$\mathscr{P}\big(X_{t\wedge\tau} \geq 1/k \, , \ \forall t \geq 0\big) \ = \ \mathscr{P}_{1/k}\big(X_t \geq 1/k \, , \ \forall t \geq 0\big)$$

Since the right hand side above equals one, then $\mathscr{P}(\tau = \infty) = 1$ and so

$$\mathscr{P} \circ (X^\tau)^{-1} \ = \ \mathscr{P} \, . \tag{3.5.5}$$

The desired result follows from (3.5.4) and (3.5.5). □

3.5.2 A Strong Markov Property

As our next step, we prove Lemma 3.5.4 below which relates any solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem with laws $\{\mathscr{P}_x\}_{x \in S\setminus\{0\}}$ we just introduced.

Let $\vartheta : \mathbb{R}_+ \times D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)$ be the measurable map defined by

$$X_t \circ \vartheta(s, \cdot) \ = \ X_{s+t}(\cdot) \, , \quad \text{for all } t, s \geq 0 \, .$$

In addition, given any $(\mathcal{G}_t)$-stopping time τ we define $\vartheta_\tau : D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)$ as

$$\vartheta_\tau(\omega) \ := \ \begin{cases} \vartheta(\tau(\omega), \omega) \, , & \text{if } \tau(\omega) < \infty \, , \\ \omega \, , & \text{otherwise} \, . \end{cases}$$

Consider the system of neighborhoods of $0 \in S$

$$A_k := \{x \in S : x < 1/k\} \, , \quad k \in \mathbb{N} \, ,$$

and their corresponding exit times

$$\sigma_k \;:=\; \inf\{t \geq 0 : X_t \in S \setminus A_k\}\,, \quad k \in \mathbb{N}\,.$$

Since A_k and $S \setminus A_k$ are closed subsets then every σ_k is a stopping time and

$$X_{\sigma_k} \;\geq\; 1/k \quad \text{on} \quad \{\sigma_k < \infty\}\,. \tag{3.5.6}$$

Lemma 3.5.4. *Let $\mathscr{P}$ be any solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem and let $k \in \mathbb{N}$. For any $\mathcal{C} \in \mathcal{G}_\infty$, we have*

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{C}\,,\, \sigma_k < \infty\} \;=\; \int_{\{\sigma_k<\infty\}} \mathscr{P}_{X_{\sigma_k}(\omega)}(\mathcal{C})\, \mathscr{P}(d\omega)\,. \tag{3.5.7}$$

(Recall observation (3.5.6).)

Proof. Fix $k \in \mathbb{N}$ and let $\{\mathscr{Q}_\omega : \omega \in D(\mathbb{R}_+, S)\}$ be a conditional probability distribution of $\mathscr{P}$ given $\mathcal{G}_{\sigma_k}$ such that for all $\omega \in D(\mathbb{R}_+, S)$ we have

$$\mathscr{Q}_\omega(\mathcal{A}) \;=\; \delta_\omega(\mathcal{A})\,, \quad \forall \mathcal{A} \in \mathcal{G}_{\sigma_k}\,. \tag{3.5.8}$$

The existence of such $\{\mathscr{Q}_\omega\}$ is established in [30, Theorem 1.3.4]for a space of continuous paths but the same proof apply for $D(\mathbb{R}_+, S)$. Taking conditional expectation with respect to $\mathcal{G}_{\sigma_k}$ in the left hand side below we have

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{C}\,,\, \sigma_k < \infty\} \;=\; \int_{\{\sigma_k<\infty\}} \mathscr{Q}_\omega\{\vartheta_{\sigma_k} \in \mathcal{C}\}\, \mathscr{P}(d\omega)\,.$$

Applying (3.5.8) we get $\mathscr{Q}_\omega\{\sigma_k = \sigma_k(\omega)\} = 1$ for all ω and so the right hand side above equals

$$\int_{\{\sigma_k<\infty\}} \mathscr{Q}_\omega\{\vartheta_{\sigma_k(\omega)} \in \mathcal{C}\}\, \mathscr{P}(d\omega)\,. \tag{3.5.9}$$

Now, we relate $\{\mathscr{Q}_\omega\}$ to $\{\mathscr{P}_x\}_{x \in S\setminus\{0\}}$. For each $f \in \mathscr{D}_0$, we know that the process (M^f_t) defined in (3.5.1) is a $\mathscr{P}$-martingale. Then, in virtue of [30, Theorem 1.2.10], for each $f \in \mathscr{D}_0$ there exists some $\mathcal{A}_f \in \mathcal{G}_{\sigma_k}$ with $\mathscr{P}[\mathcal{A}_f] = 1$ such that, for all $\omega \in \mathcal{A}_f \cap \{\sigma_k < \infty\}$,

$$(M^f_t) \text{ is a } \mathscr{Q}_\omega\text{-martingale after time } \sigma_k(\omega), \tag{3.5.10}$$

i.e. $\mathscr{Q}_\omega[M^f_{t_2} | \mathcal{G}_{t_1}] \overset{\mathscr{Q}_\omega\text{-a.s}}{=} M^f_{t_1}$, whenever $\sigma_k(\omega) \leq t_1 < t_2$, where $\mathscr{Q}_\omega[\,\cdot\,|\,\cdot\,]$ stands for conditional expectation with repect to $\mathscr{Q}_\omega$. It follows from (3.5.10) that,

$$(M^f_t) \text{ is a } \mathscr{Q}_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1}\text{-martingale}\,. \tag{3.5.11}$$

Let us consider the countable subset of $\mathscr{D}_0$

$$\tilde{\mathscr{D}}_0 \;:=\; \{f \in \mathscr{D}_0 : f(x) \text{ is a rational number for all } x \in S\}$$

and denote $\mathcal{A} := \bigcap_{f \in \tilde{\mathscr{D}}_0} \mathcal{A}_f$. Then, (3.5.11) implies that, for all $\omega \in \mathcal{A} \cap \{\sigma_k < \infty\}$,

$$\mathscr{Q}_\omega \circ (\vartheta_{\sigma_k(\omega)})^{-1} \text{ is a solution of the } (\tilde{\mathscr{D}}_0, \mathscr{L})\text{-martingale problem.}$$

But, given any $f \in \mathscr{D}_0$, $\exists\ (f_n)$ in $\tilde{\mathscr{D}}_0$ such that $f_n \to f$ and $\mathscr{L} f_n \to \mathscr{L} f$, both pointwise, and such that

$$\sup_{n\geq 1}\max_{x\in S}\big(|f_n(x)|+|\mathscr{L}f_n(x)|\big)<\infty\,.$$

By using this approximation it is easy to conclude that, for all $\omega\in\mathcal{A}\cap\{\sigma_k<\infty\}$,

$$\mathscr{Q}_\omega\circ(\vartheta_{\sigma_k(\omega)})^{-1}\text{ is a solution of the }(\mathscr{D}_0,\mathscr{L})\text{-martingale problem.}\tag{3.5.12}$$

On the other hand, for all $\omega\in\{\sigma_k<\infty\}$,

$$\mathscr{Q}_\omega\circ(\vartheta_{\sigma_k(\omega)})^{-1}\{X_0=X_{\sigma_k}(\omega)\}\;=\;\mathscr{Q}_\omega\{X_{\sigma_k(\omega)}=X_{\sigma_k}(\omega)\}\;=\;1$$

(we applied (3.5.8) in the last equality.) Namely, for all $\omega\in\{\sigma_k<\infty\}$,

$$\mathscr{Q}_\omega\circ(\vartheta_{\sigma_k(\omega)})^{-1}\text{ is starting at }X_{\sigma_k}(\omega)\in S\setminus\{0\}\,,\tag{3.5.13}$$

where we used observation (3.5.6) for the last assertion. We may now conclude from (3.5.12), (3.5.13) and the uniqueness result established in Lemma 3.5.3 that

$$\mathscr{Q}_\omega\circ(\vartheta_{\sigma_k(\omega)})^{-1}\;=\;\mathscr{P}_{X_{\sigma_k}(\omega)}\,,\quad\forall\omega\in\mathcal{A}\cap\{\sigma_k<\infty\}\,.$$

Since $\mathscr{P}(\mathcal{A})=1$, this last assertion implies that (3.5.9) equals

$$\int_{\{\sigma_k<\infty\}}\mathscr{P}_{X_{\sigma_k}(\omega)}(\mathcal{C}\,)\,\mathscr{P}(d\omega)\,.$$

This concludes the proof. □

3.5.3 A Solution Starting at $0\in S$

From now on, we shall denote by $\mathscr{P}_0$ the law of $(\mathscr{X}_t)$ (defined in (1.2.3)) so that we have now the *complete* set of laws $\{\mathscr{P}_x : x\in S\}$. Obviously $\mathscr{P}_0$ starts at 0. We prove now that $\mathscr{P}_0$ is a solution of the $(C^1(S),\mathscr{L})$-martingale problem. Recall the sequence $(T_n)_{n\geq 2}$ of independent random variables considered in (1.2.2). For each $k\in\mathbb{N}$ define the process $(\mathscr{X}^k_t)$ as

$$\mathscr{X}^k_t\;=\;\begin{cases}1/k\,, & 0\leq t<T_k\,,\\ 1/(k-1)\,, & T_k\leq t<T_k+T_{k-1}\,,\\ \vdots & \vdots\\ 1/2\,, & \sum_{n=3}^k T_n\leq t<\sum_{n=2}^k T_n\,,\\ 1\,, & t\geq\sum_{n=2}^k T_n\,,\end{cases}$$

for all $t\geq 0$. Clearly, the law of $(\mathscr{X}^k_t)$ is $\mathscr{P}_{1/k}$. Also, observe that $(\mathscr{X}^k_t)$ is related to $(\mathscr{X}_t)$ by

$$\mathscr{X}^k_t\;=\;\mathscr{X}_{S_k+t}\,,\quad\forall t\geq 0\,,\quad\text{where }\;S_k:=\sum_{n=k+1}^{\infty}T_n\,.$$

In particular, for all $t\geq 0$,

$$f(\mathscr{X}^k_t)\xrightarrow{a.s.}f(\mathscr{X}_t)\quad\text{and}\quad\mathscr{L}f(\mathscr{X}^k_t)\xrightarrow{a.s.}\mathscr{L}f(\mathscr{X}_t)\,,\quad\text{as }k\uparrow\infty\,.\tag{3.5.14}$$

Fix an arbitrary $f \in C^1(S)$, a continuous function $G : S^m \to \mathbb{R}$ and a finite set of times $0 \le s_1 < \cdots < s_m \le s < t$. In virtue of Remark 3.5.2, we have

$$E\Big[G(\mathscr{X}^k_{s_1}, \ldots, \mathscr{X}^k_{s_m})\{f(\mathscr{X}^k_t) - f(\mathscr{X}^k_s) - \int_s^t \mathscr{L}f(\mathscr{X}^k_r)dr\}\Big] \;=\; 0\,, \tag{3.5.15}$$

for all $k \ge 1$. Letting $k \uparrow \infty$ in (3.5.15) and using (3.5.14) we get

$$E\Big[G(\mathscr{X}_{s_1}, \ldots, \mathscr{X}_{s_m})\{f(\mathscr{X}_t) - f(\mathscr{X}_s) - \int_s^t \mathscr{L}f(\mathscr{X}_r)dr\}\Big] \;=\; 0\,. \tag{3.5.16}$$

We have thus shown that $\mathscr{P}_0$ is a solution of the $(C^1(S), \mathscr{L})$-martingale problem.

3.5.4 Uniqueness Starting at $0 \in S$

In this subsection we prove the uniqueness result that we used in Section 3.4. Let σ stand for the exit time from $0 \in S$, i.e.

$$\sigma \;:=\; \inf\{t \ge 0 : X_t \ne 0\}\,. \tag{3.5.17}$$

Clearly, $\sigma_k \downarrow \sigma$ pointwise. Notice that σ is not a $(\mathcal{G}_t)$-stopping time.

Proposition 3.5.5. *There exists a unique probability measure $\mathscr{P}$ on $(D(\mathbb{R}_+, S), \mathcal{G}_\infty)$ such that $\mathscr{P}\{X_0 = 0, \sigma = 0\} = 1$ and*

$$f(X_t) - \int_0^t \mathscr{L}f(X_s)ds\,, \quad t \ge 0$$

is a $\mathscr{P}$-martingale for every $f \in \mathscr{D}_0$.

Existence is, of course, a consequence of Lemma 3.4.1. Nevertheless, it follows from the conclusion of the previous subsection that $\mathscr{P}_0$ fulfils all the requirements. In order to show uniqueness we first improve the result obtained in Lemma 3.5.4.

Proposition 3.5.6. *Let $\mathscr{P}$ be a solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem starting at $0 \in S$. If $\mathscr{P}\{\sigma = 0\} = 1$ then*

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{C}\} \;=\; \mathscr{P}_{1/k}(\mathcal{C})\,, \quad \forall\, k \ge 1 \text{ and } \mathcal{C} \in \mathcal{G}_\infty\,.$$

Proof. We start showing that

$$\mathscr{P}\{\sigma_m < \infty\,,\, \forall m \in \mathbb{N}\} \;=\; 1\,. \tag{3.5.18}$$

Let us denote

$$\mathcal{A} \;:=\; \{\sigma_m < \infty\,,\, \forall m \in \mathbb{N}\} \;=\; \{\sigma_1 < \infty\}\,.$$

Since $\mathscr{P}_{1/n}(\mathcal{A}) = 1$ for any $n \in \mathbb{N}$ then applying equation (3.5.7) for $\mathcal{C} = \mathcal{A}$ and using observation (3.5.6) we get

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{A}\,,\, \sigma_k < \infty\} \;=\; \mathscr{P}\{\sigma_k < \infty\}\,, \quad \forall k \in \mathbb{N}\,.$$

But $\sigma_k + \sigma_1 \circ \vartheta_{\sigma_k} = \sigma_1$ and so $\{\vartheta_{\sigma_k} \in \mathcal{A}\,,\, \sigma_k < \infty\} = \mathcal{A}$. Using this observation in the last displayed equation we get

$$\mathscr{P}(\mathcal{A}) \;=\; \mathscr{P}\{\sigma_k < \infty\}\,, \quad \forall k \in \mathbb{N}\,.$$

Since $\{\sigma_k < \infty\} \uparrow \{\sigma < \infty\}$ then, letting $k \uparrow \infty$ in the previous equation, we get $\mathscr{P}(\mathcal{A}) = \mathscr{P}\{\sigma < \infty\}$ which equals one by assumption.

As second step, we prove that

$$\mathscr{P}\{X_{\sigma_m} = 1/m\,,\ \forall m \in \mathbb{N}\} \;=\; 1\,. \tag{3.5.19}$$

For it, consider the events

$$\mathcal{B}_n \;:=\; \{X_0 = 1/n \ \text{ and } \ X_{\sigma_m} = 1/m \ \text{ for all } 1 \le m \le n\}\,, \quad n \in \mathbb{N}$$

and $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Since $\mathscr{P}_{1/n}(\mathcal{B}_n) = 1$ for all $n \ge 1$, then, for all $k \in \mathbb{N}$, we have

$$\mathscr{P}_{X_{\sigma_k}(\omega)}(\mathcal{B}) \;=\; 1\,, \quad \forall \omega \in \{\sigma_k < \infty\}\,.$$

Applying (3.5.7) for $\mathcal{C} = \mathcal{B}$ along with this last observation we get

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{B}\,,\ \sigma_k < \infty\} \;=\; \mathscr{P}\{\sigma_k < \infty\} \;=\; 1\,, \quad \forall k \in \mathbb{N}\,.$$

We used (3.5.18) in the last equality. Therefore,

$$\mathscr{P}\{\vartheta_{\sigma_k} \in \mathcal{B} \text{ and } \sigma_k < \infty\,, \text{ for all } k \ge 1\} \;=\; 1\,. \tag{3.5.20}$$

Now (3.5.19) follows from (3.5.20), assumption $\mathscr{P}\{X_0 = 0\,,\ \sigma = 0\} = 1$ and the following observation

$$\{X_0 = 0\,,\ \sigma = 0\,,\ \forall k \ge 1\,,\ \vartheta_{\sigma_k} \in \mathcal{B}\,,\ \sigma_k < \infty\} \;\subseteq\; \{X_{\sigma_m} = 1/m\,,\ \forall m \in \mathbb{N}\}$$

To prove this inclusion, fix some ω in the event of the left hand side and fix an arbitrary $m' \in \mathbb{N}$. Since $\sigma_k(\omega) \downarrow \sigma(\omega) = 0$ then $X_{\sigma_k}(\omega) \to X_0(\omega) = 0$ as $k \uparrow \infty$ and so

$$\exists\, k' \in \mathbb{N} \text{ such that } X_{\sigma_{k'}}(\omega) < 1/m'\,. \tag{3.5.21}$$

On the other hand, $\vartheta_{\sigma_k}(\omega) \in \mathcal{B}$ for all $k \in \mathbb{N}$ and so $\exists\, n' \in \mathbb{N}$ such that

$$\vartheta_{\sigma_{k'}}(\omega) \in \mathcal{B}_{n'}\,. \tag{3.5.22}$$

In virtue of (3.5.21) and (3.5.22) we necessarily have

$$m' \;<\; n' \;\le\; k'$$

because

$$1/k' \overset{(3.5.6)}{\le} X_0 \circ \vartheta_{\sigma_{k'}}(\omega) \overset{(3.5.22)}{=} 1/n' \;=\; X_0 \circ \vartheta_{\sigma_{k'}}(\omega) \overset{(3.5.21)}{<} 1/m'\,.$$

From (3.5.22) it follows that

$$X_{\sigma_m} \circ \vartheta_{\sigma_{k'}}(\omega) \;=\; 1/m\,, \quad \forall\, 1 \le m \le n'\,.$$

Since $m' < n'$ in particular we have

$$X_{\sigma_{m'}} \circ \vartheta_{\sigma_{k'}}(\omega) \;=\; 1/m'\,.$$

But $X_{\sigma_{m'}} \circ \vartheta_{\sigma_{k'}}(\omega) = X_{\sigma_{m'}}(\omega)$ since $m' < k'$ and so $X_{\sigma_{m'}}(\omega) = 1/m'$. This concludes the proof of the desired inclusion.

Finally, the desired result follows from (3.5.19) and (3.5.7). □

Proof of Proposition 3.5.5. Let $\mathscr{P}$ be a probability satisfying the stated assumptions and let E and $E_{1/k}$ stand for expectation with respect to $\mathscr{P}$ and $\mathscr{P}_{1/k}$ respectively. Fix an arbitrary $n \in \mathbb{N}$ some $0 \le t_1 < t_2 < \cdots < t_n$ and a bounded continuous function $F : S^n \to \mathbb{R}$. In virtue of (3.5.6) we have

$$E\big[F(X_{\sigma_k+t_1}, \dots, X_{\sigma_k+t_n})\big] \;=\; E_{1/k}\big[F(X_{t_1}, \dots, X_{t_n})\big]\,, \quad \forall k \in \mathbb{N}\,.$$

But $(X_{\sigma_k+t_1}, \dots, X_{\sigma_k+t_n}) \to (X_{t_1}, \dots, X_{t_n})$ $\mathscr{P}$-a.s. as $k \uparrow \infty$ and so

$$E\big[F(X_{t_1}, \dots, X_{t_n})\big] \;=\; \lim_{k\to\infty} E_{1/k}\big[F(X_{t_1}, \dots, X_{t_n})\big]\,.$$

This guarantees the desired uniqueness. □

3.5.5 Proof of Proposition 3.1.1

In virtue of Remark 3.5.2 and Lemma 3.5.3, in order to conclude the proof of Proposition 3.1.1, it remains to prove that $\mathscr{P}_0$ is the unique solution of the $(C^1(S), \mathscr{L})$-martingale problem starting at $0 \in S$.

Observe that $fg \in C^1(S)$ for all $f, g \in C^1(S)$. We shall make use of the *carré du champ* corresponding to $(C^1(S), \mathscr{L})$:

$$\Gamma(f,g) \;:=\; \mathscr{L}(fg) - g\mathscr{L}f - f\mathscr{L}g\,, \quad \text{for every } f,g \in C^1(S)\,.$$

Clearly, $\Gamma(f,g)$ turns out to be continuous for each $f, g \in C^1(S)$. Since $\mathscr{L}$ acts as a derivation at $0 \in S$ we have

$$\Gamma(f,g)(0) \;=\; 0\,, \quad \forall f,g \in C^1(S)\,. \tag{3.5.23}$$

Recall definition of (M_t^f) given in (3.5.1) for each $f \in C^1(S)$.

Lemma 3.5.7. *Let $\mathscr{P}$ be any solution of the $(C^1(S), \mathscr{L})$-martingale problem. For all $f, g \in C^1(S)$, the process*

$$M_t^f M_t^g - \int_0^t \Gamma(f,g)(X_s)\, ds\,, \quad t \ge 0\,,$$

is a $\mathscr{P}$-martingale with respect to $(\mathcal{G}_t)$.

Proof. Fix some $f, g \in C^1(S)$. Denote

$$V_t^f \;:=\; \int_0^t \mathscr{L}f(X_s)\, ds \quad \text{and} \quad V_t^g \;:=\; \int_0^t \mathscr{L}g(X_s)\, ds\,, \quad t \ge 0\,,$$

so that, for all $t \ge 0$,

$$M_t^f + V_t^f \;=\; f(X_t) \quad \text{and} \quad M_t^g + V_t^g \;=\; g(X_t)$$

By multiplying these equalities we get

$$M_t^f M_t^g + V_t^f V_t^g + M_t^f V_t^g + V_t^f M_t^g \;=\; (fg)(X_t)\,. \tag{3.5.24}$$

By using

$$(fg)(X_t) \;=\; M_t^{fg} + \int_0^t \mathscr{L}(fg)(X_s) ds\,, \quad t \ge 0\,,$$

along with

$$V_t^f V_t^g = \int_0^t V_s^f dV_s^g + \int_0^t V_s^g dV_s^f , \quad t \geq 0 ,$$

in equality (3.5.24) we get

$$M_t^f M_t^g + M_t^f V_t^g + V_t^f M_t^g$$
$$= M_t^{fg} + \int_0^t \mathscr{L}(fg)(X_s) ds - \int_0^t V_s^f dV_s^g - \int_0^t V_s^g dV_s^f . \quad (3.5.25)$$

If we denote, for all $t \geq 0$,

$$M_t^1 := M_t^f V_t^g - \int_0^t M_s^f dV_s^g \quad \text{and} \quad M_t^2 := M_t^g V_t^f - \int_0^t M_s^g dV_s^f , \quad (3.5.26)$$

then equality (3.5.25) can be rewritten as

$$M_t^f M_t^g + M_t^1 + M_t^2 = M_t^{fg} + \int_0^t \Gamma(f,g)(X_s) ds . \quad (3.5.27)$$

By assumption, (M_t^{fg}) is a $\mathscr{P}$-martingale. In addition, in virtue of [30, Theorem 1.2.8], (M_t^1) and (M_t^2) are also $\mathscr{P}$-martingales. Therefore the desired result follows from (3.5.27). □

We now use observation (3.5.23) to prove that $0 \in S$ is an instantaneous state for any solution starting at 0.

Lemma 3.5.8. *For any solution $\mathscr{P}$ of the $(C^1(S), \mathscr{L})$-martingale problem starting at $0 \in S$ we have $\mathscr{P}\{\sigma = 0\} = 1$.*

Proof. Let $\mathscr{P}$ be a probability satisfying the assumptions. Define $f : S \to \mathbb{R}$ as the inclusion function i.e. $f(x) = x$, for $x \in S$. Clearly $f \in C^1(S)$ and so

$$M_t := X_t - \int_0^t (\mathscr{L}f)(X_s)\, ds , \quad t \geq 0 \quad (3.5.28)$$

is a $\mathscr{P}$-martingale. Since σ_k is a stopping time then it follows from Lemma 3.5.7 that

$$(M_{t\wedge\sigma_k})^2 - \int_0^{t\wedge\sigma_k} \Gamma(f,f)(X_s)\, ds , \quad t \geq 0 ,$$

is a $\mathscr{P}$-martingale. In particular, for all $t \geq 0$ we have

$$E\big[(M_{t\wedge\sigma_k})^2\big] = E\Big[\int_0^{t\wedge\sigma_k} \Gamma(f,f)(X_s)\, ds\Big] , \quad \forall k \in \mathbb{N} , \quad (3.5.29)$$

(since $M_0 = 0$, $\mathscr{P}$-a.s.) where E represents the expectation with respect to $\mathscr{P}$. By the bounded convergence theorem, letting $k \uparrow \infty$ in (3.5.29) we get

$$E\big[(M_{t\wedge\sigma})^2\big] = E\Big[\int_0^{t\wedge\sigma} \Gamma(f,f)(X_s)\, ds\Big] , \quad \forall t \geq 0 . \quad (3.5.30)$$

Since $\{s < \sigma\} \subseteq \{X_s = 0\}$, the right hand side in the above equation equals

$$E[t \wedge \sigma]\, \Gamma(f,f)(0)$$

which vanishes as noticed in observation (3.5.23). Therefore, from (3.5.30) we conclude that

$$\mathscr{P}[M_{t\wedge\sigma} = 0\,,\ \forall t \geq 0] \ = \ 1\,.$$

Using this fact in (3.5.28) we get that, $\mathscr{P}$-a.s.,

$$X_{t\wedge\sigma} \ = \ \int_0^{t\wedge\sigma} (\mathscr{L}f)(X_s)\,ds \ = \ \frac{1}{2}\big(t\wedge\sigma\big)\ , \quad \forall t \geq 0\,.$$

But, for any $t > 0$, we have on $\{t < \sigma\}$ that

$$X_{t\wedge\sigma} \ = \ X_t \ = \ 0 \ \neq \ \frac{1}{2}\big(t\wedge\sigma\big)\ .$$

Hence $\mathscr{P}\{t < \sigma\} = 0$, $\forall\ t > 0$ and we are done. □

It follows from Lemma 3.5.8 and Proposition 3.5.5 that $\mathscr{P}_0$ is the only solution of the $(C^1(S), \mathscr{L})$-martingale problem.

Chapter 4

From Finite Coalescing Transitive Markov Chains to Kingman's Coalescent*

Abstract

Let ξ be an irreducible and transitive Markov chain in continuous time, over a finite state space. Fix $n \geq 2$ and suppose that $\xi^1, \ldots, \xi^n$ are i.i.d. copies of ξ. Each time two chains meet, they stay together and follow the motion of the one with the smaller label. This mechanism induces a process in the set of partitions of $\{1, 2, \ldots, n\}$. Starting from the invariant measure, we find conditions under which a sequence of these processes, in an appropriate scale of time, converges to the Kingman's coalescent that starts with n equivalence classes. In particular, we prove this convergence in the reversible case under a condition that involves the relaxation time.

4.1 Notation and Results

Let S be a metric space. As usual, we denote by $D(\mathbb{R}_+, S)$ the set of càdlàg paths $w : \mathbb{R}_+ := [0, +\infty) \to S$ and we allways consider in $D(\mathbb{R}_+, S)$ the Skorokhod topology. When S is finite, unless we say otherwise, we consider in it the discrete metric. For every $t \in \mathbb{R}_+$ there is the projection $\mathbf{p}_t : D(\mathbb{R}_+, S) \to S$ defined by

$$\mathbf{p}_t(w) := w(t), \quad \forall\, w \in D(\mathbb{R}_+, S).$$

Given a process $\zeta : \Omega \to D(\mathbb{R}_+, S)$ we denote by ζ_t the composition $\zeta \circ \mathbf{p}_t$ for all $t \in \mathbb{R}_+$, and call them the marginals of ζ.

*Joint work with Johel Beltrán

For every $n \in \mathbb{N} := \{1,2,\dots\}$ we denote $[n] := \{1,2,\dots,n\}$. We call $\mathcal{P}_n$ the set of partitions of $[n]$, and given $\pi \in \mathcal{P}_n$, we denote by $\#(\pi)$ the number of equivalence classes of π, for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n$, we define $\boldsymbol{i}(\pi) := (i^1,\dots,i^{\#(\pi)}) \in [n]^{\#(\pi)}$ as the vector verifying

$$i^1 \;<\; i^2 \;<\; \dots \;<\; i^m \,, \tag{4.1.1}$$

where $m = \#(\pi)$, and $i^k = \min\{i \in \overline{i^k}\}$ for all $k \in [m]$, where $\overline{i^k}$ stands for the equivalence class to which i^k belongs. Clearly $i^1 = 1$ for every partition.

4.1.1 Kingman's Coalescent

Fix $n \geq 1$. We define the Kingman's coalescent over $\mathcal{P}_n$ as the process $\mathfrak{K}^n$ with trajectories in $D(\mathbb{R}_+,\mathcal{P}_n)$, determined by the following generator

$$\mathscr{L}^n f(\pi) \;:=\; \mathbf{1}_{\{\#(\pi)\geq 2\}} \sum_{1\leq i<j\leq \#(\pi)} [f((i,j)[\pi]) \;-\; f(\pi)] \,, \tag{4.1.2}$$

for every $f : \mathcal{P}_n \to \mathbb{R}$. Where, in (4.1.2), for a partition π with $\#(\pi) \geq 2$, and $j,k \in \mathbb{N}$ such that $1 \leq j < k \leq \#(\pi)$, we define $(j,k)[\pi]$ as follows. Suppose that $\boldsymbol{i}(\pi) = (i^1,\dots,i^{\#(\pi)})$, then

$$(j,k)[\pi] \;:=\; \{\overline{i^m} : m \notin \{j,k\}\} \cup \{\overline{i^j} \cup \overline{i^k}\} \,.$$

In other words, $(j,k)\pi$ is the partition obtained from π by coalescing $\overline{i^j}$ and $\overline{i^k}$.

4.1.2 Coalescence

Here we introduce the n-Kingman's approximation, a sequence of processes that under conditions **(H1)**, **(H2)** and **(H3)**, converges to the Kingman's coalescent as we state more precisely in Theorem 4.1.3

Independent Markov Chains

Let $\xi : \Omega \to D(\mathbb{R}_+, E)$ be an irreducible Markov chain in continuous time, with generator Q, over the finite set E. Fix $n \in \mathbb{N}$ and $x = (x^1,\dots,x^n) \in E^n$. Increasing Ω if necessary, we define the process $\boldsymbol{\xi}^{Q,n} : (\Omega, \mathrm{P}^Q_x) \to D(\mathbb{R}, E^n)$ as the coupling

$$\boldsymbol{\xi}^{Q,n} \;:=\; (\xi^1,\dots,\xi^n) \,,$$

where P^Q_x is a probability, defined in the sigma-algebra $\sigma(\boldsymbol{\xi}^{Q,n}_t : t \geq 0)$, such that the processes $\xi^1,\dots,\xi^n$ are independent copies of ξ satisfying $\mathrm{P}^Q_x[\boldsymbol{\xi}^{Q,n}_0 = x] = 1$.

Given a probability measure μ on E^k we let P^Q_μ stand for the respective probability under which $\boldsymbol{\xi}^{Q,n}$ has initial distribution μ, i.e.

$$\mathrm{P}^Q_\mu[\,\cdot\,] \;=\; \int \mathrm{P}^Q_x[\,\cdot\,]\mu(dx) \,.$$

We denote by $\boldsymbol{m}^n$ the stationary distribution of $\boldsymbol{\xi}^{Q,n}$. Hence, under $\mathrm{P}^Q_{\boldsymbol{m}^n}$ the process $\boldsymbol{\xi}^{Q,n}$ is the coupling of n i.i.d. copies of ξ, each of them starting from its stationary probability distribution. Also, we let $\langle f\rangle_\mu$ stand for the μ-integral of some real-valued function f.

Coalescing Markov Chains

Fix $n \in \mathbb{N}$, a set E, and $x = (x^1, \dots, x^n) \in E^n$. We denote by $\pi(x) \in \mathcal{P}_n$ the partition induced by the equivalence relation

$$i \sim j \iff x^i = x^j .$$

This permit us to define the function $\Psi_{E,n} : E^n \to \mathcal{P}_n$ by $\Psi_{E,n}(x) := \pi(x)$.

We associate to $\boldsymbol{\xi}^{Q,n}$ the process $\boldsymbol{\eta}^{Q,n} : \Omega \to D(\mathbb{R}_+, E^n)$ with marginals

$$\boldsymbol{\eta}_t^{Q,n} = (\eta_t^1, \eta_t^2, \dots, \eta_t^n) , \quad t \geq 0 ,$$

defined as follows. First, we set $\eta^1 := \xi^1$. Now, suppose that processes $\eta^1, \dots, \eta^{m-1}$ have been defined for some $m \leq n$. After denoting

$$T := \min\{t \geq 0 : \xi_t^m = \eta_t^j \text{ for some } j < m\} \quad \text{and} \quad j_m := \min\{j : \xi_T^m = \eta_T^j\} ,$$

we define

$$\eta_t^m := \begin{cases} \xi_t^m , & \text{for } t < T , \\ \eta_t^{j_m} , & \text{for } t > T . \end{cases}$$

Finally, we define the process $X^{Q,n} : \Omega \to D(\mathbb{R}_+, \mathcal{P}_n)$ by

$$X_t^{Q,n} := \Psi_{E,n}(\boldsymbol{\eta}_t^{Q,n}) , \quad \forall t \geq 0 .$$

4.1.3 Main Result

Let $n \in \mathbb{N}$ and consider $\boldsymbol{\eta}^{Q,n}$ as defined in Subsection 4.1.2. We define the stopping times

$$T_m^{Q,n} := \inf\{t \geq 0 : \#\big(X_t^{Q,n}\big) = m\} , \quad m \in [n] ,$$

Often, when the superscript Q, n is understood we do not write it. We apply this convention to all the stopping times defined in this work.

Definition 4.1.1 (Coalescing hypotheses)**.** *Consider a sequence of irreducible Markov chains $(\xi^N)_{N\in\mathbb{N}}$ with their respective generators Q_N, over finite state spaces. Define the scale of time $(\theta_N)_{N\in\mathbb{N}}$ by $\theta_N := \mathrm{E}^N_{\boldsymbol{m}^2}[T_1]$, and fix $n \in \mathbb{N} \setminus \{1\}$. We say that $(\xi^N)_{N\in\mathbb{N}}$ fulfills the* coalescing hypotheses starting with n particles *if the following conditions are satisfied*

*(**H1**) With probability converging to one, two independent particles starting from the stationary distribution do not coalesce in a scale of time smaller than $(\theta_N)_{N\in\mathbb{N}}$:*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^{Q_N}_{\boldsymbol{m}^2}[T_1 \leq \delta\theta_N] = 0 .$$

*(**H2**) There exist $p > 1$ and a scale of time $(\alpha_N)_{N\in\mathbb{N}} \subset \mathbb{R}_+$ smaller than $(\theta_N)_{N\in\mathbb{N}}$, where $\boldsymbol{\xi}^{Q_N,n}$ exhibits a local ergodic behavior in the sequences of functions uniformly bounded in $L^p(\boldsymbol{m}^n)$:*

$$\lim_{N\to\infty} \frac{\alpha_N}{\theta_N} = 0 ,$$

and

$$\lim_{N\to\infty} \mathrm{E}^{Q_N}_{\boldsymbol{m}^n}\left[\left|\frac{1}{\alpha_N}\int_0^{\alpha_N} f^N(\boldsymbol{\xi}_s^{Q_N,n})\, ds\right|\right] = 0 ,$$

provided that $(f^N : E_N^n \to \mathbb{R})_{N\in\mathbb{N}}$ *is uniformly bounded in* $L^p(\boldsymbol{m}^n)$*, and satisfies* $\langle f^N\rangle_{\boldsymbol{m}^n} = 0$ *for all* $N \in \mathbb{N}$.

*(***H3***) There exists* $\varepsilon > 0$ *such that the sequences of scaled times* $(T_1^{Q_N,2}/\theta_N)_{N\in\mathbb{N}}$ *is uniformly bounded in* $L^{1+\varepsilon}(\mathrm{P}^{Q_N}_{\boldsymbol{m}^2})$*:*

$$\mathrm{E}^{Q_N}_{\boldsymbol{m}^2}\Big[\Big(\frac{T_1}{\theta_N}\Big)^{1+\varepsilon}\Big] \;\leq\; C(n)\,, \quad \textit{for some } \varepsilon > 0\,,$$

where $C(n)$ *is a constant only, possibly depending on* n.

Clearly, in Hypothesis (H2), for the sequence of functions $(f^N)_{N\in\mathbb{N}}$, the condition $\langle f^N\rangle_{\boldsymbol{m}^n} = 0$ for all $N \in \mathbb{N}$ can be replaced by $\langle f^N\rangle_{\boldsymbol{m}^n} \to 0$ as $N \to \infty$.

Suppose that we have a sequence $(\xi^{Q_N})_{N\in\mathbb{N}}$ of irreducible Markov chains, like in Definition 4.1.1. For the time scale $\boldsymbol{\theta} := (\theta_N)_{N\in\mathbb{N}} \subset \mathbb{R}_+$ and a natural number n, we define the $(\boldsymbol{\theta}, n)$*-Kingman's approximation associated to* $(\xi^{Q_N})_{N\in\mathbb{N}}$ as the sequence of processes $(\mathbb{X}^{\boldsymbol{\theta},Q_N,n})_{N\in\mathbb{N}}$ defined by

$$\mathbb{X}_t^{\boldsymbol{\theta},Q_N,n} \;=\; X^{Q_N,n}_{t\theta_N}\,, \quad \forall t \geq 0\,,\, N \in \mathbb{N}\,.$$

When $\boldsymbol{\theta}$ is the time scale introduced in Definition 4.1.1 we write $(\mathbb{X}^{Q_N,n})_{N\in\mathbb{N}}$ instead of $(\mathbb{X}^{\boldsymbol{\theta},Q_N,n})_{N\in\mathbb{N}}$ and call it just the *n-Kingman's approximation associated to* $(\xi^{Q_N})_{N\in\mathbb{N}}$

To make notation simpler, when we consider a sequence of Markov chains like in the last paragraph, in all the previously established and future notation, we agree to replace the subscripts or subscripts Q_N with simply N. In this way, we shall write $\boldsymbol{\xi}_t^{N,n}$, $\mathrm{P}^N_{\boldsymbol{m}^n}$, etc. in place of the respectives $\boldsymbol{\xi}_t^{Q_N,n}$, $\mathrm{P}^{Q_N}_{\boldsymbol{m}^n}$, etc. Also, when there where no room for confusion, we shall omit the number n that designates the number of Markov chains considered to generate $\boldsymbol{\eta}^{Q,n}$. For example, when n is understood it will be usual for us to write $\boldsymbol{\eta}^Q$ and X^Q instead of $\boldsymbol{\eta}^{Q,n}$ and $X^{Q,n}$, respectively.

We are ready to state our main theorem; but before that we remember the notion of transitivity for Markov Chains

Definition 4.1.2. *Let* $W : \Omega \to D(\mathbb{R}_+, S)$ *be a Markov chain, and denote by* $\mathbb{P}_x$*,* $x \in S$*, the probabilities in* $D(\mathbb{R}_+, S)$ *such that* $\mathbb{P}_x[W_0 = x] = 1$*. We say that* W *is* transitive *if for every* $x, y \in S$ *there is a bijection* $\varphi_{x,y} : S \to S$ *such that* $\varphi_{x,y}(x) = y$*, and the induced bijection* $\widehat{\varphi}_{x,y} : D(\mathbb{R}_+, S) \to D(\mathbb{R}_+, S)$ *defined by*

$$\widehat{\varphi}_{x,y}(w)(t) \;=\; \varphi(w(t))\,, \quad \forall w \in D(\mathbb{R}_+, S)\,, \forall t \geq 0\,,$$

preserves the law of the chain, i.e. $(\widehat{\varphi}_{x,y})^{-1} \circ \mathbb{P}_x = \mathbb{P}_y$.

Theorem 4.1.3. *Let* $(\xi^N)_{N\in\mathbb{N}}$ *be a sequence of transitive, irreducible Markov chains over finite state spaces, and fix a natural number* $n \geq 2$*. Suppose that* $(\xi^N)_{N\in\mathbb{N}}$ *fulfills the coalescing hypotheses starting with n particles. Then, under* $\mathrm{P}^N_{\boldsymbol{m}^n}$*, the n-Kingman's approximation associated to* $(\xi^N)_{N\in\mathbb{N}}$ *converges in distribution to the Kingman's coalescent* $\mathfrak{K}^n$ *starting at* π_n*, the partition of* $[n]$ *with n equivalence classes.*

For an irreducible Markov Chain ξ with generator Q, we denote by $T_\pi^{Q,n}$, $\pi \in \mathcal{P}_n$, the stopping times defined by

$$T_\pi^{Q,n} \;:=\; \inf\{t \geq 0 :\; X_t^{Q,n} = \pi\}\,, \quad \pi \in \mathcal{P}_n\,;$$

and we call $\widetilde{T}_m^{Q,n}$ the first time $X^{Q,n}$ has at most m equivalence classes:

$$\widetilde{T}_m^{Q,n} := \inf\{t \geq 0 : \#(X_t^{Q,n}) \leq m\} .$$

Compare this with the definition of $T_m^{Q,n}$ made at the beginning of Section 4.1.3.

As we show in Subsection 4.4.1. To prove Theorem 4.1.3, we do not need Hypothesis **(H2)** in its full extension. For us it is enough to have the local ergodic behavior for the sequences $(f^{N,\pi})_{N\in\mathbb{N}}$, $\pi \in \mathcal{P}_{\#(\pi)+1}$ with $\#(\pi) \leq n-1$, defined by

$$f^{N,\pi}(x^1,\dots,x^n) := \mathrm{E}^{Q_N}_{(x^1,\dots,x^{\#(\pi)+1})}\left[\frac{\widetilde{T}_{\#(\pi)}}{\theta_N} - \mathbf{1}_{\{\widetilde{T}_{\#(\pi)}=T_\pi\}}\right] , \tag{4.1.3}$$

for all $(x^1,\dots,x^n) \in E_N^n$ and $N \in \mathbb{N}$. Thanks to Hypothesis **(H3)**, these sequences are uniformly bounded in $L^{1+\varepsilon}(\boldsymbol{m}^n)$. For this reason, reducing ε if necessary, we can assume that $1-p \geq \varepsilon$. In all the remaining work we make this this assumption without mention it.

The Reversible Case

Let ξ be an irreducible Markov chain with generator Q, over a finite state space. Suppose that ξ is reversible and denote by γ_Q its relaxation time.

Consider a sequence $(\xi^N)_{N\in\mathbb{N}}$ of transitive, irreducible and reversible Markov chains such that the size of the state spaces goes to infinite as $N \to \infty$. We define the following condition

$$\lim_{N\to\infty} \frac{\gamma_N}{\theta_N} = 0 , \tag{H'}$$

where $(\theta_N)_{N\in\mathbb{N}}$ is defined by $\theta_N = \mathrm{E}^N_{\boldsymbol{m}^2}[T_1]$, for all $N \in \mathbb{N}$. As we discuss in Subsection 4.4.3, from the work of Aldous [2] follows that **(H')** implies **(H1)** and **(H3)**; whereas we prove that **(H')** also imples **(H2)**. Therefore, we have the following corollary of Theorem 4.1.3.

Corollary 4.1.4. *Let $(\xi^N)_{N\in\mathbb{N}}$ be a sequence of transitive, irreducible and reversible Markov chains over finite state spaces. Fix a natural number $n \geq 2$, suppose that condition* **(H')** *holds. Then, under $\mathrm{P}^N_{\boldsymbol{m}^n}$, the n-Kingman's approximation associated to $(\xi^N)_{N\in\mathbb{N}}$ converges in distribution to the Kingman's coalescent $\mathfrak{K}^n$ starting at π_n, the partition of $[n]$ with n equivalence classes.*

4.1.4 Sketch of the Proof

Fix a natural number $n \geq 2$ and consider $(\xi^N)_{N\in\mathbb{N}}$, a sequence of transitive, irreducible Markov chains over finite state spaces that fulfills the coalescing hypotheses starting with n particles. Our strategy to prove Theorem 4.1.3 is summarized in the next two steps:

1. We prove that the n-Kingman's approximation is tight.
2. Then we show that every limit process of the n-Kingman's approximation solves a martingale problem with unique solution.

To achieve step (1) we use the transitivity of $(\xi^N)_{N\in\mathbb{N}}$ and the Hypothesis **(H1)**. These conditions allow us to show that asymptotically, when $N \to \infty$, the jumps in $\mathbb{X}^N$ do not happen instantly. Thanks to this, we are able to find a sequence of explicit partitions of the time that makes the modified modulus of continuity as small as we want, in probability. This sequence of partitions is determined by the times where the process jumps, i.e. it is generated by $\{T^N_{n-1} \leq T^N_{n-2} \leq \cdots \leq T^N_1\}$.

For step (2) we prove a "replacement condition" between $(\mathbb{X}^N)_{N\in\mathbb{N}}$ and $\mathfrak{K}^n$. More specifically, we find that for any pair of positive numbers $t_1 < t_2$, the conditional expectation, at time t_1, of

$$\Big| \int_{t_1}^{t_2} (\mathscr{L}^n f)\mathbb{X}_s^N - \theta_N L_N^n (f\circ\Psi_N)(\boldsymbol{\eta}_{s\theta_N}^N)\, ds \Big|$$

goes to zero when $N\to\infty$, for all $f:\mathcal{P}_n\to\mathbb{R}$. This allow us to proof that, in the limit, the martingals defined by

$$f(\mathbb{X}_t^N) - \int_0^1 \theta_N L_N^n (f\circ\Psi_N)(\boldsymbol{\eta}_{s\theta_N}^N)\, ds\,, \quad t\geq 0\,,$$

become the martingals associated to the Kingman's coalescent generated by $\mathscr{L}^n$. This completes our strategy. Then, the main point in our path to show Theorem 4.1.3 is the proof of the replacement condition. Here is where we use the local ergodic Hypothesis **(H3)**, which permit us to neglect averages in time when they are made in a scale smaller than $(\theta_N)_N$, and when the functions subject to this averages have zero mean. To do it we perform a series of transformations on the replacement condition, until reduce it to the computation of the expectations $\mathrm{E}_{\boldsymbol{m}^m}^N[T_{m-1}]$, when $N\to\infty$ for all $2\leq m\leq n$. In a few words, we reduce the proof of the replacement condition to the study of the first coalescence in $\mathbb{X}^N$, when N tends to infinity.

4.1.5 Extra Notation

Finally, here we introduce more notation that we use in the next sections. Fix $n\in\mathbb{N}$ and $m\leq n$. We denote by $I_{n,m}$ the set of all vectors $(i^1,\dots,i^m)\in[n]^m$ satisfying (4.1.1). We also set

$$I_n := \bigcup_{m=1}^n I_{n,m}\,.$$

Assume that we are in the setting of Subsection 4.1.2. Let $\boldsymbol{\zeta}$ be a process whose trajectories are in $D(\mathbb{R}_+,E^n)$ and take $\boldsymbol{i}=(i^1,\dots,i^m)\in I_n$. Suppose that $\boldsymbol{\zeta}_t=(\zeta_t^1,\dots,\zeta_t^n)$ for all $t\geq 0$. Then we denote

$$\boldsymbol{\zeta}_t(\boldsymbol{i}) := (\zeta_t^{i^1},\dots,\zeta_t^{i^m})\,. \tag{4.1.4}$$

Using this notation it is clear that

$$\boldsymbol{\eta}_t^{Q,n}(\boldsymbol{i}(X_t^{Q,n})) = \boldsymbol{\xi}_t^{Q,n}(\boldsymbol{i}(X_t^{Q,n}))\,, \quad \text{for all } t\geq 0\,. \tag{4.1.5}$$

Now we define some stopping times. For $1\leq i<j\leq n$ we denote by $\tau_{i,j}^{Q,n}$ the first time when $\xi_t^i=\xi_t^j$:

$$\tau_{i,j}^{Q,n} := \inf\{t\geq 0:\ \xi_t^i=\xi_t^j\}\,.$$

Observe that, when $n\geq 2$, every $T_m^{Q,n}$, when it is finite, coincides with $\tau_{\mathrm{i,j}}^{Q,n}$ for some random indexes $1\leq \mathrm{i}<\mathrm{j}\leq k$. Also $T_m^{Q,n}$ coincides with $T_{\boldsymbol{\pi}}^{G,k}$ for some random $\boldsymbol{\pi}\in\mathcal{P}_n$ such taht $\#(\pi)=m$.

To conclude, we distinguish the following sets in E^n

$$\mathscr{E}_{E,n}^{\pi} := \{x\in E^n:\ \Psi_{E,k}(x)=\pi\}\,,$$

for all $\pi\in\mathcal{P}_n$ and $m\in[n]$.

4.2 Some General Tools

This section is independent from the others in the present chapter. Here we develop some general theory, mostly related with the notion of maringale problem and convergence of processes, that we use in the following sections.

4.2.1 About Martingale-problem Solutions

Here we examine some results related to martingale-problem solutions. We start by fixing some notation.

Let E be a metric space. We denote by $\mathcal{M}_b(E)$ the space of measurable and bounded functions $f : E \to \mathbb{R}$. Similarly, we call $\mathcal{C}_b(E)$ the space of continuous and bounded functions $f : E \to \mathbb{R}$. Clearly $\mathcal{C}_b(E) \subset \mathcal{M}_b(E)$. Like in the other sections $D(\mathbb{R}_+, E)$ denotes the space of cadlag functions $\alpha : \mathbb{R}_+ \to E$. In $D(\mathbb{R}_+, E)$, we always consider the sigma-algebra generated by the Skorokhod topology.

Definition 4.2.1. *Fix a pair $(\mathcal{D}, L)$, where $\mathcal{D} \subset \mathcal{M}_b(E)$ and $L : \mathcal{D} \to \mathcal{M}_b(E)$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration of $\mathcal{F}$, and $X : \Omega \to D(\mathbb{R}_+, E)$ be a measurable process, adapted to $(\mathcal{F}_t)_{t\geq 0}$. We say that X is a solution for the $(\mathcal{D}, L)$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$, if for every $f \in \mathcal{D}$*

$$M_t^f \ := \ f(X_t) - \int_0^t (Lf)(X_s)\, ds \ , \quad t \geq 0 \ ,$$

defines a $(\mathbf{P}, (\mathcal{F}_t)_{t\geq 0})$-martingale. In addition, we say that X starts at $x \in E$ when $\mathbf{P}[X_0 = x] = 1$. When $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by X we agree not to mention it, we assume it understood.

Under the conditions of Definition 4.2.1, we say that the $(\mathcal{D}, L)$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$ has a *unique solution starting from $x \in E$* if the induced laws in $D(\mathbb{R}_+, E)$, of all the solutions starting from x, are the same.

A Product of Martingale-problem Solutions

In this subsection all the state spaces we consider are discrete topological spaces. Now fix two pairs $(\mathcal{D}, L)$ and $(\widetilde{\mathcal{D}}, \widetilde{L})$, where $\mathcal{D} \subset \mathcal{M}_b(E)$, $L : \mathcal{D} \to \mathcal{M}_b(E)$, $\widetilde{\mathcal{D}} \subset \mathcal{M}_b(\widetilde{E})$, and $\widetilde{L} : \widetilde{\mathcal{D}} \to \mathcal{M}_b(\widetilde{E})$. Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t)_{t\geq 0}$, a filtration of $\mathcal{F}$.

Suppose that the processes $X : \Omega \to D(\mathbb{R}_+, E)$ and $Y : \Omega \to D(\mathbb{R}_+, \widetilde{E})$, adapted to $(\mathcal{F}_t)_{t\geq 0}$, are solutions for the $(\mathcal{D}, L)$- martingale problem and$(\widetilde{\mathcal{D}}, \widetilde{L})$-martingale problem, respectively, both associated to$(\mathcal{F}_t)_{t\geq 0}$. The purpose of this subsection is to establish that, under certain conditions, the coupled process $(X, Y) : \Omega \to D(\mathbb{R}_+, E \times \widetilde{E})$ is a solution for the $(\mathcal{D} \otimes \widetilde{\mathcal{D}}, L \otimes \widetilde{L})$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$. Here $\mathcal{D} \otimes \widetilde{\mathcal{D}} \subset \mathcal{M}_b(E \times \widetilde{E})$ denotes the set of functions $f \otimes g : E \times \widetilde{E} \to \mathbb{R}$ defined by

$$f \otimes g(x,y) \ := \ f(x)g(y) \ , \quad x \in E,\ y \in \widetilde{E},\ f \in \mathcal{M}_b(E),\ g \in \mathcal{M}_b(\widetilde{E}),$$

and $L \otimes \widetilde{L} : \mathcal{D} \otimes \widetilde{\mathcal{D}} \to \mathcal{M}_b(E \times \widetilde{E})$ stands for the operator defined by

$$(L \otimes \widetilde{L}) f \otimes g \ := \ f(x)\widetilde{L}(y) + g(y) Lf(x) \ , \quad \forall x, y \in E \, .$$

To this end, we shall perform a series of computations. But first we introduce the following definition

Definition 4.2.2. *In the previous setting. We say that the $(\mathcal{D} \otimes \widetilde{\mathcal{D}}, L \otimes \widetilde{L})$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$ is the* product *of the $(\mathcal{D}, L)$-martingale problem with the $(\widetilde{\mathcal{D}}, \widetilde{L})$-martingale problem, both associated to the same filtration, $(\mathcal{F}_t)_{t\geq 0}$.*

Remark 4.2.3. *Given three martingale problems defined by the pairs $(\mathcal{D}_i, L_i)$, $i = 1,2,3$, associated to the same filtration, it is easy to see that*

$$(L_1 \otimes L_2) \otimes L_3 \;=\; L_1 \otimes (L_2 \otimes L_3)\,.$$

This allows us to define the the product of a finite number of martingale problems defined by $(\mathcal{D}_i, L_i)$, $i = 1, \dots, n$, and associated to the same filtration. We call this product, the

$$(\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n, L_1 \otimes \cdots \otimes L_n)\text{-martingale problem,}$$

associated to the same filtration.

Take $f \in \mathcal{D}$, $g \in \widetilde{\mathcal{D}}$ and denote

$$V_t \;:=\; \int_0^t Lf(X_s)\,ds\,, \quad W_t \;:=\; \int_0^t \widetilde{L}(Y_s)\,ds\,,$$

then we have the martingales $M_t := f(X_t) - V_t$ and $N_t := g(Y_t) - W_t$, $t \geq 0$. Observe that

$$f(X_t)g(Y_t) \;=\; M_t W_t + N_t W_t + V_t W_t + M_t N_t. \tag{4.2.1}$$

Take into account that the differences $M_t W_t - \int_0^t M_s\,dW_s$ and $N_t V_t - \int_0^t N_s\,dY_s$ are martingales, and that $V_t W_t = \int_0^t V_s\,dW_s + \int_0^t W_s\,dV_s$. Then, replacing these in (4.2.1), we deduce that

$$\begin{aligned} f(X_t)g(Y_t) - M_t N_t - \int_0^t (M_s + V_s)\,dW_s - \int_0^t (N_s + W_s)\,dV_s \\ = f \otimes g(X_t, Y_t) - M_t N_t - \int_0^t (L \otimes \widetilde{L}) f \otimes g(X_s, Y_s)\,ds\,, \end{aligned} \tag{4.2.2}$$

defines a martingale.

On the other hand, because $t \mapsto V_t$ and $t \mapsto W_t$ are continuous with finite total variation, $[M_t, N_t] = [f(X_t), g(Y_t)]$; hence $M_t N_t - [f(X_t), g(Y_t)]$ is a martingale. This together with (4.2.2) imply that

$$f \otimes g(X_t, Y_t) - [f(X_t), g(Y_t)] - \int_0^t (L \otimes \widetilde{L}) f \otimes g(X_s, Y_s)\,ds\,, \tag{4.2.3}$$

defines a martingale.

From these computations we deduce the following lemma.

Lemma 4.2.4. *Let X and Y be solutions of two martingale problems associated to the same filtration. Suppose that X and Y do not jump at the same time, then the coupled process (X, Y) is a solution of the product of the previous martingale problems.*

Proof. Consider X and Y as before and assume that they do not jump simultaneously. Then

$$[f(X_t), g(Y_t)] \;=\; 0\,, \quad \forall t \geq 0\,,$$

for all $f \in \mathcal{D}$ and $g \in \widetilde{\mathcal{D}}$. To conclude observe that, thanks to (4.2.3), calling $F = f \otimes g$

$$F(X_t, Y_t) \;-\; \int_0^t (L \otimes \widetilde{L}) F(X_s, Y_s)\,ds, \quad t \geq 0\,, \tag{4.2.4}$$

defines a martingale for all $F \in \mathcal{D} \otimes \widetilde{\mathcal{D}}$. □

Remark 4.2.5. *Let X_i, $i = 1, \dots, n$, be solutions of the respective $(\mathcal{D}_i, L_i)$-martingale problems, $i = 1, \dots, n$, associated to the same filtration. Then, if any pair of this solutions do not jump at the same time, then the coupled process $(X_1, \dots, X_n)$ is a solution of the $(\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n, L_1 \otimes \cdots \otimes L_n)$-martingale problem associated to the same filtration.*

Corollary 4.2.6. *Two Markov processes over finite state spaces that share the same filtration are independent, if and only if they do not jump at the same time.*

Proof. Let $X : \Omega \to D(\mathbb{R}_+, E)$ and $Y : \Omega \to D(\mathbb{R}_+, \widetilde{E})$ be two Markov processes over the finite sets E and $\widetilde{E}$. Suppose that the generator of X and Y are L and $\widetilde{L}$, respectively. We know that X is the unique solution of the $(\mathcal{M}_b(E), L)$-martingale problem and, similarly, Y is the unique solution of the $(\mathcal{M}_b(\widetilde{E}), \widetilde{L})$-martingale problem, both problems associated to the same filtration. In what follows, for simplicity, we do not mention the filtration since all the processes and martingales are adapted to this.

Suppose that X and Y do not jump at the same time. Thanks to Lemma 4.2.4

$$f \otimes g(X_t, Y_t) - \int_0^t (L \otimes \widetilde{L}) f \otimes g(X_s, Y_s)\, ds\,, \quad t \geq 0\,, \tag{4.2.5}$$

defines a martingale for every pair of functions $f \in \mathcal{M}_b(E)$ and $g \in \mathcal{M}_b(\widetilde{E})$. Observe that any function $F \in \mathcal{M}_b(E \times \widetilde{E})$ can be written as

$$F = \sum_{(x,y) \in E \times \widetilde{E}} F(x,y) \mathbf{1}_x \otimes \mathbf{1}_y\,,$$

where $\mathbf{1}_z$ stands for the indicator function of the set $\{z\}$, included in E or $\widetilde{E}$ as appropriate. This, together with the linearity of $L \otimes \widetilde{L}$ implies that the expression in (4.2.5) is still a martingale if we replace $f \otimes g$ by any function in $\mathcal{M}_b(E \times \widetilde{E})$. This means that, (X, Y) is the unique solution of the $(\mathcal{M}_b(E \times \widetilde{E}), L \otimes \widetilde{(E)})$-martingale problem, therefore X and Y are independent.

The converse in clear. □

This lemma and an obvious inductive procedure prove the following corollary.

Corollary 4.2.7. *Any finite number of Markov processes over finite state spaces that share the same filtration are independent, if and only if any pair of them do not jump at the same time.*

Convergence to a Martingale-problem Solution

Here we show that, under a replacement condition, the limit in distribution of a sequence of processes solves a martingale problem. For simplicity, all the metric spaces that we consider in this subsection are at most countable.

The previous topological assumption ensures that in the spaces we shall work the open sets are well approximated by closed sets from inside, and by open sets from outside. As a consequence, the indicators of all open sets are pointwise-approximated by continuous functions uniformly bounded by one. If we want more generality, last condition can replace the topological assumption we made in the previous paragraph.

Let $(\Omega_N, \mathcal{F}^N, \mathrm{P}^N)$, $N \in \mathbb{N}$, be a sequence of probability spaces where we have defined the processes

$$\varkappa^N : \Omega_N \to D(\mathbb{R}_+, E), \quad Y^N : \Omega_N \to D(\mathbb{R}_+, \widetilde{S}), \quad N \in \mathbb{N}\,.$$

In last expression, E and $\widetilde{S}$ are metric spaces. We denote by $(\mathcal{F}_t^N)_{t \geq 0}$, $N \in \mathbb{N}$, the filtration defined by

$$\mathcal{F}_t^N \;:=\; \sigma(Y_s^N, s \leq t)\,, \quad \forall t \geq 0\,, \quad N \in \mathbb{N}\,.$$

We assume that each $\varkappa^N$ is a solution of the $(\mathcal{D}_N, L_N)$-martingale problem associated to $(\mathcal{F}_t^N)_{t\geq 0}$, for some operators $L_N : \mathcal{D}_N \to \mathcal{M}_b(E)$, $N \in \mathbb{N}$. Consider $\varphi_N : E \to S$, $N \in \mathbb{N}$, a sequence of measurable functions between E and the metric space S, and define the processes $X^N : \Omega^N \to D(\mathbb{R}_+, S)$ by

$$X_t^N := \varphi_N(\varkappa_t^N), \quad t \geq 0, \quad \forall N \in \mathbb{N}.$$

Suppose that $(X^N, Y^N)_N$ converges in distribution to (X, Y) as $N \to \infty$, where $(X, Y) : \Omega \to D(\mathbb{R}_+, S \times \widetilde{S})$ is a process defined in the probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Finally we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration defined by

$$\mathcal{F}_t := \sigma(Y_s, s \leq t), \quad \forall t \geq 0.$$

Under this framework we have the following result.

Lemma 4.2.8. *Let $\varkappa_N$, Y_N, X_N, X and Y be like before, and let $L : \mathcal{D} \subset \mathcal{C}_b(S) \to \mathcal{C}_b(S)$ be an operator such that $f \circ \varphi_N \in \mathcal{D}_N$ for all $f \in \mathcal{D}$ and $N \in \mathbb{N}$. Suppose that for all $n \in \mathbb{N}$ and for all continuous and bounded functions $H : \widetilde{S}^n \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \mathrm{E}^N \Big[H(Y_{s_1}^N, \ldots, Y_{s_n}^N) \int_{t_1}^{t_2} \{(Lf) \circ \varphi_N - L_N(f \circ \varphi_N)\}(\varkappa_s^N)\, ds \Big] = 0, \tag{4.2.6}$$

for every $(s_1, \ldots, s_n, t_1, t_2) \in (0, \infty)^{n+2}$ such that $s_1 < s_2 < \cdots < s_n \leq t_1 < t_2$, and all $f \in \mathcal{D}$. Then X is a solution of the $(\mathcal{D}, L)$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$. We call expression (4.2.6) a replacement condition between $(X^N)_N$ *and X.*

Proof. We take $\mathbb{T} \subset \mathbb{R}_+$ for which each projection $\mathbf{p}_t : D(\mathbb{R}_+, S \times \widetilde{S})$, $t \in \mathbb{T}$, is continuous except at points forming a set of P^N-probability zero, for all $N \in \mathbb{N}$. Like it is shown in [9, Section 13], the set $\mathbb{T}$ contains 0, and its complement in $\mathbb{R}_+$ is at most countable.

Denote by $(M_t^N)_t$ the martingale given by

$$M_t^N := f(X_t^N) - \int_0^t L_N(f \circ \varphi_N)(\varkappa_s^N)\, ds. \tag{4.2.7}$$

Let $n \in \mathbb{N}$, fix $(s_1, \ldots, s_n, t_1, t_2) \in [(0, \infty) \cap \mathbb{T}]^{n+2}$ such that $s_1 < s_2 < \cdots < s_n \leq t_1 < t_2$ and take a continuous and bounded function $H : \widetilde{S}^n \to \mathbb{R}$. From the hypothesis

$$0 = \mathrm{E}^N[H(Y_{s_1}^N, \ldots, Y_{s_n}^N)\{M_{t_2}^N - M_{t_1}^N\}]. \tag{4.2.8}$$

Then, calling

$$\mathfrak{M}_t^N := f(X_t^N) - \int_0^t (Lf)(X_s^N)\, ds,$$

and replacing in (4.2.8) we have

$$\begin{aligned} 0 = {} & \mathrm{E}^N[H(Y_{s_1}^N, \ldots, Y_{s_n}^N)\{\mathfrak{M}_{t_2}^N - \mathfrak{M}_{t_1}^N\}] \\ & + \mathrm{E}^N \Big[H(Y_{s_1}^N, \ldots, Y_{s_n}^N) \int_{t_1}^{t_2} \{(Lf) \circ \varphi_N - L_N(f \circ \varphi_N)\}(\varkappa_s^N)\, ds \Big]. \end{aligned}$$

We let $N \to \infty$ in both sides of last equality and use (4.2.6) to obtain

$$\begin{aligned} 0 = & \lim_{N\to\infty} \mathrm{E}^N[H(Y^N_{s_1},\dots,Y^N_{s_n})\{\mathfrak{M}^N_{t_2} - \mathfrak{M}^N_{t_1}\}] \\ = & \lim_{N\to\infty} \mathrm{E}^N\Big[H(Y^N_{s_1},\dots,Y^N_{s_n})\Big\{f(X^N_{t_2}) - f(X^N_{t_1}) - \int_{t_1}^{t_2}(Lf)(X^N_s)\,ds\Big\}\Big] \\ = & \, \boldsymbol{E}\Big[H(Y_{s_1},\dots,Y_{s_n})\Big\{f(X_{t_2}) - f(X_{t_1}) - \int_{t_1}^{t_2}(Lf)(X_s)\,ds\Big\}\Big] \,. \end{aligned}$$

Finally, because n, H and $(s_1,\dots,s_n,t_1,t_2) \in [(0,\infty)\cap\mathbb{T}]^{n+2}$ such that $s_1 < s_2 < \cdots < s_n \leq t_1 < t_2$ were arbitrary, and thanks to the topological assumption we made at the beginning of the subsection, we conclude that X is a solution of the $(\mathcal{D},L)$-martingale problem associated to $(\mathcal{F}_t)_{t\geq 0}$. □

Observe that when $(Y^N)_N = (X^N)_N$ for all $N \in \mathbb{N}$, we only need the convergence in distribution of $(X^N)_N$. In this case the hypothesis of Lemma 4.2.8 implies that X is a solution of the $(\mathcal{D},L)$-martingale problem (associated to the filtration generated by X).

4.2.2 The Jump Function

Let (S,ρ) be a metric space where ρ take its values in $\mathbb{Z}_{\geq 0}$. We define the *jump function* in S, $\mathscr{J} : D(\mathbb{R}_+,S) \to \mathbb{R}$, by

$$\mathscr{J}(w) \;:=\; \sup_{t\geq 0} \rho(w(t),w(t-))\,, \quad \forall w \in D(\mathbb{R}_+,S)\,. \tag{4.2.9}$$

Our objective here is to prove that $\mathscr{J}$ is continuous. As usual, in $D(\mathbb{R}_+,S)$ we consider the Skorokhod topology.

In this subsection $\|\cdot\|$ denotes the uniform norm of bounded functions taking values in $\mathbb{R}$. Fix $m \in \mathbb{N}$. We define Λ_m as the set of all continuous increasing bijections $\lambda : [0,m] \to [0,m]$, and denote by Id_m the identity map of the interval $[0,m]$. For $w \in D(\mathbb{R}_+,S)$ it is also convenient to write

$$\mathscr{J}_s(w) \;:=\; \sup_{t\in[0,s]} \rho(w(t),w(t-))\,, \quad \forall s > 0\,.$$

Remember that the Skorokhod topology can be generated by the metric d defined by

$$d(u,v) \;:=\; \sum_{m=1}^{\infty} \frac{1}{2^m} d^m(g^m u, g^m v)\,,$$

where $g^m : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$g^m(t) \;:=\; \begin{cases} 1 & \text{if} \quad t \leq m-1 \\ m-t & \text{if} \quad m-1 < t \leq m \\ 0 & \text{otherwise} \end{cases}\,, \quad \forall m \in \mathbb{N}\,,$$

and $d^m : D(\mathbb{R}_+,S) \times D(\mathbb{R}_+,S) \to \mathbb{R}$ is defined by

$$d^m(u,v) \;:=\; \inf_{\lambda\in\Lambda_m} \big\{\|\mathrm{Id}_m - \lambda\| \,\vee\, \sup\{\rho\big(u(t), v\circ\lambda(t)\big) : t \in [0,m]\}\big\}\,,$$

for all $m \in \mathbb{N}$.

Lemma 4.2.9. *The jump function $\mathscr{J}$ defined in (4.2.9) is continuous.*

Proof. It is sufficient to prove that the sets $\mathscr{J}^{-1}(\{k\})$ are closed for all $k \in \mathbb{Z}_{\geq 0}$.

Fix $k \in \mathbb{Z}_{\geq 0}$, and take a sequence $(w_N)_{N\in\mathbb{N}} \subset \mathscr{J}^{-1}(\{k\})$ converging to $w \in D(\mathbb{R}_+, S)$, i.e.

$$\lim_{N\to\infty} d(w_N, w) \;=\; 0\,.$$

From the definition of d we deduce

$$\lim_{N\to\infty} d^m(w_N, w) \;=\; 0\,, \quad \forall m \in \mathbb{N}\,.$$

Then we fix $M \geq 2$ such that $\mathscr{J}_{M-2}(w_N) = k$, and take an N sufficiently large to verify $d^M(w_N, w) < 1/4$. This implies that there exists $\lambda_M \in \Lambda_M$ satisfying

$$\|\mathrm{Id}_M - \lambda_M\| < \frac{1}{4} \quad \text{and} \quad \sup_{t\in[0,M-5/4]} \rho\big(w_N(t), w\circ\lambda_M(t)\big) < \frac{1}{4}\,.$$

Then

$$|\,\mathscr{J}_{M-5/4}(w_N) \;-\; \mathscr{J}_{M-5/4}(w\circ\lambda_M)| \;<\; 1/2\,,$$

hence $k = \mathscr{J}_{M-5/4}(w_N) = \mathscr{J}_{M-5/4}(w\circ\lambda_M) = \mathscr{J}_{\lambda_M(M-5/4)}(w)$. Note that

$$M - 3/2 \;<\; \lambda_M(M-5/4) \;<\; M-1\,.$$

Therefore, since M can be arbitrarily large, we deduce that $\mathscr{J}(w) = k$. □

4.3 The first Coalescence

The present section is devoted to the study of the asymptotic behavior of the first coalescence in $(\boldsymbol{\xi}_t^{N,n})$, under $\mathrm{P}^N_{\boldsymbol{m}^n}$, in an appropriate scale of time.

We start by fixing some notation. Consider ξ, an irreducible Markov chain with generator Q over the finite state space E. Fix a natural number $n \geq 2$. We consider the coupling $\boldsymbol{\xi}^{Q,n} : \Omega \to D(\mathbb{R}_+, E)$ of n i.i.d. copies of ξ, described in Subsection 4.1.2, and define the process $\boldsymbol{\zeta}^{Q,n} = (\zeta^{Q,(i,j)})_{(i,j)\in I_{n,2}} : \Omega \to D(\mathbb{R}_+, (E^2)^{I_{n,2}})$ as follows

$$\zeta_t^{Q,(i,j)} \;:=\; (\xi_t^i, \mathbf{1}_{\{\tau_{i,j}^{Q,n}\leq t\}}\xi_t^i + \mathbf{1}_{\{\tau_{i,j}^{Q,n}> t\}}\xi_t^j)\,, \quad \forall t \geq 0\,,\ (i,j) \in I_{k,2}\,.$$

Note that $\boldsymbol{\zeta}^{Q,n}$ is a Markov process with respect to its own sigma-algebra, we call it $(\mathcal{H}_t^{Q,n})_{t\geq 0}$:

$$\mathcal{H}_t^{Q,n} \;:=\; \sigma(\boldsymbol{\zeta}_s^{Q,n}, 0 \leq s \leq t)\,, \quad t \geq 0\,.$$

This implies that all the coordinates are Markov processes with respect to the same sigma-algebra. This observation is crucial in this section. Note also, that each coordinate $\zeta^{Q,(i,j)}$ has essentially the same behavior that $\boldsymbol{\xi}^{Q,2}$. In fact, if the generator Q is given by

$$Qg(x) \;:=\; \sum_{y\in E} r(x,y)[g(y) - g(x)]\,,$$

for all $g : E \to \mathbb{R}$, the generator of each $\zeta^{Q,(i,j)}$ is defined by

$$(\widetilde{L}_Q f)(a,b) := \mathbf{1}_{\{a\neq b\}} \sum_{c\in E} \{r(a,c)[f(c,b) - f(a,b)] + r(b,c)[f(a,c) - f(a,b)]\} + \mathbf{1}_{\{a=b\}} \sum_{c\in E} r(a,c)[f(c,c) - f(a,b)] , \quad (4.3.1)$$

for all $f : E^2 \to \mathbb{R}$. We also define $Y^{Q,n} = (Y^{Q,(i,j)})_{(i,j)\in I_{n,2}} : \Omega \to D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$ as

$$Y_t^{Q,(i,j)} := \mathbf{1}_{\{\tau_{i,j}^{Q,n} \leq t\}} , \quad \forall t \geq 0 , \ (i,j) \in I_{n,2} .$$

This process can be obtained from $\zeta^{Q,n}$ by composing each of its coordinates with the function $\widetilde{\Psi}_Q : E^2 \to \{0,1\}$ defined by $\widetilde{\Psi}_Q(a,b) = \mathbf{1}_{\{a=b\}}$, i.e. $Y_t^{Q,(i,j)} = \widetilde{\Psi}_Q(\zeta_t^{Q,(i,j)})$ for all $t \geq 0$, for all $(i,j) \in I_{n,2}$. The metric we consider in $\{0,1\}^{I_{n,2}}$ is d_n defined by

$$d_n(x,y) := \sum_{(i,j)\in I_{n,2}} |x^{(i,j)} - y^{(i,j)}| ,$$

for all $x = (x^{(i,j)})_{(i,j)\in I_{n,2}}, y = (y^{(i,j)})_{(i,j)\in I_{n,2}} \in \{0,1\}^{I_{n,2}}$. This is the metric that induces the Skorokhod topology we take into account in $D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$.

On the other hand, we denote by $\mathfrak{Y}$ the process with trajectories in $\{0,1\}$ that jumps from zero to one in an exponential time of mean one, and stay in one forever. More specifically, the generator $\widetilde{\mathscr{L}}$ of $\mathfrak{Y}$ is defined by

$$(\widetilde{\mathscr{L}} f)(a) := \mathbf{1}_{\{a=0\}}[f(1) - f(0)] ,$$

for all $f : \{0,1\} \to \mathbb{R}$. Given $n \in \mathbb{N}$ we call $\mathfrak{Y}^{(n)}$ the process with trajectories in $\{0,1\}^{I_{n,2}}$ defined as the coupling of $\binom{n}{2}$ independent copies of $\mathfrak{Y}$.

Now, for the rest of the section we fix $(\xi_N)_{N\in\mathbb{N}}$, a sequence of irreducible Markov chains over finite state spaces, and a natural number $n \geq 2$. For a time scale $\boldsymbol{\theta} = (\theta_N)_{N\in\mathbb{N}} \subset \mathbb{R}_+$ we define $\mathbb{Y}^{\boldsymbol{\theta},N,n}$ by

$$\mathbb{Y}_t^{\boldsymbol{\theta},N,n} = Y_{t\theta_N}^{N,n} , \quad t \geq 0 , \quad N \in \mathbb{N} .$$

When $\boldsymbol{\theta}$ is like in Definition 4.1.1, we write simply $\mathbb{Y}^{N,n}$, instead of $\mathbb{Y}^{\boldsymbol{\theta},N,n}$. The main result of this section reads as follows.

Proposition 4.3.1. *Fix a natural number $n \geq 2$. Suppose that $(\xi_N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with two particles. Then, under $\mathrm{P}_{\boldsymbol{m}^n}^N$, the sequence $(\mathbb{Y}^{N,n})_{N\in\mathbb{N}}$ converges in distribution to $\mathfrak{Y}^{(n)}$ starting in $(1,\ldots,1) \in \{0,1\}^{I_{n,2}}$.*

As a consequence we have the following corollary

Corollary 4.3.2. *Fix $n \geq 2$. Under the hypotheses of Proposition 4.3.1, the sequence of scaled times $(\widetilde{T}_{n-1}^{N,n}/\theta_N)_{N\in\mathbb{N}}$ converges in distribution to an exponential random variable of mean $\binom{n}{2}^{-1}$. Moreover*

$$\lim_{N\to\infty} \mathrm{E}_{\boldsymbol{m}^n}^N \left[\frac{\widetilde{T}_{n-1}^{N,n}}{\theta_N} \right] = \binom{n}{2}^{-1} . \quad (4.3.2)$$

Proof. Since $(\mathbb{Y}^{N,n})_N$ converges in distribution to $\mathfrak{Y}^{(n)}$, the vector of stopping times $(\tau_{i,j}^N/\theta_N, (i,j) \in I_{n,2})$ converges in distribution, when $N \to \infty$, to the random vector

$(\boldsymbol{e}_{i,j}, (i,j) \in I_{n,2})$, where all the variables $\boldsymbol{e}_{i,j}, (i,j) \in I_{n,2}$ are i.i.d. with exponential distribution of mean one. This implies that $(\widetilde{T}^{N,n}_{n-1}/\theta_N)_{N\in\mathbb{N}}$ converges in distribution, when $N \to \infty$, to

$$\exp\binom{n}{2} \;:=\; \min\{\boldsymbol{e}_{i,j} : (i,j) \in I_{n,2}\}\,,$$

which is an exponential random variable of mean $\binom{n}{2}^{-1}$.

Finally, Hypothesis **(H3)** implies that $(\widetilde{T}^{N,n}_{n-1}/\theta_N)_{N\in\mathbb{N}}$ is uniformly integrable, therefore (4.3.2) holds. □

Now we start with the proof of Proposition 4.3.1. This proof is divided several steps across the following subsections.

4.3.1 Replacement Condition

Fix $(i,j) \in I_{n,2}$. In this subsection we proof the following result

Lemma 4.3.3. *Suppose that the sequence $(\xi^N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with two particles. Fix an integer $k \geq 1$, some $(s_1,\dots,s_k,t_1,t_2) \in (0,\infty)^{k+2}$ such that $s_1 < s_2 < \cdots < s_k \leq t_1 < t_2$ and a continuous function $K : \big(\{0,1\}^{I_{n,2}}\big)^k \to \mathbb{R}$. Then*

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Bigg[K(Y^N_{s_1\theta_N},\dots,Y^N_{s_k\theta_N})\int_{t_1\theta_N}^{t_2\theta_N}\Bigg\{\frac{(\widetilde{\mathscr{L}}f)\circ\widetilde{\Psi}_N}{\theta_N} - \widetilde{L}_N(f\circ\widetilde{\Psi}_N)\Bigg\}(\zeta^{N,(i,j)}_s)\,ds\Bigg] \;=\; 0\,,$$

for any $f : \{0,1\} \to \mathbb{R}$.

To keep notation simple we denote $K^N := K(Y^N_{s_1\theta_N},\dots,Y^N_{s_k\theta_N})$.

Localizing to a Well

As our first step here we show that Lemma 4.3.3 follows from the following result which we check in the subsequent parts of this subsection.

Lemma 4.3.4. *Suppose that the sequence $(\xi^N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with two particles. Then for all $t \geq s_k$ we have*

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[K^N\mathbf{1}_{\{t\theta_N<\tau_{i,j}\}}\Big\{\frac{\tau_{i,j}-t\theta_N}{\theta_N} - 1\Big\}\Big] \;=\; 0\,. \tag{4.3.3}$$

We define the following sets

$$\mathscr{E}^1_N \;:=\; \{(a,a) \in E^2_N : a \in E_N\} \quad\text{and}\quad \mathscr{E}^0_N \;:=\; E^2_N \setminus \mathscr{E}^1_N\,,$$

for all $N \in \mathbb{N}$. Given $f : \{0,1\} \to \mathbb{R}$, a direct computation shows that

$$\widetilde{L}_N(f\circ\widetilde{\Psi}_N) \;\equiv\; \mathbf{1}_{\mathscr{E}^0_N}\big\{[f(1)-f(0)]\widetilde{L}_N\mathbf{1}_{\mathscr{E}^1_N}\big\}\,.$$

Then, to prove Lemma 4.3.3 it suffices to verify that

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[K(Y^N_{s_1\theta_N},\dots,Y^N_{s_k\theta_N})\int_{t_1\theta_N}^{t_2\theta_N}\mathbf{1}_{\mathscr{E}^0_N}(\zeta^{N,(i,j)}_s)R_N(\zeta^{N,(i,j)}_s)\,ds\Big] \;=\; 0\,, \tag{4.3.4}$$

where

$$R_N(a,b) \;:=\; \frac{1}{\theta_N} - \big(\widetilde{L}_N \mathbf{1}_{\mathscr{E}_N^1}\big)(a,b)\;, \quad (a,b)\in\mathscr{E}_N^0\;.$$

Thanks to the indicator $\mathbf{1}_{\mathscr{E}_N^0}(\cdot)$ we may rewrite the integral in (4.3.4) as

$$\int_{t_1\theta_N}^{t_2\theta_N} \mathbf{1}_{\mathscr{E}_N^0}(\zeta_s^{N,(i,j)}) R_N(\boldsymbol{\eta}_s^N)\,ds \;=\; \mathbf{1}_{\{t_1\theta_N<\tau_{i,j}\}}\int_{t_1\theta_N}^{\tau_{i,j}} R_N(\zeta_s^{N,(i,j)})\,ds - \mathbf{1}_{\{t_2\theta_N<\tau_{i,j}\}}\int_{t_2\theta_N}^{\tau_{i,j}} R_N(\zeta_s^{N,(i,j)})\,ds\;. \tag{4.3.5}$$

Therefore, (4.3.4) follows from the limit

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[K(Y^N_{s_1\theta_N},\dots,Y^N_{s_k\theta_N})\mathbf{1}_{\{t\theta_N<\tau_{i,j}\}}\int_{t\theta_N}^{\tau_{i,j}} R_N(\zeta_s^{N,(i,j)})\,ds\Big] \;=\; 0\;, \tag{4.3.6}$$

for any $t\ge s_k$. Recall that, for each $N\ge 1$,

$$\mathfrak{M}_t^N \;:=\; \mathbf{1}_{\mathscr{E}_N^1}(\zeta_t^{N,(i,j)}) - \int_0^t \big(\widetilde{L}_N\mathbf{1}_{\mathscr{E}_N^1}\big)(\zeta_s^{N,(i,j)})\,ds\;, \quad t\ge 0\;,$$

is a martingale with respect to the filtration $(\mathcal{H}_t^N)$. To deal with (4.3.6), notice that

$$\int_{t\theta_N}^{\tau_{i,j}} R_N(\zeta_s^{N,(i,j)})\,ds \;=\; \frac{\tau_{i,j}-t\theta_N}{\theta_N} - 1 + \big(\mathfrak{M}^N_{\tau_{i,j}} - \mathfrak{M}^N_{t\theta_N}\big)\;.$$

Since $K^N\mathbf{1}_{\{t\theta_N<\tau_{i,j}\}}$ is $\mathcal{H}^N_{t\theta_N}$-measurable, expectation in (4.3.6) coincides with

$$\mathrm{E}^N_{\boldsymbol{m}^n}\Big[K^N\mathbf{1}_{\{t\theta_N<\tau_{i,j}\}}\Big\{\frac{\tau_{i,j}-t\theta_N}{\theta_N} - 1\Big\}\Big]\;. \tag{4.3.7}$$

Time Average

In order to prove Lemma 4.3.4 we shall apply the Markov property to expectation in (4.3.7). But before that, we need to write (4.3.7) as a time average. For it, to keep notation simple it is convenient to write $W := \{t\theta_N < \tau_{i,j}\}$.

Take a scale of time $(\alpha_N)_{N\in\mathbb{N}}\subset\mathbb{R}_+$ such that

$$\lim_{N\to\infty}\frac{\alpha_N}{\theta_N} \;=\; 0\;. \tag{4.3.8}$$

The difference between expression in (4.3.7) and

$$\frac{1}{\alpha_N}\int_0^{\alpha_N}\mathrm{E}^N_{\boldsymbol{m}^n}\Big[K^N\mathbf{1}_{W\cap\{t\theta_n+\ell<\tau_{i,j}\}}\Big\{\frac{\tau_{i,j}-t\theta_N}{\theta_N} - 1\Big\}\Big]\,d\ell \tag{4.3.9}$$

is bounded above by

$$\Big\{\frac{\alpha_N}{\theta_N}+1\Big\}\|K\|\;\mathrm{P}^N_{\boldsymbol{m}^n}[\tau_{i,j}\le\alpha_N]\;, \tag{4.3.10}$$

where $\|K\| = \max_{x\in(\{0,1\}^{I_{n,2}})^k}|K(x)|$. At this point we assume that $(\zeta^N)_N$ satisfies Hypothesis **(H1)**. Then, thanks to (4.3.8) we get that (4.3.10) vanishes as $N\to\infty$. In turn, the difference between expression in (4.3.9) and

$$\frac{1}{\alpha_N}\int_0^{\alpha_N}\mathrm{E}^N_{\boldsymbol{m}^n}\Big[K^N\mathbf{1}_{W\cap\{\ell<\tau_{i,j}\}}\Big\{\frac{\tau_{i,j}-(t\theta_n+\ell)}{\theta_N} - 1\Big\}\Big]\,d\ell \tag{4.3.11}$$

is bounded by $\|K\|\alpha_N/\theta_N$ which vanishes as $N \to \infty$. So, in order to get (4.3.3) it suffices to show that (4.3.11) vanishes as $N \to \infty$.

Markov Property

We define $f^N : E_N^2 \to \mathbb{R}$ as

$$f^N(x) := \mathrm{E}_x^N\left[\frac{T_1}{\theta_N} - 1\right], \quad x \in E_N^2 .$$

Then, applying the Markov property, expectation in (4.3.11) can be written as

$$\mathrm{E}_{\boldsymbol{m}^n}^N\left[K^N \mathbf{1}_{W\cap\{t\theta_N+\ell<\tau_{i,j}\}} f^N(\zeta_{t\theta_N+\ell}^{N,(i,j)})\right] . \tag{4.3.12}$$

Notice that on the event $\{t\theta_N + \ell < \tau_{i,j}\}$ we have $\zeta_{t\theta_N+\ell}^{N,(i,j)} = (\xi_{t\theta_N+\ell}^i, \xi_{t\theta_N+\ell}^j)$. Then (4.3.12) equals to

$$\mathrm{E}_{\boldsymbol{m}^n}^N\left[K^N \mathbf{1}_{W\cap\{t\theta_N+\ell<\tau_{i,j}\}} f^N(\xi_{t\theta_N+\ell}^i, \xi_{t\theta_N+\ell}^j)\right] .$$

Now, in addition to Hypothesis **(H1)**, we assume that $(\xi^N)_{N\in\mathbb{N}}$ is such that Hypothesis **(H3)** also holds for some $\varepsilon > 0$. In order to get rid of the indicator of the event $\{t\theta_N + \ell < \tau_{i,j}\}$ we observe that

$$\left|\mathrm{E}_{\boldsymbol{m}^n}^N\left[K^N \mathbf{1}_{W\cap\{\tau_{i,j}\le t\theta_N+\ell\}} f^N(\xi_{t\theta_N+\ell}^i, \xi_{t\theta_N+\ell}^j)\right]\right| \tag{4.3.13}$$

is bounded above by

$$\|K\|\left\{\mathrm{P}_{\boldsymbol{m}^n}^N[t\theta_N < \tau_{i,j} \le t\theta_N + \ell]^{\frac{\varepsilon}{1+\varepsilon}} \mathrm{E}_{\boldsymbol{m}^2}^N\left[\left(\frac{T_1}{\theta_N}\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} \right.$$
$$\left. + \ \mathrm{P}_{\boldsymbol{m}^n}^N[t\theta_N < \tau_{i,j} \le t\theta_N + \ell]\right\}, \tag{4.3.14}$$

From the Markov property

$$\mathrm{P}_{\boldsymbol{m}^n}^N[t\theta_N < \tau_{i,j} \le t\theta_N + \ell] \ \le \ \mathrm{P}_{\boldsymbol{m}^2}^N[T_1 \le \ell] \ \le \ \mathrm{P}_{\boldsymbol{m}^2}^N[T_1 \le \alpha_N] .$$

Hence, using last inequality in (4.3.14), we deduce that (4.3.13) is bounded by

$$\|K\|\mathrm{P}_{\boldsymbol{m}^2}^N[T_1 \le \alpha_N]^{\frac{\varepsilon}{1+\varepsilon}} \left(\mathrm{E}_{\boldsymbol{m}^2}^N\left[\left(\frac{T_1}{\theta_N}\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} + 1\right) .$$

Last expression vanishes when $N \to \infty$ thanks to **(H1)** and **(H3)**. This proves that (4.3.12) equals

$$\mathrm{E}_{\boldsymbol{m}^n}^N\left[K^N \mathbf{1}_W f^N(\xi_{t\theta_N+\ell}^i, \xi_{t\theta_N+\ell}^j)\right] \ + \ o_N .$$

This means that, replacing in (4.3.11), (4.3.3) follows from

$$\lim_{N\to\infty} \frac{1}{\alpha_N} \int_0^{\alpha_N} \mathrm{E}_{\boldsymbol{m}^n}^N\left[K^N \mathbf{1}_W f^N(\xi_{t\theta_N+\ell}^i, \xi_{t\theta_N+\ell}^j)\right] d\ell \ = \ 0 . \tag{4.3.15}$$

Since $K^N \mathbf{1}_W \in \mathcal{H}_{t\theta_N}^N$, applying the Markov property in (4.3.15) we deduce that (4.3.3) is consequence of

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[K^N \mathbf{1}_W \mathrm{E}^N_{(\xi^i_{t\theta_N+\ell},\xi^j_{t\theta_N+\ell})}\left[\frac{1}{\alpha_N}\int_0^{\alpha_N} f^N(\xi^i_\ell,\xi^j_\ell)\,ds\right]\right] = 0\,. \tag{4.3.16}$$

Proof of Lemma 4.3.4

Thanks to what we have done in the previous parts of this subsection, to prove Lemma 4.3.4, it only remains to show (4.3.16).

We have

$$\left|\mathrm{E}^N_{\boldsymbol{m}^n}\left[K^N \mathbf{1}_W \mathrm{E}^N_{(\xi^i_{t\theta_N+\ell},\xi^j_{t\theta_N+\ell})}\left[\frac{1}{\alpha_N}\int_0^{\alpha_N} f^N(\xi^i_\ell,\xi^j_\ell)\right]ds\right]\right| \\ \le \|K\|\mathrm{E}^N_{\boldsymbol{m}^2}\left[\left|\frac{1}{\alpha_N}\int_0^{\alpha_N} f^N(\xi^i_\ell,\xi^j_\ell)\,ds\right|\right]. \tag{4.3.17}$$

Finally we assume that $(\xi^N)_N$ satisfies Hypothesis **(H2)** for the time scale $(\alpha_N)_{N\in\mathbb{N}}$. Thanks to this hypothesis, it only remains to prove that $\langle f^N\rangle_{\boldsymbol{m}^2} = 0$ for all $N \in \mathbb{N}$; but this is a direct consequence of the definition of f^N; remember that $\theta_N = \mathrm{E}^N_{\boldsymbol{m}^2}[T_1]$ for all $N \in \mathbb{N}$.

4.3.2 Proof of Proposition 4.3.1

Here we prove Porposition 4.3.1. The terminology we use is precised in Subsection 4.2.1, in particular we use the notions of a martingale problem and maringale problem solution established in Definition 4.2.1. First we notice that taking a convenient function K in Lemma 4.3.3, or performing the same computations of its proof with a function $R : I_2^k \to \mathbb{R}$ in place of K, we obtain the following replacement condition.

Corollary 4.3.5. *Suppose that the sequence $(\xi^N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with two particles. Fix an integer $k \ge 1$, some $(s_1,\ldots,s_k,t_1,t_2) \in (0,\infty)^{k+2}$ such that $s_1 < s_2 < \cdots < s_k \le t_1 < t_2$ and a continuous function $R : I_2^k \to \mathbb{R}$. Then*

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[R(Y^{N,(i,j)}_{s_1\theta_N},\ldots,Y^{N,(i,j)}_{s_k\theta_N})\int_{t_1\theta_N}^{t_2\theta_N}\left\{\frac{(\widetilde{\mathscr{L}}f)\circ\widetilde{\Psi}_N}{\theta_N} \\ - \widetilde{L}_N(f\circ\widetilde{\Psi}_N)\right\}(\zeta^{N,(i,j)}_s)\,ds\right] = 0\,,$$

for any $f : \{0,1\} \to \mathbb{R}$, and all $(i,j) \in I_{n,2}$.

Let $(i,j) \in I_{n,2}$ and take $Y^{*,(i,j)}$ a limit process of $(\mathbb{Y}^{N,(i,j)})_N$. Thanks to Corollary 4.3.5 we can use Lemma 4.2.8 to deduce that $Z^{*,(i,j)}$ is a solution of the $(\mathcal{M}_b(\{0,1\}),\widetilde{\mathscr{L}})$-martingale problem starting at 0. On the other hand, the tightness of $(\mathbb{Y}^{N,(i,j)})_N$ follows from the tightness of $(\mathbb{Y}^{N,n})_N$. Therefore, since the $(\mathcal{M}_b(\{0,1\}),\widetilde{\mathscr{L}})$-martingale problem starting at 0 has a unique solution, we conclude that $(\mathbb{Y}^{N,(i,j)})_N$ converges to some process

$$Y^{(i,j)} : (\Omega_{(i,j)},\mathrm{P}^{(i,j)}) \to D(\mathbb{R}_+,\{0,1\})\,.$$

This defines the probability $\mathrm{P}^{(i,j)} \circ (Z^{(i,j)})^{-1}$ in $D(\mathbb{R}_+,\{0,1\})$. Now, we call $\mathbf{p}^{(i,j)}$:

$D(\mathbb{R}_+, (\{0,1\})^{I_{n,2}}) \to D(\mathbb{R}_+, \{0,1\})$ the projection defined by

$$\mathbf{p}^{(i,j)}\big((w^{(\ell,m)})_{(\ell,m)\in I_{n,2}}\big) \;:=\; w^{(i,j)} \;,$$

and consider in $D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$ the sigma-algebra induced by $\mathbf{p}^{(i,j)}$. Then, we define the probability $\mathcal{P}^{(i,j)}$ in $\big(D(\mathbb{R}_+, \{0,1\}^{I_{n,2}}\big)$ by

$$\mathcal{P}^{(i,j)}\big(\{\mathbf{p}^{(i,j)}\}^{-1}(\mathcal{B})\big) \;:=\; \mathrm{P}^{(i,j)} \circ (Y^{(i,j)})^{-1}(\mathcal{B}) \;,$$

for all Borel set $\mathcal{B} \subset D(\mathbb{R}_+, \{0,1\})$. This probability is the unique probability in $D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$ that makes $\mathbf{p}^{(i,j)}$ a solution of the $(\mathcal{M}_b(\{0,1\}), \widetilde{\mathscr{L}})$-martingale problem starting at 0, for all $(i,j) \in I_{n,2}$. We denote by $(\mathscr{F}_t^{(i,j)})_{t\geq 0}$, $(i,j) \in I_{n,2}$, the filtration defined by $\mathbf{p}^{(i,j)}$:

$$\mathscr{F}_t^{(i,j)} \;:=\; \sigma(\mathbf{p}_s^{(i,j)}, 0 \leq s \leq t) \;, \quad \forall t \geq 0 \,.$$

And we call $(\mathscr{F}_t)_{t\geq 0}$ the filtration generated by all the $(\mathscr{F}_t^{(i,j)})_{t\geq 0}$, $(i,j) \in I_{n,2}$:

$$\mathscr{F}_t \;:=\; \sigma\big(\mathscr{F}_t^{(i,j)}, (i,j) \in I_{n,2}\big) \;, \quad \forall t \geq 0 \,.$$

With this notation we are ready to start the proof.

Proof of Proposition 4.3.1. We prove in Section 4.5 that $(\mathbb{Y}^N)_N$ is tight under $\mathrm{P}^N_{\boldsymbol{m}^n}$. Take $Y^* : (\Omega_*, \mathrm{P}^*) \to D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$ a limit process of $(\mathbb{Y}^N)_N$. To conclude we have to prove that the induced probability in $D(\mathbb{R}_+, \{0,1\}^{I_{k,2}})$, that we denote by $\mathcal{P}^* := \mathrm{P}^* \circ (Y^*)^{-1}$, is uniquely determined.

Assume that $\mathbb{N}^* \subset \mathbb{N}$ is the infinite subset through which $(\mathbb{Y}^N)_N$ converges in distribution to Y^*. Thanks to Lemma 4.2.9, the set

$$U \;:=\; \{w \in D([0,+\infty), \{0,1\}^{I_{n,2}}) : \mathscr{J}(w) > 1\}$$

is open, hence

$$0 \;=\; \liminf_{N\in\mathbb{N}^*} \mathrm{P}^N_{\boldsymbol{m}^n}[\mathbb{Y}^N \in U] \;\geq\; \mathcal{P}^*[U] \,.$$

This means that with $\mathcal{P}^*$-probability one, in $D(\mathbb{R}_+, \{0,1\}^{I_{n,2}})$ there are not two different projections $\mathbf{p}^{(i,j)}$ and $\mathbf{p}^{(\ell,m)}$ that jump at the same time.

On the other hand, Lemma 4.3.3 permit us to use Lemma 4.2.8 and deduce that

$$\mathbf{p}^{(i,j)} : (D(\mathbb{R}_+, \{0,1\}^{I_{k,2}}), \mathcal{P}^*) \to D(\mathbb{R}_+, \{0,1\})$$

is a solution of the $(\mathcal{M}_b(\{0,1\}), \widetilde{\mathscr{L}})$-martingale problem starting at 0, associated to the filtration $(\mathscr{F}_t)_{t\geq 0}$, for all $(i,j) \in I_{n,2}$. Then, in virtue of Corollary 4.2.7, we deduce that under $\mathcal{P}^*$ all the projections $\mathbf{p}^{(i,j)}$, $(i,j) \in I_{n,2}$, are independent. This concludes the proof because we also have that each restriction $\mathcal{P}^*|_{\sigma(\mathbf{p}^{(i,j)})}$ coincides with $\mathcal{P}^{(i,j)}$, $(i,j) \in I_{n,2}$. Observe that these conditions determine uniquely the probability $\mathcal{P}^*$. □

4.4 Convergence to the Kingman's Coalescent

The goal of this section is to prove Theorem 4.1.3, our main result. We start with some observations about the processes defined in Subestion 4.1.2.

Fix $n \in \mathbb{N}$ and $\xi : \Omega \to D(\mathbb{R}_+, E)$, an irreducible Markov chain with generator Q given by

$$Qg(x) := \sum_{y \in E} r(x,y)[g(y) - g(x)] ,$$

for all functions $g : E \to \mathbb{R}$. Then, the generator L^n_Q of $\boldsymbol{\eta}^{Q,n}$ is defined by

$$(L^n_G f)(x) := \mathbf{1}_{\#(\Psi_{E,n}(x) \geq 2)} \sum_{(i,j) \in I_{n,2}} \{ r(x^i, x^j)[f(x_{i,j}) - f(x)] + r(x^j, x^i)[f(x_{j,i}) - f(x)] \} + \sum_{y \approx x} r(x^{i(x,y)}, y^{i(x,y)})[f(y) - f(x)] , \quad (4.4.1)$$

for all $f : E^n \to \mathbb{R}$. Where, in equation (4.4.1), $x = (x^1, \dots, x^n)$, $y = (y^1, \dots, y^n)$. Also, for $i, j \in [n]$ with $i \neq j$, $x_{i,j} = (a^1, \dots, a^n) \in E^n$ denotes the vector defined by

$$a^k = x^k , \quad \text{for all } k \notin \{i, j\} \quad \text{and } a^i = a^j = x^j .$$

Additionally $y \approx x$ means that $\Psi_{Q,n}(x) = \Psi_{Q,n}(y)$, and there is an index $i(x,y) \in [n]$ such that $x^j = y^j$ for all $\bar{j} \neq \overline{i(x,y)}$ in $\Psi_{Q,n}(x)$ and $x^{i(x,y)} \sim y^{i(x,y)}$. Notice that $\boldsymbol{\eta}^{Q,n}$ is Markov with respect to its own filtration $(\mathcal{G}^{Q,n}_t)_{t \geq 0}$ defined by

$$\mathcal{G}^{Q,n}_t := \sigma(\boldsymbol{\eta}^{Q,n}_s, \ 0 \leq s \leq t) , \quad t \geq 0 ;$$

whereas $\boldsymbol{\xi}^{Q,n}$ is Markov with respect to the filtration $(\mathcal{F}^{Q,n}_t)_{t \geq 0}$ defined by

$$\mathcal{F}^{Q,n}_t := \sigma(\boldsymbol{\xi}^{Q,n}_s, \ 0 \leq s \leq t) , \quad t \geq 0 ;$$

On the other hand, in respect to the process $\boldsymbol{\xi}^{Q,n}$, it is important to take into account the following observation

Remark 4.4.1. *Suppose that $n \geq 2$. Let $m \in [n]$ and fix the indexes*

$$1 \leq i^1 < \cdots < i^m \leq n .$$

Observe that, when ξ is transitive, under $\mathrm{P}^Q_{\boldsymbol{m}^n}$ the random vector

$$(\xi^{i^1}_{\tau^{Q,n}_{i,j}}, \dots, \xi^{i^m}_{\tau^{Q,n}_{i,j}}) \quad \textit{has the distribution of} \quad \boldsymbol{m}^m ,$$

for all $1 \leq i < j \leq n$ such that $\{i, j\} \not\subset \{i^1, \dots, i^m\}$.

Now we start with the proof of Theorem 4.1.3. To this end we fix $n \in \mathbb{N}$ and $(\xi^N : \Omega_N \to D(\mathbb{R}_+, E_N)$, a sequence of irreducible Markov chains over finite state spaces, with Q_N, $N \in \mathbb{N}$, their respective generators. Our strategy is very similar to the one performed in Section 4.3, to prove Proposition 4.3.1. The main differences are due to the more complicated nature of the process X^N in comparison with the coordinates of Y^N. Like in the case of Proposition 4.3.1, the proof of Theorem 4.1.3 is divided in several steps across the following subsections.

4.4.1 Replacement Condition

In this subsection we proof the following replacement condition

Lemma 4.4.2. *Suppose that the sequence $(\xi^N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with n particles. Fix an integer $k \geq 1$, some $(s_1, \ldots, s_k, t_1, t_2) \in (0,\infty)^{k+2}$ such that $s_1 < s_2 < \cdots < s_k \leq t_1 < t_2$ and a continuous function $H : \mathcal{P}_n^k \to \mathbb{R}$. Then*

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[H(X^N_{s_1\theta_N}, \ldots, X^N_{s_k\theta_N}) \int_{t_1\theta_N}^{t_2\theta_N} \left\{\frac{(\mathscr{L}^n f)\circ\Psi_N}{\theta_N} - L_N^n(f\circ\Psi_N)\right\}(\boldsymbol{\eta}^N_s)\, ds\right] \;=\; 0\,,$$

for any $f : \mathcal{P}_n \to \mathbb{R}$.

To keep notation simple we denote $H^N := H(X^N_{s_1\theta_N}, \ldots, X^N_{s_k\theta_N})$.

Localizing to a Well

As our first step here we show that Lemma 4.4.2 follows from the following result which we check in the subsequent parts of this subsection. Remember the notation established in Section 4.1.

Lemma 4.4.3. *Suppose that the sequence $(\xi^N)_{N\in\mathbb{N}}$ is transitive and fulfills the coalescing hypotheses starting with n particles. Then, for any partition $\pi \in \mathcal{P}_n$ with $\#(\pi) \geq 2$, and $t > s_k$ we have*

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_{\{t_1\theta_N < T_\pi \leq t_2\theta_N\}}\left\{\frac{T_{\#(\pi)-1} - T_\pi}{\theta_N} - \mathbf{1}_{\{T_{\#(\pi)-1} = T_{(i,j)[\pi]}\}}\right\}\right] \;=\; 0\,, \tag{4.4.2}$$

and

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_{\{T_\pi \leq t\theta_N < T_{\#(\pi)-1}\}}\left\{\frac{T_{\#(\pi)-1} - t\theta_N}{\theta_N} - \mathbf{1}_{\{T_{\#(\pi)-1} = T_{(i,j)[\pi]}\}}\right\}\right] \;=\; 0\,, \tag{4.4.3}$$

for all $(i,j) \in I_{\#(\pi),2}$.

Given $f : \mathcal{P}_n \to \mathbb{R}$, for $\pi \in \mathcal{P}_n$ with $\#(\pi) \geq 2$ we have

$$L_N(f\circ\Psi_N) \;\equiv\; \sum_{(i,j)\in I_{\#(\pi),2}} [f((i,j)[\pi]) - f(\pi)]\big(L_N \mathbf{1}_{\mathscr{E}_N^{(i,j)[\pi]}}\big) \quad \text{on } \mathscr{E}_N^\pi\,.$$

In virtue of this observation, to prove Lemma 4.4.2 it suffices to fix an arbitrary $\pi \in \mathcal{P}_n$ with $\#(\pi) \geq 2$ and verify that

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[H(X^N_{s_1\theta_N}, \ldots, X^N_{s_k\theta_N}) \int_{t_1\theta_N}^{t_2\theta_N} \mathbf{1}_{\mathscr{E}_N^\pi}(\boldsymbol{\eta}^N_s) R_N^{(i,j)}(\boldsymbol{\eta}^N_s)\, ds\right] \;=\; 0\,, \tag{4.4.4}$$

for all $(i,j) \in I_{\#(\pi),2}$, where

$$R_N^{(i,j)}(x) \;:=\; \frac{1}{\theta_N} - \big(L_N \mathbf{1}_{\mathscr{E}_N^{(i,j)[\pi]}}\big)(x)\,, \quad x \in \mathscr{E}_N^\pi\,.$$

Thanks to the indicator $\mathbf{1}_{\mathscr{E}_N^\pi}(\cdot)$ we may rewrite the integral in (4.4.4) as the sum of three terms as follows

$$\int_{t_1\theta_N}^{t_2\theta_N} \mathbf{1}_{\mathscr{E}_N^\pi}(\boldsymbol{\eta}_s^N) R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds \;=\; \mathbf{1}_{\{t_1\theta_N < T_\pi \le t_2\theta_N\}} \int_{T_\pi}^{T_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds$$
$$+\; \mathbf{1}_{\{T_\pi \le t_1\theta_N < T_{\#(\pi)-1}\}} \int_{t_1\theta_N}^{T_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds$$
$$-\; \mathbf{1}_{\{T_\pi \le t_2\theta_N < T_{\#(\pi)-1}\}} \int_{t_2\theta_N}^{T_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds\,. \tag{4.4.5}$$

Therefore, (4.4.4) follows from the following two limits

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[H(X^N_{s_1\theta_N},\dots,X^N_{s_k\theta_N})\mathbf{1}_{\{t_1\theta_N < T_\pi \le t_2\theta_N\}} \int_{T_\pi}^{T_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds\Big] \;=\; 0\,, \tag{4.4.6}$$

and, for any $t > s_k$,

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[H(X^N_{s_1\theta_N},\dots,X^N_{s_k\theta_N})\mathbf{1}_{\{T_\pi \le t\theta_N < T_{\#(\pi)-1}\}} \int_{t\theta_N}^{T_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds\Big] \;=\; 0\,. \tag{4.4.7}$$

Recall that, for each $N \ge 1$,

$$\mathfrak{M}_t^{N,(i,j)} \;:=\; \mathbf{1}_{\mathscr{E}_N^{(i,j)[\pi]}}(\boldsymbol{\eta}_t^N) - \int_0^t \big(L_N \mathbf{1}_{\mathscr{E}_N^{(i,j)[\pi]}}\big)(\boldsymbol{\eta}_s^N)\, ds\,, \quad t \ge 0\,,$$

is a martingale with respect to the filtration $(\mathcal{G}_t^N)$. To deal with (4.4.6), notice that

$$\int_{\widetilde{T}_{\#(\pi)}}^{\widetilde{T}_{\#(\pi)-1}} R_N^{(i,j)}(\boldsymbol{\eta}_s^N)\, ds \;=\; \frac{\widetilde{T}_{\#(\pi)-1} - \widetilde{T}_{\#(\pi)}}{\theta_N}$$
$$-\; \mathbf{1}_{\{\widetilde{T}_{\#(\pi)-1} = T_{(i,j)[\pi]}\}} \;+\; \big(\mathfrak{M}^{N,(i,j)}_{\widetilde{T}_{\#(\pi)-1}} - \mathfrak{M}^{N,(i,j)}_{\widetilde{T}_{\#(\pi)}}\big)\,.$$

Since $H^N \mathbf{1}_{\{t_1\theta_N < T_\pi \le t_2\theta_N\}}$ is $\mathcal{G}^N_{\widetilde{T}_{\#(\pi)}}$-measurable, expectation in (4.4.6) coincides with

$$\mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N \mathbf{1}_{\{t_1\theta_N < T_\pi \le t_2\theta_N\}}\Big\{\frac{T_{\#(\pi)-1} - T_\pi}{\theta_N} \;-\; \mathbf{1}_{\{T_{\#(\pi)-1} = T_{(i,j)[\pi]}\}}\Big\}\Big]\,. \tag{4.4.8}$$

Applying the same argument we get that expectation in (4.4.7) equals to

$$\mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N \mathbf{1}_{\{T_\pi \le t\theta_N < T_{\#(\pi)-1}\}}\Big\{\frac{T_{\#(\pi)-1} - t\theta_N}{\theta_N} \;-\; \mathbf{1}_{\{T_{\#(\pi)-1} = T_{(i,j)[\pi]}\}}\Big\}\Big]\,. \tag{4.4.9}$$

Time Average

In order to prove Lemma 4.4.3 we shall apply Markov property to expectations in (4.4.8) and (4.4.9). But before that, we need to write (4.4.8) and (4.4.9) as time averages. For it, to keep notation simple it is convenient to write

$$U \;:=\; \{t_1\theta_N < T_\pi \le t_2\theta_N\} \quad \text{and} \quad V \;:=\; \{T_\pi \le t\theta_N < T_{\#(\pi)-1}\}$$

for the events involved in such expressions. Take a scale of time $(\alpha_N)_{N\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\lim_{N\to\infty} \frac{\alpha_N}{\theta_N} \;=\; 0\,. \tag{4.4.10}$$

The difference between expression in (4.4.8) and

$$\frac{1}{\alpha_N}\int_0^{\alpha_N} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N \mathbf{1}_{U\cap\{T_\pi+\ell<T_{\#(\pi)-1}\}} \times \Big\{\frac{T_{\#(\pi)-1}-T_\pi}{\theta_N} - \mathbf{1}_{\{T_{\#(\pi)-1}=T_{(i,j)[\pi]}\}}\Big\}\Big]\, d\ell \tag{4.4.11}$$

is bounded above by

$$\Big\{\frac{\alpha_N}{\theta_N}+1\Big\}\|H\|\ \mathrm{P}^N_{\boldsymbol{m}^n}[T_{\#(\pi)-1}\le T_{\#(\pi)}+\alpha_N]\,, \tag{4.4.12}$$

where $\|H\| = \max_{x\in\mathcal{P}_n^k}|H(x)|$. At this point we assume that $(\xi^N)_N$ satisfies Hypothesis **(H1)**. Then, in virtue of Lemma 4.5.3 along with (4.4.10) we get that (4.4.12) vanishes as $N\to\infty$. In turn, the difference between expression in (4.4.11) and

$$\frac{1}{\alpha_N}\int_0^{\alpha_N} \mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N \mathbf{1}_{U\cap\{T_\pi+\ell<T_{\#(\pi)-1}\}} \times \Big\{\frac{T_{\#(\pi)-1}-(T_\pi+\ell)}{\theta_N} - \mathbf{1}_{\{T_{\#(\pi)-1}=T_{(i,j)[\pi]}\}}\Big\}\Big]\, d\ell \tag{4.4.13}$$

is bounded by $\|H\|\alpha_N/\theta_N$ which vanishes as $N\to\infty$. So, in order to get (4.4.2) it suffices to show that (4.4.13) vanishes as $N\to\infty$. The same argument permits us to conclude that (4.4.3) follows from the limit

$$\frac{1}{\alpha_N}\int_0^{\alpha_N} \mathrm{E}^N_{\boldsymbol{m}^k}\Big[H^N \mathbf{1}_{V\cap\{t\theta_N+\ell<T_{\#(\pi)-1}\}} \times \Big\{\frac{T_{\#(\pi)-1}-(t\theta_N+\ell)}{\theta_N} - \mathbf{1}_{\{T_{\#(\pi)-1}=T_{(i,j)[\pi]}\}}\Big\}\Big]\, d\ell \ \to\ 0\,, \tag{4.4.14}$$

as $N\uparrow\infty$.

Markov Property

Fix $\pi\in\mathcal{P}_n$ with $m:=\#(\pi)\ge 2$, $(i,j)\in I_{m,2}$, and define $g^N: E_N^m\to\mathbb{R}$ as

$$g^N(x) \ :=\ \mathrm{E}^N_x\Big[\frac{T}{\theta_N} - \mathbf{1}_{\{T=T^{N,m}_{(i,j)[\pi_m]}\}}\Big]\,,\qquad x\ \in\ E^m_N\,,$$

where $T:=\widetilde{T}^{N,m}_{m-1}=\min\{\tau^{N,m}_{\hat\imath,\hat\jmath}:(\hat\imath,\hat\jmath)\in I_{m,2}\}$ and $\pi_m\in\mathcal{P}_m$ is the partition with $\#(\pi_m)=m$. Then, applying the Markov property, expectation in (4.4.13) can be written as

$$\mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N\mathbf{1}_{U\cap\{T_\pi+\ell<T_{m-1}\}}g^N\big(\boldsymbol{\eta}^N_{T_\pi+\ell}\big(\boldsymbol{i}(X^N_{T_\pi+\ell})\big)\big)\Big]\,. \tag{4.4.15}$$

We apply observation (4.1.5), and notice that on the event $\{T_\pi+\ell<T_{m-1}\}$ we have $X^N_{T_\pi+\ell}=\pi$, to show that (4.4.15) equals to

$$\mathrm{E}^N_{\boldsymbol{m}^n}\Big[H^N\mathbf{1}_{U\cap\{T_\pi+\ell<T_{m-1}\}}g^N\big(\boldsymbol{\xi}^N_{T_\pi+\ell}\big(\boldsymbol{i}(\pi)\big)\big)\Big]\,.$$

Now, in addition to Hypothesis **(H1)**, we assume that $(\xi^N)_{N\in\mathbb{N}}$ is such that Hypothesis **(H3)** also holds for some $\varepsilon>0$. In order to get rid of the indicator of the event $\{T_\pi+\ell<$

$T_{m-1}\}$ we observe that

$$\left| \mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \mathbf{1}_{U \cap \{T_{m-1} \leq T_\pi + \ell\}} g^N \left(\boldsymbol{\xi}^N_{T_\pi + \ell}(\boldsymbol{i}(\pi)) \right) \right] \right| \tag{4.4.16}$$

is bounded above by

$$\begin{aligned} &\|H\| \, \mathrm{E}^N_{\boldsymbol{m}^n} \left[\mathbf{1}_{\{T_{m-1} \leq T_m + \ell\}} \left\{ \mathbf{1}_{\{T_m = T_\pi < \infty\}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_m+\ell}(\boldsymbol{i}(\pi))} \left[T/\theta_N \right] \right\} + 1 \right] \\ &\quad \leq \; \|H\| \left\{ \mathrm{P}^N_{\boldsymbol{m}^n} [T_{m-1} \leq T_m + \ell]^{\frac{\varepsilon}{1+\varepsilon}} \mathrm{E}^N_{\boldsymbol{m}^n} \left[\mathbf{1}_{\{T_m = T_\pi < \infty\}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_m+\ell}(\boldsymbol{i}(\pi))} \left[T/\theta_N \right]^{1+\varepsilon} \right]^{\frac{1}{1+\varepsilon}} \right. \\ &\qquad\qquad \left. + \; \mathrm{P}^N_{\boldsymbol{m}^n} [T_{m-1} \leq T_m + \ell] \right\} . \end{aligned} \tag{4.4.17}$$

We express $\{T_m = T_\pi < \infty\}$ as the disjoint union of the events

$$A_{\hat{\imath},\hat{\jmath}} \;:=\; \{T_m = T_\pi < \infty\} \cap \{T_m = \tau^N_{\hat{\imath},\hat{\jmath}}\} \, , \quad 1 \leq \hat{\imath} < \hat{\jmath} \leq n \, ,$$

such that $\hat{\imath}$ is one of the coordinates of $\boldsymbol{i}(\pi)$, and $\hat{\jmath}$ belongs to the equivalence class of $\hat{\imath}$ in π. Then we use Remark 4.4.1 to obtain

$$\begin{aligned} \mathrm{E}^N_{\boldsymbol{m}^n} \left[\mathbf{1}_{\{T_m = T_\pi < \infty\}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_m+\ell}(\boldsymbol{i}(\pi))} \left[T/\theta_N \right]^{1+\varepsilon} \right] &= \sum_{A_{\hat{\imath},\hat{\jmath}}} \mathrm{E}^N_{\boldsymbol{m}^n} \left[\mathbf{1}_{A_{\hat{\imath},\hat{\jmath}}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{\tau^N_{\hat{\imath},\hat{\jmath}}+\ell}(\boldsymbol{i}(\pi))} \left[T/\theta_N \right]^{1+\varepsilon} \right] \\ &\leq \; n^2 \mathrm{E}^N_{\boldsymbol{m}^{1+\varepsilon}} \left[\left(\frac{T_1}{\theta_N} \right)^{1+\varepsilon} \right] \; \leq \; n^2 C(n) \, , \end{aligned}$$

where in last inequality we used the Hypothesis **(H3)** for some constant $C(n)$. Then, using this in (4.4.17) we deduce that (4.4.16) is bounded above by

$$\widehat{C}(n) \; \|H\| \; \mathrm{P}^N_{\boldsymbol{m}^n} [T_{m-1} \leq T_m + \ell]^{\frac{\varepsilon}{1+\varepsilon}} \; ,$$

where $\widehat{C}(n)$ is a constant depending on n. Hence, in virtue of Lemma 4.5.3, expression (4.4.15) equals

$$\mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \, \mathbf{1}_U \, g^N \left(\boldsymbol{\xi}^N_{T_\pi+\ell}(\boldsymbol{i}(\pi)) \right) \right] \; + \; o_N \, .$$

This means that, replacing in (4.4.13), (4.4.2) follows from

$$\lim_{N\to\infty} \frac{1}{\alpha_N} \int_0^{\alpha_N} \mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \, \mathbf{1}_U \, g^N \left(\boldsymbol{\xi}^N_{T_\pi+\ell}(\boldsymbol{i}(\pi)) \right) \right] d\ell \;=\; 0 \, . \tag{4.4.18}$$

To conclude, we define

$$h^N(\boldsymbol{\xi}^{N,\hat{n}}, \hat{\pi}) \;:=\; \frac{1}{\alpha_N} \int_0^{\alpha_N} g^N \left(\boldsymbol{\xi}^{N,\hat{n}}_\ell (\boldsymbol{i}(\hat{\pi})) \right) d\ell \, ,$$

for all $\hat{n} \geq m$ and $\hat{\pi} \in \mathcal{P}_{\hat{n}}$ with $\#(\hat{\pi}) = m$. Since $H^N \mathbf{1}_U \in \mathcal{F}^N_{\widetilde{T}_{\#(\pi)}}$, applying the Markov property, the expression inside the limit in (4.4.18) equals

$$\mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \mathbf{1}_U \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_\pi}} [h^N(\boldsymbol{\xi}^{N,n}, \pi)] \right] \;=\; \mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \mathbf{1}_U \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_\pi}(\boldsymbol{i}(\pi))} [h^N(\boldsymbol{\xi}^{N,m}, \pi_m)] \right] .$$

Therefore (4.4.18) is equivalent to

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n} \left[H^N \mathbf{1}_U \mathrm{E}^N_{\boldsymbol{\xi}^N_{T_\pi}(\boldsymbol{i}(\pi))} [h^N(\boldsymbol{\xi}^{N,m}, \pi_m)] \right] \;=\; 0 \, . \tag{4.4.19}$$

Similarly, (4.4.3) follows from

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_V \mathrm{E}^N_{\boldsymbol{\xi}^N_{t\theta_N}(\boldsymbol{i}(\pi))}[h^N(\xi^{N,m},\pi_m)]\right] = 0\,. \tag{4.4.20}$$

Proof of Lemma 4.4.3

Thanks to what we have done in the previous parts of this subsection, to prove Lemma 4.4.3, it only remains to show (4.4.19) and (4.4.20). We continue with the notation established in these parts.

We have

$$\begin{aligned}\left|\mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_U \mathrm{E}^N_{\xi^N_{T_\pi}(\boldsymbol{i}(\pi))}[h^N(\boldsymbol{\xi}^{N,m},\pi_m)]\right]\right| \\ \leq \|H\|\, \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_U \mathrm{E}^N_{\xi^N_{T_\pi}(\boldsymbol{i}(\pi))}[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|]\right]\,.\end{aligned} \tag{4.4.21}$$

The event U is included in $\{T_m = T_\pi < \infty\}$, then we can express U as the disjoint union of the events

$$B_{\hat{\imath},\hat{\jmath}} := U \cap \{T_\pi = \tau^N_{\hat{\imath},\hat{\jmath}}\}\,, \quad 1 \leq \hat{\imath} < \hat{\jmath} \leq n\,,$$

such that $\hat{\imath}$ is one of the coordinates of $\boldsymbol{i}(\pi)$, and $\hat{\jmath}$ belongs to the equivalence class of $\hat{\imath}$ in π. Then we use Remark 4.4.1 to bound (4.4.21) as follows

$$\begin{aligned}&\|H\|\, \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_U \mathrm{E}^N_{\xi^N_{T_\pi}(\boldsymbol{i}(\pi))}[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|]\right] \\ &\quad = \|H\| \sum_{B_{\hat{\imath},\hat{\jmath}}} \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_{B_{\hat{\imath},\hat{\imath}}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{\tau^N_{\hat{\imath},\hat{\imath}}}(\boldsymbol{i}(\pi))}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right]\right] \\ &\quad \leq \|H\| \sum_{B_{\hat{\imath},\hat{\jmath}}} \mathrm{E}^N_{\boldsymbol{m}^m}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right] \\ &\quad \leq n^2\, \|H\|\, \mathrm{E}^N_{\boldsymbol{m}^m}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right]\,.\end{aligned}$$

This proves

$$\left|\mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_U \mathrm{E}^N_{\xi^N_{T_\pi}(\boldsymbol{i}(\pi))}[h^N(\boldsymbol{\xi}^{N,m},\pi_m)]\right]\right| \leq n^2\, \|H\|\, \mathrm{E}^N_{\boldsymbol{m}^m}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right]\,. \tag{4.4.22}$$

And similarly we obtain

$$\left|\mathrm{E}^N_{\boldsymbol{m}^n}\left[H^N \mathbf{1}_V \mathrm{E}^N_{\xi^N_{t\theta_N}(\boldsymbol{i}(\pi))}[h^N(\boldsymbol{\xi}^{N,m},\pi_m)]\right]\right| \leq n^2\, \|H\|\, \mathrm{E}^N_{\boldsymbol{m}^m}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right]\,. \tag{4.4.23}$$

Finally we assume that $(\xi^N)_N$ satisfies Hypothesis **(H2)** for the time scale $(\alpha_N)_{N\in\mathbb{N}}$. Thanks to what we have done, it only remains to proof that

$$\lim_{N\to\infty} \mathrm{E}^N_{\boldsymbol{m}^m}\left[|h^N(\boldsymbol{\xi}^{N,m},\pi_m)|\right] = 0\,.$$

Using **(H2)**, this is a consequence of proving the local ergodic behavior for the sequence of functions $(f^{N,(i,j)[\pi_m]})_{N\in\mathbb{N}}$ described in (4.1.3). More precisely, we are at the point where Lemma 4.4.3 is a consequence of the following result. Remember the definition of the functions $f^{N,\pi}$ made in (4.1.3).

Lemma 4.4.4. *For any* $2 \leq m \leq n$ *and* $1 \leq i < j \leq m$ *we have*

$$\lim_{N\to\infty} \langle f^{N,(i,j)[\pi_m]} \rangle_{\boldsymbol{m}^n} = 0\,.$$

Proof. We can assume that i, j are the indexes we fix previously in this section, then

$$\begin{aligned}\langle f^{N,(i,j)[\pi_m]} \rangle_{\boldsymbol{m}^n} &= \langle g^N \rangle_{\boldsymbol{m}^m} \\ &= \mathrm{E}^N_{\boldsymbol{m}^m} \left[\frac{T}{\theta_N} - \mathbf{1}_{\{T = T^{N,m}_{(i,j)[\pi]}\}} \right] = \mathrm{E}^N_{\boldsymbol{m}^m} \left[T/\theta_N \right] - \binom{m}{2}^{-1} + o_N \, ;\end{aligned}$$

and we conclude thanks to Corollary 4.3.2. □

4.4.2 Proof of Theorem 4.1.3

In this subsection we finish the proof of Theorem 4.1.3. Suppose that the sequence $(\xi^N)_{N \in \mathbb{N}}$ we fix at the beginning of the section is transitive and fulfills the coalescing hypotheses starting with n particles.

The Hypothesis **(H1)** along with the transitivity of $(\xi^N)_{N \in \mathbb{N}}$ prove, as we show in Section 4.5, that the sequence $(\mathbb{X}^N)_{N \in \mathbb{N}}$ is tight.

Take X^* a limit process of $(\mathbb{X}^N)_{N \in \mathbb{N}}$. Lemma 4.4.2 allows us to use Lemma 4.2.8. This proves that X^* is a solution of the $(\mathcal{M}_b(\mathcal{P}_n), \mathscr{L}^n)$-martingale problem starting at π_n, the partition of $[n]$ with $\#(\pi_n) = n$. This martingale problem has as its unique solution the Kingman's coalescent $\mathfrak{K}^n$ starting at π_n. Therefore, since X^* was arbitrary, we conclude that under $\mathrm{P}^N_{\boldsymbol{m}^n}$, $(\mathbb{X}^{\boldsymbol{\theta},N})_N$ converges to $\mathfrak{K}^n$ starting at π_n.

4.4.3 The Reversible Case

Here we proof Corollary 4.1.4. As we mention in Subsection 4.1.3 it is enough to prove that condition (**H'**) implies the coalescing hypotheses starting with n particles, when we take a sequence of irreducible, transitive and reversible Markov chains.

Fix ξ, an irreducible, transitive and reversible Markov chain with generator Q, over a finite state space E. We have the following result. Remember that γ_Q denotes the relaxation time.

Lemma 4.4.5. *For $n \geq 1$, $t \geq 0$ we have*

$$\mathrm{E}^Q_{\boldsymbol{m}^n} \left[\left(\frac{1}{t} \int_0^t f(\boldsymbol{\xi}^{Q,n}_s) \, ds \right)^2 \right] \leq \frac{2\gamma_Q \langle f^2 \rangle_{\boldsymbol{m}^n}}{t}$$

for any $f : E^n \to \mathbb{R}$ with $\langle f \rangle_{\boldsymbol{m}^n} = 0$.

Proof. Fix $n \geq 1$ and let $\mathcal{B} = \{g_0, g_1, \ldots, g_{|E|^n}\}$ be an orthonormal basis for $L^2(\boldsymbol{m}^n)$ composed by eigenfunctions of the generator of $\boldsymbol{\xi}^{Q,n}$, where g_0 is the eigenfunction corresponding to 0. In this proof, for simplicity, we write $\boldsymbol{\xi}^Q$ instead of $\boldsymbol{\xi}^{Q,n}$.

Take $g \in \mathcal{B}$ and $x \in E^n$; we know that

$$\mathrm{E}^Q_x[g(\boldsymbol{\xi}^Q_t)] = g(x) \, e^{\lambda t} \, ,$$

where λ is the eigenvalue corresponding to g. It follows, from this and the Markov property, that for any $0 \leq s < r$

$$\begin{aligned}\mathrm{E}^Q_{\boldsymbol{m}^n}[g(\boldsymbol{\xi}^Q_s) g(\boldsymbol{\xi}^Q_r)] &= \mathrm{E}^G_{\boldsymbol{m}^n}[g(\boldsymbol{\xi}^G_s) g(\boldsymbol{\xi}^Q_s) e^{\lambda(r-s)}] \\ &= e^{\lambda(r-s)} \, .\end{aligned}$$

Therefore, integrating $e^{\lambda|r-s|}$ on the square $[0,t]^2$

$$\int_0^t\int_0^t \mathrm{E}^Q_{\boldsymbol{m}^n}[g(\boldsymbol{\xi}^Q_s)g(\boldsymbol{\xi}^Q_r)]\,ds\,dr \;=\; \frac{2}{\lambda^2}(e^{\lambda t}-\lambda t-1)\,,$$

for all $g\in\mathcal{B}\setminus\{g_0\}$. Hence, bounding the right hand side,

$$\Big|\frac{1}{t^2}\int_0^t\int_0^t \mathrm{E}^Q_{\boldsymbol{m}^n}[g(\boldsymbol{\xi}^Q_s)g(\boldsymbol{\xi}^Q_r)]\,ds\,dr\Big| \;\le\; \frac{2}{-\lambda t}\,, \tag{4.4.24}$$

for all $g\in\mathcal{B}\setminus\{g_0\}$.

Finally, we take $f:E^n\to\mathbb{R}$ such that $\langle f\rangle_{\boldsymbol{m}^n}=0$. We can write $f=\sum_{j=1}^{|E|^n}a_jg_j$ for some constants $a_1,\dots,a_{|E|^n}$. Then

$$\begin{aligned}
\mathrm{E}^Q_{\boldsymbol{m}^n}\Big[\Big(\frac{1}{t}\int_0^t f(\xi^Q_s)\,ds\Big)^2\Big] &= \frac{1}{t^2}\int_0^t\int_0^t \mathrm{E}^Q_{\boldsymbol{m}^n}[f(\boldsymbol{\xi}^Q_s)f(\boldsymbol{\xi}^Q_r)]\,ds\,dr\\
&= \frac{1}{t^2}\sum_{j=1}^{|E|^n}a_j^2\int_0^t\int_0^t \mathrm{E}^Q_{\boldsymbol{m}^n}[g_j(\boldsymbol{\xi}^G_s)g_j(\boldsymbol{\xi}^G_r)]\,ds\,dr\\
&\le \frac{2\gamma_Q\mathrm{Var}(f)}{t},
\end{aligned}$$

where in the second equality we used that $\mathrm{E}^Q_{\boldsymbol{m}^n}[g_i(\boldsymbol{\xi}^Q_s)g_j(\boldsymbol{\xi}^Q_r)]=0$, for all $i\neq j$ and $r,s\in[0,t]$, and in last inequality we used (4.4.24). □

On the other hand. A direct application of [2, Proposition 3.23] give us the following bound

$$\mathrm{P}^Q_{\boldsymbol{m}^2}[T_1\le\delta\theta^r_Q] \;\le\; \frac{\gamma_Q}{\theta_Q} + (1-e^{-\delta})\,,\quad \forall\delta\ge 0\,. \tag{4.4.25}$$

And from the proof of this proposition we also obtain

$$\mathrm{E}^Q_{\boldsymbol{m}^2}\Big[\Big(\frac{T_1}{\theta^r_Q}\Big)^2\Big] \;\le\; \frac{2\gamma_Q}{\theta^r_Q} + 2\,. \tag{4.4.26}$$

Now consider $(\xi^N)_{N\in\mathbb{N}}$, a sequence of transitive, irreducible and reversible Markov chains over finite state spaces. Fix $n\in\mathbb{N}\setminus\{1\}$ and suppose that **(H')** holds. Then it is possible to take $(\alpha_N)_{N\in\mathbb{N}}\subset\mathbb{R}_+$ such that

$$\lim_{N\to\infty}\frac{\gamma_N}{\alpha_N} = 0 \quad\text{and}\quad \lim_{N\to\infty}\frac{\alpha_N}{\theta_N} = 0.$$

Lemma 4.4.5 implies **(H2)** for $p=2$ and the scale of times $(\alpha_N)_{N\in\mathbb{N}}$. Actually, we have a stronger condition with convergence in $L^2(\mathrm{P}^N_{\boldsymbol{m}^n})$ instead of just in $L^1(\mathrm{P}^N_{\boldsymbol{m}^n})$. Condition **(H1)** follows from (4.4.25), taking the respective limits and using **(H')**. Finally, **(H3)** with $\varepsilon=2$ follows from (4.4.26).

4.5 Tightness

Fix $n\ge 2$, a sequence of irreducible Markov chains $(\xi^N)_{N\in\mathbb{N}}$, and a scale of time $\boldsymbol{\theta}=(\theta_N)_N$, not necessarily the scale introduced in Definition 4.1.1. Here we prove the tightness of $(\mathbb{X}^{\boldsymbol{\theta},N})_N$ and $(\mathbb{Y}^{\boldsymbol{\theta},N})_N$ under $\mathrm{P}^N_{\boldsymbol{m}^n}$, assuming that all the the Markov chains ξ^N, $N\in\mathbb{N}$, are transitive an that the Hypothesis **(H1)** holds for the scale $\boldsymbol{\theta}$. For simplicity, in this section we write $(\mathbb{X}^N)_N$ and $(\mathbb{Y}^N)_N$ instead of $(\mathbb{X}^{\boldsymbol{\theta},N})_N$ and $(\mathbb{Y}^{\boldsymbol{\theta},N})_N$, respectively.

Remember that for a trajectory $w \in D([0,+\infty),S)$, where (S,ρ) is a metric space, the *modified modulus of continuity* is defined as

$$\tilde{\omega}(w,t,\delta) \ := \ \inf_{\Delta} \max_{m} \sup_{t_m \le r,s < t_{m+1}} \rho(w(s),w(r))\,, \quad t>0\,, \quad \delta>0\,,$$

where the infimum extends over all partitions $\Delta = \{0 = t_0 < t_1 < \cdots < t_\ell < t\}$ such that $t_{m+1} - t_m \ge \delta$ for $m = 1,\dots,\ell-1$. It is well known (see for instance [19, Theorem 4.8.1]) that the tightness of a sequence of processes $(Z^N)_N$ with trajectories in $D([0,+\infty),S)$, under $\mathrm{P}^N_{\boldsymbol{m}^k}$, follows from

1. for any $t \ge 0$, the sequence $\big(Z^N_t\big)_N$ is tight in S; and
2. for all $\varepsilon > 0$, $t > 0$,
$$\lim_{\delta\to 0} \sup_N \mathrm{P}^N_{\boldsymbol{m}^k}[\tilde{\omega}(Z^N,t,\delta) > \varepsilon] \ = \ 0\,. \tag{4.5.1}$$

In our case, since $\mathbb{X}^N_t \in \mathcal{P}_n$, $\mathbb{Y}^N_t \in \{0,1\}^{I_{n,2}}$ for all $t \ge 0$, and $\mathcal{P}_n$, $\{0,1\}^{I_{n,2}}$ are compact, condition (1) holds immediately thanks to Prohorov's criterion. For simplicity, in this section we call σ_j, $1 \le j \le n$, the time when $\mathbb{X}^N$ has j coordinates:

$$\sigma_j \ := \ \inf\{t \ge 0 : \ \#(\mathbb{X}^N_t) = j\}\,.$$

And we call $\nu_{i,j}$, $(i,j) \in I_{n,2}$, the time when the coordinate (i,j) of $\mathbb{Y}^N_t$ hits 1:

$$\nu_{i,j} \ := \ \inf\{t \ge 0 : \ Y^{N,(i,j)}_{t\theta_N} = 1\}\,.$$

Clearly $\sigma_j = T^N_j/\theta_N$ for all $j \in [n]$, and $\nu_{i,j} = \tau^N_{i,j}/\theta_N$ for all $(i,j) \in I_{n,2}$.

Lemma 4.5.1. *For the processes $(\mathbb{X}^N)_N$, condition (2) follows from*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^N_{\boldsymbol{m}^n}\big[\sigma_{j-1} - \sigma_j \le \delta\big] \ = \ 0\,, \quad \forall\, j \ge 2\,. \tag{4.5.2}$$

Proof. Assume that (4.5.2) holds, fix $\varepsilon > 0$, $t > 0$, $\epsilon > 0$. Our assumption implies that there are $\delta_0 > 0$ and $M \in \mathbb{N}$ such that

$$\mathrm{P}^N_{\boldsymbol{m}^k}[\sigma_{j-1} - \sigma_j \ge \delta_0, \text{ for all } j \in \{2,3,\dots,n\}] \ > \ 1 - \epsilon/2, \quad \forall\, N > M.$$

Denote $m := \min\{j \in [n] : \sigma_j < t\}$ and define the random partition $\Delta := \{0 = t_0 < t_1 = \sigma_{n-1} < \cdots < t_\ell = \sigma_m < t\}$. Using this partition we deduce that

$$\tilde{\omega}(\mathbb{X}^N,t,\delta) = 0 < \varepsilon\,, \quad \forall\, \delta < \delta_0\,, \quad N > M\,,$$

on the event

$$\big\{\sigma_{j-1} - \sigma_j \ge \delta_0, \text{ for all } j \in \{2,3,\dots,n\}\big\}\,,$$

that has probability at least $1 - \epsilon/2$. Hence

$$\sup_{N>M} \ \mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{X}^N,t,\delta) > \varepsilon] < \epsilon/2\,, \quad \forall \delta < \delta_0\,.$$

On the other hand, it is clear that there is $\delta_1 > 0$ such that

$$\mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{X}^N,t,\delta) > \varepsilon] \ < \ \epsilon/2\,, \quad N \le M\,, \quad \forall \delta < \delta_1.$$

Therefore

$$\sup_N \mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{X}^N, t, \delta) > \varepsilon] < \epsilon\,, \quad \forall\, \delta < \min\{\delta_0, \delta_1\}\,, \tag{4.5.3}$$

which completes the proof, since $\epsilon > 0$ was arbitrary. □

Similarly, in regard to $(\mathbb{Y}^N)_N$

Lemma 4.5.2. *For the processes $(\mathbb{Y}^N)_N$, condition (2) follows from*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^N_{\boldsymbol{m}^n}\left[|\nu_{i,j} - \nu_{\ell,m}| \le \delta\right] \;=\; 0\,, \tag{4.5.4}$$

for all $(i,j),(\ell,m) \in I_{n,2}$ such that $(i,j) \neq (\ell,m)$.

Proof. We proceed like in Lemma 4.5.1. Assume that (4.5.4) holds, fix $\varepsilon > 0$, $t > 0$, $\epsilon > 0$. Our assumption implies that there are $\delta_0 > 0$ and $M \in \mathbb{N}$ such that

$$\mathrm{P}^N_{\boldsymbol{m}^n}\big[|\nu_{i,j} - \nu_{\ell,m}| \ge \delta_0,\ \forall (i,j),(\ell,m) \in I_{n,2} \\ \text{such that } (i,j) \neq (\ell,m)\big] \;>\; 1 - \epsilon/2, \quad \forall\, N > M.$$

We order all the $K := \binom{n}{2}$ random times as follows

$$\nu_{\mathbf{i}^1,\mathbf{j}^1} \;<\; \nu_{\mathbf{i}^2,\mathbf{j}^2} \;<\; \ldots \;<\; \nu_{\mathbf{i}^K,\mathbf{j}^K}\,.$$

Of course $\mathbf{i}^k, \mathbf{j}^k$, $k \in [K]$, are random indexes. Denote

$$\ell := \min\{k \in [K] : \nu_{\mathbf{i}^k,\mathbf{j}^k} < t\}\,,$$

and define the random partition $\Delta := \{0 = t_0 < t_1 = \nu_{\mathbf{i}^1,\mathbf{j}^1} < \cdots < t_\ell = \nu_{\mathbf{i}^\ell,\mathbf{j}^\ell} < t\}$. Using this partition we deduce that

$$\tilde{\omega}(\mathbb{Y}^N, t, \delta) = 0 < \varepsilon\,, \quad \forall\, \delta < \delta_0\,, \quad N > M\,,$$

on the event

$$\big\{|\nu_{i,j} - \nu_{\ell,m}| \ge \delta_0,\ \forall (i,j),(\ell,m) \in I_{n,2} \text{ such that } (i,j) \neq (\ell,m)\big\}\,,$$

that has probability at least $1 - \epsilon/2$. Hence

$$\sup_{N>M} \mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{Y}^N, t, \delta) > \varepsilon] < \epsilon/2\,, \quad \forall \delta < \delta_0\,.$$

On the other hand, it is clear that there is $\delta_1 > 0$ such that

$$\mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{Y}^N, t, \delta) > \varepsilon] \;<\; \epsilon/2\,, \quad N \le M\,, \quad \forall \delta < \delta_1.$$

Therefore

$$\sup_N \mathrm{P}^N_{\boldsymbol{m}^n}[\tilde{\omega}(\mathbb{Y}^N, t, \delta) > \varepsilon] < \epsilon\,, \quad \forall\, \delta < \min\{\delta_0, \delta_1\}\,, \tag{4.5.5}$$

which completes the proof, since $\epsilon > 0$ was arbitrary. □

In virtue of Lemma 4.5.1, and the definition of the stopping times σ_j, the tightness of $(\mathbb{X}^N)_N$ under $\mathrm{P}^N_{\boldsymbol{m}^n}$ follows from next lemma.

Lemma 4.5.3. *Suppose that the Markov chains ξ^N are transitive for all $N \in \mathbb{N}$, and that Hypothesis (**H1**) holds for the scale of time $(\theta_N)_N$. Then for every $2 \le m \le k$ we have*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathrm{P}^N_{\boldsymbol{m}^n} \left[T^N_{m-1} - T^N_m \le \delta\theta_N \right] \;=\; 0\,.$$

Proof. Fix $2 \le m \le n$. Note that $\mathrm{P}^N_{\boldsymbol{m}^n}[T^N_m = \infty]$ goes to 0 as $N \to \infty$. Then we divide $\{T^N_m < \infty\}$ into the disjoint sets

$$A^\pi_{i(\pi),j(\pi)} \;:=\; \{T^N_m < \infty\} \cap \{T^N_m = T^N_\pi\} \cap \{T^N_\pi = \tau^N_{i(\pi),j(\pi)}\}\,,$$

where $\pi \in \mathcal{P}_n$ is such that $\#(\pi) = m$, and $\underline{1 \le} \; i(\pi) < j(\pi) \le m$ are such that $i(\pi)$ is one of the coordinates of $\boldsymbol{i}(\pi)$, and $j(\pi) \in \overline{i(\pi)}$ in π. Then using the Markov property, observation (4.1.5), and Remark 4.4.1 we have

$$\begin{aligned}
\mathrm{P}^N_{\boldsymbol{m}^n}\left[T^N_{m-1} - T^N_m \le \delta\theta_N\right] &= \sum_{A^\pi_{i(\pi),j(\pi)}} \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_{A^\pi_{i(\pi),j(\pi)}} \mathrm{E}^N_{\eta^N_{T^N_m}\left(\boldsymbol{i}(X^N_{T^N_m})\right)}\left[\mathbf{1}_{\{T^{N,m}_{m-1} \le \delta\theta_N\}}\right]\right] \;+\; o_N \\
&= \sum_{A^\pi_{i(\pi),j(\pi)}} \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_{A^\pi_{i(\pi),j(\pi)}} \mathrm{E}^N_{\xi^N_{\tau^N_{i(\pi),j(\pi)}}(\boldsymbol{i}(\pi))}\left[\mathbf{1}_{\{T^{N,m}_{m-1} \le \delta\theta_N\}}\right]\right] \;+\; o_N \\
&\le\; C(n)\mathrm{P}^N_{\boldsymbol{m}^m}\left[T^{N,m}_{m-1} \le \delta\theta_N\right] \;+\; o_N\,,
\end{aligned}$$

where $C(n)$ is a constant depending on n. On the other hand $\{T^{N,m}_{m-1} \le \delta\theta_N\} = \bigcup_{i<j}\{\tau^{N,m}_{i,j} \le \delta\theta_N\}$. Then using the union bound, and taking into account that $\mathrm{P}^N_{\boldsymbol{m}^m}[\tau^{N,m}_{i,j} \le \delta\theta_N] = \mathrm{P}^N_{\boldsymbol{m}^2}[T^{N,2}_1 \le \delta\theta_N]$, we obtain

$$\mathrm{P}^N_{\boldsymbol{m}^n}\left[T^N_{m-1} - T^N_m \le \delta\theta_N\right] \;\le\; \widehat{C}(n)\mathrm{P}^N_{\boldsymbol{m}^2}\left[T^{N,2}_1 \le \delta\theta_N\right] \;+\; o_N\,,$$

where $\widehat{C}(n)$ is a constant depending only on n. We conclude thanks to Hypothesis (**H1**). □

Similarly, the tightness of $(\Upsilon^N)_N$ under $\mathrm{P}^N_{\boldsymbol{m}^n}$ follows from next lemma.

Lemma 4.5.4. *Suppose that the Markov chains ξ^N are transitive for all $N \in \mathbb{N}$, and that Hypothesis (**H1**) holds for the scale of time $(\theta_N)_N$. Then for all $(i,j),(\ell,m) \in I_{n,2}$ such that $(i,j) \ne (\ell,m)$ we have*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathrm{P}^N_{\boldsymbol{m}^n}\left[|\tau^N_{i,j} - \tau^N_{\ell,m}| \le \delta\theta_N\right] \;=\; 0\,.$$

Proof. Fix $(i,j),(\ell,m) \in I_{n,2}$ such that $(i,j) \ne (\ell,m)$. Using the Markov property

$$\begin{aligned}
\mathrm{P}^N_{\boldsymbol{m}^n}\left[|\tau^N_{i,j} - \tau^N_{\ell,m}| \le \delta\theta_N\right] \;&=\; \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_{\{\tau^N_{i,j} \le \tau^N_{\ell,m}\}} \mathrm{E}^N_{\xi^N_{\tau^N_{i,j}}}\left[\tau^N_{\ell,m} \le \delta\theta_N\right]\right] \\
&\quad + \mathrm{E}^N_{\boldsymbol{m}^n}\left[\mathbf{1}_{\{\tau^N_{i,j} > \tau^N_{\ell,m}\}} \mathrm{E}^N_{\boldsymbol{\xi}^N_{\tau^N_{\ell,m}}}\left[\tau^N_{i,j} \le \delta\theta_N\right]\right]\,.
\end{aligned}$$

Then, in virtue of Remark 4.4.1, we deduce

$$\mathrm{P}^N_{\boldsymbol{m}^n}\left[|\tau^N_{i,j} - \tau^N_{\ell,m}| \le \delta\theta_N\right] \;\le 2\mathrm{P}^N_{\boldsymbol{m}^2}\left[T^{N,2}_1 \le \delta\theta_N\right]\,.$$

Therefore, like in the proof of 4.5.3, we conclude thanks to Hypothesis (**H1**). □

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