Instituto Nacional de Matemática Pura e Aplicada

REDUCTION OF INCENTIVE CONSTRAINTS IN BIDIMENSIONAL ADVERSE SELECTION AND APPLICATIONS

Jose Braulio Calagua Mendoza

Doctoral Thesis

Rio de Janeiro
April 2017

Instituto Nacional de Matemática Pura e Aplicada

Jose Braulio Calagua Mendoza

REDUCTION OF INCENTIVE CONSTRAINTS IN BIDIMENSIONAL ADVERSE SELECTION AND APPLICATIONS

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.
Advisor: Aloísio Araújo
Co-advisor: Carolina Parra

Rio de Janeiro
April 2017

A mi Soledad

Agradecimentos

Agradezco a mis padres José Calagua y Zoila Mendoza, a mis hermanas Valeria, Natalia, Cecilia y Jimena, así como a mis sobrinos y sobrinas por el afecto, el apoyo y la confianza. Con gran cariño agradezco a Soledad Pápel, mi ahora esposa, por creer en mi, entenderme y alentarme en este camino que decidí tomar hace unos años.

Mi gratitud a mis profesores de la UNI e IMCA, entre ellos a Yboon García por recomendarme para llegar al IMPA. Mención especial para el profesor Ramón García-Cobián quien gracias al rigor matemático y claridad de las interpretaciones económicas en los cursos de maestría fue determinante para decidirme en hacer doctorado en economía matemática.

Agradeço profundamente ao professor Aloísio Araújo por ter aceitado trabalhar comigo como orientador de tese, pelas sugestões e dedicação no meu trabalho. Agradecimentos a Carolina Parra com quem discuti as ideias e detalhes das provas, ao Mauricio Villalba pela revisão detalhada e correcções, ao Sergei Vieira e ao Paulo Kingler Monteiro pelo seu tempo para discutir tópicos puntuais da tese, e ao Alexandre Belloni pela leitura minuciosa e recomendações em parte do trabalho.

Sou muito grato com os amigos que fiz e me acompanharam nesse periodo, entre eles: Juan Pablo Luna, Ruben Lizarbe, Eric Biagioli, Aldo Zang, Enrique Chávez, Andrés Chirre, Gabriel Muñoz, Percy Abanto, Juan Pablo Gamma-Torres e Lorenzo Bastianello. Menção especial merece meu amigo Liev Maribondo, com quem estudei para aprovar o exame de qualificação e passamos simultáneamente o processo de fazer uma tese.

Finalmente, minha gratidão com os funcionários e trabalhadores do IMPA, especialmente com o pessoal do Departamento de Ensino, pela eficiência da sua labor. O presente trabalho foi realizado graças ao apoio financeiro do Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq.

Abstract

In this work, we study a bidimensional adverse selection problem in the framework of nonlinear pricing by a monopolist, where the firm produces a one-dimensional product and customers' preferences are described by two dimensions of uncertainty.

We prove that it is sufficient to consider, for each type of customer, incentive compatibility constraints over a one-dimensional set rather than the entire two-dimensional set as required by definition. For this purpose, we introduce a pre-order among types to compare their marginal valuation of consumption and we also take account possible shape of isoquants. As a consequence, the discretized problem is computationally tractable for relative fine discretizations.

Due to we extend the ideas applied in the unidimensional case with finite types when single-crossing condition is satisfied, our main assumption is the validity of single-crossing over each direction of uncertainty. Thus, we are able to have well-educated insights of the solution for a large class of valuation function and types' distributions.

Keywords: incentive compatibility, bidimensional types, monopolist's problem, regulation model.

Resumo

Neste trabalho, estudamos o problema de seleção adversa bidimensional no marco de referencia de preços não lineares por um monopolista, onde a empresa produz um produto unidimensional e as preferências dos consumidores são descritas por duas dimensões de incerteza.

Provamos que é suficiente considerar, para cada tipo de consumidor, restrições de compatibilidade de incentivo sobre um conjunto numa dimensão em vez de todo o conjunto bidimensional como é exigido por definição. Isto é feito introduzindo um pré-ordem comparando tipos de acordo com a valoração marginal de consumo e levando em consideração a possível forma das isoquantas. Com esse resultado, o problema discretizado é computacionalmente tratável para discretizações relativamente finas.

Devido a que estendemos as idéias aplicadas no caso unidimensional com finitos tipos quando a condição de single-crossing é satisfeita, nossa suposição principal é a validade de single-crossing em cada direção de incerteza. Assim, somos capazes de ter noções bem educadas da solução para uma grande classe de funções de valoração e distribuições dos tipos.

Palabras-chave: compatibilidade de incentivos, tipos bidimensionaies, problema do monopolista, modelo de regulação.

Contents

Chapter 1

Introduction

Adverse selection models (or screening models) refer to a kind of asymmetric information problems, where an informed part (the agent) possesses its private knowledge before a transaction takes place with an uninformed part (the principal) who design a contract establishing the terms in which the relationship works. This contract is a menu of options from which the agent chooses his preferred action and the principal is commit to the contract offered.

Since the late 1970s, the development of the theory of screening models has been one of the major advances in economic theory, being notably applied to diverse issues such as optimal taxation, nonlinear pricing, regulatory policy of a monopolist firm, and auctions. The majority of these applications have assumed that agents' preferences can be ordered by a single parameter of private information, the well-know *single-crossing condition*, which facilitates finding the optimal solution.

Nevertheless, a one-dimensional parameter does not seem to reflect in an appropriate manner agents' private information in many economic environments. For instance, when establishing a price for its product a monopolist firm could be uncertain about the parameters describing customers' demand function (as Laffont et al. (1987) who considered linear demand curves with unknowing slope and intercep) or maybe wish to include socio-economic and demographic parameters such as wealth, age, etc. In the design of regulatory policy at least two dimensions of private cost information (fix and marginal costs) naturally arise, as was noticed by Baron and Myerson (1982). In the same framework Lewis and Sappington (1988b) have considered a regulator uncertain about both cost and demand functions of the monopolist firm he is regulating. In labor market, several unobservable worker's characteristics such as ability, effort or leisure preferences could be considered by the employer.

In this work we focus in adverse selection models where agents characteristics are captured by a two-dimensional vector and the principal disposes of one-dimensional instrument. Initially, we concentrate in the nonlinear pricing by a monopolist firm framework but later we analyze a model in the regulation setting.

Thus, consider a monopolistic firm producing a one-dimensional quality product and customer's private preferences described by two dimensions of uncertainty. The firm would like to design a contract extracting maximum benefit from each customer's type, but typically customer will not choose this contract selecting rather a contract designed to another type of customer. Therefore, in order to maximize revenue, the firm designs a contract in a way that ensures customers do not misrepresent their preferences strategically. This kind of restriction, called *incentive compatibility* (IC), is central in the theory and arises by the asymmetry of information.

In the one-dimensional version of the problem when single-crossing condition holds, since there is complete order of preferences, IC constraints binding for each type are determined by local conditions. The resulting optimal quality allocation increases according with types' order, and just customers' type with highest valuation for the product gets same quality as when there is not asymmetry of information, which is the case when firm sets efficient prices at marginal cost level. To give and economic intuition of this result, we appeal to the excellent explanation given by Matthews and Moore (1987):

"The intuition behind its solution starts with the observation that profit is potencially greatest on contracts designed for 'high' type consumers, those with a high evaluation of quality. Because high type consumers cannot be prevent from choosing contracts meant for low types, this profit can only be realized by distorting the contracts meant for low types in a direction that makes them relatively unattractive to high types. All but the highest type of consumer should therefore receive products of inefficiently low quality."

In the multidimensional version, and particulary in the bidimensional case we are concern with, there is not general treatment to deal with the problem. The essential difficulty is the lack of an exogenous complete order of preferences, which implies that we need to look for which IC constraints are binding in a far larger set of global constraints, for each type of customer.

Some techniques have been developed to reach the solution, for example by Laffont et al. (1987), Basov (2001), Deneckere and Severinov (2015) and Araujo and Vieira (2010). However, with these techniques we are limitated to use simple forms of customers' valuation function (generally linear-quadratic forms or extensions) and type's distribution (usually uniform distribution).

Therefore, because in general we cannot obtain closed-form solutions, we could attempt to get numerical approximations of the solution discretizing agents' type set, as the model is formulated in a continuous way. However, due to the IC constraints, computational difficulties are severe when types are multidimensional. For instance when agents' type set is a square in $\mathbb{R}^2$ taking n points over each dimension derives on $n^4 - n^2$ (usually non-convex) IC constraints, which may lead to memory storage problems if discretization is fine enough (Parra (2014) has reported computer's memory exhausted for tests with $n = 14$).

Some numerical methods to deal with the problem are described in Wilson (1995). Although these methods were formulated allowing multidimensional types and product, they were designed to solve a *relax* version of the problem in which *only local* incentive compatibility constraints are assume to be binding. Even when local IC constraints are sufficient in one-dimension, this is not the usual case in multidimensions, so we cannot rely on these approximations as the solution of the complete problem.

The main contribution of this work is to prove that is sufficient to consider, for each type, IC constraints over a unidimensional set instead of the whole bidimensional type set as it is required by definition. With this result the number of IC constraints in the discretized problem is of order n^3, making it computationally tractable for relative fine discretization. Thus, we are able to have well educated predictions about some features of the solution, such as optimal quality, agents' surplus and optimal tariff shapes, as well as the participation set and how types are bunching. Specially when valuation function is such that IC constraints are convex, as we will see in two applications.

We are extending the ideas being applied in the unidimensional case with finite types when single-crossing condition holds. With the aim of extend the order induce by single-crossing in one-dimension, we introduce a pre-order among types by their marginal valuation for the product. We also consider the possible shape of isoquants using a PDE derived in Araujo and Vieira (2010). Our main assumption is the validity of single-crossing over each direction of uncertainty. Therefore, our approach is valid for a large class of valuation function and types' distributions.

This work is organized as follows. In Chapter 2 we describe the general model, review the main contributions from the literature and discuss with more detail two topics that will be important for further chapters. Chapter 3 is dedicated to our main contribution, first explaining the ideas we are extending from unidimensional case, and then justifying the reduction of IC constraints. We test our approach comparing numerical and analytic solutions of three examples already solved in the literature. Additionally, three new examples are proposed. In Chapter 4 we use our approach to analyze and give additional insights in two models already studied. In the first one we provide some conclusions about monopolist's behavior varying types' distribution. In the second one we provide the numerical solution of a model with unknown solution, which allows us to give relevant conclusions. We hope that this work will be useful for the kind of analysis that this chapter contains.

Chapter 2

Preliminaries

The aim of this chapter is to describe the problem we are concerned with and, because shall be important for further chapters, to expose some topics with more detail. In Section 1 we establish the formal model in the framework of nonlinear pricing by a monopolist and review some main results from the literature in Section 2. Armstrong's result about the optimality of exclusion is the subject of Section 3. Section 4 presents the methodology used by Araujo and Vieira (2010) who derived a necessary condition of optimality. This section also contains our first contribution.

2.1 The Model

Consider a monopolistic firm (the principal) supplier of N different goods $q \in Q \subset \mathbb{R}^N_+$ at cost $C(q)$ facing a population of customers (the agents). Customers' characteristics, reflecting their preferences over the products, are captured by a vector $\theta \in \Theta \subset \mathbb{R}^M$ which we will refer as their *type*. This type is private information of each customer, but the monopolist knows the customers' population distribution over the set Θ according to a strictly positive density function f.

A common assumption is that agents' utility is quasilinear (linear in money transfer), i.e., the utility of type θ for consumption q and payment $t \in \mathbb{R}_+$ is

$$v(q, \theta) - t$$

where $v(q, \theta)$ is the $\theta-$type agent's valuation when consumes q.

The firm is able to design a menu of options to offer to the agent specifying the quantity and corresponding payment according with customers' type revealed. This menu of options $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$ it is call a *contract*. The monopolist would wish to extract the maximum payment from each agent ensuring that agent's benefit is at least his reservation utility (the utility of the outside option); this is the *individual rationality* (IR) constraint. Also, the monopolist tries to avoid that customers take advantage of their private information because distortion of true preferences could lead to nonoptimal firm's income. By the Revelation Principle (Myerson (1979)), this can be done restricting to a class of contracts where true-telling is the best response for the agents. That is, the contract offered must ensure that agents have not the incentive of misrepresent their true type; this is the *incentive compatibility* (IC) constraint.

Thus, because firm's objective is to maximize expected net income, the monopolist's problem is

$$\max_{q(\cdot),t(\cdot)} \int_{\Theta} \{t(\theta) - C(q(\theta))\} f(\theta) d\theta$$

subject to

$$\begin{aligned} &(\text{IR}) \quad v(q(\theta),\theta) - t(\theta) \geq 0 && \forall\, \theta \in \Theta \\ &(\text{IC}) \quad v(q(\theta),\theta) - t(\theta) \geq v(q(\widehat{\theta}),\theta) - t(\widehat{\theta}) && \forall\, \theta, \widehat{\theta} \in \Theta \end{aligned}$$

In IR constraints we are assuming that reservation utility is type independent and normalized it to be zero. Note that IC constraints can be written as

$$\theta \in \operatorname*{argmax}_{\widehat{\theta}\in\Theta} \{v(q(\widehat{\theta}),\theta) - t(\widehat{\theta})\} \;\; \forall\, \theta \in \Theta$$

then, given (q, t) agents are maximizing their own utility. We say that $q : \Theta \to \mathbb{R}_+$ is *implementable* if there exists $t : \Theta \to \mathbb{R}_+$ such that (q, t) satisfies the IC constraints.

The Taxation Principle (Guesnerie (1981), Rochet (1985)) establishes that any incentive compatible contract (q, t) can be implemented by a *tariff* $T : Q \to \mathbb{R}_+$ s.t. $T(q(\theta)) = t(\theta)$. Faced with a tariff T, the $\theta-$agent will choose the bundle that maximizes his utility, then the IC constraints can be written as:

$$q(\theta) \in \underset{q \in Q}{\operatorname{argmax}}\{v(q,\theta) - T(q)\} \quad \forall\, \theta \in \Theta$$

Given an incentive compatible contract (q, t), *agents' surplus* or *informational rent* is defined as

$$\begin{aligned} V(\theta) &= \max_{\widehat{\theta} \in \Theta}\{v(q(\widehat{\theta}), \theta) - t(\widehat{\theta})\} \\ &= v(q(\theta), \theta) - t(\theta) \end{aligned}$$

We can use the informational rent to replace transfers in the principal's objective function and transform the monopolist's problem

$$\max_{q(\cdot), V(\cdot)} \int_{\Theta} \{v(q(\theta), \theta) - C(q(\theta)) - V(\theta)\} f(\theta) d\theta$$

subject to

$$\begin{array}{lll} \text{(IR)} & V(\theta) \geq 0 & \forall\, \theta \in \Theta \\ \text{(IC)} & V(\theta) - V(\widehat{\theta}) \geq v(q(\widehat{\theta}), \theta) - v(q(\widehat{\theta}), \widehat{\theta}) & \forall\, \theta, \widehat{\theta} \in \Theta \end{array}$$

With this change of variables the new expression of the objective function can be seen as the expected social value of trade $(\{v(q(\theta), \theta) - t(\theta)\} + \{t(\theta) - C(q(\theta))\})$ minus the expected informational rent of the agents.

2.2 A brief survey

2.2.1 Unidimensional case

The unidimensional case $(N = 1, M = 1)$ of this problem has been extensively studied in the literature[1]. When Spence-Mirrless or single-crossing condition $v_{q\theta} > 0$ is satisfied, first and second order necessary conditions for agent's maximization

[1] Seminal papers are Mussa and Rosen (1978), Maskin and Riley (1984)

problem are also sufficient. These conditions are equivalent to $V'(\theta) = v_\theta(q(\theta), \theta)$ for a.e. $\theta \in [\underline{\theta}, \overline{\theta}]$ and to q being non-decreasing function of θ. From the former condition, $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_\theta(q(\xi), \xi) d\xi$ then, after replacing V into firm's objective via integration by parts, the constant $V(\underline{\theta})$ negatively affects monopolist' expected income, therefore making $V(\underline{\theta}) = 0$ is the best firm's option. The result is the following problem

$$\max_{q(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \{v(q(\theta), \theta) - C(q(\theta)) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q(\theta), \theta)\} f(\theta) d\theta$$

subject to

$$q(\cdot) \quad \text{non-decreasing}$$

where F is the cumulative distribution of f. What we have done is decompose the problem: first, agents' surplus V is expressed as a function of q and second, the allocation q that maximizes firms' revenue net of the expected agents' surplus computed previously is determined. This is known as the direct approach of the problem.

If we relax the condition of q being non-decreasing we have a classic calculus variation problem and by the Euler's equation[2] we obtain a function that may be non-decreasing. If this is not the case, by the Ironing Procedure[3] a non-decreasing function is defined, which is the solution of the complete program.

As we have seen, the unidimensional problem can be solved by simple considerations if the single-crossing property is valid, because global incentive compatibility constraints are determined by local conditions and implementable allocations can be completely characterized. Araujo and Moreira (2010) studied the model when the single-crossing condition is relaxed allowing $v_{q\theta}$ to change the sign over $Q \times \Theta$. Then, local IC constraints are not sufficient anymore for implementability and nonlocal constraints can be binding. In this case, implementable allocations q may not be monotonic and they concluded that types getting the same allocation receives the same marginal tariff.

---

[2] See Kamien and Schwartz (1981)

[3] See Mussa and Rosen (1978), or Fundenberg and Tirole (1991) for details.

2.2.2 Multidimensional case

In multidimensional problems, McAfee and McMillan (1988) introduced a generalized single-crossing property and concluded that first and second-order necessary conditions are also sufficient for implementability; however, it forces the set of types getting the same allocation to be hyperplanes.

In a celebrated paper, Rochet (1987) provides the following characterization: q is implementable if and only if for all finite cycles $\{\theta_0, \theta_1, \ldots, \theta_{J+1}\} \subset \Theta$, with $\theta_{J+1} = \theta_0$

$$\sum_{k=0}^{J} \Big(v(q(\theta_k), \theta_{k+1}) - v(q(\theta_k), \theta_k) \Big) \leq 0$$

This result is quite general since it not assumes neither special structure of Θ nor regularity conditions of v; however, in practice it is not useful. When v is linear in types and Θ convex, Rochet (1987) also shows that q is implementable if and only if there exists a convex function V such that

$$\nabla_\theta v(q(\theta), \theta) = V'(\theta) \quad \text{a.e. } \theta$$

Carlier (2001) generalizes previous result in terms of $v-$convex functions and, with this equivalence, he obtains an existence result for the problem although with strong conditions on v.

At this point it is important to mention the work of Monteiro and Page (1998) where another existence result is provided. They considered customers with budget constraints which derive in compactness considerations. In their model, existence is guarantee only if other goods are available and only if the monopolist's goods are nonessential relative to other goods.

Rochet and Choné (1998) analyzed the especial case that quality dimension equals types dimension ($N = M$) and agents' valuation is $v(q, \theta) = \langle \theta, q \rangle$. In this model, Rochet's implementability characterization derives on $\nabla_\theta v(q, \theta) = q = V'(\theta)$, so the authors expressed firm's objective in terms of V and V' with the constraint of V being non-negative and convex. Thus, agents' informational rent is used as the instrument chosen by the monopolist; this is known as the dual approach. Their main contributions were to prove existence of the optimal contract in both the relax (i.e. without the convex constraint) and complete problem, provide its characterization, and introduce the Sweeping Procedure which generalizes the Ironing Procedure to multidimensions.

One of the main differences between unidimensional and multidimensional cases is the optimality of exclusion, discovered by Armstrong (1996) (next section explains with more detail this result). In that work some multidimensional examples were explicit solved, as well in Wilson (1993) [4]. However, as was noticed by Rochet and Stole (2003) [5], in all these examples types set can be partitioned a priori into unidimensional subsets, i.e., are special cases of problems that can be reduced.

Basov (2001) introduced the Hamiltonian approach and showed that, when $N > M$, qualitative features of the solution are alike of those found by Rochet and Choné (1998). Additionally, when $N < M$ full separation of types is not possible over open sets of Θ, and q may be discontinuous in the lower boundary of participation set. In Basov (2005) some examples are solved with that technique.

In this work we are concerned with the specific case $N = 1$ and $M = 2$, i.e., one-dimension of the product offered and two dimensions of customers' attributes. For this case, Laffont et al. (1987) have found the explicit solution when customers' demand function is linear, and the monopolist is uncertain about both the slope and intercep that define such individual demand function. They have shown that the optimal tariff is convex. In this particular problem customers' valuation of consume v has linear-quadratic form, a class of functions that has been explored in several examples.

2.3 Desirability of Exclusion

Armstrong (1996) demonstrated that excluding types with low product's valuation is generally optimal for the firm in the multidimensional case. This is a salient result because in one-dimesional setup it is optimal to serve all types.

The intuiton of that result is straightforward: If it were the case that it is optimal to serve all customers, increasing the tariff by $\varepsilon > 0$ the monopolist could get extra gain from types who remain in the market but has to assume the lost (not more

[4] In Wilson (1993) the called *demand-profile* approach is broadly explained.

[5] Rochet and Stole (2003) is one of the main surveys of multidimensional screening.

income) from types who will exit (those with surplus $< \varepsilon$). As it was proved in that paper, the former is greater than the latter for small values of ε, so exclusion will be optimal. The theorem goes as follow:

Theorem 2.3.1 (Armstrong(1996))**.** *Let v be such that $v(0,\theta) = 0$, $v(q,0) = 0$, $v(q,\cdot)$ convex, increasing and homogenous of degree one, and consider $\Theta \subset \mathbb{R}^M$ ($M \geq 2$) closed, strictly convex, and of full dimension in $\mathbb{R}^M$. Then, at the optimum the set*

$$\{\theta \in \Theta : V(\theta) = 0\}$$

has positive Lebesgue measure.

The proof of this theorem can be find in Armstrong (1996) or Basov (2005).

Barelli et al. (2014) have extended Armstrong's result providing alternative sufficient conditions not relying on any form of convexity, and then have shown that exclusion is obtained generically. Deneckere and Severinov (2015) have provided necessary and sufficient conditions to ensure full participation, although this means a strictly positive quantity (common in nonlinear pricing) instead of strictly positive informational rent.

2.4 Characteristic Curves

This section expose a summary of the methodology used by Araujo and Vieira (2010)[6] to derive a necessary condition for optimality when $N = 1$ and $M = 2$. They have assumed validity of single-crossing on each axis, with perfect negative correlation between the two dimentions: $v_{qa} > 0$, $v_{qb} < 0$. They have also assumed $v_a > 0$, $v_b < 0$.

Let (q,t) be an incentive compatible contract. Suppose that q and t are continuous and a.e. twice continuously differentiable functions. For any $(a,b) \in [0,1]^2$ we have

$$(a,b) \in \underset{(\widehat{a},\widehat{b}) \in [0,1]^2}{\operatorname{argmax}} \{v(q(\widehat{a},\widehat{b}),a,b) - t(\widehat{a},\widehat{b})\}$$

[6] See also Vieira (2008)

The first-order necessary condition imply

$$v_q(q(a,b),a,b)q_a(a,b) = t_a(a,b)$$

$$v_q(q(a,b),a,b)q_b(a,b) = t_b(a,b)$$

At any point (a,b) of twice continuous differentiability by the Young's theorem $t_{ab} = t_{ba}$, then an implementable allocation rule q must satisfy a.e.[7]

$$-\frac{v_{qb}}{v_{qa}}q_a + q_b = 0 \tag{2.1}$$

Following the Characteristic Method to solve previous PDE, if an initial curve $\phi(r)$ is fixed over the segment $\{(r,0) : r \in [\overline{r},1]\}$ for some $\overline{r} \in (0,1)$, there is a family of plane characteristics curves $(a(r,s),b(r,s))$ forming a partition on the participation set's interior, and satisfying:

$$\begin{aligned} a_s(r,s) &= -\frac{v_{qb}}{v_{qa}}(\phi(r),a(r,s),b(r,s)) \quad , \quad a(r,0) = r \\ b_s(r,s) &= 1 \qquad\qquad\qquad\qquad\qquad\quad , \quad b(r,0) = 0 \end{aligned}$$

For a fixed r, all types over the characteristic curve $(a(r,s),b(r,s))$ gets the same allocation $\phi(r)$. That is, plane characteristic curves are the isoquants of $q(\cdot,\cdot)$. Moreover, all the isoquants are strictly increasing in view of $-\frac{v_{qb}}{v_{qa}} > 0$.

Now, in view of (a,b) can be expressed in terms of new variables (r,s), after replacing into the expected profit the monopolist's problem can be set as

$$\max_{\phi(\cdot)} \int_{\underline{r}}^{1} \int_{0}^{s(r)} G(\phi(r),a(r,s),b(r,s)) \Big| \frac{\partial(a,b)}{\partial(r,s)} \Big| ds\, dr$$

where G is the virtual surplus[8] and $s(r)$ is such that $a(r,s(r)) = 1$.

[7] The same PDE is derived with the necessary implementability condition given by Rochet (1987) (see Proposition 3 in that paper): Assume $\exists\ A \in \mathbb{R}^{2\times 2}$ s.t. $\forall\ (q,(a,b)) \in Q\times[0,1]^2$ $\nabla^2_\theta v(q,(a,b)) - A$ is positive semi-definite. Then, for q to be implementable, it is necessary that $(a,b) \to \nabla_\theta v(q(a,b),a,b)$ is a.e. differentiable, and $\text{rot}[\nabla_\theta v(q(a,b),a,b)] = 0$ a.e.
With this result, those strong differentiability assumptions of q and t are not needed.

[8] In Appendix A we show how to determine virtual surplus G

Since $\left|\dfrac{\partial(a,b)}{\partial(r,s)}\right| = a_r(r,s)$ depends on $\phi'(r)$ (because in general $a(r,s)$ is a function of $\phi(r)$), after considering the auxiliary function

$$H(r,\phi,\phi') = \int_0^{s(r)} G(\phi(r), a(r,s), b(r,s)) a_r(r,s) ds$$

and by the Euler's equation $H_\phi - \frac{d}{dr}H_{\phi'} = 0$, the authors have derived the next necessary optimality condition when isoquants intersects the line $y = 0$ [9].

$$\int_0^{s(r)} \frac{G_q}{v_{qa}}(\phi(r), a(r,s), b(r,s)) ds = 0 \tag{2.2}$$

On the other hand, if characteristic curves intersects participation's boundary $\{(r,\beta(r)) : r \in [r_1, r_2]\}$ taking this boundary as the initial curve to solve the PDE (2.1), the authors have established the existence of a function $\lambda(\cdot)$ defined over some interval $[r_1, r_2]$ such that[10]:

$$\int_{\beta(r)}^{\overline{s}(r)} \frac{G_q}{v_{qa}}(\phi(r), a(r,s), b(r,s)) ds = \lambda(r) \tag{2.3}$$

$$\frac{G}{v_b}(\phi(r), r, \beta(r)) = \lambda'(r) \tag{2.4}$$

where, in this case, $a(r,s)$, $b(r,s)$ are the solutions of

$$\begin{aligned} a_s(r,s) &= -\frac{v_{qb}}{v_{qa}}(\phi(r), a(r,s), b(r,s)) \quad , \quad a(r,0) = r \\ b_s(r,s) &= 1 \quad , \quad b(r,0) = \beta(r) \end{aligned}$$

and $\overline{s}(r)$ is such that $a(r,\overline{s}(r)) = 1$ (if isoquants intersects $x = 1$) or $b(r,\overline{s}(r)) = 1$ (if isoquants intersects $y = 1$).

Thus, we dispose of necessary conditions for optimality in two different cases: when isoquants intersects axis X or when isoquants intersects participation's boundary.

[9] This is Theorem 1 in Araujo and Vieira Araujo and Vieira (2010)

[10] See Theorem 2 in Araujo and Vieira Araujo and Vieira (2010) for a detailed explanation

2.4.1 Special Case

If characteristic curves are concurrent at some point $(\overline{x}, \overline{y})$, we establish a necessary condition for optimality, not established by Araujo and Vieira (2010) but with the same methodology. This is the case when type $(\overline{x}, \overline{y})$ is indifferent between any quantity in some interval $[\underline{q}, \overline{q}]$.

Following the characteristic method to solve the PDE (2.1), and considering that isoquants intersects the line $y = 1$ [11], we fix $\{(r, 1) : r \in [R_1, R_2]\}$ as initial curve. Then, we have to solve:

$$\begin{aligned} a_s(r,s) &= -\frac{v_{qb}}{v_{qa}}(\phi(r), a(r,s), b(r,s)) \quad , \quad a(r,1) = r \\ b_s(r,s) &= 1 \quad , \quad b(r,1) = 1 \end{aligned} \tag{2.5}$$

where $\phi : [R_1, R_2] \to [\underline{q}, \overline{q}]$ describes the quantity (or quality) allocated to $(r, 1)$. This function is strictly increasing in view of the assumption $v_{qa} > 0$.

If $a(r,s) = A(\phi(r), r, s)$, $b(r,s) = s$ are solutions of (2.5), and $\varphi : [\underline{q}, \overline{q}] \to [R_1, R_2]$ is the inverse of ϕ, a and b can be expressed in terms of new variables q and s (q for quantity and s for the position on the characteristic curve of (a, b)). Hence

$$a(q,s) = A(q, \varphi(q), s) \quad , \quad b(q,s) = s$$

where $q \in [\underline{q}, \overline{q}]$, $s \in [\overline{y}, 1]$.

Besides, the fact that type $(\overline{x}, \overline{y})$ is indifferent between any $q \in [\underline{q}, \overline{q}]$ gives us a special restriction: Fix any $q \in [\underline{q}, \overline{q}]$. Due to $(\overline{x}, \overline{y})$ and $(\varphi(q), 1)$ gets the same allocation q, such q is the solution of both

$$\max_{\tilde{q}}\{v(\tilde{q}, \overline{x}, \overline{y}) - T(\tilde{q})\} \quad , \quad \max_{\tilde{q}}\{v(\tilde{q}, \varphi(q), 1) - T(\tilde{q})\}$$

from which $v_q(q, \overline{x}, \overline{y}) = v_q(q, \varphi(q), 1)$, $\forall\, q \in [\underline{q}, \overline{q}]$. Then

$$\int_{\underline{q}}^{\overline{q}} v_q(q, \overline{x}, \overline{y}) - v_q(q, \varphi(q), 1)\, dq = 0$$

[11] The case of isoquants intersecting the line $x = 1$ is analogous

so this new restriction must be considered into the optimization problem:

$$\begin{aligned} &\max_{\varphi(\cdot)} \int_{\underline{q}}^{\overline{q}} \int_{\overline{y}}^{1} G(q, A(q, \varphi(q), s), s)(A_\varphi \varphi' + A_q)\, ds\, dq \\ &\text{subject to} \\ &\int_{\underline{q}}^{\overline{q}} v_q(q, \overline{x}, \overline{y}) - v_q(q, \varphi(q), 1)\, dq = 0 \end{aligned} \tag{2.6}$$

As we see, an isoperimetric problem arise in case an agent is indifferent between any allocation in a range. Thus, the integral constraint constitute the main difference with cases analyzed in Araujo and Vieira (2010).

From this problem, we derive the following necessary optimal condition

Proposition 2.4.1. *If $\varphi = \varphi(q)$ is optimal, then $\exists\ \lambda \in \mathbb{R}$ such that*

$$\int_{\overline{y}}^{1} \frac{G_q}{v_{qa}}(q, A(q, \varphi, s), s)\, ds = \lambda \tag{2.7}$$

The proof is left to Appendix A.

2.4.2 Example

This subsection is dedicated to give a complete solution of the following example, which is a generalization of the example solved in Laffont et al. (1987) [12]

$$v(q, a, b) = aq - (b + c)\frac{q^2}{2}\ ,\ \text{with } c \in (0, \tfrac{1}{2})\quad ,\quad C(q) = 0\quad ,\quad f(a, b) = 1$$

Araujo and Vieira (2010), using the necessary conditions (2.2), (2.3) and (2.4) given by them, have partially found the solution by dividing the analysis into two cases

1. Isoquants intersects line $y = 0$ and line $x = 1$.
   In this case the optimal allocation for $(r, 0)$ types is

$$\phi^I(r) = \frac{3r - 1}{2c}\ ,\quad r \in [\underline{r}, 1] \quad \text{where } \underline{r} \in (\tfrac{2c+1}{2c+3}, \tfrac{1}{2})$$

[12] The same example was also analized in Deneckere and Severinov (2015) with another technique.

2. Isoquants intersects the participation's boundary $\beta(r)$ and line $x = 1$.
   In this case the participation's boundary is

$$\beta(r) = \frac{2r^3 - 4r^2 + 2r}{K_0} + K_1 \quad (K_0, K_1 \text{ constants})$$

   and the optimal allocation rule for $(r, \beta(r))$ types is

$$\phi^{II}(r) = \frac{K_0}{3r^2 - 4r + 1} \quad , \quad r \in [\underline{r}, \overline{x}]$$

Setting $\overline{y} = \beta(\overline{x})$, we are able to complete the solution of this example by the following claims:

Claim 1.
There are no isoquants intersecting participation's boundary and $y = 1$ line.

Claim 2.
There are no isoquants concurrent at the point $(\overline{x}, \overline{y})$ and intersecting the line $x = 1$.

Claim 3.
There exists isoquants concurrent at the point $(\overline{x}, \overline{y})$ and intersecting the line $y = 1$. In this case $\overline{y} = \dfrac{1-2c}{3}$ and the optimal allocation rule for $(r, 1)$ types is

$$\phi^{III}(r) = \frac{r - \overline{x}}{1 - \overline{y}} \quad , \quad r \in [\overline{x}, 1]$$

In order to determine the constants K_0, K_1, $\underline{r}$, and $\overline{x}$ note that, by continuity of the allocation rule and boundary conditions, it must be true:

1. $\phi^I(\underline{r}) = \phi^{II}(\underline{r})$
2. $\phi^{II}(\overline{x}) = \phi^{III}(1)$
3. $\beta(\underline{r}) = 0$
4. $\beta(\overline{x}) = \overline{y}$

Conditions 1. and 3. implies

$$K_0 = \frac{-(1-\underline{r})(3\underline{r}-1)^2}{2c} \quad , \quad K_1 = \frac{4c\underline{r}(1-\underline{r})}{(3\underline{r}-1)^2} \tag{2.8}$$

Because of 4., we can write $\phi^{III}(r) = (r - \overline{x})/(1 - \beta(\overline{x}))$, so condition 2. yields on $\beta(\overline{x}) = 1 - \frac{(1-3\overline{x})(1-\overline{x})^2}{K_0}$. Then, using the expression of β and (2.8)

$$(1 - \overline{x})^3 = \frac{(9 + 4c)\underline{r}^3 - (15 + 8c)\underline{r}^2 + (7 + 4c)\underline{r} - 1}{2c} \tag{2.9}$$

On the other hand, by conditions 2. and 4.

$$(1 - 3\overline{x})(\overline{x} - 1)^2 = K_0(1 - \overline{y}) \quad \text{and} \quad 2\overline{x}(\overline{x} - 1)^2 = K_0(\overline{y} - K_1)$$

dividing (seeing that $\frac{1}{3} < \frac{2c+1}{2c+3} < \overline{x} < 1$), clearing $\overline{x}$, and using (2.8), we get

$$1 - \overline{x} = \frac{(54 + 24c)\underline{r}^2 - (36 + 24c)\underline{r} + 6}{(63 + 18c)\underline{r}^2 - (42 + 24c)\underline{r} + (7 - 2c)}$$

therefore, by (2.9), we have that $\underline{r}$ is the solution on $(\frac{2c+1}{2c+3}, \frac{1}{2})$ of

$$\Big(\frac{(54 + 24c)r^2 - (36 + 24c)r + 6}{(63 + 18c)r^2 - (42 + 24c)r + (7 - 2c)}\Big)^3 = \frac{(9 + 4c)r^3 - (15 + 8c)r^2 + (7 + 4c)r - 1}{2c}$$

and with that $\underline{r}$, we obtain

$$\overline{x} = \frac{(9 - 6c)\underline{r}^2 - 6\underline{r} + 1 - 2c}{(63 + 18c)\underline{r}^2 - (42 + 24c)\underline{r} + (7 - 2c)} \tag{2.10}$$

Thus, all the elements defining ϕ^I, ϕ^{II}, ϕ^{III}, and β are determined as well as the special point $(\overline{x}, \overline{y})$. This type $(\overline{x}, \overline{y})$ is indifferent between any quantity in the interval $[0, \frac{3(1-\overline{x})}{2(1+c)}]$, while the optimal quantity allocation range is $[0, \frac{1}{c}]$.
To express the optimal quantity in terms of (a, b), note that the type set $[0, 1]^2$ can be partitioned into four sets Z^0, Z^I, Z^{II} and Z^{III} defined as:

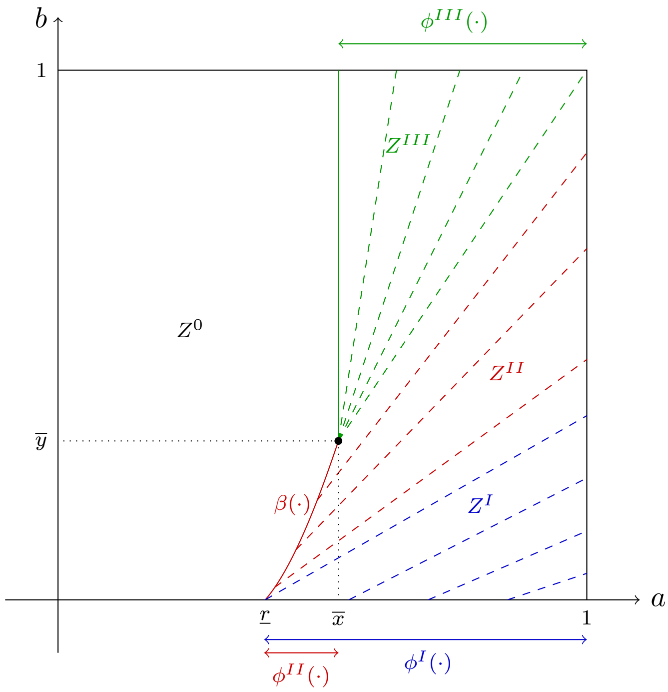
$$\begin{aligned}
Z^0 &= \{(a, b) \in [0, 1]^2 : a < \overline{x} \ \wedge \ b > \beta(a)\} \\
Z^I &= \{(a, b) \in [0, 1]^2 : b \leq (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1}\} \\
Z^{II} &= \{(a, b) \in [0, 1]^2 : a \geq \overline{x} \ \wedge \ b > (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1} \ \wedge \ b \leq (\tfrac{1-\overline{y}}{1-\overline{x}})a + \tfrac{\overline{y}-\overline{x}}{1-\overline{x}}\} \\
&\quad \cup \{(a, b) \in [0, 1]^2 : a < \overline{x} \ \wedge \ b > (\tfrac{2c}{3\underline{r}-1})a - \tfrac{\underline{r}}{3\underline{r}-1} \ \wedge \ b \leq \beta(a)\} \\
Z^{III} &= \{(a, b) \in [0, 1]^2 : a \geq \overline{x} \ \wedge \ b > (\tfrac{1-\overline{y}}{1-\overline{x}})a + \tfrac{\overline{y}-\overline{x}}{1-\overline{x}}\}
\end{aligned}$$

here Z^0 is the exclusion region, so $q(a,b)=0$ if $(a,b)\in Z^0$. Given $(a,b)\in[0,1]^2\setminus Z^0$, $r(a,b)$ is defined as the solution of

$$\begin{aligned} a = \phi^I(r)b + r \quad & \text{if } (a,b)\in Z^I \\ a = \phi^{II}(r)(b-\beta(r)) + r \quad & \text{if } (a,b)\in Z^{II} \\ a = \phi^{III}(r)(b-1) + r \quad & \text{if } (a,b)\in Z^{III} \end{aligned}$$

finally, $q(a,b)$ is defined as $q(a,b)=\phi^k(r(a,b))$ if $(a,b)\in Z^k$, $k=I,II,III$.

Figure 2.1: Isoquant curves of the solution



2.5 Appendix A

Determining virtual surplus G:
Here we show how the virtual surplus G for the bidimensional case can be determined if the optimal quantity allocated to types $(0,b)$ is the exclude option quantity q^{out}, that is $q(0,b) = q^{\text{out}}$. Also, assume $v(q^{\text{out}}, a, b)$ is constant and distributions f over a and g over b are independent, so $\rho(a,b) = f(a)g(b)$. We will show

$$G(q,a,b) = \Big(v(q,a,b) - C(q) - \frac{1-F(a)}{f(a)} v_a(q,a,b)\Big) f(a)g(b)$$

Note that, by the Fundamental Theorem of Calculus and the Envelope Theorem

$$\begin{aligned} V(a,b) - V(0,1) &= V(a,b) - V(0,b) + V(0,b) - V(0,1) \\ &= \int_0^a V_a(\tilde{a},b)d\tilde{a} + \int_1^b V_b(0,\tilde{b})d\tilde{b} \\ &= \int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a} + \int_1^b v_b(q(0,\tilde{b}),0,\tilde{b})d\tilde{b} \end{aligned}$$

since $V(0,1) = 0$ and $v_b(q(0,\tilde{b}),0,\tilde{b}) = v_b(q^{\text{out}},0,\tilde{b}) = 0$ (because $v(q^{\text{out}},a,b)$ is constant), we have

$$V(a,b) = \int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}$$

Then, through integration by parts

$$\begin{aligned} &\int_0^1 \int_0^1 V(a,b)\rho(a,b)\, da\, db \\ &= \int_0^1 \int_0^1 \Big(\int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}\Big) f(a)g(b)\, da\, db \\ &= \int_0^1 \int_0^1 \Big(\int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}\Big)\, dF(a)\, g(b)\, db \\ &= \int_0^1 \Big[\Big(\int_0^a v_a(q(\tilde{a},b),\tilde{a},b)d\tilde{a}\Big) F(a)\Big|_{a=0}^{a=1} - \int_0^1 F(a) v_a(q(a,b),a,b)da\Big]\, g(b)\, db \\ &= \int_0^1 \Big[\int_0^1 v_a(q(a,b),a,b)\, da - \int_0^1 F(a)v_a(q(a,b),a,b)da\Big]\, g(b)\, db \\ &= \int_0^1 \int_0^1 \Big(\frac{1-F(a)}{f(a)} v_a(q(a,b),a,b)\Big) f(a)g(b)\, da\, db \end{aligned}$$

Thus, the expected income

$$\int_0^1 \int_0^1 \Big(v(q(a,b),a,b) - C(q(a,b)) - V(a,b) \Big) f(a)g(b)\, da\, db$$

can be written as

$$\int_0^1 \int_0^1 G(q(a,b),a,b)\, da\, db$$

□

Proof of Proposition 2.4.1.
Because $a(q,s) = A(q,\varphi(q),s)$, $b(q,s) = s$, we have

$$\frac{\partial(a,b)}{\partial(q,s)} = \begin{vmatrix} a_q & b_q \\ a_s & b_s \end{vmatrix} = \begin{vmatrix} a_q & 0 \\ a_s & 1 \end{vmatrix} = a_q = A_\varphi \varphi' + A_q$$

At this point, the aditional assumption $a_q > 0$ is requiered [13]. Then, the revenue can be written as

$$\int_{\underline{q}}^{\overline{q}} \int_{\overline{y}}^1 G(q, A(q,\varphi(q),s),s) \times \left| \frac{\partial(a,b)}{\partial(q,s)} \right| ds\, dq = \int_{\underline{q}}^{\overline{q}} \int_{\overline{y}}^1 G(q, A(q,\varphi(q),s),s)(A_\varphi \varphi' + A_q)\, ds\, dq$$

which explain objective's function in (2.6) takes that form. For such isoperimetric problem, the necessary condition for optimality is [14]

$$H_\varphi - \frac{d}{dq}(H_{\varphi'}) = \lambda (F_\varphi - \frac{d}{dq}(F_{\varphi'}))$$

for some $\lambda \in \mathbb{R}$, where

$$H(q,\varphi,\varphi') = \int_{\overline{y}}^1 G(q, A(q,\varphi,s),s)(A_\varphi \varphi' + A_q)\, ds$$
$$F(q,\varphi,\varphi') = v_q(q,\overline{x},\overline{y}) - v_q(q,\varphi,1)$$

[13] Since $\varphi' > 0$, it would be sufficient to assume that $a_r > 0$ (in the original variables) which seems natural due to for any fix s , $a(r,s)$ should be increasing in r (for example, if $s = 1$ over the initial curve $\{(r,1) : r \in [R_1, R_2]\}$ first component is increasing in r)

[14] See Kamien and Schwartz (1981)

We have

$$\begin{aligned}
H_\varphi &= \int_{\overline{y}}^{1} G_a A_\varphi(A_\varphi \varphi' + A_q) + G(A_{\varphi\varphi}\varphi' + A_{\varphi q}) ds \\
H_{\varphi'} &= \int_{\overline{y}}^{1} G A_\varphi \, ds \\
\frac{d}{dq}(H_{\varphi'}) &= \int_{\overline{y}}^{1} (G_q + G_a(A_\varphi \varphi' + A_q))A_\varphi + G(A_{\varphi\varphi}\varphi' + A_{\varphi q}) \, ds \\
F_\varphi &= -v_{qa}(q, \varphi, 1) \\
F_{\varphi'} &= 0
\end{aligned}$$

Then

$$\begin{aligned}
H_\varphi - \frac{d}{dr}(H_{\varphi'}) &= -\int_{\overline{y}}^{1} G_q(q, A(q, \varphi, s), s) A_\varphi(q, \varphi, s) \, ds \\
F_\varphi - \frac{d}{dq}(F_{\varphi'}) &= -v_{qa}(q, \varphi, 1)
\end{aligned}$$

Since for any q and s fixed, $v_q(q, \varphi, 1) = v_q(q, A(q, \varphi, s), s)$, taking the derivative with respect to φ:

$$v_{qa}(q, \varphi, 1) = v_{qa}(q, A(q, \varphi, s), s) A_\varphi(q, \varphi, s)$$

thus, we can replace $A_\varphi(q, \varphi, s)$ and obtain, as a necessary optimal condition, that exists some $\lambda \in \mathbb{R}$ such that

$$-v_{qa}(q, \varphi, 1) \int_{\overline{y}}^{1} \frac{G_q}{v_{qa}}(q, A(q, \varphi, s), s) \, ds = -\lambda v_{qa}(q, \varphi, 1)$$

and since $v_{qa} > 0$

$$\int_{\overline{y}}^{1} \frac{G_q}{v_{qa}}(q, A(q, \varphi, s), s) \, ds = \lambda$$

□

Proof of Claim 1.
For this example $G(q, a, b) = (2a - 1)q - \frac{(b+c)}{2} q^2$, $v_{qa} = 1$ and $v_b(q, a, b) = -\frac{q^2}{2}$. Also $a(r, s) = s\phi(r) + r$, $b(r, s) = s + \beta(r)$ are the solutions of (2.5) system, and $\overline{s}(r) = 1 - \beta(r)$ because we are looking for isoquants intersecting $y = 1$.

Then, necessary conditions (2.3) and (2.4) yields on

$$\lambda(r) = \frac{3}{2}\beta(r)^2\phi(r) + (c-2)\beta(r)\phi(r) + \frac{1-2c}{2}\phi(r) + (2r-1)(1-\beta(r)) \quad (2.11)$$

$$\lambda'(r) = \frac{2(1-2r)}{\phi(r)} + \beta(r) + c \quad (2.12)$$

Taking the derivative on (2.11), by (2.12) we get

$$\phi'(r) = \frac{2(2-c-3\beta(r))}{(3\beta(r)+2c-1)(\beta(r)-1)} \quad (2.13)$$

Additionally $V(r, \beta(r)) = 0$ for all types over the boundary. Then, from $\frac{d}{dr}V(r, \beta(r)) = 0$ and by the Envelope Theorem, we have [15]

$$v_a(\phi(r), r, \beta(r)) + v_b(\phi(r), r, \beta(r))\beta'(r) = 0$$

for this example, this condition yields to

$$\phi(r)\beta'(r) = 2 \quad (2.14)$$

taking the derivative on (2.14), by (2.13) we obtain the following differential equation

$$\beta'' + \frac{(2-c-3\beta)}{(3\beta+2c-1)(\beta-1)}(\beta')^2 = 0$$

Thus, in case isoquants intersects the line $y = 1$ and participation's boundary β, such curve β satisfies previous differential equation.

The solutions (besides constant functions) are of the form

$$\beta(r) = \frac{e^{\sqrt{3}B_0 r}}{2\sqrt{3}B_1} + \frac{B_1(c+1)^2}{6\sqrt{3}}e^{-\sqrt{3}B_0 r} - \frac{c-2}{3}$$

with B_0, B_1 constants.

Note that, for this example, informational rent V is a convex function so the

[15]This condition was established in Araujo and Vieira (2010) before to derive the necessary conditions (2.3) and (2.4)

non-participation region $\Omega = \{(a,b) : V(a,b) = 0\}$ is a convex set and boundary curves must to be convex functions, i.e., $\beta''(r) \geq 0$ which implies $B_1 > 0$. Since $\frac{\sqrt{3}e^{\sqrt{3}B_0 r}}{(c+1)B_1} + \frac{(c+1)B_1}{\sqrt{3}e^{\sqrt{3}B_0 r}} \geq 2$ we have

$$\beta(r) = \Big(\frac{\sqrt{3}e^{\sqrt{3}B_0 r}}{(c+1)B_1} + \frac{(c+1)B_1}{\sqrt{3}e^{\sqrt{3}B_0 r}}\Big)\frac{(c+1)}{6} - \frac{c-2}{3} \geq 1$$

therefore, such curves cannot represent the boundary because are not contained in the interior of $[0,1]^2$. □

Proof of Claim 2.

First, we will establish the necessary condition in case isoquants are concurrent at the point $(\overline{x}, \overline{y})$ and intersects the $x = 1$ line. The PDE (2.1) can be written as

$$q_a + (-\frac{v_{qa}}{v_{qb}})q_b = 0$$

Considering $\{(1,r) : r \in [R_1, R_2]\}$ as initial curve, we have to solve

$$\begin{aligned} a_s(r,s) &= 1 && , \quad a(r,1) = 1 \\ b_s(r,s) &= -\frac{v_{qa}}{v_{qb}}(\phi(r), a(r,s), b(r,s)) && , \quad b(r,1) = r \end{aligned}$$

where $\phi : [R_1, R_2] \to [\underline{q}, \overline{q}]$ describes the quantity (or quality) allocated to $(1,r)$ types. In view of $v_{qb} < 0$ this function ϕ is strictly decreasing, so consider $\varphi : [\underline{q}, \overline{q}] \to [R_1, R_2]$ as the inverse of ϕ. If $a(r,s) = s$ and $b(r,s) = B(\phi(r), r, s)$ are the solutions of the previous system, a and b can be expressed in terms of q and s :

$$a(q,s) = s \quad , \quad b(q,s) = B(q, \varphi(q), s)$$

where $q \in [\underline{q}, \overline{q}]$, $s \in [\overline{x}, 1]$. With these variables, the revenue is

$$\int_{\underline{q}}^{\overline{q}} \int_{\overline{x}}^{1} G(q, s, B(q, \varphi(q), s))\Big(-(B_q + B_\varphi \varphi')\Big)\, ds\, dq$$

As before $(\overline{x}, \overline{y})$ is indifferent between any $q \in [\underline{q}, \overline{q}]$, then $v_q(q, \overline{x}, \overline{y}) = v_q(q, 1, \varphi)$.

Setting

$$H(q,\varphi,\varphi') = -\int_{\overline{x}}^{1} G(q,s,B(q,\varphi(q),s))(B_q + B_\varphi \varphi')\, ds$$
$$F(q,\varphi,\varphi') = v_q(q,\overline{x},\overline{y}) - v_q(q,1,\varphi)$$

the problem can be written as

$$\max_{\varphi(\cdot)} \int_{\underline{q}}^{\overline{q}} H(q,\varphi,\varphi')\, dq$$
$$\text{subject to}$$
$$\int_{\underline{q}}^{\overline{q}} F(q,\varphi,\varphi')\, dq = 0$$

The necessary condition for optimality is $H_\varphi - \frac{d}{dq}(H_{\varphi'}) = \lambda(F_\varphi - \frac{d}{dq}(F_{\varphi'}))$ for some $\lambda \in \mathbb{R}$, which yields to

$$\int_{\overline{x}}^{1} \frac{G_q}{v_{qb}}(q,s,B(q,\varphi,s))\, ds = \lambda \tag{2.15}$$

Next, we will see that cannot be the case of isoquants intersecting $x = 1$ be concurrent at $(\overline{x},\overline{y})$. The solutions of the system

$$\begin{array}{ll} a_s(r,s) = 1 & , \quad a(r,1) = 1 \\ b_s(r,s) = \frac{1}{\phi(r)} & , \quad b(r,1) = r \end{array}$$

are $a(r,s) = s$, $b(r,s) = \frac{s-1}{\phi(r)} + r$. Then $a(q,s) = s$, $b(q,s) = \frac{(s-1)}{q} + \varphi(q)$. For this example $G_q(q,a,b) = 2a - 1 - (b+c)q$, $v_{qb}(q,a,b) = -q$. Then, by the necessary condition (2.15), there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\overline{x}}^{1} \Big(\frac{1-2s}{q} + \frac{(s-1)}{q} + \varphi + c\Big)\, ds = \lambda$$

from which

$$\varphi(q) = \frac{1+\overline{x}}{2q} + \frac{\lambda}{1-\overline{x}} - c$$

On the other hand, $v_q(q,\overline{x},\overline{y}) = v_q(q,1,\varphi)$ implies

$$\varphi(q) = \frac{1-\overline{x}}{q} + \overline{y}$$

Thus, by comparison of terms $\overline{x} = \frac{1}{3}$ in contradiction with $\frac{1}{3} < \frac{2c+1}{2c+3} < \overline{x} < 1$. $\square$

Proof of Claim 3.
The solutions of the system

$$\begin{array}{ll} a_s(r,s) = \phi(r) & , \quad a(r,1) = r \\ b_s(r,s) = 1 & , \quad b(r,1) = 1 \end{array}$$

are $a(r,s) = (s-1)\phi(r) + r$, $b(r,s) = s$. Then $a(q,s) = (s-1)q + \varphi(q)$, $b(q,s) = s$. Also, we have $G(q,a,b) = (2a-1)q - (b+c)\dfrac{q^2}{2}$

In case the isoquants are concurrent at the point $(\overline{x},\overline{y})$ intersecting $y = 1$ line, by the Proposition 2.4.1 there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\overline{y}}^{1} 2((s-1)q + \varphi) - 1 - (s+c)q\, ds = \lambda$$

which yields on

$$\varphi(q) = (\frac{2c+3-\overline{y}}{4})q + \frac{1}{2} + \frac{\lambda}{2(1-\overline{y})}$$

Moreover,

$$v_q(q,\overline{x},\overline{y}) = v_q(q,\varphi(q),1) \Longrightarrow \varphi(q) = \overline{x} + (1-\overline{y})q$$

So, by comparison of terms we obtain

$$\overline{x} = \frac{1}{2} + \frac{3\lambda}{4(c+1)} \quad , \quad \overline{y} = \frac{1-2c}{3}$$

where λ is constant for a given $c \in (0, \frac{1}{2}]$. Also, we have

$$\phi^{III}(r) = \frac{r-\overline{x}}{1-\overline{y}}$$

where ϕ^{III} is the optimal allocation of type $(r,1)$. Because $\phi^{III}(\overline{x}) = 0$, the domain of ϕ^{III} is $[\overline{x}, 1]$. $\square$

Chapter 3

Reduction of IC Constraints in the Bidimensional Model

This chapter contains the main contribution of this work which consist in justify that, for a pair (q, V) to be incentive compatible, it is sufficient that each point verifies IC constraints with all the points over a unidimensional set instead of the whole type set as it is required by definition. With that, numerical approximations can be done with relative fine discretization. This approach, while specific for the case of bidimensional types and one-dimensional quantity product, is general in terms of the valuation function involved as well as types' distribution. The main assumption is the validity of single-crossing in each axis.

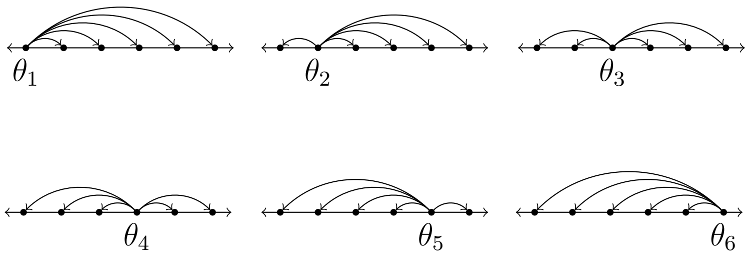
Before discuss our approach, in the following lines we illustrate the ideas being applied to deal with IC constraints in the unidimensional case with finite type set when single-crossing holds[1]. This explanation would be useful since we are extending these ideas in the bidimensional context.

Consider, for example, $\Theta = \{\theta_1, \dots, \theta_6\} \subset \mathbb{R}$ with $\theta_1 < \theta_2 < \cdots < \theta_6$ and a given (q, V). The following incentive compatibility (IC) constraints must be satisfied:

$$V(\theta) - V(\widehat{\theta}) \geq v(q(\widehat{\theta}), \theta) - v(q(\widehat{\theta}), \widehat{\theta}) \;\; \forall\, \theta, \widehat{\theta} \in \Theta$$

[1]See Laffont and Martimort (2002).

We graphically represent these constraints, where arrows' direction indicate each inequality:



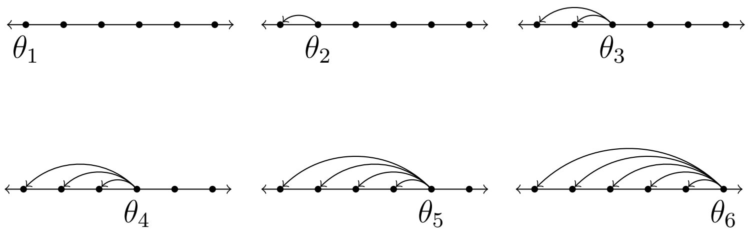
Because of single-crossing we have $v_{q\theta} > 0$, which is equivalent to

$$\theta < \hat{\theta} \Longrightarrow v_q(q,\theta) < v_q(q,\hat{\theta}) \ \forall \ q \in Q$$

thus, associate agents' demand curves (defined by $p = v_q(q,\theta)$ where p is the marginal price) can be completely ordered. Hence, types can be ranked in an increasing way.

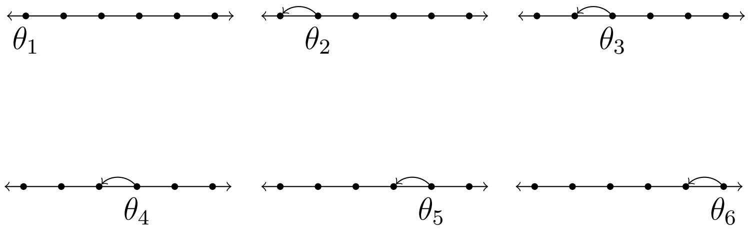
Note that $\theta-$type has no incentive to claim to be $\hat{\theta}-$type, for $\hat{\theta} > \theta$, because at any level of q the marginal $\hat{\theta}-$type's valuation of consume is greater, that is, $\widehat{\theta} - type$ is willing to pay more −for each aditional unit− than the $\theta-$type.

Therefore, we can omit a priori these upward IC constraints and checks a posteriori that the omitted constraints are indeed strictly satisfied. The new IC constraints are



Again, by the single-crossing condition, it will be sufficient to ensure that each type is binding only with the first 'worse' (from the monopolist's point of view) type of him , i.e., the type located on the left.

Thus, the IC constraints that we really have to consider are



In the continuous unidimensional case, under single-crossing, it is well known that local restrictions are sufficient, i.e., local IC constraints implies the global ones.

Section 1 states the model and the assumptions we are considering. In Section 2 a pre-order is defined with the aim to capture, in a certain way, the sorting among types explained above. Two propositions precede our central theorem. In Section 3 we formulate the problem to be solved numerically, explaining the natural limitations due to discretization. In Section 4 we test our approach comparing numerical and explicit solution of some examples from the literature, and finally we numerically solve new examples. All proofs are left to Appendix B.

3.1 Bidimensional Model

Based on the model presented in the previous chapter we will consider $N = 1$ and $M = 2$, that is, the monopolist produces a single good and types are bidimensional. We refer by q to the quality (as Mussa and Rosen (1978)) or quantity (as Maskin and Riley (1984)) of the good. Without loss of generality, let us consider $\Theta = [0,1]^2$.

In this context, the monopolist's problem is:

$$\max_{q(\cdot),V(\cdot)} \int_0^1 \int_0^1 \{v(q(a,b),a,b) - C(q(a,b)) - V(a,b)\} f(a,b) db da$$

subject to

$$\text{(IR)} \quad V(a,b) \geq 0 \qquad \forall\ (a,b) \in [0,1]^2$$

$$\text{(IC)} \quad V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q(\widehat{a},\widehat{b}),a,b) - v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) \qquad \forall\ (a,b),(\widehat{a},\widehat{b}) \in [0,1]^2$$

We assume $v \in C^3$, q and t to be continuous, and a.e. twice continuously differentiable[2]. These assumptions imply that V has the same features.

Also, the following assumptions are considered:

A1 $v_{q^2} < 0$

A2 $C(0) = 0$, $C'(q) \geq 0$ and $C''(q) \geq 0$

A3 $v_{qa} > 0$ and $v_{qb} < 0$ when $q > 0$

A4 $v_a > 0$ and $v_b < 0$ when $q > 0$

A5 $v(q^{\text{out}}, a, b)$ is constant

Assumption A1 means that each type's valuation function is strictly concave. A2 means that costs and marginal costs are non-decreasing. Assumption A3 is the single-crossing condition on each axis. We are assuming those signs for v_{qa} and v_{qb} because coincides with the assumptions in Araujo and Vieira (2010) and we are going to use later their necessary condition for optimality (2.2) in an example. Since we are interested in determine if characteristic curves are increasing or decreasing, which is given by the sign of $\frac{-v_{qb}}{v_{qa}}$, other cases are easily adapted.

Note that the assumption A4 allows us to rule out all the IR constraints providing that $V(0,1) = 0$. In fact, due to $V(a,b) = \max_{(\widehat{a},\widehat{b})}\{v(q(\widehat{a},\widehat{b}),a,b) - t(\widehat{a},\widehat{b})\}$ by the Envelope Theorem[3]

$$V_a(a,b) = v_a(q(a,b),a,b) \ , \ V_b(a,b) = v_b(q(a,b),a,b) \quad \text{a.e. } (a,b)$$

then $V_a > 0$ and $V_b < 0$ when $q > 0$, so V is strictly increasing in a and strictly decreasing in b on the interior of the participation set. Thus, it will be sufficient to impose $V(0,1) = 0$ and all the IR constraints will be satisfied.

[2]By assumption A3. stated below and using the Monotone Maximum Theorem, it can be proved that q is non-decreasing in a and non-increasing in b, and therefore a.e. differentiable. Our stronger assumptions allows us to give another (perphaps more familiar) proof. Also, those assumptions are required to establish the PDE (2.1) in section 2.4, that we will use later.

[3]See Milgrom and I.Segal (2002)

Usually assumption A5 is presented as $v(0, a, b) = 0$ because in the monopolist's problem framework the outside option is $q^{\text{out}} = 0$ and any agent assigns the value zero to this q^{out}. However, in other adverse selection problems this could be no longer true, so we assume the more general expression A5 .

Proposition 3.1.1. *If $q(\cdot, \cdot)$ is implementable, at any point (a, b) of twice continuous differentiability, we have $q_a(a, b) \geq 0$ and $q_b(a, b) \leq 0$*

Thus, q is non-decreasing in a and non-increasing in b. This result is consequence of assumption A3. Unlike the unidimensional case, this necessary condition for implementability is not longer sufficient.

3.2 Reducing IC Constraints

In bidimensional models we do not have a condition similar to the single-crossing in the unidimensional case, where all types can exogenously be ordered by their marginal valuation for consumption ($v_{q\theta} > 0$ means $\theta_1 < \theta_2 \implies v_q(q, \theta_1) < v_q(q, \theta_2)$ for any $q \in Q$ fixed). In order to be able of compare apriori two different types, at least partially, we introduce the following binary relation:

Definition 3.2.1. Given $(a, b), (\widehat{a}, \widehat{b}) \in [0, 1]^2$ we will say that (a, b) *is worse than* $(\widehat{a}, \widehat{b})$, denoted by $(a, b) \preceq (\widehat{a}, \widehat{b})$, if and only if

$$v_q(q, a, b) \leq v_q(q, \widehat{a}, \widehat{b}) \quad \forall\, q \in Q$$

Note that $\preceq$ is a *pre-order* (reflexive and transitive) on $[0, 1]^2$.

With this definition we try to capture the idea that, when $(a, b) \preceq (\widehat{a}, \widehat{b})$, the (a, b)-agent is not willing to announce to be the $(\widehat{a}, \widehat{b})-$agent, since at any level of $q \in Q$ the $(\widehat{a}, \widehat{b})-$agent has greater marginal utility, so $(\widehat{a}, \widehat{b})-$agent is willing to pay more for each aditional unit of the product.

As a direct consequence of the assumptions $v_{qa} > 0$ and $v_{qb} < 0$ we have that (a, b) is worse than any type on the southeast.

Proposition 3.2.1. *For any fixed (a, b), if $(\widehat{a}, \widehat{b})$ is such that $\widehat{a} > a$ and $\widehat{b} < b$, then* $(a, b) \preceq (\widehat{a}, \widehat{b})$

At this point, it is useful remember what we have seen in Section 2.4. For an allocation rule $q(\cdot)$ to be implementable, it must satisfy a.e.

$$-\frac{v_{qb}}{v_{qa}}q_a + q_b = 0$$

by the characteristic method to solve the previous PDE, we obtain a family of plane characteristic curves parametrized by r, s where, for a fix r, all types over the curve $(a(r,s), b(r,s))$ gets the same allocation, that is, plane characteristic curves are the isoquants of q. Also, in view of at any point of any isoquant, the tangent vector $(a_s, b_s) = (-\frac{v_{qb}}{v_{qa}}, 1)$ has both components positive, all the isoquants are strictly increasing in the participation set interior.

Note: In order to facilitate reading, sometimes we will write c.c. instead of both *characteristic curve* or *characteristic curves*.

Another way of understand Proposition 3.2.1 is by the characteristic curves. We know that the c.c. passing through the fixed type (a, b) is strictly increasing, even unknowing the exact shape. The exact shape is determined endogenously, but in any case it does never intercept the southeast region. Also, because types on that region always get greater quantity, those types can be though (for the point of view of the monopolist) as better types than the type (a, b).

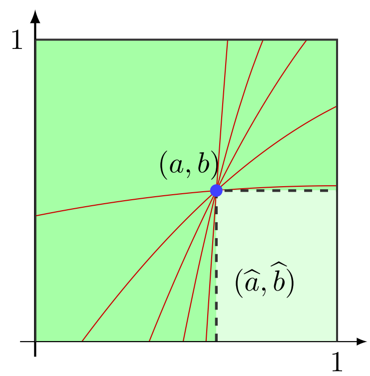


Figure 3.1: Characteristic curves passing through (a, b) and types (light green area) with which IC constraint is a priori excluded.

Now, we will exclude a priori those IC constraints for the monopolist's problem, since the difficulty comes from better types willing to claim that they are worse types, rather than reverse.

Specifically we will omit the following IC constraints, for a fixed type (a, b)

$$V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q(\widehat{a},\widehat{b}),a,b) - v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) \;\; \forall(\widehat{a},\widehat{b}) \text{ with } \widehat{a} > a, \widehat{b} < b$$

We should be able to checks a posteriori (i.e., after solution is obtained) that these omitted constraints are indeed strictly satisfied[4].

After that, we will see that it is sufficient for the monopolist to guarantee that the type (a, b) satisfies the IC constraint with the closest type to him, in terms that will be clear later.

<u>Notations</u>

- We say "(a, b) is IC with $(\widehat{a},\widehat{b})$" when

$$V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q(\widehat{a},\widehat{b}),a,b) - v(q(\widehat{a},\widehat{b}),\widehat{a},\widehat{b})$$

  that is, when the (a, b)-agent has not the incentive to announce to be the $(\widehat{a},\widehat{b})$-agent.

- $CC(\widehat{a},\widehat{b})$ is the plane characteristic curve that contains $(\widehat{a},\widehat{b})$

Proposition 3.2.2. *Let $(a, b), (\widehat{a},\widehat{b})$ be such that (a, b) is IC with $(\widehat{a},\widehat{b})$. Then (a, b) is IC with (x, y) , $\forall$ $(x, y) \in CC(\widehat{a},\widehat{b})$*

By this proposition, we just need to verify IC constraint with a representative type of each c.c., so we will focus on the border of the square.

Proposition 3.2.3. *Let $(x, y), (\widehat{a},\widehat{b}), (a, b)$ be such that (a, b) verifies IC with $(\widehat{a},\widehat{b})$ and $(\widehat{a},\widehat{b})$ verifies IC with (x, y). If $(\widehat{a},\widehat{b}) \preceq (a, b)$ and $q(x, y) \leq q(\widehat{a},\widehat{b})$ then (a, b) verifies IC with (x, y).*

Due to the kind of transitivity that this proposition shows, it is not necessary that a fixed (a, b) type verifies IC constraints with all the types (x, y) on the left of certain characteristic curve, instead, it is sufficient to verify the IC constraint with any type worse than (a, b) over such curve, ensuring that this type verifies the IC constraint with all of those (x, y).

[4]Because we are interesting on numerical approximations, the verification will also be numerical.

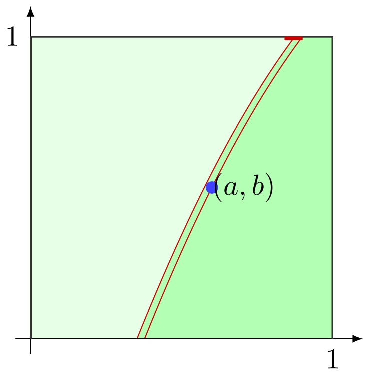


Figure 3.2: Representation of types (light green area) with which (a, b) satisfy IC constraints under Proposition 3.2.3

Taking the characteristic curve as close as possible of type (a, b), the most restrictions could be eliminated. Since c.c. are endogenously determined, but any of them passing through (a, b) intersects the border of the square $[0, 1]^2$ on the northeast of that point, previous propositions suggest that it would be sufficient to verify that (a, b) is IC with all the points over the set

$$F^{(a,b)} := \{(s, 1) \mid a \leq s \leq 1\} \cup \{(1, s) \mid b < s \leq 1\} \tag{3.1}$$

which is formalized in the following theorem.

Theorem 3.2.1. *Let (q, V) be such that*

$$\forall\ (a, b) \in [0, 1]^2\ ,\ (a, b) \text{ is IC with } (x, y), \quad \forall\ (x, y) \in F^{(a,b)}$$

then (q, V) satisfies all the incentive compatibility constraints.

This result could be understood as an analogous of the claim 'local IC constraint implies global IC constraint' true in the unidimensional case when single-crossing holds.

Until now we are not able to compare (a, b) with a type $(\widehat{a}, \widehat{b})$ on the northeast. This could be done if we know that the c.c. passing through $(\widehat{a}, \widehat{b})$ is on the right of the c.c. passing through (a, b), in which case $(a, b) \preceq (\widehat{a}, \widehat{b})$, so we would not need to consider the IC constraint in this situation. In order to obtain sufficient conditions to compare two types, some special structure on valuation function v is needed.

3.2.1 Particular valuation function

The following propositions allows us to reduce even more the IC constraints, when the valuation function v has a special structure.

Proposition 3.2.4. *Assume that v_q is concave in a and convex in b. Let (a,b) , $(\widehat{a},\widehat{b})$ be in $[0,1]^2$ with $a<\widehat{a}$, $b<\widehat{b}$. Then*

1. $(a,b) \preceq (\widehat{a},\widehat{b}) \implies \dfrac{\widehat{b}-b}{\widehat{a}-a} \leq \dfrac{-v_{qa}(q,a,\widehat{b})}{v_{qb}(q,a,\widehat{b})}$

2. $\dfrac{\widehat{b}-b}{\widehat{a}-a} \leq \dfrac{-v_{qa}(q,\widehat{a},b)}{v_{qb}(q,\widehat{a},b)} \implies (a,b) \preceq (\widehat{a},\widehat{b})$

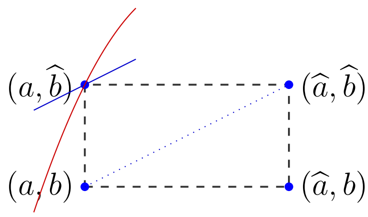
This proposition says that, in order to $(a,b) \preceq (\widehat{a},\widehat{b})$, it is necessary that $CC(a,b)$ be on the left of $CC(\widehat{a},\widehat{b})$, because at the point $(a,\widehat{b})$ the slope of $CC(a,\widehat{b})$ is greater than the slope between (a,b) and $(\widehat{a},\widehat{b})$.

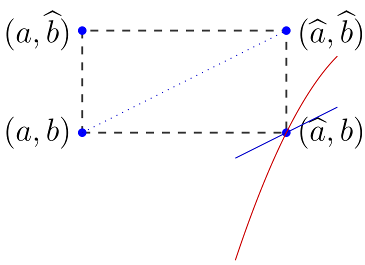
Similarly, and more useful, a sufficient condition for $(a,b) \preceq (\widehat{a},\widehat{b})$ is that the slope of $CC(\widehat{a},b)$ at the point $(\widehat{a},b)$ be greater than the slope between (a,b) and $(\widehat{a},\widehat{b})$. With that $CC(\widehat{a},\widehat{b})$ will be at the right of $CC(a,b)$.

We illustrate previous proposition in the following graphics.

Case $v_{qaa} \leq 0$ and $v_{qbb} \geq 0$

$(a,b) \preceq (\widehat{a},\widehat{b}) \implies$





$\implies (a,b) \preceq (\widehat{a},\widehat{b})$

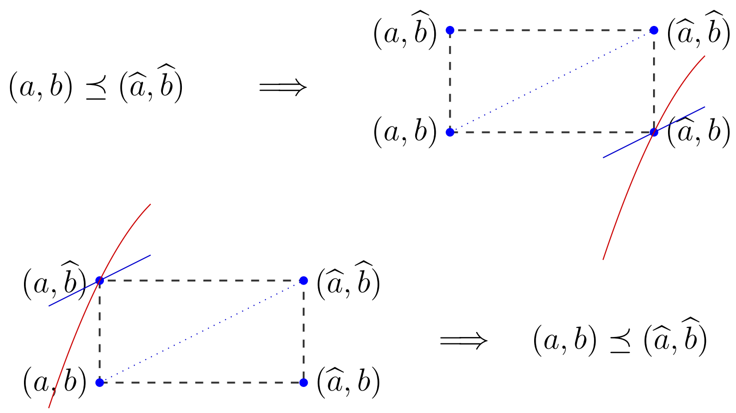
Proposition 3.2.5. *Assume that v_q is convex in a and concave in b. Let (a, b) , $(\widehat{a}, \widehat{b})$ be in $[0, 1]^2$ with $a < \widehat{a}$, $b < \widehat{b}$. Then*

1. $(a, b) \preceq (\widehat{a}, \widehat{b}) \implies \dfrac{\widehat{b} - b}{\widehat{a} - a} \leq \dfrac{-v_{qa}(q, \widehat{a}, b)}{v_{qb}(q, \widehat{a}, b)}$

2. $\dfrac{\widehat{b} - b}{\widehat{a} - a} \leq \dfrac{-v_{qa}(q, a, \widehat{b})}{v_{qb}(q, a, \widehat{b})} \implies (a, b) \preceq (\widehat{a}, \widehat{b})$

We just graphically illustrate this proposition since it is the reverse situation of the previous one.

Case $v_{qaa} \geq 0$ and $v_{qbb} \leq 0$



3.3 Numerical Formulation

By Theorem 3.2.1, it is sufficient that each point verifies IC constraints with all the points over a unidimensional set instead of the whole square. Now, we can approximate the solution of the continuous problem discretizating the type set. This section is devoted to establish such discrete problem and discuss its limitations.

Let $X_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\} \times \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$ be the grid of n^2 points on $[0, 1]^2$. For a fix (a, b) with $a < 1, b < 1$, let $\widetilde{F}^{(a,b)} := F^{(a,b)} \cap X_n$ where $F^{(a,b)}$ is defined in (3.1). Because for points over the lines $x = 1$ or $y = 1$ we cannot write the constraints with the points on the northeast, we equivalently consider

$$\widetilde{F}^{(a,1)} = \big(\{(0, s) : 0 \leq s \leq 1\} \cup \{(s, 0) : 0 \leq s < a\}\big) \cap X_n$$

$$\widetilde{F}^{(1,b)} = \big(\{(0, s) : 0 \leq s \leq b\} \cup \{(s, 0) : 0 \leq s < 1\}\big) \cap X_n$$

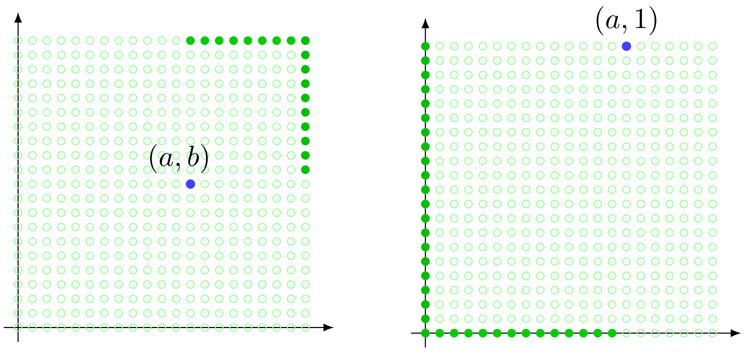


Figure 3.3: Illustration of $\widetilde{F}^{(a,b)}$ (dark green points).

The set $\widetilde{F}^{(a,b)}$ contains all the types with which (a,b) must satisfy an IC constraint.

The integral in monopolist's objective will be approximate by the trapezoidal rule, so consider the associated weights $w_{i,j}$ for each point $(a_i, b_j) \in X_n$, where

$$w = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & 1 & 1 & \dots & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Denoting $q_{i,j} = q(a_i, b_j)$ and $V_{i,j} = V(a_i, b_j)$, we are interesting in solve the following problem:

$$\max_{\{q_{i,j}, V_{i,j}\}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{i,j}(v(q_{i,j}, a_i, b_j) - V_{i,j} - C(q_{i,j}))f(a_i, b_j)$$

subject to

$$\begin{array}{ll} \text{(IR)} & V_{1,n} = 0 \\ \text{(IC)} & (a_i, b_j) \text{ is IC with } (\widehat{a}_i, \widehat{b}_j)\ ,\ \forall\ (\widehat{a}_i, \widehat{b}_j) \in \widetilde{F}^{(a_i,b_j)} \\ \text{(M)} & q_{i,j} \leq q_{i+1,j} \quad , \quad q_{i,j} \leq q_{i,j-1} \end{array} \tag{NP}$$

Remarks:

I. In the original discretized problem there are $n^4 - n^2$ (maybe nonlinear) IC constraints. After our reduction this number is of order n^3.

II. The monotonicity constraints are added in order to obtain better accuracy of the solution although, as we know, monotonicity is a necessary condition. These $2n^2$ linear restrictions added do not represent big numerical cost.

III. In case assumption A4 cannot be verifed, we just consider all the IR constraints $V_{i,j} \geq 0$.

IV. Because of the discretization, it is impossible to ensure that for each type (a, b) all the IC constraints are fulfilled. This is because there could be some points between $CC(a, b)$ and the c.c. of the first point on the border for which IC constraint is satisfied. Then, since we are not sure that (a, b) is IC with those points, for types on the right of $CC(a, b)$ the requirements of Proposition 3.2.3 may not be true and violations of IC constraints may propagate. Nevertheless, as it is shown in the next section, violations of IC constraints are asymptotically zero with finner discretizations. Next figure illustrates this issue.

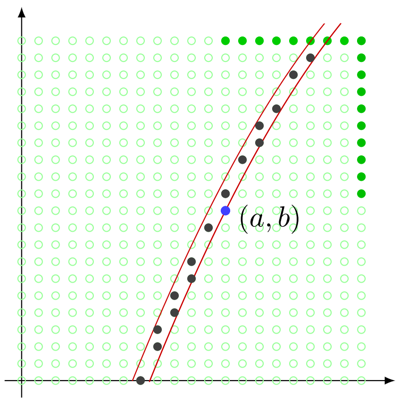


Figure 3.4: Type (a, b) might not satisfy IC constraints with black points.

V. In order to get better approximations, in view of the natural difficulties pointed out above, for the points $(a, 1)$ and $(1, b)$ (those over the lines $x = 1$ and $y = 1$), we can consider IC constraints with all the points on the southwest of those points. So, we replace the advantage of having fewer constraints (which allows fine discretization) for better accuracy on the solution keeping the IC constraints number of order n^3.

VI. When valuation function has the special 'multiplicative separable' form

$$v(q, a, b) = \psi(q) + \alpha(a, b) \times q + \beta(a, b)$$

the IC constraints become linear in $q_{i,j}$. Therefore, since IC constraints are linear in V (regardless v) and the objective function is strictly concave, the solution is unique and we can rely on numerical approximation.

3.4 Asymptotic Optimality

Next, following Belloni et al. (2010) we prove that extending the solutions of the discretized problem in an appropiate manner, all the IC violations converge uniformly to zero, and the sequence of optimal values converge to the optimal value of the continuous problem. They have considered a linear model including multiple agents and *Border* constraints[5], which are not present in our setting. In contrast, we consider a valuation function v that could be nonlinear, so we assume that v is Lipschitz[6] over $[0, 1]^2$.

As before, let X_n be the grid of n^2 points over $[0, 1]^2$. Let Q^n, V^n be the numerical solutions of the problem formulated in 3.3. These solutions are defined over the discret set X_n, so we define the extensions $\widetilde{Q}^n, \widetilde{V}^n : [0, 1]^2 \to \mathbb{R}$ as

$$\widetilde{Q}^n(a, b) := Q^n(\widehat{a}, \widehat{b}) \quad , \quad \widetilde{V}^n(a, b) := V^n(\widehat{a}, \widehat{b})$$

where $(\widehat{a}, \widehat{b}) \in X_n$ is such that

$$\widehat{a} \leq a < \widehat{a} + \tfrac{1}{n-1} \quad , \quad \widehat{b} - \tfrac{1}{n-1} < b \leq \widehat{b}$$

Proposition 3.4.1. *Given $(a, b) \in X_n$, we have*

$$\widetilde{V}^n(a, b) - \widetilde{V}^n(x, y) \geq v(\widetilde{Q}^n(x, y), a, b) - v(\widetilde{Q}^n(x, y), x, y) - O(\tfrac{1}{n-1}) \ \ \forall \ (x, y) \in F^{(a,b)}$$

---

[5]These constraints are related with allocation treated as a probability, since for their model there are N buyers and J degrees of product quality.

[6]We allow v to be continuos on $[0, 1]^2$ and differentiable on $(0, 1)^2$

That is, since $(a,b) \in X_n$ verifies IC with all the points in $\widetilde{F}^{(a,b)} = F^{(a,b)} \cap X_n$, satisfies IC with all the points in the continuous set $F^{(a,b)}$ now with some tolerance that is asymptotically zero. Following proposition shows that between any two points on the grid X_n same relaxed version of IC constraint is satisfied.

Proposition 3.4.2. *Given $(a,b),(\widehat{a},\widehat{b}) \in X_n$, we have*

$$V^n(a,b) - V^n(\widehat{a},\widehat{b}) \geq v(Q^n(\widehat{a},\widehat{b}),a,b) - v(Q^n(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) - O(\tfrac{1}{n-1})$$

Let $\delta^*(\widetilde{Q}^n, \widetilde{V}^n)$ denote the supremum over all IC constraint violations by the pair $(\widetilde{Q}^n, \widetilde{V}^n)$. That is, because of the discretization not all IC constraints are satisfied by the extensions $(\widetilde{Q}^n, \widetilde{V}^n)$ but we can be sure that

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(\widehat{a},\widehat{b}) - (v(\widetilde{Q}^n(\widehat{a},\widehat{b}),a,b) - v(\widetilde{Q}^n(\widehat{a},\widehat{b}),\widehat{a},\widehat{b})) \geq -\delta^*(\widetilde{Q}^n, \widetilde{V}^n)$$

for any $(a,b),(\widehat{a},\widehat{b}) \in [0,1]^2$

Proposition 3.4.3. *If v is Lipschitz, we have:*

$$\delta^*(\widetilde{Q}^n, \widetilde{V}^n) \leq O(\frac{1}{n-1})$$

Thus, all IC constraint violations converge uniformly to zero, which guarantees the asymptotic feasibility of the extensions. Next proposition shows that optimality can be achieved in the limit.

Proposition 3.4.4. *Let OPT_n the optimal value of the discretized problem, and OPT^* the optimal value of the continuous problem. If v is Lipschitz, we have:*

$$\liminf_{n\to\infty} OPT_n \geq OPT^*$$

If, additionally, $\exists \lim_{n\to\infty} \widetilde{Q}^n(a,b)$ and $\lim_{n\to\infty} \widetilde{V}^n(a,b)$ for any $(a,b) \in [0,1]^2$, we have

$$\lim_{n\to\infty} OPT_n = OPT^*$$

3.5 Examples

There are not many examples with closed-form solution in the literature for the case of bidimensional types and unidimensional quantity.

Laffont et al. (1987) have considered that monopolist faces customers with linear demand curves and is uncertain about both the slope and intercept of such linear demand, which yields on linear-quadratic customers' valuation $v(q,a,b) = aq - \frac{1+b}{2}q^2$. Basov (2005) proposed the Hamiltonian Approach and solved the generalization $v(q,a,b) = aq - \frac{1+b}{2}q^\gamma$ with $\gamma \geq 2$. In this case demand curves are concave. Vieira (2008) have considered that agent's characteristic might not linearly affect the valuation function. He have solved $v(q,a,b) = aq - \frac{1+b^2}{2}q^2$ usign the necessary condition (2.2). We propose an example in which demand curves are convex and use (2.2) to solve it.

In this section we test our approach comparing the numerical approximation with the analytic solution of above examples.

We have two criteria to compare our approximations. The first one is compute the *average quadratic error (a.q.e.)* between analytic quantity Q^{real} and numeric quantity Q^{num} (the same calculation is made for informational rent)

$$a.q.e.(Q^{\text{num}}, Q^{\text{real}}) = \frac{1}{n^2}\sum_{i,j=1}^{n}(Q^{\text{num}}_{i,j} - Q^{\text{real}}_{i,j})^2$$

The second criteria is just a visual comparation. Despite no being formal, in practice numerical approximations help us to formulate predictions about the functional form of the solution, like the participation set or the contour levels (i.e. how types are bunching). So, we provide graphics of the quantity, the informational rent, their contour levels and cross-section for both numerical and real solutions. We also exhibit numerical and real profits' difference for some values of n.

Furthermore, we numerically solve two additional examples that, to the best of our knowledge, have not been previously analysed in the literature. For these examples we just show the graphs of numerical solutions, contour lines, and the profit's sequence for some values of n.

As was mentioned, when we ommited IC constraints on the southeast for each type it is required to verify a posteriori if the ommited constraints are indeed strictly satisfied. It is clear that such verification can only be numeric. For this reason, we provide graphs showing whether a fix type (a_i, b_j) (in blue) is IC with all the others types in X_n, drawinall the types on the southeast of the points considered are satisfying IC.

In view of numerical optimization, as well as the limitations by the discretization pointed out in the remarks of section (3.3), it is not surprising the existence of red points (i.e. not satisfying IC constraints) in some graphs, however, the violations may be considered small. The value of $\delta = \delta_{i,j}$ in each one of the graphs indicates the minimum violation of IC constraints among the red types. That is, if we allow some tolerance $tol_{i,j} > -\delta_{i,j}$, all the IC constraints will be verified for such (a_i, b_j) blue point. Furthermore, setting $\delta^* = \min_{i,j}\{\delta_{i,j}\}$ implies that when $tol > -\delta*$ all the IC constraints will be satisfied for all the points on the square. We provide the value of δ^* in each example.

The numerical solutions were performed via Knitro/AMPL using the Active Set Algorithm. Otimization process stopped if one of the following tolerances were achieved: `maxit`$= 10^4$, `feastol`$= 10^{-15}$, `xtol`$= 10^{-15}$, `opttol`$= 10^{-15}$, where `maxit` is the maximum number of iterations, `feastol` refers to feasibility tolerance, `xtol` is the relative change of decision variables and `opttol` is the optimality KKT sttoping tolerance. In all examples, `xtol` were achieved first.

Example 1 [Laffont, Maskin & Rochet (1987)]

In Laffont et al. (1987) the authors have solved the original monopolist's problem for these data

$$v(q, a, b) = aq - \frac{(1+b)}{2}q^2 \quad , \quad C(q) = 0 \quad , \quad f(a, b) = 1$$

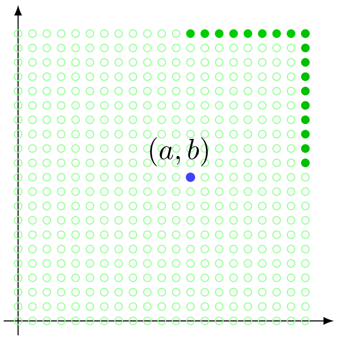
The solutions q and T they have found are:

$$q(a, b) = \begin{cases} 0 & , \quad a \leq \frac{1}{2} \\ \dfrac{4a-2}{4b+1} & , \quad \dfrac{1}{2} \leq \dfrac{a+2b}{4b+1} \leq \dfrac{3}{5} \\ \dfrac{3a-1}{2+3b} & , \quad \dfrac{3}{5} \leq \dfrac{2a+b}{2+3b} \leq 1 \end{cases}$$

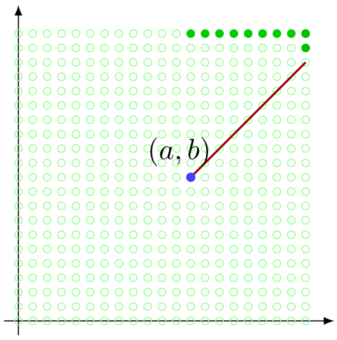
$$T(q) = \begin{cases} \dfrac{q}{2} - \dfrac{3q^2}{8} & , \quad q \leq \dfrac{2}{5} \\[2ex] \dfrac{q}{3} - \dfrac{q^2}{6} + \dfrac{1}{30} & , \quad \dfrac{2}{5} \leq q \leq 1 \end{cases}$$

Before solving the problem numerically note that v_q is linear in a and b, so we can apply Proposition 3.2.4 and reduce even more the number of constraints. Since there is no distortion at the top (the type $(1,0)$ has no distortion with respect to the contract over complete information), we must have $v_q(q(1,0),1,0) = 0$ (marginal utility equals marginal cost, which is zero) which implies $q(1,0) = 1$, then $Q = [0,1]$. Therefore $\frac{-v_{qa}}{v_{qb}} = \frac{1}{q} \geq 1$. Thus, for any (a,b), $(\widehat{a},\widehat{b})$ with $\widehat{a} > a$, $\widehat{b} > b$, it is sufficient that $\frac{\widehat{b}-b}{\widehat{a}-a} \leq 1$ to ensure that $(a,b) \preceq (\widehat{a},\widehat{b})$.

For this particular example, we can reduce the number of IC constraints for the numerical approximation as the following graphic shows.



Before Proposition 3.2.4



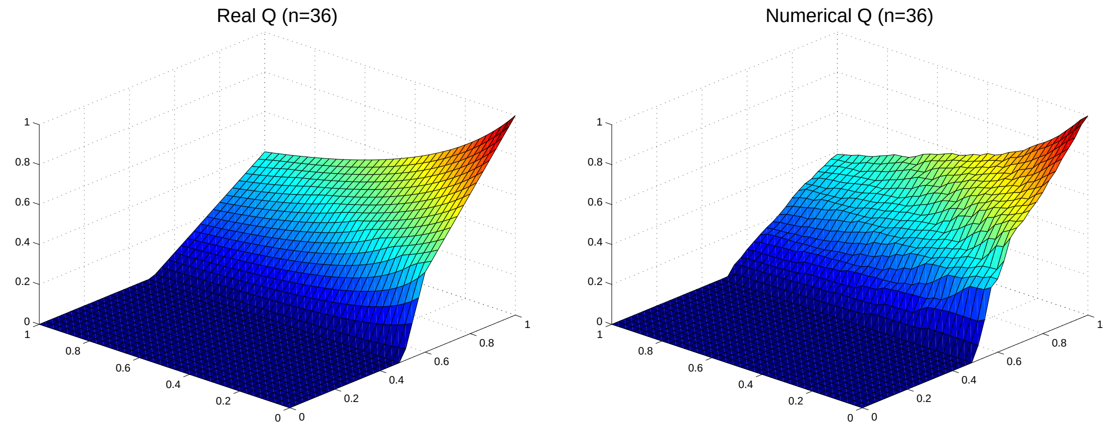
After Proposition 3.2.4

The exact number of IC constraints, after all the reductions explained above, is $\frac{1}{2}(3n^3 - 3n^2 - 4n + 4)$, instead of $n^4 - n^2$ as in the original problem. We solved the discretized problem with $n = 36$. For this value, we have to deal with 67 970 incentive compatibility constraints, having eliminated 1 610 350 of them.

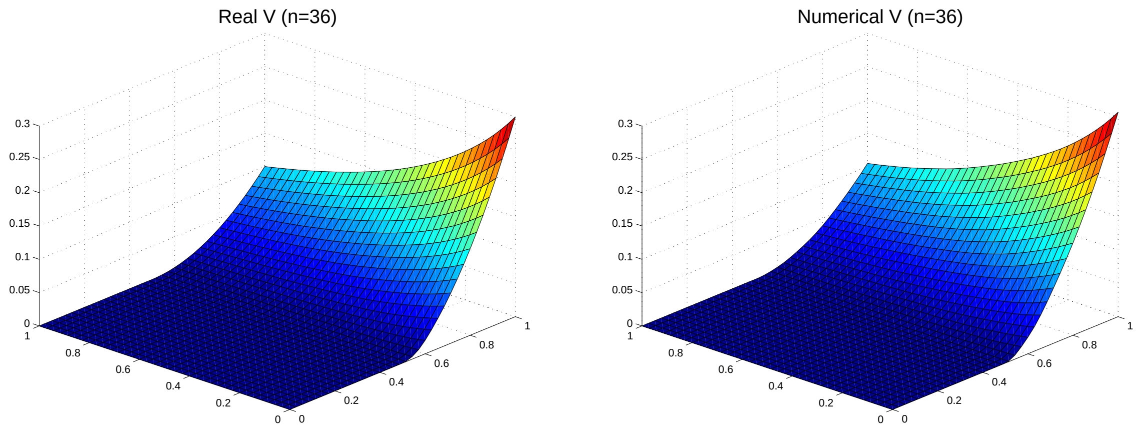
Next, we compare our result as was previously explained.

$$\begin{aligned} a.q.e.(Q^{num}, Q^{real}) &= 3.6442 \times 10^{-4} \\ a.q.e.(V^{num}, V^{real}) &= 0.1149 \times 10^{-4} \\ \left| \text{profit}^{num} - \text{profit}^{real} \right| &= 9.6668 \times 10^{-4} \end{aligned}$$

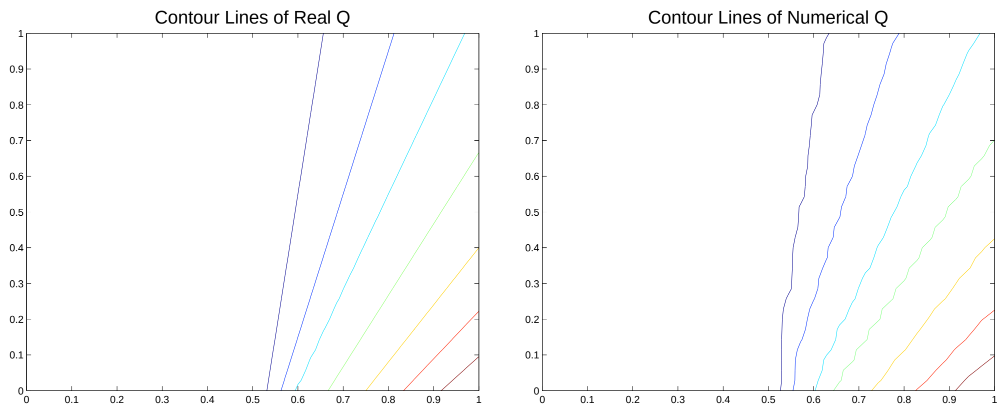
Comparing Quantity



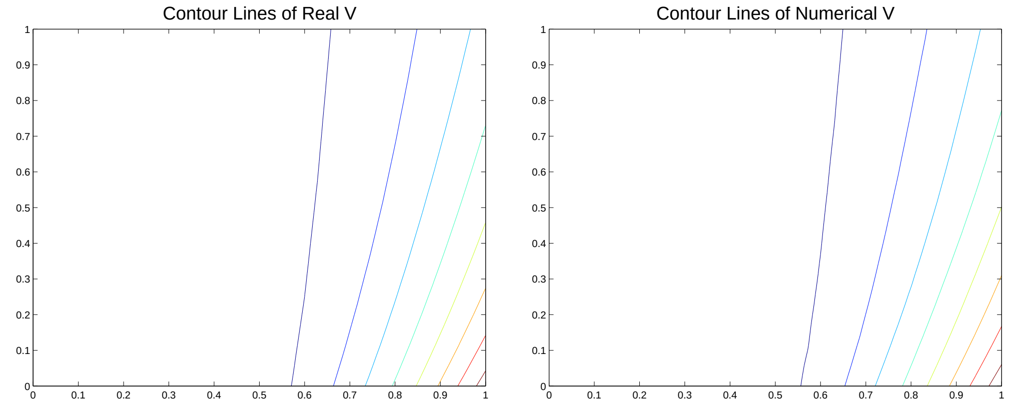
Comparing Informational Rent



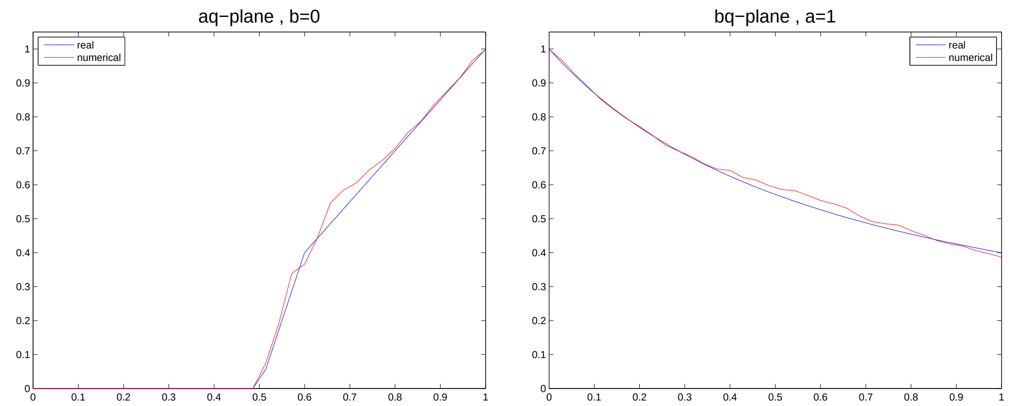
Contour Lines of Quantity



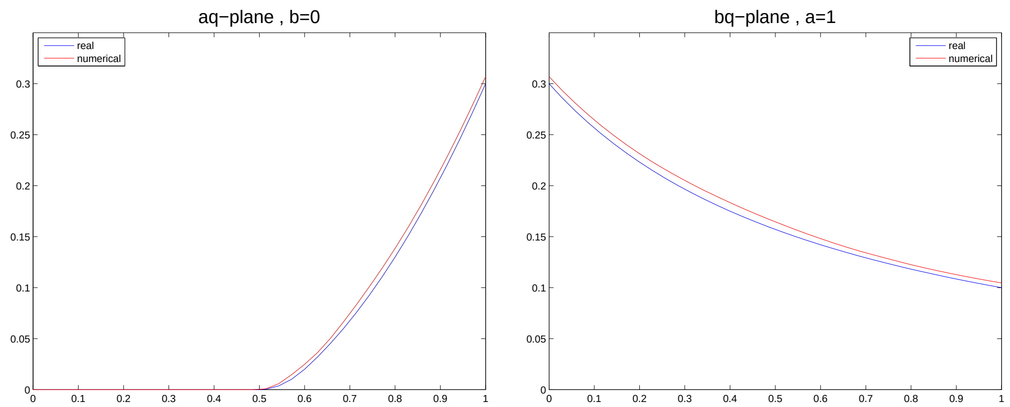
Contour Lines of Informational Rent



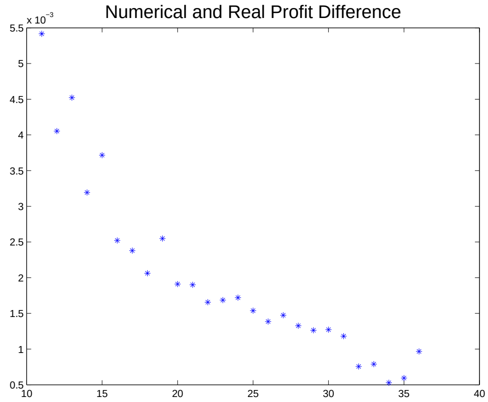
Cross-sections of Quantity



Cross-sections of Informational Rent

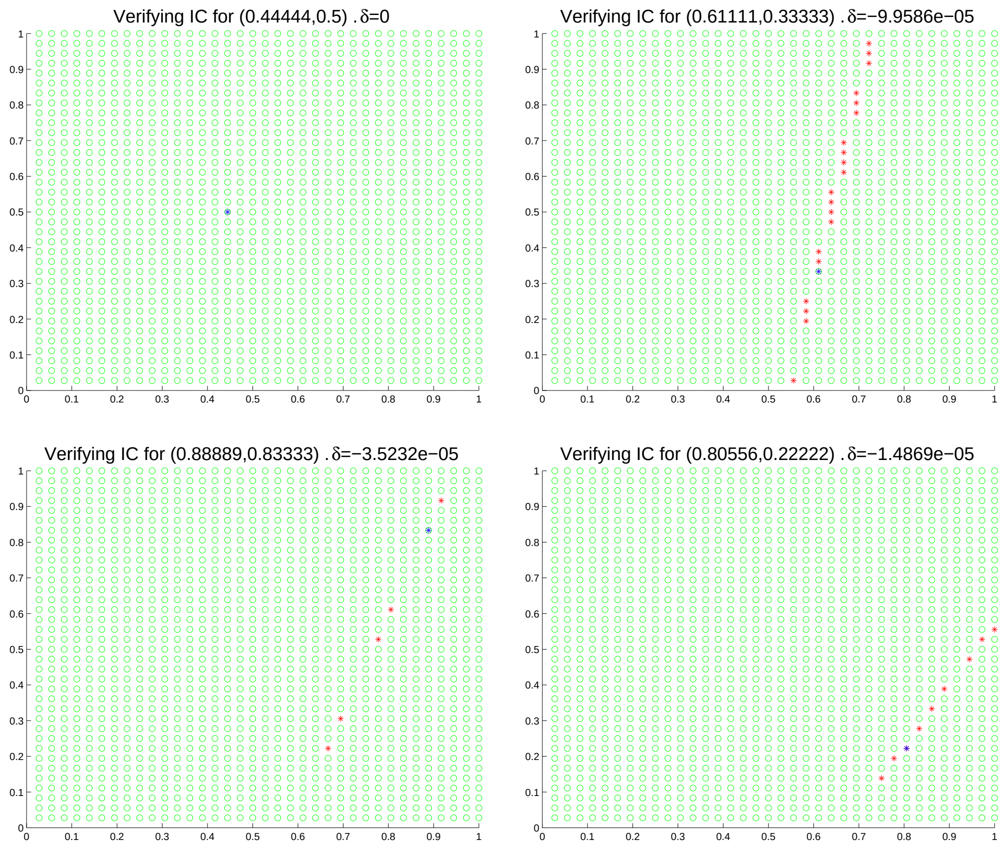


Distance to real profit



Verifying IC constraints

Here $\delta^* = -1.70804 \times 10^{-4}$. Thus, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.



Example 2 [Basov (2005)]

In Basov (2005) the author solved the original problem for the data

$$v(q,a,b) = aq - \frac{(c+b)}{\gamma} q^{\gamma} \quad , \quad C(q) = 0 \quad , \quad f(a,b) = 1$$

where $c > \frac{1}{2}$ and $\gamma \geq 2$ are constants. The solutions q and T he have found are:

$$q(a,b) = \begin{cases} 0 & , \quad a \leq \frac{1}{2} \\ \left(\frac{4a-2}{4b+2c-1}\right)^{\frac{1}{\gamma-1}} & , \quad (3+2c)a - 2b \leq 2c+1 \\ \left(\frac{3a-1}{3b+2c}\right)^{\frac{1}{\gamma-1}} & , \quad (3+2c)a - 2b > 2c+1 \end{cases}$$

$$T(q) = \begin{cases} \frac{q}{2} - \frac{(\frac{c}{2}+\frac{1}{4})}{\gamma} q^{\gamma} & , \quad q \leq \left(\frac{2}{3+2c}\right)^{\frac{1}{\gamma-1}} \\ \frac{1}{6}\left(\frac{2}{3+2c}\right)^{\frac{1}{\gamma-1}} - \frac{(\frac{c}{6}+\frac{1}{4})}{\gamma}\left(\frac{2}{3+2c}\right)^{\frac{\gamma}{\gamma-1}} + \frac{q}{3} - \frac{c}{3\gamma} q^{\gamma} & , \quad q > \left(\frac{2}{3+2c}\right)^{\frac{1}{\gamma-1}} \end{cases}$$

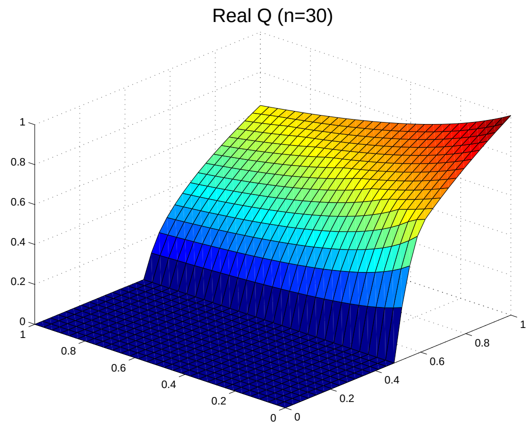
Note v_q is linear in a and b, so Proposition 3.2.4 can be applied and by analogous considerations of Example 1, we can ruled out the same additional constraints.

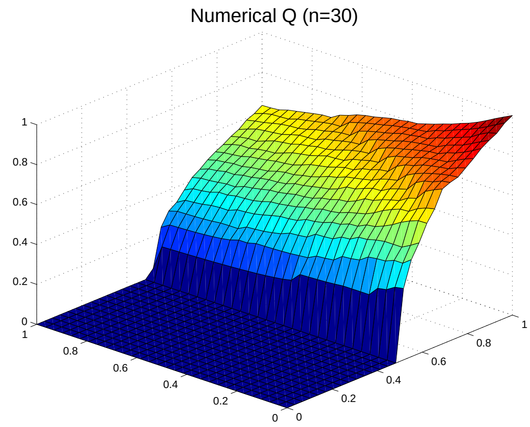
We solved the discretized problem for the case $\gamma = 3$ with $n = 30$ points. For this value, 39 092 incentive compatibility constraints were considered, and 770 008 were eliminated.

Next, we compare our result as was previously explained.

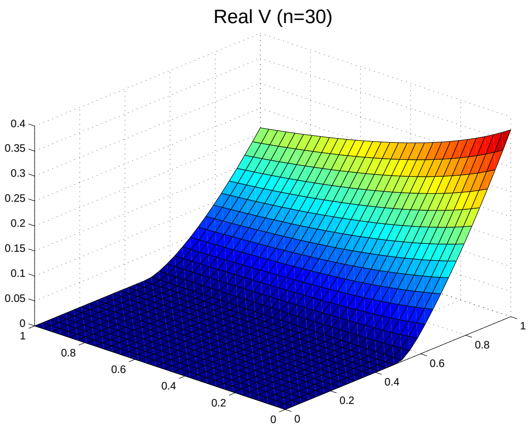
$$\begin{aligned} a.q.e.(Q^{num}, Q^{real}) &= 4.5853 \times 10^{-4} \\ a.q.e.(V^{num}, V^{real}) &= 0.0384 \times 10^{-4} \\ \left| \text{profit}^{num} - \text{profit}^{real} \right| &= 2.5717 \times 10^{-3} \end{aligned}$$

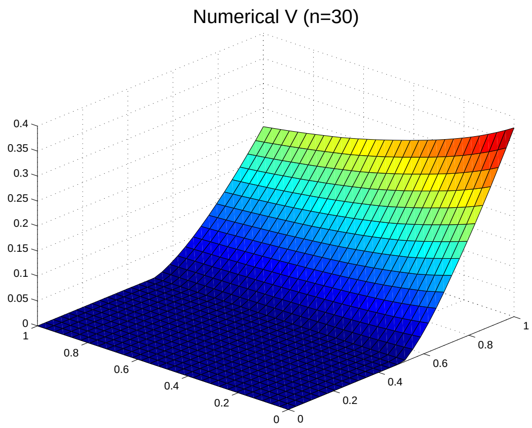
Comparing Quantity



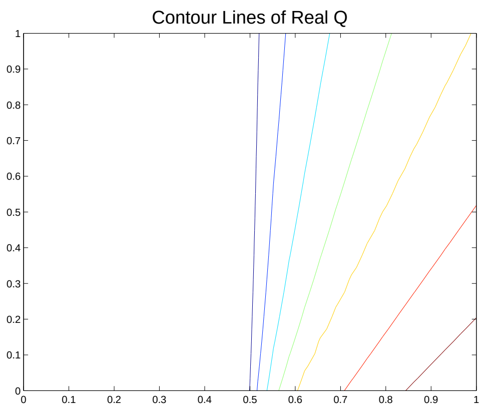


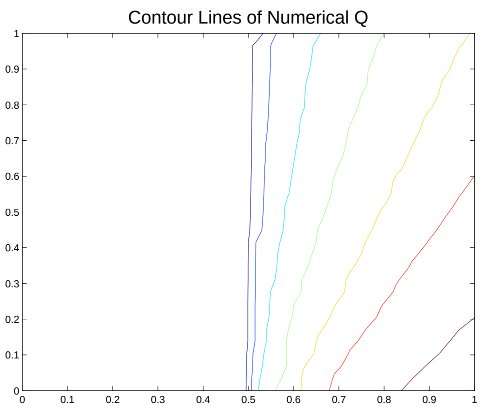
Comparing Informational Rent



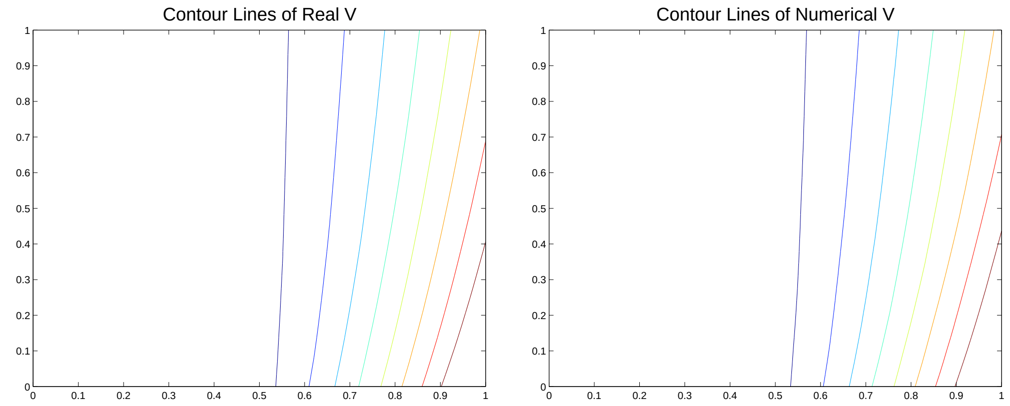


Contour Lines of Quantity

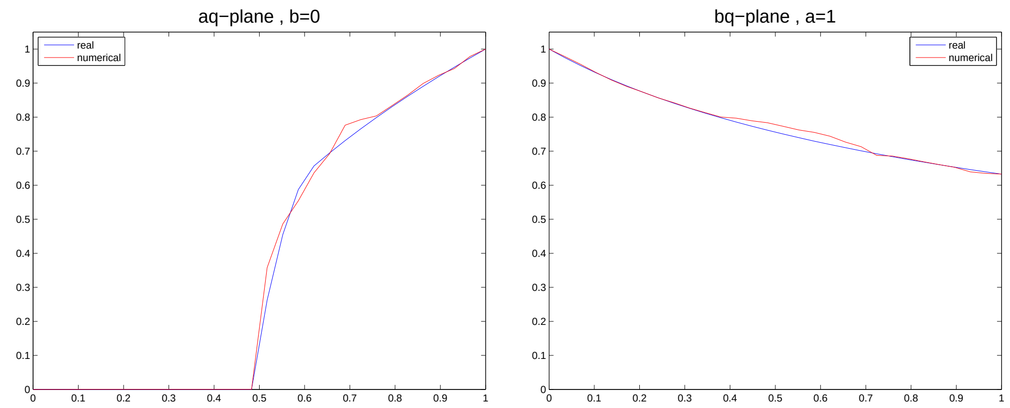




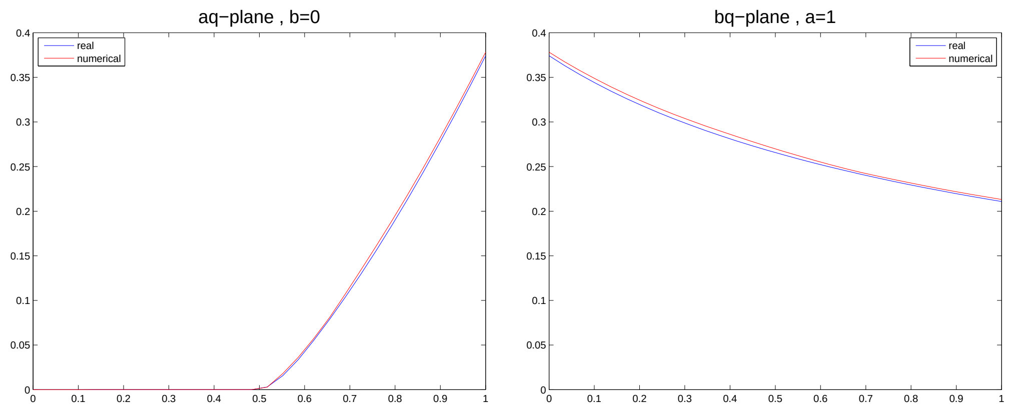
Contour Lines of Informational Rent



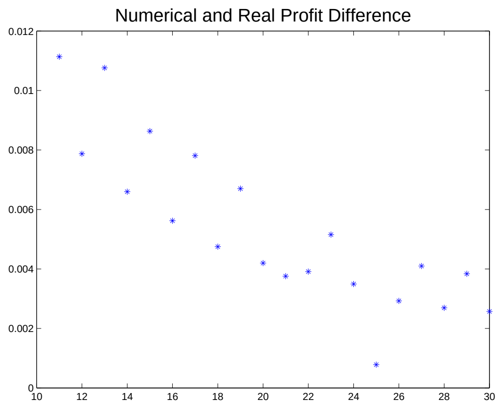
Cross-sections of Quantity



Cross-sections of Informational Rent

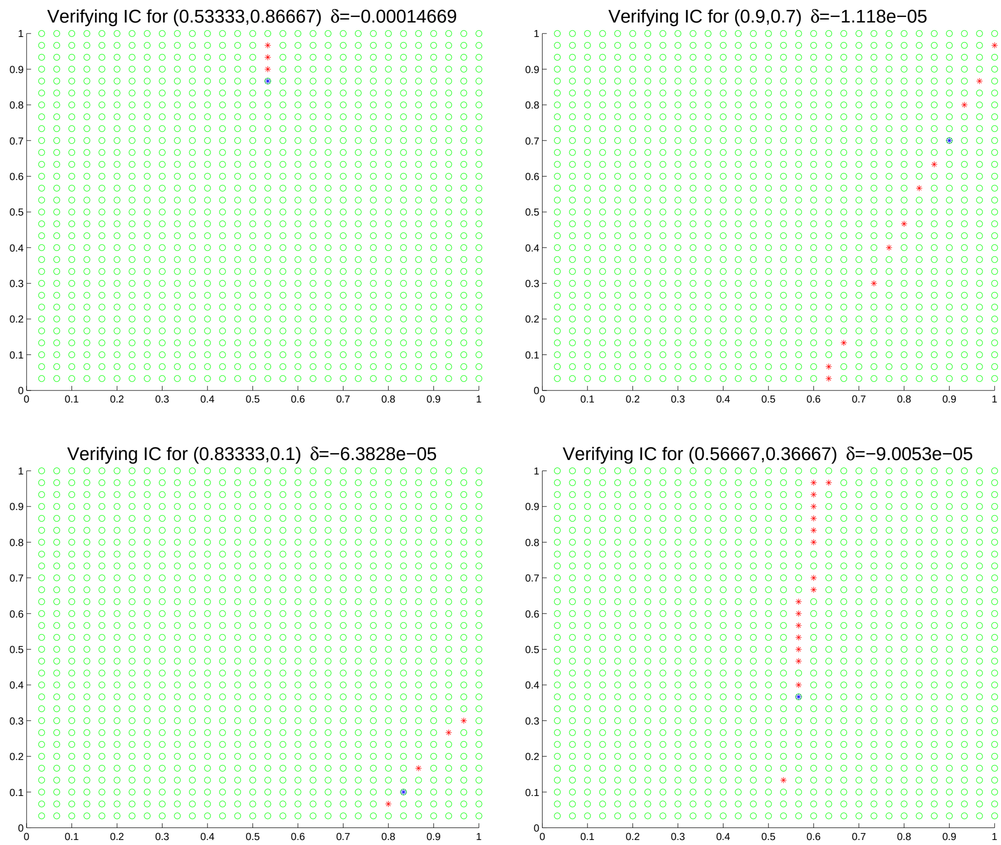


Distance to real profit



Verifying IC constraints

In this example $\delta^* = -7.82371 \times 10^{-4}$. Then, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.



Example 3 [Vieira (2008)]

In Vieira (2008) the author solved the original problem for the data

$$v(q,a,b) = aq - \frac{(1+b^2)}{2}q^2 \quad , \quad C(q) = 0 \quad , \quad f(a,b) = 1$$

The solutions q and T he have found are:

$$q(a,b) = \begin{cases} 0 & , \quad a \leq \frac{1}{2} \\[2ex] \dfrac{6a-3}{2+6b^2} & , \quad \dfrac{1}{2} \leq \dfrac{2a+3b^2}{2+6b^2} \leq \dfrac{5}{8} \\[2ex] \dfrac{5a-2}{3+5b^2} & , \quad \dfrac{5}{8} \leq \dfrac{3a+2b^2}{3+5b^2} \leq 1 \end{cases}$$

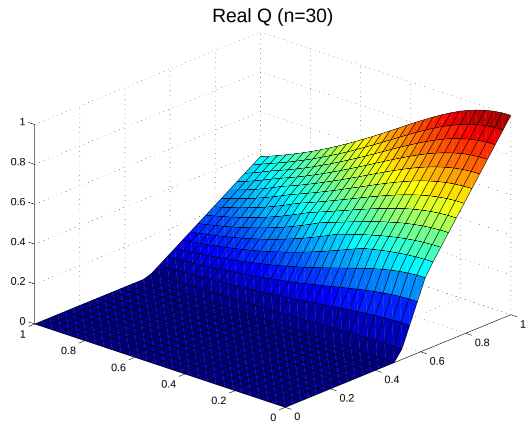
$$T(q) = \begin{cases} \dfrac{q}{2} - \dfrac{q^2}{3} & , \quad q \leq \dfrac{3}{8} \\[2ex] \dfrac{2q}{5} - \dfrac{q^2}{5} + \dfrac{3}{160} & , \quad q > \dfrac{3}{8} \end{cases}$$

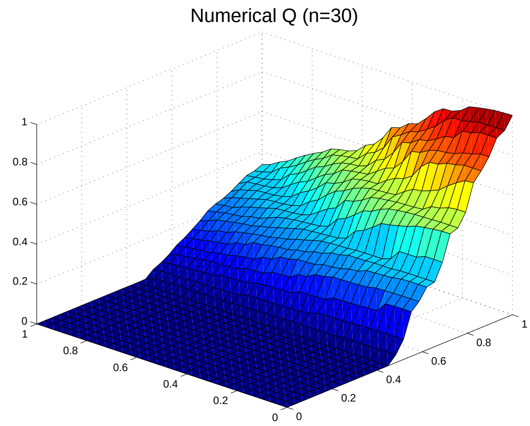
We solved the discretized problem with $n = 30$ points on each axis. For this value, 50 866 incentive compatibility constraints were considered, and 758 234 were eliminated.

Next, we compare our result as was previously explained.

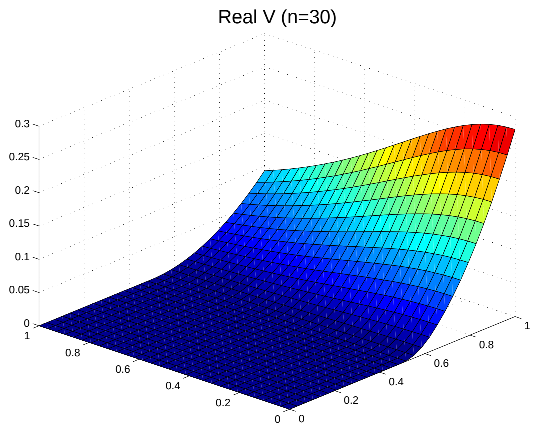
$$\begin{aligned} a.q.e.(Q^{num}, Q^{real}) &= 6.3191 \times 10^{-4} \\ a.q.e.(V^{num}, V^{real}) &= 1.2364 \times 10^{-5} \\ \left| \text{profit}^{num} - \text{profit}^{real} \right| &= 1.1325 \times 10^{-3} \end{aligned}$$

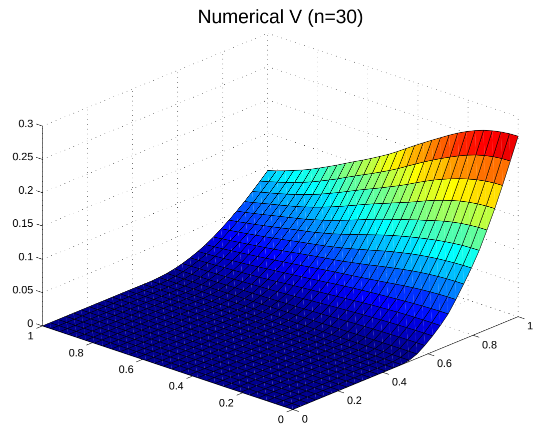
Comparing Quantity



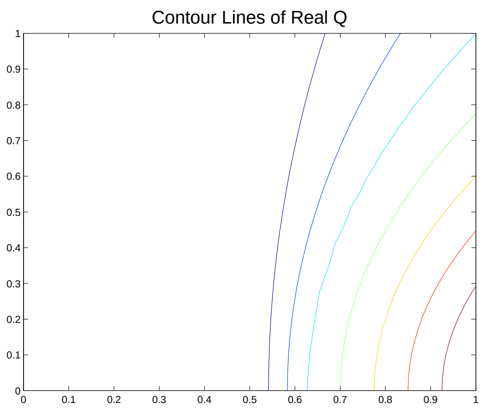


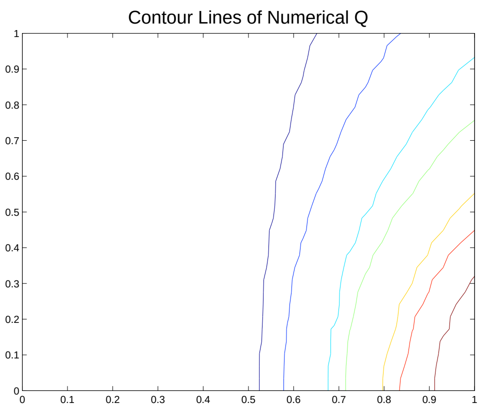
Comparing Informational Rent



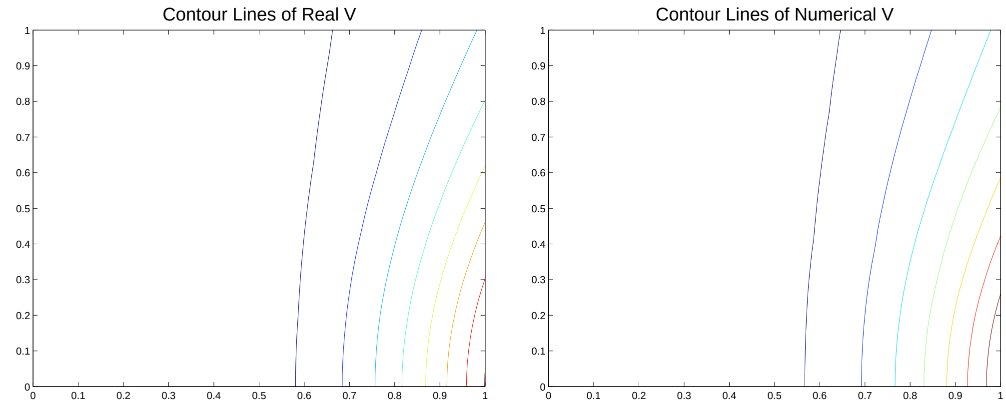


Contour Lines of Quantity

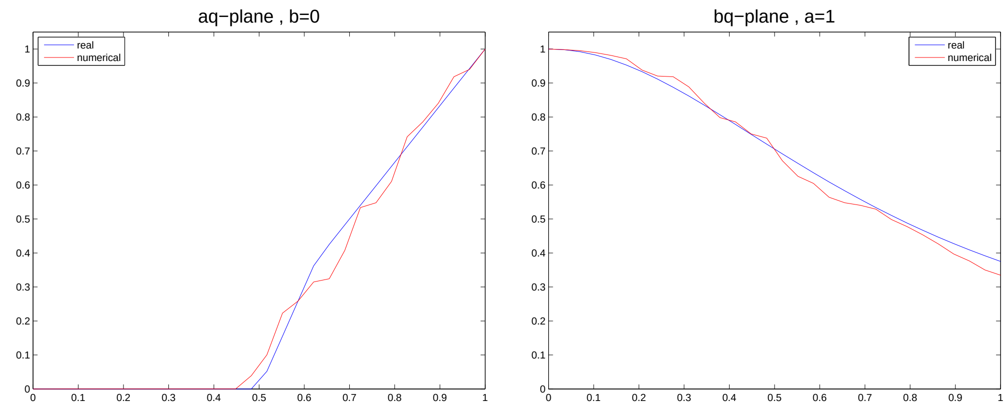




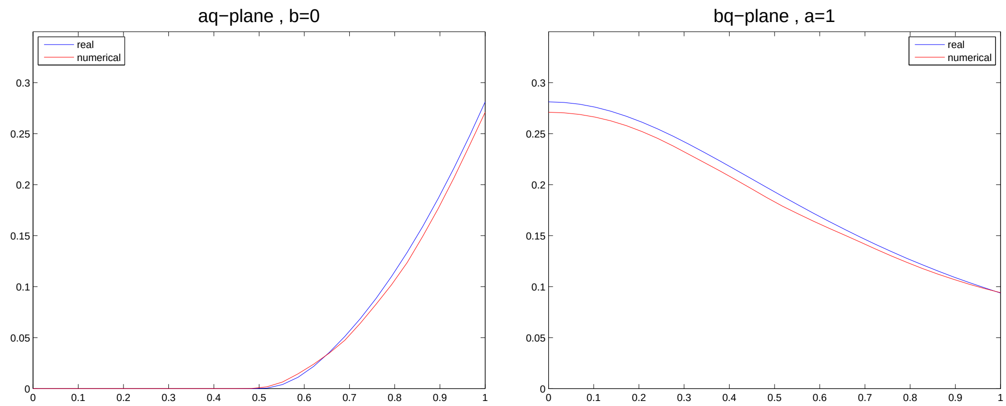
Contour Lines of Informational Rent



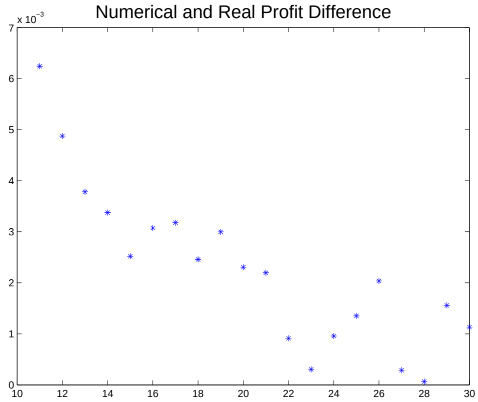
Cross-sections of Quantity



Cross-sections of Informational Rent

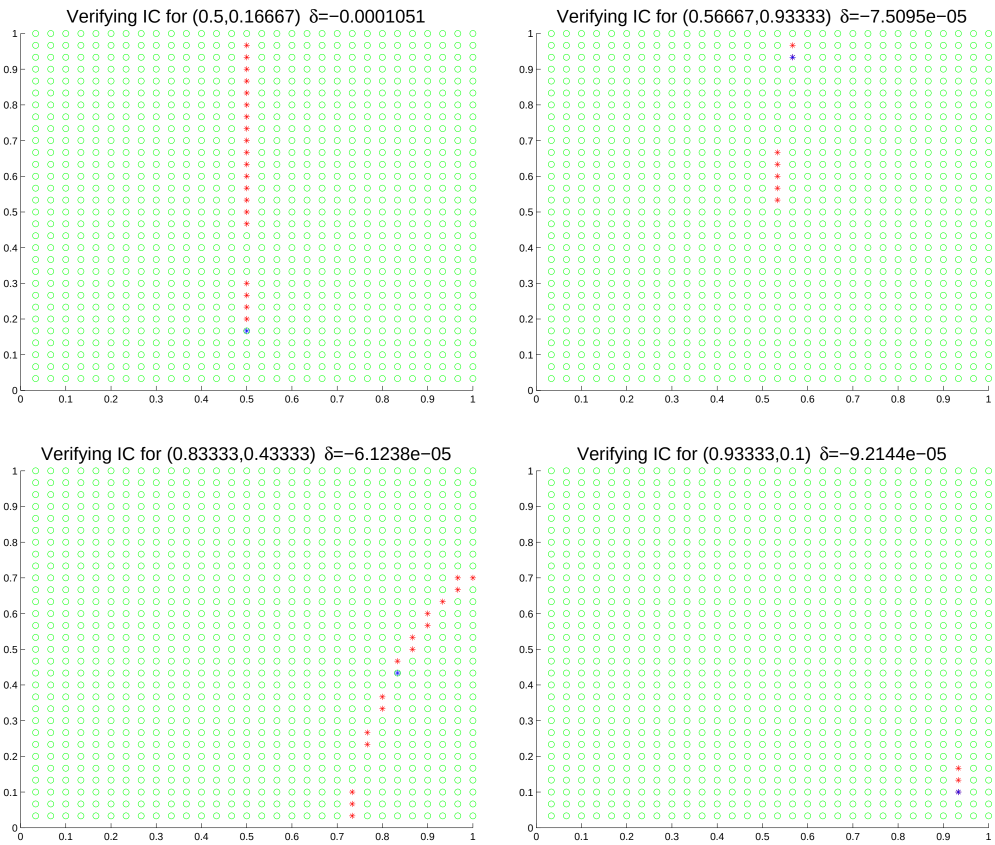


Distance to real profit



Verifying IC constraints

Here $\delta^* = -8.54331\times10^{-4}$. Thus, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.



Example 4 (Convex Demand Curves)

Consider

$$v(q,a,b) = (c-b)\log(aq+1) \quad , \quad C(q) = \lambda q \quad , \quad f(a,b) = 1$$

where $c \geq 1$ and $\lambda \in (0,1)$ are constants. A feature of this valuation function, not present in the previous ones, is that for $c = 1$ neither $(0,b)$ nor $(a,1)$ types are interested in consumption, so we could expect that is optimal for the monopolist to assign $q = 0$ over the lines $a = 0$ and $b = 1$.

The (a,b) agent's demand is defined by $p = (c-b)/(aq+1)$, then for this valuation function the monopolist faces customers with stricly convex demand curves

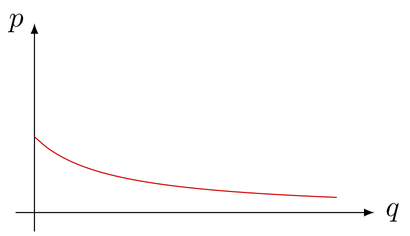


Figure 3.5: Customer's demand curve

Also, this valuation function can be seen as a log-tranformation of $(aq+1)^{(c-b)}$. We use the former v since satisfies all the assumptions, as can be easily verified.

For this problem, we have

- $G = v - \lambda q - (1-a)v_a$
- $\dfrac{G_q}{v_{qa}} = a^2q + 2a - 1 - \lambda\dfrac{(aq+1)^2}{c-b}$

The first step of the characteristic method is to solve the following initial value problem:

$$a_s(r,s) = \frac{-v_{qb}}{v_{qa}} = \frac{a(r,s)\Big(a(r,s)\phi(r)+1\Big)}{c-b(r,s)} \quad , \quad a(r,0) = r$$

$$b_s(r,s) = 1 \quad , \quad b(r,0) = 0$$

which have solutions

$$a(r,s) = \frac{c\,r}{c-(1+r\phi(r))s} \quad \text{and} \quad b(r,s) = s$$

Then, using the optimality necessary condition (2.2), we have that if $\phi(r)$ is the optimal allocation over the curve $(a(r,s), b(r,s))$ where $r \in [\underline{r}, 1]$ is fixed, then

$$\int_0^{\overline{s}(r)} \frac{G_q}{v_{qa}}(\phi(r), a(r,s), b(r,s))ds = 0$$

So, we have to solve for ϕ

$$\int_0^{\overline{s}(r)} \{\frac{c^2r^2\phi}{(c-(1+r\phi)s)^2} + \frac{2cr}{c-(1+r\phi)s} - 1 - \frac{\lambda(1+r\phi)^2(c-s)}{(c-(1+r\phi)s)^2}\}ds = 0$$

where $\overline{s}(r)$ is such that $a(r,\overline{s}(r)) = 1$, i.e. $\overline{s}(r) = \frac{c(1-r)}{1+r\phi}$
After calculations, for a fixed $r \in [\underline{r}, 1]$, $\phi(r)$ is the positive solution of

$$D(r)\phi^2 + E(r)\phi + F(r) = 0 \tag{3.2}$$

where

$$\begin{aligned} D(r) &= \lambda r(1-r) \\ E(r) &= \lambda(1-r) - \lambda r\log(r) - c\,r(1-r) \\ F(r) &= 2c\,r\log(r) + c(1-r) - \lambda\log(r) \end{aligned}$$

Since $F''(r) = \frac{2c}{r} + \frac{\lambda}{r^2}$, F is stricly convex on $(0,1)$. Note that F has a minimum over $(0,1)$ in view of $F'(r) > 0$ when $r \approx 1$ and $F'(r) < 0$ when $r \approx 0$. Besides, in view of $F(r) > 0$ for $r \approx 0$ and $F(r^*) < 0$ for some $r^* \in (0,1)$ (the minimum value must be negative in view of $F(1) = 0$), there exists a unique $r_0 \in (0,1)$ such that $F(r_0) = 0$. Since $\phi(\underline{r}) = 0$ implies $F(\underline{r}) = 0$ we must have $\underline{r} = r_0$. Therefore, $\underline{r}$ is defined as the unique solution on $(0,1)$ of

$$(2c\,r - \lambda)\log(r) + c(1-r) = 0$$

Because $F(r) < 0$ on $(\underline{r}, 1)$ (F is stricly convex, $F(1) = 0$ and $F(\underline{r}) = 0$) and

$D(r) > 0$ one solution of the quadratic equation (3.2) is always negative on $(\underline{r}, 1)$, then we can express $\phi(r)$ in the closed form:

$$\phi(r) = \frac{-E(r) + \sqrt{E(r)^2 - 4D(r)F(r)}}{2D(r)} \quad , \quad r \in (\underline{r}, 1)$$

Because $\phi(1)$ is not defined, by continuity we can define $q(1, 0)$ as

$$q(1, 0) = \lim_{r \to 1} \phi(r) = \frac{c}{\lambda} - 1$$

this value of $q(1, 0)$ solves the equation $v_q(q, 1, 0) = C'(q)$ meaning that there is not distortion at the top, as expected on the solution.

Note that $\underline{r}$ defines participation's boundary because $\phi(\underline{r}) = 0$. This boundary is given by $b = c - (c\underline{r})/a$.

Next we return to the original variables. Fix $(a, b) \in [0, 1]^2$

- If $b \geq c - (c\underline{r})/a$ then $q(a, b) = 0$ i.e., (a, b) type is excluded.
- If $b < c - (c\underline{r})/a$, $r(a, b)$ is defined as the solution of

$$\frac{c - b}{br} - \frac{c}{ab} = \frac{-E(r) + \sqrt{E(r)^2 - 4D(r)F(r)}}{2D(r)}$$

  such that $r(a, b) \in (\underline{r}, 1)$, $\phi(r(a, b)) > 0$ and $\phi'(r(a, b)) > 0$. With such $r(a, b)$ we define

$$q(a, b) = \frac{c - b}{b\, r(a, b)} - \frac{c}{ab}$$

Furthermore, the tariff as a function of r over $(\underline{r}, 1)$ can be expressed as

$$T(r) = \int_{\underline{r}}^{r} v_q(\phi(\tilde{r}), \tilde{r}, 0)\phi'(\tilde{r}) d\tilde{r}$$

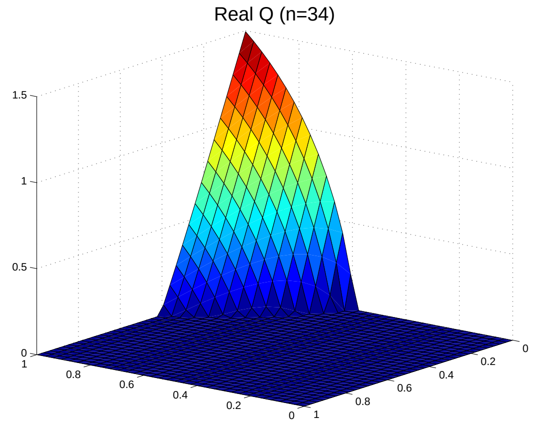
So, the type (a, b) has to transfer $t(a, b) = T(r(a, b))$ to the monopolist.

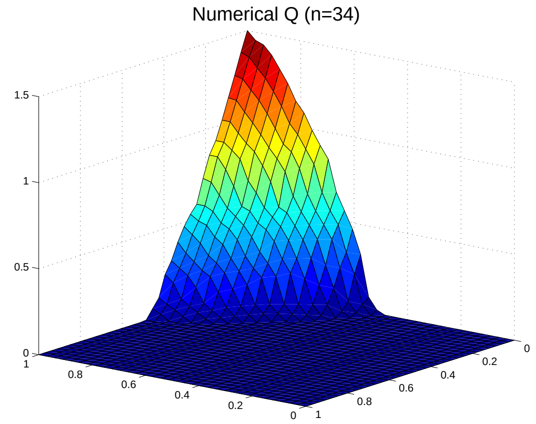
Finally, knowing $q(a, b)$ and $t(a, b)$ we can find the informational rent $V(a, b)$. For the case $c = 1$ and $\lambda = 0.4$ we did it numerically.

By the other hand, we solved the discretized problem numerically (for the same values $c = 1$, $\lambda = 0.4$) with $n = 34$, so we can compare the results.

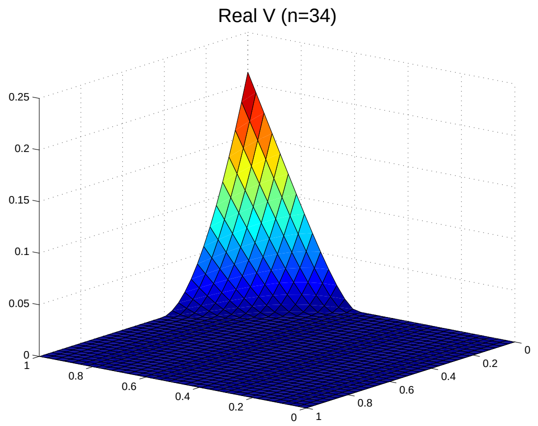
$$a.q.e.(Q^{num}, Q^{real}) = 2.6300{\times}10^{-3}$$
$$a.q.e.(V^{num}, V^{real}) = 2.6064{\times}10^{-5}$$
$$\left| \text{profit}^{num} - \text{profit}^{real} \right| = 2.3191{\times}10^{-2}$$

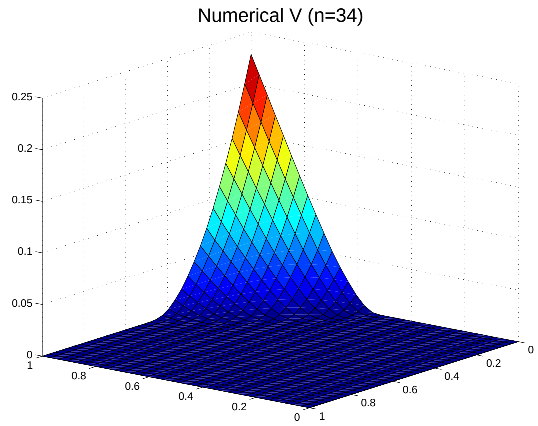
Comparing Quantity



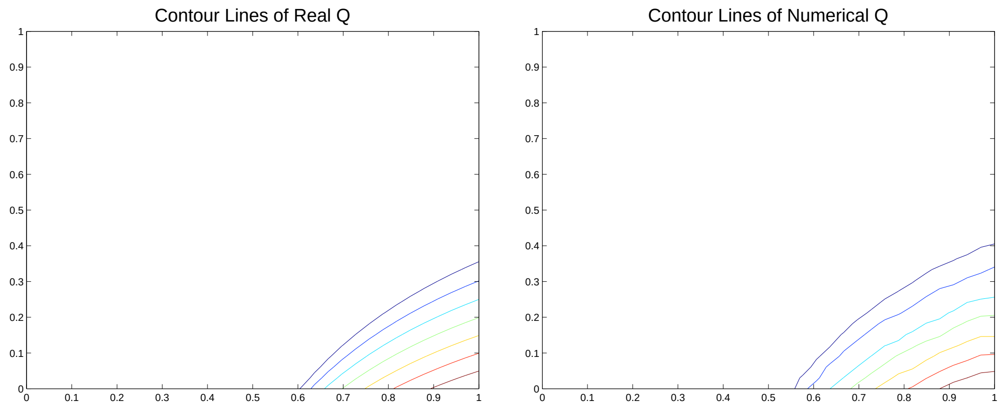


Comparing Informational Rent

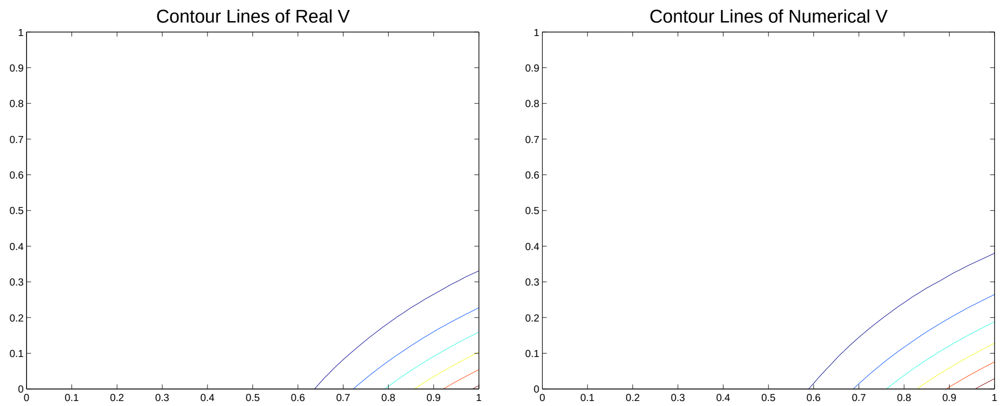




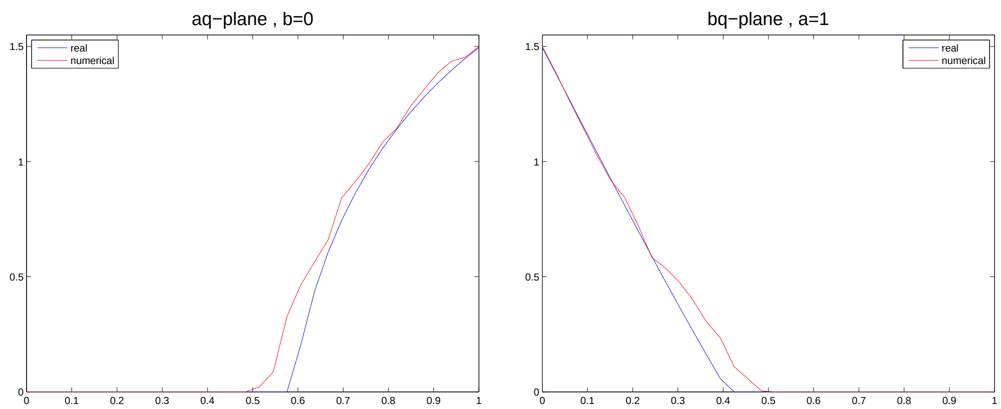
Contour Lines of Quantity



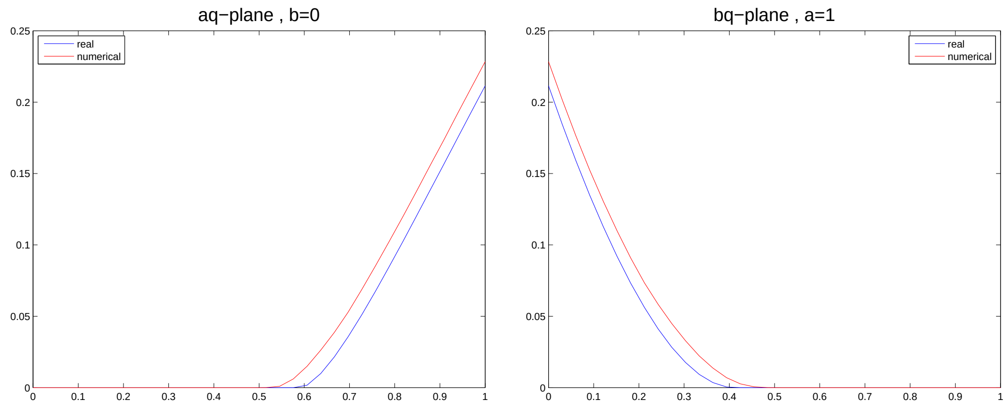
Contour Lines of Informational Rent



Cross-sections of Quantity

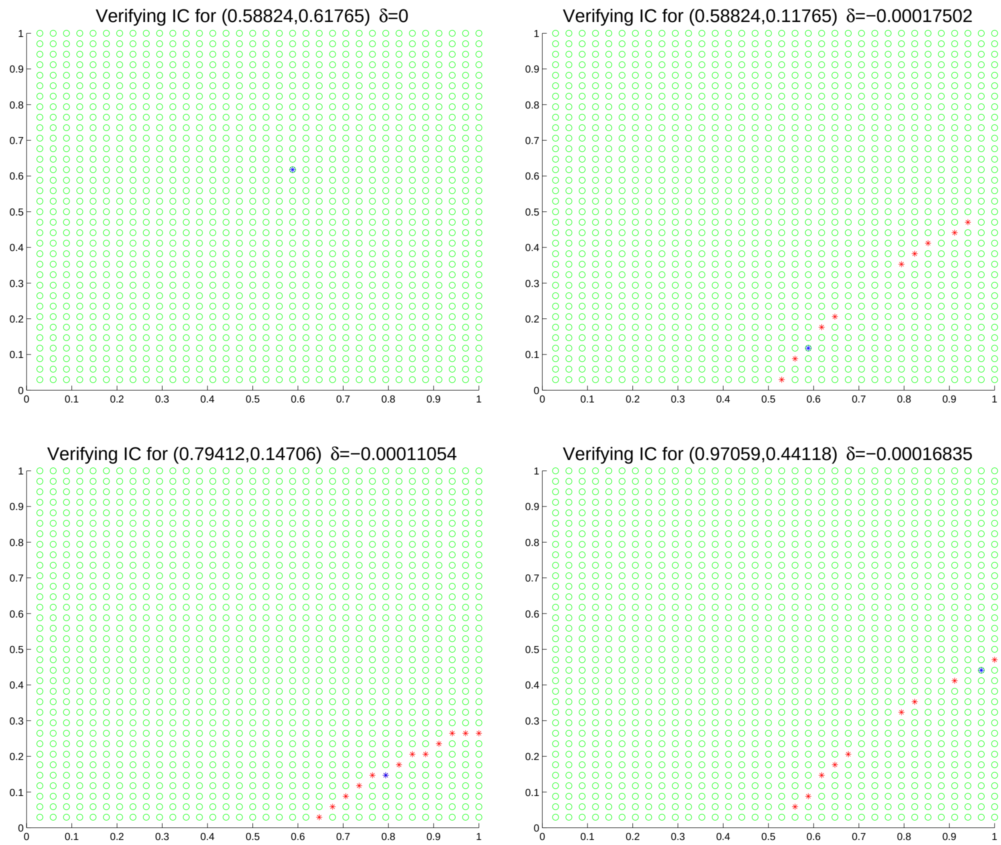


Cross-sections of Informational Rent



Verifying IC constraints

In this example $\delta^* = -5.77989 \times 10^{-4}$. Then, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.



Example 5

Consider

$$v(q,a,b) = aq + (q+1)^{1-b} - 1 \quad , \quad C(q) = q^2 \quad , \quad f(a,b) = 1$$

This valuation function is linear with respect to the parameter affecting consumption positively, while the other parameter influence consumption negatively as an exponent. All the assumptions can be easily verified.

The first step of the characteristic method is to solve the following initial value problem:

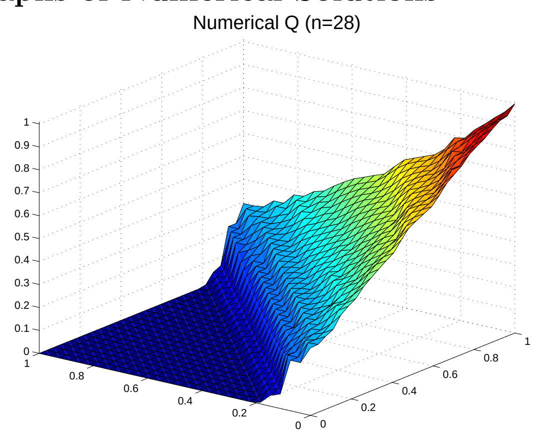
$$\begin{aligned} a_s(r,s) &= \frac{-v_{qb}}{v_{qa}} = \frac{(1+(1-b(r,s))\log(\phi(r)+1))}{(\phi(r)+1)^{b(r,s)}} \quad &, \; a(r,0) = r \\ b_s(r,s) &= 1 \quad &, \; b(r,0) = 0 \end{aligned}$$

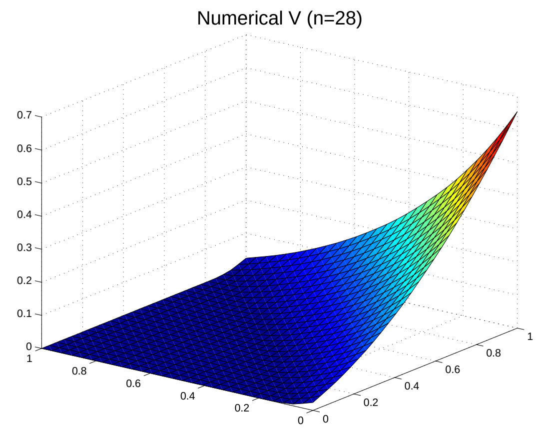
which solution cannot be express in a closed-form.

Before present the numerical approximation, we claim that types $(0,0)$ and $(1,1)$ are equally treated by the monopolist, and get a positive quantity in the optimal contract, in view of $v(q,0,0) = q = v(q,1,1)$ and $v_q(q,0,0) = 1 = v_q(q,1,1)$. Also, because there is not distorsion with respect to the first best contract for the 'best' type $(1,0)$, $q(1,0)$ must be the solution of $v_q(q,1,0) = C'(q)$, which yields on $q(1,0) = 1$. We would expect these features in the optimal allocation.

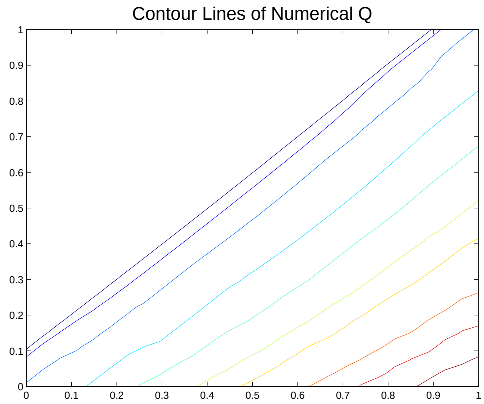
We solved the discretized problem with $n = 28$. On the following graphs we can verify the features described above.

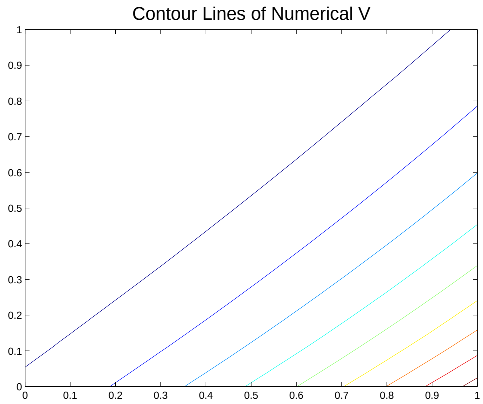
Graphs of Numerical Solutions



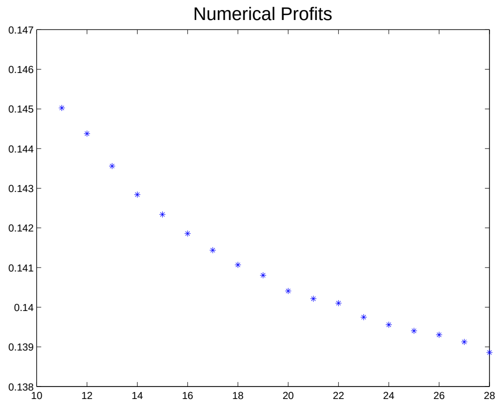


Contour Lines



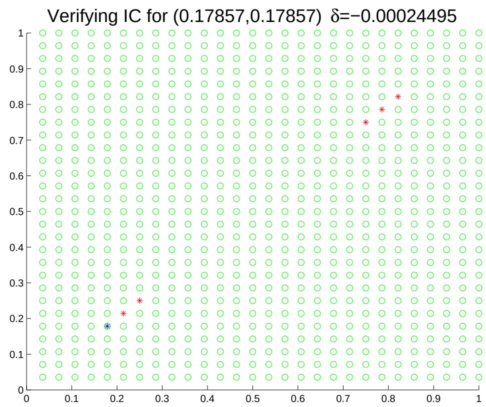


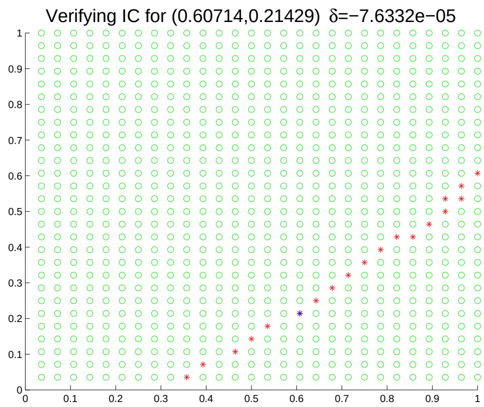
Numerical profit sequence

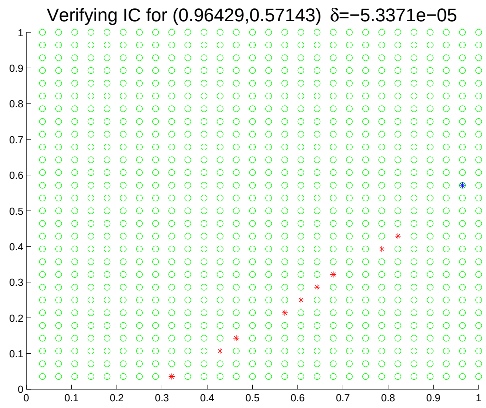


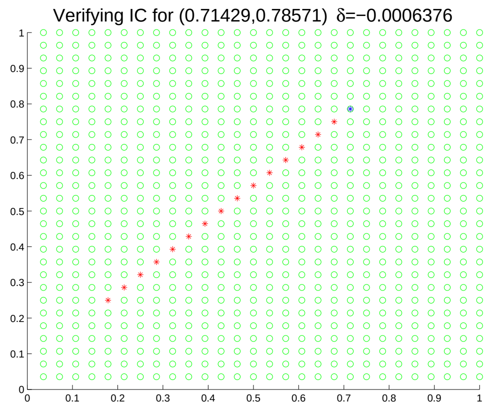
Verifying IC constraints

Here $\delta^* = -1.19598{\times}10^{-3}$. Then, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.









Example 6

Consider

$$v(q,a,b) = (q+1)^a - bq - 1 \quad , \quad C(q) = \frac{q^2}{2} \quad , \quad f(a,b) = 1$$

This valuation function is linear with respect to the parameter affecting consumption negatively, while the other parameter influence consumption positively as an exponent. All the assumptions can be easily verified.

The first step of the characteristic method is to solve the following initial value problem:

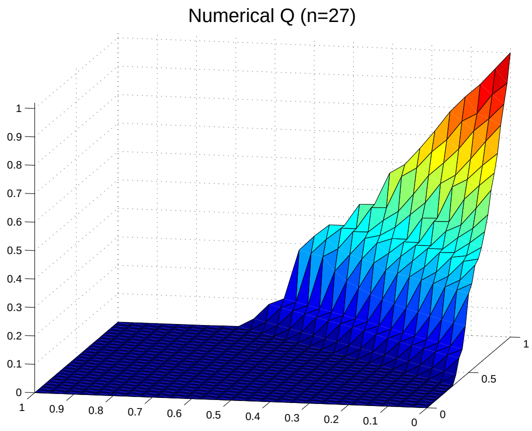
$$\begin{aligned} a_s(r,s) &= \frac{-v_{qb}}{v_{qa}} = \frac{(\phi(r)+1)^{1-a(r,s)}}{1 + a(r,s)\log(\phi(r)+1)} \quad , \; a(r,0) = r \\ b_s(r,s) &= 1 \quad , \; b(r,0) = 0 \end{aligned}$$

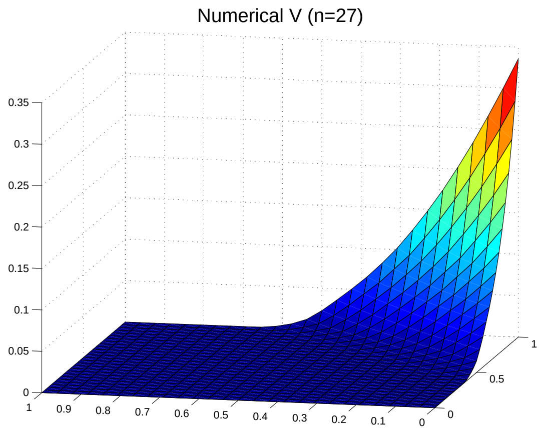
which solution cannot be express in a closed-form.

Before present the numerical approximation, we claim that types $(0,0)$ and $(1,1)$ are equally treated by the monopolist, and get zero quantity in the optimal contract, since $v(q,0,0) = 0 = v(q,1,1)$ and $v_q(q,0,0) = 0 = v_q(q,1,1)$. Also, since for the best type there is not distorsion with respect to the first best contract, $q(1,0)$ must be the solution of $v_q(q,1,0) = C'(q)$, then $q(1,0) = 1$.

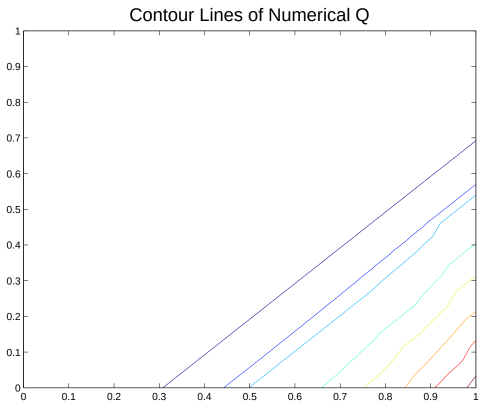
We solved the discretized problem with $n = 27$. The features described above can be verified in the following graphs of the numerical solution.

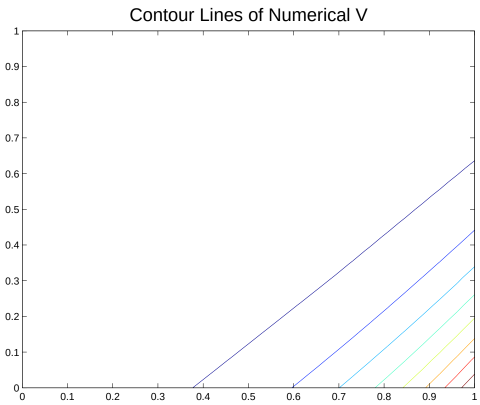
Graphs of Numerical Solutions



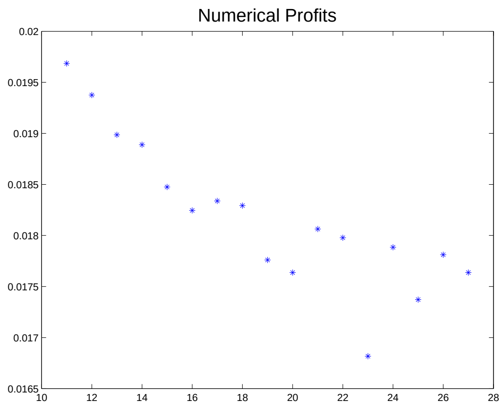


Contour Lines



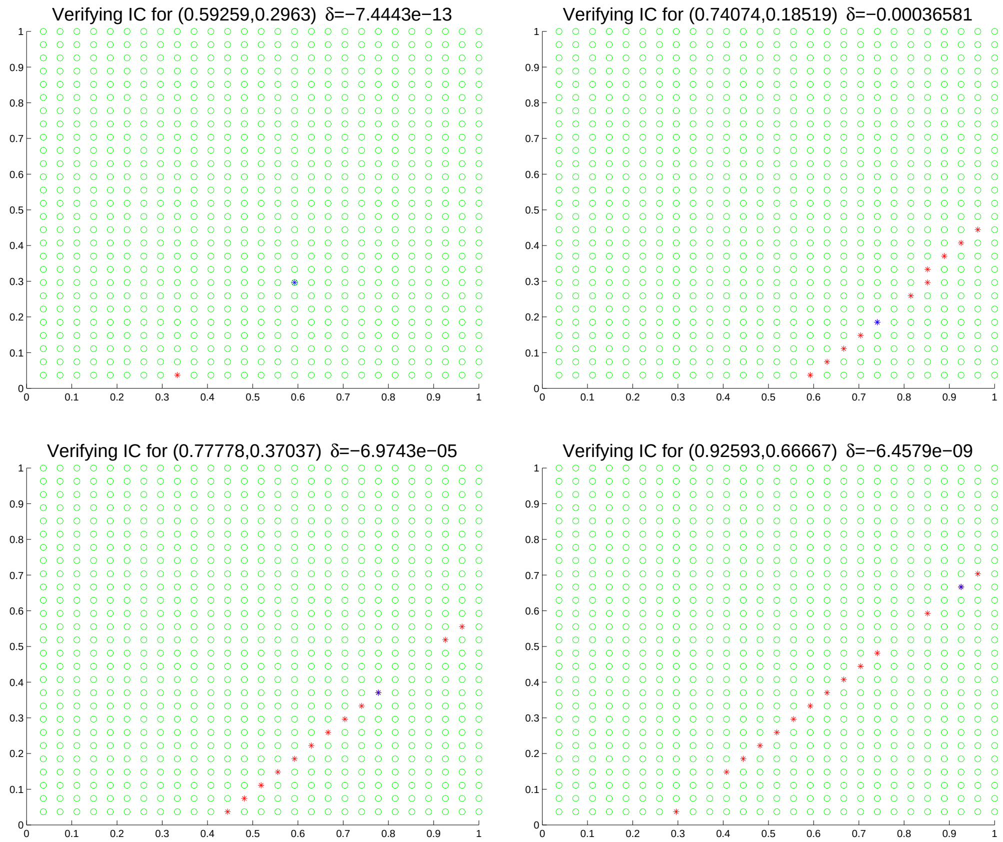


Numerical profit sequence



Verifying IC constraints

Here $\delta^* = -1.50164 \times 10^{-3}$. Then, for any tolerance $tol > -\delta^*$ all IC constraints will be satisfied.



3.6 Appendix B

Proof of Proposition 3.1.1.
Let (q,t) be an incentive compatible contract. For any $(a,b) \in [0,1]^2$:

$$(a,b) \in \underset{(\widehat{a},\widehat{b})\in[0,1]^2}{\text{argmax}} \{v(q(\widehat{a},\widehat{b}),a,b) - t(\widehat{a},\widehat{b})\}$$

Given $(a,b) \in (0,1)\times(0,1)$, consider $F(\cdot,\cdot) := v(q(\cdot,\cdot),a,b) - t(\cdot,\cdot)$. At any point of twice continuous differentiability, since (a,b) maximizes F, the necessary condition $F'(a,b) = 0$ yields on

$$t_a(a,b) = v_q(q(a,b),a,b)q_a(a,b)$$

$$t_b(a,b) = v_q(q(a,b),a,b)q_b(a,b)$$

then

$$t_{aa} = v_{qq}q_a^2 + v_{qa}q_a + v_q q_{aa}$$

$$t_{bb} = v_{qq}q_b^2 + v_{qb}q_b + v_q q_{bb}$$

$$t_{ab} = v_{qq}q_a q_b + v_{qb}q_a + v_q q_{ab}$$

$$t_{ba} = v_{qq}q_a q_b + v_{qa}q_b + v_q q_{ba}$$

By the other hand, $F''(a,b)$ must be negative semi-definite, where

$$F''(a,b) = \begin{bmatrix} v_{qq}q_a^2 + v_q q_{aa} - t_{aa} & v_{qq}q_a q_b + v_q q_{ab} - t_{ab} \\ v_{qq}q_a q_b + v_q q_{ba} - t_{ab} & v_{qq}q_b^2 + v_q q_{bb} - t_{bb} \end{bmatrix}$$

(all the arguments are (a,b)). Replacing those expressions of t_{aa}, t_{ab}, t_{ba} and t_{bb} on $F''(a,b)$ we obtain that

$$\begin{bmatrix} -v_{qa}(q(a,b),a,b)q_a(a,b) & -v_{qb}(q(a,b),a,b)q_a(a,b) \\ \\ -v_{qa}(q(a,b),a,b)q_b(a,b) & -v_{qb}(q(a,b),a,b)q_b(a,b) \end{bmatrix}$$

is negative semi-definite matrix.
Then $v_{qa}q_a \geq 0$ and $v_{qb}q_b \geq 0$. By Assumption A3 we have $v_{qa} > 0$, $v_{qb} < 0$ which implies $q_a(a,b) \geq 0$ and $q_b(a,b) \leq 0$. □

Proof of Proposition 3.2.1.
Fix $q \in Q$, by A3 $v_q(q, \cdot, b)$ is strictly increasing and $v_q(q, \widehat{a}, \cdot)$ is strictly decreasing, so $\widehat{a} > a$ and $\widehat{b} < b$ implies $v_q(q, a, b) < v_q(q, \widehat{a}, b)$ and $v_q(q, \widehat{a}, b) < v_q(q, \widehat{a}, \widehat{b})$ respectively. Thus, $v_q(q, a, b) < v_q(q, \widehat{a}, \widehat{b})$. □

Proof of Proposition 3.2.2.
If $(x, y) \in CC(\widehat{a}, \widehat{b})$ then $q(\widehat{a}, \widehat{b}) = q(x, y)$ so, by the Taxation Principle, $t(\widehat{a}, \widehat{b}) = T(q(\widehat{a}, \widehat{b})) = T(q(x, y)) = t(x, y)$. Because (a, b) is IC with $(\widehat{a}, \widehat{b})$ we have

$$\begin{aligned} v(q(a, b), a, b) - t(a, b) &\geq v(q(\widehat{a}, \widehat{b}), a, b) - t(\widehat{a}, \widehat{b}) \\ &= v(q(x, y), a, b) - t(x, y) \end{aligned}$$

that is, (a, b) is IC with (x, y). □

Proof of Proposition 3.2.3.
We have

$$V(a, b) - V(\widehat{a}, \widehat{b}) \geq v(q(\widehat{a}, \widehat{b}), a, b) - v(q(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b})$$

$$V(\widehat{a}, \widehat{b}) - V(x, y) \geq v(q(x, y), \widehat{a}, \widehat{b}) - v(q(x, y), x, y)$$

then

$$V(a, b) - V(x, y) + v(q(x, y), x, y) \geq v(q(\widehat{a}, \widehat{b}), a, b) - v(q(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b}) + v(q(x, y), \widehat{a}, \widehat{b})$$

Besides, because $v_q(q, \widehat{a}, \widehat{b}) \leq v_q(q, a, b)\ \forall\ q \in Q$, and $q(x, y) \leq q(\widehat{a}, \widehat{b})$:

$$\int_{q(x,y)}^{q(\widehat{a},\widehat{b})} v_q(q, \widehat{a}, \widehat{b}) dq \ \leq \ \int_{q(x,y)}^{q(\widehat{a},\widehat{b})} v_q(q, a, b) dq$$

then

$$v(q(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b}) - v(q(x, y), \widehat{a}, \widehat{b}) \leq v(q(\widehat{a}, \widehat{b}), a, b) - v(q(x, y), a, b)$$

Therefore

$$\begin{gathered} V(a, b) - V(x, y) + v(q(x, y), x, y) \geq \\ v(q(\widehat{a}, \widehat{b}), a, b) - v(q(\widehat{a}, \widehat{b}), \widehat{a}, \widehat{b}) + v(q(x, y), \widehat{a}, \widehat{b}) \geq v(q(x, y), a, b) \end{gathered}$$

thus

$$V(a,b) - V(x,y) \geq v(q(x,y),a,b) - v(q(x,y),x,y)$$

which means that (a,b) verifies IC with (x,y). □

Proof of Theorem 3.2.1.

Fix any $(a,b),(\widehat{a},\widehat{b}) \in [0,1]^2$. Let us prove that (a,b) is IC with $(\widehat{a},\widehat{b})$.

If $q(\widehat{a},\widehat{b}) = q^{\text{out}}$ (that is, if type $(\widehat{a},\widehat{b})$ is excluded) we have $V(\widehat{a},\widehat{b}) = 0$ so, from IR constraint $V(a,b) \geq 0$ we can write

$$V(a,b) - V(\widehat{a},\widehat{b}) \geq v(q^{\text{out}},a,b) - v(q^{\text{out}},\widehat{a},\widehat{b})$$

in view of $v(q^{\text{out}},a,b) = v(q^{\text{out}},\widehat{a},\widehat{b})$ by Assumption A5

If $q(\widehat{a},\widehat{b}) \neq q^{\text{out}}$, because $CC(\widehat{a},\widehat{b})$ is strictly increasing there are three possible cases:

Case 1 $CC(\widehat{a},\widehat{b})$ intersects $F^{(a,b)}$:

Let (x,y) the point of intersection. Because (a,b) is IC with (x,y) and $(x,y) \in CC(\widehat{a},\widehat{b})$, by Proposition 3.2.2 (a,b) is IC with $(\widehat{a},\widehat{b})$.

Case 2 $CC(\widehat{a},\widehat{b})$ intersects $\{(1,s) : 0 \leq s \leq b\}$:

Since $CC(\widehat{a},\widehat{b})$ is strictly increasing then $\widehat{b} < b$. If $\widehat{a} > a$, by Proposition 3.2.1, we have that $(a,b) \preceq (\widehat{a},\widehat{b})$, then (a,b) is IC with $(\widehat{a},\widehat{b})$. If $\widehat{a} \leq a$, consider $(x,y) \in CC(\widehat{a},\widehat{b}) \cap \text{conv}\{(a,b),(1,0)\}$ [7], then (x,y) is such that $x > a$ and $y < b$, so we are in the previous case, i.e. (a,b) is IC with (x,y) and by Proposition 3.2.2 (a,b) is IC with $(\widehat{a},\widehat{b})$.

Case 3 $CC(\widehat{a},\widehat{b})$ intersects $\{(s,1) : 0 \leq s \leq a\}$:

Since $CC(\widehat{a},\widehat{b})$ is strictly increasing then $\widehat{a} < a$. Without loss of generality, we consider that $\widehat{b} > b$ (if this is not the case, replace $(\widehat{a},\widehat{b})$ for any point in $CC(\widehat{a},\widehat{b})$ on the northwest of (a,b)). Then, by Proposition 3.2.1 $(\widehat{a},\widehat{b}) \preceq (a,b)$. Let $(x_1,1) \in CC(\widehat{a},\widehat{b}) \cap \{(s,1) : 0 \leq s \leq a\}$, and consider $(x_1,y_1) \in \{(x_1,y) : y \in \mathbb{R}\} \cap \text{conv}\{(\widehat{a},\widehat{b}),(a,b)\}$. Because $(x_1,1) \in F^{(x_1,y_1)}$ and Proposition 3.2.2, (x_1,y_1) is IC with $(\widehat{a},\widehat{b})$. Note also that $q(\widehat{a},\widehat{b}) < q(x_1,y_1)$ then, by Proposition 3.2.3, it will be sufficient that (a,b) is IC with (x_1,y_1). If this is not the case, repeat the procedure taking $(x_2,1) \in CC(x_1,y_1) \cap \{(s,1) : 0 \leq s \leq a\}$ and $(x_2,y_2) \in \{(x_2,y) : y \in$

[7] $\text{conv}\{(a,b),(1,0)\}$ is the convex hull of these points, i.e. the line segment between them.

$\mathbb{R}\} \cap \text{conv}\{(x_1, y_1), (a, b)\}$. Because $(x_2, 1) \in F^{(x_2,y_2)}$, $q(x_1, y_1) < q(x_2, y_2)$ and by Proposition 3.2.2, it will be sufficient that (a, b) is IC with (x_2, y_2). If this is not the case, we set up the point (x_3, y_3), and so on. Then, for n big enough $CC(x_n, y_n)$ intersects $F^{(a,b)}$ because the sequence (x_n, y_n) is such that $(x_n, y_n) \to (a, b)$ and $CC(x_n, y_n)$ is strictly increasing. Thus, applying n times Proposition 3.2.3, we have that (a, b) is IC with $(\widehat{a}, \widehat{b})$. □

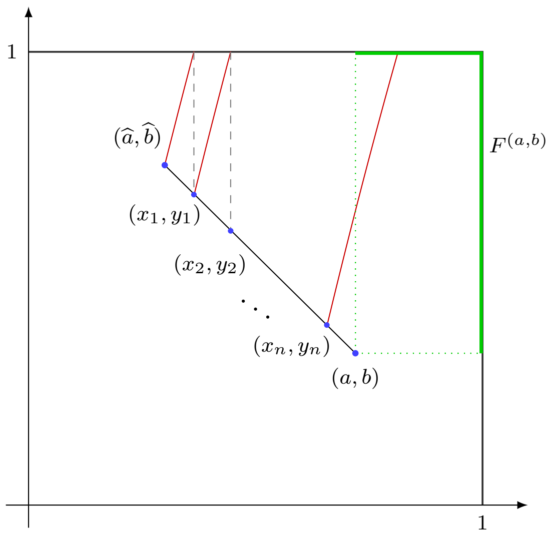


Figure 3.6: Theorem's proof illustration

Proof of Proposition 3.2.4.

1. Since $v_q(q, \cdot, \widehat{b})$ is concave and $v_q(q, a, \cdot)$ is convex:

$$v_q(q, \widehat{a}, \widehat{b}) - v_q(q, a, \widehat{b}) \leq v_{qa}(q, a, \widehat{b})(\widehat{a} - a)$$

$$v_q(q, a, b) - v_q(q, a, \widehat{b}) \geq v_{qb}(q, a, \widehat{b})(b - \widehat{b})$$

then

$$v_q(q, \widehat{a}, \widehat{b}) - v_q(q, a, b) \leq v_{qa}(q, a, \widehat{b})(\widehat{a} - a) + v_{qb}(q, a, \widehat{b})(\widehat{b} - b)$$

So, if $(a,b) \preceq (\widehat{a},\widehat{b})$ we have $0 \leq v_q(q,\widehat{a},\widehat{b}) - v_q(q,a,b)$ which implies

$$\frac{\widehat{b}-b}{\widehat{a}-a} \leq \frac{-v_{qa}(q,a,\widehat{b})}{v_{qb}(q,a,\widehat{b})}$$

in view of $\widehat{a} > a$ and $-v_{qb} > 0$.

2. Since $v_q(q,\cdot,b)$ is concave and $v_q(q,\widehat{a},\cdot)$ is convex:

$$v_q(q,\widehat{a},b) - v_q(q,a,b) \geq v_{qa}(q,\widehat{a},b)(\widehat{a}-a)$$

$$v_q(q,\widehat{a},\widehat{b}) - v_q(q,\widehat{a},b) \geq v_{qb}(q,\widehat{a},b)(\widehat{b}-b)$$

then

$$v_q(q,\widehat{a},\widehat{b}) - v_q(q,a,b) \geq v_{qa}(q,\widehat{a},b)(\widehat{a}-a) + v_{qb}(q,\widehat{a},b)(\widehat{b}-b)$$

So, if $\frac{\widehat{b}-b}{\widehat{a}-a} \leq \frac{-v_{qa}(q,\widehat{a},b)}{v_{qb}(q,\widehat{a},b)}$ then $v_{qa}(q,\widehat{a},b)(\widehat{a}-a) + v_{qb}(q,\widehat{a},b)(\widehat{b}-b) \geq 0$ thus $v_q(q,\widehat{a},\widehat{b}) - v_q(q,a,b) \geq 0$ for any $q \in Q$, that is $(a,b) \preceq (\widehat{a},\widehat{b})$

□

Proof of Proposition 3.2.5.
The proof is completely analogous of Proposition 3.2.4, so we omitted it. □

Proof of Proposition 3.4.1.
Let $(x,y) \in F^{(a,b)}$ be such that $x = 1$ (case $y = 1$ is analogous), and let $\widehat{b}$ be such that

$$\widehat{b} - \frac{1}{n-1} < y \leq \widehat{b}$$

Because (Q^n, V^n) are solutions of problem (NP), (a,b) satisfies IC with $(1,\widehat{b})$:

$$V^n(a,b) - V^n(1,\widehat{b}) \geq v(Q^n(1,\widehat{b}),a,b) - v(Q^n(1,\widehat{b}),1,\widehat{b})$$

By definition $\widetilde{Q}^n(x,y) = Q^n(1,\widehat{b})$, $\widetilde{V}^n(x,y) = V^n(1,\widehat{b})$. Also, in view of $(a,b) \in X_n$ we have $\widetilde{V}^n(a,b) = V^n(a,b)$. Then

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(x,y) \geq v(\widetilde{Q}^n(x,y),a,b) - v(\widetilde{Q}^n(x,y),1,\widehat{b})$$

On the other hand, since v is Lipschitz:

$$\left|v(\widetilde{Q}^n(x,y),1,\widehat{b}) - v(\widetilde{Q}^n(x,y),x,y)\right| \leq L||(1,\widehat{b})-(x,y)|| = O(\tfrac{1}{n-1})$$

then

$$-v(\widetilde{Q}^n(x,y),1,\widehat{b}) \geq -v(\widetilde{Q}^n(x,y),x,y) - O(\tfrac{1}{n-1})$$

Thus

$$\begin{aligned}\widetilde{V}^n(a,b) - \widetilde{V}^n(x,y) &\geq v(\widetilde{Q}^n(x,y),a,b) - v(\widetilde{Q}^n(x,y),1,\widehat{b}) \\ &\geq v(\widetilde{Q}^n(x,y),a,b) - v(\widetilde{Q}^n(x,y),x,y) - O(\tfrac{1}{n-1})\end{aligned}$$

□

Proof of Proposition 3.4.2.
If $CC(\widehat{a},\widehat{b}) \cap F^{(a,b)} = (x,y)$ we apply Proposition 3.4.1 for (a,b) with (x,y) and considering that $Q^n(\widehat{a},\widehat{b}) = Q^n(x,y)$ and $t(x,y) = t(\widehat{a},\widehat{b})$ we conclude. Another cases are treated analogously as in the proof of Theorem 3.2.1. □

Proof of Proposition 3.4.3.
Given $(a,b),(a',b') \in [0,1]^2$, it will be sufficient to prove that

$$\widetilde{V}^n(a',b') - \widetilde{V}^n(a,b) - (v(\widetilde{Q}^n(a,b),a',b') - v(\widetilde{Q}^n(a,b),a,b)) \geq -O(\tfrac{1}{n-1})$$

Let $(\widehat{a},\widehat{b}),(\widehat{a}',\widehat{b}') \in X_n$ be such that $\widehat{a} \leq a < \widehat{a} + \frac{1}{n-1}$, $\widehat{b} - \frac{1}{n-1} < b \leq \widehat{b}$ and $\widehat{a}' \leq a' < \widehat{a}' + \frac{1}{n-1}$, $\widehat{b}' - \frac{1}{n-1} < b' \leq \widehat{b}'$. Lets denote $q = \widetilde{Q}^n(a',b') = Q^n(\widehat{a}',\widehat{b}')$. Since $\widetilde{V}^n(a,b) = V^n(\widehat{a},\widehat{b})$, $\widetilde{V}^n(a',b') = V^n(\widehat{a}',\widehat{b}')$ we have

$$\begin{aligned}\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(q,a,b) - v(q,a',b')) = \\ V^n(\widehat{a},\widehat{b}) - V^n(\widehat{a}',\widehat{b}') - (v(q,\widehat{a},\widehat{b}) - v(q,\widehat{a}',\widehat{b}')) \\ + v(q,\widehat{a},\widehat{b}) - v(q,a,b) + v(q,a',b') - v(q,\widehat{a}',\widehat{b}')\end{aligned}$$

because $(\widehat{a},\widehat{b}),(\widehat{a}',\widehat{b}') \in X_n$, by Proposition 3.4.2

$$V^n(\widehat{a},\widehat{b}) - V^n(\widehat{a}',\widehat{b}') - (v(q,\widehat{a},\widehat{b}) - v(q,\widehat{a}',\widehat{b}')) \geq -O(\tfrac{1}{n-1})$$

Besides, $v_a > 0$ and $\widehat{a}' \leq a'$ implies $v(q,\widehat{a}',b') \leq v(q,a',b')$. Also $v_b < 0$ and $b' \leq \widehat{b}'$

implies $v(q, \widehat{a}', \widehat{b}') \leq v(q, \widehat{a}', b')$. Then $v(q, a', b') - v(q, \widehat{a}', \widehat{b}') \geq 0$. Hence

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(q,a,b) - v(q,a',b')) \geq v(q,\widehat{a},\widehat{b}) - v(q,a,b) - O(\tfrac{1}{n-1})$$

Because v is Lipschitz (with constant L)

$$\left| v(q,\widehat{a},\widehat{b}) - v(q,a,b) \right| \leq L||(\widehat{a},\widehat{b}) - (a,b)|| \leq O(\tfrac{1}{n-1})$$

then $v(q,\widehat{a},\widehat{b}) - v(q,a,b) \geq -O(\frac{1}{n-1})$ and finally

$$\widetilde{V}^n(a,b) - \widetilde{V}^n(a',b') - (v(q,a,b) - v(q,a',b')) \geq -O(\tfrac{1}{n-1})$$

□

Proof of Proposition 3.4.4.
Let $(\overline{Q}, \overline{V})$ denote the solution for the continuos problem, and let $(\overline{Q}^n, \overline{V}^n)$ be their restriction on the grid X_n. If (Q^n, V^n) are the solutions of the discretized problem and OPT_n is the optimal value, we have:

$$\begin{aligned} OPT_n &\geq \sum_{i=1}^{n} \sum_{j=1}^{n} w(i,j)(v(\overline{Q}^n_{i,j}, a_i, b_j) - \overline{V}^n_{i,j} - C(\overline{Q}^n_{i,j}))f(a_i,b_j) \\ &= \int_0^1 \int_0^1 (v(\overline{Q}(a,b),a,b) - \overline{V}(a,b) - C(\overline{Q}(a,b)))f(a,b)\,da\,db - O(\frac{1}{n}) \\ &= OPT^* - O(\frac{1}{n}) \end{aligned}$$

then $\liminf_{n\to\infty} OPT_n \geq OPT^*$.

On the other hand, if $\exists \lim_{n\to\infty} \widetilde{Q}^n(a,b)$ and $\lim_{n\to\infty} \widetilde{V}^n(a,b)$ for any $(a,b) \in [0,1]^2$, define:

$$\widehat{Q}(a,b) := \lim_{n\to\infty} \widetilde{Q}^n(a,b) \quad , \quad \widehat{V}(a,b) := \lim_{n\to\infty} \widetilde{V}^n(a,b)$$

By Proposition 3.4.3 $(\widehat{Q}, \widehat{V})$ is factible, then

$$\begin{aligned} OPT^* &\geq \int_0^1 \int_0^1 (v(\widehat{Q}(a,b),a,b) - \widehat{V}(a,b) - C(\widehat{Q}(a,b)))f(a,b)\,da\,db \\ &= \lim_{n\to\infty} \Big(\int_0^1 \int_0^1 (v(\widetilde{Q}^n(a,b),a,b) - \widetilde{V}^n(a,b) - C(\widetilde{Q}^n(a,b)))f(a,b)\,da\,db \Big) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} w(i,j)(v(\widetilde{Q}^n_{i,j}, a_i, b_j) - \widetilde{V}^n_{i,j} - C(\widetilde{Q}^n_{i,j}))f(a_i, b_j) + O(\tfrac{1}{n-1}) \Big) \\
&= \lim_{n\to\infty} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} w(i,j)(v(Q^n_{i,j}, a_i, b_j) - V^n_{i,j} - C(Q^n_{i,j}))f(a_i, b_j) + O(\tfrac{1}{n-1}) \Big) \\
&= \lim_{n\to\infty} \Big(OPT_n + O(\tfrac{1}{n-1}) \Big)
\end{aligned}
$$

where equalities are true by Dominated Convergence Theorem (each $\widetilde{Q}^n$, $\widetilde{V}^n$ are bounded), the finite approximation of the integral, the definition of $\widetilde{Q}^n$ and $\widetilde{V}^n$, and because (Q^n, V^n) is the solution of the discretized problem. □

Chapter 4

Applications

In this chapter we analyse two models already formulated in the literature in order to give additional insights by the numerical solution.

The first model, about a monopolist liquidity supplier in financial market, was introduced by Biais et al. (2000). Although formulated as bidimensional, it was solved as unidimensional after aggregating the two uncerntainty dimensions into one. This technique (when applicable) could generate loss of information. Thereby, it would be valuable a direct analysis in two dimensions, as we do based on approximations.

The second model concerns to regulation of a monopolistic firm with unknown demand and cost functions. It was introduced by Lewis and Sappington (1988b) and then review by Armstrong (1999) who showed that Lewis and Sappington's solution was wrong. Since the solution is unknown, numerical approximation may be relevant. The desirability of exclusion property, valid in other contexts, was attempted to extend to this model. However, as the numerical solution suggest, exclusion of a positive mass of agents should not be optimal. We provide technical and economic arguments about this feature.

4.1 Monopolist Liquidity Supplier

In Biais et al. (2000), the authors considered a monopolist who sells a risky asset to a population of potential investors. Each agent is characterized by a vector $(a, b) \in [a_l, a_u] \times [b_l, b_u]$ where a is the investor's evaluation of the asset's liquidation value (the true liquidation value is $\nu = a + \xi$ with $\xi \sim N(0, \sigma^2)$) and b is his initial position in the risky asset (the hedging needs).

When agent trades q, his final wealth is

$$W = (b + q)(a + \xi) - T(q)$$

Assuming that the informed agent has constant absolute risk aversion preferences with parameter γ, i.e. $u(W) = -e^{-\gamma W}$, the objective function of the agent is $E[W|(a,b)] - \frac{\gamma}{2}V(W|(a,b))$ which can be written as

$$\left(ab - \frac{\gamma\sigma^2}{2}b^2\right) + \left((a - \gamma\sigma^2 b)q - \frac{\gamma\sigma^2}{2}q^2 - T(q)\right)$$

The first term measures agent's reservation utility[1], while the second term measures the gains from trade with the monopolist. So, the valuation function is defined as

$$v(q,a,b) = (a - \gamma\sigma^2 b)q - \frac{\gamma\sigma^2}{2}q^2 \quad \text{with} \quad a \in [a_l, a_u]\ ,\ b \in [b_l, b_u]$$

Defining $\theta = a - \gamma\sigma^2 b$ the authors have reduced the two dimensional screening into one-dimensional problem. They made direct assumptions on the cumulative distribution $F(\theta)$ with density $f(\theta)$, and have found the existence of $\theta_b^* < 0$ and $\theta_a^* > 0$ such that the optimal trading volume offered by the monopolistic market-maker is

$$q(\theta) = \begin{cases} q^*(\theta) + \dfrac{F(\theta)}{\gamma\sigma^2 f(\theta)} & ,\ \theta \in [\underline{\theta}, \theta_b^*) \\ 0 & ,\ \theta \in [\theta_b^*, \theta_a^*] \\ q^*(\theta) - \dfrac{1 - F(\theta)}{\gamma\sigma^2 f(\theta)} & ,\ \theta \in (\theta_a^*, \overline{\theta}] \end{cases}$$

where $q^*(\theta) = \dfrac{\theta - w(\theta)}{\gamma\sigma^2}$ is the first best solution. Here $w(\theta) = E(\nu|\theta)$ is the expectation of the asset's liquidation value given θ.

They have concluded that the optimal tariff is differentiable everywhere except at 0, and that a positive measure of agents with intermediate types are rationed as a consequence of adverse selection.

[1] Setting $u_0(a,b) = ab - \frac{\gamma\sigma^2}{2}b^2$, note that $u_0(a,b) + v(q(a,b),a,b) - t(a,b) \geq u_0(a,b)$ is equivalent to $V(a,b) \geq 0$ for the usual definition of informational rent V.

On the next lines we analyse this model in the original bidimensional formulation showing that, when agents are uniformly distributed, a closed-form solution can be obtained using the optimality necessary condition (2.2). Besides, when a closed-form solution cannot be obtained, the numerical approximation allows us to make robust predictions in view of multiplicative separable valuation.

First, defining

$$\tilde{a} := \frac{a - a_l}{a_u - a_l} \quad , \quad \tilde{b} := \frac{b - b_l}{b_u - b_l}$$

we have $\tilde{a} \in [0, 1]$, $\tilde{b} \in [0, 1]$ and

$$v(q, \tilde{a}, \tilde{b}) = \Big((a_u - a_l)\tilde{a} - \gamma\sigma^2(b_u - b_l)\tilde{b} + (a_l - \gamma\sigma^2 b_l)\Big)q - \frac{\gamma\sigma^2}{2}q^2$$

setting $A = a_u - a_l$, $B = \gamma\sigma^2(b_u - b_l)$, $C = a_l - \gamma\sigma^2 b_l$ and, with some abuse of notation, redifining $a := \tilde{a}$, $b := \tilde{b}$ we obtain

$$v(q, a, b) = (Aa - Bb + C)q - \frac{\gamma\sigma^2}{2}q^2$$

Let ρ be the density over $[a_l, a_u] \times [b_l, b_u]$. Assuming independence, let f , g be the density distributions over $[a_l, a_u]$, $[b_l, b_u]$ respectively, let $\tilde{f}$, $\tilde{g}$ be the density distributions over $[0, 1]$ induced by the change of variables, and let $\widetilde{F}$, $\widetilde{G}$ be their respective cumulative distribution functions.

Because in this model a negative quantity is allowed, we separate the analysis.

<u>Case $q > 0$</u>

In view of $v_a > 0$ and $v_b < 0$, the type $(0, 1)$ is not willing to acept a positive quantity, then the virtual surplus is

$$G(q, a, b) = (v(q, a, b) - \frac{1 - \widetilde{F}(a)}{\tilde{f}(a)} v_a(q, a, b))\tilde{f}(a)\tilde{g}(b)$$

Solving the initial value problem

$$\begin{array}{ll} a_s = -\frac{v_{qb}}{v_{qa}} = \frac{B}{A} & , \quad a(r, 0) = r \\ b_s = 1 & , \quad b(r, 0) = 0 \end{array}$$

we obtain $a(r, s) = \frac{B}{A}s + r$, $b(r, s) = s$.

By the necessary condition (2.2), if ϕ^X defined over $[\underline{r}^X, 1]$ is optimal, then $\int_0^{\overline{s}(r)} \frac{G_q}{v_{qa}}(\phi^X(r), a(r,s), b(r,s))ds = 0$ where $\overline{s}(r) = \frac{A}{B}(1-r)$. This condition yields on

$$\phi^X(r) = \frac{(Ar+C)}{\gamma\sigma^2} - \frac{A\displaystyle\int_0^{\frac{A}{B}(1-r)} (1-\widetilde{F}(\tfrac{B}{A}s+r))\tilde{g}(s)\,ds}{\gamma\sigma^2\displaystyle\int_0^{\frac{A}{B}(1-r)} \tilde{f}(\tfrac{B}{A}s+r)\tilde{g}(s)\,ds} \tag{4.1}$$

for $r \in [\underline{r}^X, 1]$, where $\underline{r}^X$ is such that $\phi^X(\underline{r}^X) = 0$.

<u>Case $q < 0$</u>

Because of $v_a < 0$ and $v_b > 0$, the type $(1,0)$ is not willing to acept a negative quantity, so for this case

$$G(q,a,b) = (v(q,a,b) - \frac{1-\widetilde{G}(b)}{\tilde{g}(b)} v_b(q,a,b))\tilde{f}(a)\tilde{g}(b)$$

Solving the initial value problem

$$\begin{array}{ll} a_s = -\frac{v_{qb}}{v_{qa}} = \frac{B}{A} & , \quad a(r,0) = 0 \\ b_s = 1 & , \quad b(r,0) = r \end{array}$$

we have $a(r,s) = \frac{B}{A}s$, $b(r,s) = s + r$.

By the necessary condition (2.2), if ϕ^Y defined over $[\underline{r}^Y, 1]$ is optimal, then $\int_0^{\overline{s}(r)} \frac{G_q}{v_{qa}}(\phi^Y(r), a(r,s), b(r,s))ds = 0$ where $\overline{s}(r) = 1 - r$. This condition yields on

$$\phi^Y(r) = \frac{(-Br+C)}{\gamma\sigma^2} + \frac{B\displaystyle\int_0^{1-r} (1-\widetilde{G}(s+r))\tilde{f}(\tfrac{B}{A}s)\,ds}{\gamma\sigma^2\displaystyle\int_0^{1-r} \tilde{f}(\tfrac{B}{A}s)\tilde{g}(s+r)\,ds} \tag{4.2}$$

for $r \in [\underline{r}^Y, 1]$, where $\underline{r}^Y$ is such that $\phi^Y(\underline{r}^Y) = 0$.

Now, we will consider particular distributions.

Uniform Distributions

Here we assume that $\tilde{f}(a) = 1$ and $\tilde{g}(b) = 1$ over $[0, 1]$.

Case $q > 0$

The expression of ϕ^X established on (4.1) derives on

$$\phi^X(r) = \frac{(3r-1)A + 2C}{2\gamma\sigma^2} \quad , \quad \forall\, r \in [\frac{A-2C}{3A}, 1]$$

To return to the original variables, given (a, b) we can express $r(a, b) = a - b\frac{B}{A}$, and considering that $q(a, b) = \phi^X(r(a, b))$ when $r(a, b) \geq \underline{r}^X = \frac{A-2C}{3A}$ we obtain

$$q(a, b) = \frac{(3a-1)A - 3bB + 2C}{2\gamma\sigma^2} \quad \text{when} \quad Aa - Bb - \frac{(A-2C)}{3} \geq 0$$

Now, lets compute the tariff. For $r \in [\underline{r}^X, 1]$ we have

$$t(r, 0) = T(\phi^X(r)) = \int_{\underline{r}^X}^{r} v_q(\phi^X(\tilde{r}), \tilde{r}, 0)(\phi^X)'(\tilde{r})\, d\tilde{r}$$

after some calculus, and considering $r = (\phi^X)^{-1}(q)$ we get

$$T(q) = \frac{-\gamma\sigma^2 q^2 + 2q(A+C)}{6} \quad \text{when} \quad q > 0$$

Case $q < 0$

The expression of ϕ^Y established on (4.2) derives on

$$\phi^Y(r) = \frac{(1-3r)B + 2C}{2\gamma\sigma^2} \quad , \ \forall\, r \in [\frac{B+2C}{3B}, 1]$$

To return to the original variables, given (a, b) we can express $r(a, b) = b - a\frac{A}{B}$, and considering that $q(a, b) = \phi^Y(r(a, b))$ when $r(a, b) \geq \underline{r}^Y = \frac{B+2C}{3B}$ we obtain

$$q(a, b) = \frac{3Aa + (1-3b)B + 2C}{2\gamma\sigma^2} \quad \text{when} \quad a\frac{A}{B} - b + \frac{(B+2C)}{3B} \leq 0$$

Now, lets compute the tariff. For $r \in [\underline{r}^Y, 1]$ we have

$$t(0,r) = T(\phi^Y(r)) = \int_{\underline{r}^Y}^{r} v_q(\phi^Y(\tilde{r}), \tilde{r}, 0)(\phi^Y)'(\tilde{r})\, d\tilde{r}$$

after some calculus, and considering $r = (\phi^Y)^{-1}(q)$ we get

$$T(q) = \frac{-\gamma\sigma^2 q^2 + 2q(C-B)}{6} \quad \text{when} \quad q < 0$$

Summarizing, after return to $(a,b) \in [a_l, a_u] \times [b_l, b_u]$, the optimal allocation q is:

$$q(a,b) = \begin{cases} \big(3(a-\gamma\sigma^2 b) - (a_u - \gamma\sigma^2 b_l)\big)/(2\gamma\sigma^2) & \text{if } \ 3(a-\gamma\sigma^2 b) \geq a_u - \gamma\sigma^2 b_l \\ \big(3(a-\gamma\sigma^2 b) - (a_l - \gamma\sigma^2 b_u)\big)/(2\gamma\sigma^2) & \text{if } \ 3(a-\gamma\sigma^2 b) \leq a_l - \gamma\sigma^2 b_u \\ 0 & \text{another case} \end{cases}$$

and the corresponding tariff is:

$$T(q) = \begin{cases} \dfrac{-\gamma\sigma^2 q^2 + 2q(a_u - \gamma\sigma^2 b_l)}{6} & \text{if } \ q \geq 0 \\ \dfrac{-\gamma\sigma^2 q^2 + 2q(a_l - \gamma\sigma^2 b_u)}{6} & \text{if } \ q < 0 \end{cases}$$

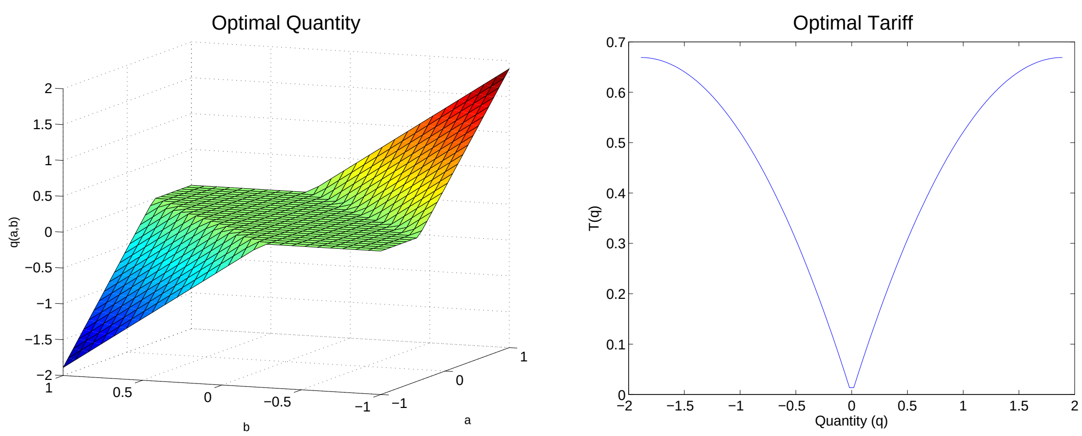


Figure 4.1: Analytical solutions graphs for the case of uniform distributions and $\gamma = 2$, $\sigma = 0.75$, $a_l = -1$, $a_u = 1$, $b_l = -1$, $b_u = 1$.

Truncated Normal Distributions

Assuming that types a, b presents truncated normal distribution over $[0,1]$ with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively, we have:

$$f(a) = \frac{e^{-\frac{(a-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}\sigma_1\Lambda_1} \quad , \quad g(b) = \frac{e^{-\frac{(b-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{2\pi}\sigma_2\Lambda_2}$$

where $\Lambda_i = \frac{1}{\sqrt{2\pi}\sigma_i}\int_0^1 e^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}} dt$, $i = 1, 2$

The expressions of ϕ^X and ϕ^Y given in (4.1) and (4.2) transforms on

$$\phi^X(r) = \frac{(Ar+C)}{\gamma\sigma^2} - \frac{A\int_0^{\frac{A}{B}(1-r)} \left(1 - \int_0^{\frac{B}{A}s+r} \frac{e^{-\frac{(t-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}\sigma_1\Lambda_1} dt\right)\left(\frac{e^{-\frac{(s-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{2\pi}\sigma_2\Lambda_2}\right) ds}{\gamma\sigma^2\int_0^{\frac{A}{B}(1-r)} \left(\frac{e^{-\left(\frac{(\frac{B}{A}s+r-\mu_1)^2}{2\sigma_1^2}+\frac{(s-\mu_2)^2}{2\sigma_2^2}\right)}}{2\pi\sigma_1\sigma_2\Lambda_1\Lambda_2}\right) ds}$$

$$\phi^Y(r) = \frac{(-Br+C)}{\gamma\sigma^2} + \frac{B\int_0^{1-r} \left(1 - \int_0^{s+r} \frac{e^{-\frac{(t-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{2\pi}\sigma_2\Lambda_2} dt\right)\left(\frac{e^{-\frac{(\frac{B}{A}s-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}\sigma_1\Lambda_1}\right) ds}{\gamma\sigma^2\int_0^{1-r} \left(\frac{e^{-\left(\frac{(\frac{B}{A}s-\mu_1)^2}{2\sigma_1^2}+\frac{(s+r-\mu_2)^2}{2\sigma_2^2}\right)}}{2\pi\sigma_1\sigma_2\Lambda_1\Lambda_2}\right) ds}$$

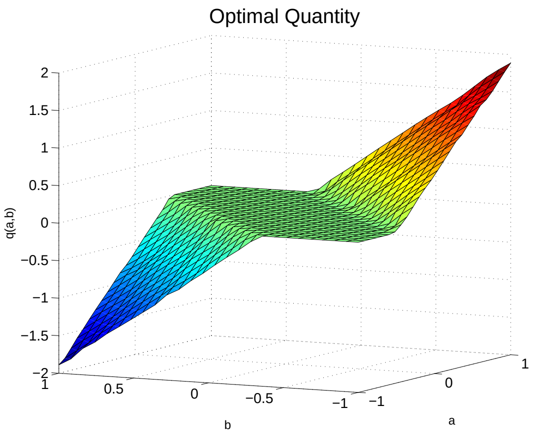
These expressions involve integrals that cannot be expressed in terms of elementary functions, so neither ϕ^X, ϕ^Y nor the derivatives (which are required to compute the tariff T) can be expressed in closed-form.

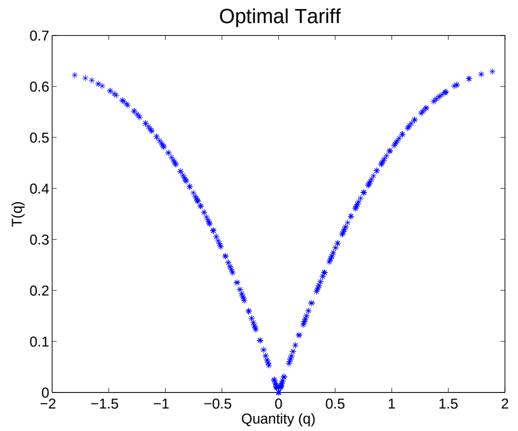
In this situation we appeal to numerical approximations solving (NP). Note that valuation function is multiplicative separable then, by remark VI in section 3.3, numerical solutions are reliable.

On the next pages we show the numerical solutions (with $n = 26$) for differents mean and variance's values. We fix $\gamma = 2$ (parameter of CARA preferences) and $\sigma = 0.75$ (variance of true liquidation). Because agents' parameter a is the assets liquidation valuation, we are calling 'optimistic' ('pessimistic') to agents with hight (low) value. Remember that b reflects the agent's initial position of the risky asset.

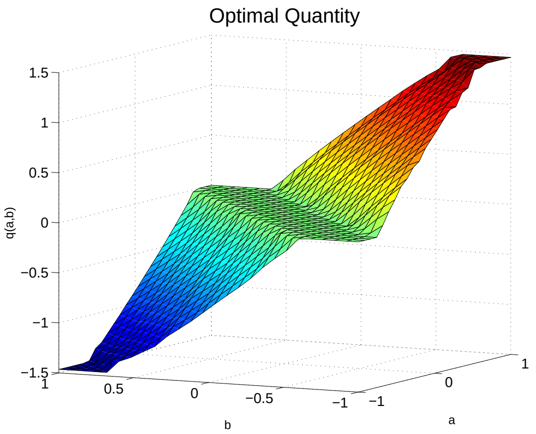
Agents are concentrated around $(\mu_1, \mu_2) = (0, 0)$

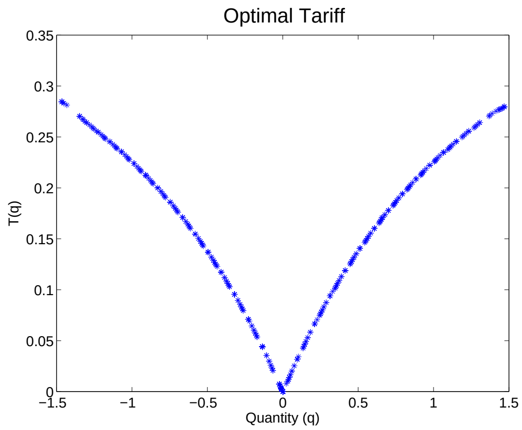
$\sigma_1^2 = \sigma_2^2 = 1$ Expected Profit=20.67×10^{-5}



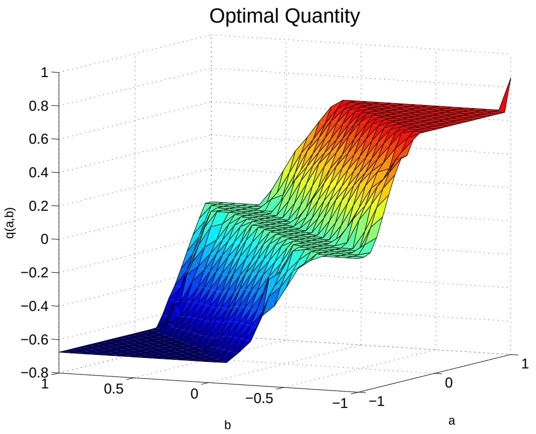


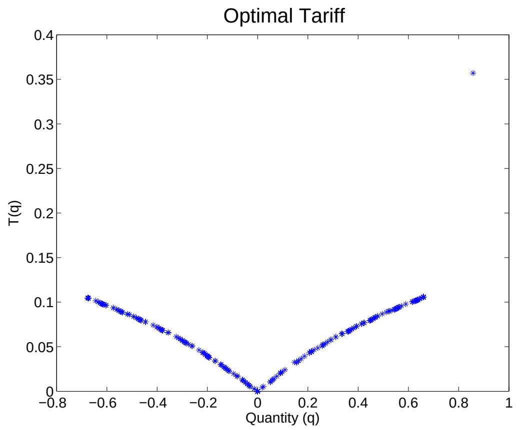
$\sigma_1^2 = \sigma_2^2 = 0.3$ Expected Profit=6.65×10^{-5}





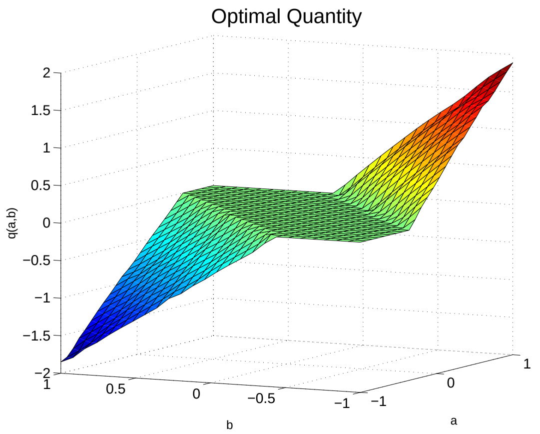
$\sigma_1^2 = \sigma_2^2 = 0.2$ Expected Profit=3.15×10^{-5}

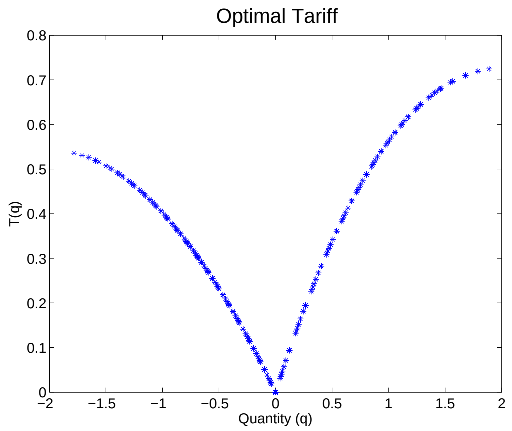




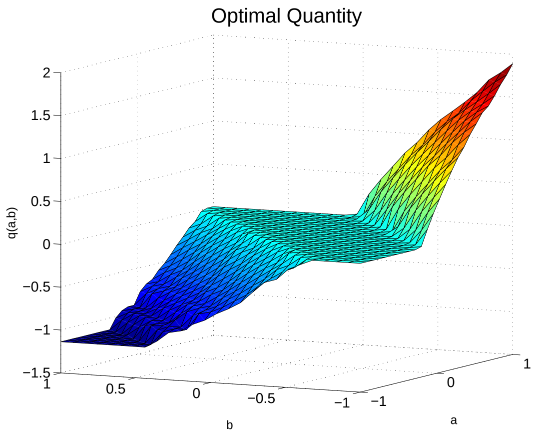
Agents are concentrated around $(\mu_1, \mu_2) = (0.75, -0.75)$

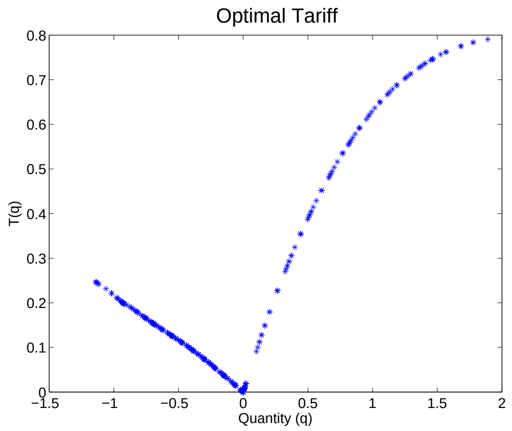
$\sigma_1^2 = \sigma_2^2 = 1$ Expected Profit=26.07×10^{-5}



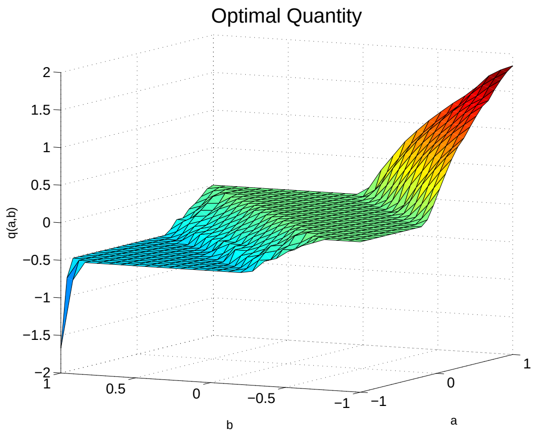


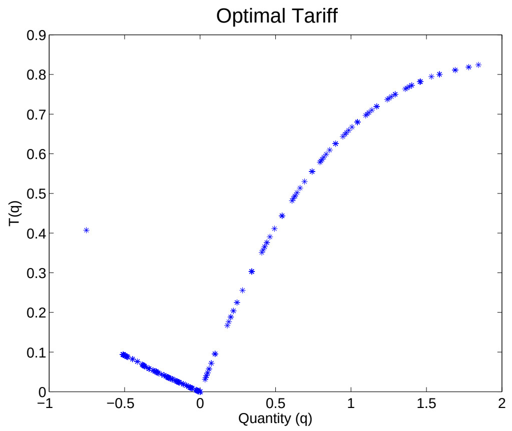
$\sigma_1^2 = \sigma_2^2 = 0.5$ Expected Profit=48.50×10^{-5}





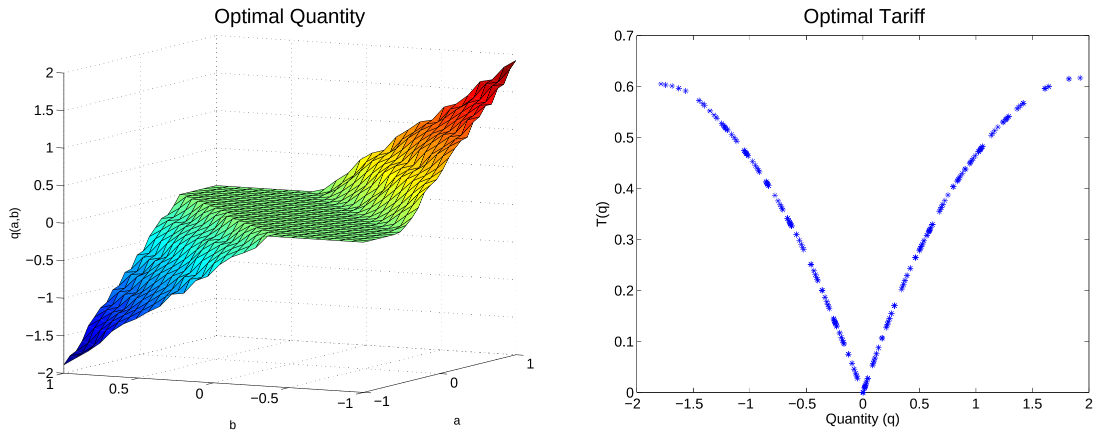
$\sigma_1^2 = \sigma_2^2 = 0.4$ Expected Profit=60.35×10^{-5}



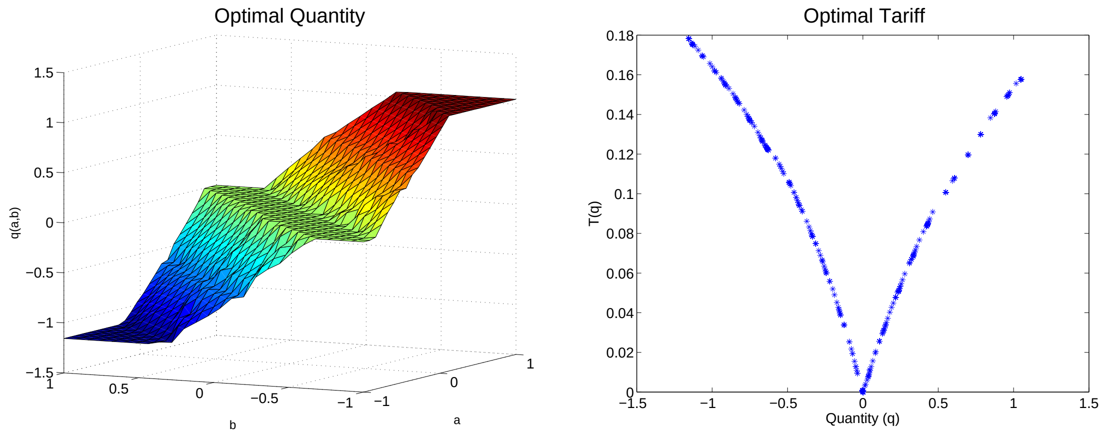


Agents are concentrated around $(\mu_1, \mu_2) = (0.75, 0.75)$

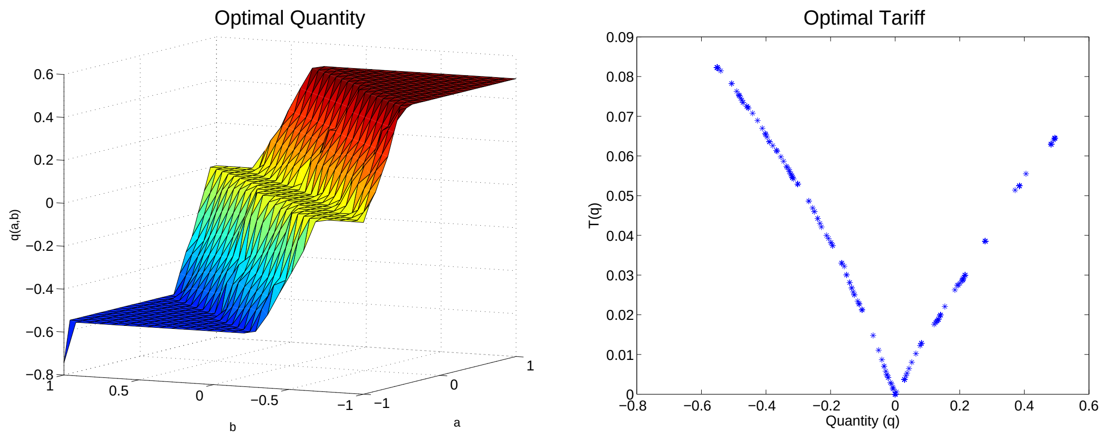
$\sigma_1^2 = \sigma_2^2 = 1$ Expected Profit=18.51×10^{-5}



$\sigma_1^2 = \sigma_2^2 = 0.3$ Expected Profit=3.79×10^{-5}

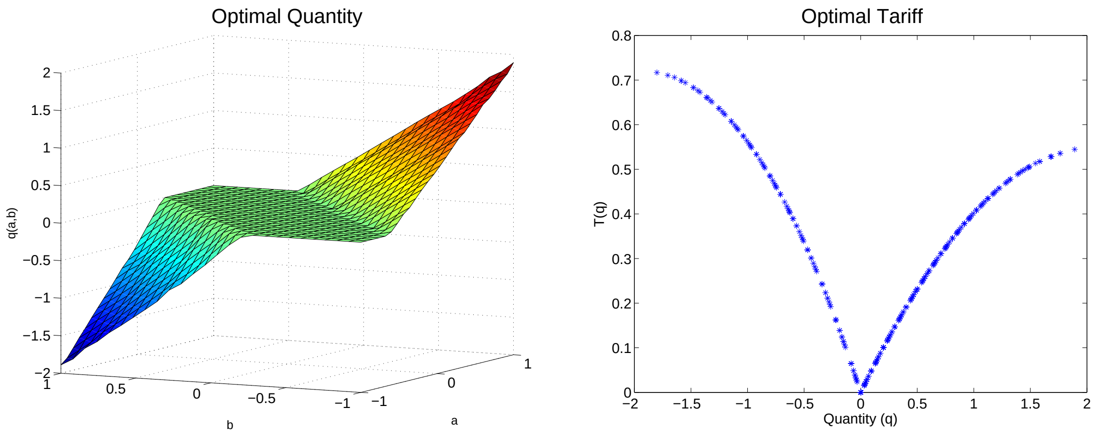


$\sigma_1^2 = \sigma_2^2 = 0.2$ Expected Profit=2.31×10^{-5}

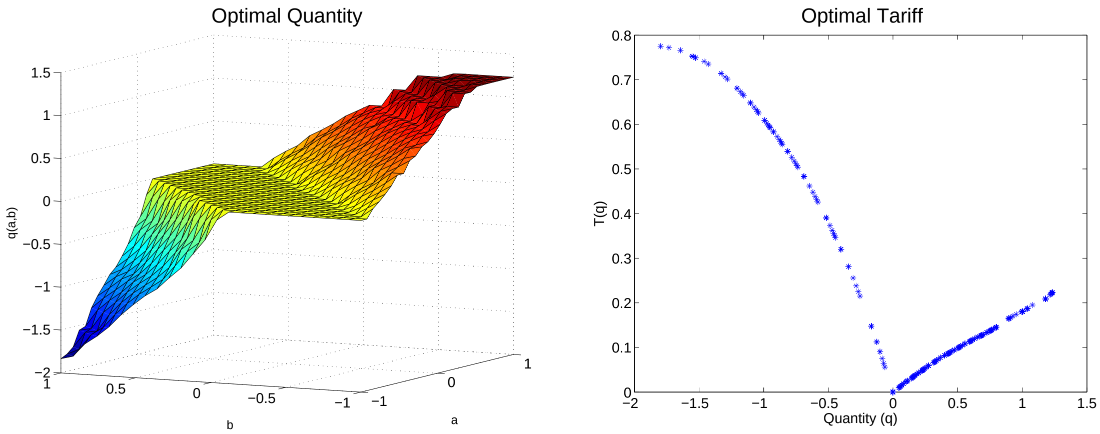


Agents are concentrated around $(\mu_1, \mu_2) = (-0.75, 0.75)$

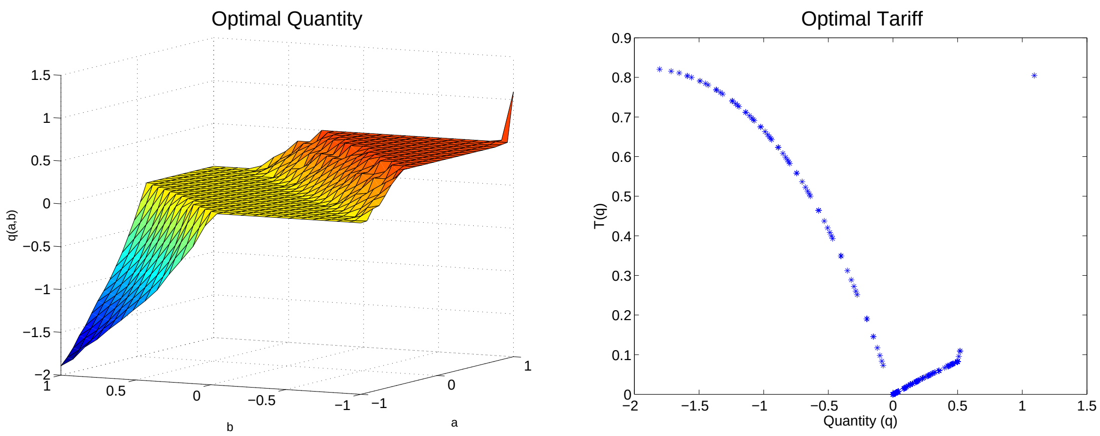
$\sigma_1^2 = \sigma_2^2 = 1$ Expected Profit=26.19×10^{-5}



$\sigma_1^2 = \sigma_2^2 = 0.5$ Expected Profit=48.67×10^{-5}



$\sigma_1^2 = \sigma_2^2 = 0.4$ Expected Profit=60.65×10^{-5}



Some insights from these solutions:

1. When agents' distribution is not too concentrate the solution does not differ too much from the uniform distribution case, no matters the point of concentration. Thus, trade will take place with two opposite sides of the market: optimistic agents with short position (a high, b low) and pessimistic agents with large position (a low, b high). With middle agents it is optimal not to trade.

2. Because type $(0, 0)$ is neither optimistic nor pessimistic about the final value of the asset ($a = 0$) and does not have the need to transact the asset ($b = 0$), when the mass of agents around this type increases (i.e. if $\sigma_1^2 \to 0$, $\sigma_2^2 \to 0$) the monopolist does not have to discriminate neither between optimistic agents with short position nor between pessimistic agents with large position, but designs a contract in which middle agents are increasingly included and discriminated. Thus, optimal quantity for the two sides becomes flat and smaller as well as the tariff for this quantity offered. As a result, expected profit decrease.

   Same features are observed when agents concentrate around $(0.75, 0.75)$ because this agent is also not willing to trade with the monopolist, due to he has large position of the asset and is optimistic about its final value.

3. The type $(0.75, -0.75)$ reflects an optimistic agent with short position, therefore a potential buyer. So, when the mass of agents around this type increases, the monopolist discriminates this side of the market whereas bunching the opposite side (pessimistic agents with large position) at a fewer quantity and tariff. Because there are more potential buyers, expected profit increases.

   Similar features are observed in the reverse situation when agents increasingly concentrate around $(-0.75, 0.75)$ (a potentially seller) since this type of agents are also willing to trade because of their large position of the asset and its pessimistic valuation.

4.2 Regulating a Monopolist Firm

In Lewis and Sappington (1988b) the authors studied the design of regulatory policy when the regulator is imperfectly informed about both cost and demand functions of the monopolist firm he is regulating.

In particular, they have considered that demand for the firm's product $q = Q(p,a)$ and cost of producing output q, $C(q,b)$ are given by

$$Q(p,a) = h(p) + a \quad , \quad C(q,b) = \widetilde{C}(q) + bq$$

where (a,b) are firm's private information parameters, distributed over $[0,1]\times[0,1]$ according to a strictly positive density function $f(a,b)$.

The regulator offers the firm a menu of contracts $(p(a,b), t(a,b))$ whereby if the firm sets unit price $p(a,b)$ for its output, it receives the subsidy $t(a,b)$ from consumers. It is assumed that regulator can ensure that the firm serves all demand at the established prices. The objective function is the expected consumer surplus net of the transfer to the firm.

Thus, the regulator's problem is to find two functions $p : [0,1]^2 \to \mathbb{R}_+$ and $t : [0,1]^2 \to \mathbb{R}_+$ in order to solve

$$\max_{p(\cdot),t(\cdot)} \int_0^1 \int_0^1 \{\Pi(Q(p(a,b),a),a) - p(a,b)Q(p(a,b),a)) - t(a,b)\} f(a,b) dbda$$

subject to

$$\begin{aligned}
&(\text{IR}) \quad p(a,b)Q(p(a,b),a)) + t(a,b) - C(Q(p(a,b),a),b) \geq 0 \\
&(\text{IC}) \quad p(a,b)Q(p(a,b),a)) + t(a,b) - C(Q(p(a,b),a),b) \geq \\
&\qquad\qquad p(\widehat{a},\widehat{b})Q(p(\widehat{a},\widehat{b}),a)) + t(\widehat{a},\widehat{b}) - C(Q(p(\widehat{a},\widehat{b}),a),b)
\end{aligned}$$

where $\Pi(Q,a) = \int_0^Q P(\xi,a)d\xi$, and $P(\cdot)$ denotes the inverse demand curve.

Lewis and Sappington considered the example

$$Q(p,a) = \alpha - p + a \quad , \quad C(q,b) = K + (c_0 + b)q$$

with α, K, c_0 positive constants and uniform distribution of types. They developed a similar idea to that exposed in section 2.4 to deal with the problem, transforming the two-dimensional maximization problem into a single-dimensional one, and have found as *'solution'* $p(a,b) = r(a+b)$ where

$$r(s) = \begin{cases} c_0 + \dfrac{s}{2} + \dfrac{s}{2\sqrt{2}} & \text{for} \quad s \in [0,1] \\ c_0 + \dfrac{s}{2} + \dfrac{2s - 1 - s^2/2}{\sqrt{2}(2-s)} & \text{for} \quad s \in [1, \overline{s}_1] \\ c_0 + 1 & \text{for} \quad s \in [\overline{s}_1, \sqrt{2}] \\ c_0 + \dfrac{s}{2} + \dfrac{2s - 1 - s^2/2}{\sqrt{2}(2-s)} - \dfrac{1}{\sqrt{2}} & \text{for} \quad s \in [\sqrt{2}, 2[\end{cases}$$

and $\overline{s}_1 \approx 1.062$.

Nevertheless, as Armstrong (1999) has noticed, this *cannot be* the solution, because $r(s) \to \infty$ as $s \to 2$, and therefore $q = \alpha + a - p(a,b) < 0$ for (a,b) close to $(1,1)$. In that paper Armstrong argued that excluding a positive mass of types should be optimal, as in the nonlinear pricing model (see section 2.3) but, because of the change of variables he used, the type set is not convex and his exclusion argument cannot strictly be applied. He also pointed out:

- "*Nevertheless, I believe that the condition that the support be convex is strongly sufficient and that it will be the usual case that exclusion is optimal...*"

- "*I have not found it possible to solve this precise example correctly...*" (in reference to the Lewis and Sappington's example above)

Then, we are facing a bidimensional adverse selection model with unknown solution, where a conjecture about optimality of exclusion were made.

Next, we show that regulator's problem can be seen as a monopolist's problem in order to apply the reduction of IC constraints and provide a numerical approximation of the solution. Setting

$$\begin{aligned} v(p,a,b) &= pQ(p,a) - C(Q(p,a),b) \\ H(p,a) &= pQ(p,a) - \Pi(Q(p,a),a) \\ V(a,b) &= v(p(a,b),a,b) + t(a,b) \end{aligned}$$

we can rewrite the regulator's problem as

$$\max_{p(\cdot),V(\cdot)} \int_0^1 \int_0^1 \{v(p(a,b),a,b) - H(p(a,b),a) - V(a,b)\} f(a,b) dbda$$

subject to

$$\begin{array}{lll} \text{(IR)} & V(a,b) \geq 0 & \forall\ (a,b) \in [0,1]^2 \\ \text{(IC)} & V(a,b) - V(\widehat{a},\widehat{b}) \geq v(p(\widehat{a},\widehat{b}),a,b) - v(p(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) & \forall\ (a,b),(\widehat{a},\widehat{b}) \in [0,1]^2 \end{array}$$

This formulation fits the standard nonlinear pricing by a monopolist problem studied in Chapter 3. That is, the regulator's problem may be understood as the monopolist's problem where −in this case− the firm's costs function is $H(p)$ and $(a,b)-$agent's valuation for 'consumption' p is $v(p,a,b)$.

Consider the following assumptions on demand and cost functions: $Q_p < 0$, $C_q > 0$, $C_{qq} \geq 0$, which means that demand decreases with prices and costs and marginal costs are increasing with output.

Given that $Q(p,a) = h(p) + a$ and $C(q,b) = \widetilde{C}(q) + bq$ we have

$$\begin{aligned} v_a &= p - C_q(Q(p,a),b) \\ v_b &= -C_b(Q(p,a),b) = -Q(p,a) < 0 \\ v_{pa} &= 1 - C_{qq}(Q(p,a),b)h'(p) > 0 \\ v_{pb} &= -h'(p) > 0 \end{aligned}$$

Assumption A4 of section 3.1 cannot be verified (the sign of v_a is endogenously determined), so the problem must consider all the IR constraints. Besides, even Assumption A3 is not valid, what really matters is the constant sign of v_{pa} and v_{pb}. In this case, $p(\cdot,\cdot)$ will be non-decreasing on a and b. Also, since $\frac{-v_{pb}}{v_{pa}} < 0$, the characteristic curves are strictly decreasing.

Seeing the regulator's problem as if it were the monopolist's problem, type $(1,1)$ would be considered the best type from the monopolist's point of view ($v_p(p,1,1) \geq v_p(p,a,b)$ for any (a,b)). Then, following the same considerations of section 3.2, it will be sufficient that each $(a,b)-$agent verifies IC constraints with all the points over the set

$$F^{(a,b)} := \{(0,s) \mid b \leq s \leq 1\} \cup \{(s,1) \mid 0 \leq s \leq a\}$$

Note that the necessary condition for optimality (2.2) established by Araujo and Vieira (2010) cannot be applied in this case, because assumption A.4 is not verified. In fact, we believe that this lack of consideration could be one of the failures on Lewis and Sappington's work, because after had transformed the problem into a single-dimesional (by incorporating local incentive compatibility constraints in the regulator's objective function), they have not considered the IR constraints, which cannot be ruled out if A.4 fails.

Consider the same example

$$Q(p,a) = \alpha - p + a \quad , \quad C(q,b) = K + (c_0 + b)q$$

with α, K, c_0 positive constants and uniform distribution of types. Then

$$\begin{aligned} v(p,a,b) &= (\alpha + a - p)(p - c_0 - b) - K \\ H(p,a) &= -\frac{(\alpha + a - p)^2}{2} \end{aligned}$$

Even when $H_{pp} < 0$, what we need to ensure strictly concavity of the objective function in the discretized problems is $v_{pp} - H_{pp} < 0$, which is true. Note also that IC constraints become linear, by the multiplicative separable form of v. Then, the discretized problem have unique solution.

We solved the discretized problem numerically for three cases:

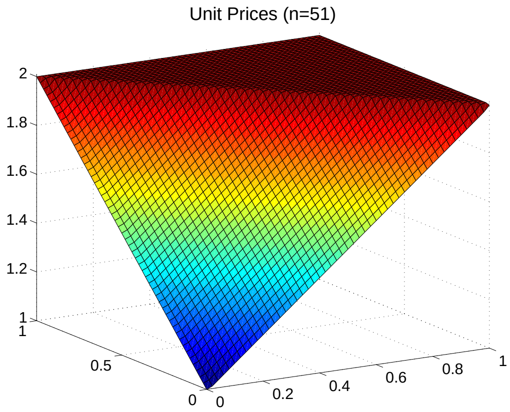
$$\begin{aligned} &c_0 = 1, \quad \alpha = 5, \quad K = 2 \\ &c_0 = 2, \quad \alpha = 4, \quad K = 4.5 \\ &c_0 = 3, \quad \alpha = 4.5, \quad K = 3 \end{aligned}$$

all of them with $n = 51$ points. For this value, 256 225 incentive compatibility constraints were considered, and 6 506 375 were eliminated.

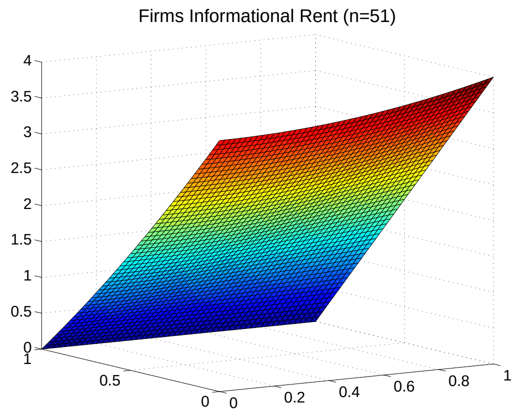
On the next pages we show the graphs of these solutions. Also, we show the numerical differences between unit prices and marginal costs on the last set of figures.

$$c_0 = 1\ ,\quad \alpha = 5\ ,\quad K = 2$$

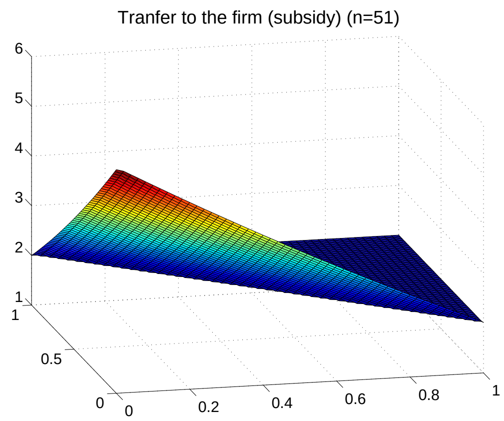
Unit Prices $p^n(\cdot,\cdot)$



Firm's Informational Rent $V^n(\cdot,\cdot)$

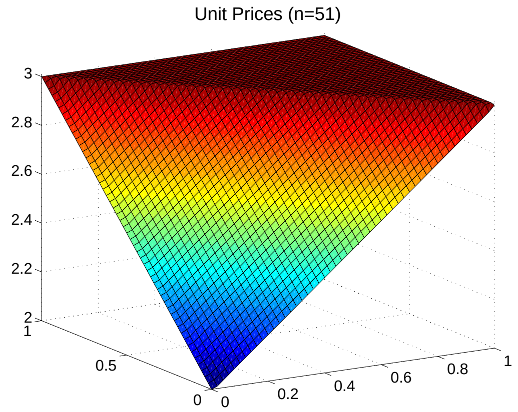


Transfer to the Firm $t^n(\cdot,\cdot)$

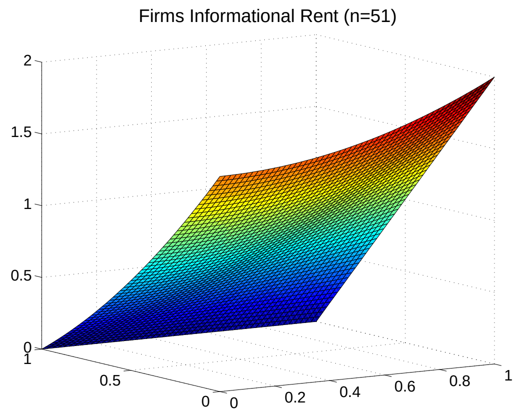


$$c_0 = 2\,, \quad \alpha = 4\,, \quad K = 4.5$$

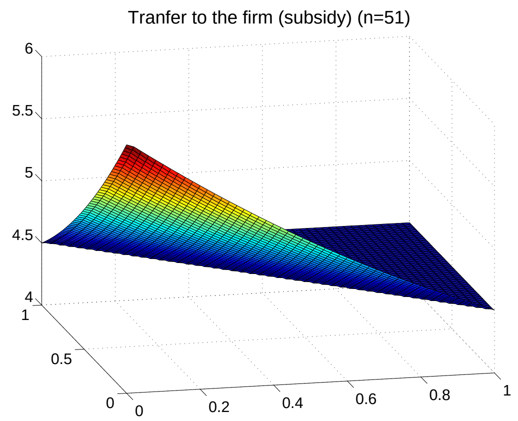
Unit Prices $p^n(\cdot,\cdot)$



Firm's Informational Rent $V^n(\cdot,\cdot)$

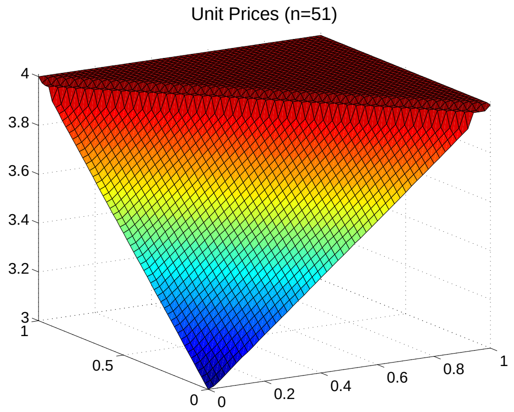


Transfer to the Firm $t^n(\cdot,\cdot)$

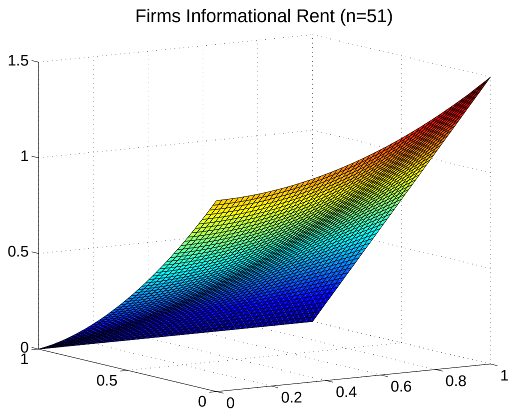


$$c_0 = 3\ ,\quad \alpha = 4.5\ ,\quad K = 3$$

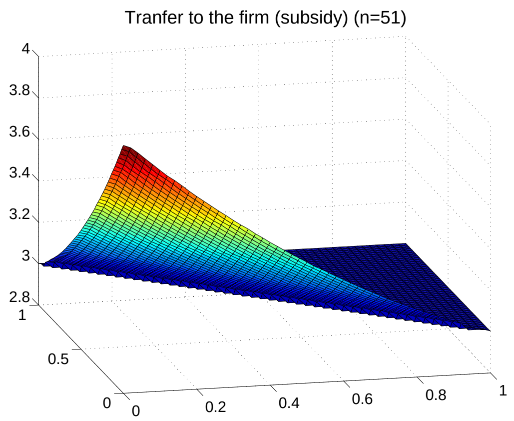
Unit Prices $p^n(\cdot,\cdot)$



Firm's Informational Rent $V^n(\cdot,\cdot)$

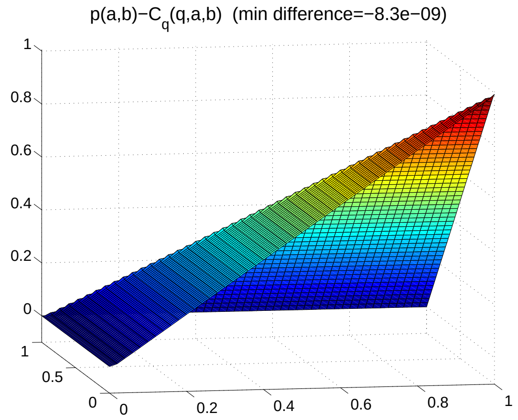


Transfer to the Firm $t^n(\cdot,\cdot)$

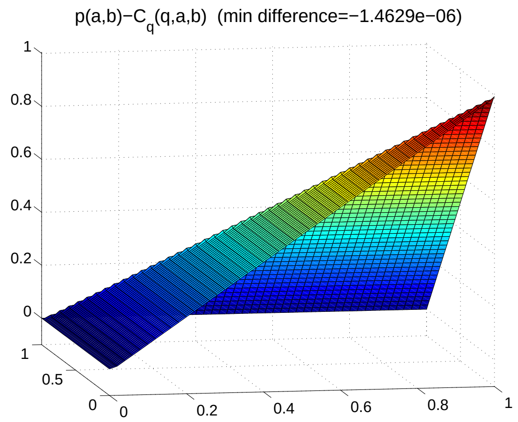


Unit prices minus marginal costs $p - C_q$

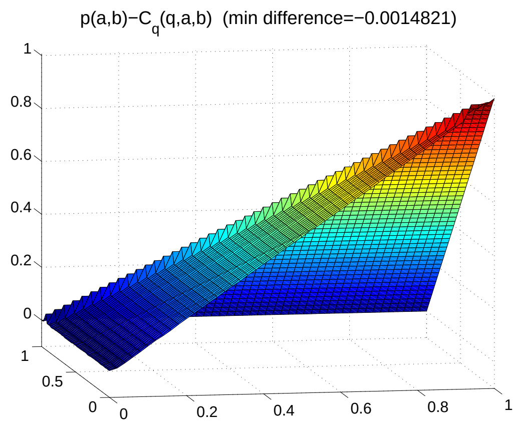
$c_0 = 1\ ,\quad \alpha = 5\ ,\quad K = 2$



$c_0 = 2\ ,\quad \alpha = 4\ ,\quad K = 4.5$



$c_0 = 3\ ,\quad \alpha = 4.5\ ,\quad K = 3$



Some insights from these solutions:

We stress that this example derives into a optimization problem with unique solution. In fact, because the objective function is quadratic and strictly concave with linear restrictions, the numerical methods to solve it are efficient.

Thus, the statements below are reliable:

1. It seems that at the optimum, all types (a, b) such that $a + b \geq 1$ are bunching with unit price $c_0 + 1$, and subsidy for them is the fix cost K. Also, unit price assigned to type $(0, 0)$ it seems to be c_0.[2]

2. In view of the numerical difference $p - C_q$, regulator induce the firm to price above marginal cost, for almost all (a, b) types other than $a = 0$ or $b = 1$, i.e. such types with lowest demand function or such types who gets zero surplus[3].

3. The numerical informational rent suggests that there is no exclusion.

4.2.1 A Discussion about Optimality of Exclusion

Perhaps the most intriguing insight from the numerical solutions of this example is that it should be optimal non-exclusion of types, contrary to the Armstrong's conjecture stated previously.

Besides, in Barelli et al. (2014) the authors have relaxed Armstrong's strong conditions (strictly convexity, homogeneity of degree one) and have proved a more general result of desirability of exclusion. For this example they have considered that prices belongs to $[c_0 + 1, \alpha]$ to conclude that their result can be applied and confirm Armstrong (1999)'s conjecture, generically. However, as can be seen, it is not true that $P \subset [c_0 + 1, \alpha]$, so their theorem should not be applied.

[2]In fact, we can conjecture the optimum price p to be $p(a, b) = c_0 + a + b$ when $a + b \leq 1$, and $p(a, b) = c_0 + 1$ when $a + b > 1$

[3] In Baron and Myerson (1982) the authors have analysed a model in which the regulator is uncertain only about the firm's cost function. At the optimum, prices are above marginal costs for all cost realizations other than the lowest. In the model of Lewis and Sappington (1988a) regulator is uncertain only about the position of demand curves. In that model, if $C''(q) \geq 0$ (like here), setting prices at the level of marginal cost for the reported demand is optimal ($p = C_q$).

We are able to provide one technical argument explaining why Armstrong's Theorem 2.3.1 about desirability of exclusion, formulated in the nonlinear pricing context, cannot be extended to this model. Also, an economic argument about why should not be optimal excluding types.

1. Because in the nonlinear pricing by a monopolist model the outside option for customers is to consume $q^{\text{out}} = 0$, the natural assumptions $v(q^{\text{out}}, a, b) = 0$, $C(q^{\text{out}}) = 0$ made in Armstrong (1996) imply the monopolist's revenue

$$v(q^{\text{out}}, a, b) - C(q^{\text{out}}) - V(a, b)$$

to be zero when $V(a, b) = 0$, that is, if type (a, b) is excluded. Then, the monopolist's penalty of causing some customers to exit the market is just not to receive income from them. On the other hand, in the regulation model the outside option is the unit price at which there is not production, i.e. p^{out} is such that $Q^{\text{out}} = 0$. With these values

$$\Pi(Q^{\text{out}}, a) - p^{\text{out}} Q^{\text{out}} - t(a, b) = -t(a, b)$$

when type (a, b) is excluded, i.e., when $t(a, b) = C(Q^{\text{out}}, b) = C(0, b)$ (because IR is binding). That is, the regulator's penalty of excluding a firm type (a, b) is to subsidy the firm's fixed cost.

Thereby, in contrast with monopolist, regulator has to assume a negative penalty whenever firm's fixed cost is positive. (In previous example, $p^{\text{out}}(a, b) = \alpha + a$, $v(p^{\text{out}}, a, b) = -K$ and $C(0, a, b) = K > 0$). Thus, the Armstrong's argument of comparing benefit (more income from agents still in the market) versus penalty (zero income from agents excluded), might not be applied in this model in view of penalty could be strictly negative.

Therefore, the main technical assumption not satisfied in Armstrong's Theorem is neither related with strictly convexity of types' set nor with homogeneity of the degree one of the valuation function (his strong technical assumptions). It is the not validity of $v(p^{\text{out}}, a, b) = 0$.

2. In the nonlinear pricing setting, when designing the contract the monopolist is faced with a population of customers with different characteristics, where the mass of customers with certain characteristics is reflected by the distribution. Then, as Armstrong have formalized, it is optimal for the monopolist not to serve customers

with low valuation of the product, because there is a mass of other customers who valuates it high.

In contrast, in the regulation model, the regulator is faced with one single firm who is going to exercise the monopoly of certain good. Here, the types's distribution reflects the probability of such firm to have certain characteristics unknown by the regulator. So, consider what will happen if regulator designs a contract with the posibility of exclusion and such unique firm choose to be excluded as its best option. As a consequence, neither production nor consumption of the good takes place in the economy and, even in this case, consumers has to subsidy (probably non-zero) fixed cost of the company. Such situation cannot be optimal.

Conclusions

Based on what have been exposed in this work, we make the following conclusions:

1) A necessary condition for optimality can be established in case isoquants are concurrent at certain point, or equivalently, in case there is a type indifferent between any allocation in an interval.

2) Comparing types according to their marginal valuation of consumption and taking account the posible shape of isoquants, allows us to reduce incentive compatibility contraints making the discretized problem numerically tractable for relative fine discretization. In applications, numerical approximations gives us the opportunity of predict features of the solution when the problem has not closed-form solution, which is the usual case. Futhermore, allows us to analyze different situations according to types' distribution.

3) In all the examples, the numerical solution of informational rent V^n was more approximate and smoother than the numerical solution of quality Q^n. This is because the problem is linear with respect to V, regardless the valuation function v.

4) We can provide the solution of the optimal contract when agents' demand curves are convex and the monopolist is uncertain about the parameters defining such curve (Example 4, Chapter 3, page 55). To the best of our knowledge, this kind of example has not been previously analyzed.

5) In the regulation model a conjecture about desirability of exclusion had been made as well as the attempt to formalize it. Nevertheless, this exclusion feature is not present on the reliable numerical approximations. The main technical assumption why Armstrong's result cannot be applied to this model is the not validity of $v(p^{\text{out}}, a, b) = 0$ and, by the nature of the model, it should not be optimal a contract with the possibility of excluding types.

Bibliography

Araujo, A. and Moreira, H. (2010). Adverse selection problems without the spence-mirrlees condition. *Journal of Economic Theory*, 145(3):1113–1141.

Araujo, A. and Vieira, S. (2010). Two-dimensional screening: a useful optimality condition. Preprint.

Armstrong, M. (1996). Multiproduct nonlinear pricing. *Econometrica*, 64(1):51–75.

Armstrong, M. (1999). Optimal regulation with unknown demand and cost functions. *Journal of Economic Theory*, 84:196–215.

Barelli, P., Basov, S., Bugarin, M., and King, I. (2014). On the optimality of exclusion in multi-dimensional screening. *Journal of Mathematical Economics*, 54:74–83.

Baron, D. and Myerson, R. (1982). Regulating a monopolist with unknown costs. *Econometrica*, 50(4):911–930.

Basov, S. (2001). Hamiltonian approach to multi-dimensional screening. *Journal of Mathematical Economics*, 36:77–94.

Basov, S. (2005). *Multidimensional Screening*. Springer-Verlag.

Belloni, A., Lopomo, G., and Shouqiang, W. (2010). Multidimensional mechanism design: Finite-dimensional approximations and efficient computation. *Operations Research*, 58(4):1079–1089.

Biais, B., Martimort, D., and Rochet, J. (2000). Competing mechanisms in a common value environment. *Econometrica*, 68(4):799–837.

Carlier, G. (2001). A general existence result for the principal-agent problem with adverse selection. *Journal of Mathematical Economics*, 35(1):129–150.

Deneckere, R. and Severinov, S. (2015). Multi-dimensional screening: A solution to a class of problems. Preprint.

Fundenberg, D. and Tirole, J. (1991). *Game Theory*. MIT Press.

Guesnerie, R. (1981). On taxation and incentives: Further reflections on the limits to redistribution. Mimeo.

Kamien, M. and Schwartz, N. (1981). *Dynamics Optimization*. New York: North-Holland.

Laffont, J. and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press.

Laffont, J., Maskin, E., and Rochet, J. (1987). Optimal nonlinear pricing with two-dimensional characteristics. In *Incentives and Economic Mechanisms (essays in honor of Leonid Hurwicz)*, pages 256–266. T.Groves, R.Radner, S. Reiter, Eds., Minneapolis.

Lewis, T. and Sappington, D. (1988a). Regulating a monopolist with unknown demand. *American Economic Review*, 78:986–998.

Lewis, T. and Sappington, D. (1988b). Regulating a monopolist with unknown demand and cost functions. *RAND Journal of Economics*, 19(3):438–457.

Maskin, E. and Riley, J. (1984). Monopoly with incomplete information. *RAND Journal of Economics*, 15:171–196.

Matthews, S. and Moore, J. (1987). Monopoly provision of quality and warranties: An exploration in the theory of multidimensional screening. *Econometrica*, 55(2):441–467.

McAfee, R. and McMillan, J. (1988). Multidimensional incentive compatibility and mechanism design. *Journal of Economic Theory*, 46(2):335–354.

Milgrom, P. and I.Segal (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, 70:583–601.

Monteiro, P. and Page, F. J. (1998). Optimal selling mechanisms for multiproduct monopolists: incentive compatibility in the presence of budget constraints. *Journal of Mathematical Economics*, 30(4):473–502.

Mussa, M. and Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory*, 18(2):301–317.

Myerson, R. (1979). Incentive compatibility and the bargaining problem. *Econometrica*, 47:61–74.

Parra, C. (2014). *Numerical and Theoretical Approaches to Adverse Selection Problems.* PhD thesis, Instituto de Matematica Pura e Aplicada.

Rochet, J. and Stole, L. (2003). *The economics of multidimensional screening*, chapter 5. Advances in Economics and Econometrics. M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, Eds.

Rochet, J.-C. (1985). The taxation principle and multi-time hamilton-jacobi equations. *Journal of Mathematical Economics*, 14:113–128.

Rochet, J.-C. (1987). A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics*, 16(2):191–200.

Rochet, J.-C. and Choné (1998). Ironing, sweeping, and multidimensional screening. *Econometrica*, 66(4):783–826.

Vieira, S. (2008). *Price discrimination without the single-crossing.* PhD thesis, Instituto de Matematica Pura e Aplicada.

Wilson, R. (1993). *Non-linear Pricing.* Oxford University Press.

Wilson, R. (1995). Nonlinear pricing and mechanism design. In Amman, H., Kendrick, D., and Rust, J., editors, *Handbook of Computational Ecnonomics*, volume 1. Elsevier Science Publishers. Revised: 10 October 1999.