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DOCTORAL THESIS

Logarithmic Modules for Chiral Differential Operators of Nilmanifolds

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"No one shall expel us from the paradise that Cantor has created."

David Hilbert

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Abstract

Instituto Nacional de Matematica Pura e Aplicada

Doctor of Philosophy

Logarithmic Modules for Chiral Differential Operators of Nilmanifolds

by Bely R MORALES

We describe explicitly the vertex algebra of (twisted) chiral differential operators on certain nilmanifolds and construct their logarithmic modules. This is achieved by generalizing the construction of vertex operators in terms of exponentiated scalar fields to Jacobi theta functions naturally appearing in these nilmanifolds. This provides with a non-trivial example of logarithmic vertex algebra modules, a theory recently developed by Bakalov.

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Contents

Chapter 1

Introduction

To any smooth manifold M (and a choice of $\omega \in H^3(M, \mathbb{Z})$) satisfying some mild topological properties (the first Pontryagin class vanishes), Malikov, Schechtmann and Vaintrob [13] and independently Beilinson and Drinfeld [5] attach a sheaf of vertex algebras $\mathcal{O}^{ch}_M$ called the sheaf of *chiral differential operators*. In the simplest case when $\omega = 0$, locally on a coordinate patch U with coordinates $\{x^i\}_{i=1,\ldots,\dim M}$, the sections $\mathcal{O}^{ch}_M(U)$ form an $\dim M$-dimensional $\beta\gamma$-system, i.e., the vertex algebra generated by fields $\{\beta_i, \gamma^i\}_{i=1,\ldots,\dim M}$ satisfying the OPE

$$\beta_i(z) \cdot \gamma^j(w) \sim \frac{\delta_{i,j}}{z-w}, \qquad \beta_i(z) \cdot \beta_j(w) \sim \gamma^i(z) \cdot \gamma^j(w) \sim 0.$$

On intersections of coordinate patches, the fields γ^i change as coordinates do while the fields β_i change as vector fields.

This construction works in the algebraic, holomorphic, real-analytic or C^∞-setting. In this work we will be mainly concerned with the C^∞-setting. Little is known about the structure of the global sections $V_M := \Gamma(M, \mathcal{O}^{ch}_M)$ of this sheaf. Only recently in the context of supermanifolds, Bailin Song proved that, in the holomorphic setting, the vertex algebra V_M coincides with the simple small $N = 4$ super-vertex algebra at central charge $c = 6$ when $M = T^*[1]N$ is the shifted cotangent bundle to a $K3$ surface N [14]. In this thesis we will describe explicitly V_M when M is the Heisenberg 3-dimensional nilmanifold.

The vertex algebra V_M (or rather its super-extension) is expected to play a central role in Mirror-Symmetry. In particular, for M and N a mirror pair of Calabi-Yau manifolds, one expects a natural isomorphism $V_M \simeq V_N$ of vertex algebras. Their characters are known to be equal by work of Borisov and Libgober [6].

If M is non-simply-connected, a subtle phenomenon arises as one needs to consider non-trivial *windings*. Aldi and Heluani showed in [1], based on ideas of C. Hull [11], that when (M, ω) is the three torus $\mathbb{T}^3$ with its generator of $H^3(\mathbb{T}^3, \mathbb{Z}) \simeq \mathbb{Z}$, or if M is its mirror dual: the Heisenberg 3-dimensional nilmanifold N with vanishing ω, the vertex algebra V_M can be naturally represented in a Hilbert space. This Hilbert space is associated to a 6-dimensional nilmanifold Y which fibers over both $\mathbb{T}^3$ and N.

For certain M, we can describe explicitly V_M in terms of a larger manifold fibering over M. Suppose that the ω-twisted Courant algebroid $TM \oplus T^*M$ of M is parallellizable. That is, there exists a global frame of vector fields $\{\beta_i\}$ and dual basis of differential forms $\{\alpha^i\}$ such that $[\beta_i, \beta_j]_{Lie} + \iota_{\beta_i}\iota_{\beta_j}\omega$ is a constant combination of β_i's and α^i's, and $\mathrm{Lie}_{\beta_i}\alpha^j$ is a constant linear combination of the α^i's. In other words, there exists a Lie algebra $\mathfrak{g}$, $\dim \mathfrak{g} = 2\dim M$, with a symmetric invariant bilinear pairing of signature $(\dim M, \dim M)$, and a trivialization $TM \oplus T^*M \simeq \mathfrak{g} \times M$. The Courant-Doffman bracket of the frame $\{\beta_i, \alpha^i\}$ is given by the bracket in $\mathfrak{g}$.

The approach, following ideas of C. Hull [11] and exploited for example in [7] is that one may try to find a manifold N with the property that $\dim N = 2\dim M$. It fibers over M, $N \twoheadrightarrow M$ and its parallelizable, such that $TN \simeq \mathfrak{g} \times N$, that is, the *Lie bracket* of vectors in a frame is identified with the Lie bracket of $\mathfrak{g}$. In this situation we consider the $\mathfrak{g}$-module $C^\infty(N)$, the Kac-Moody affinization $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ and its induced module from $C^\infty(N)$. We have an embedding $C^\infty(M) \hookrightarrow C^\infty(N)$ given by pullback, inducing the sequence of embeddings:

$$V^1(\mathfrak{g}) \subset \mathbf{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} C^\infty(M) \subset \mathcal{H} := \mathbf{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} C^\infty(N),$$

where the first module is induced from the constant function 1, coincides with the vacuum module for the algebra $\hat{\mathfrak{g}}$ and is known to be a vertex algebra. The second module coincides with the vertex algebra V_M and is here represented as a subspace of $\mathcal{H}$.

Given the situation above one would expect $\mathcal{H}$ to be naturally a module over the vertex algebra V_M. However, as we will see in this thesis, logarithms in the fields might be unavoidable. It turns out that this situation provides a natural family of examples of *logarithmic modules* over vertex algebras, as defined and studied by Bakalov in [4].

We apply the above remarks to describe $V_{\mathbb{T}^3}, V_N$ and $\mathcal{H}$ as vector spaces as follows. Let $\mathfrak{g}$ be the two step nilpotent Lie algebra of rank 3, that is a central extension of $\mathbb{R}^3$ by $\mathbb{R}^3$. It has a basis $\{\beta_i, \alpha^i\}_{i=1,2,3}$ with only non-vanishing commutators

$$[\beta_i, \beta_j] = \frac{1}{2}\sum_k \epsilon_{ijk}\alpha^k.$$

Where ϵ is the totally antisymmetric tensor. The Lie algebra $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form $\langle , \rangle$ with non-vanishing pairings

$$\left\langle \beta_i, \alpha^j \right\rangle = \delta_i^j.$$

There arises its Kac-Moody affinization $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, with non-trivial brackets

$$[a_m, b_n] = [a,b]_{m+n} + m\langle a, b\rangle \delta_{m,-n} K, \qquad a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z},$$

where $a_n = a \otimes t^n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

Let G be the unipotent Lie group with Lie algebra $\mathfrak{g}$. It is also an extension of $\mathbb{R}^3$ by $\mathbb{R}^3$. Let $\Gamma \subset G$ be the subgroup generated by a basis of the quotient $\mathbb{R}^3$ of G. It is a cocompact subgroup, the quotient $Y = G/\Gamma$ is a six-dimensional nilmanifold which is a non-trivial $\mathbb{T}^3$-fibration over $\mathbb{T}^3$. In fact we have the central extensions:

$$\begin{aligned} 0 \to \mathbb{R}^3 \to G \to \mathbb{R}^3 \to 0 \\ 0 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}^3 \to 0 \end{aligned} \tag{1.1}$$

Showing Y as a $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ fibration over $\mathbb{T}^3$.

The Lie group G acts on $L^2(Y)$ and its Lie algebra $\mathfrak{g}$ acts on $C^\infty(Y)$. We extend the $\mathfrak{g} = \mathfrak{g} \otimes t^0 \subset \hat{\mathfrak{g}}$ action on $C^\infty(Y)$ into a representation of $\hat{\mathfrak{g}}_+ := \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ by letting K act by 1 and a_n act by 0 if $n \geq 0$. The vector space $\mathcal{H}$ is the corresponding $\hat{\mathfrak{g}}$-induced module

$$\mathcal{H} := \mathbf{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} C^\infty(Y).$$

Properly speaking $\mathcal{H}$ is its L^2 completion, but we will not care about unitarity properties in this thesis.

Notice that the constant function 1 defines an embedding $V^1(\mathfrak{g}) \hookrightarrow \mathcal{H}$ of the vacuum representation of $\hat{\mathfrak{g}}$ into $\mathcal{H}$. As it is well known $V^1(\mathfrak{g})$ is a vertex algebra and this embedding makes $\mathcal{H}$ into a $V^1(\mathfrak{g})$-module. Consider now the three Torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, the morphism $Y \twoheadrightarrow \mathbb{T}^3$ provides an embedding $C^\infty(\mathbb{T}^3) \hookrightarrow C^\infty(Y)$. It is easy to see that this is an embedding of $\mathfrak{g}$-modules. The induced $\hat{\mathfrak{g}}$-module coincides with $V_{\mathbb{T}^3}$, that is

$$V^1(\mathfrak{g}) \subset V_{\mathbb{T}^3} \simeq \mathbf{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} C^\infty(\mathbb{T}^3) \subset \mathcal{H}.$$

However a little work is required to check that $\mathcal{H}$ is a vertex algebra module over $V_{\mathbb{T}^3}$. The fields associated to vectors $f \in C^\infty(\mathbb{T}^3)$ involve explicitly logarithms of the formal variable z. The situation is very similar to that of the lattice vertex algebra where the logarithms only appear exponentiated, hence appealing to the identity $\exp(\log(z)) = z$ one can get rid of them.

The situation with the Heisenberg nilmanifold is a quite different. Any line $\mathcal{L} \subset \mathbb{R}^3$ determines a central character $\chi_{\mathcal{L}} : Z(G) \simeq \mathbb{R}^3 \to \mathbb{R}$ of G. We can view $\mathcal{L}$ as a one dimensional subgroup of G (in the quotient $\mathbb{R}^3$). The subgroup $K = \ker \chi_{\mathcal{L}} \oplus \mathcal{L} \subset G$ is normal and its cokernel

$$0 \to K \to G \to \mathbf{Heis}(\mathbb{R}) \to 0,$$

is the 3-dimensional real Heisenberg group. If the line $\mathcal{L}$ is generated by an element of Γ, this sequence is compatible with Γ in the sense that there exists an analogous sequence

$$0 \to K_\Gamma \to \Gamma \to \mathbf{Heis}(\mathbb{Z}) \to 0,$$

whose quotient is now the integer Heisenberg group. This construction shows Y as a fibration over the Heisenberg nilmanifold $N = \mathbf{Heis}(\mathbb{R})/\mathbf{Heis}(\mathbb{Z})$ (it is not hard to see that the fiber is also a three torus $\mathbb{T}^3$).

We obtain thus an embedding $C^\infty(N) \hookrightarrow C^\infty(Y)$. As before it is easy to see that this is an embedding of $\mathfrak{g}$-modules. It turns out that the induced $\hat{\mathfrak{g}}$-module is also isomorphic to the vertex algebra V_N:

$$V^1(\mathfrak{g}) \subset V_N \simeq \mathbf{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} C^\infty(N) \subset \mathcal{H}.$$

This time however, logarithms are unavoidable. In fact, the fields associated to vectors of V_N have explicit logarithms of z on them when acting on $\mathcal{H}$. It is only by restricting to $V_N \subset \mathcal{H}$ that they disappear by use of the same identity $\exp(\log(z)) = z$. However, when analyzing the action of V_N on $\mathcal{H}$ these logarithms remain, making $\mathcal{H}$ a *logarithmic module* over V_N.

In order to describe explicitly the fields of V_N and their action on $\mathcal{H}$, we need to use certain results from harmonic analysis. In particular, since the representation of G in $L^2(Y)$ is unitary, it decomposes into direct sum of irreducible representations. These representations turn out to be induced from unitary irreducible representations of the real Heisenberg group, and by the Stone-von Neumann theorem they are unique once we choose a central character. One can choose explicit cyclic vectors for these representations: they are given by appropriate constant (the vacuum vector), exponential (functions from $\mathbb{T}^3$), or Jacobi theta functions. We construct vertex operators associated to these Jacobi theta functions in complete analogy as how one constructs vertex operators associated to exponential functions. These operators however, carry an explicit dependency on the logarithm of the formal variable. We show by explicit computation the locality and translation invariant property as well as the axioms for a logarithmic module as in [4].

In fact we construct vertex operators associated to any vector in $\mathcal{H}$. One would expect to have a vertex-algebra-like structure on $\mathcal{H}$ with these vertex operators. As we have already pointed out, logarithms are unavoidable. In latter years there has been an effort to include logarithmic singularities in the OPE of fields, B. Bakalov has defined *logarithmic vertex algebras* to allow for these singularities. However, in our situation, the singularities are *dilogarithmic*. If we replace $C^\infty(Y)$ in the definition of $\mathcal{H}$ by functions on the universal cover G we show that by making use of an analytic identity satisfied by the dilogarithm (analogous to that of $\exp(\log(z)) = z$) we can prove a version of locality for these vertex operators.

The structure of this thesis is as follows. In the first chapter we provide a brief summary of the theory of quantum fields, vertex algebras, logarithmic vertex algebras and logarithmic modules; in this chapter are also stated several equivalent definitions of vertex algebras, logarithmic modules and some classical results on vertex algebras as well. The second chapter is dedicated completely to study functions on the double twisted torus, initially we only consider the induced module to the whole Kac-Moody algebra from the the polynomial functions submodule and we prove that it has the structure of logarithmic module, moreover we also try to endow it with a more complicated structure (logarithmic vertex algebra) but some problems arise, finally we managed to prove that the induced module to the whole Kac-Moody algebra can be endowed with the structure of logarithmic module over some carefully chosen submodules, specifically we prove the following theorems:

Theorem 1.1 *$\mathcal{H}$ has the structure of $V_{\mathbb{T}^3}$-module.*

Theorem 1.2 *$\mathcal{H}$ has the structure of logarithmic V_N-module.*

Chapter 2

Vertex Algebras and Logarithms

2.1 Vertex Algebras

Let V be a vector space over $\mathbb{C}$, the quantum fields on V are defined as $\mathbf{Field}(V) = \mathbf{Hom}(V, V((z)))$ where $V((z)) = V[\![z]\!][z^{-1}] = V \otimes \mathbb{C}[\![z]\!][z^{-1}]$ denotes the space of Laurent series on V; i.e. a field on V is a formal series $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-1-n}$ where $a_{(n)} \in \mathbf{End}(V)$ and for each $v \in V$, $a_{(n)}v = 0$ for n large enough.
Two quantum fields $a(z_1), b(z_2)$ are called local if there is $N \in \mathbb{N}$ such that

$$(z_1 - z_2)^N [a(z_1), b(z_2)] = 0. \tag{2.1}$$

The n-product of two local fields is defined as

$$\left(a(z_1)_{(n)} b(z_2)\right)(z)v = \partial_{z_1}^{(N-1-n)} \left((z_1 - z_2)^N a(z_1) b(z_2) v\right)\Big|_{z_1=z_2=z} \tag{2.2}$$

for $v \in V, n < N$, and $\left(a(z_1)_{(n)} b(z_2)\right)(z)v = 0$ if $n \geq N$.

Notation: Given and operator A we will use the notation $A^{(k)} = \frac{A^k}{k!}$.

It can be proved that the n-product of two local field defined by 2.2 is equivalent to

$$\left(a(z_1)_{(n)} b(z_2)\right)(z)v = \mathbf{res}_{z_1} \left(\mathrm{i}_{z_1>z}(z_1 - z)^n a(z_1) b(z) v - \mathrm{i}_{z>z_1}(z_1 - z)^n b(z) a(z_1) v\right).$$

Given a field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-1-n}$ the *annihilation* and *creation* parts of $a(z)$ are defined respectively as:

$$a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-1-n},$$

$$a(z)_+ = \sum_{n\leq -1} a_{(n)} z^{-1-n};$$

The normally ordered product of two fields $a(z_1), b(z_2)$ is defined by

$$: a(z_1) b(z_2) : \; = a(z_1)_+ b(z_2) + b(z_2) a(z_1)_-.$$

Definition 2.1.1 *[9] A* vertex algebra *is the data of a vector space V called space of states, a distinguished vector $\mathbb{1} \in V$ called vacuum vector, an endomorphism $T \in$* $\mathbf{End}(V)$ *called translation and a set of fields $\mathcal{F} \subseteq$* $\mathbf{Field}(V)$ *such that:*

(vacuum axiom) $T\mathbb{1} = 0$,

(translation invariance) $[T, a(z)] = \partial_z a(z)$ *for every* $a(z) \in \mathcal{F}$,

(locality axiom) *All fields in $\mathcal{F}$ are pairwise local,*

(completeness axiom) $V = Span\left\{a^1_{(n_1)}a^2_{(n_2)}\dots a^k_{(n_k)}\mathbb{1}\right\}$.

Following [8] let $\mathfrak{g}$ be a Lie algebra with a non degenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$, for instance, every finite dimensional semisimple Lie algebra has such bilinear form. The Kac-Moody affine Lie algebra $\widehat{\mathfrak{g}}$ is defined as vector space by $\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$ and the following commutator:

$$[at^m, bt^n] = [a,b]\,t^{m+n} + m\,\langle a,b\rangle\,\delta_{m,-n}K,$$

where K is central. Let us introduce the notation $a_n = at^n$.

Consider the subalgebra of the Kac-Moody affine algebra given by $\mathfrak{g}[t]\oplus\mathbb{C}K$ and consider the one dimensional representation $\mathbb{C}\mathbb{1}$, where K acts by multiplication by a given scalar k and the elements of $\mathfrak{g}[t]$ acts by zero.

Proposition 2.1.1 *The $\widehat{\mathfrak{g}}$ module*

$$V^k(\mathfrak{g}) = \mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\mathbb{1}\simeq\mathcal{U}\left(\widehat{\mathfrak{g}}\right)\otimes_{\mathcal{U}(\mathfrak{g}[t]\oplus\mathbb{C}K)}\mathbb{C}\mathbb{1}$$

has a vertex algebra structure.

Proof:
To prove $V^k(\mathfrak{g})$ is a vertex algebra we must define a set of fields, a vacuum vector and a translation endomorphism. Consider the following set of fields $\mathcal{F} = \left\{a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-1-n}, a\in\mathfrak{g}\right\}$, the module $V^k(\mathfrak{g})$ already has a distinguished vector $1\otimes\mathbb{1}$, still denoted by $\mathbb{1}$.

Now because of the Poincare-Birkhoff-Witt theorem (PBW) and because the a_n for $n\geq 0$ act by zero, the $V^k(\mathfrak{g})$ is spanned as vector space by element of the form $a^1_{-n_{k_1}}\cdots a^r_{-n_{k_r}}\mathbb{1}$. From this last observation it is trivial that the completeness axiom is satisfied.

Now define $T : V^k(\mathfrak{g})\to V^k(\mathfrak{g})$ recursively as $T\mathbb{1} = 0$ and $[T,a_n] = -na_{n-1}$, from this definition is obvious that the translation axiom holds. Finally the last axiom to prove is the locality but if $a(z_1) = \sum_{n\in\mathbb{Z}} a_n z_1^{-1-n}$ and $b(z_2) = \sum_{n\in\mathbb{Z}} b_n z_2^{-1-n}$, then

$$\begin{aligned}
[a(z_1), b(z_2)] &= \sum_{m,n} z_1^{-1-m} z_2^{-1-n}\,[a_m, b_n] \\
&= \sum_{m,n} z_1^{-1-m} z_2^{-1-n}\,[a,b]_{m+n} + \sum_{m,n} z_1^{-1-m} z_2^{-1-n} m\,\langle a,b\rangle\,\delta_{m,-n}K \\
&= \sum_{m,n} z_1^{-1-m} z_2^{m}\left([a,b]_{m+n}\, z_2^{-1-(m+n)}\right) + \sum_{m} z_1^{-1-m} w^{-1+m} m\,\langle a,b\rangle\,K \\
&= \delta(z_1,z_2)\,[a,b]\,(z_2) + \langle a,b\rangle\,K\partial_{z_2}\delta(z_1,z_2),
\end{aligned}$$

where $\delta(z_1,z_2) = \sum_{m\in\mathbb{Z}} z_1^{-1-m} z_2^m$, and it is straightforward that $(z_1-z_2)\delta(z,w) = 0$ and $(z_1-z_2)^2\partial_w\delta(z_1,z_2) = 0$, then $(z_1-z_2)^2\,[a(z_1), b(z_2)] = 0$. □

This vertex algebra $V^k(\mathfrak{g})$ is called *the universal affine vertex algebra of level k* or *the Kac-Moody vertex algebra of level k.*

Proposition 2.1.2 (Dong's Lemma) *[12] Let $a(z), b(z), c(z)$ be pairwise local fields on V then $a(z)_{(n)}b(z), \partial_z a(z), b(z), c(z)$ $n\in\mathbb{Z}$ are also pairwise local fields.*

The set of fields in the definition of vertex algebra could be enlarged to a minimal subspace of **Field** (V) containing $\mathbf{id}_V$, closed by ∂_z and all n-products, we will still denote it by $\mathcal{F}$ then the main result of vertex algebras [9] is $V\simeq\mathcal{F}$, known as *state fields correspondence*. Then it makes sense to state a more practical definition of vertex algebra.

Definition 2.1.2 *A vertex algebra is a vector space V, a distinguished vector $\mathbb{1} \in V$ and linear map*

$$Y_z : V \to \textbf{Field}\,(V)\,, \quad v \mapsto Y(v,z),$$

such that the following axioms are satisfied:

(vacuum axiom) $Y(\mathbb{1},z) = \textbf{id}$, $Y(v,z)\mathbb{1} \in V[\![z]\!]$, $Y(v,z)\mathbb{1}|_{z=0} = v$;

(translation invariance) $[T, Y(v,z)] = \partial_z Y(v,z)$;

(locality axiom) *For every $v_1, v_2 \in V$ there is N large enough such that*

$$(z_1 - z_2)^N \left[Y_{z_1}(v_1), Y_{z_2}(v_2)\right] = 0.$$

*Where the translation endomorphism $T \in$ **End** (V) is defined as $Tv = \partial_z Y(v,z)\mathbb{1}|_{z=0}$.*

For example, in the universal affine vertex algebra the map $Y_z : V^k(\mathfrak{g}) \to \textbf{Field}\left(V^k(\mathfrak{g})\right)$ is defined as

$$Y(a_{-1}\mathbb{1}, z) = \sum_{n\in\mathbb{Z}} a_n z^{-1-n}, \quad a \in \mathfrak{g},$$

and in general for the generators of $V^k(\mathfrak{g})$

$$Y(a^1_{-n_{k_1}} \cdots a^r_{-n_{k_r}}\mathbb{1}, z) = \frac{:\partial_z^{n_{k_1}-1} Y(a^1_{-1}\mathbb{1}, z) \cdots \partial_z^{n_{k_r}-1} Y(a^r_{-1}\mathbb{1}, z):}{(n_{k_1}-1)! \cdots (n_{k_r}-1)!}. \tag{2.3}$$

Definition 2.1.3 *A module over a vertex algebra V is a vector space W equipped with a linear map $Y_z : V \to$ **Field** (W) such that:*

- $Y(\mathbb{1}) = \textbf{id}$
- $Y(a_{(n)}b) = Y(a)_{(n)}Y(b)$ *for all $n \in \mathbb{Z}$.*

There is yet another useful definition for vertex algebras using *Lie Conformal Algebras*.

Definition 2.1.4 *A Lie conformal algebra is a $\mathbb{C}[\partial]$-module R equipped with a $\mathbb{C}$-linear map (called lambda bracket) $[\cdot_\lambda \cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes R$ such that:*

a. $[\partial a_\lambda b] = -\lambda\, [a_\lambda b]$,

b. $\partial\, [a_\lambda b] = [\partial a_\lambda b] + [a_\lambda \partial b]$,

c. $[b_\lambda a] = -\, [a_{-\lambda-\partial} b]$,

d. $\left[[a_\lambda b]_{\lambda+\mu}\, c\right] = \left[a_\lambda \left[b_\mu c\right]\right] - \left[b_\lambda \left[a_\mu c\right]\right]$.

It is convenient to express $[a_\lambda b] = \sum_{n\geq 0}(a_{(n)}b)\frac{\lambda^n}{n!}$ so the expression $[a_{-\lambda-\partial}b]$ is interpreted as $\sum_{n\geq 0}(a_{(n)}b)\frac{(-\lambda-\partial)^n}{n!}$, from this it becomes clear that the definition of Lie conformal algebra can be translated into the language of n-products ($n \geq 0$). Then as proven in [9] the following definition is equivalent to the previous definitions of vertex algebras:

Definition 2.1.5 *A vertex algebra is the data of a vector space V, a distinguished vector $\mathbb{1} \in V$, an endomorphism $\partial \in$ **End** (V), a linear map $[\cdot_\lambda \cdot] : V \otimes V \to \mathbb{C}[\lambda] \otimes V$ and a linear map $:: : V \otimes V \to V$ such that*

a. $(V, \partial, [\cdot_\lambda \cdot])$ *is a Lie conformal algebra,*

b. $(V, \mathbb{1}, \partial, ::)$ *is a unital differential algebra satisfying*

$$:ab: - :ba: := \int_{\partial}^{0} [a_\lambda b]\, d\lambda$$

and

$$::ab:c: - :a:bc::=: (\int_0^{\partial} d\lambda a)\,[b_\lambda c] : + : (\int_0^{\partial} d\lambda c)\,[a_\lambda c]) :,$$

c. *The non-commutative Wick formula holds*

$$[a_\lambda :bc:] =: [a_\lambda b]c : + : b\,[a_\lambda c] : + \int_0^{\lambda} \left[[a_\lambda b]_\mu c \right] d\mu.$$

Remark: The integrals of the form $:(\int_0^{\partial} d\lambda a)\,[b_\lambda c]:$ are interpreted in the following sense: expand the lambda bracket, such that the powers of λ fall under the integral sign and then compute the formal integral, in this case would be

$$:(\int_0^{\partial} d\lambda a)\,[b_\lambda c] :=: (\int_0^{\partial} \sum_{n\geq 0} \frac{\lambda^n}{n!} d\lambda a) \left(b_{(n)}c\right) :=: \sum_{n\geq 0} \partial^{(n+1)} a \left(b_{(n)}c\right) :=: \sum_{n\geq 0} a_{(-n-2)} \left(b_{(n)}c\right) : .$$

Trivially every vertex algebra is a Lie conformal algebra, the forgetful functor has a left adjoint $R \mapsto V(R)$ which assigns to every Lie conformal algebra its universal enveloping vertex algebra [9], i.e.,

$$\mathbf{Hom}_{LCA}\,(R, V) \simeq \mathbf{Hom}_{VA}\,(V(R), V)\,.$$

Similarly to the PBW theorem which explicitly describes the structure of the universal enveloping algebra of a Lie algebra there is an analogous result for the universal enveloping vertex algebra of a Lie conformal algebra.

Proposition 2.1.3 *Any ordered basis of R freely generates $V(R)$.*

For a more detailed explanation see [9, section 1.7].

2.2 Logarithmic Fields and Logarithmic Modules

It is convenient to extend the notion of quantum fields defined before to include logarithms, i.e., it is often needed to have the notion of logarithm in the formal theory of fields, in this section the basic results of logarithmic vector field will be stated following the ideas developed by Bojko Bakalov in [4]. Let's start by introducing the formal variable $\log(z)$ which intuitively can be thought as the logarithm of z. Since there are now two formal variables we have two possible derivations

$$D_z = \partial_z + z^{-1}\partial_{\log(z)}, \qquad D_{\log(z)} = z\partial_z + \partial_{\log(z)}.$$

Notice that here we are using the derivatives D_z and $D_{\log(z)}$ instead of ∂_z and $\partial_{\log(z)}$ because the former derivations carry formally the data coded in the analytic equation "$\partial_z \log(z) = \frac{1}{z}$" while the latter derivations do not.

Let W be a vector space over $\mathbb{C}$, let $\alpha \in \mathbb{C}/\mathbb{Z}$ and define

$$\mathbf{LField}_\alpha(W) = \mathbf{Hom}\left(W, W[\log(z)][\![z]\!]z^{-\alpha}\right),$$

the space of logarithmic quantum fields on W is defined to be

$$\mathbf{LField}\,(W) = \bigoplus_{\alpha \in \mathbb{C}/\mathbb{Z}} \mathbf{LField}_{\alpha}(W).$$

Notation: The logarithmic fields will be denoted as $a(z)$ instead of $a(\log(z), z)$ when no confusion arise.

Definition 2.2.1 *Two logarithmic fields $a(z_1)$, $b(z_2)$ are local if for $N >> 0$ holds:*

$$(z_1 - z_2)^N [a(z_1), b(z_2)] = 0. \tag{2.4}$$

Definition 2.2.2 *The n-product of two local logarithmic fields $a(z_1)$ and $b(z_2)$ is defined as*

$$\left(a(z_1)_{(n)} b(z_2)\right)(z)w = D_{z_1}^{(N-n-1)} \left((z_1 - z_2)^N a(z_1) b(z_2) w\right)\Big|_{z_1=z_2=z} \tag{2.5}$$

for $w \in W$ and $n < N$. For $n \geq N$ the n-product is defined by $\left(a(z_1)_{(n)} b(z_2)\right) = 0$.

It is easy to derive the following properties from the Leibniz rule

$$\begin{aligned}
(D_z a)_{(n)} b &= -n a_{(n-1)} b, &(2.6)\\
D_z \left(a_{(n)} b\right) &= (D_z a)_{(n)} b + a_{(n)} (D_z b), &(2.7)\\
\left(\partial_{\log(z)} a_{(n)} b\right) &= \left(\partial_{\log(z)} a\right)_{(n)} b + a_{(n)} \left(\partial_{\log(z)} b\right). &(2.8)
\end{aligned}$$

Once again there is a Dong's Lemma for logarithmic fields:

Proposition 2.2.1 *Let $a(z), b(z), c(z)$ be pairwise local logarithmic fields then*

a. *$a(z)_{(n)} b(z)$ and $c(z)$ are local fields for all $n \in \mathbb{Z}$,*

b. *$D_z a(z)$, $b(z)$ and $D_{\log(z)} a(z)$ are pairwise local.*

Proof:
The part **a** is proven in [4], for the part **b** just notice that $D_z a(z) = a(z)_{(-2)}\mathbf{id}$, $D_{\log(z)} = z D_z$ then use part **a**. □

In order to define the normally ordered product for logarithmic fields some extra step is required, for $\alpha \in \mathbb{C}/\mathbb{Z}$ select a representative α_0 such that $-1 < Re(\alpha_0) \leq 0$ then any element $a(z) \in \mathbf{LField}_{\alpha}(W)$ can be uniquely expressed as

$$a(z) = \sum_{n \in \mathbb{Z}} a_n \left(\log(z)\right) z^{-n-\alpha_0},$$

where for every $w \in W$ holds $a_n(\log(z))\, w = 0$ for $n >> 0$. The annihilation and creation parts of $a(z)$ are defined:

$$a(z)_- = \sum_{n \geq 1} a_n \left(\log(z)\right) z^{-n-\alpha_0},$$

$$a(z)_+ = \sum_{n \leq 0} a_n \left(\log(z)\right) z^{-n-\alpha_0},$$

and this concepts can be extended linearly to **LField** (W); then the normally ordered product of logarithmic fields is defined by the usual formula

$$:a(z_1)b(z_2): \; = a(z_1)_+ b(z_2) + b(z_2)a(z_1)_-.$$

The *propagator* of two logarithmic fields $a(z_1)$, $b(z_2)$ is defined as

$$P(a,b;z_1,z_2) = [a(z_1)_-, b(z_2)] = a(z_1)b(z_2) - :a(z_1)b(z_2):.$$

The propagator can be used to compute the n-products [4]:

Proposition 2.2.2 *If $a(z_1)$, $b(z_2)$ are local logarithmic fields then the n-product for $n \geq 0$ can be computed by the formula*

$$\left(a(z_1)_{(n)}b(z_2)\right)(z)w = D_{z_1}^{(N-n-1)}\left((z_1-z_2)^N P(a,b;z_1,z_2)w\right)\Big|_{z_1=z_2=z}$$

where N is large enough such that the equation 2.4 holds and $n < N$.

Definition 2.2.3 *A logarithmic module over a vertex algebra V is a vector space W equipped with a linear map $Y_z : V \to$ **LField** (W) such that:*

- $Y(\mathbb{1}) = \mathbf{id}$
- $Y(a_{(n)}b) = Y(a)_{(n)}Y(b)$ *for all $n \in \mathbb{Z}$.*

Moreover, if V is a vertex algebra equipped with an automorphism φ and W is a logarithmic module over V such that $Y(\varphi a) = e^{2\pi i D_{\log(z)}}Y(a)$ holds for every $a \in V$, then W is called a *φ-twisted logarithmic module* [4].

Let W be a vector space and let $\mathcal{W} \subseteq$ **LField** (W) be a collection of logarithmic fields which are pairwise local, denote by $\overline{\mathcal{W}}$ the smallest $\mathbb{C}[D_{\log(z)}]$ submodule of **LField** (W) containing $\mathcal{W} \cup \{\mathbf{id}\}$ and closed under n-products; then, because of Proposition 2.2.1, $\overline{\mathcal{W}}$ is again a collection of pairwise local logarithmic fields.

Theorem 2.1 (Bakalov) *Let W be a vector space and $\overline{\mathcal{W}}$ be defined as above. Then $\overline{\mathcal{W}}$ with the n-product of logarithmic fields has the structure of vertex algebra where the vacuum vector is* **id** *and the translation operator is D_z.*

From this it becomes clear that W is a logarithmic module over $\overline{\mathcal{W}}$, just take the map $Y : \overline{\mathcal{W}} \to$ **LField** (W) to be the inclusion map; moreover, it is $e^{2\pi i D_{\log(z)}}$-twisted module.

Corollary 2.2.1 *Let V be a vertex algebra and W a vector space, then giving a logarithmic V-module structure on W is equivalent to give a vertex algebra morphism $V \to \overline{\mathcal{W}}$ for a local collection of logarithmic fields $\mathcal{W} \subseteq$ **LField** (W), moreover, if V is equipped with an automorphism φ then the module will be twisted if and only the associated vertex algebra morphism transforms φ into $e^{2\pi i D_{\log(z)}}$.*

2.3 Logarithmic Vertex Algebras and special functions

It is convenient now to study a new type algebra structure similar to vertex algebras involving the logarithmic fields, it is indeed the theory of *logarithmic vertex algebras* developed by Bojko Bakalov [3].

Definition 2.3.1 *A logarithmic vertex algebra is a vector space V (space of states), a distinguished vector $\mathbb{1} \in V$, linear map*

$$Y_z : V \to \textbf{LField}\,(V)\,, \quad v \mapsto Y(v,z),$$

and a locally nilpotent operator $\mathcal{N} \in \textbf{End}\,(V \otimes V)$ such that the following axioms are satisfied:

(vacuum axiom) $Y(\mathbb{1},z) = \textbf{id}$, $Y(v,z)\mathbb{1} \in V[\![z]\!]$, $Y(v,z)\mathbb{1}|_{z=0} = v$;

(translation invariance) $[T, Y(v,z)] = D_z Y(v,z)$;

(locality axiom) *For every $v_1, v_2 \in V$ there is N large enough such that*

$$(z_1-z_2)^N \mu\,(Y_{z_1} \otimes Y_{z_2})\, e^{\log(z_1-z_2)\mathcal{N}}\,(v_1 \otimes v_2) = (z_1-z_2)^N \mu\,(Y_{z_2} \otimes Y_{z_1})\, e^{\log(z_2-z_1)\mathcal{N}}\,(v_2 \otimes v_1)\,; \quad (2.9)$$

where the translation endomorphism $T \in \textbf{End}\,(V)$ is defined as $Tv = \partial_z Y(v,z)\mathbb{1}|_{z=0}$ and $\mu : \textbf{End}\,(V) \otimes \textbf{End}\,(V) \to \textbf{End}\,(V)$ denotes the composition.

Note that the role of the endomorphism $\mathcal{N}$ on the definition is to handle the logarithm in the locality axiom. Also note that every vertex algebra is trivially a logarithmic vertex algebra declaring $\mathcal{N} = 0$.
Remark: Even when the expression $\log(z)$ is a formal variable we may define formally

$$\log(xy) = \log(x) + \log(y)\,, \quad (2.10)$$

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)\,, \quad (2.11)$$

$$\log(1-x) = -\sum_{n>0} \frac{x^n}{n}, \quad (2.12)$$

therefore the expression $\log(z_1 - z_2)$ might be interpreted in the following way:

$$\log(z_1-z_2) = \log(z_1) + \log\left(1 - \frac{z_2}{z_1}\right) = \log(z_1) - \sum_{n>0} \frac{z_1^{-n} z_2^n}{n}.$$

The polylogarithm function is defined as

$$Li_p(z) = \sum_{n>0} \frac{z^n}{n^p}, \quad (2.13)$$

specifically we will be interested in the dilogarithm

$$Li_2(z) = \sum_{n>0} \frac{z^n}{n^2}. \quad (2.14)$$

Now we will proceed for the dilogarithm exactly as we did for the logarithm, were the logarithm of the product and the quotient was defined such that the usual analytic identities were satisfied. Finally we formally define

$$Li_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z) - Li_2(z). \quad (2.15)$$

Chapter 3

Functions on the double twisted torus

3.1 Kac-Moody Lie algebra and double twisted torus

Let V be a 3-dimensional real vector space and consider G the extension

$$0 \longrightarrow \wedge^2 V \longrightarrow G \longrightarrow V \longrightarrow 0$$

with internal law

$$(v, \zeta)\left(v', \zeta'\right) = \left(v + v', \zeta + \zeta' + v \wedge v'\right), \quad v, v' \in V, \zeta, \zeta' \in \wedge^2 V$$

making G into a group. Using a coordinate system $\left\{x^i, x_i^*\right\}, \quad i = 1, 2, 3$, where $\left\{x^i\right\}$ are the coordinates on the canonical basis $\left\{e^i\right\}$ of V and $\left\{x_i^*\right\}$ are coordinates on the basis $\left\{e_i^* = \epsilon_{ijk} e^j \wedge e^k\right\}$ of $\wedge^2 V$, this product translates as

$$\left(x^i, x_i^*\right)\left(y^i, y_i^*\right) = \left(x^i + y^i, x_i^* + y_i^* + \frac{1}{2}\epsilon_{ijk} x^j y^k\right),$$

where ϵ_{ijk} denotes the totally antisymmetric tensor. The *double twisted torus* Y is defined as the quotient of G modulo the subgroup Γ generated by $e^i, i = 1, 2, 3$ the standard basis of $V \simeq \mathbb{R}^3$. The tangent bundle TY is trivialized by the left invariant vector fields of G:

$$\alpha^i = \partial_{x_i^*}, \quad \beta_i = \partial_{x^i} - \frac{1}{2}\epsilon_{ijk} x^j \partial_{x_k^*},$$

being $\left[\beta_i, \beta_j\right] = \epsilon_{ijk}\alpha^k$ the only non trivial commutators, therefore they span a Lie algebra $\mathfrak{g}$; moreover this Lie algebra is equipped with a non degenerate symmetric invariant bilinear form

$$\left\langle \beta_i, \alpha^j \right\rangle = \delta_{i,j}.$$

Now consider the space of polynomials $\mathbb{C}[x^i, x_i^*]$ which is a $\mathfrak{g}$-module via the restriction of the action on $C^\infty(G)$, let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Kac-Moody Lie algebra associated to $\mathfrak{g}$ and extend the action for the elements $a_n = at^n$, $a \in \mathfrak{g}$, $n \geq 1$ by zero and make K act as the identity, then define the $\widehat{\mathfrak{g}}$-module

$$\mathcal{H} = \mathbf{Ind}_{\mathfrak{g}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}[x_i, x_i^*]. \tag{3.1}$$

Notice that $\mathcal{H}$ has naturally the structure of $V^1(\mathfrak{g})$-module.

Let's start by defining some operators on $\mathcal{H}$ that will be useful later, particularly when we try to fit $\mathcal{H}$ into an algebraic structure:

Define the operators $x_n^i := -\frac{1}{n}\alpha_n^i$ for $n \neq 0$, note that those operators commute with each other, define x_0^i acting on an element of $\mathbb{C}[x^i, x_i^*]$ as $f \mapsto x^i f$, impose that $\left[x_n^j, x_0^i\right] = 0$ and $\left[\beta_{j,n}, x_0^i\right] = \delta_{i,j}\delta_{n,0}K$; therefore x_0^i can be extended to $\mathcal{H}$.

Define the operators $W^i := \alpha_0^i$ on $\mathcal{H}$ for $i = 1, 2, 3$ and

$$P_i := \beta_{i,0} + \epsilon_{ijk} x_0^j W^k - \frac{1}{2}\epsilon_{ijk}\sum_m m x_{-m}^j x_m^k,$$

note that since all the α_n^i commute with each other then all the x_n^i and W^j commute with each other. Also define the operators

$$x_{i,n}^* := -\frac{1}{n}\left(\beta_{i,n} + \epsilon_{ijk} x_n^j W^k - \frac{1}{2}\epsilon_{ijk}\sum_m m x_{n-m}^j x_m^k\right), \quad n \neq 0,$$

the operators $x_{i,0}^*$ will be defined acting on functions as $f \mapsto x_i^* f$ with the commutation relations $\left[x_{i,0}^*, x_n^j\right] = \left[x_{i,0}^*, x_{j,0}^*\right] = 0$, $\left[x_{i,0}^*, \beta_{j,n}\right] = \frac{1}{2}\epsilon_{ijk} x_n^k$, $\left[W^j, x_{i,0}^*\right] = \delta_{i,j} K$.

Remark: The operators P_i and $x_{i,n}^*$ are well defined because even when the sum appearing in the last term runs over the integers it is actually finite since x_m^i acts by zero for m big enough.

It would be convenient to compute explicitly for later reuse all the commutators of the previously defined operators. It is obvious that

$$\left[\alpha_m^j, x_n^i\right] = 0, \tag{3.2}$$

$$\left[\beta_{j,m}, x_n^i\right] = \delta_{i,j}\delta_{n,-m}K, \tag{3.3}$$

$$\left[\alpha_m^i, W^j\right] = 0, \tag{3.4}$$

$$\left[\beta_{j,m}, W^i\right] = 0, \tag{3.5}$$

$$\left[\alpha_m^j, x_{i,n}^*\right] = \delta_{i,j}\delta_{n,-m}K \tag{3.6}$$

$$\left[\alpha_m^j, P_i\right] = 0 \tag{3.7}$$

$$\left[P_i, W^j\right] = 0. \tag{3.8}$$

For β_m^j and $x_{i,n}^*$ with $n \neq 0$ it holds

$$
\begin{aligned}
\left[\beta_{j,m}, x^*_{i,n}\right] &= -\frac{1}{n}\left[\beta_{j,m}, \beta_{i,n}\right] - \frac{\epsilon_{ipq}}{n}\left[\beta_{j,m}, x^p_n W^q\right] + \frac{\epsilon_{ipq}}{2n}\sum_s s\left[\beta_{j,m}, x^p_{n-s}x^q_s\right] \\
&= \frac{\epsilon_{ijk}}{n}\alpha^k_{n+m} - \frac{\epsilon_{ipq}}{n}\delta_{j,p}\delta_{m,-n}W^q + \frac{\epsilon_{ipq}}{2n}\sum_s s\left[\beta_{j,m}, x^p_{n-s}\right]x^q_s + \frac{\epsilon_{ipq}}{2n}\sum_s s x^q_s\left[\beta_{j,m}, x^p_{n-s}\right] \\
&= \frac{\epsilon_{ijk}}{n}\alpha^k_{n+m} - \frac{\epsilon_{ijk}}{n}\delta_{m,-n}W^k + \frac{\epsilon_{ipq}}{2n}\sum_s s\delta_{j,p}\delta_{m,s-n}x^q_s + \frac{\epsilon_{ipq}}{2n}\sum_s s\delta_{j,q}\delta_{m,-s}x^p_{n-s} \\
&= \frac{\epsilon_{ijk}}{n}\alpha^k_{n+m} - \frac{\epsilon_{ijk}}{n}\delta_{m,-n}W^k + \frac{\epsilon_{ijk}}{2n}(m+n)x^k_{m+n} + \frac{\epsilon_{ijk}}{2n}m x^k_{m+n} \\
&= (1-\delta_{m,-n})\tfrac{\epsilon_{ijk}}{n}\alpha^k_{n+m} + \delta_{m,-n}\tfrac{\epsilon_{ijk}}{n}\alpha^k_{n+m} - \tfrac{\epsilon_{ijk}}{n}\delta_{m,-n}W^k + (1-\delta_{m,-n})\tfrac{\epsilon_{ijk}}{2n}(m+n)x^k_{m+n} + (1-\delta_{m,-n})\tfrac{\epsilon_{ijk}}{2n}m x^k_{m+n} + \delta_{m,-n}\tfrac{\epsilon_{ijk}}{2n}m x^k_{m+n} \\
&= (1-\delta_{m,-n})\left(\tfrac{\epsilon_{ijk}}{n}\alpha^k_{n+m} + \tfrac{\epsilon_{ijk}}{2n}(m+n)x^k_{m+n} + \tfrac{\epsilon_{ijk}}{2n}m x^k_{m+n}\right) + \delta_{m,-n}\left(\tfrac{\epsilon_{ijk}}{n}\alpha^k_{n+m} - \tfrac{\epsilon_{ijk}}{n}W^k + \tfrac{\epsilon_{ijk}}{2n}m x^k_{m+n}\right) \\
&= (1-\delta_{m,-n})\left(-\tfrac{\epsilon_{ijk}}{n}(m+n)x^k_{n+m} + \tfrac{\epsilon_{ijk}}{2n}(m+n)x^k_{m+n} + \tfrac{\epsilon_{ijk}}{2n}m x^k_{m+n}\right) + \delta_{m,-n}\left(\tfrac{\epsilon_{ijk}}{n}\alpha^k_0 - \tfrac{\epsilon_{ijk}}{n}W^k - \tfrac{\epsilon_{ijk}}{2}x^k_{m+n}\right) \\
&= -(1-\delta_{m,-n})\frac{\epsilon_{ijk}}{2}x^k_{m+n} - \delta_{m,-n}\frac{\epsilon_{ijk}}{2}x^k_{m+n} \\
&= -\frac{\epsilon_{ijk}}{2}x^k_{m+n},
\end{aligned}
$$

notice that $x^*_{i,0}$ was defined in a way such that the previous formula is also satisfied. For β^j_m and P^i the bracket is computed as follows

$$
\begin{aligned}
\left[\beta_{j,m}, P^i\right] &= \left[\beta_{j,m}, \beta_{i0}\right] + \epsilon_{ipq}\left[\beta_{j,m}, x^p_0 W^q\right] - \frac{\epsilon_{ipq}}{2}\sum_n n\left[\beta_{j,m}, x^p_{-n}x^q_n\right] \\
&= -\epsilon_{ijk}\alpha^k_m + \epsilon_{ipq}\left[\beta_{j,m}, x^p_0\right]W^q - \frac{\epsilon_{ipq}}{2}\sum_n n\left[\beta_{j,m}, x^p_{-n}\right]x^q_n - \frac{\epsilon_{ipq}}{2}\sum_n n x^p_{-n}\left[\beta_{j,m}, x^q_n\right] \\
&= -\epsilon_{ijk}\alpha^k_m + \epsilon_{ipq}\delta_{j,p}\delta_{m,0}W^q - \frac{\epsilon_{ipq}}{2}\sum_n n\delta_{j,p}\delta_{m,n}x^q_n - \frac{\epsilon_{ipq}}{2}\sum_n n\delta_{j,q}\delta_{m,-n}x^p_{-n} \\
&= -\epsilon_{ijk}\alpha^k_m + \epsilon_{ijk}\delta_{m,0}W^k - \frac{\epsilon_{ijk}m}{2}x^k_m + \frac{\epsilon_{ikj}m}{2}\delta_{m,-n}x^k_m \\
&= -\epsilon_{ijk}\alpha^k_m + \epsilon_{ijk}\delta_{m,0}W^k - \epsilon_{ijk}m x^k_m \\
&= \epsilon_{ijk}(1-\delta_{m,0})(-\alpha^k_m - m x^k_m) + \epsilon_{ipk}z\delta_{m,0}(-\alpha^k_0 + W^k) \\
&= \epsilon_{ijk}(1-\delta_{m,0})(m x^k_m - m x^k_m) + \epsilon_{ipk}z\delta_{m,0}(-W^k + W^k) \\
&= 0.
\end{aligned}
$$

Similarly, the remaining commutators can be computed obtaining:

$$\left[\beta_{j,m}, x^*_{i,n}\right] = -\frac{\epsilon_{ijk}}{2}x^k_{m+n}, \tag{3.9}$$

$$\left[\beta_{j,m}, P_i\right] = 0, \tag{3.10}$$

$$\left[P_i, P_j\right] = -\epsilon_{ijk}W^k, \tag{3.11}$$

$$\left[x^*_{i,n}, x^*_{j,0}\right] = \frac{\epsilon_{ijk}}{2n}x^k_n, \tag{3.12}$$

$$\left[x_{i,m}, x^*_{j,n}\right] = \frac{m+n}{2mn}\epsilon_{ijk} + \frac{1}{m^2}\epsilon_{ijk}W^k\delta_{m,-n}. \tag{3.13}$$

Let us define the fields

$$\alpha^i(z) = \sum_n \alpha^i_n z^{-1-n},$$

$$\beta_i(z) = \sum_n \beta_{i,n} z^{-1-n},$$

and the logarithmic fields

$$x^i(z) = W^i \log(z) + \sum_{n\in\mathbb{Z}} x^i_n z^{-n},$$

$$x^*_i(z) = P_i \log(z) + \sum_{n\in\mathbb{Z}} x^*_{in} z^{-n} + \frac{\log(z)}{2}\epsilon_{ijk} W^j x^k(z),$$

Note that following differential equation holds

$$D_z x^i(z) = \alpha^i(z), \tag{3.14}$$

it will be useful to find a similar equation for the derivative of $x^*_i(z)$:

$$\begin{aligned}
\partial_z x^*_i(z) &= \partial_z\left(P_i \log(z)\right) + \partial_z\left(\sum_n x^*_{i,n}(z) z^{-n}\right) + \frac{\epsilon_{ijk}}{2}\partial_z\left(\log(z) W^j x^k(z)\right) \\
&= P_i z^{-1} - \sum_n n x^*_{i,n}(z) z^{-1-n} + \frac{\epsilon_{ijk}}{2} x^k(z) W^j z^{-1} + \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \beta_{i,0} z^{-1} + \epsilon_{ijk} x^j_0 W^k z^{-1} - \frac{\epsilon_{ijk}}{2}\sum_m x^j_{-m} x^k_m z^{-1} \\
&+ \sum_{n\neq 0}\beta^i_n z^{-1-n} + \epsilon_{ijk}\sum_{n\neq 0} x^j_n z^{-1-n} W^k - \frac{\epsilon_{ijk}}{2}\sum_{n\neq 0}\sum_m m x^j_{n-m} x^k_m z^{-1-n} \\
&+ \frac{\epsilon_{ijk}}{2} x^k(z) W^j z^{-1} + \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \sum_n \beta^i_n z^{-1-n} + \epsilon_{ijk}\sum_n x^j_n z^{-n} W^k z^{-1} - \frac{\epsilon_{ijk}}{2}\sum_n\sum_m x^j_{n-m} z^{-(n-m)} m x^k_m z^{-1-m} \\
&+ \frac{\epsilon_{ikj}}{2} x^j(z) W^k z^{-1} + \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \beta^i(z) + \epsilon_{ijk}\tilde{x}^j(z) W^k z^{-1} - \frac{\epsilon_{ijk}}{2}\left(\sum_n x^j_n z^{-n}\right)\left(\sum_n n x^k_n z^{-1-n}\right) \\
&+ \frac{\epsilon_{ikj}}{2} x^j(z) W^k z^{-1} + \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \beta^i(z) + \epsilon_{ijk}\tilde{x}^j(z) W^k z^{-1} + \frac{\epsilon_{ijk}}{2}\tilde{x}^j(z)\partial_z\tilde{x}^k(z) - \frac{\epsilon_{ijk}}{2} x^j(z) W^k z^{-1} \\
&+ \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \beta^i(z) + \frac{\epsilon_{ijk}}{2}\tilde{x}^j(z) W^k z^{-1} + \frac{\epsilon_{ijk}}{2}\tilde{x}^j(z)\partial_z x^k(z) - \frac{\epsilon_{ijk}}{2} x^j(z) W^k z^{-1} \\
&+ \frac{\epsilon_{ijk}}{2}\log(z) W^j \partial_z x^k(z) \\
&= \beta^i(z) + \frac{\epsilon_{ijk}}{2} x^j(z)\partial_z x^k(z) - \frac{\epsilon_{ijk}}{2} W^j W^k \log(z) z^{-1} \\
&= \beta^i(z) + \frac{\epsilon_{ijk}}{2} x^j(z)\partial_z x^k(z).
\end{aligned}$$

Here we denote $\tilde{x}^j(z) = x^j(z) - W^j\log(z)$, and note that $\epsilon_{ijk}W^jW^k = 0$ since W^j commutes with W^k. So finally we obtained a differential equation for $x^*_i(z)$:

$$D_z x_i^*(z) = \beta_i(z) + \frac{1}{2}\epsilon_{ijk} x^j(z) D_z x^k(z). \tag{3.15}$$

3.1.1 $\mathcal{H}$ as a logarithmic module

The goal now is to endow $\mathcal{H}$ with the structure of logarithmic module over a vertex algebra.

Define R to be the Lie conformal algebra

$$R = \frac{\bigoplus_{i=1,2,3} \mathbb{C}[\partial]\beta_i \bigoplus_{i=1,2,3} \mathbb{C}[\partial]x^i \bigoplus \mathbb{C}[\partial]K}{\partial K}$$

with lambda bracket defined by

$$\begin{aligned} \left[\beta_{i\,\lambda} x^j\right] &= \delta_{i,j} K, \\ \left[\beta_{i\,\lambda} \beta_j\right] &= \epsilon_{ijk} \partial x^k. \end{aligned}$$

Define the logarithmic fields

$$\begin{aligned} \beta_i(z) &= \sum_{n\in\mathbb{Z}} \beta_{i,n} z^{-1-n}, \\ x^i(z) &= W^i \log(z) + \sum_{n\in\mathbb{Z}} x^i_n z^{-n}. \end{aligned}$$

The collection of fields $\mathcal{W} = \left\{x^i(z), \beta_i(z)\right\}_{i=1,2,3} \subseteq \mathbf{LField}\,(\mathcal{H})$ is local: $(x^i(z), x^j(z))$ are local because W^i and x^i commute with each other, $(\beta_i(z), \beta_j(z))$ are local because from the proof of Proposition 2.1.1 it is known that

$$(z_1 - z_2)\left[\beta_i(z_1), \beta_j(z_2)\right] = 0,$$

and for $(\beta_j(z), x^i(z))$

$$\begin{aligned} \left[\beta_i(z_1), x^j(z_2)\right] &= \left[\sum_n \beta_{i,n} z_1^{-1-n}, W^j \log(z_2) + \sum_m x^j_m z_2^{-m}\right] \\ &= \sum_{n,m} \left[\beta_{i,n}, x^j_m\right] z_1^{-1-n} z_2^{-m} \\ &= \sum_{n,m} \delta_{i,j}\delta_{n,-m} z_1^{-1-n} z_2^{-m} \\ &= \delta_{i,j} \sum_n z_1^{-1-n} z_2^n \\ &= \delta_{i,j}\delta(z_1, z_2), \end{aligned}$$

and so $(z_1 - z_2)\left[\beta_i(z_1), x^j(z_2)\right] = 0$.

Because of Theorem 2.1 the set $\overline{\mathcal{W}}$ has the structure of vertex algebra. Define the following map

$$\begin{aligned} \varphi : \quad R &\to \overline{\mathcal{W}} \\ \beta_i &\mapsto \beta_i(z) \\ x^i &\mapsto x^i(z) \\ K &\mapsto \mathbf{id}. \end{aligned}$$

Let us check that φ is a morphism of Lie conformal algebras, i.e., that φ is compatible with all n-products for ($n \geq 0$). Trivially, $\beta_i(z_1)_{(n)}x^j(z_2) = \beta_i(z_1)_{(n)}\beta_j(z_2) = 0$ for $n \geq 1$, the other cases are

$$\begin{aligned}
\left(\beta_i(z_1)_{(0)}x^j(z_2)\right)(z) &= \left((z_1 - z_2)P(\beta_i, x^j; z_1, z_2)\right)\Big|_{z_1=z_2=z} \\
&= (z_1 - z_2)\sum_{n\geq 0, m\in\mathbb{Z}} \left[\beta_{i,n}, x^j_m\right] z_1^{-1-n} z_2^{-m}\Bigg|_{z_1=z_2=z} \\
&= (z_1 - z_2)\delta_{i,j}\sum_{n\geq 0} z_1^{-1-n} z_2^{n}\Bigg|_{z_1=z_2=z} \\
&= (z_1 - z_2)\delta_{i,j}\frac{1}{z_1 - z_2}\Bigg|_{z_1=z_2=z} \\
&= \delta_{i,j}\mathbf{id} = \varphi\left(\beta_{i(0)}x^j\right).
\end{aligned}$$

$$\begin{aligned}
\left(\beta_i(z_1)_{(0)}\beta_j(z_2)\right)(z) &= \left((z_1 - z_2)P(\beta_i, \beta_j; z_1, z_2)\right)\Big|_{z_1=z_2=z} \\
&= \epsilon_{ijk}\alpha^k(z) \\
&= \epsilon_{ijk}D_z x^k(z) \\
&= \varphi(\epsilon_{ijk}\partial x^k) \\
&= \varphi(\beta_{i(0)}\beta_j).
\end{aligned}$$

Therefore $\varphi : R \to \overline{\mathcal{W}}$ is indeed a morphism of Lie conformal algebras and because of the corollary of Theorem 2.1 there is a vertex algebras morphism that will be denoted again by $\varphi : V_1(R) \to \overline{\mathcal{W}}$ from the universal enveloping algebra of R (modulo $K = 1$) into $\overline{\mathcal{W}}$,i.e., $\mathcal{H}$ is a logarithmic module over $V_1(R)$.

3.2 An algebraic structure for $\mathcal{H}$

The main goal of this section is trying to equip $\mathcal{H}$ with a logarithmic vertex algebra structure according to Bakalov's definition (see Section 2.3), we define a locally nilpotent endomorphism and compute the locality condition for logarithmic vertex algebras (equation 2.9) and we shall see that it does not close to form a logarithmic vertex algebra.

First we take the vacuum vector as $\mathbb{1} := 1 \otimes 1$, then we must give a logarithmic field for each vector in $\mathcal{H} = \mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[\![z]\!]\oplus\mathbb{C}K}\mathbb{C}[x^i, x^*_i]_{i=1,2,3}$, the elements on $\mathcal{H}$ can be obtained from the elements: $a_{-n_{k_1}} \cdots a_{-n_{k_r}} \otimes 1$, where $a_{-n_{k_r}} \in \widehat{\mathfrak{g}}$, $x^i := 1 \otimes x^i$ and $x^*_i := 1 \otimes x^*_i$, then the fields associated to elements $a_{-n_{k_1}} \cdots a_{-n_{k_r}} \otimes 1$ would be exactly the same fields used to define the Kac-Moody vertex algebra in Section 2.3, i.e., they would be fields without logarithms; for the other two type of elements we use the fields defined in section 3.1:

$$Y(x^i, z) = x^i(z) = W^i \log(z) + \textstyle\sum_{n\in\mathbb{Z}} x^i_n z^{-n},$$

$$Y(x^*_i, z) = x^*_i(z) = P_i \log(z) + \textstyle\sum_{n\in\mathbb{Z}} x^*_{i,n} z^{-n} + \frac{\log(z)}{2}\epsilon_{ijk}W^j x^k(z).$$

Then Y_z can be extended to a linear map $Y_z : \mathcal{H} \to \mathbf{LField}(\mathcal{H})$ via the normally ordered product of the above fields.

It is also required to define a locally nilpotent endomorphism of $\mathcal{H}$. We define $\mathcal{N} \in \mathbf{End}\,(\mathcal{H} \otimes \mathcal{H})$ acting on elements of $\mathbb{C}[x_i, x_i^*]_{i=1,2,3}$ as:

$$\mathcal{N} = -\sum_{i=1,2,3} \left(\partial_{x_i^*} \otimes \left(\partial_{x^i} + \frac{1}{2}\epsilon_{ijk} x^j \partial_{x_k^*} \right) + \left(\partial_{x^i} + \frac{1}{2}\epsilon_{ijk} x^j \partial_{x_k^*} \right) \otimes \partial_{x_i^*} \right). \tag{3.16}$$

The vacuum axiom is clearly satisfied.

For the translation axiom we define $T : \mathcal{H} \to \mathcal{H}$ on the elements α_n^i and $\beta_{i,n}$ exactly as we did for the Kac-Moody vertex algebra, i.e., we define $T\mathbb{1} = 0$ and extend it by the formulas $\left[T, \alpha_n^i\right] = -n\alpha_{n-1}^i$, $[T, \beta_{i,n}] = -n\beta_{i,n-1}$, for x_n^i we define

$$T(x^i) = \alpha_{-1}^i \mathbb{1},$$

and extend it forcing the commutation $\left[T, x_n^i\right] = \alpha_{n-1}^i$. We have just defined T such that the equation

$$\left[T, x^i(z)\right] = \alpha^i(z) = D_z x^i(z)$$

holds. We would like to continue defining T such that the equation $[T, x_i^*(z)] = D_z x_i^*(z)$ be valid, but since P_i, and $x_{i,n}^*$ (for $n \neq 0$) are defined in terms of x_k^i, α_k^i, $\beta_{i,k}$ and W^i there is no need to define $\left[T, x_{i,n}^*\right]$ and $[T, P_i]$, moreover since T is defined so that the translation invariance is valid for $x^i(z)$ we get that the equation

$$[T, x_i^*(z)] = \beta_i(z) + \frac{1}{2}\epsilon_{ijk} x^j(z) D_z x^k(z) = D_z x_i^*(z)$$

is almost valid possibly failing only on the term $\left[T, x_{i,0}^*\right]$, because it was not yet defined, so in order to make the last equation valid it is only needed to define

$$\left[T, x_{i,0}^*\right] = \beta_{i,-1} + \frac{1}{2}\epsilon_{ijk} \sum_m m x_m^j x_{-m}^k.$$

Let us prove the locality condition starting with the fields $x^i(z_1), x^j(z_2)$. Note that in this case the locality condition (2.9) is just the commutator; then

$$\left[x^i(z_1), x^j(z_2)\right] = \left[W^i \log(z_1) + \sum_{n\in\mathbb{Z}} x_n^i z_1^{-n}, W^j \log(z_2) + \sum_{m\in\mathbb{Z}} x_m^j z_2^{-m}\right] = 0$$

because W^i, W^j, x_n^i, x_m^j commute with each other, therefore the fields $x^i(z_1)$ and $x^j(z_2)$ are local.

Let us try to prove that the fields $x^r(z_1), x_i^*(z_2)$ are local. Notice that

$$e^{\log(z_1-z_2)\mathcal{N}}(x^r \otimes x_i^*) = \mathbf{id} + \log(z_1 - z_2)\,\mathcal{N}(x^r \otimes x_i^*) = \mathbf{id} - \log(z_1 - z_2)\,\delta_{r,i},$$

$$e^{\log(z_2-z_1)\mathcal{N}}(x_i^* \otimes x^r) = \mathbf{id} + \log(z_2 - z_1)\,\mathcal{N}(x_i^* \otimes x^r) = \mathbf{id} - \log(z_2 - z_1)\,\delta_{r,i},$$

therefore the locality condition (2.9) is:

$$[x^r(z_1), x_i^*(z_2)] - \delta_{r,i} \log(z_1 - z_2) + \delta_{r,i} \log(z_2 - z_1)$$

$$= \left[W^r \log(z_1) + \sum_{n\in\mathbb{Z}} x_n^r z_1^{-n}, P_i \log(z_2) + \sum_{m\in\mathbb{Z}} x_{im}^* z_2^{-m} + \frac{1}{2}\log(z_2)\,\epsilon_{ijk} W^j x^k(z_2)\right]$$

$$-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1),$$

now because $W^r, W^j, x^r_n, x^k(z_2)$ commute the last equation is reduced to

$$=\left[W^r\log(z_1)+\sum_{n\in\mathbb{Z}}x^r_n z_1^{-n}, P_i\log(z_2)+\sum_{m\in\mathbb{Z}}x^*_{i,m}z_2^{-m}\right]$$

$$-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1)$$

$$=[W^r,P_i]\log(z_1)\log(z_2)+\log(z_2)\sum_{n\in\mathbb{Z}}[x^r_n,P_i]\,z_1^{-n}+\log(z_1)\sum_{m\in\mathbb{Z}}\left[W^r,x^*_{i,m}\right]z_2^{-m}$$

$$+\sum_{n,m\in\mathbb{Z}}\left[x^r_n,x^*_{i,m}\right]z_1^{-n}z_2^{-m}-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1),$$

because W^r, P_i commute, $\left[W^r,x^*_{i,0}\right]=-[x^r_0,P_i]=\delta_{ri}$, $\left[x^r_0,x^*_{i,0}\right]=0$ and $\left[x^r_k,P_i\right]=\left[W^r,x^*_{i,k}\right]=0$, $\left[x^r_s,x^*_{i,k}\right]=\frac{1}{k}\delta_{r,i}\delta_{s,-k}$ for $k\neq 0$ it follows:

$$=-\delta_{r,i}\log(z_2)+\delta_{r,i}\log(z_1)+\delta_{ri}\sum_{m\neq 0}\frac{z_1^m z_2^{-m}}{m}-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1)$$

$$=-\delta_{r,i}\left(\log(z_2)-\textstyle\sum_{m\geq1}\frac{z_1^m z_2^{-m}}{m}\right)+\delta_{r,i}\left(\log(z_1)-\textstyle\sum_{m\geq1}\frac{z_1^{-m}z_2^m}{m}\right)-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1)$$

$$=-\delta_{r,i}\log(z_2-z_1)+\delta_{r,i}\log(z_1-z_2)-\delta_{r,i}\log(z_1-z_2)+\delta_{r,i}\log(z_2-z_1)=0.$$

Finally it is only left to prove the locality of the fields $x^*_i(z_1), x^*_j(z_2)$. Let us start by analyzing the condition (2.9) in this case $\mathcal{N}$ does not act trivially on $x^*_i\otimes x^*_j$ so

$$\mathcal{N}(x^*_i\otimes x^*_j)=\frac{1}{2}\epsilon_{ijk}1\otimes x^k-\frac{1}{2}\epsilon_{ijk}x^k\otimes 1,$$

$$\mathcal{N}(x^*_j\otimes x^*_i)=-\frac{1}{2}\epsilon_{ijk}1\otimes x^k+\frac{1}{2}\epsilon_{ijk}x^k\otimes 1,$$

so the condition of locality in this case can be translated as:

$$\left[x^*_i(z_1),x^*_j(z_2)\right]+\frac{1}{2}\epsilon_{ijk}\log(z_1-z_2)\,x^k(z_2)-\frac{1}{2}\epsilon_{ijk}\log(z_1-z_2)\,x^k(z_1)$$

$$+\frac{1}{2}\epsilon_{ijk}\log(z_2-z_1)\,x^k(z_1)-\frac{1}{2}\epsilon_{ijk}\log(z_2-z_1)\,x^k(z_2)=0.$$

The commutator $\left[x^*_i(z_1),x^*_j(z_2)\right]$ was already computed by M. Aldi and R. Heluani in [1], they found:

$$\left[x^*_i(z_1),x^*_j(z_2)\right]=\epsilon_{ijk}W^k\left(Li_2(\tfrac{z_2}{z_1})+Li_2(\tfrac{z_1}{z_2})\right)+\tfrac{1}{2}\epsilon_{ijk}W^k(\log(z_2)-\log(z_1))\log\left(1-\tfrac{z_2}{z_1}\right)$$

$$+\tfrac{1}{2}\epsilon_{ijk}W^k(\log(z_1)-\log(z_2))\log\left(1-\tfrac{z_1}{z_2}\right)+\tfrac{1}{2}\epsilon_{ijk}(\log(z_1-z_2)-\log(z_2-z_1))(\textstyle\sum_m x^k_m z_1^{-m}-\sum_m x^k_m z_2^{-m}).$$

Then the locality condition is

$$\epsilon_{ijk}W^k\left(Li_2(\tfrac{z_2}{z_1})+Li_2(\tfrac{z_1}{z_2})\right)+\tfrac{1}{2}\epsilon_{ijk}W^k(\log(z_2)-\log(z_1))\log\left(1-\tfrac{z_2}{z_1}\right)+\tfrac{1}{2}\epsilon_{ijk}W^k(\log(z_1)-\log(z_2))\log\left(1-\tfrac{z_1}{z_2}\right)$$

$$+\tfrac{1}{2}\epsilon_{ijk}(\log(z_1-z_2)-\log(z_2-z_1))(\textstyle\sum_m x^k_m z_1^{-m}-\sum_m x^k_m z_2^{-m})+\tfrac{1}{2}\epsilon_{ijk}\log(z_1-z_2)\,x^k(z_2)$$

$$-\tfrac{1}{2}\epsilon_{ijk}\log(z_1-z_2)\,x^k(z_1)+\tfrac{1}{2}\epsilon_{ijk}\log(z_2-z_1)\,x^k(z_1)-\tfrac{1}{2}\epsilon_{ijk}\log(z_2-z_1)\,x^k(z_2),$$

substituting $x^k(z) = W^k \log(z) + \sum_m x^k_m z_1^{-m}$ and simplifying it is equal to

$$= \epsilon_{ijk} W^k \left(Li_2(\tfrac{z_2}{z_1}) + Li_2(\tfrac{z_1}{z_2}) \right) + \tfrac{1}{2}\epsilon_{ijk} W^k (\log(z_2) - \log(z_1)) \log\left(1 - \tfrac{z_2}{z_1}\right) + \tfrac{1}{2}\epsilon_{ijk} W^k (\log(z_1) - \log(z_2)) \log\left(1 - \tfrac{z_1}{z_2}\right)$$

$$+\tfrac{1}{2}\epsilon_{ijk} W^k \log(z_1 - z_2)\log(z_2) - \tfrac{1}{2}\epsilon_{ijk} W^k \log(z_1 - z_2)\log(z_1) + \tfrac{1}{2}\epsilon_{ijk} W^k \log(z_2 - z_1)\log(z_1) - \tfrac{1}{2}\epsilon_{ijk} W^k \log(z_2 - z_1)\log(z_2),$$

writing $\log(z_1 - z_2) = \log(z_1) + \log\left(1 - \frac{z_2}{z_1}\right)$, rearranging and using 2.15 we get:

$$\epsilon_{ijk} W^k Li_2\left(\frac{z_2}{z_1}\right) + \epsilon_{ijk} W^k (\log(z_2) - \log(z_1)) \log\left(1 - \frac{z_2}{z_1}\right) + \epsilon_{ijk} W^k Li_2\left(\frac{z_1}{z_2}\right) +$$

$$+\epsilon_{ijk} W^k (\log(z_1) - \log(z_2)) \log\left(1 - \frac{z_1}{z_2}\right) - \frac{1}{2}\epsilon_{ijk} W^k \left(\log(z_1) - \log(z_2)\right)^2$$

$$= \tfrac{\pi^2}{6}\epsilon_{ijk} W^k - \epsilon_{ijk} W^k Li_2\left(1 - \tfrac{z_1}{z_2}\right) + \tfrac{\pi^2}{6}\epsilon_{ijk} W^k - \epsilon_{ijk} W^k Li_2\left(1 - \tfrac{z_2}{z_1}\right) - \tfrac{1}{2}\epsilon_{ijk} W^k \left(\log(z_1) - \log(z_2)\right)^2$$

$$= \frac{\pi^2}{3}\epsilon_{ijk} W^k - \epsilon_{ijk} W^k Li_2\left(1 - \frac{z_1}{z_2}\right) - \epsilon_{ijk} W^k Li_2\left(1 - \frac{z_2}{z_1}\right) - \epsilon_{ijk} W^k \frac{1}{2}\left(\log(z_1) - \log(z_2)\right)^2$$

and using the formula $Li_2(1-z) + Li_2(1-z^{-1}) = -\frac{1}{2}\left(\log(z)\right)^2$ we have

$$\epsilon_{ijk} W^k \frac{\pi^2}{3} + \frac{1}{2}\epsilon_{ijk} W^k \left(\log(z_1) - \log(z_2)\right)^2 - \frac{1}{2}\epsilon_{ijk} W^k \left(\log(z_1) - \log(z_2)\right)^2 = \frac{\pi^2}{3}\epsilon_{ijk} W^k.$$

It turns out that $\mathcal{H}$ is not a logarithmic vertex algebra failing the locality axiom for the logarithmic fields corresponding to the vectors x_i^*, x_j^*. The natural way to solve this is to change the space, because if we have fields of the form $e^{x_i^*(z)}$, $e^{x_j^*(z)}$ will be able to get rid of the W^k since it would act diagonally.

There is another deeper problem in this attempt that can not be solve: the formula

$$Li_2(1-z) + Li_2(1-z^{-1}) = -\frac{1}{2}\left(\log(z)\right)^2$$

was never claimed to be true, moreover it looks to be false in the theory of logarithmic fields developed by Bojko Bakalov [4]. Right now I do not have a canonical way to enlarge Bakalov's theory of logarithmic vertex algebras in a way that this equation, which is true as an analytic formula would be valid. Now I prefer to attack the problem of studying functions on the double twisted torus with another algebraic structure (logarithmic modules).

3.3 Functions on the double twisted torus as logarithmic module

Our goal in this section will be proving that $\mathbf{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} C^\infty(G/\Gamma)$ is a module over the vertex algebra $V_{\mathbb{T}^3}$.

Let us go back to the point we started, i.e., let us focus again on the double twisted torus G/Γ but generalizing what was done previously in Section 3.1, now for smooth functions on the double twisted torus instead of polynomials.

As it was noticed at the beginning of the chapter, the group G acts on $L^2(G/\Gamma)$ as left translations and therefore the Lie algebra $\mathfrak{g}$ acts on smooth functions on G/Γ as left invariant vector fields, i.e., $\mathfrak{g}$ acts on a dense subspace of $L^2(G/\Gamma)$; so similarly to the case of polynomials it is possible to obtain a $\widehat{\mathfrak{g}}$-module out of it inducing

$$\mathcal{H} = \mathbf{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} C^{\infty}(G/\Gamma),$$

where as usual α_n^i and $\beta_{i,n}$ act as zero when $n > 0$ and K acts as the identity.

The space of smooth functions on G/Γ is much more complex than the space of polynomials, but every function $f \in C^{\infty}(G/\Gamma)$ can be interpreted as a rapidly decreasing smooth function in six variables $f = f(x^i, x_i^*)$ invariant under the action of Γ on the right, i.e. for every $(\gamma^i, \gamma_i^*) \in \Gamma$ holds $f(x^i, x_i^*) = f(x^i + \gamma^i, x_i^* + \gamma_i^* + \frac{1}{2}\epsilon_{ijk}x^j\gamma^k)$ Such functions can be decomposed in a *Fourier series* with respect to the orthonormal system $\{e^{2\pi\mathbf{i}\omega_i x_i^*}\}_{\omega\in\mathbb{Z}^3}$ as:

$$f(x^i, x_i^*) = \sum_{\omega\in\mathbb{Z}^3} e^{2\pi\mathbf{i}\omega_i x_i^*} f_\omega(x^i),$$

where f_ω satisfies $f_\omega(x^i + \gamma^i) = e^{-\pi\mathbf{i}\epsilon_{ijk}\omega_i x^j\gamma^k} f_\omega(x^i)$.

Define

$$C_\omega = \left\{ e^{2\pi\mathbf{i}\omega_i x_i^*} f; \quad f : \mathbb{R}^3 \to \mathbb{C}, \quad f(x^i + \gamma^i) = e^{-\pi\mathbf{i}\epsilon_{ijk}\omega_i x^j\gamma^k} f(x^i) \right\},$$

then

$$L^2(G/\Gamma) \simeq \bigoplus_{\omega\in\mathbb{Z}^3} C_\omega,$$

specifically for $\omega = 0$ we have

$$C_0 = \left\{ f : f(x^i + \gamma^i) = f(x^i) \right\} = \bigoplus_{\rho\in\mathbb{Z}^3} \mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i}.$$

Define also

$$V_{\mathbb{T}^3} = \mathbf{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} C_0 = \mathbf{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} \bigoplus_{\rho\in\mathbb{Z}^3} \mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i} = \bigoplus_{\rho\in\mathbb{Z}^3} \mathbf{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i}.$$

Theorem 3.1 *$V_{\mathbb{T}^3}$ is a vertex algebra.*

Proof:

Define the vacuum vector $\mathbb{1}$ as the constant 1 function, and consider the state field correspondence map

$$Y : V_{\mathbb{T}^3} \to \mathbf{Field}\,(V_{\mathbb{T}^3})$$

$$e^{2\pi\mathbf{i}\rho_i x^i} \mapsto :e^{2\pi\mathbf{i}\rho_i x^i(z)}: = e^{2\pi\mathbf{i}\rho_i x_0^i} z^{2\pi\mathbf{i}\rho_i W^i} \exp\left(2\pi\mathbf{i}\rho_i \sum_{n<0} x_n^i z^{-n}\right) \exp\left(2\pi\mathbf{i}\rho_i \sum_{n>0} x_n^i z^{-n}\right),$$

$$\alpha_{-1}^i \mathbb{1} \mapsto \alpha^i(z),$$

$$\beta_{i,-1}\mathbb{1} \mapsto \beta_i(z).$$

Let us quickly check the vertex algebra axioms :

$$Y(e^{2\pi\mathbf{i}\rho_i x^i}, z)\mathbb{1}\Big|_{z=0} = e^{2\pi\mathbf{i}\rho_i x_0^i} z^{2\pi\mathbf{i}\rho_i W^i} \exp\left(2\pi\mathbf{i}\rho_i \sum_{n<0} x_n^i z^{-n}\right) \exp\left(2\pi\mathbf{i}\rho_i \sum_{n>0} x_n^i z^{-n}\right) \mathbb{1}\Big|_{z=0} = e^{2\pi\mathbf{i}\rho_i x^i}.$$

The fields of the form $Y(e^{2\pi\mathbf{i}\rho_i x^i}, z)$ commute with each other because all the x_n^i and W^i commute, therefore they are local. The fields of the form $\alpha^i(z)$, $\beta_i(z)$ are pairwise local. Now because $\left[\alpha_m^j, x^i(z_2)\right] = 0$ it is deduced that $\alpha^j(z_1)$ and $Y(e^{2\pi\mathbf{i}\rho_i x^i}, z_2)$ commute.

The locality for the fields $\beta^j(z_1)$ and $e^{2\pi \mathbf{i}\rho_i x^i(z_2)}$ is checked as follows:

$$\left[\beta_{j,n}, x^i(z_2)\right] = \delta_{i,j} K z_2^n,$$

then

$$\left[\beta_{j,n}, e^{2\pi \mathbf{i}\rho_i x^i(z_2)}\right] = \delta_{i,j} 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i(z_2)} z_2^n,$$

from this follows

$$\left[\beta_j(z_1), e^{2\pi \mathbf{i}\rho_i x^i(z_2)}\right] = \sum_n \left[\beta_{j,n}, e^{2\pi \mathbf{i}\rho_i x^i(z_2)}\right] z_1^{-1-n}$$

$$= \delta_{i,j} \sum_n 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i(z_2)} z_1^{-1-n} z_2^n = \delta_{i,j} 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i(z_2)} \delta(z_1, z_2),$$

therefore the fields $\beta^j(z)$ and $e^{2\pi \mathbf{i}\rho_i x^i(z)}$ are local. The locality for any other pair of fields follows from Dong's Lemma 2.1.2.

The last condition remaining to be proved is the translation invariance of the fields, let us define the translation endomorphism T in $V_{\mathbb{T}^3}$. Initially it is convenient to define T acting on x^i, $T(x^i)$ should be a vector such that $Y(T(x^i), z) = \partial_z Y(x^i, z)$, but this equation is satisfied by $\alpha^i(z)$ because of the equation (3.14), so it becomes natural to define $T(x^i) = \alpha^i_{-1}\mathbb{1}$.

Now it easy to define T on any function as

$$T\left(e^{2\pi \mathbf{i}\rho_i x^i}\right) = 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i} T(x^i) = 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i} \alpha^i_{-1}\mathbb{1}. \tag{3.17}$$

We define $T(\mathbb{1}) = 1$ and in the same way as it is done in the Kac-Moody algebra we extend T recursively by the formula $[T, a_n] = -n a_{n-1}$ and impose the commutation relation $[T, x_0^i] = \alpha_{-1}$, finally T extends to the whole $V_{\mathbb{T}^3}$ as a derivation of the normally ordered product. Note that T was defined in a way so it satisfies translation invariance for the fields $\alpha^i(z)$ and $\beta_i(z)$, and for $x^i(z)$ holds

$$\left[T, x^i(z)\right] = \left[T, W^i\right] \log(z) + \sum_n \left[T, x_n^i\right] z^{-n} = \left[T, \alpha_0^i\right] \log(z) + \sum_n -\frac{1}{n}\left[T, \alpha_n^i\right] z^{-n}$$

$$= \sum_n -\frac{1}{n}(-n)\alpha^i_{n-1} z^{-n} = \sum_n \alpha_n^i z^{-1-n} = \alpha^i(z) = \partial_z x^i(z),$$

so now we can compute

$$\left[T, e^{2\pi \mathbf{i}\rho_i x^i(z)}\right] = 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i(z)} \left[T, x^i(z)\right] = 2\pi \mathbf{i}\rho_i e^{2\pi \mathbf{i}\rho_i x^i(z)} \partial_z x^i(z) = \partial_z e^{2\pi \mathbf{i}\rho_i x^i(z)}.$$

□

Recall that the operators x_n^i and $x^*_{i,m}$ $(m \neq 0)$ are still well defined using the same formal formulas described in 3.1.

Now it becomes natural to claim

Theorem 3.2 *The space $\mathcal{H}$ has the structure of $V_{\mathbb{T}^3}$-module.*

Proof:
We must define a field for each vector of $V_{\mathbb{T}^3}$. Set

$$\begin{aligned} Y(e^{2\pi \mathbf{i}\rho_i x^i}, z) &= e^{2\pi \mathbf{i}\rho_i x^i(z)}, \\ Y(\alpha^i_{-1}\mathbb{1}, z) &= \alpha^i(z), \\ Y(\beta_{i,-1}\mathbb{1}, z) &= \beta_i(z), \end{aligned}$$

and for the rest of the elements define the associated field by the normally ordered product, exactly as in the Kac-Moody algebra, for example

$$Y(\alpha^i_{-1} e^{2\pi \mathbf{i}\rho_j x^j}) = :\alpha^i(z) e^{2\pi \mathbf{i}\rho_j x^j(z)}: .$$

Note that we are defining the fields exactly as in the proof of theorem 3.1 and proving the locality condition for those fields we never used the fact that the logarithmic terms disappeared, i.e., what we actually prove there was that those logarithmic fields are pairwise local.

Now in this way because of 2.6 and because the normally ordered product of two fields is the -1-product, the function $Y : V_{\mathbb{T}^3} \to \mathbf{Field}\,(\mathcal{H})$ commutes with the n-products when $n < 0$. For any two pairs of fields of the form $\alpha^i(z)$ and $\beta^j(z)$, it is trivial to see that the n-product condition holds. So it is only left to check it for pair of fields $\left(\alpha^i(z), e^{2\pi\mathbf{i}\rho_j x^j(z)}\right)$, $\left(\beta_i(z), e^{2\pi\mathbf{i}\rho_j x^j(z)}\right)$, $\left(e^{2\pi\mathbf{i}\rho_i x^i(z)}, e^{2\pi\mathbf{i}\rho_i x^i(z)}\right)$ and for $n \geq 0$.

As $e^{2\pi\mathbf{i}\rho_i x^i(z)}$ and $e^{2\pi\mathbf{i}\rho_j x^j(z)}$ commute we know that $e^{2\pi\mathbf{i}\rho_i x^i(z)}{}_{(n)}e^{2\pi\mathbf{i}\rho_j x^j(z)} = 0$ for $n \geq 0$ but $e^{2\pi\mathbf{i}\rho_i x^i}{}_{(n)}e^{2\pi\mathbf{i}\rho_j x^j} = 0$ as well, the same situation repeats for the fields $\left(\alpha^i(z), e^{2\pi\mathbf{i}\rho_j x^j(z)}\right)$.

For the last pair of fields $\left(\beta_i(z), e^{2\pi\mathbf{i}\rho_j x^j(z)}\right)$ we only need to compute for $n = 0$,

$$\begin{aligned}
\beta_i(z)_{(0)}e^{2\pi\mathbf{i}\rho_j x^j(z)} &= (z_1 - z_2)\left[\beta_i(z_1)_-, e^{2\pi\mathbf{i}\rho_j x^j(z_2)}\right]\Big|_{z_1=z_2=z} \\
&= 2\pi\mathbf{i}\delta_{i,j}\rho_j e^{2\pi\mathbf{i}\rho_j x^j(z)} \\
&= Y\left(2\pi\mathbf{i}\rho_j\delta_{i,j}e^{2\pi\mathbf{i}\rho_j x^j}, z\right) \\
&= Y\left(\beta_{i,-1}\mathbb{1}_{(0)}e^{2\pi\mathbf{i}\rho_j x^j}, z\right).
\end{aligned}$$

□

3.4 Functions on the double twisted torus and the Heisenberg nilmanifold

The goal of this section is proving that $\mathcal{H} = \mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K} C^\infty(G/\Gamma)$ is a logarithmic module over another vertex algebra, we will do this by carefully restricting to a subspace of $C^\infty(G/\Gamma)$ such the vector fields identify with the Heisenberg Lie algebra. In order to achieve that it will be required to use some techniques from harmonic analysis on the Heisenberg group to deduce the structure of the space and therefore define the quantum vectors (logarithmic) fields for the algebra and for the module $\mathcal{H}$. The reader interested in a deeper study of harmonic analysis of the Heisenberg group may consult [2].

Define the symbol ξ_{ijk} as $\xi_{123} = \xi_{231} = \xi_{312} = 1$ and $\xi_{ijk} = 0$ for the remaining cases. Notice that it holds that $\xi_{ijk} - \xi_{ikj} = \epsilon_{ijk}$.

It is convenient to change the coordinates

$$\begin{aligned}
x^i &\mapsto x^i, \\
x^*_i &\mapsto x^*_i + \frac{1}{2}\xi_{ijk}x^j x^k,
\end{aligned}$$

so that the group law turns into

$$(x^i, x^*_i)(y^i, y^*_i) = (x^i + y^i, x^*_i + y^*_i + \xi_{ijk}x^j y^k),$$

and the action of α^i and β_j in these coordinates looks like

$$\begin{aligned}\alpha^i &= \partial_{x_i^*}\\ \beta_i &= \partial_{x^i} + \xi_{ijk}x^k\partial_{x_j^*}.\end{aligned}$$

We still have the Fourier type decomposition $L^2(G/\Gamma) \simeq \bigoplus_{\omega\in\mathbb{Z}^3} C_\omega$ but this time C_ω is given by

$$C_\omega = \left\{ e^{2\pi\mathbf{i}\omega_i x_i^*} f; \quad f : \mathbb{R}^3 \to \mathbb{C}, \quad f(x^i+\gamma^i) = e^{-2\pi\mathbf{i}\omega_i\xi_{ijk}x^j\gamma^k} f(x^i) \right\}.$$

Consider vectors $\omega \in \mathbb{Z}$ of the form $\omega = (0,0,n)$

$$C_{(0,0,n)} = \left\{ e^{2\pi\mathbf{i}nx_3^*} f_{0,0,n} : \quad f_{0,0,n}(x^i+\gamma^i) = e^{-2\pi\mathbf{i}nx^1\gamma^2} f_{0,0,n}(x^i) \right\},$$

now those functions $f_{0,0,n}$ can be decomposed into Fourier series once again

$$f_{0,0,n}(x^1,x^2,x^3) = \sum_{m\in\mathbb{Z}} e^{2\pi\mathbf{i}mx^3} f_{0,0,n,m}(x^1,x^2),$$

from this it follows the decomposition of $C_{(0,0,n)} = \bigoplus_{m\in\mathbb{Z}} C_{(0,0,n,m)}$. Let us take $m=0$ and consider the elements from $C_{(0,0,n,0)}$ for all $n \in \mathbb{Z}$, i.e. the functions $f_n : \mathbb{R}^2 \to \mathbb{C}$ such that $f_n(x^1+\gamma^1, x^2+\gamma^2) = e^{-2\pi\mathbf{i}nx^1\gamma^2} f_n(x^1,x^2)$. Define

$$\mathfrak{C}_0 = \bigoplus_{n\in\mathbb{Z}} C_{(0,0,n,0)} = \left\{ \sum_n e^{2\pi\mathbf{i}nx_3^*} f_n(x^1,x^2) \right\} \subset L^2(G/\Gamma).$$

Note that restricting to elements of G of the form $(x^1,x^2,0,0,0,x_3^*)$ is the same as working in $\mathbf{Heis}(\mathbb{R})$, the (polarized) Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x^1 & x_3^* \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

so $\mathbf{Heis}(\mathbb{R})$ acts on $C_{0,0,n,0}$ by right translations, i.e., looking at the functions on Y depending only on the variables x^1, x^2, x_3^* is the same as looking at the Heisenberg nilmanifold $N = \mathbf{Heis}(\mathbb{R})/\mathbf{Heis}(\mathbb{Z})$.

Remark: The space $\mathfrak{C}_0$ constructed above using Fourier analysis can be described intrinsically as follows: since the Heisenberg group is a central extension

$$1 \to \mathbb{R} \to \mathbf{Heis}(\mathbb{R}) \to \mathbb{R}^2 \to 1,$$

and if we take a one dimensional complex representation of the center, i.e., a central character $\chi_n : \mathbb{R} \to \mathbb{C}^*$, $\chi_n(x_3^*) = e^{2\pi\mathbf{i}nx_3^*}$ then $C_{(0,0,n,0)} = \mathbf{CoInd}_{\mathbb{R}}^{\mathbf{Heis}(\mathbb{R})}\mathbb{C}_{\chi_n}$ and $\mathfrak{C}_0$ can be expressed as

$$\mathfrak{C}_0 = \bigoplus_{n\in\mathbb{Z}} C_{(0,0,n,0)} = \bigoplus_{n\in\mathbb{Z}} \mathbf{CoInd}_{\mathbb{R}}^{\mathbf{Heis}(\mathbb{R})}\mathbb{C}_{\chi_n}.$$

Define for $n \neq 0$ and $m \in \{0, 1, \ldots |n| - 1\}$ the k-linear map $\Theta_m : L^2(\mathbb{R}) \to C_{(0,0,n,0)}$ as

$$\Theta_m(g)(x^1, x^2, 0, 0, 0, x_3^*) = e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i}(nk+m)x^1} g(x^2 + k), \tag{3.18}$$

and for $\gamma^1, \gamma^2 \in \mathbb{Z}$ it holds

$$\Theta_m(g)(x^1 + \gamma^1, x^2 + \gamma^2, 0, 0, 0, x_3^*) = e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i}(nk+m)(x^1+\gamma^1)} g(x^2 + \gamma^2 + k)$$

$$= e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i}(nk+m)x^1} g(x^2 + \gamma^2 + k),$$

making $s = k + \gamma^2$

$$= e^{-2\pi \mathbf{i} n x^1 \gamma^2} e^{2\pi \mathbf{i} n x_3^*} \sum_{s \in \mathbb{Z}} e^{2\pi \mathbf{i}(ns+m)x^1} g(x^2 + s) = e^{-2\pi \mathbf{i} n x^1 \gamma^2} \Theta_k(g)(x^1, x^2, 0, 0, 0, x_3^*),$$

then $\Theta_m(g) \in C_{0,0,n,0}$, so Θ_m is a well defined linear, because of the orthogonality relations between the exponentials the maps Θ_m are monomorphisms so then they define a unique monomorphism

$$\Theta : L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|} \to C_{(0,0,n,0)}.$$

Let $e^{2\pi \mathbf{i} n x_3^*} f$ be an element in $C_{(0,0,n,0)}$ then because $f(x^1 + 1, x^2) = f(x^1, x^2)$ it can be decomposed into Fourier series

$$e^{2\pi \mathbf{i} n x_3^*} f(x^1, x^2) = e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i} k x^1} f_k(x^2) = \sum_{m=0}^{|n|-1} e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i}(nk+m)x^1} f_{kn+m}(x^2),$$

from the property $f(x^1, x^2 + \gamma^2) = e^{-2\pi \mathbf{i} n x^1 \gamma^2} f(x^1, x^2)$ it follows $f_{nk+m}(x^2) = f_m(x^2 + k)$ so

$$e^{2\pi \mathbf{i} n x_3^*} f(x^1, x^2) = \sum_{m=0}^{|n|-1} e^{2\pi \mathbf{i} n x_3^*} \sum_{k \in \mathbb{Z}} e^{2\pi \mathbf{i}(nk+m)x^1} f_m(x^2 + k),$$

this means that any function in $C_{0,0,n,0}$ is uniquely determined by $f_0, f_1, \ldots, f_{|n|-1}$, i.e., there is a linear injection

$$\begin{aligned} \Phi : C_{(0,0,n,0)} &\to L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|} \\ f &\mapsto \left(f_0, \ldots, f_{|n|-1}\right), \end{aligned}$$

moreover Φ and Θ_m are inverse functions, so we can make $L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|}$ a $\mathbf{Heis}(\mathbb{R})$-module.

Now because of the Stone-von Neumann theorem[10], $L^2(\mathbb{R})$ is the only irreducible unitary representation of $\mathbf{Heis}(\mathbb{R})$ with the central character $\chi_n(t) = e^{2\pi \mathbf{i} n t}$ and there is a unique decomposition $C_{(0,0,n,0)} \simeq L^2(\mathbb{R}) \otimes \mathbb{C}^{p_n}$ being Θ an isomorphism and $|n| = p_n$. A fully detailed proof of this can be found in [2].

Proposition 3.4.1 *If $n \neq 0$ then $C_{(0,0,n,0)} \simeq L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|}$ and for $n = 0$ it holds that $C_{(0,0,n,0)} \simeq L^2(\mathbb{T}^2)$, i.e.,*

$$\mathfrak{C}_0 \simeq L^2(\mathbb{T}^2) \oplus \bigoplus_{n \neq 0} L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|} \simeq \bigoplus_{\rho \in \mathbb{Z}^2} \mathbb{C} e^{2\pi \mathbf{i} \rho_i x^i} \oplus \bigoplus_{n \neq 0} L^2(\mathbb{R}) \otimes \mathbb{C}^{|n|}.$$

Proposition 3.4.1 means that the G-module $\mathfrak{C}_0$ identifies with the $\mathfrak{g}$-module

$$\bigoplus_{\rho\in\mathbb{Z}^2} \mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i} \oplus \bigoplus_{n\neq 0} S(\mathbb{R})\otimes\mathbb{C}^{|n|},$$

where $S(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing smooth functions in $\mathbb{R}$. We will also denote this space by $\mathfrak{C}_0$. Since one space is the completion of the other one and it will always be clear to distinguish which one we are using.

Define the action of elements of $at^n \in t\mathfrak{g}[t]$ on $\mathfrak{C}_0$ by zero and the action of K as the identity so it is possible now to induce

$$V_N = \mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\mathfrak{C}_0 = \mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\left(\bigoplus_{\rho\in\mathbb{Z}^2} \mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i} \oplus \bigoplus_{n\neq 0} S(\mathbb{R})\otimes\mathbb{C}^{|n|}\right)$$
$$= \bigoplus_{\rho\in\mathbb{Z}^2}\mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\mathbb{C}e^{2\pi\mathbf{i}\rho_i x^i} \oplus \bigoplus_{n\neq 0}\mathbf{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K} S(\mathbb{R})\otimes\mathbb{C}^{|n|}$$

Theorem 3.3 *The space V_N has a vertex algebra structure.*

Remark. Notice that after the change of coordinates previously done the fields $x^i(z)$ remain invariant but the fields $x_i^*(z)$ don not, specifically $x_3^*(z)$ transforms into

$$x_3^*(z) = P_3\log(z) + \sum_{i\in\mathbb{Z}} x^*_{3,i}z^{-i} + \log(z)W^1\sum_{i\in\mathbb{Z}} x_i^2 z^{-i} + \frac{1}{2}\sum_{i,j\in\mathbb{Z}} x_i^1 x_j^2 z^{-i-j} + \frac{1}{2}W^1W^2\left(\log(z)\right)^2,$$

Moreover P_3 and W^i act trivially on V_N so all the terms with logarithms in $x_3^*(z)$ and $x^i(z)$ are zero.

Proof:

The vacuum vector $\mathbb{1}\in V_N$ will be the 1 constant function, let us start defining fields for the basis elements: once again the fields associated to elements of the form $a_{-n_{k_1}}\cdots a_{-n_{k_r}}\otimes 1$ with $a_{-n_{k_1}}\in\widehat{\mathfrak{g}}$ will be the *same* as for the Kac-Moody vertex algebra, for elements $e^{2\pi\mathbf{i}\rho_i x^i}$ we define

$$Y(e^{2\pi\mathbf{i}\rho_i x^i}) =: e^{\left(2\pi\mathbf{i}\rho_i x^i(z)\right)} := \exp\left(2\pi\mathbf{i}\rho_i x^i(z)_+\right)\exp\left(2\pi\mathbf{i}\rho_i x^i(z)_-\right),$$

which can also be written as

$$Y(e^{2\pi\mathbf{i}\rho_i x^i}) = e^{2\pi\mathbf{i}\rho_i x_0^i}\exp\left(2\pi\mathbf{i}\rho_i\sum_{n<0}x_n^i z^{-n}\right)\exp\left(2\pi\mathbf{i}\rho_i\sum_{n>0}x_n^i z^{-n}\right).$$

Denote $F_m\in\mathfrak{C}_0$ the image of $e^{-(x^2)^2}$ by Θ_m, i.e.

$$F_m = \Theta_m(e^{-(x^2)^2}) = \sum_{k\in\mathbb{Z}} e^{2\pi\mathbf{i}(nk+m)x^1}e^{2\pi\mathbf{i}nx_3^*}e^{-(x^2+k)^2},$$

and define

$$Y(F_m,z) = \sum_{k\in\mathbb{Z}} :\exp\left(2\pi\mathbf{i}nx_3^*(z)\right)\exp\left(2\pi\mathbf{i}(nk+m)x^1(z)\right)\exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right):,$$

since $\left[x_3^*(z)_\pm, x^i(z)_\mp\right] = \left[x_3^*(z)_\pm, \left(x^2(z)\right)_\mp\right] = \left[x^i(z)_\pm, \left(x^2(z)\right)_\mp\right] = 0$ for $i=1,2$ one can write

$$Y(F_m,z) = \sum_{k\in\mathbb{Z}}\exp\left(2\pi\mathbf{i}nx_3^*(z)_+\right)\exp\left(2\pi\mathbf{i}(nk+m)x^1(z)_+\right)\exp\left(-(x^2(z))_+^2 - 2kx^2(z)_+ - k^2\right)\cdot$$

$$\exp\left(2\pi\mathbf{i}nx_3^*(z)_-\right)\exp\left(2\pi\mathbf{i}(nk+m)x^1(z)_-\right)\exp\left(-(x^2(z))_-^2-2kx^2(z)_-\right),$$

so for every $g \in \mathfrak{C}_0$ it holds $Y(F_m)g \in V_N((z))$. Now for $a = \beta_1$, $a = \beta_2$ or $a = \alpha^3$ we would like to define $Y(f)$ for every $f \in \bigoplus_{n\neq 0} S(\mathbb{R}) \otimes \mathbb{C}^{|n|}$ imposing the following equation

$$[a(z_1), Y(f,z_2)] = Y(af,z_2)\delta(z_1,z_2). \tag{3.19}$$

Define $Y(aF_m)$ by the formula

$$Y(aF_m) = a_0Y(F_m) - Y(F_m)a_0,$$

clearly if $g \in \mathfrak{C}_0$ then $Y(aF_m)g \in V_N((z))$, note that since $L^2(\mathbb{R})$ is an irreducible $\mathbf{heis}(\mathbb{R})$-mod then every element $f \in \bigoplus_{n\neq 0} S(\mathbb{R}) \otimes \mathbb{C}^{|n|}$ is obtained acting successively on F_m with $\{\beta_1, \beta_2, \alpha^3\}$ and adding, so with the above formula it is proven that $Y(f)g \in V_N((z))$ for every functions f, g. It remains to prove that $Y(f)v \in V_N((z))$ for every function f and every $v = a_{n_k} \ldots a_{n_1} g \in V_N$, this can be done by induction on k, the base case ($k = 0$) is already proven, for the recursive case just write $v = a_q w$ where $Y(f)w \in V_N((z))$, so expanding the equation 3.19 we get

$$Y(f,z_2)a(z_1)w = a(z_1)Y(f,z_2)w - Y(a_0f,z_2)\delta(z_1,z_2)w,$$

multiplying by z_1^q and taking residues we get

$$\begin{aligned}
\mathbf{res}_{z_1} z_1^q Y(f,z_2)a(z_1)w &= \mathbf{res}_{z_1} z_1^q a(z_1)Y(f,z_2)w - \mathbf{res}_{z_1} z_1^q Y(a_0f,z_2)\delta(z_1,z_2)w,\\
Y(f,z_2)a_q w &= a_qY(f,z_2)w - z_2^qY(a_0f,z_2)\delta(z_1,z_2)w,\\
Y(f,z_2)x &= a_qY(f,z_2)w - z_2^qY(a_0f,z_2)\delta(z_1,z_2)w.
\end{aligned}$$

but $a_qY(f,z_2)w \in V_N((z))$ and $z_2^qY(a_0f,z_2)\delta(z_1,z_2)w \in V_N((z))$ because the induction hypothesis and then follows the desired result.

From this $Y : V_N \to \mathbf{Field}\,(V_N)$ is fully determined since the field for the remaining vectors is determined by taking the normally ordered product of the above fields and by linearity.

From the previous analysis it is also deduced that $Y(F_m)\mathbb{1} \in V_N[[z]]$ and from this it follows that $Y(v,x)\mathbb{1} \in V_N[[z]]$ for all $v \in V_N$, and it is clear that $Y(F_m,z)\mathbb{1}|_{z=0} = F_m$ so it holds for every element in V_N.

The computations to check the locality condition for the fields $\alpha^i(z), \beta_i(z), Y(e^{2\pi\mathbf{i}\rho_i x^i}, z)$ are analogous to the ones done for the Kac-Moody vertex algebra and the Theorem 3.1. Since the field $x^3(z)$ commute with itself, the fields $Y(F_{m_1}, z_1)$ and $Y(F_{m_2}, z_2)$ commute, therefore the only remaining pairs to check are $\{\alpha^i(z_1), Y(F_m,z_2)\}$ and $\{\beta^i(z_1), Y(F_m,z_2)\}$.

From $\left[\alpha_r^i, x_3^*(z_2)\right] = \delta_{i,3}z_2^r$ it follows that

$$\left[\alpha_r^i, e^{2\pi\mathbf{i}nx_3^*(z_2)}\right] = 2\pi\mathbf{i}nz_2^r e^{2\pi\mathbf{i}nx_3^*(z_2)}\delta_{i,3},$$

and so

$$\begin{aligned}
\left[\alpha_r^i, Y(F_m,z_2)\right] &= \left[\alpha_r^i, \textstyle\sum_{k\in\mathbb{Z}} \exp\left(2\pi\mathbf{i}nx_3^*(z_2)\right)\exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\exp\left(-(x^2(z_2))^2-2kx^2(z_2)-k^2\right)\right]\\
&= \textstyle\sum_{k\in\mathbb{Z}} \left[\alpha_r^i, \exp\left(2\pi\mathbf{i}nx_3^*(z_2)\right)\right]\exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\exp\left(-(x^2(z))^2-2kx^2(z_2)-k^2\right)\\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp\left(2\pi\mathbf{i}nx_3^*(z_2)\right)\left[\alpha_r^i, \exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\right]\exp\left(-(x^2(z))^2-2kx^2(z_2)-k^2\right)\\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp\left(2\pi\mathbf{i}nx_3^*(z_2)\right)\exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\left[\alpha_r^i, \exp\left(-(x^2(z_2))^2-2kx^2(z_2)-k^2\right)\right]\\
&= \textstyle\sum_{k\in\mathbb{Z}} \left[\alpha_r^i, \exp\left(2\pi\mathbf{i}nx_3^*(z_2)\right)\right]\exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\exp\left(-(x^2(z_2))^2-2kx^2(z)-k^2\right)\\
&= 2\pi\mathbf{i}nz_2^r\delta_{i,3}\textstyle\sum_{k\in\mathbb{Z}} e^{2\pi\mathbf{i}nx_3^*(z_2)}\exp\left(2\pi\mathbf{i}(nk+m)x^1(z_2)\right)\exp\left(-(x^2(z_2))^2-2kx^2(z_2)-k^2\right)\\
&= 2\pi\mathbf{i}nz_2^r\delta_{i,3}Y(F_m,z_2),
\end{aligned}$$

so finally this leads to

$$\begin{aligned}
\left[\alpha^i(z_1), Y(F_m, z_2)\right] &= \sum_{r\in\mathbb{Z}} \left[\alpha^i_r, Y(F_m, z_2)\right] z_1^{-1-r} \\
&= 2\pi\mathbf{i}n\delta_{i,3}Y(F_m, z_2) \sum_{r\in\mathbb{Z}} z_2^r z_1^{-1-r} = 2\pi\mathbf{i}n\delta_{i,3}Y(F_m, z_2)\delta(z_1, z_2).
\end{aligned}$$

So the fields $\alpha^i(z_1), Y(F_m, z_2)$ are a local pair.
From 3.2 and 3.9 follows that $\beta_3(z_1)$ and $Y(F_m, z_2)$ are a local pair.
Let's prove that $\beta_2(z_1)$ and $Y(F_m, z_2)$ are local, from the relations proven in section 3.1 we deduce

$$\begin{aligned}
[\beta_{2,r}, x_3^*(z_2)] &= W^1 z_2^r \log(z_2), \\
\left[\beta_{2,r}, x^1(z_2)\right] &= 0, \\
\left[\beta_{2,r}, x^2(z_2)\right] &= z_2^r K, \\
\left[\beta_{2,r}, \left(x^2(z_2)\right)^2\right] &= z_2^r x^2(z_2),
\end{aligned}$$

from this follows

$$\begin{aligned}
\left[\beta_{2,r}, e^{2\pi\mathbf{i}nx_3^*(z_2)}\right] &= 2\pi\mathbf{i}nW^1 e^{2\pi\mathbf{i}nx_3^*(z_2)} z_2^r \log(z_2), \\
\left[\beta_{2,r}, e^{2\pi\mathbf{i}(nk+m)x^1(z_2)}\right] &= 0, \\
\left[\beta_{2,r}, e^{-2kx^2(z_2)}\right] &= -2kz_2^r e^{-2kx^2(z_2)}, \\
\left[\beta_{2,r}, e^{-\left(x^2(z_2)\right)^2}\right] &= -2z_2^r x^2(z_2) e^{-\left(x^2(z_2)\right)^2},
\end{aligned}$$

which finally leads to

$$\begin{aligned}
\left[\beta_2(z_1), e^{2\pi\mathbf{i}nx_3^*(z_2)}\right] &= 2\pi\mathbf{i}nW^1 e^{2\pi\mathbf{i}nx_3^*(z_2)} \log(z_2)\,\delta(z_1, z_2), \\
\left[\beta_2(z_1), e^{2\pi\mathbf{i}(nk+m)x^1(z_2)}\right] &= 0, \\
\left[\beta_2(z_1), e^{-2kx^2(z_2)}\right] &= -2ke^{-2kx^2(z_2)}\delta(z_1, z_2), \\
\left[\beta_2(z_1), e^{-\left(x^2(z_2)\right)^2}\right] &= -2x^2(z_2)e^{-\left(x^2(z_2)\right)^2}\delta(z_1, z_2).
\end{aligned}$$

Now we can compute the brackets

$$\begin{aligned}
[\beta_2(z_1), Y(F_m, z_2)] &= [\beta_2(z_1), \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z_2))^2 - 2kx^2(z_2) - k^2)] \\
&= \textstyle\sum_{k\in\mathbb{Z}} [\beta_2(z_1), \exp(2\pi\mathbf{i}nx_3^*(z_2))] \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) [\beta_2(z_1), \exp(2\pi\mathbf{i}(nk+m)x^1(z_2))] \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) [\beta_2(z_1), \exp(-(x^2(z_2))^2)] \exp(-2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z_2))^2) [\beta_2(z_1), \exp(-2kx^2(z_2))] \exp(-k^2) \\
&= 2\pi\mathbf{i}nW^1 \log(z_2)\,\delta(z_1, z_2) \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ -2\delta(z_1, z_2)x^2(z_2) \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ -2\delta(z_1, z_2) \textstyle\sum_{k\in\mathbb{Z}} k \exp(2\pi\mathbf{i}nx_3^*(z_2)) \exp(2\pi\mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&= 2\pi\mathbf{i}nW^1 Y(F_m, z_2) \log(z_2)\,\delta(z_1, z_2) - 2Y\left(\Theta_m(xe^{-x^2}), z_2\right)\delta(z_1, z_2),
\end{aligned}$$

therefore $(z_1 - z_2)\,[\beta_2(z_1), Y(F_m, z_2)] = 0$, so they are local.

Let's prove that $\beta_1(z_1)$ and $Y(F_m, z_2)$ are local, once again from the relations proven in section 3.1 we get

$$[\beta_{1,r}, x_3^*(z_2)] = z_2^r(x^2(z_2) - W^2 \log(z_2)) = z_2^r \tilde{x}^2(z_2),$$

here we use the notation $\tilde{x}^i(z) = x^i(z) - W^i \log(z) = \sum_n x_n^i z^{-n}$, it holds

$$\begin{aligned}
\left[\beta_{1,r}, x^1(z_2)\right] &= z_2^r K, \\
\left[\beta_{1,r}, x^2(z_2)\right] &= 0, \\
\left[\beta_{1,r}, \left(x^2(z_2)\right)^2\right] &= 0,
\end{aligned}$$

which means

$$\begin{aligned}
\left[\beta_{1,r}, e^{2\pi \mathbf{i} n x_3^*(z_2)}\right] &= 2\pi \mathbf{i} n z_2^r \tilde{x}^2(z_2) e^{2\pi \mathbf{i} n x_3^*(z_2)}, \\
\left[\beta_{1,r}, e^{2\pi \mathbf{i}(nk+m)x^1(z_2)}\right] &= 2\pi \mathbf{i}(nk+m) z_2^r e^{2\pi \mathbf{i}(nk+m)x^1(z_2)}, \\
\left[\beta_{1,r}, e^{-2kx^2(z_2)}\right] &= 0, \\
\left[\beta_{1,r}, e^{-\left(x^2(z_2)\right)^2}\right] &= 0,
\end{aligned}$$

and this translates into

$$\begin{aligned}
\left[\beta_1(z_1), e^{2\pi \mathbf{i} n x_3^*(z_2)}\right] &= 2\pi \mathbf{i} n \tilde{x}^2(z_2) \delta(z_1, z_2), \\
\left[\beta_1(z_1), e^{2\pi \mathbf{i}(nk+m)x^1(z_2)}\right] &= 2\pi \mathbf{i}(nk+m) e^{2\pi \mathbf{i}(nk+m)x^1(z_2)} \delta(z_1, z_2), \\
\left[\beta_1(z_1), e^{-2kx^2(z_2)}\right] &= 0, \\
\left[\beta_1(z_1), e^{-\left(x^2(z_2)\right)^2}\right] &= 0.
\end{aligned}$$

Now the commutator of the fields is computed

$$\begin{aligned}
[\beta_1(z_1), Y(F_m, z_2)] &= [\beta_1(z_1), \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z_2))^2 - 2kx^2(z_2) - k^2)] \\
&= \textstyle\sum_{k\in\mathbb{Z}} [\beta_1(z_1), \exp(2\pi \mathbf{i} n x_3^*(z_2))] \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) [\beta_1(z_1), \exp(2\pi \mathbf{i}(nk+m)x^1(z_2))] \exp(-(x^2(z_2))^2 - 2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) [\beta_1(z_1), \exp(-(x^2(z_2))^2)] \exp(-2kx^2(z_2) - k^2) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z_2))^2) [\beta_1(z_1), \exp(-2kx^2(z_2))] \exp(-k^2) \\
&= 2\pi \mathbf{i} \tilde{x}^2(z_2) \delta(z_1, z_2) \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&+ 2\pi \mathbf{i}(nk+m) \delta(z_1, z_2) \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&= 2\pi \mathbf{i} n \tilde{x}^2(z_2) Y(F_m, z_2) \delta(z_1, z_2) + 2\pi \mathbf{i} m Y(F_m, z_2) \delta(z_1, z_2) - n x^2(z_2) Y(F_m, z_2) \delta(z_1, z_2) \\
&+ n 2\pi \mathbf{i}(x^2(z_2) + k) \delta(z_1, z_2) \textstyle\sum_{k\in\mathbb{Z}} \exp(2\pi \mathbf{i} n x_3^*(z_2)) \exp(2\pi \mathbf{i}(nk+m)x^1(z_2)) \exp(-(x^2(z))^2 - 2kx^2(z_2) - k^2) \\
&= -2\pi \mathbf{i} n W^2 Y(F_m, z_2) \log(z_2)\, \delta(z_1, z_2) + 2\pi \mathbf{i} m Y(F_m, z_2) \delta(z_1, z_2) + 2\pi \mathbf{i} n Y(\Theta_m(x e^{-x^2}), z_2) \delta(z_1, z_2).
\end{aligned}$$

The locality condition for the remaining fields follows from Dong's Lemma.

Now it is only left to prove the translation invariance of the fields, we will proceed similarly to the proof of Theorem 3.1, let us define the translation endomorphism T in V_N. We already know how to define $T(x^i)$ and proceeding exactly as in the proof of 3.1 we get that the fields $e^{2\pi \mathbf{i} \rho_i x^i(z)}$ satisfy the translation invariance condition.

For $T(x_3^*)$ the situation is similar but slightly more complicated, once again a vector such that $Y(T(x_3^*), z) = \partial_z Y(x_3^*, z)$ is needed, but unfortunately the equation 3.15 is a little bit more complicated. We start noticing that after the change of coordinates we made the equation 3.15 was transformed into

$$\partial_z x_i^*(z) = \beta_i(z) - \xi_{ijk} x^k(z) \partial_z x^j(z),$$

so taking $i = 3$, acting on the vacuum vector and evaluating $z = 0$ it becomes clear that $T(x_3^*)$ should be defined as

$$T(x_3^*) = \beta_{3,-1}\mathbb{1} - \alpha_{-1}^1 x_0^2 \mathbb{1},$$

and force the commutation relation

$$\left[T, x_{3,0}^*\right] = \beta_{3,-1} - \frac{\epsilon_{3jk}}{2} \sum_m m x_{-1-m}^j x_m^k.$$

Now it is easy to define T on any function as

$$T(f) = \partial_{x^1} f T(x^1) + \partial_{x^2} f T(x^2) + \partial_{x_3^*} f T(x_3^*).$$

To make computations easier here we will actually use the fact that W^1, W^2 and P^3 act by zero so we have no logarithms in the fields.

Consider the field $\tilde{x}_3^*(z) = \sum_n x_3^* z^{-n}$, it is convenient to prove that translation invariance holds for the field $\tilde{x}_3^*(z)$, for $n \neq 0$ we have

$$\begin{aligned}
[T, x_{3.n}^*] &= \left[T, \frac{-\beta_{3,n}}{n} + \frac{\epsilon_{3jk}}{2n} \sum_m m x_{n-m}^j x_m^k\right] \\
&= -\frac{1}{n}[T, \beta_{3,n}] + \frac{\epsilon_{3jk}}{2n} \sum_m m \left[T, x_{n-m}^j x_m^k\right] \\
&= \beta_{3,n-1} + \frac{\epsilon_{3jk}}{2n} \sum_m m \left[T, x_{n-m}^j\right] x_m^k + \frac{\epsilon_{3jk}}{2n} \sum_m m x_{n-m}^j \left[T, x_m^k\right] \\
&= \beta_{3,n-1} + \frac{\epsilon_{3jk}}{2n} \sum_m m \alpha_{n-m-1}^j x_m^k + \frac{\epsilon_{3jk}}{2n} \sum_m m x_{n-m}^j \alpha_{m-1}^k \\
&= \beta_{3,n-1} - \frac{\epsilon_{3jk}}{2n} \sum_m m(n-m-1) x_{n-m-1}^j x_m^k - \frac{\epsilon_{3jk}}{2n} \sum_m m(m-1) x_{n-m}^j x_{m-1}^k \\
&= \beta_{3,n-1} - \frac{\epsilon_{3jk}}{2n} \sum_m m(n-m-1) x_{n-m-1}^j x_m^k - \frac{\epsilon_{3jk}}{2n} \sum_m (m+1) m x_{n-m-1}^j x_m^k \\
&= \beta_{3,n-1} - \frac{\epsilon_{3jk}}{2} \sum_m m x_{n-m-1}^j x_m^k,
\end{aligned}$$

then $[T, \tilde{x}_3^*(z)]$ expands as

$$\begin{aligned}
[T, \tilde{x}_3^*(z)] &= \sum_n \left[T, x_{3,n}^*\right] z^{-n} \\
&= \sum_n \beta_{3,n-1} z^{-n} - \frac{\epsilon_{3jk}}{2} \sum_n \sum_m m x_{n-m-1}^j x_m^k z^{-n} \\
&= \sum_n \beta_{3,n} z^{-n-1} - \frac{\epsilon_{3jk}}{2} \sum_n \sum_m m x_{n-m}^j x_m^k z^{-n-1} \\
&= \beta_3(z) - \frac{\epsilon_{3jk}}{2} \sum_n \sum_m x_{n-m}^j z^{-n+m} m x_m^k z^{-m-1} \\
&= \beta_3(z) + \frac{\epsilon_{3jk}}{2} \left(\sum_n x_n^j z^{-n}\right) \left(\sum_n -n x_n^k z^{-n-1}\right) \\
&= \beta_3(z) + \frac{\epsilon_{3jk}}{2} x^j(z) \partial_z x^k(z) \\
&= \partial_z \tilde{x}_3^*(z),
\end{aligned}$$

here the last equality hold because 3.15 and $\tilde{x}_3^*(z)$ coincides with the $x_3^*(z)$ as defined in section 3.1 when setting the formal variable $\log(z) = 0$, i.e., when deleting all terms with P_3, W^1 and W^2. Now the field $x_3^*(z)$ after the change of coordinates (without the logarithmic terms) can be written as:

$$x_3^*(z) = \tilde{x}_3^*(z) + \frac{1}{2}x^1(z)x^2(z),$$

but now it becomes easy to prove translation invariance for $x_3^*(z)$ as we already know it holds for $x^1(z)$ and $x^2(z)$

$$\begin{aligned}
[T, x_3^*(z)] &= [T, \tilde{x}_3^*(z)] + \frac{1}{2}\left[T, x^1(z)x^2(z)\right] \\
&= \partial_z \tilde{x}_3^*(z) + \frac{1}{2}\left[T, x^1(z)\right] x^2(z) + \frac{1}{2}x^1(z)\left[T, x^2(z)\right] \\
&= \partial_z \tilde{x}_3^*(z) + \frac{1}{2}\partial_z x^1(z)x^2(z) + \frac{1}{2}x^1(z)\partial_z x^2(z) \\
&= \partial_z \left(\tilde{x}_3^*(z) + \frac{1}{2}x^1(z)x^2(z)\right) \\
&= \partial_z x_3^*(z).
\end{aligned}$$

Finally we have the tools for proving the translation invariance condition for the fields $Y(F_m, z)$

$$\begin{aligned}
[T, Y(F_m, z)] &= \left[T, \sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right)\right] \\
&= \sum_{k\in\mathbb{Z}} \left[T, \exp\left(2\pi \mathbf{i} n x_3^*(z)\right)\right] \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \left[T, \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right)\right] \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \left[T, \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right)\right] \\
&= \textstyle\sum_{k\in\mathbb{Z}} 2\pi \mathbf{i} n \left[T, x_3^*(z)\right] \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} 2\pi \mathbf{i}(nk+m) \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \left[T, x^1(z)\right] \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} -2(x^2(z)+k) \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \left[T, x^2(z)\right] \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&= \textstyle\sum_{k\in\mathbb{Z}} 2\pi \mathbf{i} n \partial_z \left(x_3^*(z)\right) \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} 2\pi \mathbf{i}(nk+m) \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \partial_z \left(x^1(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} -2(x^2(z)+k) \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \partial_z \left(x^2(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&= \textstyle\sum_{k\in\mathbb{Z}} \partial_z \left(\exp\left(2\pi \mathbf{i} n x_3^*(z)\right)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \partial_z \left(\exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right) \\
&+ \textstyle\sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \partial_z \left(\exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right)\right) \\
&= \partial_z \left(\sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right)\right) \\
&= \partial_z Y(F_m, z).
\end{aligned}$$

□

Knowing that V_N is a vertex algebra it is natural to endow $\mathcal{H}$ with the structure of logarithmic module

Theorem 3.4 *The space $\mathcal{H}$ has the structure of logarithmic V_N-module.*

Proof:
We must define a logarithmic module for each vector of V_N, set

$$\begin{aligned}
Y(e^{2\pi \mathbf{i}\rho_i x^i}, z) &= e^{2\pi \mathbf{i}\rho_i x^i(z)}, \\
Y(\alpha^i_{-1}\mathbb{1}, z) &= \alpha^i(z), \\
Y(\beta_{i,-1}\mathbb{1}, z) &= \beta_i(z), \\
Y(F_m, z) &= \sum_{k\in\mathbb{Z}} \exp\left(2\pi \mathbf{i} n x_3^*(z)\right) \exp\left(2\pi \mathbf{i}(nk+m)x^1(z)\right) \exp\left(-(x^2(z))^2 - 2kx^2(z) - k^2\right),
\end{aligned}$$

Now we extend Y to any function $f \in \bigoplus_{n\neq 0} S(\mathbb{R}) \otimes \mathbb{C}^{|n|}$ exactly as we did in the previous theorem 3.3, i.e., through the formula 3.19, in particular for every $a \in \mathfrak{g}$ the logarithmic field $Y(aF_m, z)$ is defined by the formula $Y(aF_m, z) = [a_0, F_m]$ and finally we extend Y to the rest of the vectors via the normally ordered product.

Notice that during all the analysis made in the proof of 3.3 to show that $Y(f)$ was actually a field was never used the fact that the logarithmic terms acted by zero, so what we actually prove back there was that the $Y(f)$ for any function was actually a logarithmic field. Similarly we proceeded, on propose, when proving that the fields were pairwise local, so what we actually prove was that those are pairwise local logarithmic fields.

Let's prove that the function $Y : V_N \to \mathbf{LField}(\mathcal{H})$ preserves the n-products, because of the way we defined the fields and equation 2.6 it is clear that Y preserves all negative n-products. For positive n-products involving only the fields $e^{2\pi \mathbf{i}\rho_i x^i(z)}$, $\alpha^i(z)$, $\beta_i(z)$ the n-product condition holds, the analysis is complete analogous to the one made in the proof of theorem 3.2. It is also clear that for $n \geq 0$

$$Y({F_{m_1}}_{(n)} F_{m_2}, z) = 0 = Y(F_{m_1}, z)_{(n)} Y(F_{m_2}, z)$$

because $Y(F_{m_1}, z)$ and $Y(F_{m_2}, z)$ commute.

For $\alpha^i(z)$ and $Y(F_m, z)$ we have

$$\begin{aligned}
\alpha^i(z)_{(0)} Y(F_m, z) &= (z_1 - z_2)\left[\alpha^i(z_1)_-, Y(F_m, z_2)\right]\Big|_{z_1=z_2=z} \\
&= 2\pi \mathbf{i} n \delta_{i,3} Y(F_m, z) \\
&= Y(\alpha^i_{(o)} F_m, z).
\end{aligned}$$

For the fields $\beta_3(z)$ and $Y(F_m, z)$ there is nothing to prove since they commute.

For the fields $\beta_2(z)$ and $Y(F_m, z)$ holds

$$\begin{aligned}
\beta_2(z)_{(0)} Y(F_m, z) &= (z_1 - z_2)\left[\beta_2(z_1)_-, Y(F_m, z_2)\right]\big|_{z_1=z_2=z} \\
&= 2\pi \mathbf{i} n W^1 Y(F_m, z_2) \log(z_2) - 2Y\left(\Theta_m(xe^{-x^2}), z_2\right) \\
&= [\beta_{2,0}, Y(F_m, z)] \\
&= Y({\beta_2}_{(0)} F_m, z).
\end{aligned}$$

Finally for $\beta_1(z)$ and $Y(F_m, z)$ holds

$$\begin{aligned}
\beta_1(z)_{(0)} Y(F_m, z) &= (z_1 - z_2)\left[\beta_1(z_1)_-, Y(F_m, z_2)\right]\big|_{z_1=z_2=z} \\
&= -2\pi \mathbf{i} n W^2 Y(F_m, z_2) \log(z_2) + 2\pi \mathbf{i} m Y(F_m, z_2) + 2\pi \mathbf{i} n Y(\Theta_m(xe^{-x^2}), z_2). \\
&= [\beta_{1,0}, Y(F_m, z)] \\
&= Y({\beta_1}_{(0)} F_m, z).
\end{aligned}$$

Therefore $\mathcal{H}$ is a logarithmic V_N-module.

□

Chapter 4

Conclusions and Future work

In this thesis we computed explicitly the vertex algebra of (twisted) chiral differential operator on a nilmanifold and we found its logarithmic module which is a highly non-trivial example of logarithmic vertex algebra modules. However from the analysis we made two new problems for future research appear:

1. In section 3.2 we were unable to prove that the induced space of $\mathbb{C}[x^i, x_i^*]$ was a logarithmic vertex algebra, essentially because we could not prove that the identity
$$Li_2(1-z) + Li_2(1-z^{-1}) = -\frac{1}{2}\left(\log(z)\right)^2$$
is valid in the theory of logarithmic quantum fields. Any attempt of forcing the equation by brute force (i.e. taking a quotient) looks unclean, would rise many other problems but above all it would be desirable to enlarge Bojko Bakalov's theory in a canonical way such that all analytics equations mixing logarithms and polylogarithms would be true. It is a fantastic and non trivial problem for future work.

2. In this thesis we considered only a degree one fibration $\mathbb{S}^1 \to N_1 \to \mathbb{T}^2$ so the natural generalization of this work is analyzing the structure for degree k fibration and a general $\omega \in \mathbb{Z} = H^3(N, \mathbb{Z})$.

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