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YAMABE-TYPE PROBLEMS ON SMOOTH METRIC MEASURE SPACES

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À Deus

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"I love inequalities. So if somebody shows me a new inequality, I say, "Oh, that's beautiful, let me think about it," and I may some ideas connected with it."

L. Nirenberg, *Interview with Louis Nirenberg.* Notice of the AMS. 49 (2002), pp. 441-449.

Abstract

This thesis deals with a general version of The Yamabe problem introduced by Case in [5] and a general version of Escobar-Riemann mapping type problem introduced as follows.

The Yamabe Problem in compact closed Riemannian manifolds is concerned with finding a metric, with constant scalar curvature in the conformal class of a given metric. It is well known that the Yamabe problem was solved by the combined work of Yamabe, Trudinger, Aubin and Schoen. In particular, we mention that Aubin solved the case when the Riemannian manifold is compact, non-locally conformally flat and with dimension equal and greater than 6.

In [5], Case considered a Yamabe type problem in the setting of smooth measure space in manifolds without boundary and for a parameter m, which generalize the original Yamabe problem when $m = 0$. In the context of Euclidean space this generalization consists to find the functions that satisfy the sharp Gagliardo, Nirenberg, Sobolev inequalities. Case also solved this problem when the parameter m is natural. In this work we are able to generalize Aubin's result for non-locally conformally flat manifolds, with dimension equal and greater than 7 and every parameter m.

On the other hand, the Escobar-Riemann problem in manifolds with boundary is concerned with finding a metric, with scalar curvature identically null in the interior and with constant mean curvature on the boundary, in the conformal class of initial metric. This problem in the Euclidean half-space reduces to the Trace Sobolev inequality. Then we generalize this inequality and we consider Escobar-Riemann type problem for m parameter and smooth measure space with boundary which generalize the Escobar-Riemann problem when $m = 0$. We resolve the Escobar-Riemann type problem when the weighted Escobar constant is negative. Also, we prove for compact manifolds that the weighted Escobar constant is always less or equal than the weighted Escobar constant for the Euclidean half-space.

Keywords: Yamabe problem, smooth measure space, Trace Sobolev inequality, Escobar-Riemann problem, existence of minimizer.

Resumo

Esta tese trata de uma versão geral do problema de Yamabe introduzida por Case em [5] e uma versão geral do tipo de problema de Escobar-Riemann introduzida por nós.

O Problema de Yamabe em variedades Riemannianas compactas sem bordo está relacionado com encontrar uma métrica com curvatura escalar constante na classe conforme de uma dada métrica. É bem conhecido que o problema de Yamabe foi resolvido pelo trabalho combinado de Yamabe, Trudinger, Aubin e Schoen. Em particular, mencionamos que Aubin resolveu o caso em que a variedade Riemanniana é não localmente conformemente plana e tem dimensão maior ou igual a 6.

Case em [5] considerou um problema tipo Yamabe no contexto de variedades ponderadas sem bordo e para um parâmetro m, que generaliza o problema original de Yamabe quando $m = 0$. No contexto do espaço Euclidiano esta generalização reduz-se a encontrar funções que satisfaçam desigualdades ótimas de Gagliardo, Nirenberg, Sobolev. Case também resolveu este problema quando o parâmetro m é natural. Neste trabalho generalizamos o resultado de Aubin para uma variedade Riemanniana non-locally conformally flat, com dimensão maior ou igual a 7 e qualquer parâmetro m.

Por outro lado, o problema de Escobar-Riemann em variedades Riemannianas com bordo está relacionado a encontrar uma métrica com curvatura escalar identicamente nula no interior e com curvatura média constante no bordo, na classe conforme da métrica inicial. Como este problema no semiespaço Euclidiano se reduz à desigualdade do Traço de Sobolev nós generalizamos essa desigualdade e consideramos o problema de tipo Escobar-Riemann para o parâmetro m numa variedade ponderada com bordo que generaliza o problema de Escobar-Riemann quando $m = 0$. Nós resolvemos o problema tipo Escobar-Riemann quando a constante de Escobar com peso é negativa. Também provamos para variedades compactas com bordo que a constante de Escobar com peso é menor ou igual a constante de Escobar com peso do semiespaço Euclidiano.

Palavras-chave: problema de Yamabe, variedades ponderadas, desigual-

dade do Traço de Sobolev, problema de Escobar-Riemann, existência de minimizadores.

Contents

Introduction

Let (M^n, g) be an n-dimensional compact Riemannian manifold and R_g the scalar curvature associated to the metric g. If M is closed the Yamabe problem is concerned with finding a metric of constant scalar curvature in the conformal class of g. It is well known that the Yamabe problem was solved by the combined work of Yamabe [15], Trudinger [14], Aubin [1], and Schoen [13], for an comprehensive presentation of this topic see [10]. In particular, we mention that Aubin in [1] solved the problem under the hypothesis that the Riemannian manifold is compact, non-locally conformally flat and with dimension $n \geq 6$.

In [4], Case considered the weighted Yamabe constants which are a one-parameter family of geometric invariants that interpolate between the Yamabe constant and the ν-entropy. This Yamabe type problem coincides with the classical Yamabe problem when the parameter is zero. These invariants are natural as curved analogues of the sharp constants for the Gagliardo-Nirenberg inequalities studied by Del Pino and Dolbeault [6]. In order to introduce Case's work, we start by mentioning Del Pino and Dolbeault's result.

Theorem 1 ([6]). *Fix $m \in [0, \infty]$. For all $w \in W^{1,2}(\mathbb{R}^n) \cap L^{\frac{2(m+n-1)}{m+n-2}}(\mathbb{R}^n)$ it holds that*

$$\widetilde{\Lambda}_{m,n} \left(\int_{\mathbb{R}^n} |w|^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m+n-2}{n}} \leq \left(\int_{\mathbb{R}^n} |\nabla w|^2 \right) \left(\int_{\mathbb{R}^n} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{2m}{n}} \quad (1)$$

where the constant $\widetilde{\Lambda}_{n,m}$ is given by

$$\widetilde{\Lambda}_{m,n} = \frac{n\pi(m+n-2)^2}{(2m+n-2)} \left(\frac{2(m+n-1)}{(2m+n-2)}\right)^{\frac{2m}{n}} \left(\frac{\Gamma(\frac{2m+n}{2})}{\Gamma(m+n)}\right)^{\frac{2}{n}}. \tag{2}$$

Moreover, equality holds in (1) *if and only if* w *is a constant multiple of the function* w_{ϵ,x_0} *defined on* $\mathbb{R}^n$ *by*

$$w_{\epsilon,x_0}(x) := \left(\frac{2\epsilon}{\epsilon^2 + |x-x_0|^2}\right)^{\frac{m+n-2}{2}} \tag{3}$$

where $\epsilon > 0$*, and* $x_0 \in \mathbb{R}^n$*.*

Before we explain Case's results, we introduce some terminology. A *smooth metric measure space without boundary* is a four-tuple $(M^n, g, e^{-\phi}dV_g, m)$ of a Riemannian manifold (M^n, g), a smooth measure $e^{-\phi}dV_g$ determined by a function $\phi \in C^\infty(M)$ and by Riemannian volume element dV_g associated to g, and a dimensional parameter $m \in [0,\infty]$. The *weighted scalar curvature* R^m_ϕ of a smooth metric measure space for $m = 0$ is $R^m_\phi = R_g$ and for $m \neq 0$ is the function $R^m_\phi := R_g + 2\Delta\phi - \frac{m+1}{m}|\nabla\phi|^2$, where Δ is the Laplacian associated to the metric g. The *weighted Yamabe quotient* is the functional $\widetilde{\mathcal{Q}}[M^n, g, e^{-\phi}dV_g, m] : C^\infty(M) \to \mathbb{R}$ defined by

$$\widetilde{\mathcal{Q}}(w) = \frac{\int_M (|\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R^m_\phi w^2) e^{-\phi} dV_g \left(\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} e^{-\frac{(m-1)\phi}{m}} dV_g\right)^{\frac{2m}{n}}}{\left(\int_M |w|^{\frac{2(m+n)}{m+n-2}} e^{-\phi} dV_g\right)^{\frac{2m+n-2}{n}}}. \tag{4}$$

The *weighted Yamabe constant* is the number

$$\widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m] = \inf\{\widetilde{\mathcal{Q}}[M^n, g, e^{-\phi}dV_g, m](w) : w \in H^1(M, e^{-\phi}dV_g)\}. \tag{5}$$

For $m = \infty$, Case defined the weighted Yamabe quotient as the limit of (4) when m goes to infinity and the weighted Yamabe constant as (5). Note that in the case $m = 0$ the weighted Yamabe constant coincide with the Yamabe constant and in the case $m = \infty$ this is equivalent to Perelman's entropy (see [12]).

In particular Case proved in [5] an Aubin-type criterion for the existence of a minimizer of the Yamabe quotient. The exact statement is:

Theorem 2 ([5])**.** *Let* $(M^n, g, e^{-\phi}dV_g, m)$ *be a compact smooth metric measure space such that* $m > 0$*. Then*

$$\widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m] \leq \widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m]. \tag{6}$$

Moreover, if the inequality (6) *is strict, then there exists a positive function* $w \in C^\infty(M)$ *such that*

$$\widetilde{\mathcal{Q}}(w) = \widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m].$$

Also, Case proved in [5] the strict inequality in (6) when $m \in \mathbb{N} \cup \{0\}$ together with a characterization for the equality in (6).

Theorem 3 ([5])**.** *Let* $(M^n, g, e^{-\phi}dV_g, m)$ *be a compact smooth metric measure space such that* $m \in \mathbb{N} \cup \{0\}$*. If*

$$\widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m] = \widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m] \tag{7}$$

then $m \in \{0, 1\}$ *and the smooth metric measure space* $(M^n, g, e^{-\phi}dV_g, m)$ *is conformally equivalent to* (S^n, g_0, dV_{g_0}, m) *for* g_0 *a metric of constant sectional curvature. Therefore, there exists a positive function* $w \in C^\infty(M)$ *such that*

$$\widetilde{\mathcal{Q}}(w) = \widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m].$$

In contrast with the Yamabe problem, for which the minimizer always exists, the weighted Yamabe constant is not always achieved by a function.

Theorem 4 ([5])**.** *There does not exist a minimizer for the weighted Yamabe constant of* $(S^n, g_0, dV_{g_0}, \frac{1}{2})$*.*

As a consequence of Theorem 2 and Theorem 4, the equality in (6) holds for $(S^n, g_0, dV_{g_0}, \frac{1}{2})$. For non-locally conformally flat manifolds with dimension $n \geq 7$ and every non-negative number m we prove that inequality (6) is strict. Then by Theorem 2 the existence of a minimizer of the weighted Yamabe problem follows. This result is a generalization of the Aubin existence Theorem and generalization of Case existence result for m non-integer.

Theorem A. *Let* $(M^n, g, e^{-\phi}dV_g)$ *be a compact smooth metric measure space without boundary,* $m \geq 0$ *and* $n \geq 7$*. If* (M, g) *is non-locally conformally flat then*

$$\widetilde{\Lambda}[M^n, g, e^{-\phi}dV_g, m] < \widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m]. \tag{8}$$

Therefore, there exists a minimizer of the weighted Yamabe quotient.

To prove Theorem A, we use similar Aubin's arguments in [1], these arguments involve test functions in the Yamabe quotient with support in a neighborhood of a point where the Weyl Tensor is non-zero, this point exists because in a non-locally conformally flat manifold with dimension $n \geq 4$ the Weyl Tensor is not identically zero. However, when we restrict to the case $m = 0$, we use different test functions to the ones used in Aubin in [1]. For this reason, our proof is a different proof of Aubin's Theorem for $n \geq 7$. On the other hand, our proof does not work for general $m > 0$ in the case $n = 6$ (cf. Remark 7).

When (M^n, g) is a Riemannian manifold with boundary, we denote by ∂M the boundary of M and by H_g the trace of the second fundamental form of ∂M. The Escobar-Riemann problem for manifolds with boundary is concerned with finding a metric $\bar{g}$ with scalar curvature $R_{\bar{g}} \equiv 0$ in M and $H_{\bar{g}}$ constant on ∂M, in the conformal class of the initial metric g. Since this problem in the Euclidean half-space reduces to finding the minimizers in the sharp Trace Sobolev inequality then we generalize this Trace Sobolev inequality and prove an analogue of Del Pino and Dolbeaut result, finding optimal constants and minimizers of these inequalities. Let $\mathbb{R}^n_+ = \{(x,t) : x \in \mathbb{R}^{n-1}, t \geq 0\}$ denote the half-space and denoted its boundary by $\partial\mathbb{R}^n_+ = \{(x,0) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}\}$. We identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ whenever necessary.

Theorem B. *Fix $m \in \mathbb{N} \cup \{0\}$. For all $w \in W^{1,2}(\mathbb{R}^n_+) \cap L^{\frac{2(m+n-1)}{m+n-2}}(\mathbb{R}^n_+)$ it holds that*

$$\Lambda_{m,n} \left(\int_{\partial\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{2m+n-2}{m+n-1}} \leq \left(\int_{\mathbb{R}^n_+} |\nabla w|^2 \right) \left(\int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{m}{m+n-1}} \tag{9}$$

where the constant $\Lambda_{n,m}$ is given by

$$\Lambda_{m,n} = (m+n-2)^2 \left(\frac{Vol(S^{2m+n-1})^{\frac{1}{2m+n-1}}}{2(2m+n-2)} \right)^{\frac{2m+n-1}{m+n-1}} \left(\frac{\Gamma(2m+n-1)}{\pi^m \Gamma(m+n-1)} \right)^{\frac{1}{m+n-1}} \tag{10}$$

and $Vol(S^{2m+n-1})$ is the volume of the $2m+n-1$ dimensional unit sphere. Moreover, equality holds if and only if w is a constant multiple of the function w_{ϵ,x_0} defined on $\mathbb{R}^n_+$ by

$$w_{\epsilon,x_0}(x,t) := \left(\frac{2\epsilon}{(\epsilon+t)^2 + |x-x_0|^2} \right)^{\frac{m+n-2}{2}} \tag{11}$$

where $\epsilon > 0$ and $x_0 \in \mathbb{R}^{n-1}$.

Note that in Theorem B m is a non-negative integer. This result should be true for $m \geq 0$ since it is analogous to Del Pino and Dolbeaut's result and because there exists a proof for the latter using probabilistic techniques. However, at this moment, this kind of techniques escape from the author's knowledge.

Following Case's ideas we will consider an Escobar-Riemann mapping type problem. We introduce the notion of *smooth metric measure space with boundary* defined by a five-tuple $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ where $d\sigma_g$ is the volume form on the boundary ∂M induced by the metric g. The *weighted Escobar quotient* for this smooth measure is defined by

$$\mathcal{Q}(w) = \frac{\int_M (|\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m w^2) e^{-\phi} dV_g + \int_{\partial M} \frac{m+n-2}{2(m+n-1)} H_\phi^m w^2 e^{-\phi} d\sigma_g}{(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} e^{-\phi} d\sigma_g)^{\frac{2m+n-2}{m+n-1}} (\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} e^{-\frac{(m-1)\phi}{m}} dV_g)^{-\frac{m}{m+n-1}}}, \tag{12}$$

where we denoted by $H_\phi^m = H_g + \dfrac{\partial \phi}{\partial \eta}$ the Gromov mean curvature and $\dfrac{\partial}{\partial \eta}$ is the outer normal derivative.

The *weighted Escobar constant* $\Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] \in \mathbb{R} \cup \{-\infty\}$ is defined by

$$\Lambda := \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] = \inf\{\mathcal{Q}(w) : w \in H^1(M, e^{-\phi}dV_g)\}. \tag{13}$$

If $m = 0$ we require $\phi = 0$ and this quotient coincides with the Sobolev quotient considered by Escobar in the Escobar-Riemann problem. For this reason and also in order to avoid confusions with the weighted Yamabe quotient and weighted Yamabe constant we call the functional $\mathcal{Q}$ as the weighted Escobar quotient and the constant Λ as the weighted Escobar constant.

We prove the existence of a minimizer of the weighted Escobar constant when this constant is negative. The exact statement is:

Theorem C. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary, $m \in \mathbb{N} \cup \{0\}$ and negative weighted Escobar constant. Then there exists a positive function $w \in C^\infty(M)$ such that*

$$\mathcal{Q}(w) = \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m].$$

Using Theorem B we prove a similar result to Theorem 2 finding that the weighted Escobar constant for a compact smooth measure space with boundary is always less or equal than the weighted Escobar constant of the model case $(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)$.

Theorem D. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary such that $m \in \mathbb{N} \cup \{0\}$. Then*

$$\Lambda[M^n, g, e^{-\phi}dV_g, m] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}. \tag{14}$$

We recall that in Theorem 2 if the inequality (6) is strict it follows the existence of the minimizer for the Yamabe type problem, the same result is expected for the Escobar type problem for that reason we conjecture that

Conjecture. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary such that $m \in \mathbb{N} \cup \{0\}$ and*

$$\Lambda[M^n, g, e^{-\phi}dV_g, m] < \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}. \tag{15}$$

Then there exists a positive function $w \in C^\infty(M)$ such that

$$\mathcal{Q}(w) = \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m].$$

Organization

The chapters are organized as follows. In Chapter 1, we prove Theorem B. In Chapter 2 we introduce some basic concepts for the Yamabe type problem and Escobar-Riemann type problem. In Chapter 3 we prove Theorem A. In Chapter 4 we prove Theorem C and Theorem D

CHAPTER 1

New Trace inequality

In this chapter we prove Theorem B. As we mentioned in the introduction, the new trace inequality prepare the way to introduce our Escobar-Riemann type problem. Therefore, the purpose of presenting this chapter first is to unify later some considerations for the Yamabe and the Escobar-Riemann type problem.

1.1 Trace inequality

In this section, we prove Theorem B and state some remarks. We only prove it for m being a positive integer because the argument depends on the Sobolev trace inequality in $\mathbb{R}^{n+2m}$ and its minimizers. This kind of ideas are due to Bakry et al. (see [2]).

Remark 1. *In the case $m = 0$ in the inequality* (9) *we recover (see [3], [7]) the Sobolev trace inequality*

$$\Lambda_{0,n}\left(\int_{\partial\mathbb{R}^n_+}|w|^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} \leq \left(\int_{\mathbb{R}^n_+}|\nabla w|^2\right), \tag{1.1}$$

where $\Lambda_{0,n} = \frac{n-2}{2}(vol(S^{n-1}))^{\frac{1}{n-1}}$. Equality in (1.1) *holds if and only if w is a positive constant multiple of the functions w of the form*

$$w = \left(\frac{\epsilon}{(\epsilon+t)^2+|x-x_0|^2}\right)^{\frac{n-2}{2}}. \tag{1.2}$$

Lemma 1. *Let p, q, B, C be positive numbers and define $h(\tau) = B\tau^p + C\tau^{-q}$ for $\tau > 0$. Then h attains the infimum in $\tau_0 = (\frac{qB}{pA})^{\frac{1}{p+q}}$ and*

$$\inf_{\tau>0} h(\tau) = h(\tau_0) = B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} \left(\frac{q}{p}\right)^{\frac{p}{p+q}} \left(\frac{q+p}{p}\right).$$

Proof. Since h is a positive continuous function for $\tau > 0$ and

$$\lim_{\tau\to 0^+} h(\tau) = \lim_{\tau\to\infty} h(\tau) = \infty$$

it follows that h attains the infimum for some $\tau_0 > 0$. A direct computation shows that $h'(\tau) = \tau^{p-1}(pB - qC\tau^{-p-q})$. Therefore $\tau_0 = (\frac{qC}{pB})^{\frac{1}{p+q}}$ and

$$\begin{aligned} h((\tfrac{qC}{pB})^{\frac{1}{p+q}}) &= B(\tfrac{qC}{pB})^{\frac{p}{p+q}} + C(\tfrac{qC}{pB})^{\frac{-q}{p+q}} \\ &= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} + B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{-q}{p+q}} \\ &= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} (1 + \tfrac{p}{q}) \\ &= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} (\tfrac{p+q}{q}). \end{aligned} \tag{1.3}$$

□

Remark 2. *If $m \to \infty$, the inequality (9) takes the form*

$$\Lambda_{\infty,n} \left(\int_{\partial\mathbb{R}^n_+} |w|^2\right)^2 \leq \left(\int_{\mathbb{R}^n_+} |\nabla w|^2\right)\left(\int_{\mathbb{R}^n_+} |w|^2\right) \tag{1.4}$$

where $\lim_{m\to\infty} \Lambda_{m,n} = \Lambda_{\infty,n}$.

The inequality (1.4) is equivalent to the trace inequality $H^1(M) \to L^2(\partial M)$

$$2(\Lambda_{\infty,n})^{\frac{1}{2}} \left(\int_{\partial\mathbb{R}^n_+} |w|^2 dx\right) \leq \int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt + \int_{\mathbb{R}^n_+} |w|^2 dxdt. \tag{1.5}$$

In fact, suppose inequality (1.5) holds. For $\tau > 0$ define the function $w_\tau(x,t) = w(\frac{1}{\tau}(x,t))$. The change of variable $(y,s) = \frac{1}{\tau}(x,t)$ implies

$$\int_{\partial\mathbb{R}^n_+} |w_\tau|^2(x,0)dx = \tau^{n-1} \int_{\partial\mathbb{R}^n_+} |w|^2(y,0)dy,$$

$$\int_{\mathbb{R}^n_+} |\nabla w_\tau|^2(x,t)dxdt = \tau^{n-2} \int_{\mathbb{R}^n_+} |\nabla w|^2(y,s)dyds$$

and

$$\int_{\mathbb{R}^n_+} |w_\tau|^2(x,t)dxdt = \tau^n \int_{\mathbb{R}^n_+} |w|^2(y,s)dyds.$$

Then using w_τ *and the equalities above in inequality* (1.5) *we get*

$$2(\Lambda_{\infty,n})^{\frac{1}{2}} \left(\int_{\partial\mathbb{R}^n_+} |w|^2(y,0)dy \right) \leq \tau B + \tau^{-1} C, \tag{1.6}$$

where $B = \int_{\mathbb{R}^n_+} |w|^2(y,s)dyds$ *and* $C = \int_{\mathbb{R}^n_+} |\nabla w|^2(y,s)dyds$. *Lemma 1 yields that for* $\tau_0 = (\frac{C}{B})^{\frac{1}{2}}$, *it holds*

$$\tau_0 B + \tau_0^{-1} C = 2B^{\frac{1}{2}} C^{\frac{1}{2}} = 2 \left(\int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} |w|^2 dxdt \right)^{\frac{1}{2}}. \tag{1.7}$$

Since inequality (1.6) *is true for every* $\tau > 0$, *in particular it is true for* $\tau_0 = (\frac{C}{B})^{\frac{1}{2}}$ *and by* (1.7) *we have*

$$2(\Lambda_{\infty,n})^{\frac{1}{2}} \left(\int_{\partial\mathbb{R}^n_+} |w|^2 \right) \leq 2 \left(\int_{\mathbb{R}^n_+} |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} |w|^2 \right)^{\frac{1}{2}}, \tag{1.8}$$

which is equivalent to (1.4).

Now, suppose that inequality (1.4) *holds, then inequality* (1.8) *holds. Inequality* (1.5) *is a consequence of inequality* $2ab \leq a^2 + b^2$.

In our proof for the Theorem B we use the following Lemma, which was taken from [5].

Lemma 2. *Fix* k, $l \geq 0$, $2m \in \mathbb{N}$, *and constants* a, $\tau > 0$. *Then*

$$\int_{\mathbb{R}^{2m}} \frac{|y|^{2l} dy}{(a + \frac{|y|^2}{\tau})^{2m+k}} = \frac{\pi^m \Gamma(m+l)\Gamma(m+k-l)\tau^{m+l}}{\Gamma(m)\Gamma(2m+k)a^{m+k-l}}.$$

Proof of Theorem B. We are able to prove inequality (9) only for $m \in \mathbb{N}$. For this purpose consider the inequality (1.1) for $\mathbb{R}^{n+2m}_+$. The proof's idea consists in using this inequality for the special function

$$f(y,x,t) := \left(w^{\frac{-2}{m+n-2}}(x,t) + \frac{|y|^2}{\tau} \right)^{-\frac{2m+n-2}{2}} \in C^\infty(\mathbb{R}^{n+2m}_+). \tag{1.9}$$

where $(x,t) \in \mathbb{R}^n_+$, $y \in \mathbb{R}^{2m}$ and $\tau > 0$.

Suppose f is of the form (1.9). First we analyze the term on the left hand side of (1.1). Fixing (x,t) we note that $f^{\frac{2(2m+n-1)}{2m+n-2}}$ takes the form of the function considered in Lemma 2 with $a = w^{\frac{-2}{m+n-2}}(x,t)$. Therefore Fubini's Theorem, Lemma 2 with $k = n-1$ and $l = 0$, and some calculation yield

$$\int_{\partial\mathbb{R}^{2m+n}_+} f^{\frac{2(2m+n-1)}{2m+n-2}} dxdy = \frac{\pi^m \Gamma(m+n-1)\tau^m}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}^n_+} w^{\frac{2(m+n-1)}{m+n-2}} dx. \tag{1.10}$$

In order to analyze the term on the right hand side of (1.1) we compute

$$|\nabla f|^2 = \frac{\left(\frac{2m+n-2}{2}\right)^2 \left(\left(\frac{2}{m+n-2}\right)^2 w^{-\frac{2(m+n)}{m+n-2}} |\nabla w|^2 + 4\frac{|y|^2}{\tau^2}\right)}{\left(w^{-\frac{2}{m+n-2}} + \frac{|y|^2}{\tau}\right)^{2m+n}}.$$

Lemma 2 leads to

$$\begin{aligned}\int_{\mathbb{R}^{2m+n}_+} |\nabla f|^2 dydxdt \;&= \left(\frac{2m+n-2}{m+n-2}\right)^2 \left(\frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)}\right) \int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt \\ &+ \left(\frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)}\right) \int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt.\end{aligned} \tag{1.11}$$

Using equalities (1.10) and (1.11) in inequality (1.1) we get that

$$\begin{aligned}&\Lambda_{2m+n,0}\left(\frac{\pi^m \Gamma(m+n-1)\tau^m}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}^n_+} w^{\frac{2(m+n-1)}{m+n-2}} dx\right)^{\frac{2m+n-2}{2m+n-1}} \\ &\leq \left(\frac{2m+n-2}{m+n-2}\right)^2 \left(\frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)}\right) \int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt \\ &+ \left(\frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)}\right) \int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt.\end{aligned} \tag{1.12}$$

Rewriting (1.12) we obtain

$$\Lambda_{2m+n,0}\left(\frac{\pi^m \Gamma(m+n-1)}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}^n_+} w^{\frac{2(m+n-1)}{m+n-2}} dx\right)^{\frac{2m+n-2}{2m+n-1}} A \leq h(\tau). \tag{1.13}$$

where

$$A=\frac{\Gamma(2m+n)}{(2m+n-2)^2\pi^m\Gamma(m+n)},$$

$$h(\tau)=B\tau^{\frac{m}{2m+n-1}}+C\tau^{-\frac{m+n-1}{2m+n-1}},$$

$$B=\frac{1}{(m+n-2)^2}\int_{\mathbb{R}^n_+}|\nabla w|^2dxdt,$$

and

$$C=\frac{m}{m+n-1}\int_{\mathbb{R}^n_+}|w|^{\frac{2(m+n-1)}{m+n-2}}dxdt.$$

Lemma 1 implies that the function h minimize for $\tau_0=(\frac{(m+n-1)C}{mB})^{\frac{m+n-2}{2m+n-1}}$ and

$$\Lambda_{2m+n,0}\left(\frac{\pi^m\Gamma(m+n-1)}{\Gamma(2m+n-1)}\int_{\partial\mathbb{R}^n_+}w^{\frac{2(m+n-1)}{m+n-2}}dx\right)^{\frac{2m+n-2}{2m+n-1}}A\leq h(\tau_0). \tag{1.14}$$

Inequality (1.14) proves inequality (9) with $\Lambda_{m,n}$ as in (10). Next, we characterize the functions that achieve equality in (9). Note that for $\mathbb{R}^{n+2m}_+$ and f defined in (1.9), the equality in (1.1) holds if and only if

$$f(y,x,t)=\left(\frac{(t+\epsilon)^2+|x-x_0|^2+|y|^2}{\tau}\right)^{-\frac{2m+n-2}{2}}, \quad \text{for} \quad \tau>0,$$

i.e

$$w^{\frac{-2}{m+n-2}}(x,t)=\tau((t+\epsilon)^2+|x-x_0|^2)$$

(see Escobar [7] and Beckner [3]). Then the family of functions $\{w_{\epsilon,x_0}\}$ in (11) satisfies the equality in (9). ∎

CHAPTER 2

Preliminaries

In this chapter, we introduce some preliminary notions for the Yamabe type problem and our Escobar-Riemann type problem. In the first section, we present some unified preliminaries for both. In the two following sections, we present preliminaries for the Yamabe type problem and Escobar-Riemann type problem, respectively.

2.1 Preliminaries for Yamabe and Escobar-Riemann type problems

In this section, we just present preliminaries for manifolds with boundary because all of them work for manifolds without boundary disregarding the boundary terms. Our approach is based on [4] and [5]. The first step is to introduce the correct definition of a smooth metric measure space with boundary. The smooth metric measure spaces appear for example in Perelman's work (see [12]) and Lott and Villani's which involves the Ricci Tensor and theory of optimal transport (see [11]).

Definition 1. *Let (M^n, g) be a Riemannian manifold with boundary and let us denote by dV_g and $d\sigma_g$ the volume form induced by g in M and ∂M, respectively. Set a function ϕ such that $\phi \in C^\infty(M)$ and $m \in [0, \infty]$ be a dimensional parameter. In the case $m = 0$, we require that $\phi = 0$. A smooth metric measure space with boundary is the five-tuple $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$.*

As in [5], sometimes we denote a smooth metric measure space with boundary by the four-tuple $(M^n, g, v^m dV_g, v^m d\sigma_g)$ where v and ϕ are related by $v^m = e^{-\phi}$. We denote by R_g the scalar curvature of M, η the outer normal on ∂M and $\dfrac{\partial}{\partial\eta}$ the normal derivative. Also, we denote by h_{ij}, $H_g := g^{ij}h_{ij}$ and $h_g = \frac{H_g}{n-1}$; the second fundamental form, the trace of the second fundamental form and the mean curvature on the boundary ∂M, respectively. In the following definitions, we consider the case $m = \infty$ as the limit case of the parameter m.

Definition 2. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a smooth metric measure space with boundary. The Bakry-Emery Ricci curvature Ric^m_ϕ, the weighted scalar curvature R^m_ϕ and the Gromov mean curvature H^m_ϕ are the tensors*

$$Ric^m_\phi := Ric + \nabla^2\phi - \frac{1}{m}d\phi \otimes d\phi, \tag{2.1}$$

$$R^m_\phi := R_g + 2\Delta\phi - \frac{m+1}{m}|\nabla\phi|^2, \tag{2.2}$$

and

$$H^m_\phi = H_g - \frac{\partial\phi}{\partial\eta}, \tag{2.3}$$

respectively.

Definition 3. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ and $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$ be two smooth metric measure spaces with boundary. We say they are pointwise conformally equivalent if there is a function $\sigma \in C^\infty(M)$ such that*

$$(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m) = (M^n, e^{\frac{2\sigma}{m+n-2}}g, e^{\frac{m+n}{m+n-2}\sigma}e^{-\phi}dV_g, e^{\frac{m+n-1}{m+n-2}\sigma}e^{-\phi}d\sigma_g, m). \tag{2.4}$$

$(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ and $(\hat{M}^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$ are conformally equivalent if there is a diffeomorphism $F : \hat{M} \to M$ such that the new smooth measure space $(F^{-1}(M), F^*g, F^*(e^{-\phi}dV_g), F^*(e^{-\phi}d\sigma_g), m)$ is pointwise conformally equivalent to $(\hat{M}^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$.

Note that in the equality (2.4) the terms $e^{\frac{m+n}{m+n-2}\sigma}$ and $e^{\frac{m+n-1}{m+n-2}\sigma}$ appear in the smooth measure in M and ∂M, respectively. In particular, the numerators of the quotients involved differ by 1. In other words, the smooth measure in M changes like a $(m+n)$-dimensional manifold and the smooth measure in ∂M changes like a $(m+n-1)$-dimensional manifold.

Definition 4. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a smooth metric measure space with boundary. The weighted Laplacian $\Delta_\phi : C^\infty(M) \to C^\infty(M)$ is an operator defined by*

$$\Delta_\phi u = \Delta u - \nabla u \cdot \nabla\phi$$

where $u \in C^\infty(M)$, Δ is the usual Laplacian associated to the metric g and ∇ is gradient calculated in the metric g.

Definition 5. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a smooth metric measure space with boundary. The weighted conformal Laplacian (L^m_ϕ, B^m_ϕ) is given by the interior operator and boundary operator*

$$\begin{aligned} L^m_\phi &= -\Delta_\phi + \frac{m+n-2}{4(m+n-1)}R^m_\phi \qquad in \quad M,\\ B^m_\phi &= \frac{\partial}{\partial\eta} + \frac{m+n-2}{2(m+n-1)}H^m_\phi \qquad on \quad \partial M. \end{aligned} \tag{2.5}$$

Proposition 1. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ and $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$ be two pointwise conformally equivalent smooth metric measure space such that $\hat{g} = e^{\frac{2\sigma}{m+n-2}}g$ and $\hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi$. Let us denote by (L^m_ϕ, B^m_ϕ) and $(\hat{L}^m_{\hat{\phi}}, \hat{B}^m_{\hat{\phi}})$ their respective weighted conformal Laplacian. Similarly, we denote with hat all quantities computed with respect to the smooth metric measure space $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$. Then we have $\hat{v} = e^{\frac{\sigma}{m+n-2}}v$ and the following transformation rules*

$$\hat{L}^m_{\hat{\phi}}(w) = e^{-\frac{m+n+2}{2(m+n-2)}\sigma}L^m_\phi(e^{\frac{\sigma}{2}}w), \qquad \hat{B}^m_{\hat{\phi}}(w) = e^{-\frac{m+n}{2(m+n-2)}\sigma}B^m_\phi(e^{\frac{\sigma}{2}}w), \tag{2.6}$$

$$\nabla_{\hat{g}}\hat{v} = e^{-\frac{2\hat{\sigma}}{m+n-2}}\left(e^{\frac{\hat{\sigma}}{m+n-2}}\nabla_{\tilde{g}}v + \frac{v}{m+n-2}e^{\frac{\hat{\sigma}}{m+n-2}}\nabla_{\tilde{g}}\hat{\sigma}\right) \tag{2.7}$$

and

$$\begin{aligned} \Delta_{\hat{g}}\hat{v} \;&= e^{-\frac{\hat{\sigma}}{m+n-2}}\left(\frac{n-1}{(m+n-2)^2}\tilde{v}|\nabla_{\tilde{g}}\hat{\sigma}|^2_{\tilde{g}} + \frac{\tilde{v}}{m+n-2}\Delta_{\tilde{g}}\hat{\sigma}\right.\\ &\left.\qquad + \Delta_{\tilde{g}}\tilde{v} + \frac{n}{m+n-2}\nabla_{\tilde{g}}\hat{\sigma}\nabla_{\tilde{g}}\tilde{v}\right). \end{aligned} \tag{2.8}$$

We mention that the first identity in (2.6) appears in [5].

In order to fix some notation, we denote by $(w, \varphi)_M = \int_M w.\varphi\, v^m dV_g$ the inner product in $L^2(M, v^m dV_g)$ and by $(w, \varphi)_{\partial M} = \int_{\partial M} w.\varphi\, v^m d\sigma_g$ the inner product in $L^2(\partial M, v^m d\sigma_g)$. If no confusion occurs we use the same notation (w, φ) for $(w, \varphi)_M$ and $(w, \varphi)_{\partial M}$.

We denote by $||.||_{2,M}$ and $||.||_{2,\partial M}$ the norm in the space $L^2(M, v^m dV_g)$ and $L^2(\partial M, v^m d\sigma_g)$, respectively. Also, it will cause no confusion we use the same notation $||.||$ for both norms.

Moreover, $H^1(M, v^m dV_g)$ denotes the closure of $C^\infty(M)$ with respect to the norm

$$\int_M |\nabla w|^2 + |w|^2.$$

Here and subsequently the integrals in M are computed using the measure $v^m dV_g$. Similarly, we will omit the smooth measure $v^m d\sigma_g$ when we compute integrals on ∂M.

2.2 Preliminaries for Yamabe type problem

In this section, we recall some concepts necessary to state the Yamabe type problem in a smooth measure space without boundary $(M^n, g, v^m dV_g)$. They are taken from [5]. We define the weighted Yamabe quotient which generalizes the Sobolev quotient in the case $m = 0$ and we consider a suitable $\mathcal{W}$-functional. Following the presentation in [5], we also define the energies of these functionals and we give some of their properties.

2.2.1 The weighted Yamabe quotient

We denote with tilde the terms associated to the Yamabe type problem in order to avoid confusions with the Escobar-Riemann type problem. We start with the definition of the Yamabe quotient.

Definition 6. *Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space without boundary. The weighted Yamabe quotient $\widetilde{\mathcal{Q}} : C^\infty(M) \to \mathbb{R}$ is defined by*

$$\widetilde{\mathcal{Q}}[M^n, g, v^m dV_g](w) = \frac{(L_\phi^m w, w)_M \left(\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} v^{-1}\right)^{\frac{2m}{n}}}{\left(\int_M |w|^{\frac{2(m+n)}{m+n-2}}\right)^{\frac{2m+n-2}{n}}}. \tag{2.9}$$

The weighted Yamabe constant $\tilde{\Lambda}[M^n, g, v^m dV_g] \in \mathbb{R}$ of $(M^n, g, v^m dV_g, m)$ is defined by

$$\widetilde{\Lambda}[M^n, g, v^m dV_g] = \inf\{\widetilde{\mathcal{Q}}[M^n, g, v^m dV_g](w) : w \in H^1(M, v^m dV_g)\}. \tag{2.10}$$

Remark 3. *In some cases, when the context is clear, we will not write the dependence of the smooth metric measure space with boundary, for example we write $\widetilde{\mathcal{Q}}$ and $\widetilde{\Lambda}$ instead of $\widetilde{\mathcal{Q}}[M^n, g, v^m dV_g]$ and $\widetilde{\Lambda}[M^n, g, v^m dV_g]$ respectively. We note that since $C^\infty(M)$ is dense in $H^1(M, v^m dV_g)$ and $\widetilde{\mathcal{Q}}(|w|) = \widetilde{\mathcal{Q}}(w)$, we may equivalently define the weighted Yamabe constant by minimizing over the space of positive smooth functions on M, as we shall often do without further comment.*

Next, we observe that the weighted Yamabe quotient is continuous in $m \in [0, \infty]$ and it is conformal in the sense of Definition 3.

Proposition 2 ([5]). *Let (M^n, g) be a compact Riemannian manifold. Fix $\phi \in C^\infty(M)$ and $m \in [0, \infty]$. Given any $w \in C^\infty(M)$, it holds that*

$$\lim_{k \to m} \widetilde{\mathcal{Q}}[M^n, g, e^{-\phi} dV_g, k](w) = \widetilde{\mathcal{Q}}[M^n, g, e^{-\phi} dV_g, m](w). \tag{2.11}$$

Proposition 3 ([5]). *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space without boundary. For any σ, $w \in C^\infty(M)$ it holds that*

$$\widetilde{\mathcal{Q}}[M^n, e^{\frac{2}{m+n-2}\sigma} g, e^{\frac{m+n}{m+n-2}\sigma} v^m dV_g](w) = \widetilde{\mathcal{Q}}[M^n, g, v^m dV_g](e^{\frac{\sigma}{2}} w). \tag{2.12}$$

Note that the integral $\int\limits_M |w|^{\frac{2(m+n)}{m+n-2}} v^m dV_g$ measures the interior volume $\int_M \hat{v}^m dV_{\hat{g}}$ of

$$(M^n, \hat{g}, \hat{v}^m dV_{\hat{g}}, m) = (M^n, w^{\frac{4}{m+n-2}} g, w^{\frac{2(m+n)}{m+n-2}} v^m dV_g, m). \tag{2.13}$$

In order to simplify computations and to avoid the trivial non-compactness of the weighted Yamabe problem, we give the next definition:

Definition 7. *Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space without boundary. We say that a positive function $w \in C^\infty(M)$ is volume-normalized in the interior if*

$$\int\limits_M |w|^{\frac{2(m+n)}{m+n-2}} v^m dV_g = 1.$$

2.2.2 $\widetilde{\mathcal{W}}$-functional

Let us start with the definition of the $\widetilde{\mathcal{W}}$-functional considered by Case in [5].

Definition 8. *Let* $(M^n, g, v^m dV_g)$ *be a compact smooth metric measure space without boundary. The* $\widetilde{\mathcal{W}}$*-functional,* $\widetilde{\mathcal{W}} : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$*, is defined by*

$$\widetilde{\mathcal{W}}(w,\tau) = \tau^{\frac{m}{m+n}}(L_\phi^m w, w) + m\int_M \left(\tau^{-\frac{n}{2(m+n)}} w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} - w^{\frac{2(m+n)}{m+n-2}}\right)$$

when $m \in [0,\infty)$.

The functional $\widetilde{\mathcal{W}}$ satisfies the following property

Proposition 4 ([5])**.** *Let* $(M^n, g, v^m dV_g)$ *be a compact smooth metric measure space without boundary. Then*

$$\lim_{k\to m} \widetilde{\mathcal{W}}[M^n, g, e^{-\phi}dV_g, k](w,\tau) = \widetilde{\mathcal{W}}[M^n, g, e^{-\phi}dV_g, m](w,\tau).$$

Also, $\widetilde{\mathcal{W}}$ satisfies the following conformal property as Proposition 3.10 in [5].

Proposition 5 ([5])**.** *Let* $(M^n, g, v^m dV_g)$ *be a compact smooth metric measure space without boundary. The* $\widetilde{\mathcal{W}}$*-functional is conformally invariant in its first component:*

$$\widetilde{\mathcal{W}}[M^n, e^{2\sigma}g, e^{(m+n)\sigma}v^m dV_g](w,\tau) = \widetilde{\mathcal{W}}[M^n, g, v^m dV_g](e^{\frac{(m+n-2)}{2}\sigma}w,\tau) \quad (2.14)$$

for all σ, $w \in C^\infty(M)$ *and* $\tau > 0$. *It is scale invariant in its second component:*

$$\widetilde{\mathcal{W}}[M^n, cg, v^m dV_{cg}](w,\tau) = \widetilde{\mathcal{W}}[M^n, g, v^m dV_g](c^{\frac{n(m+n-2)}{4(m+n)}}w, c^{-1}\tau). \quad (2.15)$$

Since, we are interested in minimizing the Yamabe quotient, it is natural to define the following energies as infima of the $\widetilde{\mathcal{W}}$-functional. It is also natural to relate one of these energies with the weighted Yamabe constant.

Definition 9. *Let* $(M^n, g, v^m dV_g)$ *be a compact smooth metric measure space without boundary. Given* $\tau > 0$*, the* τ*-energy* $\widetilde{\nu}[M^n, g, v^m dV_g](\tau) \in \mathbb{R}$ *is defined by*

$$\widetilde{\nu}[M^n, g, v^m dV_g](\tau) = \inf\left\{\widetilde{\mathcal{W}}(w,\tau) : w \in H^1(M, v^m dV_g), \int_M w^{\frac{2(m+n)}{m+n-2}} = 1\right\}.$$

The energy $\widetilde{\nu}[M^n, g, v^m dV_g] \in \mathbb{R}\cup\{-\infty\}$ *is defined by*

$$\widetilde{\nu}[M^n, g, v^m dV_g] = \inf_{\tau>0} \widetilde{\nu}[M^n, g, v^m dV_g](\tau).$$

The conformal invariance property in the $\widetilde{\mathcal{W}}$-functional is transferred to the energies.

Proposition 6 ([5]). *Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space without boundary. Then*

$$\widetilde{\nu}[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}](c\tau) = \widetilde{\nu}[M^n, g, v^m dV_g,](\tau),$$

$$\widetilde{\nu}[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}] = \widetilde{\nu}[M^n, g, v^m dV_g]$$

for all $\sigma \in C^\infty(M)$, and for all $c > 0$.

The following proposition shows that it is equivalent to considering the energy instead of the weighted Yamabe constant when the latter is non-negative.

Proposition 7 ([5]). *Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space with boundary and denote by $\widetilde{\Lambda}$ and $\widetilde{\nu}$ the weighted Yamabe constant and the energy, respectively.*

- $\widetilde{\Lambda} \in [-\infty, 0)$ *if and only if* $\widetilde{\nu} = -\infty$;
- $\widetilde{\Lambda} = 0$ *if and only if* $\widetilde{\nu} = -m$; *and*
- $\widetilde{\Lambda} > 0$ *if and only if* $\widetilde{\nu} > -m$. *Moreover, in this case we have*

$$\widetilde{\nu} = \frac{2m+n}{2}\left[\frac{2\widetilde{\Lambda}}{n}\right]^{\frac{n}{2m+n}} - m \tag{2.16}$$

and if w is a interior volume-normalized, we have w is a minimizer of $\widetilde{\Lambda}$ if and only if (w, τ) is a minimizer of $\widetilde{\nu}$ for

$$\tau = \left[\frac{n\int_M w^{\frac{2(m+n-1)}{m+n-2}}v^{-1}}{2(L_\phi^m w, w)}\right]^{\frac{2(m+n)}{2m+n}}. \tag{2.17}$$

Next we consider the Euler-Lagrange equation of the $\widetilde{\mathcal{W}}$-functional.

Proposition 8 ([5]). *Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space without boundary. Fix $\tau > 0$ and suppose that $w \in H^1(M)$ is a non-negative critical point of the map $\xi \to \widetilde{\mathcal{W}}(\xi, \tau)$ acting on the space of volume-normalized elements of $H^1(M)$. Then w is a weak solution of*

$$\tau^{\frac{m}{m+n}} L_\phi^m w + \tfrac{m(m+n-1)}{m+n-2}\tau^{-\frac{n}{2(m+n)}} w^{\frac{m+n}{m+n-2}} = c w^{\frac{m+n+2}{m+n-2}} \quad in \quad M, \tag{2.18}$$

for some constant c. If additionally (w,τ) is a minimizer of the energy, then

$$c = \frac{(2m+n-2)(m+n)}{(2m+n)(m+n-2)}(\widetilde{\nu}+m). \tag{2.19}$$

Remark 4. *If $\widetilde{\Lambda} = 0$ then the Euler equation for the Yamabe quotient coincides with the equation for finding a new conformal smooth measure space such that $\hat{R}_\phi^m \equiv 0$. On the other hand, the problem to find a conformal smooth measure space with $\hat{R}_\phi^m \equiv C$ is solved by a direct compact argument on the functional*

$$\check{Q}(w) = \frac{(L_\phi^m w, w)_M}{(\int_M |w|^{\frac{2(m+n)}{m+n-2}})^{\frac{2m+n-2}{m+n}}}$$

due to $\frac{2(m+n)}{m+n-2} < \frac{2n}{n-2}$ for $m > 0$.

2.2.3 Euclidean space as the model space for the weighted Yamabe problem

In this sub-section, we consider a family of functions together with some of its properties which are fundamental in our proof of the Aubin type existence result for minimizers of the Yamabe quotient.

Fix $n \geq 3$ and $m \geq 0$. Let us denote by $\widetilde{c}(m,n) = \frac{m+n-1}{(m+n-2)^2}$ and define for x_0, τ the family of function $\{\varphi_{x_0,\tau}\}$ such that

$$\varphi_{x_0,\tau} = \tau^{-\frac{n(m+n-2)}{4(m+n)}} \left(1 + \frac{\widetilde{c}(m,n)}{\tau}|x-x_0|^2\right)^{-\frac{(m+n-2)}{2}}. \tag{2.20}$$

We denote the normalization of $\varphi_{x_0,\tau}$ by

$$\tilde{\varphi}_{x_0,\tau} = \tilde{V}^{-\frac{m+n-2}{2(m+n)}} \varphi_{x_0,\tau}$$

where

$$\tilde{V} = \int_{\mathbb{R}^n} \varphi_{0,1}^{\frac{2(m+n)}{m+n-2}} 1^m dx = \int_{\mathbb{R}^n} \varphi_{x_0,\tau}^{\frac{2(m+n)}{m+n-2}} 1^m dx,$$

we used the change of variables in the second equality. On the other hand, a computation shows

$$-\tau^{\frac{m}{m+n}}\Delta\varphi_{x_0,\tau}+\frac{m(m+n-1)}{m+n-2}\tau^{-\frac{n}{2(m+n)}}\varphi_{x_0,\tau}^{\frac{m+n}{m+n-2}}=\frac{(m+n)(m+n-1)}{m+n-2}\varphi_{x_0,\tau}^{\frac{m+n+2}{m+n-2}}. \tag{2.21}$$

For the definition of $\varphi_{x_0,\tau}$, the definition of $\tilde{V}$ and identity (2.21) see (5.1), (5.2) and (5.3) in [5]. On the other hand, $\tilde{\varphi}_{x_0,\tau}$ is normalized and attains the infimum of the weighted Yamabe quotient, then there exists $\tilde{\tau}>0$ such that

$$\begin{aligned}\widetilde{\nu}(\mathbb{R}^n,dx^2,1^m dV_g)+m &= \widetilde{\mathcal{W}}(\mathbb{R}^n,dx^2,1^m dV_g)(\tilde{\varphi}_{x_0,\tau},\tilde{\tau})+m\\ &= \frac{\tilde{\tau}^{\frac{m}{m+n)}}}{\tilde{V}^{\frac{m+n-2}{m+n}}}\int_{\mathbb{R}^n}|\nabla\varphi_{x_0,\tau}|^2+\frac{m\tilde{\tau}^{-\frac{n}{2(m+n)}}}{\tilde{V}^{\frac{m+n-1}{m+n}}}\int_{\mathbb{R}^n}\varphi_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}}.\end{aligned} \tag{2.22}$$

Since $\tilde{\varphi}_{x_0,\tau}$ and $\varphi_{x_0,\tau}$ satisfy the equation (2.18) and (2.21), respectively, it follows that $\tilde{\tau}=\tau\tilde{V}^{-\frac{2}{2m+n}}$.

2.2.4 Relation between weighted Yamabe constants for manifolds without boundary and the Euclidean space

The next result, which corresponds to Theorem 7.1 in [5], links the weighted Yamabe constants of $(M^n,g,v^m dV_g)$ and $(M^n,g,v^{m+1}dV_g)$ with the weighted Yamabe constants for the Euclidean space with parameters m and $m+1$. This result allows us to prove the existence of a minimizer for the weighted Yamabe constant in an inductive argument for the parameter m.

Theorem 5 ([5])**.** *Let $(M^n,g,v^m dV_g)$ be a compact smooth metric measure space with non-negative weighted Yamabe constant, and suppose that there exists a smooth, positive minimizer w of the weighted Yamabe constant. Then*

$$\widetilde{\Lambda}[M^n,g,v^{m+1}dV_g]\le\widetilde{\Lambda}[\mathbb{R}^n,dx^2,dV,m+1]\frac{\widetilde{\Lambda}[M^n,g,v^m dV_g]}{\widetilde{\Lambda}[\mathbb{R}^n,dx^2,dV,m]}. \tag{2.23}$$

2.3 Preliminaries for Escobar-Riemann type problem

In this section, we define the weighted Escobar quotient which generalizes the quotient considered by Escobar in [8] for the case $m=0$. In general, the weighted Escobar quotient is not necessarily finite.

2.3.1 The weighted Escobar quotient

Definition 10. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. The weighted Escobar quotient $\mathcal{Q} : C^\infty(M) \to \mathbb{R}$ is defined by*

$$\mathcal{Q}(w) = \frac{((L_\phi^m w, w)_M + (B_\phi^m w, w)_{\partial M})(\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} v^{-1})^{\frac{m}{m+n-1}}}{(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}})^{\frac{2m+n-2}{m+n-1}}}. \tag{2.24}$$

The weighted Escobar constant $\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R}$ of the smooth measure space $(M^n, g, v^m dV_g, v^m d\sigma_g, m)$ is

$$\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g, m] = \inf\{\mathcal{Q}(w) : w \in H^1(M, v^m dV_g, v^m d\sigma_g)\}. \tag{2.25}$$

Here and subsequently, as in Remark 3, sometimes we do not write the dependence of the smooth metric measure space with boundary and we consider positive smooth functions for the weighted Escobar quotient.

On the other hand, the weighted Escobar quotient satisfies similar properties to the Yamabe quotient, for example we observe that the weighted Escobar quotient is continuous in $m \in [0, \infty]$ and it is conformal in the sense of the Definition 3.

Proposition 9. *Let (M^n, g) be a compact Riemannian manifold with boundary. Fix $\phi \in C^\infty(M)$ and $m \in [0, \infty]$. Given any $w \in C^\infty(M)$, it holds that*

$$\lim_{k \to m} \mathcal{Q}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, k](w) = \mathcal{Q}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m](w). \tag{2.26}$$

Proposition 10. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. For any σ, $w \in C^\infty(M)$ it holds that*

$$\begin{aligned} &\mathcal{Q}[M^n, e^{\frac{2}{m+n-2}\sigma} g, e^{\frac{m+n}{m+n-2}\sigma} v^m dV_g, e^{\frac{m+n-1}{m+n-2}\sigma} v^m d\sigma_g](w) \\ &= \mathcal{Q}[M^n, g, v^m dV_g, v^m d\sigma_g](e^{\frac{\sigma}{2}} w). \end{aligned} \tag{2.27}$$

Proof. A straightforward computation shows that the integrals

$$\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} v^{m-1} dV_g \quad \text{and} \quad \int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g \tag{2.28}$$

are invariant under the conformal transformation

$$(g, v^m dV_g, v^m d\sigma_g, w) \to (e^{\frac{2}{m+n-2}\sigma} g, e^{\frac{m+n}{m+n-2}\sigma} v^m dV_g, e^{\frac{m+n-1}{m+n-2}\sigma} v^m d\sigma_g, e^{-\frac{\sigma}{2}} w). \tag{2.29}$$

By Proposition 1 the term $(L^m_\phi w, w) + (B^m_\phi w, w)$ is invariant under (2.29). □

Similar to the smooth measure spaces we have some behavior for the boundary volume. Note that in the boundary the integral $\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g$ measure the boundary volume $\int_{\partial M} \hat{v}^m d\sigma_{\hat{g}}$ of

$$(M^n, \hat{g}, \hat{v}^m dV_{\hat{g}}, \hat{v}^m d\sigma_{\hat{g}}, m) = (M^n, w^{\frac{4}{m+n-2}} g, w^{\frac{2(m+n)}{m+n-2}} v^m dV_g, w^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g, m). \tag{2.30}$$

Also with the same purpose, to simplify calculus and to avoid the trivial non-compactness of the weighted Escobar-Riemann type problem, we give the next definition of the volume-normalized on the boundary.

Definition 11. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. We say that a positive function $w \in C^\infty(M)$ is volume-normalized on the boundary if*

$$\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g = 1.$$

Remark 5. *In this work, we use the same word "normalized" for functions that satisfy the Definition 7 for the volume-normalized in the interior in the Yamabe type problem and for functions that satisfy the Definition 11 for the volume-normalized on the boundary in the Escobar-Riemann type problem.*

2.3.2 $\mathcal{W}$-functional

We introduce a $\mathcal{W}$-functional with similar properties as the $\mathcal{W}$-functional considered by Case in [5] and Perelman in [12].

Definition 12. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. The $\mathcal{W}$-functional, $\mathcal{W} : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$, is defined by*

$$\begin{aligned}&\mathcal{W}(w,\tau)=\mathcal{W}[M^n,g,v^m dV_g,v^m d\sigma_g](w,\tau)\\&=\tau^{\frac{m}{2(m+n-1)}}\left((L_\phi^m w,w)+(B_\phi^m w,w)\right)+\int_M \tau^{-\frac{1}{2}}w^{\frac{2(m+n-1)}{m+n-2}}v^{-1}-\int_{\partial M}w^{\frac{2(m+n-1)}{m+n-2}}\end{aligned}\tag{2.31}$$

when $m\in[0,\infty]$.

As the weighted Escobar quotient and the $\widetilde{\mathcal{W}}$-functional, the $\mathcal{W}$-functional is continuous in m and conformally invariant. Additionally, we have one scale invariant in the variable τ.

Proposition 11. *Let* $(M^n,g,v^m dV_g,v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. Then*

$$\lim_{k\to m}\mathcal{W}[M^n,g,e^{-\phi}dV_g,e^{-\phi}d\sigma_g,k](w,\tau)=\mathcal{W}[M^n,g,e^{-\phi}dV_g,e^{-\phi}d\sigma_g,m](w,\tau).$$

Proposition 12. *Let* $(M^n,g,v^m dV_g,v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. The* $\mathcal{W}$*-functional is conformally invariant in its first component:*

$$\begin{aligned}&\mathcal{W}[M^n,e^{2\sigma}g,e^{(m+n)\sigma}v^m dV_g,e^{(m+n-1)\sigma}v^m d\sigma_g](w,\tau)\\&=\mathcal{W}[M^n,g,v^m dV_g,v^m d\sigma_g](e^{\frac{(m+n-2)}{2}\sigma}w,\tau)\end{aligned}\tag{2.32}$$

for all σ, $w\in C^\infty(M)$ *and* $\tau>0$. *It is scale invariant in its second component:*

$$\begin{aligned}&\mathcal{W}[M^n,cg,v^m dV_{cg},v^m d\sigma_{cg}](w,\tau)\\&=\mathcal{W}[M^n,g,v^m dV_g,v^m d\sigma_g](c^{\frac{(n-1)(m+n-2)}{4(m+n-1)}}w,c^{-1}\tau).\end{aligned}\tag{2.33}$$

Proof. The equality (2.32) follows as in Proposition 10 and the equality (2.33) follows by a direct computation. □

Since we are interested in minimizing the weighted Escobar quotient it is natural to define the following energies as infima using the $\mathcal{W}$-functional and relating one of these energies with the weighted Escobar constant.

Definition 13. *Let* $(M^n,g,v^m dV_g,v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. Given* $\tau>0$, *the* τ*-energy* $\nu[M^n,g,v^m dV_g,v^m d\sigma_g](\tau)$ *is the number defined by*

$$\begin{aligned}&\nu(\tau)=\nu[M^n,g,v^m dV_g,v^m d\sigma_g](\tau)\\&=\inf\left\{\mathcal{W}(w,\tau):w\in H^1(M,v^m dV_g,v^m d\sigma_g),\textstyle\int_{\partial M}w^{\frac{2(m+n-1)}{m+n-2}}=1\right\}.\end{aligned}\tag{2.34}$$

The energy $\nu[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R} \cup \{-\infty\}$ *is defined by*

$$\nu = \nu[M^n, g, v^m dV_g, v^m d\sigma_g] = \inf_{\tau>0} \nu[g, v^m dV_g, v^m d\sigma_g](\tau).$$

The conformal invariance in the $\mathcal{W}$-functional is transferred to the energies.

Proposition 13. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. Then*

$$\nu[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}, e^{(m+n-1)\sigma}v^m d\sigma_{cg}](c\tau) = \nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau),$$

$$\nu[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}, e^{(m+n-1)\sigma}v^m d\sigma_{cg}] = \nu[M^n, g, v^m dV_g, v^m d\sigma_g]$$

for all $\sigma \in C^\infty(M)$ *and* $c > 0$.

The following proposition shows that considering the energy is equivalent to considering the weighted Escobar constant when the latter is positive.

Proposition 14. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary and denote by* Λ *and* ν *the weighted Escobar constant and the energy, respectively.*

- $\Lambda \in [-\infty, 0)$ *if and only if* $\nu = -\infty$*;*
- $\Lambda = 0$ *if and only if* $\nu = -1$*; and*
- $\Lambda > 0$ *if and only if* $\nu > -1$*. Moreover, in this case we have*

$$\nu = \frac{2m+n-1}{m}\left[\frac{m\Lambda}{m+n-1}\right]^{\frac{m+n-1}{2m+n-1}} - 1 \tag{2.35}$$

and w *is a volume-normalized minimizer of* Λ *if and only if* (w, τ) *is a volume-normalized minimizer of* ν *for*

$$\tau = \left[\frac{m\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}}{(m+n-1)((L_\phi^m w, w) + (B_\phi^m w, w))}\right]^{\frac{m+n-1}{2(2m+n-1)}}. \tag{2.36}$$

Proof. If $\Lambda \in [-\infty, 0)$ then there is a volume-normalized function $w \in C^\infty(M)$ such that $(L^m_\phi w, w) + (B^m_\phi w, w) < 0$. Then, it is clear that $\mathcal{W}(w, \tau) \to -\infty$ as $\tau \to \infty$ and it follows that $\nu = -\infty$. Reciprocally, if $\nu = -\infty$ there exist a volume-normalized function w and $\tau > 0$ such that $\mathcal{W}(w, \tau) < -1$, it follows that $(L^m_\phi w, w) + (B^m_\phi w, w) < 0$ and $\Lambda \in [-\infty, 0)$.

Suppose $\Lambda \geq 0$. Lemma 1 shows that if A, $B > 0$, then

$$\inf_{x>0}\{Ax^{\frac{m}{m+n-1}} + Bx^{-1}\} = \frac{2m+n-1}{m}\left[\frac{m}{m+n-1}AB^{\frac{m}{m+n-1}}\right]^{\frac{m+n-1}{2m+n-1}} \tag{2.37}$$

for all $x > 0$, with equality if and only if

$$x = \left[\frac{mB}{(m+n-1)A}\right]^{\frac{m+n-1}{2m+n-1}}. \tag{2.38}$$

Notes that equality (2.37) is achieved in the case $A = 0$. Then we have immediately from the definitions of Λ and ν that (2.36) holds. □

2.3.3 Variational formulae for the weighted energy functionals

The next proposition contains the computation of the Euler-Lagrange equations of the minimizing of weighted Escobar quotient. We will use it in the proof of Theorem C on the regularity part.

Proposition 15. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and suppose that $0 \leq w \in H^1(M)$ is a volume-normalized minimizer of the weighted Escobar constant Λ. Then w is a weak solution of*

$$\begin{aligned} &L^m_\phi w + c_1 w^{\frac{m+n}{m+n-2}} v^{-1} = 0, && \text{in} \quad M, \\ &B^m_\phi w = c_2 w^{\frac{m+n}{m+n-2}}, && \text{on} \quad \partial M \end{aligned} \tag{2.39}$$

where

$$c_1 = \frac{m\Lambda}{m+n-2}\left(\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}\right)^{-\frac{2m+n-1}{m+n-1}}$$

and

$$c_2 = \frac{(2m+n-2)\Lambda}{m+n-2}\left(\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}\right)^{-\frac{m}{m+n-1}}.$$

Proof. This proposition follows immediately from the fact that the conformal Laplacian is self-adjoint, and the definition of the weighted Escobar constant. □

Remark 6. *If $\Lambda = 0$ then we have in the proposition above that $c_1 = 0$, $c_2 = 0$. It follows that the equations in (2.39) coincide with the equations for finding a new conformal smooth measure space such that $\hat{R}^m_\phi \equiv 0$ and $\hat{H}^m_\phi \equiv 0$. Moreover, the problem to find a conformal smooth measure space with $\hat{R}^m_\phi \equiv 0$ and $\hat{H}^m_\phi \equiv C$ is solved by a direct compact argument on the functional*

$$\check{Q}(w) = \frac{(L^m_\phi w, w)_M + (B^m_\phi w, w)_{\partial M}}{\left(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{m+n-2}{m+n-1}}}$$

due to $\frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}$ for $m > 0$.

Next, we consider the Euler Lagrange equation on the $\mathcal{W}$-functional and we will use it in the proof of Theorem D.

Proposition 16. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary, fix $\tau > 0$, and suppose that $w \in H^1(M)$ is a non-negative critical point of the map $\xi \to \mathcal{W}(\xi, \tau)$ acting on the space of volume-normalized elements of $H^1(M)$. Then w is a weak solution of*

$$\begin{aligned} \tau^{\frac{m}{2(m+n-1)}} L^m_\phi w + \tfrac{m+n-1}{m+n-2}\tau^{-\frac{1}{2}} w^{\frac{m+n}{m+n-2}} v^{-1} &= 0 && \text{in} \quad M, \\ \tau^{\frac{m}{2(m+n-1)}} B^m_\phi w &= c_3 w^{\frac{m+n}{m+n-2}} && \text{on} \quad \partial M, \end{aligned} \tag{2.40}$$

where

$$c_3 = (\nu(\tau) + 1) + \frac{\tau^{-\frac{1}{2}}}{m+n-2} \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}.$$

If additionally (w, τ) is a minimizer of the energy, then

$$c_3 = \frac{(m+n-1)(2m+n-2)}{(m+n-2)(2m+n-1)}(\nu + 1). \tag{2.41}$$

Proof. The equality (2.40) follows immediately from the definition of $\mathcal{W}$. If (w, τ) is a critical point of the map $(w, \tau) \to \mathcal{W}(w, \tau)$, then

$$\frac{m}{m+n-1}\tau^{\frac{m}{2(m+n-1)}}((L^m_\phi w, w) + (B^m_\phi w, w)) = \tau^{-\frac{1}{2}} \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}. \tag{2.42}$$

Using this identity we can express ν and c_3 in terms of $(L^m_\phi w, w) + (B^m_\phi w, w)$ and these expressions yields to (2.41). □

2.3.4 Euclidean half-space as the model space weighted Escobar problem

Theorem B gives a completed classification of the minimizers for the weighted Escobar quotient in the model space $(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)$. In this subsection we take a new (τ, x_0)-parametric family of functions as in (11) with $\tau > 0$ and $x_0 \in \mathbb{R}^{n-1}$.

To define the (τ, x_0)-parametric family of functions fix $n \geq 3$ and $m > 0$. Given any $x_0 \in \mathbb{R}^{n-1}$ and $\tau > 0$, define the function $w_{x_0,\tau} \in C^\infty(R^n_+)$ by

$$w_{x_0,\tau}(t,x) = \tau^{-\frac{(n-1)(m+n-2)}{4(m+n-1)}} \left[\left(1 + \left(\frac{c(m,n)}{\tau} \right)^{\frac{1}{2}} t \right)^2 + \frac{c(m,n)|x - x_0|^2}{\tau} \right]^{-\frac{m+n-2}{2}} \tag{2.43}$$

where $c(m,n) = \frac{m+n-1}{m(m+n-2)^2}$. By change of variables we get

$$V = \int_{\partial\mathbb{R}^n_+} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} 1^m d\sigma = \int_{\partial\mathbb{R}^n_+} w_{0,1}^{\frac{2(m+n-1)}{m+n-2}} 1^m d\sigma. \tag{2.44}$$

A straightforward computation shows that

$$\begin{aligned} -\tau^{\frac{m}{2(m+n-1)}} \Delta w_{x_0,\tau} + \tfrac{m+n-1}{m+n-2} \tau^{-\frac{1}{2}} w_{x_0,\tau}^{\frac{m+n}{m+n-2}} &= 0 && \text{in} \quad \mathbb{R}^n_+, \\ \tau^{\frac{m}{2(m+n-1)}} \frac{\partial w_{x_0,\tau}}{\partial \eta} &= \left(\tfrac{m+n-1}{m}\right)^{\frac{1}{2}} w_{x_0,\tau}^{\frac{m+n}{m+n-2}} && \text{on} \quad \partial\mathbb{R}^n_+, \end{aligned} \tag{2.45}$$

$$\sup_{(x,t)\in\mathbb{R}^n_+} w_{x_0,\tau}(x,t) = w_{x_0,\tau}(x_0, 0) = \tau^{-\frac{(n-1)(m+n-2)}{4(m+n-1)}}, \tag{2.46}$$

and for any $x \neq x_0$,

$$\lim_{\tau \to 0^+} w_{x_0,\tau}(x,t) = 0. \tag{2.47}$$

Define $\tilde{w}_{x_0,\tau} = V^{-\frac{m+n-2}{2(m+n-1)}} w_{x_0,\tau}$; with V as in (2.44). Since $\tilde{w}_{x_0,\tau}$ achieves the weighted Escobar quotient, by Proposition 14, there exits $\tilde{\tau} > 0$ such that

$$\begin{aligned} &\nu(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m) + 1 = \mathcal{W}(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)(\tilde{w}_{x_0,\tau}, \tilde{\tau}) + 1 \\ &= \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V^{\frac{m+n-2}{m+n-1}}} \textstyle\int_{\mathbb{R}^n_+} |\nabla w_{x_0,\tau}|^2 dV + \tilde{\tau}^{-\frac{1}{2}} V^{-1} \int_{\mathbb{R}^n_+} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dV. \end{aligned} \tag{2.48}$$

Then Proposition 16 yields to $\tilde{\tau} = \tau V^{-\frac{2}{2m+n-1}}$.

CHAPTER 3

An Aubin type existence theorem

In this chapter we are dedicated to prove Theorem A. Roughly speaking the proof consists of taking a point p where the Weyl tensor is non-zero and a smooth measure space conformal to the original so that the new density v is well behaved, then we take a family of functions supported in a small neighborhood of the point p. Such functions are of the form of a standard cutoff function times the family of functions $\varphi_{x_0,\tau}$ defined by (2.20). The longest part of our argument consists in estimating the functional $\widetilde{\mathcal{W}}$ in this family. Then, changing again the smooth measure space by a conformal one and taking the limit when the parameter τ goes to zero we prove that the entropy is less than the Euclidean space when $m < 1$. By Proposition 7 and Theorem 2 we get the result for $m < 1$. Finally, using Theorem 5 in an inductive argument we get the result for every positive m.

Before we prove Theorem A, we show a calculus lemma

Lemma 3. *Let $1 \leq i, j \leq n$ with $i \neq j$, then*

$$\begin{aligned} \int_{B_\epsilon^n} \frac{x_i^4 |x|^l}{(1 + \frac{\tilde{c}(m,n)}{\tau}|x|^2)^k} dx &= 3 \int_{B_\epsilon^n} \frac{x_i^2 x_j^2 |x|^l}{(1 + \frac{\tilde{c}(m,n)}{\tau}|x|^2)^k} dx \\ &= \frac{3}{n(n+2)} \int_{B_\epsilon} \frac{|x|^{4+l}}{(1 + \frac{\tilde{c}(m,n)}{\tau}|x|^2)^k} dx. \end{aligned} \tag{3.1}$$

Proof. We will use the formula

$$\int_{S_\rho^{n-1}} q\, dS_\rho = \frac{\rho^2}{d(d+n-2)} \int_{S_\rho^{n-1}} \Delta q\, dS_\rho,$$

where S_ρ^{n-1} is the sphere of radius ρ and q is a homogeneous polynomial of degree d. Then

$$\int_{S_\rho^{n-1}} x_i^4 dS_\rho = 3 \int_{S_\rho^{n-1}} x_i^2 x_j^2 dS_\rho = \frac{3}{n(n+2)}\rho^4 \int_{S_\rho^{n-1}} dS_\rho.$$

Using the last equality and polar coordinates we get the result. □

Proof of Theorem A. Here C is a positive constant which depends only on $(M^n, g, v^m dV_g)$ and possibly changes from line to line or in the same line. Since (M, g) is non-locally conformally flat there exist $p \in M$ such that the Weyl Tensor in p is non-zero, i.e. $|W|(p) \neq 0$. By (2.14) we change our original smooth measure space by $(M^n, \tilde{g}, \tilde{v}^m dV_{\tilde{g}}, \tilde{v}^m d\sigma_{\tilde{g}})$ where $\tilde{g} = e^{\frac{2\sigma}{m+n-2}} g$, $\tilde{v} = e^{\frac{2\sigma}{m+n-2}} v$, such that in p we have $\tilde{v}(p) = 1$, $\nabla_{\tilde{g}} \tilde{v}(p) = 0$ and $\Delta_{\tilde{g}} \tilde{v}(p) = 0$. We consider this new smooth measure space in order to simplify calculations in the proof. Also, we denote by tilde the terms associated to the new smooth measure space.

The underlying idea of this proof is to improve the upper bound estimated in Proposition 6.3 in [5]. For this purpose, we fix a point $p \in M$ and let $\{x_i\}$ be normal coordinates in some fixed neighborhood U, centered at $p = (0, ..., 0)$. Let $1 > \epsilon > 0$ be such that $B(p, 2\epsilon) \subset U$. Let $\eta : M \to [0, 1]$ be a cutoff function such that $\eta \equiv 1$ on B_ϵ, $supp(\eta) \subset B_{2\epsilon}$ and $|\nabla \eta|^2 < C\epsilon^{-1}$ in $A_\epsilon := B_{2\epsilon}^n \setminus B_\epsilon^n$. For each $0 < \tau < 1$, define $f_\tau : M \to \mathbb{R}$ by $f_\tau(x_1, ..., x_n) = \eta \varphi_{0,\tau}(x_1, ..., x_n)$, and set $\tilde{f}_\tau = \tilde{V}_\tau^{-\frac{m+n-2}{2(m+n)}} f_\tau$ where

$$\tilde{V}_\tau = \int_M f_\tau^{\frac{2(m+n)}{m+n-2}} \tilde{v}^m dV_{\tilde{g}}.$$

Taking $\tilde{\tau} = \tau \tilde{V}^{-\frac{2}{2m+n}}$, by the definition of $\widetilde{\mathcal{W}}$ and property (2.14) in Proposition 5 we get

$$\begin{aligned} \widetilde{\mathcal{W}}[M^n, g, v^m dV_g](e^{\frac{\sigma}{2}} \tilde{f}_\tau, \tilde{\tau}) + m &= \widetilde{\mathcal{W}}[M^n, \tilde{g}, \tilde{v}^m dV_{\tilde{g}}](\tilde{f}_\tau, \tilde{\tau}) + m \\ &= \frac{\tilde{\tau}^{\frac{m}{m+n}}}{\tilde{V}_\tau^{\frac{m+n-2}{m+n}}} \left(\int_{B_{2\epsilon}} |\nabla f_\tau|_{\tilde{g}}^2 \tilde{v}^m + \frac{m+n-2}{4(m+n-1)} R_{\tilde{\phi}}^m f_\tau^2 \tilde{v}^m dV_{\tilde{g}} \right) \\ &\quad + \frac{m \tilde{\tau}^{-\frac{n}{2(m+n)}}}{\tilde{V}_\tau^{\frac{m+n-1}{m+n}}} \int_{B_{2\epsilon}} f_\tau^{\frac{2(m+n-1)}{m+n-2}} \tilde{v}^{m-1} dV_{\tilde{g}}. \end{aligned} \tag{3.2}$$

Recall that in $\tilde{g}$-normal coordinates it holds

$$\begin{aligned} dV_{\tilde{g}} &= (1 - \tfrac{1}{6}\tilde{R}_{ij}x^i x^j - \tfrac{1}{12}\tilde{R}_{ij,k}x^i x^j x^k \\ &\quad -(\tfrac{1}{40}\tilde{R}_{ij,kl} + \tfrac{1}{180}\tilde{R}_{pijr}\tilde{R}_{pklr} - \tfrac{1}{72}\tilde{R}_{ij}\tilde{R}_{kl})x^i x^j x^k x^l + O(|x|^5))dx \end{aligned} \tag{3.3}$$

where the coefficients are evaluated in p. Thereafter, in the right hand side of every equality or inequality that involves the terms $R^m_{\tilde{\phi}}$, $\tilde{v}^m$, $R_{\tilde{g}}$, $\tilde{R}_{ij}$, $\tilde{R}_{ijkl}$ or W_{ijkl}, these functions will be calculated at p and we will omit this point from notation.

First, we estimate in the right hand side of (3.2) the term with the Bakry-Emery scalar curvature $R^m_{\tilde{\phi}}$ in the region A_ϵ. Using the changes of variable $y = \tau^{-\frac{1}{2}}\tilde{c}(m,n)^{\frac{1}{2}}x$ we obtain

$$\begin{aligned} \int_{A_\epsilon} R^m_{\tilde{\phi}} f_\tau^2 \tilde{v}^m dV_{\tilde{g}} &\leq C(1+C\epsilon)\int_{A_\epsilon} \varphi^2_{0,\tau}dx \\ &= C(1+C\epsilon)\tau^{-\frac{n(m+n-2)}{2(m+n)}} \int_{A_\epsilon} (1 + \frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-2)}dx \\ &= C(1+C\epsilon)\tau^{-\frac{n(m+n-2)}{2(m+n)}+\frac{n}{2}} \int_{A_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}} (1+|y|^2)^{-(m+n-2)}dy \\ &\leq C(1+C\epsilon)\tau^{\frac{n}{m+n}} \int_{\epsilon\tau^{-\frac{1}{2}}\tilde{c}(m,n)^{\frac{1}{2}}}^{2\epsilon\tau^{-\frac{1}{2}}\tilde{c}(m,n)^{\frac{1}{2}}} (1+r^2)^{-(m+n-2)}r^{n-1}dr \\ &\leq C(1+C\epsilon)\epsilon^{4-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}. \end{aligned} \tag{3.4}$$

Next, we estimate in the right hand side of (3.2) the term with the Bakry-Emery curvature $R^m_{\tilde{\phi}}$ in the region B^n_ϵ. For this purpose we use the Taylor expansion around p

$$R^m_{\tilde{\phi}}(x) = R_{\tilde{g}} + (R^m_{\tilde{\phi}})_i x^i + \frac{(R^m_{\tilde{\phi}})_{ij}}{2}x^i x^j + O(|x|^3), \tag{3.5}$$

$$\tilde{v}^m(x) = 1 + \frac{(\tilde{v}^m)_{ij}}{2}x^i x^j + \frac{(\tilde{v}^m)_{ijl}}{6}x^i x^j x^l + \frac{(\tilde{v}^m)_{ijlk}}{24}x^i x^j x^l x^k + O(|x|^5) \tag{3.6}$$

where we recall that the coefficients are computed in p. Using the symmetries in the ball we have

$$\begin{aligned}\int_{B_\epsilon^n} R_{\tilde\phi}^m f_\tau^2 \tilde v^m dV_{\tilde g} &= R_{\tilde g}\int_{B_\epsilon^n}\varphi_{0,\tau}^2 dx + \frac{1}{2n}(\Delta R_{\tilde\phi}^m - \frac{1}{3}R_{\tilde g}^2)\int_{B_\epsilon^n}\varphi_{0,\tau}^2|x|^2dx \\ &+\int_{B_\epsilon^n}\varphi_{0,\tau}^2 O(|x|^4)dx.\end{aligned} \tag{3.7}$$

Let us define

$$\tilde I_1 = \int_{\mathbb{R}^n}(1+|y|^2)^{-(m+n-2)}dy \tag{3.8}$$

and

$$\tilde I_6 = \int_{\mathbb{R}^n}|y|^2(1+|y|^2)^{-(m+n-2)}dy. \tag{3.9}$$

Then

$$\begin{aligned}\int_{B_\epsilon^n}\varphi_{0,\tau}^2 dx &= \tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{B_\epsilon^n}(1+\frac{\tilde c(m,n)}{\tau}|x|^2)^{-(m+n-2)}dx \\ &= \frac{\tau^{-\frac{n(m+n-2)}{2(m+n)}+\frac{n}{2}}}{c(m,n)^{\frac{n}{2}}}\int_{B^n_{\frac{\epsilon\sqrt{\tilde c(m,n)}}{\sqrt\tau}}}(1+|y|^2)^{-(m+n-2)}dy \\ &= \frac{\tau^{\frac{n}{m+n}}}{c(m,n)^{\frac{n}{2}}}(\tilde I_1 - \int_{\mathbb{R}^n\setminus B^n_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt\tau}}}(1+|y|^2)^{-(m+n-2)}dy) \\ &= \frac{\tau^{\frac{n}{m+n}}}{\tilde c(m,n)^{\frac{n}{2}}}\tilde I_1 + O(\epsilon^{4-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}).\end{aligned} \tag{3.10}$$

and similarly

$$\begin{aligned}\int_{B_\epsilon^n}\varphi_{0,\tau}^2|x|^2dx &= \tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{B_\epsilon^n}(1+\frac{\tilde c(m,n)}{\tau}|x|^2)^{-(m+n-2)}|x|^2dx \\ &= \frac{\tau^{\frac{n}{m+n}+1}}{\tilde c(m,n)^{\frac{n+2}{2}}}\tilde I_6 + O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}).\end{aligned} \tag{3.11}$$

Now, taking $q<\min\{2m+n-6,1\}$ and $0<\epsilon<1$ then for $|x|\leq\epsilon$ we get $|x|^q > |x|^2$ and

$$\begin{aligned}\int_{B_\epsilon^n}\varphi_{0,\tau}^2|x|^4dx &\leq \int_{B_\epsilon^n}\varphi_{0,\tau}^2|x|^{2+q}dx\\ &\leq \tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{B_\epsilon^n}\frac{|x|^{2+q}}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n-2}}dx\\ &\leq C\tau^{-\frac{n(m+n-2)}{2(m+n)}+\frac{n}{2}+1+\frac{q}{2}}\int_{B^n_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}}\frac{|y|^2}{(1+|y|^2)^{(m+n-2)}}dy\\ &\leq C\tau^{\frac{n}{m+n}+1+\frac{q}{2}}(C+C\int_1^\infty r^{6-2m-n+q-1}dr)\\ &\leq C\tau^{\frac{n}{m+n}+1+\frac{q}{2}}.\end{aligned} \tag{3.12}$$

Estimates (3.4), (3.7), (3.10) and (3.12) lead to

$$\begin{aligned}\int_{B_{2\epsilon}^n}R_{\tilde{\phi}}^m f_\tau^2\tilde{v}^m dV_{\tilde{g}} &= R_{\tilde{g}}\frac{\tau^{\frac{n}{m+n}}}{\tilde{c}(m,n)^{\frac{n}{2}}}\tilde{I}_1+(\Delta R_{\tilde{\phi}}^m-\tfrac{1}{3}R_{\tilde{g}}^2)\frac{\tau^{\frac{n}{m+n}+1}\tilde{I}_6}{2n\tilde{c}(m,n)^{\frac{n+2}{2}}}\\ &\quad +O(\tau^{\frac{n}{m+n}+1+\frac{q}{2}})+O(\epsilon^{4-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}).\end{aligned} \tag{3.13}$$

To estimate the gradient term in A_ϵ note that, in $\tilde{g}$-normal coordinates, the term $|\nabla f|_{\tilde{g}}^2$ in A_ϵ satisfy the following inequality

$$|\nabla f_\tau|_{\tilde{g}}^2\leq C|\nabla f_\tau|^2\leq C(\eta^2|\nabla\varphi_{0,\tau}|^2+|\nabla\eta|^2\varphi_{0,\tau}^2). \tag{3.14}$$

Also, we obtain

$$\begin{aligned}\int_{A_\epsilon}|\nabla\eta|^2\varphi_{0,\tau}^2dV_{\tilde{g}} &\leq C(1+\epsilon C)\epsilon^{-2}\int_{A_\epsilon}\varphi_{0,\tau}^2dx\\ &= C(1+C\epsilon)\epsilon^{-2}\tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{A_\epsilon}(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-2)}dx\\ &= C(1+C\epsilon)\tau^{\frac{n}{m+n}}\int_{A_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}(1+|y|^2)^{-(m+n-2)}dy\\ &\leq C(1+C\epsilon)\epsilon^{2-n-2m}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\int_{A_\epsilon}\eta^2|\nabla\varphi_{0,\tau}|^2dV_{\tilde{g}} &\leq (1+\epsilon C)\int_{A_\epsilon}|\nabla\varphi_{0,\tau}|^2dx\\ &= C(1+C\epsilon)\tau^{-\frac{n(m+n-2)}{2(m+n)}-2}\int_{A_\epsilon}(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n)}|x|^2dx\\ &= C(1+C\epsilon)\tau^{\frac{n}{m+n}-1}\int_{A_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}(1+|y|^2)^{-(m+n)}|y|^2dy\\ &\leq C(1+C\epsilon)\epsilon^{2-n-2m}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}.\end{aligned} \tag{3.16}$$

Then

$$\int_{A_\epsilon}|\nabla f_\tau|^2_{\tilde{g}}dxdt \quad = O(\epsilon^{2-n-2m}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}). \tag{3.17}$$

Next, we estimate the integral with the gradient term in B^n_ϵ in equality (3.2). For this purpose we note that in $\tilde{g}$-normal coordinates around p, we get

$$\begin{aligned}\tilde{g}^{ij} &= \delta_{ij}-\tfrac{1}{3}\tilde{R}_{iklj}x^kx^l-\tfrac{1}{6}\tilde{R}_{iklj,s}x^kx^lx^s\\ &-(\tfrac{1}{20}\tilde{R}_{iklj,su}-\tfrac{3}{45}\tilde{R}_{iklr}\tilde{R}_{jsur})x^kx^lx^sx^u+O(|x|^5)\end{aligned} \tag{3.18}$$

where $1\leq i,j,k,l,r,s,u\leq n$, again and for the last time we recall that the coefficients are computed in p. Then using the symmetries in the ball, the Taylor expansion (3.6), $\nabla_{\tilde{g}}\tilde{v}^m(p)=0$ and $\Delta_{\tilde{g}}\tilde{v}^m(p)=0$ it follows that

$$
\begin{aligned}
\int_{B_\epsilon}|\nabla f_\tau|^2_{\tilde{g}}v^m dV_{\tilde{g}} &= \int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2dx - \frac{R_{\tilde{g}}}{6n}\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2|x|^2dx\\
&-\frac{1}{3}\tilde{R}_{iklj}\int_{B^n_\epsilon}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})x^kx^ldx\\
&-(\frac{1}{6}\tilde{R}_{iklj}(\tilde{v}^m)_{su}-\frac{1}{18}\tilde{R}_{iklj}\tilde{R}_{su})\int_{B^n_\epsilon}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})x^kx^lx^sx^udx\\
&-(\frac{1}{20}\tilde{R}_{iklj,su}-\frac{3}{45}\tilde{R}_{iklr}\tilde{R}_{rsuj})\int_{B^n_\epsilon}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})x^kx^lx^sx^udx\\
&-\frac{1}{12}(\tilde{v}^m)_{ij}\tilde{R}_{kl}\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&+\frac{1}{24}(\tilde{v}^m)_{ijkl}\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&-(\frac{1}{40}\tilde{R}_{ij,kl}+\frac{1}{180}\tilde{R}_{rijs}\tilde{R}_{rkls}-\frac{1}{72}\tilde{R}_{ij}\tilde{R}_{kl})\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&+\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2O(|x|^6)dx.
\end{aligned}
\tag{3.19}
$$

For the second integral in (3.19) we have

$$
\begin{aligned}
\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2|x|^2dx &= \tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{B^n_\epsilon}\frac{|x|^4\tilde{c}(m,n)^2(m+n-2)^2}{\tau^2(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx\\
&= \tau^{\frac{n}{m+n}}\frac{(m+n-2)^2}{\tilde{c}(m,n)^{\frac{n}{2}}}\int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}|y|^4(1+|y|^2)^{-(m+n)}dy\\
&= \tau^{\frac{n}{m+n}}\frac{(m+n-2)^2}{\tilde{c}(m,n)^{\frac{n}{2}}}\tilde{I}_2+O(\epsilon^{4-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}})
\end{aligned}
\tag{3.20}
$$

where

$$
\tilde{I}_2=\int_{\mathbb{R}^n}|y|^4(1+|y|^2)^{-(m+n)}dy. \tag{3.21}
$$

Using the symmetries of the Riemann curvature tensor and Lemma 3 we get

$$\begin{aligned}\frac{\int_{B_\epsilon^n}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})\tilde{R}_{iklj}(0)x^kx^ldx}{(m+n-2)^2\tilde{c}(m,n)^2\tau^{-\frac{n(m+n-2)}{2(m+n)}-2}} &= \int_{B_\epsilon^n}\frac{\tilde{R}_{iklj}x^ix^jx^kx^l}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx \\ = \int_{B_\epsilon^n}\frac{\sum\limits_{i=1}^n\tilde{R}_{iiii}x_i^4+\sum\limits_{i\neq j}^n(\tilde{R}_{iijj}+\tilde{R}_{ijij}+\tilde{R}_{ijji})x_i^2x_j^2}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx &= 0.\end{aligned} \tag{3.22}$$

Again, symmetries of the Riemann curvature tensor yield

$$(\tilde{R}_{iklj}(0)(\tilde{v}^m)_{su}(0)-\tfrac{1}{3}\tilde{R}_{iklj}(0)\tilde{R}_{su}(0))\int_{B_\epsilon^n}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})x^kx^lx^sx^udx=0 \tag{3.23}$$

and

$$(\tfrac{1}{20}\tilde{R}_{iklj,su}(0)-\tfrac{3}{45}\tilde{R}_{iklr}(0)\tilde{R}_{rsuj}(0))\int_{B_\epsilon^n}(\partial_i\varphi_{0,\tau})(\partial_j\varphi_{0,\tau})x^kx^lx^sx^udx=0. \tag{3.24}$$

In order to compute the sixth integral in the right hand side of (3.19) we will use Lemma 3 and the symmetries of the ball, which imply

$$\begin{aligned}&\frac{\tau^{\frac{n(m+n-2)}{2(m+n)}+2}\tilde{R}_{ij}(0)(\tilde{v}^m)_{kl}(0)}{(m+n-2)^2\tilde{c}(m,n)^2}\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx \\ &= \int_{B_\epsilon^n}\frac{\tilde{R}_{ij}(\tilde{v}^m)_{kl}|x|^2x^ix^jx^kx^l}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx \\ &= \int_{B_\epsilon^n}\frac{\sum\limits_{i=1}^n\tilde{R}_{ii}(\tilde{v}^m)_{ii}x_i^4+(\sum\limits_{i\neq j}^n\tilde{R}_{ii}(\tilde{v}^m)_{jj}+\tilde{R}_{ij}(\tilde{v}^m)_{ij}+\tilde{R}_{ij}(\tilde{v}^m)_{ji}x_i^2x_j^2}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}|x|^{-2}}dx \\ &= \frac{1}{3}\sum_{i,j=1}^n(\tilde{R}_{ii}(\tilde{v}^m)_{jj}+\tilde{R}_{ij}(\tilde{v}^m)_{ij}+\tilde{R}_{ij}(\tilde{v}^m)_{ji})\int_{B_\epsilon^n}\frac{|x|^2x_i^4}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx \\ &= \frac{(\tilde{R}_{ii}(\tilde{v}^m)_{jj}+\tilde{R}_{ij}(\tilde{v}^m)_{ij}+\tilde{R}_{ij}(\tilde{v}^m)_{ji})}{n(n+2)}\int_{B_\epsilon^n}\frac{|x|^6}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx \\ &= \frac{R_{\tilde{g}}\Delta\tilde{v}^m+2\langle\text{Hess}\,\tilde{v}^m,Ric\rangle}{n(n+2)}\int_{B_\epsilon^n}\frac{|x|^6}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx.\end{aligned} \tag{3.25}$$

Since $\Delta_{\tilde{g}}\tilde{v}^m(p)=0$, we obtain

$$
\begin{aligned}
&\tilde{R}_{ij}(0)(\tilde{v}^m)_{kl}(0)\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&=\frac{2(m+n-2)^2(\langle \mathrm{Hess}\,\tilde{v}^m, Ric\rangle)}{n(n+2)\tilde{c}(m,n)^{-2}\tau^{\frac{n(m+n-2)}{2(m+n)}+2}}\int_{B_\epsilon^n}\frac{|x|^6}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx\\
&=\frac{2\tau^{\frac{n}{m+n}+1}(m+n-2)^2\langle \mathrm{Hess}\,\tilde{v}^m, Ric\rangle}{\tilde{c}(m,n)^{\frac{n+2}{2}}}\int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}\frac{|y|^6}{(1+|y|^2)^{m+n}}dy\\
&=\frac{2\tau^{\frac{n}{m+n}+1}(m+n-2)^2\langle \mathrm{Hess}\,\tilde{v}^m, Ric\rangle}{n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7+O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}})
\end{aligned}
\tag{3.26}
$$

where

$$
\tilde{I}_7=\int_{\mathbb{R}^n}|y|^6(1+|y|^2)^{-(m+n)}dy. \tag{3.27}
$$

A similar argument implies that

$$
\begin{aligned}
&(\tilde{v}^m)_{ijkl}(0)\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&=\frac{3\tau^{\frac{n}{m+n}+1}(m+n-2)^2\Delta^2\tilde{v}^m}{n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7+O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}),
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
&R_{ij,kl}(0)\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&=\frac{2\tau^{\frac{n}{m+n}+1}(m+n-2)^2(\tilde{R}_{ii,jj}+\tilde{R}_{ij,ij}+\tilde{R}_{ij,ji})}{n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7+O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}})\\
&=\frac{2\tau^{\frac{n}{m+n}+1}(m+n-2)^2\Delta R_{\tilde{g}}}{n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7+O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}),
\end{aligned}
\tag{3.29}
$$

$$\begin{aligned}
&\tilde{R}_{rijs}(0)\tilde{R}_{rkls}(0)\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx\\
&= \frac{\tau^{\frac{n}{m+n}+1}(m+n-2)^2(\tilde{R}_{rs}\tilde{R}_{rs}+\tilde{R}_{irsj}\tilde{R}_{irsj}+\tilde{R}_{irsj}\tilde{R}_{isrj})}{\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7\\
&\quad +O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}})\\
&= \frac{\tau^{\frac{n}{m+n}+1}(m+n-2)^2(|Ric_{\tilde{g}}|^2+\frac{3}{2}\tilde{R}_{irsj}\tilde{R}_{irsj})}{\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7+O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}})
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
\tilde{R}_{ij}(0)\tilde{R}_{kl}(0)\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2x^ix^jx^kx^ldx &= \frac{\tau^{\frac{n}{m+n}+1}(m+n-2)^2(R^2_{\tilde{g}}+2R_{ij}R_{ij})}{n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_7\\
&\quad +O(\epsilon^{6-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}}).
\end{aligned} \tag{3.31}$$

We used the contraction of Bianchi's identity $R_{\tilde{g}},_i=2\tilde{R}^j_{\ i,j}$ and the identity $\tilde{R}_{ijkl}\tilde{R}^{ijkl}=\frac{1}{2}\tilde{R}_{ijkl}\tilde{R}^{ilkj}$ in equalities (3.29) and (3.30), respectively. For the last integral in the right hand side of (3.19), taking $q<\min\{2m+n-6,1\}$ and $\epsilon<1$ as in (3.12), we get

$$\begin{aligned}
\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2|x|^6dx &\leq \textstyle\int_{B^n_\epsilon}|\nabla\varphi_{0,\tau}|^2|x|^{4+q}dx\\
&\leq C\tau^{-\frac{n(m+n-2)}{2(m+n)}}\int_{B^n_\epsilon}\frac{|x|^{4+q}}{\tau^2(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n}}dx\\
&\leq C\tau^{\frac{n}{m+n}+1+\frac{q}{2}}\int_{B^n_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}|y|^{2+q}(1+|y|^2)^{-(m+n)}dy\\
&\leq C\tau^{\frac{n}{m+n}+1+\frac{q}{2}}.
\end{aligned} \tag{3.32}$$

Equalities (3.17), (3.19), (3.20), (3.22), (3.23), (3.24), (3.26), (3.28), (3.29), (3.30), (3.31) and inequality (3.32) lead to

$$\begin{aligned}\int_{B_{2\epsilon}} |\nabla f_\tau|_g^2 v^m dV_{\tilde{g}} &= \int_{B_\epsilon^n} |\nabla \varphi_{0,\tau}|^2 dx + \tau^{\frac{n}{m+n}} \frac{(m+n-2)^2}{\tilde{c}(m,n)^{\frac{n}{2}}} (\frac{\Delta \tilde{v}^m}{2n} - \frac{\tilde{R}_g}{6n}) \tilde{I}_2 \\ &- \frac{\tau^{\frac{n}{m+n}+1}(m+n-2)^2}{2\tilde{c}(m,n)^{\frac{n+2}{2}} n(n+2)} \tilde{I}_7 A_1 \\ &+ O(\tau^{\frac{n}{m+n}+m+\frac{n-4}{2}} \epsilon^{2-2m-n}) + O(\tau^{\frac{n}{m+n}+1+\frac{q}{2}})\end{aligned} \tag{3.33}$$

where

$$A_1 := \frac{\langle \mathrm{Hess}\, \tilde{v}^m, Ric \rangle}{3} - \frac{\Delta^2 \tilde{v}^m}{4} + \frac{\Delta R_{\tilde{g}}}{10} - \frac{R_{\tilde{g}}^2}{36} - \frac{2|Ric_{\tilde{g}}|^2}{45} + \frac{R_{iklj} R_{iklj}}{60}. \tag{3.34}$$

In order to analyze the term A_1, we use Aubin's ideas and the following identities (see [1])

$$\tilde{T}_{ij} = \tilde{R}_{ij} - \frac{\tilde{R}}{n} g_{ij}, \tag{3.35}$$

$$\tilde{R}_{ijkl} \tilde{R}^{ijkl} = \tilde{W}_{ijkl} \tilde{W}^{ijkl} + \frac{4}{n-2} \tilde{T}_{ij} \tilde{T}^{ij} + \frac{2R_{\tilde{g}}^2}{n(n-1)} \tag{3.36}$$

and

$$\tilde{R}_{ij} \tilde{R}^{ij} = \tilde{T}_{ij} \tilde{T}^{ij} + \frac{R_{\tilde{g}}^2}{n}. \tag{3.37}$$

Then

$$\begin{aligned}A_1 = &\frac{\langle \mathrm{Hess}\, \tilde{v}^m, Ric \rangle}{3} - \frac{\Delta^2 \tilde{v}^m}{4} + \frac{\Delta R_{\tilde{g}}}{10} \\ &- \frac{(5n^2+3n-14) R_{\tilde{g}}^2}{180 n(n-1)} - \frac{(2n-7)|T_{\tilde{g}}|^2}{45(n-2)} - \frac{W_{iklj} W_{iklj}}{60}.\end{aligned} \tag{3.38}$$

Next, we analyze the last integral in the right hand side of (3.2) in the region A_ϵ

$$
\begin{aligned}
&\int_{A_\epsilon} f_\tau^{\frac{2(m+n-1)}{m+n-2}} \tilde{v}^{m-1} dV_{\tilde{g}} \leq C(1+\epsilon C)\int_{A_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dx \\
&\leq C(1+\epsilon C)\tau^{-\frac{n(m+n-1)}{2(m+n)}} \int_{A_\epsilon} (1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-1)} dx \\
&\leq C(1+\epsilon C)\tau^{\frac{n}{2(m+n)}} \int_{A_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}} (1+|y|^2)^{-(m+n-1)} dy \\
&\leq C(1+\epsilon C)\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}} \epsilon^{2-2m-n}.
\end{aligned} \tag{3.39}
$$

In order to estimate the last integral in (3.2) in the region B^n_ϵ we use the Taylor expansion around p for $\tilde{v}^{m-1}$ and the symmetries in the ball to obtain

$$
\begin{aligned}
\int_{B^n_\epsilon} f_\tau^{\frac{2(m+n-1)}{m+n-2}} \tilde{v}^{m-1} dV_{\tilde{g}} &= \int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dx - \frac{R_{\tilde{g}}}{6n}\int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} |x|^2 dx \\
&-\frac{1}{12}(\tilde{R}_{ij}(\tilde{v}^{m-1})_{kl} - \frac{1}{2}(\tilde{v}^{m-1})_{ijkl})\int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} x^i x^j x^k x^l dx \\
&-(\frac{1}{40}\tilde{R}_{ij,kl} + \frac{1}{180}\tilde{R}_{rijs}\tilde{R}_{rkls})\int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} x^i x^j x^k x^l dx \\
&+\int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} O(|x|^6) dx.
\end{aligned} \tag{3.40}
$$

Now, we deal with the second term in right hand side of (3.40)

$$
\begin{aligned}
\int_{B^n_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} |x|^2 dx &= \tau^{-\frac{n(m+n-1)}{2(m+n)}} \int_{B^n_\epsilon} |x|^2 (1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-1)} dx \\
&= \frac{\tau^{\frac{n}{2(m+n)}+1}}{\tilde{c}(m,n)^{\frac{n+2}{2}}} \int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}} |y|^2 (1+|y|^2)^{-(m+n-1)} dy \\
&= \frac{\tau^{\frac{n}{2(m+n)}+1}}{\tilde{c}(m,n)^{\frac{n+2}{2}}} \tilde{I}_3 + O(\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}} \epsilon^{4-2m-n})
\end{aligned} \tag{3.41}
$$

where

$$
\tilde{I}_3 = \int_{\mathbb{R}^n} |y|^2 (1+|y|^2)^{-(m+n-1)} dy. \tag{3.42}
$$

For the third term in the right hand side of (3.40), using $\Delta_{\tilde{g}}\tilde{v}^{m-1}(p)=0$ and Lemma 3 in a similar argument like in (3.25), we get

$$\begin{aligned}&\tilde{R}_{ij}(0)(\tilde{v}^{m-1})_{kl}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^l dx\\&=\frac{2\langle \text{Hess}\,\tilde{v}^{m-1},Ric_{\tilde{g}}\rangle}{n(n+2)\tau^{\frac{n(m+n-1)}{2(m+n)}}}\int_{B^n_\epsilon}\frac{|x|^4}{(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{m+n-1}}dx.\end{aligned}\tag{3.43}$$

Let us define

$$\tilde{I}_8=\int_{\mathbb{R}^n}|y|^4(1+|y|^2)^{-(m+n-1)}dy.\tag{3.44}$$

It follows from (3.43) that

$$\begin{aligned}\tilde{R}_{ij}(0)(\tilde{v}^{m-1})_{kl}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^l dx &=\frac{2\tau^{\frac{n}{2(m+n)}+2}\langle \text{Hess}\,\tilde{v}^{m-1},Ric_{\tilde{g}}\rangle}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}\tilde{I}_8\\&+O(\epsilon^{6-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}}).\end{aligned}\tag{3.45}$$

Also, we get

$$\begin{aligned}&(\tilde{v}^{m-1})_{ijkl}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^l dx\\&=\frac{3\tau^{\frac{n}{2(m+n)}+2}\Delta^2\tilde{v}^{m-1}}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}\tilde{I}_8+O(\epsilon^{6-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}}),\end{aligned}\tag{3.46}$$

$$\begin{aligned}&\tilde{R}_{ij,kl}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^l dx\\&=\frac{2\tau^{\frac{n}{2(m+n)}+2}\Delta R_{\tilde{g}}}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}\tilde{I}_8+O(\epsilon^{6-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}}),\end{aligned}\tag{3.47}$$

$$\begin{aligned}\tilde{R}_{rijs}(0)\tilde{R}_{rkls}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^l dx &=\frac{\tau^{\frac{n}{2(m+n)}+2}(|Ric_{\tilde{g}}|^2+\frac{3\tilde{R}_{rijs}\tilde{R}_{rijs}}{2})}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}\tilde{I}_8\\&+O(\epsilon^{6-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}})\end{aligned}\tag{3.48}$$

and

$$\tilde{R}_{ij}(0)\tilde{R}_{kl}(0)\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}x^ix^jx^kx^ldx = \frac{\tau^{\frac{n}{2(m+n)}+2}(R^2_{\tilde{g}}+2|Ric_{\tilde{g}}|^2)}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}\tilde{I}_8 +O(\epsilon^{6-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}}). \tag{3.49}$$

We used the contraction of Bianchi's identity $R_{\tilde{g}} = 2\tilde{R}^j{}_{i,j}$ and the identity $\tilde{R}_{ijkl}\tilde{R}^{ijkl} = \frac{1}{2}\tilde{R}_{ijkl}\tilde{R}^{ilkj}$ in equalities (3.47) and (3.48), respectively. On the other hand, since $\epsilon < 1$ and choose $0 < q < \min\{2m+n-6, 1\}$ then the last term in the right hand side of (3.40) is estimated as follows

$$\begin{aligned}\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}|x|^6dx &\leq \tau^{-\frac{n(m+n-1)}{2(m+n)}}\int_{B^n_\epsilon}|x|^{4+q}(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-1)}dx\\ &\leq C\tau^{\frac{n}{2(m+n)}+2+\frac{q}{2}}\int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}|y|^{2+q}(1+|y|^2)^{-(m+n-1)}dy\\ &\leq C\tau^{\frac{n}{2(m+n)}+2+\frac{q}{2}}.\end{aligned} \tag{3.50}$$

The estimates (3.39), (3.40), (3.41), (3.45), (3.46), (3.47), (3.48), (3.49) and (3.50) yield

$$\begin{aligned}\int_{B_{2\epsilon}}f_\tau^{\frac{2(m+n-1)}{m+n-2}}\tilde{v}^{m-1}dV_{\tilde{g}} &= \int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}dx - \frac{\tau^{\frac{n}{2(m+n)}+1}\tilde{R}_g}{6n\tilde{c}(m,n)^{\frac{n+2}{2}}}\tilde{I}_3\\ &-\frac{\tau^{\frac{n}{2(m+n)}+2}\tilde{I}_8}{2n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}}A_2\\ &+O(\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}}\epsilon^{4-2m-n})+O(\tau^{\frac{n}{2(m+n)}+2+\frac{q}{2}})\end{aligned} \tag{3.51}$$

where

$$\begin{aligned}A_2 &:= \frac{\langle \mathrm{Hess}\,\tilde{v}^{m-1}, Ric\rangle}{3} - \frac{\Delta^2\tilde{v}^{m-1}}{4} + \frac{\Delta R_{\tilde{g}}}{10}\\ &-\frac{(5n^2+3n-14)R^2_{\tilde{g}}}{180n(n-1)} - \frac{(2n-7)|T_{\tilde{g}}|^2}{45(n-2)} - \frac{W_{iklj}W_{iklj}}{60}.\end{aligned} \tag{3.52}$$

Now, we analyze the behavior of $\tilde{V}_\tau$ when τ is near to zero

$$\begin{aligned}\tilde{V}-\tilde{V}_{\tau} &= \int_{\mathbb{R}^n\setminus B_{2\epsilon}} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx + (\int_{A_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx - \int_{A_\epsilon} f_{0,\tau}^{\frac{2(m+n)}{m+n-2}}\tilde{v}^m dV_{\tilde{g}}) \\ &+(\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx - \int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}\tilde{v}^m dV_{\tilde{g}}).\end{aligned} \tag{3.53}$$

For the first integral in the right hand side of (3.53) we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{2\epsilon}} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx &= \tau^{-\frac{n}{2}}\int_{\mathbb{R}^n\setminus B_{2\epsilon}} (1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n)}dx \\ &= \tilde{c}(m,n)^{-\frac{n}{2}}\int_{\mathbb{R}^n\setminus B_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} (1+|y|^2)^{-(m+n)}dy \\ &\leq C\epsilon^{-n-2m}\tau^{m+\frac{n}{2}}.\end{aligned} \tag{3.54}$$

Using the expansion for the volume form (3.3) and that $\tilde{v}$ is bounded we have in the second integral in the right hand side of (3.53) that

$$\begin{aligned}\left|\int_{A_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx - \int_{A_\epsilon} f_{0,\tau}^{\frac{2(m+n)}{m+n-2}}\tilde{v}^m dV_{\tilde{g}}\right| &\leq C(1+C\epsilon)\int_{A_\epsilon} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx \\ &\leq C(1+C\epsilon)\epsilon^{-n-2m}\tau^{m+\frac{n}{2}}.\end{aligned} \tag{3.55}$$

By the expansion for the volume form (3.3), the Taylor expansion around p for $\tilde{v}^m$ and the symmetries of the ball in the third integral in the right hand side of (3.53) we get

$$\begin{aligned}\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}dx - \int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}\tilde{v}^m dV_{\tilde{g}} &= \frac{R_{\tilde{g}}}{6n}\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}|x|^2dx \\ +\frac{1}{12}(\tilde{R}_{ij}(\tilde{v}^m)_{kl} - \frac{1}{2}(\tilde{v}^m)_{ijkl})&\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}x^ix^jx^kx^l dx \\ +(\frac{1}{40}\tilde{R}_{ij,kl} + \frac{1}{180}\tilde{R}_{rijs}\tilde{R}_{rkls})\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}x^ix^jx^kx^l dx &+ \int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}}O(|x|^6)dx.\end{aligned} \tag{3.56}$$

To analyze (3.56), we consider the first integral on its right hand side

$$\begin{aligned}\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}} |x|^2 dx &= \tau^{-\frac{n}{2}} \int_{B_\epsilon^n} |x|^2 (1 + \frac{\tilde{c}(m,n)}{\tau} |x|^2)^{-(m+n)} dx \\ &= \frac{\tau}{c(m,n)^{\frac{n+2}{2}}} \int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}} |y|^2 (1+|y|^2)^{-(m+n)} dy \\ &= \frac{\tau}{c(m,n)^{\frac{n+2}{2}}} \tilde{I}_4 + O(\epsilon^{2-2m-n} \tau^{m+\frac{n}{2}})\end{aligned} \tag{3.57}$$

where

$$\tilde{I}_4 = \int_{\mathbb{R}^n} |y|^2 (1+|y|^2)^{-(m+n)} dy. \tag{3.58}$$

For the last integral in the right hand side of (3.56), recalling that $\epsilon < 1$ and $q < \min\{2m-n-4, 1\}$, we obtain

$$\begin{aligned}\int_{B_\epsilon^n} \varphi_{0,\tau}^{\frac{2(m+n)}{m+n-2}} |x|^6 dx &\leq \tau^{-\frac{n}{2}} \int_{B_\epsilon^n} |x|^{4+q} (1 + \frac{\tilde{c}(m,n)}{\tau} |x|^2)^{-(m+n)} dx \\ &\leq C\tau^{1+\frac{q}{2}} \int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}} |y|^{4+q} (1+|y|^2)^{-(m+n)} dy \\ &\leq C\tau^{2+\frac{q}{2}}.\end{aligned} \tag{3.59}$$

Equalities (3.53), (3.56), (3.57); inequalities (3.54), (3.55), (3.59) and similar arguments like we used in (3.40) to (3.51) lead to

$$\begin{aligned}\tilde{V} - \tilde{V}_\tau &= \frac{\tau R_{\tilde{g}}}{6n\tilde{c}(m,n)^{\frac{n+2}{2}}} \tilde{I}_4 + \frac{\tau^2 \tilde{I}_2 A_1}{n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}} \\ &+ O(\tau^{m+\frac{n}{2}} \epsilon^{-2m-n}) + O(\tau^{2+\frac{q}{2}}).\end{aligned} \tag{3.60}$$

Follows that the terms $\tilde{V}_\tau$ are uniformly bounded away from zero. Using estimate (3.60) and Taylor expansion for the functions $x^{-\frac{m+n-2}{m+n}}$ and $x^{-\frac{m+n-1}{m+n}}$ we obtain

$$\begin{aligned}\tilde{V}_{\tau}^{-\frac{m+n-2}{m+n}} &= \tilde{V}^{-\frac{m+n-2}{m+n}} + \frac{\tau(m+n-2)R_{\tilde{g}}}{6n(m+n)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}^{\frac{2m+2n-2}{m+n}}}\tilde{I}_4 \\ &+\frac{\tau^2(m+n-2)\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{\frac{n+4}{2}}\tilde{V}^{\frac{2m+2n-2}{m+n}}} \\ &+\frac{\tau^2(m+n-2)(m+n-1)R_{\tilde{g}}^2(\tilde{I}_4)^2}{36n^2(m+n)^2\tilde{c}(m,n)^{n+2}\tilde{V}^{\frac{3m+3n-2}{m+n}}} + O(\tau^{m+\frac{n}{2}}\epsilon^{-2m-n}) + O(\tau^{2+\frac{q}{2}})\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}\tilde{V}_{\tau}^{-\frac{m+n-1}{m+n}} &= \tilde{V}^{-\frac{m+n-1}{m+n}} + \frac{\tau(m+n-1)R_{\tilde{g}}}{6n(m+n)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}^{\frac{2m+2n-1}{m+n}}}\tilde{I}_4 \\ &+\frac{\tau^2(m+n-1)\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{\frac{n+4}{2}}\tilde{V}^{\frac{2m+2n-1}{m+n}}} \\ &+\frac{\tau^2(m+n-1)(2m+2n-1)R_{\tilde{g}}^2(\tilde{I}_4)^2}{72n^2(m+n)^2\tilde{c}(m,n)^{n+2}\tilde{V}^{\frac{3m+3n-1}{m+n}}} \\ &+O(\tau^{m+\frac{n}{2}}\epsilon^{-2m-n}) + O(\tau^{2+\frac{q}{2}}).\end{aligned} \tag{3.62}$$

On the other hand, we get

$$\begin{aligned}\frac{\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2dx}{(m+n-2)^2} &= \tau^{-\frac{n(m+n-2)}{2(m+n)}-2}\tilde{c}(m,n)^2\int_{B_\epsilon^n}|x|^2(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n)}dx \\ &= \frac{\tau^{\frac{n}{m+n}-1}}{\tilde{c}(m,n)^{\frac{n-2}{2}}}\int_{B^n_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}|y|^2(1+|y|^2)^{-(m+n)}dy \\ &= \frac{\tau^{\frac{n}{m+n}-1}}{\tilde{c}(m,n)^{\frac{n-2}{2}}}\tilde{I}_4 + O(\epsilon^{2-2m-n}\tau^{\frac{n}{m+n}+m+\frac{n-6}{2}}).\end{aligned} \tag{3.63}$$

Using equalities $\tilde{\tau} = \tau\tilde{V}^{\frac{-2}{2m+n}}$, (3.33), (3.61) and (3.63) it follows that

$$\begin{aligned}
\frac{\tilde{\tau}^{\frac{m}{m+n}}}{\tilde{V}_\tau^{\frac{m+n-2}{m+n}}}\int_{B_{2\epsilon}}|\nabla f_\tau|^2 dV_{\tilde{g}} &= \frac{\tilde{\tau}^{\frac{m}{m+n}}}{\tilde{V}^{\frac{m+n-2}{m+n}}}\int_{B_\epsilon^n}|\nabla\varphi_{0,\tau}|^2 dx \\
&+\frac{\tau(m+n-2)^3 R_{\tilde{g}}(\tilde{I}_4)^2}{6n(m+n)\tilde{c}(m,n)^n\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{2m+2n-2}{m+n}}} \\
&+\frac{\tau^2(m+n-2)^3\tilde{I}_4\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{n+1}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{2m+2n-2}{m+n}}} \\
&+\frac{\tau^2(m+n-2)^3(m+n-1)R_{\tilde{g}}^2(\tilde{I}_4)^3}{36n^2(m+n)^2\tilde{c}(m,n)^{\frac{3n+2}{2}}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{3m+3n-1}{m+n}}} \\
&-\frac{\tau(m+n-2)^2 R_{\tilde{g}}\tilde{I}_2}{6n\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{m+n-2}{m+n}}} \\
&-\frac{\tau^2(m+n-2)^3 R_{\tilde{g}}^2\tilde{I}_2\tilde{I}_4}{72n^2(m+n)\tilde{c}(m,n)^{n+1}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{2m+2n-2}{m+n}}} \\
&-\frac{\tau^2(m+n-2)^2\tilde{I}_7A_1}{2n(n+2)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{m+n-2}{m+n}}} \\
&+O(\tau^{m+\frac{n}{2}}\epsilon^{-2m-n})+O(\tau^{2+\frac{q}{2}}).
\end{aligned} \tag{3.64}$$

Similarly equalities $\tilde{\tau}=\tau\tilde{V}^{\frac{-2}{2m+n}}$, (3.13) and (3.61) yield

$$\begin{aligned}
\frac{\tilde{\tau}^{\frac{m}{m+n}}}{\tilde{V}_\tau^{\frac{m+n-2}{m+n}}}\int_{B_{2\epsilon}^n}R_{\tilde{\phi}}^m f_\tau^2\tilde{v}^m dV_g &= \frac{\tau R_{\tilde{g}}\tilde{I}_1}{\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{m+n-2}{m+n}}} \\
&+\frac{\tau^2(m+n-2)R_{\tilde{g}}^2\tilde{I}_1\tilde{I}_4}{6n(m+n)\tilde{c}(m,n)^{n+1}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{2m+2n-2}{m+n}}} \\
&+\frac{\tau^2(\Delta R_{\tilde{\phi}}^m-\frac{1}{3}R_{\tilde{g}}^2)\tilde{I}_6}{2n\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}^{\frac{2m}{(m+n)(2m+n)}+\frac{m+n-2}{m+n}}} \\
&+O(\tau^{m+\frac{n}{2}}\epsilon^{-2m-n})+O(\tau^{2+\frac{q}{2}}).
\end{aligned} \tag{3.65}$$

Now, we obtain

$$\begin{aligned}\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}dx &= \tau^{-\frac{n(m+n-1)}{2(m+n)}}\int_{B^n_\epsilon}(1+\frac{\tilde{c}(m,n)}{\tau}|x|^2)^{-(m+n-1)}dx\\ &= \frac{\tau^{\frac{n}{2(m+n)}}}{\tilde{c}(m,n)^{\frac{n}{2}}}\int_{B_{\frac{\epsilon\sqrt{\tilde{c}(m,n)}}{\sqrt{\tau}}}}(1+|y|^2)^{-(m+n-1)}dy \qquad (3.66)\\ &= \frac{\tau^{\frac{n}{2(m+n)}}}{\tilde{c}(m,n)^{\frac{n}{2}}}\tilde{I}_5+O(\epsilon^{2-2m-n}\tau^{\frac{n}{2(m+n)}+m+\frac{n-2}{2}})\end{aligned}$$

where

$$\tilde{I}_5=\int_{\mathbb{R}^n}(1+|y|^2)^{-(m+n-1)}dy. \qquad (3.67)$$

Also, equalities $\tilde{\tau}=\tau\tilde{V}^{\frac{-2}{2m+n}}$, (3.62) and (3.66) imply that

$$\begin{aligned}\frac{m\int_{B_{2\epsilon}}f_\tau^{\frac{2(m+n-1)}{m+n-2}}\tilde{v}^{m-1}dV_{\tilde{g}}}{\tilde{\tau}^{\frac{n}{2(m+n)}}\tilde{V}_\tau^{\frac{m+n-1}{m+n}}} &= \frac{m\int_{B^n_\epsilon}\varphi_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}dx}{\tilde{\tau}^{\frac{n}{2(m+n)}}\tilde{V}^{\frac{m+n-1}{m+n}}}\\ &+\frac{\tau m(m+n-1)R_{\tilde{g}}\tilde{I}_4\tilde{I}_5}{6n(m+n)\tilde{c}(m,n)^{n+1}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{2m+2n-1}{m+n}}}\\ &+\frac{\tau^2m(m+n-1)\tilde{I}_5\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{n+2}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{2m+2n-1}{m+n}}}\\ &+\frac{\tau^2m(m+n-1)(2m+2n-1)R^2_{\tilde{g}}(\tilde{I}_4)^2\tilde{I}_5}{72n^2(m+n)^2\tilde{c}(m,n)^{\frac{3n+4}{2}}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{3m+3n-1}{m+n}}}\\ &-\frac{\tau mR_{\tilde{g}}\tilde{I}_3}{2n\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{m+n-1}{m+n}}}\\ &-\frac{\tau^2m(m+n-1)R^2_{\tilde{g}}\tilde{I}_3\tilde{I}_4}{36n^2(m+n)\tilde{c}(m,n)^{n+2}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{2m+2n-1}{m+n}}}\\ &-\frac{\tau^2m\tilde{I}_8A_2}{2n(n+2)\tilde{c}(m,n)^{\frac{n+4}{2}}\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{m+n-1}{m+n}}}\\ &+O(\tau^{m+\frac{n-2}{2}}\epsilon^{-2m-n})+O(\tau^{2+\frac{q}{2}}).\end{aligned} \qquad (3.68)$$

The equality (2.22) and the estimates (3.64), (3.65) and (3.68) imply that (3.2) takes the form

$$\widetilde{\mathcal{W}}[M^n,\tilde{g},v^m dV_g]((v(p))^{-\frac{m+n-2}{2}}\tilde{f}_\tau,\tilde{\tau})+m\leq\widetilde{\nu}[\mathbb{R}^n,dx^2,dV,m]+m$$
$$+\tau A_3+\tau^2A_4+O(\tau^{m+\frac{n-2}{2}}\epsilon^{-2m-n})+O(\tau^{2+\frac{q}{2}}) \tag{3.69}$$

where

$$\begin{aligned}A_3 \quad &:= \frac{\tilde{V}^{-\frac{2m}{(m+n)(2m+n)}-\frac{m+n-2}{m+n}}R_{\tilde{g}}}{\tilde{c}(m,n)^{\frac{n}{2}}}\left(\frac{m+n-2}{4(m+n-1)}\tilde{I}_1-\frac{(m+n-2)^2}{6n}\tilde{I}_2\right.\\ &\left.-\frac{m}{6n\tilde{c}(m,n)}\tilde{I}_3+\frac{(m+n-2)^3(\tilde{I}_4)^2}{6n(m+n)\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}}+\frac{m(m+n-1)\tilde{I}_4\tilde{I}_5}{6n(m+n)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}}\right)\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}A_4 \quad &:= \frac{\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{m+n-1}{m+n}}}{\tilde{c}(m,n)^{\frac{n+2}{2}}}\left(\frac{(m+n-2)^3\tilde{I}_4\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}}\right.\\ &+\frac{(m+n-2)^3(m+n-1)R_{\tilde{g}}^2(\tilde{I}_4)^3}{36n^2(m+n)^2\tilde{c}(m,n)^n\tilde{V}^2}-\frac{(m+n-2)^3R_{\tilde{g}}^2\tilde{I}_2\tilde{I}_4}{36n^2(m+n)\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}}\\ &-\frac{(m+n-2)^2\tilde{I}_7A_1}{2n(n+2)}+\frac{(m+n-2)^2R_{\tilde{g}}^2\tilde{I}_1\tilde{I}_4}{24n(m+n)(m+n-1)\tilde{c}(m,n)^{\frac{n}{2}}\tilde{V}}\\ &+\frac{(m+n-2)(\Delta R_{\tilde{\phi}}^m-\frac{1}{3}R_{\tilde{g}}^2)\tilde{I}_6}{8n(m+n-1)}+\frac{m(m+n-1)\tilde{I}_5\tilde{I}_2A_1}{2n(n+2)(m+n)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}}\\ &+\frac{m(m+n-1)(2m+2n-1)R_{\tilde{g}}^2(\tilde{I}_4)^2\tilde{I}_5}{72n^2(m+n)^2\tilde{c}(m,n)^{n+1}\tilde{V}^2}\\ &\left.-\frac{m(m+n-1)R_{\tilde{g}}^2\tilde{I}_3\tilde{I}_4)}{36n^2(m+n)\tilde{c}(m,n)^{\frac{n+2}{2}}\tilde{V}}-\frac{m\tilde{I}_8A_2}{2n(n+2)\tilde{c}(m,n)}\right).\end{aligned} \tag{3.71}$$

We analyze the coefficients A_3 and A_4 in order to obtain the strict inequality in (6). For this purpose, we compare the integrals $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_3$, $\tilde{I}_4$, $\tilde{I}_5$, $\tilde{I}_6$, $\tilde{I}_7$, $\tilde{I}_8$ and $\tilde{V}$. This kind of comparison appeared for example in [1] and [8]. For this purpose, using polar coordinates we obtain

$$\tilde{I}_1=vol(S^{n-1})\int_0^\infty\frac{r^{n-1}}{(1+r^2)^{m+n-2}}dr, \tag{3.72}$$

$$\tilde{I}_2 = vol(S^{n-1})\int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n}}dr, \tag{3.73}$$

$$\tilde{I}_3 = vol(S^{n-1})\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-1}}dr, \tag{3.74}$$

$$\tilde{I}_4 = vol(S^{n-1})\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n}}dr, \tag{3.75}$$

$$\tilde{I}_5 = vol(S^{n-1})\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n-1}}dr, \tag{3.76}$$

$$\tilde{I}_6 = vol(S^{n-1})\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-2}}dr, \tag{3.77}$$

$$\tilde{I}_7 = vol(S^{n-1})\int_0^\infty \frac{r^{n+5}}{(1+r^2)^{m+n}}dr, \tag{3.78}$$

$$\tilde{I}_8 = vol(S^{n-1})\int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n-1}}dr \tag{3.79}$$

and

$$\tilde{V} = \frac{vol(S^{n-1})}{\tilde{c}(m,n)^{\frac{n}{2}}}\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n}}dr. \tag{3.80}$$

Integrating by parts we obtain for every $k>1$ and $l>1$

$$\int_0^\infty \frac{r^{l+1}}{(1+r^2)^k}dr = \frac{l}{2(k-1)}\int_0^\infty \frac{r^{l-1}}{(1+r^2)^{k-1}}dr, \tag{3.81}$$

which implies $\tilde{I}_4 = \frac{n}{2(m+n-1)}\tilde{I}_5$, $\tilde{I}_2 = \frac{n+2}{2(m+n-1)}\tilde{I}_3$ and $\tilde{I}_7 = \frac{n+4}{2(m+n-1)}\tilde{I}_8$. To compare $\tilde{I}_3$ with $\tilde{I}_4$, we write

$$\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-1}}dr = \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n}}dr + \int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n}}dr. \tag{3.82}$$

Using equality (3.81) in (3.82) yields

$$\frac{2m+n-4}{2(m+n-1)}\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-1}}dr = \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n}}dr. \tag{3.83}$$

Hence, $\tilde{I}_3 = \frac{2(m+n-1)}{2m+n-4}\tilde{I}_4$ and $\tilde{I}_2 = \frac{n+2}{2m+n-4}\tilde{I}_4$. Similarly

$$\int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n-1}}dr = \int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n}}dr + \int_0^\infty \frac{r^{n+5}}{(1+r^2)^{m+n}}dr. \quad (3.84)$$

Using equality (3.81) in (3.84) yields

$$\frac{2m+n-6}{2(m+n-1)}\int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n-1}}dr = \int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n}}dr. \quad (3.85)$$

Hence $\tilde{I}_8 = \frac{2(m+n-1)}{2m+n-6}\tilde{I}_2 = \frac{2(m+n-1)(n+2)}{(2m+n-4)(2m+n-6)}\tilde{I}_4$ and $\tilde{I}_7 = \frac{(n+2)(n+4)}{(2m+n-4)(2m+n-6)}\tilde{I}_4$. To compare $\tilde{I}_4$ with $\tilde{V}$, we write

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n-1}}dr = \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n}}dr + \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n}}dr. \quad (3.86)$$

Using equality (3.81) in equality above we get

$$\frac{2m+n-2}{n}\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n}}dr = \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n}}dr.$$

Therefore

$$\frac{\tilde{I}_4}{\tilde{V}} = \frac{n\tilde{c}(m,n)^{\frac{n}{2}}}{2m+n-2}. \quad (3.87)$$

Now, we compare $\tilde{I}_1$ with $\tilde{I}_4$, for this purpose observe that

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n-2}}dr = \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{m+n-1}}dr + \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-1}}dr. \quad (3.88)$$

Hence $\tilde{I}_1 = \tilde{I}_5 + \tilde{I}_3$. Therefore $\tilde{I}_1 = \frac{4(m+n-1)(m+n-2)}{n(2m+n-4)}\tilde{I}_4$. It remains to compare $\tilde{I}_6$ with $\tilde{I}_4$. We have

$$\int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-2}}dr = \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{m+n-1}}dr + \int_0^\infty \frac{r^{n+3}}{(1+r^2)^{m+n-1}}dr. \quad (3.89)$$

It follows from by equalities (3.89) and (3.81) that $\tilde{I}_6 = \frac{n}{2(m+n-2)}\tilde{I}_1 + \tilde{I}_8$. As a consequence, we get $\tilde{I}_6 = \frac{4(m+n-1)(m+n-2)}{(2m+n-4)(2m+n-6)}\tilde{I}_4$.

We are able to analyze the term A_3. Using the above comparisons for integrals and the equality $\tilde{c}(m,n) = \frac{m+n-1}{(m+n-2)^2}$ which imply

$$
\begin{aligned}
A_3 \;&= \frac{\tilde{V}^{-\frac{2m}{(m+n)(2m+n)}-\frac{m+n-2}{m+n}}(m+n-2)^2 R_{\tilde{g}}(0)\tilde{I}_4}{\tilde{c}(m,n)^{\frac{n}{2}}}\Bigg(\frac{1}{n(2m+n-4)} \\
&-\frac{n+2}{6n(2m+n-4)}-\frac{m}{3n(2m+n-4)} \\
&+\frac{m+n-2}{6(m+n)(2m+n-2)}+\frac{m(m+n-1)}{3n(m+n)(2m+n-2)}\Bigg)=0.
\end{aligned} \tag{3.90}
$$

Next, we analyze A_4. Using the fact that

$$
\begin{aligned}
\tfrac{m-5}{n(2m+n-4)(2m+n-6)} \;&= \tfrac{(m+n-2)}{2(m+n)(2m+n-2)(2m+n-4)} \\
&-\tfrac{(n+4)}{2n(2m+n-4)(2m+n-6)}+\tfrac{m(m+n-1)}{n(m+n)(2m+n-2)(2m+n-4)}
\end{aligned} \tag{3.91}
$$

we get

$$
\begin{aligned}
A_4 \;&= \frac{\tilde{I}_4(m+n-2)^2\tilde{V}^{\frac{-n}{(m+n)(2m+n)}+\frac{m+n-1}{m+n}}}{\tilde{c}(m,n)^{\frac{n+2}{2}}}\Bigg(\frac{(m-5)A_1}{n(2m+n-4)(2m+n-6)} \\
&+\frac{(m+n-2)(m+n-1)R_{\tilde{g}}^2}{36(m+n)^2(2m+n-2)^2}-\frac{(m+n-2)(n+2)R_{\tilde{g}}^2}{36n(m+n)(2m+n-2)(2m+n-4)} \\
&-\frac{(m+n-2)R_{\tilde{g}}^2}{6n(m+n)(2m+n-2)(2m+n-4)}+\frac{\Delta R_{\tilde{\phi}}^m-\frac{1}{3}R_{\tilde{g}}^2}{2n(2m+n-4)(2m+n-6)} \\
&+\frac{m(m+n-1)(2m+2n-1)R_{\tilde{g}}^2}{36n(m+n)^2(2m+n-2)^2} \\
&-\frac{m(m+n-1)R_{\tilde{g}}^2}{18n(m+n)(2m+n-2)(2m+n-4)}-\frac{mA_2}{n(2m+n-4)(2m+n-6)}\Bigg).
\end{aligned} \tag{3.92}
$$

On the other hand, we get

$$
\begin{aligned}
\langle \operatorname{Hess} v^m, Ric_{\tilde{g}}\rangle \;&= m(m-1)Ric_{\tilde{g}}(\nabla\tilde{v},\nabla\tilde{v})+m\langle \operatorname{Hess} v, Ric_{\tilde{g}}\rangle \\
&= m\langle \operatorname{Hess} v, T_{\tilde{g}}\rangle+\tfrac{m}{n}R_{\tilde{g}}\Delta\tilde{v} \\
&= m\langle \operatorname{Hess} v, T_{\tilde{g}}\rangle.
\end{aligned} \tag{3.93}
$$

To compute $\Delta_{\tilde{g}}^2\tilde{v}^m$ we will use the Ricci formula $\tilde{v}_{jij}=\tilde{v}_{jji}+\tilde{R}_{jij}^k\tilde{v}_k$ which imply

$$\begin{aligned}
&\Delta^2_{\tilde g}\tilde v^m(p) = m(m-1)(m-2)(m-3)|\nabla_{\tilde g} v|^4_{\tilde g} + 2m(m-1)(m-2)|\nabla_{\tilde g} v|^2\Delta_{\tilde g} v\\
&+4m(m-1)(m-2)\operatorname{Hess}\tilde v(\nabla_{\tilde g}\tilde v, \nabla_{\tilde g} v) + 2m(m-1)|\operatorname{Hess}\tilde v|^2_{\tilde g} + m(m-1)(\Delta_{\tilde g}\tilde v)^2\\
&+4m(m-1)\langle\nabla\tilde v, \nabla\Delta\tilde v\rangle + m\Delta^2_{\tilde g}\tilde v + 2m(m-1)Ric_{\tilde g}(\nabla\tilde v, \nabla\tilde v)\\
&= 2m(m-1)|\operatorname{Hess}\tilde v|^2_{\tilde g} + m\Delta^2_{\tilde g}\tilde v.
\end{aligned} \tag{3.94}$$

Now,

$$\begin{aligned}
\Delta^2_{\tilde g} R^m_{\tilde\phi}(p) &= \Delta^2_{\tilde g} R_{\tilde g} - 2m(m-3)|\nabla_{\tilde g} v|^2\Delta_{\tilde g} v + 2m(\Delta_{\tilde g}\tilde v)^2\\
&-2m(m-3)\langle\nabla\tilde v, \nabla\Delta\tilde v\rangle - 2m\Delta^2_{\tilde g}\tilde v - 6m(m-1)|\nabla_{\tilde g} v|^4_{\tilde g}\\
&+8m(m-1)\operatorname{Hess}\tilde v(\nabla_{\tilde g}\tilde v, \nabla_{\tilde g} v) - 2m(m-1)|\operatorname{Hess}\tilde v|^2_{\tilde g}\\
&-2m(m-1)Ric_{\tilde g}(\nabla\tilde v, \nabla\tilde v)\\
&= \Delta^2_{\tilde g} R_{\tilde g} - 2m(m-1)|\operatorname{Hess}\tilde v|^2_{\tilde g}.
\end{aligned} \tag{3.95}$$

Then, A_4 takes the form

$$A_4 = \frac{\left(-5|W|^2 + \frac{2n-7}{9(n-2)}|T_g|^2 - \frac{4}{3}m\langle T_g, \operatorname{Hess}\tilde v\rangle + \frac{m(m-1)}{2}|\operatorname{Hess}\tilde v|^2_g\right)}{n(2m+n-4)(2m+n-6)} + \frac{A_5}{36}R_g^2 \tag{3.96}$$

where

$$\begin{aligned}
A_5 &= -\tfrac{(n-7)(n-2)}{n(2m+n-4)(2m+n-6)} + \tfrac{(m+n-1)(m+n-2)}{(m+n)^2(2m+n-2)^2} - \tfrac{(m+n-2)(n+8)}{n(m+n)(2m+n-2)(2m+n-4)}\\
&+\tfrac{m(m+n-1)(2m+2n-1)}{n(m+n)^2(2m+n-2)^2} - \tfrac{2m(m+n-1)}{n(m+n)(2m+n-2)(2m+n-4)}
\end{aligned} \tag{3.97}$$

and we used to compute A_5 that

$$\begin{aligned}
-\tfrac{(n-7)(n-2)}{36n(2m+n-4)(2m+n-6)} &= -\tfrac{1}{6n(2m+n-4)(2m+n-6)}\\
\tfrac{(m-5)(5n^2+3n-14)}{180n^2(n-1)(2m+n-4)(2m+n-6)} &- \tfrac{m(5n^2+3n-14)}{180n^2(n-1)(2m+n-4)(2m+n-6)}
\end{aligned}$$

and

$$\tfrac{(m+n-2)(n+8)}{n(m+n)(2m+n-2)(2m+n-4)} = \tfrac{(m+n-2)(n+2)}{n(m+n)(2m+n-2)(2m+n-4)} + \tfrac{6(m+n-2)}{n(m+n)(2m+n-2)(2m+n-4)}.$$

On the other hand, for $n \geq 7$ and $m \geq 0$ we get

$$-\frac{(n-7)(n-2)}{n(2m+n-4)(2m+n-6)} \leq 0.$$

Since $n+8>n$, $m+n>m+n-1$ and $2m+n-2>2m+n-4$ we get

$$\begin{aligned}&\tfrac{(m+n-1)(m+n-2)}{(m+n)^2(2m+n-2)^2}-\tfrac{(m+n-2)(n+8)}{n(m+n)(2m+n-2)(2m+n-4)}\\&=\tfrac{(m+n-2)(n(m+n-1)(2m+n-4)-(n+8)(m+n)(2m+n-2))}{n(m+n)^2(2m+n-2)^2(2m+n-4)}<0.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\tfrac{m(m+n-1)(2m+2n-1)}{n(m+n)^2(2m+n-2)^2}-\tfrac{2m(m+n-1)}{n(m+n)(2m+n-2)(2m+n-4)}\\&=\tfrac{m(m+n-1)((2m+n-1)(2m+n-4)-2(m+n)(m+n)(2m+n-2))}{n(m+n)^2(2m+n-2)^2(2m+n-4)}\leq 0.\end{aligned}$$

The inequalities above imply that $A_5\leq 0$ for $n\geq 7$ and $m\geq 0$.

Next, we consider the case $0\leq m\leq 1$. Let $\hat{g}=e^{\frac{2\hat{\sigma}}{m+n-2}}\tilde{g}$ and $\hat{v}=e^{\frac{\hat{\sigma}}{m+n-2}}\tilde{v}$ be such that in the point p we have $\hat{\sigma}$ such that in p satisfy $\hat{\sigma}=0$, $\nabla_{\tilde{g}}\hat{\sigma}=0$ and $\hat{\sigma}_{ij}=\frac{m+n-2}{n-2}\tilde{T}_{ij}$. Since $\tilde{T}_{ij}$ is trace free we get in the point p that $\Delta_{\tilde{g}}\hat{\sigma}=0$ and also in this point p we have

$$\begin{aligned}\hat{R}_{ij}&=\tilde{R}_{ij}-\tfrac{n-2}{m+n-2}\hat{\sigma}_{ij}+\tfrac{n-2}{(m+n-2)^2}\hat{\sigma}_i\hat{\sigma}_j+\left(\tfrac{\Delta_{\tilde{g}}\hat{\sigma}}{(m+n-2)}-\tfrac{n-2}{(m+n-2)^2}|\nabla\hat{\sigma}|^2_{\tilde{g}}\right)\tilde{g}_{ij}\\&=\tilde{R}_{ij}-\tfrac{n-2}{m+n-2}\hat{\sigma}_{ij},\end{aligned}$$

$$R_{\hat{g}}=e^{-\frac{2\sigma}{m+n-2}}(R_{\tilde{g}}+\frac{2(n-1)}{m+n-2}\Delta_{\tilde{g}}\hat{\sigma}-\frac{(n-1)(n-2)}{(m+n-2)^2}|\nabla\hat{\sigma}|^2_{\tilde{g}})=R_{\tilde{g}}$$

and

$$\hat{T}_{ij}=\tilde{T}_{ij}-\frac{n-2}{m+n-2}\hat{\sigma}_{ij}=0.$$

On the other hand, using that in p we have $\nabla_{\tilde{g}}\tilde{v}=0$, $\Delta_{\tilde{g}}\tilde{v}=0$, $\Delta_{\tilde{g}}\hat{\sigma}=0$, transformations rules (2.7) and (2.8) yield

$$\nabla_{\hat{g}}\hat{v}=e^{-\frac{2\hat{\sigma}}{m+n-2}}(e^{\frac{\hat{\sigma}}{m+n-2}}\nabla_{\tilde{g}}v+\frac{v}{m+n-2}e^{\frac{\hat{\sigma}}{m+n-2}}\nabla_{\tilde{g}}\hat{\sigma})=0$$

and

$$\begin{aligned}\Delta_{\hat{g}}\hat{v}&=e^{-\frac{\hat{\sigma}}{m+n-2}}\left(\tfrac{n-1}{(m+n-2)^2}\tilde{v}|\nabla_{\tilde{g}}\hat{\sigma}|^2_{\tilde{g}}+\tfrac{\tilde{v}}{m+n-2}\Delta_{\tilde{g}}\hat{\sigma}+\Delta_{\tilde{g}}\tilde{v}+\tfrac{n}{m+n-2}\nabla_{\tilde{g}}\hat{\sigma}\nabla_{\tilde{g}}\tilde{v}\right)\\&=0.\end{aligned}\tag{3.98}$$

Since $\hat{\sigma}(p)=0$, $\nabla_{\hat{g}}\hat{v}(p)=0$, $\Delta_{\hat{g}}\hat{v}(p)=0$, $\hat{T}_{ij}(p)=0$, $|W|$ is conformally invariant and $|W|(p)\neq 0$ it follows that for this new metric A_4 in (3.96) takes the form

$$A_4 = \frac{\left(-5|W|^2 + \frac{m(m-1)}{2}|\mathrm{Hess}\,\hat{v}|^2_{\hat{g}}\right)}{n(2m+n-4)(2m+n-6)} + A_5 R^2_{\hat{g}} < 0. \tag{3.99}$$

Using $A_3 = 0$ and $A_4 < 0$ for $0 \leq m \leq 1$ in this new smooth measure structure, then taking ϵ small and fixed and after choosing τ small enough, inequality (3.69) yields

$$\widetilde{\nu}[M^n, g, v^m dV_g] < \widetilde{\nu}[\mathbb{R}^n, dx^2, dV, m]. \tag{3.100}$$

Proposition 7 implies

$$\widetilde{\Lambda}[M^n, g, v^m dV_g] < \widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m]. \tag{3.101}$$

Theorem 2 concludes the proof for $0 \leq m \leq 1$. Finally, Theorem 5 and an inductive argument imply that

$$\begin{aligned} \widetilde{\Lambda}[M^n, g, v^{m+1} dV_g] &\leq \frac{\widetilde{\Lambda}[M^n, g, v^m dV_g]}{\widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m]} \Lambda[\mathbb{R}^n, dx^2, dV, m+1] \\ &< \widetilde{\Lambda}[\mathbb{R}^n, dx^2, dV, m+1], \end{aligned} \tag{3.102}$$

which leads to our result for every $m > 0$. ■

Remark 7. *Note in the case $n = 6$ the proof works if $A_5 \leq 0$, which is false for a general $m > 0$.*

Remark 8. *We did not use conformal normal coordinates in Theorem A's proof as Lee and Parker used in [10] to get a simple proof of Aubin's Theorem. In our proof these coordinates do not simplify calculations because the density v^m changes conformally.*

CHAPTER 4

The Escobar type problem

In this chapter, we apply our tools for smooth measure spaces with boundary defined in Chapter 2. We develop some properties of Dirichlet eigenvalues and eigenfunctions to prove Theorem C by a direct compactness argument and also we find an upper bound for the τ-energy as τ goes to zero, Theorem D is a consequence of this estimate.

4.1 Proof of Theorem C

All functions in the family $\{w_{\epsilon,0}\}$ as in (11) are minimizers of the weighted Escobar problem. Note that these functions are not uniformly bounded in $H^1(M)$ as $\epsilon \to 0$. This fact shows that in general there is no reason to find a minimizing function by direct arguments in the weighted Escobar quotient. It is possible that if the weighted Escobar quotient is finite and we try to minimize it with a sequence of functions normalized, then the terms involved in the numerator of the weighted Escobar quotient evaluated in these functions are not bounded uniformly. The next lemma deals with the control of one of those terms from below.

Lemma 4. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and suppose that Λ is finite, then there exists a real constant C such that any volume-normalized function $\varphi \in H^1(M)$ satisfies*

$$(L_\phi^m \varphi, \varphi) + (B_\phi^m \varphi, \varphi) > C. \tag{4.1}$$

Proof. In this proof C is a real constant that depends only on the smooth measure space $(M^n, g, v^m dV_g, v^m d\sigma_g)$ and possibly changing from line to line. Suppose that there exists a sequence of functions $\{\varphi_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty}(L_\phi^m\varphi_i,\varphi_i)+(B_\phi^m\varphi_i,\varphi_i)=-\infty \quad \text{and} \quad \int_{\partial M}\varphi_i^{\frac{2(m+n-1)}{m+n-2}}=1. \tag{4.2}$$

Since Λ is finite there exists a real constant C such that every volume-normalized φ satisfies

$$C\le \Lambda(\varphi)=\left((L_\phi^m\varphi,\varphi)+(B_\phi^m\varphi,\varphi)\right)\left(\int_M\varphi^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{m}{m+n-1}}.$$

From the last inequality it follows that $\lim\limits_{i\to\infty}\int_M\varphi_i^{\frac{2(m+n-1)}{m+n-2}}=0$ and by the Hölder inequality it follows that $\int_M\varphi_i^2<C$ for any i. Similarly, using that φ_i are volume normalized and the Hölder inequality we get $\int_{\partial M}\varphi_i^2<C$. Using these L^2 estimate we obtain that

$$(L_\phi^m\varphi_i,\varphi_i)+(B_\phi^m\varphi_i,\varphi_i)>C$$

contradicting the assumption (4.2). □

In order to state the following lemma, we say that a real number ρ is an *eigenvalue type Dirichlet* on $H_0^1(M)=\{\varphi|\,\varphi\in H^1(M),\,\varphi\equiv 0 \text{ on } \partial M\}$ if ρ satisfies for some $\varphi\in H_0^1(M)$

$$L_\phi^m\varphi=\rho\varphi \quad \text{in} \quad M, \qquad \varphi\equiv 0 \quad \text{on} \quad \partial M. \tag{4.3}$$

We also call φ an *eigenfunction* if it satisfies (4.3). Let us denote by ρ_1 the first eigenvalue type Dirichlet on $H_0^{1,2}(M)$, then ρ_1 admits a variational characterization as

$$\rho_1=\inf_{\varphi\in H_0^1(M)}\frac{\int_M|\nabla\varphi|^2+\frac{m+n-2}{4(m+n-1)}R_\phi^m\varphi^2}{\int_M\varphi^2}. \tag{4.4}$$

We have ρ_1 is finite and we can choose an eigenfunction φ associated to this eigenvalue such that $\varphi\ge 0$. Moreover, using the maximum principle we can take $\varphi>0$ in $M\setminus\partial M$.

Lemma 5. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and $m>0$. Then $\Lambda=-\infty$ if and only if $\rho_1\le 0$.*

Proof. As in the previous proof, C is a real constant that depends only on the smooth measure space $(M^n, g, v^m dV_g, v^m d\sigma_g)$ and possibly changing from line to line or in the same line.

First, let us assume $\rho_1 \leq 0$. Let φ be a first eigenfunction of the problem (4.3) such that $\varphi > 0$ in $M \setminus \partial M$. Let us define

$$\psi_t = \frac{t\varphi + 1}{\sqrt{D}} \quad \text{where} \quad D = \left(\int_{\partial M} e^{-\phi} d\sigma_g\right)^{\frac{m+n-2}{(m+n-1)}}$$

and observe that for some constant $C > 0$ we have

$$\int_{\partial M} \psi_t^{\frac{2(m+n-1)}{m+n-2}} = 1 \qquad \text{and} \qquad \int_M \psi_t^{\frac{2(m+n-1)}{m+n-2}} \geq C > 0. \tag{4.5}$$

Claim 1.

$$(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) \to -\infty \quad \textit{when} \quad t \to \infty. \tag{4.6}$$

To prove this claim, we argue as Garcia and Muñoz in [9, Proposition 1]. First, we consider the case $\rho_1 < 0$, then

$$(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) = \frac{1}{D}\left[t^2\left(\rho_1 \int_M \varphi^2\right) + t\left(\frac{m+n-2}{2(m+n-1)} \int_M \varphi R^m_\phi\right) + E\right] \tag{4.7}$$

where

$$E = \frac{m+n-2}{4(m+n-1)} \int_M R^m_\phi + \frac{m+n-2}{2(m+n-1)} \int_M H^m_\phi.$$

Since $\rho_1 < 0$, the quadratic term for t on the right hand side of (4.7) is negative. Letting $t \to \infty$ it follows our claim in this case.

Now, we suppose that $\rho_1 = 0$, then

$$(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) = \frac{1}{D}\left[t\left(\frac{m+n-2}{2(m+n-1)} \int_M \varphi R^m_\phi\right) + E\right] \tag{4.8}$$

where E is defined as in the previous case. Since $\varphi \equiv 0$ on ∂M, by Hopf's Lemma, $\dfrac{\partial \varphi}{\partial \eta} < 0$. Then, integrating by parts yields

$$\frac{m+n-2}{4(m+n-1)} \int_M \varphi R^m_\phi = \int_M \Delta_\phi \varphi = \int_{\partial M} \frac{\partial \varphi}{\partial \eta} < 0.$$

Then, the linear term for t on the right hand side of (4.8) is negative. Taking $t \to \infty$ we get the conclusion in this case and we finish the claim's proof.

Finally, from the estimates (4.5) and (4.6) we get that $\mathcal{Q}(\psi_t) \to -\infty$ as $t \to \infty$, therefore we conclude $\Lambda = -\infty$.

Next, we assume that $\Lambda = -\infty$ and we prove that $\rho_1 \leq 0$. This assumption implies that R^m_ϕ is not identically zero. Let us take a minimizing sequence of functions $\{\varphi_i\}_{i=1}^{\infty}$ of Λ such that

$$\int_{\partial M} \varphi_i^{\frac{2(m+n-1)}{m+n-2}} = 1, \quad (L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) \leq 0 \quad \text{and} \quad \lim_{i\to\infty} \mathcal{Q}(\varphi_i) = -\infty.$$

Claim 2. $\displaystyle\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \to \infty$ *when* $i \to \infty$.

Arguing by contradiction, we assume that there exists a constant $C > 0$ such that $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} < C$, then by the Hölder inequality we get that $\int_M \varphi_i^2 < C$ for every i. On the other hand, we have that $(L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) \to -\infty$ when $i \to \infty$ since $\lim\limits_{i\to\infty} \mathcal{Q}(\varphi_i) = -\infty$. Using this limit, the fact that R^m_ϕ is a non-zero function and that φ_i is normalized we get $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$, which is a contradiction with the initial assumption. Hence $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \to \infty$.

Claim 3. $\displaystyle\int_M \varphi_i^2 \to \infty$ *when* $i \to \infty$.

Arguing by contradiction, suppose that there exists a constant $C > 0$ such that $\int_M \varphi_i^2 < C$. Then

$$\int_M |\nabla \varphi_i|^2 \leq (L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) + C(||\varphi_i||^2_{2,M} + ||w_i||^2_{2,\partial M}) < C. \quad (4.9)$$

On the other hand, by the Sobolev inequality we get that there exists a constant C such that

$$\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \leq C \left(\int_M |\nabla \varphi_i|^2 + \int_M \varphi_i^2 \right). \quad (4.10)$$

Then inequalities (4.9) and (4.10) yield $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \leq C$. This is a contradiction with the Claim 1 and we conclude that $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$.

Now we are able to conclude the proof of the lemma. For this purpose let us define the functions $\psi_i = \dfrac{\varphi_i}{||\varphi_i||_{2,M}}$. Arguing as in the last part of Proposition 1 in Garcia and Muñoz [9] we get that ψ_i converges weakly to a function ψ in $H^1_0(M)$ such that $||\psi||_{2,M} = 1$ and

$$\rho_1 \leq \int_M |\nabla\psi|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m \psi^2 \leq \liminf_{i\to\infty} (L_\phi^m \psi_i, \psi_i) + (B_\phi^m \psi_i, \psi_i) \leq 0.$$

□

Proof of Theorem C. Let $\{w_i\}_{i=1}^{\infty}$ be a sequence of positive functions such that $\int_{\partial M} w_i^{\frac{2(m+n-1)}{m+n-2}} = 1$, $\mathcal{Q}(w_i) \leq 0$ and $\mathcal{Q}(w_i) \to \Lambda$ when $i \to \infty$. Then

$$0 \geq (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i) \geq ||\nabla w_i||_{2,M}^2 - C(||w_i||_{2,M}^2 + ||w_i||_{2,\partial M}^2). \quad (4.11)$$

First, we consider the case $||w_i^2||_{2,M} < C$, then the last inequality yields that $\{w_i\}_{i=1}^{\infty}$ are uniformly bounded in $H^1(M)$. Recall that $m > 0$, then $1 < \frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}$, i.e. $\frac{2(m+n-1)}{m+n-2}$ is less than the critical Trace's inequality exponent. By Sobolev's and Trace's embedding Theorems, there exists a function w and a sub-sequence $\{w_i\}_{i=1}^{\infty}$ which converges to w in $L^2(M)$, $L^{\frac{2(m+n-1)}{m+n-2}}(M)$ and $L^{\frac{2(m+n-1)}{m+n-2}}(\partial M)$ and also $\{w_i\}_{i=1}^{\infty}$ converges weakly to w in $H^1(M)$. It follows that there exist a constant C such that

$$\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \geq C \quad \text{and} \quad ||w||_{\frac{2(m+n-1)}{m+n-2},\partial M} = 1.$$

Then by construction, w minimizes the weighted Escobar quotient and by Proposition 15, w is a non-negative weak solution of

$$\begin{array}{lll} L_\phi^m w + c_1 w^{\frac{m+n}{m+n-2}} v^{-1} = 0 & \text{in} & M, \\ B_\phi^m w = c_2 w^{\frac{m+n}{m+n-2}} & \text{on} & \partial M. \end{array} \quad (4.12)$$

Since $1 < \frac{m+n-1}{m+n-2} < \frac{n-1}{n-2}$, the usual elliptic regularity argument for sub-critical equations allows us to conclude that w is in fact smooth and positive, as we desired.

Following, we prove that we do not have the case when $||w_i||_{2,M} \to \infty$ is unbounded. Arguing by contradiction, we assume that $||w_i||_{2,M} \to \infty$ when $i \to \infty$. Consider the L^2 re-normalized sequence of functions $\tilde{w}_i = \frac{w_i}{||w_i||_{2,M}}$. It follows that $||\tilde{w}_i||_{\frac{2(m+n-1)}{m+n-2},\partial M} \to 0$ when $i \to \infty$. Since $\tilde{w}_i$ satisfy the inequality (4.11) for every i we know that $\{\tilde{w}_i\}_{i=1}^{\infty}$ is uniformly bounded in $H^{1,2}(M)$.

By Sobolev's and Trace's embedding Theorems, there exists a function w and a sub-sequence $\{\tilde{w}_i\}_{i=1}^{\infty}$ which converges to w in $L^2(M)$, $L^{\frac{2(m+n-1)}{m+n-2}}(M)$ and $L^{\frac{2(m+n-1)}{m+n-2}}(\partial M)$ and also weakly in $H^1(M)$. In consequence, $||w||_{2,M} = 1$ and using again that $||\tilde{w}_i||_{\frac{2(m+n-1)}{m+n-2},\partial M} \to 0$ when $i \to \infty$, we get that $w \equiv 0$ in ∂M.

On the other hand, Lemma 4 yields

$$0 > (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i) > -C.$$

Therefore $(L_\phi^m \tilde{w}_i, \tilde{w}_i) + (B_\phi^m \tilde{w}_i, \tilde{w}_i) \to 0$ when $i \to \infty$. Using w as a test function in (4.4), we conclude that

$$\rho_1 \leq \int_M |\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m w \leq \liminf_{i\to\infty} (L_\phi^m \tilde{w}_i, \tilde{w}_i) + (B_\phi^m \tilde{w}_i, \tilde{w}_i) = 0.$$

But $\rho_1 \leq 0$ contradicts Lemma 5 because Λ is finite by hypothesis. ∎

4.2 Proof of Theorem D

We prove an upper estimate of the τ-energy as τ goes to zero using Theorem B and the family $\{w_{0,\tau}\}$ in (2.43) as test functions in the $\mathcal{W}$-functional. Actually, Theorem B is the reason for which the weighted Escobar constant for the Euclidean half-space appears on the right hand side of the inequality (14).

Lemma 6. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and $m \in \mathbb{N} \cup \{0\}$, then*

$$\limsup_{\tau\to 0} \nu(\tau) \leq \nu[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m].$$

Proof. In this lemma, C is a positive constant which depend on the smooth measure space $(M^n, g, v^m dV_g, v^m d\sigma_g)$ and maybe change from line to line or in the same line. First define $\tilde{w}_{x_0,\tau} = V^{-\frac{m+n-2}{2(m+n-1)}} w_{x_0,\tau}$; with V as in (2.44). By Theorem B we know that $\tilde{w}_{x_0,\tau}$ achieves the weighted Escobar quotient, hence by Proposition 14, there exits $\tilde{\tau} > 0$ such that

$$\begin{aligned} &\nu(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m) + 1 = \mathcal{W}(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)(\tilde{w}_{x_0,\tau}, \tilde{\tau}) + 1 \\ &= \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V^{\frac{m+n-2}{m+n-1}}} \int_{\mathbb{R}^n_+} |\nabla w_{x_0,\tau}|^2 dV + \tilde{\tau}^{-\frac{1}{2}} V^{-1} \int_{\mathbb{R}^n_+} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dV. \end{aligned} \tag{4.13}$$

Then Proposition 16 yields $\tilde{\tau} = \tau V^{-\frac{2}{2m+n-1}}$.

On the other hand, fix a point $p \in \partial M$ and let (x_i, t) be the Fermi coordinates in some fixed neighborhood U of $p = (0, ..., 0)$. Let $1 > \epsilon > 0$ be such that $B(p, 2\epsilon) \subset U$. Let $\eta : M \to [0,1]$ be a cutoff function such that $\eta \equiv 1$ on B^+_ϵ, $supp(\eta) \subset B^+_{2\epsilon}$ and $|\nabla \eta|^2 < C\epsilon^{-1}$ in $A^+_\epsilon = B^+_{2\epsilon} \setminus B^+_\epsilon$. For each

$0 < \tau < 1$, define $f_\tau : M \to \mathbb{R}$ by $f_\tau(x_1, ..., x_{n-1}, t) = \eta w_{0,\tau}(x_1, ..., x_{n-1}, t)$, and set $\tilde{f}_\tau = V_\tau^{-\frac{m+n-2}{2(m+n-1)}} f_\tau$ for

$$V_\tau = \int_{\partial M} f_\tau^{\frac{2(m+n-1)}{m+n-2}}.$$

Proposition 12 implies that if w is a normalized function with the metric $v^{-2}g$, then

$$\mathcal{W}[M^n, v^{-2}g, dV_{v^{-2}g}, d\sigma_{v^{-2}g}, m](w, \tau) = \mathcal{W}[M^n, g, v^m dV_g, v^m d\sigma_g](v^{-\frac{m+n-2}{2}} w, \tau),$$

this equality allows us to consider without loss generality that $v \equiv 1$. Computing as in [10, Lemma 3.4], and using that $dV_g = (1 + O(r))dxdt$ and $d\sigma_g = (1 + O(r))dx$ we obtain

$$\begin{aligned}
&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1 \\
&= \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}} \left(\int_{B_{2\epsilon}^+} |\nabla f_\tau|_g^2 + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dV_g \right. \\
&\left. + \int_{B_{2\epsilon}^+ \cap \partial M} \frac{m+n-1}{2(m+n-2)} H_g f_\tau^2 d\sigma_g \right) + \tilde{\tau}^{-\frac{1}{2}} V_\tau^{-1} \int_{B_{2\epsilon}^+} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dV_g \\
&\leq (1 + C\epsilon) \left\{ \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}} \left(\int_{B_{2\epsilon}^+} |\nabla f_\tau|_g^2 + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dxdt \right. \right. \\
&\left. \left. + \int_{B_{2\epsilon}^+ \cap \partial M} \frac{m+n-1}{2(m+n-2)} H_g f_\tau^2 dx \right) + \tilde{\tau}^{-\frac{1}{2}} V_\tau^{-1} \int_{B_{2\epsilon}^+} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dxdt \right\}.
\end{aligned} \tag{4.14}$$

Let us recall that $c(m, n) = \frac{m+n-1}{m(m+n-2)^2}$. Fixing $\epsilon < 1$ and after taking $\sqrt{\tau} \leq \sqrt{c(m, n)} 2\epsilon$ we obtain

$$\begin{aligned}
\int_{B_{2\epsilon}^+} R_g f_\tau^2 dxdt &\leq C \int_{B_{2\epsilon}^+} w_{0,\tau}^2 dxdt \\
&= C\tau^{-\frac{(n-1)(m+n-2)}{2(m+n-1)}} \int_{B_{2\epsilon}^+} \frac{dxdt}{((1 + (\frac{c(m,n)}{\tau})^{\frac{1}{2}} t)^2 + \frac{c(m,n)}{\tau} |x|^2)^{m+n-2}} \\
&= C\tau^{\frac{n-1}{2(m+n-1)} + \frac{1}{2}} \int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \frac{dydt}{((1 + s)^2 + |y|^2)^{m+n-2}}.
\end{aligned} \tag{4.15}$$

Similar as in [10, Lemma 3.5] we get

$$\int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \frac{dydt}{((1+s)^2+|y|^2)^{m+n-2}} = \begin{cases} C & \text{if } \; 4-n-2m<0, \\ O(\tau^{m-\frac{1}{2}}) & \text{if } \; n=3,\ \frac{1}{2}-m>0 \text{ and} \\ O(\log(\tau)) & \text{if } \; n=3,\ \frac{1}{2}-m=0. \end{cases} \tag{4.16}$$

Then

$$\int_{B^+_{2\epsilon}} R_g f_\tau^2 dxdt = E_1 = \begin{cases} O(\tau^{\frac{n-1}{2(m+n-1)}+\frac{1}{2}}) & \text{if}\, 4-n-2m<0, \\ O(\tau^{\frac{n-1}{2(m+n-1)}+m}) & \text{if}\, n=3,\ m<\frac{1}{2} \text{ and} \\ O(\tau^{\tau^{\frac{n-1}{2(m+n-1)}+\frac{1}{2}}}\log(\tau)) & \text{if}\, n=3,\ \frac{1}{2}-m=0. \end{cases} \tag{4.17}$$

Now, we estimate the integrals on the right hand side in the second inequality of (4.14)

$$\begin{aligned} \int_{B^{n-1}_{2\epsilon}} H_g f_\tau^2 dx &\leq C \int_{B^{n-1}_{2\epsilon}} w_{0,\tau}^2 dx = C\tau^{\frac{n-1}{2(m+n-1)}} \int_{B^{n-1}_{2\epsilon}} (1+|y|^2)^{-(m+n-2)} dx \\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}} \end{aligned} \tag{4.18}$$

$$\int_{B^+_{2\epsilon}} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dxdt \leq \int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt. \tag{4.19}$$

Let us estimate the gradient integral in $A^+_\epsilon = B^+_{2\epsilon} \setminus B^+_\epsilon$. Observe that

$$|\nabla f_\tau|^2_{\tilde{g}} \leq C|\nabla f_\tau|^2 \leq C(\eta^2|\nabla w_{0,\tau}|^2 + |\nabla\eta|^2 w_{0,\tau}^2). \tag{4.20}$$

Now, we get

$$\begin{aligned} \int_{A^+_\epsilon} |\nabla\eta|^2 w_{0,\tau}^2 dxdt &\leq C\epsilon^{-2} \int_{A^+_\epsilon} w_{0,\tau}^2 dxdt \\ &\leq C\epsilon^{-2}\tau^{\frac{-(n-1)(m+n-2)}{2(m+n-1)}+\frac{n}{2}} \int_{A^+_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \left(\frac{1}{s^2+|y|^2}\right)^{m+n-2} dxdt \\ &\leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}} \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\int_{A_\epsilon^+}\eta^2|\nabla w_{0,\tau}|^2dxdt &\leq C\tau^{\frac{-(n-1)(m+n-2)}{2(m+n-1)}+\frac{n}{2}-1}\int_{A^+_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}}\left(\frac{1}{s^2+|y|^2}\right)^{m+n-1}dxdt\\ &\leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}}.\end{aligned} \tag{4.22}$$

Then

$$\int_{A_\epsilon^+}|\nabla f_\tau|_g^2dxdt \leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}}. \tag{4.23}$$

Since for the Fermi coordinates around p we obtain $g^{tt}=1$, $g^{ti}=0$ and $g^{ij}=\delta_{ij}+O(|x,t|)$ where $1\leq i,j\leq n-1$, it follows

$$\begin{aligned}\int_{B_\epsilon}|\nabla f_\tau|_g^2dxdt &\leq \int_{B_\epsilon}|\nabla w_{0,\tau}|^2dxdt + C\int_{B_\epsilon}|x,t|(w_{0,\tau})_i(w_{0,\tau})_j\\ &\leq \int_{B_\epsilon}|\nabla w_{0,\tau}|^2dxdt + C\tau^{\frac{n-1}{2(m+n-1)}}.\end{aligned} \tag{4.24}$$

We already have the second inequality of (4.24) because

$$\begin{aligned}\int_{B_\epsilon}|x,t|(w_{0,\tau})_i(w_{0,\tau})_j &\leq C\tau^{-\frac{(n-1)(m+n-2)}{2(m+n-1)}-2}\int_{B_\epsilon}\frac{|x,t|x_ix_jdxdt}{((1+(\frac{c(m,n)}{\tau})^{\frac{1}{2}}t)^2+\frac{c(m,n)}{\tau}|x|^2)^{m+n}}\\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}}\int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}}\frac{|y,s|^3dydt}{((1+s)^2+|y|^2)^{m+n}}\\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}}.\end{aligned} \tag{4.25}$$

Using the inequalities (4.17), (4.18), (4.23) and (4.24) in the inequality (4.14) we get that

$$\begin{aligned}&\mathcal{W}[M^n,g,dV_g,d\sigma_g,m](\tilde{f}_\tau,\tilde{\tau})+1\\ &\leq(1+C\epsilon)\left\{\frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}}\left(\int_{\mathbb{R}^n_+}|\nabla w_{0,\tau}|^2dxdt+C\tau^{\frac{n-1}{2(m+n-1)}}\right.\right.\\ &\left.\left.+C\tau^{\frac{(n-1)(2m+n-1)}{2(m+n-1)}+m+\frac{n-3}{2}}\epsilon^{2-n-2m}+E_1\right)+\tilde{\tau}^{-\frac{1}{2}}V_\tau^{-1}\int_{\mathbb{R}^n_+}w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}}dxdt\right\}.\end{aligned} \tag{4.26}$$

Now using the inequality (4.13) we conclude

$$\begin{aligned}
&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1 \\
&\leq (1 + C\epsilon)\nu[\mathbb{R}^n_+, dt^2 + dx^2, dV_g, d\sigma_g, m] \\
&+(1 + C\epsilon)\left\{\tilde{\tau}^{\frac{m}{2(m+n-1)}} V_\tau^{-\frac{m+n-2}{m+n-1}} \left(C\tau^{\frac{n-1}{2(m+n-1)}} + C\tau^{\frac{(n-1)(2m+n-1)}{2(m+n-1)}+m+\frac{n-3}{2}}\epsilon^{2-n-2m}\right.\right. \\
&\left.+E_1\right) + \tilde{\tau}^{\frac{m}{2(m+n-1)}}(V_\tau^{-\frac{m+n-2}{m+n-1}} - V^{-\frac{m+n-2}{m+n-1}}) \int_{\mathbb{R}^n_+} |\nabla w_{0,\tau}|^2 dxdt \\
&\left.+\tilde{\tau}^{-\frac{1}{2}}(V_\tau^{-1} - V^{-1}) \int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt\right\}.
\end{aligned} \tag{4.27}$$

On the other hand, we obtain

$$\begin{aligned}
V - V_\tau &\leq \int_{\mathbb{R}^{n-1}\setminus B^{n-1}_\epsilon} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dx \\
&= \tau^{-\frac{n-1}{2}} \int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_\epsilon} (1 + \frac{c(m,n)}{\tau}|x|^2)^{-(m+n-1)} dx \\
&= C \int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} (1 + |y|^2)^{-(m+n-1)} dy \\
&\leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}}.
\end{aligned} \tag{4.28}$$

In particular, we get that the constants V_τ are uniformly bounded away from zero. Using estimate (4.28) and the Taylor expansion for the functions $x^{-\frac{m+n-2}{m+n-1}}$ and x^{-1} we obtain

$$V_\tau^{-\frac{m+n-2}{m+n-1}} - V^{-\frac{m+n-2}{m+n-1}} \leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}} \tag{4.29}$$

and

$$V_\tau^{-1} - V^{-1} \leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}}. \tag{4.30}$$

Additionally, the equality (4.13) implies the following estimates

$$\tilde{\tau}^{\frac{m}{2(m+n-1)}} \int_{\mathbb{R}^n_+} |\nabla w_{0,\tau}|^2 dxdt \leq C \quad \text{and} \quad \tilde{\tau}^{-\frac{1}{2}} \int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt \leq C. \tag{4.31}$$

The substitution $\tilde{\tau} = \tilde{\tau} V^{\frac{1}{2m+n-1}}$, the inequalities (4.29), (4.30), (4.31) and (4.27) yield

$$\begin{aligned}
&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1 \\
&\leq (1 + C\epsilon)\nu[\mathbb{R}^n_+, dt^2 + dx^2, 1^m dV_g, 1^m d\sigma_g] \\
&+(1 + C\epsilon)\left\{V^{-\frac{m+n-2}{m+n-1} - \frac{m}{2(2m+n-1)(m+n-1)}}\left(C\tau^{\frac{1}{2}} + C\tau^{\frac{1}{2}+m+\frac{n-3}{2}}\epsilon^{2-n-2m}\right.\right. \\
&\left.\left.+\tau^{\frac{m}{2(m+n-1)}}E_1\right) + C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}}\right\}.
\end{aligned} \tag{4.32}$$

Finally, taking $\tau \to 0$ and after $\epsilon \to 0$ in (4.32) the conclusion follows. □

Proof of Theorem D. By the definition of ν and Lemma 6 we obtain that

$$\nu[M^n, g, v^m dV_g, v^m d\sigma_g] \leq \nu[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m]. \tag{4.33}$$

By Proposition 14 we conclude

$$\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m]. \quad \blacksquare \tag{4.34}$$

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