POSITIVITY AND ALGEBRAIC INTEGRABILITY OF HOLOMORPHIC FOLIATIONS

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Abstract. The theory of holomorphic foliations has its origins in the study of differential equations on the complex plane, and has turned into a powerful tool in algebraic geometry. One of the fundamental problems in the theory is to find conditions that guarantee that the leaves of a holomorphic foliation are algebraic. These correspond to algebraic solutions of differential equations. In this paper we discuss algebraic integrability criteria for holomorphic foliations in terms of positivity of its tangent sheaf, and survey the theory of Fano foliations, developed in a series of papers in collaboration with Stéphane Druel. We end by classifying all possible leaves of del Pezzo foliations.

1. Introduction

The theory of holomorphic foliations has its origins in the study of differential equations on the complex plane $\mathbb{C}^2$. A central problem in this theory consists in finding conditions that guarantee the existence of algebraic solutions ([Dar78], [Pai94], [Poi91]). Consider for instance the following algebraic differential equations:

$$\frac{dy}{dx} = \frac{y}{x} , \tag{1.1}$$

$$\frac{dy}{dx} = y . \tag{1.2}$$

While the solutions to equation (1.1) are algebraic, namely $y = cx$, the solutions to equation (1.2) are mostly transcendental, namely $y = ce^x$. In both cases, the algebraic differential equation defines a saturated subsheaf $\mathscr{F} \subset T_{\mathbb{C}^2}$. By saturated we mean that $T_{\mathbb{C}^2}/\mathscr{F}$ is torsion free. We call this subsheaf a *foliation* of the plane. Curves that are everywhere tangent to $\mathscr{F}$ correspond to solutions of the equation, and are called *leaves of the foliation*. We remark that classically the word foliation refers to the partition of the plane into leaves. If we extend to $\mathbb{P}^2$ the foliations $\mathscr{F} \subset T_{\mathbb{C}^2}$ defined by the equations above, we obtain the saturated subsheaves $\mathscr{O}_{\mathbb{P}^2}(1) \subset T_{\mathbb{P}^2}$ in (1.1) and $\mathscr{O}_{\mathbb{P}^2} \subset T_{\mathbb{P}^2}$ in (1.2). As we shall see, the ampleness of $\mathscr{O}_{\mathbb{P}^2}(1)$ forces the solutions of equation (1.1) to be algebraic. This is the simplest manifestation of a series of results relating properties of positivity and algebraicity of holomorphic foliations.

In general, a *foliation* on a normal variety X is a saturated nonzero coherent subsheaf $\mathscr{F} \subsetneq T_X$ that satisfies the Frobenius integrability condition: $\mathscr{F}$ is closed under the Lie bracket. The Frobenius condition guarantees that a dense open subset of X is covered by analytic submanifolds whose tangent bundles are restrictions of $\mathscr{F}$. When these submanifolds are connected and maximal, we say that they are *leaves* of $\mathscr{F}$. We refer to Section 2 for definitions and generalities about holomorphic foliations on complex varieties, including notions of singularity for foliations.

In the last decades, foliations have proved to be a powerful tool in algebraic geometry. For instance, they play an important role in the proof of Green-Griffiths conjecture for surfaces of general type with positive Segre class ([Bog77], [McQ98]). In many applications, a key problem is to find conditions that guarantee that a foliation has algebraic leaves, in which case we say that it is *algebraically integrable*, and to describe the structure of these algebraic subvarieties. We briefly mention two important instances of this.

1.1 (Miyaoka's criterion of uniruledness)**.** In [Miy87], Miyaoka proved a remarkable criterion of uniruledness in terms of numerical properties of the tangent sheaf. Namely, if X is a non-uniruled normal projective variety, then its cotangent sheaf Ω^1_X is generically semi-positive. This last condition means that for a sufficiently general complete intersection curve C on X, the restriction $(\Omega^1_X)_{|C}$ is semi-positive. This criterion in one of the main ingredients in the proof of abundance for threefolds (see [SB92]). Algebraic integrability of foliations

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plays a key role in Miyaoka's proof, which involves reduction to positive characteristic. Namely, if Ω^1_X is not generically semi-positive, then, using Harder-Narasimhan filtrations, one can construct a special foliation on X, whose restriction to a sufficiently general complete intersection curve C is ample. This foliation is shown to be algebraically integrable and covered by rational curves.

Miyaoka's algebraicity criterion has been extensively generalized. We mention Bost's arithmetic geometric counterpart ([Bos01]), Bogomolov and McQuillan's criterion, which gives rationally connectedness of general leaves (see [BM16] and also [KSCT07]), and most recently the extension by Campana and Paŭn, which considers positivity of the tangent sheaf with respect to more general movable curve classes ([CP15]). These criteria will be further discussed in Section 2.

1.2 (The structure of singular varieties with numerically trivial canonical class)**.** The Beauville-Bogomolov decomposition theorem asserts that, after étale cover, any compact Kähler manifold with numerically trivial canonical class is a product of a torus, Calabi-Yau and irreducible symplectic manifolds ([Bea83]). This structure result has been recently generalized to the singular setting in [Dru17] and [HP17]. Algebraic integrability of foliations plays a key role in the proof of this structure theorem. Namely, [GKP16b] gives a decomposition of the tangent sheaf of a singular complex projective variety X with trivial canonical class into a direct sum of foliations with strong stability properties. Druel provides in [Dru17] an algebraic integrability criterion to show that this decomposition of the tangent sheaf is induced by a product structure on an quasi-étale cover of X.

A common idea behind the algebraic integrability results for foliations discussed above is that positivity properties of foliations tend to increase algebraicity properties of the leaves. In a series of papers in collaboration with Stéphane Druel ([AD13], [AD14], [AD16], [AD17b] and [AD17a]), we have investigated foliations with positive anticanonical class, which we call *Fano foliations*. For Fano foliations, a rough measure of positivity is the *index*. The index $\iota_{\mathscr{F}}$ of a Fano foliation $\mathscr{F}$ on a complex projective manifold X is the largest integer dividing $-K_{\mathscr{F}}$ in $\mathrm{Pic}(X)$. One special property of Fano foliations is that their leaves are always covered by rational curves, even when these leaves are not algebraic. Our works on Fano foliations with high index indicated that the higher is the index, the closer it is to being algebraically integrable. First of all, we have the following general bound on the index, in analogy with Kobayachi-Ochiai's theorem on the index of Fano manifolds:

Theorem 1.3 ([ADK08, Theorem 1.1])**.** *Let $\mathscr{F} \subsetneq T_X$ be a Fano foliation of rank r on a complex projective manifold X. Then $\iota_{\mathscr{F}} \leqslant r$, and equality holds only if $X \cong \mathbb{P}^n$.*

Foliations on $\mathbb{P}^n$ attaining the bound of Theorem 1.3 are induced by linear projections $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$ ([DC05, Théorème 3.8]). These results have been generalized to the singular setting in [AD14] and [Hör14].

Next we consider Fano foliations $\mathscr{F} \subsetneq T_X$ of rank r on complex projective manifolds X with index $\iota_{\mathscr{F}} = r-1$. In analogy with the theory of Fano manifolds, we call them *del Pezzo foliations.* In contrast with the case of maximal index $\iota_{\mathscr{F}} = r$, there are examples of del Pezzo foliations on $\mathbb{P}^n$ with non-algebraic leaves. In fact, del Pezzo foliations on $\mathbb{P}^n$ were classified in [LPT13]. They are the following.

- A foliation induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^{n-r},2)$, defined by $n-r$ linear forms and one quadric form, where $\mathbb{P}(1^{n-r},2)$ denotes the weighted projective space with weights $\underbrace{1,\ldots,1}_{r \text{ times}},2$.
- The linear pullback of a foliation $\mathscr{C}$ on $\mathbb{P}^{n-r+1}$ induced by a global vector field.

In the second case, if the vector field is general, then the leaves of $\mathscr{C}$ are transcendental, as equation (1.2) above illustrates. Hence, the leaves of the del Pezzo foliation are not algebraic either. The following algebraic integrability result asserts that these are the only transcendental del Pezzo foliations on complex projective manifolds.

Theorem 1.4 ([AD13, Theorem 1.1])**.** *Let $\mathscr{F}$ be a del Pezzo foliation on a complex projective manifold $X \not\cong \mathbb{P}^n$. Then $\mathscr{F}$ is algebraically integrable, and its general leaves are rationally connected.*

In the classical setting, del Pezzo manifolds were classified by Fujita in the 1980's. Most of them are complete intersections on weighted projective spaces. One may also expect a classification of del Pezzo foliations. For example, the only del Pezzo foliations on smooth quadric hypersurfaces are those induced by the restriction of linear projections from the ambient projective space ([AD16, Proposition 3.18]). Moreover, quadrics are the only hypersurfaces that admit del Pezzo foliations ([AD13, Corollary 4.8.]). We also know examples of del Pezzo foliations on certain Grassmannians and projective space bundles over projective spaces ([AD13, Sections 4 and 9]). In Section 3 we discuss the classification of del Pezzo foliations, under restrictions on the rank or on the singularities of the foliation. A complete classification of del Pezzo foliations seems to be a difficult problem. A step in this direction is a classification of all possible leaves of del Pezzo foliations, which is given in Proposition 3.3.

We remark that codimension 1 Fano foliations of large index on complex projective spaces have been classically studied. The *degree* d of a foliation $\mathscr{F}$ of rank r on $\mathbb{P}^n$ is defined as the degree of the locus of tangency of $\mathscr{F}$ with a general linear subspace $\mathbb{P}^{n-r} \subset \mathbb{P}^n$. It satisfies $d = r - \iota_{\mathscr{F}}$. So Fano foliations of large index on $\mathbb{P}^n$ are precisely those with small degree. Codimension 1 foliations on $\mathbb{P}^n$ of degree 0 and 1 were classified in [Jou79]. Those of degree 2 were classified in [CLN96].

Theorem 1.4 is in fact a special case of a more general result that gives a lower bound for the *algebraic rank* of a Fano foliation in terms of the index. The algebraic rank $r^a_{\mathscr{F}}$ of a foliation $\mathscr{F}$ on an algebraic variety X is the maximum dimension of an algebraic subvariety through a general point of X that is everywhere tangent to $\mathscr{F}$. A foliation is said to be *purely transcendental* if its algebraic rank is 0.

Theorem 1.5 ([AD17a, Corollary 1.6.]). *Let $\mathscr{F}$ be a Fano foliation of index $\iota_{\mathscr{F}}$ on a complex projective manifold X. Then $r^a_{\mathscr{F}} \geqslant \iota_{\mathscr{F}}$, and equality holds if and only if $X \cong \mathbb{P}^n$ and $\mathscr{F}$ is the pullback under a linear projection of a purely transcendental foliation on $\mathbb{P}^{n-r^a_{\mathscr{F}}}$ with trivial canonical class.*

Fano foliations may also play a distinguished role in the emerging theory of birational geometry of foliations. Higher dimensional algebraic geometry has had a strong influence in the study of holomorphic foliations. Techniques from the minimal model program have been successfully applied to the study of global properties of holomorphic foliations, leading to the birational classification of foliations by curves on surfaces ([Bru99], [Men00], [Bru04]). In higher dimensions, very little is known and difficulties abound: Kawamata-Viehweg vanishing, abundance and resolution of singularities all fail in general for foliations. Positive results, mostly in dimension 3, include a reduction of singularities for foliations on threefolds ([Can04], [MP13]) and a cone theorem for rank 2 foliations on threefolds ([Spi17]). We also mention that there are structure results for foliations with numerically trivial canonical class ([Tou08], [LPT13], [LPT11], [PT13]).

In the context of birational geometry of foliations, it is important to develop the theory of foliations on mildly singular varieties. In Section 2 we survey some aspects of the theory of foliations in this more general setup. In Section 3 discuss the classification of del Pezzo foliations on projective manifolds.

Notation and conventions. We always work over the field $\mathbb{C}$ of complex numbers. Varieties are always assumed to be irreducible. We denote by X_{ns} the nonsingular locus of a variety X. Given a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules of generic rank r on a variety X, we denote by $\det(\mathscr{F})$ the sheaf $(\wedge^r \mathscr{F})^{**}$. If $\mathscr{G}$ is another sheaf of $\mathscr{O}_X$-modules on X, then we denote by $\mathscr{F}[\otimes]\mathscr{G}$ the sheaf $(\mathscr{F} \otimes \mathscr{G})^{**}$. When X is a normal variety, we denote by T_X the tangent sheaf $(\Omega^1_X)^*$.

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2. Foliations

In this section we define foliations on algebraic varieties, their canonical class and notions of singularities. We then discuss criteria of algebraic integrability and special properties of Fano foliations.

Definition 2.1. A *foliation* on a normal variety X is a saturated nonzero coherent subsheaf $\mathscr{F} \subsetneq T_X$ that is closed under the Lie bracket.

The *rank* r of $\mathscr{F}$ is the generic rank of $\mathscr{F}$.

The *canonical class* $K_{\mathscr{F}}$ of $\mathscr{F}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{F}}) \simeq \det(\mathscr{F})$.

We say that $\mathscr{F}$ is *Gorenstein* if $K_{\mathscr{F}}$ is Cartier.

We say that $\mathscr{F}$ is $\mathbb{Q}$-*Gorenstein* if $K_{\mathscr{F}}$ is $\mathbb{Q}$-Cartier.

Definition 2.2. Let $\mathscr{F}$ be a $\mathbb{Q}$-Gorenstein foliation of rank r on a normal variety X, and consider the induced map

$$\eta : \Omega^r_X = \wedge^r(\Omega^1_X) \to \wedge^r(T^*_X) \to \wedge^r(\mathscr{F}^*) \to \det(\mathscr{F}^*) \simeq \mathscr{O}_X(K_{\mathscr{F}}).$$

This is called a *Pfaff field* of rank r on X ([AD14, Definition 3.4]). The *singular locus* S of $\mathscr{F}$ is the closed subscheme of X whose ideal sheaf $\mathscr{I}_S$ is the image of the associated map $\Omega^r_X[\otimes]\mathscr{O}_X(-K_{\mathscr{F}}) \to \mathscr{O}_X$. On the nonsingular locus X_{ns} of X, S_{red} consists of the points at which $\mathscr{F}_{|X_{\text{ns}}}$ is not a subbundle of $T_{X_{\text{ns}}}$. When $S = \emptyset$, we say that $\mathscr{F}$ is a *regular* foliation.

An analytic subvariety $Y \subset X$ is *invariant* under $\mathscr{F}$ if it is not contained in the singular locus of $\mathscr{F}$, and the restriction $\eta_{|Y_{\text{ns}}} : \Omega^r_{X\,|Y_{\text{ns}}} \to \mathscr{O}_X(K_{\mathscr{F}})_{|Y_{\text{ns}}}$ factors through the natural map $\Omega^r_{X\,|Y_{\text{ns}}} \to \Omega^r_{Y\,|Y_{\text{ns}}}$.

A maximal invariant subvariety of dimension r is called a *leaf* of $\mathscr{F}$.

There are several notions of singularities for foliations. The notion of *reduced* foliations has been used in the birational classification of foliations by curves on surfaces (see [Bru04]). More recently, notions of singularities coming from the minimal model program have shown very useful when studying birational geometry of foliations. We introduce the notions of *canonical* and *log canonical* foliations following McQuillan's [McQ08, Definition I.1.2]. Terminal and log terminal singularities can be defined analogously.

Definition 2.3. Let $\mathscr{F}$ be a $\mathbb{Q}$-Gorenstein foliation on a normal variety X. Let $\varphi : \tilde{X} \to X$ be a projective birational morphism. There is a unique foliation $\tilde{\mathscr{F}}$ on $\tilde{X}$ that agrees with $\varphi^*\mathscr{F}$ on the open subset of $\tilde{X}$ where φ is an isomorphism, and uniquely defined rational numbers $a(E, X, \mathscr{F})$'s such that

$$K_{\tilde{\mathscr{F}}} = \varphi^* K_{\mathscr{F}} + \sum_E a(E, X, \mathscr{F})E,$$

where E runs through all exceptional prime divisors for φ. As usual, the discrepancies $a(E, X, \Delta)$'s do not depend on the birational morphism φ, but only on the valuations associated to the E's. We say that $\mathscr{F}$ is *canonical* if $a(E, X, \mathscr{F}) \geqslant 0$ for all E exceptional over X. We say that $\mathscr{F}$ is *log canonical* if $a(E, X, \mathscr{F}) \geqslant -\epsilon(E)$ for all E exceptional over X, where

$$\epsilon(E) = \begin{cases} 0 & \text{if } E \text{ is invariant by the foliation,} \\ 1 & \text{if } E \text{ is not invariant by the foliation.} \end{cases}$$

If a Gorenstein foliation is regular, then it is canonical ([AD13, Lemma 3.10]).

Definition 2.4. Let X be a normal projective variety, and $\mathscr{F}$ a $\mathbb{Q}$-Gorenstein foliation on X. We say that $\mathscr{F}$ is a *$\mathbb{Q}$-Fano foliation* if $-K_{\mathscr{F}}$ is ample. In this case, the *index* of $\mathscr{F}$ is the largest positive rational number $\iota_{\mathscr{F}}$ such that $-K_{\mathscr{F}} \sim_{\mathbb{Q}} \iota_{\mathscr{F}} H$ for a Cartier divisor H on X.

If $\mathscr{F}$ is a $\mathbb{Q}$-Fano foliation of rank r on a normal projective variety X, then, by [Hör14, Corollary 1.2], $\iota_{\mathscr{F}} \leqslant r$. Moreover, equality holds if and only if X is a generalized normal cone over a normal projective variety Z, and $\mathscr{F}$ is induced by the natural rational map $X \dashrightarrow Z$ (see also [ADK08, Theorem 1.1], and [AD14, Theorem 4.11]). In Section 3 below we discuss Fano foliations of rank r and index $\iota_{\mathscr{F}} = r - 1$. We call these *del Pezzo* foliations.

Definition 2.5. Let $\mathscr{F}$ be a foliation on a normal variety X. We say that $\mathscr{F}$ is *algebraically integrable* if it is induced by a dominant rational map $\varphi \colon X \dashrightarrow Y$ with connected fibers into a normal variety. This means that, over the smooth locus $X^\circ \subset X$ of φ, we have $\mathscr{F}_{|X^\circ} = T_{X^\circ/Y}$.

In the setting of Definition 2.5, the general leaf of $\mathscr{F}$ is a general fiber of $\varphi_{|X^\circ} \colon X^\circ \dashrightarrow Y$. The map $\varphi \colon X \dashrightarrow Y$ is unique up to birational equivalence. It is often useful to take the variety Y to be the normalization of the unique proper subvariety of the Chow variety of X whose general point parametrizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). It comes with a universal cycle and induced morphisms:

(2.1)
$$\begin{array}{ccc} Z & \xrightarrow{\ \nu\ } & X \\ {\scriptstyle\pi}\downarrow & \swarrow_{\varphi} & \\ Y. & & \end{array}$$

Here Z is normal, $\nu \colon Z \to X$ is birational and, for a general point $y \in Y$, $\nu\big(\pi^{-1}(y)\big) \subset X$ is the closure of a leaf of $\mathscr{F}$. We refer to the diagram (2.1) as the *family of leaves* of $\mathscr{F}$.

In our investigations of $\mathbb{Q}$-Gorenstein algebraically integrable foliations, it proved to be very useful to work with their *log leaves*, rather than with their leaves.

Definition 2.6 ([AD14, Definition 3.11 and Remark 3.12])**.** Let $\mathscr{F}$ be a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on a normal projective variety X. Let $i\colon F \to X$ be the normalization of the closure of a general leaf of $\mathscr{F}$. There is a canonically defined effective $\mathbb{Q}$-divisor Δ on F such that $K_F + \Delta \sim_{\mathbb{Q}} i^*K_{\mathscr{F}}$. If $\mathscr{F}$ is Gorenstein, then Δ is integral. The pair (F, Δ) is called a *general log leaf* of $\mathscr{F}$. If $y \in Y$ is a general point, then $F \cong Z_y = \pi^{-1}(y)$. Over the smooth locus of X, we have $\operatorname{Supp}(\Delta) = \operatorname{Exc}(\nu) \cap Z_y$ under this identification, where $\operatorname{Exc}(\nu)$ denotes the exceptional locus of $\nu\colon Z \to X$ ([AD17a, Lemma 2.12]). In particular, over the smooth locus of X, $F \setminus \Delta$ is smooth.

For a Cartier divisor L on X, we write $L_{|F}$ for the pullback i^*L of L.

Proposition 2.7 ([AD17a, Corollary 2.14])**.** *Let X be a smooth projective variety, and $\mathscr{F} \subsetneq T_X$ an algebraically integrable foliation on X, with general log leaf (F, Δ). Suppose that either $\rho(X) = 1$, or $\mathscr{F}$ is a Fano foliation. Then $\Delta \neq 0$.*

The following notion of log canonicity for algebraically integrable foliations is weaker than the notion introduced in Definition 2.3 (see [AD13, Proposition 3.11]).

Definition 2.8. Let X be a normal projective variety, $\mathscr{F}$ a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on X, and (F, Δ) its general log leaf. We say that $\mathscr{F}$ has *log canonical singularities along a general leaf* if (F, Δ) is log canonical.

The following is a special geometric property of algebraically integrable $\mathbb{Q}$-Fano foliation with log canonical singularities along a general leaf. It implies in particular that there is a common point in the closure of every general leaf.

Proposition 2.9 ([AD16, Proposition 3.13])**.** *Let $\mathscr{F}$ be an algebraically integrable $\mathbb{Q}$-Fano foliation on a normal projective variety X, having log canonical singularities along a general leaf. Then there is a log canonical center of the general log leaf (F, Δ) whose image in X does not vary with the log leaf.*

Remark 2.10. The log canonicity assumption in Proposition 2.9 is necessary to guarantee the existence of a common point in the closure of a general leaf. For instance, consider the Grassmannian $\mathbb{G}(1, m)$ of lines on $\mathbb{P}^m$ for $m \geqslant 3$, and the rational map $\mathbb{G}(1, m) \dashrightarrow \mathbb{G}(1, m-1)$ induced by the projection $\mathbb{P}^m \dashrightarrow \mathbb{P}^{m-1}$ from a fixed point $P \in \mathbb{P}^m$. It induces a del Pezzo foliation $\mathscr{F}$ of rank 2 on $\mathbb{G}(1, m)$ whose general log leaf (F, Δ) is isomorphic to $(\mathbb{P}^2, 2\ell)$, where ℓ is a line in $\mathbb{P}^2$ (see [AD13, Example 4.3]). More precisely, F is the $\mathbb{P}^2$ of lines contained in a plane $\Pi \cong \mathbb{P}^2$ of $\mathbb{P}^m$ that contains P, and ℓ is the line consisting of lines on Π through P. This log leaf is not log canonical, and there is no common point in the closure of a general leaf. Also, [AD13, Construction 9.10] produces del Pezzo foliations on projective space bundles over positive dimensional smooth projective varieties, which are contained in the relative tangent bundle of the fibration. Clearly there is no common point in the closure of a general leaf. The general log leaf in this case is isomorphic to a cone over $(C, 2P)$, where C is a rational normal curve and $P \in C$ is a point. Again, it is not log canonical.

It is useful to have effective algebraic integrability criteria for foliations. We recall Bogomolov and McQuillan's criterion (see also [Bos01] and [KSCT07]).

Theorem 2.11 ([BM16, Theorem 0.1])**.** *Let X be a normal projective variety, and $\mathscr{F}$ a foliation on X. Let $C \subset X$ be a complete curve disjoint from the singular loci of X and $\mathscr{F}$. Suppose that the restriction $\mathscr{F}_{|C}$ is an ample vector bundle on C. Then the leaf of $\mathscr{F}$ through any point of C is an algebraic variety, and the leaf of $\mathscr{F}$ through a general point of C is moreover rationally connected.*

This criterion can be applied to describe special properties of $\mathbb{Q}$-Fano foliations. Let X be a normal projective variety and $\mathscr{A}$ any ample line bundle on X. Consider the usual notions of slope and semi-stability with respect to $\mathscr{A}$ for torsion-free sheaves on X. Given a $\mathbb{Q}$-Fano foliation $\mathscr{F}$ of rank r on X, we have $\mu_{\mathscr{A}}(\mathscr{F}) = \frac{-K_{\mathscr{F}}\cdot\mathscr{A}^{n-1}}{r} > 0$. Let

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F} \tag{2.2}$$

be the Harder-Narasimhan filtration of $\mathscr{F}$ with respect to $\mathscr{A}$, with quotients $\mathscr{Q}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ satisfying $\mu_{\mathscr{A}}(\mathscr{Q}_1) > \mu_{\mathscr{A}}(\mathscr{Q}_2) > \ldots > \mu_{\mathscr{A}}(\mathscr{Q}_k)$. By the Mehta-Ramanathan Theorem, the Harder-Narasimhan filtration of $\mathscr{F}$ with respect to $\mathscr{A}$ commutes with restriction to a general complete intersection curve C. This generality conditions means that $C = H_1 \cap \cdots \cap H_{\dim(X)-1}$, where the H_i's are general members of linear systems $|m_i\mathscr{A}|$ for $m_i \in \mathbb{N}$ sufficiently large. It implies that each $\mathscr{F}_i$ is locally free along C. Set $s = \max\{i \geqslant 1 \,|\, \mu_{\mathscr{A}}(\mathscr{F}_i/\mathscr{F}_{i-1}) > 0\} \geqslant 1$. From the properties of the Harder-Narasimhan filtrations, it follows that $\mathscr{F}_i \subset T_X$ is a foliation for $1 \leqslant i \leqslant s$. From the slope conditions and properties of vector bundles on smooth curves ([Har71, Theorem 2.4]), it follows that each restriction $(\mathscr{F}_i)_{|C}$ is ample. By Theorem 2.11, for $1 \leqslant i \leqslant s$, $\mathscr{F}_i \subset T_X$ is an algebraically integrable foliation, and the closure of a general leaf is rationally connected. This gives the following property of $\mathbb{Q}$-Fano foliations.

Corollary 2.12. *Let $\mathscr{F}$ be a $\mathbb{Q}$-Fano foliation on a normal projective variety X. Then $\mathscr{F}$ contains an algebraically integrable subfoliation whose general leaves are rationally connected.*

Remark 2.13. Let X be a Fano manifold with $\rho(X) = 1$ and consider the Harder-Narasimhan filtration of the tangent bundle T_X as in (2.2). Since $\rho(X) = 1$, any ample line bundle $\mathscr{A}$ gives the same notion of stability. A conjecture due to Iskovskikh predicts that T_X is (semi-)stable. If T_X is not semi-stable, the first nonzero subsheaf $\mathscr{F}_1$ in its Harder-Narasimhan filtration is called the maximal destabilizing subsheaf of T_X. Arguing as before, we see that $\mathscr{F}_1$ is an algebraically integrable Fano foliation on X. The slope inequality $\mu_{\mathscr{A}}(\mathscr{F}_1) > \mu_{\mathscr{A}}(T_X)$ is equivalent to the index inequality $\frac{\iota_{\mathscr{F}_1}}{\mathrm{rank}(\mathscr{F}_1)} > \frac{\iota_X}{\dim(X)}$. So in order to prove Iskovskikh's conjecture, one must rule out the existence of Fano foliations with large index on X.

More generally, one defines the *algebraic rank* $r^a_{\mathscr{F}}$ of a foliation $\mathscr{F}$ as the maximum dimension of an algebraic subvariety through a general point of X that is everywhere tangent to $\mathscr{F}$. If $\mathscr{F}$ has rank r, then $0 \leqslant r^a_{\mathscr{F}} \leqslant r$, and $r^a_{\mathscr{F}} = r$ if and only if $\mathscr{F}$ is algebraically integrable. When $r^a_{\mathscr{F}} = 0$, we say that the foliation $\mathscr{F}$ is *purely transcendental*. Suppose that $\mathscr{F}$ is not algebraically integrable. Then there exist a normal variety Y, unique up to birational equivalence, a dominant rational map with connected fibers $\varphi\colon X \dashrightarrow Y$, and a purely transcendental foliation $\mathscr{G}$ on Y such that $\mathscr{F}$ is the pullback of $\mathscr{G}$ via φ. This means that $\mathscr{F}_{|X^\circ} = (d\varphi^\circ)^{-1}(\mathscr{G}_{|Y^\circ})$, where $X^\circ \subset X$ and $Y^\circ \subset Y$ are smooth open subsets over which φ restricts to a smooth morphism $\varphi^\circ\colon X^\circ \to Y^\circ$.

We end this section by mentioning an algebraic integrability criterion of Campana and Păun, which generalizes Theorem 2.11. The classical notion of slope-stability with respect to an ample line bundle has been extended to allow stability conditions given by movable curve classes on $\mathbb{Q}$-factorial normal projective varieties ([CP11], [GKP16a], [CP15]). In this more general setting, one still has Harder-Narasimhan filtrations as in (2.2), although the analogous of the Mehta-Ramanathan Theorem fails in general. Let $\mathscr{F}$ be a foliation on a $\mathbb{Q}$-factorial normal projective variety X, and suppose that it has positive slope with respect to movable curve class $\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$. Consider the Harder-Narasimhan filtration of $\mathscr{F}$ with respect to α as in (2.2), and set $s = \max\{i \geqslant 1 \,|\, \mu_\alpha(\mathscr{F}_i/\mathscr{F}_{i-1}) > 0\} \geqslant 1$. Then the algebraic integrability criterion of Campana and Păun ([CP15, Theorem 4.2]) implies that, for $1 \leqslant i \leqslant s$, $\mathscr{F}_i \subset T_X$ is an algebraically integrable foliation, and the closure of a general leaf is rationally connected. In particular, if $\mathscr{F}$ is a purely transcendental foliation, then $K_{\mathscr{F}}$ is pseudo-effective.

3. Classification of del Pezzo foliations

In this section we discuss classification results for del Pezzo foliations on projective manifolds.

Definition 3.1. A *del Pezzo foliation* is a Fano foliation $\mathscr{F}$ of rank $r \geqslant 2$ and index $\iota_{\mathscr{F}} = r - 1$.

In the Introduction we described all del Pezzo foliations on projective spaces and quadric hypersurfaces. We have the following classification of codimension 1 del Pezzo foliations on projective manifolds. See [AD14, Theorem 1.3] for a more general statement.

Theorem 3.2. *Let $\mathscr{F} \subset T_X$ be a codimension 1 del Pezzo foliation on a smooth projective variety X. Then one of the following holds.*

(1) *$X \cong \mathbb{P}^n$.*
(2) *X is isomorphic to a (possibly singular) quadric hypersurface in $\mathbb{P}^{n+1}$.*
(3) *There is an inclusion of vector bundles $\mathscr{K} \subset \mathscr{E}$ on $\mathbb{P}^1$, inducing a relative linear projection*

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \overset{\varphi}{\dashrightarrow} & \mathbb{P}(\mathscr{K}) \\ & \searrow \quad \swarrow_{q} & \\ & \mathbb{P}^1 & \end{array},$$

such that $X \cong \mathbb{P}(\mathscr{E})$ *and* $\mathscr{F}$ *is the pullback via* φ *of a foliation*

$$q^*\Big(\det\big(\mathscr{E}/\mathscr{K}\big)\Big) \hookrightarrow T_{\mathbb{P}(\mathscr{K})} .$$

Moreover, one of the following holds.

- $(\mathscr{E}, \mathscr{K}) \cong \big(\mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathscr{O}_{p^1}(a)^{\oplus 2}\big)$ *for some positive integer* a.
- $(\mathscr{E}, \mathscr{K}) \cong \big(\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}\big)$ *for some positive integer* a.
- $(\mathscr{E}, \mathscr{K}) \cong \big(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b), \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)\big)$ *for distinct positive integers* a *and* b.

Theorem 3.2 is the first instance of classification of del Pezzo foliations, when the ambient space is smooth and the codimension is 1. The classification problem can move in different directions. One may be interested in del Pezzo foliations on mildly singular varieties. In this direction, [AD14, Theorem 1.3] allows X to be factorial and canonical. The conclusion is the same as in Theorem 3.2, with the additional possibility of X being a cone over certain surfaces of Picard rank 1. One may be interested in classifying codimension 1 Fano foliations of slightly smaller index. Fano foliations $\mathscr{F}$ of rank $r \geq 3$ and index $\iota_{\mathscr{F}} = r - 2$ are called *Mukai foliations*. In [AD17b], we have classified codimension 1 Mukai foliations on projective manifolds. Finally, one is often interested in del Pezzo foliations of arbitrary rank. For the rest of this paper, we consider del Pezzo foliations of arbitrary rank on projective manifolds. Recall from Theorem 1.4 that, except when the ambient space is $\mathbb{P}^n$, del Pezzo foliations are always algebraically integrable. As a step in the classification problem, we give a classification of all possible general log leaves of del Pezzo foliations.

Proposition 3.3. *Let* $\mathscr{F}$ *be an algebraically integrable del Pezzo foliation of rank* $r \geqslant 2$ *on a smooth projective variety* X, *with general log leaf* (F, Δ). *Let* L *be an ample divisor on* X *such that* $-K_{\mathscr{F}} \sim (r-1)L$. *Then* $\big(F, \Delta, L_{|F}\big)$ *satisfies one of the following conditions.*

(1) $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(2), \mathscr{O}_{\mathbb{P}^r}(1))$.
(2) (F, Δ) *is a cone over* (Q^m, H), *where* Q^m *is a smooth quadric hypersurface in* $\mathbb{P}^{m+1}$ *for some* $2 \leqslant m \leqslant r$, $H \in \big|\mathscr{O}_{Q^m}(1)\big|$, *and* $L_{|F}$ *is a hyperplane under this embedding.*
(3) $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2))$.
(4) $\big(F, \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$, *and one of the following holds:*
  (a) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ *for some* $d \geqslant 2$, *and* $\Delta \sim_{\mathbb{Z}} \sigma + f$, *where* σ *is the minimal section and* f *a fiber of* $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$.
  (b) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ *for some* $d \geqslant 2$, *and* Δ *is a minimal section.*
  (c) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ *for some* $d \geqslant 1$, *and* $\Delta = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1))$.
(5) (F, Δ) *is a cone over* (C_d, B), *where* C_d *is rational normal curve of degree* d *in* $\mathbb{P}^d$ *for some* $d \geqslant 2$, $B \in \big|\mathscr{O}_{\mathbb{P}^1}(2)\big|$, *and* $L_{|F}$ *is a hyperplane under this embedding.*
(6) (F, Δ) *is a cone over the pair* (4a) *above, and* $L_{|F}$ *is a hyperplane section of the cone.*

Proof. By Proposition 2.7, $\Delta \neq 0$, and so $K_F + (r-1)L_{|F} \sim -\Delta$ is not pseudo-effective.

Let $\nu\colon \tilde{F} \to F$ be a resolution of singularities, and set $\tilde{L} = \nu^* L_{|F}$. Then $\tilde{L}$ is nef and big. In the language of [And13], $(\tilde{F}, \tilde{L})$ is a quasi-polarized variety. Moreover, $K_{\tilde{F}} + (r-1)\tilde{L}$ is not pseudo-effective. As in the proof of [Hör14, Lemma 2.5], we run a $(K_{\tilde{F}} + (r-1)\tilde{L})$-MMP, $\varphi : \tilde{F} \dashrightarrow F'$. Since $K_{\tilde{F}} + (r-1)\tilde{L}$ is not pseudo-effective, it ends with a Mori fiber space $F' \to Z$:

$$\begin{array}{ccccc} (\tilde{F}, \tilde{L}) & \dashrightarrow & (F_i, L_i) & \dashrightarrow & (F', L'). \\ \downarrow{\scriptstyle \nu} & & & & \\ F & & & & \end{array}$$

By [And13, Proposition 3.6], if (F_i, L_i) is an r-dimensional terminal $\mathbb{Q}$-factorial quasi-polarized variety, and $\mathbb{R}_{\geqslant 0}[C]$ is a $(K_{F_i} + (r-1)L_i)$-negative extremal ray of birational type, then $L_i \cdot C = 0$. Therefore, $\varphi : \tilde{F} \dashrightarrow F'$ is an MMP relative to F, and there exists a morphism $\nu'\colon F' \to F$ such that $\nu = \nu' \circ \varphi$. In particular, $L' = (\nu')^* L_{|F}$ is nef and big. Quasi-polarized varieties (F', L') with a Mori fiber space structure induced by a $(K_{F'} + (r-1)L')$-negative extremal ray were classified in [And13, Proposition 3.5]. They are the following:

(a) $(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1))$.
(b) $(Q^r, \mathscr{O}_{Q^r}(1))$, where Q^r is a quadric hypersurface in $\mathbb{P}^{r+1}$.
(c) A cone over $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$ (where the vertex is allowed to be empty).

(d) $(\mathbb{P}_B(\mathscr{E}), \mathscr{O}_{\mathbb{P}_B(\mathscr{E})}(1))$, where $\mathscr{E}$ is a nef and big vector bundle of rank r over a smooth curve B.

In case (a) we have $F' \cong F$ and $\Delta \in \left|\mathscr{O}_{\mathbb{P}^r}(2)\right|$.

In case (b) we have $F' \cong F$ and $\Delta \in \left|\mathscr{O}_{Q^r}(1)\right|$. Moreover, since $F \setminus \Delta$ is smooth, (F, Δ) is a cone over (Q^m, H), where Q^m is a smooth quadric hypersurface in $\mathbb{P}^{m+1}$ and $H \in \left|\mathscr{O}_{Q^m}(1)\right|$, for some $1 \leqslant m \leqslant r$. When $m = 1$, F is isomorphic to a cone over a conic curve, and this case will be covered under case (d) below.

In case (c), we have $F' \cong F$, (F, Δ) is a cone over the Veronese embedding of $(\mathbb{P}^2, \ell)$ in $\mathbb{P}^5$. Here ℓ is a line in $\mathbb{P}^2$ and thus Δ is a cone over a smooth conic. In particular, (F, Δ) is log canonical and Δ is its only log canonical center. By Proposition 2.9, the image of Δ does not vary with the log leaf. Suppose that the vertex V of (F, Δ) is nonempty. Then the image of V does not vary with the log leaf either. Therefore any point of X can be connected to any point in the image of V in X by a rational curve of L-degree 1. This implies that $X \cong \mathbb{P}^n$. From the classification of del Pezzo foliations on $\mathbb{P}^n$, we see that this is not possible. We conclude that $V = \emptyset$ and $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2))$.

We now consider case (d). Denote by g is the genus of B and by f a fiber of the natural morphism $\pi : F' \to B$. Write ξ for a divisor on F' such that $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \cong \mathscr{O}_{F'}(\xi)$, and e for a divisor on B such that $\mathscr{O}_B(e) \cong \det \mathscr{E}$. Then $-K_{F'} = r\xi + \pi^*(-e - K_B)$.

We consider the following possibilities:

(d-1) $F' \cong F$.
(d-2) $\nu' \colon F' \to F$ is the divisorial contraction induced by ξ.
(d-3) $\nu' \colon F' \to F$ is the small contraction induced by ξ.

In case (d-1), since F is rationally connected, we must have $B \cong \mathbb{P}^1$ and $\mathscr{E}$ is an ample vector bundle on $\mathbb{P}^1$. Write

$$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_r),$$

with $1 \leqslant a_1 \leqslant \cdots \leqslant a_r$. We have $0 \neq \Delta \in \left|\xi + (-\sum a_i + 2)f\right|$, and so

$$1 \leqslant h^0\big(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{O}_{\mathbb{P}^1}(-\sum a_i + 2)\big) = h^0\big(\mathbb{P}^1, \mathscr{E}(-\sum a_i + 2)\big).$$

This implies that $(r; a_1, \ldots, a_{r-1}) \in \big\{(2; 1), (2; 2), (3; 1, 1)\big\}$. When $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, F is a smooth quadric surface, and has already been consider. So we have one of the possibilities described in (4).

In case (d-2), F is $\mathbb{Q}$-factorial and $(\nu')^*\Delta \in \left|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*(\det \mathscr{E}^\vee \otimes \omega_B^\vee)\right|$, and so

$$1 \leqslant h^0\big(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*(\det \mathscr{E}^\vee \otimes \omega_B^\vee)\big) = h^0\big(B, \mathscr{E} \otimes \det \mathscr{E}^\vee \otimes \omega_B^\vee\big).$$

By Hartshorne's theorem ([Har71, Theorem 2.4]), $\mathscr{E}$ is nef if and only if it has no quotient with negative slope. So we must have $g \in \{0, 1\}$. Moreover, if $g = 1$, then $\det \mathscr{E}$ is the first nonzero piece of the Harder-Narasimhan filtration of $\mathscr{E}$, and $\mathcal{Q} = \mathscr{E}/\det \mathscr{E}$ is a vector bundle on B. The only member of $\left|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*(\det \mathscr{E}^\vee)\right|$ is precisely the projectivization of $\mathcal{Q}$. It is the exceptional divisor of ν', which is impossible. So we conclude that $B \cong \mathbb{P}^1$, F is a cone over a rational normal curve of degree d for some $d \geqslant 1$, and $L_{|F}$ is a hyperplane under this embedding. When $d = 1$, we have $F \cong \mathbb{P}^r$. So we may assume that $d \geqslant 2$. A straightforward computation shows that Δ is linearly equivalent to two times a ruling of the cone.

In case (d-3), $\mathscr{O}_F(L_{|F})$ pulls back to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, and Δ is the image under the small contraction of a nonzero effective divisor $\Delta' \in \left|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*(\det \mathscr{E}^\vee \otimes \omega_B^\vee)\right|$. As before, $g \in \{0, 1\}$. Moreover, if $g = 1$, then $\det \mathscr{E}$ is the first nonzero piece of the Harder-Narasimhan filtration of $\mathscr{E}$, $\mathcal{Q} = \mathscr{E}/\det \mathscr{E}$ is a vector bundle on B, and $\Delta' \cong \mathbb{P}(\mathcal{Q})$ is the only member of $\left|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*(\det \mathscr{E}^\vee)\right|$. In particular, (F, Δ) is log canonical and Δ is its only log canonical center. By Proposition 2.9, the image of Δ does not vary with the log leaf, and so the image of its singular locus V does not vary with the log leaf either. Note that V is the image of the exceptional locus of ν'. Therefore any point of X can be connected to any point of V by a rational curve of L-degree 1, and thus $X \cong \mathbb{P}^n$. From the classification of del Pezzo foliations on $\mathbb{P}^n$, we see that this is not possible. So we conclude that $B \cong \mathbb{P}^1$. As in (d-1), we see that one of the following holds:

- $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 2$, $r > 2$, and $\Delta' \sim_{\mathbb{Z}} \sigma + f$, where $\sigma = \mathbb{P}_{\mathbb{P}^1}\big(\mathscr{O}_{\mathbb{P}^1}^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)\big)$, and f a fiber of $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}^1$.
- $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 2$, $r > 2$, and $\Delta = \mathbb{P}_{\mathbb{P}^1}\big(\mathscr{O}_{\mathbb{P}^1}^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}(2)\big)$.

- $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}^{\oplus r-3} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 1$, $r > 3$, and $\Delta = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}^{\oplus r-3} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1))$.

The first case can also be described as a cone over the pair (4a) above, yielding (6).

The latter two cases, can be described as cones over the pairs (4b) and (4c) above, respectively. In these cases, (F, Δ) is log canonical and Δ is its only log canonical center. Moreover, Δ is a cone with vertex $V \neq \emptyset$ over a conic and a smooth quadric surface, respectively. By Proposition 2.9, the image of Δ does not vary with the log leaf, and so the image of V does not vary with the log leaf either. Therefore any point of X can be connected to any point in the image of V in X by a rational curve of L-degree 1, and thus $X \cong \mathbb{P}^n$. From the classification of del Pezzo foliations on $\mathbb{P}^n$, we see that this is not possible. □

We do not know examples of del Pezzo foliations with log leaves of type (3) and (6) described in Theorem 3.3. When the general log leaf of $\mathscr{F}$ is log canonical, Proposition 2.9 may be used to recover the ambient space X. For example, consider case (1), when $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(2), \mathscr{O}_{\mathbb{P}^r}(1))$. If (F, Δ) is log canonical, then Proposition 2.9 yields a common point $x \in X$ in the closure of a general leaf. Therefore any point of X can be connected to x by a rational curve of L-degree 1. This implies that $X \cong \mathbb{P}^n$. On the other hand, there are del Pezzo foliations with general log leaf of type (1) and not log canonical (see Remark 2.10).

We end this paper by reviewing a classification of del Pezzo foliations on projective manifolds under restrictions on the singularities of the foliation $\mathscr{F}$. Namely, we assume that $\mathscr{F}$ has log canonical singularities and is locally free along a general leaf. Recall from Theorem 1.4 that, if $X \not\cong \mathbb{P}^n$, then del Pezzo foliations on X are always algebraically integrable. If we remove the log canonicity assumption, we know more examples of del Pezzo foliations, as discussed in Remark 2.10. One may be able to remove the locally freeness assumption using the classification of log leaves in Proposition 3.3.

Theorem 3.4 ([AD13, 9.1 and Theorems 1.3, 9.2, 9.6])**.** *Let $\mathscr{F}$ be an algebraically integrable del Pezzo foliation of rank r on a projective manifold X. Suppose that $\mathscr{F}$ has log canonical singularities and is locally free along a general leaf. Then one of the following holds.*

1. *$X \cong \mathbb{P}^n$.*
2. *X is isomorphic to a quadric hypersurface in $\mathbb{P}^{n+1}$.*
3. *$X \cong \mathbb{P}^1 \times \mathbb{P}^k$, $r \in \{2, 3\}$ and $\mathscr{F}$ is the pullback via the second projection of a foliation on $\mathbb{P}^k$ induced by a linear projection.*
4. *There is an inclusion of vector bundles $\mathscr{K} \subset \mathscr{E}$ on $\mathbb{P}^1$, inducing a relative linear projection*

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \overset{\varphi}{\dashrightarrow} & \mathbb{P}(\mathscr{K}) \\ & \searrow \quad \swarrow_{q} & \\ & \mathbb{P}^1 & \end{array},$$

such that $X \cong \mathbb{P}(\mathscr{E})$ and $\mathscr{F}$ is the pullback via φ of a foliation

$$q^*\Big(\det\big(\mathscr{E}/\mathscr{K}\big)\Big) \hookrightarrow T_{\mathbb{P}(\mathscr{K})}.$$

Moreover, one of the following holds.
   - *$(\mathscr{E}, \mathscr{K}) \cong \big(\mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}\big)$ for some $a \geqslant 1$ and $m \geqslant 2$ ($r = 2$).*
   - *$(\mathscr{E}, \mathscr{K}) \cong \big(\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}\big)$ for some $a \geqslant 1$ and $m \geqslant 2$ ($r = 3$).*
   - *$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{K}$, where $\mathscr{K}$ is an ample vector bundle not isomorphic to $\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for any integer a ($r = 2$).*
5. *There is an inclusion of vector bundles $\mathscr{K} \subset \mathscr{E}$ on $\mathbb{P}^k$, with $k \geqslant 2$ and $\mathscr{E}/\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^k}(1)$, inducing a relative linear projection*

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \overset{\varphi}{\dashrightarrow} & \mathbb{P}(\mathscr{K}) \\ & \searrow \quad \swarrow_{q} & \\ & \mathbb{P}^k & \end{array},$$

such that $X \cong \mathbb{P}(\mathscr{E})$ and $\mathscr{F}$ is the pullback via φ of a foliation $q^*\mathscr{O}_{\mathbb{P}^k}(1) \hookrightarrow T_{\mathbb{P}(\mathscr{K})}$ ($r = 2$).

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