A liquidation risk adjustment for value at risk and expected shortfall

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Abstract

This paper proposes an intuitive and flexible framework to quantify liquidation risk for financial institutions. We develop a model of feedback where the "fundamental" dynamics of assets is modified by price impact from fund liquidations (if any). We characterize mathematically the liquidation schedule of financial institutions and study in detail the fire sales resulting endogenously from margin constraints when trading through an exchange. Our study enables to obtain tractable formulas for the value at risk and expected shortfall of a financial institution in the presence of fund liquidation. In particular, we find an additive decomposition for liquidation-adjusted risk measures which are equal to "fundamental" risk-measures plus a liquidation risk adjustment, which is proportional to the size of fund positions as a fraction of asset market depths. Our results may be used in practice by risk-management of financial institutions to better tackle liquidity events arising from fund liquidations and adjust their portfolio allocations to liquidation risk.

1 Introduction

Over the past decades, and following the impulse of new regulatory requirements, financial institutions have put a lot of emphasis of the study and development of systems for the risk management of financial portfolios. Academics and practitioners have introduced more and more statistically-sophisticated models for the dynamics of financial assets, the aim being to better reproduce empirical behavior of financial markets and better assess the risk of financial portfolios. This includes switching from Gaussian to fat-tail frameworks, introducing stochastic volatility and correlations, using complex factor models or implementing comprehensive stress tests (see McNeil et al. (2015) for a detailed review).

Despite those efforts, financial markets experience, on a regular basis, risk management fiascos where large financial institutions suffer unexpected, and sometimes unprecedented, losses, leading in some cases to their collapse. Two of the most spectacular examples are the collapse of LTCM in 1998 and the London Whale lossed of JP Morgan in 2012. LTCM was a hedge fund created in 1994, that had built a portfolio of historically-diversified strategies (diversified in terms of asset classes and geographical zones) that generated large returns, amounting typically to 40% annually. In 1998, LTCM collapsed when trying to unwind some of its positions in order to compensate for losses in one particular strategy. The failure of LTCM materialized in only a few days and was mainly due to the fact that LTCM positions were hard to unwind because of their size and the liquidity of the assets involved (Rosenfeld, 2010). The London Whale losses for JP Morgan occured in 2012 when a trader operating in London for JP Morgan Chief Investment Office and that had accumulated gigantic positions (hence the name London Whale) in credit derivatives, some amounting to hundreds of billion dollars in notional, generated a $6.2 Bn loss for JP Morgan when liquidating such positions (U.S. Senate, 2013; JP Morgan, 2014).

In most cases, and typically in the case of LTCM and the London Whale, such unexpected losses materialize when the financial institution is forced to liquidate large and/or illiquid positions. Those

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forced sales are called fire sales (Shleifer & Vishny, 2011) and are triggered after an adverse market scenario by constraints such as capital requirements (Greenwood et al., 2015; Cont & Schaanning, 2016; Braouezec & Wagalath, 2016, 2017) or margin requirements (Cont & Wagalath, 2016a). They generate endogenous price movements that further deteriorate a given fund's positions.
Whereas fire sales are recognized as a potentially destabilizing factor for financial institutions, most of the risk management systems developed in the past do not incorporate this liquidation risk, and rather focus on the statistical modeling of mark-to-market losses. As such, they do not take into account the size nor the liquidity of a fund's positions and hence cannot account for the endogenous adverse impact of fund liquidations. The use of such exogenous representation of asset prices naturally leads to a linear vision of risk: for example, value at risk and expected shortfall, which are the two most widely used risk measures, scale linearly in fund size. As a consequence, they underestimate, sometimes dramatically, the risk of a portfolio as they give no consideration to the concentration of the portfolio positions. The London Whale losses in 2012 gave a remarkable illustration of the understimation of liquidation risk: while the portfolio was monitored by "classical" homogenous risk measures that scale linearly in fund size and was expected to generate a $500 Mn loss when being liquidated, the final loss for JP Morgan amounted to $6.2 Bn, more than 12 times what was anticipated.
While the underestimation of liquidation risk is now acknowledged by regulators and financial institutions, most of the actions taken to mitigate that risk seem arbitrary and ad-hoc, with no relation to a comprehensive model. For example, in line with Basel 3 regulation, regulators have imposed add-hoc additional capital requirements for banks classified as globally systemically important banks (a classification which is itself arbitrary) in order to mitigate, among other things, their liquidation risk.
In the recent (and not so recent) past, academics have introduced models that included liquidation risk in order to price liquidation losses (Almgren & Chriss, 2000; Almgren & Lorenz, 2006) or model the realized correlation matrix (Cont & Wagalath, 2013, 2016a) and its principal components properties (Cont & Wagalath, 2016b). Avellaneda & Cont (2013) is, to our knowledge, one of the first papers to study the consequences of fire sales in terms of portfolio risk measures. Contrary to the approach that we develop here, they calculate the additional liquidation risk for a fund that liquidates large positions slowly enough so that it does not impact asset prices.
The contribution of this paper is to develop a simple model that takes into account the feedback from fire sales following adverse market scenarios, and to quantify the resulting liquidation risk explicitly. In particular, contrary to existing studies on the topic, we are able to derive in a tractable manner a liquidation risk adjustment for risk measures such as value at risk and expected shortfall, and we show how to apply this liquidation risk adjustment on empirical examples.

We first develop a model which takes into account liquidation risk in a single asset setting and we quantify the value at risk and the expected shortfall in the presence of liquidation risk for a specific type of liquidation schedule. We show that such risk measures can be decomposed as the sum a "fundamental" risk and a liquidation risk adjustment, that depends on the size and liquidity of positions. We illustrate the liquidation risk adjustment in the context of margin constraints imposed by a clearinhouse. We then extend our model to a multi asset setting and introduce a generic liquidation risk adjustment for a given portfolio of assets. We find that depending on the portfolio size and the asset market depths, the risk of a portfolio may be driven more by fundamentals or liquidation risk. We illustrate that additional liquidation risk adjustment empirically, by studying the risk of portfolios on US sector ETFs.

Section 2 presents a model which takes into account liquidation risk in a single asset framework. Section 3 applies this single asset setting to the case of margin constraints. Section 4 studies the multi-dimensional framework whose applications are illustrated empirically in Section 5.

2 Modeling liquidation risk in a single asset setting

2.1 Fundamental asset dynamics and risk-measures

Consider a market with a single asset S whose "fundamentals" dynamics over a given period of time (arbitrarily taken equal to 1 in the sequel) is:

$$S_1 = S_0 (1 + \sigma\xi) \tag{1}$$

where ξ is a centered random variable with variance 1. The dynamics given in equation (1) is called the "fundamental" dynamics in the sense that it exhibits the asset movement in the absence of systematic feedback effect. This would be the price dynamics if S moved due to exogenous factors only.
Risk management in financial institutions essentially focuses on the modeling of the random variable ξ, in order to characterize mark-to-market loss distributions. This enables to compute risk measures for the portfolios of financial institutions, typically value-at-risk or expected shortfall, which are the two most widely-used risk measures in financial institutions, and are the focus of this paper. For example, in a (simplistic) Black and Scholes framework, ξ would be Gaussian. It is well-known that tail risk is underestimated in such a Gaussian setting and financial institutions have commonly been using random variables with fat tails, such as Student random variables, to account for tail risk (McNeil et al., 2015).
Consider a fund initially holding δ_0 units of asset S. The mark-to-market loss for the fund is equal to:

$$\text{MtM Loss} = \delta_0(S_0 - S_1) = -\delta_0 S_0 \sigma\xi \tag{2}$$

Assume that the random variable ξ is continuous and denote its cumulative distribution function:

$$\mathcal{N}(x) = \mathbb{P}(\xi \leq x) \tag{3}$$

The value-at-risk and expected shortfall at level α (typically α is equal to 0.95 or 0.99) associated with the mark-to-market loss defined in equation (2) are respectively given by:

$$\underline{VaR}_\alpha = \delta_0 S_0 \sigma \times \left(-\mathcal{N}^{-1}(1-\alpha)\right) \tag{4}$$

$$\underline{ES}_\alpha = \delta_0 S_0 \sigma \times \frac{\mathbb{E}\left(-\xi \;\; \& \;\; \xi \leq \mathcal{N}^{-1}(1-\alpha)\right)}{1-\alpha} \tag{5}$$

Equations (4) and (5) show that value-at-risk and expected shortfall scale linearly in fund size δ_0. This would actually be the case for any risk-measure with the homogeneity property. In that context, "risk" is perceived as being linear and, for example, the "risk" for a fund with positions $1000 \times \delta_0$ would be equal to 1000 times the "risk" of a fund with positions δ_0. This is a natural consequence of the fact that such risk measures only focus on mark-to-market losses and explicitely assume that the institution does not react to a price movement and further impact prices. In particular, this type of risk calculation does not take into account asset liquidity which, as discussed in Section 1, proves to be a crucial determinant of portfolio risk.

2.2 Feedback from liquidation in (extreme) loss scenarios

As discussed in Section 1, risk measures are used to compute margin/capital requirements, which act as a cushion of safety in case of extreme losses. However, when a fund faces large mark-to-market losses, it may be forced to liquidate a sizeable portion of its holdings in order to comply with exogenous constraints – typically margin constraints set by the exchange or capital constraints set by the regulator – and the realized loss for the fund may be significantly larger than expected, i.e. larger than the initial mark-to-market loss, due to liquidation costs. As such, risk-measures based on mark-to-market losses may underestimate the "real" risk of the fund's positions as they do not take into account the potential feedback from the fund's own liquidations.
In this section, we model this feedback effect in a generic manner. Consider a fund with holding δ_0 units of asset S. Denote the percentage (mark-to-market) loss for the fund by:

$$X = \frac{S_0 - S_1}{S_0} = -\sigma\xi \tag{6}$$

Empirical studies show that when this percentage loss is small enough the fund does not liquidate assets. However, when the loss exceeds a threshold, the fund is forced to liquidate a positive portion of its positions. Indeed, the larger the mark-to-market loss, the larger the portion of the fund sold. When the loss is extreme, the fund is insolvent and is fully liquidated by the competent authority.
We model the liquidation of the fund by introducing a liquidation schedule function f such that $f(X)$ is equal to the portion of the fund liquidated after a percentage mark-to-market loss X i.e.:

$$f(X) = \frac{\delta_0 - \delta_1}{\delta_0} \in [0, 1] \tag{7}$$

where δ_1 denotes the fund's holdings at date 1, after potential deleveraging. f is equal to 0 on $]-\infty, \underline{\gamma}[$ and 1 on $]\overline{\gamma}, +\infty[$ and is increasing on $[\underline{\gamma}, \overline{\gamma}]$. The precise form of f depends on the constraints of the fund. In the next section, we will present explicit liquidation schedules which result from margin constraints. Cont & Wagalath (2016c) characterizes the liquidation schedule resulting from capital ratio constraints. We assume that if the fund needs to liquidate assets, such sales occur right after the fundamental asset movement, that is, at date 1. When the fund sells a large block of assets, it may impact the asset price. The amplitude of the feedback from the fund's fire sales depends on the asset liquidity. We introduce a (linear) liquidity parameter Φ, that is the market depth of asset S (citer). At date 1 and after potential rebalancing by the fund, the asset price is given by

$$S_1^* = S_0 \left(1 + \sigma\xi - \frac{\delta_0 - \delta_1}{\Phi}\right) = S_0 \left(1 - X - \frac{\delta_0}{\Phi} f(X)\right) \tag{8}$$

When the initial fund loss X is small enough ($X < \underline{\gamma}$), the fund does not have to liquidate assets ($f(X) = 0$) and $S_1^* = S_1$. Similarly, when asset S is infinitely liquid ($\Phi = \infty$), there is no feedback from the fund's liquidation on the asset price ($S_1^* = S_1$). However, when the fund is forced to liquidate a sizeable portion of its positions in an asset with finite liquidity, then it is going to impact the asset price. In that case, the fund's asset sales decrease the asset price from S_1 to S_1^*. Assuming that the fund is able to sell assets only at price S_1^*, the fund's loss is equal to

$$\text{Loss} = \delta_0 S_0 \left(X + \frac{\delta_0}{\Phi} f(X)\right) \tag{9}$$

which is no longer linear in fund size. Remark that the assumption that the fund sells assets at price $S_1^* = S_1$ is conservative: in practice, the fund gets intermediary prices between S_1 and S_1^* as it makes the asset price move from S_1 to S_1^*.

2.3 Case of a 0-1 liquidation schedule

In this section, we examine the case of a simple liquidation schedule, which will be useful for studying more realistic examples in the sequel. f is defined as follows:

$$f(x) = 0 \quad \text{if } x \leq \gamma \tag{10}$$

$$f(x) = 1 \quad \text{if } x > \gamma \tag{11}$$

where $\gamma > 0$. This means that, as long as the fund's percentage loss does not exceed γ, the fund does not need to rebalance its positions. In the case where the fund loses more than γ, it is fully liquidated (typically because it is insolvent).
The fund's loss, given in equation (9), can be simplified:

$$\text{Loss} = -\delta_0 S_0 \sigma\xi \quad \text{if } \xi \geq \frac{-\gamma}{\sigma} \tag{12}$$

$$\text{Loss} = -\delta_0 S_0 \sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \quad \text{if } \xi < \frac{-\gamma}{\sigma} \tag{13}$$

Proposition 2.1. *The value-at-risk and the expected shortfall for a fund whose loss distribution is given by equations* (12) *and* (13) *are equal respectively to:*

1. *if* $\gamma > -\mathcal{N}^{-1}(1-\alpha)\sigma$*:*

$$VaR_\alpha = \underline{VaR}_\alpha \tag{14}$$

$$ES_\alpha = \underline{ES}_\alpha + \frac{\delta_0^2 S_0}{\Phi}\frac{\mathcal{N}(-\frac{\gamma}{\sigma})}{1-\alpha} \tag{15}$$

2. *if* $\gamma \leq -\mathcal{N}^{-1}(1-\alpha)\sigma$*:*

$$VaR_\alpha = \underline{VaR}_\alpha + \frac{\delta_0^2 S_0}{\Phi} \tag{16}$$

$$ES_\alpha = \underline{ES}_\alpha + \frac{\delta_0^2 S_0}{\Phi} \tag{17}$$

where $\underline{VaR}_\alpha$ *and* $\underline{ES}_\alpha$ *are the "fundamental" value-at-risk and expected shortfall for the mark-to-market loss, defined respectively in equations* (4) *and* (5)*.*

Proof. Let us first remark that the fund's loss is always larger or equal to $-\delta_0 S_0 \sigma\xi$, which implies that $VaR_\alpha \geq \underline{VaR}_\alpha$ and $ES_\alpha \geq \underline{ES}_\alpha$.
Assume $\gamma > -\mathcal{N}^{-1}(1-\alpha)\sigma$. This implies:

$$\mathbb{P}(\text{Loss} \leq \underline{Var}_\alpha) = \mathbb{P}(\text{Loss} \leq \underline{Var}_\alpha \ \& \ -\sigma\xi \leq \gamma) + \underbrace{\mathbb{P}(\text{Loss} \leq \underline{Var}_\alpha \ \& \ -\sigma\xi > \gamma)}_{=0 \text{ because } \gamma > -\mathcal{N}^{-1}(1-\alpha)\sigma} \tag{18}$$

$$= \mathbb{P}(-\delta_0 S_0 \sigma\xi \leq \underline{Var}_\alpha \ \& \ -\sigma\xi \leq \gamma) = \mathbb{P}(-\delta_0 S_0 \sigma\xi \leq \underline{Var}_\alpha) = \alpha \tag{19}$$

This implies that $VaR_\alpha = \underline{VaR}_\alpha$. In addition, we have:

$$ES_\alpha = \mathbb{E}\left(\text{Loss} \mid \text{Loss} \geq \underline{VaR}_\alpha\right) = \frac{\mathbb{E}\left(\text{Loss} \ \& \ \text{Loss} \geq \underline{VaR}_\alpha\right)}{1-\alpha} \tag{20}$$

$$= \frac{1}{1-\alpha}\left[\mathbb{E}\left(\text{Loss} \ \& \ \text{Loss} \geq \underline{VaR}_\alpha \ \& \ -\sigma\xi \leq \gamma\right) + \mathbb{E}\left(\text{Loss} \ \& \ \text{Loss} \geq \underline{VaR}_\alpha \ \& \ -\sigma\xi > \gamma\right)\right] \tag{21}$$

$$= \frac{1}{1-\alpha}\left[\mathbb{E}\left(-\delta_0 S_0 \sigma\xi \ \& \ \text{Loss} \geq \underline{VaR}_\alpha \ \& \ -\sigma\xi \leq \gamma\right) + \mathbb{E}\left(-\delta_0 S_0 \sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \ \& \ \text{Loss} \geq \underline{VaR}_\alpha \ \& \ -\sigma\xi > \gamma\right)\right] \tag{22}$$

$$= \frac{1}{1-\alpha}\left[\mathbb{E}\left(-\delta_0 S_0 \sigma\xi \ \& \ \text{Loss} \geq \underline{VaR}_\alpha\right) + \mathbb{E}\left(\frac{\delta_0^2 S_0}{\Phi} \ \& \ \text{Loss} \geq \underline{VaR}_\alpha \ \& \ -\sigma\xi > \gamma\right)\right] \tag{23}$$

$$= \underline{ES}_\alpha + \frac{1}{1-\alpha}\mathbb{E}\left(\frac{\delta_0^2 S_0}{\Phi} \ \& \ -\sigma\xi > \gamma\right) \tag{24}$$

$$ES_\alpha = \underline{ES}_\alpha + \frac{\delta_0^2 S_0}{\Phi}\frac{\mathcal{N}(-\frac{\gamma}{\sigma})}{1-\alpha} \tag{25}$$

Assume now that $\gamma \leq -\mathcal{N}^{-1}(1-\alpha)\sigma$. We then have:

$$\mathbb{P}(\text{Loss} \leq VaR_\alpha) = \underbrace{\mathbb{P}(\text{Loss} \leq VaR_\alpha \ \& \ -\sigma\xi \leq \gamma)}_{=0} + \mathbb{P}(\text{Loss} \leq VaR_\alpha \ \& \ -\sigma\xi > \gamma) \tag{26}$$

$$= \mathbb{P}\left(-\delta_0 S_0 \sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \leq VaR_\alpha \ \& \ -\sigma\xi > \gamma\right) \tag{27}$$

$$= \mathbb{P}\left(-\delta_0 S_0 \sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \leq VaR_\alpha\right) \tag{28}$$

which implies that $VaR_\alpha = \underline{VaR}_\alpha + \frac{\delta_0^2 S_0}{\Phi}$. Finally, we calculate:

$$ES_\alpha = \frac{1}{1-\alpha}\mathbb{E}\left(-\delta_0 S_0 \sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \ \& \ -\delta_0 S_0\sigma\xi + \frac{\delta_0^2 S_0}{\Phi} \geq \underline{VaR}_\alpha + \frac{\delta_0^2 S_0}{\Phi}\right) \tag{29}$$

$$= \frac{\delta_0^2 S_0}{\Phi} - \frac{\delta_0 S_0 \sigma}{1-\alpha}\mathbb{E}\left(\xi \ \& \ -\delta_0 S_0 \sigma\xi \geq \underline{VaR}_\alpha\right) \tag{30}$$

$$= \frac{\delta_0^2 S_0}{\Phi} + \underline{ES}_\alpha \tag{31}$$

□

Proposition 2.1 shows that in the presence of feedback effects from the fund's liquidation, the value-at-risk and expected shortfall are larger than their fundamental value. When the liquidation function is equal to 0 or 1, the correction term is quadratic in fund size. As a consequence, risk-measures which take into account the liquidation risk for the fund are no longer linear in fund size. Here, we obtain a quadratic correction term as we used a linear price impact model. The difference between the risk-measures and their theoretical values reflect the liquidation costs or the concentration of positions of the fund. Indeed, the larger the fund's positions compared to the asset market depth, the larger this term of correction. On the contrary, risk measures are equal to their fundamental value in a perfectly liquid market ($\Phi = \infty$). Note that in the case where $\gamma > -\mathcal{N}^{-1}(1-\alpha)\sigma$, the value-at-risk remains unchanged. This can be interpreted naturally as follows: as γ is "large", the fund will engage in liquidations in scenarios which occur with a probability lower than $1-\alpha$, which implies that the $1-\alpha$ percentile loss i.e. the value-at-risk is unchanged. However, realized losses in those scenarios will be larger in the presence of feedback, which is translated into a larger expected shortfall.
In our framework, we obtain a liquidation adjustment for value-at-risk and expected shortfall which is quadratic in fund size, that is proportional to δ_0^2. This is a consequence of the linear price impact assumption. More generally, if we use a price impact of the form $\frac{\Delta S}{S} = \left(\frac{Q}{\Phi}\right)^\beta$ where Q is the size of the order, following the exact same steps as done in the proof of Proposition 2.1, we would obtain a liquidation adjustment proportional to $\delta_0^{1+\beta}$. Note that the case of linear price impact corresponds to $\beta = 1$. Interestingly, in the case of a square root price impact, that is $\beta = 0.5$, which is another popular form of price impact (citations), the liquidation adjustement scales as $\delta_0^{1.5}$. Other studies, such as Avellaneda & Cont (2013) and which take a different approach compared to the one used in the current paper, find the same scaling of liquidation risk. In those studies, a fund that needs to liquidate δ_0 units of assets S will implement such liquidation over a time horizon $T = \frac{\delta_0}{\delta_{max}}$, where δ_{max} is the daily maximum amount that the fund can trade in the asset without impacting its price, and which is typically equal to a fraction of the daily volume. As such, the resulting value-at-risk for the fund is equal to $\delta_0 S_0 \sigma\sqrt{T}(-\mathcal{N}^{-1}(1-\alpha)) = \frac{\delta_0^{1.5}}{\delta_{max}} S_0\sigma(-\mathcal{N}^{-1}(1-\alpha))$, which scales as $\delta_0^{1.5}$. Those studies do not consider any price impact parameter and assume that the fund has enought time to liquidate the asset without impacting its price. On the contrary, we assume here that the fund needs to liquidate assets over a fixed period of time, which may generate price impact. In practice, funds are in between those two situations: most funds are subject to exogenous constraints (capital, leverage, margin, liquidity) that they need to comply with at regular time intervals (e.g. on a daily basis for margin calls in a clearinghouse). In that case, our framework seems better-adapted as it will reproduce the additional risk generated endogenously by the price impact of the fund when liquidating assets in a bad scenario in order to comply with its exogenous constraints. However, in the case of an extreme market event that leads the fund to be unable to comply with such constraints, the fund will generally be liquidated with no particular time constraint and frameworks such as Avellaneda & Cont (2013) will be more adapted to model those effects.

3 Liquidation risk in the context of margin constraints

In this section, we examine the case of a fund, initially holding δ_0 units of the asset S described in the previous section. We assume that the fund trades through an exchange and is hence subject to margin requirements. We denote by C the total cash that the fund has in reserve (typically in its bank account). When the fund suffers a mark-to-market loss, it has to pay this loss to the clearinghouse. To that end, it can use the cash C it has in its bank account. If this is not enough, it may then liquidate part of its positions in order to comply with capital requirements.

3.1 Endogenous rebalancing due to margin requirements

We first assume that there is no feecback from the fund liquidation (i.e. $\Phi = \infty$). Let us characterize $\underline{f}$ the "fundamental" liquidation function associated to the fund's margin constraints.
The fund's mark-to-market loss is given in equation (2). Denote δ_1 the quantity that the fund holds after the potential liquidation. Depending on the magnitude of the loss, the fund is faced may face various situations:

1. If $\delta_0(S_0 - S_1) \leq C$, the fund can pay the margin with the cash it possesses and does not need to sell assets. This implies that $\delta_1 = \delta_0$.

2. If $\delta_0(S_0 - S_1) > C$, the fund needs to liquidate part of its positions in order to pay its margin call. However, if the initial loss was too large, it may be impossible to meet the margin requirement even if the fund liquidates all its positions (we rule out the possibility of short selling):

   (a) If $\delta_0 S_1 \geq \delta_0(S_0 - S_1) - C$, then it is possible for the fund to liquidate a quantity $\delta_0 - \delta_1$ such that the cash raised pays the rest of the margin: $(\delta_0 - \delta_1)S_1 = \delta_0(S_0 - S_1) - C$.

   (b) If $\delta_0 S_1 < \delta_0(S_0 - S_1) - C$, it is not possible for the fund to pay the margin call by liquidating its positions. We assume that the fund goes bankrupt. The exchange then collects the fund's portfolio and liquidates it ($\delta_1 = 0$).

As a consequence, the "fundamental" liquidation function $\underline{f}$ is given as follows:

$$\underline{f}(x) = 0 \quad \text{if } x \leq \frac{C}{\delta_0 S_0} \tag{32}$$

$$\underline{f}(x) = \frac{\delta_0 S_0 x - C}{\delta_0 S_0 (1 - x)} \quad \text{if } \frac{C}{\delta_0 S_0} \leq x \leq \frac{C + \delta_0 S_0}{2\delta_0 S_0} \tag{33}$$

$$\underline{f}(x) = 1 \quad \text{if } \frac{C + \delta_0 S_0}{2\delta_0 S_0} \leq x \tag{34}$$

Figure 3.1 displays the liquidation schedule $\underline{f}$ for the case $C = 20\%\delta_0 S_0$.

3.2 Rebalancing in the presence of price impact

We now take into account the price impact of fund liquidation. We assume that when the fund liquidates a quantity $\delta_0 - \delta_1$ of assets S it impacts the asset price and is only able to sell at a price S_1^* given in equation (8). Note that the liquidation process further increases the fund's loss and increases the required margin. The fund may be faced with the following situations:

1. If $\delta_0(S_0 - S_1) \leq C$ the fund can pay the margin with the cash it possesses and does not need to sell assets. This implies that $\delta_1 = \delta_0$.

2. If $\delta_0(S_0 - S_1) > C$:

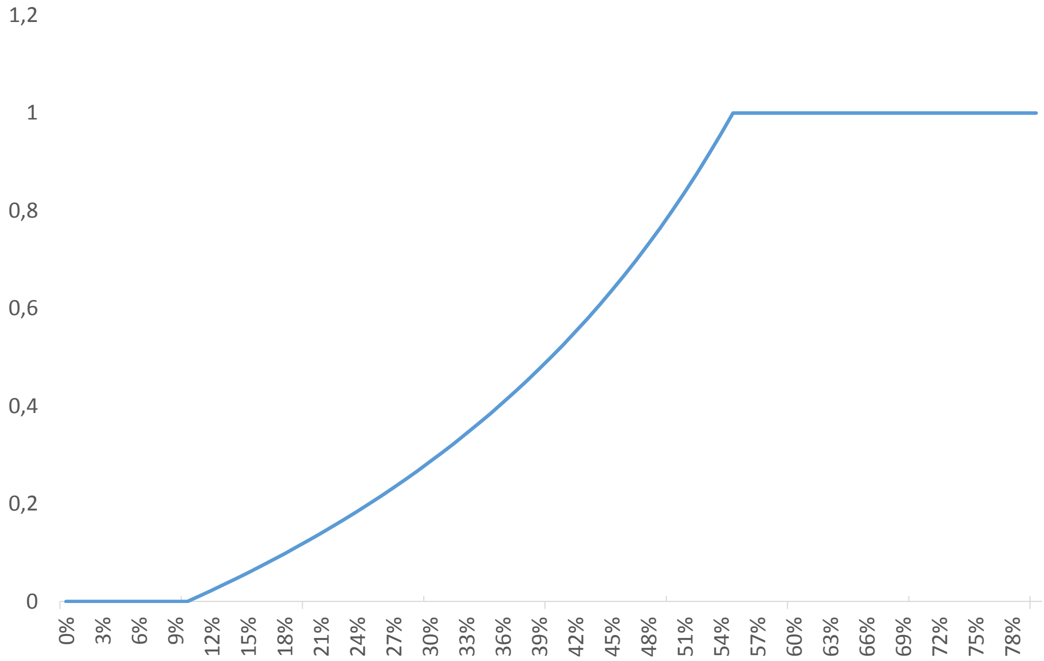

Figure 1: Liquidation schedule for a fund with $C = 20\%\delta_0 S_0$. X axis: percentage loss x; Y axis: $\underline{f}(x)$

(a) If $\delta_0 S_1 \geq \delta_0(S_0 - S_1) - C$, the fund has to engage in fire sales in order to meet its margin requirement. However, contrary to the case without feedback effect, the presence of price impact may generate a spiral to insolvency during the liquidation process. The fund has to liquidate a quantity $\delta_0 - \delta_1$ such that $C + (\delta_0 - \delta_1)S_1^* = \delta_0(S_0 - S_1^*)$ with $S_1^* = S_0\left(1 + \sigma\xi - \frac{\delta_0 - \delta_1}{\Phi}\right)$. As a consequence, δ_1 should be a root of a quadratic equation, which, depending on the market depth and the other parameters, does not necessarily have real-valued roots.

i. if it is possible to find such δ_1, then the liquidation process is efficient and the fund is able to pay the margin call.

ii. if not, the fund goes into bankruptcy and is fully liquidated: $\delta_1 = 0$.

(b) If $\delta_0 S_1 < \delta_0(S_0 - S_1) - C$, it is not possible for the fund to pay the margin call by liquidating its positions. The exchange collects the fund's portfolio and liquidates it ($\delta_1 = 0$).

When $\delta_0(S_0 - S_1) \leq C$ or $\delta_0 S_1 < \delta_0(S_0 - S_1) - C$, f coincides with $\underline{f}$. The interesting situation lies when $\delta_0 S_1 \geq \delta_0(S_0 - S_1) - C$. In this case, if there was no price impact, the fund would be able to liquidate positions so as to pay its margin call. However, in the presence of feedback effect, the liquidation process itself may deteriorate the fund's realized loss and lead to the fund being unable to meet its margin requirement. In this case, the liquidation proportion $f(x)$ is solution of the quadratic equation:

$$C + f(x)\left(1 - x - \frac{\delta_0}{\Phi}f(x)\right) = \delta_0 S_0 x + \frac{\delta_0^2 S_0}{\Phi}f(x) \tag{35}$$

If this equation has a root between 0 and 1, this means that the fund is able to liquidate assets and collect enough cash so as to meet its margin requirement. Otherwise, the liquidation process itself drives the fund its insolvency. Note that when $\Phi = \infty$, we find $f(x) = \underline{f}(x)$. As $f(x)$ is solution of a quadratic equation, it admits an expansion in $\frac{1}{\Phi}$ when Φ goes to ∞. Here we find that:

$$f(x) = \underline{f}(x) + \frac{\delta_0}{\Phi} \frac{\underline{f}(x)(1 + \underline{f}(x))}{1 - x} + O\left(\frac{1}{\Phi^2}\right) \tag{36}$$

which shows that, due to price impact, the fund has to liquidate a larger proportion of its positions so as to meet is margin requirement. This first order adjustement is, as expected, increasing when the fund holds large positions compared to the liquidity of the asset.

3.3 Numerical results

In this section, we simulate the market described in Equation (8) in the case where the fund is subject to margin constraints and follows the liquidation schedule described in Section 3.2. Recall that the fund losses are given by Equation (9). We use a Gaussian ξ with mean 0 and variance 1 and the following simulation parameters:

$$S_0 = \$100 \quad \sigma = 10\% \quad \Phi = 1,000 \quad C = 0.2\delta_0 S_0 \tag{37}$$

and we simulate the market for δ_0 = 0,25,50,...,2,500. For each fund size δ_0, we make 10^6 simulations of the market and compute the 99% value-at-risk. Figure 5.2 displays the fund's 99% value-at-risk (Y axis) as a function of $\frac{\delta_0}{\Phi}$ (X axis).

We see that instead of being linear as per Equation (4), the fund's value-at-risk is convex in fund size when accounting for feedback from fire sales, which reflects the potential liquidation costs for concentrated positions. While value-at-risk traditionally assesses the mark-to-market risk of a portfolio, in our framework, by taking into account feedback effects, it also assesses liquidation risk. In the case of concentrated positions, we find that the risk of a portfolio may stem essentially from liquidation risk, which can be as high as 10 times the mark-to-market risk using reasonable liquidity parameters and fund sizes. For such concentrated positions, using the classical value-at-risk systematically underestimates the real risk of a portfolio and adjusting for liquidation risk is necessary to get a proper assessment of the portfolio risk.

4 The multi-asset model

4.1 Modeling liquidation losses in a multi-asset setting

In this section, we extend the results of Section 2 to a multi-asset setting. We consider a market with n assets, whose initial prices at date 0 are denoted $S_{0,1}, ..., S_{0,n}$. From date 0 to 1, each asset moves due to exogenous factors. The "fundamental" return of asst i during this period is modeled as follows:

$$\frac{S_{1,i} - S_{0,i}}{S_{0,i}} = [A\xi]_i \tag{38}$$

where ξ is a Gaussian vector with mean 0 and identity covariance matrix and

$$AA' = \Sigma \tag{39}$$

Σ is the fundamental covariance matrix for the market. It reflects a structural relationship between asset returns and is assumed to be constant in our framework. Note that we use a Gaussian setting and a constant fundamental covariance matrix for clarity purpose only: our results may be extended in more complex settings (ξ with fat tails, time-varying or stochastic Σ) but the interpretation in terms of endogenous liquidation risk vs exogenous mark-to-market risk stays the same.

Consider now a fund with initial positions (in shares) $\delta_{0,i}$ in each asset i. The fund's initial value is given by:

$$V = \sum_{i=1}^{n} \delta_{0,i} S_{0,i} \tag{40}$$

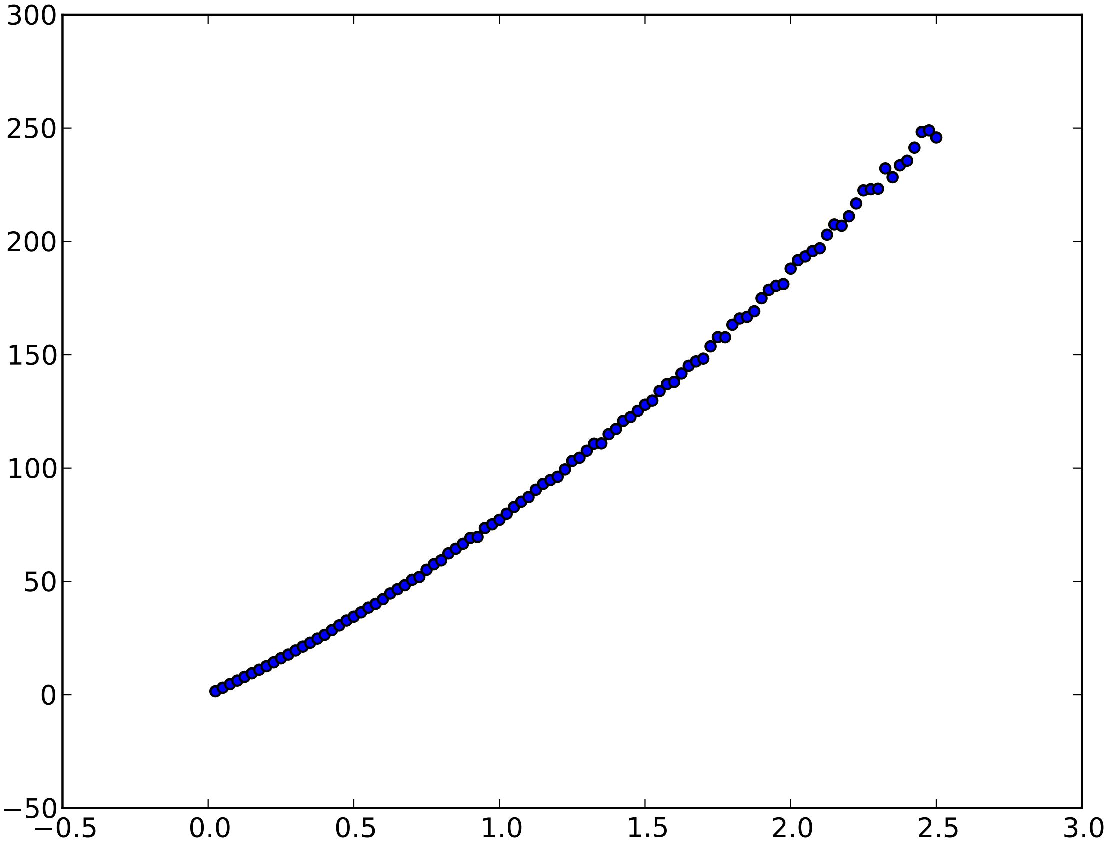

Figure 2: Fund 99% value-at-risk (Y axis) as a function of $\frac{\delta_0}{\Phi}$ (X axis)

Denote by π the vector of fund (percentage) allocations:

$$\pi' = \left(\frac{\delta_{0,1}S_{0,1}}{V}, ..., \frac{\delta_{0,n}S_{0,n}}{V}\right) \tag{41}$$

π_i corresponds to the percentage of cash invested in asset i. The fund's "fundamental" value-at-risk and expected shorfall at level α (e.g. α =0.95 or 0.99) are respectively given by:

$$\underline{VaR}_\alpha = V\sqrt{<\pi, \Sigma\pi>}\mathcal{N}^{-1}(\alpha) \tag{42}$$

$$\underline{ES}_\alpha = V\sqrt{<\pi, \Sigma\pi>}\frac{\mathbb{E}(-Z\&Z \leq \mathcal{N}^{-1}(1-\alpha))}{1-\alpha} \tag{43}$$

where $\mathcal{N}$ is the standard Gaussian cumulative distribution function and Z is a standard Gaussian random variable. As in the single asset case, those risk measures scale linearly in fund size V and fund volatility $\sqrt{<\pi, \Sigma\pi>}$.

As discussed in Sections 1 and 2, when the fund suffers a significant loss, it may need to liquidate assets, which may in turn generate larger-than-expected losses. Denote by X the percentage loss for the fund:

$$X = -<\pi, A\xi> \tag{44}$$

and by

$$f_i(X) \tag{45}$$

the proportion of asset i which is liquidated by the fund following the loss X. As done in Section 2, we assume that f_i is equal to 0 on $]-\infty, \underline{\gamma}[$ and 1 on $]\overline{\gamma}, +\infty[$ and is increasing on $[\underline{\gamma}, \overline{\gamma}]$ which reflects the fact that as long as the fund's loss is below a certain threshold, it does not need to liquidate assets; the larger the loss, the more the fund liquidates; and when the fund's loss is sufficiently large, it becomes insolvent and fully liquidated.
Denote by Φ_i the market depth for asset i. Recall that market depth is a linear measure of asset liquidity. The fund's final loss, taking into account the feedback effect from potential liquidations, is given by:

$$-\sum_{i=1}^{n} \delta_{0,i} S_{0,i} [A\xi]_i + \sum_{i=1}^{n} \frac{\delta_{0,i}^2 S_{0,i} f_i(X)}{\Phi_i} \tag{46}$$

This loss if greater or equal to the 'fundamental mark-to-market loss" $-\sum_{i=1}^{n} \delta_{0,i} S_{0,i} [A\xi]_i$, due to liquidation costs which are quantified in a tractable manner and are equal to

$$\sum_{i=1}^{n} \frac{\delta_{0,i}^2 S_{0,i} f_i(X)}{\Phi_i} \tag{47}$$

Note that, as expected intuitively, as long as there are no liquidations ($f_i(X) = 0$ for all $1 \leq i \leq n$) or market depths are infinite ($Phi_i = \infty$ for all $1 \leq i \leq n$), liquidation costs are equal to zero. More interestingly, as long as liquidations involve only assets for which the fund's positions are negligible compared to the asset market depth, those liquidation costs are also equal to zero. In the case where the fund liquidates assets in large proportions compared to market depths, liquidation costs are strictly positive and generate larger-than-expected losses. In this case, risk-measures such as the value-at-risk or expected shortfall should be adjusted to take into account the additional and endogenous liquidation costs. The next proposition quantifies this adjusment in the case of a binary liquidation strategy.

Proposition 4.1. *In the case where $f_i(x) = \mathbb{1}(x > \gamma)$, then the fund's value-at-risk and expected shortfall are given by:*

1. *if $\gamma > \mathcal{N}^{-1}(\alpha)\sqrt{< x, \Sigma x >}$*

$$VaR_\alpha = \underline{VaR}_\alpha \tag{48}$$

$$ES_\alpha = \underline{ES}_\alpha + \frac{\mathcal{N}\left(\frac{-\gamma}{\sqrt{<x,\Sigma x>}}\right)}{1-\alpha} \sum_{i=1}^{n} \frac{\delta_{0,i}^2 S_{0,i}}{\Phi_i} \tag{49}$$

2. *if $\gamma \leq \mathcal{N}^{-1}(\alpha)\sqrt{< x, \Sigma x >}$*

$$VaR_\alpha = \underline{VaR}_\alpha + \sum_{i=1}^{n} \frac{\delta_{0,i}^2 S_{0,i}}{\Phi_i} \tag{50}$$

$$ES_\alpha = \underline{ES}_\alpha + \sum_{i=1}^{n} \frac{\delta_{0,i}^2 S_{0,i}}{\Phi_i} \tag{51}$$

4.2 The liquidation risk adjustment

Following Proposition 4.1, we introduce the following definition.

Definition 4.2. *Consider a market with n assets, whose prices are denoted $S_1, ..., S_n$ and with market depths $\Phi_1, ..., \Phi_n$. The liquidation risk adjustment for a portfolio with positions δ_i in each asset i is given by:*

$$\sum_{i=1}^{n} \frac{\delta_i^2 S_i}{\Phi_i} \tag{52}$$

This liquidation risk adjustment corresponds to the liquidation costs for a portfolio with positions $(\delta_1, ..., \delta_n)$. It is expressed in dollars and is actually the sum of dollar positions in each asset $(\delta_i S_i)$ weighted by the size of those positions relative to market depths $\left(\frac{\delta_i}{\Phi_i}\right)$. Introducing V the portfolio value and π the vector of portfolio allocations, defined by:

$$V = \sum_{i=1}^{n} \delta_i S_i \tag{53}$$

and

$$\pi' = \left(\frac{\delta_1 S_1}{V}, ..., \frac{\delta_n S_n}{V}\right) \tag{54}$$

the liquidation risk adjustment given in Equation (52) can be written as

$$\sum_{i=1}^{n} \frac{\delta_i^2 S_i}{\Phi_i} = V^2 \sum_{i=1}^{n} \frac{\pi_i^2}{S_i \Phi_i} \tag{55}$$

and scales quadratically in portfolio size V. Note that the quantity $S_i \Phi_i$ can be interpreted as the dollar market depth for asset i.
Proposition 4.1 suggests the following decomposition for portfolio risk:

$$\text{Total Risk} = \text{Fundamental Risk} + \text{Liquidation Risk Adjustment} \tag{56}$$

Fundamental risk scaling linearly in fund size, while liquidation risk adjustment scales quadratically, total portfolio risk will be more driven by fundamental mark-to-market risk when portfolio size is small, and by liquidation risk for large portfolios. In the case of value at risk, total risk is given by

$$V\sqrt{<\pi, \Sigma\pi>}\mathcal{N}^{-1}(\alpha) + V^2 \sum_{i=1}^{n} \frac{\pi_i^2}{S_i \Phi_i} \tag{57}$$

and there exists a threshold portfolio size

$$V^* = \frac{\sqrt{<\pi, \Sigma\pi>}\mathcal{N}^{-1}(\alpha)}{\sum_{i=1}^{n} \frac{\pi_i^2}{S_i \Phi_i}} \tag{58}$$

above which the risk of the portfolio is more driven by liquidation risk than fundamental risk.
Consider now a portfolio with positions $\mu_1, ...\mu_n$ and with similar size, that is:

$$V = \sum_{i=1}^{n} \delta_i S_i = \sum_{i=1}^{n} \mu_i S_i \tag{59}$$

and denote

$$\psi' = \left(\frac{\mu 1 S_1}{V}, ..., \frac{\mu n S_n}{V}\right) \tag{60}$$

Assume that portfolio μ has lower fundamental risk than portfolio δ, that is

$$V\sqrt{<\psi, \Sigma\psi>}\mathcal{N}^{-1}(\alpha) < V\sqrt{<\pi, \Sigma\pi>}\mathcal{N}^{-1}(\alpha) \tag{61}$$

and larger liquidation risk adjustment, that is

$$V^2 \sum_{i=1}^{n} \frac{\psi_i^2}{S_i \Phi_i} > V^2 \sum_{i=1}^{n} \frac{\pi_i^2}{S_i \Phi_i} \tag{62}$$

then there exists a threshold fund size

$$V^* = \frac{(\sqrt{<\pi, \Sigma\pi>} - \sqrt{<\psi, \Sigma\psi>})\mathcal{N}^{-1}(\alpha)}{\sum_{i=1}^{n} \frac{\psi_i^2 - \pi_i^2}{S_i \Phi_i}} > 0 \tag{63}$$

such that when portfolio size is below (resp. above) V^*, the total risk of the portfolio μ is lower (resp. larger) than that of portfolio δ. As such our results enable to discriminate between portfolio risk taking into account fundamental mark-to-market risk but also the liquidity of the assets involved in the portfolio.

5 Empirical results

5.1 Estimating market depths

In our framework, one of the key parameters involved when quantifying liquidation risk is the asset market depth Φ, which links, in a linear manner, the return of a given asset $\frac{\Delta S}{S}$ to the size Q of a directional order on that asset:

$$\frac{\Delta S}{S} = \frac{Q}{\Phi} \tag{64}$$

Several empirical studies have developed methodologies for estimating market depths from daily data (Obizhaeva, 2011; Kyle & Obizhaeva, 2016) or high-frequency data (Cont et al., 2014). While the estimation of market depths using high-frequency data is more precise, it requires non-public (and sometimes expensive) order book data. On the contrary, the estimation using daily data only involves public and free data and is described hereafter. In the related studies (Obizhaeva, 2011; Braverman & Minca, 2014; Wagalath, 2017) the asset market depth is estimated by:

$$\Phi = \frac{1}{3}\frac{ADV}{\sigma} \tag{65}$$

where ADV is the average daily volume (expressed in shares) and σ is the realized daily volatility. The coefficient $\frac{1}{3}$ is estimated from US stock market data. Equation (65) shows, as expected intuitively, that market depth, i.e. asset liquidity, increases with average daily volumes and decreases with asset volatility. Combining Equations (64) and (65), we find that:

$$\frac{\Delta S}{S} = 3\sigma\frac{Q}{ADV} \tag{66}$$

meaning for example that a trade size of one third of the average daily volume ($Q = \frac{ADV}{3}$) would generate an asset return of one standard-deviation. As such, the scaling of the market depth given in Equation (65), enables to quantify the asset return generated by a large trade in terms of standard deviation movement.

Table (5.1) displays the estimated market depths for the US sector ETFs in 2013. Note that, in Equation (65), ADV is expressed in shares and yields a market depth expressed in shares, as used in our framework. One could also express volumes in dollars and obtain dollar market depths. Table (5.1) displays market depths in shares and dollars. The dollar market depths enable to compare liquidity between assets and we see, in the table below and as expected intuitively, that the financial sector ETF is the most liquid sector ETF, followed by the energy sector ETF. All other sector ETFs have significant lower liquidity.

5.2 The liquidation risk adjustment for a portfolio of US securities

We consider a fund investing in the US sector ETFs described in Table 5.1. The fund follows an equally-weighted fixed-mix strategy, that is, the fund invests $1/9 \approx 11.1\%$ of its wealth in each of the 9 sector ETFs. Figure 5.2 displays the mark-to-market performance of this strategy in 2013. The three worst daily losses for this strategy during that year amount to 2.4%, 2.0% and 1.5%. In comparison, assuming a Gaussian framework, the fund's 99% one-day "fundamental" value-at-risk, calculated following Equation

Name	Sector	Market depth $\times 10^8$ shares	Market depth $\times 10^9$\$
XLF	Financials	16.8	32.5
XLE	Energy	3.9	31.6
XLU	Utilities	4.4	16.8.
XLK	Technology	3.6	11.2
XLB	Materials	2.4	9.6
XLP	Consumer Staples	4.1	16.4
XLY	Consumer Discretionary	2.3	13.0
XLI	Industrials	4.0	17.8
XLV	Health Care	3.0	14.5

Table 1: Market depths of sector ETFs of the S&P 500 in 2013

(42), is equal to 1.4%. This fundamental value-at-risk is displayed in the blue bars of Figure 5.2 as a function of fund size and, as expected, grows linearly in fund size. We also compute the fund's liquidation-adjusted value-at-risk, that is the sum of the fundamental value-at-risk and the liquidation-risk adjustment, as presented in section 4.2. The results are also displayed in Figure 5.2: the red bars correspond to the fund's liquidation-risk adjustment and grow quadratically in fund size. The fund's total risk is the sum of the red bars and blue bars and can be significantly larger than what is estimated by the fundamental value-at-risk. For example, a fund with size \$10Bn has a 99% one-day liquidation-adjusted value-at-risk of 8.5%, which is significantly larger than the 1.4% fundamental value-at-risk. The difference comes from the liquidations costs for a fund which may, in bad scenarios, need to sell large blocks of assets that have a finite liquidity.
We assume now that the fund has the choice to implement one of the two following strategies: strategy 1 is the equally-weighted fixed-mix strategy decribed above, that is a strategy with a weight of $1/9 \approx 11.1\%$ in each of the nine sector ETFs; strategy 2 is a strategy with 50% in the industrial ETF (XLI) and 50% in the health care ETF (XLV), and no exposure to other ETFs. The 99% 1-day fundamental value-at-risk of strategy 2 is equal to 1.2% and is lower than that of strategy 1 (recall that it is equal to 1.4%, as discussed in the paragraph above). This is a natural consequence of of the fact that strategy 1 involves more volatile assets, such as the financial and energy sector ETFs. If the fund chooses between implementing strategy 1 and strategy 2 based on minimizing value-at-risk, calculated in the "classical" way following Equation (42), the fund would choose strategy 2. However, if the fund wants to minimize its total risk, that is the liquidation-adjusted value-at-risk, which is equal to the sum of fundamental value at risk and liquidation-risk adjustment, and that takes into account potential liquidation costs, the fund's decision is not as straightforward and will depend on its size. The reason is that, while the fundamental value at risk of strategy 2 is lower than that of strategy 1, the liquidation risk adjustment is lower for strategy 1 compared to strategy 2. This is due to the fact that strategy 2 is invested in less liquid securities. As discussed in section 4.2, there exists a threshold fund size V^*, given explicitly in Equation 63, such that if the fund size V is lower than V^*, the total risk of strategy 2 is lower than that of strategy 1, while if V is larger than V^*, strategy 1 will have the lower total risk. As such, if the fund wants to minimize its total risk and has a size larger than V^* it should actually implement strategy 1 rather than strategy 2. This is illustrated in Figure **??**, where the blue (resp. red) line represents strategy 1 (resp. strategy 2) total risk (Y axis), and we see that depending on the fund size (X axis) , the total risk of strategy 1 may be larger or lower than that of strategy 2.

6 Conclusion

References

Almgren, R. & Chriss, N. (2000). Optimal execution of portfolio transactions. *J. Risk*, 3:5–39.

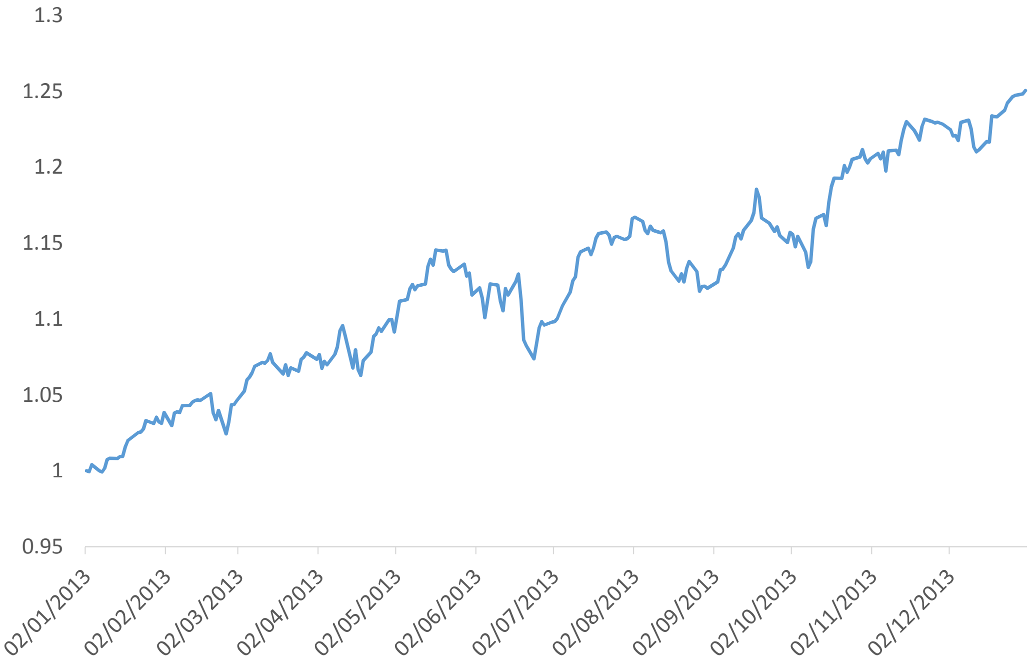

Figure 3: Performance of a strategy with equal weights on each sector ETF (Y axis) in 2013 (X axis).

Almgren, R. & Lorenz, J. (2006). Adaptive arrival price. *Algorithmic Trading: Precision, Control, Execution*, 3:59–66.

Avellaneda, M. & Cont, R. (2013). Close-out risk evaluation (core): a new risk management approach for central counterparties. *Available at SSRN 2247493*.

Braouezec, Y. & Wagalath, L. (2016). Risk-based capital requirements and optimal liquidation in a stress scenario. *Review of finance (forthcoming)*.

Braouezec, Y. & Wagalath, L. (2017). Strategic fire sales and price-mediated contagion in the banking system. *working paper*.

Braverman, A. & Minca, A. (2014). Networks of common asset holdings: Aggregation and measures of vulnerability. *working paper*.

Cont, R., Kukanov, A., & Stoikov, S. (2014). The price impact of order book events. *Journal of financial econometrics*, 12(1):47–88.

Cont, R. & Schaanning, E. F. (2016). Fire sales, indirect contagion and systemic stress-testing. *working paper*.

Cont, R. & Wagalath, L. (2013). Running for the exit: distressed selling and endogenous correlation in financial markets. *Mathematical Finance*, 23(4):718–741.

Cont, R. & Wagalath, L. (2016a). Fire sales forensics: measuring endogenous risk. *Mathematical Finance*, 26(4):835–866.

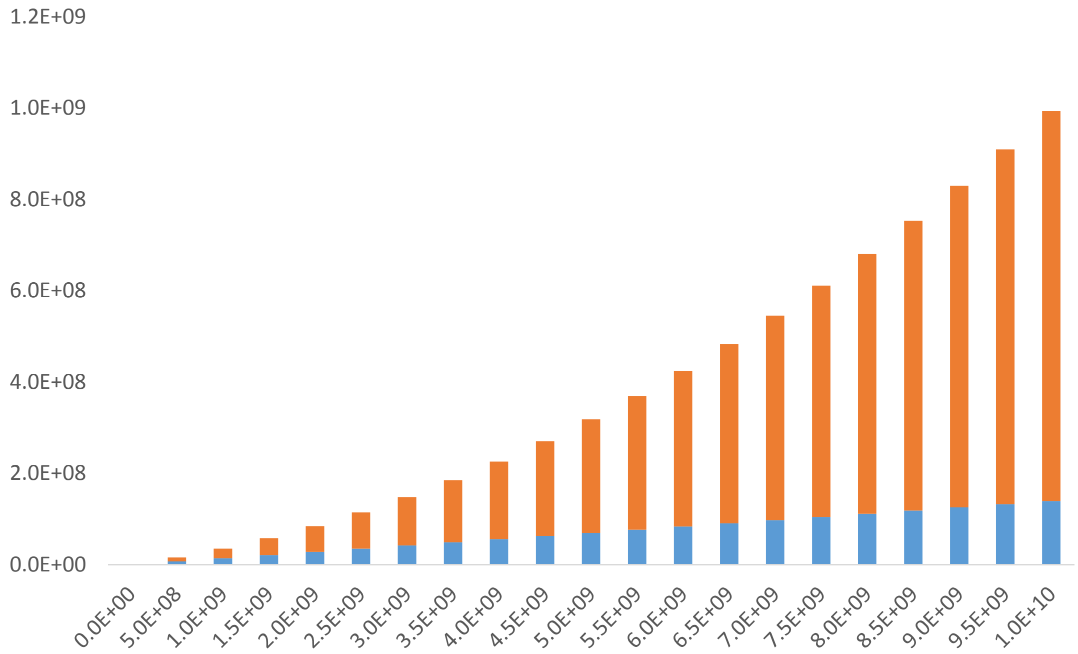

Figure 4: Value-at-risk (Y axis) for a fund investing 11.1% in each sector ETF as a function of fund size (X axis). Blue bars represent the fund's "fundamental" value-at-risk; red bars represent the fund's "liquidation risk adjustment".

Cont, R. & Wagalath, L. (2016b). Institutional investors and the dependence structure of asset returns. *International Journal of Theoretical and Applied Finance*, 19(2):37p.

Cont, R. & Wagalath, L. (2016c). Risk management for whales. *Risk*, (June):79–82.

Greenwood, R., Landier, A., & Thesmar, D. (2015). Vulnerable banks. *Journal of Financial Economics*, 115(3):471–485.

Kyle, A. S. & Obizhaeva, A. A. (2016). Market microstructure invariance: Empirical hypotheses. *Econometrica*, 84(4):1345–1404.

McNeil, A. J., Frey, R., & Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools*. Princeton university press.

Obizhaeva, A. (2011). Selection bias in liquidity estimates. *Working Paper*.

Rosenfeld, E. (2010). Long Term Capital Management. In Cont, R., editor, *Encyclopedia of Quantitative Finance*. Wiley.

Shleifer, A. & Vishny, R. (2011). Fire sales in finance and macroeconomics. *Journal of Economic Perspectives*, 25(1):29–48.

JP Morgan (2014). JP Morgan Management Task Force report regarding 2012 CIO losses. Technical report.

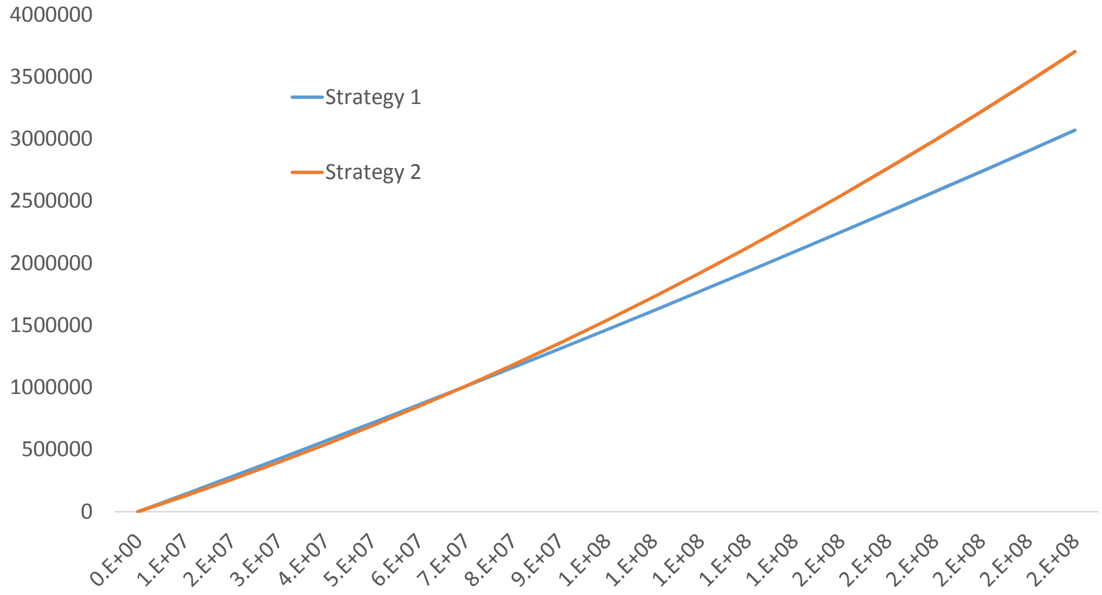

Figure 5: Total risk, including liquidation adjustment, for strategies 1 and 2 (Y axis) as a function of fund size (X axis). Strategy 1 is a strategy with equal weight on each sector ETF. Strategy 2 is a strategy with 50% in the industrial ETF and 50% in the health care ETF, and no exposure to other ETFs.

U.S. Senate (2013). JP Morgan Chase Whale trades: a case history of derivatives risk and abuses. Staff report.

Wagalath, L. (2017). Lost in contagion? building a liquidation index from covariance dynamics. *International Journal of Theoretical and Applied Finance*, 20(01):26p.