Short Wave-Long Wave Interactions in Magnetohydrodynamics

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To my parents

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Abstract

In this thesis we study several mathematical aspects of a system of equations modelling the interaction between short waves, described by a nonlinear Schrödinger equation, and long waves, described by the equations of magnetohydrodynamics for a compressible, heat conductive fluid. The system in question models an aurora-type phenomenon, where a short wave propagates along the streamlines of a magnetohydrodynamic medium. We address several problems in both the one dimensional and in the multidimensional versions of the model. Namely, existence and uniqueness of strong solutions, as well as the vanishing viscosity problem, in the 1-dimensional case; and existence of weak solutions with large data in the 2-dimensional case.

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Chapter 1

Introduction

This thesis concerns the study of several mathematical aspects of a system of equations modelling Short Wave-Long Wave Interactions between the Magnetohydrodynamics (MHD) equations and a nonlinear Schrödinger equation. The model describes the evolution of the wave function, obeying a nonlinear Schrödinger equation, along the streamlines of the fluid flow. As such, the nonlinear Schrödinger equation is stated in a different coordinate system; namely in the Lagrangian coordinates of the fluid.

The Lagrangian coordinates are characterized by being constant along the streamlines of the fluid. Accordingly, the change of variables can be defined through the flux Φ associated to the fluid's velocity field $\mathbf{u}$, given by

$$\frac{d\Phi}{dt}(t;\mathbf{x}) = \mathbf{u}(t, \Phi(t;\mathbf{x})),$$

$$\Phi(0;\mathbf{x}) = \mathbf{x},$$

and the Lagrangian transformation $\mathbf{Y}(x,t) = (y(x,t),t)$ can be defined by the relation

$$\mathbf{y}(t, \Phi(t;\mathbf{x})) = \mathbf{y}_0(\mathbf{x}),$$

where the function $\mathbf{y}_0$ is a diffeomorphism which may be chosen conveniently according to the problem. In particular, $\mathbf{y}_0$ can be chosen so that the Jacobian $J_{\mathbf{y}}(t;\mathbf{x}) := \det\left(\frac{\partial \mathbf{y}}{\partial \mathbf{z}}(t, \Phi(t;\mathbf{x}))\right)$ of the coordinate change satisfies

$$J_{\mathbf{y}}(t;\mathbf{x}) = \rho(t, \Phi(t;\mathbf{x})). \tag{1.1}$$

where, ρ is the fluid's density.

Note, that this relation implies that the Lagrangian transformation becomes singular in the presence of vacuum or concentration, that is, when the density vanishes or becomes infinity.

Having this, the nonlinear Schrödinger equation can be stated and the coupling is made through the external force term in the MHD equations and the potential term in the Schrödinger equation due to external forces.

Specifically, the full three dimensional system that we study is the following.

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p\\
&\qquad\qquad = \operatorname{div}\mathbb{S} + \beta(\nabla\times\mathbf{H})\times\mathbf{H} + \alpha\nabla(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)),\\
&\mathcal{E}_t + \operatorname{div}(\mathbf{u}(\mathcal{E} - \frac{\beta}{2}|\mathbf{H}|^2 + p)) = \operatorname{div}(\beta(\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times(\nabla\times\mathbf{H}))\\
&\qquad\qquad + \operatorname{div}(\kappa\nabla\theta) + \operatorname{div}(\mathbb{S}\mathbf{u}) + \alpha\nabla(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2))\cdot\mathbf{u},\\
&\beta\mathbf{H}_t - \beta\operatorname{curl}(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}),\\
&\operatorname{div}\mathbf{H} = 0,\\
&i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi + \tilde{\alpha}g(v)h'(|\psi|^2)\psi.
\end{aligned}
$$

Here, ρ, $\mathbf{u}\in\mathbb{R}^3$ and θ denote the fluid's density, velocity and temperature, respectively, $\mathbf{H}\in\mathbb{R}^3$ the magnetic field and $\psi=\psi(t,\mathbf{y})$ is the wave function; the total energy is

$$\mathcal{E} := \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{\beta}{2}|\mathbf{H}|^2,$$

with e being the internal energy and $\frac{1}{2}|\mathbf{H}|^2$ the magnetic energy; p denotes the pressure and $\mathbb{S}$ is the viscous stress tensor given by

$$\mathbb{S} = \lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t).$$

The viscosity coefficients λ and μ satisfy $2\mu+\lambda>0$ and $\mu>0$; κ is the heat conductivity, $\nu>0$ is the magnetic difusivity and $\beta>0$ is the magnetic permeability.

The pressure and the internal energy, in general, depend on the density and the temperature through constitutive relations of the form

$$p = p(\rho,\theta), \qquad e = e(\rho,\theta),$$

and must satisfy Maxwell's relation

$$e_\rho(\rho,\theta)=\frac{1}{\rho^2}(p-\theta p_\theta(\rho,\theta)).$$

Moreover, g and h are the coupling functions, α and $\tilde{\alpha}$ are the interaction coefficients and $v=v(t,\mathbf{y})$ is the specific volume given by

$$v(t,\mathbf{y}(t\mathbf{x}))=\frac{1}{\rho(t,\mathbf{x})}.$$

The most important feature of this coupling is that it is endowed with an energy identity, which can be stated in differential form as

$$\begin{aligned}
&\Big\{(\rho(\frac{1}{2}|\mathbf{u}|^2+e)+\beta|\mathbf{H}|^2)_t+\mathrm{div}_{\mathbf{x}}(\mathbf{u}(\rho(\frac{1}{2}|\mathbf{u}|^2+e)+p))\\
&\quad-\mathrm{div}_{\mathbf{x}}(\kappa\nabla\theta)-\mathrm{div}_{\mathbf{x}}((\lambda(\mathrm{div}\mathbf{u})\mathrm{Id}+\mu(\nabla\mathbf{u}+(\nabla\mathbf{u})^t))\cdot\mathbf{u})\\
&\qquad-\mathrm{div}_{\mathbf{x}}(\beta(\mathbf{u}\times\mathbf{H})\times\mathbf{H})-\mathrm{div}_{\mathbf{x}}(\mathbf{H}\times\nu(\nabla_{\mathbf{x}}\times\mathbf{H}))\Big\}d\mathbf{x}\\
&\quad=\frac{\alpha}{\tilde{\alpha}}\Big\{\mathrm{div}_{\mathbf{y}}(\overline{\psi}_t\nabla_{\mathbf{y}}\psi+\psi_t\nabla_{\mathbf{y}}\overline{\psi})-\tilde{\alpha}(g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2))_t\\
&\qquad\qquad-\frac{1}{2}(|\nabla_{\mathbf{y}}\psi(t,\mathbf{y})|^2)_t-\frac{1}{2}(|\psi(t,\mathbf{y})|^4)_t\Big\}d\mathbf{y}.
\end{aligned}$$

In particular, under suitable integrability conditions, this identity yields an integral form of the conservation of energy:

$$\begin{aligned}
&\frac{d}{dt}\int\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\frac{1}{2\tilde{\mu}}|\mathbf{H}|^2\right)d\mathbf{x}\\
&+\frac{d}{dt}\int\frac{\alpha}{\tilde{\alpha}}\left(\frac{1}{2}|\nabla_{\mathbf{y}}\psi(t,\mathbf{y})|^2+\frac{1}{2}|\psi(t,\mathbf{y})|^4+\tilde{\alpha}g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2)\right)d\mathbf{y}=0.
\end{aligned}$$

The phenomenon that we have in mind when we study this model is one like that of the auroras. Auroras, commonly known as polar lights, occur as fast-moving charged particles released from the sun collide with the Earth's atmosphere, channelled by Earth's magnetic field. The stream of charged particles, called solar wind, consists mainly of electrons, protons and alpha particles that, upon reaching the earth's magnetosphere, collide with atoms in the atmosphere, such as oxygen and nitrogen, imparting energy into them and thus making them excited. As the atoms return to their normal state they release photons, and when many of these collisions occur together they emit enough light for the phenomenon to be visible by the naked eye.

The aurora can thus be seen as small waves propagating along the trajectories of the particles of the atmosphere, a magnetohydrodynamic medium. Let us recall that the MHD equations describe the motion of a conductive fluid in the presence of a magnetic field. On the other hand, the nonlinear Schrödinger equation describes collective phenomena in quantum plasmas. The example of the aurora gathers many of the ingredients captured by this model.

This model was proposed and studied recently (in 2016) by Frid, Jia and Pan [28] in the three dimensional context, showing existence, uniqueness and decay rates of smooth solutions for small initial data. A similar model involving the Navier-Stokes Equations instead of the MHD equations was proposed earlier by Dias and Frid [17], and was further studied by Frid, Pan and Zhang [27].

The thesis is divided into three parts. In the first part, corresponding to Chapter 2, we do a review of some of the ideas involved in the deduction of the MHD equations and in the deduction of the SW-LW interaction model above. We also state the kind of constitutive relations we consider for the pressure and internal energy, the initial and boundary conditions for the problem and the hypotheses on the coefficients and the coupling functions. In particular, throughout this work we assume that the pressure can be decomposed into an elastic part, given by a γ-law, and a thermal part, which is linear with respect to the temperature. That is, we assume that

$$p(\rho, \theta) = a\rho^{\gamma} + \theta p_{\theta}(\rho), \tag{1.2}$$

where $a > 0$ and $\gamma > 1$ are constants, with very general assumptions on the function $p_{\theta}(\rho)$.

The second part, corresponding to Chapter 3, concerns the study of the one dimensional version of this model. The one dimensional case arises under the assumption that the flow moves in a preferable direction. That is, we assume that the three dimensional MHD flow with space variables $\mathbf{x} = (x, x_2, x_3)$ moves in the x direction and is uniform in the transverse direction (x_2, x_3). This assumption considerably simplifies the equations as well as the short wave-long wave interaction coupling, since the one dimensional Lagrangian transformation takes a very specific and plain form.

For convenience, we decompose our dependent MHD variables as

$$\rho = \rho(t, x), \qquad \theta = \theta(t, x), \qquad \mathbf{u} = (u, \mathbf{w})(t, x), \qquad \mathbf{w} = (u_2, u_3),$$

$$\mathbf{H} = (h_1, \mathbf{h})(t, x), \qquad \mathbf{h} = (h_2, h_3),$$

where u and h_1 are the longitudinal velocity and the longitudinal magnetic field, and $\mathbf{w}$ and $\mathbf{h}$ are the transverse velocity and the transverse magnetic field, respectively. Under these assumptions, we have that the partial derivatives with respect to x_2 and x_3 of the involved functions are identically zero.

With this in mind, a straightforward calculation shows that h_1 is constant (which we take to be equal to 1 without loss of generality) and our model is simplified as

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \\
&(\rho u)_t + \Big(\rho u^2 + p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho)h(|\psi \circ \mathbf{Y}|^2)\Big)_x = (\varepsilon u_x)_x, \\
&(\rho \mathbf{w})_t + (\rho u \mathbf{w} - \beta \mathbf{h})_x = (\mu \mathbf{w}_x)_x, \\
&\Big(\rho\Big(e + \tfrac{1}{2}u^2 + \tfrac{1}{2}|\mathbf{w}|^2\Big) + \tfrac{\beta}{2}|\mathbf{h}|^2\Big)_t + \Big(u\Big(\rho\Big(e + \tfrac{1}{2}u^2 + \tfrac{1}{2}|\mathbf{w}|^2\Big) + p\Big)\Big)_x \\
&\quad = (\beta \mathbf{w}\cdot\mathbf{h} - \beta u|\mathbf{h}|^2)_x + (\varepsilon u u_x + \mu \mathbf{w}\cdot\mathbf{w}_x + \nu \mathbf{h}\cdot\mathbf{h}_x)_x + (\kappa\theta_x)_x \\
&\qquad + \alpha\Big(g'(1/\rho)h(|\psi \circ \mathbf{Y}|^2)\Big)_x u, \\
&\beta \mathbf{h}_t + (\beta u \mathbf{h} - \beta \mathbf{w})_x = (\nu \mathbf{h}_x)_x, \\
&i\,\psi_t + \psi_{yy} = |\psi|^2\psi + \tilde{\alpha} g(v) h'(|\psi|^2)\psi.
\end{aligned}
\qquad ,
$$

Here, μ and $\varepsilon = \lambda + 2\mu$ are the shear viscosity and the bulk viscosity of the fluid, respectively.

In this setting, we are able to prove global existence and uniqueness of smooth solutions in a bounded open spacial domain Ω. We first prove existence and uniqueness of local solutions and then extend the local solutions to global ones based on a priori estimates.

For the local result we use a Faedo-Galerkin type method similar to that applied by Dias and Frid in [17], which in turn resembles the classic work by Kazhikhov and Shelukhin in [32] (c.f. [2, Chapter 2]). As for the global result, we develop some a priori estimates inspired by the work of Chen and Wang in [15] and by the work of Wang in [48]. In particular we show that no vacuum nor concentration develop in finite time.

Having well posedness for the one dimensional model, we turn our attention to the vanishing viscosity problem. First, we assume that the pressure has the form $p(\rho,\theta) = a\rho^\gamma + \delta\theta p_\theta(\rho)$, where $a > 0$, $\gamma > 1$, $\delta > 0$ and p_θ is a function of the density that satisfies certain growth conditions. Note that if ε, α, $\tilde{\alpha}$, δ and β are all zero we are left with a system involving Euler's equations of compressible fluid dynamics and a decoupled nonlinear Schrödinger equation. In this connection we show convergence of the sequence of solutions as ε, α, $\tilde{\alpha}$, δ and β tend to zero. More specifically, we show that if $\alpha, \tilde{\alpha} = o(\varepsilon^{1/2})$ and $\delta, \beta = o(\varepsilon)$ as $\varepsilon \to 0$, leaving $\mu > 0$ and $\nu > 0$ fixed, then the sequence of solutions to system (3.1)-(3.6) converges to a solution of the limit problem.

As the limit problem has different regularity properties than the original one (in Euler's equations shock waves are expected to occur in finite time, even if the initial data is smooth) this convergence is not a straightforward task.

To achieve this, we employ the compensated compactness method as applied by Chen and Perepelitsa in [14], where they study the problem of vanishing viscosity limit for the one dimensional Navier-Stokes equations. Due to the presence of the magnetic field and the short wave-long wave interactions we have to deduce some new estimates in order to be able to to apply the method.

It is worth mentioning that the magnetic permeability β is usually taken to be equal to 1 in the literature ([33]) since in most real world media covered by the model this constant differs only slightly from the unity. However, the only physical restriction on it is its positivity.

The third part of the thesis, contained in Chapter 4, deals with the multidimensional version of the model. The main difficulty in higher dimensions is the possible occurrence of vacuum. As the Lagrangian transformation becomes singular in the presence of vacuum an effective coupling of the fluid equations with the nonlinear Schrödinger equation cannot be made in a straightforward way. In order to overcome these difficulties, we define the interaction through a regularized system that provides a good definition for an approximate Lagrangian coordinate. Then, after showing existence of solutions, we show compactness of the sequence of solutions to the regularized system thus making sense of the desired SW-LW interaction in the limit process.

For simplicity, in the multidimensional model we focus on the isentropic case, that is, the case of a non heat-conductive fluid, which trivializes the energy equation.

In order to workaround the lack of regularity of the density we first add an artificial viscosity to the continuity equation. Fix $\varepsilon > 0$ and $\delta > 0$ and consider the following regularized MHD system

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho,\\
&(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(a\rho^\gamma + \delta\rho^\beta) + \varepsilon \nabla \mathbf{u} \cdot \nabla \rho\\
&\qquad\qquad\qquad = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{u}) + \rho \mathbf{F},\\
&\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}),\\
&\operatorname{div} \mathbf{H} = 0,
\end{aligned}
$$

where $\mathbf{F}$ is the term accounting for external forces.

Note that besides the artificial viscosity added to the continuity equation, two new terms appeared in the momentum equation (2.5). The term $\delta\rho^\beta$, where $\beta > 1$, acts as an artificial pressure and is intended to provide better estimates on the density, whereas the term $\varepsilon \nabla \mathbf{u} \cdot \nabla \rho$ is set to equate the unbalance in the energy estimates of the MHD equations caused by the introduction of the artificial viscosity. This approximate system resembles the one employed by Hu and Wang in [29] where they study the existence of weak solutions to the three dimensional MHD equations. A similar approximation was introduced by Feireisl, et al. in [24] in the study of the Navier-Stokes equations, who, in turn, followed the pioneering ideas by Lions in [38]. Recall that ε and δ are small constants and the analysis that we develop provides insights that justify the accuracy to which this regularized model approximates the desired SW-LW interaction.

Now, as it turns out, even in this regularized setting the velocity field might not be smooth enough to provide a good enough definition of Lagrangian transformation that we can work with. More specifically, in the present situation there is no a priori bound available for Jacobian of the Lagrangian transformation, as it depends on the L^∞ norm of $\operatorname{div}\mathbf{u}$. For this reason we replace the velocity by a suitable smooth approximation $\mathbf{u}^N$ (which tends to $\mathbf{u}$ as $N \to \infty$) in the definition of the Lagrangian transformation. Thus obtaining an approximate Lagrangian coordinate with $\mathbf{u}$ replaced by $\mathbf{u}_N$ in its definition.

Although we now have a smoothed Lagrangian coordinate, we lose relation (1.1) and instead we have

$$J_y(t) = e^{-\int_0^t \operatorname{div}\, u^N(s,\Phi(s,x))ds}.$$

With this our regularized SW-LW interactions model reads as:

$$
\begin{aligned}
&\rho_t + \mathrm{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho,\\
&(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(a\rho^\gamma + \delta\rho^\beta) + \varepsilon\nabla\mathbf{u}\cdot\nabla\rho\\
&\quad = \nabla(\alpha\frac{J_y}{\rho}g'(1/\rho)h(|\psi|^2)) + (\nabla\times\mathbf{H})\times\mathbf{H} + \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}),\\
&\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}),\\
&\mathrm{div}\,\mathbf{H} = 0.\\
&i\psi_t + \Delta_y\psi = |\omega|^2\psi + \alpha g(v)h'(|\psi|^2)\psi,
\end{aligned}
$$

Regarding this new system, we prove the existence of solutions on any finite time interval provided that $\varepsilon^2/\alpha \gg 1$ and show the convergence of the approximate solutions when the artificial viscosity ε together with the interaction coefficients α tend to 0 and as N tends to ∞. Then, we make δ tend to zero and show convergence to a renormalized solution of the system formed by the MHD equations together with the decoupled nonlinear Schrödinger equation. As emphasized before, the proposed approximation scheme has the purpose to legitimize the coordinates of the limiting Schrödinger equation to be considered as the Lagrangian coordinates of the fluid in a generalized sense.

Let us remark that our results hold in a smooth bounded open spacial domain in $\mathbb{R}^2$. The only restriction that does not allow us to proceed in the full three dimensional case comes from the lack of solvability of the nonlinear Schrödinger equation in this setting. However, assuming this our methods can be adapted to the three dimensional case. Also, our result covers large initial data and permits vacuum at the price of obtaining only weak solutions.

These results, both in the one dimensional and in the multidimensional cases, are the result of the research developed during this Ph.D. program, under the supervision of prof. Hermano Frid. Let us mention that the results on the multidimensional case are product of an ongoing collaboration with prof. Hermano Frid, as well as with prof. Ronghua Pan.

Chapter 2

Physical considerations and deduction of the equations

The aim of this chapter is to give a detailed description of the Short Wave-Long Wave Interactions model to be studied throughout this thesis. We first review of some of the ideas involved in the deduction of the MHD equations, then introduce the SW-LW interactions coupling and finally specify the structural conditions, constitutive relations and general assumptions under which we develop our analysis.

2.1 The Magnetohydrodynamics equations

Magnetohydrodynamics (MHD) concerns the dynamics of a compressible conducting fluid in the presence of a magnetic field. This interaction is described by a coupling between the Navier-Stokes equations, modelling the hydrodynamic part, and the Maxwell's equations, which describe the electromagnetic effects.

Let us begin by reviewing the general ideas behind the deduction of the MHD coupling.

2.1.1 The general fluid equations

In continuum mechanics the motion of a body is described by a family of one to one mappings

$$\mathbf{X}(t,\cdot):\Omega\to\Omega,\qquad t\in I,$$

where $I\subseteq\mathbb{R}$ is an interval (representing time) and $\Omega\subseteq\mathbb{R}^n$ is a spatial domain occupied by the body. The *Continuum hypothesis* requires $X(t,\cdot)$ to be a diffeomorfism for any fixed $t \in I$. Regarding $\mathbf{X}(t, \cdot)$ as the evolution of the motion in time, it is convenient to choose a reference configuration $\mathbf{X}(t_I, \mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$ at a certain time $t_I \in I$; that is, an initial setting. According to this configuration the curve $t \to \mathbf{X}(t, \mathbf{x})$ describes the trajectory of a particle starting from the position $\mathbf{x}$ at time t_I.

Granted the smoothness of the motion, $\mathbf{X}$ can be completely determined by the velocity field $\mathbf{u} : I \times \Omega \to \Omega$ given by the ordinary differential equation

$$\frac{\partial \mathbf{X}(t, \mathbf{x})}{\partial t} = \mathbf{u}(t, \mathbf{X}(t, \mathbf{x})), \quad \mathbf{X}(t_I, \mathbf{x}) = \mathbf{x}, \quad \text{for } \mathbf{x} \in \Omega,\ t \in I.$$

By applying physical laws, it is possible to deduce certain relations between the motion and the physical properties of the body under consideration. These relations are usually expressed in terms of integral equations which, in turn, can be restated as partial differential equations provided that the motion is smooth.

We are interested in the mathematical aspects of fluid dynamics; when the body in motion is a fluid. We adopt the macroscopic description of the motion which regards a fluid as a *continuum* occupying a certain domain Ω. This is in contrast with the microscopic point of view that considers the fluid as a collection of molecules and describes its motion through the dynamics of each individual particle. Accordingly, the dynamics are completely determined by the *velocity field* denoted by $\mathbf{u}$.

When studying the dynamics of fluids, aside from the velocity field, other quantities are taken into consideration; namely, its *density* ρ and *temperature* θ, regarded as functions of the time $t \in \mathbb{R}$ and the spatial variable $\mathbf{x} \in \Omega$. The general description of the dynamics can be summarized in a system of partial differential equations relating these quantities[1]: the *Navier-Stokes equations.*

The general Navier-Stokes equations are a system consisting of:

- the *continuity equation*:
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0,$$

- the *momentum equation*:
$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}\mathbb{T} = \mathbf{f},$$

[1]Sometimes, the entropy is considered instead of the temperature. These quantities, however, are related by the second law of thermodynamics expressed through Gibbs relation, and provide equivalent descriptions of the dynamics.

- and the *energy equation*:

$$\frac{\partial}{\partial t}\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)\right)+\mathrm{div}\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)\mathbf{u}\right)+\mathrm{div}\mathbf{q}=\mathrm{div}(\mathbb{T}\mathbf{u})+\mathbf{f}\cdot\mathbf{u}.$$

Here, as aforementioned, ρ, $\mathbf{u}$ and θ denote the density, the velocity field and the temperature of the fluid respectively. Additionally, e denotes the *internal energy*, $\mathbb{T}$ is a *stress tensor*, $\mathbf{q}$ is the *energy flux* and $\mathbf{f}$ is an *external force*. The symbol $\otimes$ stands for the tensor product $[\mathbf{u}\otimes\mathbf{u}]_{i,j} := u_i u_j$.

Each one of these equations is derived from particular physical considerations. The continuity equation is a consequence of the *mass conservation principle* stating that mass is preserved along the motion. The momentum equation comes from *Newton's second law of motion*, relating the inertial nature of the fluid to the forces acting on it, decomposed into a stress tensor and external forces. The energy equation is the result of thermodynamics considerations that link changes in the energy of the system due to the motion to the heat flux of the fluid.

The stress tensor in the momentum equation can be written as

$$\mathbb{T}=\mathbb{S}-p\mathrm{Id},$$

where Id is the identity matrix, p is a scalar function called *pressure* and $\mathbb{S}$ is the *viscous stress tensor*. According to the *principle of material frame indifference*, the viscous stress tensor must depend on the velocity field $\mathbf{u}$ and possibly other state variables like ρ and θ. Assuming that the physical properties of the fluid are *isotropic* ("uniform in all orientations") and assuming that $\mathbb{S}$ is a linear function of $\nabla\mathbf{u}$, it can be shown that $\mathbb{S}$ necessarily can be written as

$$\mathbb{S}=\lambda(\mathrm{div}\mathbf{u})Id+2\mu\mathbf{D}(\mathbf{u}), \tag{2.1}$$

where λ and μ are real scalar coefficients called *viscosity coefficients* that may depend on the values of other state variables like ρ and θ, and

$$\mathbf{D}(\mathbf{u})=\frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\top}\right). \tag{2.2}$$

A fluid satisfying (2.1) and (2.2) is called *Newtonian*. The viscosity coefficients are often required to satisfy the relations[2]

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0.$$

In general the pressure is a function of the density and the temperature of the fluid and is expressed through a constitutive relation of the form

$$p = p(\rho, \theta). \tag{2.3}$$

For instance, in the case of *perfect gases* the pressure satisfies *Boyle's law*:

$$p(\rho, \theta) = R\rho\theta,$$

where R is a constant. When the time comes, we will specify the kind of constitutive relation we are going to consider. For now we will stick to the general relation (2.3).

Moving on to the state variables involved in the energy equation, we must consider a constitutive relation for the internal energy as well. Given the respective relation (2.3) for the pressure, the internal energy $e = e(\rho, \theta)$ must satisfy the *second law of thermodynamics*, which in particular implies *Maxwell's relation*

$$\frac{\partial e}{\partial \rho} = \frac{1}{\rho^2}\left(p(\rho, \theta) - \theta p_\theta\right).$$

Concerning the energy flux, we consider the constitutive relation

$$\mathbf{q} = -\kappa \nabla \theta,$$

where κ is a nonnegative scalar function called *heat conductivity coefficient* which may depend on ρ and θ. This relation is known as *Fourier's law*.

Gathering all this information we arrive to Navier-Stokes-Fourier system of equa-

[2]The fluid dynamics equations make mathematical sense in n dimensions. In this general case the number 3 in the denominator of the second relation should be replaced by n. We, however, restrict ourselves to the 3-dimensional case.

tions for a Newtonian fluid, given by

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{2.4}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{div}\left(\lambda (\mathrm{div}\mathbf{u})\mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) + \mathbf{f}, \tag{2.5}$$

$$\begin{aligned}\left(\rho \left(\frac{1}{2}|\mathbf{u}|^2 + e \right) \right)_t + \mathrm{div}\left(\mathbf{u} \left(\rho \left(\frac{1}{2}|\mathbf{u}|^2 + e \right) + p \right) \right) &= \mathrm{div}\left(\kappa \nabla \theta \right) \\ &+ \mathrm{div}\left(\left(\lambda (\mathrm{div}\mathbf{u})\mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) \cdot \mathbf{u} \right) + \mathbf{f} \cdot \mathbf{u}. \end{aligned} \tag{2.6}$$

In order to conclude this Section and for future reference, we point out that, in view of (2.4) and (2.5), the energy equation (2.6) is equivalent to

$$(\rho e)_t + \mathrm{div}(\rho e \mathbf{u}) + p \mathrm{div}\mathbf{u} = \mathrm{div}(\kappa \nabla \theta) + \lambda (\mathrm{div}\mathbf{u})^2 + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) : \nabla \mathbf{u}. \tag{2.7}$$

2.1.2 The equations of electromagnetism

Magnetohydrodynamics (MHD) concerns the dynamics of conducting fluids in a magnetic field. While the hydrodynamic part is described by the Navier-Stokes equations, the electromagnetic effects are governed by the *Maxwell's equations*. The general MHD equations consist of a coupling between the two.

In practice, this coupling is not stated in its most general form. Several assumptions that simplify the model are made, often motivated by physical considerations such as empirical data of actual materials. For instance, the *magnetic permeability*, which is a parameter characteristic of each particular material, is usually assumed to be constant and equal to 1 since it differs only slightly from the unity in most real world media covered by the MHD model (see [33]). For our purposes, however, the magnetic permeability will be important and we need to keep track of it in the deduction of the coupling in order to state correctly the equations we are going to work with. With this in mind, we now turn our attention to the electromagnetic description.

Let us recall Maxwell's equations of electromagnetism.

- the *Maxwell-Ampère equation*:

$$-\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \mathbf{j},$$

- the *Maxwell-Coulomb equation*:

$$\operatorname{div} \mathbf{D} = \rho_c,$$

- the *Maxwell-Faraday equation*:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0,$$

- and the *Maxwell-Gauss equation*:

$$\operatorname{div} \mathbf{B} = 0$$

Here, the three-dimensional vector fields $\mathbf{D}, \mathbf{B}, \mathbf{E}, \mathbf{H}, \mathbf{j}$ are the *electric induction*, the *magnetic induction*, the *electric field*, the *magnetic field* and the *current density*, respectively, while the scalar field ρ_c denotes the charge density (not to be confused with the fluid's density ρ of the previous section).

Similarly to the fluid equations of motion, the Maxwell's equations come from particular physical principles. They are related to *Ampère's law*, *Gauss' law for electric fields*, *Faraday's law* and *Gauss' law for magnetism*, respectively.

According to the physical properties of the medium where the electromagnetic fields propagate, some relations that link the vector fields $\mathbf{D}, \mathbf{B}, \mathbf{E}$ and $\mathbf{H}$ can be formulated. Specifically, these relations are of the form

$$\begin{cases} D = \tilde{\varepsilon}\mathbf{E} \\ H = \tilde{\mu}^{-1}\mathbf{B}, \end{cases} \tag{2.8}$$

for some $\tilde{\varepsilon}$ and $\tilde{\mu}$ called *electric permitivity* (or dielectric constant) and *magnetic permeability* of the medium. These parameters may depend on $\mathbf{E}$ and $\mathbf{B}$ respectively (and may also depend on other quantities such as the density or the temperature in the case of fluids) and are, in general, tensor valued. However, in the simple isotropic, homogeneous case, both $\tilde{\varepsilon}$ and $\tilde{\mu}$ can be assumed to be scalar and constant and the medium is called a *perfect medium.*

Accordingly, the Maxwell's equations take the following form:

$$-\frac{\partial(\tilde{\varepsilon}\mathbf{E})}{\partial t} + \nabla \times \left(\frac{1}{\tilde{\mu}}\mathbf{B}\right) = \mathbf{j},$$

$$\operatorname{div}(\tilde{\varepsilon}\,\mathbf{E}) = \rho_c,$$
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0,$$
$$\operatorname{div}\mathbf{B} = 0$$

In order to close the system one more relation is needed. Such relation will be provided by *Ohm's law*, which relates $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$. As it also involves the velocity field of the medium we treat it in the following section where we talk about the MHD coupling.

2.1.3 The MHD Coupling

MHD models the interaction of conductive fluids and electromagnetic fields. When a conducting fluid moves in a magnetic field, electric fields and electric currents develop. Meanwhile, the magnetic field exerts forces on these currents which affect the motion of the fluid. Such an interaction is described by a coupling between the equations of fluid mechanics and the equations of electromagnetism. In order to complete the MHD model it is necessary to specify the body force exerted by the magnetic field, through the external force term in the momentum equation (2.5). Furthermore, a term must be added to the energy equation (2.6) due to mechanical work exerted by the magnetic field that dissipates into heat in the conductor.

In our current situation (a homogeneous Newtonian conductive fluid in the presence of a magnetic field) the body force can be decomposed as

$$\mathbf{f} = \mathbf{j} \times \mathbf{B} + \mathbf{f}_{\text{ext}},$$

where the first term is the *Lorentz force*, owning to the electric current $\mathbf{j}$ within the magnetic field $\mathbf{H}$, which is related to $\mathbf{B}$ through (2.8), and the second term is due to possible further external forces.

Additionally, in the present setting Ohm's law can be stated as

$$\mathbf{j} = \tilde{\sigma}(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where $\tilde{\sigma}$ denotes the *electric conductivity* of the field.

Moreover, according to *Joule's law*, the energy dissipation in a conductor when a given current flows in it (the *Joule heat*) is given by

$$\mathbf{j} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

and must be added to the right hand side of equation (2.7) (see [33]). As a result we obtain the equation

$$\begin{aligned}&(\rho e)_t + \mathrm{div}(\rho e \mathbf{u}) + p \,\mathrm{div}\mathbf{u} \\ &\qquad = \mathrm{div}(\kappa \nabla \theta) + \lambda (\mathrm{div}\mathbf{u})^2 + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) : \nabla \mathbf{u} + \mathbf{j} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B}).\end{aligned}$$

Gathering all this information, we obtain the following general system for MHD:

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{2.9}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{div}\left(\lambda (\mathrm{div}\mathbf{u}) \mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) + \mathbf{j} \times \mathbf{B} + \mathbf{f}_{\mathrm{ext}}, \tag{2.10}$$

$$\begin{aligned}&(\rho e)_t + \mathrm{div}(\rho e \mathbf{u}) + p \,\mathrm{div}\mathbf{u} \\ &\qquad = \mathrm{div}(\kappa \nabla \theta) + \lambda (\mathrm{div}\mathbf{u})^2 + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) : \nabla \mathbf{u} + \mathbf{j} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B}).\end{aligned} \tag{2.11}$$

$$-\frac{\partial (\tilde{\varepsilon} \mathbf{E})}{\partial t} + \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) = \mathbf{j}, \tag{2.12}$$

$$\mathrm{div}(\tilde{\varepsilon} \mathbf{E}) = \rho_c, \tag{2.13}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{2.14}$$

$$\mathrm{div}\, \mathbf{B} = 0, \tag{2.15}$$

$$\mathbf{j} = \tilde{\sigma} (\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{2.16}$$

Note that, in view of equations (2.9), (2.10) and (2.16), equation (2.11) is equivalent to

$$\begin{aligned}&\left(\rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \right)_t + \mathrm{div} \left(\mathbf{u} \left(\rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) + p \right) \right) = \mathrm{div} \left(\kappa \nabla \theta \right) \\ &\qquad + \mathrm{div}\left(\left(\lambda (\mathrm{div}\mathbf{u}) \mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) \cdot \mathbf{u} \right) + \mathbf{j} \cdot \mathbf{E} + \mathbf{f}_{\mathrm{ext}} \cdot \mathbf{u}.\end{aligned} \tag{2.17}$$

This is the most general system modelling a compressible magnetohydrodynamic flow. Given its great complexity, we adopt a commonly used simplification of the model: it is often assumed that the first term $\partial(\tilde{\varepsilon} \mathbf{E})/\partial t$ of the Maxwell-Ampère equation, called the *displacement current*, is small and can be neglected (see [33]), so that, from equation (2.12) we obtain

$$\mathbf{j} = \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right).$$

Also, from Ohm's law and (2.14) we have

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \left(\frac{1}{\tilde{\sigma}} \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) \right) = \nabla \times (\mathbf{u} \times \mathbf{B}).$$

Since, $\mathbf{j}$ and $\mathbf{E}$ (and therefore ρ_c) are completely determined by $\mathbf{B}$ and $\mathbf{u}$, we can drop equations (2.12), (2.13) and (2.16). Moreover, we have that

$$\begin{aligned} \mathbf{j} \cdot \mathbf{E} &= \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) \cdot \left(\frac{1}{\tilde{\sigma}} \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) - \mathbf{u} \times \mathbf{B} \right) \\ &= \operatorname{div} \left(\frac{1}{\tilde{\mu}} \mathbf{B} \times \left(\frac{1}{\tilde{\sigma}} \nabla \times \frac{1}{\tilde{\mu}} \mathbf{B} \right) \right) + \nabla \times \left(\frac{1}{\tilde{\sigma}} \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) \right) \cdot \frac{1}{\tilde{\mu}} \mathbf{B} \\ &\quad + \operatorname{div} \left((\mathbf{u} \times \mathbf{B}) \times \frac{1}{\tilde{\mu}} \mathbf{B} \right) - \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \frac{1}{\tilde{\mu}} \mathbf{B}, \end{aligned}$$

where we used the identity $\operatorname{div}(\mathbf{V} \times \mathbf{W}) = (\nabla \times \mathbf{V}) \cdot \mathbf{W} - \mathbf{V} \cdot (\nabla \times \mathbf{W})$ from vector calculus. Using this, taking the inner product of (2.14) with $\tilde{\mu}^{-1}\mathbf{B}$ and adding the resulting equation to (2.17) we deduce the energy equation for the simplified system:

$$\begin{aligned} &\left(\rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) + \frac{1}{2\tilde{\mu}} |\mathbf{B}|^2 \right)_t + \operatorname{div} \left(\mathbf{u} \left(\rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) + p \right) \right) = \operatorname{div} (\kappa \nabla \theta) \\ &\quad + \operatorname{div} \left(\left(\lambda (\operatorname{div} \mathbf{u}) \mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) \cdot \mathbf{u} \right) + \operatorname{div} \left((\mathbf{u} \times \mathbf{B}) \times \frac{1}{\tilde{\mu}} \mathbf{B} \right) \\ &\quad + \operatorname{div} \left(\frac{1}{\tilde{\mu}} \mathbf{B} \times \left(\frac{1}{\tilde{\sigma}} \nabla \times \frac{1}{\tilde{\mu}} \mathbf{B} \right) \right) + \mathbf{f}_{\text{ext}} \cdot \mathbf{u}. \end{aligned} \tag{2.18}$$

Accordingly, the momentum equation results in

$$\begin{aligned} &(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p \\ &\quad = \operatorname{div} \left(\lambda (\operatorname{div} \mathbf{u}) \mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \right) + \left(\nabla \times \frac{1}{\tilde{\mu}} \mathbf{B} \right) \times \mathbf{B} + \mathbf{f}_{\text{ext}}, \end{aligned} \tag{2.19}$$

the conservation of mass remains the same:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, , \tag{2.20}$$

and the Maxwell equations yield

$$\mathbf{B}_t + \nabla \times \left(\frac{1}{\tilde{\sigma}} \nabla \times \left(\frac{1}{\tilde{\mu}} \mathbf{B} \right) \right) = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{2.21}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2.22}$$

thus obtaining the simplified MHD system consisting of equations (2.18), (2.19), (2.20), (2.21) and (2.22).

From this point on we are going to restrict ourselves to this system and are going to address it simply as the *MHD system*.

2.2 Short Wave-Long Wave Interactions

Up to this point we have only reviewed some of the ideas involved in the deduction of the MHD system. Our main goal is to study certain mathematical aspects of the interaction between the MHD system and a nonlinear Schrödinger equation in the context of *Short Wave-Long Wave Interactions*. Our work is mainly inspired by three papers that pursue similar objectives.

The first paper written by J. P. Dias and H. Frid in 2011 where, inspired by the work of Benney on short wave-long wave interactions in [5], they propose a model consisting of a coupling between the Navier-Stokes equations for a compressible isentropic fluid and a nonlinear Schrödinger equation, studying existence and uniqueness of global solutions and the problem of vanishing viscosity and interaction coefficient limit in the one space dimensional context (see [17]).

The second paper, by H. Frid, R. Pan and W. Zhang in 2014 ([27]) which addresses the problem of global existence of smooth solutions to the Cauchy problem, when the initial data are smooth small perturbations of an equilibrium state, for a similar coupling; this time in the full 3D case.

More recently, in 2016, H. Frid, J. Jia and R. Pan extended the results of this last paper for a similar short short wave-long wave interactions coupling only this time involving the MHD equations instead of the Navier Stokes system ([28]).

Our main goal here is to study this same model in several other contexts and to give answers that are still open such as existence of global strong solutions in the planar 1D case, existence of weak solutions in the full 3D case, vanishing viscosity, etc.

In the model, the Schrödinger equation is coupled to the MHD system along particle paths, meaning that the former is stated in a different coordinate system; namely in *Lagrangian coordinates.*

The MHD system we deduced earlier is stated in the so called *Eulerian coordinates*, where the motion is described from an outsider's point of view. In the Eulerian description, the spatial variable and the temporal one are independent. The Lagrangian coordinate, in contrast, is constant along the trajectories. In other words, the Lagrangian description follows the flow, as if the observer is on a boat following the stream lines. As the Schrödinger equation is stated in Lagrangian coordinates we must first give a precise definition of this coordinate system.

2.2.1 Lagrangian Coordinates and Coupling

Given a velocity field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ in $\mathbb{R}^d$ for $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$, the Lagrangian coordinate related to u can be defined in the following way.

For $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^d$, let $\Phi(t, \mathbf{x})$ be the solution of the initial value problem

$$\frac{d\Phi}{dt}(t; \mathbf{x}) = \mathbf{u}(t, \Phi(t; \mathbf{x})), \tag{2.23}$$

$$\Phi(0; \mathbf{x}) = \mathbf{x}.$$

Then, the Jacobian $J_\Phi(t; \mathbf{x}) = \det\left(\frac{\partial \Phi}{\partial \mathbf{x}}(t; \mathbf{x})\right)$ of the transformation $x \mapsto \Phi(t; \mathbf{x})$ satisfies

$$\frac{dJ_\Phi}{dt}(t; \mathbf{x}) = \operatorname{div} \mathbf{u}(t, \Phi(t; \mathbf{x})) J_\Phi(t; \mathbf{x}), \tag{2.24}$$

$$J_\Phi(0; \mathbf{x}) = 1.$$

As aforementioned the Lagrangian coordinate is characterized by being constant along particle paths. In the notation introduced above, the *Lagrangian transformation* $\mathbf{Y}(t, \mathbf{x}) = \mathbf{Y}(t, \mathbf{y}(t, \mathbf{x}))$ can thus be defined by the relation

$$\mathbf{y}(t, \Phi(t; \mathbf{x})) = \mathbf{y}_0(\mathbf{x}), \tag{2.25}$$

where the function $\mathbf{y}_0$ is a diffeomorphism which may be chosen conveniently according to the problem.

In the 3-dimensional case of the MHD system, we can define the Lagrangian coordinate through the velocity field $\mathbf{u}$ of the fluid, by defining

$$\mathbf{y}_0(\mathbf{x}) := \Big(x_1, x_2, \int_0^{x_3} \rho_0(x_1, x_2, s)ds\Big), \tag{2.26}$$

where, ρ_0 is the initial density $\rho_0(\mathbf{x}) = \rho(0, \mathbf{x})$. This is in accordance with [27] and [28].

By (2.25), we have that

$$\frac{\partial \mathbf{y}}{\partial \mathbf{z}}(t, \Phi(t; \mathbf{x})) \cdot \frac{\partial \Phi}{\partial \mathbf{x}}(t; \mathbf{x}) = \frac{\partial \mathbf{y}_0}{\partial \mathbf{x}}(\mathbf{x}).$$

Defining $J_{\mathbf{y}}(t; \mathbf{x}) := \det\left(\frac{\partial \mathbf{y}}{\partial \mathbf{z}}(t, \Phi(t; \mathbf{x}))\right)$, we consequently have

$$J_{\mathbf{y}}(t; \mathbf{x}) J_{\Phi}(t; \mathbf{x}) = \det \frac{\partial \mathbf{y}_0}{\partial \mathbf{z}}(\mathbf{x}).$$

Taking derivative with respect to t we find that

$$\left(\frac{d}{dt} J_{\mathbf{y}}(t; \mathbf{x})\right) J_{\Phi}(t; \mathbf{x}) + J_{\mathbf{y}}(t; \mathbf{x}) \left(\frac{d}{dt} J_{\Phi}(t; \mathbf{x})\right) = 0,$$

and using (2.24) we get

$$\frac{d}{dt} J_{\mathbf{y}}(t; \mathbf{x}) = -(\operatorname{div} \mathbf{u}) J_{\mathbf{y}}(t; \mathbf{x}).$$

Therefore from equation (2.20) we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho(t, \Phi(t; \mathbf{x}))}{J_{\mathbf{y}}(t; \mathbf{x})}\right) \\
&= \frac{[\rho_t(t, \Phi(t; \mathbf{x})) + \nabla \rho(t, \Phi(t; \mathbf{x})) \cdot \mathbf{u}(t, \Phi(t; \mathbf{x}))] J_{\mathbf{y}}(t; \mathbf{x}) + \rho(t, \Phi(t; \mathbf{x}))(\operatorname{div} \mathbf{u}) J_{\mathbf{y}}(t; \mathbf{x})}{J_{\mathbf{y}}(t; \mathbf{x})^2} \\
&= \frac{[-\rho(t, \Phi(t; \mathbf{x}))(\operatorname{div} \mathbf{u}) J_{\mathbf{y}}(t; \mathbf{x}) + \rho(t, \Phi(t; \mathbf{x}))(\operatorname{div} \mathbf{u}) J_{\mathbf{y}}(t; \mathbf{x})}{J_{\mathbf{y}}(t; \mathbf{x})^2} \\
&= 0.
\end{aligned}$$

And since $J_{\mathbf{y}}(0;\mathbf{x}) = \rho_0(\mathbf{x})$, we conclude that

$$\det\left(\frac{\partial \mathbf{y}}{\partial \mathbf{z}}(t,\Phi(t;\mathbf{x}))\right) = J_{\mathbf{y}}(t;\mathbf{x}) = \rho(t,\Phi(t;\mathbf{x})), \tag{2.27}$$

i.e.

$$\det\left(\frac{\partial \mathbf{y}}{\partial \mathbf{z}}(t,\mathbf{z})\right) = \rho(t,\mathbf{z}), \tag{2.28}$$

for all $(t,\mathbf{z}) \in [0,\infty) \times \mathbb{R}^3$.

We are interested in a coupling, involving a nonlinear Schrödinger equation along particle paths. To that end, we consider the following Schrödinger equation

$$i\frac{d\psi}{dt} + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi + G\psi, \tag{2.29}$$

where ψ is the complex valued wave function, G is a real valued function corresponding to a potential due to external forces and $\mathbf{y}$ is the Lagrangian coordinate as defined above. Recall that the momentum equation (2.19) also has a term accounting for possible external forces.

As in [17], [27] and [28] we propose to model the short wave-long wave interaction by taking $\mathbf{f}_{\text{ext}}$ in (2.19) and G in (2.29) as

$$\mathbf{f}_{\text{ext}} = \alpha\nabla(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)), \qquad G = \tilde{\alpha}(g(v)h'(|\psi|^2)), \tag{2.30}$$

where α and $\tilde{\alpha}$ are positive constants, $\mathbf{Y}(t,\mathbf{x}) = (t,\mathbf{y}(t,\mathbf{x}))$ is the Lagrangian transformation as before, $v(t,\mathbf{y})$ is the *specific volume* defined by

$$v(t,\mathbf{y}(t,\mathbf{x})) = \frac{1}{\rho(t,\mathbf{x})}, \tag{2.31}$$

and $g,h : [0,\infty) \to [0,\infty)$ are nonnegative smooth functions with $h(0) = 0$.

The most important feature of this coupling is that it is endowed with an energy identity. Indeed, having defined the external force by (2.30), the last term of equation (2.18) reads

$$\mathbf{f}_{\text{ext}}\cdot\mathbf{u} = \alpha\mathrm{div}(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\mathbf{u}) - \alpha g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\mathrm{div}\mathbf{u}.$$

Multiplying (2.20) by $-(1/\rho)\alpha g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)$ we deduce that

$$-\alpha g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\mathrm{div}\,\mathbf{u}=-\alpha(g(1/\rho)_t+\mathbf{u}\cdot\nabla_{\mathbf{x}}g(1/\rho))h(|\psi\circ\mathbf{Y}|^2)\rho.$$

Observe that from the definition of $\mathbf{Y}$ we have the conversion formula between Eulerian and Lagrangian coordinates:

$$\beta(t,\mathbf{y})_t=(\beta\circ\mathbf{Y}(t,\mathbf{x}))_t+\mathbf{u}\cdot\nabla_{\mathbf{x}}(\beta\circ\mathbf{Y}(t,\mathbf{x})),$$

or synthetically,

$$\beta(t,\mathbf{y})_t=\beta_t(t,\mathbf{x})+\mathbf{u}\cdot\nabla_{\mathbf{x}}\beta(t,\mathbf{x}).$$

Keeping in mind the previously deduced formula for the Jacobian of the Lagrangian transformation synthesized by the identity $d\mathbf{y}=\rho(t,\mathbf{x})d\mathbf{x}$, we multiply equation (2.29) by $\overline{\psi}_t$ (the complex conjugate of ψ_t), take real part and incorporate the definition of G to obtain

$$\begin{aligned}
&-\alpha(g(1/\rho)_t+\mathbf{u}\cdot\nabla_{\mathbf{x}}g(1/\rho))h(|\psi\circ\mathbf{Y}|^2)\rho\,d\mathbf{x}\\
&\quad=-\alpha g(v(t,\mathbf{y}))_t\,h(|\psi(t,\mathbf{y})|^2)\,d\mathbf{y}\\
&\quad=-\alpha\Big\{\Big(g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2)\Big)_t-g(v(t,\mathbf{y}))\,h(|\psi(t,\mathbf{y})|^2)_t\Big\}d\mathbf{y}\\
&\quad=\frac{\alpha}{\tilde\alpha}\Big\{\mathrm{div}_{\mathbf{y}}(\overline{\psi}_t\nabla_{\mathbf{y}}\psi+\psi_t\nabla_{\mathbf{y}}\overline{\psi})-\tilde\alpha\Big(g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2)\Big)_t\\
&\qquad\qquad-\frac12(|\nabla_{\mathbf{y}}\psi(t,\mathbf{y})|^2)_t-\frac12(|\psi(t,\mathbf{y})|^4)_t\Big\}d\mathbf{y}.
\end{aligned}$$

Putting all of this information together and replacing it in the energy equation (2.18) we arrive at the following differential form of the conservation of energy

$$\begin{aligned}
&\Big\{\Big(\rho\Big(\frac12|\mathbf{u}|^2+e\Big)+\frac{1}{2\tilde\mu}|\mathbf{B}|^2\Big)_t+\mathrm{div}_{\mathbf{x}}\Big(\mathbf{u}\Big(\rho\Big(\frac12|\mathbf{u}|^2+e\Big)+p\Big)\Big)\\
&\quad-\mathrm{div}_{\mathbf{x}}(\kappa\nabla\theta)-\mathrm{div}_{\mathbf{x}}\Big(\Big(\lambda(\mathrm{div}\mathbf{u})\mathrm{Id}+\mu\Big(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\Big)\Big)\cdot\mathbf{u}\Big)\\
&\qquad-\mathrm{div}_{\mathbf{x}}\Big((\mathbf{u}\times\mathbf{B})\times\frac{1}{\tilde\mu}\mathbf{B}\Big)-\mathrm{div}_{\mathbf{x}}\Big(\frac{1}{\tilde\mu}\mathbf{B}\times\Big(\frac{1}{\tilde\sigma}\nabla_{\mathbf{x}}\times\frac{1}{\tilde\mu}\mathbf{B}\Big)\Big)\Big\}d\mathbf{x}\\
&\qquad\quad=\frac{\alpha}{\tilde\alpha}\Big\{\mathrm{div}_{\mathbf{y}}(\overline{\psi}_t\nabla_{\mathbf{y}}\psi+\psi_t\nabla_{\mathbf{y}}\overline{\psi})-\tilde\alpha\Big(g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2)\Big)_t\\
&\qquad\qquad\quad-\frac12(|\nabla_{\mathbf{y}}\psi(t,\mathbf{y})|^2)_t-\frac12(|\psi(t,\mathbf{y})|^4)_t\Big\}d\mathbf{y}.
\end{aligned}\tag{2.32}$$

In particular, under suitable integrability conditions, this identity yields an integral form of the conservation of energy:

$$\frac{d}{dt}\int\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\frac{1}{2\tilde{\mu}}|\mathbf{B}|^2\right)d\mathbf{x}$$
$$+\frac{d}{dt}\int\frac{\alpha}{\tilde{\alpha}}\left(\frac{1}{2}|\nabla_{\mathbf{y}}\psi(t,\mathbf{y})|^2+\frac{1}{2}|\psi(t,\mathbf{y})|^4+\tilde{\alpha}g(v(t,\mathbf{y}))h(|\psi(t,\mathbf{y})|^2)\right)d\mathbf{y}=0.$$

2.2.2 The Final System

As a result we are left with the following system of equations:

$$\rho_t+\operatorname{div}(\rho\mathbf{u})=0, \tag{2.33}$$

$$(\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p-\alpha\nabla\Big(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\Big)$$
$$=\operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id}+\mu\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right)\Big)+\left(\nabla\times\frac{1}{\tilde{\mu}}\mathbf{B}\right)\times\mathbf{B}, \tag{2.34}$$

$$\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+\frac{1}{2\tilde{\mu}}|\mathbf{B}|^2\right)_t+\operatorname{div}\left(\mathbf{u}\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)+p\right)\right)=\operatorname{div}\left(\kappa\nabla\theta\right)$$
$$+\operatorname{div}\Big(\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id}+\mu\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right)\Big)\cdot\mathbf{u}\Big)+\operatorname{div}\left((\mathbf{u}\times\mathbf{B})\times\frac{1}{\tilde{\mu}}\mathbf{B}\right)$$
$$+\operatorname{div}\left(\frac{1}{\tilde{\mu}}\mathbf{B}\times\left(\frac{1}{\tilde{\sigma}}\nabla\times\frac{1}{\tilde{\mu}}\mathbf{B}\right)\right)+\alpha\nabla\Big(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\Big)\cdot\mathbf{u}, \tag{2.35}$$

$$\mathbf{B}_t+\nabla\times\left(\frac{1}{\tilde{\sigma}}\nabla\times\left(\frac{1}{\tilde{\mu}}\mathbf{B}\right)\right)=\nabla\times(\mathbf{u}\times\mathbf{B}), \tag{2.36}$$

$$\operatorname{div}\mathbf{B}=0, \tag{2.37}$$

$$i\psi_t+\Delta_{\mathbf{y}}\psi=|\psi|^2\psi+\tilde{\alpha}g(v)h'(|\psi|^2)\psi. \tag{2.38}$$

Here, the Schrödinger equation (2.38) is stated in the Lagrangian coordinates (hence the subindex $\mathbf{y}$ in the differential operator) and the rest of the equations are stated in Eulerian coordinates.

This system is also endowed with the energy identity (2.32). Let us not forget about the constitutive relations:

$$p=p(\rho,\theta),\qquad e=e(\rho,\theta),$$

and Maxwell's relation:

$$e_\rho(\rho,\theta)=\frac{1}{\rho^2}(p(\rho,\theta)-\theta p_\theta(\rho,\theta)). \tag{2.39}$$

Of course ρ is always a nonnegative function and the Lagrangian transformation $\mathbf{Y} = \mathbf{Y}(t, \mathbf{y}(t, \mathbf{x}))$ satisfies

$$\left|\det \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(t, \mathbf{x})\right| = \rho(t, \mathbf{x}), \qquad \text{for all } (t, \mathbf{x}).$$

As a result the Lagrangian transformation is nonsingular as long as the density is strictly positive and finite (away from *vacuum* and *concentration*).

2.3 Structural conditions

In this section we state the conditions under which we base our entire analysis on. These conditions include growth conditions on the pressure, internal energy and heat conductivity, initial and boundary conditions and hypotheses on the coupling functions.

Although some of the results in this work may be stated in a more general setting, in order to maintain a more clean presentation we are going to restrict our analysis to a specific set of constitutive relations which we specify below. Our assumptions cover a variety of physical cases and agree with several other references in the literature.

2.3.1 Constitutive relations

In this work, we consider a general constitutive relation for the pressure of the form

$$p(\rho, \theta) = p_e(\rho) + \theta p_\theta(\rho). \tag{2.40}$$

That is, we assume that the pressure can be decomposed into an elastic part p_e and a thermal part θp_θ which depends linearly on the temperature. Note that (2.40) can be viewed as the first two terms of a Taylor expansion

$$p(\rho, \theta) = p(\rho, \theta_0) + (\theta - \theta_0) p_\theta(\rho, \theta_0) + O((\theta - \theta_0)^2),$$

for a given $\theta > 0$. Such constitutive relation agrees with the one considered in [23] and we refer to it for a wide discussion on its physical relevance.

For our purposes, we are going to assume that the elastic part of the pressure is given by a γ-law:

$$p_e(\rho) = a\rho^\gamma, \tag{2.41}$$

for some $a > 0$ and $\gamma > 1$. This last constraint may be relaxed as in [23], but we choose to restrict ourselves to this more simple case, since it illustrates satisfactorily the methods used in our study.

Now, according to Maxwell's relation (2.39) the internal energy can be written in the form

$$e(\rho, \theta) = P_e(\rho) + Q(\theta), \tag{2.42}$$

with P_e given by

$$P_e(\rho) = \frac{a}{\gamma - 1}\rho^{\gamma-1}, \tag{2.43}$$

and $Q(\theta)$ given by

$$Q(\theta) = \int_0^\theta C_\vartheta(z)dz, \tag{2.44}$$

where $C_\vartheta(\theta) := \partial e/\partial\theta$ is the *specific heat at constant volume*, which depends only on the temperature.

Under these assumptions, using equation (2.33) it is easy to see that equation (2.35) can be rewritten as

$$\begin{aligned}&(\rho Q(\theta))_t + \mathrm{div}(\rho Q(\theta)\mathbf{u}) + \theta p_\theta(\rho)\mathrm{div}\mathbf{u}\\ &\quad = \mathrm{div}(\kappa\nabla\theta) + \lambda(\mathrm{div}\mathbf{u})^2 + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) : \nabla\mathbf{u} + \tfrac{1}{\tilde{\sigma}}\left|\nabla\times(\tfrac{1}{\tilde{\mu}}\mathbf{B})\right|^2.\end{aligned} \tag{2.45}$$

Let us also introduce the *specific entropy* $s = s(\rho, \theta)$ through the thermodynamic relations

$$\theta s_\rho = e_\rho - \frac{p}{\rho^2}, \qquad \theta s_\theta = e_\theta,$$

that is

$$s(\rho, \theta) = \int_1^\theta \frac{C_\vartheta(z)}{z}dz - P_\vartheta(\rho), \tag{2.46}$$

where,

$$P_\vartheta(\rho) := \int_1^\rho \frac{p_\theta(z)}{z^2}dz. \tag{2.47}$$

In connection with (2.45), the entropy then satisfies the following equation

$$\begin{aligned}&(\rho s)_t + \mathrm{div}(\rho s\mathbf{u}) - \mathrm{div}\left(\frac{\kappa\nabla\theta}{\theta}\right)\\ &\quad = \frac{\kappa|\nabla\theta|^2}{\theta^2} + \frac{\lambda}{\theta}(\mathrm{div}\mathbf{u})^2 + \frac{\mu}{\theta}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) : \nabla\mathbf{u} + \frac{1}{\theta}\left|\tfrac{1}{\tilde{\sigma}}\nabla\times\tfrac{1}{\tilde{\mu}}\mathbf{B})\right|^2.\end{aligned} \tag{2.48}$$

2.3.2 Initial and boundary conditions

Throughout this work we are going to assume that the fluid under consideration occupies a domain $\Omega \subseteq \mathbb{R}^3$ during a time interval $[0,T]$, where $T>0$ is arbitrary. We focus on the case where Ω is a smooth and bounded domain. Accordingly, equations (2.33)-(2.38) must be supplemented with a set of initial and boundary conditions in order to obtain a well posed problem. Regarding the boundary conditions, we assume that

$$\begin{aligned}
&(\mathbf{u}, \nabla_{\mathbf{x}}\theta \cdot \mathbf{n}, \mathbf{B})|_{\partial\Omega} = 0, && \\
&\psi|_{\partial\Omega_{\mathbf{y}}} = 0, && t \in [0,T].
\end{aligned}$$

Here, $\mathbf{n}$ is the outer normal vector of Ω, and $\Omega_{\mathbf{y}}$ is the corresponding domain of the Lagrangian coordinate $\mathbf{y}$. Note that, by virtue of (2.25) and the boundary condition on the velocity, the Lagrangian transformation $Y(t,\cdot) : \Omega \to \mathbb{R}^3$ is a diffeomorphism onto $\Omega_{\mathbf{y}} := \mathbf{y}_0(\Omega)$ for every $t \in [0,T]$.

As for the initial conditions, we assume that

$$\begin{aligned}
&(\rho, \mathbf{m}, \rho Q(\theta), \mathbf{B})(0,\mathbf{x}) = (\rho_0, \mathbf{m}_0, \chi_0, \mathbf{B}_0)(\mathbf{x}), && \mathbf{x} \in \Omega \\
&\psi(0,\mathbf{y}) = \psi_0(\mathbf{y}), && \mathbf{y} \in \Omega_{\mathbf{y}},
\end{aligned}$$

where $\mathbf{m} = \rho\mathbf{u}$ is the momentum of the fluid.

The reason why we specify the initial conditions in terms of $\mathbf{m}$ and ρQ is to include regimes where the density ρ may vanish. Although our model depends on the fact that the density is strictly positive, since the Lagrangian coordinate becomes singular in the presence of vacuum, we will discuss the possible existence of weak solutions of our system where the Lagrangian coordinate makes sense in an approximate way. On the other hand, equations (2.34) and (2.35) (and respectively (2.45)) become singular whenever ρ vanishes. This problem will be dealt with in time. For now, we simply demand that $\mathbf{m}_0$ and χ_0 satisfy the following compatibility condition

$$(\mathbf{m}_0, \chi_0) = 0, \text{ on the set } \{x \in \Omega : \rho_0(x) = 0\}. \tag{2.49}$$

2.3.3 Growth conditions and coupling functions

Finally, let us stablish the growth conditions on the pressure, thermal energy and heat conductivity; as well as the hypotheses on the coupling functions g and h. The conditions below are in accordance with those in [23] and also agree with those imposed in [15] and in [48], where a more general constitutive relation for the pressure is considered.

Pressure:

As already mentioned, we are considering a pressure function of the form

$$p(\rho,\theta) = p_e(\rho) + \theta p_\theta(\rho), \tag{2.50}$$

where the elastic part p_e is given by a γ-law:

$$p_e(\rho) = a\rho^\gamma, \tag{2.51}$$

with $a > 0$ and $\gamma > 1$. Concerning the thermal part of the pressure we assume that p_θ satisfies the following conditions:

$$\begin{cases} p_\theta \in C[0,\infty) \cap C^1(0,\infty), & p_\theta(0) = 0 \\ p_\theta \text{ is a nondecreasing function of } \rho \in [0,\infty] & \\ p_\theta(\rho) \le p_0(1+\rho^\Gamma), & \text{for all } \rho \ge 0, \end{cases} \tag{2.52}$$

for some $p_0 \ge 0$ and $\Gamma \le \frac{\gamma}{2}$.

Internal energy:

As aforementioned, this particular choice of pressure function and Maxwell's relation (2.39) force the internal energy e to have the form

$$e(\rho,\theta) = P_e(\rho) + Q(\theta), \tag{2.53}$$

where

$$P_e(\rho) = \frac{a}{\gamma-1}\rho^{\gamma-1}, \qquad Q(\theta) = \int_0^\theta C_\vartheta(z)dz.$$

Concerning the function C_ϑ we assume that:

$$\begin{cases} C_\vartheta \in C^1[0,\infty), & \inf_{z\in[0,\infty)} C_\vartheta(z) > 0 \\ e_1(1+\theta^r) \le C_\vartheta(\theta) \le e_2(1+\theta^r), & \end{cases} \tag{2.54}$$

where $r \in [0,1]$ and e_1 and e_2 are appropriate positive constants.

Heat conductivity:

As in [15, 23, 48] we need to impose some growth conditions on the heat conductivity coefficient κ for our results to hold. In the most general case all the viscosity coefficients (μ and λ), electromagnetic coefficients ($\tilde{\mu}$ and $\tilde{\sigma}$) and heat conductivity (κ) may depend on both the density and the temperature. For our purposes, however, we take them all to be constant, except for the heat conductivity, which we assume to depend on the temperature, satisfying some growth rates. Although, dependence on the density may be assumed for generality, the dependence on the temperature is necessary for our analysis. Specifically, we assume that $\kappa = \kappa(\theta)$ depends on the temperature and satisfies:

$$\begin{cases} \kappa \in C^2([0,\infty)) \\ k_1(1+\theta^q) \leq \kappa(\theta) \leq k_2(1+\theta^q), & \text{for all } \theta \geq 0 \\ \kappa_\theta(\theta) \leq k_2(1+\theta^{q'}), & \text{for all } \theta \geq 0. \end{cases} \tag{2.55}$$

Here, $k_1 > 0$, $q \geq 2+2r$, $q' \geq 0$ and r is the same as in (2.54).

Coupling:

Finally we impose some conditions on the functions involved in the coupling describing the short wave-long wave interaction, which agree with those in [17, 27, 28]

$$\begin{cases} g, h : [0,\infty) \to [0,\infty), \text{ smooth with } g(0) = h(0) = 0, \\ \operatorname{supp} g' \text{ compact in } (0,\infty), \\ \operatorname{supp} h' \text{ compact in } [0,\infty). \end{cases} \tag{2.56}$$

In the upcoming chapters we are going to discuss several questions on our model under these structural conditions, such as existence and uniqueness of solutions. Given the complexity of the model, we begin by studying the one dimensional case, where solutions are well behaved, and then move on to the more complicated multidimensional case.

Chapter 3

SW-LW Interactions in Planar MHD

3.1 Planar equations

Considering the complexity of the model we deduced in the previous chapter, we analyse first a simplified version of it. Namely, we are going to study several aspects of the model under the assumption that the flow moves in a preferable direction. We are going to assume that the three dimensional MHD flow with space variables $\mathbf{x} = (x, x_2, x_3)$ moves in the x direction and is uniform in the transverse direction (x_2, x_3). This assumption considerably simplifies the equations as well as the short wave-long wave interaction coupling, since the one dimensional Lagrangian transformation takes a very specific and plain form. Taking advantage of this, we can write the whole system in Lagrangian coordinates in order to carry out a straightforward (although not necessarily simple) analysis.

Our approach is motivated by the work of Dias and Frid in [17], as was already mentioned, but also relies on Chen and Wang's work in [15] and on the work of Wang in [48] on the existence and uniqueness of solutions for the planar MHD system. We also incorporate Chen and Perepelitsa's results from [14], where they study the vanishing viscosity problem for the Navier Stokes equations, and adapt them to our model. More details will be given in the respective sections.

3.1.1 Planar MHD and Lagrangian coordiantes

Let us consider a three dimensional MHD flow with spatial variables $\mathbf{x} = (x, x_2, x_3)$. Let us assume that it moves in the x direction and is uniform in the transverse direction (x_2, x_3). For convenience, let us decompose our dependent MHD variables as

$$\rho = \rho(t, x), \qquad \theta = \theta(t, x), \qquad \mathbf{u} = (u, \mathbf{w})(t, x), \qquad \mathbf{w} = (u_2, u_3),$$

$$\mathbf{B} = (b_1, \mathbf{b})(t, x), \qquad \mathbf{b} = (b_2, b_3),$$

where u and b_1 are the longitudinal velocity and the longitudinal magnetic induction, and $\mathbf{w}$ and $\mathbf{b}$ are the transverse velocity and the transverse magnetic induction, respectively.

Under our assumptions, we have that the partial derivatives with respect to x_2 and x_3 of all the functions involved in our system are zero. With this in mind, a straightforward calculation shows that (2.36) takes the form

$$b_{1t} = 0,$$

$$\mathbf{b}_t + (u\mathbf{b} - b_1\mathbf{w})_x = \left(\frac{1}{\tilde{\alpha}\tilde{\mu}}\mathbf{b}_x\right)_x.$$

Also, (2.37) implies

$$b_{1x} = 0.$$

As a result b_1 is constant and we can take it to be equal to 1 (that is $b_1 \equiv 1$).

For convenience, and for later applications, in what follows we are going to write all of the equations in terms of the magnetic field $\mathbf{H}$ instead of the magnetic induction $\mathbf{B}$. Recall that $\mathbf{B}$ and $\mathbf{H}$ are related by the identity

$$\mathbf{H} = \tilde{\mu}^{-1}\mathbf{B}.$$

Consequently, writing $\mathbf{H} = (h_1, \mathbf{h})$, the one (space) dimensional version of equations

(2.33)-(2.38) is

$$\rho_t + (\rho u)_x = 0, \tag{3.1}$$

$$(\rho u)_t + \Big(\rho u^2 + p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\Big)_x = (\varepsilon u_x)_x, \tag{3.2}$$

$$(\rho\mathbf{w})_t + (\rho u\mathbf{w} - \beta\mathbf{h})_x = (\mu\mathbf{w}_x)_x, \tag{3.3}$$

$$\begin{aligned}\Big(\rho\Big(e + \tfrac12 u^2 + \tfrac12|\mathbf{w}|^2\Big) + \tfrac{\beta}{2}|\mathbf{h}|^2\Big)_t + \Big(u\Big(\rho\Big(e + \tfrac12 u^2 + \tfrac12|\mathbf{w}|^2\Big) + p\Big)\Big)_x&\\ = (\beta\mathbf{w}\cdot\mathbf{h} - \beta u|\mathbf{h}|^2)_x + (\varepsilon u u_x + \mu\mathbf{w}\cdot\mathbf{w}_x + \nu\mathbf{h}\cdot\mathbf{h}_x)_x + (\kappa\theta_x)_x&\\ + \alpha\Big(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\Big)_x u,&\end{aligned} \tag{3.4}$$

$$\beta\mathbf{h}_t + (\beta u\mathbf{h} - \beta\mathbf{w})_x = (\nu\mathbf{h}_x)_x, \tag{3.5}$$

$$i\,\psi_t + \psi_{yy} = |\psi|^2\psi + \tilde{\alpha} g(v)h'(|\psi|^2)\psi. \tag{3.6}$$

Here, $\varepsilon = \lambda + 2\mu$ is the bulk viscosity, $\beta = \tilde{\mu}$ is the magnetic permeability and $\nu = \tilde{\sigma}^{-1}$ is the *electric resistivity.* The change in notation regarding the magnetic permeability is to prevent confusion with the shear viscosity parameter μ and to avoid the overload of notation caused by the tilde.

Let us also recall that $p = p(\rho,\theta)$ and $e = e(\rho,\theta)$ are given by (2.50) and (2.53), respectively; and according to (2.45), equation (3.4) is equivalent to

$$(\rho Q(\theta))_t + (\rho Q(\theta)\mathbf{u})_x + \theta p_\theta(\rho)u_x = (\kappa\theta_x)_x + \varepsilon u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{h}_x|^2. \tag{3.7}$$

It is worth mentioning that in this case the Lagrangian transformation $\mathbf{Y}(t,x) = (t, y(t,x))$ can be defined in a simpler way by the identities

$$\frac{\partial y}{\partial x} = \rho, \qquad \frac{\partial y}{\partial t} = -\rho u \qquad y(0,x) = \int_0^x \rho_0(z)dz, \tag{3.8}$$

clarifying any ambiguity in the system above. In addition, the one dimensional version of the energy identity (2.32) continues to hold. Namely, we have

$$\left\{ \left(\rho\left(e+\tfrac{1}{2}u^2+\tfrac{1}{2}|\mathbf{w}|^2\right)+\frac{\beta}{2}|\mathbf{h}|^2\right)_t + \left(u\left(\rho\left(e+\tfrac{1}{2}u^2+\tfrac{1}{2}|\mathbf{w}|^2\right)+p\right)\right)_x \right.$$
$$\left. -(\varepsilon u u_x+\mu\mathbf{w}\cdot\mathbf{w}_x+\nu\mathbf{h}\cdot\mathbf{h}_x)_x-(\kappa\theta_x)_x-(\beta\mathbf{w}\cdot\mathbf{h}-\beta u|\mathbf{h}|^2)_x \right\} dx$$
$$= \frac{\alpha}{\tilde{\alpha}}\left\{(\overline{\psi}_t\psi_y+\psi_t\overline{\psi}_y)_y-\left(\tilde{\alpha}g(v)h(|\psi|^2)+\tfrac{1}{2}|\psi_y|^2+\tfrac{1}{4}|\psi|^4\right)_t\right\}d\mathbf{y}. \tag{3.9}$$

Furthermore, we can rewrite the whole system in Lagrangian coordinates as

$$v_t-u_y=0, \tag{3.10}$$
$$u_t+\left(p+\frac{\beta}{2}|\mathbf{h}|^2-\alpha g'(v)h(|\psi|^2)\right)_y=\left(\frac{\varepsilon u_y}{v}\right)_y, \tag{3.11}$$
$$\mathbf{w}_t-\beta\mathbf{h}_y=\left(\frac{\mu\mathbf{w}_y}{v}\right)_y, \tag{3.12}$$
$$\left[e+\tfrac{1}{2}(u^2+|\mathbf{w}|^2+\beta v|\mathbf{h}|^2)+\frac{\alpha}{\tilde{\alpha}}\left(\tilde{\alpha}g(v)h(|\psi|^2)+\tfrac{1}{2}|\psi_y|^2+\tfrac{1}{2}|\psi|^4\right)\right]_t$$
$$+\left(u\left(p+\frac{\beta}{2}|\mathbf{h}|^2-\alpha g'(v)h(|\psi|^2)\right)-\beta\mathbf{h}\cdot\mathbf{w}-(\psi_t\overline{\psi}_y+\overline{\psi}_t\psi_y)\right)_y$$
$$=\left(\frac{\kappa\theta_y}{v}+\frac{\varepsilon u u_y}{v}+\frac{\mu\mathbf{w}\cdot\mathbf{w}_y}{v}+\frac{\nu\mathbf{h}\cdot\mathbf{h}_y}{v}\right)_y, \tag{3.13}$$
$$(\beta v\mathbf{h})_t-\beta\mathbf{w}_y=\left(\frac{\nu\mathbf{h}_y}{v}\right)_y, \tag{3.14}$$
$$i\psi_t+\psi_{yy}=|\psi|^2\psi+\tilde{\alpha}g(v)h'(|\psi|^2)\psi. \tag{3.15}$$

where, v is the specific volume given by (2.31). Accordingly, equation (3.7) results in

$$Q(\theta)_t+\theta p_\theta(\rho)u_y=(\frac{\kappa\theta_y}{v})_y+\frac{\varepsilon u_y^2}{v}+\frac{\mu|\mathbf{w}_y|^2}{v}+\frac{\nu|\mathbf{h}_y|^2}{v}. \tag{3.16}$$

Of course, this change of variables is justified only when ρ is finite and strictly positive.

3.1.2 The initial-boundary value problem

In what follows we are going to study several aspects of the systems (3.1)-(3.6) and (3.10)-(3.15), such as the vanishing bulk viscosity and interaction coefficients limit. However, before we get there, we must first ensure that the proposed system is well posed. For this reason we dedicate a couple of sections to the existence and uniqueness of solutions.

Let us state precisely the problem we are going to focus our attention on. Consider the initial-boundary value problem for the system (3.1)-(3.6) in a bounded spatial domain, which we can assume to be $(0,1)$ without loss of generality, with the following initial and boundary conditions

$$\begin{cases} (\rho, u, \mathbf{w}, \mathbf{h}, \theta, \psi)|_{t=0} = (\rho_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)(x), \quad x \in (0,1), \\ (u, \mathbf{w}, \mathbf{h}, \theta_x)|_{x=0,1} = 0, \qquad \psi|_{\partial\Omega} = 0 \end{cases} \tag{3.17}$$

where the initial data satisfes the respective compatibility conditions.

In order to show well-posedness of this problem we first do it for the system in Lagrangian coordinates. In the process, we show that no vacuum nor concentration of mass develop in finite time, which also implies well posedness of the original problem in Eulerian coordinates.

From (3.8) we have that

$$y(t,x) = \int_0^x \rho(t,z)dz.$$

Using equation (3.1) and the boundary conditions we see that

$$y(t,1) = y(0,1) = \int_0^1 \rho_0(z)dz.$$

Up to a scaling we can assume that

$$\int_0^1 \rho_0(z)dz = 1,$$

so that $0 < y < 1$. With this, the initial-boundary value problem (3.1)-(3.6), (3.17) in Eulerian (t,x) coordinates is transformed into the initial-boundary value problem for the system (3.10)-(3.15) in Lagrangian coordinates (t,y) for $y \in \Omega := (0,1)$ and $t \geq 0$

with the following initial and boundary conditions

$$\begin{cases} (v, u, \mathbf{w}, \mathbf{h}, \theta, \psi)|_{t=0} = (v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)(y), \quad y \in \Omega, \\ (u, \mathbf{w}, \mathbf{h}, \theta_y, \psi)|_{\partial\Omega} = 0. \end{cases} \tag{3.18}$$

Remember that we are assuming the structural and growth conditions from Section 2.3. In connection with (2.50) and (2.53), by an abuse of notation we have that $p = p(v, \theta)$ is given by

$$p(v, \theta) = p_e(v) + \theta p_\theta(v), \tag{3.19}$$

where the elastic part p_e is given by

$$p_e(\rho) = av^{-\gamma}, \tag{3.20}$$

with $a > 0$ and $\gamma > 1$. Concerning the thermal part of the pressure p_θ we assume that

$$\begin{cases} p_\theta \in C(0, \infty) \cap C^1(0, \infty), & \lim_{v\to\infty} p_\theta(v) = 0 \\ p_\theta \text{ is a nonincreasing function of } v \in (0, \infty) & \\ p_\theta(v) \le p_0(1 + v^{-\Gamma}), & \text{for all } \rho \ge 0, \end{cases} \tag{3.21}$$

for some $p_0 \ge 0$ and $\Gamma \le \frac{\gamma}{2}$.

Accordingly, the internal energy $e = e(v, \theta)$ is given by

$$e(v, \theta) = P_e(v) + Q(\theta), \tag{3.22}$$

where

$$P_e(v) = \frac{a}{\gamma - 1} v^{1-\gamma}, \qquad Q(\theta) = \int_0^\theta C_\vartheta(z)dz. \tag{3.23}$$

Concerning the function C_ϑ we assume (2.54).

As aforementioned, the heat conductivity κ must depend on θ and satisfy (2.55). Moreover, we assume that the coupling functions g and h satisfy (2.56).As for the parameters $\varepsilon, \mu, \nu, \beta, \alpha$ and $\tilde{\alpha}$, we take them to be fixed positive constants. Let us in fact take $\alpha = \tilde{\alpha}$ from this point on, since the analysis developed below does not change otherwise, but may be clouded by the overload of notation.

Under these conditions we can prove the following result.

Theorem 3.1. *Supose that there are positive constants $m < M$ such that*

$$m \leq v_0(y), \theta_0(y) \leq M, \qquad y \in \Omega, \tag{3.24}$$

and that

$$v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0 \in H^1(\Omega), \quad \psi_0 \in H^2(\Omega; \mathbb{C}), \tag{3.25}$$

and $v_0 \in W^{1,\infty}(\Omega)$. Then, problem (3.10)-(3.15), (3.18) has a unique global solution $(v, u, \mathbf{w}, \mathbf{h}, \theta, \psi)(t, y)$ such that for any fixed $T > 0$

$$\begin{aligned}
&v \in C([0,T]; H^1(\Omega)) \cap L^\infty(0,T; W^{1,\infty}(\Omega)),\\
&(u, \mathbf{w}, \mathbf{h}) \in C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),\\
&\theta \in C([0,T]; H^1(\Omega)), \quad \theta_y \in L^2(0,T; H_0^1(\Omega))\\
&\psi \in C([0,T]; H_0^1(\Omega; \mathbb{C})) \cap L^\infty(0,T; H^2(\Omega; \mathbb{C})).
\end{aligned}$$

Also, for each $(t, y) \in [0,T] \times \Omega$ we have

$$C^{-1} \leq v(t,y), \theta(t,y) \leq C,$$

where $C > 0$ is a constant depending only on T, m, M and the initial data. Moreover, solutions depend continuously on the initial data.

In order to prove this theorem we first prove existence of local solutions and then extend the local solutions to global ones based on a priori estimates.

For the local result we use a Faedo-Galerkin type method similar to that applied by Dias and Frid in [17], which is in turn resembles the classic work by Kazhikhov and Shelukhin in [32] (c.f. [2, Chapter 2]). As for the global result, we develop some a priori estimates inspired by the work of Chen and Wang in [15] and by the work of Wang in [48].

The uniqueness of solutions is proved by analysing the equations satisfied by the difference of two possible solutions that have the same initial values and conclude by an application of Gronwall's inequality. In this part we incorporate some of the ideas by Chen and Wang in [16] and adapt them to our needs.

Note that the results in Theorem 3.1 for the problem in Lagrangian coordinates imply the corresponding results for problem (3.1)-(3.6), (3.17) in Eulerean coordinates. More precisely we have the following theorem.

Theorem 3.2. *Supose that there are positive constants $m < M$ such that*

$$m \leq \rho_0(x), \theta_0(x) \leq M, \qquad\qquad x \in (0,1), \tag{3.26}$$

and that

$$\rho_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0 \in H^1((0,1)), \quad \psi_0 \in H^2(\Omega;\mathbb{C}), \tag{3.27}$$

and $\rho_0 \in W^{1,\infty}((0,1))$. Then, problem (3.1)-(3.6), (3.17) has a unique global solution $(\rho, u, \mathbf{w}, \mathbf{h}, \theta)(t,x)$, $\psi(t,y)$ such that for any fixed $T > 0$

$$\begin{aligned}
&\rho \in C([0,T]; H^1((0,1))) \cap L^\infty(0,T; W^{1,\infty}((0,1))),\\
&(u, \mathbf{w}, \mathbf{h}) \in C([0,T]; H_0^1((0,1))) \cap L^2(0,T; H^2((0,1))),\\
&\theta \in C([0,T]; H^1((0,1))), \quad \theta_y \in L^2(0,T; H_0^1((0,1)))\\
&\psi \in C([0,T]; H_0^1(\Omega;\mathbb{C})) \cap L^\infty(0,T; H^2(\Omega;\mathbb{C})).
\end{aligned}$$

Also, for each $(t,x) \in [0,T] \times (0,1)$ we have

$$C^{-1} \leq \rho(t,x), \theta(t,x) \leq C,,$$

where $C > 0$ is a constant depending only on T, m, M and the initial data.

3.2 Existence and Uniqueness of solutions

Our main goal in this section is to prove Theorems 3.1 and 3.2. Note that Theorem 3.2 follows from Theorem 3.1 by changing back to the original coordinate system once we show that the coordinate change is nonsingular. As observed before, the Lagrangian transformation is nonsingular as long as ρ (or equivalently v) is strictly positive and finite. As this is part of the conclusion of Theorem 3.1 we need only prove this theorem.

Let us begin by showing existence of local solutions.

3.2.1 Local solutions: Galerkin method

Let us assume that the initial data $(v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)(y)$ satisfies

$$m \leq v_0(y), \theta_0(y) \leq M, \qquad\qquad y \in \Omega, \tag{3.28}$$

and that

$$v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0 \in H^1(\Omega), \quad \psi_0 \in H^1(\Omega; \mathbb{C}). \tag{3.29}$$

Then, we have the following local result.

Lemma 3.1. *There exists $T > 0$ and a solution of (3.10)-(3.15), (3.18) satisfying*

$$\begin{aligned}
&v \in C([0,T]; H^1(\Omega)), \quad \frac{m}{4} \leq v \leq 4M \\
&(u, \mathbf{w}, \mathbf{h}) \in C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\
&\theta \in C([0,T]; H^1(\Omega)), \quad \theta_y \in L^2(0,T; H_0^1(\Omega)), \quad \theta > 0 \\
&\psi \in C([0,T]; H_0^1(\Omega; \mathbb{C})), \\
&v_t, u_t, \mathbf{w}_t, \mathbf{h}_t, \theta_t \in L^2(0,T; L^2(\Omega)).
\end{aligned}$$

The rest of this section is devoted to the proof of this lemma.

Let us construct a sequence of approximate solutions $(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi_n)$ where $(u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$ are of the form

$$\begin{aligned}
u^n(t,y) &= \sum_{k=1}^n u_k^n(t) sin(k\pi y), \\
\mathbf{w}^n(t,y) &= \sum_{k=1}^n \mathbf{w}_k^n(t) sin(k\pi y), \\
\mathbf{h}^n(t,y) &= \sum_{k=1}^n \mathbf{h}_k^n(t) sin(k\pi y), \qquad\qquad n = 1, 2, ... \\
\theta^n(t,y) &= \sum_{j=0}^n \theta_j^n(t) cos(j\pi y), \\
\psi^n(t,y) &= \sum_{k=1}^n \psi_k^n(t) sin(k\pi y).
\end{aligned} \tag{3.30}$$

Note that each approximation is written as a sum of either sines or cosines so that they match the desired boundary conditions (for example, $\theta_y^n|_{\partial\Omega} = 0$).

In order to determine the coefficients $u_k^n(t), \mathbf{w}_k^n(t), \mathbf{h}_k^n(t), \theta_j^n(t), \psi_k^n(t)$, $j = 0, 1, ..., n$, $k = 1, ..., n$, we demand that equations (3.11)-(3.15) be satisfied in an approximate way. To this end, we consider the spaces

$$\begin{aligned}
\mathcal{S}_n &:= \text{span}_{\mathbb{C}}\{sin(k\pi y) : k = 1, ..., n\}, \\
\mathcal{C}_n &:= \text{span}_{\mathbb{C}}\{cos(j\pi y) : j = 0, 1, ..., n\},
\end{aligned}$$

with respective projections

$$P_n^{\mathcal{S}} : L^2(\Omega) \to \mathcal{S}_n, \qquad\qquad P_n^{\mathcal{C}} : L^2(\Omega) \to \mathcal{C}_n.$$

By virtue of (3.10) we take

$$v^n(t,y) := v_0(y) + \int_0^t u_y^n(y,s)ds, \tag{3.31}$$

so that,

$$v_t^n = u_y^n, \qquad v^n|_{t=0} = v_0.$$

With the notation above, we consider the following system.

$$u_t^n = P_n^{\mathcal{S}}\left[\left(-p(v^n,\theta^n) - \frac{\beta}{2}|\mathbf{h}^n|^2 + \alpha g'(v^n)h(|\psi^n|^2)\frac{\varepsilon u_y^n}{v^n}\right)_y\right], \tag{3.32}$$

$$\mathbf{w}_t^n = P_n^{\mathcal{S}}\left[\beta \mathbf{h}_y^n + \left(\frac{\mu \mathbf{w}_y^n}{v^n}\right)_y\right], \tag{3.33}$$

$$\beta \mathbf{h}_t^n = P_n^{\mathcal{S}}\left[\frac{1}{v^n}\left(-\beta u_y^n \mathbf{h}^n + \beta \mathbf{w}_y^n + \left(\frac{\nu \mathbf{h}_y^n}{v^n}\right)_y\right)\right], \tag{3.34}$$

$$\begin{aligned}\theta_t^n = P_n^{\mathcal{C}}\Bigg[\frac{1}{C_\vartheta(\theta^n)}\Bigg(&-\theta^n p_\theta(v^n)u_y^n + \left(\frac{\kappa(\theta^n)\theta_y^n}{v^n}\right)_y \\ &+ \frac{\varepsilon|u_y^n|^2}{v^n} + \frac{\mu|\mathbf{w}_y^n|^2}{v^n} + \frac{\nu|\mathbf{h}_y^n|^2}{v^n}\Bigg)\Bigg],\end{aligned} \tag{3.35}$$

$$i\,\psi_t^n = P_n^{\mathcal{S}}\left[-\psi_{yy}^n + |\psi^n|^2\psi^n + \alpha g(v^n)h'(|\psi^n|^2)\psi^n\right]. \tag{3.36}$$

This "approximate problem" is defined in such a way that (heuristically) a limit $(v,u,\mathbf{w},\mathbf{h},\theta,\psi)$ of the sequence (or a subsequence of) $(v^n,u^n,\mathbf{w}^n,\mathbf{h}^n,\theta^n,\psi^n)$ satisfy the system

$$v_t = u_y, \tag{3.37}$$

$$u_t = \left(-p(v,\theta) - \frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(v)h(|\psi|^2)\frac{\varepsilon u_y}{v}\right)_y, \tag{3.38}$$

$$\mathbf{w}_t = \beta\mathbf{h}_y + \left(\frac{\mu\mathbf{w}_y}{v}\right)_y, \tag{3.39}$$

$$\beta\mathbf{h}_t = \frac{1}{v}\left(- \beta u_y\mathbf{h} + \beta\mathbf{w}_y + \left(\frac{\nu\mathbf{h}_y}{v}\right)_y\right), \tag{3.40}$$

$$\theta_t = \frac{1}{C_\vartheta(\theta)}\left(- \theta p_\theta(v)u_y + \left(\frac{\kappa(\theta)\theta_y}{v}\right)_y + \frac{\varepsilon|u_y|^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{h}_y|^2}{v}\right), \tag{3.41}$$

$$i\,\psi_t = -\psi_{yy} + |\psi|^2\psi + \alpha g(v)h'(|\psi|^2)\psi, \tag{3.42}$$

which is equivalent to our original system (3.10)-(3.15).

Now, system (3.32)-(3.36) poses a system of ODE's for the coefficients $u_k^n(t)$, $\mathbf{w}_k^n(t)$, $\mathbf{h}_k^n$, $\theta_j^n(t)$, $\psi_k^n(t)$, $k = 1, 2, ..., n$, $j = 0, 1, ..., n$. Namely,

$$\frac{d}{dt}u_k^n(t) = 2\int_0^1 \left(-p(v^n,\theta^n) - \frac{\beta}{2}|\mathbf{h}^n|^2 + \alpha g'(v^n)h(|\psi^n|^2)\frac{\varepsilon u_y^n}{v^n}\right)_y sin(k\pi y)dy, \tag{3.43}$$

$$\frac{d}{dt}\mathbf{w}_k^n(t) = 2\int_0^1 \left(\beta\mathbf{h}^n + \frac{\mu\mathbf{w}_y^n}{v^n}\right)_y sin(k\pi y)dy, \tag{3.44}$$

$$\beta\frac{d}{dt}\mathbf{h}_k^n(t) = \int_0^1 \frac{1}{v^n}\left(- \beta u_y^n\mathbf{h}^n + \beta\mathbf{w}_y^n + \left(\frac{\nu\mathbf{h}_y^n}{v^n}\right)_y\right) sin(k\pi y)dy, \tag{3.45}$$

$$\begin{aligned}\frac{d}{dt}\theta_j^n(t) = \int_0^1 \frac{1}{C_\vartheta(v^n,\theta^n)}&\left(- \theta^n p_\theta(v^n)u_y^n + \left(\frac{\kappa(\theta^n)\theta_y^n}{v^n}\right)_y\right. \\ &\left. + \frac{\varepsilon|u_y^n|^2}{v^n} + \frac{\mu|\mathbf{w}_y^n|^2}{v^n} + \frac{\nu|\mathbf{h}_y^n|^2}{v^n}\right) cos(j\pi y)dy,\end{aligned} \tag{3.46}$$

$$\frac{d}{dt}\psi_k^n(t) = \int_0^1 \left(-\psi_{yy}^n + |\psi^n|^2\psi^n + \alpha g(v^n)h'(|\psi^n|^2)\psi^n\right) sin(k\pi y)dy. \tag{3.47}$$

Regarding the initial conditions, we impose that

$$(u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)|_{t=0} = (u_0^n, \mathbf{w}_0^n, \mathbf{h}_0^n, \theta_0^n, \psi_0^n), \tag{3.48}$$

where the latter satisfy

$$u_0^n, \mathbf{w}_0^n, \mathbf{h}_0^n, \psi_0^n \in \mathcal{S}_n, \qquad \theta_0^n \in \mathcal{C}_n, \tag{3.49}$$

and

$$(u_0^n, \mathbf{w}_0^n, \mathbf{h}_0^n, \theta_0^n, \psi_0^n) \to (u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0) \tag{3.50}$$

in $H^1(\Omega)$ (and, therefore, uniformly). For instance, θ_0^n can be defined by

$$\theta_0^n(y) = -\sum_{j=1}^{n} \frac{\widehat{\theta_{0y}}(j)}{j\pi} cos(j\pi y) + \left(\theta_0(0) - \sum_{j=1}^{n} \frac{\widehat{\theta_{0y}}(j)}{j\pi}\right),$$

where $\widehat{\theta_{0y}}(j)$, $j = 1, 2, ...$ are the coefficients of the sine Fourier series of θ_{0y}; that is

$$\theta_{0y}(\cdot) = \sum_{j=1}^{\infty} \widehat{\theta_{0y}}(j) sin(j\pi\cdot), \text{ in } L^2(\Omega).$$

Taking the coefficients of the newly defined approximate initial data as initial conditions for the respective coefficients and taking into account relation (3.31), the existence and uniqueness of solutions of (3.43)-(3.47) are guaranteed by the well known classical results on the theory of ordinary differential equations. From this, the existence and uniqueness of solutions of the form (3.30) for the system (3.32)-(3.36), (3.48) follow.

Having a sequence of approximate solutions we now need some uniform estimates that allow us to take a convergent subsequence to a solution of the original problem (3.10)-(3.15), (3.18).

Observe that each one of the approximate solutions is merely a local one. That is, each solution $(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$ exists only on a time interval $[0, t_n]$. So, we not only have to bound properly the norms of the involved functions, but have to guarantee that they are all defined on a uniform small enough interval $[0, t_0]$.

From the theory of ODE's we know that whenever one has existence and uniqueness of solutions to an ODE

$$\frac{d}{dt} X = F(X), \qquad X(0) = X_0,$$

where, $F : W \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then given an initial condition $X_0 \in W$, there is a maximal interval of existence (t_-, t_+), $t_- < 0 < t_+$. Such an interval is characterized by the fact that whenever $t_\pm$ is finite then as $t \to t_\pm$ one of the following two possibilities hold:

- either $X(t)$ tends to the boundary ∂W of W, or
- $|X(t)|$ tends to infinity.

With this in mind, we see that in order to guarantee the existence of $t_0 > 0$ that bounds t_n from below for all n, we have to ensure that the coefficients of $(u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$ do not blow up before t_0 and also that (v^n, θ^n) remain in a compact subset of the domain of the functions p, e and κ. That is, we have to show that $(v^n, \theta^n)(t)$ does not leave a certain compact subset of $(0,\infty) \times [0,\infty)$.

In order to avoid this last constraint we are going to make two remarks.

First we are going to assume that

$$\frac{m}{2} \leq v^n(y,t) \leq 2M, \qquad y \in \Omega, t \in [0, t_n]. \tag{3.51}$$

This is certainly true on a possibly smaller time interval, which we will show later on contains $[0, t_0]$ for some uniform $t_0 > 0$.

Second, since we do not seem to be able to show directly that θ^n is nonnegative on a uniform over n time interval, we consider smooth extensions of the functions

$$C_\vartheta(\theta), p(v,\theta), \kappa(\theta)$$

for $(v,\theta) \in (0,\infty) \times \mathbb{R}$ such that for all $v \in [m/4, 4M]$ and all $\theta \leq 0$

$$0 \leq p(v,\theta) \leq \tilde{p}_0(1+|\theta|), \qquad \tilde{\kappa}_0 \leq \kappa(\theta), \tag{3.52}$$

$$\tilde{e}_1 \leq C_\vartheta(\theta) \leq \tilde{e}_2, \tag{3.53}$$

$$|p_\theta(v,\theta)| \leq \tilde{p}_1, \qquad |p_v(v,\theta)| \leq \tilde{p}_2(1+|\theta|), \tag{3.54}$$

$$\kappa(\theta), |\kappa_\theta(v,\theta)| \leq \tilde{\kappa}_1(1+|\theta|^{\tilde{q}_1}), \tag{3.55}$$

where $\tilde{p}_0, \tilde{\kappa}_0, \tilde{e}_1, \tilde{p}_1, \tilde{p}_2, \tilde{\kappa}_1$ and $\tilde{e}_2$ are positive constants. These conditions are in agreement with our previous assumptions on the growth of the functions p, e and κ.

Such extensions are not difficult to construct. It suffices to consider a reflection with respect to the axis $(0,\infty) \times \{\theta = 0\}$ and some manipulation (cutoff) in the case of C_ϑ so that (3.53) is satisfied.

Thus, from this point on we are going to work on the system (3.32)-(3.36), (3.48) with the extended C_ϑ, p and κ (maintaining the same notation for the extensions).

As mentioned before, we seek to ensure the existence of a convergent (in some sense) subsequence of $(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$ so that its limit satisfies (3.37)-(3.42).

Once we have achieved this, since θ_0 is bounded away from zero, the limit θ of the (sub)sequence θ^n will also be positive, at least on a small enough time interval, and therefore our original system (3.10)-(3.15) (with the original C_ϑ, p and κ) will be satisfied, thus proving Lemma 3.1. Let us remark that, in fact, there is a maximum principle available for the limit equation (3.41) which guarantees the strict positivity of θ, but we leave this argument to the a priori estimates from Section 3.2.4

3.2.2 Estimates for the approximate problem

Let us begin by pointing out some useful remarks (Recall that we are assuming (3.51)).

First, the simplest case of Sobolev imbeddings applied to θ^n implies

$$\max_{y\in\Omega} |\theta^n(t,y)| \leq ||\theta^n(t)||_{L^2(\Omega)} + ||\theta^n_y(t,y)||_{L^2(\Omega)}. \tag{3.56}$$

Our growth conditions hypotheses on p and e and also (3.52)-(3.55) then imply

$$\begin{aligned}
&0 \leq p(v^n,\theta^n) \leq C(1 + ||\theta^n||_{L^2(\Omega)} + ||\theta^n_y||_{L^2(\Omega)}),\\
&C_\vartheta(\theta^n) \geq C^{-1}, \qquad \kappa(\theta^n) \geq C^{-1},\\
&|p_\theta(v^n,\theta^n)| \leq C,\\
&|p_v(v^n,\theta^n)| \leq C(1 + ||\theta^n||_{L^2(\Omega)} + ||\theta^n_y||_{L^2(\Omega)}),\\
&|\kappa(\theta^n)| + |\kappa_\theta(\theta^n)| \leq C(1 + ||\theta^n||^{\tilde q}_{L^2(\Omega)} + ||\theta^n_y||^{\tilde q}_{L^2(\Omega)}).
\end{aligned}$$

Here, and in what follows, C denotes a positive constant independent of n.

Second, for any $Z \in H^1(\Omega)$ such that $Z(y_0) = 0$ for some $y_0 \in \Omega$, we have

$$\max_{y\in\Omega} |Z(y)| \leq \min\{||Z_y||_{L^2(\Omega)}, 2||Z||^{1/2}_{L^2(\Omega)}||Z_y||^{1/2}_{L^2(\Omega)}\}.$$

In particular, this holds for all the functions in the set

$$\{u^n(t,\cdot), \mathbf{w}^n(t,\cdot), \mathbf{h}^n(t,\cdot), \psi(t,\cdot), u^n_y(t,\cdot), \mathbf{w}^n_y(t,\cdot), \mathbf{h}^n_y(t,\cdot), \theta^n_y(t,\cdot) : t \in [0,t_n]\}.$$

Finally, from (3.31) we have that

$$m - t^{1/2}\left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}ds\right)^{1/2} \leq v^n(t,y) \leq M + t^{1/2} + t^{1/2}\left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}ds\right)^{1/2}, \tag{3.57}$$

for a.e. $y \in \Omega$ and $t \in [0, t_n]$. Also,

$$||v_y^n(t)||_{L^2(\Omega)} \leq C + C\left(\int_0^t ||u_{yy}^n(s)||_{L^2(\Omega)}^2 ds\right)^{1/2}, \tag{3.58}$$

for $t \in [0, \min\{t_n, 1\}]$.

With these observations at hand we are going to prove that as long as (3.51) holds (which is certainly true at $t = 0$) we have the following inequality

$$\frac{d}{dt}\eta_n(t) \leq C_1(1 + \eta_n(t)^{q_1}), \tag{3.59}$$

for a certain $q_1 > 0$ where,

$$\eta_n(t) = ||(u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)(t)||_{H^1(\Omega)}^2 + \int_0^t ||(u_{yy}^n, \mathbf{w}_{yy}^n, \mathbf{h}_{yy}^n, \theta_{yy}^n)(s)||_{L^2(\Omega)}^2 ds. \tag{3.60}$$

Since $\eta_n(0)$ is bounded by a constant $C_2 > 0$, then from (3.59) we conclude that

$$\eta_n(t) \leq \phi(t)$$

for all $n = 1, 2, ...$ and all $0 \leq t < t^*$, where ϕ is the solution of the ODE

$$\frac{d}{dt}X(t) = C_1(1 + X(t)^{q_1})$$
$$X(0) = C_2,$$

and t^* is the maximal time of existence of the solution to this ODE relative to the initial condition C_2.

After this, by (3.57) we can choose $0 < t_0 < t^*$ small enough so that (3.51) holds for all $t \in [0, t_0]$ and all n.

With this in mind, let us assume (3.51) and begin the proof of (3.59). Multiply (3.36) by $\overline{\psi^n}$ (the complex conjugate of ψ^n), take imaginary part and integrate over Ω to obtain

$$\frac{d}{dt}||\psi^n(t)||_{L^2\Omega}^2 = 0,$$

which immediately implies that $||\psi^n(t)||_{L^2\Omega}^2 \leq C$.

Next, multiply (3.32) by $u^n - u^n_{yy}$ and integrate by parts and use (3.58) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega |u^n|^2 dy + \int_\Omega |u^n_y|^2 dy\right] + \varepsilon\int_\Omega \frac{|u^n_y|^2}{v^n}dy + \varepsilon\int_\Omega \frac{|u^n_{yy}|^2}{v^n}dy + \int_\Omega \alpha g'(v^n)h(|\psi^n|^2)u^n_y dy\\
&\quad = \int_\Omega \left(-p(v^n,\theta^n) - \frac{\beta}{2}|\mathbf{h}^n|^2\right)u^n_y dy + \int_\Omega \Bigg(p_v(v^n,\theta^n)v^n_y + p_\theta(v^n,\theta^n)\theta^n_y\\
&\qquad\quad + \beta\mathbf{h}^n\mathbf{h}^n_y - \alpha g''(v^n)h(|\psi^n|^2)v^n_y - \alpha g'(v^n)h'(|\psi^n|^2)\mathrm{Re}(\psi^n\overline{\psi^n_y}) + \frac{\varepsilon u^n_y v^n_y}{|v^n|^2}\Bigg)u^n_{yy}dy\\
&\quad \le C\left(1 + ||\theta^n||_{L^2(\Omega)} + ||\theta^n_y||_{L^2(\Omega)} + ||\mathbf{h}^n_y||^2_{L^2(\Omega)}\right)||u^n_y||_{L^2(\Omega)}\\
&\qquad + C\left(1 + ||\theta^n||_{L^2(\Omega)} + ||\theta^n_y||_{L^2(\Omega)}\right)\left(1 + \int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}ds\right)^{1/2}||u^n_{yy}||_{L^2(\Omega)}\\
&\qquad + C||\theta^n_y||_{L^2(\Omega)}||u^n_{yy}||_{L^2(\Omega)}\\
&\qquad + C||\mathbf{h}^n_y||^2_{L^2(\Omega)}||u^n_{yy}||_{L^2(\Omega)} + C||\psi^n_y||^2_{L^2(\Omega)}||u^n_{yy}||_{L^2(\Omega)}\\
&\qquad + C\left(1 + \left(\int_0^t ||u^n_{yy}(s)||_{L^2(\Omega)}ds\right)^{1/2}\right)||u^n_y||_{L^2(\Omega)}||u^n_{yy}||^{3/2}_{L^2(\Omega)}\\
&\quad \le C_{\delta_1}\Bigg(1 + ||\theta^n||^4_{L^2(\Omega)} + ||\theta^n_y||^4_{L^2(\Omega)} + ||\mathbf{h}^n_y||^4_{L^2(\Omega)} + ||\psi^n_y||^4_{L^2(\Omega)} + ||u^n_y||^8_{L^2(\Omega)}\\
&\qquad\qquad + \left(\int_0^t ||u^n_{yy}(s)||_{L^2(\Omega)}ds\right)^4\Bigg) + \delta_1||u^n_{yy}||^2_{L^2(\Omega)}. \qquad (3.61)
\end{aligned}$$

Here, $\delta_1 > 0$ is arbitrary and $C_{\delta_1} > 0$ is a constant which depends on δ_1.

By (3.31), we see that $g'(v^n)u^n_y = g(v^n)_t$ and therefore the fifth term on the left hand side can be rewritten as

$$\begin{aligned}
\int_\Omega \alpha g'(v^n)h(|\psi^n|^2)u^n_y dy = \frac{d}{dt}\left(\int_\Omega \alpha g(v^n)h(|\psi^n|^2)dy\right)&\\
- \int_\Omega \alpha g(v^n)h'(|\psi^n|^2)2\mathrm{Re}(\psi^n\overline{\psi^n_t})dy.&
\end{aligned}$$

Multiplying (3.36) by $\frac{\alpha}{\tilde\alpha}\overline{\psi^n_t}$, taking real part and integrating by parts we have

$$-\int_\Omega \alpha g(v^n)h'(|\psi^n|^2)2\mathrm{Re}(\psi^n\overline{\psi^n_t})dy = \frac{d}{dt}\frac{\alpha}{2\tilde\alpha}\int_\Omega \left(|\psi^n_y|^2 + |\psi^n|^4\right)dy.$$

Replacing this in (3.61), using our assumption (3.51) and taking $\delta_1 > 0$ small enough we have

$$\frac{d}{dt}\Big(||u^n||^2_{L^2(\Omega)} + ||u^n_y||^2_{L^2(\Omega)} + ||\psi^n||^4_{L^4(\Omega)} + ||\psi^n_y||^2_{L^2(\Omega)} \\ + ||\alpha g(v^n)h(|\psi^n|^2)||_{L^1(\Omega)}\Big) + ||u^n_{yy}||^2_{L^2(\Omega)} \\ \leq C\Bigg(1 + ||\theta^n||^4_{L^2(\Omega)} + ||\theta^n_y||^4_{L^2(\Omega)} + ||\mathbf{h}^n_y||^4_{L^2(\Omega)} + ||\psi^n_y||^4_{L^2(\Omega)} \\ + ||u^n_y||^8_{L^2(\Omega)} + \Big(\int_0^t ||u^n_{yy}(s)||_{L^2(\Omega)}ds\Big)^4\Bigg). \tag{3.62}$$

Similarly, multiplying (3.33) and (3.34) by $w^n - w^n_{yy}$ and $\mathbf{h}^n - \mathbf{h}^n_{yy}$ respectively and integrating by parts, after some manipulation we get

$$\frac{d}{dt}\left(||\mathbf{w}^n||^2_{L^2(\Omega)} + ||\mathbf{w}^n_y||^2_{L^2(\Omega)}\right) + ||\mathbf{w}^n_{yy}||^2_{L^2(\Omega)} \\ \leq C\left(1 + ||\mathbf{w}^n||^8_{L^2(\Omega)} + ||\mathbf{h}^n_y||^2_{L^2(\Omega)} + \left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}\right)^4\right), \tag{3.63}$$

and

$$\frac{d}{dt}\left(||\mathbf{h}^n||^2_{L^2(\Omega)} + ||\mathbf{h}^n_y||^2_{L^2(\Omega)}\right) + ||\mathbf{h}^n_{yy}||^2_{L^2(\Omega)} \\ \leq C\left(1 + ||u^n_y||^4_{L^2(\Omega)} + ||\mathbf{h}^n_y||^8_{L^2(\Omega)} + ||\mathbf{w}^n_y||^2_{L^2(\Omega)} + ||\mathbf{h}^n||^2_{L^2(\Omega)} + \left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}\right)^4\right). \tag{3.64}$$

Next, multiplying (3.35) by $\theta^n - \theta^n_{yy}$ and integrating we have

$$\frac{d}{dt}\left(\frac{1}{2}||\theta^n||^2_{L^2(\Omega)} + \frac{1}{2}||\theta^n_y||^2_{L^2(\Omega)}\right) + \int_\Omega \frac{\kappa(\theta^n)|\theta^n_{yy}|^2}{v^n C_\vartheta(\theta^n)}dy \\ = \int_\Omega \frac{1}{C_\vartheta(\theta^n)}\Bigg[-\theta^n p_\theta(v^n,\theta^n)u^n_y + \frac{\kappa_\theta(\theta^n)}{v^n}|\theta^n_y|^2 \\ - \frac{\kappa(\theta^n)}{|v^n|^2}v^n_y\theta^n_y + \frac{\varepsilon|u^n_y|^2}{v^n} + \frac{\mu|\mathbf{w}^n_y|^2}{v^n} + \frac{\nu|\mathbf{h}^n_y|^2}{v^n}\Bigg](\theta^n - \theta^n_{yy})dy \\ + \int_\Omega \frac{1}{C_\vartheta(\theta^n)}\frac{\kappa(\theta^n)}{v^n}\theta^n_{yy}\theta^n dy.$$

Note that our growth conditions imply that $\frac{\kappa(\theta^n)}{C_\vartheta(\theta^n)} \geq C_0^{-1}$, for some constant $C_0 >$

0, so using (3.51) we get

$$
\begin{aligned}
&\frac{d}{dt}\left(||\theta^n||^2_{L^2(\Omega)} + ||\theta^n_y||^2_{L^2(\Omega)}\right) + ||\theta^n_{yy}||^2_{L^2(\Omega)} \\
&\le \delta_2 \int_\Omega |\theta^n_{yy}|^2 dy + C_{\delta_2} \int_\Omega \Bigg[1 + (1+|\theta^n|^2)|u^n_y|^2 + (1+|\theta^n|^{2\tilde{q}})(|v^n_y|^2|\theta^n_y|^2 + |\theta^n|^2) \\
&\qquad\qquad (1+|\theta^n|^{2\tilde{q}})|\theta^n_y|^4 + |u^n_y|^4 + |\mathbf{w}^n_y|^4 + |\mathbf{w}^n_y|^4\Bigg] dy, \qquad (3.65)
\end{aligned}
$$

where $\delta_2 > 0$ is arbitrary and C_{δ_2} is a positive constant depending on δ_2.

Let us analyse this last line term by term. First,

$$
\begin{aligned}
\int_\Omega (1+|\theta^n|^2)|u^n_y|^2 dy &\le C(1 + ||\theta^n||^2_{L^2(\Omega)} + ||\theta^n_y||^2_{L^2(\Omega)})||u^n_y||^2_{L^2(\Omega)} \\
&\le C(1 + ||\theta^n||^4_{L^2(\Omega)} + ||\theta^n_y||^4_{L^2(\Omega)} + ||u^n_y||^4_{L^2(\Omega)}).
\end{aligned}
$$

Second,

$$
\begin{aligned}
&\int_\Omega (1+|\theta^n|^{2\tilde{q}})(|v^n_y|^2|\theta^n_y|^2 + |\theta^n|^2) dy \\
&\quad \le C(1 + ||\theta^n||^{2\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{2\tilde{q}}_{L^2(\Omega)})(||\theta^n||^2_{L^2(\Omega)} + \max_{y\in\Omega} |\theta^n_y|^2 ||v^n_y||^2_{L^2(\Omega)}) \\
&\quad \le C(1 + ||\theta^n||^{2\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{2\tilde{q}}_{L^2(\Omega)})(||\theta^n||^2_{L^2(\Omega)} + ||\theta^n_y||_{L^2(\Omega)}||\theta^n_{yy}||_{L^2(\Omega)}||v^n_y||^2_{L^2(\Omega)}) \\
&\quad \le C_{\delta_3}\Bigg[1 + ||\theta^n||^{8\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{8\tilde{q}}_{L^2(\Omega)} + ||\theta^n||^4_{L^2(\Omega)} + ||\theta^n_y||^8_{L^2(\Omega)} \\
&\qquad\qquad + \left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)} ds\right)^8\Bigg] + \delta_3 ||\theta^n_{yy}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\int_\Omega (1+|\theta^n|^{2\tilde{q}})|\theta^n_y|^4 dy &\le C(1 + ||\theta^n||^{2\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{2\tilde{q}}_{L^2(\Omega)})||\theta^n_y||^3_{L^2(\Omega)}||\theta^n_{yy}||_{L^2(\Omega)} \\
&\le C_{\delta_3}(1 + ||\theta^n||^{8\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{8\tilde{q}}_{L^2(\Omega)} + ||\theta^n_y||^{12}_{L^2(\Omega)}) + \delta_3||\theta^n_{yy}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_\Omega (|u^n_y|^4 + |\mathbf{w}^n_y|^4 + |\mathbf{w}^n_y|^4)dy \\
&\qquad \leq 2(||u^n_{yy}||_{L^2(\Omega)}||u^n_y||^3_{L^2(\Omega)} + ||\mathbf{w}^n_{yy}||_{L^2(\Omega)}||\mathbf{w}^n_y||^3_{L^2(\Omega)} + ||\mathbf{h}^n_{yy}||_{L^2(\Omega)}||\mathbf{w}^n_y||^3_{L^2(\Omega)}) \\
&\qquad \leq \delta_4(||u^n_{yy}||^2_{L^2(\Omega)} + ||\mathbf{w}^n_{yy}||^2_{L^2(\Omega)} + ||\mathbf{h}^n_{yy}||^2_{L^2(\Omega)}) \\
&\qquad\qquad\qquad + C_{\delta_4}(||u^n_y||^6_{L^2(\Omega)} + ||\mathbf{w}^n_y||^6_{L^2(\Omega)} + ||\mathbf{h}^n_y||^6_{L^2(\Omega)}).
\end{aligned}
$$

Putting all of this information together in (3.65) and choosing first $\delta_2 > 0$ and then $\delta_3 > 0$ small enough we have

$$
\begin{aligned}
&\frac{d}{dt}\left(||\theta^n||^2_{L^2(\Omega)} + ||\theta^n_y||^2_{L^2(\Omega)}\right) + ||\theta^n_{yy}||^2_{L^2(\Omega)} \\
&\leq C_{\delta_4}\Bigg[1 + ||\theta^n||^{q_1}_{L^2(\Omega)} + ||\theta^n_y||^{q_1}_{L^2(\Omega)} + ||u^n_y||^2_{L^2(\Omega)} + ||\mathbf{w}^n_y||^6_{L^2(\Omega)} + ||\mathbf{h}^n_y||^6_{L^2(\Omega)} \\
&\qquad\qquad + \left(\int_0^t ||u^n_{yy}(s)||^2_{L^2(\Omega)}ds\right)^8\Bigg] + \delta_4(||u^n_{yy}||^2_{L^2(\Omega)} + ||\mathbf{w}^n_{yy}||^2_{L^2(\Omega)} + ||\mathbf{h}^n_{yy}||^2_{L^2(\Omega)}),
\end{aligned}
\tag{3.66}
$$

where $q_1 = 8\tilde{q}$.

Now, adding (3.62), (3.63), (3.64) and (3.66) and choosing $\delta_4 > 0$ small enough we arrive to the inequality (3.59), which, as mentioned before, implies the existence of $t_0 > 0$ small enough and independent of n such that

$$
\max_{t\in[0,t_0]} ||(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)(t)||^2_{H^1(\Omega)} + \int_0^{t_0} ||(u^n_{yy}, \mathbf{w}^n_{yy}, \mathbf{h}^n_{yy}, \theta^n_{yy})(s)||^2_{L^2(\Omega)}ds \leq C,
\tag{3.67}
$$

$$
\frac{m}{2} \leq v^n(t,y) \leq 2M, \qquad y \in \Omega, t \in [0,t_0],
\tag{3.68}
$$

$$
|\theta^n(t,y)| \leq C, \qquad y \in \Omega, t \in [0,t_0],
\tag{3.69}
$$

for some positive constant C also independent of n.

3.2.3 Existence of local solutions

As noted earlier, estimates (3.67), (3.68) and (3.69) imply that all approximate solutions $(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$ are defined on the interval $[0, t_0]$. That is, $t_0 \leq t_n$ for all n.

Furthermore, these estimates guarantee the existence of a subsequence, which we keep denoting by $(v^n, u^n, \mathbf{w}^n, \mathbf{h}^n, \theta^n, \psi^n)$, such that

$$\left.\begin{aligned} v^n &\to v \\ (u^n, u^n_y) &\to (u, u_y) \\ (\mathbf{w}^n, \mathbf{w}^n_y) &\to (\mathbf{w}, \mathbf{w}_y) \\ (\mathbf{h}^n, \mathbf{h}^n_y) &\to (\mathbf{h}, \mathbf{h}_y) \\ (\theta^n, \theta^n_y) &\to (\theta, \theta_y) \\ \psi^n &\to \psi \end{aligned}\right\} \text{ weakly in } L^2([0,t_0]\times\Omega)$$

Having estimates (3.67)-(3.69) it is easy to prove using equations (3.32)-(3.36) that

$$\max_{t\in[0,t_0]} ||(u^n_t, \mathbf{w}^n_t, \mathbf{h}^n_t, \theta^n_t)(t)||_{L^2(\Omega)} \leq C \tag{3.70}$$

From (3.67) and (3.70) we see, in particular, that

$$(u^n_{yy}, u^n_t) \rightharpoonup (u_{yy}, u_t) \qquad \text{weakly in } L^2([0,t_0]\times\Omega),$$

which implies that

$$u \in L^2(0,t_0;H^2(\Omega)) \text{ with } u_t \in L^2(0,t;L^2(\Omega))$$

and hence (see e.g. [22])

$$u \in C([0,t_0];H^1(\Omega)).$$

Similarly,

$$\mathbf{w}, \mathbf{h} \in C([0,t_0];H^1_0(\Omega)) \cap L^2(0,t_0;L^2(\Omega)),$$

and

$$\theta \in C([0,t_0];H^1(\Omega)), \text{ with } \theta_y \in L^2(0,t_0;H^1_0(\Omega)).$$

Finally, we also have the estimates

$$\sup_{t\in[0,t_0]} ||\psi^n_t(t)||_{H^{-1}(\Omega)} + \max_{t\in[0,t_0]} ||v^n_t(t)||^2_{L^2(\Omega)} + \int_0^t ||v^n_{ty}(s)|| \leq C,$$

which also imply the analogues for the limiting functions v and ψ, and hence (see e.g.

[22])

$$\begin{aligned}
&v \in C([0,t_0]; H^1(\Omega)),\\
&\psi \in L^\infty(0,t_0; H_0^1(\Omega)) \cap W^{1,\infty}(0,t_0; H^{-1}(\Omega)).
\end{aligned}$$

The last line also implies that

$$\psi \in C([0,t_0]; L^2(\Omega))$$

so the initial condition makes sense.

By construction the initial and boundary conditions (3.18) are satisfied and multiplying the equations (3.10)-(3.15) by test functions in $\mathcal{S}^k$ and $\mathcal{C}^k$ as it corresponds and taking the limit as $n \to \infty$ (leaving k fixed) we see that $(v, u, \mathbf{w}, \mathbf{h}, \theta, \psi)$ is a local solution of the system (3.37)-(3.42). Finally, since $\theta \in C(0,t_0; H^1(\Omega))$ and θ_0 is bounded away from zero, we conclude (upon redefining t_0) that $\theta(t,y) \geq C^{-1} > 0$ for all $y \in \Omega$ and $t \in [0,t_0]$ (another way to see this is observing, as we will show later when we discuss the a priori estimates, that equation (3.41) admits a maximum principle which implies the same conclusion without needing to redefine t_0). Thus our original system is satisfied (remember that we had modified the system of equations by considering extensions of the functions p, e and κ in order to account for possible negative values of θ^n).

In order to conclude, let us show that in fact

$$\psi \in C([0,t_0]; H_0^1(\Omega)).$$

For this, it suffices to show that the function $t \to ||\psi_y(t)||^2_{L^2(\Omega)}$ is continuous. From the energy identity (3.13) we have the conservation of energy:

$$\begin{aligned}
&\int_\Omega \left[e(v,\theta) + \frac{1}{2}(u^2 + |\mathbf{w}|^2 + \beta v|\mathbf{h}|^2) + \frac{\alpha}{\tilde{\alpha}}\left(\tilde{\alpha} g(v) h(|\psi|^2) + \frac{1}{2}|\psi_y|^2 + \frac{1}{2}|\psi|^4\right)\right](t,y)dy\\
&= \int_\Omega \Bigg[e(v_0,\theta_0) + \frac{1}{2}(u_0^2 + |\mathbf{w}_0|^2 + \beta v_0 |\mathbf{h}_0|^2)\\
&\qquad\qquad + \frac{\alpha}{\tilde{\alpha}}\left(\tilde{\alpha} g(v_0) h(|\psi_0|^2) + \frac{1}{2}|\psi_{0y}|^2 + \frac{1}{2}|\psi_0|^4\right)\Bigg](y)dy.
\end{aligned}$$

Isolating the term $\frac{1}{2}\int_\Omega |\psi_y(t,y)|^2 dy$, we conclude by noting that all the other terms in this identity are continuous functions of $t \in [0, t_0]$. This concludes the proof of Lemma 3.1.

3.2.4 A priori estimates and existence of global solutions

In this section we deduce some a priori estimates independent on time on the solutions of system (3.10)-(3.15) which allow us to extend the local solutions to global ones. The a priori estimates in this section are based on the analogues contained in [15] and in [48] on the study of the planar MHD equations, although with some adaptations in order to include the coupling terms.

Let $(v, u, \mathbf{w}, \mathbf{h}, \theta, \psi)$ be a solution of (3.10)-(3.15), (3.18). Let us assume that the solution is defined on a time interval $[0, T]$ where $T > 0$ is fixed and that that $v(y,t), \theta(y,t) > 0$ for all $(y,t) \in \Omega \times [0,T]$. Under these assumptions we show that v and θ are actually bounded from above and from below by positive constants. This implies that the Lagrangian transformation is nonsingular. We also deduce some estimates on the derivatives of the solutions which show that the solution does not leave the initial function space at any finite time. In particular, we show that the $L^\infty(0,T; H^1(\Omega))$-norm of the solution is bounded by a constant. Having these estimates for any $T > 0$ together with lemma 3.1, a standard argument yields global existence.

Let us begin with some energy estimates, followed by the stated bounds on the specific volume. In all of the subsequent calculations C will denote a generic positive constant that may depend on T and on the initial data.

Lemma 3.2.

$$\frac{d}{dt}\int_\Omega v(t,y)dy = 0, \tag{3.71}$$

$$\frac{d}{dt}\int_\Omega \left(e + \theta^{1+r} + \tfrac{1}{2}(u^2 + |\mathbf{w}|^2 + \beta v|\mathbf{h}|^2) + \alpha g(v) h(|\psi|^2) + \tfrac{1}{2}|\psi_y|^2 + \tfrac{1}{2}|\psi|^4\right) dy = 0, \tag{3.72}$$

$$|\psi(t,y)| \le C. \tag{3.73}$$

Proof. Estimate (3.71) follows directly from equation (3.10) and the no-slip boundary condition $u|_{\partial\Omega} = 0$ from (3.18), while (3.72) follows from the energy equation (3.13), our hypotheses on the initial data (3.25) and the growth conditions (2.54) on the internal energy.

Finally, (3.73) is a consequence of (3.72) and the Sobolev embedding $H^1(\Omega) \hookrightarrow C(\Omega)$. $\square$

Lemma 3.3.

$$C^{-1} \le v(y,t) \le C, \tag{3.74}$$

$$\int_\Omega (\theta - 1 - \log\theta)dy + \int_0^t \int_\Omega \left(\frac{\kappa\theta_y}{v\theta^2} + \theta_y^2 \right) dy\,ds \le C, \tag{3.75}$$

$$\int_0^t \int_\Omega \left(\varepsilon u_y^2 + \mu|\mathbf{w}_y|^2 + \nu|\mathbf{h}_y|^2 \right) dy\,ds \le C. \tag{3.76}$$

Proof. In connection with (2.46), we consider the entropy $s = s(v,\theta)$ given by

$$s(\rho,\theta) = \int_1^\theta \frac{C_\vartheta(z)}{z} dz - P_\vartheta(v), \tag{3.77}$$

where,

$$P_\vartheta(v) := \int_v^1 p_\theta(z)dz. \tag{3.78}$$

Then, $s(v,\theta)$ satisfies the following equation

$$s_t - \left(\frac{\kappa(\theta)\theta_y}{v\theta} \right)_y = \frac{\kappa(\theta)\theta_y}{v\theta^2} + \frac{\varepsilon u_y^2}{v\theta} + \frac{\mu|\mathbf{w}_y|^2}{v\theta} + \frac{\nu|\mathbf{h}_y|^2}{v\theta}.$$

Note that our assumptions (3.21) imply in particular, that

$$|P_\vartheta(v)| \le C(1 + v + P_e(v)) \le C(1 + v + e(v,\theta))$$

Integrating equation (3.2.4) we have

$$\int_0^t \int_\Omega \left(\frac{\kappa(\theta)\theta_y}{v\theta^2} + \frac{\varepsilon u_y^2}{v\theta} + \frac{\mu|\mathbf{w}_y|^2}{v\theta} + \frac{\nu|\mathbf{h}_y|^2}{v\theta} \right) dyds - \int_\Omega s dy = -\int_\Omega s|_{t=0} dy. \tag{3.79}$$

Now, we have that

$$
\begin{aligned}
-\int_\Omega s dy &= \int_0^1 \int_1^\theta C_\vartheta(z)\left(1+\frac{1}{z}\right) dz dy - \int_\Omega (Q(\theta)-Q(1))dy \\
&\quad + \int_\Omega P_\theta(\rho) dy \\
&\geq C^{-1}\int_\Omega (\theta - 1 - \log\theta) dy - \int_\Omega (Q(\theta)-Q(1))dy \\
&\quad - C - C\int_\Omega v dy - C\int_\Omega P_e(\rho) dy \\
&\geq C^{-1}\int_\Omega (\theta - 1 - \log\theta) dy - C - C\int_\Omega e\, dy \\
&\geq C^{-1}\int_\Omega (\theta - 1 - \log\theta) dy - C,
\end{aligned}
\tag{3.80}
$$

and this together with (3.79) imply

$$
\int_0^t \int_\Omega \left(\frac{\kappa(\theta)\theta_y}{v\theta^2} + \frac{\varepsilon u_y^2}{v\theta} + \frac{\mu|\mathbf{w}_y|^2}{v\theta} + \frac{\nu|\mathbf{h}_y|^2}{v\theta} \right) dy ds + \int_0^1 (\theta - 1 - \log\theta) dy \leq C. \tag{3.81}
$$

With this information at hand we can proceed to bound v. Using (3.10) we can rewrite equation (3.11) as

$$
(\varepsilon \log v)_{yt} = u_t + \left(p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(v) h(|\mathbf{w}|^2) \right)_y. \tag{3.82}
$$

Now, from lemma (3.71) we see that for every $t \in [0,T]$ there is a point $a = a(t) \in \Omega$ such that $0 < v(a(t),t) \leq 2C_1$, where $C_1 := \int_\Omega v_0 dy > 0$. Integrating equation (3.82) over $[0,t]$ first and then over $[a(t),y]$, and potentiating the resulting equation we get

$$
\frac{v(y,t)}{v_0(y)} Y(t) B(y,t) = \exp\left(\frac{1}{\varepsilon} \int_0^t (p(y,s) + \tfrac{1}{2}|\mathbf{h}|^2) ds \right), \tag{3.83}
$$

where

$$
B(y,t) := \exp\left[\frac{1}{\varepsilon}\int_a^y (u(\xi,t) - u_0(\xi)) d\xi + \frac{\alpha}{\varepsilon}\int_0^t g'(v(y,s)) h(|\psi(y,s)|^2) ds \right],
$$

and

$$Y(t) := \frac{v_0(a(t))}{v(a(t),t)} \exp\Bigg[\frac{1}{\varepsilon}\int_0^t (p(a(t),s) + \tfrac{1}{2}|\mathbf{h}(a(t),s)|^2)ds \\ - \frac{\alpha}{\varepsilon}\int_0^t g'(v(a(t),s))h(|\psi(a(t),s)|^2)ds\Bigg].$$

Here, $p(y,t) = p(v(y,t),\theta(y,t))$ and $p(a(t),s) = p(v(a(t),s),\theta(a(t),s))$. Next, we multiply (3.83) by $\frac{1}{\varepsilon}(p+\frac{1}{2}|\mathbf{h}|^2)$ in order to deduce the identity

$$\frac{d}{dt}\exp\left(\frac{1}{\varepsilon}\int_0^t (p(y,s)+\tfrac{1}{2}|\mathbf{h}|^2)ds\right) = \frac{1}{\varepsilon}\frac{v(y,t)}{v_0(y)}(p+\tfrac{1}{2}|\mathbf{h}|^2)Y(t)B(y,t),$$

which implies

$$\exp\left(\frac{1}{\varepsilon}\int_0^t (p(y,s)+\tfrac{1}{2}|\mathbf{h}|^2)ds\right) = 1 + \frac{1}{\varepsilon v_0(y)}\int_0^t v(p+\tfrac{1}{2}|\mathbf{h}|^2)Y(s)B(y,s)ds.$$

With this and (3.83) we deduce the following identity for $v(y,t)$:

$$v(y,t) = \frac{v_0(y) + \frac{1}{\varepsilon}\int_0^t v(p+\frac{1}{2}|\mathbf{h}|^2)Y(s)B(y,s)ds}{Y(t)B(y,t)}. \tag{3.84}$$

Now, we proceed to bound all of the terms in the right hand side of this identity. Let us begin with B.

From Lemma 3.2 and our hypothesis on the initial data we know that $||u_0||^2_L(\Omega)$, $||u(t)||^2_L(\Omega) \le C$ for almost every $t \in [0,T]$. Also, our hypotheses on the coupling functions imply, in particular, that g and h are bounded. Taking this information into account it is easy to see that

$$C^{-1} \le B(y,t) \le C,$$

for all $(y,t) \in \Omega \times [0,T]$. Now, $a(t)$ was chosen so that $0 < v(a(t),t) \le C_1$. Thus, as p is nonnegative (recall our hypotheses (3.19)-(3.21)) we have that

$$Y(t) \ge C^{-1}.$$

Next, rewrite (3.84) as

$$Y(t)v(y,t) = \frac{v_0(y) + \frac{1}{\varepsilon}\int_0^t v(p+\frac{1}{2}|\mathbf{h}|^2)Y(s)B(y,s)ds}{B(y,t)}.$$

Then, using the bounds on B, integrating over Ω and using (3.71) we have

$$Y(t) \leq C + C\int_0^t Y(s)\int_\Omega v(p+|\mathbf{h}|^2)dy\,ds,$$

Let us recall that p is given by (3.19). Furthermore, by (3.21) we have that

$$0 \leq p_\theta(v) \leq C(1+v^{-\gamma}),$$

so that by (3.22) and (3.23)

$$vp_\theta(v) \leq C(v+e).$$

Consequently,

$$\begin{aligned} Y(t) &\leq C + C\int_0^t (1+M_\theta(s))Y(s)\int_\Omega (v+e+v|\mathbf{h}|^2)dy\,ds \\ &\leq C + C\int_0^t M_\theta(s)Y(s)ds, \end{aligned}$$

where $M_\theta(t) = \max_{y\in\Omega}\theta(t,y)$ and the last inequality holds due to Lemma 3.2. Now, using (2.55) we derive the following estimate on M_θ:

$$\begin{aligned} \int_0^t M_\theta(s)ds &\leq \int_0^t \left(\int_\Omega \theta dy + \int_\Omega |\theta_y|dy\right) ds \\ &\leq C + \int_0^t \left(\int_\Omega \frac{\theta_y^2}{v}dy + \int_\Omega v\,dy\right) ds \\ &\leq C + C\int_0^t\int_\Omega \frac{\kappa\theta_y^2}{v\theta^2}dy\,ds \\ &\leq C \end{aligned} \tag{3.85}$$

Consequently, using Gronwall's inequality we get

$$Y(t) \leq C.$$

Coming back to identity (3.84) we have

$$v(y,t) \geq C^{-1}.$$

On the other hand, from (3.83) we have

$$v(y,t) \leq C \exp\left(-\frac{1}{\varepsilon}\int_0^t (p + \tfrac{1}{2}|\mathbf{h}|^2)ds\right).$$

Now, from (3.19) we have that

$$\int_0^t p(v,\theta)ds \leq C + C\int_0^t \theta ds \leq C$$

And, since $\mathbf{h}(0,t) = 0$, using Lemma 3.2 and (3.81) we have

$$\begin{aligned}
\int_0^t \frac{1}{2}|\mathbf{h}|^2 ds &\leq \int_0^t \int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|dy\,ds \\
&\leq \int_0^t \int_\Omega \theta v|\mathbf{h}|^2 dy\,ds + C\int_0^t\int_\Omega \frac{\nu|\mathbf{h}_y|^2}{v\theta}dy\,ds \\
&\leq \int_0^t M_\theta(s)\int_\Omega v|\mathbf{h}|^2 dy\,ds + C \\
&\leq C\int_0^t M_\theta(s)ds + C \leq C,
\end{aligned}$$

thus proving that

$$v(y,t) \leq C.$$

We are left with estimate (3.76). For this we use equation (3.16). Note that, similarly to (3.85), by (2.55) we have

$$\begin{aligned}
\int_0^t\int_\Omega (1+\theta^2)dy\,ds &\leq C + C\int_0^t M_\theta(s)^2 ds \\
&\leq C + C\int_0^t\int_\Omega \frac{\kappa\theta_y^2}{\theta^2}dy\,ds \leq C
\end{aligned} \tag{3.86}$$

Consequently, having (3.74) we just need to integrate equation (3.16) to obtain

$$\begin{aligned}
&\int_0^t\int_\Omega (\varepsilon u_y^2 + \mu|\mathbf{w}_y|^2 + \nu|\mathbf{h}_y|^2)dy\,ds \\
&\qquad \leq C + \frac{1}{2}\int_0^t\int_\Omega \varepsilon u_y^2 dy\,ds + C\int_0^t\int_\Omega (1+\theta^2)dy\,ds \\
&\qquad \leq C + \frac{1}{2}\int_0^t\int_\Omega \varepsilon u_y^2 dy\,ds,
\end{aligned}$$

which implies (3.76).

□

Let us now bound the L^2 norm of the derivatives of $v, u, \mathbf{w}$ and $\mathbf{h}$.

Lemma 3.4.

$$\int_\Omega (v_y^2 + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{h}_y|^2) dy \\ + \int_0^t \int_\Omega (u_{yy}^2 + |\mathbf{w}_{yy}|^2 + |\mathbf{h}_{yy}|^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{h}_y|^4) dy\, ds \leq C. \tag{3.87}$$

$$\int_\Omega (|\psi_t|^2 + |\psi_{yy}|^2) dy \leq C, \tag{3.88}$$

Proof. First we deal with the L^2 norm of v_y. Define $V(y,t) := \varepsilon \log v$. Then, from (3.10) we see that V satisfies $V_t = \varepsilon \frac{u_y}{v}$, so we can rewrite equation (3.11) as

$$(V_y - u)_t = \left(p + \frac{\beta}{2} |\mathbf{h}|^2 - \alpha g'(v) h(|\psi|^2) \right)_y . \tag{3.89}$$

Multiply the above equation by $V_y - u$ and integrate to obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega |V_y - u|^2 dy \\ &\leq C + \int_0^t \int_\Omega (p_v v_y + p_\theta \theta_y + \mathbf{h} \cdot \mathbf{h}_y + \alpha g''(v) h(|\psi|^2) v_y \\ &\qquad - \alpha g'(v) h'(|\psi|^2) \mathrm{Re}(\psi \overline{\psi}_y))(V_y - u) dy\, ds \end{aligned}$$

Therefore, observing that $|v_y| \leq C|V_y - u| + |u|$, using (3.19) and (2.55), and recalling that both g and h have compact support, we have that

$$\begin{aligned} &\frac{1}{2} \int_\Omega |V_y - u|^2 dy \\ &\leq C + C \int_0^t \int_\Omega (1+\theta) |V_y - u|^2 dy\, ds + C \int_0^t \int_\Omega (1+\theta)|u|^2 dy\, ds \\ &\quad + C \int_0^t \int_\Omega \theta_y^2 dy\, ds + C \int_0^t \int_\Omega |\mathbf{h} \cdot \mathbf{h}_y|^2 ds + C \int_0^t \int_\Omega |\psi_y|^2 dy\, ds \\ &\leq C + C \int_0^t (1 + M_\theta(s)) \int_\Omega |V_y - u|^2 dy\, ds + C \int_0^t (1 + M_\theta(s)) \int_\Omega |u|^2 dy\, ds \\ &\quad + C \int_0^t \int_\Omega \frac{\kappa \theta_y^2}{\theta^2} dy\, ds + C \int_0^t \int_\Omega |\mathbf{h} \cdot \mathbf{h}_y|^2 ds \\ &\leq C + C \int_0^t (1 + M_\theta(s)) \int_\Omega |V_y - u|^2 dy\, ds + C \int_0^t \int_\Omega |\mathbf{h} \cdot \mathbf{h}_y|^2 ds. \end{aligned}$$

And since we have (3.85), Gronwalls's inequality then implies

$$\int_\Omega |V_y - u|^2 dy \le C + C\int_0^t \int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2\, ds.$$

In particular,

$$\int_\Omega v_y^2 dy \le C + C\int_0^t \int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2\, ds. \tag{3.90}$$

Let us bound the right hand side of this inequality. First, taking into account the boundary conditions on $\mathbf{h}$, we observe the following

$$\begin{aligned}
\int_0^t \int_\Omega |\mathbf{h}|^8 dy\, ds &\le \int_0^t \max_{y\in\Omega} |\mathbf{h}|^6 \int_\Omega |\mathbf{h}|^2 dy\, ds \\
&\le C\int_0^t \max_{y\in\Omega} |\mathbf{h}|^6 ds \\
&\le C\int_0^t \int_\Omega |\mathbf{h}|^4 |\mathbf{h}\cdot\mathbf{h}_y| dy\, ds \\
&\le \frac{1}{2}\int_0^t \int_\Omega |\mathbf{h}|^8 dy\, ds + C\int_0^t \int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\, ds.
\end{aligned}$$

Thus,

$$\int_0^t \int_\Omega |\mathbf{h}|^8 dy\, ds \le C_2 \int_0^t \int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\, ds, \tag{3.91}$$

for a certain constant $C_2 \ge 0$. Now, note that

$$(v\mathbf{h})_t \cdot (|\mathbf{h}|^2\mathbf{h}) = \frac{1}{4}(v|\mathbf{h}|^4)_t + \frac{3}{4} v_t |\mathbf{h}|^4.$$

Having this, we multiply equation (3.14) by $|\mathbf{h}|^2\mathbf{h}$, integrate over $\Omega\times[0,t]$ and use (3.10) to obtain

$$\begin{aligned}
&\frac{1}{4}\int_\Omega v|\mathbf{h}|^4 dy + \frac{3}{4}\int_0^t \int_\Omega u_y |\mathbf{h}|^4 dy\, ds - \int_0^t \int_\Omega \mathbf{w}_y \cdot |\mathbf{h}|^2 \mathbf{h} dy\, ds \\
&\qquad = \int_0^t \int_\Omega \left(\frac{\nu \mathbf{h}_y}{v}\right)_y \cdot |\mathbf{h}|^2 \cdot \mathbf{h} dy\, ds + \frac{1}{4}\int_\Omega v_0 |\mathbf{h}_0|^2 dy.
\end{aligned}$$

Concerning the first term on the right hand side we integrate by parts and get

$$\begin{aligned}
\int_0^t \int_\Omega \left(\frac{\nu \mathbf{h}_y}{v}\right)_y \cdot |\mathbf{h}|^2 \cdot \mathbf{h} dy\, ds &= -\int_0^t \int_\Omega \frac{\nu \mathbf{h}_y}{v} \cdot (2(\mathbf{h}\cdot\mathbf{h}_y)\mathbf{h} + |\mathbf{h}|^2\mathbf{h}_y) dy\, ds \\
&= -\int_0^t \int_\Omega \nu v^{-1} (2|\mathbf{h}\cdot\mathbf{h}_y|^2 + |\mathbf{h}|^2 |\mathbf{h}_y|^2) dy\, ds.
\end{aligned}$$

Replacing this in the equation above and rearranging the terms we have

$$\begin{aligned}
&\frac{1}{4}\int_\Omega v|\mathbf{h}|^4 dy + \int_0^t\int_\Omega \nu v^{-1}(2|\mathbf{h}\cdot\mathbf{h}_y|^2 + |\mathbf{h}|^2|\mathbf{h}_y|^2)dy\,ds \\
&\quad = -\frac{3}{4}\int_0^t\int_\Omega u_y|\mathbf{h}|^4 dy\,ds + \int_0^t\int_\Omega \mathbf{w}_y\cdot|\mathbf{h}|^2\mathbf{h}dy\,ds + \frac{1}{4}\int_\Omega v_0|\mathbf{h}_0|^2 dy.
\end{aligned}$$

Using the bounds on v, Young's inequality with ε and the estimates from Lemmas 3.2 and 3.3 we have

$$\begin{aligned}
&\int_\Omega |\mathbf{h}|^4 dy + \int_0^t\int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\,ds \\
&\quad \le C_\delta\int_\Omega u_y^2 + \frac{\delta}{2}\int_0^t\int_\Omega |\mathbf{h}|^8 dy\,ds + \frac{1}{2}\int_0^t\int_\Omega |\mathbf{w}_y|^2 dy\,ds + \frac{1}{2}\int_0^t\int_\Omega |\mathbf{h}|^6 dy\,ds \\
&\quad \le C_\delta + \delta\int_0^t\int_\Omega |\mathbf{h}|^8 dy\,ds,
\end{aligned}$$

where, $\delta > 0$ is arbitrary and $C_\delta > 0$ is a big enough constant which depends on δ. Using (3.91) we obtain

$$\frac{1}{4}\int_\Omega v|\mathbf{h}|^4 dy + \int_0^t\int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\,ds \le C_\delta + \delta C_2\int_0^t\int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\,ds,$$

and choosing δ small enough we arrive to the estimate

$$\int_\Omega |\mathbf{h}|^4 dy + \int_0^t\int_\Omega |\mathbf{h}\cdot\mathbf{h}_y|^2 dy\,ds \le C, \tag{3.92}$$

which together with (3.90) implies

$$\int_\Omega v_y^2 dy \le C. \tag{3.93}$$

Let us now carry on with the estimates on u. First, note that

$$\begin{aligned}
\max_{y\in\Omega} u_y^2 &\le \int_\Omega u_y dy + 2\int_\Omega |u_y u_{yy}|dy \\
&\le C_{\delta_1}\int_\Omega u_y^2 dy + \delta_1\int_\Omega u_{yy}^2 dy
\end{aligned} \tag{3.94}$$

for any $\delta_1 > 0$ and a big enough $C_\delta > 0$. Next, multiply (3.11) by u_{yy} and integrate over $\Omega\times[0,t]$. After integrating by parts, using (3.86) and the uniform bounds on v

we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega u_y^2 dy + \int_0^t\int_\Omega \frac{\varepsilon u_{yy}^2}{v} dy\,ds \\
&\quad = \frac{1}{2}\int_\Omega u_{0y}^2 dy + \int_0^t\int_\Omega \Big(\frac{\varepsilon u_y v_y}{v^2} + p_v v_y + p_\theta \theta_y + \beta \mathbf{h}\cdot\mathbf{h}_y + \alpha g''(v) h(|\psi|^2) v_y \\
&\qquad\qquad + \alpha g'(v) h'(|\psi|^2)\mathrm{Re}(\psi\overline{\psi}_y) + \Big) u_{yy} dy\,ds \\
&\quad \le C + C_{\delta_2}\int_0^t\int_\Omega (|u_y v_y|^2 + (1+\theta^2) v_y^2 + \theta_y^2 + |\mathbf{h}\cdot\mathbf{h}_y|^2 + |\psi_y|^2) dy\,ds \\
&\qquad + \delta_2 \int_0^t\int_\Omega \frac{\varepsilon u_{yy}^2}{v} dy\,ds \\
&\quad \le C + C_{\delta_2}\int_0^t \max_{y\in\Omega}(1+|u_y|+\theta^2)\int_\Omega v_y^2 dy\,ds \\
&\qquad + C_{\delta_2}\int_0^t\int_\Omega \left(\frac{\kappa\theta_y^2}{\theta^2} + |\mathbf{h}\cdot\mathbf{h}_y|^2 + |\psi_y|^2\right) dy\,ds + \delta_2\int_0^t\int_\Omega \frac{\varepsilon u_{yy}^2}{v} dy\,ds \\
&\quad \le C_{\delta_2} + C_{\delta_1} C_{\delta_2}\int_0^t\int_\Omega u_y^2 dy\,ds + (\delta_2 + \delta_1 C_{\delta_2})\int_0^t\int_\Omega \frac{\varepsilon u_{yy}^2}{v} dy\,ds \\
&\quad \le C_{\delta_1,\delta_2} + (\delta_2 + \delta_1 C_{\delta_2})\int_0^t\int_\Omega \frac{\varepsilon u_{yy}^2}{v} dy\,ds,
\end{aligned}
$$

and this holds for any $\delta_2 > 0$ and certain $C_{\delta_2}, C_{\delta_1,\delta_2}$. Consequently, choosing first $\delta_2 = \frac{1}{4}$ and then $\delta_1 = \frac{1}{4}C_{\delta_2}^{-1}$ we arrive to the estimate

$$
\int_\Omega u_y^2 dy + \int_0^t\int_\Omega u_{yy}^2 dy\,ds \le C. \tag{3.95}
$$

Now, using (3.94) and (3.95) we see that

$$
\int_0^t\int_\Omega u_y^4 dy\,ds \le \int_0^t \max_{y\in\Omega} u_y^2 \int_\Omega u_y^2 dy\,ds \le C. \tag{3.96}
$$

In a similar way as we deduced estimates (3.95) and (3.96) we can multiply (3.12) by $\mathbf{w}_{yy}$ integrate and use integration by parts as well as the estimates we already have and the interpolation inequality

$$
\max_{y\in\Omega}|\mathbf{w}_y|^2 \le C_\delta \int_\Omega |\mathbf{w}_y|^2 dy + \delta\int_\Omega |\mathbf{w}_{yy}|^2 dy \tag{3.97}
$$

in order to obtain the estimates

$$
\int_\Omega |\mathbf{w}_y|^2 dy + \int_0^t\int_\Omega (|\mathbf{w}_{yy}|^2 + |\mathbf{w}_y|^4) dy\,ds \le C. \tag{3.98}
$$

After this, rewrite (3.14) as

$$\mathbf{h}_t = -v^{-1}\mathbf{h}u_y + v^{-1}\mathbf{w}_y + v^{-1}\left(\frac{\nu\mathbf{h}_y}{v}\right)_y. \tag{3.99}$$

Multiplying this equation by $\mathbf{h}_{yy}$ and using the interpolation inequality

$$\max_{y\in\Omega}|\mathbf{h}_y|^2 \le C_\delta\int_\Omega|\mathbf{h}_y|^2dy + \delta\int_\Omega|\mathbf{h}_{yy}|^2dy, \tag{3.100}$$

as we did above with u_y and $\mathbf{w}_y$, we find

$$\int_\Omega|\mathbf{h}_y|^2dy + \int_0^t\int_\Omega(|\mathbf{h}_{yy}|^2 + |\mathbf{h}_y|^4)dy\,ds \le C. \tag{3.101}$$

Finally, we are left with (3.88). Differentiate (3.15) with respect to t, multiply it by $\overline{\psi}_t$ (the complex conjugate of ψ_t), take imaginary part and integrate to obtain

$$\begin{aligned}\int_\Omega|\psi_t|^2dy &\le C + C\int_0^t\int_\Omega v_t^2dy\,ds + C\int_0^t\int_\Omega|\psi_t|^2dy\,ds\\ &= C + C\int_0^t\int_\Omega u_y^2dy\,ds + C\int_0^t\int_\Omega|\psi_t|^2dy\,ds\\ &\le C + C\int_0^t\int_\Omega|\psi_t|^2dy\,ds,\end{aligned}$$

and from Gronwall's inequality we get

$$\int_\Omega|\psi_t|^2dy \le C. \tag{3.102}$$

Note that in light of all the estimates we have deduced so far, and in view of equation (3.15), the $L^2(\Omega)$-norm of ψ_t is equivalent to the $L^2(\Omega)$-norm of ψ_{yy} (it is at this point that we use our assumption that $\psi_0 \in H^2(\Omega)$). Thus we conclude that

$$\int_\Omega|\psi_{yy}|^2dy \le C. \tag{3.103}$$

□

We now turn our attention to the a priori estimates on the derivatives of the temperature.

Lemma 3.5.

$$C^{-1} \le \theta \le C, \tag{3.104}$$

$$|v_y| \le C, \tag{3.105}$$

$$\int_\Omega \theta_y^2 dy + \int_0^t \int_\Omega (\theta_t^2 + \theta_{yy}^2) dy\, ds \le C. \tag{3.106}$$

Proof. For these last estimates we adapt the proof of an analogue lemma in [48], which is very similar to a corresponding lemma in [15]. Set

$$\Theta := \max_{(y,t)\in\Omega\times[0,t]} \theta(y,t), \qquad X := \int_0^T \int_\Omega (1+\theta^{q+r})\theta_t^2 dy\, ds$$
$$Y := \max_{t\in[0,T]} \int_\Omega (1+\theta^{2q})\theta_y^2 dy.$$

This Y should not be confused with the function defined in the proof of Lemma 3.3.

We begin by making some useful observations. First, using the interpolation inequality

$$u_y^2 \le \int_\Omega u_y^2 dy + 2\left(\int_\Omega u_y^2 dy\right)^{1/2}\left(\int_\Omega u_{yy}^2 dy\right)^{1/2},$$

we see that

$$\max_{y\in\Omega} |u_y| \le C + C\left(\int_\Omega u_{yy}^2 dy\right)^{1/4}.$$

Let q be as in (2.55). Then,

$$\begin{aligned}
\max_{y\in\Omega} \theta^{(2q+3+r)/2} &\le \left(\int_\Omega \theta dy\right)^{(2q+3+r)/2} + \frac{2q+3+r}{2}\int_\Omega \theta^{(2q+1+r)/2}|\theta_y| dy \\
&\le C + C\left(\int_\Omega \theta^{1+r} dy\right)^{1/2}\left(\int_\Omega \theta^{2q}\theta_y^2 dy\right)^{1//2} \\
&\le C + CY^{1/2}.
\end{aligned}$$

Thus,

$$\Theta \le C + CY^{\delta_1} \tag{3.107}$$

where $\delta_1 = (2q+3+r)^{-1}$.

Now, let us show that $X+Y \le C$. Set $H(v,\theta) := v^{-1}\int_0^\theta \kappa(\xi)d\xi$. Then,

$$\begin{aligned}
H_t &= H_v u_y + \frac{\kappa}{v}\theta_t, \\
H_{ty} &= H_v u_{yy} + H_{vv} u_y v_y + \left(\frac{1}{v}\right)_v \kappa\theta_t v_y + \left(\frac{\kappa}{v}\theta_y\right)_t.
\end{aligned}$$

Now let us rewrite equation (3.16) as

$$C_\vartheta(\theta)\theta_t + \theta p_\theta u_y = \left(\frac{\kappa\theta_y}{v}\right)_y + \frac{\varepsilon u_y^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{h}_y|^2}{v}. \tag{3.108}$$

Multiplying this equation by H_t and integrating by parts

$$\int_0^t\int_\Omega \left(C_\vartheta\theta_t + \theta p_\theta u_y - \frac{\varepsilon u_y^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{h}_y|^2}{v}\right) H_t dy\,ds + \int_0^t\int_\Omega \frac{\kappa\theta_y}{v}H_{ty}dy\,ds = 0. \tag{3.109}$$

Let us estimate each one of the terms in the above identity. First, from (2.54) we have

$$\int_0^t\int_\Omega C_\vartheta\theta_t\frac{\kappa}{v}\theta_t dy\,ds \geq M_1X,$$

and also

$$\int_0^T\int_\Omega \frac{\kappa\theta_y}{v}\left(\frac{\kappa}{v}\theta_y\right)_t dy\,ds = \int_0^T\frac{d}{dt}\int_\Omega\frac{\kappa^2\theta_y^2}{v^2}dy\,ds \geq M_2Y - C,$$

for positive constants M_1 and M_2. From (2.55)

$$|H_v| + |H_{vv}| \leq C\int_0^\theta(\kappa)d\xi \leq C(1+\theta^{q+1}).$$

Next, from (2.54) and the boundedness of v

$$\begin{aligned}
\int_0^t\int_\Omega C_\vartheta\theta_t H_v u_y dy\,ds &\leq C\int_0^T\int_\Omega (1+\theta^{q+1+r})|\theta_t|\,|u_y|dy\,ds \\
&\leq \frac{M_1}{8}X + C(1+\Theta^{q+2+r})\int_0^t\int_\Omega u_y^2 dy\,ds \\
&\leq \frac{M_1}{8}X + C(1+\Theta^{q+2+r}) \\
&\leq \frac{M_1}{8}X + C + CY^{(q+2+r)\delta_1} \\
&\leq C + \frac{M_1}{8}X + \frac{M_2}{8}Y,
\end{aligned}$$

also,

$$
\begin{aligned}
\int_0^t \int_\Omega \left(\theta p_\theta u_y - \frac{\varepsilon u_y^2}{v} - \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{h}_y|^2}{v}\right) H_v u_y dy\, ds
&\le C(1+\Theta^{q+2}) \int_0^t \int_\Omega (|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{h}_y|^2)|u_y| dy\, ds \\
&\le C(1+\Theta^{q+2}) \int_0^t \int_\Omega (u_y^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{h}_y|^4) dy\, ds \\
&\le C(1+\Theta^{q+2}) \\
&\le C + CY^{(q+2)\delta_1} \\
&\le C + \frac{M_2}{8} Y,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t \int_\Omega \left(\theta p_\theta u_y - \frac{\varepsilon u_y^2}{v} - \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{h}_y|^2}{v}\right) \frac{\kappa}{v}\theta_t dy\, ds
&\le C \int_0^t \int_\Omega ((1+\theta)|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{h}_y|^2)(1+\theta^q)|\theta_t| dy\, ds \\
&\le \frac{M_1}{8} X + C(1+\Theta^{q+2-r}) \int_0^t \int_\Omega (|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{h}_y|^2) dy\, ds \\
&\le C + \frac{M_1}{8} X + CY^{(q+2-r)\delta_1} \\
&\le C + \frac{M_1}{8} X + \frac{M_2}{8} Y.
\end{aligned}
$$

Using Lemma 3.4 we have

$$
\begin{aligned}
\int_0^t \int_\Omega \frac{\kappa\theta_y}{v} H_{vv} u_y v_y &\le C(1+\Theta^{(3q+3)/2}) \int_0^t \int_\Omega \frac{\kappa^{1/2}|\theta_y|}{\theta} |u_y|\,|v_y| dy\, ds \\
&\le C(1+\Theta^{(3q+3)/2}) \left(\int_0^t \max_y u_y^2 \int_\Omega v_y^2 dy\, ds\right)^{1/2} \left(\int_0^t \int_\Omega \frac{\kappa\theta_y^2}{\theta^2} dy\, ds\right)^{1/2} \\
&\le C(1+\Theta^{(3q+3)/2}) \left(1 + \int_0^t \left(\int_\Omega u_{yy}^2 dy\right)^{1/2} ds\right)^{1/2} \\
&\le C(1+\Theta^{(3q+3)/2}) \\
&\le C + CY^{(3q+3)\delta/2} \\
&\le C + \frac{M_2}{8} Y.
\end{aligned}
$$

Next, taking into account the boundary conditions on θ we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \frac{\kappa\theta_y}{v}\left(\frac{\kappa}{v}\right)_v \theta_t v_y dy\, ds \le \int_0^t \int_\Omega (1+\theta^q)|\theta_t| \left|\frac{\kappa\theta_y}{v}\right| |v_y| dy\, ds \\
&\le \frac{M_1}{16}X + C(1+\Theta^{q-r}) \int_0^t \max_y \left|\frac{\kappa\theta_y}{v}\right|^2 \int_\Omega v_y dy\, ds \\
&\le \frac{M_1}{16}X + C(1+\Theta^{q-r}) \int_0^t 2\int_\Omega \left|\frac{\kappa\theta_y}{v}\left(\frac{\kappa\theta_y}{v}\right)_y\right| dy\, ds \\
&\le \frac{M_1}{16}X + C(1+\Theta^{q+1-r}) \left(\int_0^t \int_\Omega \kappa \left(\frac{\kappa\theta_y}{v}\right)_y^2 dy\, ds\right)^{1/2} \left(\int_0^t \int_\Omega \frac{\kappa\theta_y^2}{\theta^2} dy\, ds\right)^{1/2} \\
&\le \frac{M_1}{16}X + C(1+\Theta^{q+1-r}) \left(\int_0^t \int_\Omega \kappa (C_\vartheta^2\theta_t^2 + \theta^2 p_\theta^2 u_y^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{h}_y|^4) dy\, ds\right)^{1/2} \\
&\le \frac{M_1}{16}X + C(1+\Theta^{(2q+2-r)/2})X^{1/2} + C(1+\Theta^{(3q+4-r)/2}) \left(\int_0^t \int_\Omega u_y^2 dy\, ds\right)^{1/2} \\
&\qquad + C(1+\Theta^{(3q+2-2r)/2}) \left(\int_0^t \int_\Omega (u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{h}_y|^4) dy\, ds\right)^{1/2} \\
&\le C + \frac{M_1}{8}X + C\Theta^{2q+2} \\
&\le C + \frac{M_1}{8} + CY^{(2q+2)\delta_1} \\
&\le C + \frac{M_1}{8} + \frac{M_2}{8}Y.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\int_0^t \int_\Omega \frac{\kappa\theta_y}{v} H_v u_{yy} dy\, ds &\le \int_0^t \int_\Omega (1+\theta^{(3q+4)/2}) \frac{\kappa^{1/2}|\theta_y|}{\theta} |u_{yy}| dy\, ds \\
&\le C(1+\Theta^{(3q+4)/2}) \left(\int_0^t \int_\Omega \frac{\kappa\theta_y^2}{\theta^2} dy\, ds\right)^{1/2} \left(\int_0^t \int_\Omega u_{yy}^2 dy\, ds\right)^{1/2} \\
&\le C(1+\Theta^{(3q+4)/2}) \\
&\le C + Y^{(3q+4)\delta_1/2} \\
&\le C + \frac{M_2}{8}Y.
\end{aligned}
$$

Putting all of these estimates together with (3.109) we get

$$
X + Y \le C, \tag{3.110}
$$

which leads to the estimates

$$\theta \leq C, \tag{3.111}$$

and

$$\int_\Omega \theta_y^2 dy + \int_0^t \int_\Omega \theta_t^2 dy\, ds \leq C. \tag{3.112}$$

In order to conclude estimate (3.106) we write (3.108) as

$$\frac{\kappa\theta_{yy}}{v} = C_\vartheta \theta_t + \theta p_\theta u_y - \frac{\varepsilon u_y^2}{v} - \frac{\mu|\mathbf{w}_y|^2}{v} - \frac{\mu|\mathbf{h}_y|^2}{v} - \left(\frac{\kappa}{v}\right)_v \theta_y v_y - \frac{\kappa_\theta \theta_y^2}{v}.$$

Squaring this equality, integrating and using the interpolation inequality

$$\max_y \theta_y^2 \leq 2\left(\int_\Omega \theta_y^2 dy\right)^{1/2}\left(\int_\Omega \theta_{yy}^2 dy\right)^{1/2} \leq C\left(\int_\Omega \theta_{yy}^2 dy\right)^{1/2},$$

we have

$$\begin{aligned}
\int_0^t \int_\Omega \theta_{yy}^2 dy\, ds &\leq C\int_0^t \int_\Omega (\theta_t^2 + u_y^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{h}_y|^4) \\
&\quad + C\int_0^t \max_y \theta_y^2 \int_\Omega v_y^2 dy\, ds + \int_0^t \max_y \theta_y^2 \int_\Omega \theta_y^2 dy\, ds \\
&\leq C + C\int_0^t \left(\int_\Omega \theta_{yy}^2 dy\right)^{1/2} ds \\
&\leq C + \frac{1}{2}\int_0^t \int_\Omega \theta_{yy}^2 dy\, ds
\end{aligned}$$

which yields

$$\int_0^t \int_\Omega \theta_{yy}^2 dy\, ds \leq C. \tag{3.113}$$

We finally move on to the last estimate, consisting of a lower bound for the temperature. We have to prove that

$$C^{-1} \leq \theta(y,t), \tag{3.114}$$

for a big enough constant $C > 0$. In order to show this it suffices to apply the maximum principle (see [43]) to equation (3.108). More specifically, note that θ satisfies the following inequality

$$C_\vartheta \theta_t + \frac{v}{2\varepsilon}\theta^2 p_\theta^2 - \left(\frac{\kappa}{v}\right)_v \theta_y v_y \geq \frac{\kappa\theta_{yy}}{v}. \tag{3.115}$$

This follows from equation (3.108) and Young's inequality. In order to apply the maximum principle we have to show that the coefficients of this parabolic inequality are bounded. With the estimates already obtained, we only have to show that v_y is uniformly bounded. Let V be as in (3.89). Then,

$$\begin{aligned}V_y(y,t) = V_y(y,0) + u(y,t) - u_0(y) + \int_0^t (p_v v_y + p_\theta \theta_y + \beta \mathbf{h}\cdot\mathbf{h}_y \\ - \alpha g''(v)h(|\psi|^2)v_y - 2\alpha g'(v)h'(|\psi|^2)\mathrm{Re}(\psi\overline{\psi}_y))ds\end{aligned}$$

Then, squaring this identity and using interpolation inequalities on θ_y $\mathbf{h}_y$ and ψ_y we get

$$\begin{aligned}v_y^2 &\leq C + C\int_0^t \int_\Omega (|\theta_{yy}^2 + |\mathbf{h}_{yy}|^2 + |\psi_{yy}|^2)dy\,ds + \int_0^t v_y^2 ds \\ &\leq C + C\int_0^t v_y^2 ds,\end{aligned}$$

which yields, using Gronwall's inequality that

$$|v_y| \leq C$$

.

Consequently, taking into consideration the boundary conditions on θ, the maximum principle applied to the parabolic inequality (3.115) (see [43]) implies that θ cannot be zero in finite time, which concludes the proof. □

The estimates from Lemmas 3.2 through 3.5 provide the necessary a priori estimates which, in light of the local result from Lemma 3.1, guarantee the gobal existence of solutions.

3.2.5 Uniqueness and continuous dependence

We are left with the uniqueness part of Theorem 3.1. For this part we incorporate some of the ideas by Chen and Wang in [16] on the MHD system, although some new estimates are deduced in order to account for the SW-LW interaction.

The strategy to prove uniqueness is very straightforward, although it involves several calculations. We begin by supposing that there exist two solutions with the same initial data and consider the difference of them. This new set of functions satisfies a system of equations related to the original system (3.10)-(3.15). The proof consists in making some estimates on this new system, similar to the a priori estimates from the previous section, and use Gronwall's inequality to conclude that the difference of the two solutions has to be identically equal to zero. Such estimates are presented in a few Lemmas in order to organize the presentation.

Suppose that $(v_j, u_j, \mathbf{w}_j, \mathbf{h}_j, \theta_j, \psi_j)$, $j = 1, 2$ are two solutions of (3.10)-(3.15), (3.18) with the same initial data $(v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)$ satisfying the hypotheses of Theorem 3.1. Let $(\tilde{v}, \tilde{u}, \tilde{\mathbf{w}}, \tilde{\mathbf{h}}, \tilde{\theta}, \tilde{\psi}) = (v_1 - v_2, u_1 - u_2, \mathbf{w}_1 - \mathbf{w}_2, \mathbf{h}_1 - \mathbf{h}_2, \theta_1 - \theta_2, \psi_1 - \psi_2)$. For convenience we define

$$G(t) := \sum_{j=1,2} \int_\Omega (|\partial_y(u_j, \mathbf{w}_j, \mathbf{h}_j, \theta_j, \psi_j)|^2 + |\partial_{yy}(u_j, \mathbf{w}_j, \mathbf{h}_j, \theta_j, \psi_j)|^2).$$

Note that

$$\int_0^t G(s)ds \leq C.$$

With this notation, let us carry on with the estimates.

Lemma 3.6.

$$\int_\Omega |(\tilde{v}, \tilde{u}, \tilde{\mathbf{w}})|^2 dy + \int_0^t \int_\Omega |(\tilde{u}_y, \tilde{\mathbf{w}}_y)|^2 dy\, ds \leq C \int_0^t (1 + G(s)) \int_\Omega |(\tilde{v}, \tilde{\mathbf{h}}, \tilde{\theta}, \tilde{\psi})|^2 dy\, ds. \tag{3.116}$$

Proof. As $(v_j, u_j, \mathbf{w}_j, \mathbf{h}_j, \theta_j, \psi_j)$, $j = 1, 2$ are solutions of (3.10)-(3.15), (3.18) we have, in partiular that,

$$\begin{aligned}
&\tilde{u}_t - \left(\frac{\varepsilon \tilde{u}_y}{v_1}\right)_y - \left(\varepsilon \frac{u_{2y}}{v_1 v_2} \tilde{v}\right)_y \\
&= -\left(p(v_1, \theta_1) - p(v_2, \theta_2) + \frac{\beta}{2}(|\mathbf{h}_1|^2 - |\mathbf{h}_2|^2) - \alpha g'(v_1) h(|\psi_1|^2) - \alpha g'(v_2) h(|\psi_2|^2)\right)_y .
\end{aligned} \tag{3.117}$$

Note that for any C^1 function φ we have that the function

$$(z_1, z_2) \to \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2}$$

is a continuous function. With this in mind we see that equation (3.117) can be rewritten as

$$
\begin{aligned}
&\tilde{u}_t - \left(\frac{\varepsilon \tilde{u}_y}{v_1}\right)_y \\
&= (A_1(v_1,v_2)u_{2y}\tilde{v})_y - \Big(A_2(v_1,v_2,\theta_1)\tilde{v} + p_\theta(v_2)\tilde{\theta} + \frac{\beta}{2}(\mathbf{h}_1+\mathbf{h}_2)\cdot\tilde{\mathbf{h}} \\
&\qquad\qquad + A_3(v_1,v_2)h(|\psi_1|^2)\tilde{v} + g'(v_2)A_4(|\psi_1|,|\psi_2|)\tilde{\psi}\Big)_y. \qquad (3.118)
\end{aligned}
$$

for some continuous functions A_k, $k=1,2,3,4$. Let us recall that v_j, θ_j, $\mathbf{h}_j$ and ψ_j are uniformly bounded with v_j bounded from below by a positive constant for both $j=1,2$. Consequently, multiplying equation (3.118) integrating by parts and using Young's inequality with δ we have

$$
\begin{aligned}
&\int_\Omega \tilde{u}^2 dy + \int_0^t\int_\Omega \tilde{u}_y^2 dy\, ds \\
&\qquad \le \frac{1}{2}\int_0^t\int_\Omega \tilde{u}_y^2 dy\,ds + C\int_0^t (1+\max_{y\in\Omega}|u_{1y}|^2)\int_\Omega(\tilde{v}+\tilde{\mathbf{h}}+\tilde{\theta}+\tilde{\psi})dy\,ds,
\end{aligned}
$$

and using the interpolation inequality

$$|u_{1y}|^2 \le C(||u_{1y}||^2_{L^2(\Omega)} + ||u_{1yy}||^2_{L^2(\Omega)}),$$

we obtain

$$\int_\Omega \tilde{u}^2 dy + \int_0^t\int_\Omega \tilde{u}_y^2 dy\,ds \le C\int_0^t(1+G(s))\int_\Omega(\tilde{v}^2+|\tilde{\mathbf{h}}|^2+\tilde{\theta}^2+|\tilde{\psi}|^2)dy\,ds. \qquad (3.119)$$

Next, multiplying the equation

$$\tilde{v}_t = \tilde{u}_y$$

by $\tilde{v}$, integrating and using Young's inequality with δ

$$\int_\Omega \tilde{v}^2 dy \le \delta_0\int_0^t\int_\Omega \tilde{u}_y^2 dy\,ds + C_{\delta_0}\int_0^t\int_\Omega \tilde{v}^2 dy\,ds, \qquad (3.120)$$

for any $\delta_0>0$ and some C_{δ_0} big enough. Finally, taking the inner product of the following equation

$$\tilde{\mathbf{w}}_t - \left(\frac{\mu}{v_1}\tilde{\mathbf{w}}_y\right)_y = \left(\frac{\mu \mathbf{w}_{2y}}{v_1 v_2}\tilde{v} + \beta\tilde{\mathbf{h}}\right)_y$$

with $\tilde{\mathbf{w}}$ and integrating by parts we have

$$\begin{aligned}&\int_\Omega |\tilde{\mathbf{w}}|^2 dy + \int_0^t \int_\Omega |\tilde{\mathbf{w}}_y|^2 dy\, ds \\ &\quad \leq \frac{1}{2}\int_0^t \int_\Omega |\tilde{\mathbf{w}}_y|^2 dy\, ds + C\int_0^t (1 + \max_{y\in\Omega} |\mathbf{w}_{2y}|)\int_\Omega (\tilde{v}^2 + |\tilde{\mathbf{h}}|^2) dy\, ds,\end{aligned}$$

which implies

$$\int_\Omega |\tilde{\mathbf{w}}|^2 dy + \int_0^t \int_\Omega |\tilde{\mathbf{w}}_y|^2 dy\, ds \leq C\int_0^t (1 + G(s))\int_\Omega (\tilde{v}^2 + |\tilde{\mathbf{h}}|^2) dy\, ds. \tag{3.121}$$

Adding (3.119),(3.120) and (3.121) and choosing δ_0 small enough we get (3.116). □

Lemma 3.7.

$$\begin{aligned}&\int_\Omega \tilde{\theta}^2 dy + \int_0^t \int_\Omega \tilde{\theta}_y^2 dy\, ds \\ &\quad \leq \delta_1 \int_0^t \int_\Omega (\tilde{u}_y^2 + |\tilde{\mathbf{w}}_y|^2 + |\tilde{\mathbf{h}}_y|^2) dy\, ds + C_{\delta_1}\int_0^t (1 + G(s))\int_\Omega (\tilde{v}^2 + \tilde{\theta}^2) dy\, ds,\end{aligned} \tag{3.122}$$

for any $\delta_1 > 0$ and some $C_{\delta_1} > 0$.

Proof. We have the following equation

$$\begin{aligned}&(Q(\theta_1) - Q(\theta_2))_t - \left(\frac{\kappa(\theta_1)}{v_1}\tilde{\theta}_y\right)_y \\ &= \left(\frac{\kappa(\theta_1)\theta_{2y}}{v_1 v_2}\tilde{v} - \frac{\kappa(\theta_1) - \kappa(\theta_2)}{v_1(\theta_1 - \theta_2)}\theta_{1y}\tilde{\theta}\right)_y \\ &\quad - \left(\theta_1 p_\theta(v_1)\tilde{u}_y + \theta_1 \frac{p_\theta(v_1) - p_\theta(v_2)}{v_1 - v_2} u_{2y}\tilde{v} + p_\theta(v_2) u_{2y}\tilde{\theta}\right) \\ &\qquad + \frac{\varepsilon(u_{1y} + u_{2y})\tilde{u}_y + \mu(\mathbf{w}_{1y} + \mathbf{w}_{2y})\tilde{\mathbf{w}}_y + \nu(\mathbf{h}_{1y} + \mathbf{h}_{2y})\tilde{\mathbf{h}}_y}{v_1} \\ &\qquad\quad - \frac{\varepsilon u_{2y}^2 + \mu|\mathbf{w}_{2y}|^2 + \nu|\mathbf{h}_{2y}|^2}{v_1 v_2}\tilde{v}.\end{aligned} \tag{3.123}$$

Our assumptions (2.54) imply that

$$|Q(\theta_1) - Q(\theta_2)| = \left|\int_{\theta_2}^{\theta_1} C_\vartheta(z) dz\right| \geq C^{-1}|\tilde{\theta}|.$$

By the same token and using the boundedness of θ_1 and θ_2

$$|Q(\theta_1)-Q(\theta_2)| \le C\left|\theta_1^{1+r}-\theta_2^{1+r}\right| + |\tilde\theta| = C\left(\left|\frac{\theta_1^{1+r}-\theta_2^{1+r}}{\theta_1-\theta_2}\right|+1\right)\tilde\theta \le C\tilde\theta.$$

Also, note that

$$(Q(\theta_1)-Q(\theta_2))_y = c\vartheta()\theta_1)\tilde\theta_y + \frac{C\vartheta(\theta_1)-C\vartheta(\theta_2)}{\theta_1-\theta_2}\theta_{2y}\tilde\theta.$$

With this information at hand we proceed similarly as in Lemma (3.6), multiplying (3.123) by $(Q(\theta_1)-Q(\theta_2))$ and integrating to obtain

$$\begin{aligned}\int_\Omega \tilde\theta^2 dy + \int_0^t\int_\Omega \tilde\theta_y^2 dy\,ds \le \frac{1}{2}\int_0^t\int_\Omega \tilde\theta_y^2 dy\,ds + \delta_1\int_0^t\int_\Omega (\tilde u_y^2 + |\tilde{\mathbf{w}}_y|^2 + |\tilde{\mathbf{h}}_y|^2)dy \quad ds\\ + C_{\delta_1}\int_0^t (1+G(s))\int_\Omega (\tilde v^2+\tilde\theta^2)dy\,ds,\end{aligned}$$

for any $\delta_1>0$ and some $C_{\delta_1}>0$, which yields

$$\begin{aligned}\int_\Omega \tilde\theta^2 dy + \int_0^t\int_\Omega \tilde\theta_y^2 dy\,ds \le \delta_1\int_0^t\int_\Omega (\tilde u_y^2 + |\tilde{\mathbf{w}}_y|^2 + |\tilde{\mathbf{h}}_y|^2)dy \quad ds\\ + C_{\delta_1}\int_0^t (1+G(s))\int_\Omega (\tilde v^2+\tilde\theta^2)dy\,ds,\end{aligned}$$

□

Lemma 3.8.

$$\begin{aligned}\int_\Omega |\tilde{\mathbf{h}}|^2 dy + \int_0^t\int_\Omega |\tilde{\mathbf{h}}_y|^2 dy\,ds \le \delta_2\int_0^t\int_\Omega (u_y^2 + |\tilde{\mathbf{w}}_y|^2)dy\,ds + C_{\delta_2}\int_0^t\int_\Omega \tilde v_y^2 dy\,ds\\ + C_{\delta_2}\int_0^t (1+G(s))\int_\Omega (\tilde v^2 + |\tilde{\mathbf{h}}|^2)dy\,ds. \qquad (3.124)\end{aligned}$$

Proof. This time we consider the equation

$$\beta(v_1\mathbf{h}_1 - v_2\mathbf{h}_2)_t - \left(\frac{\nu\tilde{\mathbf{h}}_y}{v_1}\right)_y = \left(\frac{\nu\mathbf{h}_{2y}}{v_1v_2}\tilde v\right)_y + \beta\tilde{\mathbf{w}}_y. \qquad (3.125)$$

We observe that

$$\begin{aligned}(v_1\mathbf{h}_1 - v_2\mathbf{h}_2)_y &= v_1\mathbf{h}_{1y} - v_2\mathbf{h}_{2y} + v_{1y}\mathbf{h}_1 - v_{2y}\mathbf{h}_2\\ &= v_1\tilde{\mathbf{h}} + \mathbf{h}_{2y}\tilde v + \mathbf{h}_1\tilde v_y + v_{2y}\tilde{\mathbf{h}}.\end{aligned}$$

Also, we have that

$$|v_1\mathbf{h}_1 - v_2\mathbf{h}_2| \geq v_1|\tilde{\mathbf{h}}| - |\mathbf{h}_2||\tilde{v}|.$$

Then, taking inner product of equation (3.125) with $(v_1\mathbf{h}_1 - v_2\mathbf{h}_2)$ and proceeding as before

$$\begin{aligned}
&\int_\Omega |\tilde{\mathbf{h}}|^2 dy + \int_0^t \int_\Omega |\tilde{\mathbf{h}}_y|^2 dy\, ds \\
&\qquad \leq \frac{1}{2}\int_0^t \int_\Omega |\tilde{\mathbf{h}}_y|^2 dy\, ds + C\int_\Omega \tilde{v}^2 dy + C_{\delta_2}\int_0^t (1+G(s))\int_\Omega (\tilde{v}^2 + |\tilde{\mathbf{h}}|^2) dy\, ds \\
&\qquad\qquad + C\int_0^t \int_\Omega \tilde{v}_y^2 dy\, ds + \delta_2 \int_0^t \int_\Omega |\tilde{\mathbf{w}}_y|^2 dy\, ds,
\end{aligned}$$

and we conclude using (3.120) and choosing $\delta_0 = \delta_2$. □

Lemma 3.9.

$$\int_\Omega (|\tilde{\psi}|^2 + |\tilde{\psi}_y|^2) dy\, ds \leq C\int_0^t (1+G(s))\int_\Omega (\tilde{v}_y^2 + |\tilde{\psi}|^2 + |\tilde{\psi}_y|^2) dy\, ds. \tag{3.126}$$

Proof. As ψ_1 and ψ_2 are solutions of equation (3.15), we have that $\tilde{\psi}$ satisfies and equation of the form

$$i\tilde{\psi}_t + \tilde{\psi}_{yy} = B_1(\psi_1,\psi_2,v_1)\tilde{\psi} + B_2(\psi_1,\psi_2,v_1)\overline{\tilde{\psi}} + B_3(\psi_2,v_1,v_2)\tilde{v}, \tag{3.127}$$

for some continuous functions B_1, B_2 and B_3. Multiplying this equation by $\overline{\tilde{\psi}_t}$ (the complex conjugate of $\tilde{\psi}_t$), taking real part and integrating

$$\begin{aligned}
&\int_\Omega |\tilde{\psi}_y|^2 dy \\
&= \mathrm{Re}\left[\int_0^t \int_\Omega (B_1\tilde{\psi} + B_2\overline{\tilde{\psi}} + B_3\tilde{v})\overline{\tilde{\psi}_t} dy\, ds\right] \\
&= \mathrm{Re}\left[\frac{-1}{i}\int_0^t \int_\Omega |B_1\tilde{\psi} + B_2\overline{\tilde{\psi}} + B_3\tilde{v}|^2 dy\, ds\right] \\
&\qquad\qquad + \mathrm{Re}\left[\frac{1}{i}\int_0^t \int_\Omega (B_1\tilde{\psi} + B_2\overline{\tilde{\psi}} + B_3\tilde{v})\overline{\tilde{\psi}_{yy}} dy\, ds\right] \\
&= \mathrm{Re}\left[\frac{-1}{i}\int_0^t \int_\Omega (B_1\tilde{\psi} + B_2\overline{\tilde{\psi}} + B_3\tilde{v})_y \overline{\tilde{\psi}_y} dy\, ds\right] \\
&\leq C\int_0^t (1 + \max_{y\in\Omega}[|\psi_{1y}|^2 + |\psi_{2y}|^2 + v_{1y}^2 + v_{2y}^2])\int_\Omega |\tilde{\psi}_y|^2 dy\, ds \\
&\qquad\qquad + C\int_0^t \int_\Omega \tilde{v}_y^2 dy\, ds + \int_0^t \int_\Omega |\tilde{\psi}|^2 dy\, ds \\
&\leq C\int_0^t (1+G(s))\int_\Omega (\tilde{v}_y^2 + |\tilde{\psi}|^2 + |\tilde{\psi}_y|^2) dy\, ds.
\end{aligned}$$

By the same token, multiplying (3.127) by $\overline{\tilde{\psi}}$, taking imaginary part and integrating we find

$$\int_\Omega |\tilde{\psi}|^2 dy \leq C \int_0^t \int_\Omega (\tilde{v}^2 + |\tilde{\psi}|^2) dy\, ds,$$

which added to the above inequality yields (3.126). □

Lemma 3.10.

$$\begin{aligned}\int_\Omega \tilde{v}_y^2 dy \leq C \int_0^t \int_\Omega \tilde{v}_y^2 dy\, ds + \delta_3 \int_0^t \int_\Omega (\tilde{u}_y^2 + \tilde{\theta}_y^2 + |\tilde{\mathbf{h}}_y|^2) dy\, ds \\ + C \int_0^t (1 + G(s)) \int_\Omega (\tilde{v}^2 + \tilde{\theta}^2 + |\tilde{\mathbf{h}}|^2 + |\tilde{\psi}|^2 + |\tilde{\psi}_y|^2) dy\, ds. \end{aligned} \tag{3.128}$$

Proof. From (3.89) we have the equation

$$\begin{aligned}&\left(\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}\right)_t + \tilde{u}_t \\ &= \left(p(v_1, \theta_1) - p(v_2, \theta_2) + \frac{\beta}{2}(|\tilde{\mathbf{h}}_1|^2 - |\tilde{\mathbf{h}}_2|^2) - \alpha g'(v_1) h(|\psi_1|^2) - \alpha g'(v_2) h(|\psi_2|^2)\right)_y .\end{aligned}$$

Then we multiply this equtation by $\left(\frac{\lambda v_{1y}}{v_1} - \frac{\lambda v_{2y}}{v_2}\right)$, integrate and proceed as before. The only term that might pose a problem is $\tilde{u}_t \left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)$, but we observe that

$$\tilde{u}_t \left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right) = (\tilde{u}(\log v_1 - \log v_2)_y)_t - (\tilde{u}(\log v_1 - \log v_2)_t)_y + \tilde{u}_y \left(\frac{u_{1y}}{v_1} - \frac{u_{2y}}{v_2}\right),$$

so that

$$\begin{aligned}\int_0^t \int_\Omega \tilde{u}_t \left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right) dy\, ds \leq \delta \int_\Omega \left(\frac{v_{1y}}{v_1} - \frac{v_{2y}}{v_2}\right)^2 dy + C_\delta \int_\Omega \tilde{u}^2 dy \\ + C_0 \int_0^t \int_\Omega \tilde{u}_y^2 dy\, ds + C \int_0^t (1 + G(s)) \int_\Omega \tilde{v}^2 dy\, ds,\end{aligned}$$

and from (3.116) we can choose $\delta > 0$ small and $C_\delta > C_0$ in order to obtain

$$\begin{aligned}&C_\delta \int_\Omega \tilde{u}^2 dy + C_0 \int_0^t \int_\Omega \tilde{u}_y^2 dy\, ds \\ &\leq (-C_\delta + C_0) \int_0^t \int_\Omega \tilde{u}_y^2 dy\, ds + \tilde{C}_\delta \int_0^t (1 + G(s)) \int_\Omega |(\tilde{v}, \tilde{\mathbf{h}}, \tilde{\theta}, \tilde{\psi})|^2 dy\, ds.\end{aligned}$$

We omit the rest of the details as they go by the same lines as the proofs of the last few lemmas. □

We now have all the elements to prove the uniqueness of solutions stated in Theorem 3.1. All we have to do is add (3.116), (3.122), (3.124), (3.126) and (3.128) and choose δ_1, δ_2 and δ_3 small enough to arrive at the following inequality

$$D(t) \leq \int_0^t (1+G(s))D(s)ds, , \tag{3.129}$$

where

$$D(s) := \int_\Omega |(\tilde{v}, \tilde{u}, \tilde{\mathbf{w}}, \tilde{\mathbf{h}}, \tilde{\theta}, \tilde{\psi}, \tilde{v}_y, \tilde{\psi}_y)|^2 dy + \int_0^t \int_\Omega |(\tilde{u}_y, \tilde{\mathbf{w}}_y, \tilde{\mathbf{h}}_y, \tilde{\theta}_y)|^2 dy\, ds.$$

Since

$$\int_0^T G(s)ds \leq C,$$

we conclude, by Gronwall's inequality, that $D(t) = 0$ for all $t \in [0,T]$, which implies uniqueness of solutions of system (3.10)-(3.15), (3.18).

Let us point out that the same proof yields continuous dependence of solutions as the same calculations show that

$$D(t) \leq C(M)D(0) + C(M)\int_0^t (1+G(s))D(s), \tag{3.130}$$

where $(v_j, u_j, \mathbf{w}_j, \mathbf{h}_j, \theta_j, \psi_j)$ are solutions of (3.10)-(3.15) with initial data $(v_{0j}, u_{0j}, \mathbf{w}_{0j}$, $\mathbf{h}_{0j}, \theta_{0j}, \psi_{0j})$, $j = 1, 2$ respectively; and inequality (3.130) holds with a constant $C(M)$ which depends on an arbitrary $M > 0$ such that

$$|||(v_{0j}, u_{0j}, \mathbf{w}_{0j}, \mathbf{h}_{0j}, \theta_{0j}, \psi_{0j})||| \leq M, \qquad M^{-1} \leq v_{0j}, \theta_{0,j} \leq M,$$

for both $j = 1, 2$; where

$$|||(v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)||| = ||((v_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0)||_{H^1(\Omega)} + ||\psi_0||_{H^2(\Omega)} + ||v_0||_{W^{1,\infty}(\Omega)}.$$

In this case, Gronwall's inequality implies

$$D(t) \leq C(M)e^{C(M)\int_0^T (1+G(s))ds} D(0) = \tilde{C}(M)D(0).$$

3.3 Vanishing bulk viscosity and interaction coefficients

We have established well posedness of the planar SW-LW interactions model. We proved this for the system expressed in the Lagrangian coordinates of the fluid and, after showing that the Lagrangian transformation was nonsingular, the respective result for the original system followed by changing back to the original coordinates. Having this, we may turn our attention to some other questions about the model.

In Chapter 2, when we were reviewing the ideas involved in the deduction of the MHD equations, we mentioned in passing that the magnetic permeability parameter β is usually assumed to be equal to 1 as it differs only slightly from the unity in most real world media covered by the MHD model. We, however, chose to keep track of it and made it explicit in the equations. Let us remark that the only physical constraint on this parameter is that it be positive (see [33]).

Now, let us introduce a new artificial small parameter δ multiplying the thermal part of the pressure. That is, we substitute relation (2.50) by

$$p(\rho,\theta) = a\rho^\gamma + \delta\theta p_\theta(\rho), \tag{3.131}$$

where δ is some positive constant. This certainly agrees with our previous assumptions and the results we have proved so far continue to hold.

Note, however, that if we take $\varepsilon = \alpha = \beta = \delta = 0$ then we are left with a decoupled system involving the compressible one dimensional Euler Equations and the nonlinear Schrödinger equation. Namely,

$$\rho_t + (\rho u)_x = 0, \quad , \tag{3.132}$$

$$(\rho u)_t + (\rho u^2 + a\rho^\gamma)_x = 0, \tag{3.133}$$

$$(\rho \mathbf{w})_t + (\rho u \mathbf{w})_x = (\mu \mathbf{w}_x)_x, \tag{3.134}$$

$$(\rho Q(\theta))_t + (\rho Q(\theta)\mathbf{u})_x = (\kappa\theta_x)_x + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{h}_x|^2, \tag{3.135}$$

$$(\nu \mathbf{h}_x)_x = 0, \tag{3.136}$$

$$i\psi_t + \psi_{yy} = |\psi|^2\psi. \tag{3.137}$$

Our next task is to study this system and its relation with our original viscous system (3.1)-(3.6). More precisely, we are going to show that the sequence of solutions to the viscous system, given by Theorem 3.2, converges to a solution of the limit problem above as $(\varepsilon, \alpha, \beta, \delta) \to 0$. This is a delicate issue as it involves several subtleties. Besides the loss of regularity in the velocity field caused by the vanishing viscosity in the momentum equation, we point out some other issues that might complicate our analysis. For instance, the Schrödinger equation (3.137) is stated in a different coordinate system, which in principle should be the Lagrangian coordinate associated to the velocity field u. Yet, solutions of the Euler equations are not expected to be smooth and a Lagrangian transformation as we defined it is likely to be singular. On the other hand, as we pointed out in the beginning of Subsection 3.1.2, given the no-slip boundary condition $u|_{x=0,1} = 0$ on the original viscous system, the Lagagrangian coordinate takes values in the domain $\Omega_y = (0, d)$ where d is the L^1 norm of the initial density ρ_0. In light of this, we have to be careful when passing to the limit equation (3.6).

As aforementioned, we are inspired by the work of Dias and Frid in [17] who pursue similar objectives on a SW-LW interactions model involving the isentropic Navier-Stokes equations and who, in turn, follow the work by Chen and Perepelitsa in [14] on the vanishing viscosity limit for the isentropic one dimensional Navier-Stokes equations. Our main contribution here is to include the thermal description as well as the electromagnetic coupling.

The tools employed by Chen and Perepelitsa include the compensated compactness method. In this direction we cannot fail to mention the following references. The compensated compactness method arose from the ideas by Tartar and Murat (see [46, 47] and [41]). Tartar gave the first applications for scalar conservation laws. This approach was extended by DiPerna in [21] to systems con two conservation laws and the first applications to fluid dynamics [20]. DiPerna's results was later extended by Chen in [9, 10], Lions, Perthame and Souganidis [39] and Lions, Perthame and Tadmor [40]. Let us also mention the work by LeFloch and Westdickenberg [35] and of course the above mentioned work by Chen and Perepelitsa [14] and the references contained in these works.

Before we begin with the specifics, we first do a quick review of the fundamental ideas behind the compensated compactness method.

3.3.1 Compensated compactness

The problem described above is twofold. On the one hand, we have to show compactness of the sequence of solutions to the viscous system. That is, we have to show that the sequence of solutions has a convergent subsequence. On the other hand, we have to show consistency, i.e., that the limit of such convergent subsequence solves the limit equations.

Through some uniform estimates on the viscous system we can deduce compactness; in some sense. By uniform estimates we mean estimates independent of the vanishing parameters. As it turns out, we encounter ourselves in a situation where the uniform estimates available are not good enough to guarantee strong compactness directly through the usual Sobolev spaces arguments, but perhaps weak compactness only. Unfortunately, limits of weak convergent sequences do not, in general, commute with nonlinear functions of the sequence; not even for continuous functions, and the nonlinearities appearing in the limit equations suggest that strong compactness of the sequence is required in order to show consistency.

These considerations apply to more general situations and a natural question that arises is whether some nonlinear function of a particular weakly convergent sequence converges to the function of the limit.

The compensated compactness method provides a way to answer this question positively in some situations. The method consists in combining the Young measures theorem, which characterizes the weak limits of functions of a given weak convergent sequence, with the Div-Curl lemma, that provides conditions under which a particular nonlinear function of the sequence actually commutes with the limit. It has been employed successfully by several authors in the study of fluid dynamics including the ones mentioned in the previous subsection, and we intend to apply their framework to our present situation. For this reason, we dedicate this small subsection to review, very superficially, the fundamental ideas behind it.

Let us begin by stating (without proof) the Young measures theorem ([46]).

Lemma 3.11 (Young measures). *Supose $K \subseteq \mathbb{R}^m$ is bounded and $\Omega \subseteq \mathbb{R}^n$ is open. Let $\{z^\varepsilon\}$ be a sequence of measurable functions, with $z^\varepsilon : \Omega \to \mathbb{R}^m$, such that $z^\varepsilon(x) \in K$ for a.e. $x \in \Omega$. Then there is a subsequence $\{z^{\varepsilon_k}\}$ and a family of probability measures ν_x, $x \in \Omega$ on $\mathbb{R}^m$ with $\operatorname{supp}\nu_x \in \overline{K}$ so that if f is a continuous function in K and*

$$\overline{f}(x) := \langle \nu_x, f(\lambda) \rangle, \qquad a.e.$$

then

$$f(z^{\varepsilon_k}) \rightharpoonup \overline{f} \qquad in\ L^\infty(\Omega)\ weakly - *.$$

This is not the most general statement available of this theorem, but it is very appropriate for our current illustrative purposes.

Note that this theorem provides not only a characterization of the limits of continuous functions of weakly convergent sequences, but also a criterion to decide if the convergence is in fact strong. In the notation of the Theorem, one may consider the function $\overline{z}(x) := \langle \nu_x, g(\lambda)\rangle$ with $g(\lambda) = \lambda$. Then, the question at hand is whether

$$\overline{f} = f(\overline{z}), \tag{3.138}$$

for some (or any) continuous function f. Now, suppose that somehow we manage to show that ν_x is a Dirac measure for all x. Then, necessarily $\nu_x = \delta_{\overline{z}(x)}$ (the Dirac measure concentrated at $\overline{z}(x)$), (3.138) holds and the convergence $f(z^{\varepsilon_k}) \to \overline{f}$ is strong.

The Young measures Theorem, then, reduces the problem of consistency to the analysis of the Young measures. In practice, this has to be done on a case by case basis, depending on the problem under consideration.

In many applications the reduction of the Young measures to Dirac masses follows by applying cleverly the Div-Curl Lemma due to Murat and Tartar. (A version of) This Lemma may be stated as follows (see [41, 46, 47]).

Lemma 3.12 (Div-Curl Lemma). *Let $\{v^\varepsilon\}$, $\{w^\varepsilon\}$ two bounded sequences in $L^2(\Omega;\mathbb{R}^n)$ such that*

- *$\{divv^\varepsilon\}$ is pre-compact in $W^{-1,2}(\Omega)$, and*
- *$\{curlw^\varepsilon\}$ is pre-compact in $W^{-1,2}(\Omega; M^{nxn})$,*

where M^{nxn} is the space of $n \times n$ matrices and

$$(curlw)_{ij} := \frac{\partial}{\partial x_j} w^i - \frac{\partial}{\partial x_i} w^j, \quad 1 \le i, j \le n.$$

Suppose, further, that $v^\varepsilon \rightharpoonup v$, $w^\varepsilon \rightharpoonup w$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Then,

$$v^\varepsilon \cdot w^\varepsilon \to v \cdot w$$

in the sense of distributions.

Let us consider a 2×2 system of conservation laws written in the form

$$\mathbf{v}_t + F(\mathbf{v})_x = 0 \tag{3.139}$$

where $\mathbf{v} = (v_1, v_2)^\top$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a given function. The compressible Euler equations (3.132), (3.133) constitute a particular example of this kind.

Assuming that $\mathbf{v}$ is smooth we can deduce a set of auxiliary equations

$$\eta(\mathbf{v})_t + q(\mathbf{v})_x = 0 \tag{3.140}$$

by taking the inner product of (3.139) with $\nabla\eta(v_1, v_2)$ as long as η and q satisfy

$$\nabla q(\mathbf{v}) = \nabla\eta(\mathbf{v})\nabla F(\mathbf{v}). \tag{3.141}$$

A pair (η, q) satisfying this relation is called an *entropy-entropy flux pair.*

Note that equation (3.140) can be written either as $\mathrm{div}_{t,x}\, G(\mathbf{v}) = 0$ or as $\mathrm{curl}_{t,x}\, H(\mathbf{v}) = 0$, by choosing $G(\mathbf{v}) = (\eta(\mathbf{v}), q(\mathbf{v}))^\top$ and $H(\mathbf{v}) = (q(\mathbf{v}), -\eta(\mathbf{v}))^\top$.

Now suppose that we have a sequence $\mathbf{v}^\varepsilon$ of solutions to a system of the form

$$\mathbf{v}^\varepsilon_t + F(\mathbf{v}^\varepsilon)_x = R^\varepsilon, \tag{3.142}$$

with $R^\varepsilon \to 0$ (in the sense of distributions, for instance) as $\varepsilon \to 0$, and that there is a whole family of pairs $\{(\eta^j, q^j)\}_{j\in I}$ that satisfy (3.141). Then, (heuristically) for each pair (η^j, q^j) we have that

$$\eta^j(\mathbf{v}^\varepsilon)_t + q^j(\mathbf{v}^\varepsilon)_x = \tilde{R}^\varepsilon_{\eta^j}. \tag{3.143}$$

Under suitable conditions one might be able to use this last equation to verify the hypotheses of the Div-Curl Lemma and also use the Young measures Theorem in order to find a subsequence $\{\mathbf{v}^{\varepsilon_k}\}_k$ such that

$$\eta^i(\mathbf{v}^{\varepsilon_\mathbf{k}})q^j(\mathbf{v}^{\varepsilon_\mathbf{k}}) - \eta^j(\mathbf{v}^{\varepsilon_\mathbf{k}})q^i(\mathbf{v}^{\varepsilon_\mathbf{k}}) \to \overline{\eta^i}\,\overline{q^j} - \overline{\eta^j}\,\overline{q^i}$$

as $k \to \infty$, for all $i, j \in I$, which means that

$$\overline{\eta^i\, q^j} - \overline{\eta^j\, q^i} = \overline{\eta^i}\,\overline{q^j} - \overline{\eta^j}\,\overline{q^i} \tag{3.144}$$

Lastly, if the family of entropies is rich enough the relations (3.144) may provide enough information to guarantee that the support of each one of the Young measures associated to the limit $\overline{\mathbf{v}}$ must be contained in a single point, thus concluding that the convergence $\mathbf{v}^{\varepsilon_k} \to \overline{\mathbf{v}}$ is strong and, consequently, that $\overline{\mathbf{v}}$ solves equation (3.139) in the sense of distributions.

A useful tool that is often used to verify the hypotheses of the Div-Curl Lemma is Murat's Lemma which can be stated as follows ([42], see also [9]).

Lemma 3.13 (Murat's Lemma). *Let* $1 < q \leq p < r \leq \infty$. *Then*

$$\{\textit{Compact of } W^{-1,q}_{loc}(\Omega)\} \cap \{\textit{Bounded of } W^{-1,r}_{loc}(\Omega)\} \subseteq \{\textit{Compact of } W^{-1,p}_{loc}(\Omega)\}.$$

In connection to Murat's Lemma the following result also proves itself useful in this framework.

Lemma 3.14. *For each* $1 \leq q < 1^* := \frac{n}{n-1}$, *(n being the dimension of the domain* $\Omega \subseteq \mathbb{R}^n$*)*

$$\mathcal{M}(\Omega) \hookrightarrow W^{-1,q}(\Omega),$$

with compact inclusion, where, $\mathcal{M}(\Omega)$ *is the space of Radon measures on* Ω. *In particular* $L^1(\Omega) \hookrightarrow W^{-1,q}$.

These are the basic ideas that compose the compensated compactness method. Of course, there are a lot of "ifs" in this framework that have to be dealt with in each specific application.

3.3.2 Limit equations

Let us come back to our main subject, which is the study of the SW-LW interactions system. Our objective here is to study the limit of solutions of our planar SW-LW interactions system as $(\varepsilon, \alpha, \beta, \delta) \to 0$. In order to deal with the convergence issues in the continuity and momentum equations we employ the framework by Chen and Perepelitsa in [14]. Regarding the convergence of solutions in the nonlinear Schrödinger equation a simple application of Aubin-Lions lemma will suffice. The magnetic description poses no problems as we can deduce good uniform estimates on the magnetic field. Lastly, for the thermal description we adapt some ideas from the study by Feireisl in [23] on the full multidimensional compressible Navier-Stokes equations. Of course, in order to achieve all this we deduce some new uniform estimates that allow us to accommodate all of the techniques in the cited references. Such estimates pose, as will be shown later, some restriction in the way that the coefficients $(\varepsilon, \alpha, \beta, \delta)$ tend to zero. Namely, $\alpha = o(\varepsilon^{1/2})$, $\beta = o(\varepsilon)$ and $\delta = o(\varepsilon)$ as $\varepsilon \to 0$. As such, we can, for simplicity, consider α, β and δ as functions of ε and denote the sequence of solutions to (3.1)-(3.6) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$ as $(\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon)$.

In the interest of analysing the limit as $(\varepsilon, \alpha, \beta, \delta) \to 0$, let us make some considerations on the limit equations. In order to fix notation we denote by Ω the spatial domain where the Eulearean coordinates take values and by Ω_y the corresponding domain of the Lagrangian coordinate. We begin by the compressible Euler equations.

Isentropic Euler equations

Let us consider the isentropic Euler equations

$$\rho_t + (\rho u)_x = 0, \quad , \tag{3.145}$$

$$(\rho u)_t + (\rho u^2 + p_e(\rho))_x = 0, \tag{3.146}$$

where the pressure $p_e(\rho)$ (denoted this way to maintain the notation of section 2.3) is given by

$$p_e(\rho) = a\rho^\gamma \tag{3.147}$$

for some $\gamma > 1$, with initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)) \in L^\infty(\Omega). \tag{3.148}$$

As it is not possible to avoid the occurrence of vacuum in this setting, it is convenient to consider the momentum $m = \rho u$ as state variable in place of the velocity. Accordingly, system (3.145), (3.146) may be written as

$$U_t + F(U)_x = 0 \tag{3.149}$$

where, $U = (\rho, m)^\top$ and $F(U) = (m, \frac{m^2}{\rho} + p_e)$.

A pair of functions $(\eta, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is called an *entropy-entropy flux pair* (or simply entropy pair) of system (3.149) provided that they satisfy

$$\nabla q(\mathbf{U}) = \nabla \eta(\mathbf{U}) \nabla F(\mathbf{U}). \tag{3.150}$$

An entropy pair for (3.145), (3.146) is said to be convex if the Hessian $\nabla^2 \eta(\rho, m) \geq 0$, and η is called a *weak entropy* if

$$\lim_{\substack{\rho \to 0, \\ \frac{m}{\rho} = \text{const.}}} \eta(\rho, m) = 0.$$

A very important example of weak entropy pair for (3.145), (3.146) is given by the *mechanical energy* η^* and the *mechanical energy flux* q^*:

$$\eta^*(\rho,m)=\frac{1}{2}\frac{m^2}{\rho}+\rho P_e(\rho),\qquad q^*(\rho,m)=\frac{1}{2}\frac{m^2}{\rho^2}+mP_e(\rho)+\rho m P_e'(\rho),\tag{3.151}$$

where (in the notation of Section 2.3) $P_e(\rho)$ is the elastic potential

$$P_e(\rho)=\frac{a}{\gamma-1}\rho^{\gamma-1}.$$

The *total mechanical energy* for (3.145), (3.146) is

$$E[\rho,u](t):=\int_\Omega \eta^*(\rho,m)dx=\int_\Omega\left(\frac{1}{2}\rho u^2+\frac{a}{\gamma-1}\rho^\gamma\right)dx.\tag{3.152}$$

Relation (3.150) may be written in the variables (ρ,u) as the wave equation

$$\begin{cases}\eta_{\rho\rho}-\frac{p_e'(\rho)}{\rho^2}\eta_{uu}=0, & \rho>0\\ \eta|_{\rho=0}=0.\end{cases}\tag{3.153}$$

and consequently, any weak entropy pair (η,q) can be represented by

$$\begin{cases}\eta^\zeta(\rho,\rho u)=\int_{\mathbb{R}}\chi(\rho;s-u)\zeta(s)ds,\\ q^\zeta(\rho,\rho u)=\int_{\mathbb{R}}(\vartheta s+(1+\vartheta)u)\chi(\rho;s-u)\zeta(s)ds, & \vartheta=\frac{\gamma-1}{2},\end{cases}$$

for any continuous function $\zeta(s)$, where $\chi(\rho,u;s)=\chi(\rho;s-u)$ is determined by

$$\begin{cases}\chi_{\rho\rho}-\frac{p_e'(\rho)}{\rho^2}\chi_{uu}=0,\\ \chi(0,u;s)=0, \qquad \chi_\rho(0,u;s)=\delta_{u=s}.\end{cases}\tag{3.154}$$

For the γ-law case, where the pressure is given by (3.147), the weak entropy kernel is given by

$$\chi(\rho,u;s)=[\rho^{2\vartheta}-(s-u)^2]_+^\Lambda,\qquad \Lambda=\frac{3-\gamma}{2(\gamma-1)},\tag{3.155}$$

and the corresponding weak entropy pairs are given by

$$\begin{cases}\eta^\zeta(\rho,\rho u)=\rho\int_{-1}^1\zeta(u+\rho^\vartheta s)[1-s^2]_+^\Lambda ds,\\ q^\zeta(\rho,\rho u)=\rho\int_{-1}^1(u+\vartheta\rho^\vartheta s)\zeta(u+\rho^\vartheta s)[1-s^2]_+^\Lambda ds.\end{cases}\tag{3.156}$$

A direct consequence of this representation is the following ([14, Lemma 2.1]), which we state for later reference.

Lemma 3.15. *For a C^2 function $\zeta : \mathbb{R} \to \mathbb{R}$ compactly supported in $[a, b]$, we have*

$$\mathit{supp}\eta^\zeta,\ \mathit{supp}q^\zeta \subseteq \{(\rho, \rho u) : \rho^\vartheta + u \geq a, u - \rho^\vartheta \leq b\}.$$

Furthermore, there exists a constant $C_\zeta > 0$ such that, for any $\rho \geq 0$ and $u \in \mathbb{R}$, we have

- *(i) For $\gamma \in (1, 3]$,*
$$|\eta^\zeta(\rho, m)| + |q^\zeta(\rho, m)| \leq C_\zeta \rho.$$
- *(ii) For $\gamma > 3$,*
$$|\eta^\zeta(\rho, m)| \leq C_\zeta \rho, \qquad |q^\zeta(\rho, m)| \leq C_\zeta \rho \max\{1, \rho^\vartheta\}.$$

(iii) If η_n^ζ is considered as a function of (ρ, m), $m = \rho u$, then

$$|\eta_m^\zeta(\rho, m)| + |\rho \eta_{mm}^\zeta(\rho, m)| \leq C_\zeta,$$

and if η_n^ζ is considered as a function of (ρ, u), then

$$|\eta_{mu}^\zeta(\rho, m)| + |\rho^{1-\vartheta} \eta_{m\rho}^\zeta(\rho, m)| \leq C_\zeta.$$

Bearing in mind the procedure described in the previous subsection, the fact that we have a very rich family of entropy pairs, with an explicit description for them, is extremely advantageous. As the solutions to our viscous system are strong solutions, from equations (3.1), (3.2) we can deduce a relation of the form (3.143) for any entropy pair given by (3.156), which fits perfectly into the compensated compactness scheme.

In light of these considerations we state the following concept, taken from [14].

Definition 3.1. *Let (ρ_0, u_0) be given initial data such that $E[\rho_0, u_0] \leq E_0 < \infty$. A pair $(\rho, u) : \Omega \times [0, T) \to [0, \infty) \times \mathbb{R}$ is called a finite-energy entropy solution of* (3.145),(3.146),(3.148) *if the following hold:*

- *There is a locally bounded function $C(E, t) \geq 0$ such that*
$$E[\rho, u](t) \leq C(E_0, t).$$

- (ρ, u) *satisfies* (3.145) *and* (3.146) *in the sense of distributions and, more generally,*
$$\eta^\zeta(\rho, u)_t + q^\zeta(\rho, u)_x \leq 0,$$
in the sense of distributions, for the test functions $\zeta(s) \in \{\pm 1, \pm s, s^2\}$.

- *The initial data are attained in the sense of distributions.*

The first step in our analysis is to show strong convergence of a subsequence of $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ to a finite-energy entropy solution to (3.145),(3.146). For this we rely heavily on [14] and also on [17].

Transverse velocity field and magnetic field

We move on to the limit equations (3.134) and (3.136) for the transverse velocity field and the magnetic field. As μ and ν are left fixed independently of ε we can deduce some satisfactory uniform estimates on $\mathbf{w}_x$ and on $\mathbf{h}_x$ that permit the passage to the limit for the sequence $(\mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon)$ without any major complications, once we have shown that ρ^ε and $\rho^\varepsilon u^\varepsilon$ converge strongly.

Regarding equation (3.136), we see that we are left with a stationary equation and therefore the initial condition loses its meaning. However, note that from equation (3.5) we have that

$$\int_\Omega \beta \mathbf{h}^\varepsilon \varphi dx - \int_\Omega \beta \mathbf{h}_0^\varepsilon \varphi|_{t=0} ds - \int_0^t \int_\Omega (\beta u^\varepsilon \mathbf{h}^\varepsilon - \beta \mathbf{w}^\varepsilon)\varphi_x dx\, ds = -\int_0^t \int_\Omega \nu \mathbf{h}_x^\varepsilon \varphi_x dx\, ds,$$

for any smooth test function φ with compact support in $\Omega \times [0, T)$.

A couple of energy estimates based on the energy identity (3.9) and on equation (3.7) (which will be deduced later) as well as an interpolation inequality for u^ε and for $\mathbf{w}^\varepsilon$ show that $\beta \mathbf{h}^\varepsilon \to 0$ in $L^\infty(0, T; L^2(\Omega))$ and $\beta(u^\varepsilon \mathbf{h}^\varepsilon, \mathbf{w}^\varepsilon) \to 0$ in $L^1(\Omega \times (0, T))$ as $\varepsilon \to 0$ with $\beta = o(\varepsilon)$. By the same token we can a assume that $\mathbf{h}_x^\varepsilon$ converges weakly to $\mathbf{h}_x$ for some $\mathbf{h} \in L^2(0, T; H_0^1(\Omega))$. As a result, in the limit we have

$$\lim_{\varepsilon \to 0} \int_\Omega \beta \mathbf{h}_0^\varepsilon \varphi|_{t=0} ds = \int_0^t \int_\Omega \nu \mathbf{h}_x \varphi_x dx\, ds.$$

For this reason, we are compelled to impose that $\beta \mathbf{h}_0^\varepsilon \to 0$ in the sense of distributions, in which case we would have that $\mathbf{h}_x = 0$, thus forcing $\mathbf{h}$ to be identically equal to zero.

As for the limit equation (3.134), the same energy estimates allow us to assume that $\mathbf{w}^\varepsilon \to \mathbf{w}$ weakly in $L^2(0,T;H_0^1(\Omega))$ and provided that ρ^ε and $\rho^\varepsilon u^\varepsilon$ converge strongly, we can conclude that the limit equation (3.134) is satisfied in the sense of distributions.

Thermal energy

The uniform estimates that we obtain further ahead, guarantee that $\varepsilon|u^\varepsilon|^2$, $\mu|\mathbf{w}^\varepsilon|^2$ and $\nu|\mathbf{h}^\varepsilon|^2$ are bounded in $L^1(\Omega\times(0,T))$. Nonetheless, this is the best uniform estimate that we can hope to obtain on the derivatives of u, $\mathbf{w}$ and $\mathbf{h}$. This means that, consistency becomes an issue in the thermal energy limit equation (3.135) as we cannot guarantee that the sequence (or any subsequence of) $\varepsilon|u^\varepsilon|^2+\mu|\mathbf{w}^\varepsilon|^2+\nu|\mathbf{h}^\varepsilon|^2$ converges to anything other than possibly a positive Radon measure. For this reason we do not expect equation (3.135) to be satisfied and the best we can aim to obtain when taking the limit as $\varepsilon\to 0$ in equation (3.7) is an inequality.

On the bright side we note that given a nonnegative smooth test function φ, the function $f \to \int_0^t\int_\Omega |f|^2\varphi dx\,ds$ defined for $f\in L^2(\Omega\times(0,T))$ and taking values in $[0,\infty)$ may be regarded as the squared norm in the weighted L^2_φ space. As we have that the sequence $(\mathbf{w}_x^\varepsilon,\mathbf{h}_x^\varepsilon)$ is weakly convergent in $L^2(\Omega\times(0,T))$ (and therefore, also in $L^2_\varphi(\Omega\times(0,T))$) we see that

$$\liminf_{\varepsilon\to0}\int_0^t\int_\Omega(\mu|\mathbf{w}_x^\varepsilon|^2+\nu|\mathbf{h}_x^\varepsilon|^2)\varphi dx\,ds \ge \int_0^t\int_\Omega(\mu|\mathbf{w}_x|^2+\nu|\mathbf{h}_x|^2)\varphi dxds.$$

Recall that, in fact, the limit magnetic field $\mathbf{h}$ has to be equal to zero. With this in mind, we intend to show that, in the limit, the following inequality

$$(\rho Q(\theta))_t + (\rho Q(\theta)\mathbf{u})_x \ge (\kappa\theta_x)_x + \mu|\mathbf{w}_x|^2, \tag{3.157}$$

is satisfied in the sense of distributions by the limit functions. In the process we are going to show that the following inequality also holds

$$\begin{aligned}\int_\Omega \rho\left(P_e(\rho)+Q(\theta)+\frac{1}{2}|u|^2+\frac{1}{2}|\mathbf{w}|^2\right)(t)dx \\ \le \int_\Omega\left(\rho_0P_e(\rho_0)+\rho_0Q(\theta_0)+\frac{1}{2}\frac{m_0^2}{\rho_0}+\rho_0\frac{1}{2}|\mathbf{w}|^2\right)dx,\end{aligned} \tag{3.158}$$

which is nothing other that to say in the notation of (3.152) that

$$E[\rho, u](t) + ||(\rho Q(\theta), \rho|\mathbf{w}|^2)(t)||_{L^1(\Omega)} \leq E[\rho_0, u_0](0) + ||(\rho_0 Q(\theta_0), \rho_0|\mathbf{w_0}|^2)||_{L^1(\Omega)},$$

which compensates, in some way, the “loss of information” resulting from considering an inequality instead of an identity in the limit thermal energy equation. This is in accordance with the definition of variational solution of the thermal energy equation considered by Feireisl in [23, Definition 4.5].

Let us point out, that even by considering the inequality (3.157) in place of (3.135), the task of showing consistency is not simple as Q and κ are nonlinear functions of θ. This means that we have to show strong convergence of the sequence θ^ε.

For this we adapt an idea in [23] which can be divided into two steps. First, using uniform estimates and some careful analysis we can show that $Q(\theta^\varepsilon)$ converges pointwise to some limit $\overline{Q}$, in the set where ρ (the limit density) is positive. As Q is a strictly increasing function, we can write $\overline{Q}$ as $\overline{Q} = Q(\overline{\theta})$, i.e., $\overline{\theta} = Q^{-1}(\overline{Q})$. Then, using (2.54) we see that

$$0 = \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (Q(\theta^\varepsilon) - Q(\overline{\theta}))(\theta^\varepsilon - \overline{\theta}) \mathbb{1}_{\{\rho>0\}} dx\, ds \geq \lim_{\varepsilon\to 0} C^{-1} \int_0^T \int_\Omega (\theta^\varepsilon - \overline{\theta})^2 \mathbb{1}_{\{\rho>0\}} dx\, ds,$$

so that θ^ε also converges pointwise to $\overline{\theta}$ in the set $\{\rho > 0\}$. After this, we adapt a clever argument from [23] to show that the function $\mathcal{K}(\theta) := \int_0^\theta \kappa(z)dz$ converges weakly to some $\overline{\mathcal{K}}$. Accordingly, $\overline{\mathcal{K}} = \mathcal{K}(\overline{\theta})$ in the set where $\rho > 0$. Thus, if we define $\theta := \mathcal{K}^{-1}(\overline{\mathcal{K}})$ we then have that $\theta = \overline{\theta}$ in the set $\{\rho > 0\}$ and we can pass to the limit in equation (3.7) in order to conclude that θ satisfies inequality (3.157).

We will fill in the details of this procedure later.

Nonlinear Schrödinger equation

Finally, we consider the limit equation (3.137). Let us recall that $\psi : \Omega_y \times (0, T) \to \mathbb{C}$ is called a weak solution of (3.137) with initial data $\psi|_{t=0} = \psi_0$ if $\psi \in L^\infty(0, T; H_0^1(\Omega_y)) \cap W^{1,\infty}(0, T; H^{-1}(\Omega_y))$, (3.137) is satisfied in $H^{-1}(\Omega_y)$ for each $t \in (0, T)$ and the initial data is attained in the sense of distributions. Existence and uniqueness of global weak solutions to (3.137) with initial data $\psi_0 \in H^1(\Omega)$ is a well known result (see [30],[7]).

Assuming, as before, that $\alpha = o(\varepsilon^{1/2})$, the energy identity (3.9) yields uniform estimates on the $L^\infty(0, T; L^4(\Omega_y) \cap H_0^1(\Omega_y))$ norm of ψ^ε. In view of our hypotheses (2.56) on the coupling, functions a direct application of Aubin-Lions lemma (see [3, 36]) allows us to pass to the limit in equation (3.15). (A version of) Aubin-Lions lemma may be stated as (see [45])

Lemma 3.16 (Aubin-Lions Lemma)**.** *Let X_0, X and X_1 be Banach spaces such that*

$$X_0 \subset X \subset X_1$$

Supose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq \alpha_0, \alpha_1 \leq \infty$, let

$$W := \{v \in L^{\alpha_0}(0,T;X_0), \frac{dv}{dt} \in L^{\alpha_1}(0,T;X_1)\},$$

under the norm

$$||v||_W = ||v||_{L^{\alpha_0}(0,T;X_0)} + \left\| \frac{dv}{dt} \right\|_{L^{\alpha_1}(0,T;X_1)}.$$

Then,

(i) If $\alpha_0 < \infty$, then the embedding of W into $L^{\alpha_0}(0,T;X)$ is compact;

(ii) If $\alpha_0 = \infty$ and $\alpha_1 > 1$, then the embedding of W into $C([0,T];X)$ is compact.

Of course, in order to apply this lemma we have to deal with one technicality that presents itself. The wave function ψ^ε is defined on the space $\Omega_y \times (0,T)$ where Ω_y is the spatial domain of the Lagrangian coordinate. As we pointed out earlier, we have that this domain actually depends on ε and we have $\Omega_y = (0, d^\varepsilon)$, where d^ε is the L^1 norm of the initial density ρ_0^ε. Now, if we assume that $\|\rho_0^\varepsilon\|_{L^1(\Omega)} \to d := \|\rho_0\|_{L^2(\Omega)}$ then there is some $\varepsilon_0 > 0$ such that $d^\varepsilon \leq d+1$ for all $\varepsilon \leq \varepsilon_0$. Extending each ψ^ε by zero to the spatial domain $(0, d+1)$ and in view of (3.6) and by the estimates on ψ^ε we can apply Aubin-Lions Lemma with $X_0 = H^1((0,d+1))$, $X = L^2((0,d+1))$ and $X_1 = H^{-1}((0,d+1))$ in order to conclude strong convergence of $\psi^\varepsilon \to \psi$ in $L^2(0,T;L^2((0,d+1))$. Furthermore, $\psi \in L^\infty(0,T;H_0^1((0,d^\varepsilon))) \cap L^\infty(0,T;H_0^1((0,d+1)))$. Finally, as $d^\varepsilon \to d$ we conclude that $\psi \in L^\infty(0,T;H_0^1((0,d)))$ and constitutes a weak solution for the limit equation (3.137) on the spatial domain $\Omega_y = (0,d)$.

As for the initial data, we only have to assume that $\psi_0^\varepsilon \to \psi_0$ in H_0^1 as $\varepsilon \to 0$ for the argument above to hold.

3.3.3 Uniform estimates

Our goal now is to deduce some uniform estimates that allow us to proceed as sketched above. They are divided into several lemmas. Lemmas 3.17 through 3.20 are inspired by their analogues contained in [14]. In order to avoid the overload of notation, in this subsection we denote by $(\rho, u, \mathbf{w}, \mathbf{h}, \theta, \psi)$ a solution of (3.1)-(3.6) with initial conditions $(\rho_0, u_0, \mathbf{w}_0, \mathbf{h}_0, \theta_0, \psi_0)$. The estimates below are uniform in the sense that the bounding constants do not depend on ε (and hence nor on β, α or δ). To this end, in what follows C will stand for a universal constant independent of ε. We also assume that $\alpha = o(\varepsilon^{1/2})$, $\beta = 0(\varepsilon)$ and $\delta = o(\varepsilon)$ as $\varepsilon \to 0$, and that μ and ν are fixed positive constants independent of ε and that κ satisfies (2.55), also independently of ε.

We begin with the following basic energy estimate.

Lemma 3.17. *Assume that*

$$C_0^{-1} \leq \int_0^1 \rho_0 dx \leq C_0, \qquad -\int_\Omega \rho_0 s(\rho_0, \theta_0) dx \leq C_0$$

where s is the entropy given by (2.46), *and that*

$$\int_\Omega \left(\rho_0 \left(e(\rho_0, \theta_0) + \frac{1}{2}{}_0 u^2 + \frac{1}{2}|\mathbf{w}_0|^2 \right) + \frac{\beta}{2}|\mathbf{h}_0|^2 \right) dx + \int_{\Omega_y} \left(\frac{1}{2}|\psi_{0y}|^2 + \frac{1}{4}|\psi_0|^4 + \alpha g(v_0) h(|\psi_0|^2) \right) dy \leq C_0,$$

where $C_0 > 0$ is independent of ε. Then, there exists $C = C(C_0) > 0$, independent of ε such that

$$\int_\Omega \left(\rho \left(e(\rho, \theta) + \frac{1}{2}u^2 + \frac{1}{2}|\mathbf{w}|^2 \right) + \frac{\beta}{2}|\mathbf{h}|^2 \right) dx + \int_{\Omega_y} \left(\frac{1}{2}|\psi_y|^2 + \frac{1}{4}|\psi|^4 + \alpha g(v) h(|\psi|^2) \right) dy + \int_\Omega \rho(\theta - 1 - \log\theta) dx + \int_0^t \int_\Omega \left(\frac{\kappa\theta_y^2}{\theta^2} + \varepsilon u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{h}_x|^2 \right) dx\, ds \leq C \tag{3.159}$$

Proof. First, from the energy identity (3.9) we have that

$$\int_\Omega \left(\rho\left(e(\rho,\theta)+\frac{1}{2}u^2+\frac{1}{2}|\mathbf{w}|^2\right)+\frac{\beta}{2}|\mathbf{h}|^2\right)dx \\ +\int_{\Omega_y}\left(\frac{1}{2}|\psi_y|^2+\frac{1}{4}|\psi|^4+\alpha g(v)h(|\psi|^2)\right)dy \le C_0. \tag{3.160}$$

Second, from equation (3.1) we have that

$$\int_\Omega \rho dx = \int_\Omega \rho_0 dx, \tag{3.161}$$

Also, note that the one dimensional version of equation (2.46) yields

$$(\rho s)_t + (\rho u s)_x - \left(\frac{\kappa\theta_x}{\theta}\right)_x = \frac{\kappa\theta_x^2}{\theta^2}+\frac{\varepsilon u_x^2}{\theta}+\frac{\mu|\mathbf{w}_x|^2}{\theta}+\frac{\nu|\mathbf{h}|_x^2}{\theta} \tag{3.162}$$

From the definition of s and using (2.42) and (2.54) we have, similarly as in (3.80), that

$$\begin{aligned}-\int_\Omega \rho s dx &\ge C^{-1}\int_\Omega(\theta-1-\log\theta)dx - C - C\int_\Omega \rho e(\rho,\theta)dx\\ &\ge C^{-1}\int_\Omega(\theta-1-\log\theta)dx - C.\end{aligned}$$

Then, integrating equation (3.162) over $\Omega\times(0,T)$ we get

$$\begin{aligned}\int_\Omega \rho(\theta-1-\log\theta)dx + \int_0^t\int_\Omega\left(\frac{\kappa\theta_y^2}{\theta^2}+\frac{\varepsilon u_x^2}{\theta}+\frac{\mu|\mathbf{w}_x|^2}{\theta}+\frac{\nu|\mathbf{h}|_x^2}{\theta}\right)dx\,ds&\\ \le C - C\int_\Omega \rho_0 s(\rho_0,\theta_0)dx&\\ \le C.&\end{aligned}$$

Next, integrating equation (3.7) and using (3.160) together with (3.131) and our assumption that $\delta = o(\varepsilon)$

$$
\begin{aligned}
\int_0^t \int_\Omega (\varepsilon u_x^2 + |\mathbf{w}_x|^2 + \nu |\mathbf{h}_x|^2) dx\, ds \\
&\le \int_\Omega \rho e dx + \int_\Omega \rho_0 e(\rho_0, \theta_0) dx + \int_0^t \int_\Omega \delta \theta p_\theta(\rho) u_x dx\, ds \\
&\le C + C \int_0^t \int_\Omega \delta \theta (1 + \rho^{\gamma/2}) |u_x| dx\, ds \\
&\le C + C \int_0^t M_\theta(s)^2 \int_\Omega (1 + \rho^\gamma) dx\, ds + \frac{\varepsilon}{2} \int_0^t \int_\Omega u_x^2 dx\, ds.
\end{aligned}
$$

Here, $M_\theta(t) = \max_{x\in\Omega} \theta(x,t)$. Now, according to (2.54) and using (3.160) we have that $\int_\Omega \rho\theta dx \le C$. Now, for any $t \in [0,T]$ there is a point $b = b(t) \in \Omega$ such that $\theta(b(t),t) = \left(\int_\Omega \rho dx\right)^{-1} \int_\Omega \rho\theta dx \le C$. Thus, similarly as in (3.86), using (2.55) we have

$$
\int_0^T M_\theta(s)^2 ds \le C + C \int_0^T \int_\Omega \frac{\kappa \theta_x^2}{\theta^2} \le C. \tag{3.163}
$$

Also notice that by (2.53)

$$
\int_\Omega \rho^\gamma dx \le C \int_\Omega \rho e dx\, ds \le C,
$$

and hence,

$$
\int_0^t \int_\Omega (\varepsilon u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{h}_x|^2) dx\, ds \le C.
$$

□

We now establish an estimate for the spatial derivative of the density.

Lemma 3.18. *Let ρ_0, u_0 and $\mathbf{h}_0$ be such that*

$$
\varepsilon^2 \int_\Omega \frac{\rho_{0x}^2}{\rho_0^3} dx + \varepsilon\beta^2 \int_\Omega \frac{|\mathbf{h}_0|^2}{\rho_0} dx \le C_0,
$$

and that

$$
C_0^{-1} \le \int_\Omega \rho_0 dx \le C_0.
$$

where C_0 is independent of ε. Then, there exists $C = C(C_0)$ such that

$$
\varepsilon^2 \int_\Omega \frac{\rho_x^2}{\rho^3} dx + \varepsilon\beta^2 \int_\Omega \frac{|\mathbf{h}|^2}{\rho} dx + \varepsilon \int_0^t \int_\Omega \rho_x^2 \rho^{\gamma-3} dx\, ds + \varepsilon\beta \int_0^t \int_\Omega \frac{|\mathbf{h}_x|^2}{\rho} dx\, ds \le C. \tag{3.164}
$$

Proof. As in [14] we deduce the following equation for $v(x,t) = 1/\rho(x,t)$:

$$(\rho v_x^2)_t + (\rho u v_x^2)_x = 2v_x u_{xx}. \tag{3.165}$$

Using equation (3.2) we have

$$\begin{aligned} 2v_x u_{xx} &= \frac{2}{\varepsilon} v_x (p_x + (\rho u)_t + (\rho u^2)_x) + \frac{\beta}{\varepsilon} v_x (|\mathbf{h}|^2)_x - 2\frac{\alpha}{\varepsilon} v_x (g'(v) h(|\psi|^2))_x \\ &= \frac{2}{\varepsilon} v_x p_x + \frac{2}{\varepsilon}((\rho u v_x)_t + [\rho u (u v_x)_x - \rho u (v u_x)_x + v_x (\rho u^2)_x]) \\ &\quad + \frac{2}{\varepsilon} v_x (|\mathbf{h}|^2)_x - 2\frac{\alpha}{\varepsilon} v_x (g'(v) h(|\psi \circ \mathbf{Y}|^2))_x. \end{aligned} \tag{3.166}$$

Denoting by J the expression in square brackets, by integration by parts, we have

$$\begin{aligned} \int_\Omega J dx &= \int_\Omega (v u_x (\rho u)_x - u v_x (\rho u)_x + v_x (u(\rho u)_x + \rho u u_x)) dx \\ &= \int_\Omega (v u_x (\rho u)_x + \rho u v_x u_x) dx = \int_\Omega u_x^2 dx. \end{aligned}$$

Next, bearing in mind our assumption (3.131) we see that

$$v_x p_x = -a\gamma \rho^{\gamma-3} \rho_x^2 - \delta \frac{\rho_x}{\rho^2} \theta_x p_\theta(\rho) - \delta \frac{\rho_x^2}{\rho^2} \theta p'_\theta(\rho).$$

In order to deal with the term $v_x(|\mathbf{h}|^2)_x$ we first rewrite (3.1) as

$$v_t + v_x u = v u_x.$$

Multiply this equation by $\beta|\mathbf{h}|^2$ to obtain

$$\beta v_t |\mathbf{h}|^2 + \beta v_x u |\mathbf{h}|^2 - \beta v u_x |\mathbf{h}|^2 = 0.$$

Now, multiply (3.5) by $2v\mathbf{h}$ and add the resulting equation to the above to obtain

$$\beta(v|\mathbf{h}|^2)_t + 2\nu v |\mathbf{h}_x|^2 + \beta(vu|\mathbf{h}|^2)_x - (2\nu v \mathbf{h} \cdot \mathbf{h}_x)_x + 2\beta v \mathbf{h} \cdot \mathbf{w}_x = -\nu v_x (|\mathbf{h}|^2)_x.$$

In this way,

$$v_x(|\mathbf{h}|^2)_x = -\frac{\beta}{\nu}(v|\mathbf{h}|^2)_t - 2v|\mathbf{h}_x|^2 + \frac{\beta}{\nu}(vu|\mathbf{h}|^2)_x + (2v\mathbf{h} \cdot \mathbf{h}_x)_x - \frac{2\beta}{\nu} v \mathbf{h} \cdot \mathbf{w}_x.$$

Gathering this information in (3.165), multiplying by ε^2 and integrating over $\Omega \times$

$(0,t)$ we get

$$\begin{aligned}
&\varepsilon^2 \int_\Omega \frac{\rho_x^2}{\rho^3} dx \\
&= \varepsilon^2 \int_\Omega \frac{\rho_{0,x}^2}{\rho_0^3} dx - 2a\gamma\varepsilon \int_0^t \int_\Omega \rho^{\gamma-3}\rho_x^2 dx\, ds - 2\varepsilon\delta \int_0^t \int_\Omega \left(\frac{\rho_x}{\rho^2}\theta_x p_\theta(\rho) + \frac{\rho_x^2}{\rho^2}\theta p_\theta'(\rho) \right) dx\, ds \\
&\quad - 2\varepsilon \int_\Omega \frac{\rho_x}{\rho} u dx + 2\varepsilon \int_\Omega \frac{\rho_{0x}}{\rho_0} u_0 dx + 2\varepsilon \int_0^t \int_\Omega u_x^2 dx \\
&\quad - \frac{\varepsilon\beta^2}{\nu} \int_\Omega \frac{1}{\rho} |\mathbf{h}|^2 dx + \frac{\varepsilon\beta^2}{\nu} \int_\Omega \frac{1}{\rho_0} |\mathbf{h}_0|^2 dx - 2\varepsilon\beta \int_0^t \int_\Omega \frac{1}{\rho} |\mathbf{h}_x|^2 dx\, ds + \frac{2\varepsilon\beta^2}{\nu} \int_0^t \int_\Omega \frac{1}{\rho} \mathbf{b}\cdot\mathbf{w}_x dx \\
&\quad + 2\alpha\varepsilon \int_0^t \int_\Omega \frac{\rho_x}{\rho^2} (g'(1/\rho) h(|\psi\circ\mathbf{Y}|^2))_x dx\, ds. \qquad (3.167)
\end{aligned}$$

Concerning the third integral on the right hand side, by virtue of (2.52), we have that

$$-2\varepsilon\delta \int_0^t \int_\Omega \left(\frac{\rho_x}{\rho^2}\theta_x p_\theta(\rho) + \frac{\rho_x^2}{\rho^2}\theta p_\theta'(\rho) \right) dx\, ds \le 2C\varepsilon\delta \int_0^t \int_\Omega \frac{|\rho_x|}{\rho^2} |\theta_x| (1+\rho^{\gamma/2}) dx \quad ds.$$

Observe that since $\int_\Omega \rho dx = \int_\Omega \rho_0 dx$, then for each $t \in (0,T)$ there is a point $b(t) \in \Omega$ such that $\rho(b(t),t) = \int_\Omega \rho_0 dx \ge C_0^{-1}$. Therefore,

$$\max_{z\in\Omega} \frac{1}{\sqrt{\rho(z,t)}} \le \sqrt{C_0} + \int_\Omega \left| \left(\frac{1}{\sqrt{\rho}} \right)_x \right| dx \le C + C \left(\int_\Omega \frac{\rho_x^2}{\rho^3} dx \right)^{1/2}.$$

Thus, taking (2.55) into consideration we see that

$$\begin{aligned}
&2\varepsilon\delta \int_0^t \int_\Omega \frac{|\rho_x|}{\rho^2} |\theta_x| (1+\rho^{\gamma/2}) dx\, ds \\
&\le C\varepsilon \int_0^t \max_{x\in\Omega} \frac{1}{\rho^{1/2}} \int_\Omega \left(\delta \frac{|\rho_x|}{\rho^{3/2}} |\theta_x| + \delta |\theta_x|\, |\rho_x| \rho^{(\gamma-3)/2} \right) dx\, ds \\
&\le C\varepsilon \int_0^t \left(C + C \left(\int_\Omega \frac{\rho_x^2}{\rho^3} dx \right)^{1/2} \right) \left(\int_\Omega \frac{\kappa\theta_x}{\theta^2} dx \right)^{1/2} \left[\left(\int_\Omega \varepsilon^2 \frac{\rho_x^2}{\rho^3} dx \right)^{1/2} + \left(\int_\Omega \varepsilon^2 \rho_x^2 \rho^{\gamma-1} \right)^{1/2} \right] ds \\
&\le \frac{a\gamma\varepsilon}{4} \int_0^t \int_\Omega \rho_x^2 \rho^{\gamma-3} dx\, ds + C \int_0^t \left(1 + \int_\Omega \frac{\kappa\theta_x^2}{\theta^2} dx \right) \left(1 + \int_\Omega \varepsilon^2 \frac{\rho_x^2}{\rho^3} dx \right) ds.
\end{aligned}$$

We already know from Lemma 3.17 that

$$\varepsilon \int_0^t \int_\Omega u_x^2 dx\, ds \le C.$$

Concerning the fourth integral on the right hand side

$$2\varepsilon\int_\Omega \frac{\rho_x}{\rho}u dx \le \frac{\varepsilon^2}{4}\int_\Omega \frac{\rho_x^2}{\rho^3}dx + C\int_\Omega \rho u^2 dx \le \frac{\varepsilon^2}{4}\int_\Omega \frac{\rho_x^2}{\rho^3}dx + C.$$

We continue with (recall that $\beta = o(\varepsilon)$)

$$\begin{aligned}
&\frac{2\varepsilon\beta^2}{\nu}\int_0^t\int_\Omega \frac{1}{\rho}\mathbf{h}\cdot\mathbf{w}_x dx\,ds \\
&\le \frac{2\varepsilon\beta^2}{\nu}\int_0^t \max_{x\in\Omega}\frac{1}{\rho^{1/2}}\left(\int_\Omega \frac{1}{\rho}|\mathbf{h}|^2dx\right)^{1/2}\left(\int_\Omega |\mathbf{w}_x|^2dx\right)^{1/2} ds \\
&\le C\frac{\varepsilon\beta^2}{\nu}\int_0^t\left(1+\left(\int_\Omega \frac{\rho_x^2}{\rho^3}dx\right)^{1/2}\right)\left(\int_\Omega \frac{1}{\rho}|\mathbf{h}|^2dx\right)^{1/2}\left(\int_\Omega |\mathbf{w}_x|^2dx\right)^{1/2} ds \\
&\le C\int_0^t\left(\varepsilon+\left(\int_\Omega \varepsilon^2\frac{\rho_x^2}{\rho^3}dx\right)^{1/2}\right)\left(\int_\Omega \frac{\varepsilon\beta^2}{\nu}\frac{1}{\rho}|\mathbf{h}|^2dx\right)^{1/2}\left(\int_\Omega \frac{\beta^2}{\varepsilon\nu}|\mathbf{w}_x|^2dx\right)^{1/2} ds \\
&\le C\int_0^t\left(1+\int_\Omega \mu|\mathbf{w}_x|^2dx\right)\left(1+\int_\Omega \varepsilon^2\frac{\rho_x^2}{\rho^3}dx+\int_\Omega \varepsilon\beta^2\frac{1}{\rho}|\mathbf{h}|^2dx\right)ds.
\end{aligned}$$

Finally, recalling (3.8) we see that $(\psi\circ\mathbf{Y})_x = \rho\psi_y$. We also know that the Jacobian of the Lagrangian coordinate change is equal to ρ. Therefore, using (2.56) and Lemma 3.17 we see that

$$\begin{aligned}
2\alpha\varepsilon\int_0^t\int_\Omega \frac{\rho_x}{\rho^2}(g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)))_x dx\,ds &\le \frac{a\gamma\varepsilon}{8}\int_0^t\int_\Omega \rho_x^2\rho^{\gamma-3}dx\,ds + C\int_0^t\int_\Omega |\psi_x|^2dx\,ds \\
&\le \frac{a\gamma\varepsilon}{8}\int_0^t\int_\Omega \rho_x^2\rho^{\gamma-3}dx\,ds + C.
\end{aligned}$$

Putting all of these estimates together with (3.167) we deduce the inequality

$$\begin{aligned}
&\varepsilon^2\int_\Omega \frac{\rho_x^2}{\rho^3}dx + \varepsilon\beta^2\int_\Omega \frac{|\mathbf{h}|^2}{\rho}dx + \varepsilon\int_0^t\int_\Omega \rho_x^2\rho^{\gamma-3}dx\,ds + \varepsilon\beta\int_0^t\int_\Omega \frac{|\mathbf{h}_x|^2}{\rho}dx\,ds \\
&\le C + C\int_0^t\left[1+\int_\Omega\left(\frac{\kappa\theta_x^2}{\theta^2}+\mu|\mathbf{w}_x|^2\right)dx\right]\left(1+\varepsilon^2\int_\Omega \frac{\rho_x^2}{\rho^3}dx+\varepsilon\beta^2\int_\Omega \frac{1}{\rho}|\mathbf{h}|^2dx\right)ds,
\end{aligned}$$

with $C > 0$ independent of ε. And since,

$$\int_0^t\left[1+\int_\Omega\left(\frac{\kappa\theta_x^2}{\theta^2}+\mu|\mathbf{w}_x|^2\right)dx\right]ds \le C,$$

Gronwall's inequality yields (3.164). □

We now deduce some higher integrability estimates for the density.

Lemma 3.19. *Let*

$$\int_\Omega \rho_0 e(\rho_0,\theta_0)dx + \int_\Omega \rho_0 u_0^2 dx \leq C_0$$

where C_0 is independent of ε. Then, there is a constant $C = C(C_0)$, independent of ε such that

$$\int_0^t \int_\Omega (\rho^{\gamma+1} + \delta\rho\theta p_\theta(\rho) + \beta\rho|\mathbf{h}|^2)dx\,ds \leq C. \tag{3.168}$$

Let us point out that according to the growth conditions (2.53), Lemma 3.17 yields only uniform boundedness of ρ in the space $L^\infty(0,T;L^\gamma(\Omega))$. Let us carry on the proof.

Proof. Let $b \in \{0,1\}$ (recall that we are assuming that $\Omega = (0,1)$ without loss of generality) and let $\sigma(x)$ be a smooth function such that

$$\sigma(b) = 0, \qquad \sigma(1-b) = 0 \quad \text{and} \quad 0 \leq \sigma \leq 1. \tag{3.169}$$

Multiplying equation (3.2) by σ and integrating from b to x (with respect to the space variable) we have

$$\begin{aligned} p\sigma + \frac{\beta}{2}|\mathbf{h}|^2 =& -\rho u^2\sigma + \varepsilon u_x\sigma + \alpha g'(1/\rho)h(|\psi|^2)\sigma - \left(\int_b^x \rho u\sigma d\xi\right)_t \\ &+ \int_b^x \left[\left(\rho u^2 + p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho)h(|\psi|^2)\right)\sigma_x - \varepsilon u_x\sigma_x\right]d\xi. \end{aligned}$$

Multiply this identity by $\rho\sigma$ and use (3.1) to obtain

$$\begin{aligned} &\rho p\sigma^2 + \frac{\beta}{2}\rho|\mathbf{h}|^2\sigma^2 \\ &= -\rho^2u^2\sigma^2 + \varepsilon\rho u_x\sigma^2 + \alpha g'(1/\rho)h(|\psi|^2)\rho\sigma^2 - \left(\rho\sigma\int_b^x \rho u\sigma d\xi\right)_t - (\rho u)_x\sigma\int_b^x \rho u\sigma d\xi \\ &\quad + \rho\sigma\int_b^x \left[\left(\rho u^2 + p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho)h(|\psi|^2)\right)\sigma_x - \varepsilon u_x\sigma_x\right]d\xi \\ &= \varepsilon\rho u_x\sigma^2 + \alpha g'(1/\rho)h(|\psi|^2)\rho\sigma^2 - \left(\rho\sigma\int_b^x \rho u\sigma d\xi\right)_t - \left(\rho u\sigma\int_b^x \rho u\sigma d\xi\right)_x \\ &\quad + \rho u\sigma_x\int_0^x \rho u\sigma d\xi + \rho\sigma\int_b^x \left[\left(\rho u^2 + p + \frac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho)h(|\psi|^2)\right)\sigma_x - \varepsilon u_x\sigma_x\right]d\xi. \end{aligned}$$

Integrating over $\Omega \times (0,t)$ we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \rho p \sigma^2 dx + \frac{\beta}{2} \int_0^t \int_\Omega \rho |\mathbf{h}|^2 dx\, ds \\
&= \alpha \int_0^t \int_\Omega g'(1/\rho) h(|\psi|^2) \rho \sigma^2 dx\, ds + \varepsilon \int_0^t \int_\Omega \rho u_x \sigma^2 dx\, ds \\
&\quad + \int_\Omega \rho\sigma \left(\int_b^x \rho u \sigma dxi \right) dx + \int_\Omega \rho_0 \sigma \left(\int_b^x \rho_0 u_0 \sigma dxi \right) dx + r_1(t),
\end{aligned}
$$

where

$$
\begin{aligned}
r_1(t) = &\int_0^t \int_\Omega \rho u \sigma_x \left(\int_b^x \rho u \sigma d\xi \right) dx\, ds \\
&+ \int_0^t \int_\Omega \rho\sigma \int_b^x \left[\left(\rho u^2 + p + \frac{\beta}{2} |\mathbf{h}|^2 - \alpha g'(1/\rho) h(|\psi|^2) \right) \sigma_x - \varepsilon u_x \sigma_x \right] d\xi dx\, ds
\end{aligned}
$$

Note that,

$$
\begin{aligned}
\varepsilon \int_0^t \int_\Omega \rho u_x \sigma^2 dx\, ds &\le \frac{\varepsilon}{\delta_1} \int_0^t \int_\Omega u_x^2 dx\, ds + \delta_1 \int_0^t \int_\Omega \rho^2 \sigma^2 dx\, ds \\
&\le C_{\delta_1} + C\delta_1 \int_0^t \int_\Omega \rho^2 \sigma^2 dx\, ds.
\end{aligned}
$$

Now,

$$
\left| \int_b^x \rho u \sigma d\xi \right| \le \left(\int_\Omega \rho dx \right)^{1/2} \left(\int_\Omega \rho u^2 dx \right)^{1/2} \le C.
$$

And then,

$$
\left| \int_\Omega \rho\sigma \left(\int_b^x \rho u \sigma dxi \right) dx \right| \le C.
$$

Similarly,

$$
\begin{aligned}
\left| \int_0^t \int_\Omega \rho u \sigma_x \left(\int_b^x \rho u \sigma d\xi \right) dx\, ds \right| + \left| \int_0^t \int_\Omega \rho \sigma \left(\int_b^x \rho u^2 \sigma_x d\xi \right) dx\, ds \right| \\
+ \left| \int_0^t \int_\Omega \rho \sigma \left(\int_b^x \varepsilon u_x \sigma_x d\xi \right) dx\, ds \right| \le C
\end{aligned}
$$

Recall that we are assuming (3.131). Also, from Lemma 3.17 we have

$$
\int_\Omega \rho^\gamma dx \le C.
$$

Hence, taking into account (3.163), we see that

$$\begin{aligned}\left|\int_0^t \int_\Omega \rho\sigma \left(\int_b^x p\sigma_x d\xi\right) dx\, ds\right| &\le C \int_0^t (1+M_\theta(s)^2) \int_\Omega \rho\sigma \left(\int_\Omega (\rho^\gamma+1) d\xi\right) dx\, ds \\ &\le C \int_0^t (1+M_\theta(s)^2) \int_\Omega \rho dx\, ds \\ &\le C \int_0^t (1+M_\theta(s)^2) ds \le C.\end{aligned}$$

We continue with

$$\begin{aligned}\left|\int_0^t \int_\Omega \rho\sigma \left(\int_b^x \frac{\beta}{2}|\mathbf{h}|^2 \sigma_x d\xi\right) dx\, ds\right| &\le C \int_0^t \int_\Omega \rho \left(\int_0^1 \frac{\beta}{2}|\mathbf{h}|^2 d\xi\right) dx\, ds \\ &\le C \int_0^t \int_\Omega \rho dx\, ds \le C.\end{aligned}$$

Lastly,

$$\left|\alpha \int_0^t \int_\Omega \rho\sigma^2 g'(1/\rho) h(|\psi|^2) dx\, ds\right| \le C,$$

and

$$\left|\alpha \int_0^t \int_\Omega \rho\sigma \left(\int_b^x g'(1/\rho) h(|\psi|^2)\sigma_x dxi\right) dx\, ds\right| \le C$$

Choosing $\delta_1 > 0$ small enough, we get

$$\int_0^t \int_\Omega \left(\rho p + \frac{\beta}{2}\rho|\mathbf{h}|^2\right) \sigma^2 dx\, ds \le C,$$

and using the hypotheses (2.50) on p

$$\int_0^t \int_\Omega \left(\rho^{\gamma+1} + \delta\theta\rho p_\theta(\rho) + \frac{\beta}{2}\rho|\mathbf{h}|^2\right) \sigma^2 dx\, ds \le C,$$

and this holds for any σ that satisfies (3.169) (of course, the constant C in this last inequality depends on σ). Choosing $\sigma_1(x) = x$ and $\sigma_2(x) = 1 - x$ (again, $\Omega = (0, 1)$) we get

$$\int_0^t \int_\Omega \left(\rho^{\gamma+1} + \delta\theta\rho p_\theta(\rho) + \frac{\beta}{2}\rho|\mathbf{h}|^2\right) (x^2 + (1-x)^2) dx\, ds \le C.$$

Since $\min_{x\in(0,1)}(x^2 + (1-x)^2) = 1/2$ we conclude that

$$\int_0^t \int_\Omega \left(\rho^{\gamma+1} + \delta\theta\rho p_\theta(\rho) + \frac{\beta}{2}\rho|\mathbf{h}|^2\right) dx\, ds \le C.$$

□

We now improve the integrability of the velocity.

Lemma 3.20. *Let* $(\rho_0, u_0, \mathbf{h}_0, \theta_0)$ *satisfy*

$$\int_\Omega \rho_0 dx \geq C^{-1}, \qquad \varepsilon^2 \int_\Omega \frac{\rho_{0,x}^2}{\rho_0^3} dx + \varepsilon\beta^2 \int_\Omega \frac{1}{\rho_0} |\mathbf{h}_0|^2 dx \leq C_0,$$

and

$$\int_\Omega \rho_0 (e(\rho_0, \theta_0) + u_0^2) dx \leq C_0$$

where, C_0 *is a constant independent of* ε*. Then*

$$\int_0^t \int_\Omega (\rho |u|^3 + \rho^{\gamma+\vartheta}) dx\, ds \leq C, \tag{3.170}$$

where, $\vartheta = \frac{\gamma-1}{2}$ *and* $C >$ *is a constant independent of* ε*.*

Proof. Let $\zeta_\#(z) = \frac{1}{2} z|z|$. Then, the corresponding weak entropy pair $(\eta^\#, q^\#) := (\eta^{\zeta_\#}, q^{\zeta_\#})$ satisfies

$$|\eta^\#(\rho, m)| \leq C(\rho|u|^2 + \rho^\gamma), \quad C^{-1}(\rho|u|^3 + \rho^{\gamma+\vartheta}) \leq q^\#(\rho, m) \leq C(\rho|u|^3 + \rho^{\gamma+\vartheta}), \tag{3.171}$$

$$|\eta_m^\#(\rho, m)| \leq C(|u| + \rho^\vartheta), \qquad |\eta_{mm}^\#(\rho, m) \leq C\rho^{-1}. \tag{3.172}$$

and, regarding $\eta_m^\#$ in the coordinates (ρ, u)

$$|\eta_{mu}^\#(\rho, \rho u)| \leq C, \qquad |\eta_{m\rho}^\#(\rho, u)| \leq C\rho^{\vartheta-1}, \tag{3.173}$$

for all $\rho \geq 0$ and all $u \in \mathbb{R}$. This is a consequence of the representation formulas (3.156).

Multiply (3.1) by $\eta_\rho^\#$ and (3.2) by $\eta_u^\#$ and add the resulting equations to obtain

$$\eta^\#(\rho, m)_t + q^\#(\rho, m)_x = \left(-\frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(1/\rho) h(|\psi|^2) + \varepsilon u_x\right)_x \eta_m^\#(\rho, m). \tag{3.174}$$

Define the function

$$f(x,t) := \Big[q^\#(\rho, m) + \Big(\tfrac{\beta}{2}|\mathbf{h}|^2 - \alpha g'(1/\rho) h(|\psi \circ Y|^2) - \varepsilon u_x\Big)\eta_m^\#(\rho, m)\Big](x,t). \tag{3.175}$$

We claim that there is a function $a(t)$ taking values in Ω such that

$$\int_0^t |f(a(s),s)|ds \le C, \tag{3.176}$$

for some $C > 0$ independent of ε.

Assuming this for now, we integrate (3.174) over (a,x) first and then over $(0,t)$ and get

$$\begin{aligned}
&\int_a^x (\eta^{\#}(\rho,m) - \eta^{\#}(\rho_0,m_0))d\xi + \int_0^t q^{\#}(\rho,m)ds \\
&= \int_0^t f(a(s),s)ds + \int_0^t \left(-\frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(1/\rho)h(|\psi\circ Y|^2) + \varepsilon u_x\right)\eta^{\#}_m ds \\
&\quad - \int_0^t \int_a^x \left(-\frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(1/\rho)h(|\psi|^2) + \varepsilon u_x\right)(\eta^{\#}_{m\rho}\rho_x + \eta^{\#}_{mu}u_x)dx\,ds. \qquad (3.177)
\end{aligned}$$

First, from (3.171) we see that

$$\int_0^t \int_\Omega q^{\#}(\rho,m)dx\,ds \ge C^{-1}\int_0^t \int_\Omega (\rho|u|^3 + \rho^{\gamma+\vartheta}).$$

Second, from Lemma 3.17 and (3.171)

$$\int_0^t \int_\Omega (|\eta^{\#}(\rho,m)| + |\eta^{\#}(\rho_0,m_0)|)dx \le C.$$

Next, using the fact that $\mathbf{h}|_{x=0} = 0$ we see that

$$\beta^{1/2}|\mathbf{h}|^2 \le 2\left(\beta\int_\Omega |\mathbf{h}|^2\right)^{1/2}\left(\int_\Omega |\mathbf{h}_x|^2\right)^{1/2} \le C\left(\int_\Omega |\mathbf{h}_x|^2\right)^{1/2}. \tag{3.178}$$

Similarly,

$$\varepsilon^{1/2}|u| \le \int_\Omega \varepsilon^{1/2}|u_x|dx \le C\left(\varepsilon\int_\Omega u_x^2 dx\right)^{1/2}. \tag{3.179}$$

Using these two observations along with (3.172) and Lemma 3.17

$$\begin{aligned}
&\int_0^t \int_\Omega \left(-\frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(1/\rho)h(|\psi|^2) + \varepsilon u_x\right)\eta^{\#}_m dx\,ds \\
&\le C + C\int_0^t \int_\Omega (\nu|\mathbf{h}_x|^2 + \varepsilon u_x^2)dx\,ds + C(\beta + \alpha^2 + \varepsilon)\int_0^t \int_\Omega (u_x^2 + \rho^{\gamma-1})dx\,ds \\
&\le C.
\end{aligned}$$

Finally, by the same reasoning and using (3.173) and Lemma 3.19 we see that

$$\begin{aligned}
&\int_0^t \int_\Omega \left(-\frac{\beta}{2}|\mathbf{h}|^2 + \alpha g'(1/\rho)h(|\psi|^2) + \varepsilon u_x\right)(\eta^\#_{m\rho}\rho_x + \eta^\#_{mu}u_x)dx\,ds \\
&\quad \le C + C\int_0^t \int_\Omega \nu|\mathbf{h}_x|^2 dx\,ds + C(\beta+\alpha^2+\varepsilon)\int_0^t\int_\Omega u_x^2 dx\,ds \\
&\qquad + C(\beta+\alpha^2+\varepsilon)\int_0^t\int_\Omega \rho_x^2\rho^{\gamma-3}dx\,ds \\
&\quad \le C.
\end{aligned}$$

Taking this information into account, integrating (3.177) over Ω and using (3.176) we obtain (3.170).

In order to complete the proof we have to prove our claim. For this, fix $k \in \mathbb{N}$ large enough so that $\gamma \ge \max\{1+\frac{2}{2k-3}, 1+\frac{1}{2(k-1)}\}$ and observe that

$$\rho(x,t)\min_{z\in\Omega}|f(z,t)|^{1/k} \le \rho(x,t)|f(x,t)|^{1/k} \le \rho(x,t)\max_{z\in\Omega}|f(z,t)|^{1/k}.$$

Integrating over Ω and using (3.161) we see that for a.e. t there is a point $a = a(t)\in\Omega$ such that

$$|f(a(t),t)| = \left(\int_\Omega \rho_0 dx\right)^{-k}\left(\int_\Omega \rho(x,t)|f(x,t)|^{1/k}dx\right)^k.$$

Let us show that

$$\int_0^t \left(\int_\Omega \rho(x,s)|f(x,s)|^{1/k}dx\right)^k ds \le C. \tag{3.180}$$

On the one hand, since k was chosen so that $\gamma \ge 1+\frac{2}{2k-3}$ (which implies that $\frac{1}{2k} \le \frac{\gamma-1}{3\gamma-1}$), we can use (3.171) and (3.160) in order to show that

$$\begin{aligned}
&\int_\Omega \rho|q^\#(\rho,m)|^{1/k} \\
&\le C\int_\Omega \left(\rho^{(2k-1)/2k}(\rho u^2)^{3/2k} + \rho^{1+(3\gamma-1)/2k}\right)dx \\
&\le C\left(\int_\Omega \rho^{(2k-1)/(2k-3)}dx\right)^{(2k-3)/2k}\left(\int_\Omega \rho u^2 dx\right)^{3/2k} + C\int_\Omega \rho^{1+(3\gamma-1)/2k}dx \\
&\le C\left(1+\int_\Omega \rho^\gamma dx\right)^{(2k-3)/2k}\left(\int_\Omega \rho u^2 dx\right)^{3/2k} + C\left(1+\int_\Omega \rho^\gamma dx\right) \le C
\end{aligned}$$

On the other hand, since $\gamma \geq 1 + \frac{1}{2(k-1)}$ we have that

$$\begin{aligned}
&\int_\Omega \rho |\varepsilon u_x \eta_m^{\#}|^{1/k} dx \\
&\leq \varepsilon^{1/2k} \int_\Omega \rho^{1-1/2k} (\varepsilon u_x^2)^{1/2k} ((\rho u^2)^{1/2k} + \rho^{\gamma/2k}) dx \\
&\leq C \left(\int_\Omega \rho^{1+1/2(k-1)} dx \right)^{(k-1)/k} \left(\int_\Omega \varepsilon u_x^2 dx \right)^{1/2k} \left(\int_\Omega (\rho u^2 + \rho^\gamma) dx \right)^{1/2k} \\
&\leq C \left(1 + \int_\Omega \rho^\gamma dx \right)^{(k-1)/k} \left(1 + \int_\Omega \varepsilon u_x^2 dx \right)^{1/k} \\
&\leq C \left(1 + \int_\Omega \varepsilon u_x^2 dx \right)^{1/k}.
\end{aligned}$$

With this, we conclude that

$$\int_0^t \left(\int_\Omega \rho |\varepsilon u_x \eta_m^{\#}|^{1/k} dx \right)^k ds \leq C.$$

Finally, by the same reasoning, we see that

$$\begin{aligned}
&\int_\Omega \rho \left| \left(\frac{\beta}{2} |\mathbf{h}|^2 - \alpha g'(1/\rho) h(|\psi \circ Y|^2) \right) \eta_m^{\#} \right|^{1/k} dx \\
&\qquad \leq C \left(1 + \nu \int_\Omega |\mathbf{h}_x|^2 dx \right)^{1/k} \left(\int_\Omega \rho dx \right)^{(2k-1)/2k} \left(\int_\Omega (\rho u^2 + \rho^\gamma) dx \right)^{1/2k} \\
&\qquad \leq C \left(1 + \nu \int_\Omega |\mathbf{h}_x|^2 dx \right)^{1/k},
\end{aligned}$$

which implies that

$$\int_0^t \left(\int_\Omega \rho \left| \left(\frac{\beta}{2} |\mathbf{h}|^2 - \alpha g'(1/\rho) h(|\psi \circ Y|^2) \right) \eta_m^{\#} \right|^{1/k} dx \right)^k ds \leq C,$$

thus proving (3.180). □

These last four Lemmas provide the necessary uniform estimates that allow us apply Chen and Perepelitsa's compensated compactness scheme in order to deal with the convergence in the continuity equation (3.1) and in the momentum equation (3.2). They also suffice to handle the convergence issues in equations (3.3), (3.5) and (3.6) to the extent that was explained in Subsection 3.3.2. Yet, in order to address the convergence issues in the thermal energy equation (3.7) we need one more estimate that reads as follows.

Lemma 3.21. *Let*

$$C^{-1} \le \int_\Omega \rho_0 dx \le C_0, \qquad \int_\Omega \rho_0 e(\rho_0, \theta_0) dx \le C_0$$

for some $C_0 > 0$ *independent of* ε. *Then,*

$$\int_0^t \int_\Omega (\theta^{q+1} + |(\theta^{q/2})_x|^2) dx\, ds \le C, \tag{3.181}$$

where, $C > 0$ *is constant independent of* ε.

Proof. Let us define $\mathcal{K}$ as in Subsection 3.3.2 as

$$\mathcal{K}(\theta) := \int_0^\theta \kappa(z) dz.$$

Then, from (2.55) we have that

$$C^{-1}(1 + \theta^{q+1}) \le \mathcal{K}(\theta) \le C(1 + \theta^{q+1}).$$

Also, note that equation (3.7) can be rewritten as

$$\mathcal{K}_{xx} = (\rho Q(\theta))_t + (\rho u Q(\theta))_x + \delta\theta p_\theta(\rho) u_x - \varepsilon u_x^2 - \mu |w_x|^2 - \nu |\mathbf{h}_x|^2. \tag{3.182}$$

Realizing that $(\mathcal{K}_x)|_{x=0} = 0$, we integrate this equation over $(0, x) \times (0, t)$ to obtain

$$\begin{aligned} \int_0^t \mathcal{K}_x ds = \int_0^x (\rho Q(\theta) - \rho_0 Q(\theta_0)) d\xi + \int_0^t \rho Q(\theta) u ds + \delta \int_0^t \int_0^x \theta p_\theta u_x d\xi\, ds \\ - \int_0^t \int_0^x (\varepsilon u_x^2 + \mu |w_x|^2 + \nu |\mathbf{h}_x|^2) d\xi\, ds. \end{aligned}$$

Let us choose some $b = b(t) \in \Omega$ such that $\theta(b(t), t) = \left(\int_\Omega \rho_0 dx\right)^{-1} \left(\int_\Omega \rho\theta dx\right) \le C$. Then, integrating the above equality from $b(t)$ to x (with respect to the space variable) and using Lemma 3.17 we obtain

$$\int_0^t \mathcal{K} ds \le C.$$

In particular,

$$\int_0^t \int_\Omega \theta^{q+1} dx\, ds \le C.$$

In order to conclude, we observe that

$$\int_0^t \int_\Omega |\nabla(\theta^{q/2})_x|^2 dx\, ds \le C + C\int_0^t \int_\Omega \frac{\kappa\theta_x^2}{\theta^2} dx\, ds \le C.$$

□

With these estimates at hand we are ready to pass to the limit as $\varepsilon \to 0$

3.3.4 Limit in the continuity and momentum equations

Let $(\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon)$ be the unique global solution of (3.1)-(3.6). Let us consider the sequence $(\rho^\varepsilon, u^\varepsilon)$. The first step in the sketch outlined in Subsection3.3.1 is to apply the Young measures theorem to the sequence $(\rho^\varepsilon, u^\varepsilon)$. However, we do not have any uniform estimates that guarantee that this sequence takes values on a fixed compact of $\mathbb{R}^2$. We can only ensure that they take values in $\mathcal{H} := \{(\rho, u) \in \mathbb{R}^2 : \rho > 0\}$.

Fortunately, there is a stronger version of the Young measures theorem which allows us to assume that the set in which the sequence takes values is a compact metric space; and the conclusion of the theorem is the same ([1, Theorem 2.4], also [4]). With this in mind, and following [14] (cf. [35]) we can consider a compactification $\mathbb{H}$ of $\mathcal{H}$ such that the space $C(\mathbb{H})$ is isometrically isomorphic to the space of continuous functions $\phi \in C(\overline{\mathcal{H}})$ satisfying that $\phi(\rho, u)$ is constant on the vacuum $\{\rho = 0\}$ and that the map $(\rho, u) \to \lim_{s\to\infty} \phi(s\rho, su)$ belong to $C(\mathbb{S}) \cap \mathcal{H}$, where $\mathbb{S} \subseteq \mathbb{R}^2$ is the unit circle. Of course, $\mathcal{H}$ is naturally embedded in $\mathbb{H}$ (note that the vacuum line $V = \{\rho = 0\}$ is identified to a single point in $\mathbb{H}$).

By the Young measures theorem there exists a subsequence, still denoted $(\rho^\varepsilon, u^\varepsilon)$ and a weakly measurable mapping from $\Omega \times [0, \infty)$ to $\mathrm{Prob}(\mathbb{H})$, the space of probability measures in $\mathbb{H}$, $(x, t) \to \nu_{x,t}$ such that for all $\phi \in C(\mathbb{H})$

$$\phi(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \int_{\mathbb{H}} \phi(\rho, u) d\nu_{x,t}, \qquad \text{weakly} - * \text{ in } L^\infty(\Omega \times [0, \infty)).$$

As aforementioned, in order to show that the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ converges to a finite-energy weak solution $(\rho, \rho u)$ of the isentropic Euler equations (3.145), (3.146) it suffices to show that

$$\nu_{x,t} = \delta_{(\rho(x,t), \rho(x,t)u(x,t))}. \tag{3.183}$$

Now, we have a rich family of entropy pairs given by (3.156) and from earlier exposition we know that they may provide enough information that allow us to conclude

(3.183).

Just as in [14] we can show the following.

Proposition 3.1. *The following statements hold:*

(i)

$$\int_{\mathcal{H}} (\rho^{\gamma+1} + \rho|u|^3) d\nu_{x,t} \in L^1_{loc}(\Omega \times [0, \infty)). \tag{3.184}$$

(ii) Let $\phi(\rho, u)$ be a function such that

(a) ϕ is continuous on $\overline{\mathcal{H}}$ and zero on $\partial\mathcal{H}$ (in the vacuum);

(b) $supp\phi \subseteq \{(\rho, u) : \rho^\vartheta + u \geq -a, u - \rho^\vartheta \leq a\}$ for some constant $a > 0$;

(c) $|\phi(\rho, u)| \leq \rho^{\beta(\gamma+1)}$ for all (ρ, u) with large ρ and some $\beta \in (0, 1)$.

Then, ϕ is $\nu_{x,t}$-integrable and

$$\phi(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \int_{\mathcal{H}} \phi d\nu_{x,t} \qquad \text{in } L^1_{loc}(\Omega \times [0, \infty)). \tag{3.185}$$

(iii) For $\nu_{x,t}$ viewed as an element of $(C(\mathbb{H}))^*$

$$\nu_{x,t}[\mathbb{H} \setminus (\mathcal{H} \cup V)] = 0,$$

meaning that $\nu_{x,t}$ is concentrated at $\mathcal{H}$ and or the vacuum $V = \{\rho = 0\}$.

In view of this proposition, and by Lemma 3.15 the entropy pairs are $\nu_{x,t}$-integrable and we can use them as test functions in the hope to deduce the relations of the form (3.144). The proof follows the same ideas as the analogue in [14] and goes as follows.

Proof. For each k, let $\omega_k(\rho, u)$ be a nonnegative and continuous cutoff function such that $\omega_k = 1$ on $\{(\rho, u) : k^{-1}\rho^\vartheta \leq k, -k \leq u \leq k\}$ and with $\mathrm{supp}\omega_k \subseteq \{(\rho, u) : (2k)^{-1} \leq \rho^\vartheta \leq 2k, -2k \leq u \leq 2k\}$. Then, as

$$\int_0^T \int_\Omega ((\rho^\varepsilon)^{\gamma+1} + \rho^\varepsilon |u^\varepsilon|^3) \omega_k(\rho^\varepsilon, u^\varepsilon) dxds \to \int_0^T \int_\Omega \left(\int_{\mathcal{H}} (\rho^{\gamma+1} + \rho|u|^3) \omega_k(\rho, u) d\nu_{x,t} \right) dxds,$$

by virtue of Lemmas 3.19 and 3.20 and the monotone convergent theorem (3.184) follows.

In order to prove (ii), let $K \subseteq \Omega$ be any compact subset and take ϕ satisfying (a), (b) and (c) and consider the same cutoff functions as above. Then $\omega_k\phi \in C(\overline{H})$ and by the Lebesgue dominated convergence theorem and (3.184)

$$\lim_{k\to\infty}\int_0^T\int_K\left(\int_{\mathcal{H}}\phi\omega_k d\nu_{x,t}\right)dx\,ds=\int_0^T\int_K\left(\int_{\mathcal{H}}\phi d\nu_{x,t}\right)dx\,ds,$$

which is to say the same as

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0}\int_0^T\int_K\phi(\rho^\varepsilon,u^\varepsilon)\omega_k(\rho^\varepsilon,u^\varepsilon)dx\,ds=\int_0^T\int_K\left(\int_{\mathcal{H}}\phi d\nu_{x,t}\right)dx\,ds$$

Hence, proving (3.185) becomes justifying the interchange in the order of the limits in this last equation. For this, it is enough to show that

$$\int_0^T\int_K\phi(\rho^\varepsilon,u^\varepsilon)(\omega_{k_1}(\rho^\varepsilon,u^\varepsilon)-\omega_{k_2}(\rho^\varepsilon,u^\varepsilon))dx\,ds\to 0$$

as $k_1,k_2\to\infty$ uniformly in ε, but this follows from (a)-(c) and Lemma (3.19).

Finally, to prove (iii) we consider, for each $d,k>0$, a test function $\phi_{k,d}\in C(\mathbb{H})$ such that

- $0\le\phi_{k,d}\le 1$,
- $\phi_{k,d}=1$ on the set $\{\rho^2+u^2\ge k+1,\rho\ge 2d|u|\}$, and
- $\phi_{k,d}=0$ on the set $\{\rho^2+u^2\le k\}\cup\{\rho\le d|u|\}$.

By the monotone convergence theorem we have that

$$\lim_{d\to 0}\lim_{k\to\infty}\int_0^T\int_\Omega\left(\int_{\mathbb{H}}\phi_{k,d}(\rho,u)d\nu_{x,t}\right)dxdt=\int_0^T\int_\Omega\nu_{x,t}[\mathbb{H}\setminus(\mathcal{H}\cup V)]dxdt.$$

On the other hand, by Lemma 3.19, we see that

$$\begin{aligned}\int_0^T\int_\Omega\left(\int_{\mathbb{H}}\phi_{k,d}(\rho,u)d\nu_{x,t}\right)dxdt&=\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\phi_{k,d}(\rho^\varepsilon,u^\varepsilon)dxdt\\&\le\frac{(1+d^{-2})}{k}\int_0^T\int_\Omega\rho^{\gamma+1}dxdt\\&\le C\frac{(1+d^{-2})}{k}\end{aligned}$$

Taking the limit as $k\to\infty$ first and then as $d\to 0$ we conclude that

$$\int_0^T\int_\Omega\nu_{x,t}[\mathbb{H}\setminus(\mathcal{H}\cup V)]dxdt=0,$$

which implies that $\nu_{x,t}[\mathbb{H}\setminus(\mathcal{H}\cup V)]=0$ for a.e. (x,t). □

As we recall the next step to take is to apply the Div-Curl Lemma in order to find a set of relations of the form (3.144). As a matter of fact, we have the following.

Proposition 3.2. *Let ζ be any compactly supported C^2 function and let (η^ζ, q^ζ) be the corresponding entropy pair given by* (3.156). *Then the entropy dissipation measures*

$$\eta^\zeta(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_t + q^\zeta(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x$$

belong to a compact of $H^{-1}_{loc}(\Omega \times [0,\infty))$.

Proof. Multiplying (3.1) by η^ζ_ρ and (3.2) by η^ζ_m and adding the resulting equations we obtain

$$\begin{aligned}
&\eta^\zeta(\rho^\varepsilon, m^\varepsilon)_t + q^\zeta(\rho^\varepsilon, m^\varepsilon)_x \\
&= \varepsilon(\eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x)_x - \varepsilon\eta_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\zeta_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x \\
&\quad - (\delta\theta^\varepsilon p_\theta(\rho^\varepsilon)\eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon))_x + \delta\theta^\varepsilon p_\theta(\rho^\varepsilon)(\eta^\zeta_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x + \eta^\zeta_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x) \\
&\quad - \left(\frac{\beta}{2}|\mathbf{h}^\varepsilon|^2 - \alpha g'(1/\rho^\varepsilon)h(|\psi^\varepsilon|^2)\right)_x \eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon).
\end{aligned} \tag{3.186}$$

Using using Hölder inequality and Lemmas 3.15, 3.17 and 3.18 we see that

$$\begin{aligned}
&||\varepsilon\eta_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\zeta_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x||_{L^1((\Omega\times(0,T))} \\
&\qquad\qquad \le C_\zeta||(\varepsilon^{1/2}u^\varepsilon_x, \varepsilon^{1/2}(\rho^\varepsilon)^{\frac{\gamma-3}{2}}\rho^\varepsilon_x)||_{L^2(\Omega\times(0,T))} \le C,
\end{aligned}$$

Similarly, using (3.163) and (2.52)

$$||\delta\theta^\varepsilon p_\theta(\rho^\varepsilon)(\eta^\zeta_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x + \eta^\zeta_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x)||_{L^1((\Omega\times(0,T))} \le C,$$

and using (3.178), (2.56) and Lemmas 3.17 and 3.18

$$\left\|\left(\frac{\beta}{2}|\mathbf{h}^\varepsilon|^2 - \alpha g'(1/\rho^\varepsilon)h(|\psi^\varepsilon|^2)\right)_x \eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\right\|_{L^1((\Omega\times(0,T))} \le C$$

Also, note that

$$\|\varepsilon\eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x\|_{L^2((\Omega\times(0,T))} \le C,$$

and

$$\|\delta\theta^\varepsilon p_\theta(\rho^\varepsilon)\eta^\zeta_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\|_{L^2((\Omega\times(0,T))} \le C.$$

With this estimates we can use Lemma 3.14 and equation (3.3.4) in order to conclude that

$$\eta^\zeta(\rho^\varepsilon, m^\varepsilon)_t + q^\zeta(\rho^\varepsilon, m^\varepsilon)_x \text{ are confined in a compact subset of } W_{loc}^{-1,q_1}(\Omega \times [0,\infty)).$$

for some $1 < q_1 < 2$.

On the other hand, using the bounds in Lemma 3.15 and the estimates in Lemmas 3.19 and 3.20 we have that

$$\eta^\zeta(\rho^\varepsilon, m^\varepsilon), q^\zeta(\rho^\varepsilon, m^\varepsilon) \text{ are uniformly bounded in } L^2_{loc}(\Omega \times [0,\infty)),$$

which implies that

$$\eta^\zeta(\rho^\varepsilon, m^\varepsilon)_t + q^\zeta(\rho^\varepsilon, m^\varepsilon)_x \text{ are uniformly bounded in } W_{loc}^{-1,q_2}(\Omega \times [0,\infty))$$

where $q_2 = \gamma + 1 > 2$, when $\gamma \in (1,3]$ and $q_2 = \frac{\gamma+\vartheta}{1+\vartheta} > 2$ when $\gamma > 3$.

Finally, using Lemma 3.13 we conclude that

$$\eta^\zeta(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_t + q^\zeta(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x$$

belong to a compact of $H^{-1}_{loc}(\Omega \times [0,\infty))$. □

Let us introduce the following notation. First, in order to avoid the overload of notation we omit the first two arguments (ρ and u) in the entropy kernel (recall (3.155)) and denote

$$\chi(s) = [\rho^{2\vartheta} - (u-\xi)^2]_+^\Lambda.$$

Second, given any function $f(\rho, u)$ with growth slower than $\rho|u|^3 + \rho^{\gamma+\max\{1,\vartheta\}}$, we denote

$$f(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \overline{f(\rho,u)}(x,t) := \langle \nu_{x,t}, f(\rho,u)\rangle.$$

In other words, the overline stands for integration with respect to the young measure.

In the next proposition, we show that our parametrized Young measure $\nu_{x,t}$ satisfies the same commutator relation than the one in [14] (cf. [11, 19, 20, 39, 40]). It is based on this relation that one can conclude that the support of each $\nu_{x,t}$ is reduced to a point or else contained in the vacuum and, as pointed out earlier, this is enough to conclude that $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ converges to a finite-energy entropy solution to (3.145), (3.146), (3.148). As the argument that reduces the support of $\nu_{x,t}$ relies only on the commutator relation and not on the approximating sequence we only have to prove that the commutator relation holds. Remember that we are assuming that $\alpha = o(\varepsilon^{1/2})$, $\beta = o(\varepsilon)$ and $\delta = o(\varepsilon)$.

Proposition 3.3. *For each test function $\zeta(s) \in \{\pm 1, \pm s, s^2\}$ we have that*

$$\langle \nu_{t,x}, \eta^\zeta \rangle_t + \langle \nu_{x,t}, q^\zeta \rangle_x \le 0, \quad \langle \nu_{x,t}, \eta^\zeta \rangle (0, \cdot) = \eta(\rho_0, \rho_0 u_0), \tag{3.187}$$

in the sense of distributions. Moreover, $\nu_{x,t}$ satisfies the following commutator relation

$$\vartheta(\xi_2 - \xi_1)(\overline{\chi(\xi_1)\chi(\xi_2)} - \overline{\chi(\xi_1)}\ \overline{\chi(\xi_2)}) = (1-\vartheta)(\overline{u\chi(\xi_2)}\ \overline{\chi(\xi_1)} - \overline{u\chi(\xi_1)}\ \overline{\chi(\xi_2)}), \tag{3.188}$$

where, as before, $\vartheta = (\gamma - 1)/2$.

Proof. First we prove (3.187). Multiplyigng (3.1) by η^ζ_ρ and (3.2) by η^ζ_m and adding the resulting equations we obtain

$$\begin{aligned}
&\eta^\zeta(\rho^\varepsilon, m^\varepsilon)_t + q^\zeta(\rho^\varepsilon, m^\varepsilon)_x \\
&\quad = (\varepsilon \eta^\zeta_m(\rho^\varepsilon, m^\varepsilon) u^\varepsilon_x - \delta \theta^\varepsilon p_\theta(\rho^\varepsilon))_x \\
&\qquad - \Big(\varepsilon u^\varepsilon_x - \delta \theta^\varepsilon p_\theta(\rho^\varepsilon)\Big)\Big(\eta^\zeta_{mu}(\rho^\varepsilon, m^\varepsilon) u^\varepsilon_x + \eta^\zeta_{m\rho}(\rho^\varepsilon, m^\varepsilon)\rho^\varepsilon_x\Big) \\
&\qquad\quad - \eta^\zeta_m(\rho^\varepsilon, m^\varepsilon)\Big(\frac{\beta}{2}|\mathbf{h}^\varepsilon|^2 - \alpha g'(1/\rho^\varepsilon) h(|\psi^\varepsilon|^2)\Big)_x.
\end{aligned} \tag{3.189}$$

Let us show that all the terms on the RHS tend to zero as $\varepsilon \to 0$ in the sense of distributions, except possibly for the term $\varepsilon \eta^\zeta_m(\rho^\varepsilon, m^\varepsilon)|u^\varepsilon_x|^2$, which turns out to be nonpositive anyway.

From (3.156), given any $\zeta \in C^2(\mathbb{R})$ we have

$$\eta^\zeta_m(\rho, \rho u) = \int_{-1}^1 \zeta'(u + \rho^\vartheta s)[1 - s^2]^\Lambda_+ ds.$$

Hence

$$\eta^\zeta_{mu}(\rho, \rho u) = \int_{-1}^1 \zeta''(u + \rho^\vartheta s)[1 - s^2]^\Lambda_+ ds,$$

and also

$$\eta^\zeta_{m\rho} = (\rho, \rho u) = \vartheta \rho^{\vartheta - 1} \int_{-1}^1 \zeta''(u + \rho^\vartheta s) s [1 - s^2]^\Lambda_+ ds.$$

Take $\zeta(s) \in \{\pm 1, \pm s, s^2\}$. Then, $|\eta^\zeta_m(\rho^\varepsilon, m^\varepsilon)| \le C(1 + |u^\varepsilon|)$, $\eta^\zeta_{m\rho}(\rho^\varepsilon, m^\varepsilon) = 0$ and $0 \le \eta^\zeta_{mu}(\rho^\varepsilon, m^\varepsilon) \le C$ (note that $\int s[1 - s^2]^\Lambda_+ ds = 0$).

Let us recall (3.178) and (3.179). Then,

$$\begin{aligned}
\int_0^T\int_\Omega\left|\eta_m^\zeta(\rho^\varepsilon,m^\varepsilon)\left(\frac{\beta}{2}|\mathbf{h}|^2\right)_x\right|dx\,ds &\le C\int_0^T\int_\Omega(1+|u^\varepsilon|)\beta|\mathbf{h}^\varepsilon\cdot\mathbf{h}_x^\varepsilon|dx\,ds\\
&\le C\left(1+\left(\int_0^T\int_\Omega|u_x^\varepsilon|^2dx\,ds\right)^{1/2}\right)\left(\beta\int_0^T\int_\Omega|\mathbf{h}^\varepsilon\cdot\mathbf{h}_x^\varepsilon|^2dx\,ds\right)^{1/2}\\
&\le C\left(1+\left(\int_0^T\int_\Omega|u_x^\varepsilon|^2dx\,ds\right)^{1/2}\right)\left(\beta^{1/2}\int_0^T\int_\Omega|\mathbf{h}^\varepsilon|_x^2dx\,ds\right)\\
&\le C\frac{\beta^{1/2}}{\varepsilon^{1/2}}\left(\varepsilon^{1/2}+\left(\varepsilon\int_0^T\int_\Omega|u_x^\varepsilon|^2dx\,ds\right)^{1/2}\right)\\
&\le C\frac{\beta^{1/2}}{\varepsilon^{1/2}},
\end{aligned}$$

which tends to zero as $\varepsilon\to 0$.

Similarly, recalling (2.56), that $\psi_x=\rho\psi_y$ and that the Jacobian of the coordinate change equals ρ, from Lemmas 3.17 and 3.18 we have

$$\begin{aligned}
&\int_0^T\int_\Omega\left|\eta_m^\zeta(\rho^\varepsilon,m^\varepsilon)\left(\alpha g'(1/\rho^\varepsilon)h(|\psi^\varepsilon|^2)\right)_x\right|dx\,ds\\
&\qquad\le C\alpha\int_0^T\int_\Omega(1+|\rho^\varepsilon|^{1/2}|u^\varepsilon|)(|\rho^\varepsilon|^{\frac{\gamma-3}{2}}|\rho_x^\varepsilon|+|\psi_x^\varepsilon g'(1/\rho^\varepsilon)|)dx\,ds\\
&\qquad\le C\frac{\alpha}{\varepsilon^{1/2}},
\end{aligned}$$

which also tends to zero as $\varepsilon\to 0$.

Next, using (3.163)

$$\begin{aligned}
&\int_0^T\int_\Omega|\delta\theta^\varepsilon p_\theta(\rho^\varepsilon)(\eta_{mu}^\zeta(\rho^\varepsilon,m^\varepsilon)u_x^\varepsilon+\eta_{m\rho}^\zeta(\rho^\varepsilon,m^\varepsilon)\rho_x)|dxds\\
&\qquad\le C\frac{\delta}{\varepsilon^{1/2}}\left(\int_0^T M_\theta(s)^2\int_\Omega\rho^\gamma dxds\right)^{1/2}\left(\varepsilon\int_0^T\int_\Omega u_x^2dxds\right)^{1/2}\\
&\qquad\le C\frac{\delta}{\varepsilon^{1/2}}
\end{aligned}$$

which, tends to zero as well.

Finally, by the same token we have that

$$\eta_m^\zeta(\rho^\varepsilon,m^\varepsilon)(\varepsilon u_x^\varepsilon-\delta\theta^\varepsilon p_\theta(\rho^\varepsilon))\to 0$$

in the sense of distributions. Thus, taking $\varepsilon\to 0$ in (3.189) we obtain (3.187).

Let us now prove (3.188). In view of Lemmas 3.19 and 3.20 and Proposition 3.2, we can apply the Div-Curl Lemma (Lemma 3.12) in order to conclude that for any C^2 compactly supported functions ζ and ϕ we have

$$\overline{\eta^\zeta q^\phi} - \overline{\eta^\phi q^\zeta} = \overline{\eta^\zeta}\ \overline{q^\phi} - \overline{\eta^\phi}\ \overline{q^\zeta}.$$

Consequently,

$$\begin{aligned}
&\int \zeta(\xi_1)\overline{\chi(\xi_1)}d\xi_1 \int \phi(\xi_2)\overline{\vartheta\xi_2 + (1-\vartheta)u)\chi(\xi_2)}\,d\xi_2 \\
&\quad - \int \phi(\xi_2)\overline{\chi(\xi_2)}d\xi_2 \int \zeta(\xi_1)\overline{\vartheta\xi_1 + (1-\vartheta)u)\chi(\xi_1)}\,d\xi_1 \\
&\qquad = \int \zeta(\xi_1)\phi(\xi_2)\overline{\chi(\xi_1)(\vartheta\xi_2 + (1-\vartheta)u)\chi(\xi_2)}d\xi_1 d\xi_2 \\
&\qquad\quad - \int \zeta(\xi_1)\phi(\xi_2)\overline{(\vartheta\xi_1 + (1-\vartheta)u)\chi(\xi_1)\chi(\xi_2)}d\xi_1 d\xi_2.
\end{aligned}$$

As this holds for any ζ and ϕ we have

$$\overline{\chi(\xi_1)}\ \overline{\vartheta\xi_2 + (1-\vartheta)u)\chi(\xi_2)} - \overline{\chi(\xi_2)}\ \overline{\vartheta\xi_1 + (1-\vartheta)u)\chi(\xi_1)} = \vartheta(\xi_2 - \xi_1)\overline{\chi(\xi_1)\chi(\xi_2)},$$

which implies (3.188). $\square$

With this proposition, the argument to reduce the Young measures in [15] applies and we have shown the following.

Theorem 3.3. *Let the initial functions $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$ be smooth and satisfy the following conditions:*

(i) $\rho_0^\varepsilon \geq c_0^\varepsilon > 0$, $M_0^{-1} \leq \int_\Omega \rho_0^\varepsilon dx \leq M_0$, $\int_\Omega \rho_0^\varepsilon u_0^\varepsilon dx \leq M_0$, $-\int_\Omega \rho_0^\varepsilon s(\rho_0^\varepsilon, \theta_0^\varepsilon) dx \leq M_0$;

(ii) $\int_\Omega (\rho_0^\varepsilon(|u_0^\varepsilon|^2 + |\mathbf{w}_0^\varepsilon|^2) + |\mathbf{h}_0^\varepsilon|^2)dx + \int_{\Omega_y}(|\psi_{0y}^\varepsilon|^2 + |\psi_0^\varepsilon|^2)dy \leq M_0$;

(iii) $\varepsilon^2 \int_\Omega |\rho_{0x}^\varepsilon|^2 |\rho_0^\varepsilon|^{-3} dx + \varepsilon\beta^2 \int_\Omega |\mathbf{h}_0^\varepsilon|^2 (\rho_0^\varepsilon)^{-1} dx \leq M_0$;

(iv) $(\rho_0^\varepsilon, \rho_0^\varepsilon u_0^\varepsilon) \to (\rho_0, \rho_0 u_0)$ *as* $\varepsilon \to 0$ *in the sense of distributions, with* $\rho_0 \geq 0$ *a.e.*

Let $((\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon))$ *be the solution of* (3.1)-(3.6) *with* p *given by* (3.131) *and with initial data* $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$. *Assume, further that* $\alpha = o(\varepsilon^{1/2})$, $\beta = o(\varepsilon)$ *and* $\delta = o(\varepsilon)$. *Then, we may extract a subsequence (not relabelled) of* $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ *that converges in* $L^1_{loc}(\Omega \times (0,\infty))$ *to a finite-energy entropy solution* $(\rho, \rho u)$ *of the compressible Euler equations* (3.145), (3.146) *with initial data* $(\rho_0, \rho_0 u_0)$.

3.3.5 Limit for the transverse velocity, magnetic field and wave function

With Theorem 3.3 at hand, the passage to the limit in equations (3.3) and (3.5) becomes a straightforward exercise. As pointed out in Subsection 3.3.2, the uniform estimates in Lemma 3.17 and the fact that we are leaving μ and ν fixed independently of ε, imply that $\beta\mathbf{h}^\varepsilon$ and $\beta\mathbf{w}^\varepsilon$ tend to zero in $L^2(\Omega \times (0,T))$ and $\beta u^\varepsilon \mathbf{h}^\varepsilon$ tends to zero in $L^1(\Omega \times (0,T))$. Accordingly, we have that $\mathbf{h}^\varepsilon_{xx}$ in the sense of distributions. Nonetheless, we also have a uniform bound for the $L^2(0,T;H^1_0(\Omega))$ for $\mathbf{h}^\varepsilon$ so that we can assume that it converges to some limit $\mathbf{h}$ weakly in $L^2(0,T;H^1_0(\Omega))$. This implies that, necessarily, $\mathbf{h} = 0$ and the limit equation is satisfied trivially. Thus, for consistency, we demand that the initial data $\mathbf{h}^\varepsilon_0$ satisfies $\beta\mathbf{h}^\varepsilon_0 \to 0$ in the sense of distributions and drop equation (3.136).

Moving on to equation (3.3), the uniform estimates from Lemma 3.17 imply that $\mathbf{w}^\varepsilon$ has a subsequence (not relabelled) that converges to some limit $\mathbf{w}$ weakly in $L^2(0,T;H^1_0(\Omega))$. Since $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ converges strongly to $(\rho, \rho u)$ we have that $\rho^\varepsilon \mathbf{w}^\varepsilon$ and $\rho^\varepsilon u^\varepsilon \mathbf{w}^\varepsilon$ converge to $\rho\mathbf{w}$ and $\rho u\mathbf{w}$, respectively, in the sense of distributions. As $\beta\mathbf{h}$ converges strongly to zero, we have that the limit functions ρ, ρu and $\mathbf{w}$ solve the limit equation (3.134).

Regarding the initial data for the transverse velocity, we demand that $\rho^\varepsilon_0 \mathbf{w}^\varepsilon_0$ converge to some limit $\rho_0\mathbf{w_0}$ in the sense of distributions. Note that we specify the initial data for the limit equation in terms of the transverse momentum as vacuum is unavoidable in the limit functions. Accordingly, we have that the initial data is attained in the sense of distributions through the weak formulation of (3.134):

$$\int_0^t \int_\Omega \rho\mathbf{w}\varphi_t dxds - \int_\Omega \rho_0 \mathbf{w}_0 \varphi|_{t=0} ds - \int_0^t \int_\Omega \rho u \mathbf{w}\varphi_x dxds = -\int_0^t \int_\Omega \mu \mathbf{w}_x \varphi_x dxds$$

for any $\varphi \in C^\infty(\Omega \times ([0,\infty))$.

Lastly, the passage to the limit in the nonlinear Schrödinger equation is a direct consequence of Aubin-Lions Lemma, as explained in Subsection 3.3.2. For consistency, we assume that the initial data ψ^ε_0 converges to ψ_0 in $H^1_0(\Omega_y)$, thereby concluding that ψ^ε converges to the unique solution of the limit nonlinear Schrödinger equation (3.137).

We have thus proved the following.

Theorem 3.4. *Let the initial functions* $(\rho^\varepsilon_0, u^\varepsilon_0, \mathbf{w}^\varepsilon_0, \mathbf{h}^\varepsilon_0, \theta^\varepsilon_0, \psi^\varepsilon_0)$ *be smooth and satisfy the hypotheses of Theorem 3.3. Moreover, assume that* *(v)* $\rho_0^\varepsilon \mathbf{w}^\varepsilon \to \rho_0 \mathbf{w}_0$ *and* $\beta \mathbf{h}_0^\varepsilon \to 0$ *in the sense of distributions;*

(vi) $\psi_0^\varepsilon \to \psi_0$ *in* $H_0^1(\Omega_y)$*.*

Let $((\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon))$ *be the solution of* (3.1)-(3.6) *with* p *given by* (3.131) *and with initial data* $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$*. Assume, further that* $\alpha = o(\varepsilon^{1/2})$*,* $\beta = o(\varepsilon)$ *and* $\delta = o(\varepsilon)$*. Then, we may extract a subsequence (not relabelled) of* $(\mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \psi^\varepsilon)$ *such that*

- $(\mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon) \to (\mathbf{w}, 0)$ *weakly in* $L^2(0, T; H_0^1(\Omega))$*, and*
- $\psi^\varepsilon \to \psi$ *strongly in* $L^\infty(0, T; L^4(\Omega)$ *and weakly-* in* $L^\infty(0, T; H_0^1(\Omega))$*.*

Moreover, $(\rho, \rho u, \mathbf{w})$ *solve equation* (3.134)*with initial data* $\rho_0 \mathbf{w}_0$ *attained in the sense of distributions; and* ψ *is the unique weak solution of equation* (3.137)*.*

To conclude, we move on to discussing the limit passage in the thermal energy equation (3.7).

3.3.6 Limit in the thermal energy equation

As explained in Subsection 3.3.2 the limit process in the thermal energy equation (3.7) is not straightforward on account of the nonlinearities. Also, the loss of regularity of the longitudinal velocity u forces us to consider the inequality (3.157) instead of (3.135).

In order to justify the passage to the limit, we adapt some ideas in [23].

First, we observe that estimate (3.181) implies that $Q(\theta^\varepsilon)$ is uniformly bounded in $L^2(0, T; H^1(\Omega))$ and hence we can assume that

$$Q(\theta^\varepsilon) \rightharpoonup \overline{Q} \text{ weakly in } L^2(0, T; H^1(\Omega)).$$

By the same token, $Q(\theta^\varepsilon)$ is uniformly bounded in $L^2(0, T; L^\infty(\Omega))$ which, in light of Lemma 3.17, implies that $\rho^\varepsilon Q(\theta^\varepsilon)$ is uniformly bounded in the space $L^2(0, T; L^\gamma(\Omega))$.

Now, from equation (3.7) and using Lemmas 3.17 and 3.21 we can easily see that $(\rho^\varepsilon Q(\theta^\varepsilon))_t$ is uniformly bounded in the space $L^1(0, T; H^{-3}(\Omega))$ (recall that $(\kappa\theta_x)_x$ can be written as $\mathcal{K}_{xx}$ and that by Lemma 3.21 $\mathcal{K}$ is uniformly bounded in $L^1(\Omega \times (0, T))$). With this, a direct application of Aubin-Lions lemma (Lemma 3.16) and the fact that ρ^ε converges strongly to ρ imply that

$$\rho^\varepsilon Q(\theta^\varepsilon) \to \rho \overline{Q} \text{ strongly in } L^2(0, T; H^{-1}(\Omega)).$$

We claim that

$$\int_0^T\int_\Omega \rho^\varepsilon Q(\theta^\varepsilon)^2\varphi dxds \to \int_0^T\int_\Omega \rho\overline{Q}^2\varphi dxds, \tag{3.190}$$

as $\varepsilon \to 0$, for **any** $\varphi \in C_0^\infty(\Omega\times(0,T))$. Indeed, we have

$$\begin{aligned}&\int_0^T\int_\Omega(\rho^\varepsilon Q(\theta^\varepsilon)^2 - \rho\overline{Q}^2)\varphi dxds\\ &\quad\le \int_0^T\int_\Omega(\rho^\varepsilon Q(\theta^\varepsilon) - \rho\overline{Q})(Q(\theta^\varepsilon)+\overline{Q})\varphi dxds + \int_0^T\int_\Omega Q(\theta^\varepsilon)\overline{Q}(\rho^\varepsilon - \rho)\varphi dxds.\end{aligned}$$

On the one hand we have

$$\begin{aligned}&\int_0^T\int_\Omega(\rho^\varepsilon Q(\theta^\varepsilon) - \rho\overline{Q})(Q(\theta^\varepsilon)+\overline{Q})\varphi dxds\\ &\le \int_0^T \|(\rho^\varepsilon Q(\theta^\varepsilon) - \rho\overline{Q})(s)\|_{H^{-1}(\Omega)}\|(Q(\theta^\varepsilon)+\overline{Q})(s)\varphi\|_{H_0^1(\Omega)}ds\\ &\le C_\varphi\int_0^T \|(\rho^\varepsilon Q(\theta^\varepsilon) - \rho\overline{Q})(s)\|_{H^{-1}(\Omega)}\Big(\|Q(\theta^\varepsilon)(s)\|_{H^1(\Omega)} + \|\overline{Q}(s)\|_{H^1(\Omega)}\Big)ds\\ &\le C_\varphi\|\rho^\varepsilon Q(\theta^\varepsilon) - \rho\overline{Q}\|_{L^2(0,T;H^{-1}(\Omega))}\Big(\|Q(\theta^\varepsilon)(s)\|_{L^2(0,T;H^1(\Omega))} + \|\overline{Q}(s)\|_{L^2(0,T;H^1(\Omega))}\Big),\end{aligned}$$

which tends to zero as $\varepsilon \to 0$.

On the other hand,

$$\int_0^T\int_\Omega Q(\theta^\varepsilon)\overline{Q}(\rho^\varepsilon - \rho)\varphi dxds \to 0$$

by the dominated convergence theorem (recall that $\rho^\varepsilon \to \rho$ a.e. in $\Omega\times(0,T)$), thus proving the claim.

Now, from (3.190) we have that

$$\begin{aligned}&\int_0^T\int_\Omega \rho Q(\theta^\varepsilon)\varphi dxds\\ &\qquad = \int_0^T\int_\Omega(\rho-\rho^\varepsilon)Q(\theta^\varepsilon)\varphi dxds + \int_0^T\int_\Omega \rho^\varepsilon Q(\theta^\varepsilon)\varphi dxds\\ &\qquad \to \int_0^T\int_\Omega \rho\overline{Q}\varphi dxds.\end{aligned}$$

also by the dominated convergence theorem.

This can be interpreted as convergence of norms in a weighted $L^2_{\rho\varphi}$ space. In particular, we have

$$Q(\theta^\varepsilon) \to \overline{Q} \text{ a.e. in } \{(x,t) \in \Omega \times (0,T) : \rho(x,t) > 0\}. \tag{3.191}$$

Since Q is strictly increasing (recall our hypotheses (2.54)) we can define $\overline{\vartheta} := Q^{-1}(\overline{Q})$ and we have that

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega (Q(\theta^\varepsilon) - \overline{Q})(\theta^\varepsilon - \overline{\theta}) \mathbb{1}_{\{\rho > 0\}} dx ds \\ &\geq C^{-1} \int_0^T \int_\Omega (\theta^\varepsilon - \overline{\theta})^2 \mathbb{1}_{\{\rho > 0\}} dx ds, \end{aligned}$$

and hence,

$$\theta^\varepsilon \to \overline{\theta} \text{ in } L^2(\{\rho > 0\}).$$

This last bit of information guarantees that we can pass to the limit in the first two terms of equation (3.7) (remember that $\rho^\varepsilon u^\varepsilon \to \rho u$ strongly). Regarding the third term in that equation, we are assuming that there is a coefficient $\delta = o(\varepsilon)$ multiplying it, and by the estimates in Lemma 3.17 it converges to zero in the sense of distributions.

All there is left to do, then, is justify the passage to the limit in the second order term on the right hand side. For this we need the following lemma (see [23, Proposition 2.1]).

Lemma 3.22. *Let $O \subseteq \mathbb{R}^M$ be a bounded open set. Let $\{v_n\}_{n=1}^\infty$ be a sequence of measurable functions,*

$$v_n : O \to \mathbb{R}^N,$$

such that

$$\sup_{n \geq 1} \int_O \Phi(|v_n|) d\xi < \infty$$

for a certain continuous function $\Phi : [0,\infty) \to [0,\infty)$.

Then, there exists a subsequence (not relabelled) such that

$$\zeta(v_n) \to \overline{\zeta(v)} \text{ weakly in } L^1(O)$$

for all continuous functions $\zeta : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\lim_{|\mathbf{z}| \to \infty} \frac{\zeta(\mathbf{z})}{\Phi(\mathbf{z})} = 0.$$

Fix $0 < \omega < 1$ and choose $\zeta : [0,\infty) \to [0,\infty)$ as $\zeta(z) = \frac{1}{(1+z)^\omega}$. Then, multiplying

(3.7) by $\zeta(\theta^\varepsilon)$ and using equation (3.1) we have

$$\begin{aligned}(\rho^\varepsilon Q_\zeta(\theta^\varepsilon))_t + (\rho^\varepsilon u^\varepsilon Q_\zeta(\theta^\varepsilon))_x + \ \delta\theta^\varepsilon p_\theta(\rho^\varepsilon)\zeta(\theta^\varepsilon)u^\varepsilon_x \\ = (\mathcal{K}_\zeta(\theta^\varepsilon))_{xx} + \frac{\omega\kappa(\theta^\varepsilon)|\theta^\varepsilon_x|^2}{(1+\theta^\varepsilon)^\omega} + \frac{\varepsilon|u^\varepsilon_x|^2 + \mu|\mathbf{w}^\varepsilon_x|^2 + \nu|\mathbf{h}^\varepsilon_x|^2}{(1+\theta^\varepsilon)^\omega},\end{aligned} \tag{3.192}$$

where, Q_ζ and $\mathcal{K}_\zeta$ are given by

$$Q_\zeta(\theta) := \int_0^\theta \frac{C_\vartheta(z)}{(1+z)^\omega}dz, \qquad \mathcal{K}_\zeta(\theta) := \int_0^\theta \frac{\kappa(z)}{(1+z)^\omega}dz$$

From the strong convergence of ρ^ε, $\rho^\varepsilon u^\varepsilon$ and the strong convergence of θ^ε in $\{\rho > 0\}$ and the uniform estimates we see that

$$\left.\begin{aligned} &\rho^\varepsilon Q_\zeta(\theta^\varepsilon) \to \rho Q_\zeta(\overline{\theta}) \\ &\rho^\varepsilon u^\varepsilon Q_\zeta(\theta^\varepsilon) \to \rho u Q_\zeta(\overline{\theta}) \\ &\delta\theta^\varepsilon p_\theta(\rho^\varepsilon)\zeta(\theta^\varepsilon)u^\varepsilon_x \to 0 \end{aligned}\right\} \text{ weakly in } L^1(\Omega\times(0,T)).$$

Next, using Lemma 3.22 we see that

$$\mathcal{K}_\zeta(\theta^\varepsilon) \to \overline{\mathcal{K}}_\zeta \text{ weakly in } L^1(\Omega\times(0,T)),$$

for some $\overline{\mathcal{K}}_\zeta$ that satisfies

$$\rho\overline{\mathcal{K}}_\zeta = \rho\mathcal{K}_\zeta(\overline{\theta}), \text{ on } \Omega\times(0,T).$$

Now, let φ be a test function such that

$$\varphi \geq 0, \quad \varphi \in W^{2\infty}(\Omega\times(0,T)), \quad \psi_x|_{\partial\Omega} = 0, \quad \text{supp}\varphi \subseteq \overline{\Omega}\times[0,T). \tag{3.193}$$

For any such test function we have that

$$\int_0^T\int_\Omega \frac{\mu|\mathbf{w}_x|^2}{(1+\overline{\theta})^\omega}\varphi dxds \leq \liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\mu|\mathbf{w}^\varepsilon_x|^2}{(1+\theta^\varepsilon)^\omega}\varphi dxds.$$

Thus, multiplying (3.192) by φ, integrating and taking the limit as $\varepsilon \to 0$ we obtain

$$\int_0^T \int_\Omega \Big(\rho Q_\zeta(\overline{\theta})\varphi_t + \rho u Q_\zeta(\overline{\theta})\varphi_x + \overline{\mathcal{K}}_\zeta \varphi_{xx}\Big) dxds \\ \leq - \int_0^T \int_\Omega \frac{\mu |\mathbf{w}_x|^2}{(1+\overline{\theta})^\omega} \varphi dxds - \int_\Omega \rho_0 Q_\zeta(\theta_0)\varphi|_{t=0} dx. \tag{3.194}$$

For this last term we are assuming that $\rho_0^\varepsilon Q(\theta_0^\varepsilon) \to \rho_0 Q(\theta_0)$.

Now, note that

$$\frac{1}{(1+z)^\omega} \nearrow 1, \text{ as } \omega \to 0,$$

then, using the monotone convergence theorem we see that

$$\overline{\mathcal{K}}_\zeta \nearrow \overline{K},$$

where,

$$\rho \overline{\mathcal{K}} = \rho \mathcal{K}(\overline{\theta}),$$

and

$$\int_0^T \int_\Omega \overline{\mathcal{K}} dxds \leq \liminf_{\varepsilon \to 0} \int_0^T \int_\Omega \mathcal{K}(\theta^\varepsilon) dxds.$$

Finally, we can define $\theta := \mathcal{K}^{-1}(\overline{\mathcal{K}})$ and take the limit as $\omega \to 0$ in (3.194) in order to conclude that the nonnegative function θ satisfies

$$\int_0^T \int_\Omega \Big(\rho Q(\theta)\varphi_t + \rho u Q(\theta)\varphi_x + \mathcal{K}(\theta)\varphi_{xx}\Big) dxds \\ \leq - \int_0^T \int_\Omega \mu |\mathbf{w}_x|^2 \varphi dxds - \int_\Omega \rho_0 Q(\theta_0)\varphi|_{t=0} dx, \tag{3.195}$$

for any test function that satisfies (3.193); which is the weak formulation of inequality (3.157).

Finally, let us show that (3.158) holds. From the energy identity (3.9) we have

$$\int_\Omega \left(\rho^\varepsilon \left(e(\rho^\varepsilon,\theta^\varepsilon) + \tfrac{1}{2}|u^\varepsilon|^2 + \tfrac{1}{2}|\mathbf{w}^\varepsilon|^2\right) + \frac{\beta}{2}|\mathbf{h}^\varepsilon|^2\right) dx \\ + \int_{\Omega_y} \left(\alpha g(v^\varepsilon) h(|\psi^\varepsilon|^2) + \tfrac{1}{2}|\psi^\varepsilon_y|^2 + \tfrac{1}{4}|\psi^\varepsilon|^4\right) dy \\ = \int_\Omega \left(\rho_0^\varepsilon \left(e(\rho_0^\varepsilon,\theta_0^\varepsilon) + \tfrac{1}{2}|u_0^\varepsilon|^2 + \tfrac{1}{2}|\mathbf{w}_0^\varepsilon|^2\right) + \frac{\beta}{2}|\mathbf{h}_0^\varepsilon|^2\right) dx \\ + \int_{\Omega_y} \left(\alpha g(v_0^\varepsilon) h(|\psi_0^\varepsilon|^2) + \tfrac{1}{2}|\psi^\varepsilon_{0y}|^2 + \tfrac{1}{4}|\psi_0^\varepsilon|^4\right) dy.$$

By assumption, the right hand side tends to

$$\int_\Omega \rho_0 \left(e(\rho_0,\theta_0) + \tfrac{1}{2}|u_0|^2 + \tfrac{1}{2}|\mathbf{w}_0|^2\right) dx + \int_{\Omega_y} \left(\tfrac{1}{2}|\psi_{0y}|^2 + \tfrac{1}{4}|\psi_0|^4\right) dy.$$

By lower semi-continuity we have that

$$\int_\Omega \left(\tfrac{1}{2}\rho|\mathbf{w}|^2\right) dx \leq \int_\Omega \left(\tfrac{1}{2}\rho^\varepsilon|\mathbf{w}^\varepsilon|^2\right) dx$$

and by strong convergence in $\Omega \times (0,T)$ we have that

$$\int_\Omega \rho \left(e(\rho,\theta) + \tfrac{1}{2}|u|^2 + \tfrac{1}{2}|\mathbf{w}|^2\right)(t) dx + \int_{\Omega_y} \left(\tfrac{1}{2}|\psi_y|^2 + \tfrac{1}{4}|\psi|^4\right)(t) dy. \\ \leq \int_\Omega \rho_0 \left(e(\rho_0,\theta_0) + \tfrac{1}{2}|u_0|^2 + \tfrac{1}{2}|\mathbf{w}_0|^2\right) dx + \int_{\Omega_y} \left(\tfrac{1}{2}|\psi_{0y}|^2 + \tfrac{1}{4}|\psi_0|^4\right) dy.$$

for a.e. $t \in (0,T)$.

Finally, since the unique solution of the nonlinear Schrödinger equation has conservation of energy:

$$\int_{\Omega_y} \left(\tfrac{1}{2}|\psi_y|^2 + \tfrac{1}{4}|\psi|^4\right)(t) dy = \int_{\Omega_y} \left(\tfrac{1}{2}|\psi_{0y}|^2 + \tfrac{1}{4}|\psi_0|^4\right) dy$$

we conclude that

$$\int_\Omega \rho \left(e(\rho,\theta) + \tfrac{1}{2}|u|^2 + \tfrac{1}{2}|\mathbf{w}|^2\right)(t) dx \leq \int_\Omega \rho_0 \left(e(\rho_0,\theta_0) + \tfrac{1}{2}|u_0|^2 + \tfrac{1}{2}|\mathbf{w}_0|^2\right) dx. \quad (3.196)$$

We can sum up the results found in this Section through the following theorem:

Theorem 3.5. *Let the initial functions* $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$ *be smooth and satisfy the hypotheses of Theorem 3.3 and Theorem 3.4.*

Let $(\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon)$ *be the solution of* (3.1)-(3.6) *with* p *given by* (3.131) *and with initial data* $(\rho_0^\varepsilon, u_0^\varepsilon, \mathbf{w}_0^\varepsilon, \mathbf{h}_0^\varepsilon, \theta_0^\varepsilon, \psi_0^\varepsilon)$. *Assume, further that* $\alpha = o(\varepsilon^{1/2})$, $\beta = o(\varepsilon)$ *and* $\delta = o(\varepsilon)$. *Then, we may extract a subsequence (not relabelled) of* $(\rho^\varepsilon, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon, \theta^\varepsilon, \psi^\varepsilon)$ *such that as* $\varepsilon \to 0$ *we have*

- $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ *converges in* $L^1_{loc}(\Omega \times (0,\infty))$ *to a finite-energy entropy solution* $(\rho, \rho u)$ *of the compressible Euler equations* (3.145), (3.146) *with initial data* $(\rho_0, \rho_0 u_0)$;

- $(\mathbf{w}^\varepsilon, \mathbf{h}^\varepsilon) \to (\mathbf{w}, 0)$ *weakly in* $L^2(0,T; H_0^1(\Omega))$ *and* $(\rho, \rho u, \mathbf{w})$ *solve equation* (3.134)*with initial data* $\rho_0 \mathbf{w}_0$ *attained in the sense of distributions;*

- $\psi^\varepsilon \to \psi$ *strongly in* $L^\infty(0,T; L^4(\Omega)$ *and weakly-* in* $L^\infty(0,T; H_0^1(\Omega))$, *where* ψ *is the unique weak solution of equation* (3.137)

- $\rho^\varepsilon Q(\theta^\varepsilon)$ *converges strongly to* $\rho Q(\theta)$ *in* $L^1_{loc}(\Omega \times (0,\infty))$ *and* $(\rho, \rho u, \mathbf{w}, \theta)$ *constitute a variational solution of equation* (3.135) *in the sense of inequality* (3.195), *also satisfying* (3.196).

Chapter 4

Higher dimensions

We now move on to the multidimensional case. The main difficulty in higher dimensions is the possible occurrence of vacuum. As the Lagrangian transformation becomes singular in the presence of vacuum an effective coupling of the fluid equations with the nonlinear Schrödinger equation can not be made in a straightforward way. In order to overcome these difficulties, we define the interaction through a regularized system that provides a good definition for an approximate Lagrangian coordinate. Then, after showing existence of solutions, we show compactness of the sequence of solutions to the regularized system thus making sense of the desired SW-LW interaction in the limit process.

For simplicity, in the multidimensional model we focus on the isentropic case, that is, the case of a non heat-conductive fluid, which trivializes the energy equation (2.35).

Let us remark that the results that we present here hold in a smooth bounded open spacial domain in $\mathbb{R}^2$. The only restriction that does not allow us to proceed in the full three dimensional case comes from the lack of solvability of the nonlinear Schrödinger equation in this setting. However, assuming this our methods can be adapted to the three dimensional case. Also, our result covers large initial data at the price of obtaining only weak solutions.

Let us mention that the results on the multidimensional case are product of an ongoing collaboration with prof. Hermano Frid, as well as with prof. Ronghua Pan.

4.1 Regularized problem

We now consider the two-dimensional model for an isentropic magentohydrodynamic flow. Similarly to the planar case, the two-dimensional MHD equations are deduced from the full three-dimensional ones under the assumption that all the involved functions are independent of the third variable. Accordingly, we assume that our state variables ρ, $\mathbf{u}$ and $\mathbf{H}$ are functions of $(\mathbf{x}, t) \in \Omega \times [0, T]$, with Ω a smooth bounded domain of $\mathbb{R}^2$ and $T > 0$ arbitrary. As we are dealing with an isentropic flow, the energy equation is trivialized and we end up with the following system

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{4.1}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right)\Big) + (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f}_{\text{ext}}, \tag{4.2}$$

$$\mathbf{H}_t + \operatorname{curl}\left(\nu \operatorname{curl}(\mathbf{H})\right) = \operatorname{curl}(\mathbf{u} \times \mathbf{H}), \tag{4.3}$$

$$\operatorname{div} \mathbf{H} = 0, \tag{4.4}$$

where $p = p(\rho) = a\rho^\gamma$. Let us point out that, in this case, we are assuming that the magnetic permeability is constant and equal to 1, as is usual in the literature.

Since we allow for large initial data, we work with weak solutions. As a result, the Lagrangian transformation as defined before may become singular due to the possible occurrence of vacuum in finite time.

In order to workaround the lack of regularity of the density we first add an artificial viscosity to the continuity equation (4.1). Fix $\varepsilon > 0$ and $\delta > 0$ and consider the following regularized system

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \tag{4.5}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(a\rho^\gamma + \delta\rho^\beta) + \varepsilon \nabla \mathbf{u} \cdot \nabla \rho = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right)\Big) + \mathbf{f}_{\text{ext}}, \tag{4.6}$$

$$\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \tag{4.7}$$

$$\operatorname{div} \mathbf{H} = 0. \tag{4.8}$$

Note that besides the artificial viscosity added to the continuity equation, two new terms appeared in the momentum equation (4.2). The term $\delta\rho^\beta$, where $\beta > 1$, acts as an artificial pressure and is intended to provide better estimates on the density, whereas the term $\varepsilon \nabla \mathbf{u} \cdot \nabla \rho$ is set to equate the unbalance in the energy estimates of the MHD equations caused by the introduction of the artificial viscosity. This approximate system resembles the one employed by Hu and Wang in [29] where they study the existence of weak solutions to the three dimensional MHD equations. A similar approximation was introduced by Feireisl, et al. in [24] in the study of the Navier-Stokes equations, who, in turn, followed the pioneering ideas by P.-L. Lions in [38]. Recall that ε and δ are small constants and the analysis that we intend to develop will provide insights that justify the accuracy to which this regularized model approximates the desired SW-LW interaction.

Now, as it turns out, even in this regularized setting the velocity field might not be smooth enough to provide a good enough definition of Lagrangian transformation that we can work with. More specifically, in the present situation there is no a priori bound available for the Jacobian of the Lagrangian transformation, as it depends on the L^∞ norm of div$\mathbf{u}$. For this reason we replace the velocity by a suitable smooth approximation $\mathbf{u}^N$ (which tends to $\mathbf{u}$ as $N \to \infty$) in the definition of the Lagrangian transformation. Thus obtaining an approximate Lagrangian coordinate defined as before with $\mathbf{u}$ replaced by $\mathbf{u}^N$.

In order to define such an approximation of the velocity we consider the following subspaces of $L^2(\Omega)$. For each $n \in \mathbb{N}$ consider the space $X_n \subseteq L^2(\Omega; \mathbb{R}^3)$ defined as

$$X_n := E_n \times E_n \times E_n,$$

where, $E_n = \operatorname{span}\{\eta_j : j = 1, ..., n\}$ and $\eta_1, \eta_2, \cdots$ is the complete collection of normalized eigenvectors of the Laplacian with zero boundary condition in Ω; with respective projection

$$P_n : L^2(\Omega) \to X_n,$$

With this notation, given $N \in \mathbb{N}$ we define $\mathbf{u}^N$ as

$$\mathbf{u}^N = P_N \mathbf{u}. \tag{4.9}$$

Note that for any $\mathbf{u}(\mathbf{x}, t)$ that satisfies $\mathbf{u}(\cdot, t) \in L^2(\Omega)$ for a.e. t, $\mathbf{u}^N$ thus defined is smooth and can be written as

$$\mathbf{u}^N(\mathbf{x}, t) = \sum_{j=1}^{N} \mathbf{u}_j^N(t) \eta_j(\mathbf{x}), \tag{4.10}$$

for some vector valued coefficients $\mathbf{u}_j^N(t)$, $j = 1, \cdots, N$; and satisfies,

$$||\mathbf{u}_N(t)||_{L^2(\Omega)} = \left(\sum_{j=1}^{N} |\mathbf{u}_j^N(t)|^2 \right)^{1/2}. \tag{4.11}$$

In fact, in light of (4.10) we have that

$$||\nabla \mathbf{u}^N||_{L^\infty(\Omega)} \le C_N ||\mathbf{u}^N||_{L^2(\Omega)} \le C_N ||\mathbf{u}||_{L^2(\Omega)}, \tag{4.12}$$

where

$$C_N := N \max_{j=1,\cdots,N} ||\nabla \eta_j||_{L^\infty(\Omega)}. \tag{4.13}$$

With this in mind, we define the Lagrangian transformation $Y(t, \mathbf{x}) = Y(t, \mathbf{y}(t, \mathbf{x}))$ through (2.23), (2.25) with the fluids velocity $\mathbf{u}$ replaced by $\mathbf{u}^N$. Recall that we have a certain flexibility in the choice of the function $\mathbf{y}_0$. In the previous Chapters we chose it in terms of the initial density as it yielded a convenient expression for the Jacobian of the Lagrangian transformation, namely (2.26).

In the present situation, however, as we allow for vacuum, even in the initial data, we go another direction and choose

$$\mathbf{y}_0(\mathbf{x}) = \mathbf{x}.$$

With this choice for the initial diffemorphism, we see that for every $t \ge 0$ the coordinate change is a diffeomorphism from Ω into itself as well, and this holds for any N. This is due to the zero boundary conditions satisfied by each approximate velocity field $\mathbf{u}^N$.

With these modifications we now have a smoothed Lagrangian coordinate. Nonetheless, with the new definition we lose relation (2.27) and instead we have

$$J_{\mathbf{y}}(t) = \exp\left[-\int_0^t \operatorname{div} u^N(s, \Phi(s, x)) ds\right]. \tag{4.14}$$

Note that, by Poincaré's inequality, (4.12) implies

$$|J_{\mathbf{y}}(t)| \le \exp\left[C_N \left(t + \int_0^t ||u(s)||^2_{H_0^1(\Omega)} ds\right)\right], \tag{4.15}$$

provided that $\mathbf{u} \in L^2(0, T; H_0^1(\Omega))$, which is to be expected for the kind of solutions that we work with.

Now that we have a Lagrangian coordinate we can talk about the SW-LW interactions. To that end, we consider the following nonlinear Schrödinger equation stated in the newly defined Lagrangian coordinates

$$i\psi_t + \Delta_{\mathbf{y}} \psi = |\psi|^2 \psi + G\psi, \tag{4.16}$$

where ψ is the complex valued wave function and G is a real valued function corresponding to a potential. In order to complete our regularized model we have to define the coupling terms through the external force term $\mathbf{f}_{\text{ext}}$ in (4.2) and the potential G. As before we choose G as

$$G = \alpha g(v)h'(|\psi|^2). \tag{4.17}$$

Regarding $\mathbf{f}_{\text{ext}}$ we choose

$$\mathbf{f}_{\text{ext}} = \alpha\nabla\left(\frac{J_{\mathbf{y}}}{\rho}g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)\right). \tag{4.18}$$

Note that this coincides with our previous choice (2.30) once we realize that in our original model we had that $J_{\mathbf{y}} = \rho$. Note that, although vacuum is permitted in our new model, the fact that g is compactly supported in $(0,\infty)$ clarifies any ambiguity in the definition of $\mathbf{f}_{\text{ext}}$.

As a result we end up with the following system of equations:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho, \tag{4.19}$$

$$\begin{aligned}&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(a\rho^\gamma + \delta\rho^\beta) + \varepsilon\nabla\mathbf{u}\cdot\nabla\rho\\ &\quad= \nabla(\alpha\frac{J_{\mathbf{y}}}{\rho}g'(1/\rho)h(|\psi\circ\mathbf{Y}|^2)) + (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)\Big),\end{aligned} \tag{4.20}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \tag{4.21}$$

$$\operatorname{div}\mathbf{H} = 0. \tag{4.22}$$

$$i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi + \alpha g(v)h'(|\psi|^2)\psi, \tag{4.23}$$

Regarding this new system, we prove the existence of solutions on a time interval $[0, T^N]$, where T^N depends on ε, α and N. After this, we show the convergence of the approximate solutions when the artificial viscosity ε together with the interaction coefficients α tend to zero and as N tends to infinity at a specific rate at which T^N tends to infinity. Then, we make δ tend to zero and show convergence (on an arbitrary time interval $[0,T]$) to a solution of the system formed by the MHD equations together with the decoupled nonlinear Schrödinger equation. In other words, we find a solution to the limit decoupled system, consisting of the MHD equations and a nonlinear Schrödinger equation, as the limit of a sequence of solutions of the regularized SW-LW interactionsd system.

As emphasized before, the proposed approximation scheme has the purpose to legitimize the coordinates of the limiting Schrödinger equation to be considered as the Lagrangian coordinates of the fluid in a generalized sense.

4.2 Solutions to the regularized system

We consider the initial-boundary value problem for system (4.19)-(4.23) with initial data

$$(\rho, \rho\mathbf{u}, \mathbf{H})(\mathbf{x}, 0) = (\rho_0, \mathbf{m}_0, \mathbf{H}_0)(\mathbf{x}), \qquad \psi(y, 0) = \psi_0(\mathbf{y}), \tag{4.24}$$

where $\mathbf{m}_0$ is the initial momentum. Again, as vacuum is possible, it is better to regard the initial data in terms of the momentum instead of the velocity field.

With respect to the boundary conditions we demand that

$$(\nabla\rho \cdot \mathbf{n}, \mathbf{u}, \mathbf{H})|_{\partial\Omega} = 0, \qquad \psi|_{\partial\Omega_{\mathbf{y}}} = 0. \tag{4.25}$$

Note that a Neumann boundary condition was added for the density as a result of the introduction of the artificial viscosity in the continuity equation.

Theorem 4.1. *Let $T > 0$ be given and $N \in \mathbb{N}$ be fixed. Suppose that the initial data is smooth and that*

$$M_0^{-1} \leq \rho_0 \leq M_1, \tag{4.26}$$

for some positive constants M_0 and M_1. Assume, further, that β is big enough.

Then, if ε and α are small and satisfy $\frac{\varepsilon^2}{\alpha} \gg 1$, there exists a solution $(\rho, \mathbf{u}, \mathbf{H}, \psi)$ of (4.19)-(4.23) with initial and boundary conditions (4.24), (4.25). Moreover there is some $1 < r < 2$, independent of N, ε, α and δ such that

1. *ρ is nonnegative and*

$$\rho \in L^r(0, T; W^{2,r}(\Omega)) \cap L^{\beta+1}(\Omega \times (0, T)), \qquad \rho_t \in L^r(\Omega \times (0, T)); \tag{4.27}$$

2. *$\mathbf{u}, \mathbf{H} \in L^2(0, T; H_0^1(\Omega))$;*

3. *$\psi \in L^\infty(0, T; H_0^1(\Omega))$*

4. *the initial and boundary conditions are satisfied in the sense of traces.*

Furthermore, we have that

$$E_\varepsilon(t) + \varepsilon \int_0^t \int_\Omega (a\gamma\rho^{\gamma-2} + \delta\beta\rho^{\beta-2})|\nabla\rho|^2 d\mathbf{x}\, ds \le E_\varepsilon(0) + \varepsilon^{1/2} E_1, \tag{4.28}$$

for a.e. $t \in [0, T]$, *where*

$$\begin{aligned} E_\varepsilon(t) = &\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma + \frac{\delta}{\beta-1}\rho^\beta + \frac{1}{2}|\mathbf{H}|^2 \right) d\mathbf{x} \\ &+ \int_{\Omega_\mathbf{y}} \left(\frac{1}{2}|\nabla_\mathbf{y}\psi|^2 + \frac{1}{4}|\psi|^4 + \alpha g(v) h(|\psi|^2) \right) d\mathbf{y} \\ &\quad + \int_0^t \int_\Omega (\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(div\mathbf{u})^2 + \nu|\nabla\mathbf{H}|^2) d\mathbf{x} ds, \end{aligned} \tag{4.29}$$

and

$$E_1 := \varepsilon||\rho_0||_{W^{2,r}(\Omega)} + ||\rho_0||^2_{H^1(\Omega)} + E_0 + 1.$$

Let us make some remarks on the statement of this theorem. First, the largeness assumed on β is to be understood in the following sense. Theorem 4.1 holds, as will be shown later, with $r \in (1, 2)$ as long as $\beta > \max\{\frac{2r}{2-r}, \frac{2r}{r-1}\}$. Second, Theorem 4.1 does not actually assert the existence of global solutions to the regularized SW-LW interactions. It affirms that given a prefixed $T > 0$, there is a solution in the time interval $[0, T]$ satisfying (4.28) as long as $\frac{\varepsilon^2}{\alpha}$ is big enough. Remember that ε is an artificial small parameter we introduced in order to regularize the continuity equation. The reason for this hypothesis is to control uniformly in N the Jacobian of the regularized Lagrangian transformation (which may explode as $N \to \infty$). More specifically, we are going to show that (4.28) holds as long as $T \le T^N$, where $T^N = T^N(\alpha, \varepsilon)$ is defined in terms of C^N from (4.12) as

$$T^N := \frac{1}{C_N} \log\left(\frac{\varepsilon^2}{\alpha}\right) - \frac{1}{\mu}(E_0 + \varepsilon^{1/2} E_1), \tag{4.30}$$

whenever the right hand side is positive, which is the case, in particular, for α and ε small enough satisfying $\frac{\varepsilon^2}{\alpha} \gg 1$.

We intend to analyse convergence of solutions to the regularized system as $(\varepsilon, \alpha, N) \to (0, 0, \infty)$ and we do it based on the energy estimate (4.28). Thus, if we are looking for convergence to a **global** solution of the limit problem we simply have to ensure that this T^N covers any given bounded interval for big enough N and small enough ε and α. This is the case if, for instance, we take the limit $(\varepsilon, \alpha, N) \to (0, 0, \infty)$ ate any rate that satisfies

$$\left(\frac{\varepsilon^2}{\alpha}\right)^{1/C_N} \to \infty. \tag{4.31}$$

The proof of this theorem consists in a Faedo-Galerkin method, only slightly different than the one employed in the planar case. In the present situation we are going to apply Shauder's fixed point theorem in the finite-dimensional space X_n in order to solve the momentum equation, having solved all the other equations in terms of the velocity. This provides a local approximate solution of the regularized system. Then, we deduce an energy estimate, corresponding to (4.28), that allows us to extend the local approximate solutions to the time interval $[0, T^N]$. As mentioned before, our analysis is based on the work by Hu and Wang in [29] in the study of the multidimensional MHD equations and also on the work by Feireisl, et al. in [24] and the work of P.-L. Lions in [38] in the study of the Navier-Stokes equations, although we had to develop new estimates in order to include the SW-LW interactions.

The rest of this section is devoted to the proof of this theorem.

4.2.1 Approximate solutions, Faedo-Galerkin scheme

Let us now fix ε, α, δ, β and N as in the statement of Theorem 4.1. For each $n \in \mathbb{N}$, we consider the space X_n as defined before. We are going to apply Schauder's fixed point theorem in order to find a function $\mathbf{u}_n \in C(0, T; X_n)$ that satisfies equation (4.20) in an approximate way. In order to achieve this, we must first show that given a function $\mathbf{u} \in X_n$ all the other equations (4.19), (4.21), (4.22) and (4.23) can be solved in terms of it.

Let us begin with the solvability of the continuity equation in terms of the velocity. Specifically, we consider the problem

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho, & \text{on } \Omega \times (0,T) \\ \nabla\rho \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \\ \rho = \rho_0, & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{4.32}$$

Lemma 4.1. *Let $\rho_0 \in C^{2+\zeta}(\Omega)$, $\zeta > 0$ and $u \in C([0,T]; C_0^2(\overline{\Omega}))$ be given. Assume, further, that $\nabla\rho_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$.*

Then, problem (4.32) has a unique classical solution ρ such that

$$\partial_t\rho \in C([0,T]; C^{\zeta}(\overline{\Omega})), \qquad \rho \in C([0,T]; C^{2+\zeta}(\Omega)). \tag{4.33}$$

Moreover, suppose that the initial function ρ_0 is positive and let

$$\mathbf{u} \to \rho[\mathbf{u}]$$

be the solution mapping which assigns to any $\mathbf{u} \in C([0,T]; C_0^2(\Omega))$ the unique solution ρ of (4.32).

Then, this mapping takes bounded sets in the space $C([0,T]; C_0^2(\Omega))$ into bounded sets in the space

$$V := \{\partial_t \rho \in C([0,T]; C^{\varsigma}(\overline{\Omega})), \rho \in C([0,T]; C^{2+\varsigma}(\Omega))\}$$

and

$$\mathbf{u} \in C([0,T]; C_0^2(\Omega)) \to \rho[u] \in C^1([0,T] \times \overline{\Omega})$$

is continuous.

For the proof of this Lemma, we refer to [23, Proposition 7.1](cf. [24, Lemma 2.2]). Let us point out that solutions of the parabolic problem (4.32) obey the maximum principle which implies that

$$\begin{aligned}(\inf_{\mathbf{x}\in\Omega} \rho_0(\mathbf{x},0)) \exp\left(-\int_0^t ||\operatorname{div}\mathbf{u}||_{L^\infty(\Omega)} ds\right) &\le \rho(\mathbf{x},t) \\ &\le (\sup_{\mathbf{x}\in\Omega} \rho_0(\mathbf{x},0)) \exp\left(\int_0^t ||\operatorname{div}\mathbf{u}||_{L^\infty(\Omega)} ds\right),\end{aligned} \tag{4.34}$$

for all $t \in [0,T]$ and all $\mathbf{x} \in \Omega$.

We also have to consider the following problem for the magnetic field

$$\begin{cases} \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), & \text{on } \Omega \times (0,T) \\ \operatorname{div} \mathbf{H} = 0, & \text{on } \Omega \times (0,T) \\ \mathbf{H} = 0, & \text{on } \partial\Omega \\ \mathbf{H} = \mathbf{H}_0, & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{4.35}$$

Regarding this problem we have the following result as presented by Hu and Wang (see [29, Lemma 3.2]):

Lemma 4.2. *Assume that $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}))$ is given. Then, problem* (4.35) *has a* *unique solution* $\mathbf{H}$ *that satisfies*

$$\mathbf{H} \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{4.36}$$

which solves (4.35) *in the weak sense and satisfies the initial and boundary conditions in the sense of traces. Moreover, let*

$$\mathbf{u} \to \mathbf{H}[\mathbf{u}]$$

be the solution operator which assigns to $\mathbf{u} \in C([0,T];C^2(\overline{\Omega}))$ *the unique solution* $\mathbf{H}$ *of* (4.35)*. Then, this mapping maps bounded sets in* $C([0,T];C_0^2(\overline{\Omega}))$ *into bounded subsets of*

$$Y := L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

and

$$\mathbf{u} \in C([0,T];C^2(\overline{\Omega})) \to \mathbf{H} \in Y$$

is continuous.

Finally, we move on to the solvability of the nonlinear Schrödinger equation in terms of $\mathbf{u}$. It is this issue that poses a restriction on the dimension of Ω. To our knowledge, he global solvability of the nonlinear Schrödinger equation on a bounded domain of $\mathbb{R}^d$ with large initial data is an open problem for $d > 2$. In the two-dimensional case, however, we have the result by Brezis and Gallouet at hand (see [6]) whose proof we can addapt to our present situation.

Consider the following problem

$$\begin{cases} i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi + \alpha g(v)h'(|\psi|^2)\psi, & \text{on } \Omega_{\mathbf{y}} \times (0,T) \\ \psi = 0, & \text{on } \partial\Omega_{\mathbf{y}} \\ \psi = \psi_0, & \text{on } \Omega_{\mathbf{y}} \times \{t=0\}, \end{cases} \tag{4.37}$$

where, $v = v[\mathbf{u}]$ is given by

$$v(t,\mathbf{y}(t,\mathbf{x})) = \frac{1}{\rho[\mathbf{u}](t,\mathbf{x})},$$

$\rho[u]$ is as in Lemma 4.1 and $\mathbf{y}$ is the approximate Lagrangian coordinate associated to the approximate velocity field $\mathbf{u}^N$. Then, we can prove the following.

Lemma 4.3. *Assume that* $\psi_0 \in H^2(\Omega_{\mathbf{y}}) \cap H_0^1(\Omega_{\mathbf{y}})$ *and* $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}))$ *are given. Then, problem* (4.37) *has a unique solution* ψ *that satisfies*

$$\psi \in C([0,T]; H^2(\Omega_{\mathbf{y}}) \cap H_0^1(\Omega_{\mathbf{y}})) \cap C^1([0,T]; L^2(\Omega_{\mathbf{y}})). \tag{4.38}$$

Moreover, let

$$\mathbf{u} \to \psi[\mathbf{u}]$$

be the solution operator which assigns to $\mathbf{u} \in C([0,T]; C^2(\overline{\Omega}))$ *the unique solution* ψ *of* (4.37). *Then, this mapping maps bounded sets in* $C([0,T]; C_0^2(\overline{\Omega}))$ *into bounded subsets of*

$$Z := C(0,T; H_0^1(\Omega_{\mathbf{y}}) \cap L^2(\Omega))$$

and

$$\mathbf{u} \in C([0,T]; C^2(\overline{\Omega})) \to \psi \in Z$$

is continuous.

As this result is not explicitly covered by Brezis and Gallouet's one, we prove it next using an adaptation of their proof. For this we need the following two preliminary results.

The first one is due to Brezis and Gallouet and reads as

Lemma 4.4. *There is a constant* $C > 0$ *depending only on* Ω *such that*

$$||\psi||_{L^\infty(\Omega)} \leq C\Big(1 + \sqrt{\log[1 + ||\psi||_{H^2(\Omega)}]}\Big),$$

for every $\psi \in H^2(\Omega)$ *with* $||\psi||_{H^1(\Omega)} \leq 1$.

We refer to [6] for the proof. The second preliminary result is due to Segal (see [44]).

Lemma 4.5. *Assume* H *is a Hilbert Space and* $A : D(A) \subseteq H \to H$ *is am* m*-accretive linear operator. Assume* F *is a mapping from* $D(A)$ *into itself which is Lipschitz on every bounded subset of* $D(A)$.

Then, for every $\psi_0 \in D(A)$ *there exists a unique solution* ψ *of the equation*

$$\begin{cases} \frac{d\psi}{dt} + A\psi = F(\psi), \\ \psi(0) = \psi_0, \end{cases}$$

defined for $t \in [0, T_{max})$ *such that*

$$\psi \in C^1([0, T_{max}); H) \cap C([0, T_{max}); D(A)),$$

with the additional property that

$$\begin{cases} \textit{either } T_{max} = \infty, \\ \textit{or } T_{max} < \infty \textit{ and } \lim_{t \nearrow T_{max}}(||\psi|| + ||A\psi||) = \infty. \end{cases}$$

Proof of Lemma 4.3. We want to solve the equation (4.37). For this we apply Lemma 4.5 with $H = L^2(\Omega_{\mathbf{y}})$, $A(\psi) = \frac{1}{i}\Delta_{\mathbf{y}}\psi$, $D(A) = H^2(\Omega_{\mathbf{y}}) \cap H_0^1(\Omega_{\mathbf{y}})$ and

$$F(\psi) = \tfrac{1}{i}|\psi|^2\psi + \tfrac{\alpha}{i}g(v)h'(|\psi|^2)\psi.$$

It is enough to show that $||\psi||_{H^2(\Omega_{\mathbf{y}})}$ remains bounded on every bounded interval. Fix $T > 0$ and consider ψ solving (4.37) on the time interval $[0, T)$.

First, Multiplying (4.37) by $\overline{\psi}$, taking imaginary part and integrating we have

$$||\psi(t)||_{L^2(\Omega_{\mathbf{y}})} = ||\psi_0||_{L^2(\Omega_{\mathbf{y}})}.$$

Similarly, multiplying (4.37) by $\overline{\psi_t}$, taking real part and integrating we have

$$\frac{1}{2}\int_{\Omega_{\mathbf{y}}} |\nabla\psi|^2 d\mathbf{y} + \frac{1}{4}\int_{\Omega_{\mathbf{y}}} |\psi|^4 d\mathbf{y} = \frac{1}{2}\int_{\Omega_{\mathbf{y}}} |\nabla\psi_0|^2 d\mathbf{y} + \frac{1}{4}\int_{\Omega_{\mathbf{y}}} |\psi_0|^4 d\mathbf{y} + \int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g(v)h(|\psi|^2)_t d\mathbf{y}ds. \tag{4.39}$$

Now,

$$\begin{aligned} \int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g(v)h(|\psi|^2)_t d\mathbf{y}ds = &\int_{\Omega_{\mathbf{y}}} \alpha g(v)h(|\psi|^2)d\mathbf{y} - \int_{\Omega_{\mathbf{y}}} \alpha g(v_0)h(|\psi_0|^2)d\mathbf{y} \\ &- \int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g(v)_t h(|\psi|^2)d\mathbf{y}ds. \end{aligned}$$

Regarding the last term on the right hand side and using the definition of the Lagrangian transformation

$$\int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g(v)_t h(|\psi|^2)d\mathbf{y}ds = \\ \int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g'(1/\rho)h(|\psi \circ \mathbf{Y}|^2)\left(\Big(\frac{1}{\rho}\Big)_t + \mathbf{u}^N \cdot \nabla\Big(\frac{1}{\rho}\Big)\right) J_{\mathbf{y}} dx. \tag{4.40}$$

As $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}))$, we have that $|J_{\mathbf{y}}| \leq C$ and using (2.56) and Lemma 4.1 we have that the right hand side of (4.40) is bounded, that is

$$\left| \int_0^t \int_{\Omega_{\mathbf{y}}} \alpha g(v)_t h(|\psi|^2) d\mathbf{y} ds \right| \leq C.$$

This implies that

$$||\nabla \psi(t)||_{L^2(\Omega_{\mathbf{y}})} \leq C. \tag{4.41}$$

Next, let $S(t)$ be the isometry group generated by A. Then,

$$\psi(t) = S(t)\psi_0 + \tfrac{1}{i} \int_0^t S(t-s)\Big(|\psi(s)|^2\psi(s) - \alpha g(v) h'(|\psi(s)|^2)\psi(s)\Big) ds,$$

and, so

$$A\psi(t) = S(t)A\psi_0 + \tfrac{1}{i} \int_0^t S(t-s)A\left[\Big(|\psi(s)|^2\psi(s) - \alpha g(v) h'(|\psi(s)|^2)\psi(s)\Big)\right] ds.$$

Consequently,

$$\begin{aligned} ||A\psi(t)||_{L^2(\Omega_{\mathbf{y}})} \leq & ||A\psi_0||_{L^2(\Omega_{\mathbf{y}})} + \int_0^t ||A[|\psi(s)|^2\psi(s)]||_{L^2(\Omega_{\mathbf{y}})} ds \\ & + \alpha \int_0^t ||A\Big[g(v(s))h(|\psi(s)|^2)\psi(s)\Big]||_{L^2(\Omega_{\mathbf{y}})} ds. \end{aligned}$$

Using (4.41), Lemma 4.4 can be used to show that

$$\int_0^t ||A[|\psi(s)|^2\psi(s)]||_{L^2(\Omega_{\mathbf{y}})} ds \leq C \int_0^t ||\psi(s)||_{H^2(\Omega_{\mathbf{y}})} \Big(1 + \log[1 + ||\psi(s)||_{H^2(\Omega_{\mathbf{y}})}]\Big) ds. \tag{4.42}$$

Indeed, observe that

$$|D^2(|\psi|\psi)| \leq C(|\psi|^2|D^2\psi| + |\psi|\, |\nabla\psi|^2),$$

which implies

$$||\, |\psi|^2\psi||_{H^2(\Omega_{\mathbf{y}})} \leq C||\psi||^2_{L^\infty(\Omega_{\mathbf{y}})} ||\psi||_{H^2(\Omega_{\mathbf{y}})} + C||\psi||_{L^\infty(\Omega_{\mathbf{y}})} ||\psi||^2_{W^{1,4}(\Omega_{\mathbf{y}})}.$$

But, Gagliardo-Nirenberg Inequality implies (recall that $\Omega \subseteq \mathbb{R}^2$)

$$||\psi||_{W^{1,4}(\Omega_{\mathbf{y}})} \leq C||\psi||^{1/2}_{L^\infty(\Omega_{\mathbf{y}})} ||\psi||^{1/2}_{H^2(\Omega_{\mathbf{y}})}.$$

These two inequalities combined together with Lemma 4.4 imply (4.42).

A similar argument shows that

$$\int_0^t ||A\Big[g(v(s))h(|\psi(s)|^2)\psi(s)\Big]||_{L^2(\Omega_{\mathbf{y}})}ds$$
$$\leq C + C\int_0^t ||\psi(s)||_{H^2(\Omega_{\mathbf{y}})}\Big(1+\log[1+||\psi(s)||_{H^2(\Omega_{\mathbf{y}})}]\Big)ds.$$

Here we have used (4.41) and Lemma 4.1.

Thus we conclude that

$$||\psi(t)||_{H^2(\Omega_{\mathbf{y}})} \leq C + C\int_0^t ||\psi(s)||_{H^2(\Omega_{\mathbf{y}})}\Big(1+\log[1+||\psi(s)||_{H^2(\Omega_{\mathbf{y}})}]\Big)ds. \tag{4.43}$$

Denoting $G(t)$ the right hand side of this inequality we have that

$$G'(t) \leq CG(t)(1+\log[1+G(t)]),$$

which, implies that

$$\frac{d}{dt}\log\Big[1+\log[1+G(t)]\Big] \leq C$$

And hence we arrive at an estimate of the form

$$||\psi(t)||_{H^2(\Omega_{\mathbf{y}})} \leq e^{b_1 e^{b_2 t}},$$

for some constants b_1 and b_2 and every $t \in [0,T)$. In particular

$$||\psi(t)||_{H^2(\Omega_{\mathbf{y}})} \leq e^{b_1 e^{b_2 T}}, \text{ for every } t \in [0,T).$$

As this holds for every $T > 0$ we conclude that $T_{max} = \infty$.

In order to conclude the proof we have to show the stated continuity of the map $\mathbf{u} \to \psi[\mathbf{u}]$. Let $\{\mathbf{u}_k\}_k$ be a sequence in $C([0,T];C^2(\overline{\Omega}))$ such that $\mathbf{u}_k \to \mathbf{u}_\infty \in C([0,T];C^2(\overline{\Omega}))$, and let $v_k = v[\mathbf{u}_k]$, $v_\infty = v[\mathbf{u}_\infty]$, $\psi_k = \psi[\mathbf{u}_k]$ and $\psi_\infty = \psi[\mathbf{u}_\infty]$. In light of Lemma 4.1 and by the smoothness of the Lagrangian transformation we have that $v_k \to v_\infty$ in $C^1(\overline{\Omega}\times[0,T])$. Next, by Aubin-Lions lemma (Lemma 3.16) we have that there is a subsequence $\{\psi_{k_j}\}_j$ that converges in $C([0,T];H_0^1(\Omega))$ to a solution ψ of the limit equation (4.37) with $v = v_\infty$. By uniqueness we have that $\psi = \psi[\mathbf{u}_\infty]$ and also that the whole sequence $\{\psi_k\}_k$ converges to ψ in $C([0,T];H_0^1(\Omega))$, thus concluding the proof. $\square$

Having these results we can apply the Faedo-Galerkin method in order to find solutions to the regularized system. First, for each $n \in \mathbb{N}$, we are going to look for a function u_n that satisfies (4.20) in an approximate way. Specifically, we demand that u_n satisfies

$$\begin{aligned}
&\int_\Omega \rho_n \mathbf{u}_n \cdot \eta d\mathbf{x} - \int_\Omega \mathbf{m}_0 \cdot \eta d\mathbf{x} \\
&\quad + \int_0^t \int_\Omega \Big(\mathrm{div}(\rho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla(a\rho_n^\gamma + \delta\rho_n^\beta) + \varepsilon \nabla \mathbf{u}_n \cdot \nabla \rho_n \Big) \cdot \eta d\mathbf{x}\, ds \\
&\qquad = \int_0^t \int_\Omega \Big(\nabla(\alpha \frac{J_\mathbf{y}}{\rho_n} g'(1/\rho_n) h(|\psi_n \circ \mathbf{Y}|^2)) + (\nabla \times \mathbf{H}_n) \times \mathbf{H}_n \\
&\qquad\qquad + \mu\Delta\mathbf{u}_n + (\lambda + \mu)\nabla(\mathrm{div}\mathbf{u}_n) \Big) \cdot \eta d\mathbf{x}\, ds, \qquad (4.44)
\end{aligned}$$

for any $t \in [0, T]$ and any $\eta \in X_n$, where $\rho_n = \rho[\mathbf{u}_n]$, $\mathbf{H}_n = \mathbf{H}[\mathbf{u}_n]$, $\psi_n = \psi[\mathbf{u}_n]$ and $\mathbf{Y}$ is Lagrangian transformation associated to the velocity field $u_n^N = P_N u_n$, with Jacobian $J_\mathbf{y}$. This formulation may be interpreted as a projection of equation (4.20) onto the finite dimensional space X_n.

Let us rewrite this integral equation in a more suitable way. Given some function $\rho \in L^1(\Omega)$, consider the operator $\mathcal{M}[\rho] : X_n \to X_n^*$, where X_n^* is the dual space of X_n, given by

$$\langle \mathcal{M}[\rho]\mathbf{v}, \mathbf{w} \rangle := \int_\Omega \rho \mathbf{v} \cdot \mathbf{w}.$$

Then, the operator $\mathcal{M}$ is invertible provided that ρ is strictly positive on Ω and the map $\rho \to \mathcal{M}^{-1}[\rho]$, mapping $L^1(\Omega)$ into $\mathcal{L}(X_n^*; X_n)$, satisfies

$$||\mathcal{M}[\rho]^{-1}||_{\mathcal{L}(X_n^*;X_n)} \leq \frac{1}{\inf_\Omega \rho}. \qquad (4.45)$$

Moreover, the identity

$$\mathcal{M}[\rho^1]^{-1} - \mathcal{M}[\rho^2]^{-1} = \mathcal{M}[\rho^2]^{-1}\Big(\mathcal{M}[\rho^2] - \mathcal{M}[\rho^1]\Big)\mathcal{M}[\rho^1]^{-1},$$

can be used to obtain

$$||\mathcal{M}[\rho^1]^{-1} - \mathcal{M}[\rho^2]^{-1}||_{\mathcal{L}(X_n^*;X_n)} \leq c(n, \underline{\rho})||\rho^1 - \rho^2||_{L^1(\Omega)}, \qquad (4.46)$$

for any ρ^1 and ρ^2 such that

$$\inf_\Omega \rho^1, \inf_\Omega \rho^2 \geq \underline{\rho}.$$

In connection with (4.44) we also define the operator $\mathcal{N} : X_n \to X_n^*$ given by

$$\begin{aligned}\langle \mathcal{N}[\mathbf{u}], \eta \rangle = &- \int_\Omega \Big(\mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(a\rho^\gamma + \delta \rho^\beta) + \varepsilon \nabla \mathbf{u} \cdot \nabla \rho \Big) \cdot \eta d\mathbf{x} \\ &+ \int_\Omega \Big(\nabla(\alpha \frac{J_{\mathbf{y}}}{\rho} g'(1/\rho_n) h(|\psi|^2)) + (\nabla \times \mathbf{H}) \times \mathbf{H} \\ &\qquad + \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla(\mathrm{div}\mathbf{u}) \Big) \cdot \eta d\mathbf{x},\end{aligned}$$

with $\rho = \rho[\mathbf{u}]$, $\mathbf{H} = \mathbf{H}[\mathbf{u}]$ and $\psi = \psi[\mathbf{u}]$.

With this notation, identity (4.44) can be rewritten as

$$\mathbf{u}_n(t) = \mathcal{M}^{-1}[\rho_n(t)] \left(\mathbf{m}_0^* + \int_0^t \mathcal{N}[\mathbf{u}_n(s)] ds \right).$$

This means that we are looking for a fixed point of the application $\mathcal{T} : C([0,T]; X_n) \to C([0,T]; X_n)$ given by

$$\mathcal{T}[\mathbf{u}](t) = \mathcal{M}[\rho[\mathbf{u}](t)]^{-1} \left(\mathbf{m}_0^* + \int_0^t \mathcal{N}[\mathbf{u}(s)] ds \right).$$

Using Lemmas 4.1, 4.2 and 4.3, as well as (4.45) and (4.46) and Arzelà-Ascoli theorem it can be shown that $\mathcal{T}$ maps bounded sets in $C([0,T]; X_n)$ into precompact sets in $C([0,T]; X_n)$.

Moreover, define $\mathbf{u}_0 \in X_n$ as being the only element in X_n that satisfies

$$\int_\Omega \rho_0 \mathbf{u}_0 \cdot \eta d\mathbf{x} = \int_\Omega \mathbf{m}_0 \cdot \eta d\mathbf{x}, \qquad \text{for all } \eta \in X_n.$$

Consider a ball $\mathcal{B} := \{\mathbf{v} \in C([0,T]; X_n) : \sup_{t \in [0,T]} ||\mathbf{v}(t) - \mathbf{u}_0||_{X_n} \leq 1\}$. Then, $\mathcal{T}$ maps the ball $\mathcal{B}$ into itself, provided $T = T(n)$ is small enough. Consequently, Schauder's fixed point theorem guarantees the existence of at least one fixed point $\mathbf{u}_n$, $\mathbf{u}_n = \mathcal{T}[\mathbf{u}_n]$ which provides a solution to (4.44).

Now, we want to find a solution to the regularized system as a limit of the sequence $\mathbf{u}_n$. However, the approximate velocity field $\mathbf{u}_n$ is defined only on the time interval $[0, T(n)]$. Accordingly, we have to guarantee that this solution can be extended to a uniform over n time interval $[0, T^*]$. In order to achieve this, we deduce next some a priori estimates on the fixed point $\mathbf{u}_n$ we found above that allow us to iterate the fixed point argument a finite number of times until we reach the whole time interval $[0, T^*]$.

In the case of the MHD system and in the case of the Navier Stokes system, the conservation of energy provides good enough global a priori estimates that guarantee boundedness of the fixed point globally in time. In our present situation, however, the short wave-long wave interaction turns the estimate more difficult as the energy of the system is not well balanced. As a consequence we do not obtain a global a priori estimate. Fortunately, we are able to bound from below the maximal time during which the estimates hold by some T^N independent of n that satisfies the properties stated in Theorem 4.1.

The a priori estimates are based on the usual energy estimates for the MHD equations, but rely on a bootstrap argument in order to accommodate the unbalance in the energy caused by the short wave-long wave interactions coupling terms.

For convenience, we define $E_n(t)$ as in (4.29) with $(\rho, \mathbf{u}, \mathbf{H}, \psi)$ replaced by $(\rho_n, \mathbf{u}_n, \mathbf{H}_n, \psi_n)$. That is

$$\begin{aligned} E_n(t) = &\int_\Omega \left(\frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\rho_n^\gamma + \frac{\delta}{\beta-1}\rho_n^\beta + \frac{1}{2}|\mathbf{H}_n|^2\right) d\mathbf{x} \\ &+ \int_{\Omega_\mathbf{y}} \left(\frac{1}{2}|\nabla_\mathbf{y}\psi_n|^2 + \frac{1}{4}|\psi_n|^4 + \alpha g(v_n)h(|\psi_n|^2)\right) d\mathbf{y} \\ &\quad + \int_0^t \int_\Omega (\mu|\nabla \mathbf{u}_n|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u}_n)^2 + \nu|\nabla \mathbf{H}_n|^2) d\mathbf{x}ds \end{aligned} \tag{4.47}$$

In the notation of Theorem 4.1 we have the following estimate.

Lemma 4.6. *Let T^N be given by (4.30) and take $r \in (0,1)$. Assume that $\beta > \max\{2r/(2-r), 2r/(1-r)\}$ and that ε and α are small and satisfy $T^N > 0$. Then, for all $t \le T^N$ we have*

$$E_n(t) + \varepsilon \int_0^t \int_\Omega (a\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2 d\mathbf{x}ds \le E(0) + \varepsilon^{1/2}R. \tag{4.48}$$

Also,

$$||\varepsilon^{1/2}\nabla\rho_n||_{L^2(\Omega\times(0,T))} + ||\varepsilon^2\rho_{nt}||_{L^r(\Omega\times(0,T))} + ||\varepsilon^3\Delta\rho_n||_{L^r(\Omega\times(0,T))} \le C \tag{4.49}$$

where C is a universal constant independent of ε, α, n and N.

Proof. First, we find an energy identity in a similar way as when deducing (2.32).

Taking $\eta = \mathbf{u}_n$ in (4.44) and using equations (4.32), (4.35) we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\rho_n^\gamma + \frac{\delta}{\beta-1}\rho_n^\beta + \frac{1}{2}|\mathbf{H}_n|^2\right) d\mathbf{x} \\
+ \int_\Omega (\mu|\nabla \mathbf{u}_n|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u}_n)^2 + \nu|\nabla \mathbf{H}_n|^2) d\mathbf{x} \\
+ \varepsilon \int_\Omega (a\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2 d\mathbf{x} \\
+ \int_\Omega \alpha \frac{J_{\mathbf{y}}}{\rho_n} g'(1/\rho_n) h(|\psi_n \circ \mathbf{Y}|^2) \operatorname{div}\mathbf{u}_n d\mathbf{x} = 0.
\end{aligned} \tag{4.50}$$

As ρ_n is a solution of equation (4.32) with $\mathbf{u} = \mathbf{u}_n$ we have that

$$\frac{\operatorname{div}\mathbf{u}_n}{\rho_n} = \left(\frac{1}{\rho_n}\right)_t + \mathbf{u}_n \cdot \nabla\left(\frac{1}{\rho_n}\right) + \varepsilon\frac{\Delta\rho_n}{\rho_n^2}$$

Now, from the coordinate change and the definition of $v_n = v_n(\mathbf{y}, t)$ we have $v_{nt} = (1/\rho_n)_t + \mathbf{u}_n^N \cdot \nabla(1/\rho_n)$.

Thus,

$$\begin{aligned}
\int_\Omega \alpha \frac{J_{\mathbf{y}}}{\rho_n} g'(1/\rho_n) h(|\psi_n \circ \mathbf{Y}|^2) \operatorname{div}\mathbf{u}_n d\mathbf{x} = \int_{\Omega_{\mathbf{y}}} \alpha g(v_n)_t h(|\psi_n|^2) d\mathbf{y} \\
+ \int_\Omega \alpha g'(1/\rho_n) h(|\psi_n \circ \mathbf{Y}|^2) J_{\mathbf{y}} \left(\varepsilon\frac{\Delta\rho_n}{\rho_n^2} + (\mathbf{u}_n^N - \mathbf{u}_n)\cdot\frac{\nabla\rho_n}{\rho_n^2}\right) d\mathbf{x}
\end{aligned}$$

Now, using equation (4.37) we have that

$$\int_{\Omega_{\mathbf{y}}} \alpha g(v_n)_t h(|\psi_n|^2) d\mathbf{y} = \frac{d}{dt}\int_{\Omega_{\mathbf{y}}} \left(\frac{1}{2}|\nabla_{\mathbf{y}}\psi_n|^2 + \frac{1}{4}|\psi_n|^4 + \alpha g(v_n) h(|\psi_n|^2)\right) d\mathbf{y}.$$

Gathering this information in (4.50) we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho_n\mathbf{u}_n + \frac{a}{\gamma-1}\rho_n^\gamma + \frac{\delta}{\beta-1}\rho_n^\beta + \frac{1}{2}|\mathbf{H}_n|^2\right) d\mathbf{x} \\
+ \frac{d}{dt}\int_{\Omega_{\mathbf{y}}} \left(\frac{1}{2}|\nabla_{\mathbf{y}}\psi|^2 + \frac{1}{4}|\psi|^4 + \alpha g(v) h(|\psi|^2)\right) d\mathbf{y} \\
+ \int_\Omega (\mu|\nabla \mathbf{u}_n|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u}_n)^2 + \nu|\nabla \mathbf{H}_n|^2) d\mathbf{x} \\
+ \varepsilon \int_\Omega (a\gamma\rho_n^{\gamma-2} + \delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2 d\mathbf{x} \\
= \int_\Omega \alpha g'(1/\rho_n) h(|\psi_n \circ \mathbf{Y}|^2) J_{\mathbf{y}} \left(\varepsilon\frac{\Delta\rho_n}{\rho_n^2} + (\mathbf{u}_n^N - \mathbf{u}_n)\cdot\frac{\nabla\rho_n}{\rho_n^2}\right) d\mathbf{x}.
\end{aligned}$$

In order to estimate the right hand side of this identity we use a bootstrap argument as follows. First, recalling (4.15), we have that

$$|J_{\mathbf{y}}(t)| \leq \exp\left[C_N\left(t+\int_0^t ||u_n(s)||^2_{H^1_0(\Omega)}ds\right)\right]. \tag{4.51}$$

Next, we assume that

$$\mu\int_0^t ||u_n(s)||^2_{H^1_0(\Omega)}ds \leq E(0)+\varepsilon^{1/2}R \tag{4.52}$$

for all $t \leq T^N$. This is certainly the case for t small enough. Accordingly, the following calculations hold as long as (4.52) is satisfied.

With this in mind, using (2.56) and Poincaré's inequality, we have that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho_n\mathbf{u}_n+\frac{a}{\gamma-1}\rho_n^\gamma+\frac{\delta}{\beta-1}\rho_n^\beta+\frac{1}{2}|\mathbf{H}_n|^2\right)d\mathbf{x}\\
&\quad+\frac{d}{dt}\int_{\Omega_{\mathbf{y}}}\left(\frac{1}{2}|\nabla_{\mathbf{y}}\psi|^2+\frac{1}{4}|\psi|^4+\alpha g(v)h(|\psi|^2)\right)d\mathbf{y}\\
&\qquad+\int_\Omega(\mu|\nabla\mathbf{u}_n|^2+(\lambda+\mu)(\operatorname{div}\mathbf{u}_n)^2+\nu|\nabla\mathbf{H}_n|^2)d\mathbf{x}\\
&\qquad\quad+\varepsilon\int_\Omega(a\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2d\mathbf{x}\\
&\leq \alpha Ce^{C_N(T^N+\mu^{-1}(E(0)+\varepsilon^{1/2}R))}\int_\Omega\left(\varepsilon|\Delta\rho_n|+\mu|\nabla\mathbf{u}_n|^2+a\gamma\rho_n^{\gamma-2}|\nabla\rho_n|^2\right)d\mathbf{x}.
\end{aligned}$$

Taking (4.30) into consideration we see that

$$\begin{aligned}
&\alpha Ce^{C_N(T^N+\mu^{-1}(E(0)+\varepsilon^{1/2}R))}\int_\Omega\left(\varepsilon|\Delta\rho_n|+\mu|\nabla\mathbf{u}_n|^2+a\gamma\rho_n^{\gamma-2}|\nabla\rho_n|^2\right)d\mathbf{x}\\
&\qquad\leq C\varepsilon^3\int_\Omega|\Delta\rho_n|d\mathbf{x}+C\varepsilon^2\int_\Omega\mu|\nabla\mathbf{u}_n|^2d\mathbf{x}+C\varepsilon^2\int_\Omega a\gamma\rho_n^{\gamma-2}|\nabla\rho_n|^2d\mathbf{x},
\end{aligned}$$

and thus, if $\varepsilon \leq \min\{(2C)^{-1},(2C)^{-1/2}\}$ we have that

$$\frac{d}{dt}E_n(t)+\varepsilon\int_\Omega(a\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2})|\nabla\rho_n|^2d\mathbf{x} \leq C\varepsilon^3\int_\Omega|\Delta\rho_n|d\mathbf{x}, \tag{4.53}$$

for all $t \leq T^N$, and some constant $C>0$ independent of α, ε, n and N. In particular given $r>1$ we have that

$$\begin{aligned}
&||\sqrt{\rho}\mathbf{u}_n||^2_{L^\infty(0,T;L^2(\Omega))}+||\rho_n||^\beta_{L^\infty(0,T;L^\beta(\Omega))}+||\mathbf{u}_n||^2_{L^2(0,T;H^1_0(\Omega))}\\
&\qquad\leq E(0)+C(r)||\varepsilon^3\Delta\rho_n||_{L^r(\Omega\times(0,T))}.
\end{aligned} \tag{4.54}$$

Regarding the right hand side of this inequality, we are going to use $L^p - L^q$ estimates on the parabolic equation (4.32) in order to bound appropriately the $L^r(\Omega \times (0,T))$-norm of $\Delta\rho_n$ (for any fixed $T \leq T^N$). Said $L^p - L^q$ estimates read

$$\begin{aligned}||\rho_t||_{L^p(0,T;L^q(\Omega))} + ||\varepsilon\Delta\rho||_{L^p(0,T;L^q(\Omega))} \\ \leq c(p,q)(||\rho_0||_{W^{2,q}(\Omega)} + ||\mathrm{div}(\rho\mathbf{u})||_{L^p(0,T;L^q(\Omega))}).\end{aligned} \tag{4.55}$$

for any $1 < p, q < \infty$. Taking $p = q := r$ in (4.55) and applying it to ρ_n we have

$$\begin{aligned}&||\varepsilon\Delta\rho_n||_{L^r(\Omega\times(0,T))} \\ &\quad \leq c(r)(||\rho_0||_{W^{2,r}(\Omega)} + ||\mathrm{div}(\rho_n\mathbf{u}_n)||_{L^r(\Omega\times(0,T))}) \\ &\quad \leq c(r)(||\rho_0||_{W^{2,r}(\Omega)} + ||\mathbf{u}_n \cdot \nabla\rho_n||_{L^r(\Omega\times(0,T))} + ||\rho_n\mathrm{div}\mathbf{u}_n||_{L^r(\Omega\times(0,T))})\end{aligned} \tag{4.56}$$

On the one hand,

$$||\rho_n\mathrm{div}\mathbf{u}_n||_{L^{2\beta/(\beta+2)}(\Omega)} \leq ||\rho_n||_{L^\beta(\Omega)}||\mathbf{u}_n||_{H_0^1(\Omega)},$$

and therefore

$$||\rho_n\mathrm{div}\mathbf{u}_n||_{L^2(0,T;L^{2\beta/(\beta+2)}(\Omega))} \leq ||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\mathbf{u}_n||_{L^2(0,t;H_0^1(\Omega))}, \tag{4.57}$$

On the other hand, we need to estimate $||\nabla\rho_n \cdot \mathbf{u}_n||_{L^r(\Omega\times(0,T))}$, and for this we need a good estimate on $\nabla\rho_n$. Such an estimate is provided by the following $L^p - L^q$ estimate on equation (4.32), analogue to (4.55)

$$||\varepsilon\nabla\rho||_{L^p(0,T;L^q(\Omega))} \leq c(p,q)(||\rho_0||_{W^{1,q}(\Omega)} + ||\mathrm{div}(\rho\mathbf{u})||_{L^p(0,T;W^{-1,q}(\Omega))}). \tag{4.58}$$

At this point we choose $q = 2$ and leave p to be chosen conveniently. In connection with (4.58) we have that

$$||\varepsilon\nabla\rho_n||_{L^p(0,T;L^2(\Omega))} \leq c(p)(||\rho_0||_{H^1(\Omega)} + ||\rho_n\mathbf{u}_n||_{L^p(0,T;L^2(\Omega))}). \tag{4.59}$$

By Sobolev's embedding for any $p' \in [1,\infty)$ we have, since $\Omega \subseteq \mathbb{R}^2$, that

$$||\mathbf{u}_n||_{L^{p'}(\Omega)} \leq c(p')||\mathbf{u}_n||_{H_0^1(\Omega)}.$$

This implies that

$$||\rho_n \mathbf{u}_n||_{L^2(0,T;L^{p'}(\Omega))} \leq c(p')||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\mathbf{u}_n||_{L^2(0,T;H_0^1(\Omega))}, \tag{4.60}$$

for any $p' < \beta$. Furthermore, we have that

$$||\rho_n \mathbf{u}_n||_{L^\infty(0,T;L^{2\beta/(\beta+1)}(\Omega))} \leq ||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\sqrt{\rho_n}\mathbf{u}_n||_{L^\infty(0,T;L^2(\Omega))}.$$

Now, for $2 < p' < \beta$ we have

$$||\rho_n \mathbf{u}_n||_{L^2(\Omega)} \leq ||\rho_n \mathbf{u}_n||^{1-\sigma}_{L^{2\beta/(\beta+1)}(\Omega)}||\rho_n \mathbf{u}_n||^{\sigma}_{L^{p'}(\Omega)} \tag{4.61}$$

where, $\frac{1}{2} = (1-\sigma)\frac{\beta+1}{2\beta} + \sigma\frac{1}{p'}$ and $\sigma \in (0,1)$. Consequently, taking $p = \frac{2}{\sigma} > 2$ we obtain

$$\begin{aligned} ||\rho \mathbf{u}||_{L^p(0,T;L^2(\Omega))} &\leq ||\rho_n \mathbf{u}_n||^{1-\sigma}_{L^\infty(0,T;L^{2\beta/(\beta+1)}(\Omega))}||\rho_n \mathbf{u}_n||^{\sigma}_{L^2(0,T;L^{p'}(\Omega))} \\ &\leq ||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\sqrt{\rho_n}\mathbf{u}_n||^{1-\sigma}_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}_n||^{\sigma}_{L^2(0,T;H_0^1(\Omega))}. \end{aligned}$$

In connection with (4.59) we have that

$$\begin{aligned} &||\varepsilon\nabla\rho||_{L^p(0,T;L^2(\Omega))} \\ &\quad \leq c(p)(||\rho_0||_{H^1(\Omega)} + ||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\sqrt{\rho_n}\mathbf{u}_n||^{1-\sigma}_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}_n||^{\sigma}_{L^2(0,T;H_0^1(\Omega))}). \end{aligned}$$

Finally, we see that we can choose p' so that $r = p/2$ and we have

$$\begin{aligned} ||\varepsilon\nabla\rho_n\cdot\mathbf{u}_n||^r_{L^r(\Omega\times(0,T))} &\leq \int_0^t ||\varepsilon\rho_n||^r_{L^2(\Omega)}||\mathbf{u}_n||_{L^{2r/(2+r)}(\Omega)}ds \\ &\leq C\int_0^t ||\varepsilon\rho_n||^r_{L^2(\Omega)}||\mathbf{u}_n||^r_{H_0^1(\Omega)}ds \\ &\leq C\left(\int_0^t ||\varepsilon\rho_n||^p_{L^2(\Omega)}ds\right)^{r/p}\left(\int_0^t ||\mathbf{u}_n||^2_{H_0^1(\Omega)}ds\right)^{1/2}. \end{aligned}$$

In this way we have

$$\begin{aligned} &||\varepsilon\nabla\rho_n\cdot\mathbf{u}_n||_{L^r(\Omega\times(0,T))} \leq C||\varepsilon\nabla\rho_n||_{L^p(0,T;L^2(\Omega))}||\mathbf{u}_n||^{1/r}_{L^2(0,T;H_0^1(\Omega))} \\ &\leq C(||\rho_0||_{H^1(\Omega)} + ||\rho_n||_{L^\infty(0,T;L^\beta(\Omega))}||\sqrt{\rho_n}\mathbf{u}_n||^{1-\sigma}_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}_n||^{\sigma}_{L^2(0,T;H_0^1(\Omega))})\times \\ &\qquad\qquad \times ||\mathbf{u}_n||^{1/r}_{L^2(0,T;H_0^1(\Omega))}. \end{aligned} \tag{4.62}$$

Then, for β large enough so that $\frac{2\beta}{2+\beta} > r$ (which is equivalent to $\beta > \frac{2r}{2-r}$) we have that

$$||\rho_n \mathrm{div}\mathbf{u}_n||_{L^r(\Omega\times(0,T))} \leq C||\rho_n \mathrm{div}\mathbf{u}_n||_{L^2(0,T;L^{2\beta/(2+\beta)})}. \tag{4.63}$$

Putting this together with (4.54), (4.56), (4.57) and (4.62) we have that

$$\begin{aligned}||\varepsilon^3\Delta\rho_n||_{L^r(\Omega\times(0,T))} &\leq C\varepsilon^2||\rho_0||_{W^{2,r}(\Omega)} + C\varepsilon||\rho_0||^2_{H^1(\Omega)} \\ &\quad + C\varepsilon(E(0) + ||\varepsilon^3\Delta\rho_n||_{L^r(\Omega\times(0,T))})^{\frac{1}{\beta}+\frac{1}{2}+\frac{1}{2r}},\end{aligned}$$

and consequently, if β is large enough so that $\frac{1}{\beta}+\frac{1}{2}+\frac{1}{2r} \leq 1$ (in other words if $\beta \geq 2r/(1-r)$) and ε is small we have

$$||\varepsilon^3\Delta\rho_n||_{L^r(\Omega\times(0,T))} \leq C\varepsilon(\varepsilon||\rho_0||_{W^{2,r}(\Omega)} + ||\rho_0||^2_{H^1(\Omega)} + E(0) + 1).$$

In order to conclude, we observe that this last inequality together with (4.53) and (4.54) reconfirms our bootstrap assumption (4.52), and implies (4.48). $\square$

4.2.2 Convergence of the Faedo-Galerkin approximations

The uniform estimates from Lemma 4.6 permit us to iterate the fixed point argument a finite number of times to extend the local approximate solutions to the interval $[0, T]$ (provided that $T \leq T^N$). The next step in the proof of Theorem 4.1 consists in passing to the limit as $n \to \infty$. We point out that the convergence in the terms concerning ρ_n and $\mathbf{u}_n$ can be justified similarly as in [23, Section 7.3.6] and the terms involving $\mathbf{H}_n$ may be treated as in [29, Section 4]. Regarding the terms involving ψ_n a direct application of Aubin-Lions Lemma (Lemma 3.16) yields the desired result. The details are as follows.

Let N, ε, α and δ be fixed, $0 < T < T^N$ and $\{(\rho_n, \mathbf{u}_n, \mathbf{H}_n, \psi_n)\}_{n=1}^{\infty}$ be the approximate solution to the regularized system, defined in the time interval $[0, T]$, given by the Faedo-Galerkin method described above.

First, as ρ_n satisfies (4.32), we have that

$$||\nabla\rho_n||_{L^2(\Omega\times(0,T))} \leq C(\varepsilon),$$

for some constant that depends on ε, but is independent of n. This can be easily deduced by multiplying (4.32) by ρ_n and integrating by parts. Using (4.49) and (4.48), Aubin-Lions Lemma 3.16 implies that ρ_n has a subsequence (not relabelled) such that

$$\rho_n \to \rho \text{ in } L^\beta(\Omega \times (0,T)). \tag{4.64}$$

Furthermore, by (4.48) we can assume that

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{4.65}$$

Next, we see that $\mathbf{H}_n$ satisfies the following equation, equivalent to (4.35),

$$\begin{cases} \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = \nu \Delta \mathbf{H}, & \text{on } \Omega \times (0,T) \\ \text{div } \mathbf{H} = 0, & \text{on } \Omega \times (0,T) \\ H = 0, & \text{on } \partial\Omega \\ H = H_0, & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{4.66}$$

Consequently, by (4.48) we can also use Aubin-Lions Lemma in order to conclude that (selecting a subsequence if necessary)

$$\mathbf{H}_n \to \mathbf{H} \tag{4.67}$$

strongly in $L^2(\Omega \times (0,T))$ and weakly(-*) in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Furthermore, $\mathbf{H}$ satisfies

$$\text{div}\mathbf{H} = 0.$$

Now, from (4.48) and using the embedding we see that $\rho_n \mathbf{u}_n$ is uniformly bounded in $L^\infty(0,T;L^{m_\infty}(\Omega))$, where $m_\infty = \frac{2\gamma}{\gamma+1}$. Indeed,

$$\int_\Omega |\rho_n \mathbf{u}_n|^{m_\infty} d\mathbf{x} \leq \left(\int_\Omega \rho_n |\mathbf{u}_n|^2 d\mathbf{x} \right)^{1/2} \left(\int_\Omega \rho_n^\gamma d\mathbf{x} \right)^{1/\gamma} \leq C.$$

Thus, as the convergence in (4.64) is strong we may assume that

$$\rho_n \mathbf{u}_n \to \rho \mathbf{u} \text{ weakly-* in } L^\infty(0,T;L^{m_\infty}(\Omega)). \tag{4.68}$$

By the same token, we have that

$$(\nabla \times \mathbf{H}_n) \times \mathbf{H}_n \to (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{4.69}$$

weakly in $L^1(\Omega \times (0,T))$, and

$$\nabla(\mathbf{u}_n \times \mathbf{H}_n) \to \nabla(\mathbf{u} \times \mathbf{H}), \tag{4.70}$$

in the sense of distributions.

Next, in view of (4.37) Aubin-Lions lemma also yields

$$\psi_n \to \psi \tag{4.71}$$

strongly in $C(0,T;L^2(\Omega))$ and weakly-* in $L^\infty(0,T;H_0^1(\Omega))$.

Let us state (without proof) the following result, which is a consequence of the Ascoli-Arzelà theorem (see [23, Corollary 2.1]).

Lemma 4.7. *Let $\overline{O} \subseteq \mathbb{R}^M$ be compact and let X be a separable Banach space. Assume that $v_n : \overline{O} \to X^*$, $n = 1, 2, ...$ is a sequence of measurable functions such that*

$$ess \sup_{y \in \overline{O}} ||v_n(y)||_{X^*} \leq C \qquad \textit{uniformly in } n = 1, 2, ...$$

Moreover, let the family of (real) functions

$$\langle v_n, \Phi \rangle : y \to \langle v_n(y), \Phi \rangle, \qquad y \in \overline{O}, n = 1, 2...$$

be equi-continuous for any fixed Φ belonging to a dense subset in the space X.

Then, $v_n \in C(\overline{O}; X^*_{weak})$ for any $n = 1, 2, ...$ and there exist $v \in C(\overline{O}; X^*_{weak})$ such that

$$v_n \to v \textit{ in } C(\overline{O}; X^*_{weak}) \textit{ as } n \to \infty,$$

passing to a subsequence as the case may be.

In view of (4.44) and using (4.48) we see that the functions

$$t \to \int_\Omega \rho_n \mathbf{u}_n \eta^j d\mathbf{x}$$

form a precompact system in $C([0,T])$ for any fixed j. This implies, by Lemma 4.7 that in fact

$$\rho_n \mathbf{u}_n \to \rho \mathbf{u} \text{ in } C([0,T]; L^{2\gamma/(\gamma+1)}_{weak}(\Omega)). \tag{4.72}$$

A similar argument shows that the mapping

$$t \to \int_\Omega \mathbf{H}\varphi d\mathbf{x}$$

is continuous for any test function φ.

Now, as $\gamma > 1$, $L^{2\gamma/(\gamma+1)}_{weak}(\Omega)$ is compactly embedded into $H^{-1}(\Omega)$ and, consequently,

$$\rho_n \mathbf{u}_n \otimes \mathbf{u}_n \to \rho \mathbf{u} \otimes \mathbf{u} \tag{4.73}$$

weakly in $L^2(0,T;L^{c_2}(\Omega))$, where $c_2 = 2\gamma/(\gamma+1) >$.

Next, as ρ_n and ρ are strong solutions of (4.32), we have that

$$||\rho_n(t)||^2_{L^2(\Omega)} + 2\varepsilon \int_0^t ||\nabla \rho_n||^2_{L^2(\Omega)} ds = -\int_0^t \int_\Omega \rho_n^2 \mathrm{div} \mathbf{u}_n d\mathbf{x} ds + ||\rho_0||^2_{L^2(\Omega)},$$

and

$$||\rho(t)||^2_{L^2(\Omega)} + 2\varepsilon \int_0^t ||\nabla \rho||^2_{L^2(\Omega)} ds = -\int_0^t \int_\Omega \rho^2 \mathrm{div} \mathbf{u} d\mathbf{x} ds + ||\rho_0||^2_{L^2(\Omega)}$$

Using (4.64) and (4.65) we see that the right hands side of the former converges to its counterpart in the latter and thus,

$$||\nabla \rho_n||^2_{L^2(\Omega\times(0,T))} \to ||\nabla \rho||^2_{L^2(\Omega\times(0,T))},$$

and

$$||\rho_n(t)||^2_{L^2(\Omega)} \to ||\rho(t)||^2_{L^2(\Omega)}$$

for any $t \in [0,T]$, which implies the strong convergence

$$\nabla \rho_n \to \nabla \rho \textbf{ in } L^2(\Omega \times (0,T)).$$

With this we conclude that

$$\nabla \mathbf{u}_n \cdot \nabla \rho_n \to \nabla \mathbf{u} \cdot \nabla \rho$$

in the sense of distributions.

Finally, recalling the definition of $\mathbf{u}_n^N$ through (4.9), we note that the weak convergence in (4.65) implies the strong convergence

$$\mathbf{u}_n^N \to \mathbf{u}^N$$

which implies that the sequence Jacobians of the Lagrangian transformation $J_{\mathbf{y}n}$ defined through $\mathbf{u}_n^N$ converge strongly to the corresponding one related to $\mathbf{u}^N$.

With this we have shown that equations (4.19)-(4.23) are satisfied in the sense of distributions (equation (4.44) can be verified by taking test functions of the form $\psi(t)\eta_j(x)$, where $\psi \in C_0^\infty(0,T)$) by the limit function $(\rho, \mathbf{u}, \mathbf{H}, \psi)$ as each term appearing on those equations is the limit in the sense of distributions of the respective terms corresponding to the Faedo-Galerkin approximation $(\rho_n, \mathbf{u}_n, \mathbf{H}_n, \psi_n)$. We have also shown that the initial and boundary conditions (4.24), (4.25) are satisfied in the sense of distributions.

Lastly, inequality (4.28) is a consequence of (4.48) and this completes the proof of Theorem 4.1.

4.3 Vanishing artificial viscosity and interaction coefficients

Theorem 4.1 guarantees the existence of solutions to the Short Wave-Long Wave Interactions regularized system (4.19)-(4.23). Our next goal is to show that the sequence (or a subsequence) of solutions to this system converge to a global solution of the of the decoupled limit system when $(\varepsilon, \alpha, N, \delta) \to (0, 0, \infty, 0)$. In this Section we analyse the limit as $(\varepsilon, \alpha, N) \to (0, 0, \infty)$, leaving $\delta > 0$ fixed. As pointed out before, we can do all of of this as long as

$$\left(\frac{\varepsilon^2}{\alpha}\right)^{1/C_N} \to \infty. \tag{4.74}$$

In order to achieve this, we essentially adapt the arguments in [23, Section 7.4] and in [29].

The key point in the argument is to show that the sequence of densities converges strongly, in order to account for the nonlinearites from the pressure terms in the momentum equation (4.6). This is not straightforward, as it was in the previous section, since we loose regularity of the density as $\varepsilon \to 0$. In particular, an argument like that of Aubin-Lions lemma does not apply. In this direction, we can exploit the weak continuity properties of the effective viscous flux $p(\rho) - (\lambda + 2\mu)\mathrm{div}\mathbf{u}$, originally discovered by P.-L. Lions ([38]).

Let us point out that the terms involving the velocity field, the magnetic field and the wave function can be treated essentially as in the previous Section. Regarding the strong convergence of densities, the proof of weak continuity of the effective viscous flux found in [29] (cf. [23]) can be adapted with no major difficulties once we realize that (4.28), (4.15), (4.30) and (2.56) imply that the extra term, due to the SW-LW interactions, appearing in the momentum equation

$$\alpha\nabla(\frac{J_{\mathbf{y}}}{\rho}g'(1/\rho)h(\psi|^2))$$

tends to zero in the sense of distributions as $(\varepsilon, \alpha, N) \to (0, 0, \infty)$ satisfying (4.74). Accordingly, and to avoid the overload of notation, we may assume that N and α tend to ∞ and 0 respectively as functions of ε and denote by $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon, \psi_\varepsilon)$ the solution of the regularized system provided by Theorem 4.1.

The plan is as follows. First we show that ρ_ε is uniformly (in ε) bounded in $L^{\beta+1}_{loc}(\Omega \times (0, T))$ so that we can ensure that $\delta\rho^\beta$ and $a\rho^\gamma$ have (weakly) convergent subsequences. We know from Theorem 4.1 that $\rho_\varepsilon \in L^{\beta+1}(\Omega \times (0, T))$ for each ε, but we have not yet shown that they are uniformly bounded in this space.

Second, we prove the continuity of the effective viscous flux. And finally, we use this last result in order to show that $\overline{\rho \log \rho} = \overline{\rho} \log \overline{\rho}$ where the over line stands for a weak limit of the sequence indexed by ε. This last bit of information is enough to conclude the strong convergence of the densities due to the following result, which we state without proof (see [23, Theorem 2.11]).

Lemma 4.8. *Let $O \subseteq \mathbb{R}^N$ be a measurable set and $\{\mathbf{v}_n\}_{n=1}^\infty$ a sequence of functions in $L^1(O; \mathbb{R}^M)$ such that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(O; \mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semi-continuous convex function such that $\Phi(\mathbf{v}_n) \in L^1(O)$ for any n and

$$\Phi(\mathbf{v}_n) \to \overline{\Phi(\mathbf{v})} \text{ weakly in } L^1(O).$$

Then,

$$\Phi(\mathbf{v}) \le \overline{\Phi(\mathbf{v})} \text{ a.a. on } O.$$

If, moreover, Φ is strictly convex on an open convex set $U \subseteq \mathbb{R}^M$, and

$$\Phi(\mathbf{v}) = \overline{\Phi(\mathbf{v})} \text{ a.a. on } O, ,$$

then,

$$\mathbf{v}_n(\mathbf{y}) \to \mathbf{v}(\mathbf{y}) \text{ for a.e. } \mathbf{y} \in \{\mathbf{y} \in O : \mathbf{v}(\mathbf{y}) \in U\},$$

extracting a subsequence as the case may be.

From this point on, $T > 0$ will denote an arbitrary prefixed time and $C > 0$ will be a constant that may change from line to line being independent of ε, α and N. We also assume that $\delta > 0$ is fixed and that $(\varepsilon, \alpha, N) \to (0, 0, \infty)$ satisfying (4.74). Accordingly, we can also assume that $(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon, \psi^\varepsilon)$ are all defined in the time interval $[0, T]$ and satisfy (4.28).

4.3.1 Higher integrability of the density

This subsection is devoted to the proof of the following estimate.

Lemma 4.9. *For any compact $O \subseteq (\Omega \times (0, T))$ there is a constant $c = c(O)$ independent of ε (and α and N) such that*

$$\delta \int_O \rho^{\beta+1} d\mathbf{x} \le c(O). \tag{4.75}$$

The idea behind the proof of this Lemma is essentially the same as the one in the proof of Lemma 3.19, where we proved higher integrability of the density in the one dimensional setting. Of course, in the one dimensional setting we could easily find an explicit formula for the pressure in terms of the other state functions, directly from the momentum equation, by integration with respect to the spatial variable. In this multidimensional case this task is not so straightforward. Alternatively, the proof can be carried out by using appropriately chosen test functions.

Before going through the proof, let us introduce some preliminaries.

As in [23, 24, 29] we consider the operator $\mathcal{A}$ by its coordinates

$$\mathcal{A}_j[v] := \Delta^{-1}[\partial_{x_j} v], \qquad j = 1, 2, \tag{4.76}$$

where Δ^{-1} stands for the inverse of the Laplacian in $\mathbb{R}^2$. Equivalently, $\mathcal{A}_j$ can be defined through its Fourier symbol as

$$\mathcal{A}_j[v] = \mathcal{F}^{-1}\left[\frac{-i\xi_j}{|\xi|^2}\mathcal{F}[v]\right], \qquad j = 1, 2.$$

As shown in [23] the operator $\mathcal{A}$ has the following properties:

$$||\mathcal{A}_j v||_{W^{1,s}(\Omega)} \leq c(s,\Omega)||v||_{L^s(\mathbb{R}^2)}, \qquad 1 < s < \infty, \tag{4.77}$$

and consequently, by Sobolev's embeddings

$$||\mathcal{A}_j v||_{L^q(\Omega)} \leq c(s,\Omega)||v||_{L^s(\mathbb{R}^2)}, \qquad q \text{ finite, provided } \frac{1}{q} \geq \frac{1}{s} - \frac{1}{2}, \tag{4.78}$$

$$||\mathcal{A}_j v||_{L^\infty(\Omega)} \leq c(s,\Omega)||v||_{L^s(\Omega)}, \qquad \text{if } s > 2. \tag{4.79}$$

Let us also introduce the following standard smoothing operator

$$[v]^\omega_{\mathbf{x}}(\mathbf{z}) := (\vartheta_\omega * v)(\mathbf{z}) = \int_{\mathbb{R}^2} \vartheta_\omega(\xi - \mathbf{z})v(\xi)d\xi, \tag{4.80}$$

where, for each $\omega > 0$,

$$\vartheta_\omega(\mathbf{z}) := \frac{1}{\omega^2}\vartheta\left(\frac{|\mathbf{z}|}{\omega}\right), \qquad \mathbf{z} \in \mathbb{R}^2,$$

and $\vartheta \in C^\infty_0((-1,1))$ with

$$\vartheta(-\tau) = \vartheta(\tau), \qquad \int_{\mathbb{R}^2} \vartheta(|\mathbf{z}|)d\mathbf{z} = 1, \qquad \vartheta \text{ nonincreasing on } [0,\infty).$$

Let us also observe that from (4.28) we have, in particular, that

$$\rho_\varepsilon \text{ is bounded in } L^\infty(0,T;L^\beta(\Omega)),, \tag{4.81}$$

$$\mathbf{u}_\varepsilon \text{ is bounded in } L^2(0,T;H^1_0(\Omega)).,\tag{4.82}$$

$$\mathbf{H}_\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.83}$$

$$\psi_\varepsilon \text{ is bounded in } L^\infty(0,T;L^4(\Omega) \cap H^1_0(\Omega)). \tag{4.84}$$

Proof of Lemma 4.9. For $\omega > 0$, set

$$B_\omega = [\rho_\varepsilon]^\omega_{\mathbf{x}}.$$

Let us recall that ρ_ε and $\mathbf{u}_\varepsilon$ satisfy (4.19) a.a. on $\Omega \times (0,T)$, along with the boundary condition $(\nabla\rho_\varepsilon \cdot \mathbf{n})|_{\partial\Omega} = 0$. Then, extending ρ_ε and $\mathbf{u}_\varepsilon$ to be zero outside of Ω we have that

$$\rho_{\varepsilon t} + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \operatorname{div}(\mathbb{1}_\Omega \nabla\rho_\varepsilon) \tag{4.85}$$

in the sense of distributions in $\mathbb{R}^2 \times (0,T)$, where $\mathbb{1}_\Omega$ is the characteristic function of Ω.

Applying the smoothing operator $[\cdot]^\omega_{\mathbf{x}}$ to equation (4.85) we have

$$B_{\omega t} = f_\omega, \tag{4.86}$$

with

$$f_\omega = -\operatorname{div}([\rho_\varepsilon \mathbf{u}_\varepsilon]^\omega_{\mathbf{x}}) + \varepsilon \operatorname{div}[\mathbb{1}_\Omega \nabla\rho_\varepsilon]^\omega_{\mathbf{x}}.$$

Note that h_ω is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$.

As in [23] we choose the test function[1]

$$\varphi(\mathbf{x},t) = \zeta(t)\eta(x)\mathcal{A}[\xi(\cdot)B_\omega(\cdot,t)](\mathbf{x},t),$$

where $\eta, \xi \in C_0^\infty(\Omega)$ and $\zeta \in C_0^\infty((0,T))$, and use it in the momentum equation (4.20) to obtain

$$\int_0^T \int_\Omega \zeta\eta\xi(a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)B_\omega d\mathbf{x}ds = \int_0^T \int_\Omega \zeta\eta\mathbb{S}_\varepsilon : (\nabla\Delta^{-1}\nabla)[\xi B_\omega]d\mathbf{x}ds + \sum_{j=1}^9 I_j, \tag{4.87}$$

where, in the notation of Chapter 2, $\mathbb{S}_\varepsilon = \lambda(\operatorname{div}\mathbf{u}_\varepsilon)\mathrm{Id} + \mu(\nabla\mathbf{u}_\varepsilon + (\nabla\mathbf{u}_\varepsilon)^\top)$ is the viscous

[1] Let us recall that our two dimensional model can be regarded as the three dimensional one under the assumption that the involved functions are independent of the third variable. In particular, the velocity field takes values in $\mathbb{R}^3$. Accordingly, in order to use φ as a test function we define its third component as being identically equal to zero.

stress tensor, and

$$
\begin{aligned}
I_1 &= \int_0^T \int_\Omega \zeta \mathbb{S}_\varepsilon \nabla \eta \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds, \\
I_2 &= -\int_0^T \int_\Omega \zeta (a\rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta) \nabla \eta \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds, \\
I_3 &= -\int_0^T \int_\Omega \zeta (\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \nabla \eta \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds \\
I_4 &= -\int_0^T \int_\Omega \zeta \mathbf{u}_\varepsilon \cdot (\nabla \Delta^{-1} \nabla)[\xi B_\omega] \eta \rho_\varepsilon \mathbf{u}_\varepsilon d\mathbf{x} ds \\
I_5 &= -\int_0^T \int_\Omega \zeta \eta (\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds \\
I_6 &= -\int_0^T \int_\Omega \zeta_t \eta \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds \\
I_7 &= -\int_0^T \int_\Omega \zeta \eta \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A}[\xi f_\omega] d\mathbf{x} ds \\
I_8 &= \varepsilon \int_0^T \int_\Omega \zeta \eta \nabla \mathbf{u}_\varepsilon \nabla \rho_\varepsilon \cdot \mathcal{A}[\xi B_\omega] d\mathbf{x} ds \\
I_9 &= -\int_0^T \int_\Omega \zeta \alpha \frac{J_{\mathbf{y}}}{\rho_\varepsilon} g'(1/\rho_\varepsilon) h(|\psi_\varepsilon|^2) \Big(\eta \xi B_\omega + \nabla \eta \cdot \mathcal{A}[\xi B_\omega] \Big) d\mathbf{x} ds
\end{aligned}
$$

Note that by (4.79), we have that

$$
\mathcal{A}[\xi B_\omega] \text{ are bounded in } L^\infty(\Omega \times (0,T)), \tag{4.88}
$$

provided that $\beta > 2$. This together with (4.81) and (4.82) implies that the integrals I_1, I_2, I_3 and I_7 are bounded by a constant independent of ε and ω. Next, by (4.77) combined with (4.81) and (4.82) we have that I_4 is also bounded. Now, by the fact that $T \le T_N$ combined with (4.30), (4.15), (4.28) and (4.74) we see that

$$
\alpha |J_{\mathbf{y}}| \le \varepsilon^2,
$$

and thus, by (2.56), $I_9 \to 0$ as $\varepsilon \to 0$. In particular, I_9 is also bounded by a constant independent of ε and ω.

Regarding I_7, we see that ρ_ε, being a solution of equation (4.32), satisfies the identity

$$
||\rho_\varepsilon(t)||_{L^2(\Omega)}^2 + 2\varepsilon \int_0^t ||\nabla \rho_\varepsilon||_{L^2(\Omega)}^2 ds = -\int_0^t \int_\Omega \rho_\varepsilon^2 \operatorname{div} \mathbf{u}_\varepsilon d\mathbf{x} ds + ||\rho_0||_{L^2(\Omega)}^2,
$$

and therefore we see that

$$\varepsilon^{1/2}\nabla\rho_\varepsilon \text{ are uniformly bounded in } L^2(0,T;L^2(\Omega)).$$

In particular, by (4.77)

$$\mathcal{A}[\xi f_\varepsilon] \text{ are uniformly bounded in } L^2(\Omega\times(0,T)),$$

Thus, we conclude that I_7 is bounded by a constant independent of ε and ω. By the same token we see that I_8 is uniformly bounded as well. In fact, we have that $I_8 \to 0$ as $\varepsilon \to 0$.

Next, we see that (4.83) and (4.88) imply that I_5 is also bounded by a constant independent of ε and ω.

Finally, (4.77) and (4.82) also yield a uniform bound for the integral

$$\int_0^T \int_\Omega \zeta\eta\mathbb{S}_\varepsilon : (\nabla\Delta^{-1}\nabla)[\xi B_\omega]d\mathbf{x}ds.$$

Gathering all this information in (4.87) and letting $\omega \to 0$ we arrive at (4.75). Of course, the bounds obtained for the integrals above depend on ζ, η and ξ, which is why the result is local. □

4.3.2 The effective viscous flux

This section concerns the proof of the weak continuity of the effective viscous flux. However, before we get to it we have to make a few observations.

By (4.81), (4.82), (4.83) and (4.84) we can assume that

$$\rho_\varepsilon \to \rho \text{ weakly-* in } L^\infty(0,T;L^\beta(\Omega)) \tag{4.89}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \tag{4.90}$$

$$\mathbf{H}_\varepsilon \to \mathbf{H} \text{ strongly in } L^2(\Omega\times(0,T)) \text{ and weakly-* in } L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)) \tag{4.91}$$

$$\psi_\varepsilon \to \psi \text{ strongly in } C(0,T;L^2(\Omega)) \text{ and weakly-* in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.92}$$

where the strong convergence in (4.91) and in (4.92) is due to Aubin-Lions Lemma (Lemma 3.16).

Then, by the same arguments used to obtain (4.69), (4.72) and (4.73) we see that

$$(\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon \to (\nabla \times \mathbf{H}) \times \mathbf{H}, \text{ in the sense of distributions,} \tag{4.93}$$

$$\nabla \times (\mathbf{u}_\varepsilon \times \mathbf{H}_\varepsilon) \to \nabla \times (\mathbf{u} \times \mathbf{H}) \text{ in the sense of distributions,} \tag{4.94}$$

$$\rho_\varepsilon \mathbf{u}_\varepsilon \to \rho \mathbf{u} \text{ in } C([0,T]; L^{2\beta/(\beta+1)}_{weak}(\Omega)), \tag{4.95}$$

$$\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \rho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^2(0,T; L^{c_2}(\Omega)), \tag{4.96}$$

where, $c_2 = 2\gamma/(1+\gamma) > 1$.

As pointed out before we have that

$$\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \rho_\varepsilon \to 0 \tag{4.97}$$

and

$$\alpha \nabla \left(\frac{J_{\mathbf{y}}}{\rho_\varepsilon} g'(1/\rho_\varepsilon) h(|\psi_\varepsilon|^2) \right) \to 0 \tag{4.98}$$

in the sense of distributions.

Moreover, by (4.75) we can assume that

$$a\rho^\gamma + \delta \rho^\beta \to \overline{p} \text{ weakly in } L^{(\beta+1)/\beta}(\Omega \times (0,T)). \tag{4.99}$$

All of this information implies that the limit functions satisfy the equations

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{4.100}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p} = \operatorname{div}\Big(\lambda (\operatorname{div} \mathbf{u}) \mathrm{Id} + \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \Big) + \operatorname{curl}(\mathbf{H}) \times \mathbf{H}. \tag{4.101}$$

in the sense of distributions.

With this, we can state the result on the weak continuity of the effective viscous flux, originally discovered by P.-L. Lions (see [38]), as (cf. [23, 24, 29])

Lemma 4.10. *Let $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon, \psi_\varepsilon)$ be the solution of the regularized system provided by Theorem 4.1. Then,*

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \zeta \eta (a\rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta - (\lambda + 2\mu)\, div \mathbf{u}_\varepsilon) \rho_\varepsilon d\mathbf{x} ds \\ = \int_0^T \int_\Omega \zeta \eta (a\overline{\rho^\gamma} + \delta \overline{\rho^\beta} - (\lambda + 2\mu)\, div \mathbf{u}) \rho d\mathbf{x} ds, \end{aligned} \tag{4.102}$$

for any $\zeta \in C_0^\infty((0,T))$, *and* $\eta \in C_0^\infty(\Omega)$.

Proof. First, noting that

$$\xi \mathrm{div}([\rho_\varepsilon \mathbf{u}_\varepsilon]_{\mathbf{x}}^\omega) = \mathrm{div}(\xi[\rho_\varepsilon \mathbf{u}_\varepsilon]_{\mathbf{x}}^\omega) - \nabla\xi \cdot [\rho_\varepsilon \mathbf{u}_\varepsilon]_{\mathbf{x}}^\omega,$$

we see that I_7 in (4.87) may be rewritten as

$$I_7 = I_7^1 + I_7^2 + I_7^3,$$

where

$$\begin{aligned}
I_7^1 &= \int_0^T \int_\Omega \zeta\xi[\rho_\varepsilon \mathbf{u}_\varepsilon]_{\mathbf{x}}^\omega (\nabla\Delta^{-1}\mathrm{div})[\eta\rho_\varepsilon \mathbf{u}_\varepsilon] d\mathbf{x}ds \\
I_7^2 &= -\int_0^T \int_\Omega \zeta\eta\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{A}\Big[\nabla\xi \cdot [\rho_\varepsilon \mathbf{u}_\varepsilon]_{\mathbf{x}}^\omega\Big] d\mathbf{x}ds \\
I_7^3 &= -\varepsilon \int_0^T \int_\Omega \zeta\eta\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{A}[\xi \mathrm{div}(\mathbb{1}_\Omega \nabla\rho_\varepsilon)] d\mathbf{x}ds.
\end{aligned}$$

Therefore, passing to the limit as $\omega \to 0$ in (4.87) we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \zeta\eta\Big(\xi(a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\rho_\varepsilon - \mathbb{S}_\varepsilon : (\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]\Big) d\mathbf{x}ds = \sum_{j=1}^9 J_j^\varepsilon \\
&+ \int_0^T \int_\Omega \zeta \mathbf{u}_\varepsilon \Big(\xi\rho_\varepsilon(\nabla\Delta^{-1}\mathrm{div})[\eta\rho_\varepsilon \mathbf{u}_\varepsilon] - (\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]\eta\rho_\varepsilon \mathbf{u}_\varepsilon\Big) d\mathbf{x}ds,
\end{aligned} \tag{4.103}$$

where,

$$\begin{aligned}
J_1^\varepsilon &= \int_0^T \int_\Omega \zeta \mathbb{S}_\varepsilon \nabla\eta \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds, \\
J_2^\varepsilon &= -\int_0^T \int_\Omega \zeta(a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\nabla\eta \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds, \\
J_3^\varepsilon &= -\int_0^T \int_\Omega \zeta(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)\nabla\eta \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds \\
J_4^\varepsilon &= -\int_0^T \int_\Omega \zeta\eta(\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds \\
J_5^\varepsilon &= -\int_0^T \int_\Omega \zeta_t \eta\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds
\end{aligned}$$

$$J_6^\varepsilon = -\int_0^T \int_\Omega \zeta\eta\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{A}[\nabla\xi \cdot \rho_\varepsilon \mathbf{u}_\varepsilon] d\mathbf{x}ds$$
$$J_7^\varepsilon = -\varepsilon \int_0^T \int_\Omega \zeta\eta\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{A}[\xi \mathrm{div}(\mathbb{1}_\Omega \nabla\rho_\varepsilon)] d\mathbf{x}ds$$
$$J_8^\varepsilon = \varepsilon \int_0^T \int_\Omega \zeta\eta\nabla\mathbf{u}_\varepsilon \nabla\rho_\varepsilon \cdot \mathcal{A}[\xi\rho_\varepsilon] d\mathbf{x}ds$$
$$J_9^\varepsilon = -\int_0^T \int_\Omega \zeta\alpha\frac{J_{\mathbf{y}}}{\rho_\varepsilon} g'(1/\rho_\varepsilon)h(|\psi_\varepsilon|^2)\Big(\eta\xi\rho_\varepsilon + \nabla\eta \cdot \mathcal{A}[\xi\rho_\varepsilon]\Big) d\mathbf{x}ds$$

Now, using equations (4.100) and (4.101), a similar procedure yields

$$\int_0^T \int_\Omega \zeta\eta\Big(\xi(a\rho^\gamma + \delta\rho^\beta)\rho - \mathbb{S} : (\nabla\Delta^{-1}\nabla)[\xi\rho]\Big) d\mathbf{x}ds = \sum_{j=1}^{6} J_j$$
$$+ \int_0^T \int_\Omega \zeta\mathbf{u}\Big(\xi\rho(\nabla\Delta^{-1}\nabla)[\eta\rho\mathbf{u}] - (\nabla\Delta^{-1}\mathrm{div})[\xi\rho]\eta\rho\mathbf{u}\Big) d\mathbf{x}ds, \tag{4.104}$$

where,

$$J_1 = \int_0^T \int_\Omega \zeta\mathbb{S}\nabla\eta \cdot \mathcal{A}[\xi\rho] d\mathbf{x}ds,$$
$$J_2 = -\int_0^T \int_\Omega \zeta\overline{p}\nabla\eta \cdot \mathcal{A}[\xi\rho] d\mathbf{x}ds,$$
$$J_3 = -\int_0^T \int_\Omega \zeta(\rho\mathbf{u} \otimes \mathbf{u})\nabla\eta \cdot \mathcal{A}[\xi\rho] d\mathbf{x}ds$$
$$J_4 = -\int_0^T \int_\Omega \zeta\eta(\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathcal{A}[\xi\rho] d\mathbf{x}ds$$
$$J_5 = -\int_0^T \int_\Omega \zeta_t \eta\rho\mathbf{u} \cdot \mathcal{A}[\xi\rho] d\mathbf{x}ds$$
$$J_6 = -\int_0^T \int_\Omega \zeta\eta\rho\mathbf{u}\mathcal{A}[\nabla\xi \cdot \rho\mathbf{u}] d\mathbf{x}ds$$

Following [23, 29], we now proceed to show that all the integrals in the right hand side of (4.103) converge to their counterparts in (4.104).

As ρ_ε satisfies equation (4.19), Lemma 4.7 yields

$$\rho_\varepsilon \to \rho \text{ in } C([0,T]; L^\beta_{weak}(\Omega)), \tag{4.105}$$

and consequently, by (4.77) and the compactness of the embedding $W^{1,\beta}(\Omega) \to C(\overline{\Omega})$

(recall that $\beta > 2$) we have that

$$\mathcal{A}[\xi\rho_\varepsilon] \to \xi\rho \text{ in } C(\Omega \times (0,T)),$$

Thus, in light of (4.90), (4.99), (4.93), (4.95) and (4.96), we have that

$$J_k^\varepsilon \to J_k, \text{ for } k = 1,2,3,4,5.$$

Similarly, by (4.89) and (4.90) we have, in particular, that

$$\rho_\varepsilon \mathbf{u}_\varepsilon \text{ is bounded in } L^2(\Omega \times (0,T)), \tag{4.106}$$

and this together with (4.77) and (4.105) implies that

$$\nabla\xi \cdot \rho_\varepsilon \mathbf{u}_\varepsilon \to \nabla\xi \cdot \rho\mathbf{u} \text{ weakly in } L^2(0,T;H^1(\Omega)).$$

Consequently, taking (4.95) into account we have that

$$J_6^\varepsilon \to J_6.$$

As was already mentioned we have that

$$J_k^\varepsilon \to 0, \text{ for } j = 7,8,9.$$

In order to deal with the last term on the right hand side of (4.103) we state the following result (see [23, Corollary 6.1], also [24, Lemma 3.4]).

Lemma 4.11. *Let $O \subseteq \mathbb{R}^N$ be an arbitrary domain.*

(i) Let

$$\mathbf{v}_n \to \mathbf{v} \textit{ weaky in } L^p(O;\mathbb{R}^N), \quad \mathbf{w}_n \to \mathbf{w} \textit{ weaky in } L^q(O;\mathbb{R}^N),$$

with

$$1 < p, \quad ,q < \infty, \quad \frac{1}{p} + \frac{1}{q} \leq 1.$$

Then

$$\mathbf{v}_n \cdot (\nabla\Delta^{-1} div)[\mathbf{w}_n] - \mathbf{w}_n \cdot (\nabla\Delta^{-1} div)[\mathbf{v}_n] \to \mathbf{v} \cdot (\nabla\Delta^{-1} div)[\mathbf{w}] - \mathbf{w} \cdot (\nabla\Delta^{-1} div)[\mathbf{v}]$$

in the sense of distributions.

(ii) Under the same hypotheses, if

$$B_n \to B \text{ weakly in } L^p(O), \quad \mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^q(O;\mathbb{R}^n),$$

then

$$(\nabla\Delta^{-1}\nabla)[B_n]\mathbf{v}_n - (\nabla\Delta^{-1}div)[\mathbf{v}_n]B_n \to (\nabla\Delta^{-1}\nabla)[B]\mathbf{v} - (\nabla\Delta^{-1}div)[\mathbf{v}]B$$

The proof of this result consists in applying a particular case of the Div-Curl Lemma (Lemma 3.12). We refer to [23] for the proof.

Now, by (4.89) and (4.95), a direct application of the above Lemma implies

$$\begin{aligned}(\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon(t)]\eta\rho_\varepsilon\mathbf{u}_\varepsilon(t) - \xi\rho_\varepsilon(t)(\nabla\Delta^{-1}\mathrm{div})[\eta\rho_\varepsilon\mathbf{u}_\varepsilon(t)]&\\ \to (\nabla\Delta^{-1}\nabla)[\xi\rho(t)]\eta\rho\mathbf{u}(t) - \xi\rho(t)(\nabla\Delta^{-1}\mathrm{div})[\eta\rho\mathbf{u}(t)]&,\end{aligned}$$

weakly in $L^{2\beta/(\beta+3)}(\Omega)$, for each fixed t.

As we know $L^q(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$ for each $q > 1$ (remember that our spatial domain is a bounded open subset of $\mathbb{R}^2$). In particular,

$$\begin{aligned}(\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]\eta\rho_\varepsilon\mathbf{u}_\varepsilon - \xi\rho_\varepsilon(\nabla\Delta^{-1}\mathrm{div})[\eta\rho_\varepsilon\mathbf{u}_\varepsilon]&\\ \to (\nabla\Delta^{-1}\nabla)[\xi\rho]\eta\rho\mathbf{u} - \xi\rho(\nabla\Delta^{-1}\mathrm{div})[\eta\rho\mathbf{u}]&,\end{aligned}$$

strongly in $L^2(0,T;H^{-1}(\Omega))$. As a consequence, keeping in mind (4.90), we see that

$$\begin{aligned}&\int_0^T\int_\Omega \zeta\mathbf{u}_\varepsilon\Big(\xi\rho_\varepsilon(\nabla\Delta^{-1}\mathrm{div})[\eta\rho_\varepsilon\mathbf{u}_\varepsilon] - (\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]\eta\rho_\varepsilon\mathbf{u}_\varepsilon\Big)d\mathbf{x}ds\\ &\quad\to \int_0^T\int_\Omega \zeta\mathbf{u}\Big(\xi\rho(\nabla\Delta^{-1}\nabla)[\eta\rho\mathbf{u}] - (\nabla\Delta^{-1}\mathrm{div})[\xi\rho]\eta\rho\mathbf{u}\Big)d\mathbf{x}ds.\end{aligned}$$

All of this information put together with (4.103) and (4.104) yields

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \zeta\eta\Big(\xi(a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta)\rho_\varepsilon - \mathbb{S}_\varepsilon : (\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]\Big)d\mathbf{x}ds\\ &\quad= \int_0^T\int_\Omega \zeta\eta\Big(\xi(a\rho^\gamma + \delta\rho^\beta)\rho - \mathbb{S} : (\nabla\Delta^{-1}\nabla)[\xi\rho]\Big)d\mathbf{x}ds,\end{aligned} \tag{4.107}$$

for any $\zeta \in C_0^\infty((0,T))$ and $\eta, \xi \in C_0^\infty(\Omega)$.

In order to conclude, as in [23], we compute

$$\begin{aligned}
&\int_0^T \int_\Omega \zeta\eta\mathbb{S}_\varepsilon : (\nabla\Delta^{-1}\nabla)[\xi\rho_\varepsilon]d\mathbf{x}ds \\
&\qquad = \int_0^T \int_\Omega \zeta\xi(\nabla\Delta^{-1}\nabla) : (\eta\mathbb{S}_\varepsilon)\rho_\varepsilon d\mathbf{x}ds \\
&\qquad = \int_0^T \int_\Omega \zeta\xi(2\mu+\lambda)\mathrm{div}(\eta\mathbf{u}_\varepsilon)\rho_\varepsilon d\mathbf{x}ds \\
&\qquad\qquad - \int_0^T \int_\Omega \zeta\xi\rho_\varepsilon[2\mu(\nabla\Delta^{-2}\nabla) : (\mathbf{u}_\varepsilon \otimes \nabla\eta) + \lambda\mathbf{u}_\varepsilon \cdot \nabla\eta]d\mathbf{x}ds \\
&\qquad = \int_0^T \int_\Omega \zeta\xi\eta(2\mu+\lambda)\mathrm{div}\mathbf{u}_\varepsilon\, \rho_\varepsilon d\mathbf{x}ds \\
&\qquad\qquad - \int_0^T \int_\Omega 2\mu\zeta\xi\rho_\varepsilon[(\nabla\Delta^{-2}\nabla) : (\mathbf{u}_\varepsilon \otimes \nabla\eta) - \mathbf{u}_\varepsilon \cdot \nabla\eta]d\mathbf{x}ds
\end{aligned} \tag{4.108}$$

and similarly

$$\begin{aligned}
&\int_0^T \int_\Omega \zeta\eta\mathbb{S} : (\nabla\Delta^{-1}\nabla)[\xi\rho]d\mathbf{x}ds \\
&\qquad = \int_0^T \int_\Omega \zeta\xi\eta(2\mu+\lambda)\mathrm{div}\mathbf{u}\, \rho d\mathbf{x}ds \\
&\qquad\qquad - \int_0^T \int_\Omega 2\mu\zeta\xi\rho[(\nabla\Delta^{-2}\nabla) : (\mathbf{u} \otimes \nabla\eta) - \mathbf{u} \cdot \nabla\eta]d\mathbf{x}ds
\end{aligned} \tag{4.109}$$

Taking (4.105) into account, we see that the last integral on the right hand side of (4.108) converges to the last integral in the right hand side of (4.109). This and (4.107) imply (4.102), which concludes the proof. □

4.3.3 Strong convergence of densities, renormalized solutions

Using the results above we can show strong convergence of densities, essentially, in the same way as in [23, Section 7.4.3]. For this, we need to show first that the limit functions ρ and $\mathbf{u}$ solve the continuity equation in the sense of renormalized solutions, that is, they satisfy (4.100) in the sense of distributions, and more generally,

$$B(\rho)_t + \mathrm{div}(B(\rho)\mathbf{u}) + b(\rho)\mathrm{div}\mathbf{u} = 0, \tag{4.110}$$

also in the sense of distributions, for any functions

$$B \in C[0,\infty) \cap C^1(0,\infty), \quad b \in C[0,\infty), \text{ bounded on } [0,\infty), \quad B(0) = b(0) = 0, \tag{4.111}$$

satisfying

$$b(z) = B'(z)z - B(z). \tag{4.112}$$

Remark 4.1. *The function b in the definition of renormalized solutions does not have to be bounded. Indeed, provided that $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ and $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$, by Lebesgue's dominated convergence theorem it can be shown that* (4.110) *also holds for $b \in C[0,\infty)$ satisfying*

$$|b'(z)z| \leq cz^{\gamma/2}, \text{ for } z \text{ larger than some positive constant } z_0. \tag{4.113}$$

Now, the fact that ρ and $\mathbf{u}$ solve (4.100) in the sense of renormalized solutions is a direct consequence of the following general result (cf. [23, Corollary 4.1])

Lemma 4.12. *Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary domain. Let,*

$$\rho \in L^2(\Omega \times (0,T))$$

solve the continuity equation (4.100) *in the sense of distributions with*

$$\mathbf{u} \in L^2(0,T;H_0^1(\Omega)).$$

Then, ρ is a renormalized solution of (4.100) *on $\Omega \times (0,T)$.*

This result follows by applying the the regularizing operator $v \to [v]_{\mathbf{x}}^\omega$ given by (4.80) (that is, taking the functions ϑ_ω as test functions) to equation (4.100), multiplying by $B'(\rho)$ and taking the limit as $\omega \to 0$, wherein the convergence is justified by the integrability properties of ρ and $\mathbf{u}$ assumed as hypotheses. We omit the details.

Coming back to our present situation, as $\beta > 2$ and by virtue of (4.89) and (4.90) we can apply directly this result in order to conclude that ρ and $\mathbf{u}$ indeed satisfy (4.110).

In particular, in view of Remark 4.1 and using the fact that $\rho \in L^\infty(0,T;L^\beta(\Omega))$ we can choose $B(z) = z\log(z)$ in (4.110) to conclude that the following equation is satisfied in the sense of distributions on $\mathbb{R}^2 \times \Omega$:

$$(\rho\log(\rho))_t + \operatorname{div}(\rho\log(\rho)\mathbf{u}) + \rho\operatorname{div}\mathbf{u} = 0. \tag{4.114}$$

On the other hand, as ρ_ε satisfies (4.19) a.e. on $\Omega \times (0,T)$, we can multiply (4.19) by $B'(\rho_\varepsilon)$ to obtain

$$B(\rho_\varepsilon)_t+\mathrm{div}(B(\rho_\varepsilon)\mathbf{u}_\varepsilon)+\Big(B'(\rho_\varepsilon)\rho_\varepsilon-B(\rho_\varepsilon)\Big)\mathrm{div}\mathbf{u}_\varepsilon=\varepsilon\mathrm{div}(\mathbb{1}_\Omega\nabla B(\rho_\varepsilon))-\varepsilon\mathbb{1}_\Omega B''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2, \tag{4.115}$$

for any function $B\in C^2(\Omega)$ such that $B(0)=0$ with B' and B'' uniformly bounded.

Accordingly, if B is convex, and taking into account the boundary conditions (4.25), we have

$$\int_0^T\int_\Omega \zeta\Big(B'(\rho_\varepsilon)\rho_\varepsilon-B(\rho_\varepsilon)\Big)\mathrm{div}\mathbf{u}d\mathbf{x}ds\le\int_\Omega B(\rho_0)d\mathbf{x}+\int_0^T\int_\Omega\zeta_t B(\rho_\varepsilon)d\mathbf{x}ds,$$

for any $\zeta\in C^\infty[0,T]$ with $\zeta(0)=1$ and $\zeta(T)=0$.

Approximating the function $z\to z\log(z)$ by a sequence of convex functions B as above we conclude that

$$\int_0^T\int_\Omega\zeta\rho_\varepsilon\mathrm{div}\mathbf{u}d\mathbf{x}ds\le\int_\Omega\rho_0\log(\rho_0)d\mathbf{x}+\int_0^T\int_\Omega\zeta_t\rho_\varepsilon\log(\rho_\varepsilon)d\mathbf{x}ds.$$

Taking the limit as $\varepsilon\to 0$ we obtain

$$\int_0^T\int_\Omega\zeta\overline{\rho\mathrm{div}\mathbf{u}}d\mathbf{x}ds\le\int_\Omega\rho_0\log(\rho_0)d\mathbf{x}+\int_0^T\int_\Omega\zeta_t\overline{\rho\log(\rho)}d\mathbf{x}ds,$$

where, as before, the over line stands for a weak limit of the sequence indexed by ε. In particular, by (4.89), we can assume that $\rho_\varepsilon\log(\rho_\varepsilon)\to\overline{\rho\log(\rho)}$ weakly in $L^\infty(0,T;L^q(\Omega))$ for any $q<\beta$. As a consequence,

$$\int_0^t\int_\Omega\overline{\rho\mathrm{div}\mathbf{u}}d\mathbf{x}ds\le\int_\Omega\rho_0\log(\rho_0)d\mathbf{x}+\int_\Omega\overline{\rho\log(\rho)}(t)d\mathbf{x}, \tag{4.116}$$

for any Lebesgue point t of the function $\overline{\rho\log(\rho)}$.

Similarly, using a test function $\varphi(\mathbf{x},t)=\zeta(t)\eta(\mathbf{x})$ in (4.114), where ζ and η are smooth and $\zeta\ge 0$, $\eta\ge 0$, $\eta|_\Omega=1$, we obtain

$$\int_0^t\int_\Omega\rho\mathrm{div}\mathbf{u}d\mathbf{x}ds=\int_\Omega\rho_0\log(\rho_0)d\mathbf{x}-\int_\Omega\rho\log(\rho)(t)d\mathbf{x}, \tag{4.117}$$

for $t\in[0,T]$. Thus, from (4.116) and (4.117) we find the inequality

$$\int_\Omega\Big(\overline{\rho\log(\rho)}-\rho\log(\rho)\Big)(t)d\mathbf{x}\le\int_0^t\int_\Omega\Big(\rho\mathrm{div}\mathbf{u}-\overline{\rho\mathrm{div}\mathbf{u}}\Big)d\mathbf{x}ds, \tag{4.118}$$

for a.e. $t\in[0,T]$.

Using Lemma 4.10 we see that

$$\int_O \left(\overline{\rho \mathrm{div}\mathbf{u}} - \rho \mathrm{div}\mathbf{u}\right) d\mathbf{x}ds \geq \frac{1}{\lambda + 2\mu} \liminf_{\varepsilon \to 0} \int_O \left((a\rho_\varepsilon^{\gamma+1} + \delta\rho_\varepsilon^{\beta+1}) - \overline{p}\right)\rho d\mathbf{x}ds,$$

for any compact $O \subseteq \Omega \times (0,T)$. Recall that

$$\overline{p} = a\overline{\rho^\gamma} + \delta\overline{\rho^\beta}.$$

Now, as the function $z \to z^\beta$ is increasing we have

$$\begin{aligned} \rho_\varepsilon^{\beta+1} - \overline{\rho^\beta}\,\rho &= (\rho_\varepsilon^\beta - \rho^\beta)(\rho_\varepsilon - \rho) + \rho^\beta(\rho_\varepsilon - \rho) + (\rho_\varepsilon^\beta - \overline{\rho^\beta})\rho \\ &\geq \rho^\beta(\rho_\varepsilon - \rho) + (\rho_\varepsilon^\beta - \overline{\rho^\beta})\rho. \end{aligned}$$

Moreover, by virtue of Lemma 4.9 we have that

$$\rho_\varepsilon \to \rho \text{ weakly in } L^{\beta+1}(O), \qquad \rho_\varepsilon^\beta \to \overline{\rho^\beta} \text{ weakly in } L^{(\beta+1)/\beta},$$

as $\varepsilon \to 0$. Thus, we conclude that

$$\liminf_{\varepsilon \to 0} \int_O \left(\delta\rho_\varepsilon^{\beta+1} - \delta\overline{\rho^\beta}\,\rho\right) d\mathbf{x}ds \geq 0. \tag{4.119}$$

By the same token, we have that

$$\liminf_{\varepsilon \to 0} \int_O \left(a\rho_\varepsilon^{\gamma+1} - a\overline{\rho^\gamma}\,\rho\right) d\mathbf{x}ds \geq 0, \tag{4.120}$$

and consequently, from (4.118) we get

$$\int_\Omega \left(\overline{\rho\log(\rho)} - \rho\log(\rho)\right)(t) d\mathbf{x} \leq 0, \tag{4.121}$$

for a.e. t.

Finally, using Lemma 4.8 we conclude that

$$\overline{\rho\log(\rho)} = \rho\log(\rho),$$

which, is equivalent to the strong convergence

$$\rho_\varepsilon \to \rho \text{ in } L^1(\Omega \times (0,T)) \text{ and a.e.}. \tag{4.122}$$

In fact, by applying Lemma 4.7 we have that

$$\rho_\varepsilon \to \rho \text{ in } C([0,T];L^1(\Omega)). \tag{4.123}$$

In particular, we have that

$$a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta \to a\rho^\gamma + \delta\rho^\beta \tag{4.124}$$

in the sense of distributions.

4.3.4 Conclusion

With the strong convergence of the densities we have that all the nonlinearities present in the continuity and in the momentum equations are accounted for. Taking into account (4.89)-(4.98) and also (4.123) and (4.124) we conclude that the limit functions ρ, $\mathbf{u}$, $\mathbf{H}$ and ψ solve the following decoupled limit system

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho, \tag{4.125}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(a\rho^\gamma + \delta\rho^\beta) \\ = (\nabla\times\mathbf{H})\times\mathbf{H} + \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}), \tag{4.126}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \tag{4.127}$$

$$\operatorname{div}\mathbf{H} = 0. \tag{4.128}$$

$$i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi, \tag{4.129}$$

with initial and boundary conditions (4.24) and

$$(\mathbf{u},\mathbf{H})|_{\partial\Omega} = 0, \qquad \psi|_{\partial\Omega_{\mathbf{y}}} = 0, \tag{4.130}$$

respectively, and we have proved the following result.

Theorem 4.2. *Let* $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon, \psi_\varepsilon)$ *be the solution of the regularized system* (4.19)-(4.23) *provided by Theorem 4.1.*

Then, there is a subsequence (not relabelled) that converges to a global weak solution $(\rho, \mathbf{u}, \mathbf{H}, \psi)$ *of system* (4.125)-(4.129), *where the initial and boundary conditions are* *satisfied in the sense of distributions, as* $(\varepsilon, \alpha, N) \to (0, 0, \infty)$ *provided that*

$$\left(\frac{\varepsilon^2}{\alpha}\right)^{1/C_N} \to \infty, \tag{4.131}$$

where C_N *is given by* (4.13)*.*

Moreover, the density ρ *is nonnegative and satisfies equation* (4.125) *in the sense of renormalized solutions, meaning that* (4.110) *is satisfied in the sense of distributions with* B *and* b *as in* (4.111) *and* (4.112)*.*

Furthermore, we have that (4.89)*-*(4.98)*,* (4.123) *and* (4.124) *are satisfied along with the energy inequality*

$$E_\delta(t) \leq E_\delta(0), \tag{4.132}$$

for a.e. $t \in (0, T)$ *where,*

$$E_\delta(t) = \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\rho^\gamma + \frac{\delta}{\beta - 1}\rho^\beta + \frac{1}{2}|\mathbf{H}|^2\right) d\mathbf{x} + \int_{\Omega_\mathbf{y}} \left(\frac{1}{2}|\nabla_\mathbf{y}\psi|^2 + \frac{1}{4}|\psi|^4\right) d\mathbf{y}$$
$$+ \int_0^t \int_\Omega (\mu|\nabla\mathbf{u}|^2 + (\lambda + \mu)(div\mathbf{u})^2 + \nu|\nabla\mathbf{H}|^2) d\mathbf{x}ds. \tag{4.133}$$

Let us recall that the regularized system (4.19)-(4.23) was proposed as a regularized Short Wave-Long Wave interaction between the MHD System and the nonlinear Schrödinger equation. Due to the lack of regularity of solutions, and in particular, due to the possible occurrence of vacuum in finite time, the Short Wave-Long Wave interactions could not be made in a straightforward way, as the Lagrangian transformation becomes singular in the presence of vacuum. To workaround these difficulties we defined the Lagrangian coordinate through a smooth approximation $\mathbf{u}_N$ of the velocity field of the fluid, given by (4.9), and accordingly, by considering the limit as $N \to \infty$ satisfying (4.131), Theorem 4.2 serves the purpose to legitimize the coordinates of the limiting Schrödinger equation to be considered as the Lagrangian coordinate in a generalized sense.

In short, we have produced a finite-energy renormalized weak solution of the two dimensional MHD equations as a limit of solutions of the regularized Short Wave-Long Wave interactions.

Of course, there is one step left to complete the analysis, which consists in analysing the limit as $\delta \to 0$. Although the techniques are similar to those contained in this Section, there are a lot of limitations that have to be dealt with as we loose uniform boundedness of the sequence of densities in the space $L^\infty(0, T; L^\beta(\Omega))$. In particular, Lemma 4.12 can no longer be applied as we do not know, a priori, whether $\rho \in L^2(\Omega \times (0,T))$. Let us recall that β was chosen conveniently large in order to justify the analysis developed.

Fortunately, we are now dealing with the **decoupled** system involving the two dimensional MHD equations and the nonlinear Schrödinger equation, and the arguments in Section 5 of [29] can be followed literally line by line in order to justify the passing to the limit as $\delta \to 0$ in equations (4.125)-(4.128). Finally, a simple application of Aubin-Lions Lemma (Lemma 3.16) yields compactness of the sequence of solutions of (4.129) as $\delta \to 0$.

In order to conclude we dedicate the following Section to quickly describe the passage to the limit as $\delta \to 0$ as in [29, Section 5].

4.4 Vanishing artificial pressure

In the interest of analysing the limit as $\delta \to 0$ we consider the limit problem

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{4.134}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(a\rho^\gamma) = (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\Big), \tag{4.135}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \tag{4.136}$$

$$\operatorname{div}\mathbf{H} = 0. \tag{4.137}$$

$$i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi, \tag{4.138}$$

subject to initial and boundary conditions

$$(\rho, \rho\mathbf{u}, \mathbf{H})(\mathbf{x}, 0) = (\rho_0, \mathbf{m}_0, \mathbf{H}_0)(x), \qquad \psi(\mathbf{y}, 0) = \psi_0(\mathbf{y}), \tag{4.139}$$

and

$$(\mathbf{u}, \mathbf{H})|_{\partial\Omega} = 0, \qquad \psi|_{\partial\Omega_{\mathbf{y}}} = 0. \tag{4.140}$$

Recall that we assume the initial data to be smooth in order to carry out the Faedo-Galerkin method from Section 4.2. This constraint may be removed and we can consider more general initial data by means of approximation by smooth functions.

For system (4.134)-(4.138) above we consider initial data in (4.139) satisfying

$$\begin{cases} \rho_0 \geq 0, \quad \rho_0 \in L^\gamma(\Omega), \\ \frac{|\mathbf{m}_0|}{\rho_0} \in L^1(\Omega), \\ \mathbf{H}_0 \in L^2(\Omega), \\ \psi_0 \in H_0^1(\Omega). \end{cases} \tag{4.141}$$

Accordingly, we consider a sequence of approximate initial data $(\rho_{0\delta}, \mathbf{u}_{0\delta}, \mathbf{H}_{0\delta}, \psi_{0\delta})$ such that

(i)

$$\rho_{0\delta} \text{ is smooth and satisfies } \nabla\rho_{0\delta}\cdot\mathbf{n}, \quad 0 < \delta \leq \rho_{0\delta} \leq \delta^{-1/2\beta}, \tag{4.142}$$

$$\rho_{0\delta} \to \rho_0 \text{ in } L^\gamma(\Omega), \quad |\{x \in \Omega : \rho_{0\delta} < \rho_0\}| \to 0, \tag{4.143}$$

as $\delta \to 0$.

(ii)

$$\mathbf{m}_{0\delta}(\mathbf{x}) = \begin{cases} \mathbf{m}_0(\mathbf{x}), & \text{if } \rho_{0\delta}(\mathbf{x}) \geq \rho_0(\mathbf{x}), \\ 0, & \text{if } \rho_{0\delta}(\mathbf{x}) < \rho_0(\mathbf{x}), \end{cases} \tag{4.144}$$

(iii) $\mathbf{H}_{0\delta} \to \mathbf{H}_0$ in $L^2(\Omega)$, and

(iv) $\psi_{0\delta} \to \psi_0$ in $H_0^1(\Omega)$.

Then, we have the following result.

Theorem 4.3. *Let* $(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta, \psi_\delta)$ *be the solution of the decoupled system* (4.125)-(4.129)*,* (4.130) *with initial data*

$$(\rho_\delta, \mathbf{u}_\delta, \mathbf{H}_\delta, \psi_\delta)|_{t=0} = (\rho_{0\delta}, \mathbf{u}_{0\delta}, \mathbf{H}_{0\delta}, \psi_{0\delta})$$

provided by Theorem 4.2.

Then, as $\delta \to 0$ we have that

$$\rho_\delta \to \rho, \text{ weakly-* in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } C([0,T];L^\gamma_{weak}(\Omega)), \quad (4.145)$$

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^2(0,T;H^1_0(\Omega)), \quad (4.146)$$

$$\mathbf{H}_\delta \to \mathbf{H} \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \text{ and strongly in } C([0,T];L^2_{weak}(\Omega)), \quad (4.147)$$

$$\psi_\delta \to \psi \text{ strongly in } C([0,T];L^4(\Omega)) \text{ and weakly-* in } L^\infty(0,T;H^1_0(\Omega)), \quad (4.148)$$

subject to a subsequence as the case may be, where $(\rho, \mathbf{u}, \mathbf{H}, \psi)$ is a global weak solution of (4.134)-(4.138) *with initial data* (4.139) *satisfying* (4.141) *and boundary conditions* (4.140)*, satisfied in the sense of distributions. In fact we have that*

$$\rho_\delta \to \rho, \text{ in } C([0,T];L^1(\Omega)) \quad (4.149)$$

Moreover, ρ solves (4.134) *in the sense of renormalized solutions, meaning that* (4.110) *is satisfied in the sense of distributions for any B and b as in* (4.111) *and* (4.112)*.*

Furthermore, we have that

$$E(t) \le E(0), \quad (4.150)$$

for a.e. t with

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma + \frac{1}{2}|\mathbf{H}|^2\right) d\mathbf{x} \\ + \int_0^t \int_\Omega (\mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)(div\mathbf{u})^2 + \nu|\nabla \mathbf{H}|^2) d\mathbf{x} ds, \quad (4.151)$$

and,

$$\int_{\Omega_\mathbf{y}} \left(\frac{1}{2}|\nabla_\mathbf{y}\psi|^2 + \frac{1}{4}|\psi|^4\right) d\mathbf{y} = \int_{\Omega_\mathbf{y}} \left(\frac{1}{2}|\nabla_\mathbf{y}\psi_0|^2 + \frac{1}{4}|\psi_0|^4\right) d\mathbf{y}, \quad (4.152)$$

also for a.e. t.

As aforementioned, once we have Theorem 4.2, the proof of Theorem 4.3 follows by repeating line by line the arguments in [29, Section 5]. For completeness, we give a sketch of the proof below.

First, we observe that, in view of (4.142), the right hand side of (4.132) can be understood to be a constant independent of δ, and therefore, as in the previous sections we can conclude that (4.145)-(4.148) hold and, moreover,

$$(\nabla \times \mathbf{H}_\delta) \times \mathbf{H}_\delta \to (\nabla \times \mathbf{H}) \times \mathbf{H}, \text{ in the sense of distributions,} \tag{4.153}$$

$$\nabla \times (\mathbf{u}_\delta \times \mathbf{H}_\delta) \to \nabla \times (\mathbf{u} \times \mathbf{H}) \text{ in the sense of distributions,} \tag{4.154}$$

$$\rho_\delta \mathbf{u}_\delta \to \rho \mathbf{u} \text{ in } C([0,T]; L^{2\gamma/(\gamma+1)}_{weak}(\Omega)), \tag{4.155}$$

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \rho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^2(0,T; L^{c_2}(\Omega)), \tag{4.156}$$

where, $c_2 = 2\gamma/(1+\gamma) > 1$.

The next step is to deduce a higher order uniform estimate on the densities in order to conclude that $a\rho_\delta^\gamma$ has a weak limit and that $\delta\rho_\delta^\beta \to 0$ in the sense of distributions as $\delta \to 0$.

In this direction we have the following result corresponding to Lemma 5.1 in [29].

Lemma 4.13. *For any $\zeta \in C_0^\infty(0,T)$ we have*

$$\int_0^T \int_\Omega \zeta(\delta\rho_\delta^\beta + a\rho_\delta^\gamma)\log(1+\rho_\delta)d\mathbf{x}ds \leq C, \tag{4.157}$$

where, $C > 0$ is a constant independent of δ.

The proof is similar to that of Lemma 4.9. It consists in using a particular conveniently chosen test function in the momentum equation (4.126) equation. The fact that ρ_δ is a renormalized solution of the continuity equation (4.125) is important in order to estimate the terms involving time derivatives.

More specifically, we introduce the operator

$$B : \left\{ f \in L^p(\Omega) : \int_\Omega f d\mathbf{x} = 0 \right\} \to [W_0^{1,p}(\Omega)]^2, \tag{4.158}$$

being a bounded linear operator, that is,

$$||B[f]||_{W_0^{1,p}(\Omega)} \leq c(p)||f||_{L^p(\Omega)}, \text{ for any } 1 < p < \infty;$$

such that the function $\mathbf{W} = B[f] \in \mathbb{R}^3$ solves the problem

$$\text{div}\mathbf{W} = f \text{ in } \Omega, \quad \mathbf{W}|_{\partial\Omega} = 0. \tag{4.159}$$

Moreover, if f can be written in the form $f = \text{div}\mathbf{g}$ for some $\mathbf{g} \in L^r$, $\mathbf{g} \cot \mathbf{n}|_{\partial\Omega} = 0$, then

$$||B[f]||_{L^r} \leq c(r)||\mathbf{g}||_{L^r}.$$

With this notation, we can define the test function φ by its coordinates

$$\varphi_j = \zeta(t)B_i\left[\log(1+\rho_\delta) - \frac{1}{|\Omega|}\int_\Omega \log(1+\rho_\delta)d\mathbf{x}\right], j = 1, 2, \tag{4.160}$$

and $\varphi_3 = 0$ and use it the momentum equation (4.126) in order to obtain an identity of the form

$$\int_0^T \int_\Omega \zeta(\delta\rho_\delta^\beta + a\rho_\delta^\gamma)\log(1+\rho_\delta)d\mathbf{x}ds = \sum_j I_j \tag{4.161}$$

similar to identity (4.87). At this point, the fact that ρ_δ is a renormalized solution of the continuity equation comes into play in order to treat the integral corresponding to the term $(\rho_\delta \mathbf{u}_\delta)_t$ of (4.126). This is done in a similar way as was done in (4.87) only this time instead of (4.86) we have the identity

$$(\log(1+\rho_\delta))_t + \operatorname{div}(\log(1+\rho_\delta)\mathbf{u}_\delta) + \Big(\frac{\rho_\delta}{1+\rho_\delta} - \log(1+\rho_\delta)\Big)\operatorname{div}\mathbf{u} = 0. \tag{4.162}$$

Now, by virtue of inequality (4.132) and using the properties of the operator B, all the integrals on the right hand side of (4.161) turn out to be bounded by a constant independent of δ. We omit the details.

Estimate (4.157) can be used in order to conclude that

$$\int_0^T \int_\Omega \delta\rho_\delta^\beta d\mathbf{x}ds \to 0, \tag{4.163}$$

as $\delta \to 0$. This is shown in [29] by a clever application of the Hölder inequality in the Orlicz space associated to the Young function $s \to (1+s)\log(1+s) - s$.

Furthermore, estimate (4.157) can also be employed in order to show that the sequence $a\rho_\delta^\gamma$ has a weakly convergent subsequence. This is due to the following general result (see [23, Proposition 2.1]).

Lemma 4.14. *Let $O \subseteq \mathbb{R}^M$ be a bounded open set. Let $\{\mathbf{v}_n\}_{n=1}^\infty$ be a sequence of measurable functions,*

$$\mathbf{v}_n : O \to \mathbb{R}^N,$$

such that

$$\sup_n \int_O \Phi(|\mathbf{v}_n|)d\mathbf{y} < \infty$$

for a certain continuous function $\Phi : [0,\infty) \to [0,\infty)$.

Then, there exists a subsequence (not relabelled) such that

$$g(\mathbf{v}_n) \to \overline{g(\mathbf{v})} \text{ weakly in } L^1(O)$$

for all continuous functions $g : \mathbb{R}^N \to \mathbb{R}$ *satisfying*

$$\lim_{|z|\to\infty} \frac{|g(\mathbf{z})|}{\Phi(|\mathbf{z}|)} = 0.$$

In light of this Lemma we can assume that

$$a\rho_\delta^\gamma \to a\overline{\rho^\gamma}. \tag{4.164}$$

As a consequence, we conclude that the limit functions satisfy the following system

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{4.165}$$

$$\begin{aligned}(\rho\mathbf{u})_t &+ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + a\nabla\overline{\rho^\gamma} \\ &= (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\Big(\lambda(\operatorname{div}\mathbf{u})\mathrm{Id} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\Big),\end{aligned} \tag{4.166}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \tag{4.167}$$

$$\operatorname{div}\mathbf{H} = 0. \tag{4.168}$$

$$i\psi_t + \Delta_{\mathbf{y}}\psi = |\psi|^2\psi, \tag{4.169}$$

and all that is left to do is show strong convergence of the densities so that, in fact, $a\overline{\rho^\gamma} = a\rho^\gamma$.

As in Subsection 4.3.3, this is a consequence of the weak continuity of the effective viscous flux together with the fact that ρ is solves (4.134) in the sense of renormalized solutions. This last assertion (that ρ is a renormalized solution of the continuity equation) is not straightforward. In particular, we cannot apply Lemma 4.12 as we do not have a bound available for the $L^2(\Omega\times(0,T))$-norm of ρ. Remember that we are only assuming that $\gamma > 1$ and the best bound we have for ρ so far is the finiteness of its $L^\infty(0,T;L^\gamma(\Omega))$-norm.

Let us introduce the cut-off functions

$$T_k(z) = kT\Big(\frac{z}{k}\Big), \text{ for } z\in\mathbb{R} \text{ and } k = 1,2,...$$

where, $T \in C^\infty(\mathbb{R})$ is concave and satisfies

$$T(z) = \begin{cases} z, & z \leq 1, \\ 2, & z \geq 3. \end{cases}$$

As ρ_δ solves (4.125) in the sense of renormalized solutions we have

$$T_k(\rho_\delta)_t + \operatorname{div}(T_k(\rho_\delta)\mathbf{u}_\delta) + (T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta))\operatorname{div}\mathbf{u}_\delta = 0, \tag{4.170}$$

in the sense of distributions.

Passing to the limit as $\delta \to 0$ we have

$$\overline{T_k(\rho)}_t + \operatorname{div}(\overline{T_k(\rho)}\,\mathbf{u}) + \overline{(T_k'(\rho)\rho - T_k(\rho))\operatorname{div}\mathbf{u}} = 0, \tag{4.171}$$

in the sense of distributions, where, as usual, the over line stands for a weak limit of the sequence indexed by δ. Note that $\overline{T_k(\rho)\mathbf{u}} = \overline{T_k(\rho)}\,\mathbf{u}$ as in view of (4.170),

$$T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)),$$

with $L^\gamma(\Omega)$ being compactly embedded in $H^{-1}(\Omega)$.

Let us define φ given by

$$\varphi_j(\mathbf{x},t) = \zeta(t)\eta(\mathbf{x})\mathcal{A}_j[\xi T_k(\rho_\delta)], \text{ for } j = 1,2$$

and $\varphi_3 = 0$, where, $\zeta \in C_0^\infty(0,T)$, $\eta, \xi \in C_0^\infty(\Omega)$ and $\mathcal{A}$ is the operator introduced in (4.76). Using φ as test function in equation (4.126) and using (4.170) we find an identity similar to (4.103). Namely,

$$\int_0^T \int_\Omega \zeta\eta\xi\Big(a\rho_\delta^\gamma + \delta\rho_\delta^\beta - (\lambda+2\mu)\operatorname{div}\mathbf{u}_\delta\Big)T_k(\rho_\delta)d\mathbf{x}ds = \sum_j J_j^\delta \tag{4.172}$$

Similarly, using the test function

$$\varphi_j(\mathbf{x},t) = \zeta(t)\eta(\mathbf{x})\mathcal{A}_j[\xi T_k(\rho)], \text{ for } j = 1,2$$

and $\varphi_3 = 0$ and taking (4.171) into account we deduce the respective analogue to

(4.104):

$$\int_0^T \int_\Omega \zeta\eta\xi\Big(\overline{a\rho^\gamma} - (\lambda+2\mu)\mathrm{div}\mathbf{u}\Big)\overline{T_k(\rho)}d\mathbf{x}ds = \sum_j J_j. \tag{4.173}$$

As in the proof of Lemma 4.10 we can show that each one of the terms on the right hand side of (4.172) converges to its counterpart in (4.173) by using the the properties of the operator $\mathcal{A}$ and Lemma 4.11. Thus, we end up with the following result of weak continuity of the effective viscous flux.

Lemma 4.15. *For any $\zeta \in C_0^\infty(0,T)$ and $\eta, \xi \in C_0^\infty(\Omega)$ we have*

$$\lim_{\delta\to 0}\int_0^T \int_\Omega \zeta\eta\xi\Big(a\rho_\delta^\gamma - (\lambda+2\mu)\,div\mathbf{u}_\delta\Big)T_k(\rho_\delta)d\mathbf{x}ds$$
$$= \int_0^T \int_\Omega \zeta\eta\xi\Big(\overline{a\rho^\gamma} - (\lambda+2\mu)\,div\mathbf{u}\Big)\overline{T_k(\rho)}d\mathbf{x}ds. \tag{4.174}$$

Having this result, we can prove the following estimate, which is crucial in the proof of the fact that ρ is a renormalized solution of the continuity equation.

Lemma 4.16. *There is a constant $C>0$ independent of k such that*

$$\limsup_{\delta\to 0}||T_k(\rho_\delta) - T_k(\rho)||_{L^{\gamma+1}(\Omega\times(0,T))}dxds \le C. \tag{4.175}$$

Proof. Observe that

$$\begin{aligned}\rho_\delta^\gamma T_k(\rho_\delta) - \overline{\rho^\gamma}\,\overline{T_k(\rho)} =& (\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) - T_k(\rho)) + (\overline{\rho^\gamma} - \rho^\gamma)(T_k(\rho) - \overline{T_k(\rho)})\\ &+ (\rho_\delta^\gamma - \overline{\rho_\delta^\gamma})T_k(\rho) + \rho^\gamma(T_k(\rho_\delta) - \overline{T_k(\rho)}).\end{aligned}$$

Since the functions $\tau \to a\tau^\gamma$ and $\tau \to -T_k(\tau)$ are convex, by Lemma 4.8 we have that

$$\overline{\rho^\gamma} \ge \rho^\gamma \text{ and } T_k(\rho) \ge \overline{T_k(\rho)} \text{ a.e. on } \Omega\times(0,T).$$

Consequently $(\overline{\rho^\gamma} - \rho^\gamma)(T_k(\rho) - \overline{T_k(\rho)}) \ge 0$.

Also note that

$$(z^\gamma - y^\gamma)(T_k(z) - T_k(y)) \ge \gamma|T_k(z) - T_k(y)|^{\gamma+1}, \text{ for all } x,y\ge 0.$$

As a result

$$(\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) - T_k(\rho)) \ge \gamma|T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1},$$

and thus

$$\int_0^T \int_\Omega \Big(\rho_\delta^\gamma T_k(\rho_\delta) - \overline{\rho^\gamma}\,\overline{T_k(\rho)}\Big) d\mathbf{x}ds$$
$$\geq \int_0^T \int_\Omega \gamma |T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1} d\mathbf{x}ds + \int_0^T \int_\Omega \Big((\rho_\delta^\gamma - \overline{\rho_\delta^\gamma})T_k(\rho) + \rho^\gamma(T_k(\rho_\delta) - \overline{T_k(\rho)})\Big) d\mathbf{x}ds \tag{4.176}$$

On the other hand, observe that

$$\mathrm{div}\mathbf{u}_\delta T_k(\rho_\delta) - \mathrm{div}\mathbf{u}\overline{T_k(\rho)} = \mathrm{div}\mathbf{u}_\delta\Big(T_k(\rho_\delta) - T_k(\rho)\Big) + \mathrm{div}\mathbf{u}\Big(T_k(\rho) - \overline{T_k(\rho)}\Big).$$

Therefore

$$\begin{aligned} -\int_0^T \int_\Omega &\Big(\mathrm{div}\mathbf{u}_\delta T_k(\rho_\delta) - \mathrm{div}\mathbf{u}\overline{T_k(\rho)}\Big) d\mathbf{x}ds \\ &\geq -||\mathrm{div}\mathbf{u}_\delta||_{L^2(\Omega\times(0,T))} ||T_k(\rho_\delta) - T_k(\rho)||_{L^2(\Omega\times(0,T)))} \\ &\quad - ||\mathrm{div}\mathbf{u}||_{L^2(\Omega\times(0,T))} ||T_k(\rho) - \overline{T_k(\rho)}||_{L^2(\Omega\times(0,T)))} \end{aligned}$$

Now, note that $T_k(\rho) - \overline{T_k(\rho)}$ is a weak limit of $T_k(\rho) - T_k(\rho_\delta)$. Then we have that

$$||T_k(\rho) - \overline{T_k(\rho)}||_{L^2(\Omega\times(0,T)))} \leq \liminf_{\delta\to 0} ||T_k(\rho) - T_k(\rho_\delta)||_{L^2(\Omega\times(0,T)))}$$

Also recall that the sequence u_δ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$. Thus, since $\gamma > 1$ we see that

$$-\int_0^T \int_\Omega \Big(\mathrm{div}\mathbf{u}_\delta T_k(\rho_\delta) - \mathrm{div}\mathbf{u}\overline{T_k(\rho)}\Big) d\mathbf{x}ds$$
$$\geq -C - \frac{\gamma}{4}||T_k(\rho_\delta) - T_k(\rho)||^{\gamma+1}_{L^{\gamma+1}(\Omega\times(0,T)))} - \frac{\gamma}{4}\limsup_{\delta\to 0} ||T_k(\rho) - T_k(\rho_\delta)||^{\gamma+1}_{L^{\gamma+1}(\Omega\times(0,T)))}. \tag{4.177}$$

Adding (4.176) and (4.177), taking the limit as $\delta \to 0$ and using Lemma 4.15 we arrive at (4.175). □

With this result at hand we can finally prove the following result, which is essentially the same as Lemma 5.4 in [29] (cf. [23, Proposition 6.3]).

Lemma 4.17. *The limit functions ρ and $\mathbf{u}$ solve* (4.134) *in the sense of renormalized* *solutions. That is*

$$B(\rho)_t + div(B(\rho)\mathbf{u}) + b(\rho)\, div\mathbf{u} = 0, \tag{4.178}$$

in the sense of distributions, for any functions

$$B \in C[0,\infty) \cap C^1(0,\infty), \quad b \in C[0,\infty),\ \textit{bounded on } [0,\infty), \quad B(0) = b(0) = 0, \tag{4.179}$$

satisfying

$$b(z) = B'(z)z - B(z). \tag{4.180}$$

Proof. First we point out that it is enough to show that (4.178) holds for any

$$B \in C^1[0,\infty), \quad B'(z) = 0,\ \text{for all } z \geq z_B, \quad b(z) = B'(z)z - B(z). \tag{4.181}$$

Indeed, a simple approximation argument combined with Lebesgue dominated convergence theorem we recover the general case.

Let us assume that B and b satisfy (4.181). Recall that we have (4.171). Regularizing this identity via the smoothing operators $[\cdot]^{\omega}_{\mathbf{x}}$, multiplying by $B'(\overline{T_k(\rho)})$ and letting $\omega \to 0$ we get

$$B(\overline{T_k(\rho)})_t + \mathrm{div}(B(\overline{T_k(\rho)})\,\mathbf{u}) + b(\overline{T_k(\rho)})\mathrm{div}\mathbf{u} = B'(\overline{T_k(\rho)})\ \overline{(T_k(\rho) - T_k'(\rho)\rho)\mathrm{div}\mathbf{u}}, \tag{4.182}$$

in the sense of distributions. The idea now is to let $k \to \infty$.

Let $1 \leq p < \gamma$. By the weak lower semicontinuity of the norm we have that

$$\begin{aligned} ||\overline{T_k(\rho)} - \rho||^p_{L^p(\Omega\times(0,T))} &\leq \liminf_{\delta\to 0} ||T_k(\rho_\delta) - \rho_\delta||^p_{L^p(\Omega\times(0,T))} \\ &\leq k^{p-\gamma} \sup_{\delta} ||\rho_\delta||^p_{L^\gamma(\Omega\times(0,T))} \\ &\leq Ck^{p-\gamma}, \end{aligned} \tag{4.183}$$

where the right hand side tends to zero as $k \to \infty$. As a consequence

$$B(\overline{T_k(\rho)}) \to B(\rho), \quad b(\overline{T_k(\rho)}) \to b(\rho) \text{ in } L^r(\Omega\times(0,T)) \text{ for any } r \geq 1. \tag{4.184}$$

In order to complete the proof, we have to show that the right hand side of (4.182) tends to zero as $k \to \infty$. To this end, we estimate

$$\begin{aligned}
&\left\| B'(\overline{T_k(\rho)})\, \overline{(T_k(\rho) - T_k'(\rho)\rho)\mathrm{div}\mathbf{u}} \right\|_{L^1(\Omega\times(0,T)))} \\
&\qquad \leq \max_{z\geq 0} |B'(z)| \int_{\{\overline{T_k(\rho)}\leq z_B\}} \left| \overline{(T_k(\rho) - T_k'(\rho)\rho)\mathrm{div}\mathbf{u}} \right| d\mathbf{x}ds \\
&\qquad \leq \max_{z\geq 0} |B'(z)| \sup_{\delta} ||\mathrm{div}\mathbf{u}||_{L^2(\Omega\times(0,T))} \liminf_{\delta\to 0} ||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||_{L^2(\{\overline{T_k(\rho)}\leq z_B\})}.
\end{aligned} \tag{4.185}$$

By interpolation we have that

$$\begin{aligned}
&||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||_{L^2(\{\overline{T_k(\rho)}\leq z_B\})} \\
&\qquad\qquad \leq ||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||^{\omega}_{L^1(\Omega\times(0,T)))} ||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||^{1-\omega}_{L^{\gamma+1}(\{\overline{T_k(\rho)}\leq z_B\})}
\end{aligned}$$

for a certain $0 < \omega < 1$. Similarly as above, we have

$$||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||^{\omega}_{L^1(\Omega\times(0,T)))} \leq 2k^{1-\gamma} \sup_{\delta} ||\rho_\delta||_{L^\gamma(\Omega\times(0,T))}, \tag{4.186}$$

which tends to zero as $k \to \infty$. Finally, by virtue of Lemma 4.16

$$\begin{aligned}
&\limsup_{\delta\to 0} ||T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta||_{L^{\gamma+1}(\{\overline{T_k(\rho)}\leq z_B\})} \\
&\quad \leq 2 \limsup_{\delta\to 0} ||T_k(\rho_\delta)||_{L^{\gamma+1}(\{\overline{T_k(\rho)}\leq z_B\})} \\
&\quad \leq 2\Big(\limsup_{\delta\to 0} ||T_k(\rho_\delta) - T_k(\rho)||_{L^{\gamma+1}(\Omega\times(0,T)))} + ||T_k(\rho) - \overline{T_k(\rho)}||_{L^{\gamma+1}(\Omega\times(0,T)))} \\
&\qquad\quad + ||\overline{T_k(\rho)}||_{L^{\gamma+1}(\{\overline{T_k(\rho)}\leq z_B\})} \Big) \\
&\quad \leq 4C + 2z_B(T|\Omega|)^{1/(\gamma+1)}.
\end{aligned} \tag{4.187}$$

All of this information put together with (4.185) implies that the right hand side of (4.182) tends to zero as $k \to \infty$. This and (4.184) yield (4.179). $\square$

At last, we are ready for the final step in the proof the Theorem 4.3, which consists in showing strong convergence of the sequence of densities. We present here the proof contained in [29]. The argument is similar to the one in Section 4.3.3.

As in [29], we introduce the functions $L_k \in C^1(\mathbb{R})$, $k = 1, 2, ...$ given by

$$L_k(z) = \begin{cases} z\log(z), & \text{for } 0 \leq z \leq k, \\ z\log(k) + z\int_k^z \frac{T_k(s)}{s^2} ds, & \text{for } z \geq k. \end{cases}$$

Then, L_k can be written as

$$L_k(z) = c_k z + B_k(z),$$

where $B_k(z)$ satisfies (4.181). Since ρ_δ and $\mathbf{u}_\delta$ are renormalized solutions of equation (4.134) we have

$$L_k(\rho_\delta)_t + \operatorname{div}(L_k(\rho_\delta)\mathbf{u}_\delta) + T_k(\rho_\delta)\operatorname{div}\mathbf{u}_\delta = 0. \tag{4.188}$$

Similarly, by Lemma 4.17 we have

$$L_k(\rho)_t + \operatorname{div}(L_k(\rho)\mathbf{u}) + T_k(\rho)\operatorname{div}\mathbf{u} = 0, \tag{4.189}$$

in the sense of distributions. Taking the difference of (4.188) and (4.189), and integrating we have

$$\begin{aligned}
&\int_\Omega (L_k(\rho_\delta) - L_k(\rho))\Phi d\mathbf{x} \\
&\quad = \int_0^t \int_\Omega \Big((L_k(\rho_\delta)\mathbf{u}_\delta - L_k(\rho)\mathbf{u}) \cdot \nabla\Phi + (T_k(\rho)\operatorname{div}\mathbf{u} - T_k(\rho_\delta)\operatorname{div}\mathbf{u}_\delta)\Phi\Big) d\mathbf{x}ds,
\end{aligned} \tag{4.190}$$

for any $\Phi \in C_0^\infty(\Omega)$.

Now, as $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$ then (see [23, Theorem 4.2])

$$\frac{|u|}{dist(x,\partial\Omega)} \in L^2(\Omega \times (0,T)).$$

Considering a sequence of functions $\Phi_m \in C_0^\infty(\Omega)$ which approximate the characteristic function of Ω satisfying

$$\begin{aligned}
&0 \le \Phi \le 1, \quad \Phi(x) = 1 \text{ fox all } x \text{ such that } dist(x,\partial\Omega) \ge \frac{1}{m}, \\
&\text{and } |\nabla\Phi_m| \le 2m, \text{ for all } x \in \Omega,
\end{aligned}$$

and using them in (4.190) and letting $m \to \infty$ first and then $\delta \to 0$ we see that

$$\int_\Omega (\overline{L_k(\rho)} - L_k(\rho))d\mathbf{x} = \int_0^t \int_\Omega T_k(\rho)\operatorname{div}\mathbf{u} d\mathbf{x}ds - \int_0^t \int_\Omega \overline{T_k(\rho)\operatorname{div}\mathbf{u}} d\mathbf{x}ds. \tag{4.191}$$

Note that by (4.145) we can assume

$$\rho_\delta \log(\rho_\delta) \to \overline{\rho \log(\rho)} \text{ weakly-* in } L^\infty(0,T;L^r(\Omega)) \text{ for any } 1 \le r < \gamma.$$

Further, noting that $z \to L_k(z)$ approximates the function $z \to z\log(z)$ and also noting that (4.145) implies that

$$|\{(\mathbf{x},t) \in \Omega \times (0,T) : \rho_\delta(\mathbf{x},t) \ge k\}| \to 0 \text{ as } k \to \infty$$

uniformly in δ, as in [29], we have

$$||\overline{L_k(\rho)} - \overline{\rho\log(\rho)}||_{L^\infty(0,T;L^r(\Omega))} \le \liminf_{\delta\to 0} ||L_k(\rho_\delta) - \rho_\delta \log(\rho_\delta)||_{L^\infty(0,T;L^r(\Omega))} \to 0,$$

as $k \to \infty$.

Similarly we have

$$L_k(\rho) \to \rho\log(\rho) \text{ in } L^\infty(0,T;L^r(\Omega)), \text{ for all } 1 \le \alpha < \gamma.$$

Now, similarly as in estimatives (4.185)-(4.187), we can estimate

$$\left|\int_0^t \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}\mathbf{u} d\mathbf{x} ds\right| \le ||\mathrm{div}\mathbf{u}||_{L^2(\Omega\times(0,T))} ||T_k(\rho) - \overline{T_k(\rho)}||_{L^2(\Omega\times(0,T))},$$

where

$$\begin{aligned}
&||T_k(\rho) - \overline{T_k(\rho)}||_{L^2(\Omega\times(0,T))} \\
&\qquad \le ||T_k(\rho) - \overline{T_k(\rho)}||^{\omega}_{L^1(\Omega\times(0,T))} ||T_k(\rho) - \overline{T_k(\rho)}||^{1-\omega}_{L^{\gamma+1}(\Omega\times(0,T))},
\end{aligned}$$

for a certain $0 < \omega < 1$, wherein, similarly as in (4.183),

$$||T_k(\rho) - \overline{T_k(\rho)}||_{L^1(\Omega\times(0,T))} \to 0, \text{ as } k \to \infty$$

and by virtue of Lemma 4.16 we have that

$$||\overline{T_k(\rho)} - T_k(\rho)||_{L^{\gamma+1}(\Omega\times(0,T))} \le C$$

Thus, letting $k \to \infty$ in (4.191) we have

$$\int_\Omega (\overline{\rho\log(\rho)} - \rho\log(\rho))d\mathbf{x} = \lim_{k\to\infty}\int_0^t\int_\Omega \left(\overline{T_k(\rho)}\mathrm{div}\mathbf{u} - \overline{T_k(\rho)\mathrm{div}\mathbf{u}}\right)d\mathbf{x}ds, \tag{4.192}$$

Regarding the right hand side, we can use Lemma 4.15 and the monotonicity of the pressure function $z \to az^\gamma$, as in (4.118)-stongepsrhologrho to conclude that

$$\int_\Omega (\overline{\rho\log(\rho)} - \rho\log(\rho))d\mathbf{x} \le 0, \tag{4.193}$$

which, in light of Lemma 4.8 implies that

$$\overline{\rho\log(\rho)} = \rho\log(\rho),$$

and this, in turn, shows that

$$\rho_\delta \to \rho \text{ in } L^1(\Omega\times(0,T)) \text{ and a.e.,} \tag{4.194}$$

and the proof is complete.

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