A Discrete Wedge Product on Polygonal Pseudomanifolds

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Abstract

Discrete exterior calculus (DEC) offers a coordinate-free discretization of exterior calculus especially suited for computations on curved spaces. In this work, we present an extended version of DEC on surface meshes formed by general polygons that bypasses the construction of any dual mesh and the need for combinatorial subdivisions. At its core, our approach introduces a polygonal wedge product that is compatible with the discrete exterior derivative in the sense that it satisfies the Leibniz product rule. Based on the discrete wedge product, we then derive a novel primal–to–primal Hodge star operator, which then leads to a discrete version of the contraction operator and the Lie derivative on discrete 0-, 1-, and 2-forms. We show results of numerical tests indicating the experimental convergence of our discretization to each one of these operators.

Keywords: Discrete exterior calculus, discrete differential geometry, polygonal mesh, discrete wedge product, cup product, discrete Leibniz product rule, discrete Hodge star operator, discrete interior product, discrete Lie derivative.

Resumo

O cálculo exterior discreto eferece uma discretização do cálculo exterior que é independente de sistema de coordenadas e é especialmente adequada para computação nas variedades curvas. Neste trabalho, nós apresentamos uma versão desse cálculo estendida para as malhas poligonais gerais que evita a construção de qualquer malha dual e a necessidade de uma subdivisão combinatorial. Essencialmente, nossa abordagem introduz um novo produto wedge poligonal que é compatível com a derivada externa discreta no sentido de que a lei do produto de Leibniz é mantida válida. Com base nesse produto, nós definimos um novo operador estrela de Hodge discreto que não envolve malhas duais e que possibilita uma versão discreta do produto interior e da derivada de Lie de formas discretas. Nós incluimos os resultados de testes numéricos que indicam a convergência experimental de nossa discretização para cada um desses operadores.

Palavras-chave: Cálculo exterior discreto, geometria diferencial discreta, malha poligonal, produto wedge discreto, produto cup, regra de Leibniz discreta, operador estrela de Hodge discreto, produto interior discreto, derivada de Lie discreta.

Contents

Chapter 1

Introduction

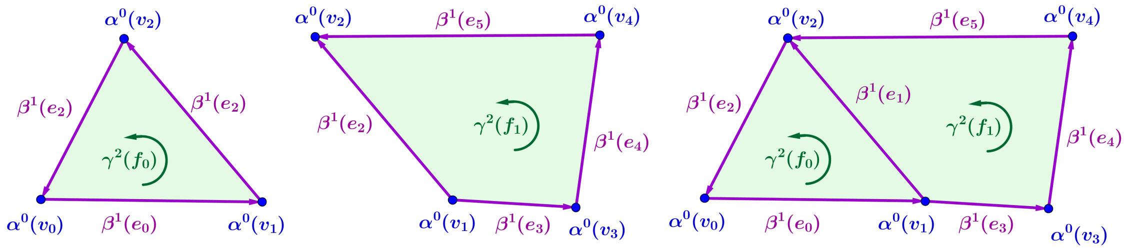

Figure 1.1: Discrete differential forms as cochains on simplicial and quadrilateral pseudomanifolds: 0–form α^0 is located on vertices, 1–form β^1 on edges, 2–form γ^2 on faces of a triangle (L), quad (C), and mixed mesh (R).

The exterior calculus of differential forms, first introduced by E. J. Cartan in 1945, has become the foundation of modern differential geometry. It is a coordinate–free calculus and therefore appropriate the analysis and calculations on curved spaces of differential manifolds.

The discretization of differential operators on surfaces is fundamental for geometry processing tasks, ranging from parameterization to remeshing. Discrete exterior calculus (DEC) is arguably one of the prevalent numerical frameworks to derive such discrete differential operators. However, the vast majority of work on DEC is restricted to simplicial meshes, and far less attention has been given to meshes formed by arbitrary polygons.

In this work, we propose a new discretization for several operators commonly associated to DEC that operate directly on polygons without involving any subdivision. Our approach offers three main practical benefits. First, by working directly with polygonal meshes, we overcome the ambiguities of subdividing a discrete surface into a triangle mesh. Second, our construction operates solely on primal elements, thus removing any dependency on dual meshes. Finally, our method includes the discretization of new differential operators such as contraction and Lie derivatives. We examined the accuracy of our numerical scheme by a series of convergence tests on flat and curved surface meshes.

1.1 Related Work and Applications

The first reference to DEC is considered to be [Bossavit 1998], where the author presents the theory of discretization through Whitney forms on a simplicial complex and thus creates the so–called Whitney complex. But some researchers believe it all started with the paper of Pinkall and Polthier

[Pinkall and Polthier 1993] on computing discrete minimal surfaces through harmonic energy minimization using the famous cotan Laplace formula. Discrete Laplace operators have been since used in mesh smoothing ([Crane et al. 2013a] and references therein), denoising, manipulation, compression, shape analysis, physical simulation, computing geodesic distances on general meshes [Crane et al. 2013b], and many other geometry processing tasks.

The first more complete textbook–like treatment about DEC on simplicial surface meshes we are aware of is the PhD thesis of [Hirani 2003], in a certain way an expansion of it is [Desbrun et al. 2005], where we can find extensions to dynamic problems. On the other hand, a concise introduction to the basic theory with the arguably most important operators can be found in [Desbrun et al. 2008]. These three publications have in common that they work with simplicial complexes only and use dual meshes. Moreover, all their wedge products are metric–dependent, except for [Hirani 2003, Definition 7.2.1], where we can find a discrete simplicial primal–primal wedge product that is actually identical to the classical cup product of simplicial forms presented in many books of algebraic topology, including [Munkres 1984] and [Massey 1991].

The SIGGRAPH 2013 Course material on Digital geometry processing with discrete exterior calculus [Crane et al. 2013c] also provides an essential mathematical background. Moreover, it shows how the geometry–processing tools on triangle meshes can be efficiently implemented within this single common framework.

An interesting but different new line of research is the so called *Subdivision Exterior Calculus* (SEC), a novel extension of DEC to subdivision surfaces that was introduced in [De Goes et al. 2016]. SEC explores the refinability of subdivision basis functions and retains the structural identities such as Stokes' theorem and Helmholtz–Hodge decomposition. It allows for computations directly on the control mesh while maintaining the properties of their inner products and differential operators recognized in [Wardetzky et al. 2007].

Next we summarize the key differences of our approach compared to existing schemes, but give slightly more attention to the cup product which, for us, is the appropriate discrete version of the wedge product.

The Cup Product and the Wedge Product

On smooth manifolds, the wedge product allows for building higher degree forms from lower degree ones. Similarly a cup product is a product of two cochains of arbitrary degree p and q that returns a cochain of degree $p+q$ located on $(p+q)$–dimensional cells. The cup product was introduced by J. W. Alexander, E. Čech, and H. Whitney [Whitney 1957] in 1930's and it became a well–studied notion in algebraic topology, mainly in the simplicial setting [Munkres 1984, Fenn 1983]. Later, the cup product was extended also to n–cubes [Massey 1991, Arnold 2012].

Influential for us was the PhD thesis of Rachel Arnold [Arnold 2012], where various cup products on simplicial and cubical n–pseudomanifolds are presented. She defines L^2 cubical Whitney forms and a cubical cup product that fits together with the wedge product of cubical Whitney forms. She then defines discrete Hodge star operators on cell complexes without reference to any dual cell complexes in order to prove the Poincaré duality in simplicial and, more importantly, in the cubical setting. She explores the extent to which the theory surrounding the Hodge star and Poincaré duality may be recovered in the absence of a dual complex. Her Hodge star operators are metric–independent, unlike in the continuous setting, which is fully sufficient for her purpose to prove the purely topological results.

In the case of the cup product on simplices, Rachel Arnold in [Arnold 2012] defines her cup product as integral of the wedge product of two simplicial Whitney forms (Definition 4.2.1 therein), where she cites Wilson ([Wilson 2007]) as the author. They defined their cup product of two cochains of any degree and on any dimension. But Xianfeng Gu already in [Gu 2002, Theorem 4.7] introduced a discrete wedge product of two simplicial 1–forms that is equivalent to the one of Wilson, yet he defined his discrete wedge product only for two 1–forms. The same discrete wedge product then appeared also in [Gu and Yau 2003].

In [Gonzalez–Diaz et al. 2011a] cup products on cubical complexes have been applied to compute cohomology rings of 3D digital images (represented as cubical complexes $Q(I)$), thus simplify the combinatorial structure of $Q(I)$ and obtain a homeomorphic cellular complex $P(I)$ with fewer cells, which improves the computational efficiency. The authors think of the cup product as the way the n–dimensional holes obtained in homology are related to each other. It is known that two objects with non isomorphic cohomology rings are not homotopic. They give an example of a torus and a wedge sum of two loops and a 2–sphere (Figure 1.2). Both these objects have two tunnels and one cavity, they have the same homology groups, but the cavity (γ) of the torus can be decomposed in the product of two tunnels (α and β), i.e., $\alpha \cup \beta = \gamma$, whereas the cavity of the second object cannot. However, for us the second object will not even be a pseudomanifold.

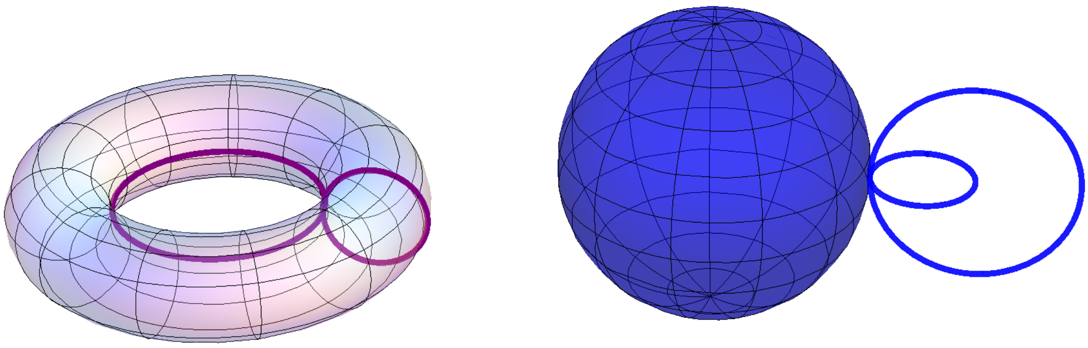

Figure 1.2: (L) Hollow torus and its two 1–dimensional holes. (R) Wedge sum of a 2–sphere and two loops. Both these objects have homology groups given as $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \times \mathbb{Z}$, and $H_2 = \mathbb{Z}$, where $H_* = \mathbb{Z}$ represents a finitely–generated abelian group with a single generator and $H_1 = \mathbb{Z} \times \mathbb{Z}$ represent the two independent generators in a finitely–generated abelian group.

Formulas for a diagonal approximation on a general polygon that are used to compute cup products on the cohomology of polyhedrons are presented in [Gonzalez–Diaz et al. 2011b]. Their cup product uses a complex structure called the AT–model (see [Gonzalez–Diaz et al. 2011b, Theorem 2]), and the coefficient of their cochains lie in the field $\mathbb{Z}_2$.

In common to previous approaches, this notion of discrete wedge product is metric–independent and satisfies core properties such as the Leibniz product rule with discrete exterior derivative, skew–commutativity, and associativity on closed forms.

A cup product on general polygons is investigated also in [Kravatz 2008], but their cup product is dependent on the choice of the so called minimal and maximal vertices and it is not clear where the result should live.

Alternative metric–dependent versions of the discrete wedge product on simplicial complexes were also suggested in [Hirani 2003, Section 7.2]. In particular, specialized expressions were necessary to address the different combination of primal and dual forms.

The Hodge Star Operator

The most common discretization of the Hodge star operator on triangle meshes is the so called diagonal approximation, which is computed based on the ratios between the volumes of primal simplices and dual cells. In contrast, we propose a Hodge star operator that is not an involution, i. e. it applied twice does not gives an identity map in general. Our Hodge star cannot even be invertible, because there is no isomorphism between groups of k–dimensional and $(2-k)$–dimensional cells of general meshes. This is a common drawback to all approaches that employ only primal meshes, we explain this further in Section 3.3.

The Inner Product

By employing our proposed discrete wedge product $\wedge$ and Hodge star operator $\star$, we define the inner product matrices by $\mathsf{M} := \wedge\star$, that turns to be identical (for the case of product of two 1–forms) to the one introduced in [Alexa and Wardetzky 2011].

The Contraction Operator and the Lie Derivative

The Lie derivative can be thought of as an extension of a directional derivative of a function to derivative of tensor fields (such as vector fields or differential forms) along a vector field. The Lie derivative is invariant under coordinate transformations, which makes it an appropriate version of a directional derivative on curved manifolds. It evaluates the change of a tensor field along the flow of another vector field and is widely used in mechanics.

Our discretization of Lie derivative of functions (0–forms) corresponds to the functional map framework of [Azencot et al. 2013], but now generalized to polygonal meshes. Our discrete Lie derivatives are thus linear operators on functions on a manifold that produce new functions, with the property that the derivative of a constant function is 0.

While maintaining the discrete exterior calculus framework, our work can also be interpreted as an extension of the Lie derivative of forms presented in [Mullen et al. 2011b] from planar regular grids to surface polygonal meshes in space.

1.2 Contributions and Overview

We start our treatment with a brief introduction to the discrete world of pseudomanifolds (Chapter 2), chains of polygonal faces, and discrete differential forms as cochains on these pseudomanifolds (see Figure 1.1). We also define the boundary and coboundary operators, the later being the discrete version of the exterior derivative.

Our contributions are presented in Chapter 3, we:

- Define a discrete wedge product as a cup product on polygonal complexes and show experimental convergence of the product of two discrete forms to the continuous wedge product of respective differential forms (Section 3.2).

- Provide a novel primal–primal discretization of the Hodge star operator (Section 3.3) that is compatible with the discrete wedge product.
- Using these two operators we derive a discrete inner product (Section 3.4).
- Employing the discrete Hodge star and wedge product we then define the contraction operator (Section 3.5). Together with the discrete exterior derivative we then use the Cartan's magic formula to derive the discrete Lie derivative and we discuss its convergence to the continuous analog in Section 3.6.

Our findings are then summarized and further research is suggested in Chapter 4.

Chapter 2

Pseudomanifolds and Discrete Differential Forms

Let us start with an informal motivation. Imagine we have a smooth manifold M and tangent bundle on it TM, which is a set of all tangent vectors in M, i.e., all tangent spaces T_xM to M at $x \in M$. A $\mathbb{R}$–valued (resp. vector–valued) differential k–form ω^k is a smooth antisymmetric k–linear map of a set of k tangent vectors to a scalar (resp. vector), that is:

$$\omega^k : TM \times \cdots \times TM \to \mathbb{R} \text{ (resp. } \mathbb{R}^n\text{)}.$$

We denote the vector space of k–forms as $\Omega^k(M)$.

For example, a 0–form is just a function (we evaluate them at points), but a 1–form maps a tangent vector to a scalar, and a 2–form assigns one scalar value to two tangent vectors $T_x \times T_x$ at some point $x \in M$.

In the discretization process it is therefore natural to define discrete 0–forms as scalars representing the value of a function at vertices, discrete 1–forms as scalars representing a differential 1–form on tangent vector field integrated along oriented edges, and discrete 2–forms as scalars encoding a density through its area integral over oriented faces.

For example, let $\alpha \in \Omega^0(M), \beta \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$ be differential forms on a differentiable 2–manifold M and let K be a cellular complex (a mesh) on the manifold M. We define the corresponding discrete differential forms (cochains, see Definition 2.3.1) $\alpha_D \in C^0(K), \beta_D \in C^1(K), \omega_D \in C^2(K)$ on each element of the mesh K as

$$\alpha_D(v) = \alpha(v), \quad \beta_D(e) = \int_e \beta, \quad \omega_D(f) = \int_f \omega,$$

where v is a vertex, e an edge, and f a face of K. We will omit the subscript $_D$ to denote discrete forms and instead we will understand that a form is discrete iff it is from the group $C^*(K)$.

Further, we do not just want any oriented edges or faces in space, we want a well–behaved set of vertices, edges, and faces: such well–behaved sets are the *(orientable) pseudomanifolds*.

A reader familiar with terms like cell complex, boundary operator, and cochains, may skip this chapter and go straight to Chapter 3, where our actual contributions are presented and new operators are defined.

2.1 CW Pseudomanifolds

Now we will go through some notions from algebraic topology to get to the definition of *orientable* and *nonorientable pseudomanifolds* (Definition 2.2.2). Orientable and nonorientable pseudomanifolds whose faces are polygons are the domains of our interest. Even though in practice we will work with even better behaved set of cells (see Remark 2.1).

Definition. A space is called a ***k*-cell** if it is homeomorphic to the unit k-ball B^k. It is called an **open cell** of dimension k if it is homeomorphic to Int B^k. The set $\dot{e}_i^k := \bar{e}_i^k - e_i^k$ for $k > 0$ is called the **boundary** of the k-cell e_i^k.

Note that an isolated point is an open and closed 0-cell since it is the whole space of dimension 0. Next we give the definition of a special class of cell complexes, called finite regular CW complexes; for more details about CW complexes see for example [Munkres 1984, §38].

Definition 2.1.1. A **finite CW complex** is a space X and a finite collection of disjoint open cells e_i whose union is X such that:

1. X is Hausdorff.
2. For each open m-cell e_i of the collection, there exist a continuous map $f_i : B^m \to X$ that maps IntB^m homeomorphically onto e_i and carries BdB^m into a union of open cells, each of dimension less than m.

If the maps f_i can be taken to be homeomorphisms, and each set $\dot{e}_i := \bar{e}_i - e_i$ equals the union of some open cells of X, then X is called a **finite regular CW complex**.

The Figure 2.1 shows an example of a regular and a non-regular CW complex.

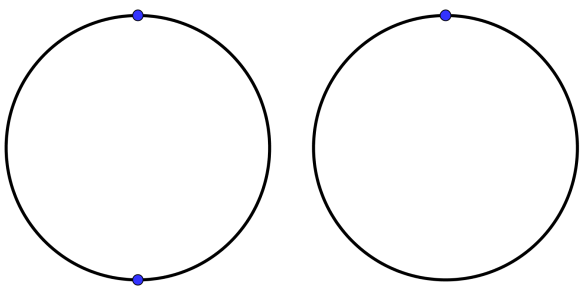

Figure 2.1: (L) A regular CW complex. (R) A non-regular CW complex: The complex consisting of a point on a sphere is not regular because both the endpoints of the 1-cell (circle) get mapped to the single 0-cell (the point), therefore the map of the boundary of the 1-cell is not a homeomorphism.

Definition. We say that e^m is a **face** of e^n if $e^m \subset \bar{e}^n$, and denote as $e^m \preceq e^n$. If $e^m \neq e^n$, then e^m is a **proper face** of e^n ($e^m \prec e^n$). The subspace X^p of X that is the union of the open cells of X of dimension at most p is called the ***p*-skeleton** of X and it is a CW complex in its own right.

Definition 2.1.2. A **CW *n*-pseudomanifold** is an n-dimensional finite regular CW complex which satisfies the following three conditions:

1. Every cell is a face of some n-cell.
2. Every $(n-1)$-dimensional cell is a face of exactly two n-cells.
3. Given any two n-cells, e_a^n and e_b^n, there exist a sequence of n-cells

$$e_0^n, e_1^n, \dots, e_k^n$$

such that $e_a^n = e_0^n$, $e_b^n = e_k^n$, and e_{i-1}^n and e_i^n have a common $(n-1)$-dimensional face ($i = 1, \dots, k$).

Remark. The definition above is due to Massey; by pseudomanifold he actually means a **pseudomanifold without boundary**. Massey calls compact the closed manifolds, i.e., by compact he means compact without boundary. As mentioned in [Massey 1991, Chapter IX], it can be shown that any regular CW complex on a closed connected n-manifold is an n-dimensional CW pseudomanifold. It is known that every compact n-manifold admits a subdivision so as to define a regular CW complex structure on it if $n \leq 3$. However, there exist compact 4-manifolds which do not admit such a subdivision. Some important properties of pseudomanifolds can be found also in [Geoghegan 2008, Section 12.3].

A pseudomanifold is a more general notion than a piecewise linear (PL) manifold, which is a topological manifold with a piecewise linear structure on it. A pseudomanifold may not be a topological manifold as it may have singularities of codimension 2. Yet it is a notion specific enough so as to allow for the definition of orientability or nonorientability (as we will see in the next section).

We complete the treatment with the following definition of an n–dimensional pseudomanifold with boundary, which is a slight modification of the given notion presented in [Seifert and Threlfall 1980, Paragraph 24].

Definition 2.1.3. A **CW *n*-pseudomanifold with boundary** is an n-dimensional finite regular CW complex which satisfies the following three conditions:

1. Every cell is a face of some n-cell.
2. Every $(n-1)$-cell is a face of at most two n-cells and there exist at least one $(n-1)$-cell which is incident with only one n-cell.
3. Given any two n-cells, e_a^n and e_b^n, there exist a sequence of n-cells

$$e_0^n, e_1^n, \dots, e_k^n$$

such that $e_a^n = e_0^n$, $e_b^n = e_k^n$, and e_{i-1}^n and e_i^n have a common $(n-1)$-dimensional face ($i = 1, \dots, k$).

2.2 Orientable Pseudomanifolds

The next definitions are actually a result of a set of theorems in [Massey 1991, Chapter IX], but for our purposes their implications are sufficient.

Definition 2.2.1. Let $K = \{K^n\}$ be a pseudomanifold on the topological space X. And let $[e^n_\lambda : e^{n-1}_\mu]$ be the **incidence number** of the cells e^n_λ and e^{n-1}_μ, for $n > 0$, such that:

1. If e^{n-1}_μ is not a face of e^n_λ, then $[e^n_\lambda : e^{n-1}_\mu] = 0$.

2. If e^{n-1}_μ is not a face of e^n_λ, then $[e^n_\lambda : e^{n-1}_\mu] = \pm 1$.

3. If e^0_α and e^0_β are the two vertices of the 1-cell e^1_λ, then $[e^1_\lambda : e^0_\alpha] + [e^1_\lambda : e^0_\beta] = 0$.

4. Let e^n_λ and e^{n-2}_ρ be cells such that $e^{n-2}_\rho \prec e^n_\lambda$; let e^{n-1}_α and e^{n-1}_β denote the unique $(n-1)$-cells e^{n-1} such that $e^{n-2}_\rho \prec e^{n-1} \prec e^{n-1}_\lambda$. Then

$$[e^n_\lambda : e^{n-1}_\alpha][e^{n-1}_\alpha : e^{n-2}_\rho] + [e^n_\lambda : e^{n-1}_\beta][e^{n-1}_\beta : e^{n-2}_\rho] = 0.$$

With these conditions it is possible to choose an **orientation** for each cell e^n_λ in one and only one way.

Thus we can specify orientations for the cells of a pseudomanifold by specifying a set of incidence numbers for the complex. Even though the definition may look slightly cumbersome, it actually gives the intuitive way to specify the orientation of cells, as we demonstrate in the Example 2.2.1.

Example 2.2.1. When using the definition, we assume we have a list of cells of K together with the information as to whether $e^{n-1}_i \prec e^n_j$ for any two cells e^{n-1}_i and e^n_j.

1. For each 1-cell e^1, choose incidence numbers between it and its two vertices such that conditions (2) and (3) hold, all other incidence numbers will be zero.

2. Now assume, inductively, that incidence numbers have been chosen between all cells of dimension $< n$. Let e^n be an n-cell. Choose a face e^{n-1}_0 of e^n, and choose $[e^n : e^{n-1}_0]$ to be +1 or -1. Using condition (4), determine $[e^n : e^{n-1}_i]$ for all $(n-1)$-cells $e^{n-1}_i \prec e^n$ which have an $(n-2)$-face in common with e^{n-1}_0. Spread over the boundary e^n by repeating this process. All other incidence numbers between e^n and $(n-1)$-cells will be zero by condition (1). And repeat this process for each n-cell of K.

For example in Figure 2.2, in the first step the incidence numbers between the 1-cell e^1_0 and its vertices were chosen to be $[e^1_0 : e^0_0] = -1$ and $[e^1_0 : e^0_1] = 1$. Next, we have chosen $[e^2_0 : e^1_0]$ to be +1, therefore the orientation of e^2_0 will be the same as that of e^1_0. But $[e^2_0 : e^1_1] = -1$.

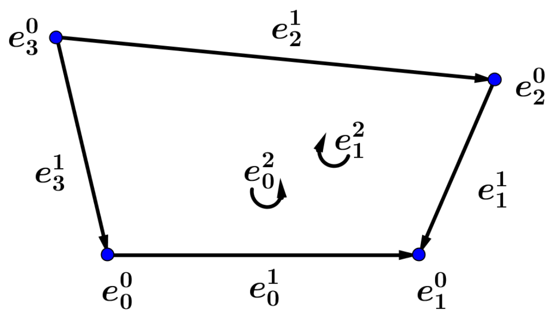

Figure 2.2: Boundary homomorphism on 2-cells e^2_0 and e^2_1.

Definition 2.2.2. Let K be an n-dimensional pseudomanifold (possibly with or without boundary), and let e_1^n, e_2^n be n-cells with a common $(n-1)$-cell e^{n-1}. We define orientations for e_1^n and e_2^n to be **coherent** (with respect to e^{n-1}) if:

$$[e_1^n : e^{n-1}] + [e_2^n : e^{n-1}] = 0.$$

A set of orientations for all the n-cells of K is said to be **coherent** if it is coherent in the above sense for any pair of n-cells with a common $(n-1)$-face.

K is said to be **orientable** if all its n-cells can be oriented such that any pair of n-cells sharing an $(n-1)$-dimensional face are oriented coherently. Otherwise it is called **nonorientable**.

The orientability or nonorientability of an n-dimensional pseudomanifold K only depends on the underlying topological space involved, and not on the choice of the regular CW complex K.

Before we define the boundary operator, we need one more definition:

Definition 2.2.3. Let K be a pseudomanifold. A ***p*-chain** on K is a function c from the set of oriented p-cells of K to the integers, such that $c(e) = -c(e')$ if e and e' are opposite orientations of the same cell. We add p-chains by adding their values, the resulting group is denoted $C_p(K)$ and is called the **chain group** of K. If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

The incidence numbers, or equivalently, the set of orientations of p-cells of K, give us such a p-chain.

Definition 2.2.4. Let $C_n(K), n \geq 0$, be the chain groups of K. The **boundary homomorphism**

$$\partial_n : C_n(K) \to C_{n-1}(K),\ n > 0,$$

is defined to be

$$\partial_n(e^n) = \sum_\lambda [e^n : e_\lambda^{n-1}] e_\lambda^{n-1}, \tag{2.1}$$

where e^n is an *oriented* n-cell of K with $n > 0$.

From the definition of incidence numbers it follows that ∂_n is well-defined and that $\partial_n(-e^n) = -\partial_n(e^n)$. Next we give an example of computation of the boundary homomorphism in a polygon:

Example 2.2.2. Let e_0^2 and e_1^2 be two 2-cells with opposite orientation as in Figure 2.2. We will check that $\partial_2 e_0^2 = -\partial_2 e_1^2$ and $\partial_2\partial_1 = 0$. We have:

$$\begin{aligned} \partial_2 e_0^2 &= +e_0^1 - e_1^1 - e_2^1 + e_3^1, \\ \partial_2 e_1^2 &= -e_0^1 + e_1^1 + e_2^1 - e_3^1, \end{aligned}$$

thus $\partial_2 e_0^2 = -\partial_2 e_1^2$. And for $\partial_1\partial_2 e_0^2$, omitting the indexes of ∂, we can write

$$\begin{aligned} \partial\partial e_0^2 &= \partial(+e_0^1 - e_1^1 - e_2^1 + e_3^1) = +\partial e_0^1 - \partial e_1^1 - \partial e_2^1 + \partial e_3^1 \\ &= e_1^0 - e_0^0 - (e_1^0 - e_2^0) - (e_2^0 - e_3^0) + e_0^0 - e_3^0 = 0. \end{aligned}$$

Definition 2.2.5. A **chain complex** $K = \{K_i, \partial_i\}$ is a sequence of abelian groups $K_i, i \in \mathbb{Z}$, and a sequence of homomorphisms $\partial_i : K_i \to K_{i-1}$ which are required to satisfy the condition

$$\partial_{i-1}\partial_i = 0 \ \ \forall i.$$

For any such chain complex $K = \{K_i, \partial_i\}$ we have that $\operatorname{im}\partial_{i+1} \subset \ker\partial_i \subset K_i$ and we can define

$$H_i(K) = \frac{\ker\partial_i}{\operatorname{im}\partial_{i+1}},$$

called the ith **homology group of** K.

If K is an n-dimensional pseudomanifold, then $K^p = \emptyset$ for $p < 0$ and $p > n$, thus $C_{n+1}(K) = 0$ and the chain complex reads:

$$0 \longrightarrow C_n(K) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{k+1}} C_k(K) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_1} C_0(K) \longrightarrow 0.$$

2.3 Discrete Differential Forms and the Exterior Derivative

Having learned about chains and the boundary homomorphism we can now define cochains and the coboundary operator on them, which correspond to our *discrete differential forms* and the *discrete exterior derivative*.

Definition 2.3.1. Let K be a pseudomanifold and $C_n(K)$ the group of oriented n-chains of K. Let G be an abelian group. The group of ***n*-dimensional cochains** of K, with coefficients in G, is the group

$$C^n(K) = \text{ Hom }(C_n(K), G).$$

The **coboundary operator** d is defined to be the dual of the boundary operator $\partial_n : C_{n+1}(K) \to C_n(K)$, i.e., it is the homomorphism

$$d : C^n(K) \to C^{n+1}(K),$$

such that $dd = 0$.

We can think of a cochain group $C^n(K)$ as dual of a chain group $C_n(K)$. We now give the formal definition of discrete differential forms and the discrete exterior derivative:

Definition 2.3.2. A **discrete *q*–form** α^q on a pseudomanifold K is an element of $C^q(K)$, the group of q–dimensional cochains of K, that is

$$\alpha^q \in C^q(K) = \text{ Hom }(C_q(K), G).$$

The **discrete exterior derivative** $d_q : C^q(K) \to C^{q+1}(K)$ is the coboundary operator and it holds:

$$(d_q\alpha^q)(c_{q+1}) = \alpha(\partial_{q+1}c_{q+1}) = \sum_{c \in C_q(K)} [c_{q+1} : c]\alpha(c). \tag{2.2}$$

The most common choices of the coefficient group G are $(\mathbb{R},+)$, $(\mathbb{C},+)$, or a vector space V together with addition, which correspond to **$\mathbb{R}$–valued**, **$\mathbb{C}$–valued**, and **vector–valued discrete differential forms**.

Similarly to the definition of chain complexes (Definition 2.2.5), we have the following:

Definition 2.3.3. A **cochain complex** $(C^*(K,R),d)$ is a sequence of abelian groups of n-dimensional cochains $C^n(K)$ together with the coboundary operators $d_n : C^n(K) \to C^{n+1}(K)$, i.e.,

$$0 \xleftarrow{d_n} C^n(K) \xleftarrow{d_{n-1}} \cdots \xleftarrow{d_k} C^k(K) \xleftarrow{d_{k-1}} \cdots \xleftarrow{d_0} C^0(K) \longleftarrow 0.$$

A q–form α is said to be a **closed form** (cocycle) if $d\alpha = 0$, and it is called an **exact form** (coboundary) if there exist a $(q-1)$–form β such that $d\beta = \alpha$. Two closed q-forms are **cohomologous** if they differ by an exact q–form, i.e.,

$$H^q(C^*) = \frac{\ker d_q(C^*)}{\operatorname{im} d_{q-1}(C^*)}.$$

Note that each exact q-form $\alpha = d\beta$ is closed, since $d\alpha = dd\beta = 0$.

If M is a smooth manifold and d is the exterior derivative, then the complex $(\Omega^*(M),d)$ is called the **de Rham cohomology complex** on M. The quotient of the real vector space of closed q–forms by the subspace of exact q–forms on M is called the ***q*–th de Rham cohomology group** of M and is denoted $H_{dR}(M)$.

Next we state an important theorem in differential geometry, called the Stokes' theorem, whose equivalent is valid also in the discrete setting.

Theorem 2.3.1 (Stokes)**.** *Let M^n be an orientable n–(pseudo)manifold with boundary ∂M^n and ω an $(n-1)$-form on M^n with compact support, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

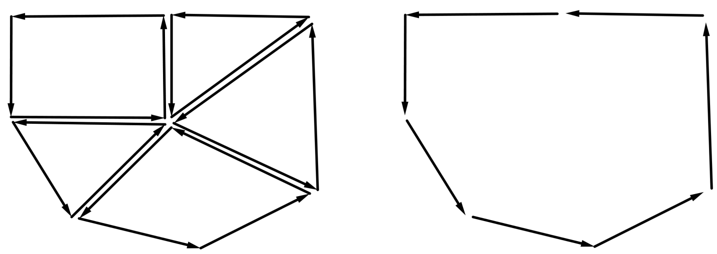

Figure 2.3: Stokes' theorem on an orientable 2–pseudomanifold: a 2–form $d\alpha$ on faces is the sum of 1–form α as in (2.2) and two faces sharing an edge have opposite directions on that edge, thus the values on interior edges cancel each other pairwise and only the contribution from the boundary edges remains.

Stokes' theorem tells us that the value of an n-form $d\omega$ over an orientable n-manifold is equal to the value of ω over its whole boundary. In the case of an orientable 2–dimensional pseudomanifold, the validity can be easily shown – see Figure 2.3.

Chapter 3

Contributions

This chapter contains the actual results of our research. Our approach is based on a discrete version of the wedge product – the cup product (Section 3.2). Please note that we constantly switch between the terms cup product and discrete wedge product, since they are equivalent for us.

3.1 Important Matrices

Throughout this report, we assume that K is a 2–dimensional pseudomanifold with polygonal faces $f \in F$. Let us further denote $v \in V$ the vertices, $e \in E$ the edges, and $h \in H$ the halfedges of the mesh K.

The boundary homomorphisms and discrete derivatives can be represented by these matrices:

$$d_0 = \partial_1 = \begin{pmatrix} [e_0 : v_0] & \cdots & [e_0 : v_{|V|-1}] \\ \vdots & \ddots & \vdots \\ [e_{|E|-1} : v_0] & \cdots & [e_{|E|-1} : v_{|V|-1}] \end{pmatrix}, \tag{3.1}$$

$$d_1 = \partial_2 = \begin{pmatrix} [f_0 : e_0] & \cdots & [f_0 : e_{|E|-1}] \\ \vdots & \ddots & \vdots \\ [f_{|F|-1} : e_0] & \cdots & [f_{|F|-1} : e_{|E|-1}] \end{pmatrix}. \tag{3.2}$$

We also denote B the incidence matrix between vertices and edges without distinction of the orientation, where $[e_i : v_j]$ is $\frac{1}{2}$ if $v_j \prec e_i$ and 0 otherwise:

$$\mathsf{B} = \frac{1}{2} d_0 \odot d_0, \tag{3.3}$$

where $\odot$ is the Hadamard (element–wise) product.

Further, the matrix fv represents the "weighted" incidence relation between vertices and faces, such that fv_{ij} is $\frac{1}{p_i}[f_i : v_j]$ for a p_i–polygonal face f_i, where $[f_i : v_j]$ is 1 if $v_j \prec f_i$ and 0 otherwise:

$$\mathsf{fv} = \begin{pmatrix} \frac{1}{p_0}[f_0 : v_0] & \cdots & \frac{1}{p_0}[f_0 : v_{|V|-1}] \\ \vdots & \ddots & \vdots \\ \frac{1}{p_{|F|-1}}[f_{|F|-1} : v_0] & \cdots & \frac{1}{p_{|F|-1}}[f_{|F|-1} : v_{|V|-1}] \end{pmatrix}. \tag{3.4}$$

To be later able to write the cup product in matrix form we need two more matrices denoted as R and L. So let us now assume that we know the **order of edges of a face** f which is a p–polygon, we define the incidence matrices $\mathsf{R} \in \mathbb{R}^{p\times p}$ and $\mathsf{L} \in \mathbb{R}^{p\times p}$ per face such that:

$$\mathsf{L}[k,j] = \begin{cases} 1 & \text{if } e_j \text{ is the } k\text{–th edge of the face } f, \\ -1 & \text{if } -e_j \text{ is the } k\text{–th edge of the face } f, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathsf{R} = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p} \right) \mathsf{R}_a,$$

where

$$\mathsf{R}_a[k,j] = \begin{cases} 1 & \text{if } e_j \text{ is the } (k+a)\text{–th edge or } -e_j \text{ is the } (k-a)\text{–th edge of } f, \\ -1 & \text{if } e_j \text{ is the } (k-a)\text{–th edge or } -e_j \text{ is the } (k+a)\text{–th edge of } f, \\ 0 & \text{otherwise.} \end{cases}$$

If we wish to express the discrete wedge product globally (for the whole mesh), we need to work with halfedges instead of edges. In such a case the matrices $\mathsf{L}, \mathsf{R} \in \mathbb{R}^{|H|\times|H|}$ read per a p–polygonal face f:

$$\mathsf{L}[k,j] = \begin{cases} 1 & \text{if } h_j \text{ is the } k\text{–th halfedge of the face } f \text{ with the same orientation as } f, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

$$\mathsf{R} = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p} \right) \mathsf{R}_a, \tag{3.6}$$

where

$$\mathsf{R}_a[k,j] = \begin{cases} 1 & \text{if } h_j \text{ is the } (k+a)\text{–th halfedge of } f \text{ with the same orientation as } f, \\ -1 & \text{if } h_j \text{ is the } (k-a)\text{–th halfedge of } f \text{ with the same orientation as } f, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

And the exterior derivative on 1–forms $d_1 \in \mathbb{R}^{|F|\times|H|}$ becomes:

$$d_1[i,j] = \begin{cases} 1 & \text{if } h_j \text{ is a halfedge of } f_i \text{ with the same orientation as } f_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

3.2 Wedge and Cup Products

In this section we first present the wedge product in the continuous world and then give a general definition of a cup product of cochains to show the clear analogy between these two. We then proceed with equations of cup products on triangles and quadrilaterals, by which we were inspired and later derived formulas for cup products on general polygons (Subsection 3.2.3).

The Wedge Product

A differential k-form α on a smooth n–manifold M is a tensor field of type $(0,k)$ that is completely antisymmetric, i.e., for $\alpha \in \Omega^k(M) \subset T_k^0(M)$ we have $\alpha(v,w) = -\alpha(w,v)$ for all $v, w \in TM$. Let $\{e_1, \dots, e_n\}$ be a basis of the tangent space T_xM to M at point $x \in M$. Let $\alpha \in T_k^0(M)$ and $\beta \in T_l^0(M)$, we define their **wedge product** $\alpha \wedge \beta \in T_{k+l}^0(M)$ at the point $x \in M$ by

$$(\alpha \wedge \beta) = \sum_{\tau} \operatorname{sign} \tau \alpha(e_{\tau(1)}, \dots, e_{\tau(k)}) \beta(e_{\tau(k+1)}, \dots, e_{\tau(k+l)}), \tag{3.9}$$

where the sum is over all permutations τ of $\{1, \dots, k+l\}$ such that $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots < \tau(k+l)$. And we have that $\operatorname{sign} \tau = +1$ if the permutation is even and $\operatorname{sign} \tau = -1$ if it is odd.

Examples of computing the wedge products can be found in [Abraham et al. 1988, Section 7.1]. The wedge product exhibits the following properties, which we maintain also in the discrete setting except the associativity (compare with the later Proposition 3.2.3):

Proposition 3.2.1. *For $\alpha \in T_k^0(M)$, $\beta \in T_l^0(M)$, and $\gamma \in T_m^0(M)$, we have*

1. *$\wedge$ is bilinear: $\alpha \wedge (c_1\beta + c_2\gamma) = c_1(\alpha \wedge \beta) + c_2(\alpha \wedge \gamma)$ and $(c_1\alpha + c_2\beta) \wedge \gamma = c_1(\alpha \wedge \gamma) + c_2(\beta \wedge \gamma)$ for some constants $c_1, c_2 \in \mathbb{R}$,*

2. *$\wedge$ is skew commutative: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$,*

3. *$\wedge$ is associative: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$,*

4. *Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.*

For the proof see [Abraham et al. 1988, Proposition 7.1.5 and Theorem 7.4.1].

A Cup Product

The wedge product allows for building higher degree forms from lower degree ones, similarly a cup product is a product of cochains of arbitrary degree p and q that returns a cochain of degree $p+q$.

Whitney in [Whitney 1957, §9 of Appendix II] gives the following abstract definition of a cup product, which also appears in [Arnold 2012, Definition 2.3.3].

Definition 3.2.1. Let X be a cell complex, the **cup product** of two cochains c^p and c^q is a bilinear operation $\cup : C^p(X) \times C^q(X) \to C^{p+q}(X)$ that satisfies the following three properties:

1. Let $\sigma_p \in C_p(X)$ and $\sigma_q \in C_q(X)$. Then $\sigma^p \cup \sigma^q$ is a $(p+q)$-cochain in $St(\sigma_p) \cdot St(\sigma_q)$, where $St(\sigma_i)$ is the union of all cells in which c_i is a face, and $A \cdot B$ denotes the union of all cells in A and B.

2. $d(c^p \cup c^q) = dc^p \cup c^q + (-1)^p c^p \cup dc^q$ (Leibniz rule).

3. If X is connected, then there exist a real number $\gamma_\cup$ such that $I^0 \cup c^p = c^p \cup I^0 = \gamma_\cup c^p$, where I^0 is the constant 0-cochain that takes value 1 on the 0-cells of X.

The interested reader is also referred to [Massey 1991, Chapter XIII]. Whitney further asserts that thanks to the Leibniz rule being valid, we have the following:

Proposition 3.2.2. *For $\alpha \in H^k(X)$, $\beta \in H^l(X)$, $\gamma \in H^m(X)$, the cup product defines a bilinear operation $H^k(X) \times H^l(X) \to H^{k+l}(X)$, which is uniquely determined (well–defined) and is*

1. *skew commutative: $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$,*
2. *associative: $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$.*

Thus the cup endows the cellular cohomology $H^*(X)$ with a ***graded commutative ring*** *structure, with the cohomology class of I^0 as unit element.*

Comparing Proposition 3.2.1 with Definition 3.2.1 and Proposition 3.2.2, we affirm that both the wedge and the cup product, respectively, are bilinear operations that take as input two forms, resp. cochains, of arbitrary degree p and q and return a form, resp. cochain, of degree $p + q$. Moreover, they both satisfy the Leibniz rule. But a general cup products is associative and skew–commutative only on cohomology. Fortunately, for cup product on polygons we will recuperate the skew–commutativity for any discrete forms (not just closed ones). However, our polygonal cup product will be associative only on closed forms, see the Proposition 3.2.3.

Even though explicit formulas for computing cup product on a general cell complex are unknown, there are well–known explicit formulas for cup products on simplicial and cubical complexes (e.g., see [Arnold 2012]), and we expand the set of known cup products with *cup products on general polygons with coefficients in $\mathbb{R}$, which is one of our contributions*.

3.2.1 The Cup Product on Simplices

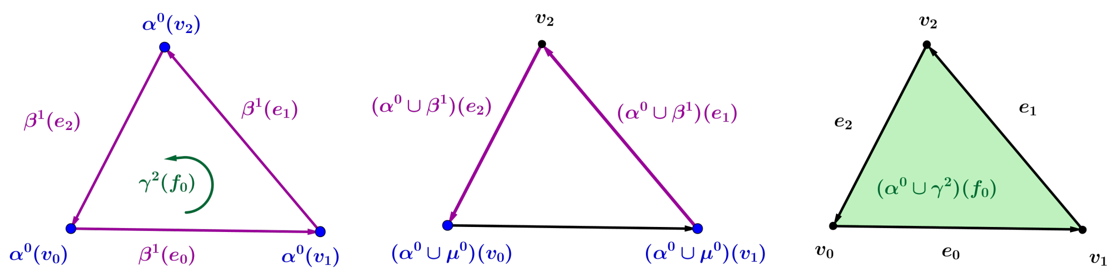

Figure 3.1: The cup product of simplicial forms is again either an 0–form located on vertices, 1–form located on edges, or an 2–form located on faces.

Arnold in [Arnold 2012, Definition 4.2.1] presents the definition of the **Wilson's cup product** $\cup : C^p(K) \times C^q(K) \to C^{p+q}(K)$ given by

$$\alpha \cup \beta(c) = \int_c W\alpha \wedge W\beta, \tag{3.10}$$

where W is the simplicial Whitney form (see [Arnold 2012, Definition 4.1.1]) and c is an $(p + q)$–simplex.

Using the equation (3.10) we computed simplicial cup products which we state as the following definition:

Definition 3.2.2. Let $(v_0, \ldots, v_k)$ be a k–simplex, the **simplicial cup product**, i.e., the cup product of forms defined on simplices reads:

$$\begin{aligned}
(\alpha^0 \cup \beta^0)(v_0) &= \alpha(v_0)\beta(v_0), \\
(\alpha^0 \cup \beta^1)(v_0, v_1) &= \frac{1}{2}(\alpha(v_0) + \alpha(v_1))\beta(v_0, v_1), \\
(\alpha^0 \cup \beta^2)(v_0, v_1, v_2) &= \frac{1}{3}(\alpha(v_0) + \alpha(v_1) + \alpha(v_2))\beta(v_0, v_1, v_2), \\
(\alpha^1 \cup \beta^1)(v_0, v_1, v_2) &= \frac{1}{6}\sum_{i=0}^{2} \alpha(v_i, v_{i+1})(\beta(v_{i+1}, v_{i+2}) - \beta(v_{i-1}, v_i)), \; i \in \mathbb{Z}/3\mathbb{Z}.
\end{aligned}$$

The action of $\cup$ is illustrated in Figures 3.1 and 3.2.

Remark. In our derivation we actually obtained $(\alpha^0 \cup \beta^2) = \frac{1}{6}(\alpha(v_0) + \alpha(v_1) + \alpha(v_2))\beta(v_0, v_1, v_2)$ and not $\frac{1}{3}(\alpha(v_0) + \alpha(v_1) + \alpha(v_2))\beta(v_0, v_1, v_2)$. But in the case of $\frac{1}{6}$, the **Leibniz rule** for $d(\alpha^0 \cup \beta^1)$ is only satisfied if $d\beta = 0$, i.e., β is a closed form. Whereas for $\frac{1}{3}$, the Leibniz rule for $d(\alpha^0 \cup \beta^1) = d\alpha \cup \beta + \alpha \cup d\beta$ is satisfied for any α and β.

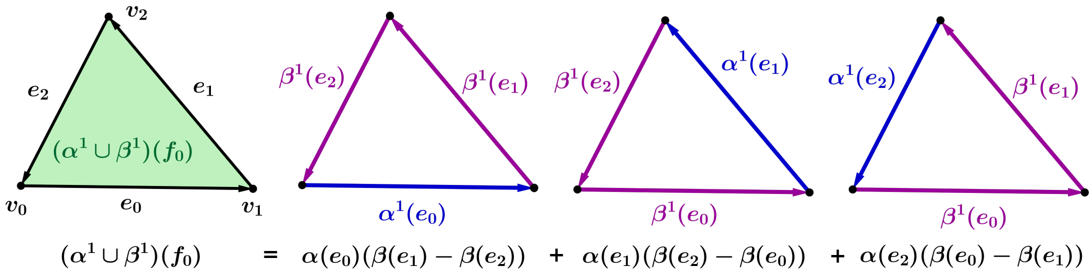

Figure 3.2: The cup product of two 1–forms is a 2–form located on faces (L).

The simplicial cup product satisfies the properties of Proposition 3.2.3, which we prove therein. Comparing the Proposition 3.2.3 with Propositions 3.2.2 and 3.2.1, we can see that our simplicial cup product is skew–commutative on any discrete forms, not only on cohomology as the generic cup product, but it is still not associative in general as the wedge product, only on cohomology.

3.2.2 The Cubical Cup Product

In [Arnold 2012, Definition 3.2.1] the author defines a cubical cup product $\cup_c$, for which the product of two cochains agrees with the wedge product of their **cubical Whitney forms** (see [Arnold 2012, Section 3.2.2]), thus the Whitney map provides a connection of cubical cohomology with de Rham cohomology ([Arnold 2012, Theorem 3.2.12]).

We do not state the definition of $\cup_c$ on an n–dimensional cubical pseudomanifold because it would involve a lot of new notation that is unnecessary for the case or our interest – the dimension 2, i.e., quadrilaterals. Therefore from [Arnold 2012, Definition 3.2.1] we computed the cup product on k–dimensional cubes for $0 \le k \le 2$ and we present the formulas in the following definition:

Definition 3.2.3. The **cubical cup product** $\cup : C^p(K) \times C^q(K) \to C^{p+q}(K)$ on a quadrilateral pseudomanifold K is defined by its action on a k–dimensional cube $(v_0, \dots, v_{2^k-1})$ as:

$$
\begin{aligned}
(\alpha^0 \cup \beta^0)(v_0) &= \alpha(v_0)\beta(v_0), \\
(\alpha^0 \cup \beta^1)(v_0, v_1) &= \frac{1}{2}(\alpha(v_0) + \alpha(v_1))\beta(v_0, v_1), \\
(\alpha^0 \cup \beta^2)(v_0, v_1, v_2, v_3) &= \frac{1}{4}\Big(\sum_{i=0}^{3} \alpha(v_i)\Big)\beta(v_0, v_1, v_2, v_3), \\
(\alpha^1 \cup \beta^1)(v_0, v_1, v_2, v_3) &= \frac{1}{4}\sum_{i=0}^{3} \alpha(v_i, v_{i+1})(\beta(v_{i+1}, v_{i+2}) - \beta(v_{i-1}, v_i)),\ i \in \mathbb{Z}/4\mathbb{Z}.
\end{aligned}
$$

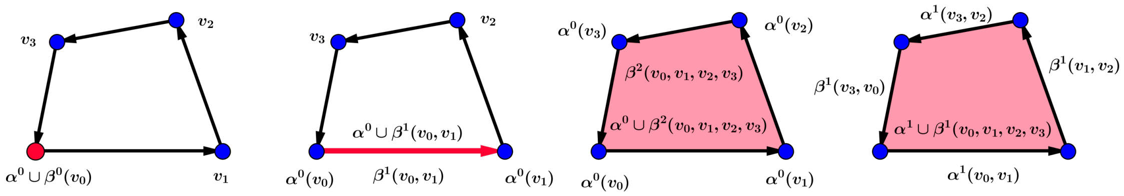

Figure 3.3: The cup product of cochains on a quadrilateral: for each degree the locations of the result of the product are colored red.

The cup product of two cubical forms is illustrated in Figure 3.3 and it has the same properties as the simplicial cup product – see Proposition 3.2.3.

3.2.3 The Cup Product on Polygons

As said earlier, we can extend the definitions of the discrete wedge product to any p–polygon such that it still satisfies the defining properties of a cup product (Definition 3.2.1) and therefore automatically also the properties of Proposition 3.2.2. Moreover, we show that our polygonal cup product is a skew–commutative bilinear operation on any polygonal forms, not just on the closed ones, it satisfies the Leibniz rule with discrete derivative, and it is associative on cohomology, viz. the Proposition 3.2.3.

Let us remember that a discrete q–form α^q is a q–cochain

$$\alpha^q = \sum_{c_q \in K} \alpha(c_q),$$

where c_q is a q–cell of a pseudomanifold K.

Definition 3.2.4. Let K be a 2–pseudomanifold whose 2–cells (faces) are polygons. The cup product $C^k(K) \times C^l(K) \to C^{k+l}(K)$ of two discrete forms α^k, β^l is a $(k+l)$–form $\alpha^k \cup \beta^l$ defined on each $(k+l)$–cell of K, that is,

$$\alpha^k \cup \beta^l = \sum_{c_{k+l} \in K} \alpha^k \cup \beta^l(c_{k+l}).$$

Let $f = (v_0, \dots, v_{p-1})$ be a p–polygonal face, (v_i, v_j) an edge, and v a vertex of K, the **polygonal cup**

product is defined for each degree and per each $(k+l)$–cell as:

$$(\alpha^0 \cup \beta^0)(v) = \alpha(v)\beta(v), \tag{3.11}$$

$$(\alpha^0 \cup \beta^1)(v_i, v_j) = \frac{1}{2}(\alpha(v_i) + \alpha(v_j))\beta(v_i, v_j), \tag{3.12}$$

$$(\alpha^0 \cup \beta^2)(f) = \frac{1}{p}\Big(\sum_{i=0}^{p-1} \alpha(v_i)\Big)\beta(f), \tag{3.13}$$

$$(\alpha^1 \cup \beta^1)(f) = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p}\right) \sum_{i=0}^{p-1} \alpha(i)(\beta((i+a)\%p) - \beta((i-a)\%p)), \tag{3.14}$$

where $\alpha(i) := \alpha(e_i) = \alpha(v_i, v_{(i+1)\%p})$ and $\%p$ means modulo p.

In Theorem 3.2.1 we prove that the polygonal cup product we have just defined actually *is* a cup product, that is, we prove that it satisfies the defining properties of a cup product (Definition 3.2.1). But we first prove, in the following Lemma, that it obeys the Leibniz rule with the discrete derivative.

Lemma 3.2.1. *The discrete wedge product from Definition 3.2.4 satisfies the Leibniz rule with the discrete derivative, that is:*

$$d_0(\alpha^0 \cup \beta^0) = d_0\alpha \cup \beta + \alpha \cup d_0\beta,$$
$$d_1(\alpha^0 \cup \beta^1) = d_0\alpha \cup \beta + \alpha \cup d_1\beta.$$

For $d(\alpha^0 \cup \beta^2)$ and $d(\alpha^1 \cup \beta^1)$ we get trivially 0 because there are no 3–dimensional chains (and thus no 3–forms) on a 2–dimensional pseudomanifold.

Proof. We will prove the Leibniz rule for each degree separately.

The case $d(\alpha^0 \cup \beta^0)$: let $e = (v_i, v_j)$ be an edge of K, then

$$\begin{aligned}
(d_0\alpha \cup \beta + \alpha \cup d_0\beta)(e) &= (\alpha(v_j) - \alpha(v_i))\frac{\beta(v_i) + \beta(v_j)}{2} + \frac{\alpha(v_i) + \alpha(v_j)}{2}(\beta(v_j) - \beta(v_i)) \\
&= \frac{\beta(v_i)(-2\alpha(v_i)) + \beta(v_j)(2\alpha(v_j))}{2} \\
&= \alpha(v_j)\beta(v_j) - \alpha(v_i)\beta(v_i) = (\alpha \cup \beta)(v_j) - (\alpha \cup \beta)(v_i) \\
&= d_0(\alpha^0 \cup \beta^0)(e).
\end{aligned}$$

The case $d(\alpha^0 \cup \beta^1)$: Let us first simplify the notation and denote $\alpha_i = \alpha^0(v_i)$, $\beta_i = \beta^1(e_i)$. Let also any index i to be understood as $i\%n$. Let $f = (v_0, \ldots, v_{n-1})$ be an n–polygonal face with edges $e_i = (v_i, v_{i+1})$. Due to the indexes being cyclic (modulo n), we can reorder the following sums in this manner:

$$\begin{aligned}
d(\alpha^0 \cup \beta^1)(f) &= \sum_{i=0}^{n-1} \frac{\alpha_i + \alpha_{i+1}}{2} \beta_i = \sum_{i=0}^{n-1} \frac{\beta_{i-1} + \beta_i}{2} \alpha_i \\
(d\alpha^0 \cup \beta^1)(f) &= \sum_{i=0}^{n-1} \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{n} \right) (\alpha_{i+1} - \alpha_i)(\beta_{i+a} - \beta_{i-a}) \\
&= \sum_{i=0}^{n-1} \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-2a}{2n} (\alpha_i)(\beta_{i+a-1} - \beta_{i-a-1} - \beta_{i+a} + \beta_{i-a}) \\
(\alpha^0 \cup d\beta^1)(f) &= \frac{1}{n} \sum_{i=0}^{n-1} \alpha_i \sum_{i=0}^{n-1} \beta_i
\end{aligned}$$

Thus we want to show that

$$\sum_{i=0}^{n-1} \alpha_i \frac{\beta_{i-1} + \beta_i}{2} = \sum_{i=0}^{n-1} \frac{\alpha_i}{2n} \Bigg[\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2a)(\beta_{i+a-1} - \beta_{i+a} + \beta_{i-a} - \beta_{i-a-1}) + 2 \sum_{j=0}^{n-1} \beta_j \Bigg],$$

but instead we will show that the following equation (3.15) holds for any $i \in \{0, \dots, n-1\}$, which implies that also the equality above is true.

$$\frac{\beta_{i-1} + \beta_i}{2} = \frac{1}{2n} \Bigg[\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2a)(\beta_{i+a-1} - \beta_{i+a} + \beta_{i-a} - \beta_{i-a-1}) + 2 \sum_{j=0}^{n-1} \beta_j \Bigg] \tag{3.15}$$

If n is even, then $a = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ is equivalent to $a = 1, 2, \dots, \frac{n}{2} - 1$, and if n is odd, then it is equivalent to $a = 1, \dots, \frac{n-1}{2}$. Thus we will first show the validity of (3.15) for n even and then for n odd.

For n even we compute:

$$\begin{aligned}
\sum_{a=1}^{\frac{n}{2}-1} (n-2a)(\beta_{k+a-1} - \beta_{k+a}) &= (n-2)(\beta_k - \beta_{k+1}) + (n-4)(\beta_{k+1} - \beta_{k+2}) \\
&\quad + (n-6)(\beta_{k+2} - \beta_{k+3}) + \cdots + 6(\beta_{k+\frac{n}{2}-4} - \beta_{k+\frac{n}{2}-3}) \\
&\quad + 4(\beta_{k+\frac{n}{2}-3} - \beta_{k+\frac{n}{2}-2}) + 2(\beta_{k+\frac{n}{2}-2} - \beta_{k+\frac{n}{2}-1}) \\
&= n\beta_k - 2(\beta_k + \beta_{k+1} + \beta_{k+2} + \beta_{k+3} + \cdots + \beta_{k+\frac{n}{2}-3} + \beta_{k+\frac{n}{2}-2} + \beta_{k+\frac{n}{2}-1}), \\
\sum_{a=1}^{\frac{n}{2}-1} (n-2a)(\beta_{k-a} - \beta_{k-a-1}) &= (n-2)(\beta_{k-1} - \beta_{k-2}) + (n-4)(\beta_{k-2} - \beta_{k-3}) \\
&\quad + (n-6)(\beta_{k-3} - \beta_{k-4}) + \cdots + 6(\beta_{k-\frac{n}{2}+3} - \beta_{k-\frac{n}{2}+2}) \\
&\quad + 4(\beta_{k-\frac{n}{2}+2} - \beta_{k-\frac{n}{2}+1}) + 2(\beta_{k-\frac{n}{2}+1} - \beta_{k-\frac{n}{2}}) \\
&= n\beta_{k-1} - 2(\beta_{k-1} + \beta_{k-2} + \beta_{k-3} + \cdots + \beta_{k+\frac{n}{2}+2} + \beta_{k+\frac{n}{2}+1} + \beta_{k+\frac{n}{2}}).
\end{aligned}$$

Thus

$$\sum_{a=1}^{\frac{n}{2}-1}(n-2a)(\beta_{k+a-1}-\beta_{k+a}+\beta_{k-a}-\beta_{k-a-1}) = n(\beta_{k-1}+\beta_k) - 2\sum_{i=k}^{k+n-1}\beta_i$$
$$= n(\beta_{k-1}+\beta_k) - 2\sum_{j=0}^{n-1}\beta_j$$

and for n even we obtain

$$\frac{1}{2n}\left[\sum_{a=1}^{\lfloor\frac{n-1}{2}\rfloor}(n-2a)(\beta_{k+a-1}-\beta_{k+a}+\beta_{k-a}-\beta_{k-a-1}) + 2\sum_{j=0}^{n-1}\beta_j\right] = \frac{\beta_{k-1}+\beta_k}{2}.$$

For n odd we proceed in a similar fashion:

$$\begin{aligned}
\sum_{a=1}^{\frac{n-1}{2}}(n-2a)(\beta_{k+a-1}-\beta_{k+a}) &= (n-2)(\beta_k-\beta_{k+1}) + (n-4)(\beta_{k+1}-\beta_{k+2}) \\
&\quad + (n-6)(\beta_{k+2}-\beta_{k+3}) + \cdots + 5(\beta_{k+\frac{n-1}{2}-3}-\beta_{k+\frac{n-1}{2}-2}) \\
&\quad + 3(\beta_{k+\frac{n-1}{2}-2}-\beta_{k+\frac{n-1}{2}-1}) + 2(\beta_{k+\frac{n-1}{2}-1}-\beta_{k+\frac{n-1}{2}}) \\
&= n\beta_k - 2(\beta_k+\beta_{k+1}+\beta_{k+2}+\beta_{k+3}+\cdots+\beta_{k+\frac{n-1}{2}-2}+\beta_{k+\frac{n-1}{2}-1}) - \beta_{k+\frac{n-1}{2}}, \\
\sum_{a=1}^{\frac{n-1}{2}}(n-2a)(\beta_{k-a}-\beta_{k-a-1}) &= (n-2)(\beta_{k-1}-\beta_{k-2}) + (n-4)(\beta_{k-2}-\beta_{k-3}) \\
&\quad + (n-6)(\beta_{k-3}-\beta_{k-4}) + \cdots + 5(\beta_{k-\frac{n-1}{2}+2}-\beta_{k-\frac{n-1}{2}+1}) \\
&\quad + 3(\beta_{k-\frac{n-1}{2}+1}-\beta_{k-\frac{n-1}{2}}) + (\beta_{k-\frac{n-1}{2}}-\beta_{k-\frac{n-1}{2}-1}) \\
&= n\beta_{k-1} - 2(\beta_{k+\frac{n-1}{2}+1}+\beta_{k+\frac{n-1}{2}+2}+\cdots+\beta_{k-3}+\beta_{k-2}+\beta_{k-1}) - \beta_{k+\frac{n-1}{2}}.
\end{aligned}$$

Thus

$$\sum_{a=1}^{\frac{n-1}{2}}(n-2a)(\beta_{k+a-1}-\beta_{k+a}+\beta_{k-a}-\beta_{k-a-1}) = n(\beta_{k-1}+\beta_k) - 2\sum_{i=k}^{k+n-1}\beta_i$$
$$= n(\beta_{k-1}+\beta_k) - 2\sum_{j=0}^{n-1}\beta_j$$

and for n odd we get

$$\frac{1}{2n}\left[\sum_{a=1}^{\lfloor\frac{n-1}{2}\rfloor}(n-2a)(\beta_{k+a-1}-\beta_{k+a}+\beta_{k-a}-\beta_{k-a-1}) + 2\sum_{j=0}^{n-1}\beta_j\right] = \frac{\beta_{k-1}+\beta_k}{2}.$$

We have shown that the equality (3.15) holds for any $i \in \{0,1,\ldots,n-1\}$, which implies $d_1(\alpha^0 \cup \beta^1) = d_0\alpha \cup \beta + \alpha \cup d_1\beta$. $\square$

Now we are ready to claim that our discrete wedge product on polygonal pseudomanifolds actually is a cup product:

Theorem 3.2.1. *The polygonal cup product from Definition 3.2.4 is a cup product, i.e., it satisfies the Definition 3.2.1.*

Proof. The equations (3.11)–(3.14) are bilinear, therefore the polygonal cup product is a *bilinear* operation.

It also satisfies item 1. of Definition 3.2.1 – the result of product of two forms is located on a face that is incident to both the elements where the two original discrete forms reside.

Due to Lemma 3.2.1, our product satisfies item 2., the Leibniz rule.

We will now prove that item 3. of Definition 3.2.1 also holds. A pseudomanifold K as we defined it is a connected cell complex, therefore we have to show that there exist a real number ξ such that $I^0 \cup \alpha^p = \alpha^p \cup I^0 = \xi\alpha^p$, where I^0 is the constant 0–form that takes value 1 on the vertices of K. Using the equations of Definition 3.2.4, we can thus write:

$$
\begin{aligned}
I^0 \cup \alpha^0 &= \sum_{v_i \in V(K)} I^0 \cup \alpha^0(v_i) = \sum_{v_i \in V} I^0(v_i)\alpha^0(v_i) = \sum_{v_i \in V} 1\alpha^0(v_i) = 1\alpha^0, \\
\alpha^0 \cup I^0 &= \sum_{v_i \in V} \alpha^0 \cup I^0(v_i) = \sum_{v_i \in V} \alpha^0(v_i) I^0(v_i) = \sum_{v_i \in V} \alpha^0(v_i)1 = 1\alpha^0, \\
I^0 \cup \alpha^1 &= \sum_{(v_i,v_j) \in E(K)} I^0 \cup \alpha^1(v_i,v_j) = \sum_{(v_i,v_j) \in E} \frac{I^0(v_i) + I^0(v_j)}{2}\alpha^1(v_i,v_j) \\
&= \sum_{(v_i,v_j) \in E} \frac{1+1}{2}\alpha^1(v_i,v_j) = 1\alpha^1, \\
\alpha^1 \cup I^0 &= \sum_{(v_i,v_j) \in E} \alpha^1 \cup I^0(v_i,v_j) = \sum_{(v_i,v_j) \in E} \alpha^1(v_i,v_j)\frac{I^0(v_i) + I^0(v_j)}{2} \\
&= \sum_{(v_i,v_j) \in E} \alpha^1(v_i,v_j)\frac{1+1}{2} = 1\alpha^1, \\
I^0 \cup \alpha^2 &= \sum_{f_i \in F(K)} I^0 \cup \alpha^2(f_i) = \sum_{f_i \in F} \sum_{v_j \prec f_i} \frac{I^0(v_j)}{p_i}\alpha^2(f_i) \\
&= \sum_{f_i \in F} \frac{p_i}{p_i}\alpha^2(f_i) = 1\alpha^2, \\
\alpha^2 \cup I^0 &= \sum_{f_i \in F} \alpha^2 \cup I^0(f_i) = \sum_{f_i \in F} \alpha^2(f_i) \sum_{v_j \prec f_i} \frac{I^0(v_j)}{p_i} \\
&= \sum_{f_i \in F} \alpha^2(f_1)\frac{p_i}{p_i} = 1\alpha^2,
\end{aligned}
$$

where p_i is the number of vertices of a face f_i. We thus conclude that $\xi = 1$ satisfies $I^0 \cup \alpha^p = \alpha^p \cup I^0 = \xi\alpha^p \;\; \forall p \in \{0,1,2\}$. ☐

Proposition 3.2.3. *The polygonal cup product from Definition 3.2.4 satisfies these properties:*

1. *Bilinearity.*
2. *Skew–commutativity* $\alpha^k \cup \beta^l = (-1)^{kl}\beta^l \cup \alpha^k$.

3. *Leibniz rule.*

4. *Associativity under the following conditions:*
   $\alpha^0 \cup (\beta^0 \cup \gamma^0) = (\alpha^0 \cup \beta^0) \cup \gamma^0$ *always,*
   $\alpha^0 \cup (\beta^0 \cup \gamma^1) = (\alpha^0 \cup \beta^0) \cup \gamma^1$ *if* $d\alpha^0 = 0$ *or* $d\beta^0 = 0$*,*
   $\alpha^0 \cup (\beta^0 \cup \gamma^2) = (\alpha^0 \cup \beta^0) \cup \gamma^2$ *if* $d\alpha^0 = 0$ *or* $d\beta^0 = 0$*,*
   $\alpha^0 \cup (\beta^1 \cup \gamma^1) = (\alpha^0 \cup \beta^1) \cup \gamma^1$ *if* $d\alpha^0 = 0$*.*

Proof. From Theorem 3.2.1 we know our product is a *bilinear* operation satisfying the *Leibniz rule* with discrete exterior derivative. We now show that it is also skew–commutative on any discrete forms and associative in the declared fashion.

With respect to the announced form of *associativity*, in the case of the product of three 0–forms we get to multiplying three real numbers on each vertex, which is indeed an associative operation. For the other degrees, we have learned in the proof of Theorem 3.2.1 that if K is connected and a 0–form α^0 is constant (which is equivalent to closed), that is $\alpha^0(v_i) = \rho \in \mathbb{R}\ \forall v_i \in K$, then the cup product of α^0 with any q–form ω^q is equal to multiplying ω^q by the real number ρ. This fact yields:

$$(\alpha^0 \cup \beta^k) \cup \gamma^l = (\rho\beta^k) \cup \gamma^l = \rho(\beta^k \cup \gamma^l) = \alpha^0 \cup (\beta^k \cup \gamma^l),$$

which proves the statement of item 4.

Concerning the *skew–commutativity*, looking at the equations (3.11)–(3.13) it is easy to see that $\alpha^0 \cup \beta^l = \beta^l \cup \alpha^0$ for $l = 0, 1, 2$. The case of the product of two 1–forms is slightly more involving. We need to prove that $(\alpha^1 \cup \beta^1)(f) = -(\beta^1 \cup \alpha^1)(f)$ on an n–polygonal face $f = (v_0, \ldots, v_{n-1})$. Let us remember that

$$(\alpha^1 \cup \beta^1)(f) = \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{n}\right) \sum_{i=0}^{n-1} \alpha(i)(\beta((i+a)\%n) - \beta((i-a)\%n),$$

where $\alpha(i) := \alpha(e_i) = \alpha(v_i, v_{(i+1)\%n})$ and $\%n$ means modulo n. Again, to simplify the notation we will omit writing the symbol $\%n$ in the indexes and by some index k we will always understand $k\%n$. We next show that

$$\sum_{i=0}^{n-1} \alpha(i)(\beta(i+a) - \beta(i-a)) = -\sum_{i=0}^{n-1} \beta(i)(\alpha(i+a) - \alpha(i-a)).$$

Due to the indexes being cyclic (modulo n), we can write:

$$\begin{aligned}
\sum_{i=0}^{n-1} \alpha(i)\beta(i+a) &= \alpha(0)\beta(a) + \alpha(1)\beta(1+a) + \cdots + \alpha(n-a-1)\beta(n-1) \\
&\quad + \alpha(-a)\beta(0) + \alpha(1-a)\beta(1) + \alpha(2-a)\beta(2) + \cdots + \alpha(n-1)\beta(n-1+a) \\
&= \sum_{j=0}^{n-1} \beta(j)\alpha(j-a), \\
\sum_{i=0}^{n-1} \alpha(i)\beta(i-a) &= \alpha(0)\beta(-a) + \alpha(1)\beta(1-a) + \cdots + \alpha(n+a-1)\beta(n-1) \\
&\quad + \alpha(a)\beta(0) + \alpha(1+a)\beta(1) + \alpha(2+a)\beta(2) + \cdots + \alpha(n-1)\beta(n-1-a) \\
&= \sum_{j=0}^{n-1} \beta(j)\alpha(j+a),
\end{aligned}$$

Thus

$$(\alpha^1 \cup \beta^1)(f) = -\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{n}\right) \sum_{i=0}^{n-1} \beta(i)(\alpha((i+a)\%n) - \alpha((i-a)\%n)) = -(\beta^1 \cup \alpha^1)(f).$$

Hence the polygonal cup product is skew–commutative on the cochain level. □

The matrix form of the cup product, using the halfedge representation and the matrices defined in Subsection 3.1, reads:

$$\alpha^0 \cup \beta^0 = \alpha^0 \odot \beta^0, \tag{3.16}$$

$$\alpha^0 \cup \beta^1 = (\mathsf{B}\,\alpha^0) \odot \beta^1, \tag{3.17}$$

$$\alpha^0 \cup \beta^2 = (\mathsf{fv}\,\alpha^0) \odot \beta^2, \tag{3.18}$$

$$\alpha^1 \cup \beta^1 = d_1(\mathsf{L}\,\alpha^1) \odot (\mathsf{R}\,\beta^1). \tag{3.19}$$

We can also express the cup product of two 1–forms on a single p–polygonal face as:

$$\alpha^1 \cup \beta^1(f) = (\mathsf{L}\,\alpha^1)^\top (\mathsf{R}\,\beta^1), \text{ for } \mathsf{R} = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p}\right) \mathsf{R}_a\,.$$

A Note about Cohomology Rings

If K is a 2–dimensional pseudomanifold with polygonal 2–cells, then by Theorem 3.2.1 and Proposition 3.2.3, our polygonal cup product defines a bilinear operation $H^k(K,\mathbb{R}) \times H^l(K,\mathbb{R}) \to H^{k+l}(K,\mathbb{R})$ that is well–defined on the cohomology level. Thus the set of cohomology classes $H^*(K,\mathbb{R})$ together with addition and the cup product forms a graded ring $(H^*(K,\mathbb{R}),+,\cup)$, called the **cohomology ring** of K, i.e.,

1. $(H^*(K,\mathbb{R}),+)$ is an abelian group,

2. $(H^*(K,\mathbb{R}),\cup)$ is a monoid,

3. and the multiplication is distributive with respect to the addition.

The cohomology ring provides us with a lot of additional information about the topology of a given manifold, more subtle than the homology groups and the Betti numbers give. It is a fact that any polyhedron of dimension ≤ 3 can be combinatorially subdivided so as to define a simplicial complex (see also **Polyhedral Hauptvermutung Conjecture** [Ranicki (ed.) 1996]). Having a polygonal surface mesh, we could thus subdivide it to get a triangle mesh and employ a simplicial cup product. However, we propose to work directly with polygonal meshes and use polygonal cup product instead.

3.2.4 Discussion

We have found a very simple formula of a cup product of cochains on general polygonal pseudomanifolds with coefficients in $\mathbb{R}$, even though any commutative ring would serve as the coefficient ring. We do not need to perform subdivision as to obtain simplicial or cubical complexes in order to compute the cup product. Also, our result is independent of the ordering of vertices, unlike the one of [Kravatz 2008] which uses a diagonal approximation that is sensitive to the choice of the so called minimal and maximal vertex. Further, unlike the cup product on polygons from [Gonzalez–Diaz et al. 2011b], our formulas are very direct and simple and we do not use any additional structure such as the AT–model of [Gonzalez–Diaz et al. 2011b, Theorem 2].

As we work directly with polygons, we overcome the ambiguities of subdividing a discrete surface into a triangle mesh in order to be able to apply the simplicial cup product. In Example 3.2.1 we demonstrate that different triangulations of the same polygonal mesh can even give slightly different results.

In the following paragraphs we present some results of several tests performed. We always compare our results to respective analytical solutions, that is why we have chosen some relatively simple differential forms: linear, quadratic, and trigonometric. For all these classes of differential forms, our cup product of discretized forms exhibit at least linear convergence to the analytically computed wedge product of corresponding continuous forms. In the case of wedge product of a 0–form with a 1–form, the convergence seems to be almost quadratic, see the Figures 3.5 and 3.8.

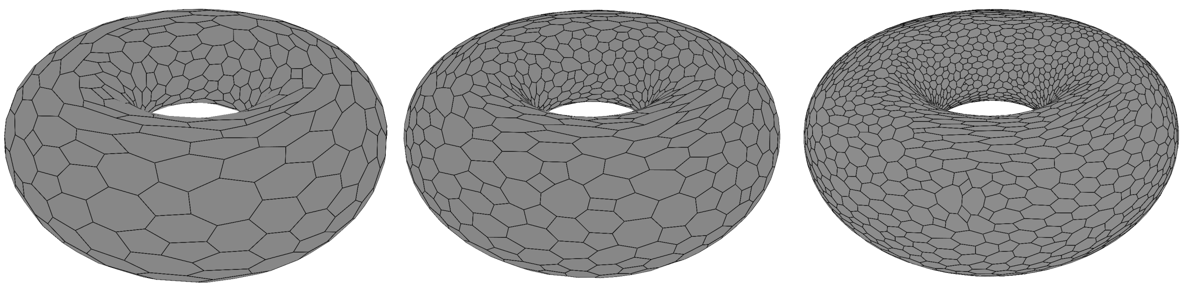

Figure 3.4: Examples of tested irregular polygonal meshes on torus (from left to right) with 1k, 2k, and 5k vertices. We can see that there are polygons with variable number of points and also the edge lengths vary significantly. Furthermore, neighborhood faces differ greatly in face areas.

3.2.5 Numerical Evaluation

The numerical evaluation of our cup product as an approximation of the continuous wedge product of differential forms was performed in the following fashion:

1. We integrate each differential l–form over all l–dimensional cells of our pseudomanifold K. That is, we define discrete forms $\alpha^0 \in C^0(K)$, $\beta^1 \in C^1(K)$, $\gamma^1 \in C^1(K)$, and $\omega^2 \in C^2(K)$, per each vertex v, edge e, and face f of K as:

$$\alpha^0(v) = A(v), \quad \beta^1(e) = \int_e B, \quad \gamma^1(e) = \int_e \Gamma, \quad \omega^2(f) = \int_f \Omega,$$

   where the Greek capital letters denote the respective differential forms.

2. Next we compute the polygonal cup products of the discrete forms on all edges e and faces f of K:

$$(\alpha^0 \cup \beta^1)(e), \quad (\alpha^0 \cup \omega^2)(f), \quad (\beta^1 \cup \gamma^1)(f).$$

3. We also calculate analytical solutions of (continuous) wedge product of the differential forms and integrate these solutions over all edges e and faces f of K:

$$A \wedge B(e) = \int_e A \wedge B, \quad A \wedge \Omega(f) = \int_f A \wedge \Omega, \quad B \wedge \Gamma(f) = \int_f B \wedge \Gamma.$$

4. We then compute the L^2–errors as

$$\begin{aligned}
\text{Error 01} &= \Big(\alpha^0 \cup \beta^1 - A \wedge B\Big)^\top M_0 \Big(\alpha^0 \cup \beta^1 - A \wedge B\Big), \\
\text{Error 02} &= \Big(\alpha^0 \cup \omega^2 - A \wedge \Omega\Big)^\top M_1 \Big(\alpha^0 \cup \omega^2 - A \wedge \Omega\Big), \\
\text{Error 11} &= \Big(\beta^1 \cup \gamma^1 - B \wedge \Gamma\Big)^\top M_2 \Big(\beta^1 \cup \gamma^1 - B \wedge \Gamma\Big).
\end{aligned}$$

   where M_2 is a diagonal $|F| \times |F|$ matrix given by

$$M_2[i,i] = \frac{1}{|f_i|}$$

   and M_0, M_1 are L^2–Hodge inner product matrices of [Alexa and Wardetzky 2011], that is, M_0 is a diagonal $|V| \times |V|$ matrix given by

$$M_0[i,i] = \sum_{f_j \succ v_i} \frac{|f_j|}{p_j},$$

   and M_1 is the inner product of two 1–forms ϵ, ξ defined per face in the sense that

$$\epsilon^\top M_1 \xi = \sum_f \epsilon|_f^\top M_f \xi|_f,$$

   where $\epsilon|_f$ denotes the restriction of ϵ to the boundary of a p–polygonal face f and M_f is a symmetric $p \times p$ matrix defined as

$$M_f := \frac{1}{|f|} B_f B_f^\top,$$

where B_f denotes a $p \times 3$ matrix with edge midpoint positions as rows (we take the barycenter of each face as the center of coordinates).

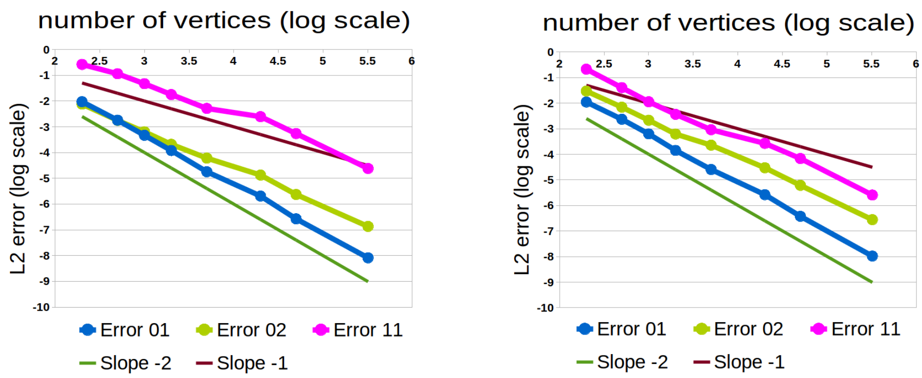

Figure 3.5: The cup product on a set of irregular meshes on a torus azimuthally symmetric around z axis: On the left, we see a plot of decreasing L^2 errors of the product of trigonometric forms (from Example 3.2.1). The graph on the right depicts the convergence rate of the product of these forms: $\alpha^0 = x^2 + y^2$, $\beta^1 = -ydx + xdy$, $\gamma^1 = -2xzdx - 2yzdy + 2(x^2 + y^2 - \sqrt{x^2+y^2})dz$, $\omega^2 = \frac{2}{\sqrt{x^2+y^2}}\big(x(\sqrt{x^2+y^2} - 1)dy \wedge dz + y(\sqrt{x^2+y^2} - 1)dz \wedge dx + z\sqrt{x^2+y^2}dx \wedge dy\big)$. The scale on both axes is $\log_{10}$. In Figure 3.4 we can see examples of the meshes tested, the implicit equation of the underlying torus is $(1 - \sqrt{x^2+y^2})^2 + x^2 - \frac{1}{4} = 0$

Example 3.2.1. We have performed several numerical tests with the following differential forms:

$$\begin{aligned}
\alpha^0 &= \sin(x)\cos(y) - 2\sin\left(z^2\right) + 1, \\
\beta^1 &= \left(\sin^2(x) - \cos(2z)\right) dx + (3\cos(x+2) + \sin(y)\cos(z))dy + (\sin(xz) + 3\cos(y))dz, \\
\gamma^1 &= (\cos(x)\sin(y) + 3)dx + (\cos(y) - \sin(z))dy + \left(\sin\left(x^2\right) + \cos(yz)\right) dz, \\
\omega^2 &= (\sin(xy) + \cos(z+1))dx \wedge dy + (\cos(3x) + \sin(2yz))dz \wedge dx + \sin(yz)dy \wedge dz.
\end{aligned}$$

In Figure 3.5 left we see the result of numerical evaluation of these forms on general polygonal meshes on a torus. And in Figure 3.6 left we show numerical convergence of the cup product on a set of polygonal meshes on a planar square and in the same figure right we illustrate that different triangulations of the same polygonal mesh can give slightly different results – we again measure the deviations from analytical solutions in the form of L^2 errors.

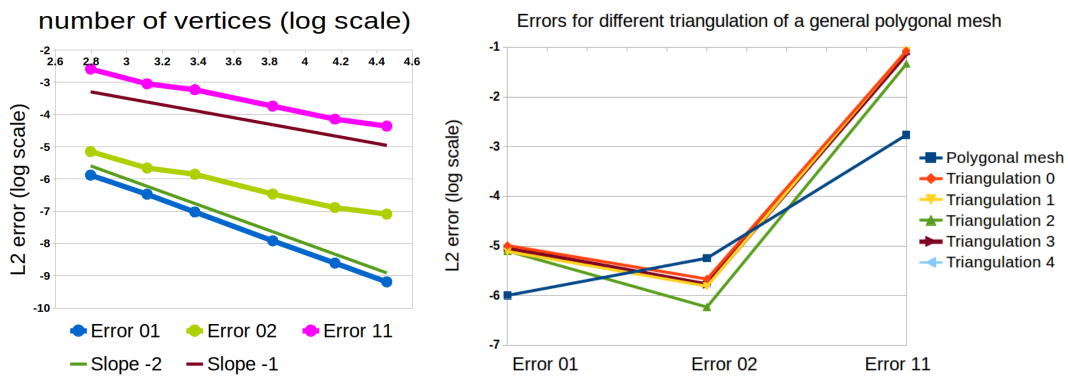

Figure 3.6: The graph on the left shows decreasing L^2 errors of the product of trigonometric forms (from Example 3.2.1) on a set of irregular meshes on a planar square. The graph on the right compares the L^2 errors for the same forms and different triangulations of a polygonal mesh with 700 vertices: we can see that the L^2 errors are sensitive to the triangulation chosen. In the case illustrated, the polygonal cup product performs slightly better than the simplicial one for product of a 0– with 1–form and product of two 1–forms. But it has a bigger error for product of a 0–form with a 2–form. In Figure 3.7 we can see the original polygonal mesh and two different triangulations of it.

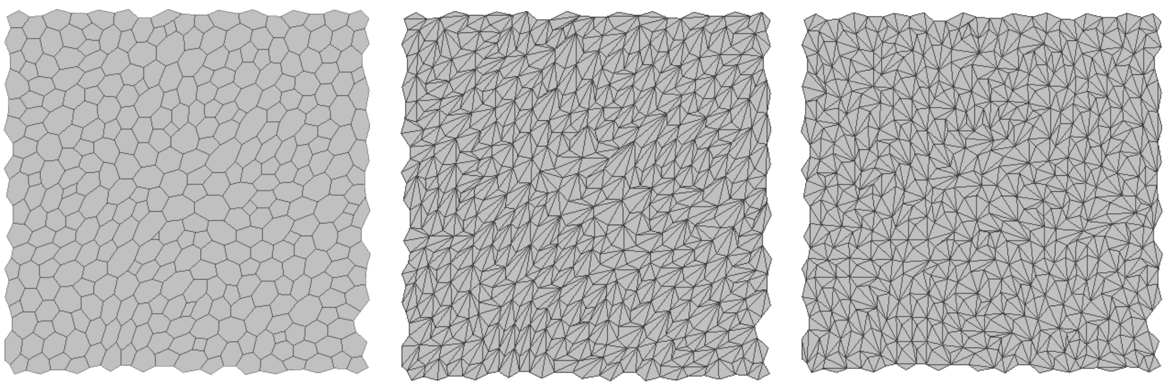

Figure 3.7: A polygonal mesh and its different triangulations: (L) the original polygonal mesh (with 700 vertices and 330 faces), (C) Triangulation 0 (700 vertices and 1289 faces), (R) Triangulation 2 (700 vertices and 1289 faces).

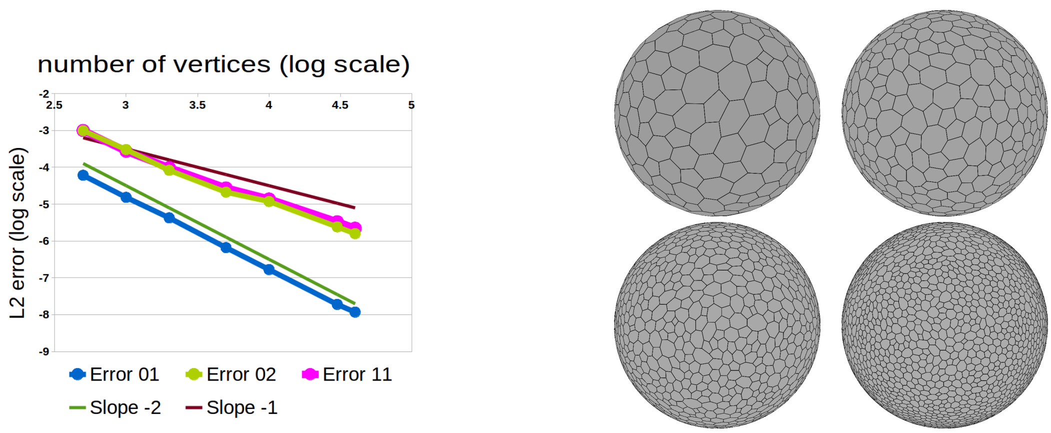

Figure 3.8: The graph on the left depicts the convergence rate of the product of these forms: $\alpha^0 = x^2+y^2$, $\beta^1 = -ydx+xdy$, $\gamma^1 = -xzdx - yzdy + (x^2+y^2)dz$, $\omega^2 = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$, on a set of general polygonal meshes on a unit sphere. On the right we can see several samples of the meshes used (from left to right and top to bottom) with 1k, 2k, 5k, and 10k vertices.

3.3 The Hodge Star Operator

We briefly introduce the Hodge star operator on Riemannian manifolds, for more detailed treatment, see [Abraham et al. 1988, Sections 7.2 and 7.5] and especially [Abraham et al. 1988, Propositions 7.2.12 and 7.2.13].

If M is a Riemannian n–manifold, i. e., a real smooth n–manifold equipped with an inner product $g : TM \times TM \to \mathbb{R}$ on the tangent spaces TM, then the Riemannian metric g also induces a contravariant (2,0)–tensor $g^{(k)}$ that acts on differential forms $\Omega^k(M)$ for every $k = 1, \ldots, n$. We will denote this inner product on k–forms by square brackets [., .] and call it pointwise inner product on differential forms (as opposite to L^2–Hodge inner product, which we define later), that is:

$$[.,.] : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}.$$

Now suppose that k is an integer such that $0 \leq k \leq n$, the **Hodge star operator** $*$ establishes a one–to–one mapping between k-forms and their dual $(n-k)$-forms. Concretely, the Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is the unique isomorphism that maps any k–form α into its dual $(n-k)$–form $*\alpha$ on M satisfying:

$$[*\alpha, \beta] = [\alpha \wedge \beta, \mu], \quad \beta \in \Omega^{n-k}(M), \tag{3.20}$$

where $\mu \in \Omega^n(M)$ denotes the volume form (a nowhere vanishing n–form) induced by the Riemannian metric. The Hodge star operator satisfies the following properties:

$$* * \alpha = (-1)^{k(n-k)}\alpha, \quad \alpha \in \Omega^k(M), \tag{3.21}$$

$$*1 = \mu, \quad *\mu = 1, \tag{3.22}$$

where 1 is the constant 0–form having value 1 at all points $x \in M$.

3.3.1 The Discrete Hodge Star Operator

We define a **discrete Hodge star operator** as a homomorphism (linear operator) from the group of p–forms to $(2-p)$–forms (cochains), i.e. $\star : C^p(K) \to C^{2-p}(K), 0 \leq p \leq 2$. But since our discrete Hodge operator is defined without a notion of any dual mesh and there is no isomorphism between the groups of p– and $(2-p)$–dimensional cells (chains), our Hodge star is not an isomorphism (invertible operator), unlike its continuous counterpart and diagonal approximations of the operator for which $**$ is an identity up to the sign (see equation (3.21)).

On the other hand, thanks to the dual forms being located on elements of our "primal" mesh, we can compute discrete wedge products of primal and dual forms and thus define a discrete inner product and discrete interior product later on (in Sections 3.4 and 3.5).

In the following paragraphs we introduce the discrete Hodge stars of each degree, our definitions were motivated by two conditions:

1. The Hodge dual of constant discrete forms on planar surfaces is exact, thus also the identities of equation (3.22) hold on planar surface meshes.

2. The discrete Hodge star operator on 1–forms leads to L^2–Hodge inner product on 1–forms identical to the one of [Alexa and Wardetzky 2011, Lemma 3].

In the Subsection 3.3.2, we will see that on curved surface meshes the Hodge star operator on constant forms is not exact in general, concretely the equation (3.22) does not hold, however this is inevitable in the way we perform the numerical tests. We take as ground truth the analytical solution, thus the volume form μ is taken as the volume form on the analytical surface and not on a polygonal approximation of that surface. And since we work with polygonal meshes, the given volume forms μ are not necessarily the volume forms on a particular polygon. Thus neither the discrete Hodge dual form of the constant form $\alpha^0 = 1$ is exact on polygonal meshes. For example, in Figure 3.10, the 2–forms ω^2 are the analytical volume forms on the underlying analytical surfaces, yet their discrete Hodge duals are not exactly 1 on our polygonal meshes as they would on the smooth underlying Riemannian surfaces.

The Hodge star operator on 2–forms is just an incidence matrix taking in account the degree p_i of the p_i–polygonal faces faces f_i and the face areas $|f_i|$. E.g., if ω^2 is a 2–form, then the 0–form $\star\omega$ on a vertex v is defined by

$$(\star_2\omega)(v) = \frac{1}{\sum\limits_{f_i \succ v} \frac{|f_i|}{p_i}} \cdot \sum_{f_i \succ v} \frac{\omega(f_i)}{p_i}, \tag{3.23}$$

i.e., it is a linear combination of values of ω on faces adjacent to v. In matrix form the operator reads

$$\star_2 = \mathsf{W}_V^{-1}\,\mathsf{fv}^\top,$$

where fv is the matrix defined in equation (3.4) and W_V is a diagonal $|V| \times |V|$ matrix given by

$$\mathsf{W}_V[i,i] = \sum_{f_k \succ v_i} \frac{|f_k|}{p_k}.$$

The Hodge star operator on 1–forms is first defined per halfedges as:

$$\star_1 = \mathsf{W}_1\,\mathsf{R}^\top, \tag{3.24}$$

where R is the matrix defined in equation (3.6) and W_1 is a symmetric matrix given per a p–polygonal face f by:

$$\mathsf{W}_1[i,j] = \frac{\langle h_i, h_j\rangle}{|f|},$$

where h_k are the halfedges incident to and having the same orientation as the face f, and $\langle\ ,\ \rangle$ denotes the dot product of the halfedges vectors.

If an edge e is not on boundary, it has two adjacent faces, thus we compute the values of a given 1–form $\star\beta$ on corresponding halfedges, sum their values with appropriate orientation sign and divide the result by 2 – see the explanation in Figure 3.9 and the Example 3.3.1.

Example 3.3.1. If we think about the mesh in the Figure 3.9, then the value of the Hodge star dual of a 1–form β on the halfedge e_0 is

$$\star\beta(e_0) = \frac{1}{4|f_0|}\Big((\langle e_0, e_1\rangle - \langle e_0, e_3\rangle)(\beta(e_0) - \beta(e_2)) + (\langle e_0, e_0\rangle - \langle e_0, e_2\rangle)(\beta(e_3) - \beta(e_1)\Big).$$

The value of the the Hodge star dual of the 1–form β on the edge e is then given by

$$\star\alpha(e) = \frac{\star\beta(e_0) - \star\beta(e_4)}{2}.$$

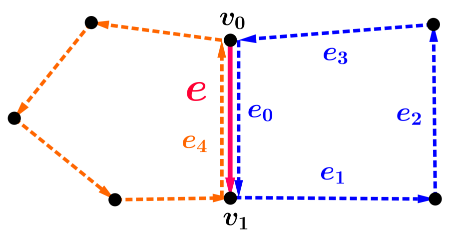

Figure 3.9: Let $\beta \in C^1$ and $e = (v_0, v_1)$ be the edge with e_0, e_1 as the corresponding halfedges, then $\star\beta(e) = \frac{\star\beta(e_0) - \star\beta(e_4)}{2}$, where $\star\beta(e_0)$ is a linear combination of values of β on dashed blue edges and $\star\beta(e_4)$ is a linear combination of values of β on dashed orange edges

The Hodge star operator on a 0–form α is a 2–form defined per a p–polygonal face f as:

$$(\star_0\alpha)(f) = \frac{|f|}{p} \sum_{v_i \prec f} \alpha(v_i), \tag{3.25}$$

and it is simply the arithmetic mean of the values of α on vertices of the given face f multiplied by the face area $|f|$. In matrix form it reads

$$\star_0 = \mathsf{W}_F \,\mathsf{fv},$$

where fv is the matrix defined in equation (3.4) and W_F is a diagonal $|F| \times |F|$ matrix with face areas as entries.

3.3.2 Numerical Evaluation and Discussion

Even though our discretizations of the Hodge star operator are not diagonal matrices, they are highly sparse and thus computationally efficient. We have numerically evaluated our approximation of the Hodge star operator in a fashion similar to the one explained at the beginning of Section 3.2.5.

We have performed several numerical tests on linear, quadratic, and trigonometric differential forms on planar and curved meshes and they exhibit the same (at least linear) convergence rate. For example in Figure 3.10 we tested different resolutions of polygonal meshes on a torus and a sphere (the same sets of meshes as in Subsection 3.2.5). We evaluated errors of approximation for each degree – for Hodge star on 2–forms (denoted as Error 2), 1–forms (Error 1), and 0–forms (Error 0). And in Figure 3.11 we show decreasing L^2 errors for trigonometric forms on a set of irregular polygonal meshes on a planar square.

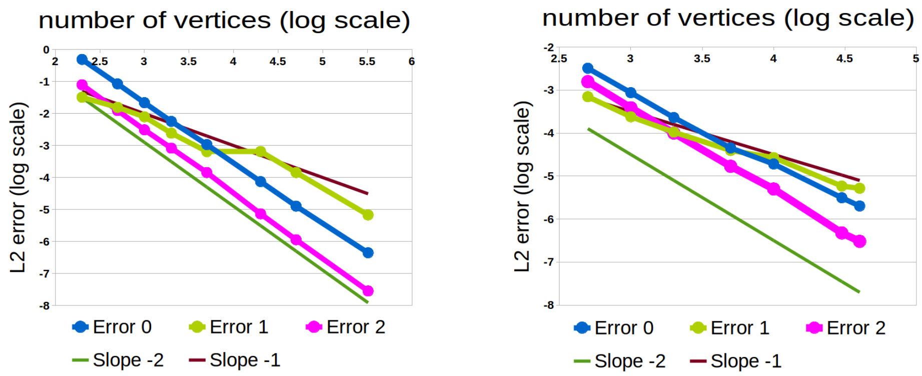

Figure 3.10: The discrete Hodge star operator on quadratic forms on the torus (L) and the sphere (R) on the same set of meshes as in the previous section. The graphs depict decreasing L^2 errors, both axes are in $\log_{10}$ scale. On the left, we computed the Hodge star duals of these forms: $\alpha^0 = x^2 + y^2$, $\beta^1 = -ydx + xdy$, $\omega^2 = \frac{2}{\sqrt{x^2+y^2}}(x(\sqrt{x^2+y^2} - 1)dy \wedge dz + y(\sqrt{x^2+y^2} - 1)dz \wedge dx + z\sqrt{x^2+y^2}dx \wedge dy)$. For the sphere and the graph on the right, we have chosen differential forms: $\alpha^0 = x^2 + y^2$, $\beta^1 = -ydx + xdy$, $\omega^2 = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. In both cases, we have chosen these forms because the analytical solutions of the Hodge star duals are known, concretely: $\star\alpha = \alpha \wedge \mu$, $\star\mu = 1$, $\star\beta = \gamma$, where for torus $\gamma = -2xzdx - 2yzdy + 2(x^2 + y^2 - \sqrt{x^2+y^2})dz$ and for sphere $\gamma = -xzdx - yzdy + (x^2 + y^2)dz$.

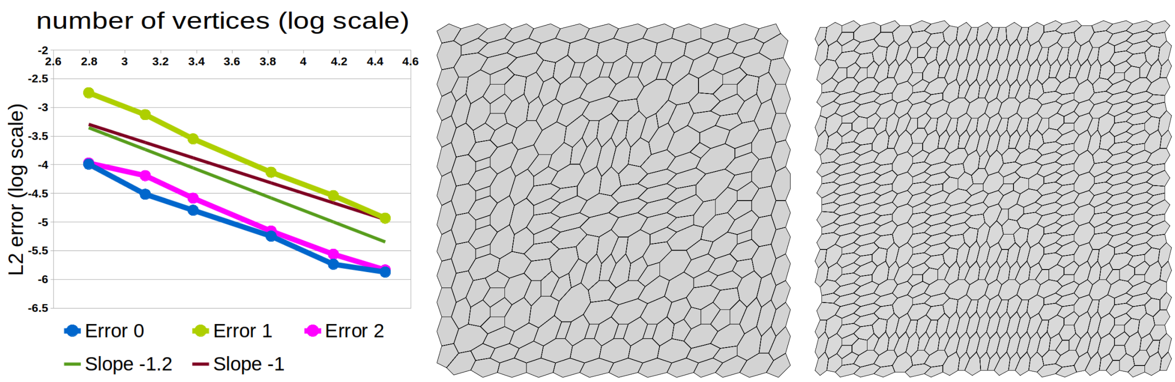

Figure 3.11: The graph on the left depicts the convergence rate of approximations of the Hodge duals for the following trigonometric differential forms: $\alpha^0 = \sin(x-1) - \cos(2y)$, $\beta^1 = (\sin(2x) + \cos(0.5y))dx + (3\sin(x) - \cos(y))dy$, and $\omega^2 = (\sin(\frac{x+1}{4}) + \cos(1 - \frac{y}{3}))dx \wedge dy$, on a set of general polygonal meshes on a planar square. In the center we can see a sample mesh with 600 vertices and on the right a mesh with 1300 vertices.

3.4 The L^2–Hodge Inner Product

The L^2–Hodge inner product of differential forms $\alpha, \beta \in \Omega^k(M)$ on a Riemannian manifold M is defined as:

$$(\alpha^k, \beta^k) := \int_M \alpha \wedge \star\beta.$$

We define a **discrete L^2–Hodge inner product** by:

$$(\alpha^k, \beta^k) := \alpha \cup \star\beta[K] = \sum_{f \in K} (\alpha \cup \star\beta)(f), \; k = 0, 1, 2,$$

where $[K]$ is the fundamental class of the surface mesh K (all faces f of K).

Thus, the discrete inner product matrices read:

$$\begin{aligned}\mathsf{M}_0 &= \mathsf{fv}^\top \, \mathsf{W}_F \, \mathsf{fv},\\ \mathsf{M}_1\,|_f &= \mathsf{R}\,\mathsf{W}_1\,\mathsf{R}^\top\,|_f,\\ \mathsf{M}_2 &= \mathsf{fv}\,\mathsf{W}_V^{-1}\,\mathsf{fv}^\top,\end{aligned}$$

where $\mathsf{M}_1\,|_f$ is the matrix of product of two 1–forms restricted to a face f.

It can be easily shown that our inner product of 1–forms on triangle meshes is identical to the one of [Gu and Yau 2003] and [Gu 2002]. And on general polygons it is equivalent to the one of [Alexa and Wardetzky 2011, Lemma 3], that is:

$$\mathsf{R}\,\mathsf{W}_1\,\mathsf{R}^\top\,|_f = \frac{1}{|f|} B_f B_f^\top, \tag{3.26}$$

where B_f is a $\mathbb{R}^{p\times 3}$ matrix with edge midpoint vectors as rows (we take the barycenter of each p–face f as the center of origin per face).

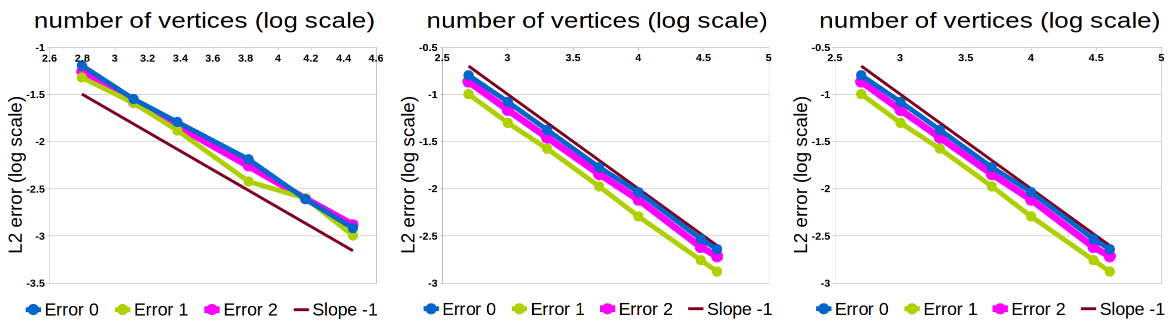

Figure 3.12: The approximation errors of L^2–Hodge inner products of 0-, 1–, and 2–forms on sets of irregular polygonal meshes on a planar square (L), sphere (C), and torus (R). The differential forms tested on torus and sphere are the same as in Figure 3.10. For the square we have chosen linear differential forms $\alpha^0 = 3x - 2y - 1$, $\beta^1 = (2x + y)dx + (-x + 3y + 1)dy$, and $\omega^2 = (x - 2y + 1)dx \wedge dy$.

To numerically evaluate our inner products, we have taken differential forms $\alpha^0, \beta^1, \omega^2$ and computed their analytical L^2 norms over the mesh K:

$$X_0 = \int_K \alpha \wedge *\alpha, \quad X_1 = \int_K \beta \wedge *\beta, \quad X_2 = \int_K \omega \wedge *\omega.$$

We have also defined discrete differential forms $\alpha_D^0, \beta_D^1, \omega_D^2$ on all elements of K as

$$\alpha_D(v) = \alpha(v), \quad \beta_D(e) = \int_e \beta, \quad \omega_D(v) = \int_f \omega,$$

and computed their discrete L^2 norms:

$$Y_0 = \alpha_D^\top \, \mathsf{M}_0 \, \alpha_D, \quad Y_1 = \beta_D^\top \, \mathsf{M}_1 \, \beta_D, \quad Y_2 = \omega_D^\top \, \mathsf{M}_2 \, \omega_D.$$

The L^2 errors are then given by

$$\text{Error } 0 = |X_0 - Y_0|, \quad \text{Error } 1 = |X_1 - Y_1|, \quad \text{Error } 2 = |X_2 - Y_2|.$$

We have numerically evaluated these errors for several linear and quadratic forms on polygonal, simplicial, and quadrilateral meshes in plane and space and they exhibit linear decrease in all cases, some examples are shown in Figure 3.12.

3.5 The Interior Product

The interior product $\mathbf{i}_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is the contraction of a differential form with a vector field X. Even though the interior product does not need a metric for its definition, if M is a Riemannian manifold, the following property holds [Hirani 2003, Lemma 8.2.1]:

Lemma 3.5.1. *Let M be a Riemannian n–manifold, $X \in \mathfrak{X}(M)$ a vector field, $\alpha \in \Omega^k(M)$, then*

$$\mathbf{i}_X \, \alpha = (-1)^{k(n-k)} * (*\alpha \wedge X^\flat),$$

where $\flat : \mathfrak{X}(M) \to \Omega(M)$ is the flat operator.

Since we already have a discrete wedge product and Hodge star operator that are compatible with each other, we can employ the lemma to define our **discrete interior product (contraction operator)** $\mathbf{i}_X : C^k(K) \to C^{k-1}(K)$ on a polygonal mesh K by:

$$\mathbf{i}_X \, \alpha = (-1)^{k(2-k)} \star (\star\alpha \cup X^\flat), \quad \alpha \in C^k(K), k = 1, 2. \tag{3.27}$$

For a vector field $X \in \mathfrak{X}(M)$ on a differentiable manifold M, the **discrete flat operator** $\flat : \mathfrak{X}(M) \to C^1(K)$ is defined on each edge e of the mesh K by:

$$X^\flat(e) = \int_e \langle X, e' \rangle d\mathbf{x} = \int_0^1 \langle X(t), e'(t) \rangle dt$$

for $e(t) = v_0 + t(v_1 - v_0)$, where v_0, v_1 are the endpoints of e.

Just like its continuous analog, our $\mathbf{i}_X$ is a linear operator that maps k–forms to $(k-1)$–forms such that $\mathbf{i}_X \, \mathbf{i}_X = 0$.

We numerically evaluate our approximation of the contraction operator in a fashion similar to the one explained at the beginning of Section 3.2.5. Concretely in Figure 3.13 we show the convergence rate of the discrete contraction of various forms on the same sets of meshes as in previous sections.

3.6 The Lie derivative

The Lie derivative evaluates the change of a differential form $\alpha \in \Omega(M)$ along the flow of a tangent vector field $X \in \mathfrak{X}(M)$ on a differentiable manifold M.

Using the Cartan's magic formula we derive the **discrete Lie derivative** $\pounds_X : C^k(K) \to C^k(K)$ as:

$$\pounds_X \alpha = \mathbf{i}_X \, d\alpha + d \, \mathbf{i}_X \, \alpha, \quad \alpha \in C^p(K), p = 0, 1, 2.$$

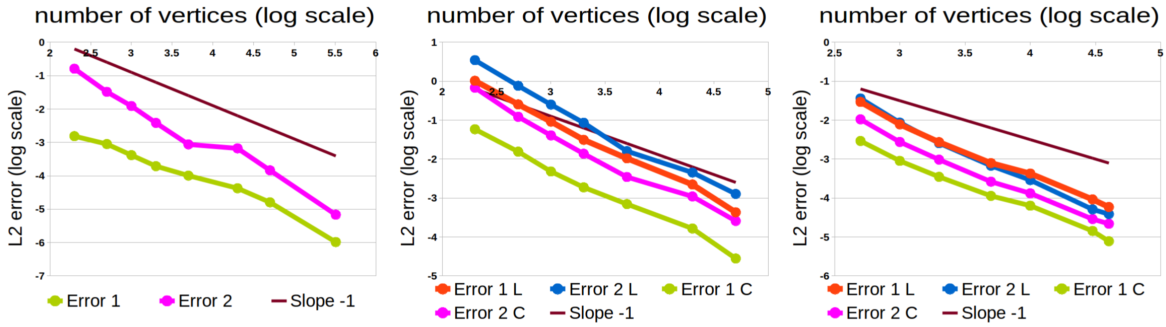

Figure 3.13: L^2 errors of the contraction operator on discrete forms compared to analytical solutions: On the left we plot the convergence for differential forms on torus given as $\beta^1 = -2xzdx - 2yzdy + 2(x^2 + y^2 - \sqrt{x^2+y^2})dz$, $\omega^2 = \frac{2}{\sqrt{x^2+y^2}}\big(x(\sqrt{x^2+y^2}-1)dy\wedge dz + y(\sqrt{x^2+y^2}-1)dz\wedge dx + z\sqrt{x^2+y^2}dx\wedge dy\big)$ wrt vector field $X = (-y, x, 0)$. The graphs in the center and on the right plot errors for linear and constant forms wrt linear vector field $X = (-y, x, 0)$ on a torus (C) and a sphere (R). The linear forms chosen are of the form $\beta^1 = (2x+y)dx + (-x+3y+z)dz$, $\omega^2 = (x-2y+z)dx\wedge dy$. And the constant forms read $\beta^1 = \frac{dx}{3} - \frac{dy}{2} + \frac{dz}{4}$ and $\omega^2 = dx\wedge dy + dy \wedge dz$. Error 1 L denotes the L^2 error on linear 1–forms, Error 1 C denotes the L^2 error on constant 1–forms, and similarly for 2–forms.

Our discrete contraction operator and Lie derivative share with their continuous counterparts some important algebraic properties such as:

$$\mathbf{i}_X\, \pounds_X = \pounds_X\, \mathbf{i}_X, \quad \pounds_X d = d\pounds_X.$$

But the Leibniz product rule of the contraction operator and Lie derivative with discrete exterior derivative is not satisfied in general. Concretely

$$\begin{aligned} \mathbf{i}_X(\alpha^k \cup \beta^l) &= (\mathbf{i}_X\, \alpha^k) \cup \beta^l + (-1)^k \alpha^k \cup (\mathbf{i}_X\, \beta^l), \\ \pounds_X(\alpha \cup \beta) &= (\pounds_X \alpha) \cup \beta + \alpha \cup (\pounds_X \beta), \end{aligned}$$

holds only if α or β is a closed 0–form.

Already in [Hirani 2003, Chapter 8] the author noticed that the Leibniz rule for Lie derivative might not hold due to the discrete wedge product not being associative in general. We confirm the observation in [Desbrun et al. 2005, Section 10] that the Leibniz rule may be satisfied only for closed forms.

3.6.1 Numerical Evaluation

We have seen converging behavior of Lie derivative of all tested forms on regular meshes, be them planar or curved: with slope -1.5 on 0–forms and at least -0.5 on 1– and 2–forms, as we illustrate in Figures 3.14 left and 3.17 left. On irregular meshes we could still see linear convergence of Lie derivative of 0–forms. However, the L^2 errors for 1– and 2–forms on very irregular meshes keep rather constant or oscillate, as we demonstrate thorough this section.

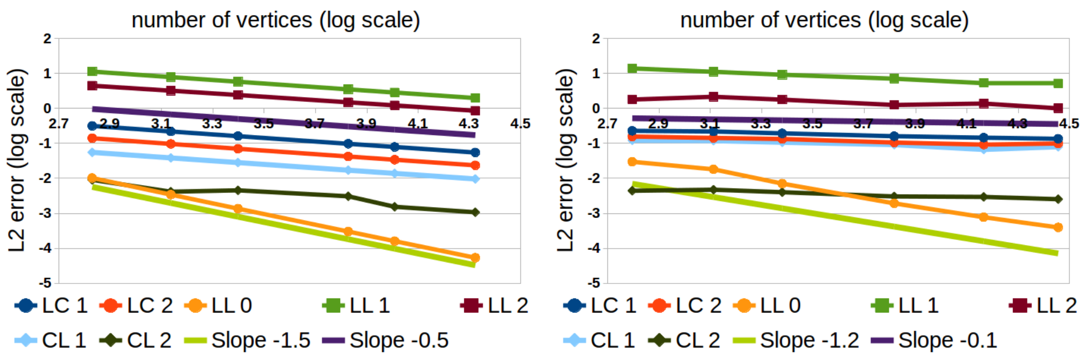

Figure 3.14: The Lie derivative on regular grids (L) and irregular polygonal meshes (R) on a unit square in plane: On constant 0–, 1–, 2– forms wrt constant vector fields, the Lie derivative is exact both on regular grid and irregular polygonal planar meshes. On 0–forms it is also exact if the vector field or the 0–form is constant. On linear 0–forms wrt linear vector fields (LL0) the derivative converges with slope -1.5 on regular grid and with slope -1.2 on irregular polygonal meshes. On 1– and 2–forms on regular grid, the L^2 errors decrease with slope -0.5, but on irregular polygonal meshes (the same set of meshes used in previous sections) the L^2 errors decrease very slowly: LL goes for linear forms wrt linear v. f., LC for linear forms wrt to constant v. f., and CL for constant forms wrt to linear v. f. The differential forms and vector fields tested are explicitly given in Example 3.6.1.

Example 3.6.1. On planar domains, we test the approximation of Lie derivatives of constant/linear differential forms wrt constant/linear vector fields. Following the notation of Figure 3.14, these are:
CL – Lie derivative of constant forms $\alpha^0 = 3$, $\beta^1 = \frac{1}{3}dx - \frac{1}{2}dy$, $\omega^2 = \frac{1}{8}dx \wedge dy$ wrt constant vector field $X = (2x, 1-y, 0)$.
LC – Lie derivative of linear forms $\alpha^0 = 3x - 2y - 1$, $\beta^1 = (2x+y)dx + (-x+3y+1)dy$, $\omega^2 = (x-2y+1)dx \wedge dy$ wrt constant vector field $X = (\frac{1}{5}, -\frac{1}{2}, 0)$.
LL – Lie derivative of linear forms $\alpha^0 = 3x - 2y - 1$, $\beta^1 = (2x+y)dx + (-x+3y+1)dy$, $\omega^2 = (x-2y+1)dx \wedge dy$ wrt linear vector field $X = (2x, 1-y, 0)$.
For example, the Lie derivatives of the linear forms wrt the linear vector field (LL) are:

$$\pounds_X \alpha^0 = 6x + 2y - 2, \quad \pounds_X \beta^1 = (8x + y + 1)dx + (-x - 6y + 2)dy, \quad \pounds_X \omega^2 = (3x - 1)dx \wedge dy.$$

We also test the influence of irregularity of a mesh on L^2 errors. Concretely, we have used **gradual jittering** on perfectly regular meshes to see how the convergence changes. The results can be seen in Figures 3.15 and 3.18. Jittering is done by moving each point of the mesh in random tangent direction, where in the case of a curved surface we project the displaced points back on the analytical surface. The displacement of each point is exactly r–times the shortest edge length l for some real $r > 0$.

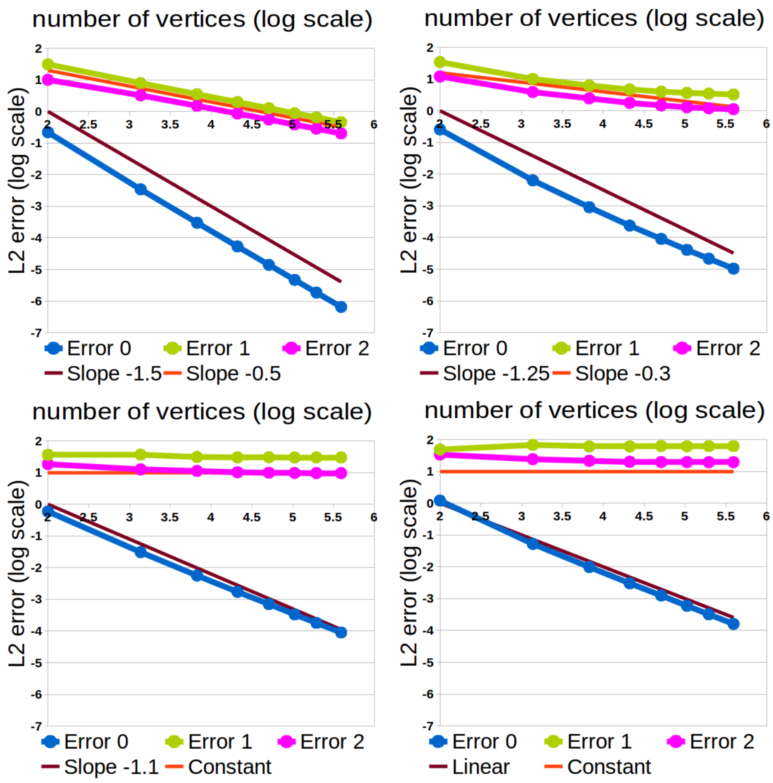

Figure 3.15: The Lie derivative of linear forms wrt linear vector fields (denoted LL in Example 3.6.1) on sets of gradually jittered quadrilateral planar meshes, as explained in Example 3.6.1. The scale on both axes is $\log_{10}$, we test meshes with 100 up to 391876 vertices. The graph on the far left top is a plot of approximation errors on completely regular planar grid. Then from left to right and top to bottom the graphs show convergence rate on a set of meshes with displacements $0.1l$, $0.2l$, $0.3l$, $0.4l$, where l is the shortest edge length of the original regular mesh. Samples of jittered meshes can be seen in Figure 3.16.

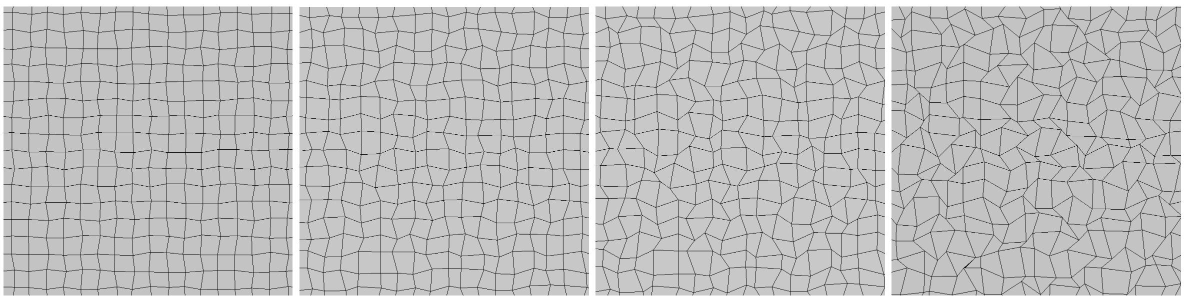

Figure 3.16: Samples of jittered meshes (with 1369 vertices): from left to right the applied displacements are $0.1l$, $0.2l$, $0.3l$, $0.4l$, where l is the shortest edge length of the original regular mesh. The mesh far right has already visible non–convex quadrilaterals.

Just like in the planar case, the Lie derivative on meshes on curved surfaces exhibit experimental convergence on regular meshes, as illustrated in Figure 3.17 left. However, on irregular meshes the discrete Lie derivative of 1– and 2–forms does not seem to converge, the L^2 errors rather oscillate around constant values – see the Figure 3.17 center and right and also Figure 3.18. Only on 0–forms the Lie derivative converges at least linearly.

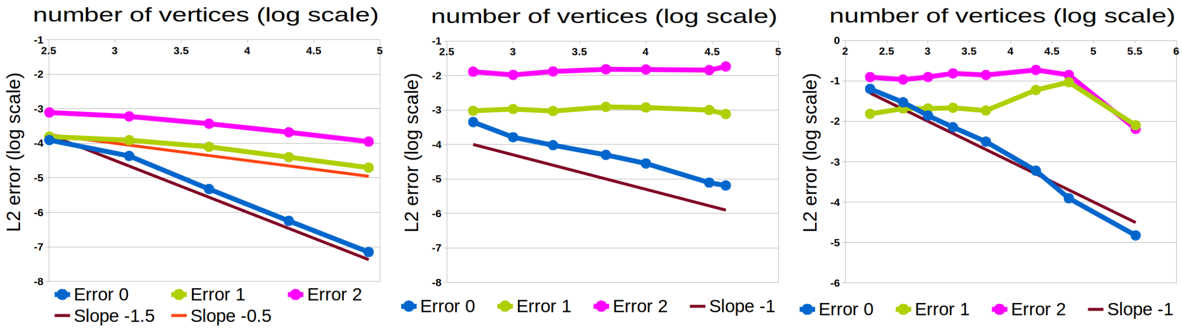

Figure 3.17: The Lie derivative on regular polygonal meshes on sphere (L), and irregular polygonal meshes on sphere (C) and torus (R). For the two graphs concerning sphere, we test differential forms $\alpha^0 = x^2+y^2$, $\beta^1 = -xzdx - yzdy + (x^2+y^2)dz$, $\omega^2 = xdy\wedge dz + ydz\wedge dx + zdx\wedge dy$ wrt vector field $X = (-y, x, 0)$. The set of irregular polygonal meshes on sphere is the same as in Figure 3.8. On torus we have tested forms $\alpha^0 = x^2+y^2$, $\beta^1 = -2xzdx - 2yzdy + 2(x^2+y^2-\sqrt{x^2+y^2})dz$, $\omega^2 = \frac{2}{\sqrt{x^2+y^2}}\big(x(\sqrt{x^2+y^2}-1)dy\wedge dz + y(\sqrt{x^2+y^2}-1)dz\wedge dx + z\sqrt{x^2+y^2}dx\wedge dy\big)$ wrt vector field $X = (-y, x, 0)$. And we have also used the same sets of meshes as in previous sections, where some samples are depicted in Figure 3.4.

In Figure 3.18 we demonstrate how jittering influences the convergence of L^2 errors of our approximation of Lie derivatives of linear forms wrt linear vector fields on triangle meshes on a unit sphere.

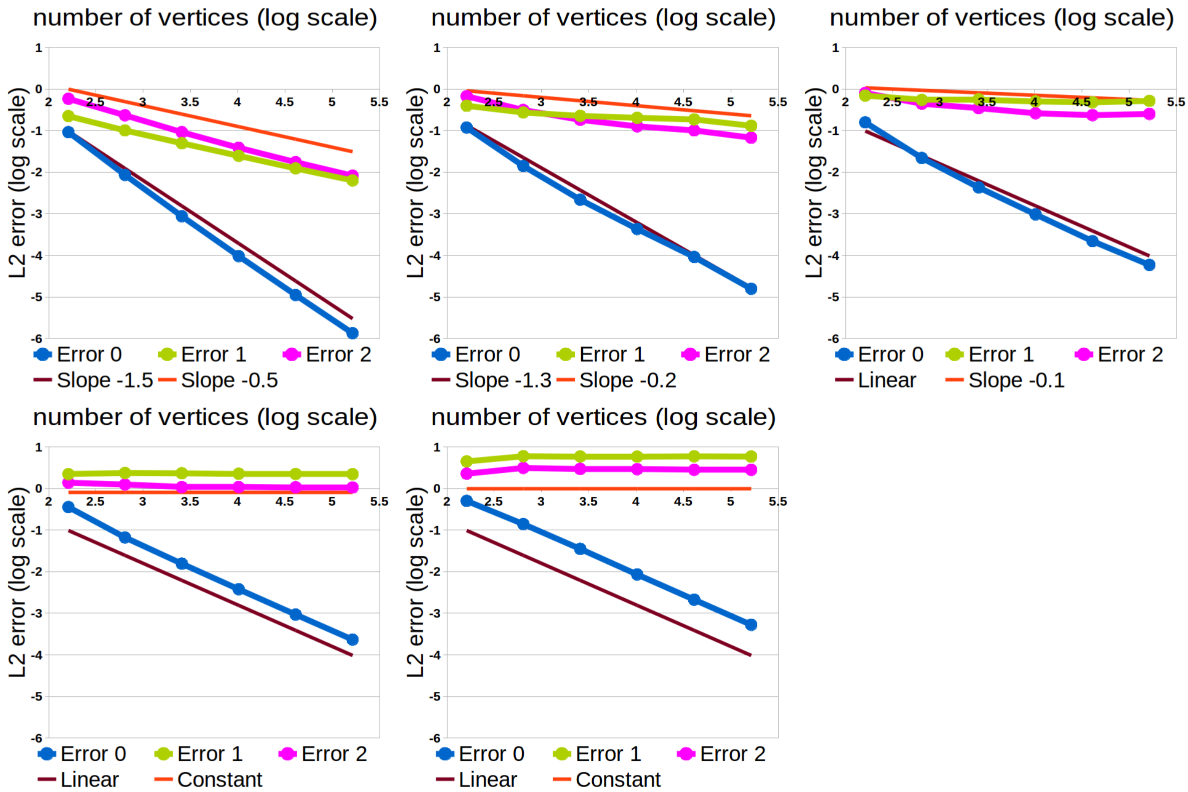

Figure 3.18: The Lie derivative of linear forms wrt linear vector fields on sets of jittered meshes on a unit sphere. The graph on the top far left is a plot of approximation errors on a regular triangle mesh. Then from left to right and top to bottom the graphs show convergence rate on a set of triangle meshes with displacements $0.05l$, $0.1l$, $0.2l$, $0.3l$, where l is the shortest edge length of the original regular mesh. Samples of jittered meshes can be seen in Figure 3.19. Here we have tested Lie derivatives of differential forms $\alpha^0 = 3x - 2y - z$, $\beta^1 = (2x+y)dx + (-x+3y+z)dy$, $\omega^2 = (x-2y+z)dx\wedge dy$ wrt vector field $X = (-y, x, 0)$.

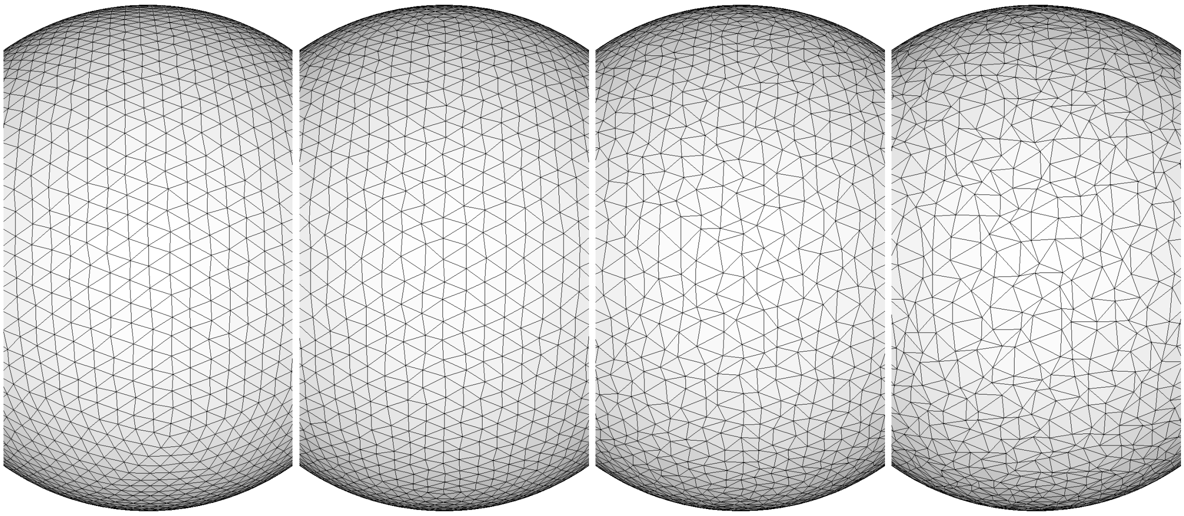

Figure 3.19: Samples of jittered meshes on a unit sphere, all with 10k vertices: from left to right the applied displacements (in tangent direction) are $0.05l$, $0.1l$, $0.2l$, $0.3l$, where l is the shortest edge length of the original regular mesh.

3.6.2 Discussion

We have seen that the discrete approximation of the Lie derivative on curved and planar surface meshes converges to the analytical solutions with slope -0.5 on 1– and 2–forms, and with slope -1.5 on 0–forms, if the meshes are regular.

On irregular meshes the evidence does not hint at converging behavior for 1– and 2–forms. However, even on highly irregular meshes, the errors kept rather constant, thus our method still may be used for specific applications in computer graphics, where we do not need a great precision. The scope of applications of our method is a subject for future research.

Chapter 4

Conclusion and Future Work

We have presented a cup product of cochains on polygonal pseudomanifolds as a discrete version of the wedge product of differential forms on smooth manifolds. Inspired by the formulas we derived from the definition of the cubical cup product of cubical cochains introduced in [Arnold 2012] and the definition of Wilson's cup product of simplicial cochains, we came with simple and straightforward formulas for the polygonal cup product of polygonal cochains, without employing any additional structure. That is, we have defined a proper cup product of cochains on polygonal pseudomanifolds and in Theorem 3.2.1 we have proved that our product satisfies the definition of a cup product. This is a new result and our main contribution.

Our polygonal cup product shares with the wedge product of differential forms defining properties such as the bilinearity, skew–commutativity, and the Leibniz product rule with (discrete) derivative. The wedge product is also associative, but our cup product is associative only on cohomology, this limitation is common to all discrete versions of wedge product we know.

We have then defined a discrete Hodge star operator that is compatible with our discrete wedge product and thus allows for definition of the L^2– Hodge inner product and also the interior product (contraction operator). For all these operations and operators we have shown that our approximations converge to the analytical solutions at least linearly.

The contraction operator together with the discrete exterior derivative then gave rise to the Lie derivative by using Cartan's magic formula. We presented an evidence that on regular meshes the Lie derivatives of 0–, 1–, and also 2–forms converge. However, the experiments show that on very irregular meshes, the Lie derivatives of 1– and 2–forms do not converge and the L^2 errors of approximation stay rather constant or oscillate.

4.1 Future Work

There are many directions of future research. We have used our discrete Hodge star operator to define a discretization of the interior product, but on Riemannian manifolds, the Hodge star operator is employed to define the codifferential operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$\delta(\alpha^k) = (-1)^k * d * \alpha.$$

Then using the codifferential operator, the Laplace–de Rham operator is given as $\Delta := \delta d + d\delta$. In the future, we want to define discrete codifferential and Laplace–de Rham operators in this manner and analyze their properties. Concretely, we aim to investigate the resulting Hodge decomposition using these two discrete operators together with the discrete exterior derivative d.

Further, we know that it is possible to define a discrete wedge product on tetrahedrons and 3–dimensional (topological) cubes in a way that it satisfies the defining properties of a wedge product such as the bilinearity, Leibniz product rule, skew–commutativity, and associativity on closed forms. Therefore we want to examine the possibility to extend the calculus we have just presented from 2–dimensional to (intrinsically) 3–dimensional manifolds, that is:

- Define DEC operators on 3–dimensional cubical and simplicial complexes, these shall include except the discrete wedge product also the Hodge star operator. Once defined these, we could derive a discrete contraction operator and a discrete Lie derivative just as we did in the 2–dimensional case.

- Look for a definition of a discrete wedge product on general 3–dimensional polytopes other than tetrahedras and cubes or at least find a subset of polytopes that allow for such a wedge product that would satisfy the Leibniz product rule and other properties. And then possibly follow the workflow above.

4.2 Conclusion

Our ultimate goal has been to define DEC operators as similar to their continuous counterparts as possible. The first step was to define the metric–independent operators and operations that would be compatible with each other and would have desired properties, we have managed to meet this target by defining the polygonal cup product that satisfies the Leibniz rule with discrete exterior derivative. The next step was to endow our pseudomanifolds with a metric that would allow for defining good approximations of metric–dependent operators such as the Hodge star. And using the discrete version of the wedge product, Hodge star operator, and exterior derivative, we derived other operators commonly associated to DEC, all of which operate directly on polygonal meshes, without any need for combinatorial subdivision or use of a dual mesh.

Geometry processing with polygonal meshes is a new developing area, maybe one of the first steps and also the most influential ones has been the definition of discrete Laplacians on general polygonal meshes by [Alexa and Wardetzky 2011]. Our objective was to continue in this venue by presenting a novel discretization of several operators and operations that act directly on general polygonal meshes and are compatible with each other. We thus extend further the DEC framework from simplicial and cubical setting to general polygonal case.

We have shown empirical convergence of our schemes even on irregular non–planar meshes, with the exception of the Lie derivative of 1–and 2–forms that exhibits a sensitivity to the quality of mesh. We believe that the generality of our framework will make it a useful tool in many geometry processing tasks and will inspire further research in the area.

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