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Non-equilibrium fluctuations of interacting particle systems

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Abstract

In the first part of the thesis, we propose an adaptation of Yau's Relative Entropy Method to the problem of proving fluctuations around the hydrodynamic limit for interacting particle systems. The method is applied to a reaction-diffusion type model introduced in [dMFL]. For this model, we establish bounds on the relative entropy between the law of the process and an approximating product measure, in any dimension. In dimension 1, we give a complete proof of the convergence of the fluctuation field to a generalized Ornstein-Uhlenbeck process. The proof makes use of mass transport notation and of concentration inequalities for subgaussian random variables.

In the second part, we establish an invariance principle for a random walk driven by simple exclusion process in one dimension. The walk has a drift to the left (resp. right) when it sits on a particle (resp. hole). The environment starts from equilibrium and is speeded up with respect to the walker. After a suitable rescaling, the random walk converges to a sum of a Brownian motion and a Gaussian process with stationary increments, independent of the Brownian motion. The proof technique approximates additive functionals of the environment process by additive functions of the exclusion process, putting the problem in the framework of [GJ].

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Contents

Chapter 1

Introduction

1.1 The Relative Entropy Method

The present thesis consists of two new results in the area of scaling limits of interacting particle systems, Theorem 2.1.1 and Theorem 3.1.3. The main tool in the proof of the first result is an inequality which is of independent interest, Theorem 2.2.1 . The first result establishes the fluctuations around the hydrodynamic limit for a reaction-diffusion process, starting from a measure which is not invariant for the dynamics. The second result is an invariance principle for a random walk on a random environment.

Both results are limit theorems at the fluctuation level for systems out of equilibrium. In the random walk model, the key step to the proof was an estimate on the relative entropy between the law of the process and the starting (non-invariant) measure, Theorem 3.2.1. We suspected that a similar estimate could also hold for other models. An entropy estimate is the core to a widespread method for proving hydrodynamic limits, Yau's Relative Entropy Method. Since we had better estimates, a natural goal was to adapt this method to the fluctuation setting. The easiest model, from a technical point of view, was the reaction-diffusion model introduced in [dMFL]. In Chapter II, we establish entropy estimates in all dimensions and give a complete proof, in dimension 1, that the density fluctuation field converges.

We begin with an overview of Yau's strategy for proving hydrodynamic limits. The reader can see the original article [Y] or [KL], Chapter 6, for a more detailed exposition.

Consider an interacting particle system on the d-dimensional torus $\mathbb{T}_n^d$ with an infinitesimal generator L_n that acts on functions $f : \mathbb{T}_n^d \to \mathbb{R}$ as

$$L_n f(\eta) := n^2 \sum_x r_x(\eta)[f(\varphi_x(\eta)) - f(\eta)],$$

where the rates r_x are non-negative and φ_x is a local function. For example, in the $1d$ exclusion process $r_x(\eta) = 1$ and $\varphi_x(\eta) = \eta^{x,x+1}$ (exchanges occupations at neighbouring sites); in Glauber dynamics, $\varphi(\eta) = \eta^x$ (flips the value of η_x).

Denote $\eta_t^n := \eta_{tn^2}$. In several models, it is possible to prove that, for an adequate set of test functions $f : [0, 1]^d \to \mathbb{R}$,

$$\lim_{n\to\infty} n^{-d} \sum_{x\in\mathbb{T}_n^d} f\left(\frac{x}{n}\right) \eta_t^n(x) = \int_{[0,1]^d} f(u)\rho(t, u)\, du \tag{1.1.1}$$

in probability, where ρ is the solution of a certain PDE, called the hydrodynamic equation. Let μ_t^n be the law of η_t^n. The first step of Yau's method is to come up with a sequence of "approximating measures", $\{\nu_t^n : t \geq 0\}_{n\in\mathbb{N}}$, and to prove the following bound on the relative entropy $H(\mu_t^n|\nu_t^n)$:

Lemma 1.1.1. *$\partial_t H(\mu_t^n|\nu_t^n) = o(n^d)$. Therefore, if $H(\mu_0^n|\nu_0^n) = o(n^d)$ then $H(\mu_t^n|\nu_t^n) = o(n^d)$ for all $t \geq 0$.*

Usually ν_t^n is chosen to be the product measure associated to the profile $\rho(t,\cdot)$ of the hydrodynamic equation. Thus, events that have small probability under ν_t^n should also have small probability under μ_t^n. The second step is to prove (1.1.1) under the approximating law ν_t^n. When the ν_t^n are product measures, the random variable $n^{-d}\sum_{x\in\mathbb{T}_n^d} f\left(\frac{x}{n}\right)\eta_t^n(x)$ concentrates around its mean, $n^{-d}\sum_{x\in\mathbb{T}_n^d} f\left(\frac{x}{n}\right)\rho\left(t,\frac{x}{n}\right)$. This assertion can be made more precise via large deviation estimates or via concentration of measure inequalities. The point is: the hydrodynamic limit follows from the combination of the estimate $H(\mu_t^n|\nu_t^n) = o(n^d)$ and an estimate on the rate of convergence in (1.1.1) under ν_t^n. An improvement in the entropy estimate does not immediately imply convergence of the fluctuation fields, but it is a step forward in this direction.

For the reaction-diffusion model, we establish a bound on the relative entropy (with respect to adequate product measures) of order 1 in dimension 1, order $\log n$ in dimension 2 and order n^{d-2} in dimensions 3 and higher. These are improvements of the $o(n^d)$ bound required in the Relative Entropy Method.

In the remaining of this introduction we lay down notation, establish some auxiliary results and sketch the proof of Theorem 2.2.1.

Yau's entropy inequality

We start with a general estimate on the entropy production. This is classical (see for example [KL], p. 120 and p. 342), but we include the proof here for completeness and state it in the form we are going to use, making explicit the appearance of the carré du champ operator in the upper bound.

Before we state the inequality we need to introduce its setting. Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain with state space Ω (finite) and infinitesimal generator $L : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ given by

$$Lf(x) := \sum_{y\in\Omega} r(x,y)[f(y) - f(x)],$$

for non-negative rates $\{r(x,y) : x, y \in \Omega, x \neq y\}$. Let Γ the *carré du champ* operator associated to L: for any $f : \Omega \to \mathbb{R}$,

$$\Gamma f(x) := \sum_{\substack{y\in\Omega \\ y\neq x}} r(x,y)[f(y) - f(x)]^2.$$

For $t \geq 0$, let μ_t denote the law of X_t and ν_t be a probability measure on Ω, which we think of as an approximation to μ_t.Let ν be a reference probability measure on Ω. Assume that $\nu_t(x) > 0$ and $\nu(x) > 0$ for all $x \in \Omega$ and all $t \geq 0$. Denote by f_t and ψ_t the densities

$$\begin{aligned} f_t(x) &:= \frac{\mu_t(x)}{\nu_t(x)} \\ \psi_t(x) &:= \frac{\nu_t(x)}{\nu(x)} \end{aligned} \quad , \text{ for all } x \in \Omega \text{ and } t \geq 0.$$

Finally, denote by $H(\mu_t|\nu_t)$ the relative entropy between the measures μ_t and ν_t:

$$H(\mu_t|\nu_t) := \int f_t \log f_t \, d\nu_t.$$

Proposition 1.1.2 (Yau's entropy inequality)**.**

$$\frac{d}{dt}H(\mu_t|\nu_t) \leq -\int \Gamma_t \sqrt{f_t}\, d\nu_t + \int L f_t - \frac{d}{dt}\log \psi_t \, d\mu_t. \tag{1.1.2}$$

Proof. For all $g : \Omega \to \mathbb{R}$, it holds [1]

$$\int \frac{d}{dt}(\psi_t f_t) g \, d\nu = \int \psi_t f_t \, Lg \, d\nu$$

Therefore

$$\begin{aligned} \frac{d}{dt}H(\mu_t|\nu_t) &= \frac{d}{dt}\int \psi_t f_t \log f_t \, d\nu \\ &= \int f_t \, L \log f_t \, d\nu_t + \int \psi_t \, \frac{d}{dt} f_t \, d\nu \\ &= \int f_t \, L \log f_t \, d\nu_t + \int \frac{d}{dt}(\psi_t f_t) - f_t(\frac{d}{dt}\psi_t) \, d\nu \end{aligned}$$

The second integral is equal to $-\int f_t(\frac{d}{dt}\psi_t)\psi_t^{-1} \, d\nu_t = -\int \frac{d}{dt} \log \psi_t \, d\mu_t$. It remains to show

$$\int f_t \, L \log f_t \, d\nu_t \leq -\Gamma\sqrt{f_t} + \int L f_t \, d\nu_t.$$

For that end, we expand $L \log f_t$ and use the inequality $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$. We obtain

$$\int f_t \, L \log f_t \, d\nu_t \leq 2 \int \sum_{y \in \Omega} r(x,y)\sqrt{f_t(x)} \left(\sqrt{f_t(y)} - \sqrt{f_t(x)} \right) d\nu_t(x). \tag{1.1.3}$$

To finish, we use the identity $2\sqrt{a}(\sqrt{b} - \sqrt{a}) = -(\sqrt{b} - \sqrt{a})^2 + (b - a)$. $\square$

The integrand in (1.1.2) is, in the case of the reaction-diffusion model, [2] a polynomial of degree two in the variables $\{\eta_x - \rho : x \in \mathbb{T}_n^d\}$. Thus, we need to estimate expectations of quantities such as $\sum_{x \in \mathbb{T}_n^d} (\eta_x - \rho)(\eta_{x+e_j} - \rho)$ in terms of the carré du champ. Similar estimates are needed in the proofs of the Boltzmann-Gibbs principle and the tightness of the fluctuation fields. We do all the estimates in two steps: first, we replace each η_x by its mean on a box around x and bound the error by the carré du champ, then we estimate the averaged polynomial using the entropy inequality and concentration of measure estimates.

[1] This is another way of writing $\frac{d}{dt}\mathbb{E}_\nu[g(X_t)] = \mathbb{E}_\nu[Lg(X_t)]$.

[2] For other systems, it will be a polynomial in the variables $(\tau_x \omega)(\eta)$ for some local function ω.

Mass transport and flows

We think of telescoping sums as mass transport. The trivial identity

$$\eta_0 - \eta_\ell = \sum_{j=1}^{\ell} \eta_{j-1} - \eta_j$$

describes the movement of a point mass from 0 to ℓ in ℓ steps: at step j, mass 1 goes from $j-1$ to j. A less obvious identity (used in the proof of the Replacement Lemma) is

$$\eta_0 - \frac{\eta_1 + \cdots + \eta_\ell}{\ell} = \sum_{j=0}^{\ell-1} \frac{\ell - j}{\ell}(\eta_j - \eta_{j+1}).$$

Here one spreads a unit mass at 0 uniformly along the interval $\{1, \dots, \ell\}$ by sending mass 1 from 0 to 1, mass $\frac{\ell-1}{\ell}$ from 1 to 2, mass $\frac{\ell-2}{\ell}$ from 2 to 3 and so on. In d dimensions, we have a similar identity. Let $\ell \in \mathbb{N}$ and $\Lambda_\ell := \{y \in \mathbb{Z}^d : 0 \le y \le \ell\}$. In Lemma 1.1.4 below, we find a function $\phi : \Lambda_\ell \to \mathbb{R}$ that satisfies [3]

$$\eta_0 - \frac{1}{\ell^d} \sum_{y \in \Lambda_\ell} \eta_y = \sum_{j=1}^{d} \sum_{0 \le y \le \ell} \phi_y(\eta_y - \eta_{y+e_j})$$

and such that $\sum_y \phi_y^2$ is small.

In the remaining of the introduction, we formalize the notion of mass flow and prove the lemma that we need in the entropy estimate.

Definition 1.1.3. Given two measures μ and ν on the finite set G, we say that $\phi : G \times G \to \mathbb{R}$ is a *flow connecting* μ *and* ν, and write $\phi : \mu \mapsto \nu$, if

(i) $\phi(x, y) = -\phi(y, x)$ for all $x, y \in G$;

(ii) $\sum_{y \in G} \phi(x, y) = \nu(x) - \mu(x)$.

We call *support* of ϕ the set of oriented edges $\{(x, y) \in G \times G : \phi(x, y) \ne 0\}$, and refer to as *cost* or *norm* of ϕ the quantity $\|\phi\|^2 := \sum_{x,y \in G} \phi(x, y)^2$.

Our goal is to construct a flow in a box of $\mathbb{Z}^d$ that connects the point mass to the uniform distribution at small cost.

Theorem 1.1.4 (Flow Lemma)**.** *Let d and ℓ be positive integers. Let Λ be a box of size ℓ in $\mathbb{Z}^d$. To fix the notation, we can take $\Lambda := \{1, \dots, \ell\}^d$.*

Then, there exists a flow $\phi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ that connects the point mass at $(1, \dots, 1)$ to the uniform distribution in Λ and is supported in nearest neighbour edges such that [4]

$$\|\phi\|^2 = \begin{cases} O(\ell) & \text{if } d = 1; \\ O(\log \ell) & \text{if } d = 2; \\ O(1) & \text{if } d \ge 3. \end{cases}$$

In addition, there is a flow that connects the point mass at zero to the uniform distribution in Λ whose cost is of the same order.

[3] As usual, e_j denotes the j-th canonical basis vector: e_j is the vector in $\mathbb{Z}^d$ whose j-th coordinate is 1 and the remaining coordinates are 0.

[4] By $\|\phi\|^2 = O(\ell)$, we mean that $\|\phi\|^2 \le C\ell$ for some constant C that does not depend on ℓ. Similarly for the other two bounds.

Remark 1.1.5. The concept of mass flow on a graph is closely related to that of current flow in electric networks. Indeed, consider an electric network where every edge has resistance 1. If a and z are distinct nodes of that network then a unit current flowing from a to z is also a mass flow connecting the point mass at a to the point mass at z.

In the remaining of the present subsection, we are going to prove Theorem 1.1.4. Our proof is going to be constructive. In one dimension, one can take

$$\phi(k, k+1) := \frac{\ell - k}{\ell} \mathbf{1}\{0 \leq k < \ell\}. \tag{1.1.4}$$

In higher dimensions, we will not give an explicit formula for the flow, but will build it instead by gluing together several copies of (1.1.4).

Consider then $d \geq 2$. We begin by introducing some notation. Let

$$\Lambda_k := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : 1 \leq x_j \leq k \text{ for all } j \leq d\},$$

and denote by U_A the uniform distribution on the finite set A, that is, the measure that assigns mass $|A|^{-1}$ to every point of A. Our goal is to connect U_{Λ_ℓ} to U_{Λ_1}.

Lemma 1.1.6. *Let $k \in \{2, \ldots, \ell\}$. There exists a mass flow ϕ_k with support in the nearest-neighbour edges of Λ_k such that*

1. $\phi_k : U_{\Lambda_k} \mapsto U_{\Lambda_{k-1}}$;
2. $\phi_k \leq d \left(\frac{2}{k}\right)^d$.

Before we prove the lemma, let us use it to prove Theorem 1.1.4. Notice that the mass flow defined by

$$\phi := \sum_{k=2}^{\ell} \phi_k,$$

connects U_{Λ_ℓ} to the point mass at $(1, \ldots, 1)$ (this can be checked directly from Definition 1.1.3).

It remains to estimate the norm of ϕ. Take a nearest-neighbour edge in Λ_ℓ, say $(x, x - e_i)$, where $x \in \Lambda_k \setminus \Lambda_{k-1}$, $i \leq d$ and $k \leq \ell$. Notice that if $j < k$ then $\phi_j(x, x - e_i) = 0$. Therefore

$$|\phi(x, x - e_i)| \leq \sum_{j=k}^{\ell} |\phi_j(x, x - e_i)| \leq \sum_{j=k}^{\ell} \frac{d2^d}{j^d} \leq \frac{d2^d}{d-1} \frac{1}{(k-1)^{d-1}}.$$

(the second inequality used Lemma 1.1.6).

Since there are less than k^{d-1} points in $\Lambda_k \setminus \Lambda_{k-1}$,

$$\|\phi\|^2 \leq c_d \sum_{k=2}^{\ell} k^{d-1} \left(\frac{1}{k^{d-1}}\right)^2,$$

for $c_d = 2^{1+d}/(d-1)$. This expression is of order $\log \ell$ when $d = 2$ and order 1 when $d \geq 3$.

Proof of Lemma 1.1.6: For each $j \in \{0, 1, \ldots, d\}$, let A_j be the set of those $(x_1, \ldots, x_d) \in \Lambda_k$ for which exactly j entries are equal to k. Thus, A_d is the corner $(k, \ldots, k)$; A_{d-1} consists of d line segments of length $k - 1$; A_{d-2} consists of $\binom{d}{2}$ squares of side length $k - 1$, and so on. The A_j are pairwise disjoint, $A_0 = \Lambda_{k-1}$ and $\bigcup_{j=1}^{d} A_j = \Lambda_k \setminus \Lambda_{k-1}$.

For each $j \in \{0, 1, \ldots, d\}$, let $m_j := U_{\Lambda_k}(A_j)$. Our strategy is to build flows $\psi_d, \psi_{d-1}, \ldots, \psi_1$ whose supports are pairwise disjoint and such that

$$\psi_j : (m_d + \cdots + m_{d-j})U_{A_j} \mapsto (m_0 + \cdots + m_{d-j-1})U_{A_{j-1}}$$

and $|\psi_j| \leq 2^d k^{-d}$ for all $j \in \{1, \ldots, d\}$. The lemma is then proved by taking $\phi_k = \psi_d + \cdots + \psi_1$.

It is helpful to think of this construction as evolving in time. First, A_d spreads its mass uniformly along A_{d-1}. Then A_{d-1} spreads its mass (plus the amount it got from A_d) across A_{d-2}. Then A_{d-2} spreads its mass (plus the amount it got from A_{d-1}) uniformly across A_{d-3}, and so on.

Let $x \in A_j$ and $m = (m_0 + \cdots + m_j)|A_j|^{-1}$ its mass at step j. Notice that $m \leq 2^d k^{-d}$. Then x has exactly j coordinates equal to k. It is adjacent to j line segments of A_{j+1}. Using the one-dimensional flux (1.1.4), we can spread mass m/j at x uniformly along each of these segments. Call ψ_j^x the superposition of these j point-to-line flows. Notice that the $\{\psi_j^x : x \in A_j\}$ have disjoint supports and that $\psi_j^x \leq m \leq 2^d k^{-d}$. We can define $\psi_j := \sum_{x \in A_j} \psi_j^x$. □

Corollary 1.1.7. *Let $\ell \in \{1, 2, \ldots, n\}$. Let $p^\ell : \mathbb{Z}_n^d \to [0,1]$ be the uniform distribution in Λ_ℓ,*

$$p^\ell(y) = \ell^{-d} \mathbf{1}\{y \in \Lambda_\ell\}.$$

and define, for $x \in \mathbb{Z}^d$,

$$q^\ell(x) = \sum_{y \in \mathbb{Z}^d} p^\ell(y) p^\ell(x - y).$$

Then there exists a mass flow

$$\psi^\ell : \delta_0 \mapsto q^\ell$$

with support in $\Lambda_{2\ell+1}$ and $\|\psi^\ell\|^2 \leq \|\psi^\ell\|^2$, where $\phi^\ell : \delta_0 \mapsto p^\ell$ is the flow constructed in Theorem 1.1.4.

Proof. One can take, for $x \in \mathbb{Z}^d$ and $j \in \{1, \ldots, d\}$.

$$\psi^\ell_{x, x+e_j} := \sum_{y \in \mathbb{Z}^d} p^\ell(y) \phi^\ell_{x-y, x-y+e_j}.$$

□

1.2 The reaction-diffusion model

The dynamics is a superposition of symmetric exclusion on the discrete torus and a birth-and-death (also called Glauber) dynamics. Its generator acts on functions $f : \mathbb{T}_n \to \mathbb{R}$ as

$$L_n f := n^2 \sum_{x \in \mathbb{T}_n} [f(\eta^{x,x+1}) - f(\eta)] + \sum_{x \in \mathbb{T}_n} c_x(\eta)[f(\eta^x) - f(\eta)],$$

where $\{c_x(\eta) : x \in \mathbb{T}_n\}$ is a family of translation invariant local functions (that is, $f(x, \eta) = f(0, \tau_x \eta)$ for all $x \in \mathbb{T}_n$ and $\eta \in \{0,1\}^{\mathbb{T}_n}$).

The model was introduced in [dMFL]. In this article, the authors proved that the hydrodynamic equation is a heat equation with a forcing term, given by $F(\rho) := \int c_x(\eta)\, d\nu_\rho(\eta)$.

$$\begin{cases} \partial_t \rho(t,u) &= \partial_{uu}\rho(t,u) + F(\rho(t,u)) & \text{for all } t \in [0,T], u \in \mathbb{T} \\ \rho(0,u) &= \rho_0(u) & \text{for all } u \in \mathbb{T}. \end{cases}$$

In the same article, the authors prove convergence of the density fluctuation field under the stationary measure.

We chose this model as a first test for our approach to non-equilibrium fluctuations. The birth-and-death rates $c_x(\eta)$ are chosen so that the product measure ν_ρ is not invariant for the dynamics. However, for a conveniently chosen value of the density, the product measure turns out to be a good approximation (as measured by the relative entropy) to the evolution of the system.

The study of large deviations is more recent. We point the reader to [FLT] and [LT] for this problem.

1.3 Slowed random walk over symmetric exclusion

Informal description of the model, Law of Large Numbers

In the second part of the thesis, we study a model of random walk in random environment (RWRE) in one dimension. The environment is a simple symmetric exclusion process, and the walker jumps at times given by a Poisson process independent of the environment. If, at the moment of jumping, the walker stands on a particle, it jumps with higher probability to its left neighbour than to its right neighbour (say the jump rates are β to the left and α to the right, $0 < \alpha < \beta$). The rates are reversed if the walker is on a hole at the moment of jumping (the rates are then β to the right and α to the left). We assume that the exclusion process starts at equilibrium and look at the scaling where the environment is spedeed up by n^2 and the walker by n (see (3.1.2) for the infinitesimal generator). The model was introduced in [AFJV] and [AJV], where the authors proved a law of large numbers and a large deviation principle.

Denoting by x_t^n the position of the random walk at time t and by $\rho \in (0,1)$ the initial density of the environment (precise definitions in Section 3.1), [AFJV] proves

$$\lim_{n\to\infty} \frac{x_t^n}{n} = (\beta-\alpha)(1-2\rho) \text{ in probability.} \tag{1.3.1}$$

Their result conforms to intuition: since the environment moves much faster than the random walker, one expects that it has plenty of time to mix between one jump and the next. Therefore the walk should behave, in the limit, as if the environment were refreshed after each jump. The asymptotic speed would then be

$$(\rho\alpha + (1-\rho)\beta) - (\rho\beta + (1-\rho)\alpha)) = (\beta-\alpha)(1-2\rho).$$

In other words: at the level of the law of large numbers, the random walk does not feel the influence of the random environment. It is natural, then, to ask about a central limit theorem. It turns out that at the level of fluctuations the random environment does influence the limiting process. Our result says that the sequence of processes

$$\left\{ \frac{x_t^n - (\beta-\alpha)(1-2\rho)nt}{\sqrt{n}} : t \le T \right\}_{n\in\mathbb{N}}$$

converges to the sum of a Brownian motion and a Gaussian process with stationary increments, independent of the Brownian motion. When $\rho = 1/2$ this Gaussian process is a fractional Brownian motion of Hurst exponent $3/4$. When $\rho \neq 1/2$ we do not have qualitative information about the process, only a formula for the variance of its increments, see (3.1.7).

For random walks in static random environments, several scaling limit results are known, see for example [Z]. The study of dynamical random environments is more recent. Most results require good mixing properties of the environment. We point the reader to [A1] and the references therein for an overview. In the setting of "slow mixing" random environments, no general techniques are known. Ours is one of the several recent works that use a well-studied Markov chain as random environment, thereby allowing the use of model-specific techniques to obtain information about the random walk. See [AT] for a discussion of these issues, for simulations and for conjectures. The work [A2] also uses the exclusion process as random environment. In the results we mentioned, the scaling limit is Brownian motion, what does not happen in our case.

The tools we use in our proof come from the field of hydrodynamic limits. In fact, our theorem can also be viewed as a variation on the problem of the tagged particle. The seminal article on this problem is [KV], where a powerful method for establishing scaling limits of tagged particles was introduced. The method considers the *environment as seen from the particle*, $\xi_t(x) := \eta_t(x+x_t)$ (η_t is the particle system and x_t is the tagged particle) and writes the position of the tagged particle as a martingale plus an integral term of the form $\int_0^t f(\xi_s)\,ds$, for a suitable function f (this is called an *additive functional*). The martingale part can be handled by the Martingale Functional Central Limit Theorem (MFCLT), see Theorem 3.1.5. The problem reduces, therefore, to studying the scaling limit of the additive functional. [KV] gives sufficient conditions to approximate this additive functional by a martingale, thus establishing, when the conditions are met, Brownian motion as the scaling limit of the tagged particle. We point the reader to [KLO] for a comprehensive exposition of the martingale approximation technique. See also [A2] for an application of the technique in the context of RWRE. In our model, the additive functional does not converge to Brownian motion. It is, instead, similar to the functionals studied in [GJ].

Our proof faces two main difficulties. The first is that we don't know the invariant measures of the environment process. To handle this problem, we prove an estimate on the relative entropy between the environment process at time t and the initial (not invariant) measure ν_ρ. This estimate tells us that the Bernoulli product measure, though not invariant, is close enough to invariant for the usual hydrodynamic limit techniques to work. Another challenge is to find the law of the limiting process. Recall that we start by writing the (centered and rescaled) position of the random walk as $M_t^n + A_t^n$, where $M_\cdot^n$ is a martingale that converges to Brownian motion and $A_\cdot^n$ is an additive functional. In the aforementioned tagged particle results, it is possible to approximate $A_\cdot^n$ by a martingale $N_\cdot^n$ and use the MFCLT to show that $M_\cdot^n + N_\cdot^n$ converges to Brownian motion. This does not work in our case. The solution is to approximate, for each $t > 0$, A_t^n by $N_{t,t}^n$, where $\{N_{s,t}^n : s \le t\}$ is a martingale. Moreover, we can show that $N_{\cdot,t}^n$ is (asymptotically) orthogonal to $M_\cdot^n$ and use an argument based on the MFCLT to show that the limiting processes $M_\cdot$ (Brownian motion) and $A_\cdot$ are independent.

Invariance Principle: Sketch of the Proof

First Step: Martingale decomposition

Recall that x_t^n denotes the position of the random walk at time t, when the exclusion process runs at speed n^2 and the walk at speed n. We analyse the process by means of the *environment as seen by the walker*, which is a process $\{\xi_t^n : t \ge 0\}$ with state space $\{0,1\}^{\mathbb{Z}}$. This process is defined by $\xi_t^n(x) := \eta_t^n(x + x_t^n)$ for all $x \in \mathbb{Z}$. Then

$$\frac{x_t^n - (\beta-\alpha)(1-2\rho)nt}{\sqrt{n}} = M_t^n + 2\sqrt{n}(\beta-\alpha)\int_0^t \xi_s^n(0) - \rho\,ds, \tag{1.3.2}$$

where M_t^n is a martingale with predictable quadratic variation given by $\langle M_t^n \rangle = 2(\beta-\alpha)t$.

To see why this is reasonable, notice that, at time s, the walker may jump to the right at rate $n\beta(1-\eta_s^n(x_t^n)) + n\alpha\eta_s^n(x_s^n) = n\beta - n(\beta-\alpha)\xi_s^n(0)$. Therefore, the moments where the walk jumps to the right form a Poisson process in $\mathbb{R}_+$ with (random) intensity $n(\beta-(\beta-\alpha)\xi_s^n(0))\,ds$. The number of jumps to the right up to time t can thus be written as a martingale $M_t^{n,+}$ plus $n\int_0^t\{\beta-(\beta-\alpha)\xi_s^n(0))\,ds$. Besides, $\langle M_t^{n,+}\rangle$ is also equal to $n\int_0^t\{\beta-(\beta-\alpha)\xi_s^n(0))\,ds$, because the jumps have size 1. An analogous statement holds for the process that counts the number of jumps to the left. After subtracting, centering and scaling we arrive at (1.3.2).

For a rigorous proof, one can write down the Dynkin martingales for the $\mathbb{N}\times\{0,1\}^{\mathbb{N}}$-valued Markov chains $(N^{n,+},\xi^n)$ and $(N^{n,-},\xi^n)$, where $N_t^{n,+}$ counts how many times the walker jumped to the right up to time t. This martingale decomposition is standard in the interacting particle systems literature, in the context of *tagged particle* problems. The interested reader can consult [L], Proposition 4.1.

Second Step: Replacement Lemma

Convergence of the (predictable) quadratic variations to a linear function is enough to ensure that the sequence M^n in (1.3.2) converges to Brownian motion. Therefore, we only need to deal with the additive functional. The main step in the proof is the so-called *replacement lemma*, where we estimate the error in the approximation of $\xi_s^n(0)$ by $(\varepsilon n)^{-1}(\xi_s^n(1)+\cdots+\xi_s^n(\varepsilon n))$.

In Section 3.3), we prove that the sequence of additive functionals is tight and that its limit points are continuous trajectories.

In the next display we summarize the steps in the characterization of the limit points of $\int_0^t\sqrt{n}(\xi_s^n(0)-\rho)\,ds$. We use the symbol "$\approx$" to mean that both sides have the same distributional limit as first $n\to\infty$ then $\varepsilon\to 0$. The first step is a consequence of the Replacement Lemma, and Y^n denotes the density fluctuation field of the simple symmetric exclusion process, see (3.1.4).

$$
\begin{aligned}
\int_0^t\sqrt{n}(\xi_t^n(0)-\rho)\,ds &\approx \sqrt{n}\int_0^t\frac{(\xi_s^n(1)-\rho)+\cdots+(\xi_s^n(\varepsilon n)-\rho)}{\varepsilon n}\,ds\\
&= \sqrt{n}\int_0^t\frac{(\eta_s^n(x_s^n+1)-\rho)+\cdots(\eta_s^n(x_s^n+\varepsilon n)-\rho)}{\varepsilon n}\,ds\\
&\approx \int_0^t Y_s^n\left(\varepsilon^{-1}1\left[\frac{x_s^n}{n},\frac{x_s^n}{n}+\varepsilon\right]\right)\,ds.
\end{aligned}
$$

By the Law of Large Numbers (1.3.1), the last integral has the same limit as

$$
\int_0^t Y_s^n\left(\varepsilon^{-1}1\left[v(\rho),v(\rho)+\varepsilon\right]\right)\,ds,
$$

where $v(\rho):=(\beta-\alpha)(2\rho-1)$ is the asymptotic speed of the random walk.

We are thus in the framework of [GJ], where the authors prove that the additive functional above converges (as first $n\to\infty$ then $\varepsilon\to 0$) to a Gaussian process with stationary increments, which, in the case where $\rho=1/2$, is a fractional Brownian motion of exponent $3/4$.

Third Step: Independence

We used different arguments for the convergence of each term in (1.3.2) and it is not obvious what the joint law of their limits should be. We prove that they are independent, based on the following fact: martingales that do not jump together are orthogonal and their joint limit has independent marginals. The martingale M_t^n in (1.3.2) jumps only when the random walk does, whereas the (limit of) the additive functional depends only on the underlying exclusion process. Therefore, we need to approximate the additive functional by a martingale orthogonal to M^n. We know, however, that the limit of of the additive functional is not Brownian, therefore it is not possible to approximate it by a martingale.

The trick to overcome this difficulty is to use a different approximation for each t. That is, we build a sequence $\{N_{s,t}^n : s,t\in[0,T]\}_{n\in\mathbb{N}}$ of stochastic processes such that, for each $t\in[0,T]$, the processes $N^n_{t,t}$ and $\sqrt{n}\int_0^t(\xi^n_s(0)-\rho)\,ds$ are close and such that $\{N_{s,t}: s \le t\}$ is a martingale orthogonal to M^n.

To find a good candidate for $N_{s,t}$, we write down Dynkin martingales for the fluctuation process. For smooth test functions $H:[0,T]\times\mathbb{R}\to\mathbb{R}$, the process $\{N^n_t(H): t\in[0,T]\}$, defined by [5]

$$N^n_t(H) := Y^n_t(H_t) - Y^n_0(H_0) - \int_0^t n^2 L^{ex}_s(Y^n_s(H_s))\,ds \tag{1.3.3}$$

is a martingale. In addition, $n^2L^{ex}_s(Y^n_s(H_s)) \approx Y^n_s((\partial_s+\Delta)H_s)$. For a fixed $t\in[0,T]$, we take as test function the solution H^t of

$$\begin{cases} (\partial_s+\Delta)H^t(s,u) &=& \varepsilon^{-1}1\,[v(\rho)s, v(\rho)s+\varepsilon] & \text{for all } s\in[0,t], u\in\mathbb{R} \\ H^t(t,u) &=& 0 & \text{for all } u\in\mathbb{R}. \end{cases}$$

Using H^t as test function in (1.3.3), we get that the process $\{N^n_{s,t}: s\in[0,t]\}$, defined as

$$\left\{N^n_{s,t} := Y^n_s(H^t_s) - Y^n_0(H^t_0) - \int_0^s n^2 L^{ex}_r(Y^n_r(H^t_r))\,dr : s\in[0,t]\right\}$$

is a martingale. Notice that it jumps only when the exclusion process jumps, and that

$$N^n_{t,t} \approx -Y^n_0(H^t_0) - \int_0^t Y^n_r\left(\varepsilon^{-1}1\,[v(\rho)r, v(\rho)r+\varepsilon]\right)\,dr.$$

As we have seen in the Second Step, the Replacement Lemma (Theorem 3.2.3) and the Law of Large Numbers 1.3.1 imply that the second term of this sum is a good approximation to $\sqrt{n}\int_0^t(\xi^n_r(0)-\rho)\,dr$.

[5] We denote by H_t the function $u\mapsto H(t,u)$.

Chapter 2

Non-equilibrium fluctuations for a one-dimensional reaction-diffusion model

2.1 Presentation of the model, statement of the fluctuations theorem and sketch of the proof

Fix $T > 0$ and $n \in \mathbb{N}$. Let $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$. We name *reaction-diffusion process* the Markov Process $\{\eta_t^n : t \in [0,T]\}$ with state space $\{0,1\}^{\mathbb{T}_n}$ and infinitesimal generator L_n, which we write down below. The generator acts on functions $f : \mathbb{T}_n \to \mathbb{R}$ as

$$\begin{aligned} L_n f(\eta) &:= n^2 \sum_{x\in\mathbb{T}_n} [f(\eta^{x,x+1}) - f(\eta)] + \sum_{x\in\mathbb{T}_n} [c_x^+(\eta)(1-\eta_x) + c_x^-(\eta)\eta_x] \cdot [f(\eta^x) - f(\eta)] \\ &:= n^2 L^{ex} f(\eta) + L^r f(\eta). \end{aligned} \tag{2.1.1}$$

We are going to work with the rates $c_x^-(\eta) = 1$ and $c_x^+(\eta) = 1 + b\eta_{x-1}\eta_{x+1}$. This is one of the simplest choices for which Bernoulli product measures are not invariant for the dynamics.

The hydrodynamic limit was studied in [dMFL]. If the initial distribution η_0^n is associated [1] to a smooth profile $u_0 : \mathbb{R} \to [0,1]$, then the empirical measure [2] converges to the solution of

$$\begin{cases} \partial_t u(t,y) &= \partial_y^2 u(t,y) + F(u(t,y)), \\ u(0,y) &= u_0(y). \end{cases}$$

In the above equation, $F(m) = \mathbb{E}_{\nu_m}[c_0^+(\eta)(1-\eta_0) + c_0^-(\eta)\eta_0]$. For our choices of c_x^+ and c_x^-, we have $F(m) = (1+bm^2)(1-m) - m$. Notice that there is some $\rho \in (0,1)$ for which $F(\rho) = 0$. We take ν_ρ as the starting measure. Notice that none of the product measures $\{\nu_m : m \in [0,1]\}$ is invariant. [3] We expect the product measure associated to the hydrodynamic equation to be a reasonable approximation to the distribution of the system.

Theorem 2.1.1. *Let the reaction-diffusion process $\{\eta_t^n : t \in [0,T]\}$ with generator given by (2.1.1) start from the product measure ν_ρ, where ρ satisfies $\mathbb{E}_{\nu_\rho}[c_0^+(\eta)(1-\eta_0) + c_0^-(\eta)\eta_0] = 0$.*

[1] Meaning that the random variables $\{\eta_0^n(x) : x \in \mathbb{T}_n\}$ are independent and $\mathbb{P}(\eta_0^n(x) = 1) = u_0(\frac{x}{n})$.

[2] The empirical measure is the random measure in $\mathbb{T}_n$ induced by the process η^n, regarded as a particle system in $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ where each particle has mass $\frac{1}{n}$. Formally, we set $\pi_t^n := \frac{1}{n}\sum_{x\in\mathbb{T}_n} \eta_t^n(x)\delta_{\frac{x}{n}}$.

[3] One can verify, for instance, that $\int L_n(\eta_0\eta_1)\, d\nu_m \neq 0$ for all m. Notice, however, that $F(\rho) = 0$ implies $\int L_n\eta_0\, d\nu_\rho = 0$.

Define the density fluctuation field *as the process* $\{X_t^n : t \in [0,T]\}$ *that acts on smooth functions* $f : \mathbb{T} \to \mathbb{R}$ *as*

$$X_t^n(f) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{T}_n} f\left(\frac{x}{n}\right)(\eta_t^n(x) - \rho).$$

Then, as $n \to \infty$*, the sequence* X^n *converges, with respect to the* J_1*-Skorohod topology of* $D_{[0,T]}\mathscr{H}_{-2}$*, to the weak solution of the infinite dimensional Ornstein-Uhlenbeck equation*

$$dX_t = (\Delta - (2 + b\rho(2-\rho)))X_t\, dt + \nabla d\mathscr{M}_t.$$

That is: for every smooth $f : \mathbb{T} \to \mathbb{R}$*, the processes*

$$M_t(f) := X_t(f) - X_0(f) - \int_0^t X_s(\Delta f - (2 + b\rho(2-\rho))f)\, ds \tag{2.1.2}$$

and

$$N_t(f) := M_t(f)^2 - 2t\chi(\rho)\, \|\nabla f\|_{L^2(\mathbb{T})}^2 \tag{2.1.3}$$

are martingales with respect to the filtration $\mathscr{F}_t := \sigma\{X_s(g) : s \le t \text{ and } g \in C^\infty(\mathbb{T})\}$.

There is a general framework for proving convergence theorems such as Theorem 2.1.1, but each model presents its own challenges. Now we lay out this general framework. The remaining chapters deal with the model-specific parts of the proof.

Step 1: Martingale decomposition and convergence of the martingale part

Let $f : \mathbb{T} \to \mathbb{R}$ be a smooth function. Define the process $\{M_t^n(f), t \in [0,T]\}$ by

$$X_t^n(f) = X_0^n(f) + M_t^n(f) + \int_0^t L_n X_s^n(f)\, ds. \tag{2.1.4}$$

then $M^n(f)$ is a martingale. We also have an explicit formula for its quadratic variation, proved in Section 4.1.

Lemma 2.1.2 (Quadratic Variation)**.** *The predictable quadratic variation of* $M^n(f)$ *is given by*

$$\begin{aligned}\langle M_t^n(f)\rangle = &\int_0^t n^2 \sum_{x \in \mathbb{T}_n} \frac{1}{n} \left\{ f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right\}^2 (\eta_x(s) - \eta_{x+1}(s))^2\, ds \\ &+ \int_0^t c_x(\eta(s)) \sum_{x \in \mathbb{T}_n} \frac{1}{n} f\left(\frac{x}{n}\right)^2 ds,\end{aligned}$$

where $c_x(\eta) = \eta_x + (1 - \eta_x)(1 + b\eta_{x-1}\eta_{x+1})$*. Moreover,*

$$\lim_{n \to \infty} \langle M_t^n(f)\rangle = 2t\, \chi(\rho) \|\nabla f\|_{L^2(\mathbb{T})}^2.$$

Therefore, a direct application of the Martingale Functional Central Limit Theorem (a good reference is [W], Theorem 2.1) gives convergence of the sequence $\{M_t^n : t \in [0,T]\}$ with respect to the J_1-Skorohod topology of $D_{[0,T]}\mathbb{R}$ to a Brownian motion of covariance $2\chi(\rho)\|\nabla f\|_{L^2(\mathbb{T})}^2$.

Step 2: Boltzmann-Gibbs Principle

Fix $f : \mathbb{T} \to \mathbb{R}$ smooth. Assume that we have tightness for the sequence $\{X_t^n : t \in [0,T]\}_{n \in \mathbb{N}}$. If the term $L_n X_s^n(f)$ inside the integral in (2.1.4) were a function of X^n, say $X_s^n(Bf)$ for some operator B, then we could pass to the limit and arrive at a martingale characterization of the limit points. What we are going to show is that we can replace $L_n X_t^n(f)$ by a function of X^n, asymptotically.

Proposition 2.1.3. *Let $f : \mathbb{T} \to \mathbb{R}$ be a smooth funtion and $\delta > 0$. Then, for all $t \in [0,T]$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\Big| \int_0^t L_n X_s^n(f) - X_s^n(\Delta f - (2 + b\rho(2-\rho))f)\, ds \Big| > \delta \right) = 0 \qquad (2.1.5)$$

The proof is in Section 2.3. The hardest part is to show that functionals of the form $\int_0^t n^{-1/2} \sum_{x\in\mathbb{T}_n} \overline{\eta}_{x-1}(s)\overline{\eta}_x(s)\overline{\eta}_{x+1}(s)\, ds$ vanish in the limit.

Step 3: Tightness of the additive functional process

In Section 2.4, we prove that, for every smooth test function $f : \mathbb{T} \to \mathbb{R}$, the sequence of additive functionals

$$\left\{ \int_0^t L_n X_s^n(f)\, ds : t \in [0,T] \right\}_{n\in\mathbb{N}}$$

is tight in $C([0,T];\mathbb{R})$. We have already seen that the sequence of martingales $\{M^n(f)\}_{n\in\mathbb{N}}$ converges. An application of Mitoma's Theorem ([M], Theorem 3.1) yields then tightness of the distribution-valued sequence $\{X_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$.

It turns out that, in dimension 1, tightness is a simple consequence of the Bounded Differences Inequality.

Step 4: Putting the proof together

In [HS] it is proven that this martingale problem given by (2.1.2) and (2.1.3) has only one solution. We have to verify that the limit points of the sequence $\{X_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ are solutions to this martingale problem and find the law of X_0.

By the Boltzmann-Gibbs Principle, $M_t^n(f)$ has the same limit as the sequence

$$\tilde{M}_t^n(f) := X_t^n(f) - X_0^n(f) - \int_0^t X_s^n(\Delta f - (2 + b\rho(2-\rho))f)\, ds.$$

As we remarked in Step 2, it follows from the Martingale FCLT that $M^n(f)$ converges to a Brownian motion of variance $2t\chi(\rho)\|\nabla f\|^2_{L^2(\mathbb{T})}$. This verifies that the limit points solve the martingale problem given by (2.1.2) and (2.1.3).

It remains to determine the law of X_0. Since the initial distribution is product, the characteristic function argument in [KL], Corollary 11.2.2 applies. The random field X_0 is a Gaussian field with covariance given by $\mathbb{E}[X_0(f)X_0(g)] = \chi(\rho)\int_{\mathbb{T}} f g\, du$.

2.2 Entropy bound

Theorem 2.2.1. *Let $\{\eta_t^n : t \geq 0\}$ be the reaction-diffusion process in $\mathbb{T}_n^d$. Assume η_0^n is distributed as ν_ρ, where ρ satisfies $\int c_x(\eta)\, d\nu_\rho = 0$. Denote by $H_n(t)$ the relative entropy between (the law of) η_t^n and ν_ρ.*

Then there exists $C > 0$ that does not depend on n such that, for all $t \in [0,T]$,

$$\begin{cases} H_n(t) & \leq & Ct & \text{if } d = 1, \\ H_n(t) & \leq & Ct\log n & \text{if } d = 2, \\ H_n(t) & \leq & Ct\, n^{d-2} & \text{if } d \geq 3. \end{cases}$$

We will not estimate the entropy directly, we will estimate its time derivative instead. The bound asserted on the statement will then follow from the assumption that the entropy at time zero is null and a Gronwall type argument.

Our tool for estimating the time derivative is Yau's Inequality, Proposition 1.1.2. In our setting, it says

$$\frac{d}{dt}H_n(t) \leq -\int \Gamma_n \sqrt{f_t^n}\, d\nu_\rho + \int f_t^n \cdot L_n^* \mathbf{1}\, d\nu_\rho. \tag{2.2.1}$$

In the inequality above, f_t^n denotes the Radon-Nykodym density of (the law of) η_t^n and the approximating measure ν_ρ and Γ_n denotes the carré du champ associated to the generator L_n.

The hard work resides in estimating the last term in (2.2.1). It turns out that the density f_t^n does not play a special role. The function $L_n^*\mathbf{1}$ is a polynomial of degree at least 2 in the variables $\{\overline{\eta}_x := \eta_x - \rho\}_{x\in\mathbb{T}_n^d}$ (see Proposition 4.1.2), and this is all the input we need from the model. The choice of ν_ρ as approximating measure was guided by the goal of killing the linear term in $L_n^*\mathbf{1}$.

We do the estimate in two steps. The first step is to prove several inequalities of the form

$$\int g \cdot \sum_{x\in\mathbb{T}_n^d} \prod_{y\in\Lambda} \overline{\eta}_{x+y}\, d\nu_\rho \leq \int \Gamma_n\left(\sqrt{g}\right)\, d\nu_\rho + (\text{error}),$$

one for each term of $L_n^*\mathbf{1}$. These inequalities hold for general ν_ρ-densities $g : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}_+$ and finite sets $\Lambda \subset \{\mathbb{T}_n^d\}$ with $|\Lambda| \geq 2$. We are going to do the case $\Lambda = \{-e_1, 0, e_1\}$, but the proof carries for other sets. It uses the Flow Lemma to glue together several applications of a simple integration by parts inequality. The second step uses concentration inequalities to control all the error terms that pop out in the first step.

Static Replacement

Given $x, y \in \mathbb{T}_n^d$ and $\eta \in \{0,1\}^{\mathbb{T}_n^d}$, denote by $\eta^{x,y}$ the configuration that exchanges the values of η_x and η_y.

Lemma 2.2.2 (Integration by parts)**.** *Let g and h be functions on the configuration space $\{0,1\}^{\mathbb{T}_n^d}$ and $x, y \in \mathbb{T}_n^d$. Assume h is invariant under the change of variables $\eta \mapsto \eta^{x,y}$. Then, for any positive a, the following inequality holds:*

$$\int g \cdot h(\eta_x - \eta_y)\, d\nu_\rho \leq an^2 \int \left(\sqrt{g\left(\eta^{x,y}\right)} - \sqrt{g(\eta)}\right)^2 d\nu_\rho(\eta) + \frac{1}{an^2}\int h^2 \cdot g\, d\nu_\rho.$$

Proof. Denote $g^{x,y}(\eta) := g(\eta^{x,y})$. Since ν_ρ is invariant under the change of variables $\eta \mapsto \eta^{x,y}$,

$$\int g \cdot h(\eta_x - \eta_y)\, d\nu_\rho = \frac{1}{2}\int h(g - g^{x,y})(\eta_x - \eta_y)\, d\nu_\rho.$$

Now we factor $g - g^{x,y} = (\sqrt{g} - \sqrt{g}^{x,y})(\sqrt{g} + \sqrt{g}^{x,y})$ and apply the elementary inequality $uv \leq 2an^2u^2 + \frac{v^2}{2an^2}$. To finish the proof, we use $(\sqrt{g}^{x,y} + \sqrt{g})^2 \leq 2(g^{x,y} + g)$ and recall that $h^{x,y} = h$ by assumption.

□

Recall the definitions of the boxes Λ_ℓ and measures p^ℓ and q^ℓ from Theorem 1.1.4 and Corollary 1.1.7. Given $x \in \mathbb{T}_n^d$, denote $\overrightarrow{\eta}_x^{\ell} := \sum_{y\in\mathbb{Z}^d} q^\ell(y)\eta_{x+y}$.

Proposition 2.2.3. *Let ψ^ℓ be a mass flow connecting the point mass at 0 to q^ℓ Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\nu_\rho)$. Let $(h_x)_{x\in\mathbb{T}_n^d}$ be a family of local functions. Then, for any $a>0$,*

$$\left\langle f, \sum_{x\in\mathbb{T}_n^d}\left(\eta_x - \overrightarrow{\eta}_x^\ell\right)h_x\right\rangle \le a\int\Gamma_n\left(\sqrt{f}\right)d\nu_\rho + \frac{a}{dn^2}\sum_{j=1}^d\sum_{z\in\mathbb{T}_n^d}\left\langle f, \left(\sum_{y\in\mathbb{T}_n^d}h_{z-y}\psi^\ell_{y,y+e_j}\right)^2\right\rangle$$

under the assumption that the support of h_x does not intersect $\Lambda_{2\ell+1}$.

Proof. We start with the telescoping identity

$$\overline{\eta}_x - \overrightarrow{\eta}_x^\ell = \sum_{j=1}^d\sum_{y\in\mathbb{T}_n^d}(\overline{\eta}_{x+y}-\overline{\eta}_{x+y+e_j})\psi^\ell_{y,y+e_j},$$

which yields

$$\begin{aligned}\left\langle f, \sum_{x\in\mathbb{T}_n^d}\left(\eta_x - \overrightarrow{\eta}_x^\ell\right)h_x\right\rangle &= \sum_{j=1}^d\left\langle f, \sum_{x\in\mathbb{T}_n^d}h_x\sum_{y\in\mathbb{T}_n^d}(\overline{\eta}_{x+y}-\overline{\eta}_{x+y+e_j})\psi_{y,y+e_j}\right\rangle\\ &= \sum_{j=1}^d\left\langle f\cdot\sum_{z\in\mathbb{T}_n^d}(\overline{\eta}_z-\overline{\eta}_{z+e_j})\sum_{y\in\mathbb{T}_n^d}h_{z-y}\psi_{y,y+e_j}\right\rangle,\end{aligned}$$

and finish by applying the Integration by Parts Lemma, 2.2.2.

□

Concentration of measure estimates

In the present section, as previously, $\ell\le n$ is an integer and ψ^ℓ is the mass flow constructed in Corollary 1.1.7. In addition, $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\nu_\rho)$ and $f:\{0,1\}^{\mathbb{T}_n^d}\to\mathbb{R}_+$ denotes a ν_ρ-density. Finally, $H(f):=\int f\log f\,d\nu_\rho$ denotes the relative entropy of $f\,d\nu_\rho$ with respect to ν_ρ.

Given a finite set $A\subset\mathbb{T}_n^d$, define

$$\overline{\eta}_A = \prod_{y\in A}\overline{\eta}_y.$$

Lemma 2.2.4. *Fix an integer ℓ_0 and a finite set $A\subset-\Lambda_{\ell_0}$. Then there exists a positive $\kappa=\kappa(d,A)$ such that*

$$\left\langle f, \alpha\sum_{z\in\mathbb{T}_n^d}\left(\sum_{y\in\mathbb{T}_n^d}\overline{\eta}_{A+z-y}\psi^\ell_{y,y+e_j}\right)^2\right\rangle \le \frac{\ell^d\|\psi^\ell\|^2}{c}\left(H(f)+2\alpha c\frac{n^d}{\ell^d}\right) \tag{2.2.2}$$

whenever the positive numbers c and α satisfy $c\alpha<(2\kappa)^{-1}$.

Proof. For $z\in\mathbb{T}_n^d$, define the random variable

$$(\psi\star\overline{\eta}_A)_z = \sum_{y\in\mathbb{Z}^d}\overline{\eta}_{A+z-y}\psi^\ell_{y,y+e_j}.$$

Let $c>0$. By the entropy inequality, the lefthand side of (2.2.2) is bounded by

$$\frac{\ell^d g_d(\ell)}{c} H(f) + \frac{\ell^d g_d(\ell)}{c} \log \mathbb{E}_{\nu_\rho}\left[\exp\left\{\frac{c\alpha}{\ell^d g_d(\ell)} \sum_{z\in\mathbb{T}_n^d} (\psi \star \overline{\eta}_A)_z^2\right\}\right]. \tag{2.2.3}$$

Notice that $(\psi \star \overline{\eta}_A)_z$ is independent of $(\psi \star \overline{\eta}_A)_w$ whenever $|v_i - w_i| > 2\ell + \ell_0 + 1$ for some $i \in \{1, \dots, d\}$. Besides, there exist a partition $\mathbb{T}_n^d := \cup_{i\in\mathscr{I}} B_i$ and a positive $\kappa = \kappa(d, A)$such that $|\mathscr{I}| \leq \kappa\ell^d$ and the random variables $\{(\overline{\eta}_A \star \psi^\ell)_z : z \in B_i\}$ are independent.

To split the sum over $\mathbb{T}_n^d$ into sums over the $\{B_i : i \in \mathscr{I}\}$, we use Hölder's inequality:

$$\begin{aligned}
&\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{\frac{c\alpha}{\ell^d g_d(\ell)} \sum_{i\in\mathscr{I}} \sum_{x\in B_i} \left(\overline{\eta}_A \star \psi^\ell\right)_x^2\right\}\right] \\
\leq &\frac{1}{\kappa\ell^d} \sum_{i\in\mathscr{I}} \log \mathbb{E}_{\nu_\rho}\left[\exp\left\{\frac{c\alpha\kappa}{g_d(\ell)} \sum_{x\in B_i} \left(\overline{\eta}_A \star \psi^\ell\right)_x^2\right\}\right]
\end{aligned}$$

By independence, the last term is equal to

$$\frac{1}{\kappa\ell^d} \sum_{x\in\mathbb{T}_n^d} \log \mathbb{E}_{\nu_\rho}\left[\exp\left\{\frac{c\alpha\kappa}{g_d(\ell)} \left(\overline{\eta}_A \star \psi^\ell\right)_x^2\right\}\right].$$

By Lemma 4.2.4, if

$$2c\alpha\kappa < 1,$$

then the logarithm in the last term is bounded by $2c\alpha\kappa$. This gives

$$\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{\frac{A}{g_d(\ell)\ell^d} \sum_{x\in\mathbb{T}_n^d} \left(\sum_{y\in\mathbb{T}_n^d} \overline{\eta}_{A+z-y} \psi_{y,y+e_j}^\ell\right)^2\right\}\right] \leq \frac{2A}{\kappa} \frac{n^d}{\ell^d}$$

for $2\kappa A < 1$.Substituting into (2.2.3), we finish the proof.

□

Lemma 2.2.5. *Let $\ell_0, \ell \in \mathbb{N}$ and $A \subset -\Lambda_{\ell_0}$. Then there exists a positive $\kappa' = \kappa'(d, \ell_0)$ such that*

$$\left\langle f, \beta \sum_{x\in\mathbb{T}_n^d} \overline{\eta}_{A+x} \overrightarrow{\eta}_{x+e_1}^\ell \right\rangle \leq \frac{1}{c}\left(H(f) + 2\beta c \left(\frac{n}{\ell}\right)^d\right). \tag{2.2.4}$$

whenever the positive numbers c and β satisfy $\beta c < 1/\kappa'$.

Proof. Denote $\tilde{p}^\ell(y) := p^\ell(-y)$ Denote by $\overline{\eta} \star p^\ell$ and $\overline{\eta} \star \tilde{p}^\ell$ the averages on the boxes Λ_ℓ and $-\Lambda_\ell$, respectively. That is,

$$\begin{aligned}
\left(\overline{\eta} \star p^\ell\right)_x &:= \sum_{y\in\mathbb{T}_n^d} p^\ell(y) \overline{\eta}_{x+y} \\
\left(\overline{\eta} \star \tilde{p}^\ell\right)_x &:= \sum_{y\in\mathbb{T}_n^d} \tilde{p}^\ell(y) \overline{\eta}_{x+y}.
\end{aligned}$$

We need the following indentity:

$$\sum_{x\in\mathbb{T}_n^d} \overline{\eta}_{A+x} \left(\overline{\eta} \star q^\ell\right)_{x+e_1} = \sum_{x\in\mathbb{T}_n^d} \left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x \cdot \left(\overline{\eta} \star p^\ell\right)_{x+e_1}.$$

For any $c > 0$, the entropy inequality (4.4.1) bounds the lefthand side by

$$\frac{1}{c}H(f) + \frac{1}{c}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta \sum_{x\in\mathbb{T}_n^d}\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x \cdot \left(\overline{\eta}\star p^\ell\right)_{x+e_1}\right\}\right]. \tag{2.2.5}$$

There exist a partition $\mathbb{T}_n^d := \cup_{i\in\mathscr{I}} B_i$ and a positive $\kappa' = \kappa'(d, A)$such that $|\mathscr{I}| \leq \kappa'\ell^d$ and for each $i \in \mathscr{I}$ the random variables $\{\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_z \cdot \left(\overline{\eta}\star p^\ell\right)_{z+e_1} : z \in B_i\}$ are independent.

As in the previous Lemma, we use Hölder's inequality to take the sum outside of the logarithm, at the cost of putting $\kappa'\ell^d$ inside the exponent. Expression (2.2.5) is bounded by

$$\frac{1}{c}H(f) + \frac{1}{c\kappa'\ell^d}\sum_{x\in\mathbb{T}_n^d}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta\kappa'\ell^d\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x \cdot \left(\overline{\eta}\star p^\ell\right)_{x+e_1}\right\}\right].$$

Now we estimate the exponential moments of $\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x \cdot \left(\overline{\eta}\star p^\ell\right)_{x+e_1}$ for fixed $x \in \mathbb{T}_n^d$. A computation based on Cauchy-Schwarz inequality gives

$$\begin{aligned}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta\kappa'\ell^d\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x \cdot \left(\overline{\eta}\star p^\ell\right)_{x+e_1}\right\}\right] &\leq \frac{1}{2}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta\kappa'\ell^d\left(\overline{\eta}_A \star \tilde{p}^\ell\right)_x^2\right\}\right] \\ &\quad + \frac{1}{2}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta\kappa'\ell^d\left(\overline{\eta}\star p^\ell\right)_{x+e_1}^2\right\}\right].\end{aligned}$$

To bound the last expression, we use Lemma 4.2.4. Notice that $\sum_{y\in\mathbb{T}_n^d} p^\ell(y)^2 \leq \ell^{-d}$.

$$\frac{1}{2}\log \mathbb{E}_{\nu_\rho}\left[\exp\left\{c\beta\kappa'\ell^d\left(\overline{\eta}\star p^\ell\right)_{x+e_1}^2\right\}\right] \leq c\beta\kappa'$$

whenever $c\beta\kappa' < 1$. An analogous inequality holds for the other term. We have to be a bit careful, though, because the random variables $\{\overline{\eta}_{A+y} : y \in \mathbb{T}_n^d\}$ are not independent. Their dependence is of finite-range, however, so we can use Hölder's inequality to arrive at the same bound, at the cost of increasing κ'. Substituting these inequalities back into (2.2.5) we finish the proof.

□

Proof of Theorem 2.2.1:

We claim that there exists $C > 0$ that does not depend on ℓ nor on n such that [4]

$$\partial_t H_n(t) \leq C\left(1 + \frac{\ell^d}{n^2}\|\phi^\ell\|^2\right)\left(H_n(t) + \left(\frac{n}{\ell}\right)^d\right). \tag{2.2.6}$$

Let us finish the proof assuming the last inequality. We combine the assumption that η_0^n has law ν_ρ at time zero, Gronwall's inequality and inequality (2.2.6) with the appropriate choices of ℓ: by the Flow Lemma 1.1.4, we know that $\|\phi^\ell\|^2 = O(\ell)$ when $d = 1$, so that we can choose ℓ of order n; that $\|\phi^\ell\|^2 = O(\log \ell)$ when $d = 2$, so that we can choose ℓ of order $\frac{n}{\sqrt{\log n}}$; and that $\|\phi^\ell\|^2 = O(1)$ when $d \geq 3$, so that we can choose ℓ of order $n^{2/d}$.

Now it remains to prove (2.2.6). We start with Yau's Inequality 1.1.2: if f_t^n is the Radon-Nykodym density of the law of η_t^n with respect to ν_ρ then

$$\partial_t H_n(t) \leq \int \Gamma_n\left(\sqrt{f_t^n}\right) d\nu_\rho + \langle f_t^n, L_n^*\mathbf{1}\rangle, \tag{2.2.7}$$

[4]The constant C depends on the model though, through the coefficients of $L_n^*\mathbf{1}$ and the number of terms in its expression.

where L_n^* denotes the adjoint of L_n in $L^2(\nu_\rho)$ and Γ_n denotes the carré du champ operator associated to L_n. We chose ρ in such a way that L_n^* is a polynomial in the variables $\{\overline{\eta}_x := \eta_x - \rho : x \in \mathbb{T}_n^d\}$ of order bigger than 1, see Proposition 4.1.2. It is enough to prove that the integral against a ν_ρ-density $f : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}_+$ of each term in the expression for $L_n^* 1$ is bounded by

$$a \int \Gamma_n\left(\sqrt{f}\right) \mathrm{d}\nu_\rho + C(a)\left(1 + \frac{\ell^d}{n^2}\|\phi^\ell\|^2\right)\left(H(f) + \left(\frac{n}{\ell}\right)^d\right), \tag{2.2.8}$$

where $C(a)$ does not depend on n nor on ℓ. We need the freedom in the choice of a so that we can sum the bounds for each term in $L_n^* 1$ to cancel the carré du champ in Yau's Inequality.

From now on we don't need any more input from the model. We can put together the inequalities of the present section to bound $\int f \cdot \sum_{x\in\mathbb{T}_n^d} \overline{\eta}_{x-e_1}\overline{\eta}_x\overline{\eta}_{x+e_1}\, d\nu_\rho$ by the expression in (2.2.8), the proofs for the other terms differing only in notation. Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\nu_\rho)$.

Applying Proposition 2.2.3, we get

$$\left\langle f, \sum_{x\in\mathbb{T}_n^d} \overline{\eta}_{x-e_1}\overline{\eta}_x \left(\overline{\eta}_{x+e_1} - \overrightarrow{\eta}^\ell_{x+e_1}\right)\right\rangle \leq an^2 \mathscr{D}\left(\sqrt{f}\right) + \frac{1}{an^2 d}\left\langle f, W^\ell \right\rangle,$$

where

$$W^\ell(\eta) = \sum_{j=1}^d \sum_{z\in\mathbb{T}_n^d} \left(\sum_{y\in\mathbb{T}_n^d} \overline{\eta}_{z-y}\overline{\eta}_{z-y-e_1}\psi^\ell_{y,y+e_j}\right)^2.$$

Applying Lemma 2.2.4, we find that, for $c > 0$ small enough,

$$\frac{1}{an^2 d}\left\langle f, W^\ell \right\rangle \leq \frac{1}{ac}\frac{\ell^d}{n^2} g_d(\ell)\left(H(f) + \left(\frac{n}{\ell}\right)^d\right).$$

Combining the last inequality with (2.2.4), we get the bound (2.2.7) for each term in the expression for $L_n^* 1$ and thus finish the proof of Theorem 2.2.1.

□

2.3 Boltzmann-Gibbs Principle

The present section is devoted to the proof of Proposition 2.1.3.

Recall formula (4.1.2) for the integrand $L_n X_t^n(f)$. Our goal is to replace each term in the formula by a function of X_t^n. The degree 1 terms, $\overline{\eta}_{x-1}$ and $\overline{\eta}_{x+1}$, can be replaced by $\overline{\eta}_x$, giving rise to the multiple of f that appears in (2.1.5). To see that, notice

$$\left|\sum_x f\left(\frac{x}{n}\right)(\overline{\eta}_x - \overline{\eta}_{x-1})\right| = \left|\sum_x (f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right))\overline{\eta}_x\right| \leq ||f'||_\infty.$$

Since our test functions are three times continuously differentiable, $|\Delta_n f - \Delta f|$ is of order n^{-1}. Therefore, $|X^n(\Delta_n f) - X^n(\Delta f)|$ is of order $n^{-1/2}$.This allows us to replace $X^n(\Delta_n f)$ by $X^n(\Delta f)$.

The difficult step is to replace the terms of degree 2 and 3. In the remaining of this section, we are going to prove that the degree 3 term vanishes in the limit. The same proof works for the degree 2 terms.

For the proof of the Boltzmann-Gibbs principle, we are going to use the log-Sobolev inequality for simple exclusion and the assumption that the paramenter b in the generator is small. This is a restriction of our proof technique, not an essential feature of the model. With a different technique (which we use in Section 3.3) it is possible to prove the Boltzmann-Gibbs principle without resorting to the log-Sobolev inequality and without any further assumptions on the paramenter b.

Lemma 2.3.1.

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \int_0^t \frac{1}{\sqrt{n}} \sum_x f\left(\frac{x}{n}\right) \overline{\eta}_s^n(x-1)\overline{\eta}_s^n(x)\overline{\eta}_s^n(x+1)\, ds \right| > \delta \right) = 0.$$

An analogous statement holds for the terms with $\overline{\eta}_{x-1}\overline{\eta}_{x+1}$ and $\overline{\eta}_{x-1}\overline{\eta}_x$.

Denote

$$V_n(\eta) := \frac{1}{\sqrt{n}} \sum_x f\left(\frac{x}{n}\right) \overline{\eta}_{x-1}\overline{\eta}_x\overline{\eta}_{x+1}.$$

It is enough to prove $\lim_{n\to\infty} \mathbb{P}(\int_0^t V_n(\eta_s)\, ds > \delta) = 0$, and the analogous limit with $-V_n$ playing the role of V_n.

The first step combines Bernstein's trick and Feynman-Kac's Inequality: for any positive $\theta > 0$,

$$\log \mathbb{P}\left(\int_0^t V_n(\eta_s)\, ds > \delta \right) \le -\theta\delta + \sup_g \left\{ \langle \theta V_n, g\rangle + \langle L_n \sqrt{g}, \sqrt{g}\rangle \right\},$$

where the supremum is taken with respect to all probability densities with respect to ν_ρ and $\langle g_1, g_2\rangle$ denotes the inner product in $L^2(\nu_\rho)$.

Recall the entropy production inequality $\langle L_n\sqrt{g}, \sqrt{g}\rangle \le -n^2\mathscr{D}(\sqrt{g}) + \langle g, L_n^* 1\rangle$ (see (1.1.3)). We are using the more traditional notation $\mathscr{D}(g)$ for the expectation of the carré du champ.

In the remaining of the proof, we are going to find a positive C, independent of n, such that the following estimates hold for all $\theta > 0$ and ν_ρ-density g:

$$\begin{aligned} \langle \theta V_n, g\rangle &\le \tfrac{n^2}{2}\mathscr{D}(\sqrt{g}) + C\left(\tfrac{\theta^2}{n} + 1\right) \\ \text{and}\quad \langle g, L_n^* 1\rangle &\le \tfrac{n^2}{2}\mathscr{D}(\sqrt{g}) + C. \end{aligned} \tag{2.3.1}$$

From the inequalities above, $\limsup_{n\to\infty} \log \mathbb{P}(\int_0^t V_n(\eta_s)\, ds > \delta) \le -\theta\delta + 2C$. Since $\theta > 0$ is arbitrary, it follows $\lim_{n\to\infty} \mathbb{P}(\int_0^t V_n(\eta_s)\, ds > \delta) = 0$.

The integrals in (2.3.1) are the same that we needed to estimate in the proof of the entropy bound. There, we established an upper bound involving the Dirichlet form and the entropy $H(f)$. For the Boltzmann-Gibbs principle, however, we cannot use the entropy as an upper bound. To overcome this, we apply log-Sobolev to replace the entropy by a multiple of the Dirichlet form. It is in this step that we need the extra assumption on the parameter b, to ensure that the constant in front of $n^2\mathscr{D}(\sqrt{f})$ is smaller than 1.

Here a technical problem shows up: we cannot apply log-Sobolev directly because it doesn't work for the measure ν_ρ, only for its conditioning on the sets $\Omega_k := \{\eta \in \{0,1\}^{\mathbb{T}_n} : \sum_x \eta_x = k\}$. Therefore, we need to redo all the computations in the proof of the entropy bound, but now taking as reference measure the conditioning of ν_ρ on the sets Ω_k, not ν_ρ itself.

Let $\nu_{n,k}$ be the uniform measure in Ω_k, that is,

$$\nu_{n,k}(\eta) := \frac{\nu_\rho(\eta)}{\nu_\rho(\Omega_k)} \cdot 1\{\eta \in \Omega_k\}.$$

The relationship between the ν_ρ and $\nu_{n,k}$ integrals is given by the following conditioning identity:

$$\int h\, d\nu_\rho = \sum_{k=0}^{n} \nu_\rho(\Omega_k) \int_{\Omega_k} h\, d\nu_{n,k}. \tag{2.3.2}$$

We start with the estimate of $\langle \theta V_n, g\rangle$, that is easier because the polynomial is simpler than the one that appears in $L_n^* 1$ and because the $n^{-\frac{1}{2}}$ in front of the expression helps. The estimate is done in two steps:

Step 1: Recall the definition of the measure q^ℓ from Corollary 1.1.7. Denote

$$V_n^\ell(\eta) := \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_n} f\left(\frac{x}{n}\right) \overline{\eta}_{x-1}\overline{\eta}_x \overrightarrow{\eta}^\ell_{x+1},$$

where

$$\overrightarrow{\eta}^\ell_{x+1} = \sum_{y\in\mathbb{Z}} q^\ell(y)\overline{\eta}_{x+1+y}.$$

Then there is a constant C that depends only on $\|f\|_\infty$ such that

$$\left\langle V_n^\ell, g\right\rangle \le C\frac{\theta\sqrt{n}}{\ell}\left(n^2\mathscr{D}\left(\sqrt{g}\right)+1\right).$$

Step 2: Let $\ell\in\mathbb{N}$, $\ell\le n$. Then, for any $a>0$,

$$\left\langle \theta\left(V_n - V_n^\ell\right), g\right\rangle \le a n^2 \mathscr{D}\left(\sqrt{g}\right) + \frac{2\theta^2}{a}\left(\frac{\ell}{n}\right)^2.$$

Choosing ℓ of order n, for example $\ell = \lfloor \frac{n}{2}\rfloor$, we get the first inequality in (2.3.1) for large enough n.

Proof of Step 1: We split the expectation $\langle\theta\sum_{x\in\mathbb{T}_n} f\left(\frac{x}{n}\right)\overline{\eta}_{x-1}\overline{\eta}_x\overrightarrow{\eta}^\ell_{x+1}, g\rangle$ into the sets Ω_k, according to (2.3.2). Set $Z_k(g) := \int_{\Omega_k} g\, d\nu_{n,k}$, so that $\frac{g}{Z_k(g)}$ is a probability density with respect to $\nu_{n,k}$.

$$\left\langle \theta V_n^\ell, g\right\rangle = \theta\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)\int_{\Omega_k} V_n^\ell \frac{g}{Z_k(g)}\, d\nu_{n,k}.$$

Let $\beta>0$. Applying the entropy inequality, we bound the last expression by

$$\frac{\theta\sqrt{n}}{\beta\ell}\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)\left(H_k\left(\frac{g}{Z_k(g)}\right) + \log\int_{\Omega_k} e^{\frac{\beta\ell}{\sqrt{n}}V_n^\ell}\, d\nu_{n,k}\right), \tag{2.3.3}$$

where H_k denotes relative entropy with respect to $\nu_{n,k}$.

To estimate the first term, we apply the log-Sobolev inequality and (2.3.2), obtaining

$$\begin{aligned}\frac{\theta\sqrt{n}}{\beta\ell}\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)H_k\left(\frac{g}{Z_k(g)}\right) &\le \frac{\theta\sqrt{n}}{\beta\ell}C_{LS}\, n^2\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)\mathscr{D}_k\left(\sqrt{\frac{g}{Z_k(g)}}\right)\\ &= \frac{\theta\sqrt{n}}{\beta\ell}C_{LS}\, n^2\mathscr{D}\left(\sqrt{g}\right).\end{aligned}$$

To estimate the second term in (2.3.3), notice that $\sum_k \nu_\rho(\Omega_k)Z_k(g) = 1$ and use Jensen's inequality to take the logarithm outside the sum, then apply (2.3.2) to recover the expectation with respect to ν_ρ.

$$\frac{\theta\sqrt{n}}{\beta\ell}\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)\log\int_{\Omega_k}e^{\frac{\beta\ell}{\sqrt{n}}V_n^\ell}\,d\nu_{n,k}\le\frac{\theta\sqrt{n}}{\beta\ell}\log\sum_{k=0}^{n}\nu_\rho(\Omega_k)Z_k(g)\int_{\Omega_k}e^{\frac{\beta\ell}{\sqrt{n}}V_n^\ell}\,d\nu_{n,k}$$
$$=\frac{\theta\sqrt{n}}{\beta\ell}\log\int e^{\frac{\beta\ell}{\sqrt{n}}V_n^\ell}\,d\nu_\rho.$$

It is possible to prove that, for small enough β (depending only on $\|f\|_\infty$), the logarithm in the above display is bounded by 1. The heuristic is that each term in the sum that defines V_n^ℓ concentrates like $\frac{1}{\sqrt{n\ell}}(\frac{\overline{\eta}_1+\cdots+\overline{\eta}_\ell}{\sqrt{\ell}})^2$, which, under ν_ρ, concentrates like the square of a Gaussian of variance $n^{-\frac{1}{4}}\ell^{-\frac{1}{2}}$. A rigorous proof can be done making use of Hölder's inequality to get rid of the sum and Hoeffding's inequality to estimate the exponential moment. A similar computation is in the proof of Lemma 2.2.5.

□

Proof of Step 2: Denote $h_x := \overline{\eta}_{x-1}\overline{\eta}_x f\left(\frac{x}{n}\right)$. Let ψ^ℓ be the mass flow from Corollary 1.1.7, that connects δ_0 to q^ℓ. Then

$$\left\langle g,\frac{\theta}{\sqrt{n}}\sum_{x\in\mathbb{T}_n}h_x\left(\overline{\eta}_{x+1}-\overrightarrow{\eta}^{\ell}_{x+1}\right)\right\rangle=\left\langle g,\frac{\theta}{\sqrt{n}}\sum_{x\in\mathbb{T}_n}h_x\sum_{j\in\mathbb{Z}}\psi_{j,j+1}(\overline{\eta}_{x+j}-\overline{\eta}_{x+j+1})\right\rangle$$
$$=\frac{\theta}{\sqrt{n}}\left\langle g,\sum_{y\in\mathbb{T}_n}(\overline{\eta}_{x+j}-\overline{\eta}_{x+j+1})\sum_{j\in\mathbb{Z}}\psi_{j,j+1}h_{y-j}\right\rangle.$$

Using the Cauchy-Schwartz inequality, we bound the last term by

$$an^2\mathscr{D}(\sqrt{g})+\frac{1}{an^2}\cdot\frac{\theta^2}{n}\sum_{y\in\mathbb{T}_n}\left\langle g,\left(\sum_{j\in\mathbb{Z}}\psi_{j,j+1}h_{y-j}\right)^2\right\rangle \tag{2.3.4}$$

Recall that, by construction, $\sum_{j\in\mathbb{Z}}\psi_{j,j+1}\le\ell$. Besides, $|h_y|\le\|f\|_\infty$ for all $y\in\mathbb{T}_n$. Applying Cauchy-Schwarz to the sum above, we arrive at

$$\left\langle\frac{\theta}{\sqrt{n}}\sum_{x\in\mathbb{T}_n}f\left(\frac{x}{n}\right)\overline{\eta}_{x-1}\overline{\eta}_x\left(\overline{\eta}_{x+1}-\overrightarrow{\eta}^{\ell}_{x+1}\right),g\right\rangle\le an^2\mathscr{D}(\sqrt{g})+\frac{\theta^2}{a}\|f\|_\infty^2\left(\frac{\ell}{n}\right)^2.$$

□

It remains to estimate the integral $\langle L_n^*1,g\rangle$. We are going to follow the same steps as in the estimate of $\langle\theta V_n,g\rangle$, but now we have to be more careful with the constant that appears in front of $n^2\mathscr{D}(\sqrt{g})$.

Recall formula (4.1.4) for the adjoint of L_n. Given $\ell\le n$, define

$$W_n^\ell:=\frac{b}{\rho}\sum_{x\in\mathbb{T}_n}\overline{\eta}_{x-1}\overline{\eta}_x\overrightarrow{\eta}^{\ell}_{x+1}+2b\sum_{x\in\mathbb{T}_n}\overline{\eta}_x\overrightarrow{\eta}^{\ell}_{x+1}.$$

For the estimate on $\langle L_n^*1,g\rangle$, we are going to need $\int e^{\gamma W_n^\ell}\,d\nu_\rho<\infty$ for a large (but fixed) γ.

Lemma 2.3.2. *Let, a and γ be positive numbers. Let C_{LS} be the log-Sobolev constant of the simple symmetric exclusion. Then, for sufficiently small γb,*

$$\langle L_n^*1,g\rangle\le\left(2a+\frac{C_{LS}}{an^2}\frac{n\ell}{\gamma}+\frac{C_{LS}}{\gamma}\frac{n}{\ell}\right)n^2\mathscr{D}(\sqrt{g})+\left(\frac{1}{an^2}\frac{n\ell}{\gamma}+1\right).$$

Therefore, choosing $\ell = n$ and assuming b is small enough (depending on C_{LS}), we can choose a such that $\langle L_n^* 1, g\rangle \le \frac{n^2}{2}\mathscr{D}(\sqrt{g}) + O(1)$, as we needed.

Proof. Write $\langle L_n^* 1, g\rangle$ as $\langle L_n^* 1 - W_n^\ell, g\rangle + \langle W_n^\ell, g\rangle$. Repeating the computations that led to (2.3.4) (telescoping and integration by parts), it is possible to prove that, for any $a > 0$,

$$\left\langle g, L_n^* 1 - W_n^\ell \right\rangle \le 2an^2\mathscr{D}(\sqrt{g}) + \frac{1}{an^2}\left\langle g, \tilde{W}_n^\ell \right\rangle,$$

where $\tilde{W}_n^\ell$ is defined by

$$\tilde{W}_n^\ell := \sum_{y\in\mathbb{T}}\left(\frac{b}{\rho}\sum_{j\in\mathbb{Z}}\psi_{j,j+1}\overline{\eta}_{y-j-1}\overline{\eta}_{y-j}\right)^2 + \sum_{y\in\mathbb{T}}\left(2b\sum_{j\in\mathbb{Z}}\psi_{j,j+1}\overline{\eta}_{y-j}\right)^2.$$

Applying the entropy inequality and the log-Sobolev inequality (following the procedure of Step 1) it is possible to prove that, for $\gamma_1 > 0$,

$$\left\langle L_n^* 1 - W_n^\ell, g\right\rangle \le \left(2a + \frac{C_{LS}}{an^2}\frac{n\ell}{\gamma}\right) n^2\mathscr{D}(\sqrt{g}) + \frac{1}{an^2}\frac{n\ell}{\gamma}\log\int e^{\frac{\gamma}{n\ell}\tilde{W}_n^\ell}\, d\nu_\rho.$$

To estimate the exponential moment, we first apply Hölder to get rid of the sums over i and y and then use independence and Lemma 4.2.2. We are not providing the details because the computation is analogous to that of Lemma 2.2.4. It follows that the logarithm above is bounded by 1 provided γb^2 is sufficiently small.

To estimate $\langle g, W_n^\ell\rangle$ we can repeat the computations we did in Step 1 for V_n^ℓ, but using $n/\gamma\ell$ instead of $\sqrt{n}/\ell$ in the entropy inequality. We find that, if $b\gamma$ is small enough,

$$\left\langle g, W_n^\ell\right\rangle \le \frac{1}{\gamma}\frac{n}{\ell}\left(C_{LS}n^2\mathscr{D}(\sqrt{g}) + 1\right).$$

□

2.4 Tightness

The proof uses the Kolmogorov-Centov criterion, see Problem 2.4.11 in [KS].

Proposition 2.4.1. *Assume that the sequence of stochastic processes $\{Y_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ satisfies*

$$\overline{\lim_{n\to\infty}}\, \mathbb{E}[|Y_t^n - Y_s^n|^\lambda] \le C|t-s|^{1+\lambda'}$$

for some positive constants λ, λ' and C and for all $s, t \in [0,T]$. Then it also satisfies

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\,\mathbb{P}\left(\sup_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}}|Y_t^n - Y_s^n| > \varepsilon\right) = 0, \text{ for all } \varepsilon > 0.$$

More precisely, we will prove the following:

Theorem 2.4.2. *For any $\lambda > 1$, there exists a constant $C = C(\lambda, f)$ such that*

$$\mathbb{E}\left[\Big|\int_s^t L_n X_r^n(f)\,dr\Big|^\lambda\right] \le C(t-s)^\lambda.$$

Tightness follows by choosing $\lambda > 1$ and applying Proposition 2.4.1.

Proof. We start by estimating $\nu_\rho(L_n X^n(f) > \delta)$, and for that we use the Bounded Differences Inequality, Proposition 4.2.1. Recall expression (4.1.2) for $L_n X_s^n(f)$. When the occupation at site y is flipped, the expression changes by at most a constant (that depends on $||f||_\infty$ and $||f'''||_\infty$) times $n^{-1/2}$. Call this constant C_f. [5]

Applying the Bounded Differences Inequality, we get

$$\log \nu_\rho(L_n X^n(f) > \delta) \le -\frac{2\delta^2}{C_f^2}.$$

Recall from Section 2.2 that the entropy is of order 1. Plugging the last bound into the entropy inequality (4.4.1) we find $K_f > 0$ that depends only on T and on f such that, for all $t \in [0, T]$,

$$\mu_t^n\left(|L_n X_r^n(f)| > \delta\right) \le \frac{K_f}{\delta^2}.$$

Applying Lemma 3.3.3, we get

$$\mathbb{E}[|L_n X_t^n|^\lambda] \le K_f^{\lambda/2} \text{ for all } t \in [0, T].$$

We finish the proof with an application of Jensen's inequality:

$$\begin{aligned}\mathbb{E}\left[\Big|\int_s^t L_n X_r^n(f)|\,dr\Big|^\lambda\right] &\le (t-s)^\lambda \cdot \frac{1}{t-s}\int_s^t \mathbb{E}[|L_n X_r^n(f)|^\lambda]\,ds\\ &\le K_f^{\lambda/2}\cdot(t-s)^\lambda.\end{aligned}$$

□

[5] We use that $|\eta_x| \le 1$ for all $x \in \mathbb{T}_n$ and that $|X^n(\Delta_n f) - X^n(\Delta f)| \le 2||f'''||_\infty n^{-1/2}$.

Chapter 3

Invariance principle for a slowed random walk over symmetric exclusion

3.1 Notation and Results

Model and Result

Consider a simple symmetric exclusion process in $\mathbb{Z}$, that is, a process $(\eta_t)_{t\geq 0}$ taking values in $\{0,1\}^{\mathbb{Z}}$ with generator L^{ex}, where

$$L^{ex} f(\eta) := \sum_{x\in\mathbb{Z}} [f(\eta^{x,x+1}) - f(\eta)] \tag{3.1.1}$$

for local functions f. In this definition, $\eta^{x,x+1}$ stands for the configuration η after interchanging the values of $\eta(x)$ and $\eta(x+1)$. For any $\rho \in (0,1)$, the measure ν_ρ on $\{0,1\}^{\mathbb{Z}}$, under which the random variables $\{\eta(x) : x \in \mathbb{Z}\}$ are independent and $\nu_\rho(\eta(x) = 1) = \rho$, is invariant for this process.

The SSEP will be our dynamic random environment. Let $0 < \alpha < \beta$. On top of the SSEP we put a random walk $(x_t^n)_{t\geq 0}$ that moves as follows: the walker waits an exponential time of rate $n(\alpha+\beta)$, independent of the environment, and flips a coin. If the coin comes up heads (probability $\frac{\alpha}{\alpha+\beta}$) the walker jumps to the right or to the left with equal probabilities. If the coin comes up tails, the walker looks at the environment: if he sits on a particle, he jumps to the left; otherwise, he jumps to the right.

We can write down the infinitesimal generator of the process $\{(\eta_t^n, x_t^n) : t \in [0,T]\}$ as

$$\begin{aligned} L_n f(\eta, x) := n^2 \sum_{z\in\mathbb{Z}} \big[f(\eta^{z,z+1}, x) - f(\eta, x)\big] + n\big[\beta + (\alpha-\beta)\eta(x)\big]\big[f(\eta, x+1) - f(\eta, x)\big] \\ + n\big[\alpha + (\beta-\alpha)\eta(x)\big]\big[f(\eta, x-1) - f(\eta, x)\big], \end{aligned} \tag{3.1.2}$$

In [AFJV], the authors proved a law of large numbers for the trajectory of the random walk. Here we state the special case that we will need.

Theorem 3.1.1 (Law of Large Numbers)**.** *Fix $T > 0$ and $\rho \in (0,1)$. Consider $x_0^n = 0$ and the process $\{\eta_t^n : t \geq 0\}$ started from ν_ρ. Then the sequence of processes $\{x_t^n : t \in [0,T]\}$ converges in probability (with respect to the J_1-Skorohod topology) to the deterministic process $\{(\beta-\alpha)(1-2\rho)t : t \in [0,T]\}$.*

Remark 3.1.2. Here is an heuristic for the speed $(\beta-\alpha)(1-2\rho)$: since the environment is much faster than the random walk, we expect that in the limit it reaches equilibrium between any two consecutive jumps of the walk, so that, at each step, the walk jumps to the right with probability $\beta(1-\rho)+\alpha\rho$ and to the left with probability $\alpha(1-\rho)+\beta\rho$, independently of the past. Its mean drift is thus $(\beta-\alpha)(1-2\rho)$.

We are now ready to state our result.

Theorem 3.1.3 (Central Limit Theorem). *Fix $T>0$ and $\rho\in(0,1)$. Consider the Markov process $\{(\eta_t^n,x_t^n):t\in[0,T]\}$ whose infinitesimal generator is given by (3.1.2). Assume that x^n starts from 0 and that the exclusion process η^n starts from ν_ρ. Then the sequence*

$$\left\{\frac{x_t^n-(1-2\rho)nt}{\sqrt{n}}:t\in[0,T]\right\}_{n\in\mathbb{N}}$$

converges in distribution with respect to the J_1- Skorohod topology on $D_{[0,T]}\mathbb{R}$ to a continuous stochastic process, which is a sum of a Brownian motion of variance $\alpha+\beta$ and a Gaussian process with stationary increments, independent of the Brownian motion.

Remark 3.1.4. As will be shown in Section 3.4, the variance of the the limiting Gaussian process can be computed explicitly. It turns out that this limit is the same as the limit of the occupation time of the origin for a weakly asymmetric exclusion process, see Theorem 6.4 in [GJ]. When the initial density ρ equals $1/2$, this process is a fractional Brownian motion of Hurst exponent $3/4$.

Our proof follows a classic strategy that started in the context of proving scaling limits for tagged particles in interacting particle systems. The proof starts by considering the *environment process*, defined by $\xi_t^n(x):=\eta_t^n(x+x_t^n)$, $x\in\mathbb{Z}$. That is the environment as seen by the particle. Its dynamics consists of a simple symmetric exclusion process speeded up by n^2 and superposed with random shifts of the whole configuration (occurring at rate $n(\alpha+\beta)$), that account for the jumps of the random walk. It is a Markov process with generator

$$\begin{aligned}\mathscr{L}_n f(\xi) &= n^2\mathscr{L}^{ex}f+n[\beta+(\alpha-\beta)\xi(0)]\big[f(\tau_1\xi)-f(\xi)\big]\\ &\quad +n[\alpha+(\beta-\alpha)\xi(0)]\big[f(\tau_{-1}\xi)-f(\xi)\big]\\ &=: n^2L^{ex}f(\xi)+nL^{rw}f(\xi)\end{aligned}\tag{3.1.3}$$

for any local function $f:\Omega\to\mathbb{R}$, where $\tau_y f(x):=f(x+y)$.

The starting point is to write the position of the random walk as a sum of a martingale and an additive functional of the environment process.[1] We can write

$$\frac{x_t^n-(1-2\rho)nt}{\sqrt{n}}=M_t^n-2\int_0^t\sqrt{n}(\xi_s^n(0)-\rho)\,ds,$$

where $(M_t^n)_{t\geq 0}$ is a martingale with predictable quadratic variation $\langle M_t^n\rangle=(\alpha+\beta)t$.

We can apply the Martingale Functional Central Limit Theorem (MFCLT) to show that, as $n\to\infty$, the sequence of martingales $\{M_t^n:t\in[0,T]\}_{n\in\mathbb{N}}$ converges in distribution with respect to the J_1- Skorohod topology on path space $D([0,T],\mathbb{R})$ to a Brownian motion of variance $\alpha+\beta$.

The next step is to prove that the sequence of processes $\{A_t^n:t\in[0,T]\}_{n\in\mathbb{N}}$ defined by

$$A_t^n:=\int_0^t\sqrt{n}(\xi_s^n(0)-\rho)\,ds$$

converges in distribution to a Gaussian process $\{A_t:t\in[0,T]\}$ with stationary increments.

[1] This is a standard result. A proof (for the tagged particle in the SSEP) can be found in [L].

Finally, we need to show that the limiting processes M^n and A^n are asymptotically independent. In other words, the sequence $\{(M^n_t, A^n_t) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution to a process with independent marginals.

Notation

Stochastic Processes:

- random walk: $\{x^n_t : t \in [0,T]\}$, starts from 0.
- exclusion process: $\{\eta_t : t \in [0,T]\}$, generator $\mathscr{L}^{ex}$ given by (3.1.1) , starts from ν_ρ.
- time scaling: $\eta^n_t := \eta_{tn^2}$.
- environment as seen from the walker: $\xi^n_s(x) := \eta^n_s(x + x^n_s)$, infinitesimal generator $L_n = n^2L^{ex} + nL^{rw}$ given by (3.1.3).
- density fluctuation field: $\mathscr{Y}^n_t := n^{-1/2}\sum_{x\in\mathbb{Z}} f(x/n)(\eta^n_t(x) - \rho)$.

Functions and operators:

- mean in a box: $\xi^\ell(x) := (\xi(x + 1) + \cdots + \xi(x + \lfloor\ell\rfloor))/\lfloor\ell\rfloor$.
- approximations of the identity: $i_\varepsilon := \varepsilon^{-1}\mathbf{1}_{(0,\varepsilon)}$.
- translations: $\tau_x f(u) := f(x + u)$.
- functions: given $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, we denote $u(t, x)$ by $u_t(x)$.
- discrete laplacian: given $f : \mathbb{R} \to \mathbb{R}$ and $u \in \mathbb{R}$, we denote $\Delta_n f(u) := n^2[f(u + n^{-1}) + f(u - n^{-1}) - 2f(u)]$.
- Dirichlet form, $\mathscr{D}(f)$: for the SSEP, (3.2.3); for the reaction-diffusion model, (**??**).

Other:

- Bernoulli product measure: for $\rho \in (0, 1)$, ν_ρ denotes the measure on $\{0,1\}^{\mathbb{Z}}$ under which the random variables $\eta(x)$ are i.i.d. and $\mathbb{P}(\eta(x) = 1) = \rho$.
- configurations: $\xi^{x,y}(z)$ denotes the element of $\{0,1\}^{\mathbb{Z}}$ obtained from ξ by interchanging the values of $\xi(x)$ and $\xi(y)$.
- η^x denotes the element of $\{0,1\}^{\mathbb{Z}}$ obtained from η by changing the value of η_x (that is, $\eta^x_z = (1 - \eta_x)\mathbf{1}_{x=z} + \eta_z\mathbf{1}_{x\neq z}$.
- for $\xi \in \{0,1\}^{\mathbb{Z}}$ and $x \in \mathbb{Z}$, we sometimes denote $\xi(x)$ by ξ_x.
- $\chi(\rho) := \rho(1 - \rho)$ and $\nu(\rho) := (\beta - \alpha)(1 - 2\rho)$.

Background Material

Invariance Principle for Martingales

Our first tool is the following theorem. Proofs and a more general statement can be found in [W] and [EK].

Theorem 3.1.5 (Martingale FCLT). *Let $\{M_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ be a sequence of square-integrable martingales. Assume that*

i) The sequence of the predictable quadratic variation processes $\{\langle M_t^n \rangle : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution to an increasing function $H(t)$;

ii) The size of the largest jump of M^n converges in probability to 0.

Then $\{M_t^n; t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution to a continuous martingale of quadratic variation H.

Moreover, if $\{N_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ is a sequence of square-integrable martingales such that M^n is orthogonal to N^n for all $n \in \mathbb{N}$ (that is, $\langle M^n, N^n \rangle = 0$) and $(N^n)_{n\in\mathbb{N}}$ also satisfies assumptions (i) and (ii), then the limiting martingales are independent.

Equilibrium Fluctuations and Ornstein-Uhlenbeck Processes

We will need two facts about equilibrium fluctuations. The first is that the distribution valued fluctuation field Y^n actually takes values in a metric subspace of $\mathscr{S}'(\mathbb{R})$, namely the Sobolev space $\mathscr{H}_{-2}$. A discussion can be found in [KL], Chapter 11 (for zero-range process) and in [C].

The second fact is the equilibrium fluctuations theorem itself. A proof can be found in [C] (for a generalized exclusion model on the torus) and in [FGN] (for an exclusion process with slow bond in $\mathbb{Z}$).

Theorem 3.1.6 (Equilibrium Fluctuations of SSEP). *Let $T > 0$, $\rho > 0$. Denote $\chi(\rho) := \rho(1-\rho)$. Consider the SSEP starting from ν_ρ. The* density fluctuation field *Y_t^n is a random element of the Sobolev space $\mathscr{H}_{-2}(\mathbb{R})$ that acts on test functions as*

$$Y_t^n(f) := \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} f\left(\frac{x}{n}\right)(\eta_{tn^2} - \rho). \tag{3.1.4}$$

Then the sequence of $\mathscr{H}_{-2}$-valued processes $\{Y_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution, with respect to the J_1-Skorohod topology on $\mathscr{D}_{[0,T]}\mathscr{H}_{-2}$, to the stationary solution $\{Y_t : t \in [0,T]\}$ of the Ornstein-Uhnlenbeck equation

$$dY_t = \Delta Y_t \, dt + \sqrt{\chi(\rho)} \nabla d\mathscr{M}_t. \tag{3.1.5}$$

This means that

1. *$Y_t(f)$ has a Gaussian distribution with mean 0 and variance $\chi(\rho)||\nabla f||^2_{L^2(\mathbb{R})}$, for all $f \in \mathscr{S}(\mathbb{R})$ and $t \in [0,T]$.*

2. *For any smooth function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process*

$$\left\{ Y_t(u_t) - Y_0(u_0) - \int_0^t Y_s((\partial_s + \Delta)u_s) \, ds : t \in [0,T] \right\}$$

is a martingale of quadratic variation $\chi(\rho) \int_0^t ||\nabla u_s||^2_{L^2(\mathbb{R})} \, ds$.

Finally, we will also need to consider the Ornstein-Uhlenbeck equation with drift:

Definition 3.1.7. Let $a > 0$. We say that a $\mathscr{S}'(\mathbb{R})-$valued process $\{\mathscr{Y}_t : t \in [0,T]\}$ is a solution of the equation

$$dY_t = \Delta Y_t\, dt + a(1-2\rho)\nabla Y_t\, dt + \sqrt{\chi(\rho)}\nabla d\mathscr{M}_t \tag{3.1.6}$$

if Condition 1 of Theorem 3.1.6 is satisfied and, for any smooth function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, the real valued process

$$\left\{ Y_t(u_t) - Y_0(u_0) - \int_0^t Y_s((\partial_s + \Delta - a(1-2\rho)\nabla)u_s)\, ds : t \in [0,T] \right\}$$

is a martingale of quadratic variation $\chi(\rho) \int_0^t ||\nabla u_s||^2\, ds$.

Scaling Limits of Additive Funtionals

During the proof we will need to work with a family of mollifiers. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative function which vanishes outside $(0,1)$ and has integral 1. For $\varepsilon > 0$, denote $\varphi_\varepsilon(u) := \varepsilon^{-1}\varphi(u/\varepsilon)$.

We will need the following result from [GJ]:

Theorem 3.1.8. *Let $\{\tilde{Y}_t; t \in [0,T]\}$ be the stationary solution of (3.1.6). For $\varepsilon \in (0,1)$ and $t \in [0,T]$, define $\mathscr{Z}_t^\varepsilon$ as*

$$\mathscr{Z}_t^\varepsilon := \int_0^t \tilde{Y}_s(\varphi_\varepsilon) ds.$$

Then, as $\varepsilon \to 0$, the sequence of processes $\{\mathscr{Z}_t^\varepsilon; t \in [0,T]\}_{\varepsilon > 0}$ converges in distribution, with respect to the uniform topology of $C_{[0,T]}\mathbb{R}$, to a Gaussian process $\{\mathscr{Z}_t; t \in [0,T]\}$ of stationary increments, such that

$$\mathbb{E}\big[\mathscr{Z}_t^2\big] = \chi(\rho)\sqrt{\frac{2}{\pi}} \int_0^t \frac{(t-s)e^{-(a(1-2\rho))^2 s/2}}{\sqrt{s}} ds. \tag{3.1.7}$$

3.2 Replacement Lemma and Entropy Bound

Recall that the environment process starts from the Bernoulli product measure ν_ρ. Making an abuse of notation, we denote

$$H_n(t) := H(\xi_t^n | \nu_\rho),$$

where ξ_t^n above denotes the probability measure in $\{0,1\}^{\mathbb{Z}}$ induced by the random configuration ξ_t^n. The notation $H(\mu|\nu)$ stands for the relative entropy between the measures μ and ν, also known in the literature as the Kullback-Leibler divergence.

Our main task in this section is to show

Theorem 3.2.1. *There is a constant $C = C(\alpha, \beta, \rho)$ such that, for every $n \in \mathbb{N}$*

$$H_n(t) \leq Ct. \tag{3.2.1}$$

The proof is divided in two steps.
Step 1:

$$H_n(t) \leq (\beta - \alpha)\mathbb{E}\left[n \int_0^t \xi_s^n(1) - \xi_s^n(-1)\, ds\right].$$

Proof. Denote by $f_t : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}_+$ the Radon-Nykodym derivative of ξ_t^n with respect to ν_ρ .
Using Theorem $A1.9.2$ in [KL] we get (all inner products are $L^2(\nu_\rho)$ inner products)

$$H_n'(t) \leq 2\langle \sqrt{f_t}, \mathscr{L}_n \sqrt{f_t}\rangle. \tag{3.2.2}$$

Now we break the generator into its exclusion and random walk parts, $L_n = n^2 L^{ex} + nL^{rw}$. For the exclusion part we can explicitly compute the Dirichlet form:

$$n^2\langle \sqrt{f}, \mathscr{L}^{ex}\sqrt{f}\rangle = -n^2\mathscr{D}^{ex}(f) := -n^2\sum_{x\in\mathbb{Z}}\int (\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)})^2\nu_\rho(d\eta). \tag{3.2.3}$$

In view of (3.2.2), we only need to control $\langle \sqrt{f_t}, \mathscr{L}^{rw}\sqrt{f_t}\rangle$. In the remaining of the proof, we will show that

$$\langle \sqrt{f}, \mathscr{L}^{rw}\sqrt{f}\rangle \leq \frac{\beta-\alpha}{2}\langle f, \xi_1 - \xi_{-1}\rangle \tag{3.2.4}$$

for any density f with respct to ν_ρ. But, before starting the proof of (3.2.4), we show how to use this inequality to finish the proof of Step 1.

Specializing to $f = f_t$, we get

$$\langle \sqrt{f_t}, n\mathscr{L}^{rw}\sqrt{f_t}\rangle \leq n\,\frac{\beta-\alpha}{2}\mathbb{E}\left[\xi_t^n(1) - \xi_t^n(-1)\right].$$

Looking back at (3.2.2) and integrating,

$$H_n(t) \leq -2n^2\int_0^t \mathscr{D}^{ex}(f_s)\, ds + (\beta-\alpha)\mathbb{E}\left[n\int_0^t \xi_s^n(1) - \xi_s^n(-1)\, ds\right],$$

what finishes Step 1.

Now we prove (3.2.4). During this proof, we'll adopt the notation $f_j(\eta) := f(\tau_j\eta)$ for the translations of a function f. We start by splitting the generator: $nL^{rw} = \alpha L_n^1 + (\beta-\alpha)L_n^2$ where

$$\begin{aligned} L_n^1 f &:= n(f_1 + f_{-1} - 2f) \\ L_n^2 f &:= n(\xi_0, f_{-1} - f) + n(1-\xi_0, f_1 - f). \end{aligned}$$

The generator L_n^1 captures the part of the dynamics that does not look at the environment.

Notice that, for any ν_ρ-density f,

$$\langle L_n^1\sqrt{f}, \sqrt{f}\rangle \leq 0,$$

so it suffices to show

$$\langle L_n^2\sqrt{f}, \sqrt{f}\rangle \leq \frac{1}{2}\langle \xi_1 - \xi_{-1}, nf\rangle.$$

Using the translation invariance of the measure ν_ρ we get

$$\begin{aligned}\langle L_n^2\sqrt{f},\sqrt{f}\rangle &= \langle n\xi_0\sqrt{f},\sqrt{f_{-1}}-\sqrt{f}\rangle + \langle n(1-\xi_0)\sqrt{f},\sqrt{f_1}-\sqrt{f}\rangle\\ &= -n + \langle n(1-\xi_0+\xi_1),\sqrt{ff_1}\rangle.\end{aligned}$$

Using that $1-\xi_0+\xi_1 \geq 0$ and the inequality $ab \leq (a^2+b^2)/2$, we get

$$\langle L_n^2\sqrt{f},\sqrt{f}\rangle \leq -n + \frac{1}{2}\langle 1-\xi_0+\xi_1, nf\rangle + \frac{1}{2}\langle 1-\xi_0+\xi_1, nf_1\rangle.$$

To finish the proof we only need to use the translation invariance of ν_ρ and make the change of variables $\xi \mapsto \tau_{-1}\xi$ in the second term.

□

Step 2: There exists a positive constant $C = C(\alpha,\beta,\rho)$ such that

$$\mathbb{E}\left[n\int_0^t \xi_s^n(1)-\xi_s^n(-1)\,ds\right] \leq Ct. \tag{3.2.5}$$

Proof. Applying Jensen's inequality we get, for any $A>0$,

$$\mathbb{E}\left[n\int_0^t \xi_s^n(1)-\xi_s^n(-1)\,ds\right] \leq \frac{1}{A}\log\mathbb{E}\left[\exp\left\{An\int_0^t \xi_s^n(1)-\xi_s^n(-1)\,ds\right\}\right].$$

Using Feynman-Kac inequality, we can bound the right hand side by

$$t\cdot\sup_f\left\{\langle nf,\xi_1-\xi_{-1}\rangle + \frac{1}{A}\langle L_n\sqrt{f},\sqrt{f}\rangle\right\} =: t\cdot\sup_f \Gamma_n(f),$$

where the supremum is taken over all $\nu_\rho-$densities f.

Here we can use (3.2.3) and (3.2.4) to get

$$\Gamma_n(f) \leq \left(1+\frac{\beta-\alpha}{2A}\right)\langle nf,\xi_1-\xi_{-1}\rangle - \frac{n^2}{A}\mathscr{D}^{ex}(f). \tag{3.2.6}$$

In the next computation, we use that the measure ν_ρ is invariant with respect to the transformation $\xi\mapsto\xi^{x,y}$, the inequality $ab \leq Ba^2+b^2/4B$ and the notation $f^{x,y}(\xi) := f(\xi^{x,y})$.

$$\begin{aligned}&\langle \xi_1-\xi_{-1}, nf\rangle\\ &= \langle\xi_1-\xi_0,nf\rangle + \langle\xi_0-\xi_{-1},nf\rangle\\ &= \frac{n}{2}\langle\xi_1-\xi_0, f-f^{0,1}\rangle + \frac{n}{2}\langle\xi_0-\xi_{-1},f-f^{-1,0}\rangle\\ &\leq Bn^2\mathscr{D}^{ex}(f) + \frac{1}{16B}\langle(\xi_1-\xi_0)^2,\left(\sqrt{f}+\sqrt{f^{0,1}}\right)^2\rangle\\ &\qquad\qquad + \frac{1}{16B}\langle(\xi_0-\xi_{-1})^2,\left(\sqrt{f}+\sqrt{f^{-1,0}}\right)^2\rangle\\ &\leq Bn^2\mathscr{D}^{ex}(f) + \frac{1}{2B}.\end{aligned} \tag{3.2.7}$$

Choosing $A=1$, $B=(1+(\beta-\alpha)/2)^{-1}$ and substituting into (3.2.6) we conclude the proof, with $C=\frac{1}{2}(1+(\beta-\alpha)/2)^2$. □

Remark 3.2.2. We will need a slight variation of the bound (3.2.5) for the tightness proof in Section 3.3. Namely, we need $\mathbb{E}\left[n\int_s^t \xi_r^n(1) - \xi_r^n(-1)\,dr\right] \le Ct$ for $s < t$. To prove this, write the integral as $n\int_0^t \mathbf{1}_{[s,t]}[\xi_r^n(1) - \xi_r^n(-1)]\,dr$ and proceed as in the proof of (3.2.5), which applies almost word for word.

Now we proceed to the proof of the Replacement Lemma. Given an integer ℓ and a configuration $\xi \in \{0,1\}^{\mathbb{Z}}$,denote by $\xi^\ell(0)$ the density of particles in a box of size ℓ to the right of site 0, that is

$$\xi^\ell(0) := \frac{\xi_1 + \cdots + \xi_\ell}{\ell}.$$

Making an abuse of notation, we use ξ^ℓ to stand for $\xi^{\lfloor \ell \rfloor}$ even when ℓ is not an integer.

Lemma 3.2.3 (Replacement Lemma)**.** *For any $t \in [0,T]$,*

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{n\to\infty} \mathbb{E}\Big|\int_0^t \sqrt{n}(\xi_s^n(0) - \xi_s^{n,\varepsilon n}(0))\,ds\Big| = 0.$$

Proof. During this proof we will denote $\lfloor \varepsilon n \rfloor$ by ℓ. Let $A > 0$. Using Jensen's inequality, we can bound the expectation in the statement by

$$\frac{1}{A}\log \mathbb{E}\left[\exp\Big|\int_0^t A\sqrt{n}(\xi_s^n(0) - \xi_s^{n,\ell}(0))\,ds\Big|\right].$$

To estimate this expectation, we use the well-known trick of applying the inequality $e^{|a|} \le e^a + e^{-a}$ and Feynman-Kac inequality and reduce the proof to the task of showing

$$\inf_{A>0}\varlimsup_{\varepsilon\to 0}\varlimsup_{n\to\infty}\cdot\sup_f\left\{\langle \xi_0 - \frac{\xi_1+\cdots+\xi_\ell}{\ell}, \sqrt{n}f\rangle + \frac{1}{A}\langle L_n\sqrt{f},\sqrt{f}\rangle\right\} = 0, \tag{3.2.8}$$

where the supremum is taken over all densities with respect to ν_ρ.

We begin with an elementary manipulation of the first inner product:

$$\langle \xi_0 - \frac{\xi_1+\cdots+\xi_\ell}{\ell}, \sqrt{n}f\rangle = \frac{1}{\ell}\langle \sqrt{n}f, (\xi_0-\xi_1)+\cdots+(\xi_0-\xi_\ell)\rangle. \tag{3.2.9}$$

Now we repeat the computations in (3.2.7), but choosing a different weight in the Cauchy-Schwarz inequality: for any choice of $B_k > 0$, $k \in \{1,\ldots,\ell\}$,

$$\begin{aligned}\langle \xi_0 - \xi_k, \sqrt{n}f\rangle &= \sum_{j=1}^k \frac{\sqrt{n}}{2}\langle \xi_{j-1}-\xi_j, (\sqrt{f}-\sqrt{f^{j,j-1}})(\sqrt{f^{j,j-1}}+\sqrt{f})\rangle \\ &\le nB_k\mathscr{D}^{ex}(f) + \frac{k}{4B_k}.\end{aligned}$$

Plugging into (3.2.9) we get

$$\langle \xi_0 - \frac{\xi_1+\cdots+\xi_\ell}{\ell}, \sqrt{n}f\rangle \le \frac{n}{\ell}\cdot\mathscr{D}^{ex}(f)\cdot\sum_{k=1}^\ell B_k + \frac{1}{4\ell}\sum_{k=1}^\ell \frac{k}{B_k}.$$

Choosing $B_k = n\sqrt{k}/2A\sqrt{\ell}$ in the last inequality,

$$\langle \xi_0 - \frac{\xi_1+\cdots+\xi_\ell}{\ell}, \sqrt{n}f\rangle \le \frac{n^2}{2A}\cdot\mathscr{D}^{ex}(f) + \frac{A\ell}{2n}. \tag{3.2.10}$$

On the other hand, we can combine (3.2.3), (3.2.4) and (3.2.7) to get, for any $B > 0$,

$$\langle \mathscr{L}_n \sqrt{f}, \sqrt{f} \rangle \leq -n^2 \mathscr{D}^{ex}(f) + \frac{\beta - \alpha}{2} \left(B n^2 \mathscr{D}^{ex}(f) + \frac{1}{2B} \right).$$

Using this inequality together with (3.2.10), we can bound the supremum in (3.2.8) by $\frac{A\ell}{n} + \frac{(\beta-\alpha)^2}{A}$. Optimizing in A, we arrive at the bound $(1 + (\beta - \alpha)^2)\sqrt{\frac{\ell}{n}}$, from which (3.2.8) follows since $\ell = \lfloor \varepsilon n \rfloor$. □

We'll finish this section with a variation on the Replacement Lemma. We can write the mean $\xi_t^{n,\varepsilon n}(0)$ as $n^{-1/2} \sum_{x \in \mathbb{Z}} \xi_t^n(x) i_\varepsilon(x/n)$, where $i_\varepsilon := \varepsilon^{-1} \mathbf{1}_{(0,\varepsilon)}$. For technical reasons, we will need to use smooth versions of i_ε.

Theorem 3.2.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative function which vanishes outside $(0,1)$ and has integral 1. Let $\varphi_\varepsilon(u) := \varepsilon^{-1}\varphi(u/\varepsilon)$. Then*

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{E} \Big| \int_0^t \sqrt{n}(\xi_s^n(0) - \rho) - n^{-1/2} \sum_{x \in \mathbb{Z}} \varphi_\varepsilon(x/n)(\xi_t^n(x) - \rho)\, ds \Big| = 0.$$

Proof. Using the smoothness and compact support of φ, it is possible to show

$$\Big| 1 - n^{-1} \sum_{x \in \mathbb{Z}} \varphi_\varepsilon(x/n) \Big| \leq \frac{C_\varepsilon}{n},$$

with $C_\varepsilon = \varepsilon \cdot \sup_{u \in \mathbb{R}} |\varphi_\varepsilon'(u)|$. Therefore, it is enough to show

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{E} \Big| \int_0^t n^{-1/2} \sum_{x \in \mathbb{Z}} \varphi_\varepsilon(x/n)(\xi_s^n(x) - \xi_s^n(0))\, ds \Big| = 0.$$

The proof is analogous to that of Lemma 3.2.3. □

3.3 Tightness

In this section we prove that the sequence of additive functionals $\{A_t^n : t \in [0,T]\}_{n \in \mathbb{N}}$, with

$$A_t^n := \int_0^t \sqrt{n}(\xi_s^n(0) - \rho)\, ds$$

is tight in $C_{[0,T]}\mathbb{R}$. Since $A_0^n = 0$ for all $n \in \mathbb{N}$, we only need to prove equicontinuity.

The proof uses the Kolmogorov-Centov criterion, see Problem 2.4.11 in [KS].

Proposition 3.3.1. *Assume that the sequence of stochastic processes $\{X_t^n : t \in [0,T]\}_{n \in \mathbb{N}}$ satisfies*

$$\varlimsup_{n \to \infty} \mathbb{E}[|X_t^n - X_s^n|^\lambda] \leq C|t-s|^{1+\lambda'}$$

for some positive constants λ, λ' and C and for all $s, t \in [0,T]$. Then it also satisfies

$$\lim_{\delta \to 0} \varlimsup_{n \to \infty} \mathbb{P} \left(\sup_{\substack{|t-s| \leq \delta \\ s,t \in [0,T]}} |X_t^n - X_s^n| > \varepsilon \right) = 0, \text{ for all } \varepsilon > 0.$$

More precisely, we will prove the following:

Theorem 3.3.2. *For any $\lambda \in (0,2)$, there exists a constant $C = C(\lambda)$ such that*

$$\mathbb{E}\left[\left|\sqrt{n}\int_s^t (\xi_r^n(0)-\rho)\,dr\right|^\lambda\right] \le C|t-s|^{3\lambda/4}$$

holds for every $s,t \in [0,T]$ and for every $n \in \mathbb{N}$. In particular, by choosing $\lambda \in (\frac{4}{3},2)$ and using Proposition 3.3.1, we see that the sequence A^n is tight in $C_{[0,T]}\mathbb{R}$.

We break the estimate into two. estimating $\mathbb{E}[|\int_s^t \sqrt{n}(\xi_r^n(0)-\xi_r^{n,\ell}(0))\,dr|^\lambda]$ for an appropriate ℓ and $\mathbb{E}[|\int_s^t \sqrt{n}(\xi_r^{n,\ell}(0)-\rho)\,dr|^\lambda]$. The second estimate is easier, because, when ℓ is large, $\xi_0^{n,\ell}$ is very close to its mean ρ (recall that at time zero the random variables $\xi^n(x)$ are i.i.d. Bernoulli). We use our estimate on the entropy, Theorem 3.2.1, to compare $\xi_r^{n,\ell}$ with $\xi_0^{n,\ell}$. This is done in Lemma 3.3.4. Of course, this approach does not take the time cancellations into account. They appear when we deal with $\int_s^t \sqrt{n}(\xi_r^{n,\ell}(0)-\xi_r^n(0))\,dr$. As in the proof of the Replacement Lemma and the entropy bound, the main tool is Feynman-Kac's inequality. We don't know how to use it to estimate the moments directly, because it only gives a bound on the exponential moments. The solution is to work with the tail probabilities $P(|\int_s^t \sqrt{n}(\xi_r^{n,\ell}(0)) - \xi_r^n(0))\,dr| > \delta)$ instead of the moments.

We start with an elementary lemma that quantifies the relationship between tail bounds and moment bounds. Its proof is in the Appendix.

$$\int_0^1 f(x)\,\mathrm{d}x$$

Lemma 3.3.3. *Let X be a nonnegative random variable. Assume that $\mathbb{P}(|X| > \delta) \le C/\delta^2$ for any $\delta > 0$. Then, for any $\lambda \in (0,2)$, there exists an universal constant $C(\lambda)$ such that $\mathbb{E}[|X|^\lambda] \le C(\lambda)\cdot C^{\lambda/2}$.*

Proof of Lemma 3.3.3. Fix $\varepsilon > 0$. Then

$$\begin{aligned}\mathbb{E}[X^\lambda] &= \int_0^\infty \lambda\delta^{\lambda-1}\mathbb{P}(X>\delta)\,d\delta \\ &\le \varepsilon^\lambda + \int_\varepsilon^\infty \lambda C\delta^{\lambda-3}\,d\delta \\ &= \varepsilon^\lambda + C\frac{\lambda}{2-\lambda}\varepsilon^{\lambda-2}.\end{aligned}$$

Choosing $\varepsilon = C^{1/2}$ we get $\mathbb{E}[X^\lambda] \le (1+\lambda/(2-\lambda))C^{\lambda/2}$. □

Lemma 3.3.4. *Fix $0 \le s < t \le T$. Then for all $\lambda \in (0,2)$ there exists a positive constant $C(\lambda)$ such that*

$$\mathbb{E}\left[\left|\sqrt{n}\int_s^t (\xi_r^{n,n|t-s|^{1/2}}(0)-\rho)\,dr\right|^\lambda\right] \le C(\lambda)|t-s|^{3\lambda/4}$$

for all $n > |t-s|^{-1/2}$.

Proof. By Jensen's inequality,

$$\mathbb{E}\left[\left|\sqrt{n}\int_s^t (\xi_r^{n,n|t-s|^{1/2}}(0)-\rho)\,dr\right|^\lambda\right] \le (t-s)^{\lambda-1}\int_s^t \mathbb{E}\left[\left|\sqrt{n}(\xi_r^{n,n|t-s|^{1/2}}(0)-\rho)\right|^\lambda\right]\,dr.$$

Therefore, it is enough to show

$$\mathbb{E}\left[\sqrt{n}\left|\xi_r^{n,n|t-s|^{1/2}}(0)-\rho\right|^{\lambda}\right] \le C(\lambda)|t-s|^{-\lambda/4}. \tag{3.3.1}$$

In order to obtain (3.3.1), it suffices to find a universal constant C such that

$$\mathbb{P}\left(\sqrt{n}\left|\xi_0^{n,n|t-s|^{1/2}}(0)-\rho\right|>\delta\right) \le \frac{C}{\delta^2|t-s|^{1/2}}. \tag{3.3.2}$$

We compare the probability at time r with the probability at time 0 using the entropy inequality (4.4.1):

$$\mathbb{P}\left(\sqrt{n}\left(\xi_r^{n,n|t-s|^{1/2}}(0)-\rho\right)>\delta\right) \le \frac{H_n(r)+\log 2}{\log\left(1+\mathbb{P}\left(\sqrt{n}\left(\xi_r^{n,n|t-s|^{1/2}}(0)-\rho\right)>\delta\right)^{-1}\right)}.$$

Using the entropy bound (3.2.1) and the last inequality, we reduce the proof of (3.3.2) to that of

$$\mathbb{P}\left(\sqrt{n}\left(\xi_r^{n,n|t-s|^{1/2}}(0)-\rho\right)>\delta\right) \le \exp\left(-\delta^2|t-s|^{1/2}\right).$$

But since the random variables $\xi_0^n(x)$ are i.i.d. Bernoulli under ν_ρ, this follows from Hoeffding's inequality, see Corollary 4.2.4. □

Remark 3.3.5. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a smooth, nonnegative and compactly supported function. We can reuse the above proof to show that the sequence

$$\left\{\int_0^t n^{-1/2}\sum_{x\in\mathbb{Z}}\varphi(x/n)(\xi_s^n(x)-\rho)\,ds : t\in[0,T]\right\}_{n\in\mathbb{N}}$$

is tight $C_{[0,T]}\mathbb{R}$. The only additional information needed is that $\sum_{x\in\mathbb{Z}}\varphi(x/n)\le Cn$ for some $C>0$. We will need this fact in the next two sections.

Lemma 3.3.6. *Fix $0\le s<t\le T$. Assume $t-s<1$. Then*

$$\mathbb{E}\left[\sqrt{n}\left|\int_s^t \xi_r^{n,n|t-s|^{1/2}}(0)-\xi_s^n(0)\,dr\right|^{\lambda}\right] \le C(\lambda)|t-s|^{3\lambda/4},$$

for all $\lambda\in(0,2)$.

Proof. In view of Lemma 3.3.3, we only need to prove

$$\mathbb{P}\left(\left|\int_s^t \sqrt{n}(\xi_r^{n,n|t-s|^{1/2}}(0)-\xi_r^n(0))\,dr\right|>\delta\right) \le C\frac{(t-s)^{3/2}}{\delta^2} \text{ for some constant } C. \tag{3.3.3}$$

A natural idea would be to mimic the proof of Lemma 3.3.4, using the entropy bound (3.2.1) and the entropy inequality (4.4.1) to reduce the proof of (3.3.3) to that of

$$\mathbb{P}\left(\left|\int_0^{t-s} \sqrt{n}(\xi_r^{n,n|t-s|^{1/2}}(0)-\xi_r^n(0))\,dr\right|>\delta\right) \le \exp\left(\frac{-\delta^2}{C(t-s)^{3/2}}\right).$$

However, if we try to use Feynman-Kac's inequality directly, the bound obtained will not be good enough. What hinders the computation is the term $\langle \xi_1 - \xi_{-1}, nf \rangle$ in the variational problem. A trick to overcome this difficulty is to define

$$\tilde{A}_t(\gamma) := \int_0^t \sqrt{n}(\xi_r^{n,n\sqrt{t}}(0) - \xi_r^n(0))\, dr - \gamma \frac{\beta - \alpha}{2} \int_0^t n(\xi_r^n(1) - \xi_r^n(-1))\, dr,$$

where γ is a parameter to be chosen later. This allows us to write the integral in (3.3.3) as

$$\tilde{A}_t(\gamma) - \tilde{A}_s(\gamma) + \gamma \frac{\beta - \alpha}{2} \int_s^t n(\xi_r^n(1) - \xi_r^n(-1)\, dr.$$

To show (3.3.3), it is enough to show the two easier estimates

$$\mathbb{P}(|\tilde{A}_t(\gamma) - \tilde{A}_s(\gamma)| > \delta) \leq C \frac{(t-s)^{3/2}}{\delta^2} \tag{3.3.4}$$

and

$$\mathbb{P}\left(\left| \gamma \frac{\beta - \alpha}{2} \int_s^t n(\xi_r^n(1) - \xi_r^n(-1))\, dr \right| > \delta \right) \leq C \frac{(t-s)^{3/2}}{\delta^2}. \tag{3.3.5}$$

We start with (3.3.4). Using the entropy bound (3.2.1) and the entropy inequality, we see that it is enough to show

$$\mathbb{P}\left(\tilde{A}_{t-s}(\gamma) > \delta \right) \leq \exp\left(-\delta^2/(t-s)^{3/2} \right) \tag{3.3.6}$$

for an appropriate choice of γ.

During the proof of (3.3.6), we will denote $\lfloor n\sqrt{t-s} \rfloor$ by ℓ. Using Feynman-Kac inequality, we can bound $\mathbb{E}\left[\exp\left(\frac{1}{\gamma} \tilde{A}_t(\gamma) \right) \right]$ by

$$\exp\left(t \cdot \sup_f \{ \langle \xi^\ell - \xi_0, \frac{\sqrt{n}}{\gamma} f \rangle + \frac{\beta - \alpha}{2} \langle \xi_1 - \xi_{-1}, nf \rangle + \langle L_n \sqrt{f}, \sqrt{f} \rangle \} \right),$$

where the supremum is taken over ν_ρ-densities f and $\xi^\ell := (\xi_1 + \cdots + \xi_\ell)/\ell$. Using (3.2.4) and (3.2.10) we can bound the last expression by $\exp((t-s)\ell/4n\gamma^2)$. Using Markov's inequality, we get

$$\mathbb{P}\left(|\tilde{A}_{t-s}(\gamma)| > \delta \right) \leq \exp\left(-\frac{\delta}{\gamma} + \frac{(t-s)\ell}{4n\gamma^2} \right).$$

Choosing $\gamma = (t-s)^{3/2}/\delta$ and using $\ell \leq n\sqrt{t-s}$, we arrive at (3.3.6).

To finish the proof of Lemma 3.3.6, it remains only to show (3.3.5). The additive functional is the same that shows up in the proof of the entropy bound, see (3.2.5). As observed in Remark 3.2.2, that proof also yields a uniform (that is, independent of n) bound for $\mathbb{E}\left[n \int_s^t \xi_r^n(1) - \xi_r^n(-1)\, dr \right]$, and that is all we need to prove (3.3.5), by an application of Markov's inequality.

□

3.4 Limit Points of the Additive Functional

In the previous section we proved that the sequence of additive functionals A^n is tight. In this section we identify its limit points. We rely strongly on the results of [GJ]. There it was proved that, for a general class of interacting particle systems $(\eta_s)_{s\geq 0}$ which includes the SSEP it holds (recall the notation $\eta_s^n := \eta_{sn^2}$)

$$\left(\int_0^t \sqrt{n}(\eta_s^n(0) - \rho)\, ds\right)_{t\geq 0} \longrightarrow (\mathscr{Z}_t)_{t\geq 0} \text{ as } n \to \infty, \tag{3.4.1}$$

where $\mathscr{Z}$ is a fractional Brownian motion of Hurst parameter $3/4$. The class of particle systems for which (3.4.1) holds is large but has a serious constraint: the Bernoulli product measures ν_ρ must be invariant for such processes, what does not hold for our environment process.

The proof of (3.4.1) given in [GJ] is based on a Local Replacement Lemma very similar to our Lemma 3.2.3. It makes it possible to replace $\eta_s^n(0)$ by its average $\eta_s^{n,\varepsilon n}(0)$ and to approximate the additive functional by a function of the density fluctuation field: $\int_0^t \sqrt{n}(\eta_s^{\varepsilon n}(0) - \rho)\, ds \approx \int_0^t Y_s^n(i_\varepsilon)\, ds$. This is combined with their theorem

$$\left(\int_0^t Y_s(i_\varepsilon)\, ds\right)_{t\geq 0} \longrightarrow (\mathscr{Z}_t)_{t\geq 0} \text{ as } \varepsilon \to 0. \tag{3.4.2}$$

In our case the replacement gives $\int_0^t \sqrt{n}(\xi_s^{n,\varepsilon n}(0) - \rho)\, ds \approx \int_0^t Y_s^n(\tau_{x_s^n/n} i_\varepsilon)\, ds$, where Y^n stands for the density fluctuation field of the SSEP. It was proved in [AFJV] that, if we start from ν_ρ, the rescaled random walk x_s^n/n converges to $(\beta-\alpha)(1-2\rho)s =: \nu(\rho)s$, a deterministic trajectory. Therefore, we expect $\int_0^t \sqrt{n}(\xi_s^n(0) - \rho)\, ds$ to behave like $\int_0^t Y_s(\tau_{\nu(\rho)s} i_\varepsilon)\, ds$. When $\rho = 1/2$, $\nu(\rho) = 0$ and we can apply (3.4.2). When $\rho \neq 1/2$, we cannot apply (3.4.2) directly. The trick is to relate the "moving field" $Y_s(\tau_{\nu(\rho)s} i_\varepsilon)$ with the fluctuation field of an *asymmetric* exclusion process. The weakly asymmetric exclusion process was also studied in [GJ]. The limit process in (3.4.2) is not a fractional Brownian motion anymore, but is still a Gaussian process and can be explicitly described. In the remaining of this section we'll implement this plan.

For technical reasons, we will not work with the discontinuous functions i_ε, using the smooth mollifiers φ_ε instead. Those were introduced before Theorem 3.1.8.

Proposition 3.4.1. *Let $\{A_t : t \in [0,T]\}$ be a limit point of the sequence A^n and $\mathscr{Z}^\varepsilon$, $\mathscr{Z}$ be the processes defined in Theorem 3.1.8, with $a = \beta - \alpha$. Then A and $\mathscr{Z}$ have the same finite-dimensional distributions.*

We begin with a lemma that allows us to write $\mathscr{Z}^\varepsilon$ in a more convenient way.

Lemma 3.4.2. *Let Y be the stationary solution of the Ornstein-Uhlenbeck equation (3.1.5) and $\nu(\rho) := (\beta - \alpha)(1 - 2\rho)$. Then the process $\{\tilde{Y}_t : t \in [0,T]\}$ defined by*

$$\tilde{Y}_t(f) := Y_t(\tau_{\nu(\rho)t} f)$$

is a solution of the equation (3.1.6), with $a = (\beta - \alpha)$.

Proof. We want to show that, for any sufficiently smooth $H : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process $\{\tilde{\mathscr{M}}_t(H) : t \in [0,T]\}$ with

$$\tilde{\mathscr{M}}_t(H) := \tilde{Y}_t(H_t) - \tilde{Y}_0(H_0) - \int_0^t \tilde{Y}_s(\partial_s + \nu(\rho)\nabla + \Delta)H_s\, ds$$

is a martingale with quadratic variation $\{\int_0^t \rho(1-\rho)||\nabla H_s||^2\, ds : t \in [0,T]\}$. Using the elementary identity $\partial_s(\tau_{\nu(\rho)s} H_s) = \tau_{\nu(\rho)s}(\partial_s H_s) + \nu(\rho)\tau_{\nu(\rho)s}\nabla H_s$, we obtain

$$\tilde{\mathscr{M}}_t(H) = Y_t(\tau_{\nu(\rho)t} H_t) - Y_0(H_0) - \int_0^t Y_s(\partial_s(\tau_{\nu(\rho)s} H_s) + \Delta(\tau_{\nu(\rho)s} H_s))\, ds.$$

By the definition of Y, this is a martingale with quadratic variation

$$\langle \tilde{\mathscr{M}}_t(H) \rangle = \int_0^t \rho(1-\rho) ||\nabla(\tau_{\nu(\rho)s} H_s)||^2\, ds = \int_0^t \rho(1-\rho)||\nabla H_s||^2\, ds,$$

as we wanted. □

Proof of Proposition 3.4.1. Let Y^n denote the density fluctuation field associated to the SSEP, and Y its limit. Consider the auxiliary processes $\{A_t^{n,\varepsilon} : t \in [0,T]\}$ and $\{\tilde{\mathscr{Z}}_t^\varepsilon : t \in [0,T]\}$ defined by

$$A_t^{n,\varepsilon} := \int_0^t Y_s^n(\tau_{x_s^n/n} \varphi_\varepsilon)\, ds$$

$$\tilde{\mathscr{Z}}_t^\varepsilon := \int_0^t Y_s(\tau_{\nu(\rho)s} \varphi_\varepsilon)\, ds.$$

In the last lemma we saw that $\mathscr{Z}^\varepsilon$ and $\tilde{\mathscr{Z}}^\varepsilon$ have the same law.

We claim that $A^{n,\varepsilon}$ converges to $\tilde{\mathscr{Z}}^\varepsilon$ as $n \to \infty$ in the sense of finite-dimensional distributions. To prove this, consider, for each $t \in [0,T]$, the function $F_t : \mathscr{D}_{[0,T]}\mathscr{H}_{-2} \times \mathscr{D}_{[0,T]}\mathbb{R} \to \mathbb{R}$ given by $F_t(\mathscr{X}, x) := \int_0^t \mathscr{X}_s(\tau_{x_s} \varphi_\varepsilon)\, ds$. Notice that all trajectories in $C_{[0,T]}\mathscr{H}_{-2} \times C_{[0,T]}\mathbb{R}$ are continuity points for F_t (here is the only place in the proof where we need to use the smoothness of φ_ε). We can write $A_t^{n,\varepsilon} = F_t(Y^n, x^n/n)$. Using the continuity of F_t and the convergences of Y^n and x^n/n we conclude that $A_t^{n,\varepsilon}$ converges weakly to $\tilde{\mathscr{Z}}_t^\varepsilon$. In the same way, we can prove convergence of the remaining finite-dimensional distributions.

Now, fixing $\varepsilon > 0$ and $t > 0$, the pair $(A_t^n, A_t^{n,\varepsilon})$ is tight in $\mathbb{R}^2$ (see Remark 3.3.5) and its limit points[2] $(A_t, \mathscr{Z}_t^\varepsilon)$ satisfy $\mathbb{E}|A_t - \tilde{\mathscr{Z}}_t^\varepsilon| \leq \overline{\lim}_n \mathbb{E}|A_t^n - A_t^{n,\varepsilon}|$. In the same way, the family $(A_t, \mathscr{Z}_t^\varepsilon)_{\varepsilon>0}$ is tight in $\mathbb{R}^2$, and its limit points $(A_t, \mathscr{Z}_t)$ satisfy $\mathbb{E}|A_t - \mathscr{Z}_t| \leq \overline{\lim}_\varepsilon |A_t - \mathscr{Z}_t^\varepsilon|$. By the Replacement Lemma 3.2.4, $\mathbb{E}|A_t - \mathscr{Z}_t| = 0$. This shows that the processes A and $\mathscr{Z}$ have the same marginals. An analogous (but notationally more cumbersome) argument takes care of the remaining finite dimensional distributions. □

3.5 Asymptotic Independence

In the previous sections we wrote the position of the scaled random walk, $\frac{x_t^n - \nu(\rho)nt}{\sqrt{n}}$, as a sum of a martingale M_t^n and an additive functional $-2A_t^n$. We saw that the martingale part converges to Brownian motion and that the additive functional converges to a Gaussian process with stationary increments. In this section we show that these limiting processes are independent, or, putting it more precisely, that the pair (M^n, A^n) converges weakly to a product measure on $(C_{[0,T]}\mathbb{R})^2$.

We start by noticing that the sequence of random vectors (M^n, A^n) is tight. Let (M, A) be a limit point. We already know the marginal distributions M and A, so we only need to show that their finite-dimensional distributions are independent.

[2]Here we abuse the notation when we denote the limit point by $(A_t, \mathscr{Z}_t^\varepsilon)$, because A_t and $\mathscr{Z}_t^\varepsilon$ are not defined in the same probability space. However, the second coordinate of each limit point of the pair does have the same law as $\mathscr{Z}_t^\varepsilon$.

First we tackle the problem of proving that M_t is independent of A_t for each $t \in [0,T]$. In view of the Replacement Lemma 3.2.3 and the Law of Large Numbers 3.1.1 , we expect A_t to behave like $\int_0^t \sqrt{n}(\eta_s^{n,\varepsilon n}(\nu(\rho)s) - \rho)\,ds$, which depends only on the environment. Our strategy is to construct a martingale $\{N_{s,t}^n : s \leq t\}$ such that $N_{t,t}^n$ approximates this integral. Such martingale is a function of the environment process alone, and therefore does not jump together with the walker. On the other hand, M^n jumps only when the walker jumps, so $\{M_{s,t}^n : s \leq t\}$ and $\{N_{s,t}^n : s \leq t\}$ are orthogonal. If in addition $\langle N_{s,t}^n \rangle$ converges to an increasing function of s, we can apply the Martingale FCLT to conclude that $\{(M_s^n, N_{s,t}^n) : s \leq t\}_{n\in\mathbb{N}}$ converges to a pair of independent continuous martingales M and N. In particular, M_t is independent of $N_{t,t} = A_t$.

Lemma 3.5.1. *Let (M, A) be a limit point of the sequence (M^n, A^n) and $t \in [0,T]$. Then M_t is independent of A_t.*

Proof. All computations in this proof are standard in the field of scaling limits of interacting particle systems, so we will just indicate most of them.

Recall the definitions of $A^{n,\varepsilon}$ from the proof of Proposition 3.4.1 and $\mathscr{Z}^\varepsilon$ from Theorem 3.1.8. The sequence $(A^{n,\varepsilon})_{n\in\mathbb{N}}$ is tight[3] in $C_{[0,T]}\mathbb{R}$, and we proved (in the beginning of the proof of Proposition 3.4.1) that its finite-dimensional distributions converge to those of $\mathscr{Z}^\varepsilon$. Therefore $(A^{n,\varepsilon})_{n\in\mathbb{N}}$ converges weakly to $\mathscr{Z}^\varepsilon$. We also know from Proposition 3.4.1 and Theorem 3.1.8 that $(\mathscr{Z}^\varepsilon)_{\varepsilon>0}$ converges weakly to A.

The proof proceeds in several steps:

Step 1: It is enough to show that M_t is independent of A_t^ε, for each $\varepsilon > 0$.

Step 2: Fix $\varepsilon > 0$. Let $H : [0,t] \times \mathbb{R}$ be the solution of

$$\begin{cases} \partial_s H(s,u) + \partial_{uu} H(s,u) &= \tau_{\nu(\rho)s}\varphi_\varepsilon \quad \text{for all } s \in [0,t], u \in \mathbb{R} \\ H(t,u) &= 0 \quad \text{for all } u \in \mathbb{R}. \end{cases}$$

Let $\Delta_n f(u) := n^2[f(u+n^{-1}) + f(u-n^{-1}) - 2f(u)]$. Then

$$N_{s,t}^n := -Y_s^n(H_s) + \int_0^s Y_r^n((\partial_r + \Delta_n)H_r)\,dr, \tag{3.5.1}$$

defined for $s \leq t$, is a martingale with quadratic variation

$$\langle N_{s,t}^n \rangle = \int_0^s \frac{1}{n} \sum_{x\in\mathbb{Z}} n^2 \left(H_r\left(\frac{x+1}{n}\right) - H_r\left(\frac{x}{n}\right)\right)^2 (\eta_r^n(x+1) - \eta_r^n(x))^2\,dr.$$

Moreover, $N_{t,t}^n - A_t^{n,\varepsilon}$ converges to zero in probability.[4]

Step 3:

$$\lim_{n\to\infty} \langle N_{s,t}^n \rangle = \int_0^s 2\rho(1-\rho)||\partial_u H_r(u)||_{L^2(\mathbb{R})}^2\,ds \text{ in probability.}$$

We now sketch a way of computing this limit. Denote $\int_{x/n}^{(x+1)/n} (\partial_u H_r(u))^2\,du$ by $f_r(x,n)$. One can use a Taylor expansion to show that $\langle N_{s,t}^n \rangle$ and $\int_0^s \sum_x f_r(x,n)(\eta_r^n(x+1) - \eta_r^n(x))^2\,dr$ have the same limit.

To finish, it remains to replace $(\eta_r^n(x+1) - \eta_r^n(x))^2$ by its mean $2\rho(1-\rho)$. One can explore the elementary fact that if a sequence of random variables X_n satisfies $\mathbb{E}X_n \to 0$ and $\mathrm{Var}X_n \to 0$

[3] Here one can use stationarity and the Cauchy-Schwarz inequality to estimate $\mathbb{E}[(A_t^{n,\varepsilon} - A_s^{n,\varepsilon})^2]$.

[4] It is here that we need the smoothness of φ_ε, for this ensures smoothness of H, and therefore a $O(n^{-1})$ error in the approximation of $\partial_{uu}H$ by Δ_n.

then $X_n \to 0$ in probability. The estimate on the variance combines Cauchy-Schwarz inequality in the time integral, stationarity of ν_ρ and the inequality $f_r(n,x) \le n/x^2$, valid for large x.
Step 4: Using the Martingale FCLT, we see that $\{(M_s^n, N_{s,t}^n) : s \le t\}_{n\in\mathbb{N}}$ converges to a continuous Gaussian process $\{(M_t, N_{s,t}) : s \le t\}$ with independent increments. Since M^n is orthogonal to N^n, M_t^n is independent of $N_{t,t}^n = A_t^\varepsilon$.

□

We finish the section by indicating how to prove that the finite-dimensional distributions $(M_{t_1}, \dots, M_{t_k})$ and $(A_{t_1}^\varepsilon, \dots, A_{t_k}^\varepsilon)$ are independent. The proof builds upon the strategy used in Lemma 3.5.1.

Theorem 3.5.2. *Let (M, A) be a limit point of the sequence (M^n, A^n) and $t \in [0,T]$. Let $0 < t_1 < \dots < t_k \le T$. Then $(M_{t_1}, \dots, M_{t_k})$ and $(A_{t_1}^\varepsilon, \dots, A_{t_k}^\varepsilon)$ are independent.*

Proof. To simplify the notation, let us treat the case with just two times s and t, with $s < t$. Recall the definition of A^ε in the beginning of the proof of Lemma 3.5.1 and that it is enough to show that M is independent of A^ε, for each $\varepsilon > 0$.
Step 1: It suffices to prove that $a_1 M_s + a_2 M_t$ is independent of $b_1 A_s^\varepsilon + b_2 A_t^\varepsilon$ for any $a_1, a_2, b_1, b_2 \in \mathbb{R}$. The proof uses characteristic functions.
Step 2: Define the Dynkin martingales $\{N_{r,s}^n : r \le s\}$ and $\{N_{r,t}^n : r \le t\}$ as in (3.5.1) (notice that the test function H in used in (3.5.1) depends on t). Declare $N_{r,s}^n := N_{s,s}^n$ when $r > s$. Notice that $\{b_1 N_{r,s}^n + b_2 N_{r,t}^n : r \le t\}$ is also a Dynkin martingale. One can show that its quadratic variation converges, as $n \to \infty$, to an increasing function of r.
Step 3: Using the Martingale FCLT, we see that $\{(M_r^n, b_1 N_{r,s}^n + b_2 N_{r,t}^n) : r \le t\}_{n\in\mathbb{N}}$ converges weakly and the limit has independent marginals. In particular, $b_1 A_s^\varepsilon + b_2 A_t^\varepsilon = b_1 N_{t,s}^n + b_2 N_{t,t}^n$ is independent of M_t. □

Chapter 4

Appendix

4.1 Computations involving the generator

Lemma 4.1.1. *Consider the fluctuation field*

$$X_t^n(f) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{T}_n} f\left(\frac{x}{n}\right)(\eta_x - \rho).$$

Then

$$\begin{aligned} L_n X^n(f)(\eta) =& \frac{1}{\sqrt{n}} \sum_x \Delta_n f\left(\frac{x}{n}\right) \eta_x \\ &+ \frac{1}{\sqrt{n}} \sum_x f\left(\frac{x}{n}\right)(-b\eta_{x-1}\eta_x\eta_{x+1} + b\eta_{x-1}\eta_{x+1} - 2\eta_x + 1). \end{aligned} \tag{4.1.1}$$

If we center $\overline{\eta}_x := \eta_x - \rho$ *and use the assumption* $0 = (1+b\rho^2)(1-\rho) - \rho = -b\rho^3 + b\rho^2 - 2\rho + 1$ *we get*

$$\begin{aligned} L_n X^n(f)(\eta) =& \frac{1}{\sqrt{n}} \sum_x \Delta_n f\left(\frac{x}{n}\right) \eta_x \\ &+ \frac{1}{\sqrt{n}} \sum_x f\left(\frac{x}{n}\right)(-b\overline{\eta}_{x-1}\overline{\eta}_x\overline{\eta}_{x+1} - b\rho(\overline{\eta}_{x-1}\overline{\eta}_x + \overline{\eta}_x\overline{\eta}_{x+1}) + (b - b\rho)\overline{\eta}_{x-1}\overline{\eta}_{x+1} \\ &+ (b\rho - b\rho^2)(\overline{\eta}_{x-1} + \overline{\eta}_{x+1}) - (b\rho^2 + 2)\overline{\eta}_x). \end{aligned} \tag{4.1.2}$$

Proof. We are going to do two computations, one for the birth and death dynamics and one for the exclusion dynamics. Let us denote by L^r the generator associated to the birth and death dynamics, that is,

$$L^r f(\eta) := \sum_x \{\eta_x + (1 - \eta_x)(1 + b\eta_{x-1}\eta_{x+1})\}[f(\eta^x) - f(\eta)].$$

Let us also denote $c_x(\eta) := \eta_x + (1 - \eta_x)(1 + b\eta_{x-1}\eta_{x+1})$.

We begin by computing $L^r\eta_x$. The reader who wants to follow the computations or to try them out on his or her own should keep in mind the identities $\eta_x(1 - \eta_x) = 0$, $\eta_x^2 = \eta_x$ and $(1 - \eta_x)^2 = 1 - \eta_x$. These are true because η_x can only assume the values 0 and 1.

$$
\begin{aligned}
L^r\eta_x &= c_x(\eta)(-\eta_x + 1 - \eta_x) \\
&\qquad = -\eta_x + (1-\eta_x)(1 + b\eta_{x-1}\eta_{x+1}).
\end{aligned}
$$

This formula will be useful for the computation of the quadratic variation, so we keep it for reference. For the Boltzmann-Gibbs principle it is better to have a polynomial in the variables $\overline{\eta}_x$. To get this polynomial we expand the last expression. This gives us the expression in the last line of (4.1.1).

Now for the exclusion part. Denote by L_n^{ex} the generator of the simple symmetric exclusion process, that exchanges the occupations of sites x and $x+1$ at rate n^2. We begin by computing

$$
\eta_x^{x,x+1} - \eta_x = (1-\eta_x)(\eta_{x-1} + \eta_{x+1}) - \eta_x(1 - \eta_{x-1} + 1 - \eta_{x+1}) = \eta_{x+1} + \eta_{x-1} - 2\eta_x.
$$

Summing the last equation over $x \in \mathbb{T}_n$,

$$
\begin{aligned}
L_n^{ex} X^n(f) &= n^{-1/2} \sum_x n^2 f\left(\frac{x}{n}\right)(\overline{\eta}_{x+1} + \overline{\eta}_{x-1} - 2\overline{\eta}_x) \\
&= n^{-1/2} \sum_x n^2 \left[f\left(\frac{x+1}{n}\right) + f\left(\frac{x-1}{n}\right) - 2f\left(\frac{x}{n}\right)\right] \overline{\eta}_x \\
&= X^n(\Delta_n f).
\end{aligned}
$$

□

Proof of Lemma 2.1.2:

Let us begin by recalling a general formula: if $(Y_t)_{t\geq 0}$ is a Markov chain on a finite state space Ω, with transition rates $\{r(\eta,\xi) : \eta, \xi \in \Omega\}$, then

$$
M_t^g := g(Y_t) - g(Y_0) - \int_0^t \sum_{\xi\in\Omega} r(Y_s,\xi)[g(\xi) - g(Y_s)]\, ds
$$

is a martingale, and its quadratic variation is given by

$$
\langle M_t^f \rangle = \int_0^t \sum_{\xi\in\Omega} r(Y_s,\xi)[g(\xi) - g(Y_s)]^2\, ds.
$$

In our case, the quadratic variation will be the sum of two integral terms. The first one comes from the exclusion dynamics. The role of the function g is played by $g(\eta) := n^{-1/2}\sum_y f(y/n)\overline{\eta}_y$, and we need to compute $g(\eta^{x,x+1}) - g(\eta)$. The corresponding term in the quadratic variation is

$$
\int_0^t n^2 \sum_x \frac{1}{n}\left\{ f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right)\right\}^2 (\eta_x(s) - \eta_{x+1}(s))^2\, ds.
$$

For the part of the quadratic variation due to the reaction dynamics, notice that $[g(\eta^x) - g(\eta)]^2 = n^{-1} f\left(\frac{x}{n}\right)^2$. The corresponding term in the quadratic variation is

$$
\int_0^t c_x(\eta(s)) \sum_x \frac{1}{n} f\left(\frac{x}{n}\right)^2\, ds \tag{4.1.3}
$$

Now, let us compute the limit of the quadratic variation as $n \to \infty$. We start with (4.1.3). The usual statement of the hydrodynamic limit asserts that, for each $t > 0$,

$$\frac{1}{n}\sum_x f\left(\frac{x}{n}\right)\eta_x \longrightarrow \int f(u)\rho(t,u)\,du,$$

where ρ is the solution of the hydrodynamic equation. It is possible to generalize the above statement replacing η_x by the translation of a local funtion, $\tau_x\psi$, that is, for any local function ψ,

$$\frac{1}{n}\sum_x f\left(\frac{x}{n}\right)\tau_x\psi \longrightarrow \int f(u)\mathbb{E}_{\nu_{\rho(t,u)}}[\psi]\,du.$$

This stronger statement is proven in [dMFL], Theorem 1, and can also be deduced from a bound of order $o(n)$ on the relative entropy between the law of η_t^n and the product measure $\nu_{\rho(t,\cdot)}$ ([KL], Corollary 6.1.3). In our case, the initial density ρ was chosen so that the solution to the hydrodynamic equation is constant. Thus, the integral in (4.1.3) converges to zero (because $\int c_x\,d\nu_\rho = 0$) and the integral in (4.1.3) converges to $t\chi(\rho)\|\nabla f\|^2_{L^2(\mathbb{T})}$.

□

Proposition 4.1.2. *Let L_n^* denote the adjoint of the generator L_n in $L^2(\nu_\rho)$. The function $L_n^*1 : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$ can be written as a finite linear combination of terms of the form*

$$\sum_{x\in\mathbb{T}_n^d}\prod_{y\in\Lambda}\overline{\eta}_{x+y},$$

where $\Lambda \subset \mathbb{Z}^d$ is a finite set with $|\Lambda| \geq 2$.

Proof. We start by recalling that the exclusion part is self-adjoint with respect to ν_ρ. Writing $L_n = n^2L^{ex} + L^r$, we just need to worry about the reaction part L^r. We know that, for any $\eta, \overline{\eta} \in \{0,1\}^{\mathbb{T}_n}$, detailed balance should hold:

$$\nu_\rho(\overline{\eta})L_n(\overline{\eta},\eta) = \nu_\rho(\eta)L_n^*(\eta,\overline{\eta}).$$

Since $L^r(\eta,\overline{\eta})$ is non-zero only when $\overline{\eta} = \eta^x$ or $\overline{\eta} = \eta$, the same holds for $(L^r)^*$. Notice

$$(L^r)^*(\eta,\eta^x) = c_x(\eta^x)\frac{\nu_\rho(\eta^x)}{\nu_\rho(\eta)}$$
$$(L^r)^*(\eta,\eta) = -\sum_x c_x(\eta).$$

Since L^{ex} is self-adjoint and $L^{ex}1 = 0$, we have $L_n^*1 = (L^r)^*1$. Therefore

$$\begin{aligned} L_n^*1 &= \sum_x \left[c_x(\eta^x)\frac{\nu_\rho(\eta^x)}{\nu_\rho(\eta)} - c_x(\eta)\right] \\ &= \sum_x \left\{\eta_x\left(\frac{1-\rho}{\rho}c_x^+(\eta) - c_x^-(\eta)\right) + (1-\eta_x)\left(\frac{\rho}{1-\rho}c_x^-(\eta) - c_x^+(\eta)\right)\right\}. \end{aligned}$$

From the last formula it is possible to compute L_n^*1 explicitly and verify that it is indeed a polynomial of degree at least two in the $\overline{\eta}_x$ variables. In one dimension with L_n given by (2.1.1), the formula is

$$L_n^*1(\eta) = \frac{b}{\rho}\overline{\eta}_{x-1}\overline{\eta}_x\overline{\eta}_{x+1} + b(\overline{\eta}_{x-1}\overline{\eta}_x + \overline{\eta}_x\overline{\eta}_{x+1}). \tag{4.1.4}$$

Below we provide a different argument, that works with more general birth and death rates. To keep the notation simple, assume that $d = 1$ and $c_x^-(\eta)$ and $c_x^+(\eta)$ depend only on $\overline{\eta}_{x-1}$ and $\overline{\eta}_{x+1}$.

Denote

$$p(\eta_{x-1}, \eta_x, \eta_{x+1}) := \eta_x \left(c_x^+(\eta) \frac{1-\rho}{\rho} - c_x^-(\eta) \right) + (1 - \eta_x) \left(\frac{\rho}{1-\rho} c_x^-(\eta) - c_x^+(\eta) \right).$$

We regard p as a polynomial in the variables $\eta_{x-1}, \eta_x, \eta_{x+1}$. There is a polynomial q such that

$$p(\eta_{x-1}, \eta_x, \eta_{x+1}) = q(\overline{\eta}_{x-1}, \overline{\eta}_x, \overline{\eta}_{x+1}).$$

Notice that $c_x^+(\rho, \rho, \rho) = \int c_x^+ \, d\nu_\rho$, and an analogous identity holds for c_x^-. We claim that the degree of q is at least 2. The independent term is

$$\begin{aligned} q(0,0,0) &= p(\rho, \rho, \rho) \\ &= (1-\rho) c_x^+(\rho, \rho, \rho) - \rho + \rho - (1-\rho) c_x^-(\rho, \rho, \rho) \\ &= 0. \end{aligned}$$

Now let us look at the coefficient of $\overline{\eta}_x$. It is equal to

$$\begin{aligned} (\partial_{\overline{\eta}_x} q)(0,0,0) &= (\partial_{\eta_x} p)(\rho, \rho, \rho) \\ &= \left(\frac{1-\rho}{\rho} c_x^+(\rho, \rho, \rho) - c_x^-(\rho, \rho, \rho) \right) - \left(\frac{\rho}{1-\rho} c_x^-(\rho, \rho, \rho) - c_x^+(\rho, \rho, \rho) \right) \\ &= 0. \end{aligned}$$

In the last inequality, we used the identity $c_x^+(\rho, \rho, \rho)(1-\rho) = \rho \, c_x^-(\rho, \rho, \rho)$.

A similar computation proves that the coefficients of $\overline{\eta}_{x-1}$ and $\overline{\eta}_{x+1}$ in q are zero.

□

4.2 Concentration Inequalities

Proposition 4.2.1 (Bounded Differences Inequality, [BLM], Theorem 6.2)**.** *Assume the function $f : \{0,1\}^{\mathbb{T}_n} \to \mathbb{R}$ satisfies*

$$|f(\eta^x) - f(\eta)| \le c_x$$

for a family of constants c_x. Then

$$\log \nu_\rho \left(f(\eta) - \int f \, d\nu_\rho > \delta \right) \le -\frac{2\delta^2}{\sum_x c_x^2}.$$

Lemma 4.2.2 (Subgaussianity)**.** *Let Z be a non-negative random variable and $c_1, c_2 > 0$.*

1. *If*

$$\mathbb{E}[e^{\theta Z}] \le c_1 e^{c_2 \theta^2/2} \text{ for all } \theta > 0$$

then

$$\mathbb{P}(Z > \lambda) \le c_1 e^{-\lambda^2/2c_2} \text{ for all } \lambda > 0.$$

2. If

$$\mathbb{P}(Z > \lambda) \le c_1 \, e^{-c_2\lambda^2} \text{ for all } \lambda > 0,$$

then

$$\log \mathbb{E}[e^{\theta Z^2}] \le \frac{c_1\theta}{c_2-\theta}, \text{ for all } \theta \in (0, c_2).$$

Proof. The first assertion uses Markov's inequality. For any $\theta > 0$

$$P(Z > \lambda) \le c_1 e^{-\theta\lambda} e^{c_2\theta^2/2}.$$

The expression $-\theta\lambda + c_2\theta^2/2$ attains its minimum at $\theta = \lambda/c_2$, where it takes the value $-\lambda^2/2c_2$.

For the second assertion, we assume $\theta < c_2$. Then

$$\begin{aligned}
\mathbb{E}[e^{\theta Z^2}] &= 1 + \int_0^\infty 2\theta u \, e^{\theta u^2} \mathbb{P}(Z \ge u) \, du \\
&\le 1 + \int_0^\infty 2\theta u \, c_1 e^{-(c_2-\theta)u^2} \, du \\
&= 1 + \left. \frac{\theta c_1 \, e^{-(c_2-\theta)u^2}}{-(c_2-\theta)} \right|_0^{u=+\infty} \\
&= 1 + \frac{c_1\theta}{c_2-\theta}.
\end{aligned}$$

To finish, we apply the inequality $\log(1+x) \le x$.

□

Theorem 4.2.3 (Hoeffding's Inequality)**.** *Let X be a mean zero random variable taking values in the interval $[a, b]$. Then*

$$\mathbb{E}[e^{\theta X}] \le e^{\theta^2(b-a)^2/8} \text{ for all } \theta > 0,$$

$$\mathbb{P}(|X| > \lambda) \le 2e^{-2\lambda^2/(b-a)^2}$$

and

$$\log \mathbb{E}[e^{\theta X^2}] \le \frac{2\theta}{\frac{2}{(b-a)^2} - \theta} \text{ for all } \theta < \frac{2}{(b-a)^2}.$$

Proof. We are going to prove only the first inequality. The first step is to write X as a convex combination of a and b, say $X = \Lambda a + (1-\Lambda)b$, for a random variable $\Lambda \in [0, 1]$. From $\mathbb{E}[X] = 0$, we get $\mathbb{E}[\Lambda] = b/(b-a)$. Since we are looking for bounds in terms of $b-a$, let us denote $\lambda := b-a$, so that $a = b - \lambda$.

The second step is to apply Jensen's inequality:

$$\mathbb{E}[e^{\theta X}] \le \mathbb{E}[\Lambda e^{\theta a} + (1-\Lambda)e^{\theta b}] = \frac{b}{\lambda} e^{\theta(b-\lambda)} + \frac{\lambda - b}{\lambda} e^{\theta b}.$$

Let us denote $y := b/\lambda$ to see the convex combination more clearly. We want to prove

$$y\,e^{\lambda(y-1)\theta} + (1-y)e^{\lambda y\theta} \le e^{\lambda^2\theta^2/8}, \text{ for all } \theta > 0 \text{ and } y \in (0,1).$$

To prove the last inequality, we compare Taylor expansions.

□

Corollary 4.2.4. *Let $X_1, \ldots, X_n$ be independent random variables and $a \in \mathbb{R}^n$. Denote $\|a\| := (\sum_{j=1}^n a_j^2)^{1/2}$ Assume $|X_j| \le 1$ for all j. Then, for all $\theta > 0$ and for $c < (2\|a\|^2)^{-1}$,*

$$\mathbb{E}\left[e^{\theta \sum_{j=1}^n a_j X_j}\right] \le e^{\theta^2 \|a\|^2}$$

and

$$\log \mathbb{E}\left[e^{c\left(\sum_{j=1}^n a_j X_j\right)^2}\right] \le 2c\|a\|^2.$$

4.3 Log-Sobolev Inequality for the Simple Symmetric Exclusion Process

This inequality says that the entropy of a density g in configuration space is bounded by n^2 times the Dirichlet form of $\sqrt{g}$. The catch is that both the entropy and the Dirichlet form cannot be taken with respect to the product measure ν_ρ, as can be seen by taking a density that is a function of the number of particles in the system. However, it is true when instead of ν_ρ we use the uniform distribution on some hyperplane $\Omega_k := \{\eta \in \{0,1\}^{\mathbb{T}_n} : \sum_x \eta_x = k\}$. We denote this measure by $\mu_{n,k}$, and the log Sobolev inequality reads

Theorem 4.3.1. *There exists a universal constant C such that*

$$\int_{\Omega_k} g \log g \, d\mu_{n,k} \le Cn^2 \sum_{x \in \mathbb{T}} \int \left(\sqrt{g^{x,x+1}} - \sqrt{g}\right)^2 \, d\mu_{n,k}.$$

The log-Sobolev inequality was proved in [Y97].

4.4 Entropy Inequalitites

Proposition 4.4.1. *Let μ and ν be probability measures on some finite set Ω. Let $f : \Omega \to \mathbb{R}$ be a function and $H(\mu|\nu)$ the relative entropy between μ and ν. Then, for all $\gamma > 0$,*

$$\int f \, d\mu \le \frac{1}{\gamma} H(\mu|\nu) + \frac{1}{\gamma} \log \int e^{\gamma f} \, d\nu$$

and

$$\mu(A) \le \frac{H(\mu|\nu) + \log 2}{\log\left(1 + \frac{1}{\nu(A)}\right)}. \tag{4.4.1}$$

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