Metric in the moduli of SU(2) monopoles from spectral curves and Gauss-Manin connection in disguise.

Marcus A. C. Torres^{1, a)}

Instituto Nacional de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110,22460-320, Rio de Janeiro,RJ, Brazil

DTuzii

(Dated: 8 May 2018)

We show here that from the metric of the manifold M_2^0 , i.e., the reduced moduli of SU(2) 2-monopoles in Yang-Mills-Higgs theory, one can recover the respective moduli of spectral curves using the method Gauss-Manin connection in disguise. This work is a step towards creating a inverse process of finding the metric of any M_k^0 , from spectral curves. This is a thirty years old problem that we hope to shed some light in it.

^{a)}mtorres@impa.br; www.impa.br/~mtorres

I. INTRODUCTION

The study of instantons and monopoles in three and four dimension are among the most comprehensive research areas at the interface of physics and mathematics. From the physics point of view, they relate to solitons, dualities and non-perturbative Yang-Mills theories. From the mathematical point of view, it involves knowledge of Analysis, Differential Geometry, Algebraic Geometry and Twistor theory. In this work, we add elements of Hodge theory to relate the metric of the moduli space of charge k monopoles M_k^0 and its spectral curves in SU(2) Yang-Mills-Higgs (YMH) theory in three spacial dimensions (static monopoles).

YMH monopoles in three dimensions are equivalent to instanton solutions of Yang-Mills theory in Euclidian four dimensions constrained by the fact that gauge fields do not depend on the fourth direction. In this equivalence, the Higgs field is the fourth component of the gauge field in four dimensions. In this way, the twistor methods applied in instantons were also adapted to YMH monopoles¹.

In a recent work², while revising the metric of M_2^0 reduced moduli of 2-monopoles, the present author noticed that the moduli of enhanced elliptic curves obtained from the self-dual metric should correspond to the moduli of spectral curves of 2-monopoles in its enhanced version. Notice¹ that the spectral curve of a k-monopole is an algebraic curve of genus $(k-1)^2$ and for k=2, the spectral curve is an elliptic curve³.

The moduli of enhanced elliptic curves appears from the metric of M_2^0 using the method Gauss-Manin connection in disguise developed by Movasati^{4–6} which shows that the set of Darboux-Halphen differential equations obtained from self-duality of the metric of M_2^0 are a vector field in the moduli of an enhanced elliptic curve.

In here, the guess made in our previous collaboration² is proved.

In sections II and III we review basic elements of SU(2) monopoles and spectral curves following closely some original articles^{1,3,7} and reference books^{8,9}. In section IV we quickly review the program Gauss-Manin connection in disguise for elliptic curves⁶ and in section V we show how the moduli of spectral curves of 2-monopoles emerge from the metric of M_2^0 . In VI we summarize this article and comment about the cases k > 2 and the issue of different parametrization of universal families of curves and the weights of the respective set of modular-type functions attached to such curves.

II. K-MONOPOLES

A k-monopole or BPS-monopole of charge k in Yang-Mills-Higgs theory is a static soliton in \mathbb{R}^3 that is a solution of the Bogomolny equation¹⁰:

$$F = \star D\phi, \quad \text{with} \tag{1}$$
$$F := \mathrm{d}A + A \wedge A \quad \text{and} \quad D := \mathrm{d} + A,$$

where A is the gauge field or connection form on a principal SU(2)-bundle over \mathbb{R}^3 , F is its curvature 2-form or field strength, D is the covariant exterior derivative or connection and the Higgs field ϕ is a section of the associated $\mathfrak{su}(2)$ -bundle. \star is the Hodge dual operation and the Bogomolny equation (1) is part of the self-duality equations for the related instantons in four dimensions.

The monopole solution has also to satisfy the finite action condition $\int |F|^2 < \infty$ and the boundary condition

$$|\phi| = 1 - \frac{k}{2r} + O(r^{-2}) \quad \text{as} \ r \to \infty, \tag{2}$$

where the charge k is an integer number.

A clever treatment of monopoles was given by Hitchin¹ where he applied the twistor methods in the space $\tilde{\mathbf{T}}$ of oriented straight lines (geodesics) in \mathbb{R}^3 . $\tilde{\mathbf{T}}$ has a holomorphic structure given by cross product and $\tilde{\mathbf{T}} \equiv \mathbf{TP}_1(\mathbb{C})$ the holomorphic tangent bundle of the projective line.

Then the solutions of Bogomolny equations were restated in terms of complex geometry of $\tilde{\mathbf{T}}$ where the spectral curves were introduced^{1,11}.

First consider E the rank 2 complex vector bundle on \mathbb{R}^3 associated to the principal SU(2) bundle. Now one defines a rank 2 complex vector bundle \tilde{E} by defining at each point $z \in \tilde{\mathbf{T}}$ a fiber E_z . To each point $z \in \tilde{\mathbf{T}}$ there is a corresponding oriented line $l_z \in \mathbb{R}^3$. E_z is given by the space of sections s of E with support on l_z such that

$$(u^j D_j - i\phi)s = 0, (3)$$

u is the unit tangent vector pointing (in the positive direction) along the oriented line l_z . It follows from Bogomolny equations that \tilde{E} has a natural holomorphic structure¹. Conversely, from a holomorphic vector bundle \tilde{E} on $\tilde{\mathbf{T}}$ one reconstructs the solution (A, ϕ) to Bogomolny equations. But not all section s of \tilde{E}_l satisfy the boundary conditions, which is the vanishing of s at both ends of l.

III. SPECTRAL CURVE OF A K-MONOPOLE

For each oriented line $l \in \mathbb{R}^3$, the space of solutions (3) which decay at $+\infty$ is onedimensional. This space is a holomorphic line bundle and a subbundle of \tilde{E}_l and it belongs to a class of ansätze \mathcal{A}_k according to the charge k of the monopole⁷. Furthermore, the set of lines for which equation (3) has a solution decaying to zero at both ends forms a compact algebraic curve S in $\mathbb{TP}_1(\mathbb{C})$. S is called the spectral curve and it has genus $(k-1)^2$.

Inhomogeneous coordinates (η, ζ) on $\mathbb{P}_1(\mathbb{C})$ gives local coordinates on $\tilde{\mathbf{T}} : (\eta, \zeta) \to \eta \partial / \partial \zeta$. S in terms of such local coordinates is given by

$$p(\eta,\zeta) = \eta^{k} + a_{1}(\zeta)\eta^{k-1} + \dots a_{k}(\zeta) = 0,$$
(4)

where $a_i(\zeta)$ is a polynomial of degree 2i.

The polynomial $p(\eta, \zeta)$ is preserved by an antiholomorphic involution $\tau(\eta, \zeta) = (-\overline{\eta}/\overline{\zeta}^2, -\overline{\zeta}^{-1})$, a real structure on $\mathbb{TP}_1(\mathbb{C})$. Therefore $p(\eta, \zeta)$ depends on $(k+1)^2 - 1$ real parameters. Since S is constrained by its genus (transcendental or ES constraint¹²), the parameter space has dimension

$$(k+1)^2 - 1 - (k-1)^2 = 4k - 1.$$
(5)

This is the dimension of the moduli space of k-monopoles M_k . Out of these parameters, the center of mass position of a k-monopole in \mathbb{R}^3 can be translated to the origin and the remaining parameter space corresponds to the reduced moduli space M_k^0 with dimension 4k-4.

A. Spectral curve for k=2

This case was extensively studied by Hurtubise³. The spectral curve S is an elliptic curve of genus 1. The real structure of S imposes via Weierstrass \mathfrak{p} -function the complex structure τ of the corresponding torus \mathbb{C}/Λ to be purely imaginary and its corresponding lattice Λ to be rectangular.

Factoring out six parameters from translation action and SO(3) action from the polynomial (4) for the spectral curve of 2-monopoles, two real parameters remain:

$$\eta^2 = r_1 \zeta^3 - r_2 \zeta^2 - r_1 \zeta, \ r_i \in \mathbb{R}, r_1 \ge 0$$
(6)

The genus constraint is enforced by matching the above equation to a normal form of the elliptic curve.

First notice that when $r_1 = 0$, the spectral curve degenerates to two k = 1 spectral curves.

$$\eta = i\frac{\pi}{2}\zeta \quad \text{and} \quad \eta = -i\frac{\pi}{2}\zeta.$$
 (7)

In this case $r_2 = \pi^2/4$, as we show below (15), and there is no free parameter. This is the case^{13,14} where a 2-monopole is simply a superposition of two 1-monopoles both centered at the origin of \mathbb{R}^3 and it agrees with the fact that the dimension of the reduced moduli M_1^0 is zero. The spectral curves (7) are two complex lines tangent to two different points in $\mathbb{P}_1 \equiv S^2$. The symmetries of the spectral curve determines a symmetry of the monopole. In this case, the isotropy group $S^1 \times \mathbb{Z}_2$ of the two k = 1 spectral curves corresponds to the axial symmetry of the two 1-monopoles solution and the exchanging the two 1-monopoles.

For $r_1 > 0$, the spectral curve can be reparametrized to:

$$\tilde{\eta}^{2} = 4\tilde{\zeta}^{3} - g_{2}(\Lambda)\tilde{\zeta} - g_{3}(\Lambda), \text{ where,}$$

$$\tilde{\eta} = \eta (4/r_{1})^{1/2}, \quad \tilde{\zeta} = \zeta - \frac{r_{2}}{3r_{1}},$$

$$g_{2}(\Lambda) = 60G_{4}(\Lambda) = 12(r_{2}/3r_{1})^{2} + 4 \quad \text{and,}$$

$$g_{3}(\Lambda) = 140G_{6}(\Lambda) = 8(r_{2}/3r_{1})^{3} + 4(r_{2}/3r_{1}).$$
(9)

and G_4 and G_6 are Eisenstein series of weight 4 and 6, respectively, functions of the retangular lattice Λ with real generator $l_r = \sqrt{4r_1}$ and imaginary generator l_i .

A homothetic scaling of the lattice transform g_2 and g_3 :

$$g_i(m\Lambda) = m^{-2i}g_i(\Lambda), \, i = 2, 3, \quad m \in \mathbb{R}^*$$
(10)

and the polynomial (8) is preserved if we reparametrize $(\tilde{\eta}, \tilde{\zeta})$ to absorb such scaling

$$\tilde{\eta} \to m^{-3} \tilde{\eta} \quad \text{and} \quad \tilde{\zeta} \to m^{-2} \tilde{\zeta}.$$
 (11)

Therefore we should consider modular functions such as $I = 27g_3^2/g_2^3$. This function will be invariant to scaling of the lattice Λ and it will only depend on the ratio of the generators $\tau = l_i/l_r$, a purely imaginary number. To make this dependency explicit we can reparametrize the variables as above with $m = l_r^{-1}$ and obtain $g_2(\tau) = 16r_1^2g_2(\Lambda) = 60G_4(\tau)$ and $g_3(\tau) = (4r_1)^3g_3(\Lambda) = 140G_6(\tau)$. In terms of the normalized Eisenstein series E_{2i} and Riemann zeta functions, $G_{2i}(\tau) = 2\zeta(2i)E_{2i}(\tau)$ with,

$$E_{2i}(q) := 1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1} \right) q^n, \quad i = 1, 2, 3,$$
(12)

with $(b_1, b_2, b_3) = (-24, 240, -504)$ and $q = e^{i2\pi\tau}, \tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) \geq 0\}$ for convergency of the series. When τ is purely imaginary, $E_{2i}(\tau)$ take values in \mathbb{R} .

From (8), I depends on the ratio $(r_1/r_2)^2$. Notice that in the limit $r_1 \to 0$, the discriminant of the elliptic curve $\Delta = g_2^3 - 27g_3^2 = 0$ and $I(\tau) = 1$. This corresponds to the limit $\tau \to i\infty$. In order to proceed showing that $r_2 = \pi^2/4$ in this limit, we explore I near 1.

$$\frac{1 - I(\tau)}{27} = \frac{64}{j(\tau)} = 2^{12} 3^3 (q - 744q^2 + \dots) \quad \text{where,}$$
(13)
$$j(\tau) = \frac{1728g_2^3}{\Delta} \text{ is the Klein modular function }.$$

From (12),

$$\frac{1-I}{27} = \frac{r^4(\frac{1}{4}+r^2)}{(1+3r^2)^3}, \quad \text{with } r = \frac{r_1}{r_2}.$$
(14)

We see that the limit $r_1, r \to 0$ coincides with $\tau \to i\infty$ or $q \to 0$. Near this limit, we keep only the first term of the Eisenstein series $G_4(\tau)$. From (12):

$$g_2(\Lambda) = \frac{1}{16r_1^2} \frac{64}{3} r_2^2 (1+3r^2) = \frac{1}{16r_1^2} 60G_4(\tau) \xrightarrow{q \to 0} \frac{1}{16r_1^2} \frac{(2\pi)^4}{12}.$$
 (15)

Therefore,
$$r_2 \xrightarrow{q \to 0} \pi^2/4.$$
 (16)

Hence, the point $(r_1, r_2) = (0, \pi^2/4)$ corresponds to the singular point $(\Delta = 0)$ of the real elliptic curve S factored out by SO(3) action and \mathbb{R}^3 translations, corresponding to $\tau = i\infty$. The total space of parameters has one real dimension and it corresponds to purely imaginary $\tau, -i\tau \in \mathbb{R}_{\geq 0} \cup \infty$.

IV. GAUSS MANIN CONNECTION IN DISGUISE

In^{15,16} Movasati realized that the Ramanujan relations between Eisenstein series can be computed using the Gauss-Manin connection of families of elliptic curves. Later in a private communication, Pierre Deligne called "Gauss-Manin connection in disguise" the vector field that best express the property of Griffths transversality^{17,18} of a Gauss-Manin connection. Since then the method Gauss-Manin connection in disguise has been applied in many families of algebraic curves and relating them to differential equations and automorphic forms or modular-type functions^{4,19–21}.

Our interest are in finding differential equations in the universal families of spectral curves of k-monopoles. The method developed for the elliptic curve^{6,16} still need to be thought through for spectral curves because of the reality condition on the spectral curves, but our general argument is that the reality condition is lifted for the sake of finding the Gauss-Manin connection and the respective vector field and later the reality condition is imposed on the domain of solutions of the vector field equations.

We present here the two known cases of families of enhanced elliptic curves which correspond to geometric expressions of Ramanujan and Darboux-Halphen differential equations. A good review is in Movasati's lectures⁶. In both cases, the idea is to define the moduli of enhanced elliptic curve by including information about its Hodge structure. Then, one calculates its Gauss-Manin connection and finds the appropriate vector field.

A. Ramanujan differential equations

We extend the one-parameter family of elliptic curves (8). Recall that the first de Rham cohomology $H^1_{dR}(E)$ of an elliptic curve E is a two-dimensional vector space. The moduli T_R of pairs $(E, [\alpha, \omega])$, where $\alpha, \omega \in H^1_{dR}(E)$ are a basis of the cohomology classes of 1-forms in E with α a regular differential 1-form on E and ω such that $\langle \alpha, \omega \rangle = 1$. In a equivalent way, T_R can be defined as the moduli of pairs $(E, [\omega]), \omega \in H^1_{dR}(E) \setminus F^1$ and there is a unique regular 1-form α in the Hodge filtration $F^1 \subset H^1_{dR}(E)$ such that $\langle \alpha, \omega \rangle = 1$. Therefore T_R is a three-dimensional space and it has a corresponding universal family of elliptic curves

$$E_t: y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3,$$
(17)

with $\alpha = \left[\frac{dx}{y}\right]$, $\omega = \left[\frac{xdx}{y}\right]$ and the moduli T_R can be expressed as

$$\mathbf{T}_{\mathbf{R}} := \{ (t_1, t_2, t_3) \in \mathbb{C}^3 | \Delta = t_2^3 - 27t_3^2 \neq 0 \}.$$
(18)

The Gauss-Manin connection of the above family E_t , written in the basis (α, ω) is given by

$$\nabla \begin{pmatrix} \alpha \\ \omega \end{pmatrix} = A \begin{pmatrix} \alpha \\ \omega \end{pmatrix}, \tag{19}$$

where

$$A = \frac{1}{\Delta} \begin{pmatrix} -\frac{3}{2}t_1\beta - \frac{1}{12}d\Delta & \frac{3}{2}\beta \\ \Delta dt_1 - \frac{1}{6}t_1d - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\beta & \frac{3}{2}t_1\beta\Delta + \frac{1}{12}d\Delta \end{pmatrix},$$
(20)
$$\beta = 3t_3dt_2 - 2t_2dt_3.$$

In T_R there is a unique vector field R such that⁶

$$\nabla_R(\alpha) = -\omega, \quad \nabla_R(\omega) = 0.$$
 (21)

The vector field R is given by the Ramanujan differential equations²²

$$\begin{cases} \frac{\partial t_1}{\partial \tau} = t_1^2 - \frac{1}{12} t_2, \\ \frac{\partial t_2}{\partial \tau} = 4 t_1 t_2 - 6 t_3, \\ \frac{\partial t_3}{\partial \tau} = 6 t_1 t_3 - \frac{1}{3} t_2^2, \end{cases}$$
(22)

where τ is a direction in the moduli T_R chosen by R.

R has been called Ramanujan vector field, modular vector field and lately, Gauss-Manin connection in disguise.

It may seem that in this process the moduli was not only enhanced but also enlarged since the moduli of elliptic curves has 1 complex dimension while $\dim(T_R) = 3$. But if we look to the solution of (22),

$$(t_1(\tau), t_2(\tau), t_3(\tau)) := \left(\frac{2\pi i}{12} E_2(\tau), \ 12\left(\frac{2\pi i}{12}\right)^2 E_4(\tau), 8\left(\frac{2\pi i}{12}\right)^3 E_6(\tau)\right), \tag{23}$$

 E_4 and E_6 are modular forms of weight k = 4, 6, respectively,

$$E_k(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k E_k(\tau), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

and E_2 is a quasi-modular form of weight 2:

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi i}c(c\tau+d).$$

Therefore, R vector field corresponds to a map of $\tau \in \mathbb{H}$ to $(t_1, t_2, t_3) \in T_{\mathbb{R}}$.

The transformation of E_2, E_4 , and E_6 , under the action of the modular group $SL(2,\mathbb{Z})$ on τ , preserving the lattice that defines the elliptic curve $E_t \equiv \mathbb{C}^2/\Lambda_{\tau}$, reveals that there is a group of isomorphisms G that acts on T_R . The quotient moduli T_R/G has one complex dimension. For $g \in G$,

$$\begin{aligned} [\alpha, \omega] \xrightarrow{\bullet g} [\alpha, \omega]g &= [c\alpha, c'\alpha + \omega/c] \\ t &= (t_1, t_2, t_3) \xrightarrow{\bullet g} t' = (c^{-2}t_1 + c'/c, c^{-4}t_2, c^{-6}t_3) \\ (x, y) \xrightarrow{\bullet g} (c^{-2}x + c'/c, c^{-3}y) \\ E_t &\equiv E_{t'} \\ \end{array} \\ = \begin{pmatrix} c & c' \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & c'/c \\ 0 & 1 \end{pmatrix}, \quad c \in \mathbb{C}^*, c' \in \mathbb{C} \end{aligned}$$

Notice that this group action preserves the intersection form $\langle \alpha, \omega \rangle = \langle c\alpha, c'\alpha + \omega/c \rangle = 1$.

B. Darboux-Halphen differential equations

g

In this case, the enhanced elliptic curve is given by a triple $(E, (P, Q), \omega)$, where E is an elliptic curve, $\omega \in H^1_{d\mathbb{R}}(E) \setminus F^1$, and P and Q are a pair of points of E that generates the 2-torsion subgroup with the Weil pairing e(P,Q) = -1. The points P and Q are given by $(T_1, 0)$ and $(T_2, 0)$. In here, the torsion data is necessary because the modular group, or group of lattice equivalence, of this enhanced curve is the congruence subgroup $\Gamma(2) \subset SL_2(\mathbb{Z})$, which has index $[SL_2(\mathbb{Z}) : \Gamma(2)] = 6$. The torsion data choose one out of six enhanced elliptic curves with same (E, ω) pairs.

For each choice of ω , there is a unique regular differential 1-form in the Hodge filtration $\omega_1 \in F^1$, such that $\langle \omega, \omega_1 \rangle = 1$ and ω , ω_1 together form a basis of $H^1_{dR}(E)$. The corresponding universal family of elliptic curves is given by

$$E_T: \quad y^2 - 4(x - T_1)(x - T_2)(x - T_3) = 0, \tag{24}$$

and moduli $T_{\rm H} = \{(T_1, T_2, T_3) \in \mathbb{C}^3 | T_1 \neq T_2 \neq T_3\}.$

In fact, this universal family patches together all six enhanced elliptic curves, separated by singularity borders $T_i = T_j$, due its symmetry under permutation of T_1, T_2 and T_3 . Hence, it is a six-fold cover of the enhanced elliptic curve (E, ω) that is isomorphic to the enhanced elliptic curve $(E, [\alpha, \omega])$ for the full modular group $SL_2(\mathbb{Z})$. The Gauss-Manin connection of the family of elliptic curves E_T written in the basis $\frac{dx}{y}$, $\frac{xdx}{y}$ is given as bellow:

$$\nabla \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} = A \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix}$$
(25)

where

$$A = \frac{dT_1}{2(T_1 - T_2)(T_1 - T_3)} \begin{pmatrix} -T_1 & 1 \\ T_2T_3 - T_1(T_2 + T_3) & T_1 \end{pmatrix} + \frac{dT_2}{2(T_2 - T_1)(T_2 - T_3)} \begin{pmatrix} -T_2 & 1 \\ T_1T_3 - T_2(T_1 + T_3) & T_2 \end{pmatrix} + \frac{dT_3}{2(T_3 - T_1)(T_3 - T_2)} \begin{pmatrix} -T_3 & 1 \\ T_1T_2 - T_3(T_1 + T_2) & T_3 \end{pmatrix}.$$

In the parameter space of the family of elliptic curves E_T there is a unique vector field H, such that

$$\nabla_H(\frac{dx}{y}) = -\frac{xdx}{y}, \ \nabla_H(\frac{xdx}{y}) = 0.$$
(26)

The vector field H is given by the Darboux-Halphen differential equation

$$\begin{cases} \frac{\partial T_1}{\partial \tau} = T_1(T_2 + T_3) - T_2 T_3, \\ \frac{\partial T_2}{\partial \tau} = T_2(T_1 + T_3) - T_1 T_3, \\ \frac{\partial T_3}{\partial \tau} = T_3(T_1 + T_2) - T_1 T_2. \end{cases}$$
(27)

where τ is a direction in $T_{\rm H}$ chosen by H. This vector field H has been called Darboux-Halphen vector field, and lately, Gauss-Manin connection in disguise. Similarly to the previous subsection, there is a group of isomorphism G' in $T_{\rm H}$ with two generators (addictive and multiplicative) and the quotient $T_{\rm H}/G'$ has one complex dimension. For $g \in G'$,

$$T = (T_1, T_2, T_3) \xrightarrow{\bullet g} T' = (c^{-2}T_1 + c'/c, c^{-2}T_2 + c'/c, c^{-2}T_3 + c'/c),$$
$$(x, y) \xrightarrow{\bullet g} (c^{-2}x + c'/c, c^{-3}y),$$
$$E_T \equiv E_{T'}, \qquad c \in \mathbb{C}^*, \ c' \in \mathbb{C}$$

C. T_R and T_H

There is a algebraic morphism between the moduli $f : T_{\rm H} \longrightarrow T_{\rm R}$ given by a match between the elliptic curves (19) and (25):

$$(T_1, T_2, T_3) \to (T, -4 \sum_{1 \le i < j \le 3} (T - T_i)(T - T_j), 4(T - T_1)(T - T_2)(T - T_3)),$$
 (28)
where $T = \frac{T_1 + T_2 + T_3}{3}$

Since the permutations of T_1, T_2 and T_3 in T_H are mapped to the same point in T_R , f is a six to one map, but if we restrict to the region $|T_1| < |T_2| < |T_3|$ in T_H , f is an isomorphism.

V. FROM THE METRIC OF THE MODULI SPACE OF 2-MONOPOLES TO THE SPECTRAL CURVE

In their book⁸, Atiyah and Hitchin showed that the reduced moduli M_2^0 of 2-monopoles is a four dimensional hyperkähler manifold and an anti-self-dual Einstein manifold. Since M_2^0 admits SO(3) isometry, the metric is a Bianch IX²³. This is consequence of the hyperkähler structure of M_2^0 which has an S^2 -parameter family of complex structures: if I, J, K are covariant constant complex structures in M_2^0 then aI + bJ + cK is also a covariant constant complex structure in M_2^0 given that $a^2 + b^2 + c^2 = 1$, $(a, b, c) \in \mathbb{R}^3$. The SO(3) isometry rotates this S^2 in a standard way. Following Atiyah and Hitchin⁸ (Chapter 8,9) and our review², the 4-dimensional Bianchi IX metric is cast in the form

$$ds^{2} = (abc)^{2}d\rho^{2} + a^{2}(\sigma_{1})^{2} + b^{2}(\sigma_{2})^{2} + c^{2}(\sigma_{3})^{2},$$
(29)

where a, b, c are real functions of ρ which parametrizes each SO(3) orbit in M_2^0 or, in other words, it parametrizes trajectories orthogonal to these orbits in M_2^0 . The SO(3) invariant 1-forms σ_i are dual to the standard basis X_1, X_2, X_3 of its Lie algebra. They obey the structure equation

$$d\sigma_i = -\sigma_j \wedge \sigma_k,\tag{30}$$

for all cyclic permutations (i, j, k) of (1, 2, 3).

The self-duality equations lead to the following equation

$$\frac{2}{a}\frac{da}{d\rho} = b^2 + c^2 - a^2 - 2bc,$$
(31)

and two other equations obtained by cyclic permutations of (a,b,c). Upon reparametrization

$$a^2 = \frac{\Omega_2 \Omega_3}{\Omega_1}, \quad b^2 = \frac{\Omega_3 \Omega_1}{\Omega_2}, \quad c^2 = \frac{\Omega_1 \Omega_2}{\Omega_3}.$$
 (32)

we obtain from (31) the three Darboux-Halphen differential equations:

$$\dot{\Omega}_i = \Omega_i (\Omega_j + \Omega_k) - \Omega_j \Omega_k, \tag{33}$$

where (i,j,k) run over cyclic permutations of (1,2,3) and the derivative (denoted by dot) is with respect to ρ , a real parameter. When we put together the fact that the k = 2 spectral curve corresponds to an elliptic curve with purely imaginary τ and consequently, real valued Eisenstein series $E_{2i}(\tau)$, we conclude that the solution (23) of Ramanujan equations (22) $(t_1(\tau), t_2(\tau), t_3(\tau))$ with purely imaginary t_1 and t_3 will only match $(\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ via fmorphism (28) if,

$$\tau = i\rho \quad \text{and} \quad \Omega_j = iT_j \tag{34}$$

From the discussion in IV B, the space

$$\mathsf{T}_{\mathfrak{Q}} := \{ (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3 | \Omega_1 \neq \Omega_2 \neq \Omega_3 \},$$
(35)

corresponds to a section of purely imaginary coordinates (T_1, T_2, T_3) of the moduli space $T_{\rm H}$ of the enhanced elliptic curve E_T . As shown below, this sectioning corresponds to the reality condition of the spectral curve. In $\tilde{\mathbf{T}}$, the reality condition is given by a real structure ς that acts by reverting the orientation of the corresponding lines in \mathbb{R}^{31} . In terms of $\tilde{\mathbf{T}}$ coordinates, it corresponds to $\varsigma(\eta, \zeta) = (-\overline{\eta}/\overline{\zeta}^2, -1/\overline{\zeta})$. Hence, a real curve in $\tilde{\mathbf{T}}$ is invariant under ς as in (6). For the reparameterized spectral curve (8) this means that the coefficients g_2, g_3 of the defining polynomial are real. Looking at their values in terms of Eisenstein series and using the group G of isomorphisms (24) of the moduli $T_{\rm R}$ with $c = (-2\pi i)^{1/2}$, we see that these real coordinates correspond to

$$g_2(\tau) = (-2\pi i)^2 t_2(\tau), \qquad g_3(\tau) = (-2\pi i)^3 t_3(\tau),$$
(36)

where $t_i(\tau)$ are the solution (23) of the Ramanujan equations (22). By G isomorphism $(0, g_2(\tau), g_3(\tau)) \equiv (t_1(\tau), t_2(\tau), t_3(\tau))$ in T_R , where an additive transformation of G sets the first coordinate to zero. The values of $t_i(\tau)$ in (23), for imaginary τ implies by the algebraic morphism f in IV C that the solution $(T_1(\tau), T_2(\tau), T_3(\tau))$ of the Darboux-Halphen equations (27) is restricted to pure imaginary values when τ is imaginary, as mentioned before. Also by the morphism f in and group isomorphisms in T_R and T_H , T_{Ω} maps to the section $\{(t_1, t_2, t_3) \in T_R | (-2\pi i)^i t_i \in \mathbb{R}\}$ of the moduli T_R of the enhanced elliptic curve E_t (17). This section of T_R satisfies the reality condition of the spectral curve, and we call it the real section of T_R .

Therefore we recognize that T_{Ω} is a six-fold cover of the moduli of the enhanced spectral curve, which we define below:

Definition 1. The moduli of an enhanced spectral curve \tilde{S}_k of a k-monopole is the real section of the moduli of the enhanced algebraic curve $(S_k^{\mathbb{C}}, \{[\alpha_i]\})$ where $S_k^{\mathbb{C}}$ is the family of algebraic curves in $\tilde{\mathbf{T}}$ given by (4) without the real structure constraints and $\{[\alpha_i]\}$ is a basis of classes of algebraic de Rham cohomology $H^1_{dR}(S_k^{\mathbb{C}})$ of differential 1-forms on $S_k^{\mathbb{C}}$ with fixed intersection matrix $\Phi_{ij} = \langle \alpha_i, \alpha_j \rangle$.

Below we summarize our findings in a form of a theorem:

Theorem 1. The moduli of enhanced spectral curves of SU(2) monopoles of charge 2 quotient by SO(3) action and translations in \mathbb{R}^3 corresponds to T_{Ω} , a real section of T_H , quotient by permutations of Ω_1, Ω_2 and Ω_3 . Furthermore, the self-dual curvature equation (31) corresponds to the Ramanujan vector field in T_R upon reparametrization (32,34) and f isomorphism (28).

Proof. This theorem is proved by f isomorphism under restriction $|T_1| < |T_2| < |T_3|$ in T_H. The SO(3) isometry in M_2^0 means that this 4-manifold can be expressed by 3-dimensional SO(3) orbits and a orthogonal trajectory parametrized by ρ . Hence, M_2^0 quotient by SO(3) action is a space of one real dimension parametrized by ρ . Accordingly, the spectral curve S_2 , quotient by SO(3) action and translations, depends on a single parameter, after genus or ES¹² constraint is imposed. When we work with the moduli of enhanced spectral curves, two extra parameters related to Hodge structure of the curve are added, but the vector field equation Gauss-Manin connection in disguise shows that the three parameters of the moduli T_H depend on a single real parameter. □

The moduli of \tilde{S}_2 does not include the point $r_1 = 0$ and $r_2 = \pi^2/4$ of zero discriminant where the curve S_2 degenerates to two S_1 . This point can be mapped to a point of zero discriminant in E_T (24) and it is given by the $\tau = i\infty$ limit in the solution of the system (33), with appropriate lattice scaling to match g_2 in (12,15):

$$\Omega_i(\rho) = \frac{\pi}{r_1} \frac{\partial}{\partial \rho} \left(\log \theta_{i+1}(i\rho) \right), \quad \text{where} \quad \rho = -i\tau \quad \text{and} \tag{37}$$

$$\begin{cases} \theta_2(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \ q = e^{2\pi i \tau}, \ \tau \in \mathbb{H}.$$
(38)

Notice that r_1/r_2 is function of $\tau = i\rho$ and in the limit $\rho \to \infty$ we can write a explicit relation

$$r_2 \to \pi^2/4$$
, and $r_1 \to \pi^2 q^{1/4}$ (39)

Therefore, near the limit $\rho \rightarrow \infty \; (q << 1)$ we have

$$\Omega_1 \approx -\frac{q^{1/4}}{4}, \quad \Omega_2 \approx -2q^{1/4}, \quad \Omega_3 \approx 2q^{1/4}$$
(40)

and the metric of M_2^0 becomes

$$ds^{2} \approx 4q^{1/4}d\rho^{2} + q^{3/4}(\sigma_{1})^{2} + \frac{q^{-1/4}}{4}\left((\sigma_{3})^{2} + (\sigma_{2})^{2}\right)$$
(41)

The metric is singular at q = 0, but asymptotically, two of the coefficients of the metric (29) become equal. In this case, the isometry grows to $SO(3) \times SO(2)^{24}$, where SO(2) action corresponds to the axial symmetry of two 1-monopole solution and it corresponds to the S^1 isotropy subgroup of the spectral curve³. In other words, the asymptotic behavior of the metric confirms the behavior of the spectral curve at $r_1 = 0$.

Furthermore, at infinite ρ distance, the $SO(3) \times SO(2)$ orbit is a 2-torus Hopf fibration of the 3-sphere S^3 , which confirms the fact that the manifold M_2^0 is an asymptotically locally Euclidean (ALE) space. This fact together with self-duality equations, characterizes it as a gravitational instanton configuration²⁵. These are elements to take in consideration for finding metrics of $M_k^0, k > 2$.

VI. CONCLUSION

We hope this article pave the way for future contributions in understanding the moduli M_k^0 of SU(2) monopoles in YMH theory. Among the obstacles, there are the growing computational challenge of Gauss-Manin Connection in Disguise for larger k and the need to

understand the homomorphism between vector fields in the enhanced spectral curves and curvature equations in the moduli M_k^0 .

The later obstacle is related to the fact that the universal families of curves can be written using different choices of parametrization, which yield different set of differential equations with different algebraic group of transformations of the moduli (where lattice scaling is one of the operations)⁶. In the well known case of elliptic curves, the different choices of parametrization of the universal families for the enhanced elliptic curves takes place according to the choices of congruence subgroups Γ of the modular group $SL_2(\mathbb{Z})^6$. The moduli parametrization of the enhanced curves are lifted to modular-type functions under algebraic group action in the moduli with distinct weights. We notice that the canonical form (4) of spectral curves of k-monopoles leads to Ramanujan type of parametrization with parameters with distinct scaling weights, while we expect that the curvature equations from M_k^0 leads to Darboux-Halphen type of parametrization with parameters with same (scaling) weight. Therefore, another future step in this projects is to find new modulartype functions attached to S_k curves that will play a role on defining the metric of M_k^0 . Such role will depend on the symmetries of the moduli that may define the behavior of the metric of M_k^0 under algebraic group of tranformations of the moduli T of the enhanced curve. In summary, the results of M_2^0 suggest that the terms of the metric of M_k^0 will be (quasi-)homogeneous polynomials or rational functions of modular-type functions that will correspond to coordinates of the moduli T of enhanced spectral curves \hat{S}_k satisfying a unique set of vector field equations in T^{20} .

ACKNOWLEDGMENTS

During the period of preparation of the manuscript MACT was fully sponsored by CNpQ-Brasil. The author profited by the rich academic environment at IMPA and by many interactions with Hossein Movasati, whose work is the basis of this project. My sincere thanks go to him and my colleagues from the project on DH equations R. Roychowdhury, Y. Nikdelan and J. A. Cruz Morales, with whom my first thoughts on this project were shared.

REFERENCES

- ¹N. J. Hitchin, Communications in Mathematical Physics 83, 579 (1982).
- ²J. A. C. Morales, H. Movasati, Y. Nikdelan, R. Roychowdhury, and M. A. C. Torres, SIGMA 14, 003 (2018), arXiv:1709.09682 [math.DG].
- ³J. Hurtubise, Communications in Mathematical Physics **92**, 195 (1983).
- ⁴M. Alim, H. Movasati, E. Scheidegger, and S.-T. Yau, Communications in Mathematical Physics **344**, 889 (2016).
- ⁵H. Movasati, *Multiple Integrals and Modular Differential Equations*, 28th Brazilian Mathematics Colloquium (Instituto de Matemática Pura e Aplicada, IMPA, 2011).
- ⁶H. Movasati, Ann. Math. Blaise Pascal **19**, 307 (2012).
- ⁷M. F. Atiyah and R. S. Ward, Communications in Mathematical Physics 55, 117 (1977).
- ⁸M. F. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles* (Princeton University Press, 2014).
- ⁹R. S. Ward and R. O. Wells, *Twistor geometry and field theory*, Vol. 4 (Cambridge University Press Cambridge, 1990).
- ¹⁰E. Bogomol'nyi, Sov. J. Nucl. Phys. (Engl. Transl.); (United States) **24:4** (1976).
- ¹¹N. J. Hitchin, Communications in Mathematical Physics 89, 145 (1983).
- ¹²N. Ercolani and A. Sinha, Commun. Math. Phys. **125**, 385 (1989).
- ¹³R. S. Ward, Commun. Math. Phys. **79**, 317 (1981).
- ¹⁴P. M. Sutcliffe, Int. J. Mod. Phys. **A12**, 4663 (1997), arXiv:hep-th/9707009 [hep-th].
- ¹⁵H. Movasati, The Ramanujan Journal **17**, 53 (2008).
- ¹⁶H. Movasati, manuscripta mathematica **139**, 495 (2012).
- ¹⁷P. A. Griffiths, American Journal of Mathematics **90**, 568 (1968).
- ¹⁸P. A. Griffiths, American Journal of Mathematics **90**, 805 (1968).
- $^{19}\mathrm{H.}$ Movasati, arXiv preprint arXiv:1411.1766 (2014).
- ²⁰H. Movasati, Surveys of Modern Mathematics, IP, Boston. Available online at http://w3. impa. br/hossein/myarticles/GMCD-MQCY3. pdf **170** (2015).
- ²¹H. Movasati and Y. Nikdelan, arXiv preprint arXiv:1603.09411 (2016).
- ²²S. Ramanujan, Trans. Cambridge Philos. Soc. **22**, 159 (1916).
- ²³L. Bianchi, General Relativity and Gravitation **33**, 2171 (2001).
- ²⁴Y. Manin and M. Marcolli, arXiv preprint arXiv:1504.04005 (2015).

 $^{25}\mathrm{G.}$ W. Gibbons and C. N. Pope, Comm. Math. Phys. 66, 267 (1979).