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Periods of Algebraic Cycles

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To my mother Gloria,
my grandmother Nieves,
and my son's mother Gabriela.

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Introduction

Research framework

This thesis is about computing periods of algebraic cycles inside smooth hypersurfaces of projective space, and its applications to variational Hodge conjecture.

Hodge conjecture was proposed in 1941 by Hodge (see [Hod41]) and reformulated in 1962 by Atiyah and Hirzebruch (see [AH62]). Despite that, it has seen few advances in its positive direction. Variational Hodge conjecture was proposed by Grothendieck as a weak version of the Hodge conjecture (see [Gro66]). And (as in the case of Hodge conjecture) few is known about its veracity. Hodge conjecture claims that every Hodge cycle inside a smooth projective variety is an algebraic cycle. On the other hand, variational Hodge conjecture claims that in all proper families of smooth projective varieties with connected base, a flat section of its de Rham cohomology bundle is an algebraic cycle at one point if and only if it is an algebraic cycle everywhere. In other words, flat deformations of an algebraic cycle remain algebraic inside the deformed smooth projective variety. In 1972, Bloch proved variational Hodge conjecture for deformations of algebraic cycles supported in local complete intersections which are semi-regular inside the corresponding smooth projective variety (see [Blo72]). Semi-regularity is a strong condition, difficult to check in concrete examples (see [DK16] for a discussion about examples of semi-regular varieties). In 2003, Otwinowska considered variational Hodge conjecture for algebraic cycles inside smooth degree d hypersurfaces X of the projective space $\mathbb{P}^{n+1}$ of even dimension n. In this context, she proved that variational Hodge conjecture is satisfied for algebraic cycles supported in one $\frac{n}{2}$-dimensional complete intersection Z of $\mathbb{P}^{n+1}$ contained in X, and $d >> 0$ (see [Otw03]). This result was improved by Dan in 2014, who removed the condition on the degree d provided that $deg(Z) < d$ (see [Dan14]). It is not known if the complete intersection subvarieties considered by Otwinowska and Dan are semi-regular inside the corresponding hypersurface.

The computation of periods of algebraic cycles was considered by Deligne in 1982. He proved that up to some constant power of $2\pi\sqrt{-1}$, the periods of algebraic cycles belong to the field of definition of the variety and the corresponding algebraic cycle (see [DMOS82]). This problem was reconsidered in 2014 by Movasati, who explained how explicit computations of periods of algebraic cycles can be used to prove variational Hodge conjecture (see [Mov17c]). His approach inspired the development of this thesis. In 2017, we computed the periods of linear cycles inside Fermat varieties and used them to prove variational Hodge conjecture for some combinations of linear cycles inside Fermat varieties (see [MV17]). In

[Ser18], Sertöz implemented an algorithm for approximating periods of arbitrary Hodge cycles inside hypersurfaces. Using this algorithm he performed reliable computations of the Picard rank of certain K3 surfaces. In 2018, we were able to compute periods of complete intersection algebraic cycles inside any smooth hypersurface (see [VL18, Theorem 1]) by determining the primitive part of the Poincaré dual of the given complete intersection cycle. This allowed us to improve the results in [MV17] to arbitrary degree and dimension by removing the computer assisted argument (see [VL18, Theorem 2]).

Main results

Consider the even dimensional smooth hypersurface of the complex projective space

$$X = \{F = 0\} \subset \mathbb{P}^{n+1},$$

given by a homogeneous polynomial with $\deg F = d$. Every $\frac{n}{2}$-dimensional subvariety Z of X determines an *algebraic cycle*

$$[Z] \in H_n(X, \mathbb{Z}).$$

Recalling from Griffiths' work [Gri69], each piece of the Hodge filtration is generated by the differential forms

$$\omega_P := res\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q}H^n_{\mathrm{dR}}(X)_{prim},$$

for $P \in \mathbb{C}[x_0, \dots, x_{n+1}]_{d(q+1)-n-2}$, where

$$\Omega := \iota_{\sum_{i=0}^{n+1} x_i \frac{\partial}{\partial x_i}}(dx_0 \wedge \cdots \wedge dx_{n+1}) = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1},$$

and $res : H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1} \setminus X) \to H^n_{\mathrm{dR}}(X)$ is the *residue map*.

We say that

$$\int_Z \omega_P \in \mathbb{C}$$

is a *period of* Z. Note that, since Z is a projective variety of positive dimension, it intersects every divisor of X, so it is impossible to find an affine chart of X where to compute the periods of Z. Since we are integrating over an algebraic cycle (consequently a Hodge cycle) we just care about the $(\frac{n}{2}, \frac{n}{2})$-part of ω_P. Thus, we will fix $q = \frac{n}{2}$, and we work with ω_P as an element of the quotient $F^{\frac{n}{2}}H^n_{\mathrm{dR}}(X)/F^{\frac{n}{2}+1}H^n_{\mathrm{dR}}(X) \simeq H^{\frac{n}{2},\frac{n}{2}}(X) \simeq H^{\frac{n}{2}}(X, \Omega_X^{\frac{n}{2}})$. After Carlson-Griffiths' work [CG80, page 7], we know

$$\omega_P = \frac{1}{\frac{n}{2}!}\left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=\frac{n}{2}} \in H^{\frac{n}{2}}(\mathcal{U}, \Omega_X^{\frac{n}{2}}). \tag{1}$$

Where $\mathcal{U}$ is the *Jacobian covering of* X. For $J = (j_0, \dots, j_{\frac{n}{2}})$,

$$F_J := F_{j_0} \cdots F_{j_{\frac{n}{2}}},$$

where $F_i := \frac{\partial F}{\partial x_i}$ for every $i = 0, \dots, n+1$, and

$$\Omega_J := \frac{\partial}{\partial x_{j_{\frac{n}{2}}}} \lrcorner \left(\cdots \frac{\partial}{\partial x_{j_0}} \lrcorner (\Omega) \cdots \right) = (-1)^{j_0 + \cdots + j_{\frac{n}{2}} + \binom{\frac{n}{2}+2}{2}} \sum_{l=0}^{\frac{n}{2}} (-1)^l x_{k_l} \widehat{dx_{k_l}}, \tag{2}$$

for $(k_0, \dots, k_{\frac{n}{2}-l})$ the multi-index obtained from $(0, 1, \dots, n+1)$ by removing the entries of J. We ω_P in Čech cohomology as in (1), but we denote the period by abuse of notation as $\int_Z \omega_P \in \mathbb{C}$, letting it be understood that we are identifying ω_P with its image under the isomorphism $H^{\frac{n}{2}}(\mathcal{U}, \Omega_X^{\frac{n}{2}}) \simeq H^{\frac{n}{2},\frac{n}{2}}(X) \subset H^n_{\mathrm{dR}}(X)$.

The first result of this thesis is the formula of periods of linear cycles inside Fermat varieties (see Theorem 2.4.1 and Corollary 2.4.1).

Theorem ([MV17])**.** *Let $X_n^d \subset \mathbb{P}^{n+1}$ be the n-dimensional Fermat variety of degree d, i.e $X_n^d = \{F := x_0^d + \cdots + x_{n+1}^d = 0\}$. Let ζ_{2d} be a $2d$-th primitive root of unity, and*

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d} x_1 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0\}.$$

Then

$$\int_{\mathbb{P}^{\frac{n}{2}}} \omega_{x_0^{i_0} \cdots x_{n+1}^{i_{n+1}}} = \begin{cases} \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1} \frac{n}{2}!} \zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n} & \text{if } i_{2l-2} + i_{2l-1} = d-2, \forall l = 1, \dots, \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases}$$

Using this result we can verify variational Hodge conjecture for combinations of linear cycles inside Fermat varieties. We divide these results into two parts. The first verification (see Theorem 3.6.1) is for algebraic cycles supported in one complete intersection subvariety of Fermat variety (this subvariety can be degenerated into a cycle that corresponds to sums of linear cycles).

Theorem ([Otw03], [Dan14], [MV17])**.** *Let $d \geq 2 + \frac{4}{n}$, $X \subset \mathbb{P}^{n+1}$ be the Fermat variety and $\delta \in H_n(X, \mathbb{Z})_{alg}$ be a complete intersection algebraic cycle $\delta = [Z]$, given by $Z = \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\}$, with*

$$x_0^d + \cdots + x_{n+1}^d = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1},$$

and deg $f_i = d_i$. Then, variational Hodge conjecture is true for

1. *$d_1 = d_2 = \cdots = d_{\frac{n}{2}+1} = 1$.*

2. *$n = 2$, $4 \leq d \leq 15$, or $n = 4$, $3 \leq d \leq 6$, or $n = 6$, $3 \leq d \leq 4$.*

These results were proved by Otwinowska in [Otw03] and by Dan in [Dan14] in more general contexts, but we provide a different proof.

The second verification of variational Hodge conjecture (see Theorem 3.6.2) is for sums of two linear cycles which are not supported in a complete intersection cycle (and so this result does not follow from Otwinowska's or Dan's work).

Theorem ([MV17]). *If $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ are two linear subspaces inside the Fermat variety X, such that $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$, then $V_{[\mathbb{P}^{\frac{n}{2}}]+[\check{\mathbb{P}}^{\frac{n}{2}}]} = V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$, for all triples (n,d,m) in the following list:*

$$
\begin{aligned}
&(2,d,-1),\ 5 \le d \le 14,\\
&(4,4,-1),(4,5,-1),(4,6,-1),(4,5,0),(4,6,0),\\
&(6,3,-1),(6,4,-1),(6,4,0),\\
&(8,3,-1),(8,3,0),\\
&(10,3,-1),(10,3,0),(10,3,1),
\end{aligned}
$$

where $\mathbb{P}^{-1}$ means the empty set. In particular, variational Hodge conjecture holds for $\delta = [\mathbb{P}^{\frac{n}{2}}] + [\check{\mathbb{P}}^{\frac{n}{2}}]$ in the above cases.

These cycles are not known to be semi-regular inside Fermat variety either. All the applications we present in this thesis rely on computer assistance, and so they are verified for certain degrees and dimensions.

The main result of this thesis is the computation of periods of algebraic cycles $[Z] \in H_n(X,\mathbb{Z})$ for $Z \subset X$ a complete intersection inside $\mathbb{P}^{n+1}$ (see Theorem 2.6.1).

Theorem ([VL18]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n given by $X = \{F = 0\}$. Suppose that $Z := \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\} \subset X$ is a complete intersection inside $\mathbb{P}^{n+1}$ (i.e. $I(Z) = \langle f_1, \dots, f_{\frac{n}{2}+1}\rangle \subset \mathbb{C}[x_0, \dots, x_{n+1}]$) and*

$$F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}.$$

Define

$$H = (h_0, \dots, h_{n+1}) := (f_1, g_1, f_2, g_2, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}).$$

Then

$$\int_Z \omega_P = \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{\frac{n}{2}!} c \cdot (d-1)^{n+2} \cdot d_1 \cdots d_{\frac{n}{2}+1}, \qquad (3)$$

where $d_i = deg\ f_i$, ω_P is given by (1), and $c \in \mathbb{C}$ is the unique number such that

$$P \cdot det(Jac(H)) \equiv c \cdot det(Hess(F)) \pmod{J^F}.$$

Where $J^F := \langle F_0, \dots, F_{n+1}\rangle \subset \mathbb{C}[x_0, \dots, x_{n+1}]$ is the Jacobian ideal associated to F.

This result says that the primitive part of the *Poincaré dual* of the algebraic cycle $[Z] \in H_n(X,\mathbb{Z})$ is given (up to a non-zero constant factor) by

$$[Z]^{\mathrm{pd}} = res\left(\frac{det(Jac(H))\Omega}{F^{\frac{n}{2}+1}}\right) \in H^{\frac{n}{2},\frac{n}{2}}(X)_{prim}.$$

Using this fact we can improve our previous verification of variational Hodge conjecture for combinations of linear cycles inside Fermat varieties to arbitrary degree and dimension (by removing the computer assisted argument).

Theorem ([VL18]). *Let $X \subset \mathbb{P}^{n+1}$ be the Fermat variety of even dimension n and degree $d \geq 2+\frac{4}{n}$. Let $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}} \subset X$ be the two linear subvarieties such that $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ given by*

$$\mathbb{P}^{n-m} := \{x_{n-2m} - \zeta_{2d}x_{n-2m+1} = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\};$$

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}x_1 = \cdots = x_{n-2m-2} - \zeta_{2d}x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m};$$

$$\check{\mathbb{P}}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}^{\alpha_0}x_1 = \cdots = x_{n-2m-2} - \zeta_{2d}^{\alpha_{n-2m-2}}x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m};$$

where $\zeta_{2d} \in \mathbb{C}$ is a primitive $2d$-root of unity, and $\alpha_0, \alpha_2, \ldots, \alpha_{n-2m-2} \in \{3, 5, \ldots, 2d-1\}$. Then, for $m < \frac{n}{2} - \frac{d}{d-2}$, $a, b \in \mathbb{Z} \setminus \{0\}$ and $\lambda := a[\mathbb{P}^{\frac{n}{2}}] + b[\check{\mathbb{P}}^{\frac{n}{2}}] \in H_n(X, \mathbb{Z})$ we have

$$V_\lambda = V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]},$$

and the Hodge locus V_λ is smooth and reduced. In particular, variational Hodge conjecture holds for λ in these cases. On the other hand, for $m \geq \frac{n}{2} - \frac{d}{d-2}$, the Zariski tangent space of V_λ has dimension strictly bigger than the dimension of $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ (which is always smooth and reduced).

We decided not to include this result in the thesis in order to keep it as short as possible, but the interested reader can find it in [VL18, Theorem 2].

Content description

In what follows we detail the content of this thesis.

Chapter 1 is introductory. Its purpose is to present Griffiths' theory on the cohomology of hypersurfaces, and introduce the language and notations we will use along the rest of the text. Since this is one of the first thesis in Hodge theory developed at IMPA, we tried to keep it self-contained. This is why we also added an Appendix recalling hypercohomology (which is highly used in our work). The reader who wants to get quickly to the main results of this thesis, and is acquainted with hypercohomology and algebraic de Rham cohomology, could just read Theorems 1.5.1, 1.5.2 and 1.6.1 before going to Chapter 2.

Griffiths' Theorems 1.5.1 and 1.5.2 (see section 1.5) are classic, and describe how to construct an explicit basis for (the primitive part of) de Rham cohomology of a given hypersurface, as the residue of meromorphic forms with pole along the hypersurface (furthermore, the order of the pole determines to which part of Hodge filtration the residue belongs).

Carlson-Griffiths' Theorem 1.6.1 (see section 1.6) is the main ingredient missing from the original formulation of infinitesimal variations of Hodge structures (developed by Carlson, Green, Griffiths and Harris), that will allow us to compute explicitly the periods of algebraic cycles. This theorem tells us explicitly how is Griffiths' basis in Čech cohomology (i.e. when we look at each element of the basis in Hodge decomposition), i.e. it computes the residue map.

For the sake of completeness we provide proofs of these results. Furthermore, we add in sections 1.1 to 1.4 a review of algebraic de Rham cohomology, including Deligne's results on logarithmic differential forms and Hodge filtration for affine varieties (we just treat the case of affine varieties obtained as the complement of a smooth hyperplane section inside a smooth projective variety). In section 1.3, we provide a different proof of Deligne's Theorem 1.3.1 relying on Carlson-Griffiths' Lemma 1.3.1. This lemma is the main ingredient in the proof of Carlson-Griffiths' Theorem 1.6.1.

Chapter 2 is the heart of this thesis, and is devoted to the computation of periods of algebraic cycles. We present two strategies to do these computations. Both rely on the reduction of the computation of the period to the computation of the integral of a top form over the projective space. In order to use Carlson-Griffiths' Theorem 1.6.1, we need to calculate every period with respect to a covering occurring in Čech cohomology.

In section 2.1 we compute the period of a standard top form of the projective space. This standard top form is described in terms of the standard covering of projective space. This is the unique period we actually compute by integration (using partitions of unity). All other periods will be computed by an adequate comparison with this period.

Our first strategy to compute periods only works for cycles that are images of linear spaces. We pull back the forms we want to integrate, to the corresponding linear space that parametrizes the cycle. For this reason, we explain in section 2.2 how to compute the pull-back of forms in algebraic de Rham cohomology.

As an application of this pull-back description, we compute in section 2.3 the periods of top forms of the projective space, described in other open coverings (different from the standard one).

In section 2.4 we use this first strategy to compute periods of linear cycles inside Fermat varieties. This computation is the heart of our results on variational Hodge conjecture (see section 3.6, Theorems 3.6.1 and 3.6.2).

Our second method to compute periods of algebraic cycles, works in general for any complete intersection (in the projective space) cycle inside a smooth hypersurface. It relies on the successive application of the coboundary map associated to Poincaré's residue sequence, for a hypersurface of a projective variety.

In section 2.5 we describe explicitly the coboundary map in Čech cohomology, together with the relation between the periods of the corresponding forms.

And in section 2.6 we use the coboundary map, to inductively reduce the computation of the period over a complete intersection algebraic cycle, to the computation of the period of a top form of the ambient projective space (here we will end up with a form described in the Jacobian covering, so our computations from section 2.3 will be needed). Our main theorem is Theorem 2.6.1, and is actually giving us an explicit description of the Poincaré dual of a complete intersection algebraic cycle, as an element of Griffiths' basis for de Rham cohomology of the given hypersurface (see Remark 2.6.3).

Finally, Chapter 3 is about applications of the results developed in Chapter 2 to variational Hodge conjecture. The results are stated (and proved) in section 3.6. There we prove variational Hodge conjecture for linear algebraic cycles inside Fermat varieties (see part 1. of Theorem 3.6.1). Using computer assistance we also prove variational Hodge conjecture for complete intersection algebraic cycles inside Fermat varieties (see part 2. of Theorem 3.6.1), and for sums of linear algebraic cycles with small intersection inside Fermat varieties (see Theorem 3.6.2).

This approach to variational Hodge conjecture using periods can be made after the development of infinitesimal variations of Hodge structures (IVHS), and its explicit description in the case of hypersurfaces (see section 3.3, Proposition 3.3.1).

Instead of proving variational Hodge conjecture, we will prove a stronger result which we call alternative Hodge conjecture. This conjecture corresponds to describe the components of a parameter space, the so called Hodge locus. We introduce Hodge locus in section 3.4 and explain its relation to IVHS in Proposition 3.4.1. This proposition tells us how to determine the Zariski tangent space of the Hodge locus using IVHS.

In section 3.5, we introduce alternative Hodge conjecture and relate it to the problem of determining components of the Hodge locus. Furthermore, we explain in Proposition 3.6.1 how we can use the periods of algebraic cycles to produce a period matrix which corresponds with IVHS. Thus, we can reduce the computation of the Zariski tangent space of the Hodge locus, to computing the rank of this period matrix (which we can do with computer assistance). Using the description of the tangent space we determine the components of the Hodge locus and prove alternative (and variational) Hodge conjecture.

Sections 3.1 and 3.2 are introductory, to prepare the ground for IVHS.

Conventions and notations

Every *ring* is commutative with unity.

If R is a ring and M is an R-module. For $S \subseteq M$, we denote by $\langle S \rangle$ the *submodule generated* by S. If $S = \{m_1, \ldots, m_k\}$, then we simply denote $\langle S \rangle$ by $\langle m_1, \ldots, m_k \rangle$.

By a *complex algebraic variety* we mean an abstract algebraic variety over $\mathbb{C}$, i.e. an integral separated scheme of finite type over $\mathbb{C}$.

When $f \in \mathbb{C}[x_1, \ldots, x_n]$ we will denote by $\{f = 0\} := \{x \in \mathbb{C}^n : f(x) = 0\}$, the zero set of f in the affine space. When $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ is a homogeneous polynomial of degree $d > 0$, we will use the same notation $\{F = 0\} := \{x \in \mathbb{P}^n : F(x) = 0\}$, for the zero set of F in the projective space. Depending on the context it will be clear if we are taking the zero set on the affine or projective space.

For X an affine variety, we denote by $\mathbb{C}[X]$ the *coordinate ring* (or the ring of regular functions), and by $I(X) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ its ideal. When $X \subseteq \mathbb{P}^n$ is projective, we denote by $I(X) \subseteq \mathbb{C}[x_0, \ldots, x_n]$ its *homogeneous ideal*.

We use the symbol $\simeq$ to denote isomorphism (in the corresponding category).

When ω is a differential form defined over a variety X and T is a vector field over X, we denote by $\iota_T(\omega)$ the *contraction of* ω *along* T.

If X is a topological space and F is a sheaf over it, then we denote by $\Gamma(F) := F(X)$ the set of global sections of this sheaf. We will refer to Γ as the *global sections functor*.

A *hyperplane section* is a divisor of a projective variety X which corresponds to the intersection of X with a hyperplane in some projective embedding. In particular, after Veronesse embedding, every hypersurface section will be called a hyperplane section.

Chapter 1

Cohomology of Hypersurfaces

Summary

As the title suggests, the main characters of this thesis are periods of algebraic cycles inside hypersurfaces (in the projective space). By this, we mean the integrals of a basis of de Rham cohomology (of a given hypersurface) over some algebraic subvariety. In this chapter we will explain how to construct explicit basis for de Rham cohomology of hypersurfaces. This was developed by Griffiths in [Gri69]. Furthermore this basis is compatible with the Hodge filtration of the given hypersurface. We will present Deligne's proof of Griffiths' theorem (as in [Del74, x9.2]). The main idea is to relate the Hodge filtration of the hypersurface, with the pole filtration of the complement of the hypersurface, via the residue map. This was possible after the work of several mathematicians (mainly Serre, Atiyah, Hodge, Griffiths, Grothendieck and Deligne) that led to the development of de Rham cohomology (and Hodge theory) in the framework of algebraic geometry. We will explain the development of these ideas, providing proofs of the main theorems.

This chapter is introductory, hence the reader who wants to see the results of this thesis should go directly to Theorem 1.5.1 (which relates the pole filtration of the complement of the hypersurface with the primitive part of the Hodge filtration of the hypersurface), read Theorem 1.5.2 (that explains how to construct Griffiths' basis for de Rham cohomology of an hypersurface via the residue map), and Theorem 1.6.1 (that computes explicitly the residue of each element of Griffiths' basis in Čech cohomology), then move to Chapter 2. In what follows we give a detailed overview of the content of this chapter.

Section 1.1 is a quick review of algebraic differential forms. We recall Euler's sequence and Bott's formula (which is a extended version of Bott's vanishing theorem).

Section 1.2 is devoted to explain how to recover de Rham cohomology groups with algebraic differential forms. This approach was pointed out by Grothendieck [Gro66], after an important result due to Atiyah and Hodge [HA55]. Using Atiyah-Hodge's theorem, we explain why we can define the *algebraic de Rham cohomology* for any smooth algebraic variety

X using hypercohomology (see Appendix) as

$$H^k_{\mathrm{dR}}(X/\mathbb{C}) := \mathbb{H}^k(X, \Omega_X^\bullet).$$

A remarkable fact about this algebraic de Rham cohomology, is that it comes with a natural filtration

$$F^i H^k_{\mathrm{dR}}(X/\mathbb{C}) := \mathrm{Im}(\mathbb{H}^k(X, \Omega_X^{\geq i}) \to \mathbb{H}^k(X, \Omega_X^\bullet)),$$

that turns out to coincide with the Hodge filtration for smooth projective varieties. Unfortunately, this filtration is not interesting for affine varieties.

In section 1.3 we introduce differential forms in X with logarithmic poles along Y. We prove the following theorem due to Deligne.

Theorem (Deligne [Del70])**.** *Let $i : U = X \setminus Y \hookrightarrow X$ be the inclusion. Then, the natural map*

$$\Omega_X^\bullet(\log Y) \hookrightarrow i_* \Omega_U^\bullet = \Omega_X^\bullet(*Y),$$

induces the isomorphism

$$\mathbb{H}^k(X, \Omega_X^\bullet(\log Y)) \simeq H^k_{dR}(U/\mathbb{C}).$$

We provide a proof based on the following lemma due to Carlson and Griffiths.

Lemma (Carlson-Griffiths [CG80])**.** *Suppose X is embedded in a projective space $\mathbb{P}^N$, and $Y = X \cap \{F = 0\}$ for some homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_N]$. Consider $\mathcal{U} = \{U_i\}_{i=0}^N$ the Jacobian covering of X given by $U_i = X \cap \{F_i \neq 0\}$, where $F_i := \frac{\partial F}{\partial x_i}$. For every $l \geq 2$ define*

$$H_l : C^q(\mathcal{U}, \Omega_X^p(lY)) \to C^q(\mathcal{U}, \Omega_X^{p-1}((l-1)Y)),$$

$$(H_l \omega)_{j_0 \cdots j_q} := \frac{1}{1-l} \frac{F}{F_{j_0}} \iota_{\frac{\partial}{\partial x_{j_0}}}(\omega_{j_0 \cdots j_q}),$$

where $\iota_{\frac{\partial}{\partial x_{j_0}}}$ denotes the usual contraction of differential forms with respect to $\frac{\partial}{\partial x_{j_0}}$. Letting $D = d + (-1)^p \delta$, then

$$DH + HD : \bigoplus_{p+q=k} \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))} \to \bigoplus_{p+q=k} \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))}$$

is the identity map.

Using this lemma we can prove Deligne's theorem in a constructive way, since the explicit operator $(1 - DH)$ reduces the pole order of forms in $H^k_{\mathrm{dR}}(U/\mathbb{C})$. This lemma will be useful to prove more results due to Griffiths in sections 1.5 and 1.6.

Using Deligne's theorem we introduce, in section 1.4, a *Hodge filtration on* U as

$$F^i H^k_{\mathrm{dR}}(U/\mathbb{C}) := Im(\mathbb{H}^k(X, \Omega_X^{\geq i}(\log Y)) \to \mathbb{H}^k(X, \Omega_X^\bullet(\log Y)) \simeq H^k_{\mathrm{dR}}(U/\mathbb{C})).$$

We use the *residue map*

$$res : H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \to H^{k}_{\mathrm{dR}}(Y/\mathbb{C})$$

to justify this is a good candidate for Hodge filtration. In fact, the residue map is a morphism of Hodge structures of type $(-1, -1)$, and furthermore the Hodge filtrations are compatible with the long exact sequence induced by the residue map, in the sense that we have the following long exact sequence

$$\cdots \to F^iH^{k+1}_{\mathrm{dR}}(X/\mathbb{C}) \to F^iH^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \xrightarrow{res} F^{i-1}H^{k}_{\mathrm{dR}}(Y/\mathbb{C}) \to F^iH^{k+2}_{\mathrm{dR}}(X/\mathbb{C}) \to \cdots.$$

Section 1.5 is about Griffiths' work on the cohomology of hypersurfaces. We use Carlson-Griffiths' lemma and Bott's formula to prove the following theorem.

Theorem (Griffiths [Gri69])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface given by $X = \{F = 0\}$, and $U := \mathbb{P}^{n+1} \setminus X$. For every $q = 0, \dots, n$, the natural map*

$$H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X)) \to H^{n+1}_{dR}(U/\mathbb{C})$$

has image equal to $F^{n+1-q}H^{n+1}_{dR}(U/\mathbb{C})$. Consequently, every piece of the Hodge filtration $F^{n-q}H^n_{dR}(X/\mathbb{C})_{prim}$, is generated by the residues of global forms with pole of order at most $q+1$ along X.

Using this theorem we prove the following result that tells us how to construct a basis for $H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim}$ compatible with the Hodge filtration.

Theorem (Griffiths [Gri69])**.** *For every $q = 0, \dots, n$ the kernel of the map*

$$\begin{aligned} \varphi : H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2)) &\to F^{n-q}H^n_{dR}(X/\mathbb{C})_{prim}/F^{n+1-q}H^n_{dR}(X/\mathbb{C})_{prim} \\ P &\mapsto res\left(\frac{P\Omega}{F^{q+1}}\right) \end{aligned}$$

is the degree $N = d(q+1) - n - 2$ part of the Jacobian ideal of F, $J^F_N \subset \mathbb{C}[x_0, \dots, x_{n+1}]_N$. Where $\Omega = \sum_{i=0}^{n+1}(-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_{n+1}$.

Finally in section 1.6, using again Carlson-Griffiths' lemma, we describe explicitly the residue map in Čech cohomology.

Theorem (Carlson-Griffiths [CG80])**.** *Let $q \in \{0, 1, \dots, n\}$, $P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2))$. Then*

$$res\left(\frac{P\Omega}{F^{q+1}}\right) = \frac{(-1)^{n(q+1)}}{q!}\left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=q} \in H^q(U, \Omega^{n-q}_X).$$

Where $\Omega_J := \frac{\partial}{\partial x_{j_q}}(\cdots \frac{\partial}{\partial x_{j_0}}(\Omega)\cdots)$, $F_J := F_{j_0} \cdots F_{j_q}$ and $U = \{U_i\}_{i=0}^{n+1}$ is the Jacobian covering restricted to X.

These theorems result in an explicit basis for the de Rham cohomology group of any hypersurface, which is compatible with its Hodge filtration. This basis will be of vital importance in our work, since we will use it to compute the periods of algebraic cycles.

We will use without proof some classical results in L^2 Hodge theory (such as Hodge decomposition), algebraic topology, several complex variables and sheaf cohomology (specially hypercohomology). Whenever we use one of these results for the first time, we will add a reference. Only for hypercohomology, we add the results we use in the Appendix.

1.1 Algebraic differential forms

In this section we will introduce algebraic differential forms on algebraic varieties. As we will see, these differential forms have nice algebraic descriptions that allow us to work with them and do computations explicitly in a wide range of cases.

Definition 1.1.1. Let R be a ring and S be an R-algebra. The *module of Kähler differentials of S over R* is the S-module generated by the set of symbols $\{df : f \in S\}$, subject to the relations

$$\begin{aligned} d(f \cdot g) &= f \cdot dg + g \cdot df \\ d(rf + sg) &= r \cdot df + s \cdot dg \end{aligned} \tag{1.1}$$

for all $f, g \in S$, and $r, s \in R$. In other words

$$\Omega^1_{S/R} := \left(\bigoplus_{f \in S} S \cdot df \right) / \langle N \rangle$$

where

$$N = \{d(fg) - f dg - g df, d(rf + sg) - r df - s dg : f, g \in S, r, s \in R\}.$$

Remark 1.1.1. If we consider the map $d : S \to \Omega^1_{S/R}$, the second relation in (1.1) says that d is an R-linear map. The first relation in (1.1) is called the *Leibniz' rule*, and d is a *derivation*. This map d is called the *universal R-linear derivation*, since every R-linear derivation $d' : S \to M$ to an S-module M factors as the composition of d with a unique S-linear map $\Omega^1_{S/R} \to M$. For a proof of this property and more about $\Omega^1_{S/R}$ see for instance [Eis95] Chapter 16. For us, the relevant constructions to keep in mind are summarized in the following examples:

Example 1.1.1. For a polynomial ring $S = R[x_1, \ldots, x_n]$,

$$\Omega^1_{S/R} = \bigoplus_{i=1}^{n} S \cdot dx_i.$$

Example 1.1.2. For $S = R[x_1, \ldots, x_n]/I$, $I = \langle f_1, \ldots, f_m \rangle$, then

$$\begin{aligned} \Omega^1_{S/R} &= \left(\bigoplus_{i=1}^{n} S \cdot dx_i \right) / \langle df_1, \ldots, df_m \rangle \\ &= \Omega^1_{R[x_1,\ldots,x_n]/R} / \langle I \cdot \Omega^1_{R[x_1,\ldots,x_n]/R}, df_1, \ldots, df_m \rangle. \end{aligned}$$

Example 1.1.3. If $U \subseteq S$ is a multiplicative subset, then

$$\Omega^1_{S[U^{-1}]/R} = S[U^{-1}] \otimes_R \Omega^1_{S/R};$$

in particular, localizing the module of Kähler differentials we obtain the module of Kähler differentials of the localization.

Definition 1.1.2. Let $X \subset \mathbb{C}^N$ be an affine algebraic variety. We define

$$\Omega^1_X(X) := \Omega^1_{\mathbb{C}[X]/\mathbb{C}},$$

where $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_N]/I(X)$ is the ring of regular functions on X.

Definition 1.1.3. Let X be a complex algebraic variety. For every open affine set $U \subset X$ we have an $\mathcal{O}_X(U)$-module of Kähler differentials $\Omega^1_U(U)$. For any pair $U \subset V$ of open affine subsets of X we have the natural restriction (morphism of $\mathbb{C}$-algebras)

$$\mathcal{O}_X(V) \to \mathcal{O}_X(U),$$

which induces the restriction of Kähler differentials

$$\Omega^1_V(V) \to \Omega^1_U(U).$$

Using these restrictions, we can glue these modules to define the *sheaf of algebraic differential forms on X*, which we denote Ω^1_X. This construction gives us a coherent $\mathcal{O}_X$-module. By taking the exterior power of Ω^1_X we obtain the coherent *sheaf of algebraic differential k-forms on X*

$$\Omega^k_X := \bigwedge^k \Omega^1_X.$$

When X is smooth of dimension n, these sheaves are locally free of rank $\binom{n}{k}$.

More generally, given any morphism of schemes $f : X \to Y$ we can define the *sheaf of relative algebraic differential forms* $\Omega^1_{X/Y}$ as the $\mathcal{O}_X$-module obtained by pasting the $\mathcal{O}_X(V)$-modules $\Omega^1_{S/R}$, where $U = \operatorname{Spec} R$, $V = \operatorname{Spec} S$ and $f(V) \subset U$. We can also define

$$\Omega^k_{X/Y} := \bigwedge^k \Omega^1_{X/Y}.$$

In what follows we will always consider morphisms $f : X \to Y$ between complex algebraic varieties. When $Y = \operatorname{Spec} R$ we will denote $\Omega^k_{X/R}$ instead of $\Omega^k_{X/Y}$. And when $Y = \operatorname{Spec} \mathbb{C}$ we will just denote Ω^k_X instead of $\Omega^k_{X/Y}$ or $\Omega^k_{X/\mathbb{C}}$.

Example 1.1.4. For $\varphi : X \to Y$ a morphism of schemes,

$$\Omega^k_{X/Y} \simeq \frac{\Omega^k_X}{\Omega^1_Y \wedge \Omega^{k-1}_X}.$$

Remark 1.1.2. One should not confuse Ω^k_X with the sheaf of holomorphic k-forms, we will denote the latter by $\Omega^k_{X^{\mathrm{hol}}}$. We will denote $\Omega^k_{X^\infty}$ the sheaf of (complex) C^∞ differential k-forms.

Definition 1.1.4. Consider the global vector field over $\mathbb{C}^{N+1}$

$$E := \sum_{i=0}^{N} x_i \frac{\partial}{\partial x_i}.$$

This vector field is called *Euler's vector field*. We define *Euler's sequence* to be

$$0 \to \Omega^{p+1}_{\mathbb{P}^N}(p+1) \to \mathcal{O}_{\mathbb{P}^N}^{\binom{N+1}{p+1}} \to \Omega^{p}_{\mathbb{P}^N}(p+1) \to 0, \tag{1.2}$$

where for every U_i in the standard covering of $\mathbb{P}^N$, and every $\omega = \sum_{|J|=p} a_J dx_{j_0} \wedge \cdots \wedge dx_{j_p} \in \Omega^{p+1}_{\mathbb{P}^N}(p+1)(U_i)$, the first map sends ω to $(a_J)_{|J|=p}$. While the second one is sending $(b_J)_{|J|=p}$, where each $b_J \in \mathcal{O}_{\mathbb{P}^N}(U_i)$, to the form $\eta = \iota_E(\sum_{|J|=p} b_J dx_{j_0} \wedge \cdots \wedge dx_{j_p})$.

Remark 1.1.3. In general, for $v = \sum_{i=0}^{N} v_i \frac{\partial}{\partial x_i}$ a vector field of $\mathbb{C}^{N+1}$, and $\omega = \sum_{|J|=p} b_J dx_{j_0} \wedge \cdots \wedge dx_{j_p} \in \Omega^{p+1}_{\mathbb{C}^{N+1}}(U)$ for some open set $U \subseteq \mathbb{C}^{N+1}$, the map $\iota_E(\omega)$ is the *contraction map of ω along the vector field $v|_U$*. It is defined as

$$\iota_v(\omega) := \sum_{|J|=p} b_J \cdot v_{j_0} dx_{j_1} \wedge \cdots \wedge dx_{j_p} \in \Omega^{p}_{\mathbb{C}^{n+1}}(U).$$

When $\omega \in \Omega_{\mathbb{P}^N}(V)$ for some open $V \subseteq \mathbb{P}^N$, we can also define $\iota_v(\omega)$ by lifting ω to an open set $U = \pi^{-1}(V) \subseteq \mathbb{C}^{N+1} \setminus \{0\}$, where $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ is the projection map. Then $\iota_v(\omega) := \pi(\iota_v(\pi^*\omega))$.

Remark 1.1.4. Euler's sequence (1.2) is an exact sequence. Using this exact sequence it is possible to prove the following extended version of Bott's vanishing theorem.

Theorem 1.1.1 (Bott's formula [Bot57])**.**

$$dim_{\mathbb{C}} H^q(\mathbb{P}^N; \Omega^p_{\mathbb{P}^N}(k)) = \begin{cases} \binom{k+N-p}{k}\binom{k-1}{p} & \text{if } q = 0, 0 \leq p \leq N, k > p; \\ 1 & \text{if } k = 0, 0 \leq p = q \leq N; \\ \binom{-k+p}{-k}\binom{-k-1}{N-p} & \text{if } q = N, 0 \leq p \leq N, k < p - N; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1.1.1. *Every $\omega \in H^0(\mathbb{P}^N; \Omega^{N-1}_{\mathbb{P}^N}(k))$ is of the form*

$$\omega = \sum_{i=0}^{N} T_i \, \iota_{\frac{\partial}{\partial x_i}}(\Omega),$$

where $T_i \in \mathbb{C}[x_0, \ldots, x_N]_{k-N}$, $\Omega := \iota_E(dx_0 \wedge \cdots \wedge dx_N) = \sum_{i=0}^{N}(-1)^i x_i \widehat{dx_i}$, and $\widehat{dx_i} := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N$.

Proof Twisting Euler's sequence (1.2) we get

$$0 \to \Omega^N_{\mathbb{P}^N}(k) \to \mathcal{O}^{N+1}_{\mathbb{P}^N}(k-N) \to \Omega^{N-1}_{\mathbb{P}^N}(k) \to 0. \tag{1.3}$$

Using the long exact sequence of (1.3) and Bott's formula we see that $H^0(\mathbb{P}^N;\Omega^{N-1}_{\mathbb{P}^N}(k))$ is generated by

$$\begin{aligned}
\iota_E(\sum_{i=0}^{N} P_i \widehat{dx_i}) &= \iota_E(\sum_{i=0}^{N}(-1)^i P_i \iota_{\frac{\partial}{\partial x_i}}(dx_0 \wedge \cdots \wedge dx_N));\\
&= \iota_E(\iota_{\sum_{i=0}^{N}(-1)^i P_i \frac{\partial}{\partial x_i}}(dx_0 \wedge \cdots \wedge dx_N));\\
&= -\iota_{\sum_{i=0}^{N}(-1)^i P_i \frac{\partial}{\partial x_i}}(\iota_E(dx_0 \wedge \cdots \wedge dx_N));\\
&= \sum_{i=0}^{N}(-1)^{i+1} P_i \iota_{\frac{\partial}{\partial x_i}}(\Omega);
\end{aligned}$$

where $P_i \in H^0(\mathbb{P}^N;\mathcal{O}_{\mathbb{P}^N}(k-N))$. Take $T_i := (-1)^{i+1}P_i$. ■

1.2 Algebraic de Rham cohomology

In this section we will explain how algebraic de Rham cohomology can be constructed for any smooth algebraic variety over $\mathbb{C}$. From now on, we will assume some knowledge in hypercohomology. Our main reference is [Mov17b] Chapter 3, and we will adopt his notation. Nevertheless, we will state in the Appendix the properties of hypercohomology we are interested in.

Consider X any smooth algebraic variety over $\mathbb{C}$. Recall the different sheaves of differential forms over X:

$$\begin{aligned} \Omega^k_X &= \text{sheaf of } \textit{algebraic} \text{ differential } k\text{-forms over the Zarisky topology;} \\ \Omega^k_{X^{hol}} &= \text{sheaf of } \textit{holomorphic} \text{ differential } k\text{-forms over the analytic topology;} \\ \Omega^k_{X^{\infty}} &= \text{sheaf of (complex) } C^{\infty} \text{ differential } k\text{-forms over the analytic topology.} \end{aligned}$$

The key result that allows us to define algebraic de Rham cohomology (i.e. just in terms of Ω^k_X) is the following theorem due to Atiyah and Hodge:

Theorem 1.2.1 (Atiyah-Hodge [HA55]). *Let U be a smooth affine variety over $\mathbb{C}$. Then, the canonical map*

$$H^k(\Gamma(\Omega_U), d) \xrightarrow{\sim} H^k(\Gamma(\Omega_{U^{\infty}}), d) = H^k_{dR}(U)$$

is an isomorphism.

Remark 1.2.1. Atiyah-Hodge's theorem can be stated as a composition of natural isomorphisms

$$H^k(\Gamma(\Omega_U), d) \simeq H^k(\Gamma(\Omega_{U^{hol}}), \partial) \simeq H^k(\Gamma(\Omega_{U^{\infty}}), d).$$

In fact, noticing that $(\Omega_{U^{\infty}}, d)$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$ (by Poincaré's lemma, see [BT82] Chapter 1 section 4) and furthermore it is acyclic (since they are fine sheaves, i.e. they admit partition of unity), we can apply Corollary 4.1.1 (see Appendix) to obtain the isomorphism

$$H^k(\Gamma(\Omega_{U^{\infty}}), d) \simeq H^k(U, \underline{\mathbb{C}}).$$

On the other hand, $(\Omega_{U^{hol}}, \partial)$ is also a resolution of $\underline{\mathbb{C}}$ (this can also be proved as a consequence of Poincaré's lemma), and is acyclic (since U is affine it is also Stein, see [Gun90] page 131, Theorem 2), then by Corollary 4.1.1 we conclude

$$H^k(\Gamma(\Omega_{U^{hol}}), \partial) \simeq H^k(U, \underline{\mathbb{C}}) \simeq H^k(\Gamma(\Omega_{U^{\infty}}), d).$$

Let us return to X any smooth algebraic variety over $\mathbb{C}$. After Remark 1.2.1, an immediate consequence of Atiyah-Hodge's theorem is that the natural inclusion of complexes

$$(\Omega_X, d) \hookrightarrow (\Omega_{X^{hol}}, \partial)$$

of sheaves over the Zariski topology of X is a quasi-isomorphism (see Appendix, Definition 4.1.3). In fact, Atiyah-Hodge's theorem is saying that the map induced between the cohomology pre-sheaves is an isomorphism in every affine open subset of X. Using Proposition 4.1.4 (see Appendix) we get

$$\mathbb{H}^k(X, \Omega_X) \simeq \mathbb{H}^k(X, \Omega_{X^{hol}}),$$

where $(\Omega_{X^{hol}}, \partial)$ is a complex of sheaves considered over the Zariski topology of X. But, using again the fact that every affine open set is Stein and Proposition 4.1.1 (see Appendix) we conclude that the hypercohomology of the complex $(\Omega_{X^{hol}}, \partial)$ over the analytic topology, can be computed via the affine covering, i.e. it coincides with the hypercohomology considered over the Zariski topology. On the other hand, over the analytic topology, $(\Omega_{X^{hol}}, \partial)$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$, then by Proposition 4.1.3 (see Appendix) we obtain the isomorphism

$$\mathbb{H}^k(X, \Omega_X) \simeq \mathbb{H}^k(X, \Omega_{X^{hol}}) \simeq H^k(X, \underline{\mathbb{C}}).$$

Finally, since (Ω_{X^∞}, d) is an acyclic resolution of $\underline{\mathbb{C}}$, we conclude (by Corollary 4.1.1, Appendix)

$$\mathbb{H}^k(X, \Omega_X) \simeq \mathbb{H}^k(X, \Omega_{X^{hol}}) \simeq H^k(X, \underline{\mathbb{C}}) \simeq H^k(\Gamma(\Omega_{X^\infty}), d) = H^k_{\mathrm{dR}}(X),$$

i.e. we obtained a way to recover the de Rham cohomology of any smooth complex algebraic variety X using algebraic differential forms.

Definition 1.2.1. For any morphism of schemes $X \to Y$, we define the *algebraic de Rham cohomology groups*

$$H^k_{\mathrm{dR}}(X/Y) := \mathbb{H}^k(X, \Omega_{X/Y}).$$

In the case $Y = \mathrm{Spec}\, R$ we will simply denote it by $H^k_{\mathrm{dR}}(X/R)$.

The preceding remarks show that this definition is compatible with the classical one in the case $X \to \mathrm{Spec}\, \mathbb{C}$ is a smooth complex algebraic variety, so we have accomplished our goal of recovering de Rham cohomology in the algebraic context. Our next goal is trying to recover the Hodge decomposition algebraically. Unfortunately, there is no good (i.e. canonical) candidate for a Hodge structure on $H^k_{\mathrm{dR}}(X/Y)$. But on the other hand, $H^k_{\mathrm{dR}}(X/Y)$ has a natural filtration, namely

$$F^i := Im(\mathbb{H}^k(X, \Omega^{\geq i}_{X/Y}) \to \mathbb{H}^k(X, \Omega_{X/Y})).$$

Proposition 1.2.1. *For $X \to$ Spec $\mathbb{C}$ smooth and projective variety. There is an isomorphism*

$$F^i/F^{i+1} \simeq H^{k-i}(X, \Omega^i_X).$$

Furthermore, we can identify $\quad F^i \simeq \mathbb{H}^k(X, \Omega_X^{\geq i}).$

Proof Consider U an affine finite cover of X. Using Proposition 4.1.1, we can represent every element $\omega \in \mathbb{H}^k(X;\Omega_X^{\geq i})$ as a sum $\omega = \omega^i + \omega^{i+1} + \cdots + \omega^k$, where each $\omega^l \in C^{k-l}(U;\Omega^l_X)$. Take the map

$$\begin{aligned} \mathbb{H}^k(X;\Omega_X^{\geq i}) &\longrightarrow H^{k-i}(X;\Omega^i_X). \\ \sum_{j=i}^{k} \omega_j &\longmapsto \omega_i \end{aligned}$$

To show this map is well defined, suppose ω is zero in $\mathbb{H}^k(U;\Omega_X^{\geq i})$, then $\omega = D\eta$ and $\omega^i = (-1)^i \delta\eta^i$, so ω^i is zero in $H^{k-i}(X;\Omega^i_X)$. We claim the kernel of this map is $\tilde{F}^{i+1} := \mathrm{Im}(\mathbb{H}^k(X;\Omega_X^{\geq i+1}) \to \mathbb{H}^k(X;\Omega_X^{\geq i}))$. In fact, is clear that $\tilde{F}^{i+1}$ is in the kernel. Furthermore, if $\omega^i = \delta\eta^i$, then $\omega - (-1)^i D\eta^i \in \tilde{F}^{i+1}$. Then, we have an inclusion

$$\mathbb{H}^k(X;\Omega_X^{\geq i})/\tilde{F}^{i+1} \hookrightarrow H^{k-i}(X;\Omega^i_X). \tag{1.4}$$

On the other hand, we have a natural projection

$$\mathbb{H}^k(X;\Omega_X^{\geq i})/\tilde{F}^{i+1} \twoheadrightarrow F^i/F^{i+1}. \tag{1.5}$$

These two facts imply

$$\dim_{\mathbb{C}} F^i/F^{i+1} \leq \dim_{\mathbb{C}} H^{k-i}(X;\Omega^i_X). \tag{1.6}$$

But, on the left hand side of (1.6) the dimensions add up to the dimension of $F^0 = H^k_{\mathrm{dR}}(X/\mathbb{C})$. While on the right hand side we have the Hodge numbers, and by Hodge decomposition they add up to the dimension of $H^k_{\mathrm{dR}}(X)$. Thus, (1.6) is an equality, then (1.4) and (1.5) are isomorphisms. Composing the inverse of (1.5) with (1.4) we have that the map

$$\begin{aligned} F^i/F^{i+1} &\longrightarrow H^{k-i}(X;\Omega^i_X) \\ \sum_{j=i}^{k} \omega_j &\longmapsto \omega_i \end{aligned}$$

is the desired isomorphism. Finally, to prove the isomorphism $\mathbb{H}^k(X;\Omega_X^{\geq i}) \simeq F^i$, we proceed inductively on $i = k, k-1, \ldots, 0$. For $i = k$ just use the isomorphism (1.5). For $i < k$, we assume we have the isomorphism $\mathbb{H}^k(X;\Omega_X^{\geq i+1}) \simeq F^{i+1}$, since this isomorphism factors as $\mathbb{H}^k(X;\Omega_X^{\geq i+1}) \simeq \tilde{F}^{i+1} \simeq F^{i+1}$ we can use (1.5) to conclude $\mathbb{H}^k(X;\Omega_X^{\geq i}) \simeq F^i$. ∎

Corollary 1.2.1. *Let X be a smooth projective variety, then via the natural isomorphism*

$$H^k_{dR}(X/\mathbb{C}) \simeq H^k_{dR}(X), \tag{1.7}$$

F^i corresponds with the classical Hodge filtration

$$F^i \simeq F^i H^k_{dR}(X) := \bigoplus_{p=i}^{k} H^{p,k-p}(X).$$

Proof The isomorphism (1.7) is given by the quasi-isomorphisms

$$(\Omega_X^{\bullet}, d) \hookrightarrow (\Omega_{X^{hol}}^{\bullet}, \partial), \tag{1.8}$$

$$(\Omega_{X^{hol}}^{\bullet}, \partial) \hookrightarrow (\Omega_{X^{\infty}}^{\bullet}, d). \tag{1.9}$$

Where (1.8) is over the Zariski topology of X, while (1.9) is over the analytic topology of X. If we consider the truncated complexes, we have the corresponding quasi-isomorphisms

$$(\Omega_X^{\geq i}, d) \hookrightarrow (\Omega_{X^{hol}}^{\geq i}, \partial), \tag{1.10}$$

$$(\Omega_{X^{hol}}^{\geq i}, \partial) \hookrightarrow (\bigoplus_{p \geq i} \Omega_{X^{\infty}}^{p, \bullet - p}, d). \tag{1.11}$$

Note that both complexes of (1.11) are resolutions of the sheaf of holomorphic ∂-closed i-forms. Therefore, the image of F^i inside $H^k_{\mathrm{dR}}(X)$ is exactly

$$\check{F}^i := Im(H^k(\Gamma(\bigoplus_{p \geq i} \Omega_{X^{\infty}}^{p, \bullet - p}), d) \to H^k_{\mathrm{dR}}(X)).$$

In particular, $F^i H^k_{\mathrm{dR}}(X) \subseteq \check{F}^i$. Since

$$\check{F}^i/\check{F}^{i+1} \simeq F^i/F^{i+1} \simeq H^{k-i}(X, \Omega^i_X) \simeq H^{i,k-i}(X) = F^i H^k_{\mathrm{dR}}(X)/F^{i+1} H^k_{\mathrm{dR}}(X),$$

we conclude $\check{F}^i = F^i H^k_{\mathrm{dR}}(X)$. ∎

The previous proposition and its corollary justify the following definition:

Definition 1.2.2. For $X \to Y$ smooth and projective, we define the *algebraic Hodge filtration*

$$F^i H^k_{\mathrm{dR}}(X/Y) := Im(\mathbb{H}^k(X, \Omega_{X/Y}^{\geq i}) \to \mathbb{H}^k(X, \Omega_{X/Y}^{\bullet})).$$

Remark 1.2.2. The previous definition could be made without hypothesis on the morphism $X \to Y$, but we realize that it does not make much sense for the non-projective case. In fact, if $X = U$ is affine and $Y = \operatorname{Spec} \mathbb{C}$, Atiyah-Hodge's isomorphism

$$H^k(\Gamma(\Omega_U^{\bullet}), d) \simeq \mathbb{H}^k(U, \Omega_U^{\bullet})$$

has image $F^k = Im(\mathbb{H}^k(U, \Omega_U^{\geq k}) \to \mathbb{H}^k(U, \Omega_U^{\bullet}))$, in particular $F^0 = F^1 = \cdots = F^k$. Another way to see this, is observing that the inequality

$$\dim_{\mathbb{C}} F^i/F^{i+1} \leq \dim_{\mathbb{C}} H^{k-i}(X, \Omega^i_{X/Y}),$$

holds without restrictions on $X \to Y$. Then, for the affine case, we always have $F^i/F^{i+1} = 0$ for $i = 0, \ldots, k-1$.

1.3 Logarithmic differential forms

In the next section we will introduce a Hodge filtration for affine varieties as in Deligne's work [Del71]. Before doing that, we need to restrict ourselves to a subclass of meromorphic differential forms. The so called differential forms with logarithmic poles. For simplicity, we will restrict ourselves to the following context: Let X be a smooth projective variety, and $Y \subset X$ a smooth hyperplane section. We will consider affine varieties of the form

$$U := X \setminus Y.$$

Definition 1.3.1. Let X be a smooth projective variety and $Y \subset X$ be a smooth hyperplane section. We define the *sheaf of rational p-forms with logarithmic poles along Y* as

$$\Omega^p_X(log Y) := \mathrm{Ker}(\Omega^p_X(Y) \xrightarrow{d} \Omega^p_X(2Y)/\Omega^p_X(Y)).$$

Analogously, we define the *sheaf of meromorphic p-forms with logarithmic poles along Y* as

$$\Omega^p_{X^{hol}}(\log Y) := \mathrm{Ker}(\Omega^p_{X^{hol}}(Y) \xrightarrow{d} \Omega^p_{X^{hol}}(2Y)/\Omega^p_{X^{hol}}(Y)).$$

By Serre's GAGA principle we know $\Omega^p_{X^{hol}}(\log Y)$ is the analytification of $\Omega^p_X(\log Y)$.

Theorem 1.3.1 (Deligne [Del70])**.** *In the context of the previous definition, let $i : U = X \setminus Y \hookrightarrow X$ be the inclusion. Then, the natural map*

$$\Omega^\bullet_X(\log Y) \hookrightarrow i_*\Omega^\bullet_U = \Omega^\bullet_X(*Y), \tag{1.12}$$

induces the isomorphism

$$\mathbb{H}^k(X, \Omega^\bullet_X(\log Y)) \simeq H^k_{dR}(U/\mathbb{C}). \tag{1.13}$$

Where $\Omega^p_X(*Y)$ denotes the sheaf of algebraic p-forms with poles (of arbitrary order) along Y. Theorem 1.3.1 can be proved in the general context where Y is an ample normal crossing divisor of X, by proving that (1.12) is a quasi-isomorphism (see [Del70]). We will provide a different proof, by constructing the isomorphism (1.13) explicitly, using a lemma due to Carlson and Griffiths.

Definition 1.3.2. Suppose X is embedded in a projective space $\mathbb{P}^N$, and $Y = X \cap \{F = 0\}$ for some homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_N]$. Consider $\mathcal{U} = \{U_i\}_{i=0}^N$ the *Jacobian covering* of X given by $U_i = X \cap \{F_i \neq 0\}$, where $F_i := \frac{\partial F}{\partial x_i}$. For every $l \geq 2$ define

$$H_l : C^q(\mathcal{U}, \Omega^p_X(lY)) \to C^q(\mathcal{U}, \Omega^{p-1}_X((l-1)Y)),$$

$$(H_l\omega)_{j_0 \cdots j_q} := \frac{1}{1-l}\frac{F}{F_{j_0}} \iota_{\frac{\partial}{\partial x_{j_0}}}(\omega_{j_0 \cdots j_q}).$$

Lemma 1.3.1 (Carlson-Griffiths [CG80])**.** *For every* $l \quad 2$*, letting* $D = d + (\quad 1)^p \quad$*, then*

$$DH + HD : \bigoplus_{p+q=k} \frac{C^q(U; \Omega_X^p(lY))}{C^q(U; \Omega_X^p((l \quad 1)Y))} \; ! \; \bigoplus_{p+q=k} \frac{C^q(U; \Omega_X^p(lY))}{C^q(U; \Omega_X^p((l \quad 1)Y))}$$

is the identity map. Note that D *is the is the differential map used to define the hypercohomology groups (see Appendix, Definition 4.1.1).*

Proof of Theorem 1.3.1 The inclusion $\quad : \Omega_X(\log Y) \, , ! \; \Omega_X(\quad Y)$ induces the morphism

$$' : \mathrm{H}^k(U; \Omega_X(\log Y)) \; ! \; \mathrm{H}^k(U; \Omega_X(\quad Y)):$$

Let us prove $\;'$ is an isomorphism. Given

$$! = \sum_{p=0}^{k} !^p \; 2 \bigoplus_{p+q=k} C^q(U; \Omega_X^p(\quad Y))$$

such that $D! = 0$, there exist $l \quad 1$ such that

$$! \; 2 \bigoplus_{p+q=k} C^q(U; \Omega_X^p(lY)):$$

If $l = 1$, we claim $! \; 2 \bigoplus_{p+q=k} C^q(U; \Omega_Y^p(\log Y))$. In fact, since $D! = 0$, we get

$$d!^p = (\quad 1)^p \quad !^{p+1} \; 2 \; C^q(U; \Omega_X^{p+1}(Y));$$

as desired.

For $l \quad 2$, it follows from the lemma that $!$ is cohomologous (in hypercohomology) to the D-closed element

$$:= (1 \quad DH)^{l \quad 1} ! \; 2 \bigoplus_{p+q=k} C^q(U; \Omega_X^p(Y)):$$

And it follows from the case $l = 1$ that

$$2 \bigoplus_{p+q=k} C^q(U; \Omega_Y^p(\log Y));$$

i.e. $\;'$ is surjective.

Now, for the injectivity, consider

$$! \; 2 \bigoplus_{p+q=k} C^q(U; \Omega_X^p(\log Y));$$

such that there exist

$$2 \bigoplus_{p+q=k \quad 1} C^q(U; \Omega_X^p(lY))$$

for some l 1, with

$$D = !:$$

If $l = 1$, we claim $2 \bigoplus_{p+q=k\ 1} C^q(U; \Omega^p_Y(\log Y))$. In fact, since $D = !$, we get

$$d\ \ ^{k\ 1} = !^k\ 2\ C^0(U; \Omega^k_X(Y));$$

then $^{k\ 1}\ 2\ C^0(U; \Omega^{k\ 1}_X(\log Y))$. Inductively, if we assume $^{p+1}\ 2\ C^{q\ 1}(U; \Omega^{p+1}_X(\log Y))$, then

$$d\ ^p = !^{p+1} + (\ 1)^p\ \ ^{p+1}\ 2\ C^q(U; \Omega^{p+1}_X(\log Y));$$

as a consequence $^p\ 2\ C^q(U; \Omega^p_X(\log Y))$.

Finally, for l 2, if we take

$$:= (1\ \ DH)^{l\ 1}\ :$$

Since $(1\ \ DH)\ = H! +\ \ 2 \bigoplus_{p+q=k} C^q(U; \Omega^p_X((l\ \ 1)Y))$, and $D((1\ \ DH)\ ) = D\ = !$, it is clear that $D\ = !$ and

$$2 \bigoplus_{p+q=k\ 1} C^q(U; \Omega^p_X(Y));$$

and by the case $l = 1$, we actually see that

$$2 \bigoplus_{p+q=k\ 1} C^q(U; \Omega^p_X(\log Y));$$

as desired. ■

Proof of Carlson-Griffiths' Lemma 1.3.1 First, we claim

$$dH_l + H_{l+1}d : \frac{C^q(U; \Omega^p_X(lY))}{C^q(U; \Omega^p_X((l\ \ 1)Y))}\ !\ \frac{C^q(U; \Omega^p_X(lY))}{C^q(U; \Omega^p_X((l\ \ 1)Y))}$$

is the identity map. In fact, take $!_J = \frac{J}{F^l}\ 2\ C^q(U; \Omega^p_X(lY))$, then

$$\begin{aligned}
(dH_l + H_{l+1}d)!_J &= d\left(\frac{1}{1\ \ l}\frac{F}{F_{j_0}}\ \frac{@}{@x_{j_0}}\left(\frac{J}{F^l}\right)\right)\ \ \frac{1}{l}\frac{F}{F_{j_0}}\ \frac{@}{@x_{j_0}}\left(\frac{d\ \ J}{F^l}\ \ l\frac{dF\ ^\ \ J}{F^{l+1}}\right) \\
&\quad \frac{1}{1\ \ l}\frac{dF}{F_{j_0}}\ ^\ \frac{\frac{@}{@x_{j_0}}(\ \ J)}{F^l}\ \ \frac{l}{1\ \ l}\frac{F}{F_{j_0}}\frac{dF\ ^\ \ \frac{@}{@x_{j_0}}(\ \ J)}{F^{l+1}} \\
&\quad + \frac{F}{F_{j_0}}\frac{(F_{j_0}\ \ J\ \ \ dF\ ^\ \ \frac{@}{@x_{j_0}}(\ \ J))}{F^{l+1}} \\
&= \frac{J}{F^l} = !_J;
\end{aligned}$$

where the congruence is taken mod $C^q(U;\Omega^p_X((l-1)Y))$. Using this, take $\omega=\sum_{p=0}^{k}\omega^p \in \bigoplus_{p+q=k} C^q(U;\Omega^p_X(lY))$ then

$$\begin{aligned}
\omega - DH\omega &= \sum_{p+q=k}(\omega^p - dH\omega^p - (-1)^{p-1}\delta H\omega^p)\\
&\equiv \sum_{p+q=k}(Hd\omega^p + (-1)^p H\delta\omega^p)\\
&\equiv \sum_{p+q=k} H(d\omega^p + (-1)^p \delta\omega^p) = HD\omega.
\end{aligned}$$

Where in the last congruence we used $\delta H\omega^p \equiv H\delta\omega^p$. In fact

$$\begin{aligned}
(\delta H_l\omega^p)_{j_0\cdots j_{q+1}} &= \sum_{m=0}^{q+1}(-1)^m (H_l\omega^p)_{j_0\cdots\widehat{j_m}\cdots j_{q+1}}\\
&= \frac{1}{1-l}\left(\frac{F}{F_{j_1}}\frac{\partial}{\partial x_{j_1}}(\omega^p_{j_1\cdots j_{q+1}}) + \frac{F}{F_{j_0}}\frac{\partial}{\partial x_{j_0}}\left(\sum_{m=1}^{q+1}(-1)^m\omega^p_{j_0\cdots\widehat{j_m}\cdots j_{q+1}}\right)\right)\\
&= \frac{F}{1-l}\left(\frac{\frac{\partial}{\partial x_{j_1}}(\omega^p_{j_1\cdots j_{q+1}})}{F_{j_1}} - \frac{\frac{\partial}{\partial x_{j_0}}(\omega^p_{j_1\cdots j_{q+1}})}{F_{j_0}}\right) + (H_l\delta\omega^p)_{j_0\cdots j_{q+1}}.
\end{aligned}$$

■

1.4 Hodge filtration for affine varieties

In this section we introduce a Hodge filtration for affine varieties. In order to justify this is the good candidate for the Hodge filtration, we will introduce the residue map for algebraic de Rham cohomology, and show it is compatible with Hodge filtrations. Let us recall first the classical residue map.

Let X be a smooth projective variety, $Y \subset X$ a smooth hyperplane section, and $U := X \setminus Y$. Taking the long exact sequence in cohomology of the pair (X, U), and using Leray-Thom-Gysin isomorphism (see [Mov17a] Chapter 4, section 6) we obtain the exact sequence

$$\cdots \to H^{k+1}_{\mathrm{dR}}(X) \to H^{k+1}_{\mathrm{dR}}(U) \xrightarrow{res} H^{k}_{\mathrm{dR}}(Y) \xrightarrow{\cup\theta} H^{k+2}_{\mathrm{dR}}(X) \to \cdots \tag{1.14}$$

where res is the residue map, and $\cup\theta$ corresponds to the wedge product with the polarization θ. In particular, we obtain a surjective map

$$res : H^{k+1}_{\mathrm{dR}}(U) \to H^{k}_{\mathrm{dR}}(Y)_{prim} = \mathrm{Ker}\,\cup\theta,$$

where $H^{k}_{\mathrm{dR}}(Y)_{prim}$ is the *primitive part* of de Rham cohomology, i.e. is the complementary space to $\theta^{\frac{k}{2}}$ inside $H^{k}_{\mathrm{dR}}(Y)$ (see [Mov17a] Chapter 5, section 7). We want to determine the algebraic counterpart of this map (together with its long exact sequence), and define a Hodge filtration for U, compatible with the Hodge filtration of Y via this map.

Definition 1.4.1. For $U = X \setminus Y$, where X is smooth projective, and Y is a smooth hyperplane section. We define the *(algebraic) Hodge filtration of* U

$$F^i H^k_{\mathrm{dR}}(U/\mathbb{C}) := Im(\mathbb{H}^k(X, \Omega_X^{\bullet \geq i}(\log Y)) \to \mathbb{H}^k(X, \Omega_X^{\bullet}(\log Y)) \cong H^k_{\mathrm{dR}}(U/\mathbb{C})).$$

In order to justify the previous definition of Hodge filtration for affine varieties, we need to introduce the residue map in algebraic de Rham cohomology.

Proposition 1.4.1. *Let X be a smooth projective variety of dimension n and $Y \subset X$ a smooth hyperplane section. Let $z_1, \dots, z_n$ be local coordinates on an open set V of X, such that $V \cap Y = \{z_1 = 0\}$. Then $\Omega^p_{X^{hol}}(\log Y)|_V$ is a free $O_{V^{hol}}$-module, for which $dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $\frac{dz_1}{z_1} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_{p-1}}$, $i_k, j_l \in \{2, \dots, n\}$, form a basis. In particular $\Omega^p_{X^{hol}}(\log Y)$ is locally free.*

Proof Let $\omega \in \Gamma(V, \Omega^p_{X^{hol}}(\log Y))$. Since it has pole of order 1 there exist $\eta \in \Gamma(V, \Omega^p_{X^{hol}})$ such that $\omega = \frac{\eta}{z_1}$. Furthermore, since the same happens to $d\omega$, we conclude $dz_1 \wedge \eta = 0$, i.e. $\eta = dz_1 \wedge \alpha$, where α is a holomorphic $(p-1)$-form, just depending on $dz_2, \dots, dz_n$. ■

Definition 1.4.2. Let X be a smooth projective variety of dimension n and $Y \subset X$ a smooth hyperplane section. Let $z_1, \dots, z_n$ be local coordinates on an open set V of X, such that $V \cap Y = \{z_1 = 0\}$. For $\omega \in \Gamma(V, \Omega^p_{X^{hol}}(\log Y))$ we define its *residue at* Y to be

$$Res(\omega) := \alpha|_{Y \cap V} \in \Gamma(V, j_* \Omega^{p-1}_{Y^{hol}}),$$

where $= \wedge \frac{dz_1}{z_1} +$, and , are holomorphic forms. This definition does not depend on the choice of the coordinates, and defines the *residue map*

$$Res : \Omega^p_{X\,hol}(\log Y) \to j\ \Omega^{p-1}_{Y\,hol}.$$

Which is part of the following exact sequence

$$0 \to \Omega^p_{X\,hol} \to \Omega^p_{X\,hol}(\log Y) \xrightarrow{Res} j\ \Omega^{p-1}_{Y\,hol} \to 0,$$

called *Poincaré's residue sequence.* Using Serre's GAGA principle we have its algebraic counterpart

$$0 \to \Omega^p_X \to \Omega^p_X(\log Y) \xrightarrow{Res} j\ \Omega^{p-1}_Y \to 0,$$

which we also call Poincaré's residue sequence.

This sequence gives rise to a short exact sequence of complexes

$$0 \to \Omega_X \to \Omega_X(\log Y) \xrightarrow{Res} j\ \Omega_Y^{-1} \to 0. \tag{1.15}$$

Taking the long exact sequence in hypercohomology we get the exact sequence

$$\to H^{k+1}_{dR}(X/C) \to H^{k+1}_{dR}(U/C) \xrightarrow{res} H^k_{dR}(Y/C) \to H^{k+2}_{dR}(X/C) \to \quad , \tag{1.16}$$

Which turns out to be the algebraic counterpart of the sequence (1.14). Since for every $i \geq 0$ we have the short exact sequence of complexes

$$0 \to \Omega_X^{\geq i} \to \Omega_X^{\geq i}(\log Y) \xrightarrow{Res} j\ \Omega_Y^{\geq i-1} \to 0.$$

We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\to \mathrm{H}^{k+1}(X, \Omega_X^{\geq i}) & \to & \mathrm{H}^{k+1}(X, \Omega_X^{\geq i}(\log Y)) & \to & \mathrm{H}^k(Y, \Omega_Y^{\geq i-1}) & \to \\
f \downarrow & & g \downarrow & & h \downarrow & \\
\to H^{k+1}_{dR}(X/C) & \longrightarrow & H^{k+1}_{dR}(U/C) & \xrightarrow{res} & H^k_{dR}(Y/C) & \longrightarrow
\end{array}$$

The vertical arrows of this diagram are all injective. In fact, the maps f and h are injective by Proposition 1.2.1. The injectivity of g is more delicate and is a theorem due to Deligne. For the reader who is acquainted with the theory of spectral sequences, Deligne's theorem affirm that the spectral sequence associated to the naive filtration of $\Omega_X(\log Y)$ degenerates at E_1 (see [Voi02] Theorem 8.35 or [Del71] Corollary 3.2.13). This fact is equivalent to the injectivity of g, but since we do not want to introduce the theory of spectral sequences, we skip its proof. As a consequence, the exact sequence (1.16) is compatible with Hodge filtrations, i.e. we have the exact sequence

$$\to F^i H^{k+1}_{dR}(X/C) \to F^i H^{k+1}_{dR}(U/C) \xrightarrow{res} F^{i-1} H^k_{dR}(Y/C) \to F^i H^{k+2}_{dR}(X/C) \to \quad .$$

1.5 Cohomology of hypersurfaces

As an application of the previous results, we will present here Griffiths' results on the cohomology of hypersurfaces [Gri69]. This work culminates with an explicit basis for the primitive cohomology of a hypersurface, compatible with the Hodge filtration. Griffiths' basis is fundamental for us and will reappear everywhere in the rest of the text.

Let $X \subset \mathsf{P}^{n+1}$ be a smooth hypersurface of degree d, and $U = \mathsf{P}^{n+1} \setminus X$. In order to construct the basis for $H^k_{\mathrm{dR}}(X)_{prim}$ we will give generators for $H^{k+1}_{\mathrm{dR}}(U/\mathsf{C})$, compatible with the algebraic Hodge filtration $F^{i+1}H^{k+1}_{\mathrm{dR}}(U/\mathsf{C})$. Then, we will obtain the desired basis by applying the algebraic residue map to the generators, and reduce the set of generators to a basis.

Remark 1.5.1. Since the residue map

$$H^{k+1}_{\mathrm{dR}}(U/\mathsf{C}) \xrightarrow{res} H^k_{\mathrm{dR}}(X/\mathsf{C})_{prim}$$

is an isomorphism (because $H^{k+1}_{\mathrm{dR}}(\mathsf{P}^{n+1})_{prim} = 0$). We conclude that $H^k_{\mathrm{dR}}(X/\mathsf{C})_{prim} = 0$ for all $k \neq n$ (because $H^{k+1}_{\mathrm{dR}}(U/\mathsf{C}) = 0$ for $k+1 \neq n+1$, see [Mov17a] Chapter 5, section 5). Thus, we are just interested in determining a basis for the non-trivial primitive cohomology group of X

$$H^n_{\mathrm{dR}}(X/\mathsf{C})_{prim}.$$

We start by giving a set of generators.

Theorem 1.5.1 (Griffiths [Gri69])**.** *For every $q = 0, \ldots, n$, the natural map*

$$H^0(\mathsf{P}^{n+1}, \Omega^{n+1}_{\mathsf{P}^{n+1}}((q+1)X)) \to H^{n+1}_{dR}(U/\mathsf{C})$$

has image equal to $F^{n+1-q}H^{n+1}_{dR}(U/\mathsf{C})$. Consequently, every piece of the Hodge filtration

$$F^{n-q}H^n_{dR}(X/\mathsf{C})_{prim},$$

is generated by the residues of global forms with pole of order at most $q+1$ along X.

Proof Consider $\omega^{n+1} \in H^0(\mathsf{P}^{n+1}, \Omega^{n+1}_{\mathsf{P}^{n+1}}((q+1)X))$. The natural map sends it to

$$\omega \in H^{n+1}_{\mathrm{dR}}(U/\mathsf{C}) = \mathsf{H}^{n+1}(\mathsf{P}^{n+1}, \Omega^\bullet_{\mathsf{P}^{n+1}}(*X)),$$

by letting $\omega^k = 0$ for $k = 0, \ldots, n$. To see in which part of Hodge filtration ω is, we need to write it as an element of $\mathsf{H}^{n+1}(\mathsf{P}^{n+1}, \Omega^\bullet_{\mathsf{P}^{n+1}}(\log X))$, i.e. we need to reduce the order of the pole of ω^{n+1} up to order 1. Thanks to Carlson-Griffiths Lemma 1.3.1, we know how to do this applying the operator $(1 - DH)$. In order to obtain a form with poles of order 1 we need to apply it q times, i.e. the image of ω in $\mathsf{H}^{n+1}(\mathsf{P}^{n+1}, \Omega^\bullet_{\mathsf{P}^{n+1}}(\log X))$ is represented by $(1-DH)^q\omega$. It is clear by the definition of H and D that

$$(1-DH)^l\omega \in \mathrm{Im}(\mathsf{H}^{n+1}(\mathsf{P}^{n+1}, \Omega^{\bullet \geq n+1-l}_{\mathsf{P}^{n+1}}((q+1-l)X)) \to \mathsf{H}^{n+1}(\mathsf{P}^{n+1}, \Omega^\bullet_{\mathsf{P}^{n+1}}((q+1-l)X))).$$

As a consequence, for $l = q$ we see that

$$(1 - DH)^q \omega \in F^{n+1-q} H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}).$$

Conversely, let $\omega \in F^{n+1-q} H^{n+1}_{\mathrm{dR}}(U/\mathbb{C})$. Then we can represent $\omega = \omega^{n+1-q} + \cdots + \omega^{n+1}$, where each

$$\omega^k \in C^{n+1-k}(\mathcal{U}, \Omega^k_{\mathbb{P}^{n+1}}(\log X)),$$

and $\omega^k = 0$ for $k = 0, \ldots, n-q$. We claim that for every $l = 0, \ldots, q$ we can represent $\omega \in \mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}(*X))$ as $\omega = \eta_l^{n+1-q+l} + \omega^{n+2-q+l} + \cdots + \omega^{n+1}$ with

$$\eta_l^{n+1-q+l} \in C^{q-l}(\mathcal{U}, \Omega^{n+1-q+l}_{\mathbb{P}^{n+1}}((l+1)X)).$$

We prove this claim by induction on l. The case $l = 0$ is clear taking $\eta_0^{n+1-q} = \omega^{n+1-q}$. Now, for $l > 0$ suppose $\omega = \eta_{l-1}^{n-q+l} + \omega^{n+1-q+l} + \cdots + \omega^{n+1}$. Since $D\omega = 0$, we know $\delta\eta_{l-1}^{n-q+l} = 0$. By Bott's formula 1.1.1

$$H^{q-l+1}(\mathbb{P}^{n+1}, \Omega^{n-q+l}_{\mathbb{P}^{n+1}}(lX)) = 0.$$

Then, there exist $\beta \in C^{q-l}(\mathcal{U}, \Omega^{n-q+l}_{\mathbb{P}^{n+1}}(lX))$ such that $\delta\beta = \eta_{l-1}^{n-q+l}$. Subtracting from ω the exact form in hypercohomology $(-1)^{n-q+l} D\beta$, we get the claim for l, and we finish the induction. Finally, applying the claim for $l = q$ we can write $\omega = \eta_q^{n+1}$ with

$$\eta_q^{n+1} \in H^0(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X)),$$

as desired. ∎

The previous theorem tells us that the elements of the form

$$res\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim},$$

where $P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2))$ and $q = 0, \ldots, n$, generate all $H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim}$.

The following theorem tells us how we can choose a basis from these generators.

Theorem 1.5.2 (Griffiths [Gri69]). *For every $q = 0, \ldots, n$ the kernel of the map*

$$\begin{aligned} \varphi : H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2)) &\to F^{n-q} H^n_{dR}(X/\mathbb{C})_{prim} / F^{n+1-q} H^n_{dR}(X/\mathbb{C})_{prim} \\ P &\mapsto res\left(\frac{P\Omega}{F^{q+1}}\right) \end{aligned}$$

is the degree $N = d(q+1) - n - 2$ part of the Jacobian ideal of F, $J^F_N \subset \mathbb{C}[x_0, \ldots, x_{n+1}]_N$.

Definition 1.5.1. Recall that the *Jacobian ideal of F* is the homogeneous ideal

$$J^F := \langle F_0, \ldots, F_{n+1} \rangle \subset \mathbb{C}[x_0, \ldots, x_{n+1}],$$

where, from now on, we denote

$$F_i := \frac{\partial F}{\partial x_i},$$

for $i = 0, \dots, n+1$. The *Jacobian ring of* F is

$$R^F := \mathbb{C}[x_0, \dots, x_{n+1}]/J^F.$$

Remark 1.5.2. Theorem 1.5.2 implies that to choose a basis for

$$F^{n-q}H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim}/F^{n+1-q}H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim} \simeq H^{n-q,q}(X)_{prim}$$

it is enough to take the elements of the form $res\left(\frac{P}{F^{q+1}}\right)$, for $P \in \mathbb{C}[x_0, \dots, x_{n+1}]_N$ forming a basis of R^F_N. In particular

$$h^{n-q,q}(X)_{prim} = \dim_{\mathbb{C}} R^F_N.$$

Proof of Theorem 1.5.2 By Theorem 1.5.1, it is clear that P is in the kernel of φ, if and only if, there exist $Q \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(dq - n - 2))$ such that

$$res\left(\frac{P\Omega}{F^{q+1}}\right) = res\left(\frac{Q\Omega}{F^q}\right).$$

Since the residue map is an isomorphism between $H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim}$. This is equivalent to say

$$\frac{(P - FQ)\Omega}{F^{q+1}} = 0 \in H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}). \tag{1.17}$$

Since $H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}) \simeq H^{n+1}(\Gamma(\Omega_U), d)$, (1.17) is equivalent to

$$\frac{(P - QF)\Omega}{F^{q+1}} = d\omega,$$

for some $\omega \in H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))$. Recall from Corollary 1.1.1 that every $\omega \in H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))$ is of the form

$$\omega = \frac{\sum_{i=0}^{n+1} T_i \iota_{\frac{\partial}{\partial x_i}}(\Omega)}{F^q},$$

for some $T_i \in \mathbb{C}[x_0, \dots, x_{n+1}]_{dq-n-1}$. As a consequence, P is in the kernel of φ, if and only if,

$$\frac{P\Omega}{F^{q+1}} \equiv \frac{q\sum_{i=0}^{n+1} T_i F_i \Omega}{F^{q+1}} \pmod{H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))},$$

in other words

$$P \equiv q\sum_{i=0}^{n+1} T_i F_i \pmod{F}.$$

Since $F \in J^F$ (by Euler's identity), this is equivalent to $P \in J^F$. ■

1.6 Computing the residue map

We will close this chapter with an explicit description in Čech cohomology of the residue map for the generators given by Griffiths' theorem. This was done in [CG80] as a consequence of Carlson-Griffiths Lemma 1.3.1.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface given by $X = \{F = 0\}$. Recall that $H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(n+2)) \cong \mathbb{C}$ is generated by

$$\Omega = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Theorem 1.6.1 (Carlson-Griffiths [CG80])**.** *Let $q \in \{0, 1, \ldots, n\}$, $P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2))$. Then*

$$res\left(\frac{P\Omega}{F^{q+1}}\right) = \frac{(-1)^{n(q+1)}}{q!}\left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=q} \in H^q(\mathcal{U}, \Omega_X^{n-q}). \tag{1.18}$$

Where $\Omega_J := \iota_{\frac{\partial}{\partial x_{j_q}}}(\cdots \iota_{\frac{\partial}{\partial x_{j_0}}}(\Omega) \cdots)$, $F_J := F_{j_0} \cdots F_{j_q}$ and $\mathcal{U} = \{U_i\}_{i=0}^{n+1}$ is the Jacobian covering restricted to X, given by $U_i = \{F_i \neq 0\} \cap X$.

Proof Let $U := \mathbb{P}^{n+1} \setminus X$. For $l = 0, \ldots, q$ define

$$^{(l)}\omega := (1 - DH)^l \left(\frac{P\Omega}{F^{q+1}}\right) \in H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}),$$

where H is the operator defined in Definition 1.3.2. We claim

$$^{(l)}\omega^{n+1-l} = \left\{\frac{(q-l)!(-1)^{n(l+1)}P}{q! \cdot F^{q-l}}\left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d\left((-1)^n \frac{V_J}{F_J}\right)\right)\right\}_{|J|=l} \in C^l(\mathcal{U}, \Omega^{n+1-l}_{\mathbb{P}^{n+1}}((q-l+1)X)),$$

where $V := dx_0 \wedge \cdots \wedge dx_{n+1}$, and $V_J := \iota_{\frac{\partial}{\partial x_{j_m}}}(\cdots \iota_{\frac{\partial}{\partial x_{j_0}}}(V) \cdots)$. In fact, for $l = 0$, the claim follows from the identity

$$\frac{\Omega}{F} = \frac{dF}{F} \wedge \frac{\Omega_{(i)}}{F_i} + d \cdot \frac{V_{(i)}}{F_i}.$$

(Which is obtained by contracting $\frac{\partial}{\partial x_i}$ in the equality $dF \wedge \Omega = d \cdot F \cdot V$.) Assuming the claim for $l \geq 0$, then

$$H_{q-l+1}(^{(l)}\omega^{n-l+1})_J = \frac{(q-l-1)!(-1)^{nl+l+1}P}{q! \cdot F^{q-l}} \frac{\Omega_J}{F_J}.$$

As a consequence,

$$^{(l+1)}\omega_J^{n-l} = (-1)^{n-l+1} H_{q-l+1}(^{(l)}\omega^{n+1-l})_J = \frac{(q-l-1)!}{q!}\sum_{m=0}^{l+1}(-1)^{n(l+1)+m}\frac{P\Omega_{J \setminus \{j_m\}}}{F^{q-l}F_{J \setminus \{j_m\}}}.$$

Using the following identity

$$\Omega_J \wedge dF + (-1)^n d\,F\,V_J = (-1)^n \sum_{m=0}^{l+1} (-1)^m F_{j_m} \Omega_{J\setminus\{j_m\}}; \tag{1.19}$$

(this identity is obtained by successive contraction of the identity $dF \wedge \Omega = d\,F\,V$ by $\frac{\partial}{\partial x_{j_m}}$, for $m = 0, \ldots, l+1$) we obtain the claim for $l+1$. In conclusion

$$\omega^{(q)}_{n+1-q} = \left\{ \frac{(-1)^{n(q+1)} P}{q!} \left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d\,(-1)^n \frac{V_J}{F_J} \right) \right\}_{|J|=q} \in C^q(U; \Omega^{n+1-q}_{\mathbb{P}^{n+1}}(\log X));$$

the rest is just to apply the residue map. ■

Remark 1.6.1. Using Carlson-Griffiths' lemma, it is possible to describe explicitly the residue map in all algebraic de Rham cohomology, i.e. as an element of

$$res\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C}).$$

We skip this computation since the last piece in Čech cohomology is enough for our purposes.

Chapter 2

Periods of Algebraic Cycles

Summary

The computation of periods inside algebraic varieties is a very old problem, that goes back to the study of elliptic integrals. It led to the first developments of algebraic topology and algebraic geometry. Periods of algebraic cycles appeared in Lefschetz *"L'analysis situs et la géométrie algébrique"* [Lef24]. In his work, Lefschetz was able to characterize homological cycles of codimension 2 with algebraic support, via the vanishing of its periods of first kind (i.e. periods of holomorphic differential forms), the so called Lefschetz Theorem on $(1,1)$-classes. This result led to one of the most famous open problems about periods of algebraic cycles. Namely, the Hodge conjecture, which is a generalization of Lefschetz Theorem to codimension $2k$ cycles.

Hodge conjecture advanced to the next stage with the development of infinitesimal variations of Hodge structures (IVHS) by Carlson, Green, Griffiths and Harris [CGGH83], that allows us to study the Hodge conjecture in families of varieties. The parameters where Hodge conjecture is a non-trivial problem determine the Hodge locus.

Periods of algebraic cycles play a fundamental role when we look at Hodge conjecture in families, in fact, they determine IVHS. This is our main motivation to compute those periods, and it is the central topic of this chapter (and of this thesis). The applications are left to Chapters 3. For more problems about periods of algebraic cycles see [Mov17a] Chapter 18.

Let us explain what we mean by periods of algebraic cycles. Let

$$X = \{F = 0\} \subset \mathbb{P}^{n+1}$$

be an even dimensional smooth hypersurface, given by a homogeneous polynomial with $\deg F = d$. Every $\frac{n}{2}$-dimensional subvariety Z of X determines an *algebraic cycle*

$$[Z] \in H_n(X, \mathbb{Z}).$$

Recalling Griffiths' Theorem 1.5.1, each piece of the Hodge filtration is generated by the differential forms

$$\omega_P := res\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q}H^n_{\mathrm{dR}}(X/\mathbb{C})_{prim},$$

for $P \in \mathbb{C}[x_0,\dots,x_{n+1}]_{d(q+1)-n-2}$. We say that

$$\int_Z \omega_P \in \mathbb{C}$$

is a *period of* Z. Notice that, since Z is a projective variety of positive dimension, it intersects every divisor of X, so it is impossible to find an affine chart of X where to compute the periods of Z.

Our strategy to compute the periods is to reduce the computation to a period of some projective space $\mathbb{P}^N$. In section 2.1, we consider over $\mathbb{P}^N$ the standard covering $\mathcal{U}$. We fix the top form

$$\frac{\Omega}{x_0\cdots x_{n+1}} = \frac{\sum_{i=0}^{n+1}(-1)^i x_i\widehat{dx_i}}{x_0\cdots x_{n+1}} \in H^N(\mathcal{U},\Omega^N_{\mathbb{P}^N}),$$

called the *standard top form with respect to* $\mathcal{U}$, and compute its period over $\mathbb{P}^N$. To do this we determine explicitly the image of this form via the isomorphism $H^N(\mathbb{P}^N,\Omega^N_{\mathbb{P}^N}) \simeq H^{2N}_{\mathrm{dR}}(\mathbb{P}^N)$, and we get the period.

Proposition. *For* $(\beta_0,\dots,\beta_N) \in \mathbb{Z}^{N+1}$,

$$\int_{\mathbb{P}^N} x_0^{\beta_0}\cdots x_N^{\beta_N}\Omega = \begin{cases} 0 & \text{if } (\beta_0,\dots,\beta_N) \neq (-1,\dots,-1), \\ (-1)^{\binom{N+1}{2}}(2\pi\sqrt{-1})^N & \text{if } (\beta_0,\dots,\beta_N) = (-1,\dots,-1). \end{cases}$$

In section 2.2 we describe the pull-back of differential forms in algebraic de Rham cohomology (for the definition of algebraic de Rham cohomology see section 1.2). Obtaining the following result, that can be taken as a definition.

Proposition. *Let* X *and* Y *be smooth complex algebraic varieties, and* $\mathcal{U}$ *an affine open covering of* X*. Consider an affine morphism* $\varphi : Y \to X$ *(i.e. such that* $\varphi^{-1}(U)$ *is affine, for each open affine* U *of* X*), and* $\omega \in H^k_{dR}(X/\mathbb{C})$*. Denoting* $\varphi^{-1}(\mathcal{U}) := \{\varphi^{-1}(U)\}_{U\in\mathcal{U}}$*, then* $\varphi^*\omega \in H^k_{dR}(Y/\mathbb{C})$ *is given by*

$$\varphi^*\omega = \sum_{i=0}^{k}\varphi^*\omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}),\Omega^i_Y),$$

where

$$\omega = \sum_{i=0}^{k}\omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U},\Omega^i_X),$$

$$(\varphi^*\omega^i)_{j_0\cdots j_{k-i}} := \varphi^*(\omega^i_{j_0\cdots j_{k-i}}) \in \Omega^i_Y(\varphi^{-1}(U_{j_0\cdots j_{k-i}}));$$

and

$$\varphi^*\left(\sum_I a_I dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right) := \sum_I \varphi^* a_I d(\varphi^* x_{i_1})\wedge\cdots\wedge d(\varphi^* x_{i_k})$$

(in particular, φ^* commutes with d and δ, then it also commutes with D).

In section 2.3 we get the first application of the previous propositions. Namely, we compute periods of $\mathbb{P}^N$ for differential forms described in other open coverings (such as the Jacobian covering associated to a smooth hypersurface), as follows.

Proposition. *Let $f_0,\ldots,f_N \in \mathbb{C}[x_0,\ldots,x_N]_d$ be homogeneous degree d polynomials such that*

$$\{f_0=\cdots=f_N=0\}=\emptyset\subseteq\mathbb{P}^N.$$

Consider

$$F:=(f_0:\cdots:f_N):\mathbb{P}^N\to\mathbb{P}^N,$$

and $\sigma:=(d-1)(N+1)$. For every $Q\in\mathbb{C}[x_0,\ldots,x_N]_\sigma$ the period of $\mathbb{P}^N$ for the following form (described in the open covering $\mathcal{U}_F:=F^{-1}\mathcal{U}$) is

$$\int_{\mathbb{P}^N}\frac{Q\Omega}{f_0\cdots f_N}=c\cdot d^{N+1}\cdot(-1)^{\binom{N+1}{2}}(2\pi\sqrt{-1})^N,$$

where $c\in\mathbb{C}$ is the unique number such that

$$Q\equiv c\cdot det(Jac(F)) \text{ (mod } \langle f_0,\ldots,f_N\rangle),$$

and $Jac(F)$ is the Jacobian matrix of F when we see it as a map $(f_0,\ldots,f_N):\mathbb{C}^{N+1}\to\mathbb{C}^{N+1}$.

In this proposition it is implicit the fact that

$$\frac{\mathbb{C}[x_0,\ldots,x_N]_\sigma}{\langle f_0,\ldots,f_N\rangle_\sigma}\simeq\mathbb{C},$$

which is consequence of the following theorem due to Macaulay asserting that $\langle f_0,\ldots,f_N\rangle$ is an *Artinian Gorenstein ideal of socle* σ.

Theorem (Macaulay [Mac16])**.** *Given $f_0,\ldots,f_N\in\mathbb{C}[x_0,\ldots,x_N]$ homogeneous polynomials with $deg(f_i)=d_i$ and*

$$\{f_0=\cdots=f_N=0\}=\emptyset\subseteq\mathbb{P}^N.$$

Letting

$$R:=\frac{\mathbb{C}[x_0,\ldots,x_N]}{\langle f_0,\ldots,f_N\rangle},$$

then for $\sigma:=\sum_{i=0}^N(d_i-1)$, we have that *(i) For every* $0 \leq i \leq \sigma$ *the product* $R_i \times R_{\sigma-i} \to R_\sigma$ *is a perfect pairing.*

(ii) $\dim_{\mathbb{C}} R_\sigma = 1$.

(iii) $R_e = 0$ *for* $e > \sigma$.

Another consequence of the explicit description of pull-backs is that we can compute the periods of linear cycles. We do this in section 2.4 for linear cycles inside Fermat varieties.

Theorem ([MV17]). *Let* $X_n^d \subset \mathbb{P}^{n+1}$ *be the* n*-dimensional Fermat variety of degree* d*, i.e*

$$X_n^d = \{F := x_0^d + \cdots + x_{n+1}^d = 0\}.$$

Let ζ_{2d} *be a* $2d$*-th primitive root of unity, and*

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}x_1 = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\}.$$

For every

$$i \in I_{(\frac{n}{2}+1)d-n-2} := \left\{(i_0, \dots, i_{n+1}) \in \{0, \dots, d-2\}^{n+2} : i_0 + \cdots + i_{n+1} = (\tfrac{n}{2}+1)d - n - 2\right\},$$

consider

$$\omega_i := res\left(\frac{x_0^{i_0} \cdots x_{n+1}^{i_{n+1}}\Omega}{F^{\frac{n}{2}+1}}\right) \in H^n_{dR}(X_n^d/\mathbb{C}).$$

Then

$$\int_{\mathbb{P}^{\frac{n}{2}}} \omega_i = \begin{cases} \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1}\frac{n}{2}!}\zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n} & \text{if } i_{2l-2} + i_{2l-1} = d-2, \ \forall l = 1, \dots, \frac{n}{2}+1; \\ 0 & \text{otherwise.} \end{cases}$$

This computation is simple and interesting enough to obtain new results on the variational Hodge conjecture (see [MV17] or section 3.6).

Sections 2.5 and 2.6 are devoted to the computation of periods of complete intersection algebraic cycles, which is the main theorem of this thesis.

Given the complete intersection $Z \subset X$ of dimension $\frac{n}{2}$, we construct a chain of subvarieties

$$Z = Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1},$$

where each Z_i is a hypersurface of Z_{i+1}. We inductively reduce the computation of the period of Z to a period of $\mathbb{P}^{n+1}$. In order to do this it is enough to relate every period of an hyperplane section Y of a projective variety X, with a period of X.

Recalling that for X a smooth projective variety and Y a smooth hyperplane section of X, we have the *Poincaré's residue sequence* (1.15)

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet(\log Y) \xrightarrow{Res} j_*\Omega_Y^{\bullet-1} \to 0.$$

Which induces the isomorphism

$$H^{n-1}_{\mathrm{dR}}(Y/\mathbb{C}) \simeq H^{n}_{\mathrm{dR}}(X/\mathbb{C}).$$

An explicit description of this isomorphism (together with the periods relation) is given in section 2.5 as follows.

Proposition. *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n+1$, and $Y \subset X$ a hyperplane section given by $\{F = 0\} \cap X$, for some homogeneous $F \in \mathbb{C}[x_0, \dots, x_N]_d$. Let $\omega \in C^n(X, \Omega^n_X)$ such that $\omega|_Y \in Z^n(Y, \Omega^n_Y)$. For any $\tilde{\omega} \in C^n(X, \Omega^{n+1}_X(\log Y))$ such that*

$$\tilde{\omega} \equiv \omega \wedge \frac{dF}{F} \pmod{C^n(X, \Omega^{n+1}_X)},$$

we have

$$\tau := (-1)^{n+1}\,\delta(\tilde{\omega}) \in Z^n(X, \Omega^{n+1}_X).$$

Furthermore, $\tau \in H^{n+1}(X, \Omega^{n+1}_X)$ is uniquely determined by $\omega|_Y \in H^n(Y, \Omega^n_Y)$ and

$$\int_X \tau = \frac{(-1)^{n+1}\, 2\pi\sqrt{-1}}{d} \int_Y \omega.$$

Finally, in section 2.6 we use this description to obtain the formula for periods of complete algebraic cycles.

Theorem ([VL18]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by $X = \{F = 0\}$. Suppose*

$$F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1},$$

such that $Z := \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\} \subset X$ is a complete intersection (i.e. $dim(Z) = \frac{n}{2}$). Define

$$H = (h_0, \dots, h_{n+1}) := (f_1, g_1, f_2, g_2, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}).$$

Then

$$\int_Z res\left(\frac{P\Omega}{F^{\frac{n}{2}+1}}\right) = \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{\frac{n}{2}!} c \cdot (d-1)^{n+2} - d_1 \cdots d_{\frac{n}{2}+1},$$

where $d_i = deg\ f_i$ and $c \in \mathbb{C}$ is the unique number such that

$$P \cdot det(Jac(H)) \equiv c \cdot det(Hess(F)) \pmod{J^F}.$$

Where $J^F := \langle F_0, \dots, F_{n+1} \rangle$ is the *Jacobian ideal of F*. Notice that by Macaulay's theorem J^F is an Artinian Gorenstein ideal of socle $2\sigma := (d-2)(n+2)$.

As a consequence of this theorem we obtain that if

$$\delta = \sum_{i=1}^{k} n_i [Z_i] \in H_n(X, \mathbb{Z})$$

is an algebraic cycle of *complete intersection type*, i.e. every Z_i is a complete intersection (in the sense of the previous theorem) of X. Then there exist a polynomial

$$P_\delta \in \mathbb{C}[x_0, \dots, x_{n+1}]_{\delta} ,$$

such that (up to a constant non-zero factor not depending on P)

$$\int_\delta res\left(\frac{P\Omega}{F^{\frac{n}{2}+1}}\right) = c_P,$$

where

$$P \cdot P_\delta \equiv c_P \cdot det(Hess(F)) \pmod{J^F}.$$

This is essentially the same as determining the primitive part of the *Poincaré dual of* δ

$$\delta^{\mathrm{pd}} = res\left(\frac{P_\delta \Omega}{F^{\frac{n}{2}+1}}\right) \in H^{\frac{n}{2}}(X, \Omega_X^{\frac{n}{2}})_{prim}.$$

Furthermore, the theorem provides an explicit method to compute such polynomial P_δ. As a first corollary of this result we have an algebraic (computational) criterion to determine whether a complete intersection type cycle is trivial or not (in primitive cohomology).

Corollary. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by $X = \{F = 0\}$. If $\delta \in H_n(X, \mathbb{Z})$ is a complete intersection type algebraic cycle, then*

$$P_\delta \in J^F \text{ if and only if } \delta = \lambda \cdot [X \cap \mathbb{P}^{\frac{n}{2}+1}] \text{ in } H_n(X, \mathbb{Q}), \text{ for some } \lambda \in \mathbb{Q}.$$

The computation of this polynomial P_δ is our main contribution to the literature.

2.1 Standard top form in $\mathbb{P}^N$

All our methods to compute periods reduce at some point to compute the period of a top form in the projective space. Since $H^{2N}_{dR}(\mathbb{P}^N) = H^{N,N}(\mathbb{P}^N) \simeq H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}) \simeq \mathbb{C}$, we just need to know the period of one generator. Since it is simpler to compare two top forms in the Čech cohomology group $H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$, we will fix the top form

$$\frac{\Omega}{x_0\cdots x_N} = \frac{\sum_{i=0}^N(-1)^i x_i \widehat{dx_i}}{x_0\cdots x_N} = \left(\frac{dx_1}{x_1}-\frac{dx_0}{x_0}\right)\wedge\cdots\wedge\left(\frac{dx_N}{x_N}-\frac{dx_0}{x_0}\right)\in H^N(\mathcal{U},\Omega^N_{\mathbb{P}^N}),$$

and compute its period. Here $\mathcal{U}=\{U_i\}_{i=0}^N$ is the standard open covering of $\mathbb{P}^N$, i.e. $U_i=\{x_i\neq 0\}$. We will refer to this form as the *standard top form of* $\mathbb{P}^N$ *associated to the covering* $\mathcal{U}$.

Proposition 2.1.1.

$$\int_{\mathbb{P}^N}\frac{\Omega}{x_0\cdots x_N} = (-1)^{\binom{N+1}{2}}(2\pi\sqrt{-1})^N.$$

Remark 2.1.1. Since we have natural isomorphisms $H^N(\mathbb{P}^N,\Omega^N_{\mathbb{P}^N})\simeq \mathbb{H}^{2N}(\mathbb{P}^N,\Omega^\bullet_{(\mathbb{P}^N)^\infty})\simeq H^{2N}_{dR}(\mathbb{P}^N)$. The element $\frac{\Omega}{x_0\cdots x_N}\in H^N(\mathbb{P}^N,\Omega^N_{\mathbb{P}^N})$ corresponds to a top form $\omega\in H^{2N}_{dR}(\mathbb{P}^N)$. By abuse of notation we will denote

$$\int_{\mathbb{P}^N}\frac{\Omega}{x_0\cdots x_N} := \int_{\mathbb{P}^N}\omega.$$

We always use this identification when we talk about periods.

Proof Let us determine the image of $\frac{\Omega}{x_0\cdots x_N}$ via the isomorphism

$$H^N(\mathbb{P}^N,\Omega^N_{\mathbb{P}^N})\simeq \mathbb{H}^{2N}(\mathbb{P}^N,\Omega^\bullet_{(\mathbb{P}^N)^\infty})\simeq H^{2N}_{dR}(\mathbb{P}^N). \tag{2.1}$$

Under the first isomorphism of (2.1) we obtain the element in hypercohomology represented by the sum

$$\eta=\sum_{i=0}^{2N}\eta^i\in\bigoplus_{i=0}^{2N}C^{2N-i}(\mathcal{U},\Omega^i_{(\mathbb{P}^N)^\infty}),$$

where

$$\eta^i=\begin{cases}\frac{\Omega}{x_0\cdots x_N} & \text{if } i=N,\\ 0 & \text{if } i\neq N.\end{cases}$$

In order to determine the image of η under the second isomorphism of (2.1), we need to find another representative of the form

$$\omega=\sum_{i=0}^{2N}\omega^i\in\bigoplus_{i=0}^{2N}C^{2N-i}(\mathcal{U},\Omega^i_{(\mathbb{P}^N)^\infty}),$$

where

$$\omega^i = \begin{cases} \omega^{2N} & \text{if } i = 2N, \\ 0 & \text{if } i \neq 2N. \end{cases}$$

Then ω^{2N} will be the d-closed global $2N$-form on $\mathbb{P}^N$ representing the image of $\frac{}{x_0 \cdots x_N}$ in $H^{2N}_{\mathrm{dR}}(\mathbb{P}^N)$. In order to construct this element ω, we will construct inductively elements

$$_j = \sum_{i=0}^{2N} {}^i_j \in \bigoplus_{i=0}^{2N} C^{2N-i}(U, \Omega^i_{(\mathbb{P}^N)^\infty}),$$

with

$$^i_j = \begin{cases} ^{N+j}_j & \text{if } i = N+j, \\ 0 & \text{if } i \neq N+j, \end{cases} \tag{2.2}$$

such that they all represent the same element in hypercohomology, and $_0 =$, then the desired element will correspond to $\omega = {}_N$.

Let $\{a_i\}_{i=0}^N$ be a partition of unity subordinated to $\{U_i\}_{i=0}^N$. Let us denote

$$U_{i_1 \cdots i_k} := U_{i_1} \cap \cdots \cap U_{i_k}, \text{ and } U_{i_1 \cap i_k} := \textstyle\bigcap_{h \notin \{i_1, \ldots, i_k\}} U_h.$$

Lemma 2.1.1. *For every $j = 0, \ldots, N$ we can take $_j$ as in (2.2) given by*

$$(^{N+j}_j)_{i_1 \cap i_j} := j!\,(-1)^{i_1 + \cdots + i_j + Nj + \binom{j+1}{2}} da_{i_1} \wedge \cdots \wedge da_{i_j} \wedge \frac{\Omega}{x_0 \cdots x_N} \in \Omega^{N+j}_{(\mathbb{P}^N)^\infty}(U_{i_1 \cap i_j}).$$

Proof We proceed by induction on j. For $j = 0$ the lemma is clear. Suppose it holds for $j \geq 0$, let us define

$$= \sum_{i=0}^{2N-1} {}^i \in \bigoplus_{i=0}^{2N-1} C^{2N-1-i}(U, \Omega^i_{(\mathbb{P}^N)^\infty}),$$

with

$$^i = \begin{cases} ^{N+j} & \text{if } i = N+j, \\ 0 & \text{if } i \neq N+j \end{cases}$$

where

$$^{N+j}_{i_0 \cap i_j} := \frac{j!\,(-1)^{i_0 + \cdots + i_j + N(j+1) + \binom{j+2}{2}} (\sum_{l=0}^{j} (-1)^l a_{i_l} \widehat{da_{i_l}}) \wedge \Omega}{x_0 \cdots x_N} \in \Omega^{N+j}_{(\mathbb{P}^N)^\infty}(U_{i_0 \cap i_j}).$$

We have that $(^{N+j}) = (-1)^{N+j+1} {}^{N+j}_j$, thus we can take $_{j+1} := {}_j + D( )$. Then

$$(^{N+j+1}_{j+1})_{i_0 \cap i_j} = d\, ^{N+j}_{i_0 \cap i_j} = \frac{(j+1)!\,(-1)^{i_0 + \cdots + i_j + N(j+1) + \binom{j+2}{2}} da_{i_0} \wedge \cdots \wedge da_{i_j} \wedge \Omega}{x_0 \cdots x_N} \in \Omega^{N+j+1}_{(\mathbb{P}^N)^\infty}(U_{i_0 \cap i_j})$$

as claimed. ∎

In particular, we have that the image of $\frac{\Omega}{x_0\cdots x_N}$ in $H^{2N}_{\mathrm{dR}}(\mathbb{P}^N)$ is represented by the global closed $2N$-form $\omega^{2N} = \omega^{2N}_N$ given by

$$(\omega^{2N})_i = N!\,(-1)^{N-i}da_0\wedge\cdots\wedge\widehat{da_i}\wedge\cdots\wedge da_N\wedge\frac{\Omega}{x_0\cdots x_N}\in\Omega^{2N}_{(\mathbb{P}^N)^{\infty}}(U_i).$$

Since $Supp(\omega^{2N})\subset U_{0\cdots N}$. In order to integrate ω^{2N} over $\mathbb{P}^N$, is enough to do it on any affine chart. Setting $z_i = x_i/x_0$ for $i = 1,\dots,N$

$$\int_{\mathbb{P}^N}\frac{\Omega}{x_0\cdots x_N} = \int_{U_0}(\omega^{2N})_0 = N!\,(-1)^N\int_{\mathbb{C}^N}da_1\wedge\cdots\wedge da_N\wedge\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_N}{z_N}.$$

Without loss of generality, we can take the partition of unity such that

$$a_i = \begin{cases} 0 & \text{if } |z_i|\leq 1 \\ 1 & \text{if } |z_i|\geq 2, |z_j|\leq 1\ \forall j\in\{1,\dots,N\}\setminus\{i\}\end{cases}$$

and

$$a_1+\cdots+a_N = 1 \text{ if } \exists j\in\{1,\dots,N\} : |z_j|\geq 2.$$

Thus

$$Supp(da_1\wedge\cdots\wedge da_N)\subset\{z\in\mathbb{C}^N : 1\leq|z_i|\leq 2, \forall i = 1,\dots,N\} = (\overline{\mathbb{D}(0;2)}\setminus\mathbb{D}(0;1))^N.$$

For each pair of the form (I,J) where I and J are two disjoint subsets of $\{1,\dots,N\}$, we define the set

$$\Gamma_{I,J} := \{z\in(\overline{\mathbb{D}(0;2)}\setminus\mathbb{D}(0;1))^N : |z_i| = 2, |z_j| = 1, \forall i\in I, j\in J\}.$$

We consider $\Gamma_{I,J}\in H_{2N-\#I-\#J}(\mathbb{C}^N;\mathbb{Z})$ as a singular cycle of (real) dimension $2N-\#I-\#J$ of $\mathbb{C}^N$. In order to do computations we will fix an orientation on each $\Gamma_{I,J}$. On $\mathbb{C}^N$ we consider for each $j = 1,\dots,N$ polar coordinates and fix the orientation of $\mathbb{C}^N$ to be $\omega := d\rho_1\wedge d\theta_1\wedge\cdots\wedge d\rho_N\wedge d\theta_N$. In order to give an orientation to $\Gamma_{I,J}$ we order $I\cup J = \{k_1<\cdots<k_r\}$ and define the orientation to be $\iota_{n_{k_r}}(\cdots\iota_{n_{k_1}}(\omega)\cdots)$, where $n_j = \frac{\partial}{\partial x_j}+\frac{\partial}{\partial y_j}$. For each $k = 1,\dots,N$ we define the singular cycle of dimension $N+k-1$

$$\Gamma_k := \sum_{i=1}^{k}\Gamma_{\{i\},\{k+1,\dots,N\}}.$$

Let $A := \frac{(-1)^N}{N!}\int_{\mathbb{P}^N}\frac{\Omega}{x_0\cdots x_N}$ and $V := \frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_N}{z_N}$.

Lemma 2.1.2. *For every* $k = 1,\dots,N$

$$A = \int_{(-1)^{\binom{N-k+1}{2}}\Gamma_k}(-1)^{\binom{N}{2}-\binom{k}{2}}\frac{a_1^{N+1-k}}{(N+1-k)!}da_2\wedge\cdots\wedge da_k\wedge V.$$

Proof In fact, since

$$A = \int_{(\overline{D(0;2)} \setminus D(0;1))^N} d(a_1 da_2 \wedge \quad \wedge da_N \wedge V);$$

it follows from Stokes' theorem that

$$A = \int_{\sum_{i=1}^{N} \quad \{i\};\emptyset \quad \emptyset;\{i\}} a_1 da_2 \wedge \quad \wedge da_N \wedge V.$$

Noticing that $a_i\big|_{\quad \emptyset;\{i\}} \quad 0$, we get

$$A = \int_{\quad N} a_1 da_2 \wedge \quad \wedge da_N \wedge V.$$

Proceeding inductively, if we assume the claim for k we will show it for $k \quad 1$. The first thing to observe is that $a_j\big|_{\quad k} \quad 0; \forall j > k.$ Thus, over Γ_k we have the relation $a_1 + \quad + a_k = 1$, then $da_k = \quad da_1 \quad \quad da_{k \quad 1}$, and

$$\begin{aligned} A &= \int_{(\quad 1)^{\binom{N \quad k+1}{2}} \quad k} (\quad 1)^{\binom{N}{2} \quad \binom{k}{2}} \frac{a_1^{N+1 \quad k}}{(N+1 \quad k)!} da_2 \wedge \quad \wedge da_{k \quad 1} \wedge (\quad da_1) \wedge V \\ &= \int_{(\quad 1)^{\binom{N \quad k+1}{2}} \quad k} (\quad 1)^{\binom{N}{2} \quad \binom{k \quad 1}{2}} d\left(\frac{a_1^{N+1 \quad (k \quad 1)}}{(N+1 \quad (k \quad 1))!} da_2 \wedge \quad \wedge da_{k \quad 1} \wedge V \right). \end{aligned}$$

Noting that

$$\begin{aligned} \partial\Gamma_k &= \sum_{i=1}^{k} \partial\Gamma_{\{i\};\{k+1;\ldots;N\}} \\ &= (\quad 1)^{N \quad k+1} \sum_{i=1}^{k} \left(\sum_{1 \quad j<i} (\Gamma_{\{i;j\};\{k+1;\ldots;N\}} \quad \Gamma_{\{i\};\{j;k+1;\ldots;N\}}) \quad \sum_{i<j \quad k} (\Gamma_{\{i;j\};\{k+1;\ldots;N\}} \quad \Gamma_{\{i\};\{j;k+1;\ldots;N\}}) \right) \\ &= (\quad 1)^{N \quad k+1} \sum_{1 \quad i<j \quad k} \Gamma_{\{i\};\{j;k+1;\ldots;N\}} \quad \Gamma_{\{j\};\{i;k+1;\ldots;N\}}. \end{aligned}$$

And using the fact that $a_i\big|_{\quad \{j\};\{i;k+1;\ldots;N\}} \quad 0$ and $a_j\big|_{\quad \{i\};\{j;k+1;\ldots;N\}} \quad 0$ for $1 \quad i < j \quad k \quad 1$. It follows from Stokes theorem that

$$\begin{aligned} A &= \int_{(\quad 1)^{\binom{N \quad k+1}{2} + N \quad k+1} \sum_{1 \quad i<k \quad \{i\};\{k;k+1;\ldots;N\}}} (\quad 1)^{\binom{N}{2} \quad \binom{k \quad 1}{2}} \frac{a_1^{N+1 \quad (k \quad 1)}}{(N+1 \quad (k \quad 1))!} da_2 \wedge \quad \wedge da_{k \quad 1} \wedge V \\ &= \int_{(\quad 1)^{\binom{N \quad k+2}{2}} \quad k \quad 1} (\quad 1)^{\binom{N}{2} \quad \binom{k \quad 1}{2}} \frac{a_1^{N+1 \quad (k \quad 1)}}{(N+1 \quad (k \quad 1))!} da_2 \wedge \quad \wedge da_{k \quad 1} \wedge V \end{aligned}$$

as claimed. ∎

In particular

$$\begin{aligned} A &= \int_{(-1)^{\binom{N}{2}}\Gamma_1} (-1)^{\binom{N}{2}} \frac{a_1^N}{N!} V \\ &= \int_{\Gamma_1} \frac{V}{N!}. \end{aligned}$$

Finally, by Fubini's theorem and Cauchy's integral formula we get (note that the sign appears from the orientation we are considering over Γ_1)

$$A = \frac{(-1)^{\binom{N}{2}}(2\pi\sqrt{-1})^N}{N!}.$$

This finishes the proof of Proposition 2.1.1. ■

Now it is easy to see that an element of $Z^N(\mathcal{U};\Omega^N_{\mathbb{P}^N})$ is of the form

$$\frac{P\Omega}{x_0^{\beta_0}\cdots x_N^{\beta_N}}$$

with $\beta_0,\ldots,\beta_N \in \mathbb{N}$ such that $\beta_0+\cdots+\beta_N = deg(P)+N+1$. Then it is a $\mathbb{C}$-linear combination of terms of the form

$$x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega$$

with $\alpha_0,\ldots,\alpha_N \in \mathbb{Z}$ such that $\alpha_0+\cdots+\alpha_N = -N-1$. The following proposition tells us how to compute the period of any such form:

Proposition 2.1.2. *The form*

$$x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega \in H^N(\mathcal{U};\Omega^N_{\mathbb{P}^N})$$

represents an exact top form (when is identified with an element of $H^{N,N}(\mathbb{P}^N)$) if and only if $(\alpha_0,\ldots,\alpha_N) \neq (-1,\ldots,-1)$. In particular,

$$\int_{\mathbb{P}^N} x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega = \begin{cases} 0 & \text{if } (\alpha_0,\ldots,\alpha_N) \neq (-1,\ldots,-1), \\ (-1)^{\binom{N+1}{2}}(2\pi\sqrt{-1})^N & \text{if } (\alpha_0,\ldots,\alpha_N) = (-1,\ldots,-1). \end{cases}$$

Proof The only thing left to prove is that for $(\alpha_0,\ldots,\alpha_N) \neq (-1,\ldots,-1)$ the form is exact. Using the isomorphism $H^N(\mathbb{P}^N;\Omega^N_{\mathbb{P}^N}) \cong \mathbb{H}^{2N}(\mathbb{P}^N;\Omega^\bullet_{\mathbb{P}^N})$ it is enough to show the element $\omega \in \mathbb{H}^{2N}(\mathbb{P}^N;\Omega^\bullet_{\mathbb{P}^N})$ given by

$$\omega^N = x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega,$$

and $\omega^i = 0$ for $i \neq N$, is zero in hypercohomology. Noting that if $(\alpha_0,\ldots,\alpha_N) \neq (-1,\ldots,-1)$, there exist $\alpha_i \geq 0$, i.e. $x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega \in \Omega^N_{\mathbb{P}^N}(\cap_{j\neq i}U_j)$, we can define $\eta \in \bigoplus_{j=0}^{2N-1} C^{2N-1-j}(\mathcal{U};\Omega^j_{\mathbb{P}^N})$ by

$$\eta^N_J := (-1)^{N+i}x_0^{\alpha_0}\cdots x_N^{\alpha_N}\Omega,$$

for $J = (0,\ldots,i-1,i+1,\ldots,N)$, $\eta^N_{J'} = 0$ for $J' \neq J$, and $\eta^j = 0$ for $j \neq N$. This form clearly satisfy $D\eta = \omega$ as desired. ■

2.2 Pull-back in algebraic de Rham cohomology

Given a morphism $Y \to X$ of smooth complex algebraic varieties, we have an induced homomorphism in algebraic de Rham cohomology

$$H^k_{\mathrm{dR}}(X/\mathbb{C}) \to H^k_{\mathrm{dR}}(Y/\mathbb{C}).$$

In this section we describe these pull-back homomorphisms.

Proposition 2.2.1. *Let X and Y be smooth complex algebraic varieties, and $\mathcal{U}$ an affine open covering of X. Consider a morphism $\varphi : Y \to X$, such that $\varphi^{-1}(U)$ is affine, for each open $U \in \mathcal{U}$. Denoting $\varphi^{-1}(\mathcal{U}) := \{\varphi^{-1}(U)\}_{U \in \mathcal{U}}$, then for every $\omega \in H^k_{dR}(X/\mathbb{C})$, the pull-back $\varphi^*\omega \in H^k_{dR}(Y/\mathbb{C})$ is given by*

$$\varphi^*\omega = \sum_{i=0}^{k} \varphi^*\omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_Y),$$

where

$$\omega = \sum_{i=0}^{k} \omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_X),$$

$$(\varphi^*\omega^i)_{j_0 \cdots j_{k-i}} := \varphi^*(\omega^i_{j_0 \cdots j_{k-i}}) \in \Omega^i_Y(\varphi^{-1}(U_{j_0 \cdots j_{k-i}}));$$

and

$$\varphi^*\left(\sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) := \sum_I \varphi^* a_I d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_k})$$

(in particular, φ^* commutes with d and δ, then it also commutes with D).

Proof It is easy to see that $D(\varphi^*\omega) = \varphi^*(D\omega) = 0$. Now, in order to show that $\varphi^*\omega$ corresponds to the pull-back of the form ω, we have to show that there exist a representative of the hypercohomology class of ω of the form

$$\psi = \psi^0 + \cdots + \psi^k \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_{X^\infty});$$

with $\psi^i = 0$, $\forall i = 0, \ldots, k-1$. And a representative of the hypercohomology class of $\varphi^*\omega$ of the form

$$\tilde{\psi} = \tilde{\psi}^0 + \cdots + \tilde{\psi}^k \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_{Y^\infty});$$

with $\tilde{\psi}^i = 0$, $\forall i = 0, \ldots, k-1$. Such that $\tilde{\psi}^k \in \Omega^k_{Y^\infty}(Y)$ is the pull-back of $\psi^k \in \Omega^k_{X^\infty}(X)$ as C^∞ differential k-forms. We will in fact show more, we will show inductively that for every $l = 0, \ldots, k$ there exist a representative of the hypercohomology class of ω of the form

$$\psi_l = \psi^0_l + \cdots + \psi^k_l \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_{X^\infty});$$

with $\; ^i_l = 0$, $8i = 0; :::; l \quad 1$. And a representative of the hypercohomology class of $' \;!$ of the form

$$\tilde{\;}_l = \tilde{\;}^0_l + \quad + \tilde{\;}^k_l \; 2 \bigoplus_{i=0}^{k} C^{k \quad i}(' \quad ^1(U); \Omega^i_{Y\,1});$$

with $\tilde{\;}^i_l = 0$, $8i = 0; :::; l \quad 1$. Such that, for every $j = l; :::; k$, the form $(\tilde{\;}^j_l)_{i_0 \quad i_{k \quad j}} \; 2 \; \Omega^j_{Y\,1}(' \quad ^1(U_{i_0 \quad i_{k \quad j}}))$ is the pull-back of the form $(\; ^j_l)_{i_0 \quad i_{k \quad j}} \; 2 \; \Omega^j_{X\,1}(U_{i_0 \quad i_{k \quad j}})$. In fact, the claim follows for $l = 0$ by the definition of $' \;!$. Assuming the claim for l we will show it for $l + 1$. Consider $fa_h g^N_{h=0}$ a partition of unity subordinated to the covering U. Define

$$= \; ^0 + \quad + \; ^{k \quad 1} \; 2 \bigoplus_{i=0}^{k \quad 1} C^{k \quad 1 \quad i}(U; \Omega^i_{X\,1});$$

such that

$$^i = \begin{cases} 0 & \text{if } i \not= l; \\ & \text{if } i = l; \end{cases}$$

where

$$_{i_0 \quad i_{k \quad 1 \quad l}} := \sum_{h=0}^{N} a_h(\; ^l_l)_{i_0 \quad i_{k \quad 1 \quad l} h} \; 2 \; \Omega^l_{X\,1}(U_{i_0 \quad i_{k \quad 1 \quad l}}):$$

And define

$$\tilde{\;} = \tilde{\;}^0 + \quad + \tilde{\;}^{k \quad 1} \; 2 \bigoplus_{i=0}^{k \quad 1} C^{k \quad 1 \quad i}(' \quad ^1(U); \Omega^i_{Y\,1});$$

such that

$$\tilde{\;}^i = \begin{cases} 0 & \text{if } i \not= l; \\ \tilde{\;} & \text{if } i = l; \end{cases}$$

where

$$\tilde{\;}_{i_0 \quad i_{k \quad 1 \quad l}} := \sum_{h=0}^{N} ' \; a_h(\tilde{\;}^l_l)_{i_0 \quad i_{k \quad 1 \quad l} h} \; 2 \; \Omega^l_{Y\,1}(' \quad ^1(U_{i_0 \quad i_{k \quad 1 \quad l}})):$$

Then, using that $(\; ^l_l) = 0$ we see that $(\;) = (\quad 1)^{k \quad l} \; ^l_l$. Also noticing that $(\tilde{\;}^l_l) = 0$ (and using that $f' \; a_h g^N_{h=0}$ is a partition of unity subordinated to $' \quad ^1(U)$) we get $(\tilde{\;}) = (\quad 1)^{k \quad l} \tilde{\;}^l_l$. Thus, defining $\; _{l+1} := \; _l + (\quad 1)^{k+1} D \;$, and $\tilde{\;}_{l+1} := \tilde{\;}_l + (\quad 1)^{k+1} D \tilde{\;}$, the claim follows for $l + 1$ since $d \tilde{\;} = ' \;(d \;)$ (because $\tilde{\;} = ' \;(\;)$). ■

Remark 2.2.1. The morphism $' : Y \;! \; X$ considered in Proposition 2.2.1 is *affine*, i.e. $' \quad ^1(U)$ is affine for every open affine subset $U \quad X$ (not only the elements of the covering U). In fact, in order to verify if a morphism is affine it is enough to check it for a covering of X (see [Har77] Exercise II.5.17).

A first application of Proposition 2.2.1 is the description of pull-backs in Čech cohomology for each piece of the Hodge structure.

Corollary 2.2.1. *Let X and Y be smooth complex algebraic varieties, and $\mathcal{U}$ an affine open covering of X. Consider an affine morphism $\varphi : Y \to X$, and $\omega \in H^q(\mathcal{U}; \Omega^p_X)$, then the pull-back*

$$\varphi^*\omega \in H^q(\varphi^{-1}(\mathcal{U}); \Omega^p_Y),$$

is given by

$$(\varphi^*\omega)_{j_0 \cdots j_q} = \varphi^*(\omega_{j_0 \cdots j_q}) \in \Omega^p_Y(\varphi^{-1}(U_{j_0 \cdots j_q})).$$

Proof The pull-back morphism in algebraic de Rham cohomology

$$\varphi^* : H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}) \to H^{p+q}_{\mathrm{dR}}(Y/\mathbb{C})$$

is compatible with the Hodge filtration (i.e. $\varphi^*(F^k H^{p+q}_{\mathrm{dR}}(X/\mathbb{C})) \subseteq F^k H^{p+q}_{\mathrm{dR}}(Y/\mathbb{C})$), thus the pull-back in Čech cohomology is induced by

$$H^q(X;\Omega^p_X) \cong F^q H^{p+q}_{\mathrm{dR}}(X/\mathbb{C})/F^{q+1}H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}) \xrightarrow{\varphi^*} F^q H^{p+q}_{\mathrm{dR}}(Y/\mathbb{C})/F^{q+1}H^{p+q}_{\mathrm{dR}}(Y/\mathbb{C}) \cong H^q(Y;\Omega^p_Y).$$

■

2.3 Periods of top forms

In section 2.1 we computed the period of the standard top form relative to the standard covering of $\mathbb{P}^N$. Using Corollary 2.2.1 we can compute the periods of top forms in $\mathbb{P}^N$ described with other open coverings (not just the standard one), for instance the Jacobian covering associated to a smooth hypersurface.

Proposition 2.3.1. *Let $f_0, \ldots, f_N \in \mathbb{C}[x_0, \ldots, x_N]_d$ homogeneous polynomials of the same degree $d > 0$, such that*

$$\{f_0 = \cdots = f_N = 0\} = \emptyset \subset \mathbb{P}^N.$$

They define the finite morphism $F : \mathbb{P}^N \to \mathbb{P}^N$ given by

$$F(x_0 : \cdots : x_N) := (f_0 : \cdots : f_N).$$

Let $\mathcal{U}_F = \{V_i\}_{i=0}^N$ be the open covering associated, i.e. $V_i = \{f_i \neq 0\}$. Then the top form

$$\frac{\Omega_F}{f_0 \cdots f_N} := \frac{\sum_{i=0}^N (-1)^i f_i \widehat{df_i}}{f_0 \cdots f_N} \in H^N(\mathcal{U}_F, \Omega^N_{\mathbb{P}^N}),$$

has period

$$\int_{\mathbb{P}^N} \frac{\Omega_F}{f_0 \cdots f_N} = d^N \cdot (-1)^{\binom{N+1}{2}} (2\pi\sqrt{-1})^N.$$

Proof If Ω is the standard top form associated to the standard covering, applying Corollary 2.2.1 we get $F^{-1}(\mathcal{U}) = \mathcal{U}_F$ and $F^*\Omega = \Omega_F$. Then it follows from topological degree theory that

$$\int_{\mathbb{P}^N} \frac{\Omega_F}{f_0 \cdots f_N} = deg(F) \cdot \int_{\mathbb{P}^N} \frac{\Omega}{x_0 \cdots x_N} = deg(F) \cdot (-1)^{\binom{N+1}{2}} (2\pi\sqrt{-1})^N.$$

Since F is defined by a base point free linear system, the fiber of F is generically reduced by Bertini's theorem (see [Har77] page 179), and corresponds to d^N points by Bezout's theorem, i.e. $deg(F) = d^N$. ■

Before going further we will recall the following theorem due to Macaulay (for a proof see [Voi03] Theorem 6.19):

Theorem 2.3.1 (Macaulay [Mac16])**.** *Given $f_0, \ldots, f_N \in \mathbb{C}[x_0, \ldots, x_N]$ homogeneous polynomials with $deg(f_i) = d_i$ and*

$$\{f_0 = \cdots = f_N = 0\} = \emptyset \subset \mathbb{P}^N.$$

Letting

$$R := \frac{\mathbb{C}[x_0, \ldots, x_N]}{\langle f_0, \ldots, f_N \rangle},$$

then for $\rho := \sum_{i=0}^N (d_i - 1)$, we have that

(i) *For every* $0 \le i \le \sigma$ *the product* $R_i \times R_{\sigma-i} \to R_\sigma$ *is a perfect pairing.*

(ii) $\dim_{\mathbb{C}} R_\sigma = 1$.

(iii) $R_e = 0$ *for* $e > \sigma$.

Definition 2.3.1. A ring of the form $R = \mathbb{C}[x_0,\ldots,x_N]/I$ for some homogeneous ideal I is called an *Artinian Gorenstein ring of socle degree* σ, if there exist $\sigma \in \mathbb{N}$ such that R satisfies properties (i), (ii) and (iii) of Macaulay's Theorem 2.3.1. We also say that I is an *Artinian Gorenstein ideal of socle* σ.

Remark 2.3.1. It is easy to see (using Euler's identity) that

$$\Omega_F = d^{-1}\det(Jac(F))\Omega$$

where $Jac(F) = \left(\frac{\partial f_i}{\partial x_j}\right)_{0\le i,j\le N}$ is the Jacobian matrix of F. Any element of $Z^N(\mathcal{U}_F, \Omega^N_{\mathbb{P}^N})$ is of the form

$$\omega = \frac{P\Omega}{f_0^{\beta_0}\cdots f_N^{\beta_N}}$$

where $\beta_0,\ldots,\beta_N \in \mathbb{Z}_{>0}$ with $d\cdot(\beta_0+\cdots+\beta_N) = deg(P)+N+1$. Using Macaulay's Theorem 2.3.1, we see that

$$P = \sum_{d(\alpha_0+\cdots+\alpha_N)=deg(P)-d(N+1)} f_0^{\alpha_0}\cdots f_N^{\alpha_N} P_\alpha$$

with $deg(P_\alpha) = (d-1)(N+1) = \sigma$. This reduces the problem of computation of periods, to forms of the form

$$\frac{P_\alpha\,\Omega}{f_0^{\alpha_0}\cdots f_N^{\alpha_N}}, \tag{2.3}$$

with $\alpha_0,\ldots,\alpha_N \in \mathbb{Z}$ such that $\alpha_0+\cdots+\alpha_N = N+1$ and $deg(P_\alpha) = (d-1)(N+1)$. It is clear that such a form represents an exact top form of $\mathbb{P}^N$ if some α_i is non-positive (in fact, it is equivalent to show that it is zero in hypercohomology and this is clear because the form extends to a d-closed form on $V_{0\cdots\hat{i}\cdots N}$, as in the proof of Proposition 2.1.2) then we reduce the computation to forms of the form

$$\frac{Q\Omega}{f_0\cdots f_N},$$

where $deg(Q) = \sigma$.

Corollary 2.3.1. *If* $Q \in \mathbb{C}[x_0,\ldots,x_N]_\sigma$, *then*

$$\int_{\mathbb{P}^N}\frac{Q\Omega}{f_0\cdots f_N} = c\cdot d^{N+1}\cdot(-1)^{\binom{N+1}{2}}(2\pi\sqrt{-1})^N,$$

where $c \in \mathbb{C}$ *is the unique number such that*

$$Q \equiv c \cdot det(Jac(F)) \pmod{\langle f_0, \ldots, f_N \rangle}.$$

Proof To show the existence and uniqueness of $c \in \mathbb{C}$ we use item (ii) of Macaulay's Theorem 2.3.1. So, it is enough to show

$$det(Jac(F)) \notin \langle f_0, \ldots, f_N \rangle.$$

This is direct from the previous considerations and the fact

$$\frac{\Omega_F}{f_0 \cdots f_N} \in H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$$

does not represent an exact top form by Proposition 2.3.1. ■

Remark 2.3.2. In summary, the computation of the period reduces to the computation of such constant c that relates Q with $det(Jac(F))$ in R.

2.4 Periods of linear cycles inside Fermat varieties

As another application of the pull-back description given in section 2.2, we compute periods of algebraic cycles (for differential forms lying in the middle part of Hodge decomposition) in a very particular case, but interesting enough to get new results (see Theorem 3.6.2). Let us consider the Fermat variety of even dimension n and degree d

$$X_n^d : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0,$$

and the linear cycle of dimension $\frac{n}{2}$ given by

$$\mathbb{P}^{\frac{n}{2}} : x_0 - \zeta_{2d} x_1 = x_2 - \zeta_{2d} x_3 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0,$$

where ζ_{2d} is a $2d$-primitive root of unity. From Carlson-Griffiths' theorem we have a basis for the middle part of Hodge decomposition given in Čech cohomology by

$$(\omega_i)_J = \frac{x_0^{i_0} \cdots x_{n+1}^{i_{n+1}} \Omega_J}{\frac{n}{2}! \cdot d^{\frac{n}{2}+1} \cdot (x_{j_0} \cdots x_{j_{\frac{n}{2}}})^{d-1}} \in H^{\frac{n}{2}}(U, \Omega^{\frac{n}{2}}_{X_n^d}),$$

where $J = \{j_0 < \cdots < j_{\frac{n}{2}}\} \subseteq \{0, \ldots, n+1\}$, U is the standard covering of $\mathbb{P}^{n+1}$ restricted to X_n^d. Recall $\Omega \in H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(n+2))$ is defined by

$$\Omega := \sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i},$$

Ω_J is the contraction of Ω by J i.e.

$$\begin{aligned} \Omega_J &:= \iota_{\frac{\partial}{\partial x_{j_{\frac{n}{2}}}}}\left(\cdots \left(\iota_{\frac{\partial}{\partial x_{j_0}}}(\Omega)\right) \cdots \right) \\ &= (-1)^{j_0 + \cdots + j_{\frac{n}{2}} + \binom{\frac{n}{2}+2}{2}} \textstyle\sum_{l=0}^{\frac{n}{2}} (-1)^l x_{k_l} \widehat{dx_{k_l}}, \end{aligned} \tag{2.4}$$

where $K = \{k_0 < \cdots < k_{\frac{n}{2}}\} := \{0, \ldots, n+1\} \setminus J$, and

$$i \in I_{(\frac{n}{2}+1)d-n-2} := \left\{(i_0, \ldots, i_{n+1}) \in \{0, \ldots, d-2\}^{n+2} : i_0 + \cdots + i_{n+1} = (\tfrac{n}{2}+1)d - n - 2\right\}.$$

Theorem 2.4.1 ([MV17]). *For $i \in I_{(\frac{n}{2}+1)d-n-2}$ we have*

$$\int_{\mathbb{P}^{\frac{n}{2}}} \omega_i = \begin{cases} \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1} \cdot \frac{n}{2}!} \zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n} & \text{if} \quad i_{2l-2} + i_{2l-1} = d-2, \quad \forall l = 1, \ldots, \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let $\phi : \mathbb{P}^{\frac{n}{2}}_{(y_1:\cdots:y_{\frac{n}{2}+1})} \to \mathbb{P}^{n+1}_{(x_0:\cdots:x_{n+1})}$ be the closed immersion with image $\mathbb{P}^{\frac{n}{2}}$ given by

$$\phi(y_1 : \cdots : y_{\frac{n}{2}+1}) = (\zeta_{2d} y_1 : y_1 : \cdots : \zeta_{2d} y_{\frac{n}{2}+1} : y_{\frac{n}{2}+1}).$$

It follows from Corollary 2.2.1 that

$$\omega_i=\frac{1}{d^{\frac{n}{2}+1}\frac{n}{2}!}\left\{\frac{\zeta_{2d}^{i_0+i_2+\cdots+i_n}y^{i'}\Omega_J}{(x_{j_0}\cdots x_{j_{\frac{n}{2}+1}})^{d-1}}\right\}_{|J|=\frac{n}{2}}\in H^{\frac{n}{2}}(\pi^{-1}(\mathcal{U}),\Omega^{\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}), \tag{2.5}$$

where $i'=(i_0+i_1,\ldots,i_n+i_{n+1})$ and $\pi^{-1}(\mathcal{U})$ is the open covering of $\mathbb{P}^{\frac{n}{2}}$ given by the pre-images of the standard covering of $\mathbb{P}^{n+1}$ (in particular this covering has repeated open sets). Since for $\{k_1<\cdots<k_{\frac{n}{2}}\}\subset\{1,\ldots,\frac{n}{2}+1\}$ we have

$$\Omega_J\Big(\frac{\partial}{\partial y_{k_1}},\ldots,\frac{\partial}{\partial y_{k_{\frac{n}{2}}}}\Big)=\Omega(\varphi(y))\left(\frac{\partial}{\partial x_{j_0}},\ldots,\frac{\partial}{\partial x_{j_{\frac{n}{2}}}};\ \zeta_{2d}\frac{\partial}{\partial x_{2k_1-2}}+\frac{\partial}{\partial x_{2k_1-1}},\ldots,\zeta_{2d}\frac{\partial}{\partial x_{2k_{\frac{n}{2}}-2}}+\frac{\partial}{\partial x_{2k_{\frac{n}{2}}-1}}\right).$$

It follows that if $\#(J\cap\{2l-2,2l-1\})=2$ for some $l\in\{1,\ldots,\frac{n}{2}+1\}$, then $\Omega_J=0$. On the other hand, if $\#(J\cap\{2l-2,2l-1\})=1, \forall l\in\{1,\ldots,\frac{n}{2}+1\}$ then

$$\Omega_J\Big(\frac{\partial}{\partial y_{k_1}},\ldots,\frac{\partial}{\partial y_{k_{\frac{n}{2}}}}\Big)=\zeta_{2d}^{\#J_1}(-1)^{k+\binom{\frac{n}{2}}{2}+\#J_1}y_k,$$

where $\{k_1,\ldots,k_{\frac{n}{2}}\}=\{1,\ldots,\frac{n}{2}+1\}\setminus\{k\}$ and $J_1:=\{j\in J \mid j \text{ is odd}\}$. Hence

$$\Omega_J=(-\zeta_{2d})^{\#J_1}(-1)^{\binom{\frac{n}{2}}{2}+1}\Omega', \tag{2.6}$$

where

$$\Omega'=\sum_{k=1}^{\frac{n}{2}+1}(-1)^{k-1}y_k\widehat{dy_k}.$$

Since for any such J we have $(x_{j_0}\cdots x_{j_{\frac{n}{2}}})^{d-1}=\zeta_{2d}^{(d-1)(\frac{n}{2}+1-\#J_1)}(y_1\cdots y_{\frac{n}{2}+1})^{d-1}$, replacing (2.6) in (2.5) we get

$$(\pi^*\omega_i)_J=\frac{(-1)^{\binom{\frac{n}{2}+1}{2}}\zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n}y^{i'}\Omega'}{d^{\frac{n}{2}+1}\frac{n}{2}!(y_1\cdots y_{\frac{n}{2}+1})^{d-1}}\in H^{\frac{n}{2}}(\mathcal{U}',\Omega^{\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}), \tag{2.7}$$

where $\mathcal{U}'$ is the standard covering of $\mathbb{P}^{\frac{n}{2}}$. By Proposition 2.1.2, the form (2.7) is not exact if and only if $i'_l=d-2, \forall l\in\{1,\ldots,\frac{n}{2}+1\}$. The result follows from the fact that the standard top form $\frac{\Omega'}{y_1\cdots y_{\frac{n}{2}+1}}$ integrates $(-1)^{\binom{\frac{n}{2}+1}{2}}(2\pi\sqrt{-1})^{\frac{n}{2}}$ over $\mathbb{P}^{\frac{n}{2}}$. ■

Another interesting property of Fermat variety is that its automorphism group acts transitively on the set of linear subspaces of dimension $\frac{n}{2}$ inside it. This allows us to use the pull-back of these automorphisms to compute the periods of all linear cycles inside Fermat variety. Consider the groups μ_d^{n+2} and S_{n+2}, where $\mu_d^{n+2}=\mu_d\times\cdots\times\mu_d$, $\mu_d:=\{1,\zeta_d,\ldots,\zeta_d^{d-1}\}$ is the group of d-th roots of unity, and S_{n+2} is the group of permutations of $\{0,\ldots,n+1\}$. An element $a=(\zeta_d^{a_0},\ldots,\zeta_d^{a_{n+1}})\in\mu_d^{n+2}$ acts over the Fermat variety X_n^d by coordinate-wise multiplication, in particular the diagonal $\mu_d \hookrightarrow \mu_d^{n+2}$ acts trivially. By the other hand, an element $b=(b_0,\dots,b_{n+1})\in S_{n+2}$ acts over X_n^d by permutation of the coordinates. For $(a,b)\in \mu_d^{n+2}\rtimes S_{n+2}$ we define $\mathbb{P}^{\frac{n}{2}}_{a,b}:=b^{-1}(a^{-1}(\mathbb{P}^{\frac{n}{2}}))$ i.e.

$$\mathbb{P}^{\frac{n}{2}}_{a,b}:\begin{cases} x_{b_0}-\zeta_{2d}^{1+2a_1-2a_0}x_{b_1}=0,\\ x_{b_2}-\zeta_{2d}^{1+2a_3-2a_2}x_{b_3}=0,\\ x_{b_4}-\zeta_{2d}^{1+2a_5-2a_4}x_{b_5}=0,\\ \vdots \\ x_{b_n}-\zeta_{2d}^{1+2a_{n+1}-2a_n}x_{b_{n+1}}=0. \end{cases}$$

Corollary 2.4.1. *For $i\in I_{(\frac{n}{2}+1)d-n-2}$ we have*

$$\int_{\mathbb{P}^{\frac{n}{2}}_{a,b}}\omega_i=\begin{cases}\frac{\mathrm{sign}(b)\,(2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1}\,\frac{n}{2}!}\,\zeta_{2d}^{\sum_{e=0}^{\frac{n}{2}}(i_{b_{2e}}+1)\cdot(1+2a_{2e+1}-2a_{2e})} & \text{if } i_{b_{2l-2}}+i_{b_{2l-1}}=d-2,\ \forall 1\le l\le \frac{n}{2}+1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof Considering the automorphism of $\mathbb{P}^{n+1}$ given by $(a\cdot b)^{-1}=b^{-1}\cdot a^{-1}:\mathbb{P}^{n+1}\to\mathbb{P}^{n+1}$, we have that

$$\int_{\mathbb{P}^{\frac{n}{2}}}((a\cdot b)^{-1})^*\omega_i=\int_{\mathbb{P}^{\frac{n}{2}}_{a,b}}\omega_i.$$

For $J=\{j_0,\dots,j_{\frac{n}{2}}\}\subset\{0,\dots,n+1\}$ we see that

$$((b^{-1})^*\omega_i)_J=\mathrm{sign}(b)(\omega_{i_b})_{b^{-1}(J)}$$

where $b^{-1}(J)=\{b_{j_0}^{-1},\dots,b_{j_{\frac{n}{2}}}^{-1}\}$ and $i_b=(i_{b_0},\dots,i_{b_{n+1}})\in I_{(\frac{n}{2}+1)d-n-2}$. Then

$$(((a\cdot b)^{-1})^*\omega_i)_J=\mathrm{sign}(b)(a^{-1})^*((\omega_{i_b})_{b^{-1}(J)})=\mathrm{sign}(b)\,\zeta_{2d}^{2\sum_{e=0}^{n+1}a_e(i_{b_e}+1)}(\omega_{i_b})_{b^{-1}(J)}.$$

It follows from Theorem 2.4.1 that the period is non-zero for $i_{b_{2l-2}}+i_{b_{2l-1}}=d-2$, $\forall 1\le l\le \frac{n}{2}+1$, in that case we obtain that $\zeta_{2d}^{2\sum_{e=0}^{n+1}a_e(i_{b_e}+1)}=\zeta_{2d}^{\sum_{e=0}^{\frac{n}{2}}(i_{b_{2e}}+1)\cdot(2a_{2e+1}-2a_{2e})}$ and the result follows. ■

2.5 Coboundary map

In order to compute periods of complete intersection algebraic cycles, we need to compute periods of smooth hyperplane sections of a given projective smooth variety X. In other words, for $Y \hookrightarrow X$ a smooth hypersurface given by $\{F = 0\}$, we need an explicit description of the isomorphism

$$\begin{array}{ccc} H^n(Y, \Omega^n_Y) & \simeq & H^{n+1}(X, \Omega^{n+1}_X), \\ \omega & \mapsto & \hat{\omega} \end{array}$$

together with the relation of periods, i.e. the number $a \in \mathbb{C}$ such that

$$\int_X \hat{\omega} = a \int_Y \omega.$$

For this purpose recall the exact sequence (1.16)

$$\cdots \to H^{k+1}_{\mathrm{dR}}(X/\mathbb{C}) \to H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \xrightarrow{res} H^{k}_{\mathrm{dR}}(Y/\mathbb{C}) \xrightarrow{\delta} H^{k+2}_{\mathrm{dR}}(X/\mathbb{C}) \to \cdots,$$

induced by the short exact sequence of complexes

$$0 \to \Omega^\bullet_X \to \Omega^\bullet_X(\log Y) \xrightarrow{Res} j_*\Omega^{\bullet-1}_Y \to 0.$$

Since $H^{2n+1}_{\mathrm{dR}}(U) = H^{2n+2}_{\mathrm{dR}}(U) = 0$, the coboundary map is an isomorphism

$$H^{2n}_{\mathrm{dR}}(Y/\mathbb{C}) \simeq H^{2n+2}_{\mathrm{dR}}(X/\mathbb{C}).$$

Noticing these vector spaces are one dimensional, and δ preserves Hodge filtration we see, it induces the desired isomorphism

$$H^n(Y, \Omega^n_Y) \simeq H^{n+1}(X, \Omega^{n+1}_X).$$

Proposition 2.5.1. *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n+1$, and $Y \subset X$ a smooth hyperplane section given by $\{F = 0\} \cap X$, for some homogeneous $F \in \mathbb{C}[x_0, \ldots, x_N]_d$. Let $\omega \in C^n(X, \Omega^n_X)$ such that $\omega|_Y \in Z^n(Y, \Omega^n_Y)$. For any $\tilde{\omega} \in C^n(X, \Omega^{n+1}_X(\log Y))$ such that*

$$\tilde{\omega} \equiv \omega \wedge \frac{dF}{F} \ (mod\ C^n(X, \Omega^{n+1}_X)),$$

we have

$$\hat{\omega} := (-1)^{n+1} \delta(\tilde{\omega}) \in Z^{n+1}(X, \Omega^{n+1}_X).$$

Furthermore, $\hat{\omega} \in H^{n+1}(X, \Omega^{n+1}_X)$ is uniquely determined by $\omega|_Y \in H^n(Y, \Omega^n_Y)$ and

$$\int_X \hat{\omega} = \frac{(-1)^{n+1} 2\pi\sqrt{-1}}{d} \int_Y \omega. \tag{2.8}$$

Proof The map described in the proposition is the coboundary map τ, i.e. $\tau(\omega) = \tilde{\omega}$. So it is left to prove the period relation. By the fact that τ is an isomorphism of one dimensional spaces we have a constant $a_{X,Y} \in \mathbb{C}$ such that

$$\int_X \tau(\omega) = a_{X,Y} \int_Y \omega,$$

for every $\omega \in H^n(Y, \Omega_Y^n)$. Since X and Y are complete intersections, Lefschetz hyperplane section theorem (see [Mov17a]) implies we can extend ω and $\tau(\omega)$ to $\mathbb{P}^N$. If X is complete intersection of type $(d_1, \dots, d_k)$, then $[X] = d_1 \cdots d_k [\mathbb{P}^{n+1}] \in H_{2n+2}(\mathbb{P}^N, \mathbb{Z})$ and $[Y] = dd_1 \cdots d_k [\mathbb{P}^n] \in H_{2n}(\mathbb{P}^N, \mathbb{Z})$. By Stokes' theorem

$$d_1 \cdots d_k \int_{\mathbb{P}^{n+1}} \tau(\omega) = a_{X,Y}\, dd_1 \cdots d_k \int_{\mathbb{P}^n} \omega.$$

In other words

$$a_{X,Y} = \frac{a_{\mathbb{P}^{n+1},\mathbb{P}^n}}{d}.$$

Finally, to compute $a_{\mathbb{P}^{n+1},\mathbb{P}^n}$ we suppose $\mathbb{P}^n = \{x_{n+1} = 0\}$, we take $\omega \in C^n(\mathcal{U}, \mathbb{P}^{n+1})$, where $\mathcal{U}$ is the standard open covering of $\mathbb{P}^{n+1}$, and

$$\omega_J = \frac{\sum_{i=0}^n (-1)^i x_{j_i} \widehat{dx_{j_i}}}{x_{j_0} \cdots x_{j_n}}, \text{ for } |J| = n.$$

Then

$$\tilde{\omega}_J = \begin{cases} \frac{\sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i}}{x_0 \cdots x_{n+1}} & \text{if } J = \{0, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence

$$\tilde{\omega}_{0 \cdots n} = \frac{\sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i}}{x_0 \cdots x_{n+1}}.$$

It follows by Proposition 2.1.1 that $a_{\mathbb{P}^{n+1},\mathbb{P}^n} = (-1)^{n+1} 2\pi\sqrt{-1}$. ■

Remark 2.5.1. Notice that in Proposition 2.5.1, the assumption Y being a *smooth* hyperplane section, is just to simplify the exposition when talking about Ω_Y^n and $\Omega_X^{n+1}(\log Y)$. But this condition is superfluous. In fact, we can take $Y = \{F = 0\} \cap X$ not necessarily smooth and the relation (2.8) will still be true (a way to argue this is that both sides of the equation are continuous with respect to $F \in \mathbb{C}[x_0, \dots, x_N]_d$, and we already showed they are equal in a dense open set of it).

2.6 Periods of complete intersection algebraic cycles

In this section we compute periods of complete intersection algebraic cycles inside a smooth hypersurface $X\subset\mathbb{P}^{n+1}$, of even dimension n. After Carlson-Griffiths' theorem we know these periods are generated by the forms

$$\omega_P = res\left(\frac{P\Omega}{F^{\frac{n}{2}+1}}\right) = \frac{1}{\frac{n}{2}!}\left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=\frac{n}{2}} \in H^{\frac{n}{2}}(X,\Omega_X^{\frac{n}{2}}), \tag{2.9}$$

where $P\in\mathbb{C}[x_0,\dots,x_{n+1}]_\delta$, and $\delta=(d-2)(\frac{n}{2}+1)$. In order to compute these periods over a complete intersection subvariety Z of $\mathbb{P}^{n+1}$ (contained in X), the main ingredient is the explicit description of the coboundary map. Given the complete intersection $Z\subset X$ of dimension $\frac{n}{2}$, we construct a chain of subvarieties

$$Z=Z_0\subset Z_1\subset Z_2\subset\cdots\subset Z_{\frac{n}{2}+1}=\mathbb{P}^{n+1},$$

where each Z_i is a hypersurface of Z_{i+1}, and apply inductively the coboundary map, to reduce the computation of the period of Z to the computation of the integral of a top form in $\mathbb{P}^{n+1}$.

Remark 2.6.1. For us $Z\subset\mathbb{P}^{n+1}$ will be a *complete intersection inside* $\mathbb{P}^{n+1}$ of dimension $\frac{n}{2}$ if there exist $f_1,\dots,f_{\frac{n}{2}+1}\in\mathbb{C}[x_0,\dots,x_{n+1}]$ homogeneous polynomials such that

$$I(Z)=\langle f_1,\dots,f_{\frac{n}{2}+1}\rangle.$$

Theorem 2.6.1. *Let* $X\subset\mathbb{P}^{n+1}$ *be a smooth hypersurface given by* $X=\{F=0\}$. *Suppose*

$$F=f_1g_1+\cdots+f_{\frac{n}{2}+1}g_{\frac{n}{2}+1},$$

such that $Z:=\{f_1=\cdots=f_{\frac{n}{2}+1}=0\}\subset X$ *is a complete intersection. Define*

$$H=(h_0,\dots,h_{n+1}):=(f_1,g_1,f_2,g_2,\dots,f_{\frac{n}{2}+1},g_{\frac{n}{2}+1}).$$

Then

$$\int_Z\omega_P=\frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{\frac{n}{2}!}c\cdot(d-1)^{n+2}\cdot d_1\cdots d_{\frac{n}{2}+1}, \tag{2.10}$$

where $d_i=deg\ f_i$, ω_P *is given by* (2.9), *and* $c\in\mathbb{C}$ *is the unique number such that*

$$P\cdot det(Jac(H))\equiv c\cdot det(Hess(F)) \pmod{J^F}.$$

Remark 2.6.2. To understand the statement of theorem 2.6.1 recall that Macaulay's Theorem 2.3.1 implies that

$$R^F:=\frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{J^F},$$

where $J^F:=\langle F_0,\dots,F_{n+1}\rangle$ is the Jacobian ideal, is an Artinian Gorenstein ring of socle $2\delta=(d-2)(n+2)$. In particular $\dim_{\mathbb{C}}R^F_{2\delta}=1$. Furthermore, by Proposition 2.3.1 and Remark 2.3.1, $R^F_{2\delta}$ is generated by $det(Hess(F))$. As a consequence, for any pair of polynomials $P,Q\in\mathbb{C}[x_0,\dots,x_{n+1}]_\delta$ there exist a unique $c\in\mathbb{C}$ such that

$$P\cdot Q\equiv c\cdot det(Hess(F)) \pmod{J^F}.$$

Definition 2.6.1. We will say that a Hodge cycle $\delta \in H_n(X,\mathbb{Z})$ is of *complete intersection type* if

$$\delta = \sum_{i=1}^{k} n_i [Z_i],$$

for $Z_1,\dots,Z_k \subset X$ a set of $\frac{n}{2}$-dimensional subvarieties that are complete intersection inside $\mathbb{P}^{n+1}$, given by

$$Z_i = \{f_{i,1} = \cdots = f_{i,\frac{n}{2}+1} = 0\},$$

for every $i = 1,\dots,k$, such that there exist $g_{i,1},\dots,g_{i,k} \in \mathbb{C}[x_0,\dots,x_{n+1}]$ with

$$F = \sum_{j=1}^{\frac{n}{2}+1} f_{i,j} g_{i,j}.$$

For every such Hodge cycle, we define its *associated polynomial*

$$P_\delta := \sum_{i=1}^{k} c_{d_i} \cdot n_i \cdot det(Jac(H_i)) \in R^F_\sigma,$$

where $d_i := \deg Z_i$, $\sigma := (d-2)(\frac{n}{2}+1)$ and $H_i := (f_{i,1}, g_{i,1},\dots,f_{i,\frac{n}{2}+1}, g_{i,\frac{n}{2}+1})$.

Remark 2.6.3. Theorem 2.6.1 tells us that in order to compute the periods of a complete intersection type cycle δ it is enough to know its associated polynomial P_δ. In fact, we are determining the *Poincaré dual of the cycle δ in primitive cohomology*

$$\delta^{\mathrm{pd}} = res\left(\frac{P_\delta \Omega}{F^{\frac{n}{2}+1}}\right) \in H^{\frac{n}{2}}(X,\Omega_X^{\frac{n}{2}})_{prim}.$$

In the sense that it satisfies (up to some constant non-zero factor)

$$\int_\delta \omega = \int_X \omega \wedge res\left(\frac{P_\delta \Omega}{F^{\frac{n}{2}+1}}\right), \qquad \forall \omega \in H^n_{\mathrm{dR}}(X)_{prim}.$$

Remark 2.6.4. In order to prove Theorem 2.6.1, we will use Proposition 2.5.1 to construct inductively

$$\tilde{\omega}^{(0)} := \omega|_Z \in H^{\frac{n}{2}}(Z,\Omega_Z^{\frac{n}{2}}) \text{ and } Z_0 := Z.$$

Then for $l = 1,\dots,\frac{n}{2}+1$ we define

$$\tilde{\omega}^{(l)} := \tilde{\omega}^{(l-1)} \in H^{\frac{n}{2}+l}(Z_l,\Omega_{Z_l}^{\frac{n}{2}+l}) \text{ and } Z_l := \{f_{l+1} = \cdots = f_{\frac{n}{2}+1} = 0\} \subset \mathbb{P}^{n+1}.$$

Observe that $Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1}$. Since both sides of (2.10) are continuous with respect to the parameters

$$(f_1, g_1,\dots,f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in \bigoplus_{i=1}^{\frac{n}{2}+1} \mathbb{C}[x_0,\dots,x_{n+1}]_{d_i} \oplus \mathbb{C}[x_0,\dots,x_{n+1}]_{d-d_i},$$

such that $F := f_1 g_1 + \quad + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$. It is enough to prove Theorem 2.6.1 for a generic $(f_1; g_1; ::: ; f_{\frac{n}{2}+1}; g_{\frac{n}{2}+1})$. This is why we can assume each $Z_{l \ \ 1}$ is a smooth hyperplane section of Z_l, for $l = 1; ::: ; \frac{n}{2}+1$, as in the hypothesis of Proposition 2.5.1.

Lemma 2.6.1. *For each* $l = 0; ::: ; \frac{n}{2}+1$ *and* $J \quad f0; ::: ; n+1g$ *with* $jJj = \frac{n}{2}+l$

$$(?^{(l)})_J \;=\; \frac{(\ \ 1)^{\binom{\frac{n}{2}+2}{2}+\frac{n}{2}l+\binom{l+1}{2}+\ \ (J)} P\, d^l d_1 \quad d_l}{\frac{n}{2}!\ \ F_J}\left[\sum_{m=1}^{l}(\ \ 1)^{m\ \ 1} g_m \frac{\widehat{dg_m}}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l} dx_{k_r} \bigwedge_{t=1}^{l} \frac{df_t}{d_t}\right.$$
$$\left. +(\ \ 1)^l \sum_{p=0}^{\frac{n}{2}\ \ l}(\ \ 1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \widehat{dx_{k_p}} \wedge \bigwedge_{t=1}^{l} \frac{df_t}{d_t} + (\ \ 1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l} dx_{k_r} \wedge \frac{\widehat{df_q}}{d_q}\right]:$$

Where for $J = (j_0; ::: ; j_{\frac{n}{2}+l})$, $\ (J) = j_0 + \quad + j_{\frac{n}{2}+l}$, *and* $K = (k_0; ::: ; k_{\frac{n}{2}\ \ l}) = (0; 1; ::: ; n+1) n J$.

Proof We proceed by induction on l:

$\underline{l = 0}$: By (2.4) we get

$$(?^{(0)})_{j_0 \ \ j_{\frac{n}{2}}} = (!)_{j_0 \ \ j_{\frac{n}{2}}} = \frac{(\ \ 1)^{\binom{\frac{n}{2}+2}{2}+\ \ (J)} P}{\frac{n}{2}!\ \ F_J}\left[\sum_{p=0}^{\frac{n}{2}}(\ \ 1)^p x_{k_p} \widehat{dx_{k_p}}\right]:$$

$\underline{l \ ) \ l+1}$:

$$(?^{(l)})_J \wedge \frac{df_{l+1}}{f_{l+1}} \qquad \frac{(\ \ 1)^{\binom{\frac{n}{2}+2}{2}+\frac{n}{2}l+\binom{l+1}{2}+\ \ (J)} P\, d^l d_1 \quad d_{l+1}}{\frac{n}{2}!\ \ F_J \ \ f_{l+1}}\left[\sum_{m=1}^{l}(\ \ 1)^{m\ \ 1} g_m \frac{\widehat{dg_m}}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t}\right.$$
$$+(\ \ 1)^l \sum_{p=0}^{\frac{n}{2}\ \ l}(\ \ 1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \widehat{dx_{k_p}} \wedge \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t}$$
$$+(\ \ 1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l} dx_{k_r} \wedge \frac{\widehat{df_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}}$$
$$\left. +(\ \ 1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{\widehat{dg_u}}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l} dx_{k_r} \wedge \frac{\widehat{df_u}}{d_u}\right]:$$

Applying we get

$$
\begin{aligned}
!_J^{(l+1)} &= \frac{(\ 1)^{\binom{\frac{n}{2}+2}{2}+\frac{n}{2}(l+1)+\binom{l+2}{2}+\ (J)} P d^l d_1 \quad d_{l+1}}{\frac{n}{2}!\ \ F_J \ \ f_{l+1}} \\
&\Bigg[\sum_{m=1}^{l}(\ 1)^{m\ \ 1} g_m \frac{\widehat{dg_m}}{d} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p}\right) \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(\ 1)^l \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} x_{j_p}\right) \bigwedge_{s=1}^{l} \frac{dg_s}{d} \bigwedge_{q=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_q} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(\ 1)^{l+1} \sum_{p=0}^{\frac{n}{2}\ \ l\ \ 1} (\ 1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \left(\sum_{r=0}^{\frac{n}{2}+l+1} F_{j_r} dx_{j_r}\right) \wedge \widehat{dx_{k_p}} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(\ 1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p}\right) \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \wedge \frac{\widehat{df_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}} \\
&+(\ 1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{\widehat{dg_u}}{d} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p}\right) \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \wedge \frac{\widehat{df_u}}{d_u}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\ 1)^{\binom{\frac{n}{2}+2}{2}+\frac{n}{2}(l+1)+\binom{l+2}{2}+\ (J)} P d^{l+1} d_1 \quad d_{l+1}}{\frac{n}{2}!\ \ F_J \ \ f_{l+1}} \Bigg[\sum_{m=1}^{l}(\ 1)^{m\ \ 1} g_m \frac{\widehat{dg_m}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(\ 1)^l F \bigwedge_{s=1}^{l} \frac{dg_s}{d} \bigwedge_{q=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_q} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} + (\ 1)^{l+1} \sum_{p=0}^{\frac{n}{2}\ \ l\ \ 1} (\ 1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \frac{dF}{d} \wedge \widehat{dx_{k_p}} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(\ 1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \wedge \frac{\widehat{df_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}} \\
&+(\ 1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{\widehat{dg_u}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}\ \ l\ \ 1} dx_{k_r} \wedge \frac{\widehat{df_u}}{d_u}\Bigg] :
\end{aligned}
$$

Replacing $F = f_1 g_1 + \quad + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$ in the first three expressions we finish the induction. ■

Proof of Theorem 2.6.1 Using Lemma 2.6.1 for $l = \frac{n}{2}+1$ we get

$$
\begin{aligned}
(!^{(\frac{n}{2}+1)})_{0 \quad n+1} &= \frac{(\ 1)^{\ (J)} P d^{\frac{n}{2}+1} d_1 \quad d_{\frac{n}{2}+1}}{\frac{n}{2}!\ \ F_0 \quad F_{n+1}} \Bigg[\sum_{m=1}^{\frac{n}{2}+1}(\ 1)^{m\ \ 1} g_m \frac{\widehat{dg_m}}{d} \bigwedge_{t=1}^{\frac{n}{2}+1} \frac{df_t}{d_t} \\
&+(\ 1)^{n+1} \sum_{q=1}^{\frac{n}{2}+1} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \wedge \frac{\widehat{df_q}}{d_q}\Bigg] :
\end{aligned}
$$

Replacing $F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$ we obtain

$$(\omega^{(\frac{n}{2}+1)})_{0\cdots n+1} = \frac{(-1)^{\epsilon(J)} P d^{\frac{n}{2}+1} d_1 \cdots d_{\frac{n}{2}+1}}{\frac{n}{2}!\, F_0 \cdots F_{n+1}} \left[\sum_{m=1}^{\frac{n}{2}+1} (-1)^{m-1} \left(\frac{d-d_m}{d}\right) g_m \widehat{\frac{dg_m}{d}} \bigwedge_{t=1}^{\frac{n}{2}+1} \frac{df_t}{d_t} + (-1)^{\frac{n}{2}} \sum_{q=1}^{\frac{n}{2}+1} (-1)^q f_q \bigwedge_{s=1}^{\frac{n}{2}+1} \frac{dg_s}{d} \wedge \widehat{\frac{df_q}{d_q}} \right];$$

in other words

$$(\omega^{(\frac{n}{2}+1)})_{0\cdots n+1} = \frac{(-1)^{\epsilon(J)+\binom{\frac{n}{2}+2}{2}} P e_0 \cdots e_{n+1}}{\frac{n}{2}!\, F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k h_k \widehat{\frac{dh_k}{e_k}};$$

where $e_k = deg(h_k)$. Replacing $e_i h_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} \cdot x_j$ and $dh_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} dx_j$ we obtain

$$(\omega_i^{(\frac{n}{2}+1)})_{0\cdots n+1} = \frac{(-1)^{\binom{\frac{n}{2}+1}{2}} P \cdot det(Jac(H))}{\frac{n}{2}!\, F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k x_k \widehat{dx_k}.$$

The theorem follows from Proposition 2.5.1, Proposition 2.1.1, and Corollary 2.3.1. ∎

Remark 2.6.5. Notice that from the formula of Theorem 2.6.1, all periods are zero when

$$det(Jac(H)) = 0.$$

For instance if some g_i is constant. This is consistent with the fact that Z will be the complete intersection of X with a codimension $\frac{n}{2}$ complete intersection of $\mathbb{P}^{n+1}$, in fact in this case

$$Z = \{f_1 = \cdots = f_{i-1} = f_{i+1} = \cdots = f_{\frac{n}{2}+1} = 0\} \cap X.$$

Furthermore, the formula is giving us a characterization of such algebraic cycles:

Corollary 2.6.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by*

$$X = \{F = 0\}.$$

If $\delta \in H_n(X, \mathbb{Z})$ is a complete intersection type algebraic cycle, then

$$P \in J^F \text{ if and only if } \delta = \lambda[X \cap \mathbb{P}^{\frac{n}{2}+1}] \text{ in } H_n(X, \mathbb{Q}), \text{ for some } \lambda \in \mathbb{Q}.$$

Proof By Macaulay's Theorem 2.3.1, $P \in J^F$, if and only if, $P \cdot P \in J^F$ for all $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$. Theorem 2.6.1 says this is equivalent to the vanishing of all periods for $\omega \in H^n_{\mathrm{dR}}(X)_{prim}$. By Poincaré's duality we conclude this is equivalent to $\delta = \lambda[X \cap \mathbb{P}^{\frac{n}{2}+1}]$ in $H_n(X, \mathbb{Q})$, for some $\lambda \in \mathbb{Q}$. ∎

Remark 2.6.6. This corollary is giving us an algebraic (and computable) criterion to determine whether an algebraic cycle (given as a combination of complete intersection algebraic cycles) is trivial or not. Even in the case X is a surface, to determine the selfintersection of a singular divisor is not trivial.

Remark 2.6.7. Another observation we can derive from Theorem 2.6.1 is that each period is of the form $(2\pi\sqrt{-1})^{\frac{n}{2}}$ times a number in a number field k, where k is the smallest number field such that $f_1, g_1, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1} \in k[x_0, \ldots, x_{n+1}]$, i.e. the periods belong to the same field where we can decompose F as $f_1g_1 + \cdots + f_{\frac{n}{2}+1}g_{\frac{n}{2}+1}$. This was already mentioned in Deligne's work about absolute Hodge cycles (see [DMOS82] Proposition 7.1).

We close this section with an example of how Theorem 2.6.1 can be used to compute periods. We provide another proof of Theorem 2.4.1.

Proposition 2.6.1. *For* $X = \{x_0^d + \cdots + x_{n+1}^d = 0\}$ *the Fermat variety,* $\mathbb{P}^{\frac{n}{2}} = \{x_0 - \zeta_{2d}x_1 = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\}$*; and* $\delta = [\mathbb{P}^{\frac{n}{2}}]$*, its associated polynomial is*

$$P_\delta = d^{\frac{n}{2}+1}\zeta_{2d}^{\frac{n}{2}+1}\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2} x_{2j-2}^{d-2-l}\zeta_{2d}^{l}x_{2j-1}^{l}\right).$$

Proof We notice that the Jacobian matrix of H is diagonal by 2×2 blocks, and each block has determinant $d(-\zeta_{2d}x_{2j-2}^{d-1} + x_{2j-1}^{d-1})/(x_{2j-2} - \zeta_{2d}x_{2j-1})$. ■

Corollary 2.6.2. *For* X *and* $\mathbb{P}^{\frac{n}{2}}$ *as in Proposition 2.6.1, and* $i \in I_{(\frac{n}{2}+1)d-n-2}$ *we have*

$$\int_{\mathbb{P}^{\frac{n}{2}}} \omega_i = \begin{cases} \frac{(2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1}\frac{n}{2}!}\zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n} & \text{if } \quad i_{2l-2} + i_{2l-1} = d-2, \quad \forall l = 1, \ldots, \frac{n}{2}+1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof By Theorem 2.6.1 we only need to compute $c \in \mathbb{C}$ such that

$$x^iP_\delta \equiv c \cdot det(Hess(F)) \pmod{\langle x_0^{d-1}, \ldots, x_{n+1}^{d-1}\rangle}.$$

By Proposition 2.6.1

$$x^iP_\delta = d^{\frac{n}{2}+1}\zeta_{2d}^{\frac{n}{2}+1}x^i\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2} x_{2j-2}^{d-2-l}\zeta_{2d}^{l}x_{2j-1}^{l}\right),$$

$$\equiv \zeta_{2d}^{\frac{n}{2}+1+l_1+\cdots+l_{\frac{n}{2}}}(x_0 \cdots x_{n+1})^{d-2} \pmod{\langle x_0^{d-1}, \ldots, x_{n+1}^{d-1}\rangle},$$

if there exist $l_j \in \{0, \ldots, d-2\}$ such that $l_j + i_{2j-1} = d-2$ and $d-2-l_j+i_{2j-2} = d-2$, for every $j = 1, \ldots, \frac{n}{2}+1$. And is zero otherwise. This condition is equivalent to $l_j = i_{2j-2}$ and $i_{2j-2} + i_{2j-1} = d-2$. The desired result follows from the computation of the Hessian matrix for the Fermat variety

$$det(Hess(F)) = d^{n+2}(d-1)^{n+2}(x_0 \cdots x_{n+1})^{d-2}.$$

■

Chapter 3

Variational Hodge Conjecture

Summary

Roughly speaking, while Hodge conjecture asks whether every Hodge cycle is an algebraic cycle, variational Hodge conjecture asks whether the deformation of an algebraic cycle, that remains a Hodge cycle along the deformation, is an algebraic cycle. In this chapter we will introduce the space of deformation of Hodge cycles, the so called Hodge locus. In order to prove variational Hodge conjecture, we will determine some local components of Hodge locus. The study of this space is an active source of research and is far from being well understood.

When considering the Hodge locus for Hodge cycles inside surfaces, it corresponds to the classical Noether-Lefschetz locus (that corresponds to the parameter space of surfaces with non-trivial Picard number). Although Hodge conjecture is known to be true for surfaces, it says nothing about the description of Noether-Lefschetz locus. The Noether-Lefschetz locus has been studied by several mathematicians (such as Green, Voisin, Harris among others), but it is very mysterious for surfaces of degree 8 or more.

In order to study Hodge and Noether-Lefschetz loci, our main tool is infinitesimal variation of Hodge structures (IVHS), developed by Carlson, Green, Griffiths and Harris [CGGH83]. We will give an algebraic approach to IVHS. This is possible after the algebraic description of Gauss-Manin connection made by Katz and Oda [KO68].

We will close this chapter showing how to use the information of periods of algebraic cycles (developed in Chapter 2) to determine components of the Hodge locus (and prove variational Hodge conjecture). The chapter is divided as follows.

In section 3.1 we introduce the *de Rham cohomology sheaf* $F^iH^k_{dR}(X/T)$ *associated to a locally trivial family of smooth projective varieties* $\pi : X \to T$. These sheaves come with a *Hodge filtration.*

In section 3.2 we introduce *Gauss-Manin connection*

$$\nabla : H^k_{dR}(X/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^k_{dR}(X/T).$$

And we show its transversality with respect to the Hodge filtration.

$$\nabla F^i H^k_{\mathrm{dR}}(X/T) \subset \Omega^1_T \otimes_{O_T} F^{i-1} H^k_{\mathrm{dR}}(X/T).$$

The so called Giffiths' transversality theorem.

Dualizing Gauss-Manin connection we obtain

$$\overline{\nabla} : F^i H^k_{\mathrm{dR}}(X/T)/F^{i+1} H^k_{\mathrm{dR}}(X/T) \to \Omega^1_T \otimes_{O_T} F^{i-1} H^k_{\mathrm{dR}}(X/T)/F^i H^k_{\mathrm{dR}}(X/T).$$

Specializing at $t \in T$ we obtain the map

$$\overline{\nabla}_t : T_tT \to Hom_{\mathbb{C}}(H^{i,k-i}(X_t), H^{i-1,k-i+1}(X_t)).$$

In section 3.3 we describe it explicitly in the case $\pi : X \to T$ is the family of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. We show it corresponds with polynomial multiplication after we identify $T_tT \simeq \mathbb{C}[x_0, \dots, x_{n+1}]_d$, $H^{i,n-i}(X_t)_{prim} \simeq R^F_{d(n-i+1)-n-2}$ and $H^{i-1,n-i+1}(X_t)_{prim} \simeq R^F_{d(n-i+2)-n-2}$ (where the last two identifications are given by Griffiths' Theorems 1.5.1 and 1.5.2).

In section 3.4 we introduce the *Hodge locus* V^p associated to every $\lambda \in H_{2n-k}(X_t, \mathbb{Z})$ and every $p \in \{0, \dots, k\}$. We describe its Zariski tangent space using IVHS, obtaining the following result.

Proposition. *For every $\lambda \in H_{2n-k}(X_t, \mathbb{Z})$ such that its Poincaré dual is in $F^p H^k_{dR}(X_t)$ we define the map*

$$^0\overline{\nabla}_t(\lambda) : T_tT \to H^{n-p+1,n-k+p-1}(X_t) ;$$

given by

$$(^0\overline{\nabla}_t(\lambda)(v))(\omega) := \int_\lambda (\overline{\nabla}_t(v))(\omega).$$

The Zariski tangent space of the Hodge locus corresponding to λ is

$$T_tV^p = Ker\ ^0\overline{\nabla}_t(\lambda).$$

In section 3.5 we introduce the *global Hodge locus*

$$\mathrm{Hod}_d := \{t \in T : X_t \text{ has non-trivial Hodge cycles}\}.$$

Using the Hilbert scheme we introduce the *variational Hodge conjecture*. This problem was proposed by Grothendieck as a weaker version of Hodge conjecture. We will prove a stronger result, that we call *alternative Hodge conjecture*.

Conjecture (Alternative Hodge conjecture)**.** *Let T be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. For any $t \in Hod_d$ and $\lambda \in H_n(X_t, \mathbb{Z})_{alg}$, let λ be the induced flat* *section of $\underline{H^n_{dR}(X/T)}$ given by λ. Then there exist $P \in \mathbb{Q}[x]$ and a subvariety $\Sigma \subset \Sigma_{P,d}$ such that*

$$\text{Graph } \nabla_{V^{\frac{n}{2}}} = \varphi_P(\Sigma).$$

Where $\Sigma_{P,d} := \{(Z, X) \in Hilb_P \times T : Z \subset X\}$ is the relative Hilbert scheme, and $\varphi_P : \Sigma_{P,d} \to \underline{H^n_{dR}(X/T)}$ sends (Z, X_t) to $([Z]^{\mathrm{pd}}, t)$. In other words, for every algebraic cycle, its deformation as a Hodge cycle corresponds to an algebraic deformation of the cycle in a flat family.

Finally, in section 3.6 we show how periods of algebraic cycles (developed in Chapter 2) can be used to prove alternative Hodge conjecture (using computer assistance). Our results are summarized in the following.

Theorem. *For $d \geq 2 + \frac{4}{n}$, let $t \in Hod_d$ be the point corresponding to the Fermat variety and $\delta \in H_n(X_t, \mathbb{Z})_{alg}$ a complete intersection algebraic cycle $\delta = [Z]$, given by $Z = \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\}$, with*

$$x_0^d + \cdots + x_{n+1}^d = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1},$$

and deg $f_i = d_i$. Then, the alternative Hodge conjecture holds for δ

1. $d_1 = d_2 = \cdots = d_{\frac{n}{2}+1} = 1$.
2. $n = 2$, $4 \leq d \leq 15$, *or* $n = 4$, $3 \leq d \leq 6$, *or* $n = 6$, $3 \leq d \leq 4$.

Theorem ([MV17]). *Let T be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. Let $0 \in T$ be the point representing the Fermat variety X_0. If $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ are two linear subspaces inside X_0, such that $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$, then letting $\lambda := [\mathbb{P}^{\frac{n}{2}}] + [\check{\mathbb{P}}^{\frac{n}{2}}]$*

$$V^{\frac{n}{2}}_{\lambda} = V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} \cap V^{\frac{n}{2}}_{[\check{\mathbb{P}}^{\frac{n}{2}}]},$$

for all triples (n, d, m) in the following list:

$$(2, d, -1),\ 5 \leq d \leq 14,$$
$$(4, 4, -1), (4, 5, -1), (4, 6, -1), (4, 5, 0), (4, 6, 0),$$
$$(6, 3, -1), (6, 4, -1), (6, 4, 0),$$
$$(8, 3, -1), (8, 3, 0),$$
$$(10, 3, -1), (10, 3, 0), (10, 3, 1),$$

where $\mathbb{P}^{-1}$ means the empty set. In particular, alternative Hodge conjecture holds for λ in these cases.

3.1 De Rham cohomology sheaf associated to a family

In this section, we give an algebraic interpretation of the sheaf of sections of the de Rham cohomology bundle associated to a locally trivial fibration. This approach will be done using algebraic de Rham cohomology introduced in Chapter 1.

Let $X \xrightarrow{\pi} T$ be a family of smooth projective varieties. Assume T is smooth, connected and π is a proper submersion. If follows by Ehresmann's theorem (see [Dun18]) that π is a locally trivial fibration over T. Using the trivializations we can give a vector bundle structure to

$$\underline{H^k_{\mathrm{dR}}(X/T)} := \bigsqcup_{t\in T} H^k_{\mathrm{dR}}(X_t) \to T.$$

We denote by $\mathcal{H}^k$ the sheaf of holomorphic sections of $\underline{H^k_{\mathrm{dR}}(X/T)}$, and by H^k the sub-sheaf of sections s of $\underline{H^k_{\mathrm{dR}}(X/T)}$ such that

$$s(t) \in Im(H^k(X_t;\mathbb{Z}) \to H^k_{\mathrm{dR}}(X_t)), \forall t \in \text{Domain } s.$$

In other words, $H^k = R^k\pi_*\underline{\mathbb{Z}}$ is the higher direct image of $\underline{\mathbb{Z}}$ under π (for a reference on higher direct images of sheaves see [Har77] Chapter III, section 8), and $\mathcal{H}^k = H^k \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{T^{hol}}$. Using Ehresmann's theorem we can prove that H^k is a locally constant sheaf (or local system). In fact, given two local sections $\sigma, \tau \in H^k(U)$ such that $t \in U \subset T$, $\sigma(t) = \tau(t)$ and U is simply connected (so π is trivial over U). We claim $\sigma(t') = \tau(t')$ for all $t' \in U$. In fact, taking a path $\gamma : [0,1] \to U$ with $\gamma(0) = t$ and $\gamma(1) = t'$, and assuming $\pi : X_U \to U$ is trivial, i.e. it corresponds to $\mathrm{pr}_2 : X_U = X_t \times U \to U$. Then $\sigma^{\mathrm{pd}} \circ \gamma$ is a homotopy between $(\sigma(t))^{\mathrm{pd}}$ and $(\sigma(t'))^{\mathrm{pd}}$, while $\tau^{\mathrm{pd}} \circ \gamma$ is a homotopy between $(\tau(t))^{\mathrm{pd}} = (\sigma(t))^{\mathrm{pd}}$ and $(\tau(t'))^{\mathrm{pd}}$, so $(\sigma(t'))^{\mathrm{pd}} = (\tau(t'))^{\mathrm{pd}}$ in $H_{2n-k}(X_{t'};\mathbb{Z})$, then $\sigma(t') = \tau(t')$ in $H^k_{\mathrm{dR}}(X_{t'})$. In other words, $\sigma(t')$ is obtained from $\sigma(t)$ by *parallel transport*.

On the other hand, recall that in section 1.2, Definition 1.2.1, we defined the algebraic de Rham cohomology associated to $X \to T$ as

$$H^k_{\mathrm{dR}}(X/T) := \mathrm{H}^k(X, \Omega^\bullet_{X/T}).$$

This is an $\mathcal{O}_T(T)$-module. For every pair of affine open sets $U_1 \subset U_2$ of T, we have the $\mathcal{O}_T(U_i)$-module $H^k_{\mathrm{dR}}(X_{U_i}/U_i)$, where $X_{U_i} = \pi^{-1}(U_i)$ for $i = 1, 2$. The inclusion map $U_1 \hookrightarrow U_2$ induces a restriction map $H^k_{\mathrm{dR}}(X_{U_2}/U_2) \to H^k_{\mathrm{dR}}(X_{U_1}/U_1)$. Putting all these maps together we obtain a quasi-coherent $\mathcal{O}_T$-module that we denote

$$\mathcal{H}^k_{\mathrm{dR}}(X/T).$$

Noting that

$$\mathrm{H}^k(X, \Omega^\bullet_{X/T}) \otimes_{\mathcal{O}_{T,t}} k_{T,t} = \mathrm{H}^k(X_t, \Omega^\bullet_{X_t}),$$

we get the specialization of $\mathcal{H}^k_{\mathrm{dR}}(X/T)$ at every $t \in T$ is

$$\mathcal{H}^k_{\mathrm{dR}}(X/T)_t \otimes_{\mathcal{O}_{T,t}} k_{T,t} \simeq H^k_{\mathrm{dR}}(X_t/\mathbb{C}).$$

As a consequence, it follows by Ehresmann's theorem (and [Har77] exercise II.5.8) that $H^k_{\mathrm{dR}}(X/T)$ is a locally free sheaf of constant rank $r = b^k(X_t)$ for any $t \in T$. Furthermore, the analytification of $H^k_{\mathrm{dR}}(X/T)$ (in Serre's GAGA correspondence) is in fact $\mathcal{H}^k$ (for a proof see [KO68]).

Definition 3.1.1. For every affine open set $U \subset T$ we have

$$H^k_{\mathrm{dR}}(X/T)(U) = H^k_{\mathrm{dR}}(X_U/U).$$

Collating the submodules $F^i H^k_{\mathrm{dR}}(X_U/U) \subset H^k_{\mathrm{dR}}(X_U/U)$ (see Definition 1.2.2), we define a coherent (in fact locally free) subsheaf of $H^k_{\mathrm{dR}}(X/T)$ denoted by

$$F^i H^k_{\mathrm{dR}}(X/T).$$

These sheaves determine a decreasing *Hodge filtration for* $H^k_{\mathrm{dR}}(X/T)$.

Proposition 3.1.1. *Let $X \xrightarrow{\pi} T$ be a family of smooth projective varieties, such that T is smooth, connected and π is a proper submersion. Then*

$$F^i H^k_{dR}(X/T)/F^{i+1} H^k_{dR}(X/T) \simeq R^{k-i}\pi_* \Omega^i_{X/T}.$$

In other words, for $U \subset T$ affine open set we have

$$F^i H^k_{dR}(X_U/U)/F^{i+1} H^k_{dR}(X_U/U) \simeq H^{k-i}(X_U, \Omega^i_{X_U/U}).$$

Proof Recalling the proof of Proposition 1.2.1, the inclusion (1.4) holds for any morphism $X \to Y$. In particular, we have

$$\mathbb{H}^k(U, \pi_*\Omega^{\geq i}_{X_U/U})/\tilde{F}^{i+1}_U \hookrightarrow H^{k-i}(X_U, \Omega^i_{X_U/U}), \tag{3.1}$$

where $\tilde{F}^{i+1} := \mathrm{Im}(\mathbb{H}^k(U, \pi_*\Omega^{\geq i+1}_{X_U/U}) \to \mathbb{H}^k(U, \pi_*\Omega^{\geq i}_{X_U/U}))$. On the other hand, letting $R = \mathcal{O}_T(U)$, we know $H^{k-i}(X_U, \Omega^i_{X_U/U})$ is a finitely generated R-module (see [Har77] Theorem III.8.8). Then by Nakayama's lemma (see [Eis95] Corollary 4.8) it is enough to show this injection (3.1) specializes to an isomorphism on every maximal ideal of R, to conclude it is in fact an isomorphism. But this is exactly what is proved in Proposition 1.2.1. Considering now the natural projection

$$\mathbb{H}^k(U, \pi_*\Omega^{\geq i}_{X_U/U})/\tilde{F}^{i+1} \to F^i H^k_{\mathrm{dR}}(X_U/U)/F^{i+1} H^k_{\mathrm{dR}}(X_U/U), \tag{3.2}$$

again by Nakayama's lemma it is enough to show (3.2) specializes to an isomorphism on every maximal ideal of R, to conclude it is an isomorphism. And this also was showed in the proof of Proposition 1.2.1. ■

3.2 Gauss-Manin connection

In this section we introduce Gauss-Manin connection in the algebraic sheaves $H^k_{\mathrm{dR}}(X/T)$. We also prove Griffiths' transversality theorem in this context.

Let $\pi : X \to T$ be a proper submersion representing a family of smooth projective varieties over a connected smooth variety T. Recall the analytic sheaf $\mathcal{H}^k = H^k \otimes \mathcal{O}_{T^{hol}}$, where H^k is the local system given by $H^k := R^k \pi_* \underline{\mathbb{Z}}$. This is the sheaf of holomorphic sections of the vector bundle

$$\underline{H^k_{\mathrm{dR}}(X/T)} = \bigsqcup_{t \in T} H^k_{\mathrm{dR}}(X_t) \to T.$$

Where the bundle structure on $\underline{H^k_{\mathrm{dR}}(X/T)}$ is induced by the trivializations given by Ehresmann's theorem.

Using this identification, for every $U \subset T$ open, $H^k(U)$ corresponds to sections $s \in \underline{H^k_{\mathrm{dR}}(X/T)}(U)$ such that

$$s(t) \in Im(H^k(X_t; \mathbb{Z}) \to H^k_{\mathrm{dR}}(X_t)), \quad \forall t \in U.$$

The *(analytic) Gauss-Manin connection*

$$\nabla : \mathcal{H}^k \to \Omega^1_{T^{hol}} \otimes_{\mathcal{O}_{T^{hol}}} \mathcal{H}^k;$$

is the flat connection which makes every locally constant section of H^k a flat section. In other words, for every polydisc $\Delta \subset T$ centered at $t \in T$ take a basis $\gamma_1, \ldots, \gamma_s \in H^k(X_t; \mathbb{C})$, using the trivialization given by Ehresmann's theorem, extend it to $\gamma_1, \ldots, \gamma_s \in H^k_{\mathrm{dR}}(X_\Delta/\Delta)$. Then define

$$\nabla\Big(\sum_{i=1}^{s} f_i \gamma_i\Big) := \sum_{i=1}^{s} df_i \otimes \gamma_i;$$

for every $f_i \in \mathcal{O}_{T^{hol}}(\Delta)$ and $i = 1, \ldots, s$. Now we will describe the algebraic counterpart of this connection.

Definition 3.2.1. Let $\pi : X \to T$ be a proper submersion representing a family of smooth projective varieties over a connected smooth variety T. The *(algebraic) Gauss-Manin connection* is

$$\nabla : H^k_{\mathrm{dR}}(X/T) \to \Omega^1_T \otimes_{\mathcal{O}_T} H^k_{\mathrm{dR}}(X/T)$$

defined locally for every $t \in T$ in the following way: Since Ω^1_T is locally free, let $t_1, \ldots, t_r \in \mathfrak{m}_{T,t}$ be a coordinate system (i.e. $\mathfrak{m}_{T,t} = \langle t_1, \ldots, t_r \rangle$ and $\dim T = r$), then $(\Omega^1_T)_t = \bigoplus_{i=1}^r \mathcal{O}_{T,t} dt_i$. Let $\omega \in H^k_{\mathrm{dR}}(X/T)_t$. Consider U an affine neighbourhood of t such that $\Omega^1_T(U) = \bigoplus_{i=1}^r \mathcal{O}_T(U) dt_i$, $U = \mathrm{Spec}\ R$ and $\omega \in H^k_{\mathrm{dR}}(X_U/U)$. Take $\mathcal{U} = \{U_i\}_{i \in I}$ an affine covering of X_U, then

$$\omega = \sum_{j=0}^{k} \omega^j \in \bigoplus_{j=0}^{k} C^{k-j}(\mathcal{U}, \Omega^j_{X_U/R});$$

with $D\omega = 0$. Let $p : \Omega^j_{X_U} \to \Omega^j_{X_U/R}$ be the natural projection, then there exist

$$\tilde{\omega} = \sum_{j=0}^{k} \tilde{\omega}^j \in \bigoplus_{j=0}^{k} C^{k-j}(U; \Omega^j_{X_U});$$

such that $p(\tilde{\omega}) = \omega$. Since $p(D\tilde{\omega}) = D\omega = 0$ it follows

$$D\tilde{\omega} = \sum_{j=0}^{k+1} \left(\sum_{i=1}^{r} dt_i \wedge \eta_i^{j-1} \right) \in \bigoplus_{j=0}^{k} C^{k+1-j}(U; \pi^*\Omega^1_U \wedge \Omega^{j-1}_{X_U});$$

where $\eta_i^{-1} = 0$ for every $i = 1, \ldots, r$. Finally, letting

$$\eta_i := \sum_{j=0}^{k} \eta_i^j \in \bigoplus_{j=0}^{k} C^{k-j}(U; \Omega^j_{X_U});$$

we define

$$\nabla \omega := \sum_{i=1}^{r} dt_i \otimes p(\eta_i) \in (\Omega^1_T)_t \otimes_{O_{T,t}} H^k_{\mathrm{dR}}(X/T)_t.$$

To see this is well defined, we have to show first that each

$$p(\eta_i) \in H^k_{\mathrm{dR}}(X/T)_t;$$

in other words we have to show that $D\eta_i = 0$ in the sheaf of relative differential forms, after localizing at t. Since $DD\tilde{\omega} = 0$, we have

$$\sum_{i=1}^{r} dt_i \wedge D\eta_i = 0.$$

After localizing, since $dt_1, \ldots, dt_r$ are a base for $(\Omega^1_T)_t$ it follows that

$$D\eta_i \in \bigoplus_{j=0}^{k} C^{k-j}(U; \pi^*\Omega^1_U \wedge \Omega^{j-1}_{X_U})_t;$$

in other words $p(D\eta_i) = D(p(\eta_i)) = 0$, as desired. Now it is routine to check this definition does not depend on the choices made. It is also an exercise to check for every $r \in O_{T,t}$ that

$$\nabla(r \otimes \omega) = r\nabla\omega + dr \otimes \omega;$$

i.e. ∇ is a connection.

Theorem 3.2.1 (Griffiths' transversality [Gri68]). *Let $\pi : X \to T$ be a family of smooth projective varieties satisfying the hypothesis of Definition 3.2.1, then*

$$\nabla F^i H^k_{dR}(X/T) \subset \Omega^1_T \otimes_{O_T} F^{i-1} H^k_{dR}(X/T).$$

Proof We can assume $T = \operatorname{Spec} R$ is affine, and take U an affine covering of X. Let

$$\omega = \sum_{j=i}^{k} \omega^j \in F^i H^k_{\mathrm{dR}}(X/R),$$

where each $\omega^j \in C^{k-j}(U, \Omega^j_{X/R})$. Taking $\omega = p(\tilde{\omega})$, with

$$\tilde{\omega} = \sum_{j=i}^{k} \tilde{\omega}^j \in \bigoplus_{j=i}^{k} C^{k-j}(U, \Omega^j_X),$$

we see that $D(\tilde{\omega})^j = 0$ for $j < i$, i.e.

$$D\tilde{\omega} = \sum_{j=i}^{k+1} \left(\sum_{l=1}^{r} dt_l \wedge \eta_l^{j-1} \right) \in \bigoplus_{j=i}^{k+1} C^{k+1-j}(U, \pi^*\Omega^1_T \wedge \Omega^{j-1}_X).$$

As a consequence,

$$\nabla \omega = \sum_{l=1}^{r} dt_l \otimes \left(\sum_{j=i}^{k+1} p(\eta_l^{j-1}) \right) \in \Omega^1_R \otimes_R F^{i-1} H^k_{\mathrm{dR}}(X/R).$$

■

3.3 Infinitesimal variations of Hodge structures

In this section we use Gauss-Manin connection to introduce the infinitesimal variations of Hodge structures (IVHS). This will be our main tool to study the tangent space at each point of the Hodge locus (to be defined in the next section).

From Griffiths' transversality theorem, Gauss-Manin's connection induces

$$\nabla : F^iH^k_{\mathrm{dR}}(X/T)/F^{i+1}H^k_{\mathrm{dR}}(X/T) \to \Omega^1_T \otimes_{O_T} F^{i-1}H^k_{\mathrm{dR}}(X/T)/F^iH^k_{\mathrm{dR}}(X/T).$$

Using Proposition 3.1.1 we get

$$\nabla : R^{k-i}\pi_*\Omega^i_{X/T} \to \Omega^1_T \otimes_{O_T} R^{k-i+1}\pi_*\Omega^{i-1}_{X/T}.$$

Dualizing this morphism, we have

$$\overline{\nabla} : \Theta_T \to Hom_{O_T}(R^{k-i}\pi_*\Omega^i_{X/T}, R^{k-i+1}\pi_*\Omega^{i-1}_{X/T}).$$

Specializing at every $t \in T$ we obtain

$$\overline{\nabla}_t : T_tT \to Hom_{\mathbb{C}}(H^{i,k-i}(X_t), H^{i-1,k-i+1}(X_t)).$$

The following proposition tells us how is this map for $\pi : X \to T$ the family of all smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$.

Proposition 3.3.1. *Let $X_t = \{F = 0\}$ be any smooth degree d hypersurface of $\mathbb{P}^{n+1}$, and $t \in T \subset \mathbb{C}[x_0, \dots, x_{n+1}]_d$ the corresponding parameter. After identifying $T_tT \simeq \mathbb{C}[x_0, \dots, x_{n+1}]_d$, and using the identifications given by Carlson-Griffiths' Theorem 1.6.1, the Gauss-Manin connection (restricted to the primitive part of each piece of the Hodge structure)*

$$\overline{\nabla}_t : \mathbb{C}[x_0, \dots, x_{n+1}]_d \to Hom_{\mathbb{C}}(R^F_{d(n-i+1)-n-2}, R^F_{d(n-i+2)-n-2}),$$

is, up to a constant non-zero factor, the multiplication of polynomials.

Proof Let $X \to T$ be the universal family of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. Let $I = \{(\alpha_0, \dots, \alpha_{n+1}) \in \mathbb{Z}^{n+1}_{\geq 0} : \sum_{i=0}^{n+1} \alpha_i = d\}$ and $s \in T$ be the point corresponding to $F = \sum_{\alpha \in I} s_\alpha x^\alpha$, i.e. $X_s = \{F = 0\}$ (we are changing t by s, to use t as a variable without making confusing notation). Given $v = \sum_{\alpha \in I} c_\alpha x^\alpha \in T_sT = \mathbb{C}[x_0, \dots, x_{n+1}]_d$ and $P \in R^F_{d(n-i+1)-n-2}$ we want to determine $(\overline{\nabla}_s(v))(P) \in R^F_{d(n-i+2)-n-2}$. Writing (using Carlson-Griffiths Theorem 1.6.1)

$$\omega_P = \frac{(-1)^{n(n-i+1)}}{(n-i)!}\left\{\frac{P\Omega_J}{F_J}\right\}_J \in H^{n-i}(U_{X_s}, \Omega^i_{X_s});$$

we have to lift it to a neighbourhood of X_s inside X. It is easy to see that

$$X = \{(x, t) \in \mathbb{P}^{n+1} \times T : f(x, t) := \sum_{\alpha \in I} t_\alpha x^\alpha = 0\}.$$

Let us take the covering $\mathcal{U} = \{U_i\}_{i=0}^{n+1}$ of X given by $U_i := \{(x,t) \in X : \frac{\partial f}{\partial x_i}(x,t) \neq 0\}$. This is a covering since X_t is smooth for every $t \in T$. Let $\beta \in I$ such that $s_\beta \neq 0$. Consider

$$\omega := \frac{(-1)^{n(n-i+1)}}{(n-i)!}\left\{\frac{t^{|J|}}{s^{|J|}}\frac{P\Omega_J}{f_J}\right\}_J \in C^{n-i}(\mathcal{U}, \Omega^i_X).$$

It is clear that the specialization of ω at $s \in T$ is ω_P. Furthermore, using the identity (1.19) we get

$$\omega = \frac{(-1)^{n(n-i+1)}}{(n-i)!}\left\{\frac{t^{|J|}}{s^{|J|}}\frac{P((-1)^n\Omega_J \wedge (\sum_{i=0}^{n+1} f_i dx_i) + (\sum_{i=0}^{n+1} f_i x_i)V_J)}{f_J}\right\}_J \in H^{n-i+1}(\mathcal{U}, \Omega^i_X).$$

Since $X = \{f = 0\}$ and $df = \sum_{i=0}^{n+1} f_i dx_i + \sum_{\beta \in I} x^\beta dt_\beta$, we obtain

$$\omega = \frac{(-1)^{n(n-i+1)+n+1}}{(n-i)!}\left\{\frac{t^{|J|}}{s^{|J|}}\frac{P\Omega_J \wedge (\sum_{\beta\in I} x^\beta dt_\beta)}{f_J}\right\}_J \in H^{n-i+1}(\mathcal{U}, \Omega^i_X).$$

Finally

$$\left(\overline{\nabla}_s\left(\sum_{\beta\in I} c_\beta \frac{\partial}{\partial t_\beta}\right)\right)(\omega_P) = \frac{(-1)^{n(n-i+1)}}{(n-i)!}\left\{\frac{vP\Omega_J}{F_J}\right\}_J \in H^{n-i+1}(\mathcal{U}_Y, \Omega^{i-1}_Y).$$

In conclusion

$$(\overline{\nabla}_s(v))(P) = (-1)^n(n-i+1)v \cdot P \in R^F_{d(n-i+2)-n-2}.$$

■

Remark 3.3.1. It is clear from Proposition 3.3.1 that $\overline{\nabla}_t$ factors trough R^F_d, i.e. there exist two maps

$$\rho_t : T_tT \to R^F_d,$$

and

$$r : R^F_d \to Hom_{\mathbb{C}}(R^F_{d(n-i+1)-n-2}, R^F_{d(n-i+2)-n-2}),$$

such that

$$\overline{\nabla}_t = r \circ \rho_t.$$

Here it is hidden the *Kodaira-Spencer map*

$$\kappa_t : T_tT \to H^1(X_t, \Theta_{X_t}),$$

which is induced as the specialization of the coboundary map in the following short exact sequence

$$0 \to \Theta_{X/T} \to \Theta_X \to \pi^*\Theta_T \to 0.$$

Where Θ_X and Θ_T are the *sheaf of vector fields* on X and T respectively, while $\Theta_{X/T}$ is *the sheaf of vector fields tangent to* .

Whenever $n \geq 2$ and $(n, d) \neq (2, 4)$, Kodaira-Spencer map is surjective and Ker $\rho_t = J_d^F$ (see [Voi03], Lemma 6.15). Therefore, we can identify

$$H^1(X_t, \Theta_{X_t}) \simeq R_d^F,$$

and we obtain the map

$$\nabla : H^1(X_t, \Theta_{X_t}) \to Hom_{\mathbb{C}}(H^{i,n-i}(X_t), H^{i-1,n-i+1}(X_t)),$$

which is the so called *infinitesimal variations of Hodge structures (IVHS)* introduced by Carlson, Green, Griffiths and Harris in [CGGH83].

3.4 Hodge locus

In this section we will introduce Hodge locus as an analytic scheme. We will show how to determine its Zariski tangent space using IVHS.

Definition 3.4.1. Let $: X \to T$ be a family of smooth projective varieties as in section 3.1. Recall the local system $H^k := R^k \ \underline{Z}$ over T. For every $\in \Gamma(T, H^k)$ and $p \in \{0, \ldots, k\}$ we define its associated *Hodge locus*

$$V^p := \{t \in T : (t) \in F^pH^k_{\mathrm{dR}}(X_t)\}.$$

Remark 3.4.1. Recalling $H^k := H^K \ O_{T^{hol}}$, we can define F^pH^k as the subsheaf given by sections s with $s(t) \in F^pH^k_{\mathrm{dR}}(X_t)$ for every $t \in$ Domain s. Looking at as a global section of the analytic sheaf H^k/F^pH^k, we see V^p is the zero set of , thus it has a natural structure of analytic sub-scheme of T. It is a deep result due to Cattani, Deligne and Kaplan that V^p is in fact an algebraic subset of T (see [CDK95]).

Remark 3.4.2. For every $t \in V^p$ the germ of analytic scheme (V^p, t) is determined just by $(t) \in H^k(X_t, Z) \cap F^pH^k_{\mathrm{dR}}(X_t)$. In fact, since H^k is locally constant, for every pair $, {}' \in \Gamma(T, H^k)$ such that $(t) = {}'(t)$ we have $|_U = {}'|_U$ for some open neighbourhood U of t (see section 3.1). Conversely, given any $\in H^k(X_t, Z) \cap F^pH^k_{\mathrm{dR}}(X_t)$ there exist $\in \Gamma(U, H^k)$ for some neighbourhood U of t, such that $(t) = $. For this reason we will use the notation

$$V^p := V^p := (V^p, t),$$

where $\in H_{2n\ k}(X_t, Z)$ is the Poincaré dual of in the sense that

$$\int ! = \int_{X_t} ! \wedge \ , \qquad \forall ! \in H^{2n\ k}_{\mathrm{dR}}(X_t).$$

Note that if $_u \in H_{2n\ k}(X_u, Z)$ is the Poincaré dual of (u) for $u \in U$, then $_u$ is the cycle obtained by monodromy from $_t = $ using the trivialization given by Ehresmann's theorem.

Proposition 3.4.1. *For every* $\in H_{2n\ k}(X_t, Z)$ *such that its Poincaré dual is in* $F^pH^k_{dR}(X_t)$ *we define the map*

$$^0\overline{\nabla}_t(\ ) : T_tT \to H^{n\ p+1, n\ k+p\ 1}(X_t) \ ,$$

given by

$$(^0\overline{\nabla}_t(\ )(v))(\ ) := \int (\overline{\nabla}_t(v))(\ ).$$

The Zariski tangent space of the Hodge locus corresponding to is

$$T_tV^p = Ker\ ^0\overline{\nabla}_t(\ ).$$

Proof Let U be a polydisc around t. Let $\omega_1, \ldots, \omega_r \in \Gamma(U; F^{p-1}\mathcal{H}^k / F^p\mathcal{H}^k)$ and $\omega_{r+1}, \ldots, \omega_s \in \Gamma(U; \mathcal{H}^k / F^{p-1}\mathcal{H}^k)$ such that $\omega_1(u), \ldots, \omega_r(u)$ form a basis of $H^{p-1,k-p+1}_{\mathrm{dR}}(X_u)$ and $\omega_{r+1}(u), \ldots, \omega_s(u)$ form a basis of $H^k_{\mathrm{dR}}(X_u)/F^{p-1}H^k_{\mathrm{dR}}(X_u)$, for every $u \in U$. Let $\alpha \in \Gamma(U; \mathcal{H}^k)$ such that $\alpha(t)$ is Poincaré dual to γ. Then

$$\alpha(t) = \sum_{i=1}^{s} f_i(t)\,\omega_i(t),$$

for some $f_i \in \Gamma(U; \mathcal{O}_{T^{hol}})$. Letting $\gamma_u \in H_{2n-k}(X_u; \mathbb{Z})$ be the Poincaré dual to $\alpha(u) \in H^k(X_u; \mathbb{Z})$, and considering $\delta_i \in \Gamma(U; \mathcal{H}^{2n-k})$ such that $\delta_i(u)$ is dual (respect to the wedge product) to $\omega_i(u)$ for every $u \in U$. We get

$$f_i(u) = \sum_{j=1}^{s} f_j \int_{X_u} \delta_i(u) \wedge \omega_j(u) = \int_{X_u} \delta_i(u) \wedge \alpha(u) = \int_{\gamma_u} \delta_i(u),$$

then

$$V^p = \left(\left\{u \in U : \int_{\gamma_u} \delta_1(u) = \cdots = \int_{\gamma_u} \delta_s(u) = 0\right\}, t\right).$$

As a consequence

$$T_t V^p = \{v \in T_t T : (df_1)_t(v) = \cdots = (df_s)_t(v) = 0\}.$$

Note that

$$\begin{aligned}
(df_i)_t(v) &= \int_{X_t} \delta_i(t) \wedge \left(\sum_{j=1}^{s} (df_j)_t(v)\,\omega_j(t)\right) \\
&= -\int_{X_t} \sum_{j=1}^{s} f_j(t)\,\delta_i(t) \wedge (\overline{\nabla}_t(v))(\omega_j(t)) \\
&= \int_{X_t} \sum_{j=1}^{s} f_j(t)(\overline{\nabla}_t(v))(\delta_i(t)) \wedge \omega_j(t) \\
&= \int_{X_t} (\overline{\nabla}_t(v))(\delta_i(t)) \wedge \alpha(t) \\
&= \int_{\gamma} (\overline{\nabla}_t(v))(\delta_i(t)).
\end{aligned}$$

Where in the second equality we used that $(\overline{\nabla}_t(v))\alpha(t) = 0$, and in the third one we used that $(\overline{\nabla}_t(v))(\alpha \wedge \beta) = (\overline{\nabla}_t(v))(\alpha) \wedge \beta + \alpha \wedge (\overline{\nabla}_t(v))(\beta)$. Then $(df_i)_t = 0$, for all $i = r+1, \ldots, s$, since $\gamma^{\mathrm{pd}} \in F^p H^k_{\mathrm{dR}}(X_t)$ and $(\overline{\nabla}_t(v))(\delta_i(t)) \in F^{n-p+1} H^{2n-k}_{\mathrm{dR}}(X_t)$. In consequence

$$T_t V^p = \left\{v \in T_t T : \int_{\gamma} (\overline{\nabla}_t(v))(\delta_1(t)) = \cdots = \int_{\gamma} (\overline{\nabla}_t(v))(\delta_r(t)) = 0\right\}.$$

The result follows from the fact $\delta_1(t), \ldots, \delta_r(t)$ form a basis of $H^{n-p+1, n-k+p-1}(X_t)$. ■

Remark 3.4.3. Note that in the previous proof we showed that the structure of analytic scheme of V^p is induced by the holomorphic functions $f_i \in \Gamma(U, O_{T^{hol}})$ given by

$$f_i(u) = \int_{\delta_u} {}_i(u),$$

where ${}_1, \ldots, {}_s \in F^{n-p+2}H^{2n-k}(U)$ form a basis at each fiber.

Definition 3.4.2. Let X be any smooth projective variety of dimension n. A cycle $\in H_{2n-2k}(X, \mathbb{Z})$ is called an *integral Hodge cycle* if

$$^{\mathsf{pd}} \in H^{2k}(X, \mathbb{Z}) \cap F^k H^{2k}_{\mathrm{dR}}(X).$$

Since $^{\mathsf{pd}}$ is a real class (i.e. it is invariant under complex conjugation), this is equivalent to

$$^{\mathsf{pd}} \in H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X).$$

A *Hodge cycle* is the analogous definition with $\mathbb{Q}$ instead of $\mathbb{Z}$. We will usually work with integral Hodge cycles, but we will reefer to both (integral and rational) as Hodge cycles, leaving the prefix understood by the context. We denote the subgroup of Hodge cycles as $\mathrm{Hodge}_{2n-2k}(X, \mathbb{Z})$. We say $\in \mathrm{Hodge}_{2n-2k}(X, \mathbb{Z})$ is a *trivial Hodge cycle* if $^{\mathsf{pd}} = 0 \in H^{2k}_{\mathrm{dR}}(X)_{prim}$.

Definition 3.4.3. Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n. Considering X as a fiber of the universal family of smooth degree d hypersurfaces, we define for every Hodge cycle $\in \mathrm{Hodge}_n(X, \mathbb{Z})$ its associated *Hodge locus* to be $V^{\frac{n}{2}}$. This $V^{\frac{n}{2}}$ corresponds to the locus of hypersurfaces obtained by deformation of X where the corresponding deformation of (obtained by monodromy or parallel transport) is still a Hodge cycle.

Corollary 3.4.1. *Let $X \to T$ be the universal family of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. Suppose n is even and $d \geq 2 + \frac{4}{n}$. For every $t \in T$ and every non-trivial $\in \mathrm{Hodge}_n(X_t, \mathbb{Z})$, the corresponding Hodge locus $V^{\frac{n}{2}}$ is properly contained in (T, t). In fact, the Zariski tangent space $T_t V^{\frac{n}{2}}$ is properly contained in $T_t T$.*

Proof Suppose $T_t V^{\frac{n}{2}} = T_t T$. Then by Proposition 3.4.1 we have

$$\int (\overline{\nabla}_t(v))(\ ) = 0, \qquad \forall \in H^{\frac{n}{2}+1, \frac{n}{2}-1}(X_t).$$

By Proposition 3.3.1 we can identify the map

$$\overline{\nabla}_t : \mathbb{C}[x_0, \ldots, x_{n+1}]_d \otimes R^t_{d\frac{n}{2}-n-2} \to R^t_{d(\frac{n}{2}+1)-n-2}$$

with polynomial multiplication. Since $d\frac{n}{2} - n - 2 \geq 0$, the $\mathbb{C}$-vector space generated by the image of $\overline{\nabla}_t$ is all $R^t_{d(\frac{n}{2}+1)-n-2}$, as a consequence

$$\int \omega = 0, \qquad \forall \omega \in H^{\frac{n}{2}, \frac{n}{2}}(X_t)_{prim},$$

i.e. $^{\mathrm{pd}} = 0 \in H^n_{\mathrm{dR}}(X_t)_{prim}$, contradicting the choice of . ■

Remark 3.4.4. It follows from the previous corollary that the Hodge locus $V^{\frac{n}{2}}$ depends only on the primitive part of

$$^{\mathrm{pd}} \in H^n(X_t; \mathbb{Z}) \cap H^{\frac{n}{2},\frac{n}{2}}(X_t)_{prim}.$$

In fact if $= [X_t \cap \mathbb{P}^{\frac{n}{2}+1}] \in H_n(X_t; \mathbb{Z})$, then

$$V^{\frac{n}{2}}_{+c} \cap V^{\frac{n}{2}} = V^{\frac{n}{2}} \cap V^{\frac{n}{2}} = V^{\frac{n}{2}}$$

for all $c \in \mathbb{C}$ and $\in \mathrm{Hodge}_n(X_t; \mathbb{Z})$. In consequence

$$V^{\frac{n}{2}} = V^{\frac{n}{2}}_{+c} \quad \forall c \in \mathbb{C}.$$

3.5 Variational Hodge conjecture

In this section we will recall some facts about the Hilbert scheme (for a reference see [Ser07]). Using the Hilbert scheme we introduce variational Hodge conjecture (VHC) and a stronger conjecture, which we call alternative Hodge conjecture (AHC).

Definition 3.5.1. Let $N > 0$ be a fixed natural number and $P \in \mathbb{Q}[t]$ the Hilbert polynomial of a subscheme of $\mathbb{P}^N$. The *Hilbert functor* is

$$Hilb_P : \mathbf{Sch}/\mathbb{C} \to \mathbf{Sets},$$

given by

$$Hilb_P(S) := \{\varphi : X \to S : \varphi \text{ is projective, flat and } \forall s \in S, X_s \text{ has Hilbert polynomial } P\}.$$

And to every morphism $T \to S$ and $\varphi : X \to S \in Hilb_P(S)$ associates the pull-back $\varphi' : X \times_S T \to T \in Hilb_P(T)$.

Theorem 3.5.1 (Grothendieck [Gro61])**.** *The Hilbert functor $Hilb_P$ is representable by a projective $\mathbb{C}$-scheme, called the Hilbert scheme and denoted Hilb_P.*

Example 3.5.1. When we consider P the Hilbert polynomial of a degree d hypersurface of $\mathbb{P}^{n+1}$, $\mathrm{Hilb}_P = \mathbb{P}^N$ for $N = \binom{n+1+d}{d} - 1$. In other words, it is the parameter space of degree d hypersurfaces of $\mathbb{P}^{n+1}$.

At follows we introduce a subvariety of Hilbert scheme we are interested in.

Definition 3.5.2. Let

$$\Sigma_{P,d} := \{(Z, X) \in Hilb_P \times T : Z \subset X\},$$

be the *relative Hilbert scheme of subvarieties with Hilbert polynomial P inside smooth degree d hypersurfaces of* $\mathbb{P}^{n+1}$. Consider a multi-degree $\underline{d} = (d_1, \dots, d_{\frac{n}{2}+1})$ and a polynomial P corresponding to the Hilbert polynomial of a complete intersection of type $(d_1, \dots, d_{\frac{n}{2}+1})$ inside $\mathbb{P}^{n+1}$. We define

$$\Sigma_{\underline{d}} := \overline{\Sigma^0} \subset \Sigma_{P,d},$$

where Σ^0 consists of pairs (Z, X), such that $X = \{F = 0\}$, $Z = \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\}$ and there exist $g_i \in \mathbb{C}[x_0, \dots, x_{n+1}]$ for $i = 1, \dots, \frac{n}{2} + 1$, such that

$$F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}.$$

Proposition 3.5.1. *In the context of the previous definition, let $t \in T$ be the point corresponding to X. Identifying $T_t T \simeq \mathbb{C}[x_0, \dots, x_{n+1}]_d$ we have*

$$\langle f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1} \rangle_d \subset T_t pr_2(\Sigma_{\underline{d}}).$$

Proof Let $S := \mathbb{C}[x_0, \dots, x_{n+1}]$. Consider the map

$$\Phi : S_{d_1} \oplus \cdots \oplus S_{d_{\frac{n}{2}+1}} \oplus S_{d-d_1} \oplus \cdots \oplus S_{d-d_{\frac{n}{2}+1}} \to \mathrm{pr}_2(\Sigma_{\underline{d}}),$$

taking $(r_1, \dots, r_{\frac{n}{2}+1}, s_1, \dots, s_{\frac{n}{2}+1})$ to $\{\sum_{i=1}^{\frac{n}{2}+1} r_i s_i = 0\} \subset \mathbb{P}^{n+1}$. It is clear that for $p = (f_1, \dots, f_{\frac{n}{2}+1}, g_1, \dots, g_{\frac{n}{2}+1})$

$$\Phi'(p)(v_1, \dots, v_{\frac{n}{2}+1}, w_1, \dots, w_{\frac{n}{2}+1}) = f_1 v_1 + \cdots + f_{\frac{n}{2}+1} v_{\frac{n}{2}+1} + g_1 w_1 + \cdots + g_{\frac{n}{2}+1} w_{\frac{n}{2}+1},$$

where $v_i \in S_{d_i}$ and $w_i \in S_{d-d_i}$ for $i = 1, \dots, \frac{n}{2}+1$. ■

Now we introduce variational Hodge conjecture using the Hilbert scheme. This conjecture was proposed by Grothendieck as a weaker version of Hodge conjecture. Let us recall first Hodge conjecture (and its integral version).

Conjecture 3.5.1 (Integral Hodge conjecture)**.** *Every Hodge cycle* $\gamma \in$ *Hodge*$_{2k}(X, \mathbb{Z})$ *(recall Definition 3.4.2) is an algebraic cycle, i.e. there exist subvarieties* $Z_i \subset X$ *of dimension* k *and integers* $n_i \in \mathbb{Z}$ *for* $i = 1, \dots, k$ *such that*

$$\gamma = \sum_{i=1}^{k} n_i [Z_i].$$

Denoting $H_{2k}(X, \mathbb{Z})_{alg}$ *the group of algebraic cycles we can resume IHC by stating*

$$\mathit{Hodge}_{2k}(X, \mathbb{Z}) = H_{2k}(X, \mathbb{Z})_{alg}, \forall k = 0, \dots, n.$$

Remark 3.5.1. Integral Hodge conjecture was originally asked by Hodge in [Hod41]. This conjecture is known to be false. The first counterexamples were provided by Atiyah and Hirzebruch in [AH62]. In that work, they suggest to modify the conjecture, stating the so called Hodge conjecture.

Conjecture 3.5.2 (Hodge conjecture)**.** *For* X *a smooth projective variety of dimension* n*,*

$$\mathit{Rank}\ \mathit{Hodge}_{2k}(X, \mathbb{Z}) = \mathit{Rank}\ H_{2k}(X, \mathbb{Z})_{alg}, \forall k = 0, \dots, n.$$

In other words,

$$\mathit{Hodge}_{2k}(X, \mathbb{Q}) = H_{2k}(X, \mathbb{Q})_{alg}, \forall k = 0, \dots, n.$$

Where we define rational Hodge and algebraic cycles in the same way we did for integral cycles but now with rational coefficients. Another way to state Hodge conjecture is saying for every Hodge cycle $\gamma \in$ *Hodge*$_{2k}(X, \mathbb{Z})$ *there exist a* $n \in \mathbb{Z}_{>0}$ *such that* $n\gamma$ *is an algebraic cycle.*

Remark 3.5.2. It is clear that IHC implies HC. When X is a hypersurface, we know by Lefschetz hyperplane section theorem, Poincaré duality and Picard-Lefschetz theory (see [Mov17a] Chapters 5 and 6) that

$$H^m(X;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}^k & \text{if } m = n; \\ \mathbb{Z} & \text{if } 0 \leq m \leq 2n \text{ is even and } m \neq n; \\ 0 & \text{otherwise.} \end{cases}$$

This implies that for n odd, Hodge conjecture is true, while for n even to verify Hodge conjecture reduces to check it for the middle cohomology group. On the other hand, even when Hodge conjecture holds, IHC is non-trivial, in fact Kollár (in [Kol92]) has shown IHC fails for very general degree 48 hypersurfaces of $\mathbb{P}^4$ (see [Tot13]).

Definition 3.5.3. Let T be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$ for n even. The *(global) Hodge locus of degree d hypersurfaces of $\mathbb{P}^{n+1}$* is

$$\mathrm{Hod}_d := \{t \in T : X_t \text{ has non-trivial Hodge cycles}\}.$$

Recall that $\delta \in \mathrm{Hodge}_{2n-2k}(X_t,\mathbb{Z})$ is *trivial* if $\delta^{\mathrm{pd}} = 0 \in H^{2k}_{\mathrm{dR}}(X_t)_{prim}$. In particular, every non-trivial Hodge cycle of X_t belongs to $\mathrm{Hodge}_n(X_t,\mathbb{Z})$.

Remark 3.5.3. Recall that in section 3.4, we associated a (local) Hodge locus $V^{\frac{n}{2}}_\delta$ to every non-trivial Hodge cycle $\delta \in \mathrm{Hodge}_n(X_t,\mathbb{Z})_{prim}$ at $t \in T$. We can see the global Hodge locus Hod_d as a union of these local analytic spaces. In consequence, we have an induced analytic structure on Hod_d. Furthermore, it follows from Corollary 3.4.1 that (Hod_d,t) is a countable union of properly contained analytic subvarieties of (T,t). In particular, a generic smooth degree d hypersurface of $\mathbb{P}^{n+1}$ of even dimension n, only has trivial Hodge cycles (then it satisfies Hodge conjecture). If we assume Hodge conjecture, it can be proved that Hod_d is in fact an *algebraic subvariety* of T. A deep theorem due to Cattani, Deligne and Kaplan proves that Hod_d is an algebraic subvariety of T without assuming Hodge conjecture. This is one of the strongest evidences supporting this conjecture.

Remark 3.5.4. For $n = 2$ integral Hodge conjecture holds (by Lefschetz (1,1) theorem, see [GH94]), and we have the equality between the *Noether-Lefschetz locus*

$$\mathrm{NL}_d := \{S \subset \mathbb{P}^3 : S \text{ is a smooth degree } d \text{ surface with Picard number bigger than } 1\},$$

and the Hodge locus. Thus, the local Hodge locus V^1_δ describe the local (analytic) branches of Noether-Lefschetz locus and can be used to study it (this approach has been exploited by Green [Gre89], Ciliberto-Harris-Miranda [CHM88], Voisin [Voi89], [Voi90], among others).

Remark 3.5.5. Recalling from section 3.1, H^n is the sheaf of holomorphic sections of the vector bundle

$$\underline{H^n_{\mathrm{dR}}(X/T)} = \bigsqcup_{i \in T} H^n_{\mathrm{dR}}(X_t) \to T.$$

Recall from section 3.5, for every $P \in \mathbb{Q}[x]$ the relative Hilbert scheme is

$$\Sigma_{P,d} := \{(Z, X) \in \mathrm{Hilb}_P \quad T : Z \quad X\},$$

and we have a natural map

$$\varphi_P : \Sigma_{P,d} \to \underline{H^n_{\mathrm{dR}}(X/T)},$$

sending each (Z, X_t) to $([Z]^{\mathrm{pd}}, t)$. To know whether this map is surjective or not is an interesting problem in the spirit of Hodge conjecture. This is the essence of variational Hodge conjecture, which we state at follows.

Conjecture 3.5.3 (Variational Hodge conjecture)**.** *For every polynomial $P \in \mathbb{Q}[x]$ consider the natural map*

$$\varphi_P : \Sigma_{P,d} \to \underline{H^n_{dR}(X/T)}.$$

For any $t \in Hod_d$ and $\in H_n(X_t, \mathbb{Z})_{alg}$, let be the induced section of $\underline{H^n_{dR}(X/T)}$ given by (here we mean that (t) is Poincaré dual to , and furthermore it extends to a neighbourhood of $t \in T$ by monodromy, see Remark 3.4.2). Then

$$\mathit{Graph} \quad j_{V_{\frac{n}{2}}} \quad \langle \{\varphi_P(\Sigma_{P,d})\}_{P \in \mathbb{Q}[x]} \rangle. \tag{3.3}$$

Where the right hand side expression corresponds to the $\mathbb{C}$-vector space generated by the germs of sections of $\underline{H^n_{dR}(X/T)}$ coming from $\Sigma_{P,d}$ at $t \in T$. In other words, (3.3) *means that the deformation of every algebraic cycle as a Hodge cycle is still an algebraic cycle.*

We will prove the following conjecture, that is stronger than variational Hodge conjecture.

Conjecture 3.5.4 (Alternative Hodge conjecture)**.** *For any $t \in Hod_d$ and $\in H_n(X_t, \mathbb{Z})_{alg}$, let be the induced section of $\underline{H^n_{dR}(X/T)}$ given by . Then there exist $P \in \mathbb{Q}[x]$ and a subvariety $\Sigma \quad \Sigma_{P;d}$ such that*

$$\mathit{Graph} \quad j_{V_{\frac{n}{2}}} = \varphi_P(\Sigma).$$

In other words, for every algebraic cycle, its deformation as a Hodge cycle corresponds to an algebraic deformation of the cycle in a flat family.

Remark 3.5.6. Each one AHC or HC implies VHC. Furthermore, AHC is equivalent to determine the local branches of the global Hodge locus. In particular, for $n = 2$ the global Hodge locus corresponds to the Noether-Lefschetz locus, where VHC is known to hold (by Lefschetz $(1,1)$ theorem), but AHC is a highly non-trivial conjecture.

3.6 Using periods to prove variational Hodge conjecture

In this final section we show how to prove alternative Hodge conjecture (AHC) by computer assistance. The main ingredients are the periods of algebraic cycles computed in Chapter 2.

Definition 3.6.1. Let $X=\{F=0\}\subset\mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n. For every Hodge cycle $\delta\in \mathrm{Hodge}_n(X,\mathbb{Z})$, we define its *period matrix*

$$P(\delta):=\left[\int_\delta res\left(\frac{P_iQ_j\Omega}{F^{\frac{n}{2}+1}}\right)\right]_{i\in I,j\in J}.$$

Where $\{P_i\}_{i\in I}$ form a basis of R^F_d and $\{Q_j\}_{j\in J}$ form a basis of $R^F_{d\frac{n}{2}-n-2}$. Recall that $R^F:=\mathbb{C}[x_0,\dots,x_{n+1}]/J^F$ is the Jacobian ring, and $J^F:=\langle F_0,\dots,F_{n+1}\rangle$ is the Jacobian ideal of F.

Proposition 3.6.1. *Let $X\to T$ be the family of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$, n an even number and $(n,d)\neq(2,4)$. For every $t\in T$ and every Hodge cycle $\delta\in H_n(X_t,\mathbb{Z})$*

$$\mathit{Codim}_{\,T_tT}\,T_tV_\delta^{\frac{n}{2}}=\mathit{Rank}\,P(\delta).$$

Proof By Proposition 3.4.1

$$\mathrm{Codim}_{\,T_tT}\,T_tV_\delta^{\frac{n}{2}}=\mathrm{Rank}\ {}^0\overline{\nabla}_t(\delta).$$

By Remark 3.3.1 we can factor this map by Kodaira-Spencer's map

$${}^0\overline{\nabla}_t(\delta)={}^0\nabla_t(\delta)\circ\kappa_t.$$

Since Kodaira-Spencer's map is surjective for $(n,d)\neq(2,4)$, then

$$\mathrm{Codim}_{\,T_tT}\,T_tV_\delta^{\frac{n}{2}}=\mathrm{Rank}\ {}^0\nabla_t(\delta).$$

Finally, we use Proposition 3.3.1 to represent ${}^0\nabla_t(\delta)$ by a matrix, and we notice this matrix is the period matrix $P(\delta)$ up to a constant non-zero factor. ■

Definition 3.6.2. Let $\underline{a}=(a_1,\dots,a_{2s})\in\mathbb{N}^{2s}$, we define the number

$$C_{\underline{a}}:=\binom{n+1+d}{n+1}-\sum_{k=1}^{2s}(-1)^{k-1}\sum_{a_{i_1}+\cdots+a_{i_k}\le d}\binom{n+1+d-a_{i_1}-\cdots-a_{i_k}}{n+1}.$$

Proposition 3.6.2. *Let T be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. Consider $t\in T$ a point corresponding to a hypersurface $X=\{F=0\}$, such that*

$$F=f_1g_1+\cdots+f_{\frac{n}{2}+1}g_{\frac{n}{2}+1},$$

for $f_i \in \mathbb{C}[x_0,\ldots,x_{n+1}]_{d_i}$ *and* $g_i \in \mathbb{C}[x_0,\ldots,x_{n+1}]_{d-d_i}$. *If we let* $Z := \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\}$, $\underline{d} = (d_1,\ldots,d_{\frac{n}{2}+1})$ *and* $\delta := [Z] \in H_n(X,\mathbb{Z})$, *then*

$$V_\delta^{\frac{n}{2}} = pr_2(\Sigma_{\underline{d}}),$$

if

$$\text{Rank } P(\delta) = C_{\underline{d}}.$$

In that case, AHC holds for δ, *and furthermore* $V_\delta^{\frac{n}{2}}$ *is smooth and reduced.*

Proof We know by Proposition 3.5.1 that

$$\langle f_1,g_1,\ldots,f_{\frac{n}{2}+1},g_{\frac{n}{2}+1}\rangle_d \subset T_t\mathrm{pr}_2(\Sigma_{\underline{d}}) \subset T_tV_\delta^{\frac{n}{2}}. \tag{3.4}$$

Suppose first that $t \in \mathrm{pr}_2(\Sigma_{\underline{d}})$ is general, so it is smooth and the ideal $\langle f_1,g_1,\ldots,f_{\frac{n}{2}+1},g_{\frac{n}{2}+1}\rangle$ is generated by a regular sequence. Then, we can use its Koszul complex to determine a free resolution of it (see [Eis95] Chapter 17). Using this resolution we conclude that

$$\mathrm{Codim}_{\mathbb{C}[x_0,\ldots,x_{n+1}]_d}\langle f_1,g_1,\ldots,f_{\frac{n}{2}+1},g_{\frac{n}{2}+1}\rangle_d = C_{\underline{d}} \geq \mathrm{Codim}_T\ \mathrm{pr}_2(\Sigma_{\underline{d}}).$$

Now, for any $t \in \mathrm{pr}_2(\Sigma_{\underline{d}})$ (not necessarily smooth), if

$$\text{Rank } P(\delta) = C_{\underline{d}},$$

the equality in (3.4) follows by Proposition 3.6.1, and furthermore

$$\dim \mathrm{pr}_2(\Sigma_{\underline{d}}) = \dim T_t\mathrm{pr}_2(\Sigma_{\underline{d}}) = \dim V_\delta^{\frac{n}{2}} = \dim T_tV_\delta^{\frac{n}{2}}.$$

■

Theorem 3.6.1. *For* $d \geq 2+\frac{4}{n}$, $\delta \in H_n(X,\mathbb{Z})$ *as in Proposition 3.6.2 and* $t = 0 \in T$ *the Fermat variety, AHC holds for* δ

1. $d_1 = d_2 = \cdots = d_{\frac{n}{2}+1} = 1$.
2. $n = 2$, $4 \leq d \leq 15$, *or* $n = 4$, $3 \leq d \leq 6$, *or* $n = 6$, $3 \leq d \leq 4$.

Proof The proof of 2. is by computer assistance, verifying in each case that Rank $P(\delta) = C_{\underline{d}}$. In order to prove 1. consider

$$f_i = x_{2i-2} - \zeta_{2d}x_{2i-1}.$$

Define

$$I_N := \left\{(i_0,i_1,\ldots,i_{n+1}) \in \mathbb{Z}^{n+2} \mid 0 \leq i_e \leq d-2,\ i_0+i_1+\cdots+i_{n+1} = N\right\},$$

and

$$L := \{i \in I_{(\frac{n}{2}+1)d-n-2} \mid i_{2l-2}+i_{2l-1} = d-2, \forall l = 1,\ldots,\frac{n}{2}+1\}.$$

By Theorem 2.4.1, the period matrix is

$$P(\lambda) = c \cdot [p_{i+j}],$$

for

$$p_i = \begin{cases} \zeta^{\frac{i_0+i_2+\cdots+i_n}{2d}} & \text{if } i \in L, \\ 0 & \text{otherwise,} \end{cases}$$

and $c \in \mathbb{C}^*$ a non-zero constant. Let

$$A := \{i \in I_{\frac{n}{2}d-n-2} \mid i_0 = i_2 = \cdots = i_n = 0\},$$

$$B := \{j \in I_d \mid j_0 = j_2 = \cdots = j_n = 0\}.$$

Consider the map $\ : B \to A$ given by $\ (j)_{2l-2} = 0$, $\ (j)_{2l-1} = d-2-j_{2l-1}$, for $l = 1, \ldots, \frac{n}{2}+1$. It is easy to see that $\ $ is a bijection, thus

$$\#A = \#B = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2 = C_{(1,\ldots,1)}.$$

We affirm that the rows $p_{i+\cdot}, i \in A$ form a base for the image of $[p_{i+j}]$. Indeed, since for $(i,j) \in A \times B$

$$p_{i+j} = \begin{cases} 1 & \text{if } i = \ (j), \\ 0 & \text{otherwise,} \end{cases}$$

it follows that these rows are linearly independent. In order to see that they generate the image, it is enough to show they generate all the rows. Let $i \in I_{\frac{n}{2}d-n-2}$. If $i_{2l-2}+i_{2l-1} > d-2$ for some $l \in \{1, \ldots, \frac{n}{2}+1\}$, then $p_{i+\cdot} = 0$. If not, then $\exists ! j \in B : i+j \in L$, in fact $j_{2l-2} = 0$, $j_{2l-1} = d-2-i_{2l-2}-i_{2l-1}$, for $l = 1, \ldots, \frac{n}{2}+1$. We claim that

$$p_{i+\cdot} = \zeta^{\frac{i_0+i_2+\cdots+i_n}{2d}} \, p_{\ (j)+\cdot}.$$

In fact, if $h \in I_d$ is such that $p_{\ (j)+h} = 0$, then $\ (j)+h \notin L$, so

$$\exists l \in \{1, \ldots, \frac{n}{2}+1\} : \ (j)_{2l-2} + \ (j)_{2l-1} + h_{2l-2} + h_{2l-1} > d-2.$$

Since

$$\ (j)_{2l-2} + \ (j)_{2l-1} = i_{2l-2} + i_{2l-1}, \tag{3.5}$$

it follows that $p_{i+h} = 0$. On the other hand, if $h \in I_d$ is such that $\ (j)+h \in L$, then by (3.5) $i+h \in L$ and

$$p_{i+h} = \zeta^{\frac{(i_0+h_0)+\cdots+(i_n+h_n)}{2d}} = \zeta^{\frac{i_0+\cdots+i_n}{2d}} \zeta^{\frac{h_0+\cdots+h_n}{2d}} = \zeta^{\frac{i_0+\cdots+i_n}{2d}} \, p_{\ (j)+h}.$$

■

Remark 3.6.1. The first part of Theorem 3.6.1 says that $V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]}$ is a local component of the Hodge locus, smooth and reduced at $0 \in T$ the Fermat variety. Furthermore, we know its codimension in T is

$$\text{Codim}\,_T V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2.$$

It was showed by Movasati in [Mov17c] this is a lower bound for all components of the Hodge locus passing through Fermat variety. Thus, this bound is sharp. Furthermore, it can be shown this is the unique component of minimal codimension passing trough Fermat (this will be proved in an article under preparation). In the case $n = 2$, the problem of characterizing the special components of the Noether-Lefschetz locus is a classical one, partially solved by Voisin in small degrees [Voi89], [Voi90]. The smallest codimension components of the Noether-Lefschetz locus were characterized by Green [Gre89] and independently by Voisin [Voi89]. In higher dimension, to determine the sharp lower bound is still open, not to mention the characterization of the smallest codimension components.

Remark 3.6.2. Theorem 3.6.1 holds for every (n, d) such that $d \geq 2 + \frac{4}{n}$, and every $t \in T$ as in Proposition 3.6.2 (not just the Fermat variety). This fact was proved by Dan in [Dan14] for $deg(Z) < d$. Another proof of this fact (without restrictions on the degree) was provided by Movasati in [Mov17b] Chapter 7.

We close this section with another proof of AHC for sums of linear cycles inside Fermat variety.

Theorem 3.6.2 ([MV17])**.** *Let T be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. Let $0 \in T$ be the point representing the Fermat variety X_0. If $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ are two linear subspaces inside X_0, such that $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$, then letting $\delta := [\mathbb{P}^{\frac{n}{2}}] + [\check{\mathbb{P}}^{\frac{n}{2}}]$*

$$V^{\frac{n}{2}}_{\delta} = V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} \cap V^{\frac{n}{2}}_{[\check{\mathbb{P}}^{\frac{n}{2}}]},$$

for all triples (n, d, m) in the following list:

$$(2, d, -1),\ 5 \leq d \leq 14,$$
$$(4, 4, -1), (4, 5, -1), (4, 6, -1), (4, 5, 0), (4, 6, 0),$$
$$(6, 3, -1), (6, 4, -1), (6, 4, 0),$$
$$(8, 3, -1), (8, 3, 0),$$
$$(10, 3, -1), (10, 3, 0), (10, 3, 1),$$

where $\mathbb{P}^{-1}$ means the empty set. In particular, alternative Hodge conjecture holds for δ in these cases.

Proof It is enough to show that

$$\dim T_0 V^{\frac{n}{2}}_{\delta} = \dim V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} \cap V^{\frac{n}{2}}_{[\check{\mathbb{P}}^{\frac{n}{2}}]}.$$

In order to compute the dimension of $V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} \cap V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]}$, we notice that it is the germ of variety of the following set

$$S := \{t \in T : X_t \text{ contains two linear cycles } \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}} \text{ such that } \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m\}.$$

And so, it is enough to compute the dimension of S. Let $\mathbb{G}$ be the Grassmannian of two codimension $\frac{n}{2}+1$ linear subvarieties $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ inside $\mathbb{P}^{n+1}$ such that they intersect in a m dimensional linear subvariety $\mathbb{P}^m = \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}}$. Consider the incidence variety

$$S' := \{(t, (\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})) \in T \times \mathbb{G} : \mathbb{P}^{\frac{n}{2}} \cup \check{\mathbb{P}}^{\frac{n}{2}} \subset X_t\}.$$

Since, for every $t \in T$, $\mathrm{pr}_1^{-1}(t) \cap S'$ is a finite set, $\dim S' = \dim S$. In order to compute the dimension of S', we fix a point $(\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}) \in \mathbb{G}$ and compute the dimension of its fiber under pr_2. By a linear change of coordinates we notice all the fibers have the same dimension, thus we may assume that

$$\mathbb{P}^{\frac{n}{2}} : x_0 = \cdots = x_{\frac{n}{2}} = 0,$$

$$\check{\mathbb{P}}^{\frac{n}{2}} : x_0 = \cdots = x_m = x_{\frac{n}{2}+m+2} = \cdots = x_{n+1} = 0.$$

Every $F \in \mathbb{C}[x_0, \dots, x_{n+1}]_d$, such that both linear cycles are inside $\{F = 0\}$ can be written as

$$F = \sum_{j=0}^{m} x_j f_j + \sum_{k=1}^{\frac{n}{2}-m} x_{m+k} \left(\sum_{l=1}^{\frac{n}{2}-m} x_{\frac{n}{2}+m+1+l} g_{k,l} \right),$$

with f_j not depending on $x_0, \dots, x_{j-1}$, and $g_{k,l}$ not depending on the variables $x_0, \dots, x_{m+k-1}$ nor on $x_{\frac{n}{2}+m+2}, \dots, x_{\frac{n}{2}+m+l}$. Thus, the fiber of S' with respect to pr_2 has dimension

$$\sum_{j=0}^{m} \binom{n+1-j+d-1}{d-1} + \sum_{k,l=1}^{\frac{n}{2}-m} \binom{n+2-m-k-l+d-2}{d-2} = \binom{n+1+d}{d} + \binom{m+d}{d} - 2\binom{\frac{n}{2}+d}{d}.$$

Since the dimension of $\mathbb{G}$ is $(m+1)(n-m+1) + (n+2)(\frac{n}{2}-m) = 2(\frac{n}{2}+1)^2 - (m+1)^2$, we conclude that

$$\begin{aligned} \mathrm{Codim} V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} \cap V^{\frac{n}{2}}_{[\mathbb{P}^{\frac{n}{2}}]} = \mathrm{Codim}\, S &= 2\binom{\frac{n}{2}+d}{d} - 2(\frac{n}{2}+1)^2 - \binom{m+d}{d} + (m+1)^2 \\ &= 2C_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}} - C_{1^{n+1-m},(d-1)^{m+1}}. \end{aligned} \tag{3.6}$$

Note that we are denoting $C_{1^a,(d-1)^b}$ meaning that it is $C_{\underline{e}}$ for $\underline{e} = (1, \dots, 1, d-1, \dots, d-1)$ where the first a entries are equal to 1, and the last b entries are equal to $d-1$.

In order to compute the codimension of $T_0 V^{\frac{n}{2}}$, we compute the period matrix $P(\ )$ using the period formula given by Corollary 2.4.1, and then we apply Corollary 3.6.1. We compute the rank of the period matrix $P(\ )$ with computer assistance, and we verify in each case that it is equal to (3.6). ∎

Remark 3.6.3. Theorem 3.6.2 holds for a general $t \in T$ such that X_t contains two $\frac{n}{2}$-dimensional linear subvarieties intersecting each other in a m-dimensional linear space, for (n, d, m) such that $m < \frac{n}{2} - \frac{d}{d-2}$, see [VL18] Theorem 2 and Remark 3.

Chapter 4

Appendix

4.1 Hypercohomology

In this appendix we recall (without proofs) the basic properties of hypercohomology we use along the text. Our main reference is [Mov17b] Chapter 3.

Definition 4.1.1. Let X be a topological space, $\mathcal{U}$ be an open covering of X, and $(\mathcal{S}^\bullet, d)$ be a complex of sheaves over X. Consider the Čech cochains groups

$$\mathcal{S}^i_j := C^j(\mathcal{U}, \mathcal{S}^i).$$

Let

$$\mathcal{L}^k := \bigoplus_{i+j=k} \mathcal{S}^i_j,$$

and define $D_k : \mathcal{L}^k \to \mathcal{L}^{k+1}$ as $D_k|_{\mathcal{S}^i_j} := d + (-1)^i \delta$. These maps determine a complex of abelian groups $(\mathcal{L}^\bullet, D)$. We define the *hypercohomology of the complex* $(\mathcal{S}^\bullet, d)$ *relative to the covering* $\mathcal{U}$ as

$$\mathbb{H}^k(\mathcal{U}, \mathcal{S}^\bullet) := H^k(\mathcal{L}^\bullet, D).$$

In particular each element $\omega \in \mathbb{H}^k(\mathcal{U}, \mathcal{S}^\bullet)$ is represented by a sum

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^k,$$

where each $\omega^i \in C^j(\mathcal{U}, \mathcal{S}^i)$, $\delta\omega^0 = 0$, $d\omega^i = (-1)^i \delta\omega^{i+1}$ for $i = 0, \ldots, k-1$, and $d\omega^k = 0$. The *hypercohomology of the complex* $(\mathcal{S}^\bullet, d)$ is

$$\mathbb{H}^k(X, \mathcal{S}^\bullet) := \varinjlim_{\mathcal{U}} \mathbb{H}^k(\mathcal{U}, \mathcal{S}^\bullet),$$

where the direct limit is taken over the set of coverings directed by the refinement relation.

Definition 4.1.2. Let X be a topological space, and $\mathcal{S}$ a sheaf over X. We say $\mathcal{S}$ is *acyclic* if

$$H^q(X, \mathcal{S}) = 0, \quad \forall q > 0.$$

Proposition 4.1.1. *Let X be a topological space, $\mathcal{U}$ be an open covering of X, and $(\mathcal{S}^\bullet; d)$ be a complex of sheaves over X. If $\mathcal{U}$ is a locally finite cover, that is acyclic with respect to every $\mathcal{S}^i$, i.e.*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_r}; \mathcal{S}^i) = 0,$$

for all $q, r > 0$ and $i \geq 0$. Then

$$\mathbb{H}^k(X; \mathcal{S}^\bullet) \simeq \mathbb{H}^k(\mathcal{U}; \mathcal{S}^\bullet).$$

Proposition 4.1.2. *When every $\mathcal{S}^i$ is acyclic, we have*

$$\mathbb{H}^k(X; \mathcal{S}^\bullet) \simeq H^k(\Gamma(\mathcal{S}^\bullet); d).$$

Proposition 4.1.3. *Let X be a topological space, $\mathcal{S}$ a sheaf of abelian groups over X. If*

$$0 \to \mathcal{S} \to \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \cdots,$$

is an exact sequence of sheaves (in other words $(\mathcal{S}^\bullet; d) : 0 \to \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{f_1} \cdots$ is a resolution of $\mathcal{S}$), then

$$H^k(X; \mathcal{S}) \simeq \mathbb{H}^k(X; \mathcal{S}^\bullet).$$

Corollary 4.1.1. *Let X be a topological space, $\mathcal{S}$ a sheaf of abelian groups over X. If $(\mathcal{S}^\bullet; d) : 0 \to \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{f_1} \cdots$ is an acyclic resolution of $\mathcal{S}$, then*

$$H^k(X; \mathcal{S}) \simeq H^k(\Gamma(\mathcal{S}^\bullet); d).$$

Definition 4.1.3. A morphism between complexes of sheaves over X

$$\Phi : (\mathcal{S}^\bullet; d) \to (\mathcal{G}^\bullet; d)$$

is called a *quasi-isomorphism*, if it induces isomorphisms between each cohomology sheaf. In other words, the morphism of sheaves

$$\mathcal{H}^k(\Phi) : \mathcal{H}^k(\mathcal{S}^\bullet; d) \to \mathcal{H}^k(\mathcal{G}^\bullet; d)$$

are isomorphisms for all $k \geq 0$. Notice $\mathcal{H}^k(\mathcal{S}^\bullet; d) = \mathrm{Ker}\, d_k / \mathrm{Im}\, d_{k-1}$ is a sheaf.

Proposition 4.1.4. *Whenever $\Phi : (\mathcal{S}^\bullet; d) \to (\mathcal{G}^\bullet; d)$ is a quasi-isomorphism*

$$\mathbb{H}^k(X; \mathcal{S}^\bullet) \simeq \mathbb{H}^k(X; \mathcal{G}^\bullet),$$

for every $k \geq 0$.

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