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How Systoles Grow

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Mathematics is really very small, not big, it is small. There are not that many great ideas and people use the same ideas over and over again in different contexts.

Peter Sarnak

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ABSTRACT

In this thesis we study the growth of systoles along sequences of coverings of a fixed closed hyperbolic surface. We also study the distribution of the set of arithmetic closed hyperbolic surfaces in the moduli space of closed hyperbolic surfaces and a quantitative form of the Ehrenpreis conjecture. The main results of the thesis include the following theorems.

We show that the set of arithmetic closed hyperbolic surface of genus g is not well distributed in the moduli space of closed hyperbolic surfaces of genus g, i.e. by using an auxiliary metric in the moduli space we show that for any compact set we can find surfaces arbitrarily far away from any arithmetic surface if the genus is sufficiently large.

In contrast, we show that for any sequence of closed hyperbolic surfaces with different genera we can find a sequence of arithmetic surfaces in the corresponding genera such that the logarithms of their systoles have the same growth.

For any fixed arithmetic closed hyperbolic surface Buser–Sarnak and Katz–Schaps–Vishne constructed a sequence of coverings of such surface with logarithmic growth of the systoles with an explicit constant. We generalize these constructions for semi-arithmetic surfaces admitting a modular embedding.

Keywords: Arithmetic surfaces, moduli spaces, systoles, Weil-Petersson metric.

RESUMO

Nesta tese estudamos o crescimento de sístoles ao longo de sequencias de recobrimentos de uma superfície hiperbólica fechada fixada. Estudamos também a distribuição do conjunto das superfícies aritméticas no espaço moduli de superfícies hiperbólicas fechadas e uma forma quantitativa da conjectura de Ehrenpreis. Os principais resultados desta tese incluem os seguintes teoremas.

Mostramos que o conjunto de superfícies aritméticas fechadas de gênero g não é bem distribuído no espaço moduli de superfícies hiperbólicas fechadas de gênero g, i.e. , usando uma métrica auxiliar no espaço moduli mostramos que para qualquer conjunto compacto podemos encontrar superfícies arbitrariamente distante de qualquer superfície aritmética se o gênero é suficientemente grande.

Em constraste, mostramos que para qualquer sequencia de supefícies hiperbólicas com gêneros diferentes, podemos achar uma sequencia de superfícies aritméticas com o mesmos gêneros correspondentes tal que os logaritmos de suas sístoles tem o mesmo crescimento.

Para cada superfície aritmética fechada Buser–Sarnak e Katz–Schaps–Vishne construíram uma sequencia de recobrimentos de tal superfície com crescimento logaritmico da sístole com uma constante explícita. Generalizamos essas construções para superfícies semi-aritméticas que admitem mergulho modular.

Palavras-chave: Superfícies aritméticas, espaços moduli, sístoles, métrica de Weil-Petersson.

CHAPTER 1

INTRODUCTION

This thesis is about the growth of systoles along sequences of coverings of a fixed closed hyperbolic surface. A hyperbolic surface is a two-dimensional manifold with a complete Riemannian metric of constant curvature -1. The *systole* of a non simply connected Riemannian manifold is the shortest length of closed curves that are not homotopic to a point.

After the publication of the seminal paper *Filling Riemannian Manifolds* [17] of Gromov, the study of systoles of Riemannian manifolds has been rapidly increasing because of its connection with many others branches of mathematics.

In this work we focus in the interaction of systoles with the study of *moduli spaces of closed hyperbolic surfaces.* Recall that the moduli space $\mathcal{M}_{\mathrm{g}}$ is the space of all closed hyperbolic surfaces of genus g up to isometries.

The space $\mathcal{M}_{\mathrm{g}}$ has a natural topology where any suitable metric invariant is a continuous function with respect to such topology. In particular, the systole, the diameter and the first positive eigenvalue of the Laplace-Beltrami operator are continuous.

On $\mathcal{M}_{\mathrm{g}}$ we define the function $\Psi = \log(\mathrm{sys}) : \mathcal{M}_{\mathrm{g}} \to \mathbb{R}$ which associates for each closed hyperbolic surface S the logarithm of its systole. The remarkable *Mahler compactness theorem* due to Mumford implies that Ψ is a proper function. Moreover, by considering the Teichmuller space $\mathcal{T}_{\mathrm{g}}$ as the universal covering of $\mathcal{M}_{\mathrm{g}}$ with the covering map $\pi : \mathcal{T}_{\mathrm{g}} \to \mathcal{M}_{\mathrm{g}}$, Akrout proved in [1] that the map $\Psi \circ \pi : \mathcal{T}_{\mathrm{g}} \to \mathbb{R}$ is a topological Morse function (see Chapter 4 for more details).

The facts that Ψ is proper and $\Psi \circ \pi$ is a Morse function tell us that the systoles have great importance for understanding the topology of $\mathcal{M}_{\mathrm{g}}$. The study of the structure of $\mathcal{M}_{\mathrm{g}}$ has intrigued a lot of mathematics in different areas.

Given a closed hyperbolic surface, there is a relationship between its systole and its topology. In fact, if we fix a genus, there exists a natural upper bound for the systole of any closed hyperbolic surface of genus g which depends on g in the following way:

$$\mathrm{sys}(S) \lesssim 2\log(g), \quad \text{when } g \text{ goes to infinity.}$$

The critical points of Ψ are precisely the surfaces with maximal systole. Few examples of such critical points are known. In [44], Schmutz gave some characterizations of surfaces with maximal systole and some examples. Moreover, some examples of Schmutz of surfaces with maximal systoles in low genus are arithmetic surfaces.

For high genus, the best estimates for the values of maximal systoles known so far come from *congruence coverings of closed arithmetic surfaces.* Such surfaces were used firstly by Buser and Sarnak in [11] in order to study the *Schottky problem.* The construction of Buser and Sarnak was generalized by Katz, Schaps and Vishne in [29].

The constructions in [11] and [29] give for each closed arithmetic surface S a sequence of coverings S_i with arbitrarily large genus g_i, such that

$$\mathrm{sys}(S_i) \gtrsim \frac{4}{3}\log(g_i), \quad \text{when } i \text{ goes to infinity.}$$

This constant $\frac{4}{3}$ is the best obtained so far. It is expected that, for high genus, there exist arithmetic closed hyperbolic surfaces which are criticial points for the function Ψ.

More generally, in other geometric problems about closed hyperbolic surfaces, the arithmetic ones usually appear as examples of extremal solutions. For example, the Hurwitz upper bound for the cardinality of the isometry group of a compact hyperbolic surface is attained only by arithmetic surfaces (see [4]). See [40] for more related problems.

Examples like these motivated Schmutz to consider in [45] the following Hypothesis in the context of closed hyperbolic surfaces: *"The definition of the best metric should be chosen such that (some) arithmetic surfaces are among the surfaces with the best metric".*

With these considerations in mind it is natural to try to understand the behaviour of the arithmetic closed hyperbolic surfaces and their systoles in $\mathcal{M}_{\mathrm{g}}$.

We can now state the first problem of this thesis.

Problem 1. *How are the arithmetic closed hyperbolic surfaces of genus g distributed in $\mathcal{M}_{\mathrm{g}}$?*

For any $g \geq 2$ we can define $\mathcal{AS}_g$ to be the set of arithmetic closed hyperbolic surfaces of genus g. In [8], Borel showed that for any g the set $\mathcal{AS}_g$ is finite.

Let $|\,\mathcal{AS}_g\,|$ be the cardinality of $\mathcal{AS}_g$. The asymptotic growth of $\log|\,\mathcal{AS}_g\,|$ was investigated in [5]. Belolipetsky, Gelander, Lubotzky and Shalev showed that

$$\lim_{g\to\infty} \frac{\log|\,\mathcal{AS}_g\,|}{g\log g} = 2.$$

On the other hand, the space $\mathcal{M}_{\mathrm{g}}$ carries a Riemannian metric with finite volume. Indeed, we can take the Weil-Petersson metric on $\mathcal{M}_g$ (see Chapter 4) with distance function d_{wp} and volume measure vol_{wp}. In [48] Schumacher and Trapani described the asymptotic growth of $\log \mathrm{vol}_{wp}(\mathcal{M}_g)$ with respect to this metric. In fact, they proved that

$$\lim_{g\to\infty} \frac{\log \mathrm{vol}_{wp}(\mathcal{M}_g)}{g\log g} = 2.$$

When I first found this equation, I thought that the arithmetic closed hyperbolic surfaces should be uniformly coarsely dense in $\mathcal{M}_{\mathrm{g}}$, i.e. I thought that there should exist constants $\varepsilon, \mu > 0$ such that for any closed hyperbolic surface S of genus $g \geq 2$ and systole at least μ, it could be possible to find an arithmetic closed hyperbolic surface S' of the same genus such that $d_{wp}(S, S') \leq \varepsilon$.

It turns out, though, that this result is far from being true.

Theorem A. *For any $L, \mu > 0$ fixed, there exists $g_0 = g_0(L, \mu) \geq 2$ such that for any $g \geq g_0$ there exists a closed hyperbolic surface S of genus g and systole at least μ with $d_{wp}(S, S') \geq L$ for any arithmetic closed hyperbolic surface $S' \in \mathcal{AS}_g$.*

The proof of Theorem A is based on volume estimates of metric balls in $\mathcal{M}_{\mathrm{g}}$ using comparison with spaces of constant negative curvature and lower bound for the sectional curvature $\mathcal{M}_{\mathrm{g}}$ with respect to the Weil-Petersson metric. We combine an upper bound for the volume of such balls with the upper bound for the number of arithmetic closed hyperbolic surfaces of genus g in order to give an upper bound for the volume of the thick part of $\mathcal{M}_{\mathrm{g}}$ and to get a contradiction when assuming that the theorem is false.

Although the arithmetic points are not coarsely dense, we are mainly interested in the maximal points of the function Ψ. Thus we can be more specific in our question.

Problem 2. *Fix a constant $\varepsilon > 0$. Is there an absolute constant $\delta > 0$ such that for any closed hyperbolic surface of genus g and systole at least ε, we can find an arithmetic closed hyperbolic surface S' of genus g such that $|\Psi(S) - \Psi(S')| \leq \delta$?*

The main difficulty of this problem is that we have little information about the geometry of a generic arithmetic closed hyperbolic surface, despite the definition of an arithmetic closed surface being very simple.

Let S be a closed hyperbolic surface, let $L(S)$ be the set of lengths of all closed geodesics on S and let $L^*(S) = \{\cosh^2\left(\frac{l}{2}\right) \mid l \in L(S)\}$. We define the *invariant trace field* of S by $k(S) = \mathbb{Q}(L^*(S))$. We say that S is *arithmetic* if $k(S)$ is a totally real finite extension of $\mathbb{Q}$ with $L^*(S)$ contained in the ring of integers of $k(S)$ and if for any embedding $\sigma : k(S) \to \mathbb{R}$ with $\sigma \neq \mathrm{id}$, we have $\sigma(L^*(S)) \subset [-2, 2]$.

Let S_1 and S_2 be closed hyperbolic surfaces. We say that S_1 and S_2 are *commensurable* if there exists another closed hyperbolic surface S which covers S_1 and S_2. It is not difficult to see that if S_1 and S_2 are commensurable, then $k(S_1) = k(S_2)$. Consequently, it is not so clear how to determine the systole of an arithmetic closed hyperbolic surface using only this definition. In particular, a surface which is commensurable to an arithmetic closed hyperbolic surface is also arithmetic. Thus a natural way of constructing arithmetic surfaces is by fixing one and taking coverings with arbitrary large degree. In fact, the set of arithmetic surfaces which cannot cover any other arithmetic surface is very small in the set of all arithmetic surfaces (see [5, Theorem 1.1]).

When analyzing a covering of a closed hyperbolic surface we can use tools of geometric group theory in order to study the geometry of the correponding covering and a natural graph which appears in this context.

Indeed, let $S \to S'$ be a covering of closed hyperbolic surfaces. Consider a generator set A for $\pi_1(S')$. We can define the Schreier graph of the covering as the graph where the vertices are the cosets $\pi_1(S)\gamma$ with $\gamma \in \pi_1(S')$ and we have an edge $\{\pi_1(S)\gamma, \pi_1(S)\eta\}$ if $\eta = \gamma\tau$ for some $\tau \in A$.

If the degree of the covering is d then the Schreier graph has d vertices and is k-regular, where $k = |A|$. Geometric group theory tells us that we can compare the geometry of S with the geometry of the Schreier graph of the covering. In particular, the systole of S is comparable to the girth of this graph, where the *girth* of a graph is the shortest length of a circuit.

The strategy of comparing the geometry of Riemannian coverings and Cayley graphs (the particular case where the covering is regular) was very fruitful in the study of spectral problems. The work of Robert Brooks motivated me to try to use the same ideas in systolic problems. Although Brooks has proved some theorems about systoles of coverings using Cayley graphs, his main motivation was to compare the first eigenvalue of the Laplace-Beltrami operator on the surface and the first eigenvalue of the Laplace operator on the corresponding Cayley graph.

With these tools I was able to give the following asymptotic answer to Problem 2.

Theorem B. *There exists a universal constant $g_0 \geq 2$ such that for any $\varepsilon > 0$ we can find a constant $L > 0$ which depends on ε with the following property: For any closed hyperbolic surface S of genus $g \geq g_0$ and systole at least ε there exists an arithmetic closed hyperbolic surface S' of genus g such that*

$$\frac{1}{L} \leq \frac{\Psi(S)}{\Psi(S')} \leq L.$$

In fact this theorem is a corollary of Theorem 7.2.2 in Chapter 7. There I will prove a more general theorem which does not use arithmeticity.

The idea behind the proof of Theorem B is to construct graphs with any possible girth and then to consider coverings of a fixed hyperbolic surface with systole comparable to the girth of these graphs.

The construction of such graphs is based on special graphs such as *cages* and *Ramanujan graphs*, which have remarkable geometric properties. Moreover, the construction of coverings is based on constructing subgroups of the fundamental group of a closed surface of genus 2 via surjectives maps to the free group of rank 2 with some suitable properties.

Recall that a group G is *fully residually free* if for any finite set $X \subset G$ which does not contain the identity there exists a normal subgroup $N \lhd G$ such that $N \cap X = \emptyset$ and G/N is a free group.

Benjamim Baumslag showed that the fundamental group of a closed surface of genus $g \geq 2$ is fully residually free. The crucial part in the proof of Theorem B is to give a new proof of the result of Baumslag for the case $g = 2$ using the geometry of the groups involved and not only the algebraic aspects.

The same ideas can be used in order to obtain some interesting results about growth of systoles along coverings. For example, in Chapter 7 we prove the following theorem.

Theorem C. *Let M be the Bolza surface, i.e. the arithmetic surface of genus 2 of maximal systole in $\mathcal{M}_2$ and let $s = \mathrm{sys}(M) = 2\cosh^{-1}(1+\sqrt{2})$. Then for any $k \in \mathbb{N}$ there exists a finite covering $M_k \to M$ with $\mathrm{sys}(M_k) = ks$ and degree $\leq (uk)^{vk^2}$ for some positive constants u, v.*

Returning to the study of systoles of arithmetic closed hyperbolic surfaces, we already mentioned that asymptotically we have $\mathrm{sys}(S) \lesssim 2\log(g)$. On the other hand, Katz, Schaps and Vishne constructed for each closed arithmetic surface S a sequence of coverings S_i with arbitrarily large genus g_i, such that $\mathrm{sys}(S_i) \gtrsim \frac{4}{3}\log(g_i)$. Therefore, we can consider the following problem (see [35, Problem 1.4]).

Problem 3. *Determine the supremum of γ such that there exists a family* *of closed hyperbolic surfaces X_i with*

$$\mathrm{sys}(X_i) \gtrsim \gamma \log(\textit{genus of } X_i), \quad \textit{where genus of } X_i \textit{ goes to infinity.}$$

In [35], Makisumi showed that in the construction of Katz, Schaps and Vishne, the constant $\frac{4}{3}$ cannot be improved. Moreover, if we apply Theorem B we have a growth worse than logarithmic (see Corollary 7.2.3).

In [41], Petri constructed a sequence $\{S_k\}$ of closed hyperbolic surfaces not necessarily arithmetic with genus g_k arbitrary large and $\mathrm{sys}(S_k) \gtrsim \gamma \log(g_k)$ for some explicit constant γ. In fact, he uses some special graphs and pairs of pants in order to construct such sequences with $\gamma = \frac{4}{7}$.

In this thesis we will give a new family of constructions of sequences of closed hyperbolic surfaces which are not arithmetic and have logarithmic growth of the systole with explicit constants.

We say that a closed hyperbolic surface is a *Belyĭ surface* if it is a ramified covering of the Riemann sphere ramified over at most three points. It follows from Belyĭ's theorem [6] that the set of all Belyi surfaces of genus g is dense in $\mathcal{M}_{\mathrm{g}}$.

Furthermore, any Belyĭ surface has a ramified covering of finite degree which is a *semi-arithmetic closed hyperbolic surface admitting a modular embedding* (see Chapter 6 for the definition). Although such surfaces are not arithmetic in general, they share a lot of properties with the arithmetic ones.

In this spirit we state now the following generalization of the construction of non arithmetic closed hyperbolic surfaces with large systole.

Theorem D. *Let S be a closed semi-arithmetic hyperbolic surface admitting a modular embedding. Then there exists an integer $r > 0$ which depends on S, and a sequence of coverings $S_i \to S$ with $\mathrm{area}(S_i) \to \infty$ such that*

$$\mathrm{sys}(S_i) \geq \frac{4}{3r} \log(\mathrm{area}(S_i)) - c,$$

where $c > 0$ is a constant which does not depend on i.

It is worth noting that any arithmetic closed hyperbolic surface admits a modular embedding and the constant r in this case is equal to 1. Hence we recover the constructions that are already known in the arithmetic context.

The proof of Theorem D use the same ideas of the paper of Buser and Sarnak: the sequence of coverings is constructed from the arithmetic input in the definition of closed hyperbolic surfaces admitting modular embedding and the estimate of the systole becomes an estimate for traces of matrices with suitable properties.

Problem 2 has a "virtual" solution in the following sense. For any $\varepsilon > 0$ and any closed hyperbolic surface S of genus $g \geq 2$, there exists a covering $\tilde{S}$ of S of genus $h \geq g$ and an arithmetic closed hyperbolic surface S' of genus h with $|\Psi(\tilde{S}) - \Psi(S')| \leq \varepsilon$.

Indeed, for any $g \geq 2$, we have in the space $\mathcal{M}_{\mathrm{g}}$ the Teichmuller distance $d_T(\cdot, \cdot)$. The importance of this distance in the study of systoles of closed hyperbolic surfaces is the fact that for any $g \geq 2$, the Morse function Ψ is 1-Lipschitz with respect to d_T. The recent proof of the Ehrenpreis conjecture due to Kahn and Markovic tells us that for any $\varepsilon > 0$ and any two closed hyperbolic surfaces S_1 and S_2 we can take coverings $S_1' \to S_1$ and $S_2' \to S_2$ with $S_1', S_2' \in \mathcal{M}_{\mathrm{g}}$ for some g and $d_T(S_1, S_2) \leq \varepsilon$. Thus if we fix an arithmetic surface M of genus 2 and apply this theorem for the pair (S, M) we obtain the "virtual" solution of Problem 2.

When I tried to use the theorem of Kahn and Markovic in order to solve Problem 2 the following new question arose:

Problem 4. *Fix two closed hyperbolic surfaces S_1 and S_2 and $\varepsilon > 0$. What is the minimal $g \geq 2$ such that we can find coverings $S_1' \to S_1$ and $S_2' \to S_2$ with $S_1', S_2' \in \mathcal{M}_{\mathrm{g}}$ for some $g \geq 2$ with $d_T(S_1', S_2') \leq \varepsilon$?*

We can give a lower bound for such minimal g when S_1 and S_2 are not commensurable and share the same invariant trace field. In fact, using geometric properties of the Weil-Petersson metric with respect to the special coordinates of $\mathcal{M}_{\mathrm{g}}$ and also a comparison between d_{wp} and d_T we can prove the following theorem.

Theorem E. *Let S_1, S_2 be non commensurable arithmetic closed hyperbolic surfaces derived from quaternion algebras over a field k. Then there exists a constant $C > 0$ which depends only on k such that for any $\varepsilon > 0$ and any coverings $S_1' \to S, S_2' \to S_2$ with S_1', S_2' of genus $g < -C\log(\varepsilon)$ we have $d_T(S_1', S_2') > \varepsilon$.*

This thesis is organized as follows. In Chapter 2 we begin with a general picture introducing the locally symmetric spaces. The symmetric spaces are Riemannian manifolds constructed via algebraic objects and they have remarkable geometric properties due this interplay between algebra and geometry. The closed hyperbolic surfaces are very special examples of locally symmetric spaces and in Chapter 2 we recall the main geometric/algebraic facts about them.

In Chapter 3 we review the algebraic/arithmetic properties of quaternion algebras and algebraic number theory which is used in the constructions of arithmetic closed hyperbolic surfaces and semi-arithmetic closed hyperbolic surfaces admitting modular embbeding. These facts will be necessary in order to prove Theorem D.

In Chapter 4 we will use the geometry of closed hyperbolic surfaces in order to construct a parametrization of $\mathcal{M}_{\mathrm{g}}$. In fact, by decomposing any surface in pairs of pants we can construct the Teichmuller space $\mathcal{T}_{\mathrm{g}}$, a simply connected manifold which covers $\mathcal{M}_{\mathrm{g}}$ with ramifications with the deck group isomorphic to the *mapping class group* of the correponding closed surface. In $\mathcal{T}_{\mathrm{g}}$ we will introduce some properties of the Weil-Petersson and Teichmuller metrics which will be useful in the proofs of Theorem A and Theorem E.

In Chapter 5 we will review some facts about graphs and groups. The geometric approach to these objects will be used in order to construct regular graphs and subgroups of the fundamental group of a closed surface of genus 2 with desirable properties which will be used in the proofs of Theorems B and C.

In Chapter 6 we will present the development of mathematical ideas in the construction of closed hyperbolic surfaces with large systole from the search of the small eigenvalues of the Laplace-Beltrami operator up to Problem 3. In the last part of Chapter 6 we will give the proof of Theorem D.

We begin Chapter 7 by proving Theorem A. Next we present the proof of the main Theorem B and some other results about systoles of coverings. We will finish the chapter with the proof of Theorem E.

For the sake of completeness we give in Appendix A some basic definitions which are assumed along this thesis.

CHAPTER 2

HYPERBOLIC SURFACES

2.1 Locally symmetric spaces

Let X be a Riemannian manifold and G be its isometry group. We say that X is *homogeneous* if G acts transitively on X.

Example 1.

1. The Euclidean space $\mathbb{R}^n$ with the standard metric is homogeneous since translations preserve the metric, and any point $x \in \mathbb{R}^n$ is the image of the origin by the translation $T_x(v) = v + x$.

2. More generally, any Lie group G with a left-invariant metric is homogeneous since any element $g \in G$ is contained in the orbit of the identity.

3. The sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \,|\, |x| = 1\}$ with the metric induced by the metric on $\mathbb{R}^{n+1}$ is homogeneous since the rotations preserve the sphere and for any $x \in \mathbb{S}^n$ we can complete an orthonormal basis $\{x, x_2, \cdots, x_n\}$ of $\mathbb{R}^{n+1}$. The rotation $\Theta(e_1) = x$, $\Theta(e_{i+1}) = x_i$, is an isometry of the sphere which sends e_1 to x.

4. For each $n \geq 2$ consider the bilinear form B on $\mathbb{R}^{n+1}$ given by

$$B((x_1, \cdots, x_{n+1}), (y_1, \cdots, y_{n+1})) = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$

We define the *hyperbolic n-space* (also called the *Lobachevsky n-space*)

$$\mathcal{L}^n = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, B(x,x) = -1 \text{ and } x_{n+1} > 0\}.$$

It is easy to check that for each $x \in \mathcal{L}^n$ the bilinear form B is positive defined on the tangent space $T_x\mathcal{L}^n$. Thus, we define the hyperbolic metric on $\mathcal{L}^n$ as the Riemannian metric induced by B. The group

$$\mathrm{O}(n,1) = \{T \in \mathrm{GL}(n+1,\mathbb{R}) \,|\, B(Tv,Tv) = B(v,v) \text{ for any } v \in \mathbb{R}^{n+1}\}$$

is a Lie group which leaves $\mathcal{L}^n$ invariant, and by definition it acts by isometries on the hyperbolic n-space. Moreover, analogously with the sphere we can see that $\mathcal{L}^n$ is homogeneous.

5. Any finite product of homogeneous Riemannian manifolds is homogeneous with respect to the product metric.

The examples above are called *space forms.* Any n-dimensional Riemannian manifold of constant curvature c is locally isometric to $\mathbb{R}^n$ (if $c = 0$) or $\mathbb{S}^n$ (if $c = 1$) or $\mathcal{L}^n$ (if $c = -1$).

A Riemannian manifold X is a *symmetric space* if it is homogeneous and in its isometry group contains an involution with at least one isolated fixed point, i.e. there exists an isometry $\phi : X \to X$ such that $\phi^2 = id$ and there exist a point $x \in X$ and a neighborhood U of x, such that $\phi(x) = x$ and $\phi(y) \neq y$ for all $y \in U - \{x\}$.

Example 2. 1. Consider the isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ given by $I(x) = -x$. Note that 0 is the unique fixed point of I. Thus $\mathbb{R}^n$ is a symmetric space.

2. In the sphere $\mathbb{S}^n$, let $p = (0, 0, \cdots, 1)$ and consider the orthogonal map $R : \mathbb{S}^n \to \mathbb{S}^n$ given by $R(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, -x_n, x_{n+1})$. Note that p is the unique fixed point of R. Thus $\mathbb{S}^n$ is a compact symmetric space.

3. The hyperbolic n-space $\mathcal{L}^n$ is a symmetric space since the same map $R \in \mathrm{O}(n,1)$ and the point $p \in \mathcal{L}^n$ is the unique fixed point of R in $\mathcal{L}^n$.

4. Any finite product of symmetric spaces is a symmetric space.

Let G be a connected, noncompact, simple Lie group with finite center and let K be a maximal compact subgroup of G. The quotient space G/K is noncompact and simply connected. Moreover, G/K admits a G-invariant Riemannian metric (defined by the Killing-Cartan form) such that with this metric G/K is a symmetric space.

Consider the structure of symmetric space in G/K. With such metric G/K has non-positive curvature and is nonflat. Conversely, this is the only way to build noncompact, nonflat, irreducible symmetric spaces, where a symmetric space is irreducible if its universal covering is not isometric to any nontrivial product ([21]).

By the work of Cartan, the space of all noncompact, nonflat, irreducible symmetric spaces is classified since he gave the list of all simple Lie groups and determined which compact groups can arise in this construction.

Example 3. For any n-dimensional flat torus T of volume 1 we can associate a monomorphism $\iota : \mathbb{Z}^n \to \mathbb{R}^n$ such that $\mathbb{R}^n/\iota(\mathbb{Z}^n)$ is isometric to T. Note that ι is not unique.

We say that ι_1 and ι_2 are equivalent if there exists a linear isometry Θ of $\mathbb{R}^n$ such that $\iota_2 = \Theta \circ \iota_1$. Any equivalence class of this relation is called a *marked unit flat torus*. The space of all marked unit flat tori of a fixed dimension has a structure of symmetric space.

Indeed, it is not so difficult to see that $\mathrm{SO}(n,\mathbb{R})$ is a maximal compact subgroup of $\mathrm{SL}(n,\mathbb{R})$. Thus the quotient space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ has a structure of symmetric space.

We can give a natural bijection betweeen the space of marked unit flat tori and the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. For any ι we can associate the matrix $A_\iota \in \mathrm{SL}(n,\mathbb{R})$ whose rows are the vectors $\iota(e_1), \cdots, \iota(e_n)$. Note that if $\iota' = \Theta\iota$, then $A_{\iota'} = A_\iota\Theta^{-1}$. Thus A_ι and $A_{\iota'}$ are equivalent modulo $\mathrm{SO}(n,\mathbb{R})$.

Conversely, any class $A\,\mathrm{SO}(n,\mathbb{R})$ gives naturally a monomorphism $\iota_A : \mathbb{Z}^n \to \mathbb{R}^n$ defined by $\iota_A(e_i) = A^t(e_i)$, where A^t denotes the transpose of A. Note that if $A' = A\Theta$, then $\iota_{A'} = \Theta^{-1}\iota_A$.

The bijection $A\,\mathrm{SO}(n,\mathbb{R}) \mapsto [\iota_A]$ tells us that the space of marked unit flat tori has a homogeneous nonflat geometry.

A Riemannian manifold is locally symmetric if its universal covering is a symmetric space, so any Riemannian manifold of constant curvature (0 or 1 or -1) is locally symmetric by Example 2.

Therefore, we can factorize any locally symmetric space into a Lie group G (with a corresponding maximal compact group K) and a discrete subgroup $\Gamma < G$ such that the locally symmetric space is isometric to $\Gamma \backslash G/K$.

The hyperbolic 2-space $\mathcal{L}^2$ is one of the simplest examples of such spaces. But at the same time $\mathcal{L}^2$ does not satisfy the rigidity theorems, i.e. there are nontrivial deformations of locally symmetric spaces covered by $\mathcal{L}^2$,and this is one of the reasons for the study of this particular manifold.

2.2 The geometry of hyperbolic surfaces

There are at least three important models for the hyperbolic plane.

Proposition 2.2.1. *The following Riemannian manifolds are all mutually isometric.*

(a) The hyperboloid model $\mathcal{L}^2$ defined in Example 1.

(b) The disc model $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$ with the metric h defined by

$$h(x)(v, v) = 4\frac{|v|^2}{(1 - |x|^2)^2} \text{ for any } x \in \mathbb{D},\, v \in T_x\mathbb{D}.$$

(c) The upper half-space model $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, where $\Im(z)$ denotes the imaginary part of z. We consider on $\mathbb{H}$ the Riemannian metric ω defined by

$$\omega(z)(v) = \frac{|v|^2}{\Im(z)^2} \text{ for any } z \in \mathbb{H},\, v \in T_z\mathbb{H}.$$

For a proof of this proposition see [31, Proposition 3.5]. From now on we will call any of such metrics by the hyperbolic metric. In this work we will mainly use the upper half-space model.

The group $\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathrm{GL}(2, \mathbb{R}) \mid \det(A) = 1\}$ acts on $\mathbb{H}$ by linear-fractional transformations

$$z \mapsto \frac{az + b}{cz + d} \text{ if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is straightforward to check that the action of $\mathrm{SL}(2,\mathbb{R})$ preserves the hyperbolic metric. Moreover, any isometry of $\mathbb{H}$ which preserves the orientation of $\mathbb{H}$ coincides with a linear-fractional tranformation in $\mathrm{SL}(2,\mathbb{R})$.

Note that if $A \in \mathrm{SL}(2,\mathbb{R})$, then $-A$ has the same action. Thus we can consider the action of the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ on $\mathbb{H}$. The group $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the group of orientation-preserving isometries of $\mathbb{H}$.

Let $\alpha : I \to \mathbb{H}$ be a smooth path. We define the length of α with respect to the hyperbolic metric by the integral

$$\int_I \frac{|\alpha'(t)|}{\Im(\alpha(t))} dt.$$

Recall that a *geodesic* on a Riemannian manifold is a path which locally minimizes the distance between its endpoints. By the definition of geodesics and applying the homogeneity of $\mathbb{H}$, we can check that any vertical line and any intersection of $\mathbb{H}$ with a Euclidean circle with the center on the real axis are geodesics for the hyperbolic metric. In fact, the following theorem says that they are the only geodesics on $\mathbb{H}$.

Theorem 2.2.2. *In $\mathbb{H}$ there is a unique geodesic through any two distinct points.*

In Euclidean geometry, probably the main geometric figure is a triangle, it is common to decompose any object into triangles. On the other hand, in hyperbolic geometry there is another polygon which plays similar role.

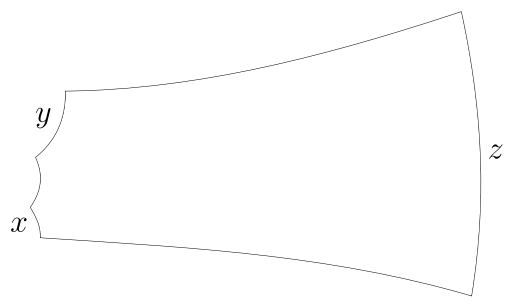


Figure 2.1: Right-angled hyperbolic hexagon

Indeed, in $\mathbb{H}$ there exist *right-angled geodesic hexagons* and they constitute the fundamental stones for the construction of any hyperbolic surface.

We can parametrize the space of all right-angled hexagons up to isometries. Indeed, we have the following important fact ([10, pag. 40]).

Theorem 2.2.3. *Let x, y, z be any positive real numbers. Then there exists a unique up to isometry right-angled geodesic hexagon with non-adjacents sides of length x, y, z respectively.*

The sides of righ-angled geodesic hexagons are related by the following formulaes (see [10, Theorem 2.4.1]).

Proposition 2.2.4. *For any convex right-angled geodesic hexagon with consecutive sides $a, \gamma, b, \alpha, c, \beta$ the following formulaes hold:*

(i) $\cosh(c) = \sinh(a)\sinh(b)\cosh(\gamma) - \cosh(a)\cosh(b)$;

(ii) $\sinh(a) : \sinh(\alpha) = \sinh(b) : \sinh(\beta) = \sinh(c) : \sinh(\gamma)$;

(iii) $\coth(\alpha)\sinh(\gamma) = \cosh(\gamma)\cosh(b) - \coth(a)\sinh(b)$.

We introduce now the main object of this work.

Definition 1. Let M be a closed surface. An atlas $\mathcal{A}$ on M is called *hyperbolic* if it has the following properties:

(i) $\phi(U) \subset \mathbb{H}$, for all $(U, \phi) \in \mathcal{A}$.

(ii) If $(U, \phi), (V, \psi) \in \mathcal{A}$, with $U \cap V \neq \emptyset$, then for each connected component W of $U \cap V$ there exists $T_W \in \mathrm{PSL}(2, \mathbb{R})$ such that $T_W = \psi \circ \phi^{-1}$ on $\phi(W)$.

Definition 2. Let M be a closed surface equipped with a hyperbolic atlas $\mathcal{A}$. We define a *closed hyperbolic surface* by the pair $S = (M, \mathcal{A})$.

It follows from the definition of hyperbolic surface that such surfaces carry a natural Riemannian metric which is locally isometric to the hyperbolic plane. Thus, any hyperbolic surface has constant curvature -1 and is a locally symmetric space.

A nice property of negative curvature is the natural relation between homotopy classes of curves on the surface and geodesics. The next theorem is a special property of hyperbolic metrics, and its proof can be found in [10, Theorem 1.5.3].

Theorem 2.2.5. *Let S be a hyperbolic surface. Let $a, b \in S$ and $c : [0,1] \to S$ a curve with $c(0) = a$ and $c(1) = b$ (if $a = b$ we require that c is homotopically non-trivial). Then the following hold;*

- *In the homotopy class of c fixing the poins a, b there exists a unique shortest curve γ. This curve is a geodesic.*
- *If $a = b$, then c is freely homotopic to a unique closed geodesic γ.*
- *If c is simple, then the geodesics in the items above are simple.*

The topology and geometry of the hyperbolic surface are related by the bijection given by the theorem above:

$$\{\text{free homotopy classes of}\, \pi_1(S)\} \leftrightarrow \{\text{closed geodesics on } S\}. \tag{2.1}$$

Since the set of free homotopy classes of the fundamental group is equal to the conjugacy classes of this group, and for any closed surface its fundamental group is finitely generated, we conclude that the set of conjugacy classes is countable. Therefore, the set of closed geodesics on a closed hyperbolic surface is countable.

Although in our definition of hyperbolic surface the atlas can be very large, we can uniformize the atlas by a unique map. In fact, we have the well known Uniformization Theorem.

Theorem 2.2.6. *For any closed hyperbolic surface S there exists a local isometry $\pi : \mathbb{H} \to S$ which is a covering map.*

This theorem tell us that any hyperbolic surface has a subatlas where any pair (U, ϕ) is of the form $(U, (\pi \restriction_{\tilde{U}})^{-1})$, where $\tilde{U}$ is a component of $\pi^{-1}(U)$ for a fixed covering map $\pi : \mathbb{H} \to S$.

Let S be a hyperbolic surface and $\pi : \mathbb{H} \to S$ be a covering. If we take the deck group $\Gamma = \{T \in \mathrm{PSL}(2, \mathbb{R}) \,|\, \pi \circ T = \pi\}$, then T acts discontinuosly and freely on $\mathbb{H}$. Moreover, the quotient space $\Gamma \backslash \mathbb{H}$ is isometric to S by the natural map $\Gamma z \mapsto \pi(z)$.

Thus we can see any closed hyperbolic surface as a quotient space $\Gamma \backslash \mathbb{H}$ where $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ is a group which acts discontinuosly and freely on $\mathbb{H}$.

Let $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ be the deck transformations group of a covering map from $\mathbb{H}$ to a closed surface S of genus g. Since Γ acts discontinuosly on $\mathbb{H}$ it is a discrete subgroup of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ ([26, Theorem 2.2.6]).

More generally, a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is called a *Fuchsian group*. The nontrivial elements of a Fuchsian group are classified in three types depending on their trace.

Definition 3. Let Γ be a Fuchsian group and $\gamma \in \Gamma$ with $\gamma \neq 1$.

1. We say that γ is *elliptic* if $|\operatorname{tr}(\gamma)| < 2$. If γ is elliptic, then γ has a unique fixed point on $\mathbb{H}$.

2. We say that γ is *parabolic* if $|\operatorname{tr}(\gamma)| = 2$. If γ is parabolic, then γ fixes a unique point on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

3. We say that γ is *hyperbolic* if $|\operatorname{tr}(\gamma)| > 2$. If γ is hyperbolic, then the set of fixed points of γ consists of a set of two points on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

These names come from the geometry of the action of the group $\langle\gamma\rangle$ in $\mathbb{R}^2$ by linear transformations. From the point of view of hyperbolic geometry, the action of each elliptic, parabolic or hyperbolic element has the following caracterization.

Consider $p \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. We define a *horocycle* centered in p as any Euclidean circle contained in $\mathbb{H}$ tangent to p if $p \in \mathbb{R}$, or any horizontal line $\{x + iy_0 \,|\, x \in \mathbb{R}\}$ if $p = \infty$.

Now let $p, q \in \partial\mathbb{H}$ be distinct points and consider the unique geodesic $L_{pq} \subset \mathbb{H}$ whose ends are p, q. We have the following proposition.

Proposition 2.2.7. *Let Γ be a Fuchsian group and $\gamma \in \Gamma$. Then we have:*

1. *If γ is elliptc, then γ is a rotation around its fixed point.*

2. *If γ is parabolic, then γ leaves invariant any horocycle centered in its fixed point in $\partial\mathbb{H}$.*

3. *If γ is hyperbolic, then γ leaves invariant the geodesic L_{pq}, where $\{p, q\}$ is the set of fixed points of γ on $\partial\mathbb{H}$.*

4. If the quotient space $\Gamma\backslash\mathbb{H}$ *is compact, then* Γ *does not contain parabolic elements.*

The proof of these basic facts can be found for example in [26] .

Example 4. Let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some real $\lambda > 1$. Note that A is a hyperbolic element and fixes the points 0 and ∞, therefore A leaves invariant the imaginary axis.

The action on this geodesic is given by $ti \mapsto \lambda^2 ti$ and if we calculate the hyperbolic distance betweeen ti and $A(ti) = \lambda^2 ti$ for any t, we conclude that A acts as a translation of displacement $2\log(\lambda)$.

We can rewrite this displacement as

$$2\log(\lambda) = 2\cosh^{-1}\left(\frac{|\operatorname{tr}(A)|}{2}\right),$$

since $\cosh^{-1}(s) = \log(s + \sqrt{s^2 - 1})$ and $\operatorname{tr}(A) = \lambda + \lambda^{-1}$.

More generally, for any hyperbolic element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ there exists an isometry $T \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\gamma = T \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} T^{-1},$$

where λ is the biggest eigenvalue of γ.

Therefore, the displacement $\operatorname{disp}(\gamma)$ of γ on its invariant geodesic L is the same as the displacement of $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ on the imaginary axis, i.e.

$$\operatorname{disp}(\gamma) = 2\log(\lambda) = 2\cosh^{-1}\left(\frac{|\operatorname{tr}(A)|}{2}\right) = 2\cosh^{-1}\left(\frac{|\operatorname{tr}(\gamma)|}{2}\right),$$

since A and γ have the same trace.

Let $S = \Gamma\backslash\mathbb{H}$ be a closed hyperbolic surface and let $\gamma \in \Gamma$ be nontrivial. Then by item 4 of Proposition 2.2.7 it follows that γ is a hyperbolic element.

Consider a nontrivial conjugacy class C of Γ and take $\gamma \in C$. The invariant geodesic $L_\gamma \subset \mathbb{H}$ of γ projects on a closed geodesic $c \subset S$. Moreover, c does not depend on the choice of $\gamma \in C$.

Conversely, by (2.1) for any closed geodesic $\alpha \subset S$ there exists a unique conjugacy class C_α of Γ such that α is the projection of L_γ for any $\gamma \in C_\alpha$.

Furthermore, if we choose arbitrary $\tilde{\alpha} \in C_\alpha$ for any closed geodesic $\alpha \in S$ of length $l(\alpha)$, then Example 4 gives us the following formula

$$|\operatorname{tr}(\tilde{\alpha})| = 2\cosh\left(\frac{l(\alpha)}{2}\right). \tag{2.2}$$

For a closed surface S, let $\chi(S)$ be the Euler characteristic of S. We have

$$\chi(S) = 2 - 2g, \quad \text{where } g \text{ is the genus of } S.$$

By the Gauss-Bonnet theorem we have for any hyperbolic surface S of genus g the following formula

$$\operatorname{area}(S) = 4\pi(g-1).$$

Therefore, a closed surface admits a Riemannian metric of constant curvature -1 only if $g \geq 2$. Moreover, the topology of such surface determines the area. Thus the area of a hyperbolic surface is not a geometric invariant if we fix a topology.

In order to understand the geometry of a hyperbolic surface we need to study their geometric invariants. In the following paragraphs we will mention two invariants, one metric and other analytical which a priori are of a very different nature but as we will see, in fact, they are two sides of the same coin.

Let M be a compact nonsimply connected Riemannian manifold. We define the *systole* of M as the infimum of the lengths of homotopically non trivial closed paths on M, and we denote this geometric invariant by $\operatorname{sys}(M)$.

Let S be a closed hyperbolic surface. Since the set of the lengths of the closed geodesics on S is discrete, we can give the following definition.

Definition 4. The *length spectrum* of S is the ordered set

$$L(S) = \{l(\gamma_1) \leq l(\gamma_2) \leq \cdots\}$$

of lengths of the closed geodesics of S.

In this context we can define the systole of S as the bottom of the length spectrum of S.

If we fix a genus $g \geq 2$, there is no lower bound for the systole of hyperbolic surfaces of genus g. Indeed, we will see in Chapter 4 ([Theorem 4.1.3)]) that for any $\varepsilon > 0$, we can find a surface S_ε of genus g and systole at most ε.

On the other hand, we have the following theorem of Gromov (see [18, 2.C.]).

Theorem 2.2.8. *Let M be a closed connected surface of genus $g(M) \geq 2$ with a Riemannian metric. Then,*

$$\mathrm{sys}(M)^2 \leq C\frac{(\log(g(M)))^2}{g(M)}\,\mathrm{area}(M),$$

where C is a universal constant.

In Gromov's proof the constant C is very large. Later on, Katz and Sabourau showed in [28, Theorem 2.2] that $C \leq \frac{1}{\pi}(1+o(1))$ when g goes to infinity.

In the particular case where S is a hyperbolic surface, if S has systole $\mathrm{sys}(S)$ it is easy to see that at each point $p \in S$ we can embed a hyperbolic disk of radius $\frac{\mathrm{sys}(S)}{2}$. Therefore,

$$\mathrm{area}\left(D\left(p, \frac{\mathrm{sys}(S)}{2}\right)\right) = 2\pi\left(\cosh\left(\frac{\mathrm{sys}(S)}{2}\right) - 1\right) \leq \mathrm{area}(S).$$

Hence,

$$\mathrm{sys}(S) \leq 2\log(\mathrm{area}(S)) + A \tag{2.3}$$

for some universal constant $A > 0$.

Remark 1. The estimate of the constant C given by Katz and Sabourau shows that for any $\varepsilon > 0$ there exists $g_0 = g_0(\varepsilon) \geq 2$ such that for any closed hyperbolic surface S of genus $g \geq g_0$,

$$\mathrm{sys}(S)^2 \leq \frac{\log(g)^2(4\pi(g-1))}{\pi g}(1+\varepsilon) \leq (2\log(g))^2(1+\varepsilon). \tag{2.4}$$

Hence inequality (2.3) is stronger than (2.4) in the hyperbolic case.

Let M be a complete n-dimensional Riemannian manifold. There exists a generalization of the Laplace operator from the Euclidean space to M. The *Laplace-Beltrami operator* is defined by

$$\Delta u = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{j,k=1}^{n} \partial_j(\mathcal{G}^{jk}\sqrt{\mathcal{G}}\partial_k u)$$

for any smooth function $u : M \to \mathbb{R}$, where $\mathcal{G} = \det(\mathcal{G}_{ij})$ is the determinant of the metric tensor, $\mathcal{G}_{ij}$ are the components of the metric tensor with respect to local coordinates and $\left(\mathcal{G}^{jk}\right) = (\mathcal{G}_{jk})^{-1}$.

If M is closed, we have the following important theorem ([10, Theorem 7.2.6]).

Theorem 2.2.9 (Spectral Theorem). *If M is a closed Riemannian manifold, then the Hilbert space $L^2(M)$ has a complete orthonormal system of C^∞-eigenfunctions $\phi_0, \phi_1, \ldots$ correponding to the eigenvalues $\lambda_0 = 0 < \lambda_1 < \cdots$ of the Laplace-Beltrami operator, i.e. $\Delta\phi_i = \lambda_i\phi_i$ for all i.*

The motivation for the definition of the length spectrum of a hyperbolic surfaces comes from the following classical definition.

Definition 5. The *spectrum* of a closed Riemannian manifold M is the ordered set

$$\Lambda(M) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$$

of the eigenvalues of the Laplace-Beltrami operator on M.

The spectrum of a closed Riemannian manifold encodes a lot of information about its geometry. For example, Weyl's asymptotic law says that for any n there exists a constant $a(n)$ such that for any closed Riemannian manifold M of dimension n, its spectrum satisfies

$$\lambda_k^{\frac{n}{2}} \sim k\frac{a(n)}{\mathrm{vol}(M)},$$

where as usual the relation $f(x) \sim g(x)$ means that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

If $n = 2$ then $a(2) = 4\pi$ and this limit shows that the spectrum of M determines the area of M (see [10, Theorem 9.2.14]).

To finish this section we will recall an important theorem due to Huber, which connects two aparently very diffent objects, the length spectrum and the spectrum of a hyperbolic surfaces (see [10, Theorem 9.2.9])

Theorem 2.2.10. *Two closed hyperbolic surfaces have the same spectrum if and only if they have the same length spectrum.*

CHAPTER 3

SEMI-ARITHMETIC AND ARITHMETIC SURFACES

3.1 Quaternion Algebras.

Let k be a field of characteristc other than 2.

Definition 6. A ring D with unit is a k-*algebra* if D has the structure of a k-vector space compatible with the operations on D, i.e.

$$\lambda(xy) = (\lambda x)y = x(\lambda y) \quad \text{for all} \quad \lambda \in k \quad \text{and} \quad x, y \in D.$$

The main example to have in mind is the algebra $M_n(k)$ of $n \times n$ matrices with entries in k.

A k-subspace $A \subset D$ is a left (respectively, right) *ideal* if $xA \subset A$ (respectively, $Ax \subset A$) for all $x \in A$. We say that D is *simple* if the only two-side (i.e. left and right at the same time) ideals of D are the zero ideal and the whole D.

The algebra D is *central* if $k = \mathcal{Z}(D)$, where

$$\mathcal{Z}(D) = \{x \in D \,|\, xy = yx \quad \text{for all} \quad y \in D\}.$$

Definition 7. A k-algebra is a *central simple algebra* if it is central and simple.

Example 5. We say that D is a *division* algebra if every non-zero element of D has a multiplicative inverse. In this case if $A \subset D$ is a left/right ideal of D and $A \neq 0$, then $1 = aa^{-1} \in A$. Thus any division algebra is simple.

Example 6. Let k be a number field, i.e. k is a finite extension of $\mathbb{Q}$. Note that k is a simple $\mathbb{Q}$-algebra but it is not central since $\mathcal{Z}(k) = k$.

A central simple algebra is a very rigid algebraic object. In fact, we have the following theorems.

Theorem 3.1.1 (Skolem-Noether). *Let k be a field, D a central simple algebra over k of finite dimension and D' a finite dimensional simple k-algebra. If $f, g : D' \to D$ are algebra homomorphisms, then there exists an element $x \in D^*$ such that*

$$f(u) = xg(u)x^{-1} \quad \textit{for all} \quad u \in D'.$$

Theorem 3.1.2 (Wedderburn). *Let D be a central simple algebra over a field k. Then there exists a central division algebra A over k such that $D \cong M_n(A)$ for some $n \geq 1$.*

A very useful corollary of Theorem 3.1.1 is

Corollary 3.1.3 (Skolem-Noether Theorem). *Every automorphism of a central simple algebra over a field k is an inner automorphism.*

Definition 8. A *quaternion algebra over* a field k is a central simple algebra over k of dimension 4.

Example 7. Consider the central simple algebra $D = M_2(k)$. Since $M_2(k)$ has dimension 4 over k, the algebra D is a quaternion algebra. If we identify the identity matrix of D and the unit of k, we have a simple presentation for D as follows: $D = \{a1 + bI + cJ + dIJ \,|\, I^2 = J^2 = 1,\ a, b, c, d \in k\}$, where

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

More generally, it follows from Theorems 3.1.2 and 3.1.1 that any quaternion algebra over a field k has a simple presentation.

Theorem 3.1.4. *Let D be a quaternion algebra over a field k of characteristc 0. Then there exists a basis $\{1, I, J, K\}$ of D such that*

$$I^2 = a, \quad J^2 = b \text{ and } IJ = -JI = K, \quad \text{where } a, b \in k^*.$$

Proof. If D is not a division algebra, then by Theorem 3.1.2 we have that $D \cong M_2(k)$. In this case we have already given a presentation in Example 7.

If D is a division algebra, let $x \in D$ such that $x \notin k$. Let $k(x)$ be the minimal subring of D containing k and x.

Since x is invertible, $k(x)$ is a field extension of k generated by 1 and x, i.e. $k(x)$ is a quadratic extension of a field of characteristc 0. Hence, there exists an element $I \in k(x)$ such that $I^2 \in k$ and $k(x) = k(I)$.

If we take the nontrival isomorphism

$$\sigma : k(I) \to k(I) \quad \sigma(I) = -I,$$

then if we see $\sigma : k(I) \to D$ as an algebra homomorphism, we can apply Theorem 3.1.1 in order to find $J \in D^*$ such that $\sigma(t) = JtJ^{-1}$. In particular,

$$-I = \sigma(I) = JIJ^{-1}.$$

We want to prove that $\{1, I, J, IJ\}$ is a basis for D and $J^2 \in k^*$.

Firstly, we can see that $J \notin k(I)$, otherwise we can apply σ on J and $\sigma(J) = J$. Hence $J \in k$ and σ is the trivial isomorphism.

Since D has dimension 4 over k, we only need to check that $\{1, I, J, IJ\}$ is linearly independent. Suppose that there exist $\alpha, \beta, \gamma \in k$ such that

$$IJ = \alpha + \beta I + \gamma J.$$

We use now the assumption that D is a division algebra in order to write

$$J = \frac{\alpha + \beta I}{I - \gamma} \in k(I).$$

This contradiction shows that $\{1, I, J, IJ\}$ is a basis of D.

The algebra D is central. Thus if we check that J^2 commutes with I it will follow that $J^2 \in \mathcal{Z}(D) = k$ and the proof will finish. By definition of J we have indeed

$$J^2 I = J(JI) = J(-IJ) = (-JI)J = (IJ)J = IJ^2.$$

□

We have the following classical corollary.

Corollary 3.1.5. *There exist up to isomorphism only two quaternion algebras over $\mathbb{R}$, namely the algebra of matrices $M_2(\mathbb{R})$ and the Hamilton algebra $\mathcal{H} = \{t + xI + yJ + zK \mid I^2 = J^2 = -1\,,\ IJ = -JI = K\}$.*

For an arbitrary D we have a generalization of trace and determinant of a matrix. Let D be a quaternion algebra over a field k with basis $\{1, I, J, K\}$ as in Theorem 3.1.4.

Definition 9. For any $x = x_0 + x_1 I + x_2 J + x_3 K \in D$ we define its *conjugate, reduced norm* and *trace*, respectively, by:

1. (conjugate): $\bar{x} = x_0 - x_1 I - x_2 J - x_3 K$.
2. (reduce norm): $\mathrm{rn}(x) = x\bar{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$.
3. (trace): $\mathrm{tr}(x) = x + \bar{x} = 2x_0$.

Remark 2. Note that a priori the definition of conjugate, reduced norm and trace depends on the choice of the basis. However, it is a straighforward manipulation to show that if we have another basis $\{1, I', J', I'J'\}$ with $I'^2, J'^2 \in k^*$, then these definitions do not depend on the choice.

Example 8. Let us choose in $M_2(k)$ the basis $\{id, I, J, IJ\}$, where

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k)$ we have

$$A = \frac{a+d}{2} id + \frac{b+c}{2} I + \frac{a-d}{2} J + \frac{c-b}{2} K,$$

with $a + d = A + \bar{A}$ and $ad - bc = (\frac{a+d}{2})^2 - (\frac{b+c}{2})^2 - (\frac{a-d}{2})^2 + (\frac{c-b}{2})^2 = A\bar{A}$.

3.2 Quaternion algebras over number fields.

Throughout this section, k will denote a *totally real* number field, i.e. k is a finite extension of $\mathbb{Q}$ such that for any Galois embedding $\sigma : k \to \mathbb{C}$ we have $\sigma(k) \subset \mathbb{R}$. The ring of integers of k will be denoted by R_k. Recall that k is the field of fractions of R_k.

Let D be a quaternion algebra over k, an *ideal* L in D is a finitely generated R_k-module of rank 4 such that any R_k-basis of L is a k-basis of D. Note that this definition of ideal is not related with the definition of ideal in previous section.

An *order* $\mathcal{O}$ in D is an ideal which is also a subring of D containing 1. An order is maximal if it is not properly contained in any other order.

Example 9. Consider $k = \mathbb{Q}$ and $D = M_2(\mathbb{Q})$. The subring $M_2(\mathbb{Z})$ is a maximal order.

Indeed, it is easy to check that $M_2(\mathbb{Z})$ is an order of D. Let $\mathcal{O}$ be an order containing $M_2(\mathbb{Z})$. If there exists $U = \begin{pmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_2} \\ \frac{a_3}{b_3} & \frac{a_4}{b_4} \end{pmatrix} \in \mathcal{O}$ with g.c.d$(a_i, b_i) = 1$ and $|b_j| > 1$ for some j, then we can suppose that $j = 1$ since we can multiply U by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the left or on the right and in at most two steps any entry of U transforms to the first one. Now we have

$$U' = L_2(UL_1 - M_1) - M_2 = \begin{pmatrix} \frac{a_1}{b_1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{O},$$

where

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & b_2 b_4 \end{pmatrix}, L_2 = \begin{pmatrix} 1 & 0 \\ 0 & b_3 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & a_2 b_4 \\ 0 & a_4 b_2 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ a_3 & 0 \end{pmatrix}.$$

Thus

$$V = U' + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{a_1}{b_1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O}.$$

The submodule $\mathbb{Z}[V]$ has finite dimension because it is contained in the ideal $\mathcal{O}$, but this implies that the extension $\mathbb{Z}[\frac{a_1}{b_1}]$ is finite, which gives a contradiction since $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

Let $n = [k : \mathbb{Q}]$ be the degree of the extension and let $\phi_1, \ldots, \phi_n : k \to \mathbb{R}$ be the Galois embeddings. Let D be a quaternion algebra over k.

Definition 10. For an arbritary embedding $\sigma : k \to \mathbb{R}$ we define the *extension of scalars* algebra $D \otimes_\sigma \mathbb{R}$ as the $(k, \mathbb{R})$-bimodule with a natural algebra structure and $\mathbb{R}$ considered as a k-vector space via the map σ, i.e. the product of an element $\lambda \in k$ on $t \in \mathbb{R}$ is given here by $\sigma(\lambda)t$.

In a more concrete way we have the following proposition.

Proposition 3.2.1. *Let D be a quaternion algebra over a totally real number field k and let $\sigma : k \to \mathbb{R}$ be a Galois embedding. If D has a basis $\{1, I, J, IJ\}$ with $IJ = -JI$, $I^2 = a$ and $J^2 = b$ where $a, b \in k^*$, then $D \otimes_\sigma \mathbb{R}$ has a basis $\{1, I', J', I'J'\}$ with $I'J' = -J'I'$, $I'^2 = \sigma(a)$ and $J'^2 = \sigma(b)$.*

Proof. Recall that the $\mathbb{R}$-structure of $D \otimes_\sigma \mathbb{R}$ is given by the product $s(x \otimes t) = x \otimes (st)$. If we take the set

$$\{1 = 1 \otimes 1,\ I' = I \otimes 1,\ J' = J \otimes 1,\ I'J' = IJ \otimes 1\},$$

then clearly it generates $D \otimes_\sigma \mathbb{R}$ as an $\mathbb{R}$-vector space.

Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be such that $\alpha_0 + \alpha_1 I' + \alpha_2 J' + \alpha_3 I'J' = 0$. If we consider a basis $\{t_\nu\}_{\nu \in \mathbb{N}}$ of $\mathbb{R}$ as a k-vector space (by σ), we can write

$$\alpha_i = \sum_\nu \sigma(\beta_{\nu i}) t_\nu \quad \text{for each} \quad i = 0, 1, 2, 3.$$

Therefore,

$$0 = \sum_\nu (\beta_{0\nu} + \beta_{1\nu} I + \beta_{2\nu} J + \beta_{3\nu} IJ) \otimes t_\nu.$$

Since the set $\{1 \otimes t_\nu, I \otimes t_\nu, J \otimes t_\nu, IJ \otimes t_\nu\}_\nu$ is a k basis of $D \otimes_\sigma \mathbb{R}$, it follows that $\beta_{i\nu} = 0$ for any pair (i, ν). Hence, $\alpha_i = 0$ for each $i = 0, 1, 2, 3$.

Note that by properties of tensor product and the definition of the structures on $D \otimes_\sigma \mathbb{R}$, we have

$$I'^2 = I^2 \otimes 1 = a \otimes 1 = 1 \otimes \sigma(a) = \sigma(a)(1 \otimes 1),$$

and

$$J'^2 = J^2 \otimes 1 = b \otimes 1 = 1 \otimes \sigma(b) = \sigma(b)(1 \otimes 1).$$

□

It follows from Corollary 3.1.5 that there are two possibilities for $D \otimes_\sigma \mathbb{R}$, either it is isomorphic to $M_2(\mathbb{R})$ or it is isomorphic to the Hamilton algebra $\mathcal{H}$. We say that D is *ramified* at σ if $D \otimes_\sigma \mathbb{R} \cong \mathcal{H}$, and that D *splits* at σ otherwise.

Proposition 3.2.2. *Let D be a quaternion algebra over a totally real number field k. If D splits at the embedding σ of k in $\mathbb{R}$, then there exists a basis $\{1, I, J, IJ\}$ of D such that $IJ = -JI$, $I^2 = a, J^2 = b$ with $a, b \in k^*$ and $\sigma(a) > 0$.*

Proof. Take a basis $\{1, I, J, IJ\}$ given by Proposition 3.1.4 with $I^2 = a', J^2 = b'$ for some $a', b' \in k^*$. We need to show that $\sigma(a') > 0$ or $\sigma(b') > 0$. Indeed, if $\sigma(a') < 0$ and $\sigma(b') < 0$, then by Proposition 3.2.1, $D \otimes_\sigma \mathbb{R}$ has a real basis $\{1, I', J', I'J'\}$ with $I'^2 = \sigma(a')$ and $J'^2 = \sigma(b')$. Hence, for any

$$x = x_0 + x_1 I' + x_2 J' + x_3 I'J' \neq 0$$

we have

$$\mathrm{rn}(x) = x_0^2 - \sigma(a')x_1^2 - \sigma(b')x_2^2 + \sigma(a'b')x_3^2 > 0.$$

Therefore, each nonzero $x \in D \otimes_k \mathbb{R}$ has an inverse $\mathrm{rn}(x)^{-1}\bar{x}$, i.e. D ramifies at the embedding σ. This contradiction implies the proposition since we can switch I and J if necessary. $\square$

Let D be a quaternion algebra over a totally real number field k such that D splits at the trivial embedding. Take in D a basis $\{1, I, J, IJ\}$ given by Proposition 3.2.2.

We can embed D in $M_2(\mathbb{R})$ by the map

$$\psi(x_0 + x_1 I + x_2 J + x_3 K) = \begin{pmatrix} x_0 - \sqrt{a}x_1 & b(x_2 - x_3\sqrt{a}) \\ x_2 - x_3\sqrt{a} & x_0 + \sqrt{a}x_1 \end{pmatrix}.$$

Note that

$$\mathrm{tr}(\psi(x)) = \mathrm{tr}(x) \quad \text{and} \quad \det(\psi(x)) = \mathrm{rn}(x),$$

for any $x \in D$. Hence, if we take an order $\mathcal{O}$ of D, the set

$$\Gamma(D, \mathcal{O}) = \{x \in \mathcal{O} \mid \mathrm{rn}(x) = 1\}$$

is a multiplicative group and it is identified via ψ with a subgroup of $\mathrm{SL}(2,\mathbb{R})$.

If we take another embedding ψ' from D into $M_2(\mathbb{R})$, then by Theorem 3.1.1 there exists an invertible matrix $A \in M_2(\mathbb{R})$ (we can suppose that $\det(A) = \pm 1$) such that

$$\psi'(x) = A\psi(x)A^{-1} \quad \text{for all} \quad x \in D.$$

3.3 Semi-arithmetic surfaces admitting modular embedding.

We say that two groups $\Gamma_1, \Gamma_2 < \mathrm{SL}(2,\mathbb{R})$ are *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. For any group $\Lambda < \mathrm{PSL}(2,\mathbb{R})$ we denote by $\tilde{\Lambda}$ the preimage of Λ in $\mathrm{SL}(2,\mathbb{R})$ by the natural projection $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$.

Definition 11. Let $S = \Gamma\backslash\mathbb{H}$ be a closed hyperbolic surface, we say that S is a *semi-arithmetic surface* if $\tilde{\Gamma}$ is commensurable to a subgroup $\Sigma < \Gamma(D, \mathcal{O})$ for some order $\mathcal{O}$ in a quaternion algebra D over a totally real number field k which splits at the trivial embedding of k.

If $\tilde{\Gamma}$ is contained in $\Gamma(D, \mathcal{O})$ we say that S is a semi-arithmetic surface *derived from a quaternion algebra.*

There is an equivalent definition of semi-arithmetic surfaces due to the work of Takeuchi from which it is often easier to check if a specific surface is semi-arithmetic or not:

Let $S = \Gamma\backslash\mathbb{H}$ be a closed hyperbolic surface. We define the *invariant trace field* of S (or Γ) as the field

$$L = \mathbb{Q}(\mathrm{tr}(\tilde{\Gamma}^2)) \quad \text{where} \quad \mathrm{tr}(\tilde{\Gamma}^2) = \{|\,\mathrm{tr}(\gamma^2)|\,|\,\gamma \in \tilde{\Gamma}\}.$$

We say that S is a semi-arithmetic surface if its invariant trace field is a totally real number field and the set $\mathrm{tr}(\tilde{\Gamma}^2)$ is contained in the ring of integers of this number field. For a proof of the equivalence of this definition and Definition 11 see [47, Proposition 1].

Definition 12. A semi-arithmetic closed hyperbolic surface $S = \Gamma\backslash\mathbb{H}$ with invariant trace field k' is said to be *arithmetic* if for any nontrivial embedding $\sigma : k' \to \mathbb{R}$ it holds that

$$\sigma(\mathrm{tr}(\tilde{\Gamma}^2)) \subset [-2, 2].$$

In the class of semi-arithmetic surfaces there exists a special subclass of surfaces, namely, the surfaces which admit a modular embedding.

Let D be a quaternion algebra over a totally real number field k of degree n with embeddings $\phi_1, \ldots, \phi_n : k \to \mathbb{R}$, with ϕ_1 being the identity and such that D is ramified only at $\phi_{r+1}, \cdots, \phi_n$ for some $r \geq 1$.

Let $\mathcal{O}$ be an order in D and let $S = \Gamma\backslash\mathbb{H}$ be a semi-arithmetic closed surface with $\tilde{\Gamma} < \psi(\Gamma(D, \mathcal{O}))$ for some injective homomorphism of algebras $\psi : D \to M_2(\mathbb{R})$.

In this case we can check easily that $k' = \mathbb{Q}(\mathrm{tr}(\tilde{\Gamma})^2) = k$, i.e. the invariant trace field of S coincides with k when S is derived from a quaternion algebra over k.

For each $i = 1, \cdots, r$, since $D \otimes_{\phi_i} \mathbb{R} \simeq M_2(\mathbb{R})$ by the Proposition 3.2.2 we can find a basis $\{1, I_i, J_i, I_iJ_i\}$ of D such that

$$I_iJ_i = -J_iI_i, \quad I_i^2 = a_i, \quad J_i^2 = b_i \quad \text{and} \quad \phi_i(a_i) > 0.$$

Hence, for each $i = 1, \cdots, r$ we can define the injective homomorphism of algebras $\psi_i : D \to M_2(\mathbb{R})$ given by

$$\psi_i(x) = \begin{pmatrix} \phi_i(x_0) - \sqrt{\phi_i(a_i)}\phi_i(x_1) & \phi_i(b_i)(\phi_i(x_2) - \phi_i(x_3)\sqrt{\phi_i(a_i)}) \\ \phi_i(x_2) - \phi_i(x_3)\sqrt{\phi_i(a_i)} & \phi_i(x_0) + \sqrt{\phi_i(a_i)}\phi_i(x_1) \end{pmatrix},$$

where $x = x_0 + x_1I_i + x_2J_i + x_3I_iJ_i \in D$. Note that ψ_i is a homomorphism of k-algebras, when we define a k-structure on $\mathbb{R}$ via the map ϕ_i (see Definition 10).

Therefore, if we fix $i = 1, \cdots, r$, then for each $\gamma \in \tilde{\Gamma}$ we can define

$$\gamma^{\phi_i} := \psi_i(x) \quad \text{if} \quad \gamma = \psi_1(x).$$

Definition 13. We say that a closed hyperbolic surface $S = \Gamma\backslash\mathbb{H}$ *admits a r-modular embedding* or simply *admits a modular embedding* if S is a semi-arithmetic surface derived from a quaternion algebra over a field k, and there exist exactly r Galois embeddings $\phi_1, \phi_2, \cdots, \phi_r : k \to \mathbb{R}$ with $\phi_j(\mathrm{tr}(\tilde{\Gamma})^2) \nsubseteq [-2, 2]$ such that for each $j = 1, \cdots, r$ the functional equation

$$F(\gamma^{\phi_j} \cdot z) = \gamma^{\phi_j} \cdot F(z) \text{ for all } z \in \mathbb{H}, \gamma \in \tilde{\Gamma}.$$

is solved by a holomorphic map $F_j : \mathbb{H} \to \mathbb{H}$.

Remark 3. The name *modular embedding* comes from the fact that the group $\Gamma(D, \mathcal{O})$ acts on $\mathbb{H}^r$ properly and discontinuosly via the map $\Psi = (\psi_1, \cdots, \psi_r)$, and up to a finite index, this action is free. If we consider the manifold $M = \Gamma(D, \mathcal{O})\backslash\mathbb{H}^r$, then M is a locally symmetric space and the map $F = (F_1, \cdots, F_r) : \mathbb{H} \to \mathbb{H}^r$ induces an embedding $\bar{F} : S \to M$. By the Schwarz-Pick Lemma, F_i is a 1-Lipschitz map for any i with respect to the hyperbolic metric. Thus F is a $\sqrt{r}$-Lipschitz map.

In the particular case of $D = M_2(k)$ and $\mathcal{O} = M_2(R_k)$, the corresponding manifold M is known as a *Hilbert Modular manifold* (see [39]).

Example 10. The simplest example of semi-arithmetic surfaces admitting modular embeddings are the arithmetic surfaces derived from quaternion algebras. Indeed, let $S = \Gamma\backslash\mathbb{H}$ with $\tilde{\Gamma} < \Gamma(D, \mathcal{O})$ for some quaternion algebra D defined over a field k. It follows from the definition of arithmetic surfaces that we need only to solve

$$F(\gamma \cdot z) = \gamma \cdot F(z) \text{ for all } z \in \mathbb{H}, \gamma \in \tilde{\Gamma}$$

for some homolomorphic map. The indentity map of $\mathbb{H}$ is a trivial solution.

Example 11. The main source of examples of semi-arithmetic surfaces admitting modular embedding are the hyperbolic triangle groups $\Delta(l, m, n)$, where $l, m, n \in \mathbb{Z}_{\geq 2}$ such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

Geometrically, these groups are orientation-preserving subgroups of the groups of isometries of $\mathbb{H}$ generated by the reflections in the sides of a hyperbolic triangle with angles $\dfrac{\pi}{l}, \dfrac{\pi}{m}, \dfrac{\pi}{n}$.

Algebraically, these groups are discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ given by the presentation

$$\Delta(l, m, n) = \langle A, B, C \mid A^l = B^m = C^n = ABC = 1\rangle.$$

Any two discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ with this presentation are conjugate (see [49, Lemma 1]). It follows from the main theorem in [12] that any torsion free finite index subgroup of a triangle group gives a closed hyperbolic surface admitting a modular embedding.

In their proof Cohen and Wolfart construct the holomorphic maps using the reflection principle and the geometric fact that such triangles give a tesselation of the hyperbolic plane.

Remark 4. It is a well known result due to Takeuchi (see [49]) that there exist only a finite number of triangle groups which contain a torsion free subgroup Γ such that the surface $S = \Gamma\backslash\mathbb{H}$ is arithmetic.

Example 11 shows that the class of semi-arithmetic surfaces admitting modular embedding is larger than the class of arithmetic surfaces. At the present moment there is no known example of a semi-arithmetic closed surface admitting modular embedding which is not arithmetic or triangular (we say that $S = \Gamma\backslash\mathbb{H}$ is triangular if Γ is a finite index subgroup of a triangular group).

A crucial fact about surfaces which admit a modular embedding is that the holomorphic maps in the definition of the modular embedding in general are not necessarily isometries. In fact, we have that with the exception of the identity, which is equivariant to the trivial embedding, the holomorphic maps which appear in the definition of modular embedding are contractions. We have the following proposition.

Proposition 3.3.1 ([47])**.** *Let $S = \Gamma\backslash\mathbb{H}$ be a closed hyperbolic surface admiting a modular embedding and let $k = \mathbb{Q}(\{\mathrm{tr}(\gamma)^2\})$ be its invariant trace field. Then for all nontrivial embeddings $\phi : k \to \mathbb{R}$ we have*

$$|\phi(\mathrm{tr}(\gamma))| < |\,\mathrm{tr}(\gamma)|,$$

for every nontrivial $\gamma \in \Gamma$.

Example 12. Let Q be a hyperbolic quadrilateral such that three of the interior angles are $\frac{\pi}{2}$ while the fourth is $\frac{\pi}{3}$.

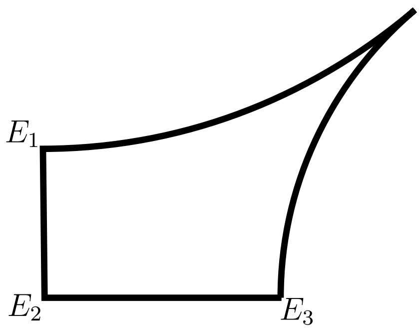


Figure 3.1: Quadrilateral Q.

Let E_i, $i = 1, 2, 3$, be the three vertices of Q correponding to angles $\frac{\pi}{2}$, in the natural order. There are a lot of choices for Q. We normalize Q so that

$$2\cosh(d(E_1, E_2)) = 7 - \sqrt{5}. \tag{3.1}$$

Hence, we have a Fuchsian group Γ_0 with the set of generators $\{U_1, U_2, U_3, V\}$, where U_1, U_2, U_3 are involutions with fixed points E_1, E_2, E_3, respectively, and V fixes the other vertex of Q.

The choice of the generators is such that a mirror image of Q obtained by the reflection in the side E_1E_2 is a fundamental domain for the action of Γ_0.

Let Γ_0' the subgroup of Γ_0 generated by the set $\{x = U_1U_2, y = U_2U_3, z = U_1U_3, V\}$. We can check using these generators (see [49]) that

$$\mathrm{tr}(\Gamma_0') \subset \mathbb{Z}[\mathrm{tr}(x), \mathrm{tr}(y), \mathrm{tr}(z), \mathrm{tr}(V)],$$

Using hyperbolic trigonometry, the construction of Q, and the normalization (3.1), we have that

$$\mathrm{tr}(x)^2 = 7 - \sqrt{5}, \quad \mathrm{tr}(y)^2 = 7 + \sqrt{5}, \quad \mathrm{tr}(z)^2 = 11, \quad \mathrm{tr}(V)^2 = 1. \tag{3.2}$$

Thus $\mathbb{Q}(\mathrm{tr}(\Gamma_0'^2)) = \mathbb{Q}(\sqrt{5}) = k$ and $\mathrm{tr}(\Gamma_0'^2) \subset R_k = \mathbb{Z}[\sqrt{5}]$, which implies that any torsion-free subgroup of Γ_0' gives a closed semi-arithmetic surface. Now supppose that $S = \Gamma\backslash\mathbb{H}$ with $\Gamma < \Gamma_0'$ admits a modular embedding.

Let $\sigma : k \to \mathbb{R}$ be the nontrivial embedding of k which switches $\sqrt{5}$ and $-\sqrt{5}$. Since k is the invariante trace field of $\tilde{\Gamma}$ and $\sigma(\mathrm{tr}\,\tilde{\Gamma}) \not\subseteq [-2.2]$, by definition we have a holomorphic map $F : \mathbb{H} \to \mathbb{H}$ such that $F(\gamma \cdot z) = \gamma^\sigma F(z)$ for all $z \in \mathbb{H}$ and $\gamma \in \tilde{\Gamma}$.

It follows from Proposition 3.3.1 that

$$|\sigma(\mathrm{tr}(\gamma))| < |\,\mathrm{tr}(\gamma)| \quad \text{for every nontrivial } \gamma.$$

By (3.2) we have,

$$\mathrm{tr}(x^2) = \mathrm{tr}(x)^2 - 2 = 5 - \sqrt{5} \quad \text{and} \quad \mathrm{tr}(y^2) = \mathrm{tr}(y)^2 - 2 = 5 + \sqrt{5}.$$

On the other hand, if we apply the formula $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$, by induction we have

$$\sigma(\mathrm{tr}(x^{2l})) = \mathrm{tr}(y^{2l}) > \mathrm{tr}(x^{2l}) > 0, \tag{3.3}$$

for any natural l.

Since $\tilde{\Gamma}$ has finite index in Γ_0, there exists $l > 1$ such that $x^{2l}, y^{2l} \in \tilde{\Gamma}$ and inequality (3.3) contradicts Proposition 3.3.1. Therefore S is a semi-arithmetic surface which does not admit a modular embedding.

3.4 Congruence coverings of semi-arithmetic surfaces.

Let $S = \Gamma\backslash\mathbb{H}$ be a semi-arithmetic closed surface derived from a quaternion algebra. Thus there exist a quaternion algebra D over a totally real number field k, a maximal order $\mathcal{O} \subset D$ and an embedding $\psi : D \to M_2(\mathbb{R})$ such that $\tilde{\Gamma} < \psi(\Gamma(D, \mathcal{O}))$.

Let $\mathfrak{a} \subset R_k$ be an ideal of the ring R_k. We can suppose that $R_k \subset \mathcal{O}$ since by assumption $\mathcal{O}$ is maximal. Thus, the product subring $\mathfrak{a}\mathcal{O} \subset \mathcal{O}$ is an order of D. Hence, the quotient ring $\mathcal{O}/\mathfrak{a}\mathcal{O}$ is finite.

Definition 14. The *principal congruence covering* $S_{\mathfrak{a}} = \Gamma(\mathfrak{a})\backslash\mathbb{H}$ of S of level $\mathfrak{a}$ is given by

$$\tilde{\Gamma}(\mathfrak{a}) = \mathrm{Ker}\left(\tilde{\Gamma} \xrightarrow{\psi^{-1}} \Gamma(D, \mathcal{O}) \to (\mathcal{O}/\mathfrak{a}\mathcal{O})^*\right),$$

where the second map is the natural projection.

Recall that for any ideal $\mathfrak{a} \subset R_k$ its norm is defined by the index $\mathrm{N}(\mathfrak{a}) := [R_k : \mathfrak{a}]$ as additive groups.

We want to understand the geometry of $S_{\mathfrak{a}}$ when $\mathfrak{a}$ varies. For this we need of the following proposition (see [29, Corollary 4.6]).

Proposition 3.4.1. *Let D be a quaternion algebra over a totally real number field k and let $\mathcal{O}$ be a maximal order of D. There exists a constant $\lambda = \lambda(D, \mathcal{O})$ such that for any group $\Lambda < \Gamma(D, \mathcal{O})$ and for any ideal $\mathfrak{a} \subset R_k$ we have*

$$[\Lambda : \Lambda(\mathfrak{a})] \leq \lambda \, \mathrm{N}(\mathfrak{a})^3,$$

where

$$\Lambda(\mathfrak{a}) = \mathrm{Ker}\left(\Gamma(D, \mathcal{O}) \to (\mathcal{O}/\mathfrak{a}\mathcal{O})^*\right).$$

We have the following immediate corollary.

Corollary 3.4.2. *Let $S = \Gamma \backslash \mathbb{H}$ be a semi-arithmetic closed hyperbolic surface derived from a quaternion algebra. There exists a constant $C > 0$ which depends only on the quaternion algebra and the order in the definition of S such that for any principal congruence covering $S_{\mathfrak{a}}$ of S we have*

$$\mathrm{area}(S_{\mathfrak{a}}) \leq C \, \mathrm{N}(\mathfrak{a})^3 \, \mathrm{area}(S).$$

CHAPTER 4

GEOMETRY OF TEICHMULLER AND MODULI SPACES OF CLOSED HYPERBOLIC SURFACES

4.1 The Teichmuller space

Denote by $\mathcal{F}_g$ the set of all Fuchsian groups $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ such that $\Gamma\backslash\mathbb{H}$ is a closed hyperbolic surface of genus g. Evidently, $\Lambda_1\backslash\mathbb{H}$ is isometric to $\Lambda_2\backslash\mathbb{H}$ by an orientation-preserving isometry if and only if Λ_1 and Λ_2 are conjugate subgroups, $\Lambda_1 = T\Lambda_2 T^{-1}$, $T \in \mathrm{PSL}(2,\mathbb{R})$.

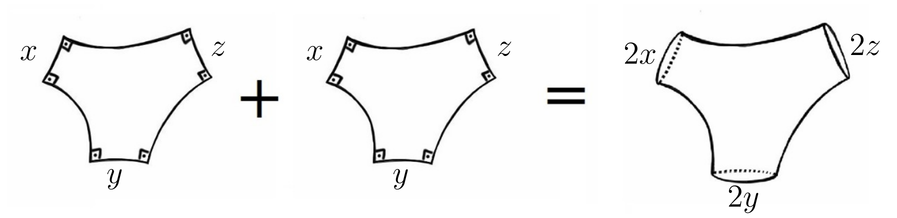
We define $\mathcal{M}_{\mathrm{g}}$ as the set of equivalence classes of closed hyperbolic surfaces of genus g where two surfaces S_1 and S_2 are equivalent if there exists an orientation-preserving isometry between them. Thus $\mathcal{M}_{\mathrm{g}}$ can be identified with the quotient of $\mathcal{F}_g$ modulo conjugation by elements of $\mathrm{PSL}(2,\mathbb{R})$.

Denote the fundamental group of a closed surface of genus g by Γ_g and consider the usual presentation of Γ_g,

$$\Gamma_g = \langle a_1, \cdots, a_{2g} \,|\, a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} \rangle.$$

Each group in $\mathcal{F}_g$ is isomorphic to Γ_g. Let $\mathrm{Hom}(\Gamma_g, \mathcal{F}_g)$ be the set of all isomorphisms from Γ_g to elements of $\mathcal{F}_g$. Two such isomorphisms ξ and ζ are said to be equivalent if they differ only by a conjugation in $\mathrm{PSL}(2,\mathbb{R})$, that is, if there exists $T \in \mathrm{PSL}(2,\mathbb{R})$ such that $T\xi(a_j) = \zeta(a_j)T$, $j = 1, \cdots, 2g$. The equivalence classes are called *Teichmuller points*, and the set of equivalence classes is the *Teichmuller space* $\mathcal{T}_g$. For each Teichmuller point we have naturally associated a closed hyperbolic surface.

In order to give a description of $\mathcal{T}_g$ as a manifold we will see a Teichmuller point as a collage of fundamental blocks called *pairs of pants*. We recall that a pair of pants with boundary $\{\gamma_1, \gamma_2, \gamma_3\}$ is the pasting of two isometric hyperbolic hexagons H_1, H_2 along three non adjacent isometric sides. The boundary of the pair of pants consists of closed geodesics with length twice the corresponding length of the sides in the pasting.



A cubic graph is a finite 3-regular graph. Let Ω be a a connected cubic graph with $2g-2$ vertices and $3g-3$ edges. It is useful to view each edge as the union of two half-edges with each half-edge emanating from one of the two connected vertices. We use the following notation. The vertices and edges of Ω are denoted by

$$v_1, \ldots, v_{2g-2} \quad \text{and} \quad e_1, \ldots, e_{3g-3}.$$

For each vertex v_i of Ω the three half-edges emanating from v_i are denoted by e_{ij} with $j = 1, 2, 3$. Thus each edge of Ω will be denoted by $e_k = (e_{ij}, e_{lm})$ for some (j, m) if e_k is formed by the vertices v_i, v_l. If we choose for any $k = 1, \cdots, 3g-3$ a positive real number x_k, then we get a collection of pairs of pants $Y_1, \cdots, Y_i, \cdots, Y_{2g-2}$ such that Y_i has boundary $\gamma_{i1}, \gamma_{i2}, \gamma_{i3}$ and $l(\gamma_{ij}) = x_k = l(\gamma_{lm})$ if $e_k = (e_{ij}, e_{lm})$.

Let $(L, A) = (x_1, \cdots, x_{3g-3}, y_1, \cdots, y_{3g-3})$ be a sequence of real numbers, where $L = (x_1, \cdots, x_{3g-3}) \in \mathbb{R}^{3g-3}_{>0}$ and $A = (y_1, \cdots, y_{3g-3}) \in \mathbb{R}^{3g-3}$. For each such (L, A) we define the hyperbolic surface $S(L, A, \Omega)$ as the quotient space

$$S(L, A, \Omega) = Y_1 \sqcup \cdots \sqcup Y_{2g-2} \operatorname{mod}(\mathcal{P}),$$

where $\mathcal{P}$ is the equivalence relation given by

$$\gamma_{ij}(t) = \gamma_{lm}(t - y_k), \quad 0 \leq t \leq 1, \quad \text{if } e_k = (e_{ij}, e_{lm}),$$

where each closed geodesic is parametrized with constant speed on the circle of length one.

Since Ω is connected, $S(L, A, \Omega)$ is a closed hyperbolic surface of genus g. Indeed, any pair of pants has Euler characteristic -1. Thus we can associate for any $(L, A) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ a representating class $\sigma_\Omega(L, A) \in \mathcal{T}_{\mathrm{g}}$. Conversely, we have the following proposition.

Proposition 4.1.1. *Let S be a closed hyperbolic surface of genus $g \geq 2$. There exist simple closed geodesics $\gamma_1, \cdots, \gamma_{3g-3}$ which decomposes S into pairs of pants.*

Proof. Let γ_1 be a systole of S, i.e. a closed geodesic of the smallest length. Necessarily, γ_1 must be simple. Cut S along γ_1. Each connected component S' of the surface thus obtained is a hyperbolic surface with boundary and genus at least one with at most two boundary components. Hence S' contains a homotopically nontrivial simple closed curve, say $\tilde{\gamma_2}$, which is not homotopic to a boundary component of S'. We can take the unique closed geodesic γ_2 freely homotopic to $\tilde{\gamma_2}$. Now cut S' along γ_2, and continue whenever each component is not a pair of pants. After finitely many steps, S is decomposed into pairs of pants and using the Euler characteristic we deduce that the number of pairs of pants is equal to $2g - 2$. $\square$

In fact, for any closed surface we have a minimal pants decomposition in the following sense.

Theorem 4.1.2 (Bers' Theorem). *Every closed hyperbolic surface of genus $g \geq 2$ has a pants decomposition with simple closed geodesics $\gamma_1, \cdots, \gamma_{3g-3}$ satisfying*

$$l(\gamma_j) \leq 26(g-1) \quad \textit{for all} \ \ j = 1, \cdots, 3g-3.$$

For a proof of this result see [10, Theorem 5.1.2]

In order to define an extra structure on $\mathcal{T}_{\mathrm{g}}$ we need of the following theorem. The proof can be deduced from Theorem 6.2.7 in [10].

Theorem 4.1.3. *Let Ω be a connected 3-regular graph of $2g-2$ vertices. Then for any Teichmuller point $[\theta : \Gamma_g \to \Sigma] \in \mathcal{T}_{\mathrm{g}}$ with $\Sigma \in \mathcal{F}_g$ there exists a unique $(L_\theta, A_\theta) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ such that $\sigma_\Omega(L_\theta, A_\theta) = [\theta]$.*

It follows from Theorem 4.1.3 that there exists a bijection $\phi_\Omega : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to \mathcal{T}_g$ associated to any 3-regular connected graph Ω of $2g-2$ vertices.

For each conjugacy class C of Γ_g we have a *length function* $L_C : \mathcal{T}_g \to \mathbb{R}$ which associates for any closed hyperbolic surface S the length of the unique closed geodesic on S whose conjugacy class is C. Therefore we can consider the first $3g-3$ components of ϕ_Ω^{-1} as the length functions of a fixed set of conjugacy classes.

Indeed, let Ω be as above. Any edge of Ω correspond to a free homotopy class of a simple closed geodesic on the surfaces constructed in Theorem 4.1.3. Consider the set of conjugacy classes $C_1, \cdots, C_{3g-3}$ of Γ_g which represent the set of such simple closed geodesics. We call such classes *the partition of* Ω.

We say that a conjugacy class in Γ_g is *simple* if in the corresponding free homotopy class on S_g there is a simple closed curve. Now, we will complete the partition of Ω in order to embed $\mathcal{T}_g$ in an analytic variety (see [10, Lemma 6.3.4]).

Proposition 4.1.4. *Let Ω be a connected 3-regular graph with $2g-2$ vertices with partition $\{C_1, \cdots, C_{3g-3}\}$. Then there exists a unique* canonical system of classes *$\{C_1, \cdots, C_{3g-3}, \cdots, C_{9g-9}\}$ formed by nontrivial and simple conjugacy classes of Γ_g such that the map*

$$\mathcal{A}_\Omega : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to \mathbb{R}^{9g-9},$$

given by

$$\mathcal{A}_\Omega(L, A) = (L_{C_1}(S), \cdots, L_{C_{9g-9}}(S))$$

has an analytic left inverse (defined on a neighborhood of the image of $\mathcal{A}_\Omega$), where $S = \phi_\Omega(L, A)$.

We want to give an analytic structure on the space $\mathcal{T}_g$. For this we need the following theorem. For a proof see [10, Theorem 6.3.5].

Theorem 4.1.5. *Fix a connected cubic graph Ω with $2g-2$ vertices. Then for any nontrival conjugacy class C of Γ_g, the function $(L, A) \mapsto L_C(\phi_\Omega(L, A))$ is real analytic.*

Theorem 4.1.6. *If two connected* 3*-regular graphs* Ω, Ω' *are given with maps* $\phi_\Omega, \phi_{\Omega'}$*, respectively, then the map*

$$\phi_\Omega^{-1} \circ \phi_{\Omega'} : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

is a real analytic diffeomorphism.

Theorem 4.1.6 justifies the following definition.

Definition 15. On $\mathcal{T}_{\mathrm{g}}$ we introduce the unique real analytic structure such that the maps ϕ_Ω are real analytic diffeomorphisms. The associated systems of coordinates $(\phi_\Omega, \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3})$ are called the *Fenchel-Nielsen coordinates.*

Remark 5. It follows from Definition 15 that any length function is real analytic on $\mathcal{T}_{\mathrm{g}}$.

Proof of Theorem 4.1.6. Consider the canonical system of conjugacy classes $\{C_1, \cdots, C_{3g-3}, \cdots, C_{9g-9}\}$ of Γ_g with respect to Ω. The map $\tilde{\mathcal{A}}_\Omega : \mathcal{T}_{\mathrm{g}} \to V_\Omega \subset \mathbb{R}^{9g-9}$ given by $\tilde{\mathcal{A}}_\Omega = \mathcal{A}_\Omega \circ \phi_\Omega^{-1}$ has coordinates $(L_{C_1}, \cdots, L_{C_{9g-9}})$, where V_Ω is the neighborhood of the image of $\mathcal{A}_\Omega$ given by Theorem 4.1.4. If we have $\phi_\Omega(L, A) = \phi_{\Omega'}(L', A')$, then

$$(L, A) = \mathcal{A}_\Omega^{-1}(L_{C_1}(\phi_{\Omega'}(L', A')), \cdots, L_{C_{9g-9}}(\phi_{\Omega'}(L', A'))).$$

Since the length functions are analytic in the coordinates (L', A') by Theorem 4.1.5 and $\mathcal{A}_\Omega^{-1}$ is analytic by Proposition 4.1.4, it follows that the bijection $\phi_\Omega^{-1} \circ \phi_{\Omega'}$ is real analytic. □

We will finish this section with a generalization of the Bers Theorem.

Proposition 4.1.7. *Let* $S = \Gamma \setminus \mathbb{H}$ *be a closed hyperbolic surface of genus* $g \geq 2$ *and systole* $\mathrm{sys}(S) \geq s$*. Then there exist a cubic graph* Ω *with* $2g-2$ *vertices, a universal constant* $a > 0$*, and a constant* $b = b(s)$ *such that*

$$S = \phi_\Omega(l_1, \cdots, l_{3g-3}, \alpha_1, \cdots, \alpha_{3g-3}) \quad \textit{with} \quad 0 \leq \alpha_j < 1,$$

and

$$\mathcal{A}_\Omega(l_1, \cdots, l_{3g-3}, \alpha_1, \cdots, \alpha_{3g-3}) = (l_1, \cdots, l_{9g-9}) \quad \textit{with} \quad l_i \leq ag + b(s)$$

for all $i = 1, \cdots, 9g-9$.

Proof. By Theorem 4.1.2 we can take a pants decomposition of S with a partition given by conjugacy classes $C_1, C_2, \cdots, C_{3g-3}$ of simple closed curves. Moreover, we can reconstruct S from the cubic graph Ω induced by the corresponding pants decomposition with twist coordinates in the interval $[0,1)$. Thus there exists a Teichmuller point $[\theta : \Gamma_g \to \Gamma]$ with coordinates

$$(l_{C_1}(S), \cdots, L_{C_{3g-3}}(S), \alpha_1(S), \cdots, \alpha_{3g-3}(S)).$$

The construction of the canonical system of conjugacy classes $\{C_1, \cdots,$ $C_{9g-9}\}$ gives simple closed geodesics

$$\{\gamma_1, \cdots, \gamma_{3g-3}, \delta_1, \cdots, \delta_{3g-3}, \eta_1, \cdots, \eta_{3g-3}\}$$

on S such that

$$L_{C_j}(S) = l(\gamma_j), \quad L_{C_{3g+j-3}}(S) = l(\delta_j), \quad L_{C_{6g+j-6}}(S) = l(\eta_j),$$

for all $j = 1, \cdots, 3g-3$. These lengths satisfy the following inequalities (see [10, Proposition 3.3.11]:

$$\cosh\left(\tfrac{l(\delta_j)}{2}\right) \le \sinh\left(\tfrac{l(\gamma_{j_p})}{2}\right)\sinh\left(\tfrac{l(\gamma_{j_q})}{2}\right)[\sinh(a_j)\sinh(b_j)\cosh(\alpha_j l(\gamma_j)) + \cosh(a_j)\cosh(b_j)]$$

and

$$\cosh\left(\tfrac{l(\eta_j)}{2}\right) \le \sinh\left(\tfrac{l(\gamma_{j_p})}{2}\right)\sinh\left(\tfrac{l(\gamma_{j_q})}{2}\right)[\sinh(a_j)\sinh(b_j)\cosh((\alpha_j+1) l(\gamma_j)) + \cosh(a_j)\cosh(b_j)]$$

where $\gamma_{j_p}, \gamma_{j_q}$ are closed geodesics in the pants decomposition and a_j (resp. b_j) is the distance from γ_j to γ_{j_p} (resp. from γ_j to γ_{j_q}).

Since $0 \le \alpha_j < 1$ and $l(\gamma_i) \le Cg$ for some universal constant C by Theorem 4.1.2, it follows from the inequalities above that

$$\cosh\left(\frac{l(\delta_j)}{2}\right) \le \cosh(Cg)\sinh\left(\frac{Cg}{2}\right)^2 \cosh(a_j + b_j)$$

and

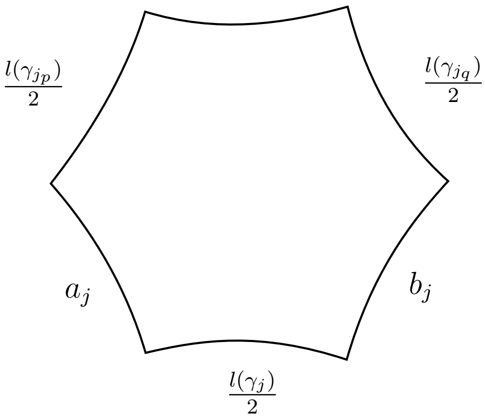
$$\cosh\left(\frac{l(\eta_j)}{2}\right) \le \cosh(2Cg)\sinh\left(\frac{Cg}{2}\right)^2 \cosh(a_j + b_j).$$

Using that $\sinh(t) < \cosh(t)$ and $\cosh(t_1)\cosh(t_2) \le \cosh(t_1 + t_2)$ for any $t, t_1, t_2 \ge 0$ we conclude that

$$l(\delta_j) \le 4Cg + 2a_j + 2b_j \quad \text{and} \quad l(\eta_j) \le 6Cg + a_j + b_j. \tag{4.1}$$

for all $j = 1, \cdots, 3g-3$.

The main geometric feature of the construction of the geodesics δ_i, η_i is that a_j, b_j are sides of a hyperbolic hexagon with non-adjacents sides of lengths $\frac{l(\gamma_j)}{2}, \frac{l(\gamma_{j_p})}{2}, \frac{l(\gamma_{j_q})}{2}$, respectively.



If we apply formula (i) from Proposition 2.2.4 twice we have,

$$\cosh(a_j) = \frac{\cosh\left(\frac{l(\gamma_{j_q})}{2}\right) + \cosh\left(\frac{l(\gamma_{j_p})}{2}\right)\cosh\left(\frac{l(\gamma_j)}{2}\right)}{\sinh\left(\frac{l(\gamma_{j_p})}{2}\right)\sinh\left(\frac{l(\gamma_j)}{2}\right)}$$

and

$$\cosh(b_j) = \frac{\cosh\left(\frac{l(\gamma_{j_p})}{2}\right) + \cosh\left(\frac{l(\gamma_{j_q})}{2}\right)\cosh\left(\frac{l(\gamma_j)}{2}\right)}{\sinh\left(\frac{l(\gamma_{j_q})}{2}\right)\sinh\left(\frac{l(\gamma_j)}{2}\right)}.$$

Since any closed geodesic on S has length at least s there exists a constant $b_1(s)$ such that for any j, k

$$\sinh\left(\frac{l(\gamma_{j_k})}{2}\right)\sinh\left(\frac{l(\gamma_j)}{2}\right) \geq \operatorname{sech}(b_1(s)).$$

Therefore,

$$\cosh(a_j) \leq 2\cosh(b_1(s))\cosh(Cg) \quad \text{and} \quad \cosh(b_j) \leq 2\cosh(b_1(s))\cosh(Cg). \qquad (4.2)$$

If we take $a := 10C$ and $b(s) := 4(b_1(s) + \cosh^{-1}(2))$ then by (4.1) and (4.2) we have

$$l(\gamma_j) \leq ag, \quad l(\delta_j) \leq ag + b(s), \quad \text{and} \quad l(\eta_j) \leq ag + b(s)$$

for all $j = 1, \cdots, 3g-3$. □

4.2 Mapping Class Groups and Moduli spaces

Let $\mathrm{Aut}(\Gamma_g)$ be the group of automorphisms of Γ_g. The inner automorphisms of Γ_g are the isomorphisms of type $\eta \mapsto \gamma\eta\gamma^{-1}$ for some $\gamma \in \Gamma_g$, and they form a normal subgroup $\mathrm{Inn}(\Gamma_g) \trianglelefteq \mathrm{Aut}(\Gamma_g)$. The quotient

$$\mathrm{Out}(\Gamma_{\mathrm{g}}) = \mathrm{Aut}(\Gamma_g)/\mathrm{Inn}(\Gamma_g)$$

is called the *outer automorphisms group* of Γ_g.

Let $\mathfrak{M}_{\mathrm{g}}{}^{\pm} = \mathrm{Diff}(S_g)/\,\mathrm{Diff}_0(S_g)$ be the *extended mapping class group* of a closed surface of genus g, where $\mathrm{Diff}(S_g)$ (respectively, $\mathrm{Diff}_0(S_g)$) is the group of all diffeomorphisms of S_g (respectively, the group of diffeomorphisms isotopic to the identity).

The *mapping class group* of a closed surface of genus g is the index two subgroup $\mathfrak{M}_{\mathrm{g}}$ of $\mathfrak{M}_{\mathrm{g}}{}^{\pm}$ given by the kernel of the homomorphism $\mathfrak{M}_{\mathrm{g}}{}^{\pm} \to \mathbb{Z}/2\mathbb{Z}$, which associates 0 for the class of orientation-preserving diffeomorphisms and 1 for the class orientation-reversing ones, i.e. we have a short exact sequence

$$1 \to \mathfrak{M}_{\mathrm{g}} \to \mathfrak{M}_{\mathrm{g}}{}^{\pm} \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

There is a natural injective homomorphism $\mathfrak{M}_{\mathrm{g}}{}^{\pm} \to \mathrm{Out}(\Gamma_{\mathrm{g}})$. In fact, we have the following theorem (see [15, Theorem 8.1]).

Theorem 4.2.1 (Dehn-Nielsen-Baer)**.** *Let $g \geq 2$. The natural homomorphism*

$$\mathfrak{M}_{\mathrm{g}}{}^{\pm} \to \mathrm{Out}(\Gamma_{\mathrm{g}})$$

is an isomorphism.

Using the natural inclusion $\mathfrak{M}_{\mathrm{g}} \hookrightarrow \mathrm{Out}(\Gamma_{\mathrm{g}})$ we have an action of the mapping class group on the Teichmuller space given by $[\phi] \cdot [\rho] = [\rho \circ \phi^{-1}]$, where $\phi : \Gamma_g \to \Gamma_g$ is an automorphism and $\rho : \Gamma_g \to \Lambda$ is an isomorphism for some $\Lambda \in \mathcal{F}_g$.

Note that this action is well-defined. However, it is not so clear that any $[\phi] \in \mathrm{Out}(\Gamma_{\mathrm{g}})$ gives an analytic map on $\mathcal{T}_{\mathrm{g}}$. In fact, a stronger result holds.

Theorem 4.2.2. *Let $g \geq 2$. The action of $\mathfrak{M}_{\mathrm{g}}$ on $\mathcal{T}_{\mathrm{g}}$ is properly discontinuous and preserves the analytic structure.*

For a proof of Theorem 4.2.2 see [15, Theorem 12.2]. Note that if a hyperbolic surface S has a nontrivial isometry ϕ and $[\rho_S]$ is a Teichmuller point giving S, then the action of $[\phi]$ on $[\rho_s]$ is trivial, i.e. $[\phi] \cdot [\rho_S] = [\rho_S]$. In fact, we have that for any Teichmuller point its isotropy group by the action of $\mathfrak{M}_{\mathrm{g}}$ is equal to the orientation-preserving isometry group of the corresponding hyperbolic surface.

On the other hand, this isotropy group is finite since by Hurwitz's theorem (see [10, Theorem 6.5.9]) the group of orientation-preserving isometries of any closed hyperbolic surface of genus g has at most $84(g-1)$ elements. This shows that the action of $\mathfrak{M}_{\mathrm{g}}$ on $\mathcal{T}_{\mathrm{g}}$ is not free. However, we have the following result (see [10, Theorem 6.5.7]).

Theorem 4.2.3. *For any $g > 2$ let $I_g \subset \mathcal{T}_{\mathrm{g}}$ be the set of all surfaces in $\mathcal{T}_{\mathrm{g}}$ which have a nontrivial isometry group. Then I_g is a proper closed real analytic subvariety of $\mathcal{T}_{\mathrm{g}}$.*

It follows from Theorems 4.2.2 and 4.2.3 that the quotient space $\mathcal{T}_{\mathrm{g}} / \mathfrak{M}_{\mathrm{g}}$ is an *orbifold* (see [51, Section 13.2] for a formal definition). Note that this quotient space is equal to the moduli space $\mathcal{M}_{\mathrm{g}}$ of closed hyperbolic surfaces of genus g. Indeed, if S_1 and S_2 are isometric by the isometry $\phi : S_1 \to S_2$ then $[\rho_{S_2}] = [\phi] \cdot [\rho_{S_1}]$, i.e. two surfaces in $\mathcal{T}_{\mathrm{g}}$ are isometric if and only if they are in the same orbit of the action of $\mathfrak{M}_{\mathrm{g}}$.

Recall that $\mathrm{PSL}(2, \mathbb{R})$ coincides with the group of conformal equivalences of $\mathbb{H}$ seen as an open subset of the complex plane. Therefore, any hyperbolic surface naturally has a structure of Riemann surface, i.e. a complex manifold of dimension 1. This interplay between complex structure and hyperbolic structure is important in the study of $\mathcal{T}_{\mathrm{g}}$ and $\mathcal{M}_{\mathrm{g}}$ that follows.

Let $p, q \in \mathcal{T}_{\mathrm{g}}$ be Teichmuller points and choose representatives $\sigma : \Gamma_g \to \Sigma$ and $\lambda : \Gamma_g \to \Lambda$ with $\Sigma, \Lambda \in \mathcal{F}_g$. We associate for each such pair σ, λ a space $H(\sigma, \lambda)$ of orientation-preserving Lipschtz homeomorphisms $F : \mathbb{H} \to \mathbb{H}$ such that

$$F(\sigma(\gamma) \cdot \tau) = \lambda(\gamma) \cdot F(\tau) \quad \text{for all} \quad \gamma \in \Gamma_g \quad \text{and} \quad \tau \in \mathbb{H}.$$

If we fix $F \in H(\sigma, \lambda)$, then F is differentiable except for a set of zero measure. Thus for almost every point $\tau \in \mathbb{H}$ the *dilatation* of F at $\tau = x + iy$

can be defined by

$$K_F(\tau) = \frac{|F_z(\tau)| + |F_{\bar{z}}(\tau)|}{|F_z(\tau)| - |F_{\bar{z}}(\tau)|},$$

where

$$F_z = \frac{1}{2}\left(\frac{\partial}{\partial x}F - i\frac{\partial}{\partial y}F\right) \quad \text{and} \quad F_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}F + i\frac{\partial}{\partial y}F\right).$$

The *dilatation* of the map F is defined to be the number

$$K_F = \sup K_F(\tau),$$

where the supremum is taken over all points τ where F is differentiable. We say that F is K-quasiconformal if $K = K_F < \infty$.

Note that $F \in H(\sigma, \lambda)$ if and only if $F^{-1} \in H(\lambda, \sigma)$. If $T, S \in \mathrm{PSL}(2, \mathbb{R})$ consider the isomorphisms

$$\sigma^T(\gamma) = T^{-1}\sigma(\gamma)T \quad \text{and} \quad \lambda^S(\gamma) = S^{-1}\lambda(\gamma)S.$$

We have a bijection

$$H(\sigma, \lambda) \to H(\sigma^T, \lambda^S), \quad F \mapsto S^{-1} \circ F \circ T.$$

Since T, S are conformal maps it follows from item (c) of the proposition below (see [15, Proposition 11.3]) that these bijections preserve the dilatation.

Proposition 4.2.4. *Let $F, G : \mathbb{H} \to \mathbb{H}$ be Lipschtz quasiconformal homeomorphisms and $A \in \mathrm{PSL}(2, \mathbb{R})$ be a conformal equivalence. Then*

(a) The composition $G \circ F$ is quasiconformal and

$$K_{G \circ F} \leq K_G K_F.$$

(b) The inverse F^{-1} is quasiconformal and

$$K_{F^{-1}} = K_F.$$

(c)

$$K_{A \circ F} = K_F = K_{F \circ A}.$$

We say that $F \in H(\sigma, \lambda)$ is a Teichmuller mapping if F is K-quasi-conformal and $K_{\sigma,\lambda} = \inf\{K_G \mid G \in H(\sigma, \lambda)\}$. It is not clear that there exists a Teichmuller mapping, however we have the Teichmuller theorems (see [15, Theorems 11.8 and 11.9]):

Theorem 4.2.5. *Let $\sigma, \lambda : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ be injective homomorphisms with images $\Sigma, \Lambda \in \mathcal{F}_g$, respectively. Then, there exists a unique K-quasiconformal map $F_{\sigma\lambda} \in H(\sigma, \lambda)$ such that*

$$K = \inf\{K_F \mid F \in H(\sigma, \lambda)\}.$$

Given two Teichmuller points $p, q \in \mathcal{T}_g$ we define the Teichmuller distance

$$d_T(p, q) = \log(\inf\{K_F \mid F \in H(\sigma, \lambda)\}),$$

for an arbitrary choice of representatives σ, λ of p, q, respectively. The discussion above shows that this infimum does not depend on the choice of representatives. Moreover, if F is 1-quasiconformal map, then $F_{\bar{z}}(\tau) = 0$ for almost every point in $\mathbb{H}$. Therefore, F is differentiable everywhere and it is conformal, i.e. F is an isometry of $\mathbb{H}$ which conjugates σ and λ. Thus by Theorem 4.2.5,

$$d_T(p, q) = 0 \quad \text{if and only if} \quad p = q.$$

The symmetry and triangular inequality of the distance follow from Proposition 4.2.4. Hence d_T defines a metric on $\mathcal{T}_g$.

Let p, p' be Teichmuller points with representatives σ, σ' and let

$$S = \sigma(\Gamma_g)\backslash\mathbb{H} \quad , \quad S' = \sigma'(\Gamma_g)\backslash\mathbb{H},$$

be the corresponding closed hyperbolic surfaces. Any quasiconformal map $F \in H(\sigma, \sigma')$ induces a homeomorphism $f : S \to S'$ whose lifting is F. From now on, a K-quasiconformal map between hyperbolic surfaces will mean a homeomorphism between the surfaces induced by a K-quasiconformal map of any of their representatives in $\mathcal{T}_g$.

The Teichmuller distance gives a basis for the topology of $\mathcal{T}_g$ and for any length function we have the following theorem (see [10, Theorem 6.4.3]).

Theorem 4.2.6. *Let S, S' be closed hyperbolic surfaces of genus $g \geq 2$. If a homeomorphism $h : S \to S'$ is K-quasiconformal and if for each conjugacy class C of Γ_g we denote by hC the conjugacy class of the image of the homomorphism induced by $h_* : \Gamma_g \to \Gamma_g$, then*

$$\frac{1}{K} L_C(S) \leq L_{hC}(S') \leq K L_C(S).$$

We can define the systole of a Teichmuller point as the systole of any hyperbolic surface corresponding to this point. Assuming the same notation of Theorem 4.2.6 we have the following corollary.

Corollary 4.2.7. *The function* $\log(\mathrm{sys}) : \mathcal{T}_{\mathrm{g}} \to \mathbb{R}$ *is* 1*-Lipschtiz with respect to Teichmuller distance.*

Proof. By abuse of notation, take the surfaces S and S' in $\mathcal{T}_{\mathrm{g}}$ and $h : S \to S'$ the K-quasiconformal map which realize the Teichmuller distance between S and S'. Consider conjugacy classes $C_S, C_{S'}$ of Γ_g such that

$$\mathrm{sys}(S) = L_{C_S}(S) \quad \text{and} \quad \mathrm{sys}(S') = L_{C_{S'}}(S').$$

Since h is a homeomorphism, there exists a conjugacy class C' such that $C_{S'} = hC'$. If we apply Theorem 4.2.6 twice we have

$$\frac{1}{K} \mathrm{sys}(S) \leq \frac{1}{K} L_{C'}(S) \leq \mathrm{sys}(S') \leq L_{hC}(S') \leq K \,\mathrm{sys}(S).$$

Now taking the log in these inequalities we conclude that

$$|\log(\mathrm{sys}(S)) - \log(\mathrm{sys}(S'))| \leq \log(K) = d_T(S, S').$$

□

The function systole is continuous in $\mathcal{T}_{\mathrm{g}}$ and is clearly invariant by the action of $\mathfrak{M}_{\mathrm{g}}$. Therefore, it descends to a continuous function $\mathrm{sys} : \mathcal{M}_{\mathrm{g}} \to \mathbb{R}$. As the first application of the function systole for the topology of $\mathcal{M}_{\mathrm{g}}$, we show that the space $\mathcal{M}_{\mathrm{g}}$ is not compact.

Indeed, we can construct for each $g \geq 2$ a sequence of points $S_i \in \mathcal{M}_{\mathrm{g}}$ such that $\mathrm{sys}(M_i) \to 0$ using pants with small boundaries, by the continuity and positivity of the function sys we conclude that this sequence has no convergent subsequence.

Let $\varepsilon > 0$ be a fixed lower bound for the systole. The $\varepsilon-$*thick part* of $\mathcal{M}_{\mathrm{g}}$ is the set

$$\mathcal{M}_{\mathrm{g}}{}^{\geq\varepsilon} = \{S \in \mathcal{M}_{\mathrm{g}} \mid \mathrm{sys}(S) \geq \varepsilon\}.$$

Although $\mathcal{M}_{\mathrm{g}}$ is not compact we have the following theorem due to Mumford (see [15, Theorem 12.6]).

Theorem 4.2.8. *Let $g \geq 2$. For each $\varepsilon > 0$, the $\varepsilon-$*thick part *of $\mathcal{M}_{\mathrm{g}}$ is compact.*

Therefore, the function systole is proper in $\mathcal{M}_{\mathrm{g}}$. However, note that in $\mathcal{T}_{\mathrm{g}}$ the function systole is not proper, since if we fix a system of Fenchel-Nielsen coordinates the subset $(0, \varepsilon] \times \mathbb{R}_{+}^{3g-4} \times \mathbb{R}^{3g-3} \subset \mathbb{R}_{+}^{3g-3} \times \mathbb{R}^{3g-3}$ is a closed, noncompact subset contained in $\mathrm{sys}^{-1}[0, \varepsilon]$.

Perhaps the main importance of the function systole is the possibility of using it for obtaining topological information about $\mathcal{M}_{\mathrm{g}}$. The reason for this is the theorem of Akrout [1] which proves that $\mathrm{sys} : \mathcal{T}_{\mathrm{g}} \to \mathbb{R}$ is a *topological Morse* function.

Let M be a topological n-manifold and let $f : M \to \mathbb{R}$ be a continuous function. A point $p \in M$ is said to be regular for f if there is a (topological) chart around $p = 0$ on which $f(x_1, \cdots, x_n) = x_1 + f(p)$. Otherwise, p is a critical point of f. A critical point $p \in M$ is said to be a nondegenerate critical point of f if in a local (topological) chart around $p = 0$ we have

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$

for some k.

The integer k does not depend on the choice of such a chart and it is called the *index* of the critical point p. The function f is called topologically Morse if all critical points are nondegenerate. The usual results of Morse theory, such as Morse inequalities and the construction of a homotopy model with cells corresponding to critical points, hold in the topological context.

We finish this section by recalling an application of the Morse theory for giving topological information about the virtual Euler characteristic of $\mathfrak{M}_{\mathrm{g}}$. Let $G < \mathfrak{M}_{\mathrm{g}}$ be a subgroup of finite index N_G such that the action of G on $\mathcal{T}_{\mathrm{g}}$ is torsion free. The set of subgroups of $\mathfrak{M}_{\mathrm{g}}$ with this property is non empty.

Let $M_G = \mathcal{T}_g/G$ be the quotient manifold and let $\chi(M_G)$ be the usual Euler characteristic of M_G. It is a fact of cohomology of groups that the rational number $\dfrac{\chi(M_G)}{N_G}$ does not depend on G (see [46]), and we define the *virtual Euler characteristic*

$$\chi(\mathfrak{M}_g) = \frac{\chi(M_G)}{N_G},$$

for any $G < \mathfrak{M}_g$ of finite index with free action.

Remark 6. In [20] Harer and Zagier established the remarkable following formula:

$$\chi(\mathfrak{M}_g) = \zeta(1-2g),$$

where $\zeta(s)$ is the Riemann zeta function.

Let $\mathcal{C} \subset \mathcal{T}_g$ be the set of critical points of the function systole and consider the subset $\mathcal{C}_1 \subset \mathcal{C}$ of representatives of the orbits by the action of $\mathfrak{M}_g$ on $\mathcal{C}$. By [46, Theorem 23] the set $\mathcal{C}_1$ is finite and applying Morse theory Schaller proved in [46] the following theorem.

Theorem 4.2.9. *For any $S \in \mathcal{C}_1$ let $|\operatorname{Isom}(S)|$ be the cardinality of the isometry group of S and $k(S)$ the index of S. Then*

$$\sum_{S \in \mathcal{C}_1} \frac{(-1)^{k(S)}}{|\operatorname{Isom}(S)|} = \chi(\mathfrak{M}_g).$$

For more information and conjectures about the relation between the function systole and the topology of $\mathcal{M}_g$, we refer to [7] and [46].

4.3 Geometry of the Weil-Petersson Metric

The Teichmuller space $\mathcal{T}_g$ has a structure of a complex manifold of dimension $3g-3$ compatible with the Fenchel-Nielsen coordinates. If $S = \Gamma\backslash\mathbb{H}$ represents a Teichmuller point in $\mathcal{T}_g$ consider the complex vector space $\mathcal{Q}(\Gamma)$ formed by holomorphic functions from $\mathbb{H}$ to $\mathbb{C}$ such that

$$\mathcal{Q}(\Gamma) = \{f : \mathbb{H} \to \mathbb{C} \mid f(T(z)) = T'(z)^{-2} f(z)\,, \text{for all } z \in \mathbb{H},\, T \in \Gamma\},$$

where T' denotes the complex derivative of T. Note that, if we change Γ by any conjugate Γ^A, for some $A \in \mathrm{PSL}(2,\mathbb{R})$, then the map

$$\mathcal{Q}(\Gamma) \to \mathcal{Q}(\Gamma^A), \quad f \mapsto A'^2(f \circ A),$$

is a complex isomorphism, where A' denotes the complex derivative of A.

We define the *space of holomorphic quadratic differential* $\mathcal{Q}(S)$ of S as the class of complex isomorphism of the vector spaces $\mathcal{Q}(\Gamma^A)$, where $A \in \mathrm{PSL}(2,\mathbb{R})$. It follows from the Riemann-Roch Theorem that $\mathcal{Q}(S)$ has complex dimension equal to $3g-3$ for any closed hyperbolic surface of genus g.

It is common to identify the cotangent space $T^*_S\mathcal{T}_{\mathrm{g}}$ of any $S = \Gamma\backslash\mathbb{H} \in \mathcal{T}_{\mathrm{g}}$ with the space $\mathcal{Q}(S)$. In order to unify the complex structure of $\mathcal{T}_{\mathrm{g}}$ and the hyperbolic geometry of the Teichmuller points we define the *Weil-Petersson metric* on $\mathcal{T}_{\mathrm{g}}$: it arises from the hermitian product $\mathcal{H}$ on $\mathcal{Q}(S)$, namely

$$\mathcal{H}(f,g) = \int_S f\bar{g}y^2 dxdy,$$

where $S = \Gamma\backslash\mathbb{H}$, the integration is over any fundamental domain of Γ and we are choosing the representative $\mathcal{Q}(\Gamma)$ for $\mathcal{Q}(S)$. This integral does not depend on the choices of the fundamental domain and the representative class for $\mathcal{Q}(S)$.

We define the *Weil-Petersson metric* $\mathcal{G}$ as the Riemannian metric on the cotangent bundle of $\mathcal{T}_{\mathrm{g}}$ induced by the Weil-Petersson metric $\mathcal{H}$, i.e.

$$\mathcal{G}(\omega,\eta) = 2\mathrm{Re}(\mathcal{H}(\omega,\eta)),$$

for any pair of covectors $\omega, \eta \in T^*\mathcal{T}_{\mathrm{g}}$. The Weil-Petersson metric has negative sectional curvature ([52]) and the group $\mathfrak{M}_{\mathrm{g}}$ acts by isometries. Thus, $\mathcal{G}$ descends to a metric (where the tangent space is well-defined) on $\mathcal{M}_{\mathrm{g}}$.

Let g_{wp} be the corresponding Riemannian metric on the tangent bundle of $\mathcal{T}_{\mathrm{g}}$. Since $T\mathcal{T}_{\mathrm{g}}$ has an involution I given by the complex structure, we can define the *Weil-Petersson form* ω_{wp} by

$$\omega_{wp} = g_{wp}(IX, Y),$$

for any pair of tangent vectors X, Y.

We say that the a Hermitian metric $\mathcal{H}$ is a *Kähler metric* if $d\omega_{wp} = 0$. We have the following theorem due to Ahlfors (see [23, Theorem 7.15]).

Theorem 4.3.1. *The Weil-Petersson metric is Kählerian.*

The Weil-Petersson form gives a sympletic structure on $\mathcal{T}_{\mathrm{g}}$ compatible with the Weil-Petersson metric. If we denote the Fenchel-Nielsen coordinates for a fixed choice of a cubic graph by $(x_1, \cdots, x_{3g-3}, y_1, \cdots, y_{3g-3})$, then by the work of Wolpert, the Weil-Petersson form has a simple expression in these coordinates (see [23, Theorem 8.6]).

Theorem 4.3.2 (Wolpert)**.** *The Weil-Petersson form is given by*

$$\omega_{wp} = \sum_{i=1}^{3g-3} dx_i \wedge dy_i.$$

We will need some results about the geometry of the Weil-Petersson metric. For the proof of item (a) of the next proposition see [53, Corollary 4.7] and for item (b) see [54, Lemma 3.16].

Proposition 4.3.3. *Consider the Teichmuller space $\mathcal{T}_{\mathrm{g}}$ equipped with the Weil-Petersson metric. For any nontrivial conjugacy class C of Γ_g let $L_C : \mathcal{T}_{\mathrm{g}} \to \mathbb{R}$ be the corresponding length function. Then the following hold:*

(a) The function L_C is convex along Weil-Petersson geodesics.

(b) If C represents a simple closed curve, then

$$|\nabla L_C(X)| \leq c' \left(L_C(X) + L_C(X)^2 \exp\left(\frac{L_C(X)}{2}\right)\right),$$

for some universal constant c'. Here ∇L_C denotes the gradient of the function L_C with respect to the Weil-Petersson metric.

In the Teichmuller metric we saw that the logarithm of the function systole is 1-Lipschtiz. The next proposition provides an analogous result for the Weil-Petersson metric (see [55, Theorem 1.3]).

Theorem 4.3.4. *There exists a universal constant $K > 0$ independent of g, such that for all $S, S' \in \mathcal{T}_{\mathrm{g}}$,*

$$|\sqrt{\mathrm{sys}(S)} - \sqrt{\mathrm{sys}(S')}| \leq K d_{wp}(S, S').$$

Since the mapping class group acts by isometries on $\mathcal{T}_{\mathrm{g}}$ with respect to the Weil-Petersson metric, it makes sense to consider the Ricci curvature $\mathrm{Ric}(S)$ for any point $S \in \mathcal{M}_{\mathrm{g}}$ as the minimum of the Ricci curvature $\mathrm{Ric}(\tilde{S})(v, v)$ for any lifting $\tilde{S}$ of S and $v \in T_{\tilde{S}} \mathcal{T}_{\mathrm{g}}$ with $g_{wp}(v, v) = 1$. We have the following useful theorem due to Teo [50, Proposition 3.2].

Theorem 4.3.5. *For each $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ which does not depend on g, such that*

$$\inf_{S \in \mathcal{M}_{\mathrm{g}}^{\geq \varepsilon}} Ric(S) \geq -C(\varepsilon).$$

Although $\mathcal{M}_{\mathrm{g}}$ is not compact, if we consider the volume measure vol_{wp} in $\mathcal{T}_{\mathrm{g}}$ with respect to the Weil-Petersson metric, then any fundamental domain $D \subset \mathcal{T}_{\mathrm{g}}$ for the action of $\mathfrak{M}_{\mathrm{g}}$ on $\mathcal{T}_{\mathrm{g}}$ has finite measure, i.e. $\mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}})$ is always finite. Moreover, Schumacher and Trapani showed in [48] the following asymptotic growth of $\log \mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}})$.

Theorem 4.3.6.

$$\lim_{g \to \infty} \frac{\log \mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}})}{g \log g} = 2.$$

The finitiness of the volume of $\mathcal{M}_{\mathrm{g}}$ means that we obtain a probability measure $\mathbb{P}_g$ on $\mathcal{M}_{\mathrm{g}}$ by defining

$$\mathbb{P}_g(A) = \frac{\mathrm{vol}_{wp}(A)}{\mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}})}$$

for every measurable set $A \subset \mathcal{M}_{\mathrm{g}}$.

Let $\varepsilon > 0$ be a constant. We define the following random variable

$$N_{g,\varepsilon} : \mathcal{M}_{\mathrm{g}} \to \mathbb{Z}_{\geq 0},$$

by

$$N_{g,\varepsilon}(S) = \text{number of primitive closed geodesics on } S \text{ with length at most } \varepsilon,$$

where a closed geodesic $\alpha \subset S$ is primitive if there does not exist a closed geodesic $\beta \subset S$ such that $\alpha = \beta^k$ for some natural k.

Combining methods of probability theory and Weil-Petersson geometry, Mirzakhani and Petri showed in [37] the following theorem.

Theorem 4.3.7. *For every $\varepsilon > 0$ the random variables $N_{g,\varepsilon}$ converges in distribution to a Poisson distributed random variable with mean λ_ε, where*

$$\lambda_\varepsilon = \int_0^\varepsilon \frac{e^t + e^{-t} - 2}{t} dt.$$

A random variable $X : \mathcal{M}_\mathrm{g} \to \mathbb{Z}_{\geq 0}$ is said to be Poisson distributed with mean $\lambda \in (0, \infty)$ if

$$\mathbb{P}_g(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for all}\, k \geq 0,$$

see Definition 17 in the next chapter.

We conclude this section with the following corollary.

Corollary 4.3.8. *For every $\varepsilon > 0$,*

$$\lim_{g\to\infty} \mathrm{vol}_{wp}(\mathcal{M}_\mathrm{g}{}^{\geq\varepsilon}) = \infty.$$

Proof. By the definition,

$$\mathcal{M}_\mathrm{g}{}^{\geq\varepsilon} \supset \{S \in \mathcal{M}_\mathrm{g} \mid N_{g,\varepsilon}(S) = 0\}.$$

Hence by Theorem 4.3.7 there exists g_0 such that if $g \geq g_0$, then

$$\frac{\mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}}{}_g^{\geq\varepsilon})}{\mathrm{vol}_{wp}(\mathcal{M}_\mathrm{g})} \geq \frac{e^{-\lambda_\varepsilon}}{2}.$$

Now the result follows from Theorem 4.3.6 since $\lim_{g\to\infty} \mathrm{vol}_{wp}(\mathcal{M}_\mathrm{g}) = \infty$. $\square$

CHAPTER 5

GROUPS AND GRAPHS

5.1 Schreier graphs and Free groups

Let G be a finitely generated group, let H be a subgroup of G, and let $\Sigma = \{g_1^{\pm 1}, \cdots, g_r^{\pm 1}\}$ be a symmetric set of generators of G.

Definition 16. The (right) *Schreier coset graph* $\Sigma(G, H)$ is defined as follows. Its vertex set is the set of right cosets of H in G. For each right coset H_i and each generator g_j there is an edge from H_i to the right coset $H_i g_j$.

If $H = \{e\}$, then the Schreier graph $\Sigma(G, \{e\})$ is called the *Cayley graph* of G with respect to the generating set Σ, and it is denoted by $Cay(G, \Sigma)$.

Remark 7. If H is a normal subgroup then the Schreier graph $\Sigma(G, H)$ is canonically isomorphic to the Cayley graph $Cay(G/H, \bar{\Sigma})$, where $\bar{\Sigma}$ is the image of Σ in G/H.

Let Ω be a graph and $k > 1$ a natural number. The graph Ω is called $k-$*regular* if every vertex of Ω has degree k. For example, any Schreier graph is regular of even degree. We will denote by $E(\Omega)$ the set of edges of Ω and by $V(\Omega)$ the set of vertices. An *s-factor* of Ω is a subgraph K of Ω which is regular of degree s and which contains every vertex of Ω. When the edges of Ω can be partitioned into s-factors, we say that Ω is *s-factorable.*

Theorem 5.1.1 (Petersen, 1891)**.** *Every regular graph (connected or not) of even degree is* 2*-factorable.*

For a short proof of this classical theorem see [14, Corollary 2.1.5].

Remark 8. Note that if Ω is $2k$-regular, then necessarily Ω is partitioned by k distinct 2-factors.

Recall that a *circuit* on a finite graph Ω is a closed path without repetitions of edges. The *girth* of Ω, denoted by $\mathrm{girth}(\Omega)$ is the length of a minimal circuit. If we consider the set $C(\Omega)$ of circuits on Ω, we can define

$$L : C(\Omega) \to \mathbb{Z}_{\geq 2}, \quad L(K) = \text{length of } K.$$

The set $L(C(\Omega)) = \{l_1 < l_2 < \cdots < l_\omega\}$ is called the length spectrum of Ω and the *j-girth* of Ω, if exists, is the number l_j. In particular, $\mathrm{girth}(\Omega)$ is the 1-girth of Ω.

Denote by F_2 the free group of rank 2 with a fixed set of generators $A = \{x^{\pm 1}, y^{\pm 1}\}$. Any element $\omega \in \mathrm{F}_2$ is a unique word in the alphabet $\{x, y, x^{-1}, y^{-1}\}$ such that it does not contain a sequence of the form

$$xx^{-1}, x^{-1}x, yy^{-1}, yy^{-1}.$$

The number of letters in such word is the *length of* ω and is denoted by $l_A(\omega)$. By convention, $l_A(1) = 0$.

Proposition 5.1.2. *For any connected* 4*-regular graph* Ω *on* n *vertices, there exists a subgroup* $\Gamma < \mathrm{F}_2$ *of index* n *with the following properties:*

1. *The Schreier graph of cosets of* Γ *on the symmetric set* A *is isomorphic to* Ω*;*

2. *If* $l = \mathrm{girth}(\Omega)$ *then* $l = \min\{l_A(\gamma) \mid \gamma \in \Gamma \setminus \{1\}\}$.

Proof. Take a minimal circuit C on Ω and a vertex v of C. Consider a bijection between $V(\Omega)$ and $\{1, \cdots, n\}$ such that v corresponds to 1. By Theorem 5.1.1, $E(\Omega) = K_1 \sqcup K_2$ where K_i is a 2-factor.

Each K_i can be written as a disjoint union of circuits $C_i^1, \cdots, C_i^{t_i}$. For each C_i^j assign an arbitrary orientation and consider the cyclic permutation $\pi_{ij} \in S_n$, which corresponds to the cyclic order in which the oriented circuit C_i^j passes through the vertices of Ω. Then, for each $i = 1, 2$ consider the permutation $\pi_i = \pi_{i1} \cdots \pi_{it_i}$ and the homomorphism $\phi : \mathrm{F}_2 \to S_n$ given by $\phi(x) = \pi_1$ and $\phi(y) = \pi_2$.

The subgroup $\Gamma = \{t \in \mathrm{F}_2 \,|\, \phi(t) \cdot 1 = 1\}$ has index n because ϕ is transitive. Indeed, since Ω is connected, for each i there exists a path joining the vertices 1 and i, but each edge of this path is contained in K_1 or K_2, and in any case we can always go from a vertex to its adjacent vertex in the path by the action of $\phi(x)^{\pm 1}$ or $\phi(y)^{\pm 1}$. Hence, the path induces a word w in $\{x^{\pm 1}, y^{\pm 1}\}$ such that $i = \phi(w)(1)$. In particular, the minimal circuit C is represented by a reduced word $W \in \mathrm{F}_2$ such that $l_A(W) = l(C) = l$. Moreover, if $\phi(W)(1) = 1$, then $W \in \Gamma$.

To see the isomorphism in part 1, take $\alpha_j \in \mathrm{F}_2$ such that $\phi(\alpha_j) \cdot 1 = j$ for any $j = 1, \cdots, n$ and choose $\alpha_1 = 1$. The Schreier graph of cosets of Γ on A has the vertices $\{\Gamma, \Gamma\alpha_2, \cdots, \Gamma\alpha_n\}$ and the bijection $\Gamma\alpha_j \longleftrightarrow j$ is an isomorphism of graphs. In fact, j is adjacent to k if and only if $k = \pi_i^{\pm 1}(j)$ and this is equivalent to $\phi(\alpha_k)(1) = \phi(a\alpha_j)(1)$ for some $a \in A$. But this means that $\Gamma\alpha_j = \Gamma\alpha_k a^{-1}$, i.e, $\Gamma\alpha_j$ and $\Gamma\alpha_k$ are adjacent.

To complete the proof it remains to show that any non-trivial element $\gamma \in \Gamma$ satisfies $l_A(\gamma) \geq l$. In fact, any non-empty reduced word $\omega \in \Gamma$ corresponds to a closed path of length $l_A(\omega)$ in the Schreier graph of cosets of Γ. But any closed path in Ω has length at least l, hence by the isomorphism we have $l_A(\omega) \geq l$. □

5.2 The space of 4-regular graphs

For each $n \geq 2$, let $\mathcal{F}_n$ be the set of 4−regular graphs on n vertices. We will define on $\mathcal{F}_n$ a probability measure introduced by Bollobás.

We consider a set with $4n$ points, each point labeled with an integer between 1 and n, each integer occurring four times. We then build a graph at random by selecting pairs of points from the set, without replacement. If at step i the numbers l_i and m_i are selected, we add to the graph an edge joining l_i and m_i.

Now we recall the notion of the asymptotic Poisson distribution in a more general situation.

Definition 17. (a) A random variable Z which takes values in the non negative integers is a *Poisson distribution with mean* λ if

$$\mathbb{P}(Z = k) = \frac{\lambda^k e^{-k}}{k!},$$

where λ is the expected value of X.

(b) Let $\{Z_n\}$ be a family of random variables on the probability spaces $\mathcal{P}_n$. The $\{Z_n\}$ are *asymptotic Poisson distributions* as $n \to \infty$ if there exists λ such that

$$\lim_{n\to\infty} \mathbb{P}(Z_n = k) = \frac{\lambda^k e^{-k}}{k!},$$

for all k.

(c) The vectors $(Z_{n,1}, \cdots, Z_{n,l})$ of random variables on the probability spaces $\mathcal{P}_n$ are asymptotically independent Poisson distributions if, for each $i = 1, \cdots, l$, the random variables $\{Z_{n,i}\}$ tend to Poission distribution Z_i as $n \to \infty$, and if the variables Z_i are independent, i.e.

$$\mathbb{P}(Z_1 = a_1, Z_2 = a_2, \cdots, Z_l = a_l) = \mathbb{P}(Z_1 = a_1) \cdots \mathbb{P}(Z_l = a_l).$$

We will need two results concerning this model. See [24, Chapter 9] for the proofs.

Theorem 5.2.1. *Consider for each i the random variable $X_{n,i}$ on $\mathcal{F}_n$ given by*

$$X_{n,i}(G) = \textit{number of closed paths in } G \textit{ of length } i.$$

Then for each l the vectors $(X_{n,1}, \cdots, X_{n,l})$ are asymptotically independent Poisson distributions with means

$$\lambda_i = \frac{3^i}{2i}.$$

A *Hamiltonian cycle* is a circuit on a graph that visits each vertex exactly once.

Theorem 5.2.2. *Let $\mathcal{H}_n \subset \mathcal{F}_n$ be the subset of 4-regular graphs on n vertices containing a Hamiltonian cycle. Then*

$$\lim_{n\to\infty} \mathbb{P}_n(\mathcal{H}_n) = 1,$$

where $\mathcal{P}_n$ denotes the probability measure on $\mathcal{F}_n$ with respect to the Bollobás model. Therefore, asymptotically almost surely the graphs on $\mathcal{F}_n$ are connected.

Now we will introduce an operation on the space of all 4-regular graphs.

Definition 18. Let G_1, G_2 be connected 4-regular graphs and let $e_i \in E(G_i)$ be edges. We define $G_1 *_{e_1,e_2} G_2$ as the graph obtained by cutting the edges e_1, e_2 and adding a vertex v that attaches to the free ends of e_1 and e_2.

Remark 9. Note that $G_1 *_{e_1,e_2} G_2$ is a connected 4-regular graph on $|G_1| + |G_2| + 1$ vertices.

Proposition 5.2.3. *Let G_1, G_2 be connected 4-regular graphs. Let g_i be the girth of G_i. There exist edges $e_i \in E(G_i)$ such that*

$$\mathrm{girth}(G_1 *_{e_1,e_2} G_2) = \min\{g_1, g_2\}.$$

Proof. We can choose a minimal circuit C_i in each G_i and take an edge e_i in the complement of C_i for $i = 1, 2$. Consider now a circuit $C \subset G_1 *_{e_1,e_2} G_2$. If $C \subset G_i \backslash e_i$ for some $i = 1, 2$, then $l(C) \geq \min\{g_1, g_2\}$.

Otherwise assume that C passes through v. We can suppose that C passes through v once. In this case we can decompose $C = (v, x) \cup L \cup (y, v)$ where x, y are the vertices of some e_i and L is a path in the corresponding $G_i \backslash e_i$ joining x and y. Note that the length of L is at least $\min\{g_1 - 1, g_2 - 1\}$, since otherwise $L \cup e_i$ would contain a circuit of length at most $g_i - 1$ in G_i. Hence we have $l(C) \geq \min\{g_1 - 1, g_2 - 1\} + 2 \geq \min\{g_1, g_2\}$. $\square$

Remark 10. In view of Proposition 5.2.3, from now on we will write

$$G_1 * G_2 := G_1 *_{e_1,e_2} G_2$$

for a suitable choice of edges such that $\mathrm{girth}(G_1 * G_2) = \min\{g_1, g_2\}$.

Let $k \geq 2$ and $g \geq 3$ be integers. A (k, g)-graph is a k-regular graph with girth g. A (k, g)-*cage* is a (k, g)-graph of minimum number of vertices. We denote by $\nu(k, g)$ the number of vertices of a (k, g)-cage.

We will need the following theorem, see [30, Thm. A] for the proof.

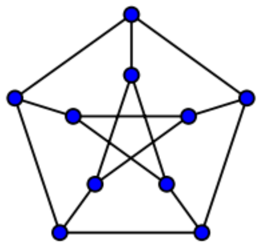


Figure 5.1: Example of a $(3, 5)$-cage.

Theorem 5.2.4. *Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then*

$$\nu(k, g) \leq 2kq^{\frac{3g}{4} - a},$$

where $a = 4, \frac{11}{4}, \frac{7}{2}, \frac{13}{4}$ for $g \equiv 0, 1, 2, 3 (\bmod 4)$, respectively.

Note that any 4-regular graph on n-vertices has girth at most $C \log(n)$ for some constant $C > 0$ which does not depend on n. Conversely, the following proposition is a coarse inverse of this upper bound.

Proposition 5.2.5. *There exist a constant $c > 0$ and an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and any $5 \leq s \leq c \log(n)$ there exists a 4-regular, connected graph $G_{n,s}$ on n vertices with girth s.*

Proof. By Theorem 5.2.4, for any $s \geq 5$, there exists a 4-regular connected graph H_s on n_s vertices with girth s such that $n_s \leq a 5^{\frac{3}{4}s}$ for some $a > 0$ which does not depend on s.

On the other hand, for any odd prime p there exists a 4-regular connected graph X_p on $p(p-1)(p+1)$ vertices with $\mathrm{girth}(X_p) \geq b \log(|X_p|+1)$ for some $b > 0$ which does not depend on p (See [13, Appendix 4]).

Arguing as in [11, p.47], there exist positive integers K and n_0 such that every $m \geq n_0$ can be written as

$$m = m_1 + \cdots + m_k,$$

for some $k \leq K$, m_j being of the form $p(p-1)(p+1) + 1$ and $m_j \geq m^{\frac{1}{2k}}$.

Whenever $s \geq 5$ satisfies $a5^{\frac{3}{4}s} < \min\{\frac{n}{2}, n-n_0\}$ it follows that $n-n_s > n_0$ and $n - n_s > \frac{n}{2}$. Therefore, there are odd primes $p_1, \cdots, p_k$ such that

$$n - n_s = m_1 + \cdots + m_k,$$

where $m_j = p_j(p_j-1)(p_j+1)+1$, $k \leq K$ and $m_j \geq (n-n_s)^{\frac{1}{2k}} \geq \left(\frac{n}{2}\right)^{\frac{1}{2K}}$.

For any $j = 1, \cdots, k$ consider the corresponding graph X_{p_j} and define

$$G_{n,s} = H_s * (X_{p_1} * \cdots * X_{p_k}).$$

Note that by induction $|G_{n,s}| = |H_s| + \sum_{j=1}^k |X_{p_j}| + k = n_s + \sum_{j=1}^k m_j = n$. Moreover, by Proposition 5.2.3 we have

$$\mathrm{girth}(G_{n,s}) = \min\{s, \mathrm{girth}(X_{p_1}), \cdots, \mathrm{girth}(X_{p_k})\}.$$

If we take $c > 0$ such that $s \leq c\log(n)$ implies $s \leq \frac{b}{2K}\log(\frac{n}{2})$ and $a5^{\frac{3}{4}s} < \min\{\frac{n}{2}, n-n_0\}$, then $s \leq b\log(m_j) \leq \mathrm{girth}(X_{p_j})$ for any $1 \leq j \leq k$ and hence $\mathrm{girth}(G_{n,s}) = s$. $\square$

Lemma 5.2.6. *Let k, l be positive integers such that $3 \leq k < l$. Then for all n sufficiently large, there exists a connected 4-regular graph on n vertices with* girth $= k$ *and* 2-girth $> l$.

Proof. By Theorem 5.2.1, for a fixed $p \geq 3$ the random variables $\{X_{n,i}\}_{i=3}^p$ are asymptotically independent Poisson random variables converging to the vector $(X_1, \cdots, X_p)$ with X_i having mean $\mu_i = \frac{3^i}{2i}$. This means that for any list $(m_3, \cdots, m_p)$ with $m_i \in \mathbb{Z}_{\geq 0}$ we have

$$\lim_n \mathbb{P}_n(X_{n,3} = m_3, \cdots, X_{n,p} = m_p) = e^{-(\mu_3 + \cdots + \mu_p)} \prod_{i=3}^p \frac{\mu_i^{m_i}}{m_i!}. \tag{5.1}$$

Now if we fix $k \geq 3$, let

$$\Theta_k^n = \{G \in \mathcal{F}_n \mid X_{n,k}(G) > 0 \,\mathrm{and}\, X_{n,j}(G) = 0 \text{ if } 3 \leq j \leq l,\, j \neq k\}$$

be the set of 4-regular graphs on n vertices of girth k and $2-$girth at least $l+1$. The measure of Θ_k^n satisfies:

$$\mathbb{P}_n(X_{n,3} = 0, \cdots, X_{n,k-1} = 0, X_{n,k} > 0, X_{n,k+1} = 0, \cdots, X_{n,l} = 0)$$

$$= \sum_{m\geq 1} \mathbb{P}_n(X_{n,3} = 0, \cdots, X_{n,k-1} = 0, X_{n,k} = m, X_{n,k+1} = 0, \cdots, X_{n,l} = 0)$$

By Fatou's Lemma we have

$$\underline{\lim}_n \left(\sum_{m\geq 1} \mathbb{P}_n(X_{n,k} = m, X_{n,j} = 0,\, j \neq k,\, j \leq l \right) \geq \sum_{m\geq 1} \left(\underline{\lim}_n \mathbb{P}_n(X_{n,k} = m, X_{n,j} = 0,\, j \neq k,\, j \leq l \right).$$

On the other hand, by (5.1)

$$\underline{\lim}_n \mathbb{P}_n(X_{n,k} = m, X_{n,j} = 0,\, j \neq k,\, 3 \leq j \leq l) = \exp\left(-\sum_{i=3}^{l} \mu_i \right) \frac{\mu_k^m}{m!}$$

Hence $\underline{\lim}_n \mathbb{P}_n(\Theta_k^n) \geq \exp\left(-\sum_{i=3}^{l} \mu_i \right) \sum_{m\geq 1} \frac{\mu_k^m}{m!} > \exp\left(-\sum_{i=3, i\neq k}^{l} \mu_i \right).$

By Theorem 5.2.2 asymptotically almost surely the graphs in $\mathcal{F}_n$ are connected. If we consider the subset of Θ_k^n where the graphs are connected, then given $k \geq 3$ for any n sufficiently large this subset has positive measure, in particular, there exists a connected graph $G_k \in \mathcal{F}_n$ with $\mathrm{girth}(G_k) = k$ and 2-girth$(G_k) > l$. □

5.3 Quantitative fully residually freedom of Γ_2.

A group G is *n-residually free*, n a fixed positive integer, if for any set $F = \{f_1, \cdots, f_n\} \subset G$ of nontrivial elements there exists a normal subgroup $N_F \lhd G$ such that

$$G/N_F \ \text{ is free and } \ F \cap N_F = \emptyset.$$

A group is said to be *fully residually free* if it is n-residually free for all positive n.

The algebraic importance of residually freedom is that some algebraic properties of free groups are still valid for residually free groups (see [3]). Another application of the residually freedom is the study of counting problems of hyperbolic manifolds with bounded volume: in any dimension there are hyperbolic manifolds of finite volume with fundamental group residually free (see [5] for more details).

In this section we will use the notation of B. Baumslag and G. Baumslag in [3] and [2], respectively. Given two elements $h, f \in \mathrm{F}_2$ we say that h *reacts* with f if $l_A(hf) < l_A(h) + l_A(f)$. If h does not react with f we will write

$$h \wedge f.$$

The word $u = [x, y] = xyx^{-1}y^{-1}$ is reduced and cyclically reduced. Moreover, if $f \in \mathrm{F}_2$ commutes with u, then $f \in \langle u \rangle$. This means that u generates its own centralizer in F_2.

Lemma 5.3.1. *Let $m \geq 3$ be an integer and $\varepsilon_q \in \{\pm 1\}$, $q = 1, 2$. Then for any $l > 1$ and $\gamma \in \mathrm{F}_2 \setminus \langle u \rangle$ with $l_A(\gamma) \leq l$ it holds*

$$u^{\varepsilon_1 ml}\, \gamma\, u^{\varepsilon_2 ml} = u^{\varepsilon_1 (m-2)l} \wedge \gamma' \wedge u^{\varepsilon_2 (m-2)l}$$

with $\gamma' \neq 1$.

Proof. If γ contains the subword $u^{\pm 1}$ at the beginning we can write $\gamma = u^{\pm 1} \wedge \gamma_1$, with a finite number of steps we can write $\gamma = u^{i_1} \wedge \tilde{\gamma}$ for some $i_1 \in \mathbb{Z}$ with $|i_1| \leq \frac{l}{4}$. Similarly, $\tilde{\gamma} = \hat{\gamma} \wedge u^{i_2}$ for some $i_2 \in \mathbb{Z}$ with $|i_2| \leq \frac{l}{4}$. Hence, we can always write $\gamma = u^{i_1} \wedge \hat{\gamma} \wedge u^{i_2}$ with $|i_1| + |i_2| \leq \frac{l}{4}$.

Note that we have the equation

$$u^{\varepsilon_1 ml} \gamma u^{\varepsilon_2 ml} = u^{\varepsilon_1 (m-2)l} \wedge u^{\varepsilon_1 2l + i_1}\, \hat{\gamma}\, u^{\varepsilon_2 2l + i_2} \wedge u^{\varepsilon_2 (m-2)l}.$$

Indeed, $|i_1| + |i_2| \leq \frac{l}{4}$ and $m \geq 3$ imply that $\varepsilon_q(m-2)l$ and $2\varepsilon_q l + i_q$ have the same sign for $q = 1, 2$.

Finally, we have that $\gamma' = u^{\varepsilon_1 2l + i_1}\, \hat{\gamma}\, u^{\varepsilon_2 2l + i_2} \neq 1$. Otherwise, $\hat{\gamma} \in \langle u \rangle$ and consequently $\gamma \in \langle u \rangle$. $\square$

We use the following notation:

$${}^{s}\delta^{t}(a_1)a_2 \ldots a_{p-1}(a_p)$$

will denote the four expressions obtained from $a_1 a_2 \cdots a_p$ by deleting or not deleting a_1 and a_p independently.

Proposition 5.3.2. *Let k be any given positive integer. Suppose that*

$$\gamma_1, \gamma_{r+1} \in \mathrm{F}_2\,, \gamma_2 \cdots, \gamma_r, \eta_1, \cdots, \eta_r \in \mathrm{F}_2 \setminus \langle u \rangle.$$

Furthermore, suppose that $\sum_{i=1}^{r+1} l_A(\gamma_i) + \sum_{j=1}^{r} l_A(\eta_j) \leq k$.

Then there exists a constant $d > 0$ *which does not depend on* k *such that*

$$^s\delta^t(\gamma_1)u^{dk}\eta_1 u^{-dk}\gamma_2 u^{dk}\eta_2 u^{-dk}\cdots u^{dk}\eta_r u^{-dk}(\gamma_{r+1}) \neq 1.$$

Proof. Consider the word

$$w = {}^s\delta^t(\gamma_1)u^{7k}\eta_1 u^{-7k}\gamma_2 u^{7k}\eta_2 u^{-7k}\cdots u^{7k}\eta_r u^{-7k}(\gamma_{r+1}).$$

By Lemma 5.3.1 we have $u^{7k}\eta_i u^{-7k} = u^{5k} \wedge \eta_i' \wedge u^{-5k}$ with $\eta_i' \neq 1$. Hence,

$$w = {}^s\delta^t(\gamma_1)u^{5k} \wedge \eta_1' \wedge u^{-5k}\gamma_2 u^{5k} \wedge \eta_2' \wedge \cdots \wedge u^{-5k}\gamma_r u^{5k} \wedge \eta_r' \wedge u^{-5k}(\gamma_{r+1}).$$

We can rewrite this word as

$$^s\delta^t(\gamma_1)u^{5k} \wedge \eta_1' \wedge u^{-2k}\left(u^{-3k}\gamma_2 u^{3k}\right)u^{2k} \wedge \cdots \wedge u^{-2k}\left(u^{-3k}\gamma_r u^{3k}\right)u^{2k} \wedge \eta_r' \wedge u^{-5k}(\gamma_{r+1}).$$

Using Lemma 5.3.1 once again we have $u^{-3k}\gamma_j u^{3k} = u^{-k} \wedge \gamma_j' \wedge u^k$ with $\gamma_j' \neq 1$ for $2 \leq j \leq r$. Hence we have

$$w = {}^s\delta^t(\gamma_1)u^{5k} \wedge \eta_1' \wedge u^{-3k} \wedge \gamma_2' \wedge u^{3k} \wedge \eta_2' \wedge u^{-3k} \wedge \cdots \wedge u^{-3k} \wedge \gamma_r' \wedge u^{3k} \wedge \eta_r' \wedge u^{-5k}(\gamma_{r+1}).$$

If we write

$$w' = u^{5k} \wedge \eta_1' \wedge u^{-3k} \wedge \gamma_2' \wedge u^{3k} \wedge \eta_2' \wedge u^{-3k} \wedge \cdots \wedge u^{-3k} \wedge \gamma_r' \wedge u^{3k} \wedge \eta_r' \wedge u^{-5k},$$

it follows that

$$l_A(w) \geq l_A(w') - l_A(\gamma_1) - l_A(\gamma_{r+1}) > 20k - 2k = 18k > 0.$$

Hence, if we take $d = 7$, then w is not trivial. $\square$

Let Γ_2 be the fundamental group of an orientable compact surface of genus 2. If we consider a new copy F_2' of F_2 with the set of generators $A' = \{x'^{\pm 1}, y'^{\pm 1}\}$ and let $v = [y', x'] \in \mathrm{F}_2'$ it is well-known that

$$\Gamma_2 \text{ is isomorphic to the quotient } \mathrm{F}_2 * \mathrm{F}_2' / \langle\langle u * v^{-1} \rangle\rangle,$$

where $\mathrm{F}_2 * \mathrm{F}_2'$ is the free product.

It is known that Γ_2 is residually free (see [2, Theorem 8]). We will need a refinement of this theorem, in fact we will give a proof based on the proof of Baumslag but using geometry of Γ_2 and F_2.

Considering the natural monomorphisms $\iota : \mathrm{F}_2 \to \Gamma_2$ and $\iota' : \mathrm{F}_2' \to \Gamma_2$, we can identify $\mathrm{F}_2, \mathrm{F}_2'$ as subgroups of Γ_2. With this identification in mind, we can take $B = A \cup A'$ as a set of generators of Γ_2 and let $l_B(\tau)$ denote the length of any element $\tau \in \Gamma_2$ with respect to B.

Proposition 5.3.3. *There exists a constant $\epsilon > 0$ such that for every positive integer k there exists an epimorphism $\psi_k : \Gamma_2 \to \mathrm{F}_2$ with the following properties:*

1. $\psi_k(\iota(t)) = t$ *for every* $t \in \mathrm{F}_2$;

2. $\psi_k(\gamma) \neq 1$ *if* $1 < l_B(\gamma) \leq k$;

3. $l_A(\psi_k(\omega)) \leq \epsilon k l_B(\omega)$ *for all* $\omega \in \Gamma_2$.

Proof. Fixed a positive integer l we define the map $\rho_l : \mathrm{F}_2' \to \mathrm{F}_2$ given by $\rho_l(x') = u^l y u^{-l}$ and $\rho_l(y') = u^l x u^{-l}$. Since $\iota(u) = u = \rho_l(v)$ it follows that the map $\tilde{\phi_l} : \mathrm{F}_2 * \mathrm{F}_2' \to \mathrm{F}_2$ given by ι and ρ_l descends to the quotient and defines a homomorphism $\phi_l : \Gamma_2 \to \mathrm{F}_2$. By the construction $\phi_l(\iota(t)) = \iota(t) = t$ for any l and any $t \in \mathrm{F}_2$.

Let $\gamma \in \Gamma_2$ be a non trivial element with $l_B(\gamma) \leq k$. Now let d be the constant given by Proposition 5.3.2, take $l = dk$ and define $\psi_k = \phi_l = \phi_{dk}$. If $\gamma \notin \mathrm{F}_2$, then writing γ as a reduced word we have

$$\gamma = \gamma_1 * \eta_1' * \gamma_2 * \cdots * \eta_r' * \gamma_{r+1},$$

where $\gamma_i \in \mathrm{F}_2$ and $\eta_j \in \mathrm{F}_2'$ are non-trivial with the possible exception of γ_1 and γ_{r+1}. Since we have $u = v$ in Γ_2, we can suppose that $\eta_j' \notin \langle v \rangle$ and $\gamma_i \notin \langle u \rangle$ for $2 \leq i \leq r$.

Hence

$$\psi_k(\gamma) = \gamma_1 u^{dk} \eta_1 u^{-dk} \gamma_2 u^{dk} \eta_2 u^{-dk} \cdots u^{dk} \eta_r u^{-dk} \gamma_{r+1}.$$

where η_j is the word η_j' with x' replaced by y and y' replaced by x. Note that $\eta_j \neq 1$ if and only if $\eta_j' \neq 1$. Furthermore, $\eta_j' \in \langle v \rangle$ if and only if $\eta_j \in \langle u \rangle$.

We also have

$$\sum_i l_A(\gamma_i) + \sum_j l_A(\eta_j) = \sum_i l_A(\gamma_i) + \sum_j l_{A'}(\eta'_j) = l_B(\gamma) \leq k.$$

Therefore, we can apply Proposition 5.3.2 to conclude that $\psi_k(\gamma) \neq 1$ if $1 < l_B(\gamma) \leq k$. Now for any $b \in B$ we have

$$l_A(\phi_k(b)) \leq 1 + 8dk \leq 9dk,$$

and hence if we take $\epsilon = 9d$, we have

$$l_A(\psi_k(\omega)) \leq \max_{b \in B}\{l_A(\psi_k(b))\} l_B(\omega) \leq \epsilon k l_B(\omega).$$

$\square$

CHAPTER 6

CONSTRUCTING NEW SEQUENCES OF CLOSED HYPERBOLIC SURFACES WITH LOGARITHMIC GROWTH OF THE SYSTOLE

In this chapter we will generalize a construction of sequences of closed hyperbolic surfaces with *systole* growing logarithmically in terms of the area. While the well known construction applies only for arithmetic surfaces, our generalization covers a much wider semi-arithmetic class.

Theorem 6.0.1. *Let S be a closed semi-arithmetic hyperbolic surface admitting an r-modular embedding. Then S has a sequence of coverings $S_i \to S$ with* $\operatorname{area}(S_i) \to \infty$ *and*

$$\operatorname{sys}(S_i) \geq \frac{4}{3r} \log(\operatorname{area}(S_i)) - c,$$

where $c > 0$ is a constant which does not depend on i.

Remark 11. When S is arithmetic we have $r = 1$ and the bound in Theorem 6.0.1 reduces to the previously known results.

6.1 Some history

In the early $70'$s, the connection between Number Theory and Hyperbolic Geometry developed by Selberg raised a new chapter with the paper

"Selberg's trace formula as applied to a compact Riemann surface" ([36]) of McKean, which stated wrongly that the spectral gap of any closed hyperbolic surface is at least $\frac{1}{4}$.

Two years after this paper, Randol ([42]) gave the first counterexample for this claim. In fact, Randol showed that for any sufficiently large integer N there exists a hyperbolic surface with N small eigenvalues (counted with multiplicity) for the Laplace-Beltrami operator on the surface, where a eigenvalue is called small if it is smaller than $\frac{1}{4}$.

Consider the function

$$\mathrm{Ev}_S(x) = \text{number of eigenvalues smaller than } x \text{ for the surface } S.$$

After the work of Randol, the study of Ev_S on a neighborhood of $\frac{1}{4}$ was of growing interest due to its importance for the Selberg's zeta function of the surface.

Although Randol had showed that $\mathrm{Ev}_S(\frac{1}{4})$ can be large, Huber showed in [22] that if we fix a genus, the ratio $\dfrac{\mathrm{Ev}_S(\frac{1}{4})}{\mathrm{Ev}_S(\frac{1}{4}+\varepsilon)}$ is very close to 0 whenever the systole of S is sufficiently large for any $\varepsilon > 0$.

This tells us that although the surface may have small eigenvalues, for any $\varepsilon > 0$ if the systole of S is sufficiently large in terms of ε, the eigenvalues in $[0, \frac{1}{4}+\varepsilon)$ concentrate in $[\frac{1}{4}, \frac{1}{4}+\varepsilon)$.

At that time there were no known examples of surfaces with sufficiently large systole. In [22] , Huber constructed cocompact Fuchsian subgroups of $\mathrm{PSL}(2, \mathbb{Q}(\sqrt{p}))$, for some prime p, and he showed that such groups give surfaces with large systole.

In the subsequent search for closed hyperbolic surfaces with large systole, in [9] Peter Buser constructed hyperbolic surfaces using pairs of pants with large geodesics in the boundary and the gluing of pants performed by special cubic graphs. The combinatorics of such graphs give surfaces of large systole. In fact, the surfaces obtained by Buser in his paper give closed surfaces of any genus g and systole at least $\sqrt{2\log(g)}$.

Recall that $\mathrm{sys}(S) \leq 2\log(g) + A$ for some constant $A > 0$ independent of the genus of g. The work of Buser left open a new question about the geometry of closed surfaces, which became independent of the problem of small eigenvalues. The question was the following: Are there hyperbolic surfaces with arbitrarily large genus and systole of the order of $\log(g)$?

In their study of the Schottky problem of describing the locus of the Jacobian of Riemann surfaces in the space of principally polarized abelian varieties, Buser and Sarnak showed in [11] that for all sufficiently large g, there exists a closed hyperbolic surface with systole at least $d\log(g)$ for some universal constant $d > 0$. In the proof of this result they constructed arithmetic hyperbolic surfaces defined over $\mathbb{Q}$ and took congruence coverings. This result solved the question above and it gave rise to a new question.

The sequence of coverings that Buser and Sarnak constructed give surfaces S_i with $\text{area}(S_i) \to \infty$ and

$$\text{sys}(S_i) \geq \frac{4}{3}\log(\text{area}(S_i)) - c,$$

where the constant c depends only on the arithmetic setup of the construction and does not depend on i. The constant $\frac{4}{3}$ was a novelty in this problem and raised the question: is $\frac{4}{3}$ the sharp constant for the problem of large systole?

In 2007 Katz, Schaps and Vishne generalized in [29] the construction of Buser and Sarnak for any arithmetic hyperbolic surface, and also for arithmetic hyperbolic 3-manifolds. A partial answer for this last question was given by Makisumi in [35]. He showed that for the surfaces constructed by Buser and Sarnak and by Katz, Schaps and Vishne the biggest constant that can occur is $\frac{4}{3}$. For surfaces in general this is still an open problem.

Our contribution to this problem is to generalize the construction of a sequence of closed hyperbolic surfaces with systole growing logarithmically with an explicit constant to a more general class of surfaces (see Remark 4). On the other hand, the constant is worse when the surface is not arithmetic, what was expected (see [45], [43]).

6.2 Proof of the Theorem

We will now construct the new sequences of closed hyperbolic surfaces with logarithmic growth of sytole. We use the same ideas of the papers [11] and [29]. More recently, Murillo ([38],[39]) gave generalizations of these constructions to high dimensional hyperbolic spaces and other symmetric spaces.

From now on, $S = \Gamma\backslash\mathbb{H}$ will denote a closed semi-arithmetic surface admitting modular embedding, i.e there exist a quaternion algebra D over a totally real number field k of degree n with exactly r Galois embeddings $\phi_1, \phi_2, \cdots, \phi_r : k \to \mathbb{R}$ with $\phi_j(\mathrm{tr}(\tilde{\Gamma})^2) \nsubseteq [-2, 2]$, a maximal order $\mathcal{O}$ of D, an embedding $\psi : \Gamma(D, \mathcal{O}) \to \mathrm{SL}(2, \mathbb{R})$ such that $\tilde{\Gamma} < \psi(\Gamma(D, \mathcal{O}))$ and r holomorphic maps $F_j : \mathbb{H} \to \mathbb{H}$ such that

$$F_j(\gamma^{\phi_j} \cdot z) = \gamma^{\phi_j} \cdot F(z) \text{ for all } z \in \mathbb{H}, \gamma \in \tilde{\Gamma} \text{ and } j = 1, \cdots, r.$$

Consider the family of congruence coverings $\{S_\mathfrak{a}\}$ of S where $\mathfrak{a}$ runs over the ideals of R_k.

The relation between closed geodesic on a hyperbolic surface and trace of hyperbolic elements in the Fuchsian group is given by the formula (2.2):

$$|\,\mathrm{tr}(\alpha)| = 2 \cosh \left(\frac{l(\alpha)}{2} \right). \tag{6.1}$$

Here α means at the same time a hyperbolic element of Γ and the corresponding closed geodesic induced on the surface S. Thus, if we want to give a lower bound for the systole of $S_\mathfrak{a}$, it is equivalent to giving a lower bound for the trace of any nontrivial element of $\Gamma(\mathfrak{a})$. In fact, we have the following proposition.

Proposition 6.2.1. *Let $S = \Gamma\backslash\mathbb{H}$ be a closed semi-arithmetic surface admitting an r-modular embedding. Consider the sequence of congruence subgroups $\Gamma(\mathfrak{a}) \trianglelefteq \Gamma$. There exists a constant $c_2 > 0$ which does not depend on $\mathfrak{a}$ such that for any nontrivial $\gamma \in \Gamma(\mathfrak{a})$ we have the inequality $|\,\mathrm{tr}(\gamma)| > c_2 \, \mathrm{N}(\mathfrak{a})^{\frac{2}{r}}$.*

Proof. Note that for any $x \in \mathcal{O}$, the submodule $R_k[x] \subset \mathcal{O}$ is finitely generated. Moreover, by definition of $\mathrm{tr}(x)$ and $\mathrm{rn}(x)$ we have

$$x^2 - \mathrm{tr}(x)x + \mathrm{rn}(x) = 0,$$

which implies that $R_k[\mathrm{tr}(x), \mathrm{rn}(x)]$ is a finite extension of R_k. Since R_k is integrally closed in k, it follows that $\mathrm{tr}(x), \mathrm{rn}(x) \in R_k$ for any $x \in \mathcal{O}$. Thus, any order is closed with respect to conjugations, since $\bar{x} = \operatorname{tr}(x) - x$ and $R_k \subset \mathcal{O}$.

If $\gamma \in \tilde{\Gamma}(\mathfrak{a})$, then up to taking the preimage by ψ we have

$$\gamma - 1 = \sum_i x_i w_i \quad \text{with} \quad x_i \in \mathfrak{a} \text{ and } w_i \in \mathcal{O}.$$

Therefore,

$$\det(\gamma - 1) = (\sum_i x_i w_i)(\sum_i x_i \bar{w}_i) = \sum_i x_i^2 \det(w_i) + \sum_{i<j} x_i x_j \operatorname{tr}(w_i \bar{w}_j) \in \mathfrak{a}^2.$$

On the other hand $\det(\gamma - 1) = 2 - \operatorname{tr}(\gamma) \in \mathfrak{a}^2$. Since $\Gamma(\mathfrak{a})$ is torsion free, we have $\operatorname{tr}(\gamma) - 2 \neq 0$ for any $\gamma \neq 1$ in $\Gamma(\mathfrak{a})$. Hence,

$$\mathrm{N}((\operatorname{tr}(\gamma) - 2)R_k) \geq \mathrm{N}(\mathfrak{a}^2).$$

For $j = r+1, \ldots, n$ we have $|\operatorname{tr}(\phi_j(\gamma))| \leq 2$ for all $\gamma \in \Gamma$. Thus

$$\begin{aligned}
\mathrm{N}(\mathfrak{a}^2) &\leq \mathrm{N}((\operatorname{tr}(\gamma) - 2)R_k) = \prod_{i=1}^{n} |\phi_i(\operatorname{tr}(\gamma) - 2)| \\
&\leq \prod_{i=1}^{r} 2\max\{2, |\phi(\operatorname{tr}(\gamma))|\} \prod_{j=r+1}^{n} 2\max\{2, |\phi_j(\operatorname{tr}(\gamma))|\} \\
&\leq 2^n \prod_{i=1}^{r} |\phi_i(\operatorname{tr}(\gamma))| 2^{n-r} \\
&< 2^{2n-r} |\operatorname{tr}(\gamma))|^r.
\end{aligned}$$

Where in the last inequality we are using Proposition 3.3.1. The constant $c_2 = \frac{1}{2^{2n-r}}$ does not depend on $\mathfrak{a}$ and the proof finishes. $\square$

Now we conclude the chapter with the proof of Theorem 6.0.1.

Proof of Theorem 6.0.1. Consider the sequence of ideals $\mathfrak{a}$ of R_k and the corresponding sequence of coverings $S_{\mathfrak{a}}$. If S has area $\operatorname{area}(S)$ and $d_{\mathfrak{a}}$ denotes the degree of the covering, then $\operatorname{area}(S_{\mathfrak{a}}) = d_{\mathfrak{a}} \operatorname{area}(S)$. By Proposition 6.2.1 and Equation (6.1) we have

$$\operatorname{sys}(S_{\mathfrak{a}}) \geq \inf_{\gamma \in \Gamma_{\mathfrak{a}} \setminus \{1\}} 2\cosh^{-1}\left(\frac{c_2 \, \mathrm{N}(\mathfrak{a})^{\frac{2}{r}}}{2}\right).$$

We consider now the infinite family of ideals $\mathfrak{a} \subset R_k$ such that $c_2 \, \mathrm{N}(\mathfrak{a})^{\frac{2}{r}} > 2$. With this assumption we can use the inequality

$$\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1}) \geq \log(t) \quad \text{for any} \quad t \geq 1$$

to show that

$$\mathrm{sys}(S_{\mathfrak{a}}) \geq 2 \log \left(\frac{c_2 \, \mathrm{N}(\mathfrak{a})^{\frac{2}{r}}}{2} \right) \geq \frac{4}{r} \log(\mathrm{N}(\mathfrak{a})) + 2 \log \left(\frac{c_2}{2} \right). \tag{6.2}$$

It follows from (6.2) that $\mathrm{sys}(S_{\mathfrak{a}})$ goes to infinity when $\mathrm{N}(\mathfrak{a})$ goes to infinity. Therefore $\mathrm{area}(S_{\mathfrak{a}})$ goes to infinity when $\mathrm{N}(\mathfrak{a})$ goes to infinity. Indeed, recall that $\mathrm{sys}(M) \lesssim \log(\mathrm{area}(M))$ for any closed hyperbolic surface M.

By Proposition 3.4.2, we have

$$\mathrm{N}(\mathfrak{a}) \geq \sqrt[3]{\frac{d_{\mathfrak{a}}}{C}} = \sqrt[3]{\frac{\mathrm{area}(S_{\mathfrak{a}})}{c_1 \, \mathrm{area}(S)}}.$$

Therefore,

$$\mathrm{sys}(S_{\mathfrak{a}}) \geq \frac{4}{3r} \log(\mathrm{area}(S_{\mathfrak{a}})) - \left(\frac{4}{3r} \log(\mathrm{area}(S)) - \left(\frac{c_2}{2} \right) \right).$$

$\square$

CHAPTER 7

DISTRIBUTION OF ARITHMETIC POINTS AND ITS SYSTOLES

7.1 Coarse density of arithmetic points

Let $\mathcal{AS}_g \subset \mathcal{M}_{\mathrm{g}}$ be the set of arithmetic compact hyperbolic surfaces of genus g. In [8] Borel showed that for any $g \geq 2$ the set $\mathcal{AS}_g$ is finite. Denote $|\,\mathcal{AS}_g\,|$ the cardinality of $\mathcal{AS}_g$. In [5] the authors investigated the asymptotic growth of these cardinalities. In fact, we have the following theorem (see [5, Corollary 1.4]).

Theorem 7.1.1.

$$\lim_{g\to\infty} \frac{\log(|\,\mathcal{AS}_g\,|)}{g\log(g)} = 2.$$

On the other hand we recall the asymptotic growth of the volume of the moduli spaces with respect to the Weil-Petersson metric (see Section 4.3) .

Theorem 7.1.2.

$$\lim_{g\to\infty} \frac{\log \mathrm{vol}_{wp}(\mathcal{M}_{\mathrm{g}})}{g\log g} = 2.$$

If we compare Theorem 7.1.1 with Theorem 7.1.2, then a natural question arises:

Are there constants $\varepsilon, \delta > 0$ such that for any closed hyperbolic surface S of genus g and systole at least ε we can find an arithmetic closed hyperbolic surface A of genus g with $d_{wp}(X, A) \leq \delta$?

An affirmative answer to this question would imply that the arithmetic hyperbolic surfaces are uniformly well distributed. In contrast we have the following theorem.

Theorem 7.1.3. *For any $\delta, \varepsilon > 0$ there exists $g_0 = g_0(\delta, \varepsilon) \geq 2$ such that for every $g \geq g_0$,*

$$\mathcal{M}_{\mathrm{g}}{}^{\geq\varepsilon} \not\subseteq \bigcup_{A \in \mathcal{AS}_g} B_{wp}(A, \delta).$$

Proof. Suppose the contrary. Then there exist $\delta, \varepsilon > 0$ and a function $\pi : \mathcal{M}_g^{\geq\varepsilon} \to \mathcal{AS}_g$ such that for any $X \in \mathcal{M}_{g,\varepsilon}$ we have $d(X, \pi(X)) \leq \delta$. Thus in this case we can write for any g,

$$\mathcal{M}_g^{\geq\varepsilon} \subset \bigcup_{A \in \pi(\mathcal{M}_g^{\geq\varepsilon})} B_{wp}(A, \delta).$$

By Theorem 4.3.4, the function $\sqrt{\mathrm{sys}}$ is K-Lipschitz for some $K > 0$ with respect to the Weil-Petersson metric for any g. We can suppose that $\varepsilon \geq (3K\delta)^2$ since

$$\varepsilon' > \varepsilon \Rightarrow \mathcal{M}_g^{\geq\varepsilon'} \subset \mathcal{M}_g^{\geq\varepsilon}.$$

Therefore, if $A \in \pi(\mathcal{M}_g^{\geq\varepsilon})$, then

$$\sqrt{\mathrm{sys}(A)} \geq \sqrt{\mathrm{sys}(X)} - |\sqrt{\mathrm{sys}(X)} - \sqrt{\mathrm{sys}(A)}| \geq 2K\delta.$$

By the same argument we can show that

$$B_{wp}(A, \delta) \subset \mathcal{M}_g^{\geq(K\delta)^2} \quad \text{for any} \quad A \in \pi(\mathcal{M}_g^{\geq\varepsilon}).$$

We can now use the argument applied by Yunhui Wu in the proof of Theorem 1.5 in [55].

By Teo's curvature bound (Theorem 4.3.5), there exists a constant $C = C(K, \delta) > 0$ such that the Ricci curvature in $B_{wp}(A, \delta)$ is bounded from below by $-C$.

We know that the set $\mathcal{M}_g^{\geq(K\delta)^2}$ is compact. If we apply the Bishop-Gromov Comparison Theorem [19, 5.3.bis], we can find a constant $C' = C'(\delta) > 0$ such that

$$\mathrm{vol}_{wp}(B_{wp}(A, \delta)) \leq C' \,\mathrm{vol}(\mathbb{S}^{6g-7})(\sqrt{r})^{7-6g} \int_0^\delta \sinh(\sqrt{r}t)^{6g-7} dt,$$

where $r = \frac{C}{6g-6}$.

Now we can use the relations

$$\Gamma(3g-3) = (3g-4)! \sim \sqrt{\pi(6g-8)}(3g-4)^{3g-4}e^{4-3g},$$

where Γ is the Gamma function, and

$$\mathrm{vol}(\mathbb{S}^{6g-7}) = \frac{2\pi^{3g-3}}{\Gamma(3g-3)}, \quad \sinh(\sqrt{rt}) \leq \frac{Dt}{\sqrt{g}} \quad \text{for all } 0 \leq t \leq \delta$$

for some constant $D = D(\delta)$. These facts together give us

$$\mathrm{vol}_{wp}(B_{wp}(A,\delta)) \leq \frac{(C'')^g}{g^{3g}},$$

for some constant C'' independent of g.

It follows from Theorem 7.1.1 that if g is sufficiently large, then

$$|\pi(\mathcal{M}_g^{\geq \varepsilon})| \leq |\,\mathcal{AS}_g\,| \leq g^{\frac{5}{2}g}.$$

Hence

$$\mathrm{vol}_{wp}(\mathcal{M}_g^{\geq \varepsilon}) \leq \sum_{A \in \pi(\mathcal{M}_g^{\geq \varepsilon})} \mathrm{vol}_{wp}(B_{wp}(A,\delta)) \leq \frac{g^{\frac{5}{2}g}(C'')^g}{g^{3g}}.$$

Thus $\mathrm{vol}_{wp}(\mathcal{M}_g^{\geq \varepsilon})$ goes to 0 when g goes to infinity.

We have a contradiction with Corollary 4.3.8 and therefore the theorem follows. $\square$

7.2 Distribution of systoles

Let (X,d) be a metric space. Let $A \subset X$ be a compact subset and $r > 0$. The *r-covering number* $\eta_X(A,r)$ is the minimal number of balls in X of radius r needed to cover A in X. By Theorem 7.1.3 we cannot cover $\mathcal{M}_{\mathrm{g}}{}^{\geq \varepsilon}$ by balls centered on $\mathcal{AS}_g$ for any radius uniformly with respect to the Weil-Petersson metric. If we consider the Teichmuller metric on $\mathcal{M}_{\mathrm{g}}$, we have the following theorem [16, Theorem 1.2].

Theorem 7.2.1. *If $\varepsilon > 0$ and $r > 0$, then*

$$\lim_{g \to \infty} \frac{\log(\eta_{\mathcal{M}_{\mathrm{g}}}(\mathcal{M}_{\mathrm{g}}{}^{\geq \varepsilon}, r))}{g \log(g)} = 2.$$

We can also ask about the distribution of arithmetic points in the thick-part of moduli space with respect to Teichmuller metric. Since the Teichmuller metric is not Riemmanian, this question seems more difficult.

However, in some sense we can prove that the set of systoles of closed arithmetic surfaces is well distributed.

We will need to introduce some notation. Let Y be a set and $f, g : Y \to \mathbb{R}$ be functions. We say that $f \ll g$ if there exists a universal positive constant $R > 0$ such that $f(y) \leq Rg(y)$ for all $y \in Y$. If the constant R depends only on a set of parameters Λ we write $f \ll_\Lambda g$. When $f \ll g$ and $g \ll f$ (respectively, $f \ll_\Lambda g$ and $g \ll_\Lambda f$) we write $f \asymp g$ (respectively, $f \asymp_\Lambda g$).

Recall that for any sequence $(X_g) \in \prod_{g\geq 2} {\mathcal{M}_{\mathrm{g}}}^{\geq \varepsilon}$, the sequence $(\mathrm{sys}(X_g))$ satisfies $\log(\varepsilon) \leq \log(\mathrm{sys}(X_g)) \leq \log(\log(g)) + d$ for some d which does not depend on g (see (2.3)). This motivates the following definition.

Definition 19. A sequence $(a_g)_{g\geq 2}$ of positive numbers is called *systolic admissible* if

$$A \leq \log(a_g) \leq \log(\log(g)) + d,$$

for some constant A.

Now we can state the main result of this thesis.

Theorem 7.2.2 (Main Theorem)**.** *Let S be a closed hyperbolic surface of genus 2. For any systolic admissible sequence $(a_g)_{g\geq 2}$ with $a_g \geq \varepsilon > 0$, there exists a sequence of coverings $S_g \to S$ with $S_g \in \mathcal{M}_{\mathrm{g}}$ such that*

$$\log(\mathrm{sys}(S_g)) \asymp_{S,\varepsilon} \log(a_g).$$

Remark 12. In [44, Theorem 5.2], P. Schaller showed that the hyperbolic surface of genus 2 which have the maximal systole in $\mathcal{M}_2$ is the Bolza surface M. On the other hand, in [27, Corollary 10.4] it was shown that the Bolza surface is an arithmetic hyperbolic surface derived from a quaternion algebra. If we apply the Theorem 7.2.2 for M we conclude that the set of logarithms of systoles of arithmetic surfaces has any admissible growth function.

Let S be a fixed orientable compact hyperbolic surface of genus 2. There exists a monomorphism $\rho : \Gamma_2 \to \mathrm{Isom}^+(\mathbb{H})$ such that $S \simeq \rho(\Gamma_2)\backslash\mathbb{H}$. We will denote $\gamma \cdot p := \rho(\gamma)(p)$ for any $\gamma \in \Gamma_2$ and $p \in \mathbb{H}$.

By the Milnor-Schwarz Lemma, fixed a point $p \in \mathbb{H}$ there exist constants $q, \beta > 0$ which depend only on the geometry of S and the set of generators B of Γ_2 such that

$$\frac{1}{q} l_B(\gamma) - \beta \leq \operatorname{dist}(p, \gamma \cdot p) \leq q l_B(\gamma) + \beta \tag{7.1}$$

for every $\gamma \in \Gamma_2$, where dist is the hyperbolic distance in $\mathbb{H}$.

Proof of Theorem 7.2.2. Let c, n_0 be as in the Proposition 5.2.5 and take $g_0 = n_0 + 1$. If we fix $g \geq g_0$, then we can take the graph $G_{n_g,h}$ with girth h and $n_g = g - 1$ vertices for any integer h with $5 \leq h \leq c \log(g-1)$.

We use now the Proposition 5.1.2 in order to get a subgroup $\Gamma_{h,n_g} < \mathrm{F}_2$ of index n_g such that

$$a_{h,n_g} = \operatorname{girth}(G_{n_g,h}) = \min_{\gamma \in \Gamma_{h,n_g} \setminus \{1\}} \{l_A(\gamma)\}.$$

Consider the constant ϵ from Proposition 5.3.3. Define $f_h = \lfloor \sqrt{\epsilon^{-1} h} \rfloor$ and take m_0' minimal such that $h \geq m_0'$ implies $f_h \geq 1$ for any h and n.

For each f_h we take the homomorphism ψ_{f_h} from Proposition 5.3.3. If we fix h, define the subgroup $\Lambda_{h,n_g} = \psi_{f_h}^{-1}(\Gamma_{h,n_g}) < \Gamma_2$. Since ψ_{f_h} is surjective, Λ_{h,n_g} has index n_g in Γ_2 and therefore the natural projection $S_{h,n_g} = \rho(\Lambda_{h,n_g}) \backslash \mathbb{H} \to S$ is a covering of degree $n_g = g - 1$. Hence S_{h,n_g} is a closed surface of genus g.

We need to estimate the systole of S_{h,n_g}. By Proposition 5.3.3, $\psi_{f_h}(v) = v$ for any $v \in \mathrm{F}_2$. Let $\gamma_0 \in \Gamma_{h,n_g}$ be the element such that $h = l_A(\gamma_0)$. Then $\gamma_0 \in \Lambda_{h,n_g}$ and $l_B(\gamma_0) \leq l_A(\gamma_0) = h$, where A is the set of generators of Γ_2 as in Proposition 5.3.3. Hence, if we consider the closed geodesic induced by γ_0 on S_{h,n_g} of length $l(\gamma_0)$ we have,

$$\operatorname{sys}(S_{h,n_g}) \leq l(\gamma_0) = \inf_z \operatorname{dist}(z, \gamma_0 \cdot z) \leq \operatorname{dist}(p, \gamma_0 \cdot p) \leq qh + \beta.$$

Let $D(p)$ be the fundamental domain of Dirichlet centered in p for the action of $\rho(\Gamma_2)$ on $\mathbb{H}$. Then the length of the systole of S_{h,n_g} can be evaluated if we take a lifting of closed geodesic which realizes the systole with the initial point in $D(p)$, because the projection of this geodesic in S is a geodesic of the same length. This means that there exist a point $p_0 \in D(p)$ and a non-trivial element $\omega \in \Lambda_{h,n_g}$ such that $\mathrm{dist}(p_0, \omega \cdot p_0) = \mathrm{sys}(S_{h,n_g})$. If δ is equal to the diameter of S we have

$$\mathrm{sys}(S_{h,n_g}) \geq \mathrm{dist}(p, \omega \cdot p) - 2\,\mathrm{dist}(p, p_0) \geq \frac{1}{q} l_B(\omega) - \beta - 2\delta.$$

Suppose that $l_B(\omega) < f_h$, where B is the basis of F_2. Then by Proposition 5.3.3 we have

$$\psi_{f_h}(\omega) \in \Gamma_{h,n_g} \quad \text{and} \quad \psi_{f_h}(\omega) \neq 1.$$

It follows from the construction of Γ_{h,n_g} and Proposition 5.3.3 again that

$$m_0' \leq h \leq l_A(\psi_{f_h}(\omega)) \leq \epsilon f_h^2 < h.$$

This contradiction implies that $l_B(\omega) \geq f_h \geq \sqrt{\epsilon^{-1} h} - 1$. Hence, if we take $\delta' = \frac{1}{q} + \beta + 2\delta$ and $q' = \epsilon^{-2} q^{-1}$ we have

$$\mathrm{sys}(S_{h,n_g}) \geq q' \sqrt{h} - \delta'.$$

Now we can choose m_0'' minimal such that $t \geq m_0''$ implies $q' - \frac{\delta'}{\sqrt{t}} > 0$. If we take $m_0 = \max\{m_0', m_0''\}$, then there exists a positive constant L_1 such that for any $h \in \mathbb{N}$ with $m_0 \leq h \leq c \log(g-1)$ we have for all $g \geq n_0$,

$$\frac{1}{L_1} \sqrt{h} \leq q' \sqrt{h} - \delta' \leq \mathrm{sys}(S_{h,n_g}) \leq qh + \beta \leq L_1 h. \tag{7.2}$$

Take a systolic admissible sequence $(a_g)_{g \geq 2}$ with $a_g \geq \varepsilon$. For each $g \geq g_0$ consider the smallest integer $h(g) \in [m_0, c \log(g-1)]$.

Since $\varepsilon \leq a_g \leq e^d \log(g)$ for all $g \geq g_0 + 1$ and $m_0 \leq h(g) \leq c \log(g-1)$, by definition of $h(g)$ there exists a constant $L_2 > 0$ such that

$$\frac{1}{L_2} \leq \frac{h(g)}{\mathrm{sys}(X_g)} \leq L_2.$$

Therefore, by (7.2) the surface $S_g := S_{h(g),n_g}$ satisfies

$$\frac{1}{L_1 L_2} \sqrt{a_g} \leq \mathrm{sys}(S_g) \leq L_1 L_2 a_g,$$

whenever $g \geq g_0$. For $g = 2, \ldots, g_0 - 1$ we take a covering S_g of S of degree $g - 1$ with $\mathrm{sys}(S_g) = \mathrm{sys}(S) = s$. There exists a constant $L_3 > 0$ which depends only on ε and S such that

$$\frac{1}{L_3} \leq \frac{s}{a_g} \leq L_3, \quad \text{for each} \quad g = 2, \ldots, g_0 - 1.$$

To conclude the proof we take logarithms in these inequalities and a positive constant R sufficiently large in terms of L_1, L_2, L_3 such that

$$\frac{1}{R}\log(a_g) \leq \log(\mathrm{sys}(S_g)) \leq R\log(a_g),$$

for all $g \geq 2$.

$\square$

Corollary 7.2.3. *Let S be a compact hyperbolic surface of genus 2. Then S admits a sequence of finite covering $S_g \to S$, where g = genus of S_g and*

$$\mathrm{sys}(S_g) \geq C\sqrt{\log(g)}.$$

For some constant C which depends on S.

Of course, better bounds are known for some sequences of surfaces as in Chapter 6, but our estimate applies to any initial surface S of genus 2 and does not use arithmeticity or semi-arithmeticity.

The following theorem gives information about the sequence of systoles which has infinite multiplicity.

Theorem 7.2.4. *Let S be a compact hyperbolic surface of genus 2 and let $L'(S) = \{a_1 < a_2 < \cdots\}$ be the* pure length spectrum *of S, i.e. is the length spectrum without multiplicities and in ascending order. Then there exists a subsequence $(a_{i_r})_{r\geq 1}$ such that for any r, there exists a sequence of finite coverings $S_{m,r} \to S$ with degree $d_m \to \infty$ and* $\mathrm{sys}(S_{m,r}) = a_{i_r}$.

Proof. Let $p_k := \lceil \epsilon k^2 \rceil$. We can apply Lemma 5.2.6 for $l = p_k$, thus there exists a connected graph $G_k \in \mathcal{F}_n$ with $\mathrm{girth}(G_k) = k$ and $2 - \mathrm{girth}(G_k) > p_k$. Now we use Proposition 5.1.2 to exhibit for any large n a subgroup $\Gamma_n < \mathrm{F}_2$ of index n such that G_n is isomorphic to the Schreier graph of Γ_n, $k = \min\{l_A(w) | w \in \Gamma_n,\, w \neq 1\}$ and $\min\{l_A(w) | w \in \Gamma_n,\, l_A(w) > k\} > p_k$.

Now consider the homomorphism ψ_k given by Proposition 5.3.3 and the sequence of subgroups $\Lambda_n = \psi_k^{-1}(\Gamma_n)$. We can suppose that $S = \rho(\Gamma_2)\backslash\mathbb{H}$ satisfies (7.1). We have a sequence of coverings $S_n = \rho(\Lambda_n)\backslash\mathbb{H} \to S$ of degree $n \to \infty$ with $\mathrm{sys}(S_n) = l(\lambda_n) = a_{i_n}$, since the systole of S_n is the length of a closed geodesic in S. For any $\lambda \in \Lambda_n$ with $\lambda \neq 1$ we have $l_B(\lambda) \geq k$. Indeed, suppose the contrary, i.e. that there exists $\omega \in \Lambda_n$, $\omega \neq 1$ and $l_B(\omega) < k$. Then $\omega \notin \mathrm{F}_2$ and by the proof of Proposition 5.3.3 we have $l_A(\psi_k(\omega)) > k$.

Hence $l_A(\psi_k(\omega)) > p_k \geq \epsilon k^2$, and by Proposition 5.3.3 again we have

$$\epsilon k^2 < l_A(\psi_k(\omega)) \leq \epsilon k l_B(\omega)$$

which gives us the desired contradiction. Hence, for any k if we argue as in the proof of Theorem 7.2.2, we have

$$q'k - \delta' = x_k \leq \mathrm{sys}(S_n) = a_{i_n} \leq y_k = qk + \beta$$

for any n sufficiently large.

Since the sequence a_{i_n} is contained in the compact interval $[x_k, y_k]$ and the set $L(S)$ is discrete, we have then $a_{i_n} = a_{t_k}$ for infinitely many values of n for some $t_k \in \mathbb{N}$. Note that if we vary k then the set $\{t_k\}$ cannot be bounded, since it is possible to take a sequence of $k_j's$ with $k_j \to \infty$ and $[x_{k_u}, y_{k_u}] \cap [x_{k_v}, y_{k_v}] = \emptyset$ whenever $u \neq v$.

To finish the proof, we take $i_r := t_{k_r}$. We showed above that for any $r \geq 1$ there exists a subsequence of finite coverings of S with constant systole a_{i_r} and unbounded degree. $\square$

We will finish this section by giving a special subset of the real line formed by systoles of arithmetic hyperbolic surfaces.

Let X be a non-empty set. We denote by $\mathrm{Perm}(X)$ the group of permutations of X. Given $f \in \mathrm{Perm}(X)$, we denote $\mathrm{supp}(f) = \{x \in X | f(x) \neq x\}$. Let $a < b$ be positive integers, we will use the notation $[a, b]$ for the set $\{a, a+1, \ldots, b\}$.

Lemma 7.2.5. *Let k, m be positive integers, $l_0 = 0, n_0 = k$, $\tau_0 = id_{\mathbb{N}}$ and $\sigma_0 = (1, 2, \ldots, k)$ the cyclic permutation in* $\mathrm{Perm}(\mathbb{N})$. *Then for every $r \geq 1$ there exist integers $n_r > l_r > n_{r-1}$ and permutations $\sigma_r, \tau_r \in$* $\mathrm{Perm}(\mathbb{N})$ *such that* $\mathrm{supp}(\sigma_r) = [n_{r-1}+1, n_r]$, $\mathrm{supp}(\tau_r) = [l_{r-1}+1, l_r]$, *and for any non-zero integer l with $|l| \leq m$ we have:*

$$x \in \mathrm{supp}(\sigma_{r-1}) \Rightarrow \tau_r^l(x) \in \mathrm{supp}(\sigma_r), \tag{7.3}$$

$$x \in \mathrm{supp}(\tau_r) \backslash \mathrm{supp}(\sigma_{r-1}) \Rightarrow \sigma_r^l(x) \in \mathrm{supp}(\sigma_r) \backslash \mathrm{supp}(\tau_r). \tag{7.4}$$

Moreover, we have the following relations for $r \geq 0$:

$$l_{r+1} = l_r + (2m+1)(n_r - l_r), \tag{7.5}$$

$$n_{r+1} = n_r + (2m+1)(2m)(n_r - l_r). \tag{7.6}$$

Proof. We will make induction on r.

For $r = 1$ we define $l_1 = (2m+1)k$, $n_1 = l_1 + 4m^2k$ and $\tau_1 \in S_{l_1}$ as follows. If we identify

$$\{1, \cdots, (2m+1)k\} \approx \{0, \cdots, 2m\} \times \{1, \cdots, k\}$$

by $(i,j) \mapsto ik + j$, then we define $\tau_1(0,j) = (1,j)$, $\tau_1(2m-1,j) = (2m,j)$, $\tau_1(2a-1,j) = (2a+1,j)$ if $1 \leq a \leq m-1$, and $\tau_1(2b,j) = (2(b-1),j)$ if $1 \leq b \leq m$. Note that $\mathrm{supp}(\tau_0) = \emptyset$ and by the construction we have $\mathrm{supp}(\sigma_0) \subset \mathrm{supp}(\tau_1)$.

Now we define $\sigma_1 : [k+1, n_1] \to [k+1, n_1]$ as follows. Again we make an identification

$$\{k+1, \cdots, n_1\} \approx \{0, \cdots, 2m\} \times \{1, \cdots, 2mk\}.$$

given by $(i,j) \mapsto k + i(2mk) + j$. In this case we define σ_1 by the same formulae of τ_1 as above. Note that for any $t \in [1,k]$ we have $\bigcup_{1 \leq |l| \leq m} \tau_1^l(t) \subset \{k+1, \cdots, (2m+1)k\}$. This implies (7.3) for $r = 1$. On the other hand, if $x \in [k+1, l_1]$, then $\sigma_1^l(x) > l_1$ for any $1 < |l| \leq m$ and therefore $\sigma_1^l(x) \in \mathrm{supp}(\sigma_1) \backslash \mathrm{supp}(\tau_1)$ which prove the lemma for $r = 1$.

Suppose we have defined $\sigma_1, \tau_1, \cdots, \sigma_r, \tau_r$ with $\mathrm{supp}(\sigma_i) = [n_{i-1}+1, n_i]$, $\mathrm{supp}(\tau_i) = [l_{i-1}+1, l_i]$, satisfying (7.3) and (7.4), and $n_i > l_i > n_{i-1}$ satisfying (7.5) and (7.6) for any $1 \leq i \leq r$. We can take $l_{r+1} = l_r + (2m+1)(n_r - l_r)$ which also can be rewritten as $l_{r+1} = n_r + 2m(n_r - l_r)$. In this case we make the identification

$$\{l_r + 1, \cdots, l_{r+1}\} \approx \{0, \cdots, 2m\} \times \{1, \cdots, (n_r - l_r)\}$$

given by $(i,j) \mapsto l_r + i(n_r - l_r) + j$. Now we define τ_{r+1} on this set by the same formulae as above.

Now we take $n_{r+1} = n_r + (2m+1)(2m)(n_r - l_r)$ and we define σ_{r+1} for $\{n_r + 1, \cdots, n_{r+1}\}$ making the identification

$$[n_r + 1, n_{r+1}] \approx [0, 2m] \times [1, (2m)(n_r - l_r)]$$

given by $(i,j) \mapsto n_r + i(2m)(n_r - l_r) + j$, and let σ_{r+1} be given by the same set of formulae as in the first case.

Note that $n_{r+1} > l_{r+1} > n_r$ and by the construction l_{r+1} and n_{r+1} satisfy (7.5) and (7.6). To finish the proof we need to check (7.3) and (7.4) for σ_{r+1} and τ_{r+1}. We fix a non-zero integer l with $|l| \leq m$. If $x \in \operatorname{supp}(\sigma_r)$, then $n_r < \tau_{r+1}^l(x) \leq l_r + (2m+1)(n_r - l_r) = l_{r+1} < n_{r+1}$ and therefore $\tau_{r+1}^l(x) \in \operatorname{supp}(\sigma_{r+1})$ which shows (7.3). To show (7.4) we take $x \in [n_r + 1, l_{r+1}] = \operatorname{supp}(\tau_{r+1}) \backslash \operatorname{supp}(\sigma_r)$ and see that by the definition $\sigma_{r+1}^l(x) \in [l_{r+1} + 1, n_{r+1}] = \operatorname{supp}(\sigma_{r+1}) \backslash \operatorname{supp}(\tau_{r+1})$. $\square$

Theorem 7.2.6. *Let M be the Bolza surface, i.e. the arithmetic surface of genus 2 of maximal systole in $\mathcal{M}_2$ mentioned in Remark 12, and let $s = \operatorname{sys}(M) = 2\cosh^{-1}(1+\sqrt{2})$. Then for any $k \in \mathbb{N}$ there exists a finite covering $M_k \to M$ with $\operatorname{sys}(M_k) = ks$ and degree $\leq (uk)^{vk^2}$ for some positive constants u, v.*

Proof. Let $\alpha \subset M$ be a systole of M. Since M is maximal, by [44, Propostion 2.6] the curve α is non-separating. We can suppose that the monomorphism $\rho : \mathrm{S}_2 \to \operatorname{Isom}^+(\mathbb{H})$ such that $M \simeq \rho(\mathrm{S}_2)\backslash\mathbb{H}$ satisfies: $\rho(x)$ represents α and $p \in \mathbb{H}$ is projected on a point of α.

Now we take $a = \lceil q(ks + \beta + 2\delta) \rceil$ and $b = \lceil \epsilon a^2 \rceil$, where ϵ is as in Proposition 5.3.3, q, β are as in (7.1), and s, δ are the systole and diameter respectively of M.

By Lemma 7.2.5, if we choose $m = r = b$, we have two permutations $\sigma = \sigma_0 \cdot \sigma_1 \cdots \sigma_b$ and $\tau = \tau_1 \cdot \tau_2 \cdots \tau_b$ in S_{N_k} where $N_k := n_b$. Note that since $i \neq j$ implies $\operatorname{supp}(\sigma_i) \cap \operatorname{supp}(\sigma_j) = \emptyset$, these permutations commute. The same holds for $\tau_i's$.

If we take the homomorphism $\xi : \mathrm{F}_2 \to S_{N_k}$ given by $\xi(x) = \sigma$ and $\xi(y) = \tau$, then the subgroup $H_k < \mathrm{F}_2$ defined by $H_k = \{w \in \mathrm{F}_2 \,|\, \xi(w) \cdot 1 = 1\}$ has index $\leq N_k$ (by the proof of Lemma 7.2.5 we have in fact an equality). Besides, if $w \in H_k \backslash \langle x \rangle$ with $l_A(w) \leq b$, then $w = x^{i_1} y^{j_1} \cdots y^{j_t} x^{i_{t+1}}$ with $j_p, i_{p'} \neq 0$ for any $1 \leq p \leq t$, $2 \leq p' \leq t$ and $2t - 1 \leq \sum_{p=1}^{t} (|i_p| + |j_p|) + |i_{t+1}| \leq b$. In particular: $t, |i_p|, |j_{p'}| \leq b$.

On the other hand, $\xi(w)(1) = \sigma^{i_1} \tau^{j_1} \cdots \tau^{j_t} \sigma^{i_{t+1}}(1)$. If we use (7.3), (7.4), the commutativity of $\sigma_i's$ and $\tau_i's$, and the fact that $\operatorname{supp}(\sigma_{r-1}) \backslash \operatorname{supp}(\tau_{r-1}) \subset$

supp(τ_r) successively for $r = 0, 1, \cdots, t$, we conclude that $\xi(w)(1) > k$. Therefore,

$$\min\{l_A(w) | w \in H_k \text{ and } w \notin \langle x \rangle\} > b. \tag{7.7}$$

Now we will apply Proposition 5.3.3. We take the homomorphism $\psi_a : \mathrm{S}_2 \to \mathrm{F}_2$ and define the group $G_k = \psi_a^{-1}(H_k)$. Let $M_k = \rho(G_k)\backslash\mathbb{H}$ be the covering of M. Since $x^k \in G_k$ and $\rho(x)$ represents the systole of M^k, we have

$$\mathrm{sys}(M_k) \leq ks.$$

Now let $\gamma \subset M_k$ be a closed geodesic γ of length $l(\gamma)$. If we repeat the same argument of the proof of Theorem 7.2.2, there exists a non-trivial element $\gamma \in G_k$ representing this geodesic such that

$$l(\gamma) \geq \frac{1}{q} l_B(\gamma) - \beta - 2\delta. \tag{7.8}$$

If γ is not a power of x, then $l_B(\gamma) > a$. Indeed, if $l_B(\gamma) \leq a$, then part 2 of Proposition 5.3.3 and the proof of Proposition 5.3.2 show that $\psi_a(\gamma) \neq 1$ and $\psi_a(\gamma) \notin \langle x \rangle$ if $\gamma \notin \langle x \rangle$. Hence, using the estimate (7.7) and part 3 of Proposition 5.3.3, we have

$$\epsilon a^2 \leq b < l_A(\psi_k(\gamma)) \leq \epsilon a l_B(\gamma) \leq \epsilon a^2,$$

which gives a contradiction. Since $l_B(\gamma) > a \geq q(ks + \beta + 2\delta)$ by (7.8) we have $l(\gamma) > ks$. Note that the minimal power of x belonging to H_k is k, the element x^k represents the systole of M_k and $\mathrm{sys}(M_k) = ks$.

To finish the proof we need to estimate the degree N_k of the covering. If we subtract (7.5) from (7.6) in Lemma 7.2.5 we have that $n_i - l_i$ is a geometric progression with ratio $t = 4b^2$ where $b = m$ and the initial term is $n_0 - l_0 = k$. Therefore, $n_i - l_i = t^i(n_0 - l_0) = (4b^2)^i k$. This implies by (7.6) that $N_k = n_b = n_{b-1} + (2b+1)(2b)(4b^2)^{b-1}k = n_{b-2} + ((4b^2)^{b-2} + (4b^2)^{b-1})(2b+1)(2b)k = \cdots = n_0 + (1 + (4b^2) + (4b^2)^2 + \cdots + (4b^2)^{b-1})(2b+1)(2b)k$. Hence

$$N_k = \left(1 + \frac{(4b^2)^b - 1}{4b - 1}(2b+1)(2b)\right) k.$$

Since $a \leq Ak$ for some constant $A > 0$ there exists a constant $B > 0$ such that $b \leq Bk^2$ for any $k \geq 1$. Therefore, there exist constants $u, v > 0$ such that $N_k \leq (uk)^{vk^2}$. □

7.3 Quantitative lower bound for the Ehrenpreis conjecture

Let S_1, S_2 be closed hyperbolic surfaces. If S_1 and S_2 are commensurable then there exists a common covering S for both. Although for a generic pair of closed hyperbolic surfaces they will not be commensurable, Leon Ehrenpreis conjectured that for a given $\delta > 0$ one can find coverings $S_1' \to S_1$ and $S_2' \to S_2$ with

$$S_1', S_2' \in \mathcal{M}_{\mathrm{g}} \quad \text{and} \quad d_T(S_1', S_2') \leq \delta.$$

This conjecture remained open for 40 years and it was solved by Jeremy Khan and Vladimir Markovic in [25]. The natural question which arises is the following: If we fix S_1 and S_2 how small g can be in terms of δ?

Definition 20. Let S_1, S_2 be incommensurable, closed hyperbolic surfaces and let $\delta > 0$. We define $g[S_1, S_2](\delta)$ as the minimal genus g such that there exist coverings $S_1' \to S_1$ and $S_2' \to S_2$ with $S_1', S_2' \in \mathcal{M}_{\mathrm{g}}$ and $d_T(S_1', S_2') \leq \delta$.

In this section we will give the first lower bounds for $g[S_1, S_2](\delta)$ when S_1 and S_2 are closed hyperbolic surfaces of a very special type.

Theorem 7.3.1. *Let S, S' be incommensurable arithmetic closed hyperbolic surfaces derived from quaternions algebras over a field k . Then there exists a constant $C > 0$ which depends only on k such that for any $\delta > 0$ we have*

$$g[S, S'](\delta) \geq -C \log(\delta).$$

In order to prove Theorem 7.3.1 we need the following lemma.

Lemma 7.3.2. *If Γ is an arithmetic Fuchsian group derived from a quaternion algebra over a number field k, then there exists a constant $c > 0$ which depends only on the field k, such that for any $\operatorname{tr}(\gamma), \operatorname{tr}(\gamma') \in \mathcal{O}_k$ with $\operatorname{tr}(\gamma) \neq \operatorname{tr}(\gamma')$ we have*

$$|\operatorname{tr}(\gamma) - \operatorname{tr}(\gamma')| > c.$$

Proof. This result is part of Lemma 2.1 in [33]. The idea of the proof is very simple, we use the fact that if k is the field of definition of the quaternion algebra which gives Γ and ϕ is any nontrivial embedding of this field into $\mathbb{C}$, then for any trace $t \in \{\mathrm{tr}(\gamma)\,|\,\gamma \in \Gamma\}$ we have $|\phi(t)| \leq 2$. If $\mathrm{tr}(\gamma) \neq \mathrm{tr}(\gamma')$, then

$$1 \leq |N_{\mathbb{Q}}^{k}(\mathrm{tr}(\gamma) - \mathrm{tr}(\gamma'))| \leq |\,\mathrm{tr}(\gamma) - \mathrm{tr}(\gamma')|(4)^{[k:\mathbb{Q}]-1}.$$

□

Theorem 7.3.1 follows directly from the following theorem.

Theorem 7.3.3. *Let $S = \Gamma\backslash\mathbb{H}$ be an arithmetic closed surface of genus $g \geq 2$ derived from a quartenion algebra over a field k. Then there exists a constant $A > 0$ independent of g such that if $S' \in \mathcal{M}_{\mathrm{g}}$ is another arithmetic closed surface derived from a quartenion algebra over k, we have*

$$d_T(S, S') \geq \exp(-Ag).$$

Proof. We fix $S \in \mathcal{M}_{\mathrm{g}}$ with $S = \Gamma\backslash\mathbb{H}$ and Γ derived from a quaternion algebra over k. It is easy to see that there is a uniform lower bound $s = s(k) > 0$ for the systoles of any arithmetic closed hyperbolic surface with invariant trace field k (see [34, Section 12.2]).

Now we can apply Proposition 4.1.7 in order to get a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of S such that the curve system $\mathcal{C} = \{\gamma_j, \delta_j, \eta_j\}$ with $j \in \{1, \ldots, 3g-3\}$ determines S completely and

$$l(\gamma_j), l(\delta_j), l(\eta_j) \leq Ag \quad \text{for all} \quad 1 \leq j \leq 3g-3,$$

where A is a positive constant which depends on s.

Take $X \in \mathcal{M}_{\mathrm{g}}$ arbitrarily and let $\Gamma_X \in \mathcal{H}_g$ be the uniformizing group of X. For any nontrivial closed curve β (here we are identifying a free homotopy class with its representative in the fundamental group), let $T_\beta(X) \in \Gamma_X$ be a hyperbolic element such that its axis projects on the unique closed geodesic in X homotopic to β. We know that $L_\beta(X)$ and $\mathrm{tr}(T_\beta(X))$ are related by the equality:

$$\mathrm{tr}(T_\beta(X)) = 2\cosh\left(\frac{L_\beta(X)}{2}\right). \tag{7.9}$$

Consider the constant c given by Lemma 7.3.2. We have

$$|\,\mathrm{tr}(\gamma) - \mathrm{tr}(\gamma')| > c, \tag{7.10}$$

whenever $\operatorname{tr}(\gamma) \neq \operatorname{tr}(\gamma')$. Define

$$\varepsilon_g = \sup\{\varepsilon > 0 \,|\, \varepsilon \sinh\left(\frac{Ag}{2} + \frac{\varepsilon}{4}\right) < c\}. \tag{7.11}$$

It follows from the definition of ε_g that there exist constants $a', \varepsilon_0 > 0$ which depend only on A and c such that

$$\exp(-a'g) \leq \varepsilon_g \leq \varepsilon_0 \quad \text{for all} \quad g \geq 2. \tag{7.12}$$

Now we take the following neighborhood of S in $\mathcal{M}_{\mathrm{g}}$,

$$V(S) = \{X \in \mathcal{M}_{\mathrm{g}}\,;\, L_\omega(S) - \frac{\varepsilon_g}{2} < L_\omega(X) < L_\omega(S) + \frac{\varepsilon_g}{2} \text{ for all } \omega \in \mathcal{C}\}.$$

Note that if $X \in V(S)$, then for any $\omega \in \mathcal{C}$ the mean value inequality applied in (7.9) gives

$$|\operatorname{tr}(T_\omega(X)) - \operatorname{tr}(T_\omega(S))| \leq \left(\sup_{L_\omega(S) - \frac{\varepsilon_g}{2} < t < L_\omega(S) + \frac{\varepsilon_g}{2}} \sinh(t/2)\right) \varepsilon_g \leq c$$

The last inequality follows from the definition of ε_g, the assumption that $L_\omega(S) \leq Ag$ for all $\omega \in \mathcal{C}$ and the basic fact that $\sinh(x)$ is increasing.

We deduce from (7.10) and the fact that the length of curves in $\mathcal{C}$ determines the surface (see 4.1.7) that there does not exist $X \in V(S) - \{S\}$ such that X is an arithmetic closed hyperbolic surface derived from a quaternion algebra over k.

Therefore, if we take another $S' \in \mathcal{M}_{\mathrm{g}}$ derived from a quaternion algebra over k we have

$$d_{wp}(S', S) \geq d(S, \partial V(S)), \tag{7.13}$$

where d_{wp} means the distance with respect to the Weil-Petersson metric and $\partial V(S)$ is the boundary of the neighborhood $V(S)$ of S in $\mathcal{M}_{\mathrm{g}}$.

Now take a point $X_0 \in \partial V(S)$ such that $d_{wp}(S, X_0) = d(S, \partial V(S))$. Note that the existence of this point is guaranted by the compactness of the closure of $V(S)$.

There exists a curve $\omega_0 \in \mathcal{C}$ with $|L_{\omega_0}(X_0) - L_{\omega_0}(S)| = \frac{\varepsilon_g}{2}$. From now on we will denote $L := L_{\omega_0}$.

Consider $\alpha : [0, \delta] \to \mathcal{M}_{\mathrm{g}}$, a Weil-Petersson geodesic parametrized by the arc length such that $\alpha(0) = X_0$, $\alpha(\delta) = S$, and $\delta = d_{wp}(X_0, S)$. The map

$t \to L(\alpha(t))$ is a convex function since the length function is convex in the Weil-Petersson metric by part (a) of Proposition 4.3.3.

Therefore,

$$L(\alpha(t)) \leq Ag + \frac{\varepsilon_g}{2} \quad \text{for all} \quad 0 \leq t \leq \delta.$$

By part (b) of Proposition 4.3.3 and the upper bound (7.12) for ε_g we have following estimate:

$$\frac{\varepsilon_g}{2} = |L(S) - L(X_0)| \leq \left(\sup_{0 \leq t \leq \delta} |\nabla L(\alpha(t))| \right) \delta \leq \sqrt{c'((Ag + \tfrac{\varepsilon_0}{2}) + (Ag + \tfrac{\varepsilon_0}{2})^2 \exp\left(\tfrac{2Ag+\varepsilon_0}{4}\right)}\ \delta$$

Therefore,

$$d_{wp}(X_0, S) = \delta \geq \frac{\varepsilon_g}{\sqrt{c'}\sqrt{((Ag+\frac{\varepsilon_0}{2})+(Ag+\frac{\varepsilon_0}{2})^2 \exp\left(\frac{2Ag+\varepsilon_0}{2}\right)}}.$$

Now if we apply the lower bound for ε_g in (7.12), we have

$$d_{wp}(X_0, S) \geq \exp(-A_1 g) \tag{7.14}$$

for some constant $A_1 > 0$ which depends only on A, a', c, c'. Recall that c' does not depend on g.

In our definition of the Teichmuller metric, the comparison theorem of the distances given by the Teichmuller metric and the Weil-Petersson metric gives (see [32])

$$d_T(X, Y) \geq \frac{2d_{wp}(X, Y)}{\sqrt{4\pi(g-1)}}. \tag{7.15}$$

for any pair $X, Y \in \mathcal{M}_{\mathrm{g}}$.

To finish the proof we apply (7.13), (7.14) and (7.15). We have a constant $B > 0$ which depends only on k such that

$$d_T(S, S') \geq \frac{2d_{wp}(S, X_0)}{\sqrt{4\pi(g-1)}} \geq \frac{2\exp(-A_1 g)}{\sqrt{4\pi(g-1)}} \geq \exp(-Bg)$$

for all $g \geq 2$.

□

APPENDIX A

BACKGROUND ASSUMED

In this appendix we will recall the main definitions and results which are assumed in the text.

A.1 Riemannian Manifolds

Definition 21. Let X be a topological space. An *n-chart* on X is a pair $\{U, \phi\}$ where $U \subset X$ is an open set and $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism. An *n-atlas* on X is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ of n-charts such that

(i) $X = \cup_{\alpha \in \Lambda} U_\alpha$;

(ii) for any $\alpha, \beta \in \Lambda$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a diffeomorphism.

A smooth *manifold* of dimension n is a Hausdorff, second countable topological space X, together with an n-atlas.

Let X be a smooth manifold of dimension n. For any $x \in X$ and chart (U_α, ϕ_α) with $x \in U_\alpha$ we can take a copy $\mathbb{R}^n_\alpha$ of $\mathbb{R}^n$. We define the *tangent space* $T_x X$ as the vector space given by the equivalence relation

$$v_\alpha \equiv w_\beta \quad \text{if and only if} \quad D(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(x)) \cdot w_\beta = v_\alpha.$$

Definition 22. A *Riemannian manifold* is a smooth manifold X, equipped with an inner product $\mathcal{G}(x)$ on each tangent space T_xX such that for any chart (U, ϕ_α) the map

$$y \mapsto \mathcal{G}(\phi_\alpha^{-1}(y))(v_\alpha, v_\alpha)$$

is smooth.

For any smooth curve $c : [a, b] \to X$ we define the length of c to be

$$l(c) := \int_a^b \sqrt{\mathcal{G}(c'(t), c'(t))}dt,$$

where $c'(t)$ is the velocity of the curve c in the time t.

Definition 23. Let X be a Riemannian manifold. We define the distance of two points p, q in X with respect to the metric $\mathcal{G}$ by

$$d(p, q) = \inf\{l(c) \mid \text{c is a smooth curve from } p \text{ to } q\}.$$

An *isometry* between Riemannian manifolds X and X' with metrics $\mathcal{G}, \mathcal{G}'$, respectively, is a diffeomorphism $f : X \to Y$ such that

$$\mathcal{G}'(f(x))(Df(x)v, Df(x)w) = \mathcal{G}(x)(v, w) \quad \text{for all} \quad x \in X,\ v, w \in T_xX.$$

Note that any isometry of Riemannian manifolds preserves the respective distances, i.e. it is an isometry of metric spaces.

A.2 Groups and Actions

Definition 24. A *topological group* is a group G equipped with a topology such that the operations are continuous maps, i.e. the maps

$$G \times G \to G \quad (g, h) \mapsto gh,$$

and

$$G \to G \quad g \mapsto g^{-1}$$

are continuous. If G is a manifold and the maps above are smooth, then we call G a *Lie group*.

Let G be a topological group and Y be a topological space.

Definition 25. • An *action* of G on Y is a continuos map $\mu : G\times Y \to Y$ such that

  - $\mu(1, y) = y$ for all $y \in Y$, and
  - $\mu(gh, y) = \mu(g, \mu(h, y))$ for all $g, h \in G$ and $y \in Y$.

  We denote the image $\mu(g, y)$ by $g \cdot y$.

- An action is *proper* if the map μ is proper, i.e. the preimage of a compact subset is always compact.
- The action is *free* if no nontrivial element of G fixes point, i.e.

$$\mu(g, x) \neq x \quad \text{if} \quad g \neq 1.$$

- The action is *properly discontinuous* if, for every compact $K \subset X$, the set

$$\{g \in G \,|\, g \cdot K \cap K \neq \emptyset\}$$

  is finite.

- For any $y \in Y$ we define
  - $G_y = \{g \in G \,|\, g \cdot y = y\}$ is the *stabilizer subgroup* of y.
  - $G \cdot y = \{g \cdot y \,|\, g \in G\}$ is the *orbit* of y.

Definition 26. Let X and Y be topological spaces. A continous map $f : X \to Y$ is said to be a covering map if for any $y \in Y$ there exists a connected open set $V_y \subset Y$ with $y \in V_y$ such that for any connected component U_α of $f^{-1}(V_y)$ the restriction $f \restriction_{U_\alpha}: U_\alpha \to V_y$ is a homeomorphism.

Proposition A.2.1. *Let Γ be a topological group and M be a manifold. If Γ acts on M properly discontinuously, then the quotient space $\Gamma\backslash M$ is a manifold and the natural projection $M \to \Gamma\backslash M$ is a covering map.*

A.3 Number fields

Definition 27.

- A complex number z is *algebraic* if there exists a polynomial $P \in \mathbb{Z}[X]$ such that $P(z) = 0$.
- A nonzero polynomial is *monic* if the leading coefficient is 1, i.e.

$$P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0.$$

- A complex number z is an *algebraic interger* if there exists a monic polynomial $P \in \mathbb{Z}[X]$ such that $P(z) = 0$.

Proposition A.3.1 ($\mathbb{Z}$ is integrally closed)**.** *If $x \in \mathbb{Q}$ is an algebraic integer, then $x \in \mathbb{Z}$.*

Definition 28.

- A *number field* is a subfield $k \subset \mathbb{C}$ such that k has finite dimension as a $\mathbb{Q}$-vector space.
- If k is a number field, we define the ring of integers of k by the set

$$R_k = \{x \in k \,|\, x \,\text{is an algebraic integer}\}.$$

Proposition A.3.2. *Let k be a number field. Then any element $x \in k$ is algebraic, moreover, R_k is a domain such that k is the field of fractions of R_k.*

Definition 29.

- Let k be a number field. We define a *Galois embedding* as any embedding of fields $\sigma : k \to \mathbb{C}$. We say that a Galois embedding σ is real if $\sigma(k) \subset \mathbb{R}$, otherwise we say that σ is complex.
- The *degree* of k is the dimension of k as a $\mathbb{Q}$-vector space.

Proposition A.3.3. *Let k be a number field of degree n. Then there exists an algebraic element $\alpha \in k$ with minimal polynomial $P \in \mathbb{Z}[X]$ of degree n (i.e. P is the irreducible polynomial with $P(\alpha) = 0$ of minimal degree) such that $k = \mathbb{Q}(\alpha)$. Furthermore, any Galois embedding of k sends α to β, where β is a root of P. Conversely, for any root β of P there exists an embedding $\sigma_\beta : k \to \mathbb{C}$ such that $\sigma_\beta(\alpha) = \beta$.*

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