Bidimensional Adverse Selection: Analytical and Numerical Solutions *

Aloisio Araujo[1,2], Braulio Calagua[†1], and Sergei Vieira[3]

[1]IMPA, Estrada Dona Castorina 110, Rio de Janeiro, Brasil
[2]EPGE-FGV, Praia de Botafogo 190, Rio de Janeiro, Brasil
[3]Ibmec, Av. Presidente Wilson 118, Rio de Janeiro, Brasil

February 9, 2018

Abstract

This paper studies adverse selection problems with a one-dimensional principal instrument and a two-dimensional agent type. We provide an optimality condition that characterizes the bunching of types that allows us to obtain analytical solutions for examples from the literature and for a new example that is far from the linear-quadratic case. Additionally, by comparing types by their marginal valuation for the instrument, we reduce the number of incentive compatibility constraints, thus making the discretized problem computationally tractable for relatively fine discretizations.

Keywords: two-dimensional screening, bunching, non single-crossing, incentive compatibility, regulation model.

JEL Classification: D42, D82.

*We are grateful to Alexandre Belloni, Carolina Parra and Mauricio Villalba for their helpful comments. This research was supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq).

†Corresponding author.

E-mail addresses: aloisio@impa.br (A. Araujo), bcalagua@impa.br (B. Calagua), srgvie@gmail.com (S. Vieira)

1 Introduction

Although the importance of multi-dimensions for modeling agents' characteristics is recognized, few bidimensional models have been analyzed in the literature since the seminal work of Laffont et al. (1987). This is due to the technical difficulties in deriving the optimal solution and to the complexity in obtaining numerical solutions.

The purpose of this paper is to provide a methodology that can be used for numerically solving a wide variety of bidimensional screening problems and to provide a necessary optimality condition that can be used to derive the analytical solution in particular examples.

In a one-dimensional context without the single-crossing property, Araujo and Moreira (2010) obtained necessary conditions for implementability and optimality that we generalize to the two-dimensional case. For implementability, they established that if $q(\theta) = q(\widehat{\theta}) = q$ with $\theta \neq \widehat{\theta}$, then $v_q(q, \theta) = v_q(q, \widehat{\theta})$, meaning that the marginal valuation must be the same for different types that receive the same allocation. This condition is extended as $\frac{d}{ds} v_q(k, a(s), b(s)) = 0$ over a contour line $q(a(s), b(s)) = k$.

For optimality, Araujo and Moreira (2010) established that if $q(\theta) = q(\widehat{\theta}) = q$ with $\theta \neq \widehat{\theta}$, then

$$\frac{g_q}{v_{q\theta}}(q, \theta) = \frac{g_q}{v_{q\theta}}(q, \widehat{\theta})$$

where g is the one-dimensional virtual surplus. This necessary condition for discrete one-dimensional pooling types is extended for two-dimensional continuum types as

$$\int_0^{\overline{s}(r)} \frac{G_q}{v_{qa}}(\phi(r), a(r, s), b(r, s)) ds = 0$$

over a contour line $q(a(r, s), b(r, s)) = \phi(r)$ for a fixed r. Function G denotes the two-dimensional virtual surplus, and function $\phi(\cdot)$ is the optimal allocation for types on axis X.

This necessary condition for optimality allows us to solve the linear-quadratic example of Laffont et al. (1987) and the variation proposed by Deneckere and Severinov (2015). We also solve a log-valuation example in the nonlinear pricing framework. This log-valuation is derived from strictly convex demand curves, which is quite different than the linear demand associated with the linear-quadratic valuation.

Next, by defining a pre-order that compares types by their marginal valuation for the instrument, we prove that it is sufficient to consider (for each type of agent) incentive compatibility constraints over a one-dimensional set rather than the entire two-dimensional set as required by the definition. As a consequence, a significant number of incentive constraints are ruled out in the discretized problem, thus making it computationally tractable for a relatively fine discretization.

With this methodology, we numerically solve the regulation model introduced by Lewis and Sappington (1988b) and then reviewed by Armstrong (1999) , who showed that Lewis and Sappington's solution was incorrect. Because this is a model with an unknown analytical solution, it is important to know the numerical solution. Armstrong (1999) has conjectured that, in this model, it is optimal to exclude a positive mass of agents as in the non-linear pricing setting. However, the numerical solution suggests that the exclusion should not be optimal in this case. We provide a technical and an economic argument about this feature.

1.1 Related Literature

As mentioned, Laffont et al. (1987) solved the case where the monopolist sells a single product and considered customers' linear demand curves with an unknown slope and intercept.

McAfee and McMillan (1988) studied the problem when the dimensionality of products is no bigger than the dimensionality of characteristics. They introduced the generalized single-crossing (GSC), which is a condition under which the first-order and second-order necessary conditions for the customer's problem are also sufficient for implementability. They have characterized the optimal contract for single products and multiple types under the GSC by generalizing the result from Laffont et al. (1987).

Rochet and Choné (1998) established the existence of the optimal contract and provided the characterization in the case that the dimensionalities of products and characteristics are the same and the customer's valuation is linear with the type. They introduced the sweeping procedure as a generalization of the ironing procedure for dealing with bunching in the multidimensional context.

Basov (2001) introduced the Hamiltonian approach as a tentative method of generalizing Rochet and Choné (1998) to the case when the numbers of products and characteristics may be different. Later on, these techniques were extended in Basov (2005) to address more general customer preferences. There are also cases where the multidimensionality can be reduced by aggregation (for example, Biais et al. (2000)) or separability (for example, Wilson (1993), Armstrong (1996)) to unidimensional problems.

Some numerical methods to address the problem are described in Wilson (1995). Although those methods were formulated to allow for multidimensional types and products, they were designed to solve a relaxed version of the problem in which just the local incentive compatibility constraints are assumed to be binding. Even when local IC constraints are sufficient in one dimension, this is not the case in multiple dimensions. Therefore, we cannot rely on these approximations as the solution of the complete problem.

1.2 Outline of the Paper

The plan of the paper is as follows. The model is presented in Section 2. In Section 3, we derive the partial differential equation related to the incentive compatibility constraints. The optimality conditions are established in Section 4, and we use them in Section 5 to analytically solve two examples. Section 6 is dedicated to explaining the reduction of incentive constraints that is used in Section 7 for numerically solving a regulation model. All proofs are relegated to the Appendix.

2 Model

To describe the model, we concentrate on the nonlinear pricing framework in the style of Mussa and Rosen (1978). The customer (agent) has quasi-linear utility $v(q, a, b) - t$, where $v(q, a, b)$ is the value of the customer's type $(a, b) \in \Theta = [0, 1] \times [0, 1]$ when it consumes $q \in \mathbb{R}_+$ and t is the monetary payment. The firm is a profit maximizing monopolist producing a single product $q \in \mathbb{R}_+$. The firm does not observe (a, b) but knows the probability distribution over Θ according to the differentiable density function $\rho(a, b) > 0$. The firm's revenue is $t - C(q)$, where $C(\cdot)$ is a C^2 cost function, with $C(\cdot) \geq 0$, $C'(\cdot) \geq 0$ and $C''(\cdot) \geq 0$.

Using the revelation principle, we can restrict our attention to direct and truthful mechanisms[1]. Thus, the monopolist's problem consists of choosing

[1] The revelation principle has been enunciated in Gibbard (1973).

the contract $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves

$$\max_{q(\cdot),t(\cdot)} \int_0^1 \int_0^1 \big(t(a, b) - C(q(a, b))\big)\rho(a, b)da\, db \tag{MP}$$

subject to *individual rationality* constraints,

$$v(q(a, b), a, b) - t(a, b) \geq 0 \tag{IR}$$

and *incentive compatibility* constraints,

$$(a, b) \in \text{argmax}_{(a',b')\in\Theta}\{v(q(a', b'), a, b) - t(a', b')\} \quad \forall\ (a, b) \in \Theta \tag{IC}$$

A contract $(q(\cdot), t(\cdot))$ is incentive compatible if it satisfies the (IC) constraints. We say that $q(\cdot)$ is implementable if we can find a monetary payment $t(\cdot)$ such that the pair $(q(\cdot), t(\cdot))$ is incentive compatible. For an incentive-compatible contract, the informational rent is defined as

$$V(a, b) = v(q(a, b), a, b) - t(a, b) \tag{1}$$

The informational rent is used to eliminate the monetary payment $t(\cdot)$ from the monopolist's problem (MP). In the single-product and single-characteristic case, by combining the integration by parts and the envelope theorem from Milgrom and Segal (2002), one can derive a new expression for the monopolist's expected profit that depends only on $q(\cdot)$.

This idea can be extended to the multi-dimensional context. Indeed, in the case with multiple characteristics, Armstrong (1996) proposed the "integration by rays" technique that also results in an expression for the monopolist's expected profit that depends only on $q(\cdot)$. However, when we have multiple characteristics, there are several paths connecting distinct customers. Therefore, instead of using "integration by rays", it may be convenient to choose a different path for the integration. This decision, of course, depends on the specific problem to be addressed.

In this paper, we are not concerned with the particular method used. We will assume that the monopolist's expected payoff is given by

$$\int_0^1 \int_0^1 g(q(a, b), a, b)\rho(a, b)da\, db \tag{2}$$

As in the one-dimensional case, we will call $g(\cdot)$ the virtual surplus. To simplify our notation, we define $G(q, a, b) = g(q, a, b)\rho(a, b)$. We make the following assumptions about the valuation function v and the function G.

Assumptions. *$v(q,a,b)$ is three times differentiable, and $G(q,a,b)$ is two times differentiable. They satisfy the following:*

1. *$v(q^{\text{out}},a,b)$ is a constant.*
2. *$v_a > 0$ and $v_b < 0$ when $q > 0$.*
3. *$v_{qa} > 0$ and $v_{qb} < 0$ when $q > 0$.*
4. *$v_{qq} < 0$ and $G_{qq} < 0$.*

Assumption 1 is usually presented as $v(0,a,b) = 0$ because in nonlinear pricing, the exit option is $q^{\text{out}} = 0$, and any agent assigns it zero value. However, in other adverse selection problems, q^{out} could be endogenously determined. Therefore, we assume this more general expression. Assumption 2 implies that the informational rent increases with a and decreases with b. It also implies that the boundary between the participation and exclusion regions is increasing. Assumption 3 is the single-crossing condition in each direction. As a consequence, it requires that an implementable $q(a,b)$ be increasing with a and decreasing with b. Finally, Assumption 4 requires the strict concavity of both $v(\cdot,a,b)$ and $G(\cdot,a,b)$ for each (a,b)-type customer. The last assumption ensures that the first-order necessary condition for a $q(a,b)$ that maximizes expression 2 is also sufficient for optimality.

Although our results do not depend on how virtual surplus is determined, under previous Assumptions 1 and 2, we are able to provide an expression of G in the case that the types are independently distributed over each direction.

Proposition 2.1. *Assume $q(0,b) = q^{\text{out}}$ and $\rho(a,b) = f(a)h(b)$. Then,*

$$G(q,a,b) = \Big(v(q,a,b) - C(q) - \frac{1-F(a)}{f(a)} v_a(q,a,b)\Big) f(a)h(b)$$

, where $F(\cdot)$ is the cumulative distribution over a.

3 Local Incentive Condition

Now, we present the partial differential equation (PDE) derived from the (IC) constraints. Consider an incentive compatible contract (q,t). Each (a,b)-type customer must solve the maximization problem

$$\max_{(a',b')\in\Theta} \{v(q(a',b'),a,b) - t(a',b')\} \tag{3}$$

The first-order necessary optimality conditions for problem 3 are

$$
\begin{aligned}
v_q(q(a,b),a,b)q_a(a,b) &= t_a(a,b)\\
v_q(q(a,b),a,b)q_b(a,b) &= t_b(a,b)
\end{aligned}
\tag{4}
$$

From the equations in (4), we can derive the cross derivatives t_{ab} and t_{ba}. Finally, by using Schwarz's integrability condition $t_{ab}(a,b) = t_{ba}(a,b)$, we obtain the following proposition.[2]

Proposition 3.1 (Quasi-Linear PDE)**.** *Suppose that the contract (q,t) is incentive compatible and twice differentiable in an open set $\Omega \supset \Theta$. Then, it satisfies the following equation*

$$
-\frac{v_{qb}}{v_{qa}}q_a + q_b = 0 \tag{5}
$$

Equation (5) is a quasi-linear first-order partial differential equation[3]. It describes the relationship between the contour lines of $q(a,b)$ and $v_q(q(a,b),a,b)$. Indeed, let $(a(s),b(s))$ be a contour line with $q(a(s),b(s)) = k$. Then, by differentiating $q(a(s),b(s))$ and $v_q(k,a(s),b(s))$ along this curve, we obtain

$$
\frac{d}{ds}q(a(s),b(s)) = q_a a_s + q_b b_s = 0 \tag{6}
$$

and

$$
\frac{d}{ds}v_q(k,a(s),b(s)) = v_{qa}a_s + v_{qb}b_s \tag{7}
$$

Finally, by using equations (5), (6) and (7), we conclude that

$$
\frac{d}{ds}v_q(k,a(s),b(s)) = 0 \tag{8}
$$

Observe that equation (8) states that if $(a(s),b(s))$ is a contour line $q(a,b) = k$, then it is also a contour line of $v_q(k,a,b)$.

We have yet another interpretation. The "taxation principle" says that we can also implement $q(a,b)$ with a tariff $T : Q = q(\Theta) \to \mathbb{R}$ such that

[2] Laffont et al. (1987) mentioned this integrability condition and the PDE for the particular case they were treating. Here, we present the general expression for this PDE.

[3] See John (1981) for a complete analysis of this kind of PDE and a description of the method of characteristic curves used to solve it.

$T(q(a,b)) = t(a,b)$ for all $(a,b) \in \Theta$. By using this tariff T, we can write the customer's problem as

$$\max_{q \in Q} \{v(q,a,b) - T(q)\}$$

Note that when T is differentiable at $q(a(s),b(s)) = k$, Proposition 3.1 simply states that all types that choose $q(a(s),b(s))$ obtain the same marginal utility $v_q(k,a(s),b(s))$. Araujo and Moreira (2010) have a similar condition (the U-condition) in the one-dimensional context. They use it in the derivation of the optimality condition. We will follow the same steps adapted to the two-dimensional case.

3.1 Solving the Quasi-Linear Equation

The solution of the quasi-linear equation (5) will provide a natural reparametrization of the types following the contour lines of $q(\cdot)$ in the participation region. After that, using calculus of variations, we will derive the optimality conditions involving types in the same contour line.

We use the method of characteristic curves to solve equation (5). This method consists of reducing a partial differential equation to a system of ordinary differential equations. Then, the system is integrated using the initial data prescribed on a curve Γ. Formally, we have the following Cauchy problem associated with equation (5):

$$\begin{aligned} -\frac{v_{qb}}{v_{qa}} q_a + q_b &= 0 \\ q_{|\Gamma} &= \phi(r) \end{aligned} \tag{9}$$

where $\Gamma = \{(\alpha_0(r), \beta_0(r)) : r \in [r_1, r_2]\}$ is a curve on the $ab-$plane. The basic idea is to prescribe the value of $q(\cdot)$ on Γ and then use the characteristic curves to propagate this information to the participation region. In this sense, because Γ is a one-dimensional curve, we are reducing the problem from two dimensions to one.

Following the method, we define the family of curves $(a(r,s), b(r,s))$ as the solution of

$$\begin{aligned} a_s(r,s) &= -\frac{v_{qb}}{v_{qa}}(\phi(r), a(r,s), b(r,s)) \quad , \quad && a(r,s_0) = \alpha_0(r) \\ b_s(r,s) &= 1 \quad , \quad && b(r,s_0) = \beta_0(r) \end{aligned} \tag{10}$$

Assuming that $(a(r,s), b(r,s))$ is invertible for all r and s such that (a,b) is in the participation region, the method provides a change of variables

$$a = a(r,s) \quad , \quad b = b(r,s)$$

such that $q(a(r,s), b(r,s)) = \phi(r)$.

To be precise, if we fix r, the curve $(a(r,s), b(r,s))$ parametrized by s is an isoquant of $q(\cdot)$ at level $\phi(r)$. Note that by fixing s, we obtain

$$q_a a_r + q_b b_r = \phi'(r) \tag{11}$$

By denoting $\overline{s}(r)$ as the maximum value of the parameter s for a given r, the contribution to the monopolist's expected profits from the types in the images of $a(r,s)$ and $b(r,s)$ is

$$\int_{r_1}^{r_2} \int_{s_0}^{\overline{s}(r)} G(\phi(r), a(r,s), b(r,s)) \left| \frac{\partial(a,b)}{\partial(r,s)} \right| ds\, dr \tag{12}$$

Because we want to maximize this contribution, by using the calculus of variations techniques, we can derive the necessary conditions for $\phi(\cdot)$ to be optimal.

By varying the initial curve $(\alpha_0(r), \beta_0(r))$, we analyze three cases that deserve separate treatments

1. Isoquants intersecting the X axis.
2. Isoquants intersecting the participation's boundary.
3. Isoquants that are concurrent at some point.

4 Optimality Conditions

4.1 Isoquants intersecting the X axis

For this case, consider $\Gamma = \{(r,0) : r \in [r_1, r_2]\}$ and $s_0 = 0$. Then, the solutions of system (10) are $a(r,s) = A(\phi(r), r, s)$ and $b(r,s) = s$ (we use $A = A(q,r,s)$ to express the dependence of $a(r,s)$ on $\phi(r)$). Because $\overline{s}(r)$ is such that $a(r, \overline{s}(r)) = 1$ or $b(r, \overline{s}(r)) = 1$, we denote $\overline{s}(r) = U(\phi(r), r)$.

Moreover, in view of $\phi'(r) > 0$ (allocations over $(r, 0)$ must be increasing), $q_a > 0$ and $b_r = 0$, by equation (11), $a_r > 0$, which implies that the Jacobian determinant is positive, i.e.,

$$\frac{\partial(a,b)}{\partial(r,s)} = \begin{vmatrix} a_r & 0 \\ a_s & 1 \end{vmatrix} = A_q\phi' + A_r > 0$$

Thus, the contribution to the monopolist's expect profit (12) can be written as

$$\int_{r_1}^{r_2} \int_0^{U(\phi(r),r)} G(\phi(r), A(\phi(r), r, s), s)\,(A_q\phi' + A_r)\, ds\, dr \tag{13}$$

Considering the function

$$H(\phi, \phi', r) := \int_0^{U(\phi,r)} G(\phi, A(\phi, r, s), s)\,(A_q\phi' + A_r)\, ds \tag{14}$$

we rewrite (13) and define the following problem

$$\max_{\phi(\cdot)} \int_{r_1}^{r_2} H(\phi(r), \phi'(r), r)\, dr$$

The Euler equation for this problem is

$$H_\phi - \frac{d}{dr} H_{\phi'} = 0$$

from which we obtain the following:

Theorem 4.1. *If $\phi(r)$ is the optimal allocation of $(r, 0)$ types, then*

$$\int_0^{U(\phi(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r), r, s), s) ds = 0 \tag{15}$$

Observe that Theorem 4.1 gives the optimality condition along the characteristic curve $\gamma(s) = (a(r, s), s)$. It is analogous to the Araujo and Moreira (2010) optimality condition now prescribed in the characteristic curve $\gamma(s)$[4]. The condition says that the average of the marginal virtual surplus G_q weighted by $1/v_{qa}$ along the characteristic curve $\gamma(s)$ is zero.

[4]In Araujo and Moreira (2010), this condition can be found in their Theorem 2 (critical U-shaped curve)

4.2 Isoquants intersecting the participation's boundary

In this case, the characteristic curves intersect the boundary between the participation and exclusion regions. We parametrize this boundary using the curve $\beta(r)$ that we assume is differentiable when $r \in (r_1, r_2)$. For all types in this boundary, the informational rent $V(r, \beta(r)) = 0$. Then,

$$\frac{d}{dr} V(r, \beta(r)) = 0 \tag{16}$$

Using the envelope theorem from Milgrom and Segal (2002), we obtain the following expression for the marginal informational rent over the curve $\beta(r)$:

$$\frac{d}{dr} V(r, \beta(r)) = v_a(\phi(r), r, \beta(r)) + v_b(\phi(r), r, \beta(r))\beta'(r) \tag{17}$$

For a simpler notation, we define the function

$$R(\phi, \beta, \beta', r) = v_a(\phi, r, \beta) + v_b(\phi, r, \beta)\beta' \tag{18}$$

Therefore, by using (17) and (18), we can write the boundary constraint (16) for the monopolist's problem as

$$R(\phi(r), \beta(r), \beta'(r), r) = 0 \tag{19}$$

For this case, $\Gamma = \{(r, \beta(r)) : r \in [r_1, r_2]\}$ and $s_0 = \beta(r)$. Hence, the solutions of the system (10) are $a(r, s) = A(\phi(r), \beta(r), r, s)$ and $b(r, s) = s$[5]. We also denote $\overline{s}(r) = U(\phi(r), \beta(r), r)$.

By equation (19), note that $\beta(\cdot)$ must be increasing because of the assumptions $v_a > 0$ and $v_b < 0$. Since characteristic curves are also increasing in the $ab-$plane, the intersections with line $x = 1$, $(1, \overline{s}(r))$, are increasingly upward and with line $y = 1$, $(a(r, 1), 1)$, are increasingly to the left. Then, the allocations over $x = 1$ and $y = 1$ ($q(1, \overline{s}(r))$ and $q(a(r, 1), 1)$, respectively) are decreasing as a function of r, so $\phi(r)$ it is.

By equation (11), in view of $\phi'(r) < 0$, $q_a > 0$ and $b_r = 0$, it is necessary that $a_r < 0$. Therefore, the Jacobian determinant is negative, i.e.,

$$\frac{\partial(a, b)}{\partial(r, s)} = \begin{vmatrix} a_r & 0 \\ a_s & 1 \end{vmatrix} = A_q\phi' + A_\beta\beta' + A_r < 0$$

[5] $s_0 = 0$ could be taken and then $\hat{s} = s + \beta(r)$ defined to obtain the same parametrization

Thus, the contribution to the monopolist's expect profits (12) can be written as

$$\int_{r_1}^{r_2}\int_{\beta(r)}^{U(\phi(r),\beta(r),r)} -G(\phi(r),A(\phi(r),\beta(r),r,s),s)\,(A_q\phi'+A_\beta\beta'+A_r)\,ds\,dr \tag{20}$$

By defining the function

$$H(\phi,\phi',\beta,\beta',r):=\int_{\beta}^{U(\phi,\beta,r)} -G(\phi,A(\phi,\beta,r,s),s)\,(A_q\phi'+A_\beta\beta'+A_r)\,ds \tag{21}$$

we can rewrite (20) as

$$\int_{r_1}^{r_2} H(\phi(r),\phi'(r),\beta(r),\beta'(r),r)\,dr$$

We still have to consider constraint (19) in the monopolist's problem. For this, we use the Lagrangian multiplier $\lambda(r)$ to append this constraint. The resulting problem is

$$\max_{\phi(\cdot),\beta(\cdot)}\int_{r_1}^{r_2}\{H(\phi(r),\phi'(r),\beta(r),\beta'(r),r)+\lambda(r)R(\phi(r),\beta(r),\beta'(r),r)\}\,dr$$

In this problem, we have to choose the optimal pair $(\phi(\cdot),\beta(\cdot))$. Thus, we have a system with two Euler equations, one for $\phi(\cdot)$ and the other for $\beta(\cdot)$, as follows:

$$H_\phi-\frac{d}{dr}H_{\phi'}+R_\phi\lambda(r)=0\ ;\quad \text{and} \tag{22}$$

$$H_\beta-\frac{d}{dr}H_{\beta'}+\lambda(r)[R_\beta-\frac{d}{dr}R_{\beta'}]-R_{\beta'}\lambda'(r)=0 \tag{23}$$

From the system of equations above, we obtain

Theorem 4.2. *Assume $R_\phi\neq 0$[6]. If $\phi(r)$ is the optimal allocation of $(r,\beta(r))$ types, then there is a function $\lambda(r)$ such that*

$$(i)\qquad \int_{\beta(r)}^{U(\phi(r),\beta(r),r)}\frac{G_q}{v_{qa}}(\phi(r),A(\phi(r),\beta(r),r,s),r,s)ds=\lambda(r) \tag{24}$$

$$(ii)\qquad \frac{G}{v_b}(\phi(r),r,\beta(r))=\lambda'(r) \tag{25}$$

Observe that in Theorem 4.2 (i), we obtain a slightly different condition compared with Theorem 4.1. Now, due to the boundary constraint (19), the integral on the left side is not zero but is equal to the Lagrange multiplier $\lambda(r)$. Additionally, in Theorem 4.2 (ii), we have another condition to control the variation of the Lagrangian multiplier. Theorem 4.2 together with the boundary condition (19) will give us a system of ordinary differential equations. We need to solve this system to find the optimal $\phi(\cdot)$ and $\beta(\cdot)$. This is illustrated in example 5.1.

4.3 Isoquants that are concurrent at some point

In the case that the planar characteristic curves are concurrent at the point $(\overline{x}, \overline{y})$, this type is indifferent between any quantity in some interval $[\underline{q}, \overline{q}]$. We analyze the specific case that the isoquants end up at the border $y = 1$. The other case of isoquants ending at $x = 1$ can be analyzed analogously.

Consider $\Gamma = \{(r, 1) : r \in [R_1, R_2]\}$ as the initial curve and $s_0 = 1$. The solutions of the system (10) are $a(r, s) = A(\phi(r), r, s)$ and $b(r, s) = s$. We have $s \in [\overline{y}, 1]$ because any isoquant starts at point $(\overline{x}, \overline{y})$, and $\overline{s}(r) = 1$.

Here, $\phi : [R_1, R_2] \to [\underline{q}, \overline{q}]$ describes the quantity allocated to $(r, 1)$ types, which is strictly increasing in view of $q_a > 0$. Then, by equation (11), $a_r > 0$.

Let $\varphi : [\underline{q}, \overline{q}] \to [R_1, R_2]$ be the inverse of ϕ. Now, a and b can be expressed in terms of new variables q and s (q for quantity and s for the position on the characteristic curve of (a, b)):

$$a(q, s) = A(q, \varphi(q), s) \quad , \quad b(q, s) = s$$

where $q \in [\underline{q}, \overline{q}]$, $s \in [\overline{y}, 1]$.

Because $a_q = a_r \varphi' > 0$, the Jacobian determinant is positive, i.e.,

$$\frac{\partial(a, b)}{\partial(q, s)} = \begin{vmatrix} a_q & 0 \\ a_s & 1 \end{vmatrix} = A_q + A_\varphi \varphi' > 0$$

Thus, the contribution to the monopolist's expected profits can be written as

$$\int_{\underline{q}}^{\overline{q}} \int_{\overline{y}}^{1} G(q, A(q, \varphi(q), s), s)(A_q + A_\varphi \varphi')\, ds\, dq$$

[6]The condition $R_\phi \neq 0$ in Theorem 4.2 is related to the method of characteristic curves. This condition is equivalent to saying that the characteristic curves and the boundary curve are not tangent when they cross each other at the point $(r, \beta(r))$. This is a requirement of the method of characteristic curves.

Furthermore, the fact that type $(\overline{x}, \overline{y})$ is indifferent between any $q \in [\underline{q}, \overline{q}]$ gives us a special restriction: Fix any $q \in [\underline{q}, \overline{q}]$. Since $(\overline{x}, \overline{y})$ and $(\varphi(q), 1)$ receive the same allocation q, this q is the solution of both

$$\max_{\tilde{q}}\{v(\tilde{q}, \overline{x}, \overline{y}) - T(\tilde{q})\} \quad \text{and} \quad \max_{\tilde{q}}\{v(\tilde{q}, \varphi(q), 1) - T(\tilde{q})\}$$

from which

$$v_q(q, \overline{x}, \overline{y}) = v_q(q, \varphi(q), 1)\ ,\ \forall\ q \in [\underline{q}, \overline{q}] \tag{26}$$

Then,

$$\int_{\underline{q}}^{\overline{q}} v_q(q, \overline{x}, \overline{y}) - v_q(q, \varphi(q), 1)\, dq = 0$$

Therefore, this new restriction must be considered in the optimization problem:

$$\max_{\varphi(\cdot)} \int_{\underline{q}}^{\overline{q}} \int_{\overline{y}}^{1} G(q, A(q, \varphi(q), s), s)(A_q + A_\varphi \varphi')\, ds\, dq$$

$$\text{subject to} \int_{\underline{q}}^{\overline{q}} v_q(q, \overline{x}, \overline{y}) - v_q(q, \varphi(q), 1)\, dq = 0$$

For this isoperimetric problem, the necessary condition for optimality is

$$H_\varphi - \frac{d}{dq}(H_{\varphi'}) = \lambda(F_\varphi - \frac{d}{dq}(F_{\varphi'})) \tag{27}$$

for some $\lambda \in \mathbb{R}$, where

$$H(q, \varphi, \varphi') = \int_{\overline{y}}^{1} G(q, A(q, \varphi, s), s)(A_q + A_\varphi \varphi')\, ds \tag{28}$$

$$F(q, \varphi, \varphi') = v_q(q, \overline{x}, \overline{y}) - v_q(q, \varphi, 1) \tag{29}$$

from which we obtain the following:

Theorem 4.3. *If $\varphi = \varphi(q)$ is optimal, then $\exists\ \lambda \in \mathbb{R}$ such that*

$$\int_{\overline{y}}^{1} \frac{G_q}{v_{qa}}(q, A(q, \varphi, s), s)\, ds = \lambda \tag{30}$$

As we have seen, an isoperimetric problem arises in the case that an agent is indifferent between any allocation in a range. This leads to the necessary condition (30) that is slightly different from condition (24) in Theorem 4.2. The difference is that the Lagrange multiplier has to be the same for all characteristic curves.

5 Examples with Analytical Solution

5.1 Linear Demand

The firm faces customers with linear demands and is uncertain about both the slope and intercept of the demand curves, thus yielding the customers' valuation

$$v(q,a,b) = aq - (b+c)\frac{q^2}{2} \tag{31}$$

with the exogenous parameter $c \in (0,1]$. Firm's costs are assumed to be zero, and the types are uniformly distributed. By Proposition 2.1, we obtain the virtual surplus

$$G(q,a,b) = (2a-1)q - (b+c)\frac{q^2}{2} \tag{32}$$

Case $1/2 \leq c \leq 1$

Laffont et al. (1987) obtained the solution for the case $c = 1$. In view of the fact that the maximum quantity is attained at $(1,0)$, we begin the analysis by looking for isoquants that intersect axis X. For this case, the change of variables is $a(r,s) = \phi(r)s + r$, $b(r,s) = s$. By Theorem 4.1, the optimal allocation $\phi = \phi(r)$ satisfies

$$(2r - 1 - c\phi)\overline{s}(r) + \frac{\overline{s}(r)^2}{2}\phi = 0$$

where $\overline{s}(r) = \frac{1-r}{\phi}$ if the isoquants end up at the line $x = 1$ and $\overline{s}(r) = 1$ if the isoquants end up at the line $y = 1$. Thus, we obtain

$$\phi(r) = \begin{cases} \frac{4r-2}{2c-1} & , \quad \underline{r} \leq r \leq r_I \\ \frac{3r-1}{2c} & , \quad r_I \leq r \leq 1 \end{cases}$$

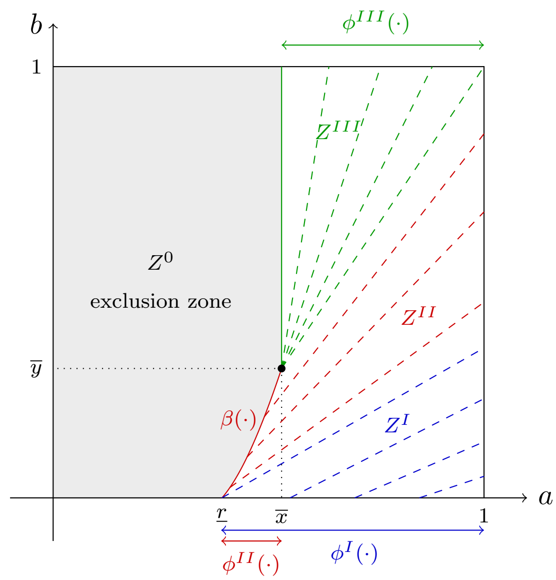
where r_I is such that $\phi(r)$ is continuous, that is, $r_I = \frac{2c+1}{2c+3}$. Note that the characteristic curve that crosses $(r_I, 0)$ also crosses the corner point $(1,1)$. Furthermore, $\underline{r} = 1/2$ because at this value, $\phi(\underline{r}) = 0$, and the participation region is determined. To return to the original variables, $r(a,b)$ is the solution of $a = r + b\phi(r)$. By defining $q(a,b) = \phi(r(a,b))$, we obtain

$$q(a,b) = \begin{cases} 0 & , \quad a \leq \frac{1}{2} \\ \frac{4a-2}{4b+2c-1} & , \quad \frac{1}{2} \leq \frac{(2c-1)a+2b}{4b+2c-1} \leq \frac{2c+1}{2c+3} \\ \frac{3a-1}{3b+2c} & , \quad \frac{2c+1}{2c+3} \leq \frac{2ac+b}{3b+2c} \leq 1 \end{cases}$$

Case $0 < c < 1/2$

Deneckere and Severinov (2015) have also analyzed this case. For a better understanding of the explanation below, in Figure 1 below, we show the isoquants of the solution.

Figure 1: Isoquants of optimal q for $c \in (0, 0.5)$ in Example 5.1



We begin the analysis as in the previous case, but now, the expression $\phi(r) = \frac{4r-2}{2c-1}$ does not make sense in view of $\frac{4r-2}{2c-1} < 0$. Therefore, there are no isoquants intersecting axis X and ending up at the line $y = 1$. As before, the optimal allocation rule for $(r, 0)$ types when isoquants end at the line $x = 1$ is

$$\phi^I(r) = \frac{3r-1}{2c} \quad , \quad r \in [\underline{r}, 1] \tag{33}$$

Note that $\frac{2c+1}{2c+3} \in (\frac{1}{3}, \frac{1}{2})$, so $\phi^I(\underline{r}) > 0$ and the participation region is not determined yet. Next, we look for isoquants intersecting the participation's boundary $\beta(r)$ (to be defined) and the line $x = 1$. In this case, the change of variables is $a(r,s) = (s - \beta(r))\phi(r) + r$, $b(r,s) = s$. We also have $\overline{s}(r) = \frac{1-r}{\phi(r)} + \beta(r)$. Then, by Theorem 4.2 and boundary condition (19), we obtain the following system of ordinary differential equations

$$\begin{aligned} (3r^2 - 4r + 1)\phi'(r) + 2(r-1)\phi^2(r)\beta'(r) + 2r\phi(r) &= 0 \\ \phi(r) - \frac{\phi^2(r)\beta'(r)}{2} &= 0 \end{aligned} \tag{34}$$

By solving this system, we obtain that the participation's boundary is of the form

$$\beta(r) = \frac{2r^3 - 4r^2 + 2r}{K_0} + K_1 \quad (K_0, K_1 \in \mathbb{R}) \tag{35}$$

and the optimal allocation rule for $(r, \beta(r))$ types is

$$\phi^{II}(r) = \frac{K_0}{3r^2 - 4r + 1} \quad , \quad r \in [\underline{r}, \overline{x}] \tag{36}$$

If we look for isoquants intersecting the participation's boundary and ending up at the line $y = 1$, the boundary derived is not compatible with our example.

Claim 1. *There are no isoquants intersecting the participation's boundary and the line* $y = 1$*.*

Note that ϕ^{II} is not zero. Hence, the participation region is not completely determined. By setting $\overline{y} = \beta(\overline{x})$, we first look for isoquants that are concurrent at the point $(\overline{x}, \overline{y})$ and intersect the line $x = 1$, but this cannot be the case.

Claim 2. *There are no isoquants concurrent at point* $(\overline{x}, \overline{y})$ *and intersecting the line* $x = 1$*.*

If we look for isoquants concurrent at the point $(\overline{x}, \overline{y})$ and ending up at the line $y = 1$, the change of variables (in terms on q and s) is $a(q, s) = (s-1)q + \varphi(q)$ and $b(r, s) = s + 1$. By Theorem 4.3 and restriction (26) we have

$$\overline{y} = \frac{1 - 2c}{3} \tag{37}$$

$$\phi^{III}(r) = \frac{r - \overline{x}}{1 - \overline{y}} \quad \text{with} \quad r \in [\overline{x}, 1] \tag{38}$$

where $\phi^{III} = \varphi^{-1}$ is the optimal allocation for $(r, 1)$ types. Note that $\phi^{III}(\overline{x}) = 0$, hence the boundary participation has a vertical part.

Next, we determine K_0, K_1, $\overline{x}$ and $\underline{r}$ by the continuity of the allocation rule and boundary conditions.

Claim 3. *We obtain* $\underline{r}$ *as the solution in* $(\frac{2c+1}{2c+3}, \frac{1}{2})$ *of*

$$\Big(\frac{(54+24c)r^2-(36+24c)r+6}{(63+18c)r^2-(42+24c)r+(7-2c)}\Big)^3 = \frac{(9+4c)r^3-(15+8c)r^2+(7+4c)r-1}{2c}$$

With such $\underline{r}$*, we obtain*

$$\overline{x} = \frac{(9-6c)\underline{r}^2-6\underline{r}+1-2c}{(63+18c)\underline{r}^2-(42+24c)\underline{r}+(7-2c)}$$

$$K_0 = \frac{-(1-\underline{r})(3\underline{r}-1)^2}{2c} \quad , \quad K_1 = \frac{4c\underline{r}(1-\underline{r})}{(3\underline{r}-1)^2}$$

Thus, all the elements defining ϕ^I, ϕ^{II}, ϕ^{III}, β and the special point $(\overline{x}, \overline{y})$ are determined. This type is indifferent between any quantity in the interval $[0, \frac{3(1-\overline{x})}{2(1+c)}]$, while the optimal allocation range is $[0, \frac{1}{c}]$.

To express the optimal quantity in terms of (a, b), note that the type set $[0,1]^2$ can be partitioned into four sets Z^0, Z^I, Z^{II} and Z^{III} that are defined as

$$\begin{aligned}
Z^0 &= \{(a,b) \in [0,1]^2 : a < \overline{x} \ \wedge \ b > \beta(a)\} \\
Z^I &= \{(a,b) \in [0,1]^2 : b \le (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1}\} \\
Z^{II} &= \{(a,b) \in [0,1]^2 : a \ge \overline{x} \ \wedge \ b > (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1} \ \wedge \ b \le (\tfrac{1-\overline{y}}{1-\overline{x}})a + \tfrac{\overline{y}-\overline{x}}{1-\overline{x}}\} \\
&\quad \cup \{(a,b) \in [0,1]^2 : a < \overline{x} \ \wedge \ b > (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1} \ \wedge \ b \le \beta(a)\} \\
Z^{III} &= \{(a,b) \in [0,1]^2 : a \ge \overline{x} \ \wedge \ b > (\tfrac{1-\overline{y}}{1-\overline{x}})a + \tfrac{\overline{y}-\overline{x}}{1-\overline{x}}\}
\end{aligned}$$

Here, Z^0 is the exclusion region. Therefore, $q(a,b) = 0$ if $(a,b) \in Z^0$.
Given $(a,b) \in [0,1]^2 \setminus Z^0$, $r(a,b)$ is defined as the solution of

$$\begin{aligned}
a = \phi^I(r)b + r \quad & \text{if } (a,b) \in Z^I \\
a = \phi^{II}(r)(b - \beta(r)) + r \quad & \text{if } (a,b) \in Z^{II} \\
a = \phi^{III}(r)(b-1) + r \quad & \text{if } (a,b) \in Z^{III}
\end{aligned}$$

Finally, $q(a,b) = \phi^k(r(a,b))$ if $(a,b) \in Z^k$, $k = I, II, III$.

5.2 Convex Demand

The firm faces customers with strictly convex demand of the form $p = (c - b)/(aq+1)$, where $c \ge 1$ is exogenous and the parameters $(a,b) \in [0,1]^2$ are the customer's private information. From this demand curve, the associated valuation function is

$$v(q,a,b) = (c-b)\log(aq+1) \tag{39}$$

Also consider the monopolist's cost function $C(q) = \lambda q$ with $\lambda \in (0,1)$ and the uniform distribution of types. By Proposition 2.1, the virtual surplus has the form $G(q,a,b) = v - \lambda q - (1-a)v_a$. As before, we begin the analysis by looking for isoquants intersecting axis X and ending up at the line $x = 1$. For this case, the change of variables is provided by the solutions of the system

$$\begin{aligned} a_s(r,s) &= \frac{a(r,s)\Big(a(r,s)\phi(r)+1\Big)}{c-b(r,s)} \qquad &, \quad a(r,0) = r \\ b_s(r,s) &= 1 \qquad &, \quad b(r,0) = 0 \end{aligned}$$

which are

$$a(r,s) = \frac{c\,r}{c-(1+r\phi(r))s} \quad , \quad b(r,s) = s$$

Then, by Theorem (4.1), if $\phi(r)$ is the optimal allocation over the curve $(a(r,s), b(r,s))$, where $r \in [\underline{r}, 1]$ is fixed ($\underline{r}$ to be determined), then

$$D(r)\phi^2 + E(r)\phi + F(r) = 0 \qquad , \text{ where} \tag{40}$$

$D(r) = \lambda r(1-r)$, $E(r) = (\lambda - c\,r)(1-r) - \lambda r \log(r)$, $F(r) = (2c\,r - \lambda)\log(r) + c(1-r)$

Note that F is strictly convex and has a minimum $r^* \in (0,1)$ in view of $F'(r) < 0$ when $r \approx 0$ and $F'(r) > 0$ when $r \approx 1$. Additionally, due to $F(r) > 0$ for $r \approx 0$ and $F(r^*) < 0$ (because $F(1) = 0$), there exists a unique $r_0 \in (0,1)$ such that $F(r_0) = 0$. Since $\phi(\underline{r}) = 0$ implies that $F(\underline{r}) = 0$, we must have $\underline{r} = r_0$. Therefore, $\underline{r}$ is defined as the unique solution on $(0,1)$ of

$$(2c\,r - \lambda)\log(r) + c(1-r) = 0 \tag{41}$$

In view of $F(r) < 0$ on $(\underline{r}, 1)$ (F is strictly convex, $F(\underline{r}) = 0$ and $F(1) = 0$) and $D(r) > 0$, one solution of the quadratic equation (40) is always negative on $(\underline{r}, 1)$. Then, we can express $\phi(r)$ in the closed form

$$\phi(r) = \frac{-E(r) + \sqrt{E(r)^2 - 4D(r)F(r)}}{2D(r)} \quad , \quad r \in (\underline{r}, 1) \tag{42}$$

Although $\phi(1) = q(1,0)$ is not defined, by continuity,

$$q(1,0) := \lim_{r\to 1} \phi(r) = \frac{c}{\lambda} - 1$$

This value solves the equation $v_q(q,1,0) = C'(q)$, meaning that there is no distortion at the top, as expected for the solution.

Note that $\underline{r}$ defines the participation's boundary since $\phi(\underline{r}) = 0$. This boundary is given by $b = c - (c\underline{r})/a$ in the $ab-$plane.

Next, we return to the original variables. Fix $(a,b) \in [0,1]^2$.

- If $b \geq c - (c\underline{r})/a$, then $q(a,b) = 0$, meaning that the (a,b) type is excluded.
- If $b < c - (c\underline{r})/a$, $r(a,b)$ is defined as the solution of

$$\frac{c-b}{br} - \frac{c}{ab} = \frac{-E(r) + \sqrt{E(r)^2 - 4D(r)F(r)}}{2D(r)}$$

  such that $r(a,b) \in (\underline{r}, 1)$, $\phi(r(a,b)) > 0$ and $\phi'(r(a,b)) > 0$. With such $r(a,b)$, we define

$$q(a,b) = \frac{c-b}{b\,r(a,b)} - \frac{c}{ab}$$

In Figure 2, we show the isoquant curves of the optimal q for $c = 1.5$ and $\lambda = 0.5$.

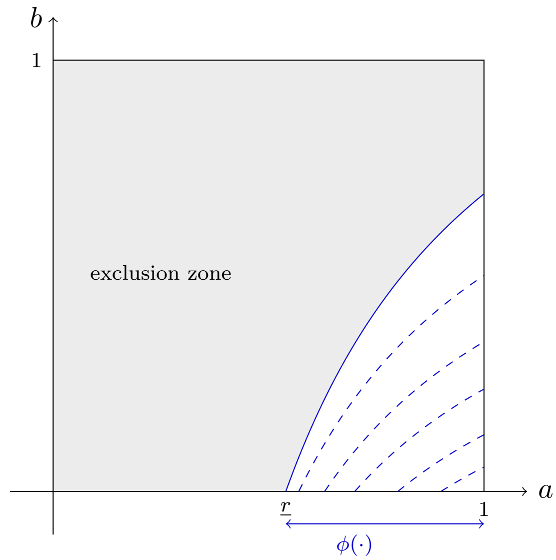
6 Reduction of Incentive Constraints

When numerically solving the problem, the main difficulty is related to the number of constraints. This is because after discretizing the type set $[0,1]^2$ into a grid of n points over each axis, there are $n^4 - n^2$ IC constraints. Therefore, fine discretizations result in memory storage problems. Next, we present a methodology that allows us to reduce the number of IC constraints. It is inspired by the ideas to address IC constraints in the unidimensional case with a finite type set when single-crossing holds[7].

In bidimensional models, we do not have a condition similar to the single-crossing in the unidimensional case where all types can be exogenously ordered by their marginal valuation for consumption. This is because $v_{q\theta} > 0$

[7]See Laffont and Martimort (2002)

Figure 2: Isoquants of optimal q in Example 5.2



is equivalent to $\theta_1 < \theta_2 \implies v_q(q, \theta_1) < v_q(q, \theta_2) \quad \forall q \in Q$. Then, to be able to compare a priori two different types at least partially, we introduce the following binary relation:

Definition 1. *Given $(a, b), (\widehat{a}, \widehat{b}) \in [0, 1]^2$, we say that (a, b) is* worse than *$(\widehat{a}, \widehat{b})$, which is denoted by $(a, b) \preceq (\widehat{a}, \widehat{b})$, if and only if*

$$v_q(q, a, b) \leq v_q(q, \widehat{a}, \widehat{b}) \quad \forall\, q \in Q$$

Note that $\preceq$ is a pre-order (reflexive and transitive) on $[0, 1]^2$. With this definition, we try to capture the idea that when $(a, b) \preceq (\widehat{a}, \widehat{b})$, the (a, b)-agent is unwilling to pretend to be the $(\widehat{a}, \widehat{b})$-agent. This is because for any $q \in Q$, the $(\widehat{a}, \widehat{b})-$agent has greater marginal valuation for consumption and is willing to pay more for each additional unit of the product.

As a direct consequence of Assumption 3, we have that (a, b) is worse than any type in the southeast.

Proposition 6.1. *Given (a, b), if $\widehat{a} > a$ and $\widehat{b} < b$, then $(a, b) \preceq (\widehat{a}, \widehat{b})$*

By fixing type (a, b), we exclude a priori IC constraints with any type in the southeast, as the difficulty comes from better agents willing to claim that they are worse agents rather than the reverse. Specifically, we will omit the following IC constraints:

$$V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q(\widehat{a},\widehat{b}),a,b) - v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) \qquad \widehat{a} > a \ , \ \widehat{b} < b \tag{43}$$

In the next proposition, we show that these constraints are indeed fulfilled when (q, V) satisfies the necessary conditions related to the envelope theorem and the monotonicity of $q(\cdot, \cdot)$ over each axis.

Proposition 6.2. *Assume (q, V) is such that*

$$V_a(a,b) = v_a(q(a,b),a,b) \ , \ V_b(a,b) = v_b(q(a,b),a,b) \ , \ q_a \geq 0 \ , \ q_b \leq 0$$

By fixing (a, b), the constraints given in (43) are satisfied.

We denote by $CC(\widehat{a},\widehat{b})$ the planar characteristic curve that contains $(\widehat{a},\widehat{b})$. Additionally, the expression "(a, b) is IC with $(\widehat{a},\widehat{b})$" means that

$$V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q(\widehat{a},\widehat{b}),a,b) - v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b})$$

The following proposition shows that we just need to verify the IC constraint with a representative type of each characteristic curve.

Proposition 6.3. *Let $(a, b), (\widehat{a},\widehat{b})$ be such that (a, b) is IC with $(\widehat{a},\widehat{b})$. Then (a, b) is IC with (x, y) , $\forall$ $(x, y) \in CC(\widehat{a},\widehat{b})$*

As a consequence, we can focus on the border of $[0, 1]^2$. The following proposition is key to reduce the restrictions.

Proposition 6.4. *Let $(x, y), (\widehat{a},\widehat{b})$, and (a, b) be such that (a, b) is IC with $(\widehat{a},\widehat{b})$ and $(\widehat{a},\widehat{b})$ is IC with (x, y). If $(\widehat{a},\widehat{b}) \preceq (a, b)$ and $q(x, y) \leq q(\widehat{a},\widehat{b})$, then (a, b) is IC with (x, y).*

Due to the kind of transitivity shown in Proposition 6.4, it is not necessary that type (a, b) verifies the IC constraints with all the types (x, y) on the left of a certain characteristic curve. Indeed, it is sufficient to verify the IC constraint with any type worse than (a, b) over such a curve, but making sure that this type verifies the IC constraints with all of those (x, y).

By taking the characteristic curve as close as possible to type (a, b), the most restrictions can be eliminated. Since the characteristic curves are endogenously determined but any of them passing through (a, b) intersects the border of the square $[0, 1]^2$ on the northeast of that point, previous propositions suggest that it would be sufficient to verify that (a, b) is IC with all the points over the set

$$F^{(a,b)} := \{(s, 1) \mid a \leq s \leq 1\} \cup \{(1, s) \mid b < s \leq 1\} \tag{44}$$

which is formalized it in the following theorem.

Theorem 6.1. *Assume* $\frac{d}{dq}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$ *and* $\frac{d}{da}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$. *Let* (q, V) *be such that*

$$\forall \ (a, b) \in [0, 1]^2 \ , \ (a, b) \textit{ is IC with } (x, y) \ \ \forall \ (x, y) \in F^{(a,b)}$$

Then, (q, V) *satisfies all the incentive compatibility constraints.*

Technical assumptions $\frac{d}{dq}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$ and $\frac{d}{da}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$ are given to avoid pathological cases. This result could be understood as analogous to the claim *local IC constraint implies global IC constraint*, which is true in the unidimensional case when single-crossing holds.

6.1 Discretized Problem

By Theorem 6.1, it is sufficient that each point satisfies the IC constraints with all points over a unidimensional set instead of the whole square. Now, we can approximate the solution of the continuous problem by discretizing the type set. This section is devoted to establishing this discrete problem and discussing its limitations.

Let $X_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\} \times \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$ be the grid of n^2 points on $[0, 1]^2$. For a fixed (a, b) with $a < 1$ and $b < 1$, let $\widetilde{F}^{(a,b)} := F^{(a,b)} \cap X_n$, where $F^{(a,b)}$ is defined in (44). Because for points over the line $x = 1$ or $y = 1$, we cannot write the constraints with the points on the northeast, we equivalently consider

$$\widetilde{F}^{(a,1)} = \big(\{(0, s) : 0 \leq s \leq 1\} \cup \{(s, 0) : 0 \leq s < a\}\big) \cap X_n$$

$$\widetilde{F}^{(1,b)} = \big(\{(0, s) : 0 \leq s \leq b\} \cup \{(s, 0) : 0 \leq s < 1\}\big) \cap X_n$$

The set $\widetilde{F}^{(a,b)}$ contains all the types with which (a,b) must satisfy an IC constraint. The integral in the monopolist's objective will be approximated by the trapezoidal rule. Therefore, we consider the associated weights $w(i,j)$ for each point $(a_i,b_j) \in X_n$. By denoting $q_{i,j} = q(a_i,b_j)$ and $V_{i,j} = V(a_i,b_j)$, we are interested in solving the following problem:

$$\begin{aligned}
&\max_{\{q_{i,j},V_{i,j}\}} \sum_{i=1}^{n}\sum_{j=1}^{n} w(i,j)(v(q_{i,j},a_i,b_j) - V_{i,j} - C(q_{i,j}))\rho(a_i,b_j) \\
&\text{subject to} \\
&\qquad \text{(IR)} \quad V_{1,n} = 0 \\
&\qquad \text{(IC)} \quad (a_i,b_j) \text{ is IC with } (\widehat{a}_i,\widehat{b}_j)\ ,\ \forall\ (\widehat{a}_i,\widehat{b}_j) \in \widetilde{F}^{(a_i,b_j)}
\end{aligned} \tag{45}$$

Remarks:

1. In the original discretized problem, there are n^4 constraints, of which n^2 are IR and $n^4 - n^2$ are (maybe nonlinear) IC. After our methodology, the number of IC constraints is of order n^3.

2. In case Assumption 2 cannot be verified, we must consider all the IR constraints $V_{i,j} \geq 0$.

3. To obtain better accuracy of the solution, we can consider the monotonicity constraints $q_{i,j} \leq q_{i+1,j}$ and $q_{i,j} \leq q_{i,j-1}$. These $2n^2$ linear restrictions do not represent large numerical costs.

4. When the valuation function has the special *multiplicative separable* structure $v(q,a,b) = \psi(q) + \alpha(a,b)q + \beta(a,b)$, the IC constraints become linear in $q_{i,j}$. Therefore, since the IC constraints are linear in $V_{i,j}$ (regardless of v) and the objective function is strictly concave, the solution is unique, and we can rely on numerical approximations.

Because of the discretization, it is impossible to ensure that for each type (a,b), all the IC constraints are fulfilled. Nevertheless, following Belloni et al. (2010), we prove that the violations of the IC constraints (that is, the terms for which these constraints are not satisfied) uniformly converge to zero with finer discretizations and the sequence of optimal values converges to the optimal value of the continuous problem. These authors have considered a linear model including multiple agents and border constraints[8], which are not present in our setting. In contrast, we consider a valuation function v that could be nonlinear.

Let (Q^n, V^n) be the solution of the discretized problem (45). Since these functions are defined on X_n, we define the extensions $\widetilde{Q}^n, \widetilde{V}^n : [0,1]^2 \to \mathbb{R}$ as

$$\widetilde{Q}^n(x,y) := Q^n(a,b) \quad , \quad \widetilde{V}^n(x,y) := V^n(a,b)$$

where $(a,b) \in X_n$ is such that $a \leq x < a + \frac{1}{n-1}$ and $b - \frac{1}{n-1} < y \leq b$.

Let $\delta^*(\widetilde{Q}^n, \widetilde{V}^n)$ be the supremum over all IC constraint violations by the pair $(\widetilde{Q}^n, \widetilde{V}^n)$. That is, although some constraints are not fulfilled, we can be sure that for any $(a,b), (a',b') \in [0,1]^2$,

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(\widetilde{Q}^n(a',b'),a,b) - v(\widetilde{Q}^n(a',b'),a',b')) \geq -\delta^*(\widetilde{Q}^n, \widetilde{V}^n)$$

To guarantee the asymptotic feasibility of extensions $(\widetilde{Q}^n, \widetilde{V}^n)$, all IC constraint violations must uniformly converge to zero, as the next proposition shows.

Proposition 6.5. *We have $\delta^*(\widetilde{Q}^n, \widetilde{V}^n) \leq O(\frac{1}{n-1})$.*

The following proposition shows that optimality can be achieved in the limit.

Proposition 6.6. *Let OPT_n be the optimal value of the discretized problem, and let OPT^* be the optimal value of the continuous problem. Then, $\liminf_{n\to\infty} OPT_n \geq OPT^*$. Additionally, if $\lim_{n\to\infty} \widetilde{Q}^n(a,b)$ and $\lim_{n\to\infty} \widetilde{V}^n(a,b)$ exist for any $(a,b) \in [0,1]^2$, then $\lim_{n\to\infty} OPT_n = OPT^*$.*

7 Numerical Solution: Regulating a Monopolist Firm

Lewis and Sappington (1988b) studied the design of regulatory policy when the regulator is imperfectly informed about both the costs and the demand functions of the monopolist firm he is regulating. They considered that

[8] These constraints are related to the allocation treated as a probability since, in their model, there are N buyers and J degrees of product quality.

demand for the firm's product $q = Q(p, a)$ and the costs of producing output q, $C(q, b)$, involve firm's private information parameters (a, b) distributed over $\Theta = [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}]$ according to a strictly positive density function $f(a, b)$.

The regulator offers the firm a menu of contracts $(p(a, b), t(a, b))$ whereby if the firm sets the unit price $p(a, b)$ for its output, it receives the subsidy $t(a, b)$. It is assumed that the regulator can ensure that the firm serves all demand at the established prices. The regulator's objective function is the expected consumer surplus net of the transfer to the firm

$$\int_{\underline{a}}^{\overline{a}} \int_{\underline{b}}^{\overline{b}} \{\Pi(Q(p(a, b), a), a) - p(a, b)Q(p(a, b), a)) - t(a, b)\} f(a, b) da\, db$$

where $\Pi(Q, a) = \int_0^Q P(\xi, a) d\xi$, and $P(\cdot)$ denotes the inverse demand curve. By setting

$$\begin{aligned} v(p, a, b) &= pQ(p, a) - C(Q(p, a), b) \\ H(p, a) &= pQ(p, a) - \Pi(Q(p, a), a) \\ V(a, b) &= v(p(a, b), a, b) + t(a, b) \end{aligned}$$

we can write the regulator's problem as

$$\max_{p(\cdot), V(\cdot)} \int_{\underline{a}}^{\overline{a}} \int_{\underline{b}}^{\overline{b}} \{v(p(a, b), a, b) - H(p(a, b), a) - V(a, b)\} f(a, b) db da$$

subject to

(IR) $V(a, b) \geq 0 \qquad \forall\ (a, b) \in \Theta$

(IC) $V(a, b) - V(\widehat{a}, \widehat{b}) \geq v(p(\widehat{a}, \widehat{b}), a, b) - v(p(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b}) \qquad \forall\ (a, b), (\widehat{a}, \widehat{b}) \in \Theta$

Note that this formulation fits the standard nonlinear pricing model. The authors have derived a solution for the particular example

$$Q(p, a) = \alpha - p + a \quad , \quad C(q, b) = K + (c_0 + b)q$$

with α, K and c_0 positive constants and a uniform distribution over $\Theta = [0, 1]^2$. However, as Armstrong (1999) has noted, Lewis and Sappington's solution for this example cannot be right. Furthermore, in that paper, Armstrong argued that excluding a positive mass of types should be optimal (as in nonlinear pricing). However, because of the change in the variables he used, the type set is not convex, and his exclusion argument cannot strictly be applied. He also expressed the following:

- "*Nevertheless, I believe that the condition that the support be convex is strongly sufficient and that it will be the usual case that exclusion is optimal...*"

- "*I have not found it possible to solve this precise example correctly...*"

Therefore, we are facing a bidimensional adverse selection model with an unknown solution where a conjecture about the optimality of exclusion was made.

Given that $v(p, a, b) = (\alpha + a - p)(p - c_0 - b) - K$, the signs of v_a and v_b are endogenously determined, and Assumption 2 cannot be verified. Thus, we must consider all the IR constraints in the discretized problem (45). Even though Assumption 3 is not valid ($v_{pa} = v_{pb} = 1$), what really matters is the constant signs of v_{pa} and v_{pb}. In this case, $p(\cdot, \cdot)$ will be non-decreasing in a and b. Additionally, since $\frac{-v_{pb}}{v_{pa}} < 0$, the characteristic curves are strictly decreasing. Following the same considerations as in Section 6, it will be sufficient that each $(a, b)-$agent verifies the IC constraints with all the points over the set

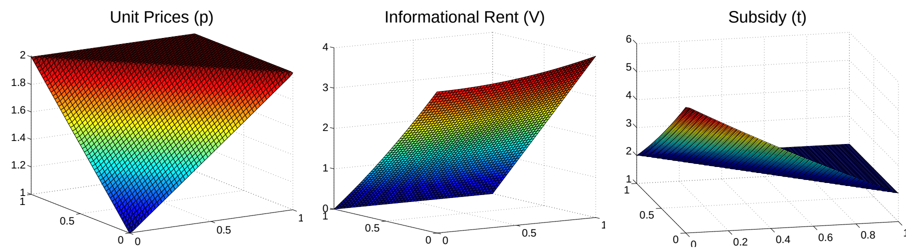
$$F^{(a,b)} := \{(0, s) \mid b \leq s \leq 1\} \cup \{(s, 1) \mid 0 \leq s \leq a\}$$

Note that the necessary conditions for optimality established in Section 4 cannot be applied in this case because Assumption 2 is not verified. In fact, we believe that this lack of consideration could be one of the failures in Lewis and Sappington's work. After they transformed the problem into a one-dimensional (by incorporating local incentive compatibility constraints into the regulator's objective function), they did not consider the IR constraints, which cannot be ruled out if Assumption 2 fails.

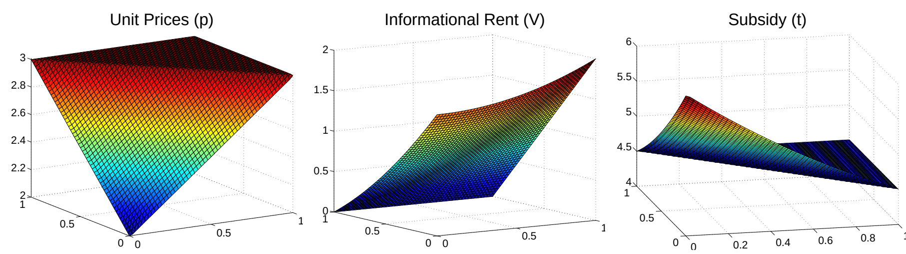
Note that the discretized problem has a unique solution in view of the linearity of IC constraints (v is multiplicative separable) and the strictly concavity of the objective function ($v_{pp} - H_{pp} < 0$ even when $H_{pp} < 0$).

We numerically solved the problem for three different cases of c_0, α and K. The type set was discretized into $n = 51$ points over each direction. The numerical solutions were obtained via Knitro/AMPL by using the active set algorithm. Next, we show the graphs of optimal prices, informational rent and subsidies.

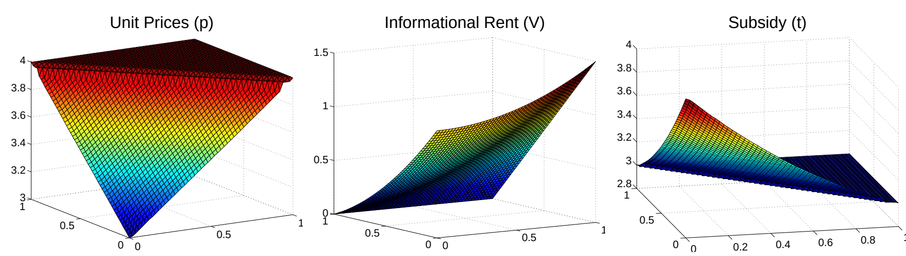
Case 1: $\boldsymbol{c_0 = 1\ ,\quad \alpha = 5\ ,\quad K = 2}$



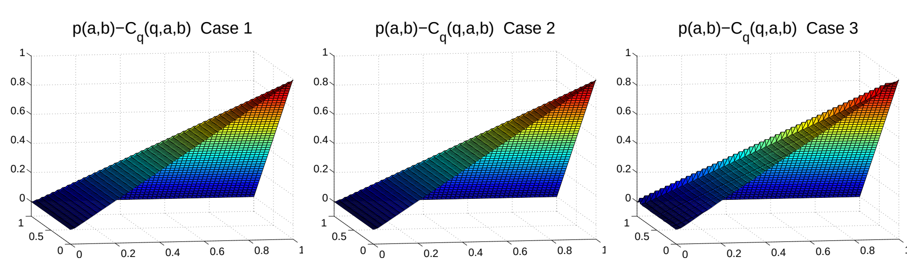
Case 2: $\boldsymbol{c_0 = 2\ ,\quad \alpha = 4\ ,\quad K = 4.5}$



Case 3: $\boldsymbol{c_0 = 3\ ,\quad \alpha = 4.5\ ,\quad K = 3}$



We also show the numerical differences between unit prices and marginal costs.



Some insights from these solutions:

We stress that this example derives into an optimization problem with a unique solution. In fact, the numerical methods to solve it are efficient. Thus, the statements below are reliable:

1. It seems that at the optimum, all types (a, b) such that $a + b \geq 1$ are bunching with unit price $c_0 + 1$, and the subsidy for them is the fixed cost K. Additionally, the unit price assigned to type $(0, 0)$ seems to be c_0.[9]

2. In view of the numerical difference $p - C_q$, the regulator induces the firm to price above marginal costs for almost all (a, b) types rather than $a = 0$ or $b = 1$ (i.e., such types with the lowest demand function or such types who obtain zero surplus)[10].

3. The numerical informational rent suggests that there is no exclusion.

7.1 A Discussion about the Optimality of Exclusion

Perhaps the most intriguing insight from the numerical solutions is that the non-exclusion of a positive mass of types should be optimal, contrary to Armstrong's conjecture stated previously.

Furthermore, in Barelli et al. (2014), the authors relaxed Armstrong's strong conditions (strict convexity and homogeneity of degree one) and proved a more general result of the desirability of exclusion. For this example, they considered that prices belong to $[c_0 + 1, \alpha]$ to conclude that their result can be applied and confirm Armstrong's conjecture. However, as can be seen, it is not true that $P \subset [c_0 + 1, \alpha]$. Therefore, their theorem should not be applied.

[9] In fact, we conjecture the optimum price p to be $p(a, b) = c_0 + a + b$ when $a + b \leq 1$ and $p(a, b) = c_0 + 1$ when $a + b > 1$

[10] In Baron and Myerson (1982), the authors analyzed a model in which the regulator is uncertain only about the firm's cost function. At the optimum, prices are above marginal costs for all cost realizations other than the lowest. In the model of Lewis and Sappington (1988a), the regulator is uncertain only about the position of the demand curves. In that model, if $C''(q) \geq 0$ (similar to here), setting prices at the level of marginal costs for the reported demand is optimal ($p = C_q$).

We are able to provide one technical argument explaining why Armstrong's Theorem about desirability of exclusion formulated in the nonlinear pricing context cannot be extended to this model. Additionally, we can provide an economic argument about why excluding types should not be optimal.

1. In nonlinear pricing, the customers' exit option is $q^{\text{out}} = 0$. Hence, the natural assumptions $v(q^{\text{out}}, a, b) = 0$ and $C(q^{\text{out}}) = 0$ imply that the monopolist's revenue $v(q^{\text{out}}, a, b) - C(q^{\text{out}}) - V(a, b)$ is zero when $V(a, b) = 0$ (that is, when type (a, b) is excluded). Then, the monopolist's penalty for causing some customers to exit the market is to not receive income from them.

On the other hand, in the regulation model, the firm's exit option is the unit price at which there is no production (i.e., p^{out} is such that $Q^{\text{out}} = 0$). Then, $\Pi(Q^{\text{out}}, a) - p^{\text{out}} Q^{\text{out}} - t(a, b) = -t(a, b)$ when type (a, b) is excluded. Additionally, $t(a, b) = C(Q^{\text{out}}, b) = C(0, b)$ because IR is binding. Thus, the regulator's penalty of excluding a firm is to subsidize the firm's fixed costs.

Thereby, in contrast with the monopolist, the regulator has to assume a negative penalty whenever firm's fixed costs are positive (in the previous example, $C(0, b) = K > 0$). Therefore, Armstrong's argument of comparing benefits (more income from agents still in the market) versus penalties (zero income from agents excluded) might not be applicable to this model. Thus, the main technical assumption not satisfied in Armstrong's Theorem is related neither to the strict convexity of the type set nor to the homogeneity of degree one of the valuation function (the strong technical assumptions), it is related to the invalidity of $v(p^{\text{out}}, a, b) = 0$.

2. When designing the contract, the monopolist is faced with a mass of customers, and the type distribution reflects the customers' different characteristics. In contrast, the regulator is faced with a single firm who is going to exercise the monopoly of some good. The type distribution reflects the probability that this firm has certain characteristics unknown by the regulator. Therefore, in the case that the regulator designs a contract with the possibility of exclusion and the firm chooses to be excluded as its best option, neither production nor consumption of the good occurs in the economy. Even in that case, consumers have to subsidize the firm's fixed costs. This situation cannot be optimal.

8 Appendix: Mathematical Proofs

Proof of Proposition 2.1. By the fundamental theorem of calculus and the envelope theorem,

$$V(a,b) - V(0,b) = \int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}$$
$$V(0,b) - V(0,1) = \int_1^b v_b(q(0,\tilde{b}),0,\tilde{b})d\tilde{b} = \int_1^b \underbrace{v_b(q^{\text{out}},0,\tilde{b})}_{=0}\,d\tilde{b} = 0$$

We have $v_b(q^{\text{out}},0,\tilde{b}) = 0$ as a consequence of Assumption 1. Moreover, by Assumption 2, we must have $V(0,1) = 0$. Hence, $V(a,b) = \int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}$. Then, through integration by parts,

$$\int_0^1 \int_0^1 V(a,b)f(a)h(b)\,da\,db$$
$$= \int_0^1 \Big[\Big(\int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}\Big)F(a)\Big|_{a=0}^{a=1} - \int_0^1 F(a)v_a(q(a,b),a,b)da\Big]\,h(b)\,db$$
$$= \int_0^1 \int_0^1 \Big(\frac{1-F(a)}{f(a)}v_a(q(a,b),a,b)\Big)f(a)h(b)\,da\,db$$

The result follows after placing the last expression into the expected income. □

Proof of Proposition 3.1. The cross derivatives t_{ab} and t_{ba} are given by

$$t_{ab} = \big(v_{qq}(q,a,b)q_b + v_{qb}(q,a,b)\big)q_a + v_q(q,a,b)q_{ab}$$
$$t_{ba} = \big(v_{qq}(q,a,b)q_a + v_{qa}(q,a,b)\big)q_b + v_q(q,a,b)q_{ba}$$

As t is twice differentiable at (a,b), we use Schwarz's integrability condition $t_{ab} = t_{ba}$ and $q_{ab} = q_{ba}$, and the result follows. □

Proof of Theorem 4.1. From the definition of function H in (14), we have

$$\begin{aligned}
H_\phi &= \int_0^U \{(G_q + G_a A_q)(A_q\phi' + A_r) + G(A_{qq}\phi' + A_{qr})\}\, ds \\
&\quad + G(\phi, A(\phi, r, U), U)\big(A_q\phi' + A_r\big)U_q \\
H_{\phi'} &= \int_0^{U(\phi,r)} G(\phi, A(\phi, r, s), s)A_q(\phi, r, s)\, ds \\
\frac{d}{dr}H_{\phi'} &= \int_0^U \{\big(G_q\phi' + G_a(A_q\phi' + A_r)\big)A_q + G(A_{qq}\phi' + A_{qr})\}\, ds \\
&\quad + G(\phi, A(\phi, r, U), U)A_q(\phi, r, U)\big(U_q\phi' + U_r\big)
\end{aligned}$$

Hence, the Euler equation $H_\phi - \frac{d}{dr}H_{\phi'} = 0$ yields

$$G(A_r U_q - A_q U_r) + \int_0^U G_q A_r\, ds = 0 \tag{46}$$

Let us show that $A_r U_q - A_q U_r = 0$. Since $U(\phi, r)$ is such that $a(r, U(\phi, r)) = 1$ or $b(r, U(\phi, r)) = 1$, we have two possibilities:

1. $A(\phi, r, U(\phi, r)) = 1$
   By differentiating with respect to ϕ and r, we obtain $A_q + A_s U_q = 0$ and $A_r + A_s U_r = 0$, respectively, from which
   $$A_r U_q - A_q U_r = A_r U_q - (-A_s U_q)U_r = U_q(A_r + A_s U_r) = 0$$
2. $U(\phi, r) = 1$
   Differentiating with respect to ϕ and r, we obtain $U_q = 0$ and $U_r = 0$.

Additionally, in the view that the marginal valuation is constant along the characteristic curve, we have $v_q(\phi, r, 0) = v_q(\phi, A(\phi, r, s), s)$. Then, by differentiating with respect to r,

$$v_{qa}(\phi, r, 0) = v_{qa}(\phi, A(\phi, r, s), s)A_r(\phi, r, s)$$

Finally, by substituting $A_r(\phi, r, s)$ into equation (46), we obtain

$$\int_0^{U(\phi,r)} v_{qa}(\phi, r, 0)\frac{G_q}{v_{qa}}(\phi, A(\phi, r, s), s)\, ds = 0 \tag{47}$$

Note that $v_{qa}(\phi, r, 0) > 0$ does not depend on s. Therefore, we obtain the result. $\square$

Proof of Theorem 4.2(i). From definition of H in (21), we have

$$H_\phi = - G(\phi, A(\phi,\beta,r,U),U)\big(A_q\phi' + A_\beta\beta' + A_r\big)U_q$$
$$- \int_\beta^U \{(G_q + G_a A_q)(A_q\phi' + A_\beta\beta' + A_r) + G(A_{qq}\phi' + A_{\beta q}\beta' + A_{rq})\}\, ds$$
$$H_{\phi'} = - \int_\beta^U GA_q\, ds$$
$$\frac{d}{dr}H_{\phi'} = - G(\phi, A(\phi,\beta,r,U),U)A_q(\phi,\beta,r,U)\big(U_q\phi' + U_\beta\beta' + U_r\big)$$
$$- \int_\beta^U \big(G_q\phi' + G_a(A_q\phi' A_\beta\beta' + A_r)\big)A_q + G(A_{qq}\phi' + A_{q\beta}\beta' + A_{qr})\, ds$$

Then,

$$H_\phi - \frac{d}{dr}H_{\phi'} = - G(A_\beta U_q - A_q U_\beta)\beta' - G(A_r U_q - A_q U_r) - \int_\beta^U (G_q A_\beta\beta' + G_q A_r)\, ds$$

Let us show that $A_\beta U_q - A_q U_\beta = 0$ and $A_r U_q - A_q U_r = 0$. From the definition of U, we have two possibilities:

1. $A(\phi,\beta,r,U(\phi,\beta,r)) = 1$
   By differentiating with respect to q, β and r, we obtain $A_q + A_s U_q = 0$, $A_\beta + A_s U_\beta = 0$ and $A_r + A_s U_r = 0$, respectively, from which
   $$A_\beta U_q - A_q U_\beta = A_\beta U_q - (-A_s U_q)U_\beta = U_q(A_\beta + A_s U_\beta) = 0$$
   $$A_r U_q - A_q U_r = A_r U_q - (-A_s U_q)U_r = U_q(A_r + A_s U_r) = 0$$

2. $U(\phi,\beta,r) = 1$
   By differentiating with respect to q, β and r, we obtain $U_q = U_\beta = U_r = 0$.

Thus, the Euler equation (22) can be written as

$$\int_\beta^U G_q\big(A_\beta\beta' + A_r\big)\, ds - R_\phi\lambda(r) = 0 \tag{48}$$

Since the marginal valuation is constant along the characteristic curve, that is, $v_q(\phi,r,\beta) = v_q(\phi, A(\phi,\beta,r,s),s)$, by differentiating with respect to β and r,

$$\begin{aligned} v_{qb}(\phi,r,\beta) &= v_{qa}(\phi, A(\phi,\beta,r,s),s)A_\beta(\phi,\beta,r,s) \\ v_{qa}(\phi,r,\beta) &= v_{qa}(\phi, A(\phi,\beta,r,s),s)A_r(\phi,\beta,r,s) \end{aligned} \tag{49}$$

Then, considering (18),

$$A_\beta \beta' + A_r = \frac{v_{qb}(\phi, r, \beta)\beta' + v_{qa}(\phi, r, \beta)}{v_{qa}(\phi, A(\phi, \beta, r, s), s)} = \frac{R_\phi}{v_{qa}(\phi, A(\phi, \beta, r, s), s)}$$

Finally, by substituting this expression into (48), we obtain

$$R_\phi \Big[\int_{\beta(r)}^{U(\phi(r),\beta(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r), \beta(r), r, s), r, s) ds - \lambda(r) \Big] = 0$$

and the result follows in view of $R_\phi \neq 0$. $\square$

Proof of Theorem 4.2(ii). From the definition of H in (21), we have

$$\begin{aligned}
H_\beta = & -G(\phi, A(\phi, \beta, r, U), U)\big(A_q\phi' + A_\beta\beta' + A_r\big)U_\beta \\
& + G(\phi, A(\phi, \beta, r, \beta), \beta)\big(A_q\phi' + A_\beta\beta' + A_r\big) \\
& - \int_\beta^U \{G_a A_\beta (A_q\phi' + A_\beta\beta' + A_r) + G(A_{q\beta}\phi' + A_{\beta\beta}\beta' + A_{r\beta})\}\, ds \\
H_{\beta'} = & -\int_\beta^U G A_\beta \, ds \\
\frac{d}{dr} H_{\beta'} = & -G(\phi, A(\phi, \beta, r, U), U) A_\beta(\phi, \beta, r, U)(U_q\phi' + U_\beta\beta' + U_r) \\
& + G(\phi, A(\phi, \beta, r, \beta), \beta) A_\beta(\phi, \beta, r, \beta)\beta' \\
& - \int_\beta^U \{\big(G_q\phi' + G_a(A_q\phi' + A_\beta\beta' + A_r)\big)A_\beta + G(A_{\beta q}\phi' + A_{\beta\beta}\beta' + A_{\beta r})\}\, ds
\end{aligned}$$

Then,

$$\begin{aligned}
H_\beta - \frac{d}{dr}H_{\beta'} = & -G(A_qU_\beta - A_\beta U_q)\phi' - G(A_rU_\beta - A_\beta U_r) \\
& + G(\phi, r, \beta)\big(A_q(\phi, \beta, r, \beta)\phi' + A_r(\phi, \beta, r, \beta)\big) + \phi' \int_\beta^U G_q A_\beta \, ds
\end{aligned}$$

As we have seen in the proof of item (i), $A_qU_\beta - A_\beta U_q = 0$ and $A_rU_\beta - A_\beta U_r = 0$. Furthermore, since $A(\phi, \beta, r, \beta) \equiv r$, we obtain $A_q(\phi, \beta, r, \beta) = 0$ and $A_r(\phi, \beta, r, \beta) = 1$. Therefore, by also considering (49), we have

$$H_\beta - \frac{d}{dr}H_{\beta'} = G(\phi, r, \beta) + \phi' v_{qb}(\phi, r, \beta) \int_\beta^U \frac{G_q}{v_{qa}}(\phi, A(\phi, \beta, r, s), s)\, ds \quad (50)$$

On the other hand, from the definition of R in (18),

$$R_\beta - \frac{d}{dr}R_{\beta'} = -\, v_{qb}(\phi, r, \beta)\phi' \tag{51}$$

Then, by substituting (50), (51) and $R_{\beta'} = v_b(\phi, r, \beta)$ into (23), we obtain

$$\phi' v_{qb}(\phi, r, \beta)\Big[\int_\beta^U \frac{G_q}{v_{qa}}(\phi, A(\phi, \beta, r, s), s)\, ds - \lambda(r)\Big] + G(\phi, r, \beta) - v_b(\phi, r, \beta)\lambda'(r) = 0$$

Note that, by item (i), the term in brackets is zero, and the result follows. $\square$

Proof of Theorem 4.3. From the definitions of H and F in (28) and (29), similar to the proofs of Theorems 4.1 and 4.2, we have

$$H_\varphi - \frac{d}{dr}(H_{\varphi'}) = -\int_{\overline{y}}^1 G_q(q, A(q, \varphi, s), s)A_\varphi(q, \varphi, s)\, ds \tag{52}$$

$$F_\varphi - \frac{d}{dq}(F_{\varphi'}) = -v_{qa}(q, \varphi, 1) \tag{53}$$

Since for any q and s fixed, $v_q(q, \varphi, 1) = v_q(q, A(q, \varphi, s), s)$, taking the derivative with respect to φ yields

$$v_{qa}(q, \varphi, 1) = v_{qa}(q, A(q, \varphi, s), s)A_\varphi(q, \varphi, s) \tag{54}$$

Thus, by substituting (52), (53) and (54) into (27), we obtain as a necessary condition that there exists some $\lambda \in \mathbb{R}$ such that

$$-v_{qa}(q, \varphi, 1)\int_{\overline{y}}^1 \frac{G_q}{v_{qa}}(q, A(q, \varphi, s), s)\, ds = -\lambda v_{qa}(q, \varphi, 1)$$

, and the result follows in view of $v_{qa} > 0$. $\square$

Proof of Claim 1. For this example, the necessary conditions (24) and (25) yield

$$\lambda(r) = \frac{3}{2}\beta(r)^2\phi(r) + (c-2)\beta(r)\phi(r) + \frac{1-2c}{2}\phi(r) + (2r-1)(1-\beta(r)) \tag{55}$$

$$\lambda'(r) = \frac{2(1-2r)}{\phi(r)} + \beta(r) + c \tag{56}$$

By taking the derivative of (55), by (56), we obtain

$$\phi'(r) = \frac{2(2-c-3\beta(r))}{(3\beta(r)+2c-1)(\beta(r)-1)} \tag{57}$$

Additionally, from boundary condition (19),

$$\phi(r)\beta'(r) = 2 \tag{58}$$

By taking the derivative of (58), by (57), we obtain

$$\beta'' + \frac{(2-c-3\beta)}{(3\beta+2c-1)(\beta-1)}(\beta')^2 = 0 \tag{59}$$

Thus, in the case that isoquants intersect line $y = 1$ and the participation's boundary β, curve β satisfies the differential equation (59). The solutions of (59) (besides constant functions) are of the form

$$\beta(r) = \frac{e^{\sqrt{3}B_0 r}}{2\sqrt{3}B_1} + \frac{B_1(c+1)^2}{6\sqrt{3}}e^{-\sqrt{3}B_0 r} - \frac{c-2}{3} \tag{60}$$

with $B_0, B_1 \in \mathbb{R}$.

Note that, for this example, informational rent V is a convex function. Hence, the non-participation region $\Omega = \{(a,b) : V(a,b) = 0\}$ is a convex set, and the boundary curves must be convex functions. That is, $\beta''(r) \geq 0$, which implies $B_1 > 0$. Since $\frac{\sqrt{3}e^{\sqrt{3}B_0 r}}{(c+1)B_1} + \frac{(c+1)B_1}{\sqrt{3}e^{\sqrt{3}B_0 r}} \geq 2$, we have

$$\beta(r) = \Big(\frac{\sqrt{3}e^{\sqrt{3}B_0 r}}{(c+1)B_1} + \frac{(c+1)B_1}{\sqrt{3}e^{\sqrt{3}B_0 r}}\Big)\frac{(c+1)}{6} - \frac{c-2}{3} \geq 1$$

Therefore, such curves cannot represent the boundary because they are not contained in the interior of $[0,1]^2$. □

Proof of Claim 2. First, we derive the necessary condition in the case that the isoquants are concurrent at the point $(\overline{x},\overline{y})$ and intersect the line $x = 1$. The PDE (5) can be written as $q_a + (-\frac{v_{qa}}{v_{qb}})q_b = 0$. Consider $\{(1,r) : r \in [R_1,R_2]\}$ as the initial curve. If $a(r,s) = s$ and $b(r,s) = B(\phi(r),r,s)$ are the solutions of

$$\begin{aligned} a_s(r,s) &= 1 \quad &, \quad a(r,1) = 1 \\ b_s(r,s) &= -\frac{v_{qa}}{v_{qb}}(\phi(r),a(r,s),b(r,s)) \quad &, \quad b(r,1) = r \end{aligned} \tag{61}$$

where $\phi : [R_1,R_2] \to [\underline{q},\overline{q}]$ describes the quantity allocated to $(1,r)$ types, a and b can be expressed as $a(q,s) = s$ and $b(q,s) = B(q,\varphi(q),s)$ with $q \in [\underline{q},\overline{q}]$

and $s \in [\overline{x}, 1]$, where $\varphi = \phi^{-1}$ (ϕ is strictly increasing due to $v_{qb} < 0$). Since type $(\overline{x}, \overline{y})$ is indifferent between any $q \in [\underline{q}, \overline{q}]$, $v_q(q, \overline{x}, \overline{y}) = v_q(q, 1, \varphi)$. By setting $H(q, \varphi, \varphi') = \int_{\overline{x}}^{1} G(q, s, B(q, \varphi(q), s))\Big(- (B_q + B_\varphi \varphi')\Big)\, ds$ and $F(q, \varphi, \varphi') = v_q(q, \overline{x}, \overline{y}) - v_q(q, 1, \varphi)$, the problem can be written as

$$\max_{\varphi(\cdot)} \int_{\underline{q}}^{\overline{q}} H(q, \varphi, \varphi')\, dq \quad \text{subject to} \quad \int_{\underline{q}}^{\overline{q}} F(q, \varphi, \varphi')\, dq = 0$$

The necessary condition $H_\varphi - \frac{d}{dq}(H_{\varphi'}) = \lambda(F_\varphi - \frac{d}{dq}(F_{\varphi'}))$ for some $\lambda \in \mathbb{R}$ implies

$$\int_{\overline{x}}^{1} \frac{G_q}{v_{qb}}(q, s, B(q, \varphi, s))\, ds = \lambda \tag{62}$$

Next, we prove Claim 2. For this case, the solutions of the system (61) are $a(r, s) = s$ and $b(r, s) = \frac{s-1}{\phi(r)} + r$. Then, $a(q, s) = s$ and $b(q, s) = \frac{(s-1)}{q} + \varphi(q)$. By condition (62), there exists $\lambda \in \mathbb{R}$ such that $\int_{\overline{x}}^{1} \Big(\frac{1-2s}{q} + \frac{(s-1)}{q} + \varphi + c\Big)\, ds = \lambda$, from which

$$\varphi(q) = \frac{1+\overline{x}}{2q} + \frac{\lambda}{1-\overline{x}} - c \tag{63}$$

On the other hand, $v_q(q, \overline{x}, \overline{y}) = v_q(q, 1, \varphi)$ implies that

$$\varphi(q) = \frac{1-\overline{x}}{q} + \overline{y} \tag{64}$$

Then, (63) and (64) imply $\overline{x} = \frac{1}{3}$, which contradicts $\frac{1}{3} < \frac{2c+1}{2c+3} < \overline{x} < 1$. □

Proof of Claim 3. By using the conditions $\phi^{I}(\underline{r}) = \phi^{II}(\underline{r})$ and $\beta(\underline{r}) = 0$, we obtain the constants K_0 and K_1 in terms of $\underline{r}$:

$$K_0 = \frac{-(1-\underline{r})(3\underline{r}-1)^2}{2c} \quad , \quad K_1 = \frac{4c\underline{r}(1-\underline{r})}{(3\underline{r}-1)^2} \tag{65}$$

Since we can write $\phi^{III}(r) = (r - \overline{x})/(1 - \beta(\overline{x}))$, from $\phi^{II}(\overline{x}) = \phi^{III}(1)$,

$$\beta(\overline{x}) = 1 - \frac{(1-3\overline{x})(1-\overline{x})^2}{K_0} \tag{66}$$

Then, by (35) and using (65),

$$(1-\overline{x})^3 = \frac{(9+4c)\underline{r}^3 - (15+8c)\underline{r}^2 + (7+4c)\underline{r} - 1}{2c} \tag{67}$$

On the other hand, since $\beta(\overline{x}) = \overline{y}$ from (66) and (35),

$$(1-3\overline{x})(\overline{x}-1)^2 = K_0(1-\overline{y}) \quad \text{and} \quad 2\overline{x}(\overline{x}-1)^2 = K_0(\overline{y}-K_1)$$

By dividing (seeing that $\frac{1}{3} < \frac{2c+1}{2c+3} < \overline{x} < 1$), clearing $\overline{x}$, using K_1 from (65) and now using $\overline{y} = (1-2c)/3$, we obtain

$$1-\overline{x} = \frac{(54+24c)\underline{r}^2 - (36+24c)\underline{r} + 6}{(63+18c)\underline{r}^2 - (42+24c)\underline{r} + (7-2c)} \tag{68}$$

Therefore, by (67) and (68), we have that $\underline{r}$ is the solution on $(\frac{2c+1}{2c+3}, \frac{1}{2})$ of

$$\Big(\frac{(54+24c)r^2 - (36+24c)r + 6}{(63+18c)r^2 - (42+24c)r + (7-2c)}\Big)^3 = \frac{(9+4c)r^3 - (15+8c)r^2 + (7+4c)r - 1}{2c}$$

with such $\underline{r}$. We obtain $\overline{x}$ from (68). □

Proof of Proposition 6.1. Fix $q \in Q$; by Assumption 2, $v_q(q,\cdot,b)$ is strictly increasing and $v_q(q,\widehat{a},\cdot)$ is strictly decreasing, so $\widehat{a} > a$ and $\widehat{b} < b$ imply $v_q(q,a,b) < v_q(q,\widehat{a},b)$ and $v_q(q,\widehat{a},b) < v_q(q,\widehat{a},\widehat{b})$, respectively. Thus, $v_q(q,a,b) < v_q(q,\widehat{a},\widehat{b})$. □

Proof of Proposition 6.2. Fix $(\widehat{a},\widehat{b})$ such that $a < \widehat{a}$ and $b > \widehat{b}$. Define $F(x,y) := V(x,y) - v(q(\widehat{a},\widehat{b}),x,y) \;\; \forall\, (x,y) \in [0,\widehat{a}] \times [\widehat{b},1]$. Then,

$$F_x = V_a(x,y) - v_a(q(\widehat{a},\widehat{b}),x,y) = v_a(q(x,y),x,y) - v_a(q(\widehat{a},\widehat{b}),x,y)$$

$$F_y = V_b(x,y) - v_b(q(\widehat{a},\widehat{b}),x,y) = v_b(q(x,y),x,y) - v_b(q(\widehat{a},\widehat{b}),x,y)$$

Conditions $q_a \geq 0$ and $q_b \leq 0$ imply that $q(x,y) \leq q(\widehat{a},\widehat{b})$. From Assumption 3, we obtain $F_x \leq 0$ and $F_y \geq 0$. Then, since $a < \widehat{a}$ and $b > \widehat{b}$, we have $F(a,b) \geq F(\widehat{a},\widehat{b})$. That is, (a,b) is IC with $(\widehat{a},\widehat{b})$. □

Proof of Proposition 6.3. If $(x,y) \in CC(\widehat{a},\widehat{b})$, then $q(\widehat{a},\widehat{b}) = q(x,y)$. Therefore, by the taxation principle, $t(\widehat{a},\widehat{b}) = T(q(\widehat{a},\widehat{b})) = T(q(x,y)) = t(x,y)$. Because (a,b) is IC with $(\widehat{a},\widehat{b})$, we have

$$v(q(a,b),a,b) - t(a,b) \geq v(q(\widehat{a},\widehat{b}),a,b) - t(\widehat{a},\widehat{b}) = v(q(x,y),a,b) - t(x,y)$$

that is, (a,b) is IC with (x,y). □

Proof of Proposition 6.4. Since (a,b) is IC with $(\widehat{a},\widehat{b})$ and $(\widehat{a},\widehat{b})$ is IC with (x,y), we have

$$\begin{aligned} V(a,b)-V(x,y)+&v(q(x,y),x,y)\geq \\ &v(q(\widehat{a},\widehat{b}),a,b)-v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b})+v(q(x,y),\widehat{a},\widehat{b}) \end{aligned} \tag{69}$$

Additionally, because $v_q(q,\widehat{a},\widehat{b})\leq v_q(q,a,b)\ \forall\ q\in Q$ and $q(x,y)\leq q(\widehat{a},\widehat{b})$, $\int_{q(x,y)}^{q(\widehat{a},\widehat{b})} v_q(q,\widehat{a},\widehat{b})dq \ \leq\ \int_{q(x,y)}^{q(\widehat{a},\widehat{b})} v_q(q,a,b)dq$. Then,

$$v(q(\widehat{a},\widehat{b}),a,b)-v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b})+v(q(x,y),\widehat{a},\widehat{b})\geq v(q(x,y),a,b) \tag{70}$$

Therefore, from (69) and (70), (a,b) is IC with (x,y). $\square$

Proof of Theorem 6.1. Fix any $(a,b),(\widehat{a},\widehat{b})\in[0,1]^2$. Let us prove that (a,b) is IC with $(\widehat{a},\widehat{b})$.

If $q(\widehat{a},\widehat{b})=q^{\text{out}}$ (that is, if type $(\widehat{a},\widehat{b})$ is excluded), we have $V(\widehat{a},\widehat{b})=0$, so from the IR constraint $V(a,b)\geq 0$, we can write

$$V(a,b)-V(\widehat{a},\widehat{b})\geq v(q^{\text{out}},a,b)-v(q^{\text{out}},\widehat{a},\widehat{b})$$

in view of $v(q^{\text{out}},a,b)=v(q^{\text{out}},\widehat{a},\widehat{b})$ by Assumption 1.

If $q(\widehat{a},\widehat{b})\neq q^{\text{out}}$, because $CC(\widehat{a},\widehat{b})$ is strictly increasing, there are three possible cases:

Case 1 $CC(\widehat{a},\widehat{b})$ intersects $F^{(a,b)}$:
Let (x,y) the point of intersection. Because (a,b) is IC with (x,y) and $(x,y)\in CC(\widehat{a},\widehat{b})$, by Proposition 6.3, (a,b) is IC with $(\widehat{a},\widehat{b})$.

Case 2 $CC(\widehat{a},\widehat{b})$ intersects $\{(1,s):0\leq s\leq b\}$:
Since $CC(\widehat{a},\widehat{b})$ is strictly increasing, then $\widehat{b}<b$. If $\widehat{a}>a$, by Proposition 6.1, we have that $(a,b)\preceq(\widehat{a},\widehat{b})$. Then, (a,b) is IC with $(\widehat{a},\widehat{b})$. If $\widehat{a}\leq a$, consider $(x,y)\in CC(\widehat{a},\widehat{b})\cap\text{conv}\{(a,b),(1,0)\}$ [11]. Then, (x,y) is such that $x>a$ and $y<b$, and we are in the previous case. That is, (a,b) is IC with (x,y), and by Proposition 6.3, (a,b) is IC with $(\widehat{a},\widehat{b})$.

Case 3 $CC(\widehat{a},\widehat{b})$ intersects $\{(s,1):0\leq s\leq a\}$ [12]:
Since $CC(\widehat{a},\widehat{b})$ is strictly increasing, $\widehat{a}<a$. Without the loss of generality,

[11] $\text{conv}\{(a,b),(1,0)\}$ is the convex hull of these points.

[12] In Figure 3, we present a graphic illustration for this case.

we consider that $\widehat{b} > b$ [13]. Let $(x_1, 1) \in CC(\widehat{a}, \widehat{b}) \cap \{(s, 1) : 0 \leq s \leq a\}$, and $(x_1, y_1) \in \{(x_1, y) : y \in \mathbb{R}\} \cap \mathrm{conv}\{(\widehat{a}, \widehat{b}), (a, b)\}$. Note that $q(\widehat{a}, \widehat{b}) < q(x_1, y_1)$ and, by Proposition 6.1, $(x_1, y_1) \preceq (a, b)$. Since (x_1, y_1) is IC with $(x_1, 1)$ (due to $(x_1, 1) \in F^{(x_1, y_1)}$), by Proposition 6.3, (x_1, y_1) is IC with $(\widehat{a}, \widehat{b})$. Then, by Proposition 6.4, it will be sufficient that $CC(x_1, y_1) \cap F^{(a,b)} \neq \emptyset$ to conclude that (a, b) is IC with $(\widehat{a}, \widehat{b})$. If $CC(x_1, y_1) \cap F^{(a,b)} = \emptyset$, repeat the procedure taking $(x_2, 1) \in CC(x_1, y_1) \cap \{(s, 1) : 0 \leq s \leq a\}$ and $(x_2, y_2) \in \{(x_2, y) : y \in \mathbb{R}\} \cap \mathrm{conv}\{(x_1, y_1), (a, b)\}$. Similarly to the above, we have $q(x_1, y_1) < q(x_2, y_2)$, $(x_2, y_2) \preceq (a, b)$ and that (x_2, y_2) is IC with (x_1, y_1). Then, by Proposition 6.4, it will be sufficient that $CC(x_2, y_2) \cap F^{(a,b)} \neq \emptyset$ to conclude that (a, b) is IC with (x_1, y_1), and therefore, by Proposition 6.4, that (a, b) is IC with $(\widehat{a}, \widehat{b})$. If $CC(x_2, y_2) \cap F^{(a,b)} = \emptyset$, we set up the point (x_3, y_3), and so on. Note that $\frac{d}{dq}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$ and $\frac{d}{da}\left(\frac{v_{qa}}{v_{qb}}\right) \geq 0$ implies $\frac{d}{dr} a_s(r, 1) \geq 0$ in view of

$$\frac{d}{dr} a_s(r, 1) = \frac{d}{dr}\Big(- \frac{v_{qb}}{v_{qa}}(q(r, 1), r, 1)\Big) = \Big(\frac{v_{qb}}{v_{qa}}\Big)^2 \Big[\frac{d}{dq}\Big(\frac{v_{qa}}{v_{qb}}\Big) \times q_a(r, 1) + \frac{d}{da}\Big(\frac{v_{qa}}{v_{qb}}\Big)\Big]$$

That is, the slope of the characteristic curves at the border $(r, 1)$ is non-decreasing, which guarantees that for a large enough n, $CC(x_n, y_n) \cap F^{(a,b)} \neq \emptyset$ because (x_n, y_n) will be close to (a, b) and $CC(x_n, y_n)$ is strictly increasing. Thus, applying Proposition 6.4 n times, we have that (a, b) is IC with $(\widehat{a}, \widehat{b})$. □

Proof of Proposition 6.5. The proof is based in the two following lemmas.

Lemma 1. *Given $(a, b) \in X_n$, $\forall$ $(x, y) \in F^{(a,b)}$, we have*

$$\widetilde{V}^n(a, b) - \widetilde{V}^n(x, y) \geq v(\widetilde{Q}^n(x, y), a, b) - v(\widetilde{Q}^n(x, y), x, y) - O(\tfrac{1}{n-1})$$

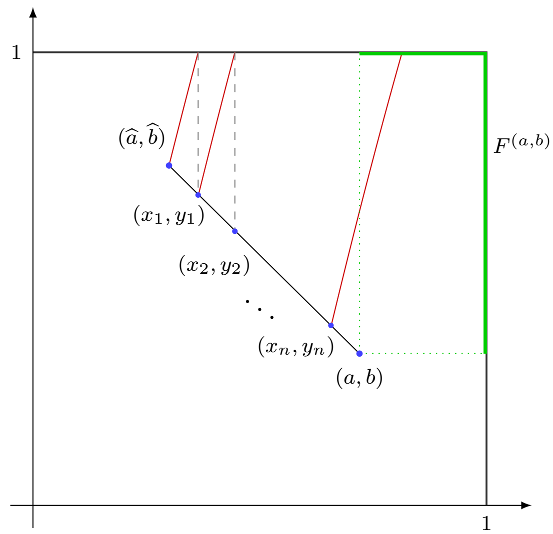
That is, since $(a, b) \in X_n$ verifies IC with all points in $\widetilde{F}^{(a,b)} = F^{(a,b)} \cap X_n$, it satisfies a relaxed IC version with all points in the continuous set $F^{(a,b)}$ with some tolerance that is asymptotically zero. The next lemma shows that between any two points on the grid X_n, the same relaxed IC version holds.

Lemma 2. *Given $(a, b), (\widehat{a}, \widehat{b}) \in X_n$, we have*

$$V^n(a, b) - V^n(\widehat{a}, \widehat{b}) \geq v(Q^n(\widehat{a}, \widehat{b}), a, b) - v(Q^n(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b}) - O(\tfrac{1}{n-1})$$

[13] If this is not the case, replace $(\widehat{a}, \widehat{b})$ for any point in $CC(\widehat{a}, \widehat{b})$ on the northwest of (a, b).

Figure 3: Illustration of Theorem 6.1 proof.



We return to the proof of Proposition 6.5. Given $(a,b),(a',b') \in [0,1]^2$, it will be sufficient to prove that

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(\widetilde{Q}^n(a',b'),a,b) - v(\widetilde{Q}^n(a',b'),a',b')) \geq -O(\tfrac{1}{n-1})$$

Let $(\widehat{a},\widehat{b}),(\widehat{a}',\widehat{b}') \in X_n$ be such that $\widehat{a} \leq a < \widehat{a} + \frac{1}{n-1}$, $\widehat{b} - \frac{1}{n-1} < b \leq \widehat{b}$ and $\widehat{a}' \leq a' < \widehat{a}' + \frac{1}{n-1}$, $\widehat{b}' - \frac{1}{n-1} < b' \leq \widehat{b}'$. Let $q = \widetilde{Q}^n(a',b') = Q^n(\widehat{a}',\widehat{b}')$. Since $\widetilde{V}^n(a,b) = V^n(\widehat{a},\widehat{b})$, $\widetilde{V}^n(a',b') = V^n(\widehat{a}',\widehat{b}')$ we have

$$\begin{aligned}\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(q,a,b) - v(q,a',b')) = \\ &V^n(\widehat{a},\widehat{b}) - V^n(\widehat{a}',\widehat{b}') - (v(q,\widehat{a},\widehat{b}) - v(q,\widehat{a}',\widehat{b}')) \\ &+ v(q,\widehat{a},\widehat{b}) - v(q,a,b) + v(q,a',b') - v(q,\widehat{a}',\widehat{b}')\end{aligned}$$

Since $(\widehat{a},\widehat{b}),(\widehat{a}',\widehat{b}') \in X_n$, by Proposition 2,

$$V^n(\widehat{a},\widehat{b}) - V^n(\widehat{a}',\widehat{b}') - (v(q,\widehat{a},\widehat{b}) - v(q,\widehat{a}',\widehat{b}')) \geq -O(\tfrac{1}{n-1})$$

Moreover, $v_a > 0$ and $\widehat{a}' \leq a'$ imply that $v(q,\widehat{a}',b') \leq v(q,a',b')$. Additionally, $v_b < 0$ and $b' \leq \widehat{b}'$ imply that $v(q,\widehat{a}',\widehat{b}') \leq v(q,\widehat{a}',b')$. Then, $v(q,a',b') -$

$v(q,\widehat{a}',\widehat{b}') \geq 0$. Hence,

$$\widetilde{V}^n(a,b)-\widetilde{V}^n(a',b')-(v(q,a,b)-v(q,a',b')) \geq v(q,\widehat{a},\widehat{b})-v(q,a,b)-O(\tfrac{1}{n-1})$$

Since v is Lipschitz (with constant L),

$$\left|v(q,\widehat{a},\widehat{b})-v(q,a,b)\right| \leq L||(\widehat{a},\widehat{b})-(a,b)|| \leq O(\tfrac{1}{n-1})$$

Then, $v(q,\widehat{a},\widehat{b})-v(q,a,b) \geq -O(\frac{1}{n-1})$. Therefore,

$$\widetilde{V}^n(a,b)-\widetilde{V}^n(a',b')-(v(q,a,b)-v(q,a',b')) \geq -O(\tfrac{1}{n-1})$$

□

Proof of Lemma 1. Let $(x,y) \in F^{(a,b)}$ be such that $x=1$ (case $y=1$ is analogous), and let $\widehat{b}$ be such that $\widehat{b}-\frac{1}{n-1} < y \leq \widehat{b}$. Since (Q^n,V^n) are the solutions of problem (45), (a,b) satisfies IC with $(1,\widehat{b})$

$$V^n(a,b)-V^n(1,\widehat{b}) \geq v(Q^n(1,\widehat{b}),a,b)-v(Q^n(1,\widehat{b}),1,\widehat{b})$$

By definition, $\widetilde{Q}^n(x,y)=Q^n(1,\widehat{b})$ and $\widetilde{V}^n(x,y)=V^n(1,\widehat{b})$. Additionally, in view of $(a,b) \in X_n$, we have $\widetilde{V}^n(a,b)=V^n(a,b)$. Then,

$$\widetilde{V}^n(a,b)-\widetilde{V}^n(x,y) \geq v(\widetilde{Q}^n(x,y),a,b)-v(\widetilde{Q}^n(x,y),1,\widehat{b}) \tag{71}$$

On the other hand, since v is Lipschitz,

$$\left|v(\widetilde{Q}^n(x,y),1,\widehat{b})-v(\widetilde{Q}^n(x,y),x,y)\right| \leq L||(1,\widehat{b})-(x,y)||=O(\tfrac{1}{n-1})$$

Then,

$$-v(\widetilde{Q}^n(x,y),1,\widehat{b}) \geq -v(\widetilde{Q}^n(x,y),x,y)-O(\tfrac{1}{n-1}) \tag{72}$$

Therefore, from (71) and (72),

$$\widetilde{V}^n(a,b)-\widetilde{V}^n(x,y) \geq v(\widetilde{Q}^n(x,y),a,b)-v(\widetilde{Q}^n(x,y),x,y)-O(\tfrac{1}{n-1})$$

□

Proof of Lemma 2. If $CC(\widehat{a},\widehat{b}) \cap F^{(a,b)}=(x,y)$, we apply Proposition 1 for (a,b) with (x,y), and considering that $Q^n(\widehat{a},\widehat{b})=Q^n(x,y)$ and $t(x,y)=t(\widehat{a},\widehat{b})$, we conclude. Other cases are treated analogously as in the proof of Theorem 6.1. □

Proof of Proposition 6.6. Let $(\overline{Q}, \overline{V})$ denote the solution for the continuous problem, and let $(\overline{Q}^n, \overline{V}^n)$ be their restriction on the grid X_n. If (Q^n, V^n) are the solutions of the discretized problem and OPT_n is the optimal value, we have

$$\begin{aligned} OPT_n &\geq \sum_{i=1}^{n}\sum_{j=1}^{n} w(i,j)(v(\overline{Q}^n_{i,j}, a_i, b_j) - \overline{V}^n_{i,j} - C(\overline{Q}^n_{i,j}))f(a_i, b_j) \\ &= \int_0^1\int_0^1 (v(\overline{Q}(a,b), a, b) - \overline{V}(a,b) - C(\overline{Q}(a,b)))f(a,b)\, da\, db - O(\frac{1}{n}) \\ &= OPT^* - O(\frac{1}{n}) \end{aligned}$$

Then, $\liminf_{n\to\infty} OPT_n \geq OPT^*$.

On the other hand, if $\exists\ \lim_{n\to\infty} \widetilde{Q}^n(a,b)$ and $\lim_{n\to\infty} \widetilde{V}^n(a,b)$ for any $(a,b) \in [0,1]^2$, define

$$\widehat{Q}(a,b) := \lim_{n\to\infty} \widetilde{Q}^n(a,b) \quad , \quad \widehat{V}(a,b) := \lim_{n\to\infty} \widetilde{V}^n(a,b)$$

Proposition 6.5 guarantees that $(\widehat{Q}, \widehat{V})$ is feasible. Hence

$$\begin{aligned} OPT^* &\geq \int_0^1\int_0^1 (v(\widehat{Q}(a,b), a, b) - \widehat{V}(a,b) - C(\widehat{Q}(a,b)))f(a,b)\, da\, db \\ &= \lim_{n\to\infty} \Big(\int_0^1\int_0^1 (v(\widetilde{Q}^n(a,b), a, b) - \widetilde{V}^n(a,b) - C(\widetilde{Q}^n(a,b)))f(a,b)\, da\, db \Big) \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} w(i,j)(v(\widetilde{Q}^n_{i,j}, a_i, b_j) - \widetilde{V}^n_{i,j} - C(\widetilde{Q}^n_{i,j}))f(a_i, b_j) + O(\tfrac{1}{n-1}) \Big) \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} w(i,j)(v(Q^n_{i,j}, a_i, b_j) - V^n_{i,j} - C(Q^n_{i,j}))f(a_i, b_j) + O(\tfrac{1}{n-1}) \Big) \\ &= \lim_{n\to\infty} \Big(OPT_n + O(\tfrac{1}{n-1}) \Big) \end{aligned}$$

where equalities are true by the dominated convergence theorem (each $\widetilde{Q}^n$ and $\widetilde{V}^n$ are bounded), by the finite approximation of the integral, by the definition of $\widetilde{Q}^n$ and $\widetilde{V}^n$, and because (Q^n, V^n) is the solution of the discretized problem. $\square$

References

Araujo, A. and Moreira, H. (2010). Adverse selection problems without the Spence-Mirrlees condition. *Journal of Economic Theory*, 145(3):1113–1141.

Armstrong, M. (1996). Multiproduct nonlinear pricing. *Econometrica*, 64(1):51–75.

Armstrong, M. (1999). Optimal regulation with unknown demand and cost functions. *Journal of Economic Theory*, 84:196–215.

Barelli, P., Basov, S., Bugarin, M., and King, I. (2014). On the optimality of exclusion in multi-dimensional screening. *Journal of Mathematical Economics*, 54:74–83.

Baron, D. and Myerson, R. (1982). Regulating a monopolist with unknown costs. *Econometrica*, 50(4):911–930.

Basov, S. (2001). Hamiltonian approach to multi-dimensional screening. *Journal of Mathematical Economics*, 36:77–94.

Basov, S. (2005). *Multidimensional Screening*. Springer-Verlag.

Belloni, A., Lopomo, G., and Shouqiang, W. (2010). Multidimensional mechanism design: Finite-dimensional approximations and efficient computation. *Operations Research*, 58(4):1079–1089.

Biais, B., Martimort, D., and Rochet, J. (2000). Competing mechanisms in a common value environment. *Econometrica*, 68(4):799–837.

Deneckere, R. and Severinov, S. (2015). Multidimensional screening: A solution to a class of problems. (mimeo) econ.ucsb.edu.

Gibbard, A. (1973). Manipulation of voting schemes: a general result. *Econometrica*, 41:587–601.

John, F. (1981). *Partial Differential Equations*. Springer Verlag.

Laffont, J. and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press.

Laffont, J., Maskin, E., and Rochet, J. (1987). Optimal nonlinear pricing with two-dimensional characteristics. In *Incentives and Economic Mechanisms (essays in honor of Leonid Hurwicz)*, pages 256–266. T.Groves, R.Radner, S. Reiter, Eds., Minneapolis.

Lewis, T. and Sappington, D. (1988a). Regulating a monopolist with unknown demand. *American Economic Review*, 78:986–998.

Lewis, T. and Sappington, D. (1988b). Regulating a monopolist with unknown demand and cost functions. *RAND Journal of Economics*, 19(3):438–457.

McAfee, R. and McMillan, J. (1988). Multidimensional incentive compatibility and mechanism design. *Journal of Economic Theory*, 46(2):335–354.

Milgrom, P. and Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, 70:583–601.

Mussa, M. and Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory*, 18(2):301–317.

Rochet, J.-C. and Choné (1998). Ironing, sweeping, and multidimensional screening. *Econometrica*, 66(4):783–826.

Wilson, R. (1993). *Non-linear Pricing*. Oxford University Press.

Wilson, R. (1995). Nonlinear pricing and mechanism design. In Amman, H., Kendrick, D., and Rust, J., editors, *Handbook of Computational Ecnonomics*, volume 1. Elsevier Science Publishers. Revised: 10 October 1999.