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ON THE RANK OF THE BAUM-BOTT MAP

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

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**THE GENERIC RANK OF THE BAUM-BOTT MAP FOR LOW DEGREE
FOLIATIONS ON HIGH-DIMENSIONAL PROJECTIVE SPACES**

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A mis mamás Eusebia y Jakeline.

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RESUMO

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Os índices de Baum-Bott são invariantes importantes em folheações holomorfas singulares por curvas com singularidades isoladas. Se a folheação tem uma singularidade não degenerada então seus índices no ponto singular podem ser facilmente calculados usando os autovalores da parte linear de um germe de campo de vetores que induz a folheação em uma vizinhança da singularidade. A aplicação Baum-Bott está definida no espaço de folheações holomorfas de dimensão um, em uma variedade complexa e compacta, com fibrado cotangente a folheação fixado. Esta aplicação associa a uma folheação seus índices de Baum-Bott em cada ponto singular. Restringimos o estudo a folheações em espaços projetivos complexos. No caso de folheações de dimensão um no plano projetivo, o posto genérico é conhecido. Damos uma cota superior para o posto genérico da aplicação Baum-Bott para folheações em espaços projetivos, o qual depende do grau da folheação e a dimensão do espaço projetivo. Mais ainda, estendemos o resultado dado para o plano projetivo e determinamos o posto genérico para folheações de grau dois definidas em espaços projetivos de dimensão par, e também para folheações de grau máximo oito, no espaço projetivo de dimensão três. Além disso, estudamos o posto na folheação de Jouanolou.

Palavras-chave: Folheações holomorfas, índice de Baum-Bott, aplicação Baum-Bott, posto genérico, folheação de Jouanolou.

ABSTRACT

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The Baum-Bott indexes are important invariants of singular holomorphic foliations by curves with isolated singularities. If a foliation has a non-degenerate singularity, its indexes at that point can be easily calculated using the eigenvalues of the linear part of a germ of a vector field, which defines the foliation at a neighborhood of the singular point. The Baum-Bott map is defined on the space of one-dimensional holomorphic foliations on a compact complex manifold with a fixed cotangent bundle to the foliation. This map associates to a foliation its Baum-Bott indexes at each singular point. We concentrate on foliations on the complex projective space. The generic rank of this map on the space of one-dimensional foliations on the projective plane is already known. We give an upper bound of the generic rank of the Baum-Bott map for foliations on projective spaces, the number depends on the degree of the foliation and the dimension of the projective space. Moreover, we extend the known results for the projective plane and determine the generic rank for degree-two foliations on even-dimensional projective spaces, as well as for degree up to eight on the three-dimensional projective space. Additionally, we study the rank at the Jouanolou foliation.

Key words: Holomorphic foliations, Baum-Bott index, Baum-Bott map, generic rank, Jouanolou foliation.

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Introduction

The Poincaré-Hopf theorem relates the Euler characteristic and the Hopf-indexes of a vector field defined on a manifold. The theorem asserts that the properly counted number of zeros of a vector field equals the Euler number of a manifold. Consequently, it relates a topological concept, the Euler characteristic, and an analytic one, the Hopf-index of a vector field.

Chern classes can be read as obstructions. On a complex manifold the Euler characteristic coincides with the top Chern class of the manifold. In this case the top Chern class gives us an obstruction to construct a vector field without singularities. In general, the Chern classes give us obstructions to construct linearly independent vector fields. This concept is generalized to vector bundles, and the Chern classes of a vector bundle give us obstructions to construct global independent sections.

Baum-Bott indexes are related with Chern classes. A one-dimensional holomorphic foliation on a complex manifold is given by a section of the tensor product of the tangent bundle and a line bundle. The Chern classes of the tensor product can be calculated by some local invariants of the foliation around the singularities. These local invariants are the Baum-Bott indexes of the holomorphic foliation. So, if we know the Baum-Bott indexes of a holomorphic foliation, we can know the Chern classes of the tensor product that defines the foliation.

In some cases, the Baum-Bott indexes are easy to compute. If the foliation has only non-degenerate singularities, the Baum-Bott indexes can be determined by considering the symmetric functions of the eigenvalues of a germ of vector field that defines the foliation near the singularity. Conversely, if we know the Baum-Bott indexes at a singular point, we can determine the eigenvalues of the linear part of the vector field at the singularity, up to projectivization of the tuple of eigenvalues.

Since the local behavior of a foliation, near a singular point, is determined in most cases by its eigenvalues, then the Baum-Bott indexes can give us great information. For instance, if the eigenvalues of the linear part of a vector field at a singular point are in the Poincaré domain, then the singularity has a behavior of a local attractor (see [10]), and if they are also non resonant, then the foliation is linearizable at a neighborhood of the singularity (see [17]).

A one-dimensional foliation \mathcal{F} , on a compact complex manifold M of dimension n , is a section of $TM \otimes L$, where L is a holomorphic line bundle on M and TM is the tangent bundle of M . The line bundle L is the cotangent bundle $T_{\mathcal{F}}^*$ of the foliation \mathcal{F} . The set of foliations on M with cotangent bundle L is denoted by $\mathcal{Fol}(M, L) = \mathbb{P}H^0(M, TM \otimes L)$.

The Baum-Bott theorem states that the Chern classes of the vector bundle $TM \otimes L$ are the sum of suitable Baum-Bott indexes of a holomorphic foliation which has only isolated singularities and cotangent bundle L . In the following theorem the i -th Chern class is denoted by c_i and the i -th elementary symmetric function of the eigenvalues by C_i .

Theorem 1 ([27]). *Let M be a compact complex manifold of dimension n , L be a holomorphic line bundle on M and ξ be a holomorphic section of $TM \otimes L$ with isolated zeros. Consider the Chern classes:*

$$c^\nu(TM \otimes L) = c_1^{\nu_1}(TM \otimes L) \dots c_n^{\nu_n}(TM \otimes L),$$

$$\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}^n \text{ and } \nu_1 + 2\nu_2 + \dots + n\nu_n = n.$$

Then

$$\int_M c^\nu(TM \otimes L) = \sum_{p: \xi(p)=0} \text{Res}_p \left\{ \frac{C^\nu(J\xi) dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\},$$

where $J\xi = \left(\frac{\partial \xi_i}{\partial z_j} \right)$ is the Jacobian matrix and the Grothendieck residue symbol $\text{Res}_p \left\{ \frac{C^\nu(J\xi) dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\}$ is the Baum-Bott index C^ν of ξ at p .

The Baum-Bott indexes are easier to estimate at non-degenerate singular points. For instance, let $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}^n$ with $\nu_1 + 2\nu_2 + \dots + n\nu_n = n$. If $p(\mathcal{F})$ is a non-degenerate singularity of a foliation \mathcal{F} on M , and $X_{\mathcal{F}}$ is a germ of vector field which defines \mathcal{F} around $p(\mathcal{F})$, then the Baum-Bott index C^ν of \mathcal{F} at $p(\mathcal{F})$ is expressed in terms of the eigenvalues of the linear part of $X_{\mathcal{F}}$ at $p(\mathcal{F})$:

$$BB_\nu(\mathcal{F}, p(\mathcal{F})) = \frac{C^\nu(DX_{\mathcal{F}}(p(\mathcal{F})))}{\det(DX_{\mathcal{F}}(p(\mathcal{F})))}.$$

A one-dimensional foliation on the projective space has tangent bundle $\mathcal{O}(1-d)$, for some non-negative integer d . This number is called the degree of the foliation. The space of one-dimensional degree- d foliations on the projective space \mathbb{P}^n is denoted by $\mathcal{Fol}(n, d)$. A foliation in that space, with only non-degenerate singularities, has $N = c_n(\mathbb{P}^n, T\mathbb{P}^n \otimes T_{\mathcal{F}}^*)$ singular points. The set of those foliations is denoted by $\mathcal{Fol}_{\text{red}}(n, d)$.

Now, we associate to each foliation its Baum-Bott indexes in the following way. We enumerate the set

$$\Delta = \{(\nu_1, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}^n \mid \nu_1 + 2\nu_2 + \dots + n\nu_n = n\} = \{N_0, N_1, \dots, N_k\},$$

where $N_0 = (0, \dots, 0, 1)$. Let \mathcal{F}_0 be a foliation with only non-degenerate singularities and with singular set $\text{Sing}(\mathcal{F}_0) = \{p_1^0, \dots, p_N^0\}$. Then, there is an open neighborhood of \mathcal{F}_0 , $U \subset \mathcal{F}\text{ol}_{\text{red}}(n, d)$, and holomorphic maps, $p_1, \dots, p_N : U \rightarrow \mathbb{P}^n$, such that $\text{Sing}(\mathcal{F}) = \{p_1(\mathcal{F}), \dots, p_N(\mathcal{F})\}$ and $p_j(\mathcal{F}_0) = p_j^0$ for $j = 1, \dots, N$. The local Baum-Bott map $BB : U \rightarrow (\mathbb{C}^k)^N$ is defined by:

$$\mathcal{F} \mapsto ((BB_{N_1}(p_1(\mathcal{F})), \dots, BB_{N_k}(p_1(\mathcal{F})), \dots, (BB_{N_1}(p_N(\mathcal{F})), \dots, BB_{N_k}(p_N(\mathcal{F}))).$$

We extend the domain of the local Baum-Bott map to $\mathcal{F}\text{ol}_{\text{red}}(n, d)$ by symmetry. More specifically, we denote by $(\mathbb{C}^k)^N/S_N$, the quotient of $(\mathbb{C}^k)^N$ by the equivalence relation which identifies the points (z_1, \dots, z_N) and $(z_{\sigma(1)}, \dots, z_{\sigma(N)})$, where $\sigma \in S_N$, $z_i \in \mathbb{C}^k$ and S_N is the group of permutations of N elements. In this way we have the map $BB : \mathcal{F}\text{ol}_{\text{red}}(n, d) \rightarrow (\mathbb{C}^k)^N/S_N$:

$$\mathcal{F} \mapsto [(BB_{N_1}(p_1(\mathcal{F})), \dots, BB_{N_k}(p_1(\mathcal{F})), \dots, (BB_{N_1}(p_N(\mathcal{F})), \dots, BB_{N_k}(p_N(\mathcal{F})))],$$

where $[z_1, \dots, z_N]$ denotes the class of (z_1, \dots, z_N) in $(\mathbb{C}^k)^N/S_N$.

The *global Baum-Bott map*, $BB : \mathcal{F}\text{ol}(n, d) \dashrightarrow (\mathbb{P}^k)^N/S_N$, is the rational map which extends the Baum-Bott map given above.

The Baum-Bott map, for foliations on the projective plane, is not dominant due to the Baum-Bott formula. In general, the Baum-Bott formula theorem states that the sum of the Baum-Bott indexes of a foliation, on complex surfaces, is the autointersection of the normal bundle of the foliation by itself:

Theorem 2 (Baum-Bott formula [2]). *Let \mathcal{F} be a holomorphic foliation with only isolated singularities on a compact surface M , then*

$$N_{\mathcal{F}}.N_{\mathcal{F}} = \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p).$$

If $M = \mathbb{P}^2$ and d is the degree of the foliation \mathcal{F} , then we get

$$(d+2)^2 = \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p).$$

Gómez-Mont and Luengo asked in [14], if there are other hidden relations between the Baum-Bott indexes of a degree- d foliation on the projective plane. This question is equivalent to finding the generic rank of the Baum-Bott map.

Lins-Neto and Pereira in [22] found the generic rank of the map for one-dimensional foliations on the projective plane:

Theorem 3 ([22]). *If $d \geq 2$, then the maximal rank of the Baum-Bott map for degree- d foliations on \mathbb{P}^2 is*

$$d^2 + d.$$

In particular, if $d \geq 2$ then the dimension of the generic fiber of the map is

$$3d + 2.$$

As a consequence, the unique relation among the Baum-Bott indexes is given by the Baum-Bott formula.

Lins-Neto, in [21], studies the generic fiber of this map for degree-two foliations on the projective plane. In this case the dimension of a generic fiber is the dimension of the automorphism group $\text{Aut}(\mathbb{P}^2)$. Then, a generic fiber is an union of orbits of the action of $\text{Aut}(\mathbb{P}^2)$ on $\mathcal{Fol}(2, 2)$. He finds the exact number of orbits.

Theorem 4 ([21]). *The generic fiber of the Baum-Bott map for degree-two foliations on the projective plane \mathbb{P}^2 contains exactly 240 orbits of the natural action of the automorphism group $\text{Aut}(\mathbb{P}^2)$.*

We are interested in the generic rank of the Baum-Bott map for foliations on high-dimensional complex projective spaces. Theorem 1 gives us some relations among the Baum-Bott indexes of a foliation. We want to know if there are other algebraic interactions between these indexes, too.

In the case of degree-one foliations, we find the maximal rank:

Proposition 1. *If $n \geq 2$, then the generic rank of the Baum-Bott map for one-dimensional foliations on \mathbb{P}^n of degree one is*

$$n - 1.$$

In particular, a generic fiber of this map is an union of orbits of the action of $\text{Aut}(\mathbb{P}^n)$ on $\mathcal{Fol}(n, 1)$ and it has dimension

$$n(n + 1).$$

We give an upper bound for the generic rank of the Baum-Bott map.

Proposition 2. *For $n \geq 2$ and $d \geq 2$, the generic rank of the Baum-Bott map for degree- d foliations on the projective space \mathbb{P}^n is at most*

$$\min\{\dim \mathcal{Fol}(n, d) - \dim \text{Aut}(\mathbb{P}^n), (N - 1)(n - 1)\}.$$

If $n = 2$, the upper bound given in the proposition above is sharp and it is equal to the maximal rank given in Theorem 3. Moreover, when $n = d = 2$, the two numbers given in the upper bound coincide. Then, we have the following conjecture.

Conjecture 1. *For $n, d \geq 2$, the generic rank of the Baum-Bott map $BB : \mathcal{Fol}(n, d) \dashrightarrow (\mathbb{P}^k)^N / S_N$ is*

$$\min\{\dim \mathcal{Fol}(n, d) - \dim \text{Aut}(\mathbb{P}^n), (N - 1)(n - 1)\}.$$

We prove that the conjecture is true for foliations of low degree on \mathbb{P}^3 :

Theorem 5. *If $d = 2, \dots, 8$, then the generic rank of the Baum-Bott map for degree- d foliations on \mathbb{P}^3 is*

$$\dim \mathcal{Fol}(3, d) - \dim \text{Aut}(\mathbb{P}^3) = \frac{(d + 1)(d + 2)(d + 4)}{2} - 16.$$

In particular, if $N(d) = d^3 + d^2 + 1$, a generic fiber of the map $BB : \mathcal{Fol}(3, d) \dashrightarrow (\mathbb{P}^2)^{N(d)} / S_{N(d)}$ is a finite union of orbits of the action of $\text{Aut}(\mathbb{P}^3)$ on $\mathcal{Fol}(3, d)$.

In general, finding the rank of the Baum-Bott map at a random foliation is difficult. We compute the rank of this map at the Jouanolou foliation in terms of some linear transformations. This foliation, on \mathbb{P}^n , is defined in the affine coordinate system $(x_1, \dots, x_n) \in \mathbb{C}^n$ by the vector field

$$\sum_{i=1}^{n-1} (x_{i+1}^d - x_i x_1^d) \partial_i + (1 - x_n x_1^d) \partial_n.$$

We need some notation. Let $J = \{j_1, \dots, j_r\}$ be an ordered set and V_j , for $j \in J$, be vectors of same dimension. We denote $[V_j]_{j \in J} = [V_{j_1} \dots V_{j_r}]$, the matrix whose column vectors are V_{j_1}, \dots, V_{j_r} . Let $n, d \in \mathbb{Z}$, and $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$, we define the linear transformation $M_{n,d}(I) : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ given by the matrix:

$$M_{n,d}(I) = \begin{bmatrix} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d + 1) \\ i_{j+3} - i_{j+2} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{bmatrix}_{1 \leq j \leq n},$$

where $i_{n+1} = d - 1 - (i_1 + \dots + i_n)$. We are identifying $i_j = i_{j \bmod (n+1)}$.

We can state:

Theorem 6. *Let $n \geq 3$ be an integer number. The rank of the local Baum-Bott map $BB : \mathcal{Fol}_{red}(n, d) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree d Jouanolou foliation is:*

$$\begin{aligned} \dim \mathcal{Fol}(n, d) + 1 - n(n+1) - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^n \\ |I| \leq d-1}} \dim \ker(M_{n,d}(I)) & \quad , \text{ if } n \text{ is even.} \\ \dim \mathcal{Fol}(n, d) + 1 - n(n+1) - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^n \\ |I| \leq d-1}} \dim \ker(M_{n,d}(I)) - \frac{(n+1)}{2} & \quad , \text{ if } n \text{ is odd.} \end{aligned}$$

We partially prove Conjecture 1 for degree-two foliations:

Theorem 7. *Let $n \geq 2$ be an even number. The rank of the local Baum-Bott map $BB : \mathcal{Fol}_{red}(n, 2) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree-2 Jouanolou foliation is*

$$\dim \mathcal{Fol}(n, 2) - \dim \text{Aut}(\mathbb{P}^n) = (n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - (n+1)^2.$$

Corollary 1. *Let $n \geq 2$ be an even number. A generic fiber of the global Baum-Bott map, defined on $\mathcal{Fol}(n, 2)$, is a finite union of orbits of the action of the automorphism group $\text{Aut}(\mathbb{P}^n)$ on the space $\mathcal{Fol}(n, 2)$.*

The rank at the Jouanolou foliation, in other cases, is strictly less than the upper bound given in Proposition 2.

Proposition 3. *If $n \geq 3$, the rank of the local Baum-Bott map $BB : \mathcal{Fol}_{red}(n, d) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree- d Jouanolou foliation, for degree d greater than two, is strictly less than the upper bound given in Proposition 2. The same holds for degree $d = 2$ with odd dimension n .*

We are able to explicitly find the rank at the Jouanolou foliation on \mathbb{P}^3 :

Theorem 8. *Let $d \geq 2$. The rank of the local Baum-Bott map at the degree- d Jouanolou foliation on the projective space \mathbb{P}^3 is*

- if d is even,

$$\dim \mathcal{Fol}(3, d) - \text{Aut}(\mathbb{P}^3) - \left(\binom{d+2}{3} - 2 \right),$$

- if $d \equiv -1 \pmod{4}$,

$$\dim \mathcal{Fol}(3, d) - \text{Aut}(\mathbb{P}^3) - \left(\binom{d+2}{3} + \frac{d-3}{2} \right),$$

- if $d \equiv 1 \pmod{4}$,

$$\dim \mathcal{Fol}(3, d) - \text{Aut}(\mathbb{P}^3) - \left(\binom{d+2}{3} + \frac{d-1}{2} \right),$$

and the dimension of the space $\mathcal{Fol}(3, d)$ is $4\binom{d+3}{3} - \binom{d+2}{3} - 1$.

The structure of the thesis is divided into three chapters:

In the First Chapter we give some concepts and results to make easier the comprehension of the thesis. We give the concept of holomorphic foliation, a glimpse on Chern classes and we state the Baum-Bott Theorem 1, which relates Baum-Bott indexes and Chern classes.

In the Second Chapter we define the Baum-Bott map and estimated its rank. We review the Baum-Bott formula, which is Theorem 2, and the theorems of the Baum-Bott map, which are Theorems 3 and 4, for foliations on the projective plane \mathbb{P}^2 . Then, we study the rank at foliations on high-dimensional projective spaces. We state Proposition 1 for degree-one foliations, and Proposition 2, which give us an upper bound for the generic rank of this map. We prove Theorem 5, which is stated for foliations of low degree on the projective space \mathbb{P}^3 .

The Third Chapter is dedicated to the Jouanolou foliation. We give some known results and remarks of this foliation. We study how the space of vector fields, generating foliations, is decomposed in terms of the automorphisms of the Jouanolou foliation. We also estimate its Baum-Bott indexes and the rank of the local Baum-Bott map at this foliation, which is given in Theorem 6. As consequences, we get Theorem 7 and Corollary 1, which estimate the generic rank and fiber of the Baum-Bott map for degree-two foliation on even-dimensional projective spaces, and Theorem 8, which give us the exact rank of this map at the Jouanolou foliation on \mathbb{P}^3 .

Finally, at the Appendix, we study the eigenspaces of an operator induced by an automorphism of the homogeneous Jouanolou vector field.

Chapter 1

Preliminares

In this chapter we review some fundamental topics. For more information, see [28] and [29].

1.1 Holomorphic foliations

A *non-singular holomorphic foliation* of dimension k on a complex manifold M is given by:

- a covering $\{U_\alpha\}_{\alpha \in A}$ of M by open sets,
- for each $\alpha \in A$, there is a biholomorphism $\psi : U_\alpha \rightarrow \mathbb{D}^k \times \mathbb{D}^{n-k}$, where $\mathbb{D} \subset \mathbb{C}$ is the unitary disc at the origin,
- if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{aligned} \psi_{\alpha\beta} : \psi_\alpha(U_{\alpha\beta}) &\rightarrow \psi_\beta(U_{\alpha\beta}) \\ (z, w) &\mapsto \psi_\beta \circ \psi_\alpha^{-1}(z, w) = (\varphi_1(z, w), \varphi_2(w)). \end{aligned}$$

Each open set U_α is called a trivializing open set of the foliation. A *plaque* is a set $\psi_\alpha^{-1}(\mathbb{D}^k \times w_0)$, where $w_0 \in \mathbb{D}^{n-k}$. On M we define a relation of equivalence: $p \equiv q$ if there are plaques P_1, \dots, P_s , for some $s \in \mathbb{N}$ such that $p \in P_1$, $q \in P_s$ and such that $P_i \cap P_{i+1} \neq \emptyset$, for $i = 1, \dots, s-1$. The equivalence class of $p \in M$ is called the *leaf* by p .

A *singular holomorphic foliation* of dimension k on a complex manifold M is a non-singular foliation of dimension k in $M \setminus S$, where S is an analytic variety on M of codimension at least two. We ask S to be minimal in the following sense, if S' is a proper analytic subset $S' \subset S$, then the foliation on $M \setminus S$ cannot be extended to $M \setminus S'$. Under that condition, S is called the *singular set* of the foliation \mathcal{F} and denoted by $\text{Sing}(\mathcal{F})$.

Every one-dimensional foliation is locally defined by a holomorphic vector field. More precisely, a holomorphic one-dimensional foliation \mathcal{F} on M is given by a covering $(U_\alpha)_{\alpha \in A}$

of M by open subsets, a collection $(X_\alpha)_{\alpha \in A}$ of holomorphic vector fields, $X_\alpha \in \mathfrak{X}(U_\alpha)$, and a multiplicative cocycle $(g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$, such that $X_\alpha = g_{\alpha\beta} X_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$.

In the case of one-dimensional foliations on the projective space, we have the following proposition:

Proposition 1.1.1 ([28: Teorema 6.4.1]). *Any one dimensional foliation \mathcal{F} on the projective space \mathbb{P}^n , $n \geq 2$, has cotangent bundle of the form $T_{\mathcal{F}}^* = \mathcal{O}(k)$, where $k \geq -1$. The integer number $d = k + 1$ is called the degree of the foliation \mathcal{F} . If the foliation \mathcal{F} has degree d , then it can be defined in an affine coordinate system $(\mathbb{C}^n, (x_1, \dots, x_n))$ by a polynomial vector field of the form*

$$X = \sum_{j=1}^n Q_j(x) \partial_j + G\mathcal{R},$$

where Q_1, \dots, Q_n are polynomials of degree at most d , G is a homogeneous polynomial of degree d and $\mathcal{R} = \sum_{j=1}^n x_j \partial_j$ is the radial vector field.

1.2 Chern classes

Let M be a C^∞ manifold and $\pi : E \rightarrow M$ be a C^∞ complex vector bundle of rank n . We denote by $\mathcal{A}^0(U)$ the \mathbb{C} -algebra $C^\infty(U, \mathbb{C})$, and $\mathcal{A}^p(U)$ the $\mathcal{A}^0(U)$ -module of complex p -forms on U . Let $\mathcal{A}^p(U, M)$ be the $\mathcal{A}^0(U)$ -module $C^\infty(U, \wedge^p(TM^{\mathbb{C}})^* \otimes E)$, i.e. $\mathcal{A}^p(U, M)$ is the set of C^∞ sections of the bundle $\wedge^p(TM^{\mathbb{C}})^* \otimes E$ on U , where $(TM^{\mathbb{C}})^*$ denotes the dual of the complexification of the real tangent bundle $T_{\mathbb{R}}M$ of M .

A *connection* in E is a \mathbb{C} -linear map

$$\nabla : \mathcal{A}^0(M, E) \rightarrow \mathcal{A}^1(M, E),$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s), \text{ for all } f \in \mathcal{A}^0(M), s \in \mathcal{A}^0(M, E).$$

The connection can be extended to a \mathbb{C} -linear map

$$\nabla : \mathcal{A}^1(M, E) \rightarrow \mathcal{A}^2(M, E),$$

such that

$$\nabla(w \otimes s) = dw \otimes s - w \wedge \nabla(s), \text{ for all } w \in \mathcal{A}^1(M) \text{ and } s \in \mathcal{A}^0(M, E).$$

The *curvature* of the connection ∇ is defined by $K_\nabla := \nabla \circ \nabla : \mathcal{A}^0(M, E) \rightarrow \mathcal{A}^2(M, E)$.

In order to obtain a local representation of the curvature, let $s^\alpha = (s_1^\alpha, \dots, s_n^\alpha)$ be a local frame of E on U_α , this means $s_i^\alpha \in \mathcal{A}^0(U_\alpha, E)$ and $\{s_1^\alpha(x), \dots, s_n^\alpha(x)\}$ is a base of E_x , for every $x \in U_\alpha$, where we suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M which trivializes $TM^{\mathbb{C}}$ and E . Then, there exists $\theta_{ij}^\alpha \in A^1(U_\alpha)$ such that

$$\nabla s_i^\alpha = \sum_{j=1}^n \theta_{ji}^\alpha s_j^\alpha.$$

Then, K_∇ is given locally by a $n \times n$ matrix of C^∞ 2-forms

$$\Theta^\alpha = d\theta^\alpha - \theta^\alpha \wedge \theta^\alpha,$$

where $\Theta_{ij}^\alpha = d\theta_{ij}^\alpha - \sum_k \theta_{ik}^\alpha \wedge \theta_{kj}^\alpha$.

An *invariant polynomial* over the space of $n \times n$ matrices $M(n, \mathbb{C})$ is a function $p : M(n, \mathbb{C}) \rightarrow \mathbb{C}$ which is a polynomial in the entries of the matrix and

$$p(g^{-1}Ag) = p(A), \text{ for all } A \in M(n, \mathbb{C}) \text{ and for all } g \in \text{GL}(n, \mathbb{C}).$$

The elementary symmetric functions of the eigenvalues of an $n \times n$ matrix are examples of invariant polynomials, and they are defined by

$$\det(tI + A) = \sum_{j=0}^n C_{n-j}(A)t^j,$$

where A is a $n \times n$ matrix. The function C_j is called the j -th *elementary symmetric function of the eigenvalues* of an $n \times n$ matrix. Given $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}^n$, we denote

$$C^\nu = C_1^{\nu_1} C_2^{\nu_2} \dots C_n^{\nu_n}.$$

Let p be an invariant polynomial of degree k . If we define $p(K_\nabla|_{U_\alpha})$ by $p(\Theta^\alpha)$ for all $\alpha \in A$, then $p(\Theta^\alpha) = p(\Theta^\beta)$ on $U_\alpha \cap U_\beta$. Hence it defines a global $2k$ -form $p(K_\nabla)$ on M , which does not depend on the trivialization and is closed, i.e., $dp(K_\nabla) = 0$, [28: Lemma 3.2.2]. This means that the $2k$ -form $p(K_\nabla)$ belongs to the de Rham cohomology $H_{dR}^{2k}(M, \mathbb{C})$. Indeed, the class $[p(K_\nabla)] \in H_{dR}^*(M, \mathbb{C})$ is independent of the connection ∇ over E . This means that it can be defined the class $[p(E)] := [p(K_\nabla)]$, which depends only on the class of C^∞ isomorphism of E . The class $[p(E)]$ is called the characteristic class of E .

Let C_j , for $j = 1, \dots, n$ be the elementary symmetric polynomials of the eigenvalues of an $n \times n$ matrix. The Chern forms of a curvature K_∇ associated to a connection ∇ over E are defined by

$$c_j(K_\nabla) = C_j \left(\frac{i}{2\pi} K_\nabla \right),$$

and the *Chern classes* of E are $c_0(E) = 1$, and

$$c_j(E) = \left[C_j \left(\frac{i}{2\pi} K_\nabla \right) \right] \in H_{dR}^{2j}(M, \mathbb{C}).$$

The $2n$ -form $c(E) = 1 + c_1(E) + \dots + c_n(E)$, is called the *total Chern class* of E .

If M is a complex manifold, then we define $c_i(M) := c_i(TM)$, where TM denotes the holomorphic tangent bundle of M . The $2i$ -th form $c_i(M)$ is called the i -th *Chern class* of M .

1.3 The Baum-Bott index formula

There is a theorem of Heinz Hopf which asserts that on a compact manifold the properly counted number of zeros of a vector field equals the Euler number of the manifold. In the article [4] Bott shows that when a vector field has non-degenerate singularities other relations appear between the characteristic number of the manifold and the local invariants of the vector field. If the vector field is holomorphic, then all the characteristic numbers are determined by the singularities of the vector field.

In [3] Bott generalizes its results to holomorphic vector fields with higher-dimensional zero sets and to bundles, instead of vector fields.

In the article [2], Baum and Bott extend the results of [4] to meromorphic vector fields, which is equivalent to give sections of $TM \otimes L$, where TM is the holomorphic tangent bundle of M , and L is a holomorphic line bundle of the manifold. They only assume that the singularities are isolated. Carrel in [8] gets the same result as a particular case. In [6], Bruzzo and Rubtsov get localization formulas which contains the Baum-Bott formula of [2] and Bott's formula of [4] as a special situation. In [11], Chern proves the theorem given in the article [2] with differential geometric tools under the assumption of non degeneracy. Soares in [27] also proves this theorem but without the non-degeneracy assumption using the relation between Grothendieck residues and the Bochner-Martinelli kernel. Guillot in [15] works with endomorphism on the projective space and gets as a special case a Baum-Bott formula like in [2] for homogeneous vector fields.

In 1972, in [1], Baum and Bott give a theorem on singularities of holomorphic foliations which includes the meromorphic vector-field theorem as a special case. In [13], Dia gives an improvement of the results. Lehman and Suwa in [19] generalize the Baum-Bott residues for singular holomorphic foliations on complex manifolds, working with a singular subvariety of the complex manifold.

Here we are going to state Soares' result in [27]. For this, we need to define the residue of a form.

Let $f_i : V \rightarrow \mathbb{C}$, $i = 1, \dots, n$, be holomorphic functions defined on the open subset $V \subset \mathbb{C}^n$, with $0 \in V$ such that $f^{-1}(0) = \{0\}$, and then define $f := (f_1, \dots, f_n)$. Consider

Γ to be the real n -cycle defined by $\Gamma = \{z \in V \mid |f_i(z)| = \epsilon, i = 1, \dots, n\}$ with orientation $d(\arg f_1) \wedge \dots \wedge d(\arg f_n) \geq 0$. Set

$$w = \frac{g(z)dz_1 \wedge \dots \wedge dz_n}{f_1(z) \dots f_n(z)},$$

be a meromorphic n -form, where g is holomorphic in V . The *residue* of w on 0 is

$$\text{Res}_0 w = \left(\frac{1}{2\pi i} \right)^n \int_{\Gamma} \frac{g(z)dz_1 \wedge \dots \wedge dz_n}{f_1(z) \dots f_n(z)}.$$

Theorem 1.3.1 ([27]). *Let M be a compact complex manifold of dimension n , L be a holomorphic line bundle on M and ξ be a holomorphic section of $TM \otimes L$ with isolated zeros. Consider the Chern classes:*

$$c^\nu(TM \otimes L) = c_1^{\nu_1}(TM \otimes L) \dots c_n^{\nu_n}(TM \otimes L),$$

$$\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}^n, \quad \nu_1 + 2\nu_2 + \dots + n\nu_n = n.$$

Then

$$\int_M c^\nu(TM \otimes L) = \sum_{p: \xi(p)=0} \text{Res}_p \left\{ \frac{C^\nu(J\xi)dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\},$$

where $J\xi = \left(\frac{\partial \xi_i}{\partial z_j} \right)$ is the Jacobian matrix, and $\text{Res}_p \left\{ \frac{C^\nu(J\xi)dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\}$ denotes the Grothendieck residue symbol and it is called the Baum-Bott index C^ν of ξ at p .

If Q is a homogeneous invariant polynomial of degree n , then Q is a linear combination with complex coefficients of the invariant polynomials C^ν , i.e., $Q = \sum_{\nu} \lambda_{\nu} C^{\nu}$, $\lambda_{\nu} \in \mathbb{C}$. If we denote by $q(TM \otimes L) \in H_{dR}^{2n}(M; \mathbb{C})$ the class corresponding to Q , then

$$\int_M q(TM \otimes L) = \sum_{p: \xi(p)=0} \text{Res}_p \left\{ \frac{Q(J\xi)dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\}.$$

We are interested in the case where M is the complex projective space \mathbb{P}^n , and the line bundle L is $\mathcal{O}(d-1)$, where d is an integer number with $d \geq 0$. In this case, the holomorphic section ξ of $T\mathbb{P}^n \otimes \mathcal{O}(d-1)$ represents a degree- d singular holomorphic foliation on the projective space \mathbb{P}^n .

Chapter 2

The Baum-Bott map

In this chapter we define the Baum-Bott map on the space of one-dimensional foliations on a compact complex manifold. When the manifold is the projective space \mathbb{P}^n we get an upper bound for the rank of the Baum-Bott map. In some cases we will see that the bound is sharp.

2.1 The Baum-Bott map

Recall that if X is a compact complex manifold of dimension n , a foliation by curves on X is a section of $TX \otimes L$, where L is a holomorphic line bundle over X . The line bundle L is the cotangent bundle $T_{\mathcal{F}}^*$ of the foliation \mathcal{F} . We denote by $\mathcal{Fol}(X, L) = \mathbb{P}H^0(X, TX \otimes L)$ the set of foliations \mathcal{F} on X with cotangent bundle $T_{\mathcal{F}}^* = L$. It is known that all foliations in $\mathcal{Fol}(X, L)$ with only non-degenerate singularities have the same number of singularities, $N = N(L) = c_n(TX \otimes L)$.

When a foliation has a non-degenerate singularity, its Baum-Bott indexes can be easily calculated. Let us denote by C_i the i th-elementary symmetric functions of the eigenvalues of a $n \times n$ matrix and $C^\alpha = C_1^{\alpha_1} C_2^{\alpha_2} \dots C_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and $\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n$. Let \mathcal{F} be a foliation with non-degenerate singularity $p(\mathcal{F})$, and let $X_{\mathcal{F}}$ be a germ of vector field around $p(\mathcal{F})$ which defines \mathcal{F} around $p(\mathcal{F})$. Then the Baum-Bott index of \mathcal{F} associated to α at $p(\mathcal{F})$ is:

$$BB_{\alpha}(\mathcal{F}, p(\mathcal{F})) = \frac{C^{\alpha}(DX_{\mathcal{F}}(p(\mathcal{F})))}{\det(DX_{\mathcal{F}}(p(\mathcal{F})))}.$$

As we saw in Theorem 1.3.1, the Baum-Bott indexes of a foliation with only isolated singularities are related with the characteristic classes of the cotangent bundle of the foliation and the tangent bundle of the manifold.

We associate to each foliation, the projectivization of eigenvalues at each singularity. Given a foliation \mathcal{F} on a compact complex manifold X with only non-degenerate

singularities $\{p_1(\mathcal{F}), \dots, p_N(\mathcal{F})\}$, let's choose X_j , a germ of vector field around $p_j(\mathcal{F})$ defining \mathcal{F} around the singularity $p_j(\mathcal{F})$. We define

$$\text{spec}(\mathcal{F}, p_j(\mathcal{F})) := [(\lambda_1(p_j(\mathcal{F})), \dots, \lambda_n(p_j(\mathcal{F})))]_{\mathbb{P}},$$

where $\lambda_i(p_j(\mathcal{F}))$ is an eigenvalue of $DX_j(p_j(\mathcal{F}))$,

$$[(x_1, \dots, x_n)] = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid \sigma \in S_n\},$$

where S_n is the permutation group $\{1, \dots, n\}$, and $[x]_{\mathbb{P}} = \{\lambda x \mid \lambda \in \mathbb{C}^*\}$. Note that, if the eigenvalues are two by two distinct, then $\text{spec}(\mathcal{F}, p_j)$ can be identified to the projectivization of the set of eigenvalues of $DX_j(p_j)$. Also, it does not depend on the germ of the chosen vector field. Let's denote by

$$\mathcal{Fol}_{\text{red}}(X, L) := \{\mathcal{F} \in \mathcal{Fol}(X, L) / \text{all the singularities of } \mathcal{F} \text{ are non-degenerate}\}.$$

Given $\mathcal{F} \in \mathcal{Fol}_{\text{red}}(X, L)$ with singularities $\{p_1(\mathcal{F}), \dots, p_N(\mathcal{F})\}$, we define

$$E(\mathcal{F}) := (\text{spec}(\mathcal{F}, p_1(\mathcal{F})), \dots, \text{spec}(\mathcal{F}, p_N(\mathcal{F}))),$$

and $\xi(\mathcal{F}) := [E(\mathcal{F})]$, where $[(x_1, \dots, x_N)] = \{(x_{\sigma(1)}, \dots, x_{\sigma(N)}) / \sigma \in S_N\}$. Then we have the map

$$\begin{aligned} \xi : \mathcal{Fol}_{\text{red}}(X, L) &\rightarrow Z := ((\mathbb{P}(\mathbb{C}^n/S_n))^N)/S_N \\ \mathcal{F} &\mapsto [(\text{spec}(\mathcal{F}, p_1(\mathcal{F})), \dots, \text{spec}(\mathcal{F}, p_N(\mathcal{F})))]. \end{aligned}$$

Note that Z is a complex variety and the map ξ is holomorphic.

The projectivization of the eigenvalues of $DX_j(p_j(\mathcal{F}))$, where the singularity $p_j(\mathcal{F})$ is non-degenerate, allows us to calculate the Baum-Bott indexes. Conversely, the projectivization of the eigenvalues associated to a foliation \mathcal{F} at a non-degenerate singularity $p(\mathcal{F})$ can be calculated in terms of $BB_{A_1}(\mathcal{F}, p(\mathcal{F})), \dots, BB_{A_{n-1}}(\mathcal{F}, p(\mathcal{F}))$, if we choose the sequence (A_1, \dots, A_{n-1}) in a good way. Here we use the fact that $C_1(DX_j(p_j(\mathcal{F}))), \dots, C_n(DX_j(p_j(\mathcal{F})))$ determine the characteristic polynomial of $DX_j(p_j(\mathcal{F}))$, and the map $\Lambda : U \rightarrow \Lambda(U)$ is a bijection, where U is an open subset of \mathbb{P}^{n-1} , and identifying λ as a diagonal matrix, then

$$\begin{aligned} \Lambda : \quad U &\rightarrow \mathbb{P}^{n-1} \\ [\lambda = (\lambda_1, \dots, \lambda_n)]_{\mathbb{P}} &\rightarrow [(C_1^n(\lambda), C_1^{n-2}(\lambda)C_2(\lambda), \dots, C_1^{n-i}(\lambda)C_i(\lambda), \dots, C_n(\lambda))]_{\mathbb{P}}. \end{aligned}$$

Let us concentrate on foliations on the projective space \mathbb{P}^n . In this case, the cotangent bundle is $\mathcal{O}(d-1)$, where $d \geq 1$ is the degree of the foliation. We denote

by $\mathcal{F}ol(n, d) = \mathbb{P}H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}(d-1))$. If $n \geq 2$ and $d \geq 1$, we have

$$\dim(\mathcal{F}ol(n, d)) = (n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - 1.$$

We denote by

$$\mathcal{F}ol_{\text{red}}(n, d) := \{\mathcal{F} \in \mathcal{F}ol(n, d) \mid \text{all the singularities of } \mathcal{F} \text{ are non-degenerate}\}.$$

Remark 2.1.1. *Note that $\mathcal{F}ol_{\text{red}}(n, d)$ is a Zariski open and dense subset of $\mathcal{F}ol(n, d)$. Moreover, if a foliation $\mathcal{F}_0 \in \mathcal{F}ol_{\text{red}}(n, d)$ then it has exactly $N = d^n + d^{n-1} + \dots + d + 1$ singularities.*

Let us define the local Baum-Bott map. If $\mathcal{F}_0 \in \mathcal{F}ol_{\text{red}}(n, d)$ with singular set $\text{Sing}(\mathcal{F}_0) = \{p_1^0, \dots, p_N^0\}$, then there is a neighborhood V of the foliation \mathcal{F}_0 , $V \subset \mathcal{F}ol_{\text{red}}(n, d)$, and holomorphic maps, $p_1, \dots, p_N : V \rightarrow \mathbb{P}^n$ such that $p_j(\mathcal{F}_0) = p_j^0$, and for any foliation $\mathcal{F} \in V$, its singular set is $\text{Sing}(\mathcal{F}) = \{p_1(\mathcal{F}), \dots, p_N(\mathcal{F})\}$. In this case we define the holomorphic map $BB : V \rightarrow (\mathbb{C}^k)^N$, the *local Baum-Bott map*, by:

$$\mathcal{F} \mapsto (BB_{N_1}(\mathcal{F}, p_1(\mathcal{F})), \dots, BB_{N_k}(\mathcal{F}, p_1(\mathcal{F})), \dots, BB_{N_1}(\mathcal{F}, p_N(\mathcal{F})), \dots, BB_{N_k}(\mathcal{F}, p_N(\mathcal{F}))),$$

where $\{(\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_{\geq 0}^n \mid \alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n\} = \{N_0, N_1, \dots, N_k\}$, with $N_0 = (0, \dots, 0, 1)$.

Let us extend the domain of the map BB to $\mathcal{F}ol_{\text{red}}(n, d)$, by introducing a symmetry on the coordinates of the complex vector space $(\mathbb{C}^k)^N$ with respect to the singularities. More precisely, let S_N be the symmetric group of order N . We denote by $(\mathbb{C}^k)^N/S_N$ the quotient of $(\mathbb{C}^k)^N$ by the equivalence relation which identifies two points (z_1, \dots, z_N) and $(z_{\sigma(1)}, \dots, z_{\sigma(N)})$, where $\sigma \in S_N$ and $z_i \in \mathbb{C}^k$. Then, we can define the map $BB : \mathcal{F}ol_{\text{red}}(n, d) \rightarrow (\mathbb{C}^k)^N/S_N$:

$$\mathcal{F} \mapsto [(BB_1(\mathcal{F}, p_1(\mathcal{F})), \dots, BB_k(\mathcal{F}, p_1(\mathcal{F})), \dots, (BB_1(\mathcal{F}, p_N(\mathcal{F})), \dots, BB_k(\mathcal{F}, p_N(\mathcal{F})))],$$

where $[z_1, \dots, z_n]$ denotes the class of (z_1, \dots, z_N) in $(\mathbb{C}^k)^N/S_N$ and $\text{Sing}(\mathcal{F}) = \{p_1(\mathcal{F}), \dots, p_N(\mathcal{F})\}$.

This map can be extended to a rational map,

$$BB : \mathcal{F}ol(n, d) \dashrightarrow (\mathbb{P}^k)^N/S_N,$$

which is called the *global Baum-Bott map*.

By the Baum-Bott Theorem 1.3.1, we see that the Baum-Bott map is not dominant.

In the case of compact surfaces, the Baum-Bott Theorem 1.3.1 give us an explicit relation between the indexes. Let \mathcal{F} be a holomorphic foliation on a compact complex

surface M , and let $N_{\mathcal{F}}$ denotes the normal bundle of \mathcal{F} . The following result, can be found in [2] and [5].

Theorem 2.1.1 ([2]). *If the foliation \mathcal{F} has only isolated singularities, then*

$$N_{\mathcal{F}}.N_{\mathcal{F}} = \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p).$$

Therefore if the complex surface M is the projective plane \mathbb{P}^2 and \mathcal{F} is a degree- d foliation, then we get

$$\sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p) = (d + 2)^2.$$

In the above statement the symbol $BB(\mathcal{F}, p)$ corresponds to the Baum-Bott index C_1^2 of \mathcal{F} at p .

Gómez-Mont and Luengo ask in [14], if there are other hidden relations between the Baum-Bott indexes of a degree d foliation on the projective plane.

In the article [22], Lins-Neto and Pereira give an answer to the question.

Theorem 2.1.2 ([22]). *If $d \geq 2$, then the maximal rank of the Baum-Bott map for degree- d foliations on the projective plane \mathbb{P}^2 is $d^2 + d$. In particular, if the degree $d \geq 2$, then the dimension of a generic fiber of the map is $3d + 2$.*

This means that the only relation between the Baum-Bott indexes, in the case of the projective plane, is the Baum-Bott relation in Theorem 2.1.1.

Lins-Neto in [21], studies the generic fiber of the map for degree-two foliations on the projective plane. In this case, the dimension of a generic fiber is eight, which is the dimension of the automorphism group $\text{Aut}(\mathbb{P}^2)$. His result is the following:

Theorem 2.1.3 ([21]). *The generic fiber of the Baum-Bott map for degree-two foliations on the projective plane \mathbb{P}^2 contains exactly 240 orbits of the natural action of the automorphism group $\text{Aut}(\mathbb{P}^2)$.*

In this work, we study the Baum-Bott map in the case of one-dimensional foliations on the projective space \mathbb{P}^n , of higher dimension.

2.2 An upper bound for the rank of the Baum-Bott map on the projective space

We denote $gr(n, d)$ the generic rank of the Baum-Bott map for degree- d foliations on the projective space \mathbb{P}^n . In particular, when the degree is zero, $gr(n, 0) = 0$. This is because

any foliation of degree zero has only one singularity and in some affine coordinate system $\mathbb{C}^n \subset \mathbb{P}^n$ is defined by the radial vector field $\mathcal{R} = \sum_{j=1}^n x_j \partial_j$.

Let us see what happens when the degree is greater than zero. First, let us find a lower bound for the dimension of a generic fiber of the Baum-Bott map.

If F is a fiber of the Baum-Bott map BB , then

$$\dim F \geq \dim \mathcal{Fol}(n, d) - \text{gr}(n, d).$$

There exist a Zariski open and dense subset $U \subset \mathcal{Fol}(n, d)$ such that

$$\dim F = \dim \mathcal{Fol}(n, d) - \text{gr}(n, d), \quad (2.2.1)$$

for all fibers F in the open set U .

We have a natural action $\text{Aut}(\mathbb{P}^n)$ on $\mathcal{Fol}(n, d)$. Recall that:

$$\text{Aut}(\mathbb{P}^n) = \{[A]/A \in \text{GL}(n+1, \mathbb{C})\} = \mathbb{P}\text{GL}(n+1, \mathbb{C}),$$

where $[A]$ is the projectivization of A . Then, $\text{Aut}(\mathbb{P}^n)$ acts on $\mathcal{Fol}(n, d)$ by:

$$\begin{aligned} \text{Aut}(\mathbb{P}^n) \times \mathcal{Fol}(n, d) &\rightarrow \mathcal{Fol}(n, d) \\ (T, \mathcal{F}) &\mapsto T^*\mathcal{F}. \end{aligned}$$

The orbit of a foliation $\mathcal{F}_0 \in \mathcal{Fol}(n, d)$ will be denoted by:

$$\begin{aligned} \mathcal{O}(\mathcal{F}_0) &:= \{\mathcal{F} \in \mathcal{Fol}(n, d) \mid \text{there exists } T \in \text{Aut}(\mathbb{P}^n) \text{ such that } T^*\mathcal{F}_0 = \mathcal{F}\} \\ &= \{T^*\mathcal{F}_0 \mid T \in \text{Aut}(\mathbb{P}^n)\}. \end{aligned}$$

Since the Baum-Bott map is invariant by the action of the automorphism group $\text{Aut}(\mathbb{P}^n)$, we have $\mathcal{O}(\mathcal{F}_0) \subset BB^{-1}\{BB(\mathcal{F}_0)\}$. Thus for a fiber F and for a foliation $\mathcal{F}_0 \in F$, we get:

$$\dim F \geq \dim \mathcal{O}(\mathcal{F}_0).$$

We replace this last inequality in (2.2.1), then for a generic foliation \mathcal{F}_0 ,

$$\dim \mathcal{O}(\mathcal{F}_0) \leq \dim \mathcal{Fol}(n, d) - \text{gr}(n, d). \quad (2.2.2)$$

Let us find the dimension of the orbit $\mathcal{O}(\mathcal{F}_0)$, for a generic foliation \mathcal{F}_0 . We know that the orbit $\mathcal{O}(\mathcal{F}_0) = \frac{\text{Aut}(\mathbb{P}^n)}{\text{Stab}(\mathcal{F}_0)}$, where $\text{Stab}(\mathcal{F}_0) = \{T \in \text{Aut}(\mathbb{P}^n) \mid T^*\mathcal{F}_0 = \mathcal{F}_0\}$ is the stabilizer of \mathcal{F}_0 . Therefore we focus on the stabilizer. We study the cases degree $d = 1$ and $d \geq 2$, separately.

In the case of degree-one foliations on the projective space \mathbb{P}^n , the foliation \mathcal{F}_0 can be given by a linear vector field in \mathbb{C}^{n+1} . Suppose that all eigenvalues of this vector field are non-zero and two by two different. In some homogeneous coordinates, the foliation is defined by the linear vector field

$$X(x_0, \dots, x_n) = \sum_{j=0}^n \lambda_j x_j \partial_j,$$

where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq \lambda_j$, $\lambda_i \neq 0$, for $i, j = 0, \dots, n$. Suppose that $[A]$ is an automorphism in $\text{Stab}(\mathcal{F}_0)$. Since the foliation \mathcal{F}_0 is given by the vector field X , we have

$$A^{-1} \cdot X \circ A = \mu X, \quad (2.2.3)$$

for some $\mu \in \mathbb{C}$ with $\mu \neq 0$. If the matrix A is given by $A = [a_{ij}]_{0 \leq i, j \leq n}$, replacing in (2.2.3) we have:

$$a_{ij}(\lambda_i - \mu \lambda_j) = 0, \text{ for } i, j = 0, \dots, n. \quad (2.2.4)$$

Since $[A] \in \text{Aut}(\mathbb{P}^n)$, there are indexes $i_0, j_0 \in \{0, \dots, n\}$ such that $a_{i_0 j_0} \neq 0$; replacing in (2.2.4), we have $\lambda_{i_0} = \mu \lambda_{j_0}$, and $a_{i_0 j} = a_{i j_0} = 0$, for $i \neq i_0$ and $j \neq j_0$. Once again, there are indexes $i_1 \neq i_0$ and $j_1 \neq j_0$ such that $a_{i_1 j_1} \neq 0$. We replace in (2.2.4), and we have $\lambda_{i_1} = \mu \lambda_{j_1}$, $a_{i_1 j} = a_{i j_1} = 0$, for $i \neq i_1$ and $j \neq j_1$. Doing so, there are indexes $i_0, \dots, i_n \in \{0, \dots, n\}$ all different, and indexes j_0, \dots, j_n with the same property such that $a_{i_k j_k} \neq 0$ and $\mu = \frac{\lambda_{i_k}}{\lambda_{j_k}}$, for $k = 0, \dots, n$. We can suppose that $\lambda_i \neq \xi \lambda_j$, for all $i, j = 0, \dots, n$, where ξ is a k -th root of the unity, for $k = 1, \dots, n+1$. Then $i_k = j_k$, for all $k = 0, \dots, n$ and $\mu = 1$. Hence A is a diagonal matrix. Therefore, the stabilizer for a generic degree-one foliation on the projective space \mathbb{P}^n is $\text{Stab}(\mathcal{F}_0) = \{[A] \in \text{Aut}(\mathbb{P}^n) \mid A \text{ is a diagonal matrix}\}$.

Consequently, for a generic degree-one foliation \mathcal{F}_0 :

$$\mathcal{O}(\mathcal{F}_0) = \frac{\text{Aut}(\mathbb{P}^n)}{\{[A] \in \text{Aut}(\mathbb{P}^n) \mid A \text{ is a diagonal matrix}\}},$$

and $\dim \mathcal{O}(\mathcal{F}_0) = (n+1)^2 - 1 - n = n^2 + n$. We know that $\dim \mathcal{Fol}(n, 1) = n(n+2) - 1$, and replacing this information in (2.2.2):

$$\text{gr}(n, 1) \leq n - 1. \quad (2.2.5)$$

In the case of generic foliations of degree $d \geq 2$, the stabilizer is the identity. For instance, a foliation \mathcal{F}_0 can be represented by a homogeneous vector field X in \mathbb{C}^{n+1} . Let $[A]$ be an automorphism in $\text{Stab}(\mathcal{F}_0)$, then $A^*X = \mu X$, for some $\mu \in \mathbb{C}^*$, hence $X \circ A = \mu A \circ X$. The singularities of the foliation \mathcal{F}_0 are points $[(p_0^i, \dots, p_n^i)]_{\mathbb{P}} \in \mathbb{P}^n$,

such that $X(p_0^i, \dots, p_n^i) = \lambda_i(p_0^i, \dots, p_n^i)$, for some $\lambda_i \in \mathbb{C}$, and generically there are $d^n + \dots + d + 1$ of such points. The set of singular points of the foliation is invariant by the automorphism $[A]$. We know that an automorphism is uniquely determined by $n + 2$ points in generic position, and the automorphism $[A]$ fixes a subset of the projective space \mathbb{P}^n that has more than $n + 2$ points. Hence for a generic foliation \mathcal{F}_0 the automorphism $[A]$ must be the identity. Therefore, generically, for a generic foliation \mathcal{F}_0 of degree $d \geq 2$:

$$\mathcal{O}(\mathcal{F}_0) = \frac{\text{Aut}(\mathbb{P}^n)}{\{[\text{Id}]\}} \cong \text{Aut}(\mathbb{P}^n).$$

Replacing this information in 2.2.2, we get

$$\text{gr}(n, d) \leq (n + 1) \binom{n + d}{n} - \binom{n + d - 1}{n} - (n + 1)^2. \quad (2.2.6)$$

Let us estimate another bound in terms of some Baum-Bott indexes. Let \mathcal{F} be a foliation with non-degenerate singularity $p(\mathcal{F})$, and let $X_{\mathcal{F}}$ be a germ of vector field around $p(\mathcal{F})$ which defines the foliation \mathcal{F} around the singularity $p(\mathcal{F})$. The Baum-Bott index of the foliation \mathcal{F} associated to $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, with $\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n$, at the singularity $p(\mathcal{F})$ is:

$$BB_{\alpha}(\mathcal{F}, p(\mathcal{F})) = \frac{C^{\alpha}(DX_{\mathcal{F}}(p(\mathcal{F})))}{\det(DX_{\mathcal{F}}(p(\mathcal{F})))},$$

where $C^{\alpha} = C_1^{\alpha_1} \dots C_n^{\alpha_n}$. Let us denote $BB_1 = BB_{(n,0,0,\dots,0)}$, $BB_2 = BB_{(n-2,1,0,\dots,0),\dots}$, $BB_{n-2} = BB_{(2,0,\dots,0,1,0,0)}$ and $BB_{n-1} = BB_{(1,0,\dots,0,1,0)}$. Observe that for $\alpha \neq (0, \dots, 0, 1)$, we have

$$BB_{\alpha}(\mathcal{F}, p(\mathcal{F})) = (BB_1^{1-(\alpha_2+\dots+\alpha_{n-1})} BB_2^{\alpha_2} BB_3^{\alpha_3} \dots BB_{n-1}^{\alpha_{n-1}})(\mathcal{F}, p(\mathcal{F})).$$

Then the Baum-Bott map, at a generic foliation \mathcal{F} , depends on those $n - 1$ indexes, at each singularity. Also, the Baum-Bott Theorem 1.3.1, tell us that we have at least $(n - 1)$ relations between those $n - 1$ indexes and this means

$$\text{gr}(n, d) \leq (n - 1)N - (n - 1) = (N - 1)(n - 1). \quad (2.2.7)$$

From 2.2.5, 2.2.6 and 2.2.7, we get an upper bound for the generic rank of the Baum-Bott map.

Proposition 2.2.1. *Let $\text{gr}(n, d)$ denote the generic rank of the Baum-Bott map for one-dimensional degree- d foliations on the projective space \mathbb{P}^n with $n \geq 2$, and $N = d^n + \dots + d + 1$. Then we have:*

1. *For degree-one foliations,*

$$\text{gr}(n, 1) = n - 1.$$

2. For $d \geq 2$

$$\begin{aligned} \text{gr}(n, d) &= d^2 + d && , \text{ if } n = 2, \\ \text{gr}(n, d) &\leq \min \{ \dim \mathcal{Fol}(n, d) - \dim \text{Aut}(\mathbb{P}^n), (N-1)(n-1) \} && , \text{ if } n \geq 3, \end{aligned}$$

$$\text{where } \dim \mathcal{Fol}(n, d) = (n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - 1.$$

Proof. For degree one, we already have $\text{gr}(n, 1) \leq n-1$ from (2.2.5). If we find a foliation such that its rank at the Baum-Bott map is $n-1$, we have the result. In fact, observe that the map $[\lambda = (\lambda_1, \dots, \lambda_n)]_{\mathbb{P}} \mapsto [C_1^n(\lambda), \dots, C_1^{n-i}(\lambda)C_i(\lambda), \dots, C_n(\lambda)]_{\mathbb{P}}$ defined on some open subset of \mathbb{P}^{n-1} is a bijection. Since a degree-one foliation on the projective space \mathbb{P}^n is generated in the affine coordinate system \mathbb{C}^n by the vector field:

$$X = \lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2 + \dots + \lambda_n x_n \partial_n,$$

for some $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}^*$, then the result follows.

For $n, d \geq 2$, from (2.2.6) and (2.2.7), we have

$$\text{gr}(n, d) \leq \min \left\{ (n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - (n+1)^2, (N-1)(n-1) \right\}.$$

If $n = 2$, we have

$$3 \binom{2+d}{2} - \binom{2+d-1}{2} - 3^2 = d^2 + d + 3(d-2),$$

and,

$$\text{gr}(2, d) \leq \min \{ d^2 + d + 3(d-2), (d^2 + d)(1) \} = d^2 + d.$$

For degree- d foliations on the projective space \mathbb{P}^2 , with $d \geq 2$, by Theorem 2.1.2, the maximum rank is $d^2 + d$. Then

$$\text{gr}(2, d) = d^2 + d.$$

For $n \geq 3$, we will show that

$$(N-1)(n-1) - \left((n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - (n+1)^2 \right) > 0. \quad (2.2.8)$$

For instance, for $n = 3$, the difference (2.2.8) is

$$\begin{aligned} & (d^3 + d^2 + d)(3 - 1) - \left(4 \binom{3+d}{3} - \binom{3+d-1}{3} - 4^2 \right) \\ &= 2(d^3 + d^2 + d) - \left(\frac{(d+1)(d+2)(d+4)}{2} - 16 \right) \\ &= \frac{3(d-2)^2 + 15(d-2)^2 + 14(d-2) + 16}{2} > 0. \end{aligned}$$

If $n \geq 4$,

$$\frac{d+n+1}{n-1} \leq \frac{d+5}{3}, \text{ and } \frac{d+k}{k-1} \leq d, \text{ for } k \geq 4,$$

we replace in the difference (2.2.8),

$$\begin{aligned} & (n-1)(d^n + d^{n-1} + d^{n-2} + \dots + d) + \\ & \quad - \left((n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - (n+1)^2 \right) \\ & \geq (n-1)d^{n-4}(d^4 + d^3 + d^2 + d) + \\ & \quad - \left(\frac{(d+1)(d+2)(d+3)}{2} \left(\frac{(d+4)}{3} \dots \frac{(d+n-1)}{n-2} \right) \frac{(d+n+1)}{n-1} - 25 \right) \\ & \geq 3(d^4 + d^3 + d^2 + d) - \frac{(d+1)(d+2)(d+3)(d+5)}{6} + 25 \\ & = \frac{17(d-2)^4 + 143(d-2)^3 + 42(d-2)^2 + 427(d-2) + 270}{6} > 0. \end{aligned}$$

■

Remark 2.2.1. Observe that $\min \{ \dim \mathcal{Fol}(n, d) - \dim \text{Aut}(\mathbb{P}^n), (N-1)(n-1) \}$ is

$$\begin{aligned} & (N-1)(n-1) = d^2 + d \quad , \text{ if } n = 2. \\ & \dim \mathcal{Fol}(n, d) - \dim \text{Aut}(\mathbb{P}^n) \quad , \text{ if } n \geq 3. \end{aligned}$$

Remark 2.2.2. We have $g(2, 1) = 1$. The Camacho-Sad index over an invariant line gives the extra condition.

We would like to know if the upper bound of the Proposition 2.2.1 is sharp for $n \geq 3$.

Remark 2.2.3. If $n \geq 3$ and $d \geq 2$, the upper bound of Proposition 2.2.1 is sharp if, and only if, the generic fiber of the Baum-Bott map is a finite union of orbits of the action of $\text{Aut}(\mathbb{P}^n)$ on the space $\mathcal{Fol}(n, d)$.

2.3 The generic rank of the Baum-Bott map for foliations of low degree on \mathbb{P}^3

Let \mathcal{F} be a degree- d foliation on the projective space \mathbb{P}^3 with only isolated and non-degenerate singularities. If $p(\mathcal{F})$ is a singularity of \mathcal{F} , then there is a germ of vector field $X_{\mathcal{F}}$, given in some affine coordinate system, such that $p(\mathcal{F})$ is its singularity and it is non-degenerate. Its Baum-Bott indexes can be calculated in terms of the indexes BB_1 and BB_2 , which are defined by:

$$BB_1(\mathcal{F}, p_i(\mathcal{F})) = \frac{(C_1(DX_{\mathcal{F}}(p_i(\mathcal{F}))))^3}{C_3(DX_{\mathcal{F}}(p_i(\mathcal{F})))},$$

$$BB_2(\mathcal{F}, p_i(\mathcal{F})) = \frac{(C_1(DX_{\mathcal{F}}(p_i(\mathcal{F}))))C_2(DX_{\mathcal{F}}(p_i(\mathcal{F})))}{C_3(DX_{\mathcal{F}}(p_i(\mathcal{F})))}.$$

Then, we are interested in the rank of the following map:

$$\begin{array}{ccc} BB & \mathcal{F} \text{ol}_{\text{red}}(3, d) & \rightarrow (\mathbb{C}^2)^N \\ & \mathcal{F} & \mapsto (BB_1(\mathcal{F}, p_1(\mathcal{F})), BB_2(\mathcal{F}, p_1(\mathcal{F})), \dots, BB_1(\mathcal{F}, p_N(\mathcal{F})), BB_2(\mathcal{F}, p_N(\mathcal{F}))), \end{array}$$

where $\mathcal{F} \text{ol}_{\text{red}}(3, d) \subset \mathcal{F} \text{ol}(3, d)$ is the open and dense set of foliations in which each element has only isolated and non-degenerate singularities. In this case, the number of singularities is $N = 1 + d + d^2 + d^3$. Observe that this map and the Baum-Bott map have the same rank. We will call this map also the Baum-Bott map and denote it by BB .

By proposition 2.2.1, the generic rank of the Baum-Bott map for $d \geq 2$ has the upper bound,

$$\text{gr}(3, d) \leq 4 \binom{3+d}{3} - \binom{2+d}{3} - (4)^2 = \frac{(d+1)(d+2)(d+4)}{2} - 16.$$

If $2 \leq d \leq 9$, we will see that

$$\text{gr}(3, d) = \frac{(d+1)(d+2)(d+4)}{2} - 16.$$

We fix $d \geq 2$. To show that the upper bound given in Proposition 2.2.1 is sharp, it is enough to find a foliation such that its rank is that number. Therefore, we are interested in computing the rank of the derivative of the Baum-Bott map at a given foliation which has only isolated and non-degenerate singularities. We show first for $3 \leq d \leq 9$.

We know that a degree- d foliation on the projective space \mathbb{P}^3 can be defined in the affine coordinate system \mathbb{C}^3 by a polynomial vector field of the form $Q_1\partial_x + Q_2\partial_y + Q_3\partial_z + G\mathcal{R}$,

where Q_1, Q_2, Q_3 are polynomials in three variables of degree at most d , G is a homogeneous polynomial of degree d and \mathcal{R} is the radial vector field, then

$$\begin{aligned} \mathbf{B}_d = \{ & x^i y^j z^k \partial_x, x^l y^m z^n \partial_y, x^p y^q z^r \partial_z, x^s y^t z^u \mathcal{R} \mid \mathcal{R} \text{ is the radial vector field,} \\ & i + j + k, l + m + n, r + s + t \leq d, s + t + u = d \text{ and} \\ & i, j, k, l, m, n, p, q, r, s, t, u \in \mathbb{Z}_{\geq 0}\}, \end{aligned}$$

is a \mathbb{C} -basis of the space of degree- d foliations on the projective space \mathbb{P}^3 .

We are going to see how the derivative matrix $DBB(\mathcal{F}_0)$ can be calculated in terms of the basis \mathbf{B}_d , at some foliation \mathcal{F}_0 . Let $X_0 = X\partial_x + Y\partial_y + Z\partial_z$ be a vector field in \mathbb{C}^3 which generates a degree- d foliation on the projective space \mathbb{P}^3 and let $V = V_1\partial_x + V_2\partial_y + V_3\partial_z$ be a vector field in \mathbb{C}^3 generated by the monomial vector fields of the basis \mathbf{B}_d . Fixing p_0 a non-degenerate singularity of X_0 , we want to calculate $\partial_t BB_1(X_0 + tV, \rho(t))$, $\partial_t BB_2(X_0 + tV, \rho(t))$, at $t = 0$, where ρ is the curve of singularities of the vector field $X_0 + tV$ with $\rho(0) = p_0$.

Recall that if $A = [a_{ij}]$ is a 3×3 matrix, then its elementary symmetric functions of the eigenvalues are:

$$C_1(A) = \text{tr}(A) = a_{11} + a_{22} + a_{33},$$

$$C_2(A) = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32},$$

$$C_3(A) = \det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Then, the Baum Bott indexes of $X_0 + tV$ are:

$$\begin{aligned} BB_1(X_0 + tV, \rho(t)) &= \frac{C_1^3(D(X_0 + tV)(\rho(t)))}{C_3(D(X_0 + tV)(\rho(t)))}, \\ BB_2(X_0 + tV, \rho(t)) &= \frac{C_1(D(X_0 + tV)(\rho(t)))C_2(D(X_0 + tV)(\rho(t)))}{C_3(D(X_0 + tV)(\rho(t)))}. \end{aligned}$$

We denote, for simplicity, $C_i(t) := C_i(D(X_0 + tV)(\rho(t)))$ and $BB_i(t) := BB_i(X_0 + tV, \rho(t))$, then

$$\begin{aligned} BB_1'(0) &= \frac{3C_1(0)^2 C_1'(0) C_3(0) - C_1(0)^3 C_3'(0)}{C_3(0)^2} \\ BB_2'(0) &= \frac{C_1'(0) C_2(0) C_3(0) + C_1(0) C_2'(0) C_3(0) - C_1(0) C_2(0) C_3'(0)}{C_3(0)^2} \end{aligned}$$

We denote $|\cdot| = \det(\cdot)$. For computing $BB'_1(0), BB'_2(0)$, we need the following equations:

$$C_1(0) = \text{tr}(DX_0(p_0)) \quad (2.3.9)$$

$$C_2(0) = \left(\left| \frac{\partial(X, Y)}{\partial(x, y)} \right| + \left| \frac{\partial(X, Z)}{\partial(x, z)} \right| + \left| \frac{\partial(Y, Z)}{\partial(y, z)} \right| \right) (p_0) \quad (2.3.10)$$

$$C_3(0) = \det(DX_0(p_0)) \quad (2.3.11)$$

$$\rho'(0) = -[DX_0(p_0)]^{-1}V(p_0) \quad (2.3.12)$$

$$C'_1(0) = \nabla(\text{tr}(DX_0))(p_0) \cdot \rho'(0) + \text{tr}(DV(p_0)). \quad (2.3.13)$$

$$\begin{aligned} C'_2(0) = & \nabla \left(\left| \frac{\partial(X, Y)}{\partial(x, y)} \right| + \left| \frac{\partial(X, Z)}{\partial(x, z)} \right| + \left| \frac{\partial(Y, Z)}{\partial(y, z)} \right| \right) (p_0) \cdot \rho'(0) + \\ & + \left(\left| \frac{\partial(X, V_2)}{\partial(x, y)} \right| + \left| \frac{\partial(X, V_3)}{\partial(x, z)} \right| + \left| \frac{\partial(Y, V_3)}{\partial(y, z)} \right| \right) (p_0) + \\ & + \left(\left| \frac{\partial(V_1, Y)}{\partial(x, y)} \right| + \left| \frac{\partial(V_1, Z)}{\partial(x, z)} \right| + \left| \frac{\partial(V_2, Z)}{\partial(y, z)} \right| \right) (p_0). \end{aligned} \quad (2.3.14)$$

$$\begin{aligned} C'_3(0) = & \nabla(\det(DX_0))(p_0) \cdot \rho'(0) + \\ & + \left(\left| \frac{\partial(V_1, Y, Z)}{\partial(x, y, z)} \right| + \left| \frac{\partial(X, V_2, Z)}{\partial(x, y, z)} \right| + \left| \frac{\partial(X, Y, V_3)}{\partial(x, y, z)} \right| \right) (p_0) \end{aligned} \quad (2.3.15)$$

We see that to calculate the rank of the Baum-Bott map at a specific vector field with only non-degenerate singularities we can use the formulas (2.3.12)-(2.3.15) and implement it in an algorithm.

ALGORITHM: Calculates the rank of the Baum-Bott map at a degree d -foliation on the projective space \mathbb{P}^3 , for foliations with non-degenerate singularities.

INPUT: Vector field (X, Y, Z) in the vector space \mathbb{C}^3 which generates a degree d -foliations on the projective space \mathbb{P}^3 .

OUTPUT: The rank of the derivative of the Baum-Bott map at the foliation generated by the given vector field.

Step 1 Define a matrix MV whose set of rows, $MV[j]$, forms the basis \mathbf{B}_d .

Step 2 Define MS a matrix whose rows, $MS[i]$, are the singularities of (X, Y, Z) , which are the solutions of $(X, Y, Z)(x, y, z) = (0, 0, 0)$.

$k \leftarrow 0$, this constant will count all the singularities that are going to be found by subsequently change of coordinates.

Step3 Verify that the singularities are non-degenerate, if there is one that is degenerate, exit.

Step 4 Calculate the matrix DBB , the derivative matrix of the Baum-Bott map with respect to the basis MV :

For $j : 1$, to the dimension row of MV

For $i : k + 1$, to the dimension row of $MS + k$

Let dS be the solution of the matrix equation $D(X, Y, Z)(MS[i])(dS) = MV[j](MS[i])$, dS is the derivative of a parametrization of the singular points of $(X, Y, Z) + tMV[j]$, around $MS[i]$.

The derivative of the Baum-Bott indexes BB_1, BB_2 at $MS[i]$ in the direction $MV[j]$:

$DBB(2i - 1, j) \leftarrow DBB_1(MS[i])(MV[j])$.

$DBB(2i, j) \leftarrow DBB_2(MS[i])(MV[j])$.

$k \leftarrow$ dimension row of $MS + k$.

Step 5 *If* we can make another change of coordinates to another affine coordinate system:

Redefine (X, Y, Z) and MV in this new affine coordinate system.

Else Go to Step 7

Step 6 *If* there are new singularities of the foliation:

Redefine MS with only new singularities.

Go to Step 3.

Else Go to Step 5.

Step 7 Calculate the rank of DBB .

First, we study foliations on the projective space \mathbb{P}^3 of degree $d \geq 3$. In the next section we will focus on $d = 2$.

We consider the degree-three foliation \mathcal{F}_0 on the projective space \mathbb{P}^3 , generated by the vector field (X, Y, Z) in the affine coordinate system \mathbb{C}^3 :

$$\begin{aligned} X &= \left(x + \frac{1}{2}y + \frac{1}{3}z + \frac{1}{4}\right)\left(\frac{1}{5}x + \frac{1}{6}y + \frac{1}{7}z + \frac{1}{8}\right)\left(\frac{1}{9}x + \frac{1}{10}y + \frac{1}{11}z + \frac{1}{12}\right), \\ Y &= \left(\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z + \frac{1}{5}\right)\left(\frac{1}{6}x + \frac{1}{7}y + \frac{1}{8}z + \frac{1}{9}\right)\left(\frac{1}{10}x + \frac{1}{11}y + \frac{1}{12}z + \frac{1}{13}\right), \\ Z &= \left(\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{6}\right)\left(\frac{1}{7}x + \frac{1}{8}y + \frac{1}{9}z + \frac{1}{10}\right)\left(\frac{1}{11}x + \frac{1}{12}y + \frac{1}{13}z + \frac{1}{14}\right). \end{aligned}$$

This vector field has 27 singularities in the affine space \mathbb{C}^3 . We use the Algorithm and find that the rank of the Baum-Bott map restricted to these singularities is 54, the rank we were looking for in the case $d = 3$, which is the upper bound given in Proposition 2.2.1.

Let \mathcal{F}_d be the degree- d foliation on the projective space \mathbb{P}^3 , generated by the vector field $X_d = (X, Y, Z)$ in the affine coordinate system \mathbb{C}^3 :

$$\begin{aligned} X &= \prod_{k=1}^d \left(\frac{1}{4k-3}x + \frac{1}{4k-2}y + \frac{1}{4k-1}z + \frac{1}{4k} \right), \\ Y &= \prod_{k=1}^d \left(\frac{1}{4k-2}x + \frac{1}{4k-1}y + \frac{1}{4k}z + \frac{1}{4k+1} \right), \\ Z &= \prod_{k=1}^d \left(\frac{1}{4k-1}x + \frac{1}{4k}y + \frac{1}{4k+1}z + \frac{1}{4k+2} \right). \end{aligned}$$

To find the singularities of the vector field, we solve the system:

$$\begin{bmatrix} \frac{1}{4k-3} & \frac{1}{4k-2} & \frac{1}{4k-1} \\ \frac{4m-2}{1} & \frac{4m-1}{1} & \frac{4m}{1} \\ \frac{1}{4n-1} & \frac{1}{4n} & \frac{1}{4n+1} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{4k} \\ -\frac{1}{4m+1} \\ -\frac{1}{4n+2} \end{bmatrix},$$

where k, m, n are natural numbers and $1 \leq k, m, n \leq d$. We see that the determinant of that matrix is

$$-\frac{(4(k-m)-1)(4(m-n)-1)(2(k-n)-1)}{16(2k-1)(4k-3)(4k-1)m(2m-1)(4m-1)n(4n-1)(4n+1)} \neq 0.$$

Hence the singularities of the vector field X_d are $(x(k, m, n), y(k, m, n), z(k, m, n))$, where $1 \leq k, m, n \leq d$ are natural numbers, and

$$\begin{aligned} x(k, m, n) &= -\frac{(4n-1)(2m-1)(4k-3)}{4k(4m+1)(2n+1)}, \\ y(k, m, n) &= \frac{3(2k-1)(4m-1)n}{k(4m+1)(2n+1)}, \\ z(k, m, n) &= -\frac{3(4k-1)m(4n+1)}{2k(4m+1)(2n+1)}. \end{aligned}$$

The singularities are all different. In fact, two singularities

$$(x(k, m, n), y(k, m, n), z(k, m, n)), (x(r, s, t), y(r, s, t), z(r, s, t))$$

with

$$(x(k, m, n), y(k, m, n), z(k, m, n)) = (x(r, s, t), y(r, s, t), z(r, s, t))$$

must fulfill one of the following conditions:

1. $k = r, m = s, n = t,$

2. $k = r$, $m = \frac{1}{4}(1 + 4t)$, $n = \frac{1}{4}(-1 + 4s)$,
3. $k = \frac{1}{4}(1 + 4s)$, $m = \frac{1}{4}(-1 + 4r)$, $n = t$,
4. $k = \frac{1}{4}(1 + 4s)$, $m = \frac{1}{4}(1 + 4t)$, $n = \frac{1}{2}(-1 + 2r)$,
5. $k = \frac{1}{2}(1 + 2t)$, $m = \frac{1}{4}(-1 + 4r)$, $n = \frac{1}{4}(-1 + 4s)$,
6. $k = \frac{1}{2}(1 + 2t)$, $m = s$, $n = \frac{1}{2}(-1 + 2r)$.

Since k, m, n, r, s, t are natural numbers, we get $k = r, m = s, n = t$. Therefore the vector field has d^3 different singularities. The singularities are non-degenerate. In fact, we have:

$$DX_d(x(r, s, t), y(r, s, t), z(r, s, t)) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \frac{1}{4r-3} & \frac{1}{4r-2} & \frac{1}{4r-1} \\ \frac{1}{4s-2} & \frac{1}{4s-1} & \frac{1}{4s} \\ \frac{1}{4t-1} & \frac{1}{4t} & \frac{1}{4t+1} \end{bmatrix},$$

where

$$\begin{aligned} a &= \prod_{k=1, k \neq r}^d \left(\frac{1}{4k-3}x(r, s, t) + \frac{1}{4k-2}y(r, s, t) + \frac{1}{4k-1}z(r, s, t) + \frac{1}{4k} \right), \\ b &= \prod_{m=1, m \neq s}^d \left(\frac{1}{4m-2}x(r, s, t) + \frac{1}{4m-1}y(r, s, t) + \frac{1}{4m}z(r, s, t) + \frac{1}{4m+1} \right), \\ c &= \prod_{n=1, n \neq t}^d \left(\frac{1}{4n-1}x(r, s, t) + \frac{1}{4n}y(r, s, t) + \frac{1}{4n+1}z(r, s, t) + \frac{1}{4n+2} \right). \end{aligned}$$

If we show $abc \neq 0$, then the singularity $(x(r, s, t), y(r, s, t), z(r, s, t))$ is non-degenerate. Otherwise, if for some k we have

$$\frac{1}{4k-3}x(r, s, t) + \frac{1}{4k-2}y(r, s, t) + \frac{1}{4k-1}z(r, s, t) + \frac{1}{4k} = 0,$$

then

$$\frac{3(k-r)(4k-4r-1)(2k-2t-1)}{4k(2k-1)(4k-3)(4k-1)r(4r+1)(2t+1)} = 0,$$

which implies $k = r$. If for some m we have

$$\frac{1}{4m-2}x(r, s, t) + \frac{1}{4m-1}y(r, s, t) + \frac{1}{4m}z(r, s, t) + \frac{1}{4m+1} = 0,$$

then

$$\frac{3(4m-4r+1)(m-s)(4m-4t-1)}{8m(2m-1)(4m-1)(4m+1)r(4s+1)(2t+1)} = 0,$$

from what we get $m = s$. Analogously, if for some n we have

$$\frac{1}{4n-1}x(r, s, t) + \frac{1}{4n}y(r, s, t) + \frac{1}{4n+1}z(r, s, t) + \frac{1}{4n+2} = 0,$$

then

$$\frac{3(2n-2r+1)(4n-4s+1)(n-t)}{4n(2n+1)(4n-1)(4n+1)r(4s+1)(2t+1)} = 0,$$

and we get $n = t$. In conclusion, $abc \neq 0$.

We restrict the derivative of the Baum-Bott map at the foliation generated by the vector field X_d to these d^3 singularities. We calculate its rank with the Algorithm, and we get that the upper bound given in the Proposition 2.2.1 is sharp, for degree $d = 4, \dots, 9$.

Thus, we have proved the following theorem.

Theorem 2.3.1. *If $3 \leq d \leq 9$, the generic rank of the Baum-Bott map for degree- d foliations on the projective space \mathbb{P}^3 is*

$$\dim \mathcal{Fol}(3, d) - \dim \text{Aut}(\mathbb{P}^3) = \frac{(d+1)(d+2)(d+4)}{2} - 16.$$

In other words, a generic fiber of the Baum-Bott map has dimension $\dim \text{Aut}(\mathbb{P}^3)$ and is a finite union of orbits of the action of the automorphism group $\text{Aut}(\mathbb{P}^3)$ on the space of one-dimensional degree- d foliations on the projective space \mathbb{P}^3 .

2.4 The generic rank of the Baum-Bott map for degree-two foliations on \mathbb{P}^3 and \mathbb{P}^4

For the case of degree-two foliations on the projective plane \mathbb{P}^2 , Guillot gives an example of a foliation with maximum rank for the Baum-Bott map in his article [16]. We would like to have explicit examples for degree-two foliations in higher dimensions.

We saw in Section 2.2 that the Baum-Bott map can be restricted to the study of the $n-1$ Baum-Bott indexes BB_1, \dots, BB_{n-1} with

$$BB_i(\mathcal{F}, p(\mathcal{F})) = \frac{(C_1^{n-i} C_i)(DX_{\mathcal{F}}(p(\mathcal{F})))}{C_n(DX_{\mathcal{F}}(p(\mathcal{F})))}, \text{ for } i = 1, \dots, n-1,$$

where $p(\mathcal{F})$ is a non-degenerate singular point of the foliation \mathcal{F} , $X_{\mathcal{F}}$ is a germ of vector field which defines the foliation around the singular point $p(\mathcal{F})$. Hence, we consider $BB = (BB_1, \dots, BB_{n-1})$.

Let us focus on a particular degree-two foliation \mathcal{F}_0 on the projective space \mathbb{P}^n , defined in \mathbb{C}^{n+1} by the following homogeneous vector field $\mathbb{X}_0 = (\mathbb{X}_0^0, \dots, \mathbb{X}_n^0)$:

$$\begin{aligned}\mathbb{X}_0^0 &= x_0(x_1 - x_2 - x_3 - \dots - x_n), \\ \mathbb{X}_1^0 &= x_1(x_2 - x_3 - \dots - x_n - x_0), \\ \mathbb{X}_i^0 &= x_i(x_{i+1} - x_{i+2} - \dots - x_n - x_0 - \dots - x_{i-1}), \text{ for } i = 2, \dots, n-1, \\ \mathbb{X}_n^0 &= x_n(x_0 - x_1 - \dots - x_{n-1}).\end{aligned}\tag{2.4.16}$$

We want to express the derivative of the Baum-Bott map at this foliation in easier terms. The stabilizer of the foliation \mathcal{F}_0 is

$$\text{Stab}(\mathcal{F}_0) = \{\mathbb{T}^i \in \text{Aut}(\mathbb{P}^n) \mid i = 0, 1, \dots, n\},$$

where \mathbb{T} is the automorphism on the projective space \mathbb{P}^n defined by

$$\mathbb{T}(x_0, x_1, \dots, x_n) = (x_1, x_2, \dots, x_n, x_0).\tag{2.4.17}$$

If $p \in \text{Sing}(\mathcal{F}_0)$, then $\mathbb{T}^{-i}(p) \in \text{Sing}(\mathcal{F}_0)$, for $i = 0, \dots, n$.

Given $I = (i_0, \dots, i_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, we set $x^I = x_0^{i_0} \dots x_n^{i_n}$, $|I| = i_0 + \dots + i_n$. Let \mathbb{B} be the set

$$\mathbb{B} = \{x^I \partial_j \mid I \in \mathbb{Z}_{\geq 0}^{n+1}, |I| = 2 \text{ and } j = 0, \dots, n\}.$$

We can consider \mathbb{B} as a generator for the space of degree-two foliations on the projective space \mathbb{P}^n by homogeneous polynomial vector fields on \mathbb{C}^{n+1} . Suppose $p_0 \in \mathbb{P}^n$ is a non-degenerate singular point of \mathcal{F}_0 and let \mathbb{V} be an homogeneous vector field in \mathbb{C}^{n+1} . In the next equations, we identify an homogeneous vector field in \mathbb{C}^{n+1} with its corresponding vector field in affine coordinate systems. If $p(t)$ is the singularity of $\mathbb{X}_0 + t\mathbb{V}$ such that $p(0) = p_0$, then

$$BB_j(\mathbb{X}_0 + t\mathbb{V}, p(t)) = BB_j(\mathbb{T}^*(\mathbb{X}_0 + t\mathbb{V}), \mathbb{T}^{-1}(p(t))) = BB_j(\mathbb{X}_0 + t\mathbb{T}^*\mathbb{V}, \mathbb{T}^{-1}(p(t))).$$

Consequently, if $I \in \mathbb{Z}_{\geq 0}^{n+1}$, we have

$$\begin{aligned}BB_j(\mathbb{X}_0 + tx^I \partial_0, p(t)) &= BB_j(\mathbb{X}_0 + tx^{\mathbb{T}^{-1}(I)} \partial_1, \mathbb{T}^{-1}(p(t))) \\ &= BB_j(\mathbb{X}_0 + tx^{\mathbb{T}^{-2}(I)} \partial_2, \mathbb{T}^{-2}(p(t))) \\ &\vdots \\ &= BB_j(\mathbb{X}_0 + tx^{\mathbb{T}^{-n}(I)} \partial_n, \mathbb{T}^{-n}(p(t)))\end{aligned}\tag{2.4.18}$$

To find the rank of the Baum-Bott map at the foliation \mathcal{F}_0 we can just consider the Baum-Bott indexes at the points whose orbits generate the singular set of the foliation

\mathcal{F}_0 . Let p_1, \dots, p_{k_n} be those generators and suppose they are all non-degenerate. Let $\mathbb{B}_0 = \{x^I \partial_0 \mid I \in \mathbb{Z}_{\geq 0}^{n+1} \text{ and } |I| = 2\}$ be given an order, and $\mathbb{B}_1 = \mathbb{T}^*(\mathbb{B}_0)$, ..., $\mathbb{B}_n = (\mathbb{T}^n)^*(\mathbb{B}_0)$. In this way, $\mathbb{B} = \mathbb{B}_0 \cup \mathbb{B}_1 \cup \dots \cup \mathbb{B}_n$. Now, for each $j = 0, \dots, n$, let's denote

$$DBB(p_1)|_{\mathbb{B}_j} = \begin{bmatrix} DBB_1(p_1)|_{\mathbb{B}_j} \\ \vdots \\ DBB_{n-1}(p_1)|_{\mathbb{B}_j} \end{bmatrix}, \dots, DBB(p_{k_n})|_{\mathbb{B}_j} = \begin{bmatrix} DBB_1(p_{k_n})|_{\mathbb{B}_j} \\ \vdots \\ DBB_{n-1}(p_{k_n})|_{\mathbb{B}_j} \end{bmatrix},$$

$$D_0 = \begin{bmatrix} DBB(p_1)|_{\mathbb{B}_0} \\ \vdots \\ DBB(p_{k_n})|_{\mathbb{B}_0} \end{bmatrix}, \dots, D_n = \begin{bmatrix} DBB(p_1)|_{\mathbb{B}_n} \\ \vdots \\ DBB(p_{k_n})|_{\mathbb{B}_n} \end{bmatrix}.$$

Then

$$\text{rank } DBB(\mathcal{F}) = \text{rank} \begin{bmatrix} D_0 & D_1 & D_2 & \dots & D_{n-2} & D_{n-1} & D_n \\ D_n & D_0 & D_1 & \dots & D_{n-3} & D_{n-2} & D_{n-1} \\ D_{n-1} & D_n & D_0 & \dots & D_{n-4} & D_{n-3} & D_{n-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ D_2 & D_3 & D_4 & \dots & D_n & D_0 & D_1 \\ D_1 & D_2 & D_3 & \dots & D_{n-1} & D_n & D_0 \end{bmatrix}.$$

We can choose the generators of the set $\text{Sing}(\mathcal{F}_0)$ that are in the affine coordinate system $(\mathbb{C}^n, (x_1, \dots, x_n))$. In this way, the sets $\mathbb{B}_0, \dots, \mathbb{B}_n$ in this affine coordinate system become B_0, \dots, B_n , with the inherited order: $B_0 = \{x^I \partial_1 \mid I \in \mathbb{Z}_{\geq 0}^n \text{ and } |I| \leq 2\}$, $B_1 = \tilde{T}^*(B_0), \dots, B_n = (\tilde{T}^*)^n(B_0)$, where $\tilde{T}^*(\cdot) = x_1 T^*(\cdot)$ and

$$T(x_1, \dots, x_n) = \left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, \frac{1}{x_1} \right).$$

The homogeneous vector field \mathbb{X}_0 , in this affine coordinate system, is given by the vector field $X_0 = (X_1^0, \dots, X_n^0)$:

$$\begin{aligned} X_1^0 &= x_1(-x_1 + 2x_2 - 1), \\ X_2^0 &= x_2(-2x_1 + x_2 + 2x_3 - 1), \\ X_i^0 &= x_i(-2x_1 + x_i + 2x_{i+1} - 1), \text{ for } i = 3, \dots, n-1, \\ X_n^0 &= x_n(-2x_1 + x_n + 1). \end{aligned} \tag{2.4.19}$$

For the case $n = 3$, the vector field X_0 in the affine coordinate system \mathbb{C}^3 is given by

$$(X, Y, Z) = (-x(1 + x - 2y), y(-1 - 2x + y + 2z), z(1 - 2x + z)).$$

This vector field generates a degree two-foliation on the projective space \mathbb{P}^3 . Its singular set is generated by

$$\{(1, 1, 1), (-1, 0, 3), (-1, 0, 0), (0, 1, 0), (0, 0, 0)\}.$$

They are non-degenerate singularities and we can find D_0, \dots, D_3 . The rank of the matrix $DBB(\mathcal{F}_0)$ is 20, which is the upper bound given in Proposition 2.2.1.

For $n = 4$, let's see with more details the Baum-Bott map. Let \mathcal{F} be a degree- d foliation on the projective space \mathbb{P}^4 with only isolated and non-degenerate singularities. Let $p(\mathcal{F})$ be a singularity of \mathcal{F} , there is a germ of vector field, $X_{\mathcal{F}}$, given in some affine coordinate system, such that $p(\mathcal{F})$ is its singularity, and it is non-degenerate. We are interested in the following Baum-Bott indexes:

$$\begin{aligned} BB_1(\mathcal{F}, p_i(\mathcal{F})) &= \frac{C_1^4(DX_{\mathcal{F}}(p_i(\mathcal{F})))}{C_4(DX_{\mathcal{F}}(p_i(\mathcal{F})))} \\ BB_2(\mathcal{F}, p_i(\mathcal{F})) &= \frac{(C_1^2(DX_{\mathcal{F}}(p_i(\mathcal{F}))))C_2(DX_{\mathcal{F}}(p_i(\mathcal{F})))}{C_4(DX_{\mathcal{F}}(p_i(\mathcal{F})))} \\ BB_3(\mathcal{F}, p_i(\mathcal{F})) &= \frac{(C_1(DX_{\mathcal{F}}(p_i(\mathcal{F}))))C_3(DX_{\mathcal{F}}(p_i(\mathcal{F})))}{C_4(DX_{\mathcal{F}}(p_i(\mathcal{F})))}. \end{aligned}$$

In this case, we have to study the rank of the following map $BB : \mathcal{Fol}_{\text{red}}(4, d) \rightarrow (\mathbb{C}^3)^N$, defined by

$$\mathcal{F} \mapsto (BB_1(\mathcal{F}, p_i(\mathcal{F})), BB_2(\mathcal{F}, p_i(\mathcal{F})), BB_3(\mathcal{F}, p_i(\mathcal{F})))_{\{i=1, \dots, N\}},$$

where $\mathcal{Fol}_{\text{red}}(4, d) \subset \mathcal{Fol}(4, d)$ is the open and dense set of foliations in which each element has only isolated and non-degenerate singularities. In this case, the number of singularities is $N = 1 + d + d^2 + d^3 + d^4$.

Now, we calculate the Baum Bott indexes. Let $X = (X_1, X_2, X_3, X_4)$ and $V = (V_1, V_2, V_3, V_4)$ be vector fields, then the elementary symmetric functions of the eigenvalues

are

$$\begin{aligned}
C_1(DX) &= \partial_1 X_1 + \partial_2 X_2 + \partial_3 X_3 + \partial_4 X_4, \\
C_2(DX) &= \left| \frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} \right| + \left| \frac{\partial(X_1, X_3)}{\partial(x_1, x_3)} \right| + \left| \frac{\partial(X_1, X_4)}{\partial(x_1, x_4)} \right| + \\
&\quad + \left| \frac{\partial(X_2, X_3)}{\partial(x_2, x_3)} \right| + \left| \frac{\partial(X_2, X_4)}{\partial(x_2, x_4)} \right| + \left| \frac{\partial(X_3, X_4)}{\partial(x_3, x_4)} \right|, \\
C_3(DX) &= \left| \frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)} \right| + \left| \frac{\partial(X_1, X_2, X_4)}{\partial(x_1, x_2, x_4)} \right| + \left| \frac{\partial(X_1, X_3, X_4)}{\partial(x_1, x_3, x_4)} \right| + \left| \frac{\partial(X_2, X_3, X_4)}{\partial(x_2, x_3, x_4)} \right|, \\
C_4(DX) &= \left| \frac{\partial(X_1, X_2, X_3, X_4)}{\partial(x_1, x_2, x_3, x_4)} \right|.
\end{aligned}$$

Let's denote $C_i(t) = C_i(D(X + tV))(\rho(t))$, where $\rho(t)$ is the singularity of the vector field $X + tV$. If the vector field X has non-degenerate singularities, then $DBB_i(X, \rho(0))(V) = \partial_t BB_i(X + tV, \rho(t))|_{t=0}$, and

$$\begin{aligned}
DBB_1(X, \rho(0))(V) &= 4 \frac{C_1^3(0)}{C_4(0)} C_1'(0) - \frac{C_1^4(0)}{C_4^2(0)} C_4'(0), \\
DBB_2(X, \rho(0))(V) &= 2 \frac{C_1(0)C_2(0)}{C_4(0)} C_1'(0) + \frac{C_1^2(0)}{C_4(0)} C_2'(0) - \frac{C_1^2(0)C_2(0)}{C_4^2(0)} C_4'(0), \\
DBB_3(X, \rho(0))(V) &= \frac{C_3(0)}{C_4(0)} C_1'(0) + \frac{C_1(0)}{C_4(0)} C_3'(0) - \frac{C_1(0)C_3(0)}{C_4^2(0)} C_4'(0).
\end{aligned}$$

For the case $n = 4$, the vector field $X_0 = (X_1^0, X_2^0, X_3^0, X_4^0)$ in the affine coordinate system \mathbb{C}^4 is given by

$$\begin{aligned}
X_1^0 &= x_1(-x_1 + 2x_2 - 1), \\
X_2^0 &= x_2(-2x_1 - x_2 + 2x_3 - 1), \\
X_3^0 &= x_4(-2x_1 + x_3 + 2x_4 - 1), \\
X_4^0 &= x_4(-2x_1 + x_4 + 1).
\end{aligned}$$

There are seven points which generate the set $\text{Sing}(\mathcal{F})$, in the affine coordinate system, these points are

$$\{(0, 0, 0, 0), (0, 0, 0, -1), (0, 0, 1, 0), (0, 0, 3, -1), (0, 1, 0, -1), (0, -5, 3, -1), (1, 1, 1, 1)\}.$$

The generators of the singular set are non-degenerate and we can use this points to find D_0, \dots, D_4 . Then we can calculate the rank.

By these two examples we can state the following theorem.

Theorem 2.4.1. *The Baum-Bott map for degree-two foliations on the projective space \mathbb{P}^3 and \mathbb{P}^4 has generic rank 20 and 45, respectively.*

Corollary 1. *A generic fiber of the Baum-Bott map for degree-two foliations on the projective space \mathbb{P}^3 and \mathbb{P}^4 is a finite union of orbits of the action of the automorphism group on the projective space on the space of one-dimensional degree-two foliations on the projective space of same dimension.*

Chapter 3

The example of the Jouanolou foliation

In this chapter we study the Jouanolou foliation and its rank at the local Baum-Bott map.

3.1 The Jouanolou foliation

The Jouanolou foliation is the first example of a one-dimensional foliation on the projective plane \mathbb{P}^2 without invariant algebraic curves. In [18] Jouanolou showed that the holomorphic foliation on \mathbb{P}^2 defined in the affine plane by the vector field:

$$x' = 1 - xy^d, \quad y' = x^d - y^{d+1}, \quad d \geq 2, \quad (3.1.1)$$

does not have any algebraic leaf.

Algebraic solutions of differential equations were studied in 1878 by Darboux. He focused on equations of first order and first degree over the complex projective plane \mathbb{P}^2 . He showed in [12] that if an equation of this kind has enough algebraic solutions, then it must have a first integral. Then, Poincaré in 1891 showed in [25] that in order to find an explicit algebraic solution to a particular equation it would be enough to find an upper bound on the degree of the solution in terms of the polynomials that define the equation. Indeed, if the equation is defined by polynomials of degree less than or equal to two, then it always has solutions of degree one in the projective plane.

The study focused on the search for bounds on the degree of the solution, which is known as *Poincaré's Problem*, and many such bounds have been found; for example see [9] from Carnicer, [7] from Cerveau and Lins-Neto. However, Jouanolou in [18], showed that these turned out to be of limited use in solving differential equations due to his following result:

Theorem 3.1.1 ([18]). *A generic foliation on the projective plane \mathbb{P}^2 of degree greater than or equal to two does not have any algebraic solution.*

As part of the proof of this theorem, Jouanolou gave an explicit example of a family of foliations with no algebraic solution, which is equation (3.1.1). Later, Lins-Neto in [20] proved that the set of foliations on the projective plane \mathbb{P}^2 of degree greater than or equal to two without algebraic leaves contains an open and dense set.

Another family of examples is presented by Żołądek in [30]. He presents a series of polynomial planar vector fields without invariant algebraic curves in the projective plane \mathbb{P}^2 .

Theorem 3.1.2 ([30: Theorem 1]). *Let the integer exponents a, b, c, d satisfy the assumptions*

- $a \leq d \leq b + c$,
- $(b + c + 2)^2 - 4N < 0$ and $2(b + c + 1) < N + 1$;
- $\gcd(a + 1, d + 1) = 1$, $\gcd(c + 1, b + c - d + 1) = 1$, $\gcd(b + 1, b + c - a + 1) = 1$

where $N = (b + 1)(c + 1) - (a - c)(d - b)$ and \gcd stands for the greatest common divisor. Then the system

$$x' = y^a - x^{b+1}y^c, \quad y' = x^d - x^b y^{c+1}$$

does not have invariant algebraic curves in the complex projective plane \mathbb{P}^2 .

Jouanolou's result was extended by Soares to the projective space \mathbb{P}^3 :

Theorem 3.1.3 ([26]). *Let X_μ , $\mu \in \mathbb{C}$, be the vector field*

$$X_\mu = (\mu x + y^d - x^{d+1})\partial_x + (\mu y + z^d - yx^d)\partial_y + (\mu z + 1 - zx^d)\partial_z$$

and let \mathcal{F}_μ be the foliation on the projective space \mathbb{P}^3 represented by the vector field X_μ . Then, for $0 < |\mu| \ll 1$ and degree $d \geq 2$, the foliation \mathcal{F}_μ has no invariant algebraic set, meaning either an algebraic solution or an algebraic surface invariant by the solution.

Theorem 3.1.4 ([26]). *Let $\mathcal{F}ol(3, d)$ denote the space of one-dimensional foliations of degree d on the projective space \mathbb{P}^3 . For each $d \geq 2$ there is a dense subset $\mathfrak{F}_d \subset \mathcal{F}ol(3, d)$ such that any foliation in \mathfrak{F}_d has no invariant algebraic set.*

On higher dimensions, Lins-Neto and Soares in [23], generalized the above results for the projective space \mathbb{P}^n . In this case, the multidimensional Jouanolou system is defined by

$$x'_0 = x_1^d, \quad x'_1 = x_2^d, \dots, \quad x'_n = x_0^d. \quad (3.1.2)$$

For n even the Jouanolou foliation doesn't have invariant algebraic curves, and for n odd, perturbed Jouanolou foliation doesn't have invariant algebraic curves. With these examples it is proven that the set of foliations on the projective space \mathbb{P}^n of a given degree and without algebraic curves contains an open and dense set:

Theorem 3.1.5 ([23]). *Consider the vector field*

$$X_0^d = \sum_{i=1}^{n-1} (x_{i+1}^d - x_i x_1^d) \partial_i + (1 - x_n x_1^d) \partial_n, \quad (3.1.3)$$

and

$$X_\mu^d = X_0^d + \mu \mathcal{R}, \quad \mu \in \mathbb{C},$$

where \mathcal{R} is the radial vector field $\mathcal{R} = \sum_{i=1}^n x_i \partial_i$, and let $\mathcal{F}_0^d, \mathcal{F}_\mu^d$ be the foliations on the projective space \mathbb{P}^n , $n \geq 2$ represented, by the vector fields X_0^d and X_μ^d respectively. Then, for degree $d \geq 2$ and n even, the foliation \mathcal{F}_0^d has no algebraic solution and, for degree $d \geq 2$ and n odd, the foliation \mathcal{F}_μ^d has no algebraic solution provided $0 < |\mu| \ll 1$.

Theorem 3.1.6 ([23]). *Let $\mathcal{F}ol(n, d)$ denote the space of one-dimensional holomorphic foliations of degree d on the projective space \mathbb{P}^n , $n \geq 2$. For each $d \geq 2$, there is an open and dense subset $\mathfrak{F}_d \subset \mathcal{F}ol(n, d)$ such that any foliation in \mathfrak{F}_d has no algebraic solution.*

In fact, they show that the foliation \mathcal{F}_0^d has no algebraic solution of geometric genus greater than zero, whether n is even or odd. If n is odd, there are $\frac{(d^{n+1} - 1)}{(d^2 - 1)}$ invariant projective lines and they are the only invariant algebraic curves. But when one adds a perturbation $\mu \mathcal{R}$, $0 < |\mu| \ll 1$, then there are no invariant algebraic curves at all.

The study then focus on invariant algebraic hypersurfaces. In [24], Maciejewski, Moulin, Nowicki and Strelcyn, stated that if $n + 1 \geq 3$ is prime and $d > 2 \frac{n}{n-1}$, then the foliation given by the equation (3.1.2) does not have invariant algebraic hypersurfaces. In particular, this holds for the primes $n + 1 \geq 5$ and $d \geq 3$ and for $n + 1 = 3$ and $d \geq 5$. Żołądek, in [31], generalized this and Theorem 3.1.3 of Soares with the following theorem.

Theorem 3.1.7 ([31]). *If $n \geq 2$ and $d \geq 2$, then the Jouanolou foliation given by (3.1.2) has no invariant algebraic hypersurfaces. This implies that the space of foliations on the projective space \mathbb{P}^n without invariant algebraic hypersurfaces is dense in the space of all foliations of given degree.*

Let us see some interesting facts about the Jouanolou foliation. As we have seen, the generalized degree- d Jouanolou foliation \mathcal{J}_d , $d \geq 2$, on \mathbb{P}^n can be defined in homogeneous coordinates by the radial vector field in \mathbb{C}^{n+1} and the homogeneous vector field (3.1.2). In affine coordinates $(x_1, \dots, x_n, 1)$, it is induced by the vector field (3.1.3). All singularities of the foliation are contained in this affine coordinate system \mathbb{C}^n and are the solutions of the system

$$x_1^N = 1, \quad x_{n-j} = x_1^{-d^{j+1} - d^j \dots - d}, \quad 0 \leq j \leq n-2,$$

where $N = d^n + d^{n-1} + \dots + d + 1$. If ξ is a primitive N -th root of unity, then the singular set is:

$$\text{Sing}(\mathcal{J}_d) = \left\{ p_i = (\xi^i, \xi^{-i(d^{n-1} + \dots + d)}, \dots, \xi^{-id}) : 1 \leq i \leq N \right\}.$$

For each i , the set of eigenvalues of the linear part of the vector field at the singular point p_i is $\text{Spec}(\mathcal{J}_d, p_i) = \{\lambda_i^j = (-1 + d\omega^j)\xi^{id} \mid 1 \leq j \leq n\}$, where ω is a primitive $(n+1)$ -th root of unity.

Let $[\alpha_0, \dots, \alpha_n] \in \mathbb{PGL}(n+1, \mathbb{C})$ denote the class of the automorphism $(x_0, \dots, x_n) \mapsto (\alpha_0 x_0, \dots, \alpha_n x_n)$. Define

$$\mathcal{H} = \{[\alpha_0, \dots, \alpha_n] : \frac{\alpha_i}{\alpha_{i+1}} = \frac{\alpha_{i+1}^d}{\alpha_{i+2}^d}, 0 \leq i \leq n-2, \frac{\alpha_{n-1}}{\alpha_n} = \frac{\alpha_n^d}{\alpha_0^d}\}$$

The subgroup $\mathcal{H} \subset \mathbb{PGL}(n+1, \mathbb{C})$ is cyclic of order N and is generated by the class

$$[\xi, \xi^{-(d^{n-1} + \dots + d)}, \dots, \xi^{-(d^2 + d)}, \xi^{-d}, 1]$$

where ξ is a primitive N -th root of unity. Moreover, \mathcal{H} acts freely and transitively on the singular set of the foliation $\text{Sing}(\mathcal{J}_d)$. This group leaves invariant the Jouanolou foliation. Moreover, the vector field in \mathbb{C}^{n+1} , which generates the Jouanolou foliation on \mathbb{P}^n , admits the symmetry groups: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/(n+1)\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$ for $N = \frac{d^{n+1} - 1}{d - 1}$.

In some cases, the Jouanolou foliation admits invariant algebraic curves. From [23], by Lins-Neto and Soares, the degree- d Jouanolou foliation \mathcal{J}_d , $d \geq 2$ and n odd, has $\frac{d^{n+1} - 1}{d^2 - 1}$ invariant lines. Every invariant line has $d+1$ singular points, and the direction of such a projective line at p_i is the eigendirection associated to the eigenvalue $-(1 + d\omega^j)\xi^{id}$, with $\omega^j = -1$. See figure 3.1.

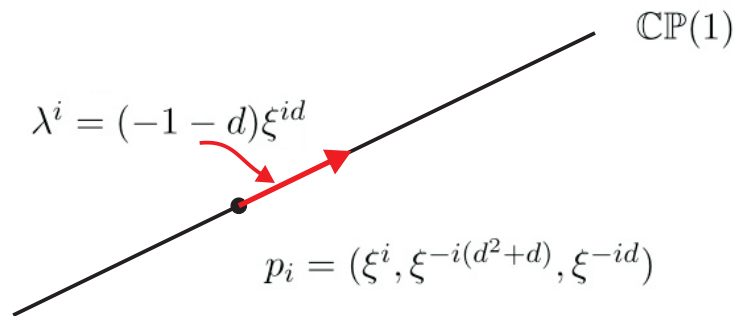


Figure 3.1: Tangent projective line is an eigendirection for the Jouanolou foliation on \mathbb{P}^3 .

3.2 Eigenspaces associated to the Jouanolou foliation

In this section we will decompose the space of vector fields in \mathbb{C}^n , which generates foliations on \mathbb{P}^n of given degree, into eigenspaces of an operator derived from an automorphism

leaving the Jouanolou foliation invariant. This will help us to find the rank of the Baum-Bott map at the Jouanolou foliation, we will see this in Section 3.4. Some proofs of this section are in Appendix A.

The degree- d Jouanolou foliation \mathcal{J}_d on the projective space \mathbb{P}^n is defined by the homogeneous vector field in \mathbb{C}^{n+1} :

$$\mathbb{X}_{\mathcal{J}_d} = (x_2^d, x_3^d, \dots, x_{n+1}^d, x_1^d).$$

This foliation is invariant by the automorphisms $\mathbb{A}, \mathbb{S} \in \text{Aut}(\mathbb{P}^n)$, which are defined by:

$$\begin{aligned} \mathbb{A}(x_1, \dots, x_{n+1}) &= (\alpha_1 x_1, \dots, \alpha_n x_n, \alpha_{n+1} x_{n+1}), \\ \mathbb{S}(x_1, \dots, x_{n+1}) &= (x_{n+1}, x_1, \dots, x_n), \end{aligned}$$

where $\alpha_1 = \xi = \xi^{-(d^n + \dots + d)}$, $\alpha_2 = \xi^{-(d^{n-1} + d^{n-2} + \dots + d)}$, ..., $\alpha_j = \xi^{-(d^{n+1-j} + \dots + d)}$, ..., $\alpha_n = \xi^{-d}$, $\alpha_{n+1} = 1$, ξ is a primitive N -th root of unity and $N = d^n + d^{n-1} + \dots + d + 1$. The automorphisms \mathbb{A} and \mathbb{S} generate cyclic subgroups of $\text{Aut}(\mathbb{P}^n)$ of order N and $n + 1$, respectively. The invariance of \mathcal{J}_d by these automorphisms follows from:

$$\mathbb{A}^*(\mathbb{X}_{\mathcal{J}_d}) = \xi^d \mathbb{X}_{\mathcal{J}_d}, \quad \mathbb{S}^*(\mathbb{X}_{\mathcal{J}_d}) = \mathbb{X}_{\mathcal{J}_d}.$$

In the affine coordinate system (x_1, \dots, x_n) the Jouanolou foliation is defined by the vector field:

$$X_{\mathcal{J}_d} = \sum_{i=1}^{n-1} (x_{i+1}^d - x_i x_1^d) \partial_i + (1 - x_n x_1^d) \partial_n,$$

and the automorphisms \mathbb{A}, \mathbb{S} are defined by the maps A and S , respectively:

$$\begin{aligned} A : \mathbb{C}^n &\rightarrow \mathbb{C}^n, & A(x_1, \dots, x_n) &= (\alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}, \alpha_n x_n), \\ S : \mathbb{C}^n &\dashrightarrow \mathbb{C}^n, & S(x_1, \dots, x_n) &= \frac{1}{x_n} (1, x_1, \dots, x_{n-1}). \end{aligned}$$

Recall that a degree- d foliation \mathcal{F} on the projective space \mathbb{P}^n is defined by a non-zero section of the twisted tangent bundle $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1)$ modulo non-zero complex multiple. The family of such foliations is parametrized by the projective space

$$\mathcal{Fol}(n, d) := \mathbb{P}(H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1))).$$

A foliation \mathcal{F} is represented in the affine coordinate system (x_1, \dots, x_n) by a vector field of the form

$$X = Q + G\mathcal{R},$$

where \mathcal{R} is the radial vector field $\mathcal{R} = x_1\partial_1 + \dots + x_n\partial_n$, G is a homogeneous polynomial of degree d and Q is a polynomial vector field of degree at most d . Given $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$, we set $x^I = x_1^{i_1} \dots x_n^{i_n}$ and $|I| = i_1 + \dots + i_n$. The space $\mathbf{V}_d = H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1))$, considered as a \mathbb{C} -vector space, is generated by

$$\mathbf{B}_d = \{x^I\partial_k, x^R\mathcal{R} \mid I, R \in \mathbb{Z}_{\geq 0}^n, |I| \leq d, |R| = d, \text{ and } k = 1, \dots, n\}.$$

The pull-back maps associated to A and S , $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ and $S^* : \mathbf{V}_d \rightarrow S^*(\mathbf{V}_d)$, are defined by $A^*(X) = DA^{-1}.X \circ A$ and $S^*(X) = DS^{-1}.X \circ S$.

We would like to observe that \mathbf{B}_d is a basis of eigenvectors of A^* . The eigenvalues are N -th roots of unity. Let $E_j = \{V \in \mathbf{V}_d \mid A^*V = \xi^j V\}$ denotes the eigenspace of the operator A^* associated to the eigenvalue ξ^j . We have the following lemma:

Lemma 3.2.1. *Every vector in \mathbf{B}_d is an eigenvector of $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ and $\mathbf{V}_d = \bigoplus_{j=1}^N E_j$. Moreover, if $x^I\partial_k, x^R\mathcal{R} \in \mathbf{B}_d$ then*

$$A^*(x^I\partial_k) = \alpha_k^{-1} \alpha^I x^I \partial_k \quad \text{and} \quad A^*(x^R\mathcal{R}) = \alpha^R x^R \mathcal{R},$$

where $\alpha^I = \alpha_1^{i_1} \dots \alpha_n^{i_n}$.

Next, we will study how the map S^* acts in \mathbf{V}_d .

Lemma 3.2.2. *Let Q be a polynomial with $\deg(Q) \leq d$ in n variables. Then*

$$S^*(Q\partial_1) = -x_n Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)\mathcal{R} \quad (3.2.4)$$

$$S^*(Q\partial_k) = x_n Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)\partial_{k-1}, \text{ for } 2 \leq k \leq n \quad (3.2.5)$$

$$S^*(Q\mathcal{R}) = -x_n Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)\partial_n \quad (3.2.6)$$

We see that the vector fields in (3.2.4), (3.2.5) and (3.2.6) in general have a pole divisor x_n^{d-1} . Therefore, we define

$$\tilde{S}^* : \mathbf{V}_d \rightarrow \mathbf{V}_d, \quad \tilde{S}^* := x_n^{d-1} S^*.$$

Let's see how it acts on \mathbf{V}_d :

Lemma 3.2.3. *Let $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$ such that $|I| \leq d$. If $i_{n+1} = d - |I|$, then*

$$\tilde{S}^*(x^I\partial_1) = -x_1^{i_2} \dots x_{n-1}^{i_n} x_n^{i_{n+1}} \mathcal{R} \quad (3.2.7)$$

$$\tilde{S}^*(x^I\partial_k) = x_1^{i_2} \dots x_{n-1}^{i_n} x_n^{i_{n+1}} \partial_{k-1}, \text{ for } 2 \leq k \leq n \quad (3.2.8)$$

$$\tilde{S}^*(x^I\mathcal{R}) = -x_1^{i_2} \dots x_{n-1}^{i_n} x_n^{i_{n+1}} \partial_n \quad (3.2.9)$$

This operator maps eigenspaces into eigenspaces, more specifically, we have the following lemma.

Lemma 3.2.4. *We have $\tilde{\mathbb{S}}^*(X_{\mathcal{J}_d}) = X_{\mathcal{J}_d}$ and $\tilde{\mathbb{S}}^*(E_k) = E_{f(k)}$, where*

$$f(k) = (dk - d^2 + d) \pmod{N}. \quad (3.2.10)$$

Moreover,

$$X_{\mathcal{J}_d} = \sum_{j=0}^n (\tilde{\mathbb{S}}^*)^j(\partial_n). \quad (3.2.11)$$

Proof. The following relation holds: $S^{-1} \circ A \circ S = A^{d^n}$. It yields:

$$S^{-1} \circ A^d \circ S = A^{d^{n+1}} = A.$$

The above relation implies: $S \circ A = A^d \circ S$, $A^* \circ S^* = S^* \circ (A^*)^d$, $x_n^{d-1} A^* \circ S^* = x_n^{d-1} S^* \circ (A^*)^d = \tilde{\mathbb{S}}^* \circ (A^*)^d$. Since $A^* \circ (x_n S^*) = \xi^{-d} x_n A^* \circ S^*$, we have $x_n^{d-1} A^* \circ S^* = A^* \circ (\xi^{d(d-1)} x_n^{d-1} S^*) = A^* (\xi^{d(d-1)} \tilde{\mathbb{S}}^*) = \xi^{d(d-1)} A^* \circ \tilde{\mathbb{S}}^*$. Hence, $A^* \circ \tilde{\mathbb{S}}^* = \xi^{-d(d-1)} \tilde{\mathbb{S}}^* \circ (A^*)^d$. If $V \in E_k$, then $(A^*)^d V = \xi^{dk} V$ and $A^*(\tilde{\mathbb{S}}^*(V)) = \xi^{-d(d-1)} \tilde{\mathbb{S}}^*(\xi^{dk} V) = \xi^{dk-d^2+d} V$, which proves (3.2.10). In particular, we get $\tilde{\mathbb{S}}^*(E_d) = E_d$. Finally $\tilde{\mathbb{S}}^*(X_{\mathcal{J}_d}) = X_{\mathcal{J}_d}$ and (3.2.11) follows from (3.2.7), (3.2.8) and (3.2.9). \blacksquare

The automorphisms \mathbb{A} and \mathbb{S} induce operators $\mathbb{A}^*, \mathbb{S}^* : \mathbb{V}_d \rightarrow \mathbb{V}_d$, where \mathbb{V}_d is the space of homogeneous vector fields in \mathbb{C}^{n+1} of degree d . Let us compare Lemmas 3.2.1 and A.1, of Appendix A. Observe that the eigenvalue of the operator \mathbb{A}^* on the vector $x_1^{i_1} \dots x_n^{i_n} x_{n+1}^{i_{n+1}} \partial_k$ is the same as the eigenvalue of the operator A^* on the vector $x_1^{i_1} \dots x_n^{i_n} \partial_k$, if $k \leq n$, and $-x_1^{i_1} \dots x_n^{i_n} \mathcal{R}$, if $k = n+1$. Also, from Lemma 3.2.3 and Lemma A.2, \mathbb{S}^* evaluated in $x_1^{i_1} \dots x_{n+1}^{i_{n+1}} \partial_k$ has the same image in \mathbb{V}_d as $\tilde{\mathbb{S}}^*$ at $x_1^{i_1} \dots x_n^{i_n} \partial_k$, if $k \leq n$, and $-x_1^{i_1} \dots x_n^{i_n} \partial_k \mathcal{R}$, if $k = n+1$. Also, we can identify the monomials of degree d in $n+1$ variables with the monomials in n variables of degree at most d with the following correspondence:

$$x_1^{i_1} \dots x_n^{i_n} x_{n+1}^{i_{n+1}} \mapsto x_1^{i_1} \dots x_n^{i_n}, \text{ where } i_1 + \dots + i_n + i_{n+1} = d.$$

These identifications are summarized in the following lemma:

Lemma 3.2.5. *Let $I = (i_1, \dots, i_n, i_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1}$ and $I' = (i_1, \dots, i_n)$. If we identify $x^I \partial_k$ with $x^{I'} \partial_k$, for $k = 1, \dots, n$, and $x^I \partial_{n+1}$ with $-x^{I'} \mathcal{R}$, then*

$$\begin{aligned} A^*(x^I \partial_k) &= \mathbb{A}^*(x^I \partial_k), & A^*(-x^{I'} \mathcal{R}) &= \mathbb{A}^*(x^I \partial_{n+1}), \\ \tilde{\mathbb{S}}^*(x^I \partial_k) &= \mathbb{S}^*(x^I \partial_k), & \tilde{\mathbb{S}}^*(-x^{I'} \mathcal{R}) &= \mathbb{S}^*(x^I \partial_{n+1}). \end{aligned}$$

Now we are able to know how many eigenspaces of each dimension exist:

Theorem 3.2.1. *The operator $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ has $\binom{n+d}{n} + n\binom{n+d-1}{n-1} - (n-1)\frac{3n+4}{2}$ non-trivial eigenspaces. More precisely:*

1. *There is one eigenspace of dimension $n+1$:*

$$E_d = \langle x_2^d \partial_1, \dots, x_n^d \partial_{n-1}, \partial_n, x_1^d \mathcal{R} \rangle.$$

2. *There are $\binom{n+d-1}{n}$ eigenspaces of dimension n , and they are:*

$$\langle x_1 x^I \partial_1, \dots, x_n x^I \partial_n \rangle,$$

where $I \in \mathbb{Z}_{\geq 0}^n$ and $|I| \leq d-1$.

3. *There are $\frac{(3n-4)(n+1)}{2}$ eigenspaces of dimension two, which are:*

- (a) $(\tilde{\mathbb{S}}^*)^k(\langle x_2^{d-1} x_n \partial_1, \partial_2 \rangle)$, $(\tilde{\mathbb{S}}^*)^k(\langle x_2^{d-1} \partial_1, x_1^d \partial_2 \rangle)$, $(\tilde{\mathbb{S}}^*)^k(\langle x_2^{d-1} x_j \partial_1, x_{1+j}^d \partial_2 \rangle)$, for $3 \leq j \leq n-1$, $k = 0, \dots, n$ and,
- (b) *if n is even:* $(\tilde{\mathbb{S}}^*)^k(\langle x_{2+j}^{d-1} \partial_1, x_1^{d-1} x_{1+j} \partial_{2+j} \rangle)$, for $1 \leq j \leq \frac{n}{2} - 1$ and $k = 0, \dots, n$,
if n is odd: $(\tilde{\mathbb{S}}^*)^k(\langle x_{2+j}^{d-1} \partial_1, x_1^{d-1} x_{1+j} \partial_{2+j} \rangle)$, for $1 \leq j \leq \frac{n-3}{2}$, $k = 0, \dots, n$
and for $j = \frac{n-1}{2}$, then $k = 0, \dots, \frac{n-1}{2}$.

4. *There are $(n+1) \left[\binom{n+d-1}{n-1} - 3(n-1) \right]$ eigenspaces of dimension 1, which are generated by: $(\tilde{\mathbb{S}}^*)^k(x^I \partial_1)$, where $x^I \partial_1 \in \mathbf{B}_d$, with $i_1 = 0$, $k = 0, \dots, n$ and $x^I \neq 1, x_2^{d-1} x_j, x_j^{d-1}, x_j^d$, for $j = 2, \dots, n$.*

The list is complete and each eigenspace of A^ corresponds to exactly one of the spaces given in the above items.*

Proof. Let E be an eigenspace of A^* . Then by the identification given in Lemma 3.2.5, E is contained in an eigenspace \mathbb{E} of \mathbb{A}^* . The spaces E and \mathbb{E} are equal, after the identification, if and only if \mathbb{E} does not contain a monomial $x^I \partial_{n+1}$, with $i_{n+1} > 0$, since this kind of vector does not belong to \mathbf{B}_d . Theorem A.1 gives the list of eigenspaces of \mathbb{A}^* . We see that the eigenspaces of \mathbb{A}^* of dimensions 1 and 2, and the eigenspace \mathbb{E}_d are eigenspaces of A^* , since $x^I \partial_{n+1}$ with $i_{n+1} > 0$ is not a monomial in those eigenspaces. Lemma A.7 and Theorem A.1 give the complete list of eigenspaces of those kinds, and we get Items 1,3 and 4. We observe from Item 1 of Theorem A.1 that $x^I \partial_{n+1} \in \mathbb{E}$, with $i_{n+1} > 0$, precisely for eigenspaces generated by $\langle x_1 x^{\bar{I}} \partial_1, \dots, x_{n+1} x^{\bar{I}} \partial_{n+1} \rangle$, where $x^{\bar{I}} = x_{n+1} x^I$. Since $E \subset \mathbb{E}$, then $E = \langle x_1 x^{\bar{I}} \partial_1, \dots, x_n x^{\bar{I}} \partial_n \rangle$, and we conclude the proof. ■

In the case of the space of polynomial vector fields in \mathbb{C}^3 generating degree d foliations on \mathbb{P}^3 , we are able to identify each eigenspace of dimension two, three and four. We identify $x = x_1$, $y = x_2$ and $z = x_3$.

Lemma 3.2.6. *Let \mathbf{V}_d be the space of polynomial vector fields defined in \mathbb{C}^3 generating degree- d foliations on the projective space \mathbb{P}^3 . Then the linear operator $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ has the following list of eigenvalues and eigenvectors which generate eigenspaces of dimension two, three and four (see Table 3.1):*

Eigenvalue \ Eigenvector in	$\mathcal{P}(\partial_x)$	$\mathcal{P}(\partial_y)$	$\mathcal{P}(\partial_z)$	$\mathcal{P}(\mathcal{R})$
ξ^d	$y^d \partial_x$	$z^d \partial_y$	∂_z	$x^d \mathcal{R}$
$\xi^{d^3+d^2+d}$	∂_x			$zx^{d-1} \mathcal{R}$
ξ^{d^3+d}	$z^d \partial_x$			$yx^{d-1} \mathcal{R}$
ξ^{d^2+d}	$zy^{d-1} \partial_x$	∂_y		
ξ^{d^2+2d}	$y^{d-1} \partial_x$	$x^d \partial_y$		
ξ^{2d}		$z^{d-1} \partial_y$	$x^d \partial_z$	
ξ^{2d+1}		$xz^{d-1} \partial_y$	$y^d \partial_z$	
ξ^{d+1}			$x \partial_z$	$y^d \mathcal{R}$
ξ^{d^3+d+1}			$y \partial_z$	$z^d \mathcal{R}$
ξ^{d^3+2d}	$z^{d-1} \partial_x$		$yx^{d-1} \partial_z$	
ξ^{d^2+d+1}		$x \partial_y$		$zy^{d-1} \mathcal{R}$
$\xi^{i-j(d^2+d)-kd}$	$x^{i+1} y^j z^k \partial_x$	$x^i y^{j+1} z^k \partial_y$	$x^i y^j z^{k+1} \partial_z$	

Table 3.1: Eigenvectors and eigenvalues of A^* , for $n = 3$, where $\mathcal{P}(V) = \{x^I V \in \mathbf{B}_d\}$.

In the case of the space of polynomial vector fields in \mathbb{C}^4 generating degree- d foliations on \mathbb{P}^4 , we are able to identify each eigenspace of dimension two, four and five.

Lemma 3.2.7. *Let \mathbf{V}_d be the space of polynomial vector fields in \mathbb{C}^4 generating degree- d foliations on the projective space \mathbb{P}^4 . Then the linear operator $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ has one eigenspace of dimension five, 20 eigenspaces of dimension two and $\binom{d+3}{4}$ of dimension four (see Table 3.2).*

3.3 The Baum-Bott indexes of the Jouanolou foliation

In this section we want to calculate the Baum-Bott indexes of the Jouanolou foliation. Recall that the Jouanolou foliation is defined in the affine coordinate system \mathbb{C}^n by the vector field

$$X_{\mathcal{J}_d} = \sum_{i=1}^{n-1} (x_{i+1}^d - x_i x_1^d) \partial_i + (1 - x_n x_1^d) \partial_n.$$

$\mathcal{P}(\partial_1)$	$\mathcal{P}(\partial_2)$	$\mathcal{P}(\partial_3)$	$\mathcal{P}(\partial_4)$	$\mathcal{P}(\mathcal{R})$
$x_2^d \partial_1$	$x_3^d \partial_2$	$x_4^d \partial_3$	∂_4	$x_1 \mathcal{R}$
$x_2^{d-1} \partial_1$	$x_1^d \partial_2$			
$x_4 x_2^{d-1} \partial_1$	∂_2			
$x_3 x_2^{d-1} \partial_1$	$x_4^d \partial_2$			
$x_3^{d-1} \partial_1$		$x_2 x_1^{d-1} \partial_3$		
$x_4^{d-1} \partial_1$			$x_3 x_1^{d-1} \partial_4$	
∂_1				$x_4 x_1^{d-1} \mathcal{R}$
$x_4^d \partial_1$				$x_3 x_1^{d-1} \mathcal{R}$
$x_3^d \partial_1$				$x_2 x_1^{d-1} \mathcal{R}$
	$x_1 x_3^{d-1} \partial_2$	$x_2^d \partial_3$		
	$x_3^{d-1} \partial_2$	$x_1^d \partial_3$		
	$x_4 x_3^{d-1} \partial_2$	∂_3		
	$x_1 x_4^{d-1} \partial_2$		$x_3 x_2^{d-1} \partial_4$	
	$x_1 \partial_2$			$x_4 x_2^{d-1} \mathcal{R}$
		$x_2 x_4^{d-1} \partial_3$	$x_3^d \partial_4$	
		$x_1 x_4^{d-1} \partial_3$	$x_2^d \partial_4$	
		$x_4^{d-1} \partial_3$	$x_1^d \partial_4$	
		$x_2 \partial_3$		$x_4 x_3^{d-1} \mathcal{R}$
			$x_3 \partial_4$	$x_4^d \mathcal{R}$
			$x_2 \partial_4$	$x_3^d \mathcal{R}$
			$x_1 \partial_4$	$x_2^d \mathcal{R}$
$x_1^{i_1+1} x_2^{i_2} x_3^{i_3} x_4^{i_4} \partial_1$	$x_1^{i_1} x_2^{i_2+1} x_3^{i_3} x_4^{i_4} \partial_2$	$x_1^{i_1} x_2^{i_2} x_3^{i_3+1} x_4^{i_4} \partial_3$	$x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4+1} \partial_4$	

Table 3.2: Eigenvectors of the linear map $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$, for $n = 4$.

In the above table, each row contains the generators of the same eigenspace.

The operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is given by $A(x_1, \dots, x_n) = (\alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}, \alpha_n x_n)$, where $\alpha_j = \xi^{-(d^{n+1-j} + d^{n-j} + \dots + d)}$, $j = 1, \dots, n$, and ξ is a primitive N -th root of unity. The points $p_i = A^{i-1}(1, \dots, 1)$, for $i = 1, \dots, N$ give all the singular points of the foliation \mathcal{J}_d . Since $A^*(X_{\mathcal{J}_d}) = \xi^d X_{\mathcal{J}_d}$, we have:

$$BB(X_{\mathcal{J}_d}, p_i) = BB(A^*(X_{\mathcal{J}_d}), A^{-1}(p_i)) = BB(X_{\mathcal{J}_d}, p_1).$$

Then, it is enough to calculate the indexes at the point $(1, \dots, 1) \in \mathbb{C}^n$.

In order to know the Baum-Bott indexes of the foliation, we must find the elementary symmetric functions of the eigenvalues of the matrix $DX_{\mathcal{J}_d}(1, \dots, 1)$. We have that

$$DX_{\mathcal{J}_d}(1, \dots, 1) = -\text{Id} + dB,$$

where B is:

$$\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Observe that $B^{n+1} = \text{Id}$ and its characteristic polynomial is $p_B(\lambda) = \lambda^n + \lambda^{n-1} + \dots + 1$. Then the characteristic polynomial of $DX_{\mathcal{J}_d}(1, \dots, 1)$ is

$$p_{\mathcal{J}_d}(\lambda) = \sum_{i=0}^n d^{n-i} (\lambda + 1)^i.$$

Therefore the elementary symmetric functions of the eigenvalues of $DX_{\mathcal{J}_d}(1, \dots, 1)$ are:

$$C_i(DX_{\mathcal{J}_d}(1, \dots, 1)) = (-1)^i \sum_{k=0}^i \binom{n-i+k}{k} d^{i-k}. \quad (3.3.12)$$

In particular, the Baum-Bott indexes are determined by:

$$\begin{aligned} BB_i(\mathcal{J}_d, (1, \dots, 1)) &= \frac{C_1^{n-i}(DX_{\mathcal{J}_d}(1, \dots, 1)) C_i(DX_{\mathcal{J}_d}(1, \dots, 1))}{C_n(DX_{\mathcal{J}_d}(1, \dots, 1))} \\ &= \frac{(d+n)^{n-i} \left(\sum_{k=0}^i \binom{n-i+k}{k} d^{i-k} \right)}{d^n + d^{n-1} + \dots + d + 1}, \quad i = 1, \dots, n-1. \end{aligned}$$

We conclude the following proposition.

Proposition 3.3.1. *Consider the degree $d \geq 2$ Jouanolou foliation \mathcal{J}_d on the projective space \mathbb{P}^n . If $p(\mathcal{J}_d) \in \text{Sing}(\mathcal{J}_d)$, then for $j = 1, \dots, n-1$, we have:*

$$BB_j(\mathcal{J}_d, p(\mathcal{J}_d)) = \frac{(d+n)^{n-j} \left(\sum_{k=0}^j \binom{n-j+k}{k} d^{j-k} \right)}{d^n + d^{n-1} + \dots + d + 1}.$$

In general,

$$\sum_{p(\mathcal{F}) \in \text{Sing}(\mathcal{F})} BB_j(\mathcal{F}, p(\mathcal{F})) = (d+n)^{n-j} \left(\sum_{k=0}^j \binom{n-j+k}{k} d^{j-k} \right),$$

for any one dimensional degree- d foliation \mathcal{F} on the projective space \mathbb{P}^n with only isolated singularities.

3.4 The rank of the local Baum-Bott map at the Jouanolou foliation

Our goal in this section is to estimate the rank of the local Baum-Bott map at the Jouanolou foliation.

In [22], Lins-Neto and Pereira study the rank of the Baum-Bott map at the Jouanolou foliation on the projective plane \mathbb{P}^2 . They stated the following theorem.

Theorem 3.4.1 ([22: Theorem 2]). *For any $d \geq 2$, the rank of the local Baum-Bott map at the Jouanolou foliation \mathcal{J}_d on the projective space \mathbb{P}^2 is at most*

$$\frac{d^2 + 7d - 6}{2}.$$

In particular, if $d = 2, 3$, then $\text{rank}(BB, \mathcal{J}_d) = d^2 + d$, and if $d \geq 4$, then $\text{rank}(BB, \mathcal{J}_d) < d^2 + d$.

We see that for $n = 2$ the rank of the local Baum-Bott map at \mathcal{J}_d is strictly less than the generic rank of the local Baum-Bott, if $d \geq 4$.

From now on, let us consider $n \geq 3$. We will consider the Baum-Bott map defined on $\mathbf{V}_d = H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1))$, which is generated as a \mathbb{C} -vector space by

$$\mathbf{B}_d = \{x^I \partial_k, x^R \mathcal{R} \mid k = 1, \dots, n, I = (i_1, \dots, i_n), R = (r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^n, \\ |I| \leq d \text{ and } |R| = d\}.$$

Since the singularities of $X_{\mathcal{J}_d}$ are non-degenerate, there exists a neighborhood U of $X_{\mathcal{J}_d}$ in \mathbf{V}_d and holomorphic maps $\gamma_j : U \rightarrow \mathbb{C}^n$, $j = 1, \dots, N$, such that $\gamma_j(X_{\mathcal{J}_d}) = p_j$ and $\text{Sing}(X) = \{\gamma_1(X), \dots, \gamma_N(X)\}$, for all $X \in U$. The local Baum-Bott map can be written on U as

$$BB(X) = (BB_1(X, \gamma_1(X)), \dots, B_{n-1}(X, \gamma_1(X)), \dots \\ \dots, BB_1(X, \gamma_N(X)), \dots, BB_{n-1}(X, \gamma_N(X)))$$

In this way, we want to compute the rank of the linear map

$$DBB(X_{\mathcal{J}_d}) : \mathbf{V}_d \rightarrow \mathbb{C}^{(n-1)N}.$$

In the following lemma, we will see that $DBB(X_{\mathcal{J}_d})$ can be encoded in one of the singular points of $X_{\mathcal{J}_d}$:

Lemma 3.4.1. *For any $V \in \mathbf{V}_d$, $k \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, N\}$, we have:*

$$DBB_k(X_{\mathcal{J}_d}, p_{j+1}) \cdot V = \xi^{-d} DBB_k(X_{\mathcal{J}_d}, p_j) \cdot A^*(V),$$

where in the above formula $p_{N+1} = p_1$.

Proof. Let $r > 0$ be such that $X_{\mathcal{J}_d} + tV \in U$, for all $|t| < r$. Set $\rho_j(t) := \gamma_j(X_{\mathcal{J}_d} + tV)$, $|t| < r$ (see Figure 3.2). Since $A^*X_{\mathcal{J}_d} = \xi^d X_{\mathcal{J}_d}$ and $BB_k(A^*X, p) = BB_k(X, A^{-1}(p))$, for all $p \in \text{Sing}(X)$, we have $BB_k(X_{\mathcal{J}_d} + tV, \rho_{j+1}(t)) = BB_k(\xi^d X_{\mathcal{J}_d} + tA^*(V), A^{-1}(\rho_{j+1}(t))) = BB_k(X_{\mathcal{J}_d} + t\xi^{-d}A^*(V), A^{-1}(\rho_{j+1}(t)))$.

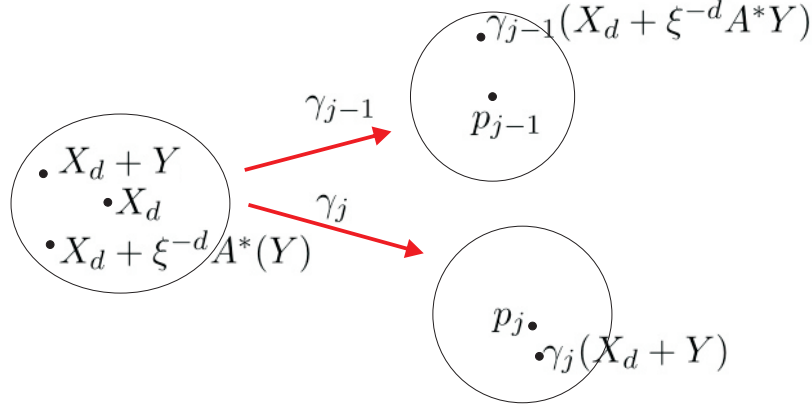


Figure 3.2: The maps γ_{j-1} and γ_j , where $Y = tV$.

Taking the derivative with respect to t at $t = 0$ in the above formula, we get

$$DBB_k(X_{\mathcal{J}_d}, p_{j+1}) \cdot V = DBB_k(X_{\mathcal{J}_d}, p_j) \cdot \xi^{-d}A^*(V) = \xi^{-d}DBB_k(X_{\mathcal{J}_d}, p_j) \cdot A^*(V).$$

■

For simplicity we denote by $T_j := (DBB_1(X_{\mathcal{J}_d}, p_j), \dots, DBB_{n-1}(X_{\mathcal{J}_d}, p_j)) : \mathbf{V}_d \rightarrow \mathbb{C}^{n-1}$, $1 \leq j \leq N$, and $DBB(X_{\mathcal{J}_d}) = T := (T_1, \dots, T_N)$.

As a consequence of Lemma 3.4.1, T can be factorized into a product of a nonsingular matrix and a block diagonal matrix. More specifically, we have:

Corollary 2. For any $j \in \{1, \dots, N\}$ and $V \in \mathbf{V}_d$ we have:

$$T_{j+1}(V) = \xi^{-d}T_j(A^*(V)).$$

In particular:

a) $A^*(\ker(T)) = \ker(T)$,

b) if we set $K_i = E_i \cap \ker(T)$, $i \in \{1, \dots, N\}$, then $\ker(T) = \bigoplus_{i=1}^N K_i$,

c) $E_i \cap \ker(T_1) = K_i$, for all $j = 1, \dots, N$.

$$d) \dim(\ker(T)) = \sum_{i=1}^N \dim(E_i \cap \ker(T_1)) \text{ or equivalently}$$

$$\text{rank } T = \sum_{E_i \neq \emptyset} \text{rank } T_1|_{E_i}.$$

Proof. Items *a)* and *b)* follow from Lemma 3.4.1.

For item *c)*, observe that $V \in E_i \cap \ker(T_1)$ if, and only if, $A^*(V) = \xi^i V$ and $0 = T_1(\xi^{ij} V) = T_1((A^*)^j(V)) = \xi^{jd} T_{1+j}(V)$, for $j = 1, \dots, n-1$, which is equivalent to $V \in E_i \cap \ker(T)$.

For item *d)*, let's consider V an eigenvector of the operator A^* associated to the eigenvalue ξ^i , from Lemma 3.4.1:

$$T_j(V) = \xi^{(j-1)(i-d)} T_1(V).$$

Choosing a base for each eigenspace E_i associated to the eigenvalue ξ^i , the derivative of the Baum-Bott map at the Jouanolou foliation is:

$$\begin{aligned} T &= \begin{bmatrix} T_1|_{E_1} & T_1|_{E_2} & \dots & T_1|_{E_N} \\ T_2|_{E_1} & T_2|_{E_2} & \dots & T_2|_{E_N} \\ \vdots & \vdots & \vdots & \vdots \\ T_N|_{E_1} & T_N|_{E_2} & \dots & T_N|_{E_N} \end{bmatrix} \\ &= \begin{bmatrix} T_1|_{E_1} & T_1|_{E_2} & \dots & T_1|_{E_N} \\ \xi^{(1-d)} T_1|_{E_1} & \xi^{2-d} T_1|_{E_2} & \dots & \xi^{N-d} T_1|_{E_N} \\ \vdots & \vdots & \vdots & \vdots \\ \xi^{(N-1)(1-d)} T_1|_{E_1} & \xi^{(N-1)(2-d)} T_1|_{E_2} & \dots & \xi^{(N-1)(N-d)} T_1|_{E_N} \end{bmatrix} \\ &= M \begin{bmatrix} T_1|_{E_1} & 0 & \dots & 0 \\ 0 & T_1|_{E_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_1|_{E_N} \end{bmatrix}, \end{aligned} \tag{3.4.13}$$

where the matrix $M \in M(\mathbb{C}, (n-1)N)$:

$$M = \begin{bmatrix} \text{Id} & \text{Id} & \dots & \text{Id} \\ \xi^{(1-d)} \text{Id} & \xi^{2-d} \text{Id} & \dots & \xi^{N-d} \text{Id} \\ \vdots & \vdots & \vdots & \vdots \\ \xi^{(N-1)(1-d)} \text{Id} & \xi^{(N-1)(2-d)} \text{Id} & \dots & \xi^{(N-1)(N-d)} \text{Id} \end{bmatrix}.$$

and Id is the $(n-1) \times (n-1)$ identity matrix. Since $\det(M) \neq 0$, we can calculate the rank of the linear map T in terms of the rank of the linear map T_1 , which proves *(d)*. ■

Corollary 2 tell us that in order to calculate the rank of T , it is enough to calculate T_1 in each eigenspace of A^* . Recall that $\tilde{S}^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$, given by (3.2.7), (3.2.8) and (3.2.9), maps eigenspaces of A^* into eigenspaces (see Lemma 3.2.4). As a consequence of this fact, it is enough to study T at some convenient eigenspaces:

Corollary 3. *We have $T(\tilde{S}^*(V)) = T(V)$, for all $V \in \mathbf{V}_d$. Moreover, $E_d \subset \ker(T)$.*

Remark 3.4.1. *Note that $x_2^d \partial_1, \dots, x_n^d \partial_{n-1}, \partial_n, x_1^d \mathcal{R} \in E_d$ are the monomials of the vector field $X_{\mathcal{J}_d}$.*

Proof. By equations (3.2.4), (3.2.5) and (3.2.6), if Q is a polynomial in n variables of degree at most d , then we get:

$$\begin{aligned} x_n^{d-1} S^*(X_{\mathcal{J}_d} + tQ\partial_1) &= X_{\mathcal{J}_d} - tQ\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \mathcal{R}, \\ x_n^{d-1} S^*(X_{\mathcal{J}_d} + tQ\partial_k) &= X_{\mathcal{J}_d} + tQ\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \partial_{k-1}, \text{ for } k \geq 2, \text{ and} \\ x_n^{d-1} S^*(X_{\mathcal{J}_d} - tQ\mathcal{R}) &= X_{\mathcal{J}_d} + tQ\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \partial_n, \end{aligned}$$

and the singularity $S^{-1}\rho_1(t)$ goes to p_1 as t goes to zero. This implies

$$\begin{aligned} T(Q\partial_1) &= T\left(-Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \mathcal{R}\right), \\ T(Q\partial_k) &= T\left(Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \partial_{k-1}\right), \text{ for } 2 \leq k \leq n, \\ T(-Q\mathcal{R}) &= T\left(Q\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) x_n^d \partial_n\right), \end{aligned}$$

and the first part of the corollary follows. Finally, since $BB(X_{\mathcal{J}_d} + tX_{\mathcal{J}_d}, \rho_1(t)) = BB(X_{\mathcal{J}_d}, p_1)$, we get $T(X_{\mathcal{J}_d}) = 0$. From (3.2.11), we have

$$T(X_{\mathcal{J}_d}) = T\left(\sum_{k=0}^n (\tilde{S}^*)^k(\partial_n)\right) = (n+1)T(\partial_n),$$

then $T(\partial_n) = 0$ and $E_d \subset \ker(T)$. ■

From now on, we are going to use the identifications made in Lemma 3.2.5. We denote $\partial_k = \partial_{k \bmod (n+1)}$. Then, the corollary above is read as:

Lemma 3.4.2. *Given $I \in \mathbb{Z}_{\geq 0}^{n+1}$, we have:*

$$T_1(x^I \partial_k) = T_1(x^J \partial_{k-1}), \text{ for } k = 1, \dots, n+1, \quad (3.4.14)$$

where $J = \mathbb{S}^{-1}(I)$.

Given a polynomial vector field V , define $\mathcal{P}(V) = \{x^I V \in \mathbf{B}_d \mid I \in \mathbb{Z}_{\geq 0}^n \text{ and } |I| \leq d\}$. By the relation above, to calculate the map T_1 on a vector, we can compute it on a more

convenient one, at some vector in $\mathcal{P}(\partial_1)$. In order to find the rank of the linear map T_1 at vectors in $\mathcal{P}(\partial_1)$, we need to find the derivative of the elementary symmetric function of the eigenvalues with respect to those vectors. The calculations are easier if the vector is zero at the singular point p_1 . Let us see it in the following lemma:

Lemma 3.4.3. *Let Q be a polynomial in n variables such that $Q(p_1) = 0$ and let t be a complex number, small enough. Let $X(t) = X_{\mathcal{J}_d} + tQ\partial_1$, with singularity at p_1 , for all $t \in \mathbb{C}$, and set $C_i(t) = C_i(DX(t)(p_1))$. Then, we have:*

$$\begin{aligned} C'_1(0) &= \partial_1 Q(p_1), \\ C'_i(0) &= (-1)^i \left(-\binom{n-1}{i-1} \partial_1 Q(p_1) + \right. \\ &\quad \left. + \sum_{k=0}^{i-2} \binom{n-i+k}{k} (\partial_2 Q(p_1) + \dots + \partial_{n-i+k+2} Q(p_1)) d^{i-k-1} \right), \end{aligned} \quad (3.4.15)$$

for $i = 2, \dots, n$.

Proof. Let $P(\lambda) = \det(\lambda Id - DX(t)(p_1))$ be the characteristic polynomial of the matrix $DX(t)(p_1)$, then it is equal to the following determinant:

$$\begin{vmatrix} (\lambda + (d+1) - t\partial_1 Q) & (-d - t\partial_2 Q) & -t\partial_3 Q & -t\partial_4 Q & \dots & -t\partial_{n-1} Q & -t\partial_n Q \\ d & (\lambda + 1) & -d & 0 & \dots & 0 & 0 \\ d & 0 & (\lambda + 1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \dots & -d & 0 \\ d & 0 & 0 & 0 & \dots & (\lambda + 1) & -d \\ d & 0 & 0 & 0 & \dots & 0 & (\lambda + 1) \end{vmatrix},$$

where each $\partial_i Q$ is evaluated at p_1 , for $i = 1, \dots, n$. Let us denote $W_1 = \lambda + (d+1) - t\partial_1 Q$, $W_2 = -d - t\partial_2 Q$, and $W_i = -t\partial_i Q$, for $i = 3, \dots, n$. We divide the last row by $\lambda + 1$, hence we have

$$P(\lambda) = (\lambda + 1) \begin{vmatrix} W_1 & W_2 & W_3 & W_4 & \dots & W_{n-1} & W_n \\ d & (\lambda + 1) & -d & 0 & \dots & 0 & 0 \\ d & 0 & (\lambda + 1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \dots & -d & 0 \\ d & 0 & 0 & 0 & \dots & (\lambda + 1) & -d \\ \left(\frac{d}{\lambda + 1}\right)^1 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

We add to the first row the n -th row multiplied by $-W_n$, and to the $(n-1)$ -th row we add the last row times d , then to the resulting $(n-1)$ -th row, we divide it by $\lambda+1$. Then

$$P(\lambda) = (\lambda+1)^2 \begin{vmatrix} \left(W_1 - \left(\frac{d}{\lambda+1}\right)W_n\right) & W_2 & W_3 & W_4 & \dots & W_{n-2} & W_{n-1} \\ d & (\lambda+1) & -d & 0 & \dots & 0 & 0 \\ d & 0 & (\lambda+1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \dots & (\lambda+1) & -d \\ \left(\frac{d}{\lambda+1} + \left(\frac{d}{\lambda+1}\right)^2\right) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Doing so, we arrive to the expression

$$\begin{aligned} P(\lambda) &= (\lambda+1)^{n-1} \begin{vmatrix} \left(W_1 - \left(\frac{d}{\lambda+1}\right)W_n - \dots - \left(\frac{d}{\lambda+1}\right)^{n-2}W_3\right) & W_2 \\ \left(\frac{d}{\lambda+1} + \dots + \left(\frac{d}{\lambda+1}\right)^{n-1}\right) & 1 \end{vmatrix} \\ &= (\lambda+1)^{n-1}W_1 - d(\lambda+1)^{n-2}W_n + \dots + \\ &\quad - (d(\lambda+1)^{n-2} + d^2(\lambda+1)^{n-3} + \dots + d^{n-2}(\lambda+1))W_3 + \\ &\quad - (d(\lambda+1)^{n-2} + d^2(\lambda+1)^{n-3} + \dots + d^{n-2}(\lambda+1) + d^{n-1})W_2. \end{aligned}$$

Therefore

$$\begin{aligned} P(\lambda) &= (\lambda+1)^n + (\lambda+1)^{n-1}(d - t\partial_1 Q) + d(\lambda+1)^{n-2}(d + t(\partial_2 Q \dots + \partial_n Q)) + \\ &\quad + d^2(\lambda+1)^{n-3}(d + t(\partial_2 Q \dots + \partial_{n-1} Q)) + \dots + \\ &\quad + d^{n-2}(\lambda+1)(d + t(\partial_2 Q + \partial_3 Q)) + d^{n-1}(d + t\partial_2 Q). \end{aligned}$$

It follows that,

$$\begin{aligned} \partial_t P(\lambda)|_{t=0} &= (\lambda+1)^{n-1}(-\partial_1 Q(p_1)) + d(\lambda+1)^{n-2}(\partial_2 Q(p_1) + \dots + \partial_n Q(p_1)) + \\ &\quad + d^2(\lambda+1)^{n-3}(\partial_2 Q(p_1) + \dots + \partial_{n-1} Q(p_1)) + \dots + \\ &\quad + d^{n-2}(\lambda+1)(\partial_2 Q(p_1) + \partial_3 Q(p_1)) + d^{n-1}\partial_2 Q(p_1). \end{aligned}$$

We know that $P(\lambda) = \sum_{i=0}^n (-1)^i \lambda^{n-i} C_i(t)$, then $\partial_t P(\lambda)|_{t=0} = \sum_{i=1}^n (-1)^i \lambda^{n-i} C'_i(0)$, and we obtain the formulas (3.4.15). ■

Now we can identify which vectors are not in the kernel of the linear map T_1 :

Lemma 3.4.4. *Let d and n be integer numbers with $d \geq 2$, and $n \geq 3$. Let Q be the monomial $Q = x_1^{i_1} \dots x_n^{i_n}$ of degree at most d . If $Q \notin \{x_2^d, x_1^{r+1}x_2^r \dots x_n^r\}$, then $Q\partial_1 \notin \ker(T_1)$. The number r is the solution of $d = (n+1)r + 1$.*

Proof. By contradiction, suppose that $Q\partial_1 \in \ker(T_1)$. Then $(Q - x_2^d)\partial_1 \in \ker(T_1)$. Let $C_i(t) = C_i(D(X_{\mathcal{J}_d} + t(Q - x_2^d)\partial_1)(p_1))$. Observe that $p_1 \in \text{Sing}(X_{\mathcal{J}_d} + t(Q - x_2^d)\partial_1)$. The following relations hold:

$$i \frac{C_1'(0)}{C_1(0)} - \frac{C_i'(0)}{C_i(0)} = 0, \text{ for } i = 2, \dots, n.$$

Denote $i_{n+1} = d - (i_1 + \dots + i_n)$, then $i_2 = d - i_1 - (i_3 + \dots + i_{n+1})$. We use equations (3.4.15) and (3.3.12). We have:

$$(i_{n+1} - i_1)d + n(i_1 + i_{n+1}) - i_1(n-1) = 0,$$

and for $i = 3, \dots, n$:

$$\begin{aligned} & (i_1(1-i) + i_{n-i+3} + \dots + i_{n+1})d^{i-1} + \sum_{k=1}^{i-2} \binom{n-i+k}{k} ((1-i)i_1 + i_{n-i+3+k} + \dots + i_{n+1}) + \\ & + n \binom{n-i+k-1}{k-1} (i_1 + i_{n-i+2+k} + \dots + i_{n+1}) d^{i-k-1} + \binom{n-2}{i-2} (ni_{n+1} + i_1) = 0. \end{aligned} \quad (3.4.16)$$

From the first equation,

$$i_1 + ni_{n+1} = (i_1 - i_{n+1})d \geq 0, \quad (3.4.17)$$

$$i_1(d-1) = i_{n+1}(d+n) \quad (3.4.18)$$

For $i = 3$, we replace (3.4.17) in (3.4.16), and we get

$$(-2i_1 + i_n + i_{n+1})d + (2i_1 + n(i_n + i_{n+1})) = 0.$$

This is equivalent to $2i_1(d-1) = (i_n + i_{n+1})(d+n)$, and using (3.4.18) give us $i_n = i_{n+1}$. For $i = 4$, we replace i_n by i_{n+1} . From (3.4.17) and (3.4.18) we conclude that $i_{n-1} = i_{n+1}$. Doing so, we get $i_3 = \dots = i_{n+1}$. Now, from (3.4.17) we obtain $(i_1 - i_{n+1})(d-1) = (n+1)i_{n+1}$. We know that $d = i_1 + i_2 + \dots + i_{n+1}$, hence $d = (i_1 - i_{n+1})d + i_2 - i_{n+1}$. If $i_1 = i_{n+1}$, then $i_2 = d$. If $i_1 > i_{n+1}$, then $i_1 = i_{n+1} + 1$ and $i_2 = i_{n+1}$, thus $d = (n+1)i_{n+1} + 1$.

■

Let us compute the rank of T_1 restricted to the eigenspaces of dimension one.

Lemma 3.4.5. *The total rank of T_1 at the eigenspaces of dimension one of $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$ is:*

$$\sum_{\dim E_i=1} \text{rank}(T_1|_{E_i}) = (n+1) \left[\binom{n+d-1}{n-1} - 3(n-1) \right].$$

Proof. Item 4 of Theorem 3.2.1 gives us the structure of the eigenspaces of dimension one. We use Corollary 3, and we have the following equality:

$$\sum_{\dim E_i=1} \text{rank}(T_1|_{E_i}) = (n+1) \sum_{x^I \notin D} \text{rank}(T_1(x^I \partial_1)),$$

where $x^I \partial_1 \in \mathbf{B}_d$, $D = \{x^I, 1, x_2^{d-1}x_j, x_j^{d-1}, x_j^d \mid I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n \text{ with } i_1 > 0, \text{ and } j = 2, \dots, n\}$. Since $x_2^d, x_1^{r+1}x_2^r \dots x_n^r \in D$, then $(n+1) \sum_{x^I \notin D} \text{rank}(T_1(x^I \partial_1))$ is the number of eigenspaces of dimension 1. \blacksquare

We will calculate the rank of the linear map T_1 restricted to eigenspaces of dimension two.

From Item 3 of Theorem 3.2.1 and Corollary 3, we have:

Lemma 3.4.6. *Consider the eigenspaces of dimension two of the operator $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$. If n is even, then*

$$\sum_{\substack{j=1 \\ \dim E_j=2}}^N \text{rank } T_1|_{E_j} = (n+1) \left\{ \sum_{r=2}^{\frac{n}{2}} \text{rank } T_1[x_{n+1}x_{r+1}^{d-1}\partial_1, x_r x_1^{d-1}\partial_{r+1}] \right. \\ \left. + \sum_{r=3}^{n+1} \text{rank } T_1[x_r x_2^{d-1}\partial_1, x_{r+1}^d \partial_2] \right\}.$$

If n is odd, then

$$\sum_{\substack{j=1 \\ \dim E_j=2}}^N \text{rank } T_1|_{E_j} = (n+1) \left\{ \sum_{r=2}^{\frac{n-1}{2}} \text{rank } T_1[x_{n+1}x_{r+1}^{d-1}\partial_1, x_r x_1^{d-1}\partial_{r+1}] + \right. \\ \left. + \sum_{r=3}^{n+1} \text{rank } T_1[x_r x_2^{d-1}\partial_1, x_{r+1}^d \partial_2] \right\} + \frac{n+1}{2} \text{rank } T_1[x_{n+1}x_{\frac{n+3}{2}}^{d-1}\partial_1].$$

In the above formulas $T_1[\cdot, \cdot]$ denotes the restriction of T_1 to the space $\langle \cdot, \cdot \rangle$.

Now, let us analyze the rank of the linear map T_1 restricted to the eigenspaces of dimension two of the form $\langle x_r x_2^{d-1} \partial_1, x_{r+1}^d \partial_2 \rangle$:

Lemma 3.4.7. *For $r = 3, \dots, n+1$, the rank of T_1 restricted to the space generated by $x_r x_2^{d-1} \partial_1, x_{r+1}^d \partial_2$ is one.*

Proof. Lemma 3.4.14 tells us that

$$\text{rank } T_1[x_r x_2^{d-1} \partial_1, x_{r+1}^d \partial_2] = \text{rank } T_1[x_r x_2^{d-1} \partial_1, x_r^d \partial_1].$$

Let us define $Q(x_1, \dots, x_n) = dx_r x_2^{d-1} - x_r^d - (d-1)x_2^d$, for $r = 3, \dots, n$, and $Q(x_1, \dots, x_n) = dx_2^{d-1} - 1 - (d-1)x_2^d$, in the case $r = n+1$. Observe that $p_1 \in \text{Sing}(X_{j_d} + tQ\partial_1)$, for all $t \in \mathbb{C}$. Then, $\det(\lambda \text{Id} + D(X_{j_d} + tQ\partial_1)(p_1)) = \det(\lambda \text{Id} + D X_{j_d}(p_1))$, this implies that $BB(X_{j_d} + tQ\partial_1, p_1) = BB(X_{j_d}, p_1)$, for all $t \in \mathbb{C}$, hence $Q\partial_1 \in \ker(T_1)$. Therefore, $\text{rank } T_1[x_r x_2^{d-1} \partial_1, x_{r+1}^d \partial_2] \leq 1$, for $r = 3, \dots, n+1$. From Lemma 3.4.4, we conclude that $\text{rank } T_1[x_r x_2^{d-1} \partial_1, x_{r+1}^d \partial_2] = 1$. ■

We can recognize which eigenspaces of dimension two have maximum rank:

Lemma 3.4.8. *Let $n \geq 3$. If $n = 2K + 1$, or $n = 2K$, for some positive integer K , then*

$$\text{rank } T_1[x_{r+1}^{d-1} x_{n+1} \partial_1, x_r x_1^{d-1} \partial_{r+1}] = 2, \text{ for } r = 2, \dots, K.$$

Proof. By relation (3.4.14), $T_1(x_r x_1^{d-1} \partial_{r+1}) = T_1(x_{n-r+2}^{d-1} x_{n+1} \partial_1)$. Since $2 \leq r \leq K$, $x_{n-r+2}^{d-1} x_{n+1}$ and $x_{r+1}^{d-1} x_{n+1}$ are different polynomials. We realize that $r+1 < n-r+2$. Suppose there exist some $\alpha \in \mathbb{C}$ such that $T_1(x_{r+1}^{d-1} x_{n+1} \partial_1) = \alpha T_1(x_{n-r+2}^{d-1} x_{n+1} \partial_1)$. Let $Q_1 = x_{r+1}^{d-1} x_{n+1} - x_2^d$, and $Q_2 = x_{n-r+2}^{d-1} x_{n+1} - x_2^d$. Then $\partial_2 Q_1 = -d, \partial_{r+1} Q_1 = d-1, \partial_2 Q_2 = -d, \partial_{n-r+2} Q_2 = d-1$. It follows that $T_1(Q_1 \partial_1) = \alpha T_1(Q_2 \partial_1)$. Observe that $p_1 \in \text{Sing}(X_{j_d} + tQ_j \partial_1)$, for $j = 1, 2, t \in \mathbb{C}$. Let us denote $C_{i, Q_j}(t) = C_i(D(X_{j_d} + tQ_j \partial_1)(p_1))$. We use (3.4.15) to get $C'_{i, Q_1}(0) = \alpha C'_{i, Q_2}(0)$, for $i = 2, \dots, n$. If $i = 2$, then $C'_{2, Q_1}(0) = d(\partial_2 Q_1 + \dots + \partial_n Q_1) = -d = C'_{2, Q_2}(0)$, hence $\alpha = 1$. If $i = r+1$, then $C'_{r+1, Q_1}(0) - C'_{r+1, Q_2}(0) = (-1)^{r+2} (d+1)d^r$, which give us a contradiction. ■

We conclude the study of eigenspaces of dimension 2 with the following lemma:

Lemma 3.4.9. *The total rank at the eigenspaces of dimension 2 of A^* is:*

$$\sum_{\substack{j=1 \\ \dim E_j=2}}^N \text{rank } T_1|_{E_j} = \begin{cases} (n+1)(2n-3) & , \text{ if } n \text{ is even.} \\ (n+1)(2n-4) + \frac{n+1}{2} & , \text{ if } n \text{ is odd.} \end{cases}$$

Proof. We replace Lemma 3.4.7 and Lemma 3.4.8 in Lemma 3.4.6. ■

Finally, let us study T_1 at eigenspaces of dimension n . We will need some notations. Let $J = \{j_1, \dots, j_r\}$ be an ordered set and V_j , for $j \in J$, be vectors of same dimension. We denote $[V_j]_{j \in J} = [V_{j_1} \dots V_{j_r}]$, the matrix whose column vectors are V_{j_1}, \dots, V_{j_r} . If $n, d \in \mathbb{Z}$, and $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$, we define the linear transformation $M_{n,d}(I) : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ given

by the matrix:

$$M_{n,d}(I) = \left[\begin{array}{c} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d+1) \\ i_{j+3} - i_{j+2} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{array} \right]_{1 \leq j \leq n},$$

where $i_{n+1} = d - 1 - (i_1 + \dots + i_n)$. We are identifying $i_j = i_{j \bmod (n+1)}$.

We can relate the rank of the linear map T_1 restricted to an eigenspace of dimension n with the exponents of the monomial that defines the eigenspace.

Lemma 3.4.10. *Let n, d be integer numbers such that $d \geq 2$ and $n \geq 3$. Then for all monomials $x^I = x_1^{i_1} \dots x_n^{i_n}$ of degree at most $d - 1$, we have*

$$\text{rank } T_1|_{\{x_j x^I \partial_j | j=1, \dots, n\}} = \text{rank } M_{n,d}(I),$$

Proof. Let i_1, \dots, i_n be non-negative integer numbers such that $i_1 + \dots + i_n \leq d - 1$. Let $x^I = x_1^{i_1} \dots x_n^{i_n}$ be the monomial and let $i_{n+1} = d - 1 - (i_1 + \dots + i_n)$. By identification (3.4.14), we have

$$\begin{aligned} T_1(x_2 x^I \partial_2) &= T_1(x_1^{i_2+1} x_2^{i_3} \dots x_n^{i_n} x_{n+1}^{i_1} \partial_1), \\ T_1(x_3 x^I \partial_3) &= T_1(x_1^{i_3+1} x_2^{i_4} \dots x_n^{i_n} x_{n+1}^{i_2} \partial_1), \dots, \\ T_1(x_n x^I \partial_n) &= T_1(x_1^{i_n+1} x_2^{i_{n+1}} \dots x_{n+1}^{i_{n-1}} \partial_1). \end{aligned}$$

Let us still denote $i_j = i_{j \bmod (n+1)}$, and let Q_j be the polynomial $Q_j(x_1, \dots, x_{n+1}) = x_1^{i_j+1} x_2^{i_{j+1}} \dots x_n^{i_{j+n-1}} x_{n+1}^{i_{j+n}} - x_2^d$, $C_i(0) = C_i(D(X_{\mathcal{J}_d}(p_1)))$, and $C_{i,Q_j}(t) = C_i(D(X_{\mathcal{J}_d} + tQ_j \partial_1)(p_1))$. Observe that $p_1 \in \text{Sing}(X_{\mathcal{J}_d} + tQ_j \partial_1)$, for $i = 1, \dots, n - 1$, $j = 1, \dots, n$ and $t \in \mathbb{C}$.

We see that the linear transformation T_1 restricted to space generated by the set of eigenvectors $\{x_j x^I \partial_j | j = 1, \dots, n\}$ has the same rank as the following matrix

$$\left[\begin{array}{cccc} \left(2 \frac{C'_{1,Q_1}(0)}{C_1(0)} - \frac{C'_{2,Q_1}(0)}{C_2(0)} \right) & \left(2 \frac{C'_{1,Q_2}(0)}{C_1(0)} - \frac{C'_{2,Q_2}(0)}{C_2(0)} \right) & \dots & \left(2 \frac{C'_{1,Q_n}(0)}{C_1(0)} - \frac{C'_{2,Q_n}(0)}{C_2(0)} \right) \\ \left(3 \frac{C'_{1,Q_1}(0)}{C_1(0)} - \frac{C'_{3,Q_1}(0)}{C_3(0)} \right) & \left(3 \frac{C'_{1,Q_2}(0)}{C_1(0)} - \frac{C'_{3,Q_2}(0)}{C_3(0)} \right) & \dots & \left(3 \frac{C'_{1,Q_n}(0)}{C_1(0)} - \frac{C'_{3,Q_n}(0)}{C_3(0)} \right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(n \frac{C'_{1,Q_1}(0)}{C_1(0)} - \frac{C'_{n,Q_1}(0)}{C_n(0)} \right) & \left(n \frac{C'_{1,Q_2}(0)}{C_1(0)} - \frac{C'_{n,Q_2}(0)}{C_n(0)} \right) & \dots & \left(n \frac{C'_{1,Q_n}(0)}{C_1(0)} - \frac{C'_{n,Q_n}(0)}{C_n(0)} \right) \end{array} \right]. \quad (3.4.19)$$

Since $Q_j(p_1) = 0$, we can use the formulas given by (3.4.15):

$$\partial_1 Q_j = i_j + 1, \partial_2 Q_j = i_{j+1} - d, \partial_k Q_j = i_{j+k-1}, \text{ for } k = 3, \dots, n+1, \text{ and } j = 1, \dots, n.$$

Therefore for $i = 2, \dots, n$ and $j = 1, \dots, n$, we get

$$\begin{aligned} iC'_i(0)C'_{1,Q_j}(0) - C_1(0)C'_{i,Q_j}(0) &= i(i_j + 1) \sum_{k=0}^i \binom{n-i+k}{k} d^{i-k} + \\ &+ (d+n) \left(\sum_{k=0}^{i-2} \binom{n-i+k}{k} (i_{j+1} - d + i_{j+2} + \dots + i_{j+n+k-i+1}) d^{i-k-1} - \binom{n-1}{i-1} (i_j + 1) \right). \end{aligned}$$

We replace $i_{j+1} - d + i_{j+2} + \dots + i_{j+n+k-i+1}$ by $-(i_j + 1 + i_{j+n+k-i+2} + \dots + i_{j+n})$ in the above expression, hence we get

$$\begin{aligned} 2C_2(0)C'_{1,Q_j}(0) - C_1(0)C'_{2,Q_j}(0) &= d((i_j + 1 - i_{j+n})d - (i_j + 1 + ni_{j+n})), \\ iC'_i(0)C'_{1,Q_j}(0) - C_1(0)C'_{i,Q_j}(0) &= d(((i-1)(i_j + 1) - (i_{j+n-i+2} + \dots + i_{j+n}))d^{i-1} + \\ &+ \sum_{k=1}^{i-2} \left(\binom{n-i+k}{k} ((i-1)(i_j + 1) - (i_{j+n+k-i+2} + \dots + i_{j+n})) + \right. \\ &\left. - n \binom{n-i+k-1}{k-1} (i_j + 1 + i_{j+n+k-i+1} + \dots + i_{j+n}) \right) d^{i-k-1} + \\ &\left. - (i_j + 1 + ni_{j+n}) \binom{n-2}{i-2} \right). \end{aligned} \quad (3.4.20)$$

We observe that we have to calculate the rank of the matrix:

$$\left[\begin{array}{c} (i_j + 1 - i_{j+n})d - (i_j + 1 + ni_{j+n}) \\ (2(i_j + 1) - (i_{j+n-1} + i_{j+n}))d^2 + ((n-2)(2(i_j + 1) - i_{j+n}) - n(i_j + 1 + i_{j+n-1} + i_{j+n}))d - (i_j + 1 + ni_{j+n})(n-2) \\ \frac{1}{d}(3C_3(0)C'_{1,Q_j}(0) - C_1(0)C'_{3,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0) - C_1(0)C'_{n,Q_j}(0)) \end{array} \right]_{1 \leq j \leq n}.$$

To the second row, we add the first row multiplied by $-(n-2)$. The resulting second row, we divide it by d , so the matrix is similar to

$$\left[\begin{array}{c} d(i_j + 1 - i_{j+n}) - (i_j + 1 + ni_{j+n}) \\ (2(i_j + 1) - (i_{j+n-1} + i_{j+n}))d - (2(i_j + 1) + n(i_{j+n-1} + i_{j+n})) \\ \frac{1}{d}(3C_3(0)C'_{1,Q_j}(0) - C_1(0)C'_{3,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0) - C_1(0)C'_{n,Q_j}(0)) \end{array} \right]_{1 \leq j \leq n}.$$

To the second row, we add the first row multiplied by -2 . The resulting second row we divide it by $(d+n)$, so the matrix is similar to

$$\begin{bmatrix} (i_j + 1 - i_{j+n})d - (i_j + 1 + ni_{j+n}) \\ i_{j+n} - i_{j+n-1} \\ \frac{1}{d}(3C_3(0)C'_{1,Q_j}(0) - C_1(0)C'_{3,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0) - C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1 \leq j \leq n}.$$

The third row of the matrix is:

$$\begin{aligned} & 3(i_j + 1) - (i_{j+n-2} + i_{j+n-1} + i_{j+n})d^3 + \\ & + ((n-3)(3(i_j + 1) - (i_{j+n-1} + i_{j+n-1} + i_{j+n})) - n(i_j + 1 + i_{j+n-2} + i_{j+n-1} + i_{j+n}))d^2 + \\ & + \left(\binom{n-2}{2}\right)(3(i_j + 1) - i_{j+n}) - n(n-3)(i_j + 1 + i_{j+n-1} + i_{j+n})d + \\ & - (i_j + 1 + ni_{j+n})\binom{n-2}{2}. \end{aligned}$$

We add $-(d^3 + (2n-3)d^2 + n(n-3)d)$ times the second row to the third one, then we add $-\binom{n-2}{2}$ times the first row to the third one, the resulting row we divide it by d . So the third row becomes

$$\begin{aligned} & (3(i_j + 1) - (i_{j+n-2} + 2i_{j+n}))d^2 + \\ & + ((n-3)(3(i_j + 1) - 2i_{j+n}) - n(i_j + 1 + i_{j+n-2} + 2i_{j+n}))d - 2(n-3)(i_j + 1 + ni_{j+n}). \end{aligned}$$

We add $-2(n-3)$ times the first row to the third one, divide the resulting row by d , then add -3 times the first row to the third one. The resulting third row we divide it by $(d+n)$ and becomes

$$i_{j+n} - i_{j+n-2}.$$

In this way, the matrix 3.4.19 is similar to

$$\begin{bmatrix} (i_j + 1 - i_{j+n})d - (i_j + 1 + ni_{j+n}) \\ i_{j+n} - i_{j+n-1} \\ i_{j+n} - i_{j+n-2} \\ \frac{1}{d}(4C_4(0)C'_{1,Q_j}(0) - C_1(0)C'_{4,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0) - C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1 \leq j \leq n}.$$

We proceed analogously with the following rows and realize that the matrix (3.4.19) is similar to

$$\left[\begin{array}{c} (i_j - i_{j+n})(d-1) + d - 1 - (n+1)i_{j+n} \\ i_{j+n} - i_{j+n-1} \\ \dots \\ i_{j+n} - i_{j+2} \end{array} \right]_{1 \leq j \leq n}.$$

Since $d-1 = i_j - i_{j+n} + i_{j+1} - i_{j+n} + \dots + i_{j+n-1} - i_{j+n} + (n+1)i_{j+n}$, then the matrix (3.4.19) is similar to

$$\left[\begin{array}{c} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d+1) \\ i_{j+3} - i_{j+2} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{array} \right]_{1 \leq j \leq n}.$$

■

By the preceding lemma, in some cases appears another eigenspace that is in the kernel of the linear map T_1 .

Lemma 3.4.11. *Let n be a integer number such that $n \geq 3$ and $d = (n+1)r + 1$, for some positive integer r . Then $x_j x_1^r x_2^r \dots x_n^r \partial_j$ is in the kernel of $T_1 : \mathbf{V}_d \rightarrow \mathbb{C}^{n-1}$, for $j = 1, \dots, n$.*

In the case $d = 2$, the linear map T_1 restricted to eigenspaces of dimension n has maximum possible rank, more specifically:

Lemma 3.4.12. *Let $n \geq 3$, $d = 2$ and E an eigenspace of dimension n of $A^* : \mathbf{V}_2 \rightarrow \mathbf{V}_2$. Then:*

$$\text{rank } T_1|_E = n - 1.$$

Proof. Let E be an eigenspace of dimension n of A^* . Then by Item 2 of Theorem 3.2.1 $E = \langle \{x_j x^I \partial_j \mid j = 1, \dots, n\} \rangle$, for some $I \in \mathbb{Z}_{\geq 0}^n$ with $|I| \leq d-1$. Lemma 3.4.10 tells us that $\text{rank } T_1|_{\{x_j x^I \partial_j \mid j=1, \dots, n\}}$ is the same as

$$\left[\begin{array}{c} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d+1) \\ i_{j+3} - i_{j+2} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{array} \right]_{1 \leq j \leq n} = \left[\begin{array}{ccc} d+1 & d & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & 0_{(n-2) \times 1} & \text{Id}_{n-2} \end{array} \right] \left[\begin{array}{c} i_{j+1} - i_j \\ i_{j+2} - i_{j+1} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{array} \right]_{1 \leq j \leq n},$$

where $i_1 + \dots + i_{n+1} = d-1$. If we demonstrate that $\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} \neq 0$, then the lemma is proved. In fact, we see that the determinant of the matrix $[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n}$ is:

$$(-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} i_{n+1} - i_n & i_n - i_{n-1} & i_{n-1} - i_{n-2} & \dots & i_4 - i_3 & i_3 - i_2 & i_2 - i_1 \\ i_1 - i_{n+1} & i_{n+1} - i_n & i_n - i_{n-1} & \dots & i_5 - i_4 & i_4 - i_3 & i_3 - i_2 \\ i_2 - i_1 & i_1 - i_{n+1} & i_{n+1} - i_n & \dots & i_6 - i_5 & i_5 - i_4 & i_4 - i_3 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ i_{n-3} - i_{n-4} & i_{n-4} - i_{n-5} & i_{n-5} - i_{n-6} & \dots & i_{n+1} - i_n & i_n - i_{n-1} & i_{n-1} - i_{n-2} \\ i_{n-2} - i_{n-3} & i_{n-3} - i_{n-4} & i_{n-4} - i_{n-5} & \dots & i_1 - i_{n+1} & i_{n+1} - i_n & i_n - i_{n-1} \\ i_{n-1} - i_{n-2} & i_{n-2} - i_{n-3} & i_{n-3} - i_{n-4} & \dots & i_2 - i_1 & i_1 - i_{n+1} & i_{n+1} - i_n \end{vmatrix}$$

Since $d = 2$, $i_1 + \dots + i_{n+1} = 1$. Let's analyze all possible subcases:

- If $i_{n+1} = 1$, then

$$\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix} \neq 0.$$

- If $i_n = 1$:

$$\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{vmatrix} \neq 0.$$

- If $i_1 = 1$:

$$\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \end{vmatrix} \neq 0.$$

- If $i_{n-1} = 1$:

$$\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{vmatrix} \neq 0.$$

- If $i_k = 1$, $2 \leq k \leq n - 2$. Let's denote:

$$M_k = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix}_{k \times k}.$$

Then

$$\det[(i_{j+i} - i_{j+i-1})]_{1 \leq i, j \leq n} = (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 0_{k \times n-k} & M_k \\ -M_{n-k}^T & 0_{(n-k) \times k} \end{vmatrix} \neq 0,$$

where M_{n-k}^T denotes the transpose of the matrix M_{n-k} . ■

Gathering all the information, we can estimate the rank of the local Baum-Bott map at the Jouanolou foliation:

Theorem 3.4.2. *The rank of the local Baum-Bott map $BB : \text{Fol}_{red}(n, d) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree- d Jouanolou foliation is:*

$$\begin{aligned} \dim \mathbf{V}_d - n(n+1) - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^n \\ |I| \leq d-1}} \dim \ker(M_{n,d}(I)) & \quad , \text{ if } n \text{ is even.} \\ \dim \mathbf{V}_d - n(n+1) - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^n \\ |I| \leq d-1}} \dim \ker(M_{n,d}(I)) - \frac{(n+1)}{2} & \quad , \text{ if } n \text{ is odd.} \end{aligned}$$

Proof. By Corollary 2, the rank of the Baum-Bott map at the Jouanolou foliation is the sum of ranks of the linear transformation $T_1 : \mathbf{V}_d \rightarrow \mathbb{C}^{n-1}$ restricted to the eigenspaces of the linear operator $A^* : \mathbf{V}_d \rightarrow \mathbf{V}_d$. By Theorem 3.2.1, there are only eigenspaces of dimension one, two, n and $n+1$, and we know how many there are of each dimension. There is only one eigenspace of dimension $n+1$, which is E_d , and by Corollary 3 it is contained in the kernel of T_1 . Each eigenspace of dimension n is generated

by $\{x_j x^I \partial_j \mid j = 1, \dots, n\}$, where $I \in \mathbb{Z}_{\geq 0}^n$ and $|I| \leq d - 1$. We use Lemma 3.4.10 to calculate the rank of T_1 restricted to those eigenspaces. Lemma 3.4.9 tell us the total rank of T_1 restricted to eigenspaces of dimension 2. Finally, for eigenspaces of dimension 1, we use Lemma 3.4.5. Then:

$$\text{rank}(T) = \sum_{\dim E_j=1} \text{rank } T_1|_{E_j} + \sum_{\dim E_j=2} \text{rank } T_1|_{E_j} + \sum_{\dim E_j=n} \text{rank } T_1|_{E_j} + \sum_{\dim E_j=n+1} \text{rank } T_1|_{E_j}.$$

We add and substract $\dim \mathbf{V}_d$ on the right side of the equality and we conclude the proof.

■

The difference of the dimension of the space of foliations and the dimension of the automorphism group, which is given by Proposition 2.2.1, in general, is greater than the rank of the local Baum-Bott map at the Jouanolou foliation, as we see in the following proposition:

Proposition 3.4.3. *The rank of the local Baum-Bott map $BB : \mathcal{Fol}_{red}(n, d) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree- d Jouanolou foliation, for degree d greater than two, is strictly less than the upper bound given in Proposition 2.2.1. The same holds for degree $d = 2$ with odd dimension n .*

Proof. By Theorem 3.2.1, there are $\binom{n+d-1}{n}$ eigenspaces of dimension n , these are generated by $\{x_j x^I \partial_j \mid j = 1, \dots, n\}$, where $I \in \mathbb{Z}_{\geq 0}^n$ and $|I| \leq d - 1$. Lemma 3.4.10 gives us the rank $T_1|_{\{x_j x^I \partial_j \mid j=1, \dots, n\}}$ in terms of I . Observe that $\text{rank } T_1|_{\{x_j x^I \partial_j \mid j=1, \dots, n\}} \leq n - 1$. Then, in Theorem 3.4.2, we have:

if n is even:

$$\text{rank } T \leq \dim \mathbf{V}_d - (n+1)n - \binom{n+d-1}{n},$$

if n is odd:

$$\text{rank } T \leq \dim \mathbf{V}_d - (n+1)n - \frac{n+1}{2} - \binom{n+d-1}{n}.$$

In any case

$$\text{rank } T \leq \dim \mathbf{V}_d - (n+1)n - \binom{n+d-1}{n}.$$

We have $(n+1)^2 < (n+1)n + \binom{n+d-1}{n}$, for $d \geq 3$, and if $d = 2$, then $(n+1)^2 = (n+1)n + \binom{n+d-1}{n}$. Hence we conclude:

If n is even and $d \geq 3$: $\text{rank } T < \dim \mathbf{V}_d - (n+1)^2$.

If n is odd and $d \geq 2$: $\text{rank } T < \dim \mathbf{V}_d - (n+1)^2 - \frac{n+1}{2}$. ■

When the dimension of the projective space is even and we are in the case of degree two foliations, we know the generic rank of the Baum-Bott map because of the Jouanolou foliation.

Theorem 3.4.4. *Let $n \geq 2$ be an even number. The rank of the local Baum-Bott map $BB : \mathcal{Fol}_{red}(n, 2) \rightarrow (\mathbb{C}^{n-1})^N$ at the degree-2 Jouanolou foliation is equal to the upper*

bound given by Proposition 2.2.1. Then, the generic rank of the Baum-Bott map for degree-2 foliations on the projective space \mathbb{P}^n is

$$\dim \mathcal{Fol}(n, 2) - \dim \text{Aut}(\mathbb{P}^n) = (n+1) \binom{n+d}{n} - \binom{n+d-1}{n} - (n+1)^2.$$

In particular a generic fiber of the Baum-Bott map is a finite union of orbits of the action of the automorphism group $\text{Aut}(\mathbb{P}^n)$ on the space $\mathcal{Fol}(n, 2)$.

Proof. By Lemma 3.4.12, the rank of the linear map T_1 at eigenspaces of dimension n is $n-1$. Theorem 3.2.1 states that there are $\binom{n+d-1}{n}$ of these eigenspaces which are generated by $\{x_j x^I \partial_j \mid j=1, \dots, n\}$, $I \in \mathbb{Z}_{\geq 0}^n$ with $|I| \leq d-1$. Lemma 3.4.10 tells us how to calculate $\text{rank } T_1|_{\{x_j x^I \partial_j \mid j=1, \dots, n\}}$ in terms of I . We replace this information in Theorem 3.4.2 and the theorem is proved. \blacksquare

On \mathbb{P}^3 , we can estimate the rank of the local Baum-Bott map at the Jouanolou foliation:

Theorem 3.4.5. *Let $d \geq 2$. The rank of the local Baum-Bott map at the degree- d Jouanolou foliation \mathcal{J}_d on the projective space \mathbb{P}^3 is*

- if d is even,

$$\text{rank}(T) = \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} - 2 \right),$$

- if $d \equiv -1 \pmod{4}$,

$$\text{rank}(T) = \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} + \frac{d-3}{2} \right),$$

- if $d \equiv 1 \pmod{4}$,

$$\text{rank}(T) = \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} + \frac{d-1}{2} \right),$$

and the dimension of the space \mathbf{V}_d is $4 \binom{d+3}{3} - \binom{d+2}{3}$.

Proof. By Lemma 3.4.2, we have to estimate the rank of the matrices

$$\left[\begin{array}{c} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d+1) \\ i_{j+3} - i_{j+2} \end{array} \right]_{1 \leq j \leq 3} = \left[\begin{array}{ccc} d+1 & d & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} i_{j+1} - i_j \\ i_{j+2} - i_{j+1} \\ i_{j+3} - i_{j+2} \end{array} \right]_{1 \leq j \leq 3},$$

where i_j' 's are integer numbers with $i_j \geq 0$, where $i_j = i_j \bmod (n+1)$ and $i_1+i_2+i_3+i_4 = d-1$. Let us analyze the determinant of the matrix

$$\begin{vmatrix} i_2 - i_1 & i_3 - i_2 & i_4 - i_3 \\ i_3 - i_2 & i_4 - i_3 & i_1 - i_4 \\ i_4 - i_3 & i_1 - i_4 & i_2 - i_1 \end{vmatrix}$$

We denote $a = i_3 - i_1$, $b = i_4 - i_2$, $c = i_3 - i_2$, then the determinant is

$$\begin{vmatrix} a - b & c & b - c \\ c & b - c & -a - b + c \\ b - c & -a - b + c & a - c \end{vmatrix}$$

We add the second row to the last one, then we add the first column to the last one, and finally we add the first row to the second one, then the determinant is $-(a+b-2c)(a^2+b^2)$.

If the determinant is equal to zero, we have two cases.

First case: $a = b = 0$. This implies $i_3 = i_1$, $i_4 = i_2$ and the matrix:

$$\begin{bmatrix} (d+1)(i_2 - i_1) + d(i_1 - i_2) & (d+1)(i_1 - i_2) + d(i_2 - i_1) & (d+1)(i_2 - i_1) + d(i_1 - i_2) \\ i_2 - i_1 & i_1 - i_2 & i_2 - i_1 \end{bmatrix}$$

can have rank one if $d = 2r + 1$, where $r = i_1 + i_2$, $i_1 \neq i_2$, $i_1 = i_3$ and $i_2 = i_4$, or it can have rank zero if $d = 4r + 1$ and $r = i_1 = i_2 = i_3 = i_4$.

Second case: $a + b - 2c = 0$. This means that $i_4 = i_3 - i_2 + i_1$ and the matrix:

$$\begin{bmatrix} (d+1)(i_2 - i_1) + d(i_3 - i_2) & (d+1)(i_3 - i_2) + d(i_1 - i_2) & (d+1)(i_1 - i_2) + d(i_2 - i_3) \\ i_1 - i_2 & i_2 - i_3 & i_2 - i_1 \end{bmatrix}$$

has rank less than two if $(i_3 - i_2)^2 + (i_1 - i_2)^2 = 0$. This yields $d = 4r + 1$, $i_1 = i_2 = i_3 = i_4 = r$ and the rank is zero.

In view of the results, if $d = 4r + 1$, there are $2r$ eigenspaces of dimension 3 such that the rank of the linear map $DBB(X_{j_d}, p_1)$ restricted to each of those eigenspaces is one, and one eigenspace of dimension three that is in the kernel of the linear map $DBB(X_{j_d}, p_1)$.

If $d = 4r + 3$, there are $2r + 2$ eigenspaces of dimension 3 such that the rank of the linear map T_1 restricted to each of those eigenspaces is one. We replace this information in Theorem 3.4.2:

if $d = 4r + 1$:

$$\text{rank } T = \dim \mathbf{V}_d - 3 \times 4 - \frac{4}{2} - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^3 \\ |I| \leq d-1}} 1 - 2r - 2,$$

if $d = 4r + 3$

$$\text{rank } T = \dim \mathbf{V}_d - 3 \times 4 - \frac{4}{2} - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^3 \\ |I| \leq d-1}} 1 - (2r + 2),$$

if $d = 2r$,

$$\text{rank } T = \dim \mathbf{V}_d - 3 \times 4 - \frac{4}{2} - \sum_{\substack{I \in \mathbb{Z}_{\geq 0}^3 \\ |I| \leq d-1}} 1,$$

and this finishes the proof of the theorem. ■

Remark 3.4.2. *When $n = 3$, we observe that the rank of the local Baum-Bott map at the Jouanolou foliation is strictly less than the generic rank of the Baum-Bott map, at least for degree $2, \dots, 8$.*

Appendix

Appendix A

Eigenspaces and the Jouanolou foliation

In this appendix, we identify the eigenspaces associated to an automorphism, which leaves invariant the Jouanolou homogeneous equation, in the space of homogeneous vector fields of fixed degree.

The degree- d Jouanolou foliation \mathcal{J}_d on the projective space \mathbb{P}^n is defined by the homogeneous vector field in \mathbb{C}^{n+1} :

$$\mathbb{X}_{\mathcal{J}_d} = (x_2^d, x_3^d, \dots, x_{n+1}^d, x_1^d). \quad (\text{A.1})$$

This foliation is invariant by some automorphisms. For instance, let ξ be a primitive N -th root of unity and $\mathbb{A}(x_1, \dots, x_{n+1}) = (\alpha_1 x_1, \dots, \alpha_n x_n, \alpha_{n+1} x_{n+1})$, where $\alpha_1 = \xi = \xi^{-(d^n + \dots + d)}$, $\alpha_2 = \xi^{-(d^{n-1} + d^{n-1} + \dots + d)}$, ..., $\alpha_j = \xi^{-(d^{n+1-j} + \dots + d)}$, ..., $\alpha_n = \xi^{-d}$, $\alpha_{n+1} = 1$, and $N = d^n + d^{n-1} + \dots + d + 1$. This automorphism generates a subgroup of order N of the automorphism group $\text{Aut}(\mathbb{P}^n)$ and the Jouanolou foliation is invariant by this subgroup, since:

$$\mathbb{A}^* \mathbb{X}_{\mathcal{J}_d} = \xi^d \mathbb{X}_{\mathcal{J}_d}.$$

The singular set of the Jouanolou foliation is determined by this automorphism:

$$\text{Sing}(\mathcal{J}_d) = \{\mathbb{A}^{i-1}[1, 1, \dots, 1] \mid i = 1, \dots, N\}.$$

Let us fix some notations. From now on, $n \geq 3$. Given $I = (i_1, \dots, i_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1}$, we set $x^I = x_1^{i_1} \dots x_{n+1}^{i_{n+1}}$, $|I| = i_1 + \dots + i_{n+1}$. Let \mathbb{V}_d be the space of homogeneous vector fields of degree d in the complex vector space \mathbb{C}^{n+1} and $\mathbb{B}_d = \{x^I \partial_k \in \mathbb{V}_d \mid |I| = d, k = 1, \dots, n+1\}$.

The space of homogenous vector fields of fixed degree can be decomposed into smaller subspaces determined by \mathbb{A} :

Lemma A.1. *Every vector in \mathbb{B}_d is an eigenvector of $\mathbb{A}^* : \mathbb{V}_d \rightarrow \mathbb{V}_d$, with eigenvalue some N th root of unity. The space \mathbb{V}_d is the sum of the eigenspaces $\mathbb{E}_j = \{V \in \mathbb{V}_d \mid$*

$\mathbb{A}^*V = \xi^j V\}$, for $j = 0, 1, \dots, N - 1$. Moreover, if $x^I \partial_k \in \mathbb{B}_d$ then

$$\mathbb{A}^*(x^I \partial_k) = \alpha^I \alpha_k^{-1} x^I \partial_k, \quad (\text{A.2})$$

where $\alpha^I = \alpha_1^{i_1} \dots \alpha_{n+1}^{i_{n+1}} = \xi^{\rho(I)}$ and $\rho(I) = -i_1(d^n + \dots + d) - i_2(d^{n-1} + \dots + d) - i_3(d^{n-2} + \dots + d) - i_j(d^{n-j+1} + \dots + d) - i_{n-1}(d^2 + d) - i_n d$.

We have another subgroup which leaves invariant the Jouanolou foliation. Let \mathbb{S} be the automorphism of the projective space \mathbb{P}^n defined by

$$\mathbb{S}[x_1, x_2, \dots, x_{n+1}] = [x_{n+1}, x_1, \dots, x_n]. \quad (\text{A.3})$$

This automorphism generates a cyclic subgroup of $\text{Aut}(\mathbb{P}^n)$ of order $(n+1)$, this map also leaves invariant the Jouanolou foliation since $\mathbb{S}^* \mathbb{X}_{\mathcal{J}_d} = \mathbb{X}_{\mathcal{J}_d}$.

The operator \mathbb{S}^* sends an eigenspace of \mathbb{A}^* in another eigenspace, except one, the eigenspace corresponding to the eigenvalue ξ^d . This eigenspace is sent to itself. We can see it in the following lemma.

Lemma A.2. *The operator $\mathbb{S}^* : \mathbb{V}_d \rightarrow \mathbb{V}_d$ maps eigenspaces of \mathbb{A}^* to eigenspaces of \mathbb{A}^* and $\mathbb{S}^*(x^I \partial_k) = x^{\mathbb{S}^{-1}(I)} \partial_{k-1}$, where $x^I \partial_k \in \mathbb{B}_d$ and $\partial_0 = \partial_{n+1}$.*

Proof. Let $x^I \partial_k, x^J \partial_{k+r} \in \mathbb{B}_d$ with r a positive integer number. Suppose $1 < k < k+r \leq n$. If the vectors belong to the same eigenspace, then by the equation (A.2) there exists an integer number K such that

$$\begin{aligned} & (i_1 - j_1)d^{n-1} + \dots + (i_1 + \dots + i_{k-1} - (j_1 + \dots + j_{k-1}))d^{n-k+1} + \\ & (i_1 + \dots + i_k - 1 - (j_1 + \dots + j_k))d^{n-k} + \\ & + (i_1 + \dots + i_{k+r-1} - 1 - (j_1 + \dots + j_{k+r-1}))d^{n-(k+r)+1} + \\ & + (i_1 + \dots + i_{k+r} - (j_1 + \dots + j_{k+r}))d^{n-(k+r)} + \\ & + (i_1 + \dots + i_n - (j_1 + \dots + j_n)) = KN. \end{aligned}$$

We multiply by d the equation above, and this yields to the following expression:

$$\begin{aligned} & (i_2 - j_2)d^{n-1} + \dots + (i_2 + \dots + i_{k-1} - (j_2 + \dots + j_{k-1}))d^{n-k+2} + \\ & (i_2 + \dots + i_k - 1 - (j_2 + \dots + j_k))d^{n-k+1} + \\ & + (i_2 + \dots + i_{k+r-1} - 1 - (j_2 + \dots + j_{k+r-1}))d^{n-(k+r)+2} + \\ & + (i_2 + \dots + i_{k+r} - (j_2 + \dots + j_{k+r}))d^{n-(k+r)+1} + \\ & + (i_2 + \dots + i_{n+1} - (j_2 + \dots + j_{n+1})) = (dK - (i_1 - j_1))N. \end{aligned}$$

This implies that the vectors $x_1^{i_2} \dots x_n^{i_{n+1}} x_{n+1}^{i_1} \partial_{k-1}$ and $x_1^{j_2} \dots x_n^{j_{n+1}} x_{n+1}^{j_1} \partial_{k+r-1}$ have the same eigenvalue, and these vectors are the image by the operator \mathbb{S}^* of the vectors considered at the beginning.

The proofs for the other cases are similar. \blacksquare

Consequently, two eigenvectors correspond to the same eigenvalue if and only if the image of these eigenvectors by the operator \mathbb{S}^* is contained in another eigenspace of the space \mathbb{V}_d . We can subsequently apply \mathbb{S}^* to a fixed eigenspace until one eigenvector is of the form $x^I \partial_1$. Then, it is enough to study the eigenspaces that contain an eigenvector $x^I \partial_1 \in \mathbb{B}_d$. We are going to do this in the following lemmas.

Lemma A.3. *Let $x^I \partial_1, x^J \partial_1 \in \mathbb{B}_d$ belong to the same eigenspace. Then $I = J$.*

Proof. Suppose by contradiction that $I \neq J$. By equation (A.2), since they belong to the same eigenspace, we have $\rho(I) - \rho(J) = 0 \pmod{N}$. This means that $\rho(J) - \rho(I) = KN$, for some $K \in \mathbb{Z}$. In particular,

$$\begin{aligned} KN = & (i_1 - j_1)d^n + (i_1 + i_2 - (j_1 + j_2))d^{n-1} + \dots + \\ & + (i_1 + \dots + i_{n-1} - (j_1 + \dots + j_{n-1}))d^2 + (i_1 + \dots + i_n - (j_1 + \dots + j_n))d. \end{aligned} \quad (\text{A.4})$$

The above relation implies that d divides K because $N = 1 \pmod{d}$. Since $\sum_{r=1}^t i_r \leq d$, $\sum_{r=1}^t j_r \leq d$, for $t = 1, \dots, n$, we have $-dN < KN < dN$. This yields $-d < K < d$. Since d divides K , we have $K = 0$. Therefore,

$$\begin{aligned} 0 = & (i_1 - j_1)d^{n-1} + (i_1 + i_2 - (j_1 + j_2))d^{n-2} + \dots + \\ & + (i_1 + \dots + i_{n-1} - (j_1 + \dots + j_{n-1}))d + (i_1 + \dots + i_n - (j_1 + \dots + j_n)). \end{aligned} \quad (\text{A.5})$$

Hence, $(i_1 + \dots + i_n - (j_1 + \dots + j_n)) = 0 \pmod{d}$. We have three cases.

First case: If $i_1 + \dots + i_n = (j_1 + \dots + j_n) + d$, then $j_1 = \dots = j_n = 0$. If we replace in A.5, we get: $i_1 d^{n-1} + (i_1 + i_2)d^{n-2} + \dots + (i_1 + \dots + i_{n-1})d + (i_1 + \dots + i_n) = 0$, then $i_1 = \dots = i_n = 0$, which is a contradiction.

Second case: If $i_1 + \dots + i_n = j_1 + \dots + j_n - d$. It is analogous to the first case, and it leads us a contradiction.

Third case: If $i_1 + \dots + i_n = j_1 + \dots + j_n$, we have

$$\begin{aligned} 0 = & (i_1 - j_1)d^{n-2} + (i_1 + i_2 - (j_1 + j_2))d^{n-3} + \dots + \\ & + (i_1 + \dots + i_{n-2} - (j_1 + \dots + j_{n-2}))d + (i_1 + \dots + i_{n-1} - (j_1 + \dots + j_{n-1})), \end{aligned} \quad (\text{A.6})$$

then $((i_1 + \dots + i_{n-1}) - (j_1 + \dots + j_{n-1})) = 0 \pmod{d}$, again we have three possibilities, but working analogously as before, $i_1 + \dots + i_{n-1} = j_1 + \dots + j_{n-1}$. Continuing this process, we get $i_1 = j_1$, and we conclude that $I = J$. \blacksquare

Lemma A.4. *Let $x^I \partial_1, x^J \partial_2 \in \mathbb{B}_d$ belong to the same eigenspace.*

1. *If $i_1 > 0$, then $x^I = x_1 x^{\tilde{I}}$, $x^J = x_2 x^{\tilde{I}}$, where $\tilde{I} \in \mathbb{Z}_{\geq 0}^{n+1}$ and $|\tilde{I}| = d - 1$.*
2. *If $i_1 = 0$, then $x^I = x_2^{d-1} x_k$ and $x^J = x_{k+1}^d$, where $2 \leq k \leq n + 1$ and $x_k = x_{k \bmod (n+1)}$.*

Proof. Let $i_1 > 0$, we have that $x^I = x_1 x^{\tilde{I}}$, for some $\tilde{I} \in \mathbb{Z}_{\geq 0}^{n+1}$. We can proof that $x_1 x^{\tilde{I}} \partial_1, x_2 x^{\tilde{I}} \partial_2$ belong to the same eigenspace, and using Lemmas A.2 and A.3, we get $x^J = x_2 x^{\tilde{I}} \partial_2$.

Let $i_1 = 0$, if $j_2 > 0$, we follow the arguments given above and we get a contradiction. Therefore, $i_1 = j_2 = 0$. By equation (A.2), there is some integer K such that

$$\begin{aligned} d((-j_1 - 1)d^{n-1} + (i_2 - j_1)d^{n-2} + (i_2 + i_3 - (j_1 + j_3))d^{n-3} + \\ + \dots (i_2 + \dots + i_n - (j_1 + j_3 + \dots + j_n))) = KN. \end{aligned} \quad (\text{A.7})$$

If $K = -d$, equation (A.7) becomes

$$\begin{aligned} d^{n-1}(d - j_1 - 1) + d^{n-2}(d - j_1 + i_2) + d^{n-3}(d - (j_1 + j_3) + i_2 + i_3) + \\ + \dots + d - (j_1 + j_3 + \dots + j_n + i_2 + \dots + i_n) + 1 = 0, \end{aligned} \quad (\text{A.8})$$

then $d = j_1$ because $\sum_{r=2}^n d^{n-r}((d - \sum_{k=1}^r j_k) + \sum_{k=2}^r i_k) + 1 > 0$, and replacing in the equation above, we get $i_2 = d - 1$, and the solution is $x_2^{d-1} x_{n+1} \partial_1, x_1^d \partial_2$.

If $K = 0$, equation (A.7) yields

$$\begin{aligned} (-1 - j_1)d^{n-1} + (i_2 - j_1)d^{n-2} + (i_2 + i_3 - (j_1 + j_3))d^{n-3} + \\ + \dots (i_2 + \dots + i_n - (j_1 + j_3 + \dots + j_n)) = 0, \end{aligned} \quad (\text{A.9})$$

and $i_2 + \dots + i_n - (j_1 + j_3 + \dots + j_n) = 0 \bmod(d)$. We have two subcases:

If $i_2 + \dots + i_n = d + j_1 + j_3 + \dots + j_n$: Replacing in the equation (A.9) and dividing by d , we must have $i_2 + \dots + i_{n-1} + 1 = d$, we subsequently divide by d the equation (A.9), until we get the exprexion $-d^2 + i_2 d + i_2 + i_3 + 1 = 0$, hence the eigenvectors are $x_2^{d-1} x_n \partial_1, x_{n+1}^d \partial_2$.

If $i_2 + \dots + i_n = j_1 + j_3 + \dots + j_n$: In equation (A.9) we get $i_2 + \dots + i_{n-1} - (j_1 + j_3 + \dots + j_{n-1}) = 0 \bmod(d)$. We procede as before, and we get the other eigenvectors. \blacksquare

Lemma A.5. *Let $r \in \mathbb{N}$ such that $2 \leq r \leq n - 1$. Let $x^I \partial_1, x^J \partial_{r+1} \in \mathbb{B}_d$ have the same eigenvalue, then*

1. *if $i_1 > 0$: $x^I = x_1 x^{\tilde{I}}$, $x^J = x_{r+1} x^{\tilde{I}}$, where $\tilde{I} \in \mathbb{Z}_{\geq 0}^{n+1}$ and $|\tilde{I}| = d - 1$.*
2. *if $i_1 = 0$: $x^I = x_{r+1}^{d-1} x_{n+1}$, $x^J = x_1^{d-1} x_r$ or $x^I = x_2^d$, $x^J = x_{r+2}^d$.*

Proof. Let $i_1 > 0$, we follow the same arguments given for the proof of Item 1 of Lemma A.4, and we get the result.

Let $i_1 = 0$. If $r \leq n - 2$, by the equation (A.2) there is some integer K such that

$$\begin{aligned} & d((-j_1 - 1)d^{n-1} + (i_2 - (j_1 + j_2) - 1)d^{n-2} + \\ & + (i_2 + \dots + i_r - (j_1 + \dots + j_r) - 1)d^{n-r} + \\ & + (i_2 + \dots + i_{r+1} - (j_1 + \dots + j_r))d^{n-r-1} + \\ & + (i_2 + \dots + i_{r+2} - (j_1 + \dots + j_r + j_{r+2}))d^{n-r-2} + \dots + \\ & + (i_2 + \dots + i_n - (j_1 + \dots + j_r + j_{r+2} + \dots + j_n)) = KN. \end{aligned} \quad (\text{A.10})$$

We have two cases:

First case: If $K = -d$, the equation (A.10) becomes

$$\begin{aligned} & d^{n-1}(d - j_1 - 1) + d^{n-2}(d - (j_1 + j_2) + i_2 - 1) + \dots + \\ & + d^{n-r}(d - (j_1 + \dots + j_r) + i_2 \dots + i_r - 1) + d^{n-r-1}(d - (j_1 + \dots + j_r) + i_2 \dots + i_{r+1}) + \\ & + d^{n-r-2}(d - (j_1 + \dots + j_r + j_{r+2}) + i_2 \dots + i_{r+2}) + \dots + \\ & + d - (j_1 + \dots + j_r + j_{r+2} + \dots + j_n) + i_2 + \dots + i_n + 1 = 0, \end{aligned} \quad (\text{A.11})$$

if $d = j_1$, we must have $i_2 + \dots + i_n = d - 1$, we replace in the equation (A.11), subsequently divide by d , and we get a contradiction. So $d = j_1 + \dots + j_r$ and $i_2 = 0$. We replace in equation (A.11), we get $i_3 + \dots + i_n = d - 1$, we subsequently divide by d the expression A.11, and we find that $i_{r+1} = d - 1, i_{n+1} = 1$, we replace in equation (A.11) and simplify. We arrive at the expression

$$d^{r-2}(d - j_1 - 1) + \dots + d(d - (j_1 + \dots + j_{r-2}) - 1) + (d - (j_1 + \dots + j_{r-1}) - 1) = 0,$$

then $d - 1 = j_1 + \dots + j_{r-1}$, we replace in the equation above, divide by d , we find that the indexes are $j_1 = d - 1, j_r = 1$, and the vectors are $x_{r+1}^{d-1} x_{n+1} \partial_1, x_1^{d-1} x_r \partial_{r+1}$.

Second case: If $K = 0$, the equation (A.10) becomes

$$\begin{aligned} & (-j_1 - 1)d^{n-1} + (i_2 - (j_1 + j_2) - 1)d^{n-2} + (i_2 + \dots + i_r - (j_1 + \dots + j_r) - 1)d^{n-r} + \\ & + (i_2 + \dots + i_{r+1} - (j_1 + \dots + j_r))d^{n-r-1} + \\ & + (i_2 + \dots + i_{r+2} - (j_1 + \dots + j_r + j_{r+2}))d^{n-r-2} + \\ & + \dots + (i_2 + \dots + i_n - (j_1 + \dots + j_r + j_{r+2} + \dots + j_n)) = 0, \end{aligned} \quad (\text{A.12})$$

then $i_2 + \dots + i_n - (j_1 + \dots + j_r + j_{r+2} + \dots + j_n) = 0 \pmod{d}$. We have two subcases:

1st subcase: If $i_2 + \dots + i_n = d + j_1 + \dots + j_r + j_{r+2} + \dots + j_n$: we replace in equation (A.12) and divide by d , we must have $i_2 + \dots + i_r = 0$, we subsequently divide by d the equation (A.12), and we get a contradiction.

2nd subcase: If $i_2 + \dots + i_n = j_1 + \dots + j_r + j_{r+2} + \dots + j_n$: in equation A.12 we get $i_2 + \dots + i_{r+2} = j_1 + \dots + j_r + j_{r+2}$, and the equation becomes

$$\begin{aligned} & (-j_1 - 1)d^{n-1} + (i_2 - (j_1 + j_2) - 1)d^{n-2} + (i_2 + \dots + i_r - (j_1 + \dots + j_r) - 1)d^{n-r} + \\ & + (i_2 + \dots + i_{r+1} - (j_1 + \dots + j_r))d^{n-r-1} = 0. \end{aligned} \quad (\text{A.13})$$

We have two possible cases:

a) If $i_2 + \dots + i_{r+1} - (j_1 + \dots + j_r) = d$, we replace in (A.13) and we get $i_2 = d$, then the eigenvectors are $x_2^d \partial_1, x_{r+2}^d \partial_{r+1}$.

b) If $i_2 + \dots + i_{r+1} = j_1 + \dots + j_r$, we replace in (A.13), this yields the equation

$$-i_{r+1} - 1 = i_2 + \dots + i_r - (j_1 + \dots + j_r) - 1 = -d,$$

which means that $i_2 + \dots + i_k - 1 \leq 0$, for $k = 2, \dots, r$, then the expression on the left side of the equation (A.13) would be negative, which is a contradiction.

The case $r = n - 1$ is analogous. ■

Lemma A.6. *Let $x^I \partial_1, x^J \partial_{n+1} \in \mathbb{B}_d$ belong to the same eigenspace.*

1. *If $i_1 > 0$, then $x^I = x_1 x^{\tilde{I}}$, $x^J = x_{n+1} x^{\tilde{I}}$, where $\tilde{I} \in \mathbb{Z}_{\geq 0}^{n+1}$ and $|\tilde{I}| = d - 1$.*
2. *If $i_1 = 0$, then $x^I = x_{k+1}^d$ and $x^J = x_1^{d-1} x_k$, where $k \in \{1, \dots, n\}$.*

Proof. Suppose $i_1 = 0$. If $x_2^{i_2} x_3^{i_3} \dots x_n^{i_n} \partial_1$ and $x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \partial_{n+1}$ belong to the same eigenspace, then $j_1 + \dots + j_n = d$, and for some integer K :

$$(j_1 + 1)d^m + (j_1 + j_2 + 1 - i_2)d^{m-1} + \dots + (j_1 + \dots + j_n + 1 - (i_2 + \dots + i_n))d = KN. \quad (\text{A.14})$$

Then $0 < K < d + 1$ and $K \equiv 0 \pmod{d}$, therefore $K = d$ and we get the equation

$$j_1 d^{m-1} + (j_1 + j_2 - i_2) d^{m-2} + \dots + (j_1 + \dots + j_n - (i_2 + \dots + i_n)) = d^m. \quad (\text{A.15})$$

We have two cases.

First case: If $j_1 + \dots + j_n = d + i_2 + \dots + i_n$:

$$j_1 d^{m-2} + (j_1 + j_2) d^{m-3} + \dots + (j_1 + \dots + j_{n-1}) + 1 = d^{m-1},$$

this implies $j_1 + \dots + j_{n-1} = d - 1$, doing so, the solution is $j_1 = d - 1, j_n = 1$. The eigenvectors are $\partial_1, x_1^{d-1} x_n \partial_{n+1}$.

Second case: If $j_1 + \dots + j_n = i_2 + \dots + i_n = d$:

$$j_1 d^{m-2} + (j_1 + j_2 - i_2) d^{m-3} + \dots + j_1 + \dots + j_{n-1} - (i_2 + \dots + i_{n-1}) = d^{m-1}.$$

- If $j_1 + \dots + j_{n-1} = i_2 + \dots + i_{n-1} + d$, we have $i_n = d$ and

$$j_1 d^{n-3} + (j_1 + j_2) d^{n-4} + \dots + j_1 + \dots + j_{n-2} + 1 = d^{n-2},$$

then $j_1 + \dots + j_{n-2} = d - 1$. The solution is $j_1 = d - 1$, $j_{n-1} = 1$ and the eigenvectors are $x_n^d \partial_1$, $x_1^{d-1} x_{n-1} \partial_{n+1}$

- If $j_1 + \dots + j_{n-1} = i_2 + \dots + i_{n-1}$:

$$j_1 d^{n-3} + (j_1 + j_2) d^{n-4} + \dots + j_1 + \dots + j_{n-2} - (i_2 + \dots + i_{n-2}) = d^{n-2}.$$

Doing so, we get

- If $j_1 + \dots + j_k = i_2 + \dots + i_k + d$, the solutions are $x_{k+1}^d \partial_1$, $x_1^{d-1} x_k \partial_{n+1}$, for $k = 3, \dots, n - 2$.

- If $j_1 + j_2 + j_3 = i_2 + i_3$:

$$j_1 d + j_1 + j_2 - i_2 = d^2.$$

If $j_1 + j_2 = i_2 + d$, the solution is $x_3^d \partial_1$, $x_1^{d-1} x_{n-1} \partial_{n+1}$.

If $j_1 + j_2 = i_2$, then $j_1 = d$ and $i_2 = d$, the solution is $x_2^d \partial_1$, $x_1^d \partial_{n+1}$.

■

From now on, we will identify x_{n+1+k} with x_k and ∂_{n+1+k} with ∂_k .

Let's identify the eigenspaces of dimension 2.

Lemma A.7. *An eigenspace of dimension 2 of $\mathbb{A}^* : \mathbb{V}_d \rightarrow \mathbb{V}_d$ has only one of the following forms:*

1. $(\mathbb{S}^*)^k (\langle x_2^{d-1} x_j \partial_1, x_{1+j}^d \partial_2 \rangle)$, for $3 \leq j \leq n + 1$, $k = 0, \dots, n$ and
2. if n is even: $(\mathbb{S}^*)^k (\langle x_{2+j}^{d-1} x_{n+1} \partial_1, x_1^{d-1} x_{1+j} \partial_{2+j} \rangle)$, for $1 \leq j \leq \frac{n}{2} - 1$ and $k = 0, \dots, n$,
if n is odd: $(\mathbb{S}^*)^k (\langle x_{2+j}^{d-1} x_{n+1} \partial_1, x_1^{d-1} x_{1+j} \partial_{2+j} \rangle)$, for $1 \leq j \leq \frac{n-3}{2}$, $k = 0, \dots, n$
and for $j = \frac{n-1}{2}$, then $k = 0, \dots, \frac{n-1}{2}$.

The list is complete, those are all the possible forms. In fact, there are $\frac{(3n-4)(n+1)}{2}$ eigenspaces of dimension 2.

Proof. Let E be an eigenspace of dimension 2, there is some k such that $(\mathbb{S}^*)^k(E) = \langle x^I \partial_1, x^J \partial_{1+j} \rangle$ for some $I, J \in \mathbb{Z}_{\geq 0}^{n+1}$ and $0 \leq j < n$. Then $E = (\mathbb{S}^*)^{n+1-k} (\langle x^I \partial_1, x^J \partial_{1+j} \rangle)$. From Lemma A.3, $j > 0$. The second item of Lemmas A.4 and A.5 give us the complete list of possible vectors $x^I \partial_1, x^J \partial_{1+j}$. Therefore, the eigenspaces of dimension 2 are $(\mathbb{S}^*)^k (\langle x_2^{d-1} x_j \partial_1, x_{1+j}^d \partial_2 \rangle)$, for $3 \leq j \leq n + 1$, $k = 0, \dots, n$, and $(\mathbb{S}^*)^k (\langle x_{2+j}^{d-1} x_{n+1} \partial_1, x_1^{d-1} x_{1+j} \partial_{2+j} \rangle)$, for $1 \leq j \leq n - 2$, $k = 0, \dots, n$. Now we want to exclude the repeated cases:

- a) If $(\mathbb{S}^*)^k(\langle x_2^{d-1}x_j\partial_1, x_{1+j}^d\partial_2 \rangle) = \langle x_2^{d-1}x_r\partial_1, x_{1+r}^d\partial_2 \rangle$, for some $3 \leq j < r \leq n+1$, $k = 1, \dots, n$. Then $\langle x_{2-k}^{d-2}x_{j-k}\partial_{1-k}, x_{1+j-k}^d\partial_{2-k} \rangle = \langle x_2^{d-1}x_r\partial_1, x_{1+r}^d\partial_2 \rangle$, but it is not possible, and Item 1 follows.
- b) If $(\mathbb{S}^*)^k(\langle x_{2+j}^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{1+j}\partial_{2+j} \rangle) = \langle x_{2+r}^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{1+r}\partial_{2+r} \rangle$, for $1 \leq j < r \leq n-2$ and $k = 1, \dots, n$. Then $x_{2+j-k}^{d-1}x_{n+1-k}\partial_{1-k} = x_1^{d-1}x_{1+r}\partial_{2+r}$, $x_{1-k}^{d-1}x_{1+j-k}\partial_{2+j-k} = x_{2+r}^{d-1}x_{n+1}\partial_1$. We realized that $k = j+1$, $r = n-j-1$. We have $(\mathbb{S}^*)^{j+1}(\langle x_{2+j}^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{1+j}\partial_{2+j} \rangle) = \langle x_{n-j+1}^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{n-j}\partial_{2+(n-j-1)} \rangle$ and $1 \leq j \leq \frac{n-1}{2}$. Then if n is even, $j = 1, \dots, \frac{n-2}{2}$ and $k = 0, \dots, n$ give us different eigenspaces. When n is odd: $j = 1, \dots, \frac{n-3}{2}$, $k = 0, \dots, n$ and $j = \frac{n-1}{2}$, $k = 0, \dots, \frac{n-1}{2}$, those cases generate different eigenspaces.

According to Lemma Lemmas A.4, there are $n-1$ eigenspaces of the form $\langle x^I\partial_1, x^J\partial_2 \rangle$ and by Lemma A.5 one of the form $\langle x^I\partial_1, x^J\partial_{2+r} \rangle$, for each $r = 1, \dots, n-2$. We can apply (\mathbb{S}^*) subsequently to those eigenspaces to get all the eigenspaces of dimension 2 and we get Table A.1, and the result follows. \blacksquare

$\mathcal{P}(\partial_k) \backslash \mathcal{P}(\partial_{k+r})$	$\mathcal{P}(\partial_1)$	$\mathcal{P}(\partial_2)$	$\mathcal{P}(\partial_3)$	\dots	$\mathcal{P}(\partial_n)$	$\mathcal{P}(\partial_{n+1})$
$\mathcal{P}(\partial_1)$		$n-1$	1	\dots	1	$n-1$
$\mathcal{P}(\partial_2)$			$n-1$	\ddots	1	1
\vdots				\ddots	\ddots	1
$\mathcal{P}(\partial_{n-1})$					$n-1$	1
$\mathcal{P}(\partial_n)$						$n-1$

$$\mathcal{P}(\partial_k) = \{x^I\partial_k \mid I \in \mathbb{Z}_{\geq 0}^{n+1}, |I| = d\}$$

Table A.1: Number of eigenspaces of dimension two with vectors in $\mathcal{P}(\partial_k)$ and in $\mathcal{P}(\partial_{k+r})$.

We gather all the information above and count the number of eigenspaces of each positive dimension.

Theorem A.1. *The operator $\mathbb{A}^* : \mathbb{V}_d \rightarrow \mathbb{V}_d$ has $\binom{n+d}{n} + n\binom{n+d-1}{n-1} - (n-1)\frac{3n+4}{2}$ non-trivial eigenspaces. More precisely:*

1. There are $\binom{n+d-1}{n} + 1$ eigenspaces of dimension $n+1$, and they are:

- $E_d = \langle x_2^d\partial_1, \dots, x_{n+1}^d\partial_n, x_1^d\partial_{n+1} \rangle$,
- $\langle x_1x^I\partial_1, \dots, x_{n+1}x^I\partial_{n+1} \rangle$, where $I \in \mathbb{Z}_{\geq 0}^{n+1}$ and $|I| = d-1$.

2. There are $\frac{(3n-4)(n+1)}{2}$ eigenspaces of dimension two, which are:

- $\langle x_{i-1}x_{k+1}^{d-1}\partial_k, x_i^d\partial_{k+1} \rangle$, for $1 \leq i, k \leq n+1$ and $i \neq k+1, k+2$,
- $\langle x_{k-1}x_{k+r}^{d-1}\partial_k, x_{k+r-1}x_k^{d-1}\partial_{k+r} \rangle$, for $1 \leq k \leq n+1$ and $2 \leq r \leq n-1$

3. There are $(n+1) \left[\binom{n+d-1}{n-1} - 3(n-1) \right]$ eigenspaces of dimension 1, which are: $(\mathbb{S}^*)^k(x^I\partial_1)$, where $x^I\partial_k \in \mathbb{B}_d$, with $i_1 = 0$, $k = 0, \dots, n$ and $x^I \neq x_jx_2^{d-1}, x_j^d, x_r^{d-1}x_{n+1}$, for $j = 3, \dots, n+1$ and $r = 3, \dots, n$.

Proof. From the first items of Lemmas A.4, A.5 and A.6, we identify for each fixed $I \in \mathbb{Z}_{\geq 0}^{n+1}$, with $|I| = d-1$, the vectors $x_jx^I\partial_j$, $j = 1, \dots, n+1$ belong to the same eigenspace. There are $\binom{n+d-1}{n}$ of this kind.

From the second items of the same Lemmas, we realized that $x_2^d\partial_1$ is in the same eigenspace as $x_{k+1}^d\partial_k$, for $k = 2, \dots, n+1$, the corresponding eigenvalue is d . These are all the eigenspaces of dimension $n+1$.

The eigenspaces of dimension 2 are given by Lemma A.7.

The eigenspaces of dimension 1 are given by $(\mathbb{S}^*)^k(x^I\partial_1)$, for $k = 0, \dots, n$, where $x^I\partial_1 \in \mathbb{B}_d$ generates an eigenspace of dimension 1, then $i_1 = 0$. Let's identify which $x^I\partial_1$, with $i_1 = 0$, belong to eigenspaces of dimension greater than one. The first items of Lemmas A.4, A.5 and A.6 tell us that there are $(n-1)$ eigenspaces of the form $\langle x^I\partial_1, x^J\partial_2 \rangle$, $n-1$ of the form $\langle x^I\partial_1, x^J\partial_{n+1} \rangle$, one $\langle x^I\partial_1, x^J\partial_{2+r} \rangle$, for each $r = 1, \dots, n-2$ and $\langle x_2^d\partial_1, \dots, x_1^d\partial_{n+1} \rangle = E_d$. Thus, we can conclude 3. ■

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