THE ROLE OF SELF LOOPS AND LINK REMOVAL IN EVOLUTIONARY GAMES ON NETWORKS

DARIO MADEO (U. SIENA), CHIARA MOCENNI (U. SIENA), JEAN CARLO MORAES (UFRGS), AND JORGE P. ZUBELLI (IMPA)

Abstract. Recently, a new mathematical formulation of evolutionary game dynamics has been introduced accounting for a finite number of players organized over a network, where the players are located at the nodes of a graph and edges represent connections between them. Internal steady states are particularly interesting in control and consensus problems, especially in a networked context where they are related to the coexistence of different strategies. In this paper we consider this model including self loops. Existence of internal steady states is studied for different kind of graph topologies. Results on the effect of removing links from central players are also presented.

1. Introduction

Many physical systems of interest can be described by evolutionary games on graphs within the more general framework of dynamical systems on complex networks [1]. For example, opinion dynamics under social network influence [2], spread of contagious diseases subject to competition and selection [3, 4, 26], crime dynamics [5], bacterial networks [6], multi-agent decision-making dynamics [28, 7], and the emergence of cooperation in networked populations [8, 9, 10, 27, 11, 12]. Among the different interaction mechanisms, the simplest ones can be modeled as two-strategy games [13, 14]. From a mathematical perspective, the evolutionary games equation on networks (EGN), introduced in [15], can be written as a set of N ordinary differential equations, where N is the number of nodes. Recently, model reduction and symmetries have been investigated by using the concept of lumpability of graphs [17].

Similarly to the standard case, the replicator equation on networks possesses different kinds of steady states: *mixed steady states* that belong to the interior of the simplex ($x_v \in (0, 1), \forall v$), *pure steady states* for which all entries of the belong to the border of the simplex ($x_v \in \{0, 1\}, \forall v$), and *pure/mixed steady states* (for which $x_v \in [0, 1], \forall v$). Mixed steady states, hereafter called *internal steady states*, are particularly important in the EGN because they represent situations where the player assumes hybrid decisions corresponding to partially agree to all available strategies. This includes the case for which some of the strategies are strongly preferred to the others, for example the probability to choose a given strategy can be very close to 1, although different. As a consequence, the probability to choose all the other strategies will be very close to 0 since the sum of all probabilities equals 1. Internal steady states are the most reasonable states for which a group of individual can be able to agree on a compromised decision.

Moreover, the importance of the internal states mentioned above lies on the fact that they represent situations where different subpopulations may coexist. Thus, studying the attractiveness of these states is connected to the possibility that subgroups of the players eventually coexist in an asymptotically stable manner [13, 15]. On the contrary, oscillations making the dynamics mode interesting will be produced only if we have unstable internal states [24].

In this paper, we study the feasibility of internal steady states in the EGN proposed in [15], by considering different situations, such as, for example, the presence of self connections in the network. This is particularly relevant for social applications, since self loops describe well how a player is able to interact also with himself, thus modeling positive or negative feedback on player decisions. We find necessary conditions for the feasibility of internal steady states of EGN. We distinguish the cases of dominant, coordination and anti-coordination payoff matrices of the underlying games. Moreover, we prove sufficient conditions for the feasibility of internal steady states when the graph is complete. Existence and feasibility of internal steady states is relevant for solving control and consensus problems. Controlling dynamical systems over networks in order to drive a population of agents towards a specific steady state has been widely studied [18, 19, 20], while the presence of adaptive networks has been tackled in [21, 22].

Furthermore, this work proves results concerning how the dynamics of the whole system is influenced by varying the network connectivity of a single node. The problem under investigation connects with the diffusion centrality issue [23], whereby the role of the central individuals in a social network is analyzed by observing indirect information flow.

Finally, the effect of link removal from central players is studied theoretically for graphs with no self edges, while numerical results are proposed to investigate the case of graphs with self edges. Including self edges in the model proposed in [15] is very promising to study the stability of internal steady states [29]. Indeed, in a previous paper it has been shown that stability is not possible for graph with no self edges. Thus, the stability strongly depends on the strength of self connections as well as on graph topology. Intuitively, the presence of self loops is related to positive or negative feedbacks in the dynamical equation.

The paper is organized as follows. In Section 2 we illustrate the basics of evolutionary games on networks and in particular on two-strategy game for graphs including self loops. Section 3 states the necessary conditions for the existence of internal steady states. Some numerical results are provided to analyze the existence of mixed steady states for a generic and heterogeneous scenario. Then, in Section 4 we present sufficient conditions in the case of complete graphs. In Section 5 we tackle the problem of link removal from a player by providing theoretical results for the case of no self loops, and numerical experiments for the case with self loops. Finally, in Section 6 we state some conclusions and suggest future developments.

2. Preliminaries

We start by considering the evolutionary game equation on a network (EGN) introduced in [15], where s belongs to a set of strategies $S = \{1, 2, \ldots, M\}$, v belongs to a set of vertices $V = \{1, 2, \ldots, N\}$ and $x_{v,s}$ is the probability that vertex v chooses strategy s,

$$\dot{x}_{v,s} = x_{v,s}(p^G_{v,s} - \phi^G_v). \tag{2.1}$$

Here, $p^G_{v,s}$ is the payoff of player v using pure strategy s, and ϕ^G_v is the average payoff over the set of strategies available to vertex v. In this paper we consider a generalization of the model where $p^G_{v,s}$ and ϕ^G_v are both defined by means of *player-to-player* payoff matrices $B_{v,u} \in \mathbb{R}^{M\times M}$, such that $B_{v,u}$ is the payoff matrix used by player v against player u. As a consequence, the model presented in [15] coincides with the special case whereby $B_{v,u} = B_v,\ \forall u, v \in V$. As usual, $p^G_{v,s}$ and ϕ^G_v depend on the graph G, which in turn is defined by means of an $N \times N$ adjacency matrix $A = (a_{v,u}) \in \mathbb{R}^{N\times N}_{\geq 0}$ with $(v, u) \in V^2$. More precisely,

$$p^G_{v,s} = \sum_{u=1}^{N} a_{v,u}\mathbf{e}_s^\top B_{v,u}\mathbf{x}_u,$$

and

$$\phi^G_v = \sum_{u=1}^{N} a_{v,u}\mathbf{x}_v^\top B_{v,u}\mathbf{x}_u,$$

where $\mathbf{e}_s$ is the s-th canonical-basis vector of $\mathbb{R}^M$ and $\mathbf{x}_v = [x_{v,1}\ x_{v,2}\ \ldots\ x_{v,M}]^T$ is the distribution of pure strategies of player v.

Moreover, we consider graphs with self loops, i.e., $a_{v,v} \geq 0$. In this regard, it is straightforward to consider also *self games* described by payoff matrices $B_{v,v}$ where a player v plays against itself. For convenience, we define $\deg(v) = \sum_{u=1}^{N} a_{v,u}$. Since we we are only concerned with graphs without isolated vertices, then $\sum_{u=1,u\neq v}^{N} a_{v,u} = \deg(v) - a_{v,v} > 0$. Hence, $0 \leq \delta_v < 1$, where $\delta_v := a_{v,v}/\deg(v)$ is the relative self connectivity (i.e., how strong is the self connection with respect to the sum of all connection weights).

Since for every v the constraint $x_{v,1} + x_{v,2} + \ldots + x_{v,M} = 1$ holds, we have that for M strategies and N vertices the EGN leads to a system with $N(M-1)$ ordinary differential equations. Furthermore, in this article we analyze the replicator equation for two-strategy games ($M = 2$). Therefore, for convenience, we drop the second index s of $x_{v,s}$, introducing the variable $y_v = x_{v,1}$, whereas $x_{v,2} = 1 - y_v$. Thus, in our case the EGN becomes a system of N ODEs:

$$\dot{y}_v = y_v(p_v^G - \phi_v^G). \tag{2.2}$$

Equation (2.2) can be rewritten in a more convenient way as follows: Let $b_{v,u,s,r}$ be the payoff of player v against u when they use strategies s and r, respectively. Then, the payoff function for vertex v against u can be defined by means of the payoff matrix:

$$B_{v,u} = \begin{pmatrix} b_{v,u,1,1} & b_{v,u,1,2} \\ b_{v,u,2,1} & b_{v,u,2,2} \end{pmatrix} . \tag{2.3}$$

We denote by $\sigma_{v,u,r} = (-1)^{r+1}(b_{v,u,1,r} - b_{v,u,2,r})$ the payoff difference of player v when u uses strategy r. According to [16], $B_{v,u}$ can be equivalently rewritten as a diagonal matrix, namely

$$B_{v,u} = \mathrm{diag}(\sigma_{v,u,1}, \sigma_{v,u,2}), \tag{2.4}$$

and Equation (2.2) reads as

$$\dot{y}_v = y_v(1 - y_v) f_v(\mathbf{y}) \ , \tag{2.5}$$

where $\mathbf{y} = (y_1, \dots, y_N)^\top$, $f_v(\mathbf{y}) = \sum_{u=1}^N a_{v,u} f_{v,u}(y_u)$ and $f_{v,u}(y_u) = y_u \mathrm{Tr}\,(B_{v,u}) - \sigma_{v,u,2}$, where $\mathrm{Tr}\,(B_{v,u})$ is the trace of matrix $B_{v,u}$. Steady states of (2.5) are very important solutions because they influence significantly the asymptotic dynamics of the system. Moreover, they can be related to the Nash equilibria of the game described by the payoff matrix of Equation (2.4).

A solution $\mathbf{y}^*$ of the EGN is a steady state if, and only if, $y_v = 0$, or $y_v = 1$, or $f_v(\mathbf{y}^*) = 0$. But,

$$f_v(\mathbf{y}^*) = \mathbf{0} \Leftrightarrow \sum_{u=1}^N a_{v,u} \mathrm{Tr}\,(B_{v,u})\, y_u = \sum_{u=1}^N a_{v,u} \sigma_{v,u,2}, \forall v \in V,$$

or equivalently

$$[(\Sigma^1 + \Sigma^2) \circ A]\mathbf{y}^* = [\Sigma^2 \circ A]\mathbf{1}, \tag{2.6}$$

where Σ^1 and Σ^2 are matrices with $\Sigma^1_{v,u} = \sigma_{v,u,1}$, $\Sigma^2_{v,u} = \sigma_{v,u,2}$, A is the adjacency matrix, $\mathbf{1}$ is the N dimensional vector with one in every entry and $\circ$ denotes the Hadamard product defined by $P \circ Q = R$ where $R = \{r_{i,j}\} := \{p_{i,j} q_{i,j}\}$.

Our goal is to study how the connectivity of the graph and the presence of self loops impacts the existence of an internal steady state. From [15] we know that if all the players have the same payoff matrix and there are no self loops, then the topology does not matter. We will start our study looking at theoretical results on the existence of mixed equilibrium in case more general than the one approached in [15].

3. Existence of the internal mixed equilibrium

Whenever $[(\Sigma^1 + \Sigma^2) \circ A]$ is invertible, then we have a unique solution:

$$\mathbf{y}^* = [(\Sigma^1 + \Sigma^2) \circ A]^{-1}[\Sigma^2 \circ A]\mathbf{1}. \tag{3.1}$$

If $\mathbf{y}^*$ satisfies Equation (2.6), then it is a steady state for the ODE system (2.5). Moreover, a steady state $\mathbf{y}^*$ satisfying (2.6) is feasible if, and only if, $y_v^* \in [0,1],\ \forall v \in V$. A feasible steady state $\mathbf{y}^*$ is said to be internal if, and only if, $y_v^* \in (0,1),\ \forall v \in V$. A feasible steady state is *pure* if $y_v^* \in \{0,1\}^N,\ \forall v \in V$. A feasible steady state is *non pure* if $y_v^* \in (0,1)$ for some v. We can relate steady states of EGN to Nash equilibria of the underlying game described by the payoff matrices (2.3). Indeed, in two-strategy games, if $\mathbf{y}^*$ is a feasible steady state, then it is also a Nash equilibrium of the underlying static game [16].

However, it is not enough to guarantee the solvability of Equation (2.6) in order for the mixed steady states to exist, because we also need that the solutions of the linear equations (2.6) belong to hypercube $\Delta_S = \{y \in \mathbb{R}^N : 0 \le y_i \le 1\ ,\ \forall i \in V\}$. We start by giving necessary conditions on the values of the σs in order for the mixed steady states to be feasible.

Suppose that for every game, the payoff matrix that player v uses when it plays with his neighbors is equal for every opponent. In other words, $\forall v \in V, B_{v,u} = B_v = \mathrm{diag}(\sigma_{v,1}, \sigma_{v,2}),\ \forall u \neq v$. In contradistinction, the self game of player v is represented by the matrix $B_{v,v} = B_v^{\circlearrowleft} = \mathrm{diag}(\sigma_{v,1}^{\circlearrowleft}, \sigma_{v,2}^{\circlearrowleft})$.

Now let us define β_v and γ_v as follows:

$$\begin{cases} T_v^{\circlearrowleft} = \beta_v T_v \\ \sigma_{v,2}^{\circlearrowleft} = \gamma_v \sigma_{v,2} \end{cases},$$

where $T_v = \mathrm{Tr}\,(B_v)$ and $T_v^{\circlearrowleft} = \mathrm{Tr}\left(B_v^{\circlearrowleft}\right)$. Note that if $T_v = 0$ for some v then either player v has a dominant strategy or it is indifferent to any strategy. In the case he has a dominant strategy no internal equilibria can be obtained, just mixed equilibria. In the case the player is indifferent, then he will always play the same strategy that he starts the game, therefore, in order to look for an internal equilibria, we could look to the network game without this player. Thus, we define $d_v = \frac{\sigma_{v,2}}{T_v}$ and $d_v^{\circlearrowleft} = \frac{\sigma_{v,2}^{\circlearrowleft}}{T_v^{\circlearrowleft}} = \frac{\gamma_v \sigma_{v,2}}{\beta_v T_v} = \frac{\gamma_v}{\beta_v} d_v$.

Proposition 3.1. *Suppose that the adjacency matrix A is non-negative ($A \in \mathbb{R}_{\geq 0}^{N \times N}$) and that each node has at least one neighbor. Moreover, suppose that $T_v \neq 0$. If $\mathbf{y}^*$ is a mixed steady state, then for any $v \in V$:*

$$\begin{cases} d_v(1 + \delta_v(\gamma_v - 1)) - y_v^* \delta_v(\beta_v - 1) \in (0,1) & \text{if} \quad a_{v,v} \neq 0 \\ d_v \in (0,1) & \text{if} \quad a_{v,v} = 0 \end{cases}.$$

Proof. Let $[A\mathbf{y}]_v$ and $[A\mathbf{1}]_v$ be the v-th component of the vector $A\mathbf{y}$ and $A\mathbf{1}$, respectively.

Since each node has at least one neighbor, then $\deg(v) > 0, \ \forall v \in V$. Therefore,

$$\begin{aligned} &[A\mathbf{y}^*]_v = \textstyle\sum_{u=1}^N a_{v,u} \cdot y_u^* \geq \min(\mathbf{y}^*) \cdot \sum_{u=1}^N a_{v,u} = \\ &= \min(\mathbf{y}^*) \cdot \deg(v) > 0, \, \forall v \in V. \end{aligned}$$

Furthermore,

$$\begin{aligned} &[A\mathbf{y}^*]_v = \textstyle\sum_{u=1}^N a_{v,u} \cdot y_u^* < \sum_{u=1}^N a_{v,u} \cdot 1 = \\ &= \deg(v), \ \forall v \in V \ . \end{aligned}$$

This implies that $0 < [A\mathbf{y}^*]_v < \deg(v)$, and hence

$$\frac{[A\mathbf{y}^*]_v}{\deg(v)} \in (0,1) \ \forall v \in V. \tag{3.2}$$

From Problem (2.6), we know that

$$\begin{array}{lll} \sum_{u=1}^N a_{v,u} \mathrm{Tr}\,(B_{v,u})\, y_u^* & = \sum_{u=1}^N a_{v,u} \sigma_{v,u,2} & \Rightarrow \\ T_v [A\mathbf{y}^*]_v + a_{v,v}(T_v^{\circlearrowleft} - T_v) y_v^* & = \sigma_{v,2} \deg(v) + a_{v,v}(\sigma_{v,2}^{\circlearrowleft} - \sigma_{v,2}) & \Rightarrow \\ T_v \left([A\mathbf{y}^*]_v + a_{v,v}(\beta_v - 1) y_v^* \right) & = \sigma_{v,2} \left(\deg(v) + a_{v,v}(\gamma_v - 1) \right) & \end{array}, \tag{3.3}$$

$\forall v \in V$. If $a_{v,v} = 0$ then

$$\frac{[A\mathbf{y}^*]_v}{\deg(v)} = d_v \in (0,1). \ \forall v \in V.$$

If $a_{v,v} \neq 0$, then dividing both sides of Equation (3.3) by $\deg(v)\, T_v$, we have:

$$\begin{array}{l} \frac{[A\mathbf{y}^*]_v}{\deg(v)} + \delta_v(\beta_v - 1) y_v^* = d_v \left(1 + \delta_v(\gamma_v - 1)\right) \Rightarrow \\ \frac{[A\mathbf{y}^*]_v}{\deg(v)} = d_v \left(1 + \delta_v(\gamma_v - 1)\right) - \delta_v(\beta_v - 1) y_v^* \in (0,1). \end{array} \ \forall v \in V.$$

$\square$

Note that this result give us a necessary condition on having an internal steady state $\mathbf{y}^*$ that depends on the weighted average of d_v and $\mathbf{y}^*$, with weights $\delta_v(\beta_v - 1)$ and $\delta_v(\gamma_v - 1)$. If we know the components of $\mathbf{y}^*$, no such condition is needed. However, the proposition allow us to prove the interesting corollary below. This result will give us a truly necessary condition for having an internal steady state in the cases where we have coordination and anti-coordination payoff matrices. It is also interesting to note that if $\delta_v = 0$ (i.e., no self loops) then we have that $d_v \in (0, 1)$, recovering the results of [15] for the uniform payoff case. In this sense the results in the previous theorem are an extension of the results presented in [15].

Corollary 3.1. *Suppose that $\mathbf{y}^*$ is a mixed steady state. Let $\ell_v = a_{v,v} - deg\,(v)$ be the v-th element of the Laplacian matrix of the graph, i.e., $L = A - diag(deg\,(v)_{v\in V})$. For each v, then:*

(a) if $\delta_v = 0$, then B_v represents a coordination game if $T_v > 0$ or an anti-coordination game if $T_v < 0$;
(b) if $\delta_v > 0$ and $1 < \beta_v < \gamma_v$, then B_v represents a coordination game if $T_v > 0$ or an anti-coordination game if $T_v < 0$;
(c) if $\delta_v > 0$, $\beta_v > 1$ and $\gamma_v < \ell_v$, then the pure strategy $y^*_v = 1$ is dominant for the game represented by B_v if $T_v > 0$, or the pure strategy $y^*_v = 0$ is dominant for the game represented by B_v if $T_v < 0$.

Proof.

- Proof of (a)
  If $\delta_v = 0$ then $a_{v,v} = 0$, by the previous proposition we have that $d_v \in (0, 1)$, then if $T_v > 0$, then B_v represents a coordination game, while for $T_v < 0$ we have an anti-coordination game.

For the cases with, $\delta_v > 0$, firstly, notice that

$$1 - \frac{1}{\delta_v} = \frac{a_{v,v} - \deg(v)}{a_{v,v}}.$$

Moreover, since $\delta_v > 0 \Rightarrow a_{v,v} = 1$, then:

$$1 - \frac{1}{\delta_v} = a_{v,v} - \deg(v) = \ell_v.$$

Secondly, if $\beta_v > 1$, then $\delta_v(\beta_v - 1) > 0$. Furthermore:

$$0 < d_v(1 + \delta_v(\gamma_v - 1)) - y_v^* \delta_v(\beta_v - 1) < d_v(1 + \delta_v(\gamma_v - 1))$$

and

$$1 > d_v(1 + \delta_v(\gamma_v - 1)) - y_v^* \delta_v(\beta_v - 1) > d_v(1 + \delta_v(\gamma_v - 1)) - \delta_v(\beta_v - 1),$$

yielding that

$$0 < d_v(1 + \delta_v(\gamma_v - 1)) < 1 + \delta_v(\beta_v - 1).$$

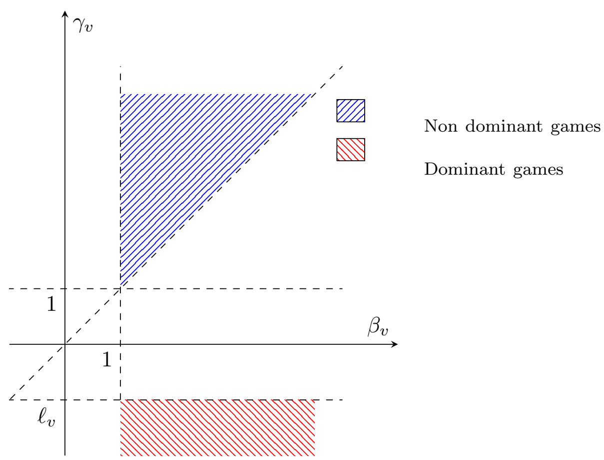

FIGURE 1. This figure reports the regions on the plane (β_v, γ_v) where conditions of Corollary 3.1 are met for a generic player v.

- Proof of (b)
  If $1 < \beta_v < \gamma_v$, then $1 + \delta_v(\gamma_v - 1) > 0$ and
  $$0 < d_v(1 + \delta_v(\gamma_v - 1)) < 1 + \delta_v(\beta_v - 1) \Rightarrow d_v \in \left(0, \tfrac{1+\delta_v(\beta_v-1)}{1+\delta_v(\gamma_v-1)}\right).$$
  Since $\gamma_v > \beta_v$ then $1 + \delta_v(\gamma_v - 1) > 1 + \delta_v(\beta_v - 1)$, which implies $d_v \in (0, 1)$ and the conclusion follows.
- Proof of (c)
  $\beta_v > 1$ and $\gamma_v < \ell_v$, then $\frac{1+\delta_v(\beta_v-1)}{1+\delta_v(\gamma_v-1)} < 0$ and
  $$0 < d_v(1 + \delta_v(\gamma_v - 1)) < 1 + \delta_v(\beta_v - 1) \Rightarrow d_v \in \left(\tfrac{1+\delta_v(\beta_v-1)}{1+\delta_v(\gamma_v-1)}, 0\right).$$
  It turns out that d_v is a negative number and hence $\sigma_{v,1}$ and $\sigma_{v,2}$ have different signs. In particular, if $T_v > 0$, then $\sigma_{v,2} < 0$ and $0 < |\sigma_{v,2}| < \sigma_{v,1}$, while $T_v < 0$

implies that $\sigma_{v,1} < 0$ and $0 < |\sigma_{v,1}| < \sigma_{v,2}$. In the former case, B_v represents a game with the pure strategy $y = 1$ dominant; instead for the latter case the pure strategy $y = 0$ is dominant.

□

Unfortunately, for the regions that were not marked in Figure 1 we do not have theoretical results and we have to investigate them numerically.

3.1. **Numerical results.** Proposition 3.1 and Corollary 3.1 relate the existence of the internal steady state with the connectivity of each player v (i.e., δ_v), the strength of its self games (i.e., β_v and γ_v), and the game nature. However, these results only provide necessary conditions for the existence of internal mixed steady states. While sufficient conditions that work for particular graph structures will be proven in the next sections, here we show a numerical example using an Erdös-Rényigraph sample with $N = 150$ nodes and average degree 10. We assume that $a_{v,v} = 1$ for all $v \in V$, i.e., each player has a self loop. For this numerical example, we divide the nodes into six groups of 25 elements each. For each group, we choose the parameters $\sigma_{v,1}$ and $\sigma_{v,2}$ in the set $\{(1, 1), (0.9, 1), (1, 0.9), (-1, -1), (-0.9, -1), (-1, -0.9)\}$. In this way, half on the nodes have a coordination game payoff matrix, while the other half play anti-coordination games. We also assume that $\beta_v = \beta$ and $\gamma_v = \gamma$ for all the players. Then, for each couple of values β and γ in the set $[-30, 30] \times [-30, 30]$, we evaluated the solution $\mathbf{y}^*$ of Equation (3.1). In Figure 2 we report in blue the couples (β, γ) for which there Equation (3.1) has no solution, or if the solution can not be classified as an internal steady state (i.e., $y^*_v \in (0, 1)\ \forall v$). Instead, if the solution $\mathbf{y}^*$ is internal, then the couple (β, γ) is depicted in light green. Finally, in dark green we report all the other steady states that satisfy the condition of Proposition 3.1.

4. Feasibility of mixed steady states for complete graphs

In this section we report some theoretical results on the feasibility for complete graphs. In order to study how connectivity plays a role on the existence of steady states we start our study with the more connected possible graph: the complete one. Our goal is to observe that more connections in the graph will imply that the payoff matrices of the players should be very similar in order the system be able to have an internal steady state.

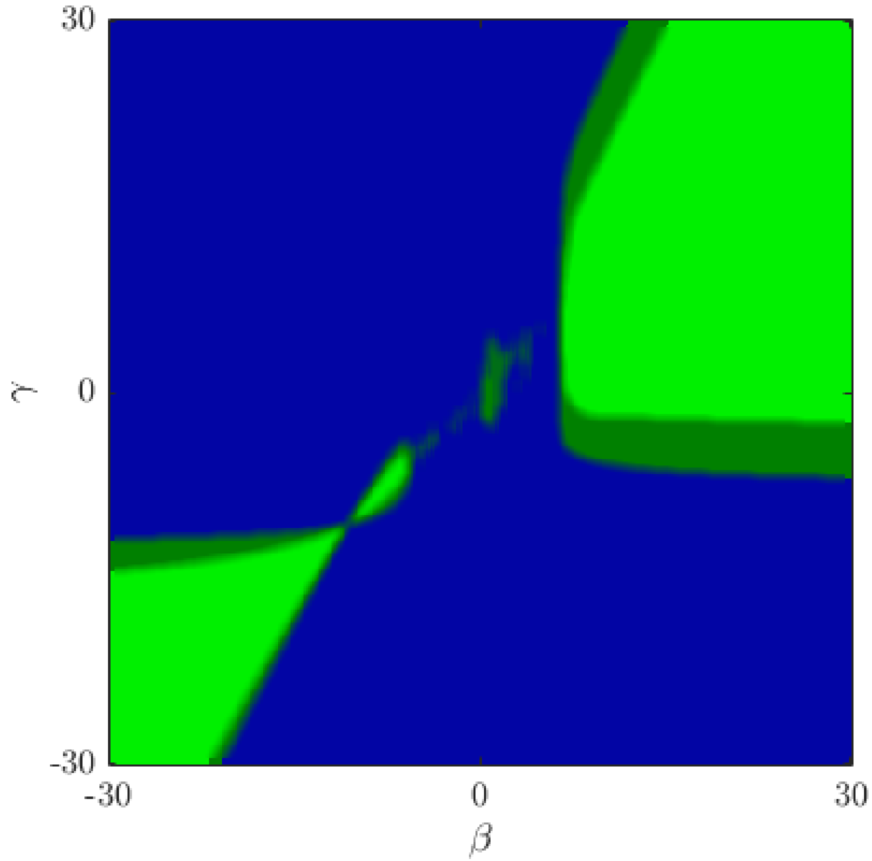

FIGURE 2. Existence of the internal mixed equilibrium in the hypothesis that $\beta_v = \beta$ and $\gamma_v = \gamma$ for all the $N = 150$ players, arranged over an Erdös-Rényi graph distribution with average degree 10. Blue region does not present any internal mixed equilibrium. Light green regions show an internal mixed equilibrium, satisfying Proposition 3.1. Dark green regions represent the value of β and γ satisfying Proposition 3.1 for non internal steady states which are solutions of Equation (3.1).

4.1. **Feasibility of mixed steady states with self-edges.** Consider a complete undirected and unweighted graph of N nodes with self-edges. Then, the adjacency matrix A is equal to $\mathbf{1}_{N\times N}$. Moreover, we consider that the self-game strengths are given by $\beta_v = \beta$ for all players. In this case, we get that $(\Sigma^1 + \Sigma^2) \circ A = \mathrm{diag}(T_v)A_\beta$ and $\Sigma^2 \circ A = \mathrm{diag}(\sigma_{v,2})A_\beta$ where $A_\beta = \mathbf{1}_{N\times N} + (\beta - 1)I_{N\times N}$.

Lemma 4.1. *Let $N \in \mathbb{N}^+$. If $\beta \neq 1$ and $\beta \neq 1-N$, then A_β is invertible and its inverse is given by $A_\beta^{-1} = [q_{v,u}]$ where*

$$\begin{cases} q_{v,v} = \frac{\beta+(N-2)}{(\beta-1)(\beta+N-1)} \\ q_{v,u} = \frac{-1}{(\beta-1)(\beta+N-1)}, \ v \neq u. \end{cases}$$

The proof of this lemma is a direct consequence of the remark that A_β is a circulant matrix.

To ease the discussion of the upcoming results, we introduce the average of a vector over a set of indices, $\langle \mathbf{x} \rangle_\Psi = \frac{1}{|\Psi|} \sum_{v\in\Psi} x_v$, where $|\Psi|$ is the cardinality of the set Ψ.

Theorem 4.2. *Let A, with $N \geq 3$ vertices, be the adjacency matrix of a complete graph with self-edges. If $T_v \neq 0$ and $\beta_v = \beta \notin \{1, 1-N\}$ for all $v \in V$, then there is at most one non-pure steady-state $\mathbf{y}^*$ for the system of ODEs in Equation (2.5) and $\langle \mathbf{y}^* \rangle_V = \frac{\gamma+N-1}{\beta+N-1} \langle \mathbf{d} \rangle_V$. Moreover, $\mathbf{y}^*$ is an internal steady-state if, and only if, for all $v \in V$:*

If sign$(\gamma + N - 1) =$ sign$(\beta - 1)$, then:

$$\frac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} < d_v < \frac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} + \frac{\beta-1}{\gamma+N-1}. \tag{4.1}$$

If sign$(\gamma + N - 1) \neq$ sign$(\beta - 1)$, then:

$$\frac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} + \frac{\beta-1}{\gamma+N-1} < d_v < \frac{N\langle \mathbf{d} \rangle_V}{\beta+N-1}. \tag{4.2}$$

Remark 4.3. For the case $N = 2$, it is easy to check that $\mathbf{y}^*$ is feasible if, and only if, $0 < d_v < 1$ for $v = 1, 2$.

Proof. For $\beta \notin \{1, 1-N\}$ and $\sigma_{v,1} + \sigma_{v,2} \neq 0$ we have that both $(\Sigma^1 + \Sigma^2) \circ A$ and $\Sigma^2 \circ A$ are invertible. Then, Equation (3.1) becomes:

$$y^* = A_\beta^{-1} D A_\gamma \mathbf{1}, \tag{4.3}$$

where $D = \text{diag}(T_v)^{-1}\text{diag}(\sigma_{v,2}) = \text{diag}(d_v)$ is a diagonal matrix.

In this case, the components of the steady state in Equation (4.3) are defined as follows:

$$\begin{aligned} y_v^* &= \frac{(\beta+N-2)(\gamma+N-1)}{(\beta-1)(\beta+N-1)} d_v - \frac{(\gamma+N-1)}{(\beta-1)(\beta+N-1)} \sum_{\substack{u=1 \\ u\neq v}}^N d_u \Rightarrow \\ y_v^* &= \frac{(\beta+N-1)(\gamma+N-1)}{(\beta-1)(\beta+N-1)} d_v - \frac{(\gamma+N-1)}{(\beta-1)(\beta+N-1)} \sum_{u=1}^N d_u \Rightarrow \\ y_v^* &= \frac{\gamma+N-1}{\beta-1} \left(d_v - \frac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} \right), \end{aligned}$$

while the average of all the components of $\mathbf{y}^*$ is:

$$\langle \mathbf{y}^* \rangle = \tfrac{\gamma+N-1}{\beta-1}\left(\langle \mathbf{d} \rangle_V - \tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} \right) = \tfrac{\gamma+N-1}{\beta+N-1}\langle \mathbf{d} \rangle_V.$$

Since each component y_v^* is in the set $(0,1)$, then:

$$0 < \tfrac{\gamma+N-1}{\beta-1}\left(d_v - \tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} \right) < 1.$$

If $\operatorname{sign}(\gamma+N-1) = \operatorname{sign}(\beta-1)$, then:

$$\tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} < d_v < \tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} + \tfrac{\beta-1}{\gamma+N-1}.$$

On the other hand, if $\operatorname{sign}(\gamma+N-1) \neq \operatorname{sign}(\beta-1)$, then:

$$\tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1} + \tfrac{\beta-1}{\gamma+N-1} < d_v < \tfrac{N\langle \mathbf{d} \rangle_V}{\beta+N-1}.$$

□

Corollary 4.4. *Under the assumptions of Theorem 4.2, if $d_v = d$ for all $v \in V$, then $\mathbf{y}^*$ is internal to the simplex if, and only if,*

$$\frac{\gamma+N-1}{\beta+N-1} d \in (0,1).$$

The proof of the corollary is straightforward by plugging $d_v = \langle \mathbf{d} \rangle_V$ in (4.1) or (4.2).

4.2. **Feasibility of mixed steady states with no self-edges.** In the following theorems we discus the feasibility of internal steady states for complete graphs with no self-edges. In this case Theorem 4.2 is simplified to the form:

Theorem 4.5. *Let A, with $N \geq 3$ vertices, be the adjacency matrix of a complete graph with no self-edges. If $T_v \neq 0$, $\forall v \in V$ then there is at most one non-pure steady-state $\mathbf{y}^*$ for the system of ODEs in Equation* (2.5) *and $\langle \mathbf{y}^* \rangle_V = \langle \mathbf{d} \rangle_V$. Moreover, $\mathbf{y}^*$ is an internal steady-state if, and only if,*

$$\tfrac{N\langle \mathbf{d} \rangle_V - 1}{N-1} \leq d_v \leq \tfrac{N\langle \mathbf{d} \rangle_V}{N-1}\ , \forall\, v\, \in V. \tag{4.4}$$

Corollary 4.6. *Under the assumptions of Theorem 4.2, if $\mathbf{y}^*$ is an internal steady-state then*

$$|d_v - \langle \mathbf{d} \rangle_V| < \tfrac{1}{N-1}, \quad \forall\, v \in V.$$

Proof. From (4.4), we have that:

$$\tfrac{\langle \mathbf{d} \rangle_V - 1}{N-1} < d_v - \langle \mathbf{d} \rangle_V < \tfrac{\langle \mathbf{d} \rangle_V}{N-1}, \quad \forall\, v \in V.$$

Since $\mathbf{y}^*$ is an internal steady state and $T_v \neq 0$, then by Theorem (4.2), $d_v \in (0,1)$, since $\beta_v = \gamma_v = 0$. Therefore,

$$-\tfrac{1}{N-1} < d_v - \langle \mathbf{d} \rangle_V < \tfrac{1}{N-1}, \quad \forall\, v \in V,$$

which is the statement of the corollary. □

The result (4.5) provides a necessary condition for the System (2.5) to have an internal steady state. If for any vertex v, the distance of d_v to the average $\bar{d}$ is greater than or equal to $\frac{1}{N-1}$, then the system can only have pure steady states. We can also see that if N is large, then we may only have internal steady states whose components are very close to each other, i.e., in a complete graph, the system can only have internal steady states if the payoff's ratio of every player d_v, does not get more than $\frac{1}{N-1}$ distant from the average of all payoff's ratios. For a large system, this will require similar payoffs for all players.

5. Feasibility of mixed steady states by varying the network connectivity

We now consider the following scenario: take a fully connected graph, choose one specific node and start deleting successively different links from this node. The general question under consideration is what is the effect in the dynamics of such procedure? In general terms, such circle of ideas has attracted the attention of other researchers. For instance, in telecommunications and computer networks this corresponds to the so-called "bond percolation" process (see Section 16.1 of [25]). In Chapter 16 of [25], a comprehensive review of percolation and network resilience can be found. In contradistinction with such approach we focus on one single node and analyze the resilience with respect to link removal in a deterministic fashion.

We study the effect of link removal from central player starting from a complete network. In particular, we report some theoretical results for the case of graphs with no self edges ($\beta_v = \gamma_v = 0, \forall v \in V$). The case of networks including self edges is then investigated by means of numerical simulations, showing that removing links can change dramatically the asymptotic behavior of the system, some times destroying the internal steady states.

5.1. **Theoretical results on games with no self-edges by removing links.**

Theorem 5.1. *Let A be the adjacency matrix of a complete graph with $N > 3$ vertices and no self-edges. Assume that the connection between vertices v_0 and u_0 is removed and $\Gamma = V \setminus \{v_0, u_0\}$. Moreover, assume that $\sigma_{v,1} + \sigma_{v,2} \neq 0$ for all v. If an internal steady* *state* $\mathbf{y}^*$ *exists, then following conditions hold*

$$\begin{cases} \frac{-2+\langle \mathbf{d}\rangle_\Gamma}{N-3} < d_{v_0} = d_{u_0} < \frac{\langle \mathbf{d}\rangle_\Gamma}{N-3}, \\ \frac{-1+d_{v_0}}{N-1} + \langle \mathbf{d}\rangle_\Gamma < d_v < \frac{d_{v_0}}{N-1} + \langle \mathbf{d}\rangle_\Gamma, \ \forall v \in \Gamma. \end{cases}$$

Proof. Without loss of generality, let us assume $v_0 = 1$, $u_0 = 2$. Writing Equation (2.6) as a system of linear equations, we get

$$\begin{cases} y_3 + y_4 + \ldots + y_N & = d_1(N-2) \\ y_3 + y_4 + \ldots + y_N & = d_2(N-2) \\ y_1 + y_2 + y_4 \ldots + y_N & = d_3(N-1) \\ & \vdots \\ y_1 + y_2 + y_3 \ldots + y_{N-1} & = d_N(N-1) \end{cases}.$$

If $d_1 \neq d_2$ then the first two equations would be incompatible, therefore $d_1 = d_2$. In this case, the system has infinite solutions with y_1 and y_2 as free variables. Let $z = y_1 + y_2$ and assume $d_1 = d_2$, then the system with N equations can be reduced to a system with $N-1$ equations

$$\begin{cases} y_3 + y_4 + \ldots + y_N & = d_1(N-2) \\ z + y_4 + \ldots + y_N & = d_3(N-1) \\ & \vdots \\ z + y_3 + \ldots + y_{N-1} & = d_N(N-1) \end{cases}.$$

This system has only one solution given by:

$$z^* = -(N-3)d_1 + (N-1)\langle \mathbf{d}\rangle_\Gamma,$$
$$y_v^* = d_1 + (N-1)\langle \mathbf{d}\rangle_\Gamma - (N-1)d_v, \ \forall v \in \Gamma.$$

If the solution is in the simplex, then for all $v \in \Gamma$, it is true that $0 < y_v^* < 1$ and $0 < z < 2$. This implies that i) and ii) must hold. $\square$

Suppose now that we start from an almost complete graph and iteratively remove additional links from the same vertex v_0. Let $\Lambda = \{v_1, v_2, \ldots, v_{M-1}\}$ be the set of vertices that have been disconnected from vertex v_0 and $\Gamma = V \backslash \big(\Lambda \cup \{v_0\}\big)$ the set of vertex that are still connected to v_0.

Theorem 5.2. *Let* A*, with* $N > 3$ *vertices, be the adjacency matrix of a complete graph where* $K-1$ *vertices have been disconnected from vertex* v_0*. Let* Λ *be the set of disconnected vertices and* Γ *the remaining set of connected vertices. Moreover, let us assume that*

$\sigma_{v,1} + \sigma_{v,2} \neq 0$ for all $v \in V$. Then, there exists an internal steady state, if, and only if, all the following conditions hold:

i) $(K-1)\langle \mathbf{d} \rangle_\Lambda - \frac{(N-1)(K-2)}{(N-2)} \langle \mathbf{d} \rangle_\Gamma < d_{v0} < (K-1)\langle \mathbf{d} \rangle_\Lambda - \frac{(N-1)(K-2)}{(N-2)} \langle \mathbf{d} \rangle_\Gamma + \frac{K-2}{N-2}$,

ii) $-1 - \frac{N-K}{K-2} d_{v0} + \frac{(N-2)(K-1)}{K-2} \langle \mathbf{d} \rangle_\Lambda < (N-2) d_v < -\frac{N-K}{K-2} d_{v0} + \frac{(N-2)(K-1)}{K-2} \langle \mathbf{d} \rangle_\Lambda,\ \forall v \in \Lambda$,

iii) $-1 + d_{v0} + (N-1)\langle \mathbf{d} \rangle_\Gamma < (N-1) d_v < d_{v0} + (N-1)\langle \mathbf{d} \rangle_\Gamma,\ \forall v \in \Gamma$.

Proof. Let us assume without loss of generality that $\Lambda = \{2, 3, \cdots, K\}$. Then, $A = (a_{v,u})_{N\times N}$ where

$$a_{v,u} = \begin{cases} 0, & \text{if } \begin{cases} v = u & \text{or} \\ v = 1, 1 \le u \le K & \text{or} \\ 1 \le v \le K, u = 1 & \end{cases} \\ 1, & \text{otherwise.} \end{cases}.$$

The matrix A is invertible for all $N > 2$ and its inverse is given by the block matrix

$$A_N^{-1} = \begin{bmatrix} R_1 & R_2 & R_3 \\ R_2^\top & R_4 & R_5 \\ R_3^\top & R_5^\top & R_6 \end{bmatrix},$$

where

$$\begin{array}{l} R_1 = \frac{N-2}{(K-2)(N-K)} \\ R_2 = -\frac{1}{K-2} \mathbf{1}_{1\times(K-1)} \\ R_3 = \frac{1}{N-K} \mathbf{1}_{1\times(N-K)} \\ R_4 = \frac{1}{K-2} \mathbf{1}_{(K-1)\times(K-1)} - I_{(K-1)} \\ R_5 = \mathbf{0}_{(K-1)\times(N-K)} \\ R_6 = \frac{1}{N-K} \mathbf{1}_{(N-K)\times(N-K)} - I_{(N-K)} \end{array}.$$

In the above formulas, I_i is the i-dimensional identity matrix, and $\mathbf{0}_{i\times j}$ and $\mathbf{1}_{i\times j}$ are the the $i \times j$ matrices of all 0 and 1 entries, respectively.

By Equation (3.1), the steady state can be expressed as

$$\begin{array}{l} y_1^* = d_1 \frac{N-2}{K-2} - \frac{(N-2)(K-1)}{(K-2)} \langle \mathbf{d} \rangle_\Lambda + (N-1)\langle \mathbf{d} \rangle_\Gamma \\ y_v^* = -\frac{(N-K)d_1 - (N-2)(K-2)d_v + (N-2)(K-1)\langle \mathbf{d} \rangle_\Lambda}{(K-2)}, v \in \Lambda \\ y_u^* = d_1 - d_u + (N-1)\langle \mathbf{d} \rangle_\Gamma, u \in \Gamma. \end{array}$$

The result thus follows from the fact the $y_v^* \in (0,1)$, for all $v \in V$. $\square$

Remark 5.3. If $d_v = d \in (0,1), \forall v \in V$, then we can show that inequalities i), ii) and iii) hold. Then, from Theorem 5.2, we have that the internal steady state exists. This is in agreement with the conclusions obtained by Theorem 1 in [15].

5.2. **Simulation results on games with self-edges by removing links.** In the following we report some numerical results on the effect of link removal in networks with self edges

In order to study the link removal from networks including self edges we develop two experiments. In both experiments we deal with complete graphs of five vertices (players), each of them is evolving on the basis of different anti-coordination payoff matrices (games). In particular, in the first experiment we set $d_1 = 0.29$, $d_2 = 0.24$, $d_3 = 0.35$, $d_4 = 0.25$, $d_5 = 0.24$, $\beta_v = 2$ and $\gamma_v = 3$, $\forall v$. In the second experiment we set $d_1 = 0.5$, $d_2 = 0.56$, $d_3 = 0.61$, $d_4 = 0.57$, $d_5 = 0.62$, $\beta_v = 2$ and $\gamma_v = -3$, $\forall v$. The two configurations satisfy the hypotheses of Theorem 4.2, i.e., the internal steady state is feasible in both cases. In Figure 1, panels 1.a and 2.a display a simulation of the complete graph for the two cases. The simulations of Figure 1, panels 2.a and 2.b are obtained by removing one link from vertex 1 ($a_{1,2} = 0$), while panels 1.c and 2.c show the simulation after removing another link from the same vertex 1 ($a_{1,2} = a_{1,3} = 0$).

One can notice that in experiment 1 the link removal does not affect the feasibility of internal steady states. Indeed, although the value of the steady state is changing, it is still reached asymptotically by the numerical solutions of Equation 2.5. The vertex mostly affected by the link removal is vertex 1 (solid blue line in the left panels), as expected. On the contrary, in experiment 2 the internal steady state is destroyed by the link removal process. In this case, the asymptotic solution converges to a steady state where vertex 1 (solid blue line in the right panels) is vanishing. For the sake of clarity, we notice that in panels 2.b and 2.c the internal steady state is not feasible anymore since it is external to the simplex. Then, the fact that it is no longer approached asymptotically is not really due to its instability. The problem of stability of internal steady state is under investigation by the authors and will be tackled in future works.

Thereafter, we conducted another numerical experiment employing 168000 random graphs with 60 nodes. Starting from a complete graph of 60 nodes, we employed 3 different removal strategies to produce 1000 different graphs which average degree is $\overline{k} \in \{60, 59, 58, \ldots, 6, 5\}$. A comment concerning the different strategies for edge removal is in order. In the so-called

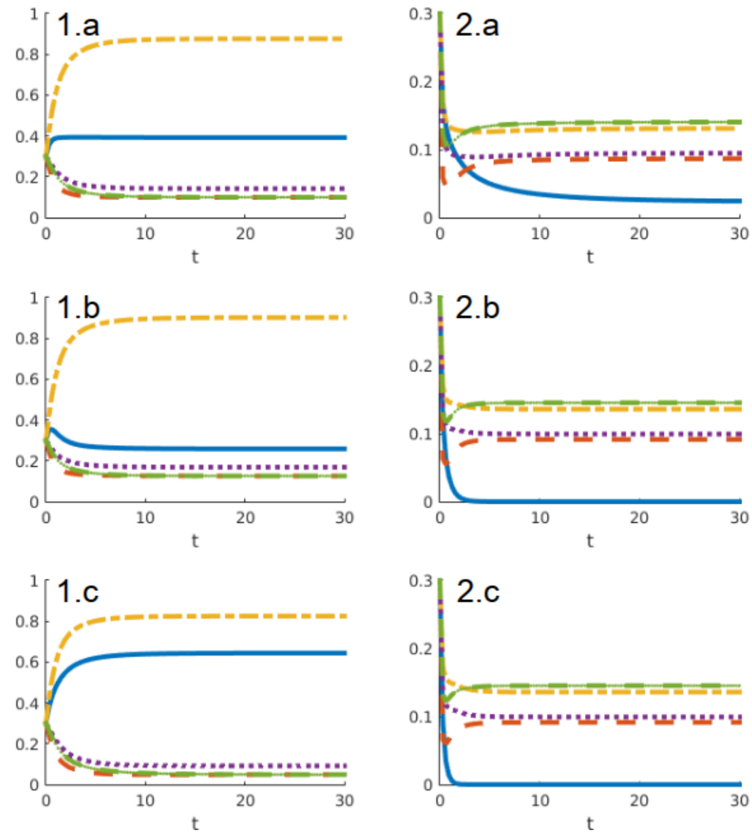

FIGURE 3. Simulations experiments for two different configurations of the game and the graph. The initial conditions for the two experiments are set equal to 0.3 $\forall v$.

random removal strategy, we started from a complete graph, and then we randomly removed 60 links, in order to obtain a new graph with average degree equal to 59. In general, starting from a graph with average degree $\overline{k}$, we removed 60 links in order to obtain a graph with average degree $\overline{k}-1$. The *random regular* removal strategy is similar to the random-removal approach, but we remove exactly one link for each node, in order to obtain at each step a random regular graph (i.e., all nodes have the same degree). Finally, the *Erdös-Rényi removal strategy* consisted of starting from an Erdös-Rényi graph sample with average degree $\overline{k}$, then removing a certain amount of links in order to obtain an Erdös-Rényi graph sample with average degree $\overline{k}$. An existing link remains with probability $\frac{\overline{k}-1}{\overline{k}}$, otherwise, it is removed. In this way we are able to build Erdös-Rényi graph samples using a removal process. For each node we fixed $\beta_v = \gamma_v = -30$. For this numerical example, we divide the nodes into six groups of 10 elements each. For each group, we choose the parameters

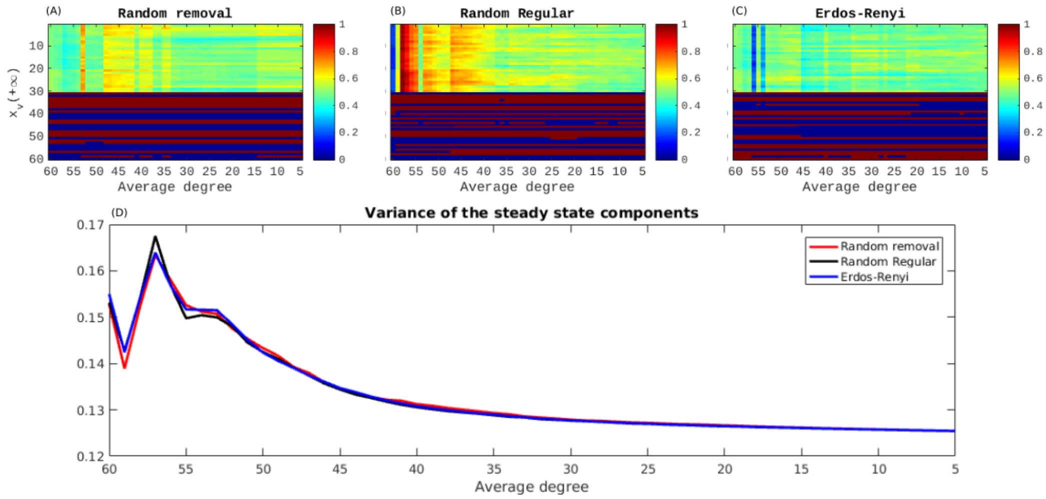

FIGURE 4. Subplots (A), (B) and (C) report as a function of the average degree of the network, the value at steady state reached by each of the 60 members of the considered population, for each link removal strategy. Subplot(D) shows the variance of the whole population steady states as a function of the average degree of the network for each link removal strategy.

$\sigma_{v,1}$ and $\sigma_{v,2}$ in the set $\{(1,1),(0.9,1),(1,0.9),(-1,-1),(-0.9,-1),(-1,-0.9)\}$. In this way, half on the nodes have a coordination game payoff matrix, while the other half play anti-coordination games. For each of the 168000 random graphs, a random initial condition has been created (i.e., $x_v(0)$ is a uniformly distributed random number in the set $(0,1)$). Thereafter, we let the system of ODE (2.5) evolves towards a steady state. An example of the reached steady states is depicted in the subplots (A), (B) and (C) of Figure (4), where a colored point represents the value of the v-th component of the steady state for a given graph, which average degree is reported in the abscissa. We can notice that, for more sparser graphs, the behavior of the bistable nodes (i.e., the first 30 nodes), becomes more regular, that is, it is easier for the whole population to reach similar steady state (i.e., consensus) as long as the neighbors size decrease. For supporting this claim, in Figure (4) (D) we report the variance of the steady steady state for each removal strategy (56000 graphs for each removal strategy). The variance decreases as the number of removed links increases.

6. Conclusions and Future Developments

In this paper we study the relationship between network topology and self loops in the Evolutionary Games Equation on Graphs. Specifically, we state some necessary results for the existence and feasibility of internal steady states. Necessary and sufficient conditions for the case of complete graphs have also been provided. Furthermore, numerical results have been presented when the graph is sampled from an Erdös-Rényi model with a large number of vertices. Then, we exploited the influence of varying the connectivity of the network by removing iteratively the edges of a single node. This link removal process has been studied starting from a complete network without and with self loops. The former is developed through theoretical results, whereas the latter through numerical simulations. The presence of self loops introduces feedbacks in the model equation. Thus, a natural continuation of this research, which is presently under investigation by the authors, concerns the stability analysis of internal steady states. This in turn, is very relevant for control and consensus problems on networks.

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Conflict of interest

The authors don't have any conflict of interest.

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