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EFFECTIVE DIVISORS IN $\overline{M_g}$ GENERATED BY WRONSKIAN CLASSES IN $\overline{M_{g,2}}$

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Introduction

One of the most important recent endeavors in the field of Algebraic Geo–metry is to describe the birational geometry of the moduli spaces associated to curves.

Fixing the genus g, a topological invariant of curves, the moduli space of curves has been constructed by Mumford in the 60's; it is denoted M_g. Even though Birational Geometry is concerned with general properties, a compactification of M_g is useful. A compactification of M_g is the Deligne–Mumford $\overline{M_g}$ ([DM]), the compactification by adding stable curves to the boundary.

In order to try to characterize the effective cone of $\overline{M_g}$ and to answer other questions related to the birational geometry of $\overline{M_g}$, several effective divisors were computed in $\mathrm{Pic}_{fun}(\overline{M_g}) \otimes \mathbb{Q}$ in terms of the so-called Harer basis. The Brill–Noether divisors were computed by Harris and Mumford [HMu] by the method of test curves. Also by the same method, Diaz [D] and Cukierman [C] computed other divisors. Farkas computed several divisors by the same method and together with Popa [FP] obtained inequalities between the first few coefficients of any effective divisor in $\overline{M_g}$ not contained in the boundary.

Recently, Cumino, Esteves and Gatto ([CEG1],[CEG2]) recomputed the Diaz and Cukierman divisors with a new approach. Instead of using test curves, the calculation was done over a general 1-parameter family of stables curves. They used the theory of limit linear series for curves of compact type introduced by Eisenbud and Harris ([EH1]), but in a slightly more ge–neral format, working for any nodal connected curves. This approach has also been taken by Abreu [A] to compute a new effective divisor in $\overline{M_g}$, in his thesis work under the guidance by Esteves.

For $g = 2n$, the divisor Abreu computed is defined as the closure of the locus of smooth curves C having a pair of points (P, Q) such that Q has ramification weight at least 2 in the linear system $H^0(\omega_C(-nP))$ and

P has ramification weight at least 2 in the linear system $H^0(\omega_C(-nQ))$. We can consider other classes of divisors which are similar to the divisor which was calculated by Abreu. For instance, for nonnegative integers a, b such that $a+b=g$, a general problem is the calculation of the class of the divisor $R_{a,b}$ which is defined as the closure of the locus of smooth curves C having a pair of points (P,Q) such that Q has ramification weight at least 2 in the linear system $H^0(\omega_C(-aP))$ and P has ramification weight at least 2 in the linear system $H^0(\omega_C(-bQ))$. Notice that Abreu's thesis work addresses the case $a=b$. A natural variant of this kind of divisors is: for each positive integer $1 \leq n \leq g-2$, consider the divisor $\overline{S^2W_n}$ which is defined as the closure of the locus of smooth curves C having a pair of points (P,Q) such that Q has ramification weight at least 3 in the linear system $H^0(\omega_C(-nP))$. Our work addresses the case $n=1$.

Thus, this work addresses the problem of computing the class in the Picard group of the functor $\mathrm{Pic}_{fun}(\overline{M_g})$ of a certain effective divisor of $\overline{M_g}$. This divisor, $\overline{S^2W_1}$ in $\overline{M_g}$, is defined as the closure of the locus of smooth curves C having a pair of points (P,Q) with Q having ramification weight at least 3 in the linear system $H^0(\omega_C(-P))$. Our approach is to combine the methods by Cumino, Esteves and Gatto and the method of test curves. For simplicity, we denote $\overline{S^2W} := \overline{S^2W_1}$.

Writing the class of the divisor we want to compute as

$$\overline{S^2W} := a\lambda - a_0\delta_0 - a_1\delta_1 - \ldots - a_{[g/2]}\delta_{[g/2]}$$

and using the method of test curves, we obtain the coefficient a_i in terms of the coefficient a_1 for every $i > 1$ and each odd integer $g \geq 5$. We find the following relations:

$$a_i = (i(g-i)/(g-1))a_1, \text{ for every } 2 \leq i \leq [g/2].$$

Also, we compute the coefficient a by using the Thom–Porteous formula and intersection theory. For each g, we get

$$a = 9g^5 - 51g^4 + 129g^3 - 207g^2 + 174g - 54.$$

In order to find a lower bound for each coefficient a_i, we need to state the following hypothesis:

Hypothesis $(*)$.

If (X, A) is a general pointed smooth curve, then for every ramification point $P \in X$ of the complete linear system $H^0(\omega_X(-(g_X-1)A))$ and for every $i \geq 1$, the complete linear system $H^0(\omega_X((i+1)A-P))$ does not have ramification points on $X-\{A\}$ having ramification weight at least 3.

Using the methods by Cumino, Esteves and Gatto, and using the hypothesis $(*)$, we find the following inequalities for each g:

$$-b_i \leq a_i \text{ for every } 1 \leq i \leq [g/2],$$

where

$$b_i := 6i^4g^2 - 6i^4g + 12i^4 - 6i^3g^3 - 3i^3g^2 - 3i^3g - 18i^3 + 3i^2g^4 + 3i^2g^2 + 12i^2g + 6i^2 - 3ig^5 + 12ig^4 - 21ig^3 + 21ig^2 - 21ig + 6i.$$

We actually have equalities above for $g=3$ and $i=1$, and for $g=4$ and $i=2$.

The 'general family' method, i.e. the method we use to obtain a lower bound for each coefficient a_i, with $i \geq 1$, is coarsely described below. Let $\pi : \mathcal{X} \to T$ be a general family of stable curves over a smooth projective curve T. We can assume that the singular curves in our family have only one node. Furthermore, we can assume that these curves are not in the divisor we want to compute. Now, let $\mathcal{Y} = \mathcal{X} \times_T \mathcal{X}$ and blow up to solve the singularities of $\mathcal{Y}$. Let $\mathcal{B}$ be this blow up. Composing with the first projection $\mathcal{Y} \to \mathcal{X}$, we obtain a map $\rho : \mathcal{B} \to \mathcal{X}$. We consider ρ as a family of curves over $\mathcal{X}$.

Let ω be the relative dualizing sheaf of $\mathcal{B}/\mathcal{X}$ and $\mathcal{L} := \omega(-\widetilde{\Delta})$, where $\widetilde{\Delta}$ is the strict transform of Δ in $\mathcal{B}$. It may be necessary to modify $\mathcal{L}$, $\mathcal{B}$ and even $\mathcal{X}$. Abusing notation, we can say that the changes must be suitable enough that we get a family ρ of nodal curves over $\mathcal{X}$ such that $h^0(\mathcal{L}\big|_F) = g-1$ for every fiber F. Thus, $\rho_*\mathcal{L}$ is locally free of rank $g-1$ and we can use relative sheaves of jets to compute the ramification points of $H^0(\mathcal{L}\big|_F)$ as the fiber F varies. By considering the natural evaluation map $u : \rho^*\rho_*\mathcal{L} \to J^{g-2}_\rho(\mathcal{L})$ and substracting excess components of the degeneracy scheme W' of u, we get a divisor W intersecting each fiber in finitely many points. If W intersects each singular fiber with multiplicity at most 2 at each point, then we have $\pi_*\rho_*(c_3(J^2_\rho(\mathcal{O}_\mathcal{B}(W)))) = [\pi]^*(\overline{S^2W})$, where $[\pi] : T \to \overline{M_g}$ is the map which is induced by π; otherwise, we may have excess points on the singular fibers and we must calculate multiplicities at certain points and substract them from $c_3(J^2(\mathcal{O}_\mathcal{B}(W)))$ to get $[\pi]^*(\overline{S^2W})$. It turns out that we can obtain a lower bound for the coefficient a_i, for every $i \geq 1$.

In order to obtain the coefficient a_i in terms of the coefficient a_1 for every $i > 1$ and every odd positive integer $g \geq 5$, we use the method of test curves. We use $[g/2] - 1$ test curves, which are induced by families of flag stable curves over $\mathbb{P}^1$, and we apply a result which is similar to [HMo], Theorem 6.65, statement 2. To be able to apply the result, we use a general result about flag curves.

Our work is organized as follows: In Chapter 1 we present some preliminaries on ramification schemes, smoothings, limit linear systems and linear series on general smooth curves. In Chapter 2 we review some facts about the construction of $\overline{M_g}$ and about its associated Picard groups. In Chapter 3 we present some needed results on linear systems on rational and elliptic curves. In Chapter 4 we introduce the divisor $\overline{S^2W}$ and compute the coefficient of λ in the expression for $\overline{S^2W}$; we do a description of the results and methods used in the following chapters, and we present our main theorem (Theorem 4.1.1). Also, we present a few results which can be useful to compute the coefficient of δ_0 in the expression for $\overline{S^2W}$. In Chapter 5 we present a general result on flag curves (Proposition 5.1.1) and we apply this result by using the method of test curves to get relations between the coefficients of $\delta_1, \ldots, \delta_{[g/2]}$ in the expression for $\overline{S^2W}$. Finally, in Chapter 6 we compute lower bounds for the coefficients of $\delta_1, \ldots, \delta_{[g/2]}$.

Chapter 1

Limit linear systems and ramification schemes

1.1 Ramification points

A *nodal curve* C is a reduced, connected, projective scheme of dimension 1 over $\mathbb{C}$ whose only singularities are nodes. The dualizing sheaf ω_C is an invertible sheaf over C. The *arithmetic genus* of C is $g_C = h^0(C, \omega_C)$.

Let C be a smooth curve, $\mathcal{L}$ an invertible sheaf on C and $V \subseteq H^0(\mathcal{L})$ a linear system of dimension $r+1$, for an integer $r \geq 0$. For each $P \in C$ and each integer $i \geq 0$, let $V(-iP) := V \cap H^0(\mathcal{L}(-iP))$, the space of sections of V vanishing at P with multiplicity at least i. The orders of vanishing at P of sections of $\mathcal{L}$ in V can be ordered in a increasing sequence $a_0, \ldots, a_r$. Define the *ramification weight* of P,

$$wt_V(P) = \sum_{i=0}^{r} (a_i - i)$$

We say that P is a *ramification point* of V if $wt_V(P) > 0$; otherwise P is said to be an *ordinary point* of V. If $wt_V(P) = 1$, we call P a *simple ramification point*; and if $wt_V(P) \geq 2$ we call P a *special ramification point.*

On the other hand, V induces a section of $\mathcal{L}^{\otimes r+1} \otimes \omega_C^{\otimes \binom{r+1}{2}}$, obtained by considering locally Wronskian determinants of a sequence of $r+1$ functions. The zero locus of this section is denoted by R_V and called the *ramification divisor* of $(V, \mathcal{L})$. Indeed, a local analysis shows that

$$R_V = \textstyle\sum_{P \in C} wt_V(P)P$$

The degree of R_V is the degree of the invertible sheaf $\mathcal{L}^{\otimes r+1} \otimes \omega_C^{\otimes \binom{r+1}{2}}$, i.e.,

$$\deg(R_V) = (r+1)(\deg(\mathcal{L}) + r(g-1)),$$

known as the *Plücker formula.*

1.2 Ramification schemes

Let $\pi : \mathcal{X} \to T$ be a flat, projective morphism whose fibers are nodal curves of genus g. We say that π is a *family of curves.* Suppose $\mathcal{X}$ is a nonsingular scheme. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ and $\mathcal{V} \subseteq \pi_*\mathcal{L}$ a locally free subsheaf of rank $r+1$, for an integer $r \geq 0$. Suppose for each $t \in T$ the composition

$$V_t = \mathcal{V}_t/(\mathfrak{m}_{T,t}\mathcal{V}_t) \longrightarrow (\pi_*\mathcal{L})_t/(\mathfrak{m}_{T,t}(\pi_*\mathcal{L})_t) \longrightarrow H^0(\mathcal{X}_t, \mathcal{L}\big|_{\mathcal{X}_t})$$

is injective. We call $\mathcal{V}$ a *relative linear system.*

There exist sheaves $J^i_\pi(\mathcal{L})$ for each integer $i \geq 0$ satisfying the following properties (see [E1], [LT])

(1) $J^0_\pi(\mathcal{L}) \cong \mathcal{L}$.

(2) $J^i_\pi(\mathcal{L})$ is locally free of rank $i+1$.

(3) There are natural evaluation maps $e_i : \pi^*\pi_*\mathcal{L} \to J^i_\pi(\mathcal{L})$.

(4) For each $i \geq 1$, there is an exact sequence of truncation

$$0 \longrightarrow \omega_\pi^{\otimes i} \otimes \mathcal{L} \longrightarrow J^i_\pi(\mathcal{L}) \xrightarrow{r_i} J^{i-1}_\pi(\mathcal{L}) \longrightarrow 0$$

where ω_π is the relative dualizing sheaf of π. The truncation maps r_i are compatible with the evaluation maps, i.e., $e_{i-1} = r_i \circ e_i$ for every $i \geq 1$.

When π is a family of smooth curves, the sheaves $J^i_\pi(\mathcal{L})$ are called *relative sheaves of principal parts* of order i of $\mathcal{L}$.

Let $\mathcal{W}'_\mathcal{V}$ be the degeneracy locus of the natural evaluation map

$$u_r : \pi^*\mathcal{V} \to \pi^*\pi_*\mathcal{L} \to J^r_\pi(\mathcal{L})$$

Notice that u_r is a morphism between locally free sheaves of rank $r+1$ over $\mathcal{X}$. Locally, $\mathcal{W}'_\mathcal{V}$ is given by the zero locus of a Wronskian determinant of a sequence of $r+1$ functions. Furthermore, $\mathcal{W}'_\mathcal{V}$ has the property that $\mathcal{W}'_\mathcal{V} \cap \mathcal{X}_t$ is the ramification divisor R_{V_t} of the linear system $V_t \subseteq H^0(\mathcal{X}_t, \mathcal{L}\big|_{\mathcal{X}_t})$ for every smooth fiber $\mathcal{X}_t$. Let $\mathcal{X}_{ns} \subseteq \mathcal{X}$ be the locus of nonsingular fibers of π. The closure $\mathcal{W}'_\mathcal{V} \cap \mathcal{X}_{ns}$ in $\mathcal{X}$ is denoted by $\mathcal{W}_\mathcal{V}$. We call $\mathcal{W}_\mathcal{V}$ the *ramification divisor* of $(\mathcal{V}, \mathcal{L})$.

In case $\mathcal{V} = \pi_*\mathcal{L}$, we say that $\mathcal{W}_{\mathcal{V}}$ is the ramification divisor of the invertible sheaf $\mathcal{L}$.

The formation of $\mathcal{W}'_{\mathcal{V}}$ is functorial in the following sense: suppose there are a morphism $\psi : \mathcal{L}' \to \mathcal{L}$ which is an injective morphism between invertible sheaves on $\mathcal{X}$ whose degeneracy divisor is D, a relative linear system $\mathcal{V}' \subseteq \pi_*\mathcal{L}'$ of rank $r+1$, a morphism $\mu : \mathcal{V}' \to \mathcal{V}$ with degeneracy scheme Y, and a commutative diagram

$$\begin{array}{ccc} \mathcal{V}' & \longrightarrow & \pi_*\mathcal{L}' \\ \mu\downarrow & & \downarrow\pi_*\psi \\ \mathcal{V} & \longrightarrow & \pi_*\mathcal{L} \end{array}$$

By using the naturality of the evaluation maps, we obtain the following commutative diagram of locally free sheaves of rank $r+1$

$$\begin{array}{ccc} \pi^*\mathcal{V}' & \longrightarrow & J^r_\pi(\mathcal{L}') \\ \pi^*\mu\downarrow & & \downarrow J^r_\pi(\psi) \\ \pi^*\mathcal{V} & \longrightarrow & J^r_\pi(\mathcal{L}) \end{array}$$

By using the truncation exact sequences, we obtain that the degeneracy divisor of $J^r_\pi(\psi) : J^r_\pi(\mathcal{L}') \to J^r_\pi(\mathcal{L})$ is $(r+1)D$; therefore, taking determinants in the commutative diagram, we obtain

$$\pi^*Y + \mathcal{W}'_{\mathcal{V}} = (r+1)D + \mathcal{W}'_{\mathcal{V}'} \qquad (1.2.1)$$

Now, we will define the *k-th special ramification locus.* The divisor $\mathcal{W}_{\mathcal{V}}$ is the zero locus of a section $w : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}(\mathcal{W}_{\mathcal{V}})$. By using the natural evaluation maps, this section induces derivatives $w^{(k)} : \mathcal{O}_{\mathcal{X}} \to J^k_\pi(\mathcal{O}_{\mathcal{X}}(\mathcal{W}_{\mathcal{V}}))$. Let $S^k\mathcal{W}_{\mathcal{V}}$ be the zero scheme of $w^{(k)}$. We say that $S^k\mathcal{W}_{\mathcal{V}}$ is the *k-th special ramification locus.* On $\mathcal{X}_{ns}$, the support of $S^k\mathcal{W}_{\mathcal{V}}$ is the set of points P having ramification weight at least $k+1$ in the linear system $V_{\pi(P)} \subseteq H^0(\mathcal{L}|_{\mathcal{X}_{\pi(P)}})$.

1.3 Smoothings

Let C be a nodal curve. A *smoothing* of C is a flat, projective morphism $p : \mathcal{C} \to \Sigma$ where $\Sigma = Spec\,\mathbb{C}[[t]]$, $\mathcal{C}$ is a regular scheme and C is isomorphic to the special fiber.

Let $p : \mathcal{C} \to \Sigma$ be a smoothing of a nodal curve C of genus g. Let $\mathcal{C}_*$ be the generic fiber, $\mathcal{L}$ an invertible sheaf on $\mathcal{C}$ and $\mathcal{V} \subseteq p_*\mathcal{L}$ a relative linear system of rank $r+1$. Let $V = H^0(\mathcal{V}) \subseteq H^0(\mathcal{L})$. As p is flat and Σ is a regular, integral scheme of dimension 1, it follows that every associated point of $\mathcal{L}$ belongs to $\mathcal{C}_*$. Then the restriction $\Gamma(\mathcal{C}, \mathcal{L}) \to \Gamma(\mathcal{C}_*, \mathcal{L}\big|_{\mathcal{C}_*})$ is injective. Indeed, suppose $s \in \Gamma(\mathcal{C}, \mathcal{L})$ satisfies $s\big|_{\mathcal{C}_*} = 0$; then we have $\overline{\text{Supp}(s)} \cap \mathcal{C}_* = \emptyset$. On the other hand, if $s \neq 0$, then we can write $\text{Supp}(s) = \overline{\{x_1\}} \cup \ldots \cup \overline{\{x_m\}}$ as a union of irreducible components and we obtain that the points $x_1, \ldots, x_m$ are associated points of $\mathcal{L}$, hence these points belong to $\mathcal{C}_*$. It follows that $s = 0$. Thus, $H^0(\mathcal{L})$ is a torsion-free $\mathbb{C}[[t]]$-module and hence free. Also, it follows that V is a free $\mathbb{C}[[t]]$-module. Notice that, since cohomology commutes with flat base change, we have the isomorphism $H^0(\mathcal{L}) \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)) \cong H^0(\mathcal{L}\big|_{\mathcal{C}_*})$. Let $V_* = V \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$. Since $\mathcal{V} \subseteq p_*\mathcal{L}$ is a relative linear system, we have an injective map $V/tV \hookrightarrow H^0(\mathcal{L})/tH^0(\mathcal{L})$, that is, $V = V_* \cap H^0(\mathcal{L})$.

Let D be a divisor on $\mathcal{C}$ with support in C. Let $V(D)_*$ be the image of V_* under the natural isomorphism $H^0(\mathcal{L}\big|_{\mathcal{C}_*}) \cong H^0(\mathcal{L}(D)\big|_{\mathcal{C}_*})$. Define $V(D) = V(D)_* \cap H^0(\mathcal{L}(D))$.

If D is an effective divisor on $\mathcal{C}$, we define $V(-D) = V \cap H^0(\mathcal{L}(-D))$. Also, if $D \subseteq C$ is a subcurve, define $V\big|_D$ as the image of Vunder the restriction map $H^0(\mathcal{L}) \to H^0(\mathcal{L}\big|_D)$.

Let $C_1, \ldots, C_n$ be the irreducible components of C. Since C is connected, for each $i = 1, \ldots, n$ there exists an invertible sheaf $\mathcal{L}_i$ on $\mathcal{C}$ of the form

$$\mathcal{L}_i = \mathcal{L}(\sum_{i=1}^{n} a_{i,l}C_l) = \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^{n} a_{i,l}C_l)$$

such that the restriction map

$$H^0(C, \mathcal{L}_i\big|_C) \to H^0(C_i, \mathcal{L}_i\big|_{C_i})$$

is injective. We say that $\mathcal{L}_i$ has focus on C_i. Let $V_i = V(\sum_{i=1}^{n} a_{i,l}C_l)$ and let $\bar{V}_i$ be the image of V_i under the restriction map

$$H^0(\mathcal{C}, \mathcal{L}_i) \to H^0(C_i, \mathcal{L}_i\big|_{C_i})$$

The dimension of $\bar{V}_i$ is $r+1$. We say that $(\bar{V}_i, \mathcal{L}_i\big|_{C_i})$ is a *limit linear system* on C_i.

Let R_i be the ramification divisor of $(\bar{V}_i, \mathcal{L}_i\big|_{C_i})$ and W the ramification divisor of $\mathcal{V}$. Then (see [E2], Theorem 7)

$$W \cap C = \sum_{i=1}^{n} R_i + \sum_{i<j} \sum_{P \in C_i \cap C_j} (r+1)(r - l_{i,j})P$$

where $l_{i,j} = a_{i,j} - a_{i,i} + a_{j,i} - a_{j,j}$. We call $W \cap C$ the *limit ramification divisor* of $(\mathcal{V}, \mathcal{L})$, and $l_{i,j}$ is called the *connecting number* between $\mathcal{L}_i$ and $\mathcal{L}_j$ with respect to C_i and C_j.

Now, we will present some facts about smoothings and general curves. The following discussion can be found in [A].

Consider $C = E \cup F$, where E, F are subcurves without irreducible components in common. We have that $V(-E) = V \cap H^0(\mathcal{L}(-E))$ induces a relative linear system $\mathcal{V}(-E) \subseteq p_*(\mathcal{L}(-E))$ of rank $r+1$. By using the equation (1.2.1), we obtain $\mathcal{W}'_{\mathcal{V}} = (r+1)E + \mathcal{W}'_{\mathcal{V}(-E)} - p^*Y$, where Y is the degeneracy divisor of $\mathcal{V}(-E) \to \mathcal{V}$. Let $\mu : V(-E) \to V$ be the inclusion map. We have that μ is a homomorphism of free $\mathbb{C}[[t]]$-modules of rank $r+1$ and $p^*Y = \mathrm{ord}_t(\det(\mu))C$. Now, we will show that $\mathrm{ord}_t(\det(\mu)) = \dim_{\mathbb{C}} \mathrm{coker}(\mu)$. Indeed, since $tV \subseteq V(-E)$, we have the natural epimorphisms of $\mathbb{C}$-vector spaces

$$V/tV \to V/V(-E) \text{ and } V(-E)/tV(-E) \to V(-E)/tV.$$

Let $m := \dim_{\mathbb{C}} V/V(-E)$ and let $g_1, \ldots, g_m$ be sections in V such that their images in $V/V(-E)$ give us a $\mathbb{C}$-basis of $V/V(-E)$. As $V/tV \to V/V(-E)$ is an epimorphism, it follows that

$$\dim_{\mathbb{C}} V(-E)/tV = \dim_{\mathbb{C}} V/tV - \dim_{\mathbb{C}} V/V(-E) = r + 1 - m$$

Let $f_1, \ldots, f_{r+1-m}$ be sections in $V(-E)$ such that their images give us a $\mathbb{C}$-basis of $V(-E)/tV$. Since the images of $tg_1, \ldots, tg_m$ in $tV/tV(-E)$ give us a $\mathbb{C}$-basis of $tV/tV(-E)$, we have that a $\mathbb{C}$-basis of $V(-E)/tV(-E)$ is given by the images of $f_1, \ldots, f_{r+1-m}, tg_1, \ldots, tg_m$ in $V(-E)/tV(-E)$. By Nakayama's lemma, $f_1, \ldots, f_{r+1-m}, tg_1, \ldots, tg_m$ span the $\mathbb{C}[[t]]$-module $V(-E)$, and since $V(-E)$ is a free $\mathbb{C}[[t]]$-module of rank $r+1$, we have that $f_1, \ldots, f_{r+1-m}, tg_1, \ldots, tg_m$ give us a $\mathbb{C}[[t]]$-basis of $V(-E)$. Analogously, we have that $f_1, \ldots, f_{r+1-m}, g_1, \ldots, g_m$ give us a $\mathbb{C}[[t]]$-basis of V. Therefore, using the bases $f_1, \ldots, f_{r+1-m}, tg_1, \ldots, tg_m$ and $f_1, \ldots, f_{r+1-m}, g_1, \ldots, g_m$ of $V(-E)$ and V respectively, we get that $\mathrm{ord}_t(\det(\mu)) = m = \dim_{\mathbb{C}} \mathrm{coker}(\mu)$.

Therefore

$$\begin{aligned}\mathcal{W}'_{\mathcal{V}} &= (r+1)E + \mathcal{W}'_{\mathcal{V}(-E)} - dim_{\mathbb{C}} coker(\mu) C \\ &= (r+1)E + \mathcal{W}'_{\mathcal{V}(-E)} - dim_{\mathbb{C}}(V\big|_E)C\end{aligned}$$

Now, let $\{P_1, \dots, P_n\} := E \cap F$. Using the exact sequence

$$0 \to V\big|_F(-P_1 - \ldots - P_n) \to V\big|_C \to V\big|_E \to 0,$$

where $V\big|_F(-P_1 - \ldots - P_n) := V\big|_F \cap H^0(\mathcal{L}\big|_F(-P_1 - \ldots - P_n))$, we get $\dim_{\mathbb{C}} V\big|_F(-P_1 - \ldots - P_n) + \dim_{\mathbb{C}} V\big|_E = \dim_{\mathbb{C}} V\big|_C = \dim_{\mathbb{C}} V/tV = r+1$. Then

$$\begin{aligned}\mathcal{W}'_{\mathcal{V}} &= (r+1)E + \mathcal{W}'_{\mathcal{V}(-E)} - dim_{\mathbb{C}}(V\big|_E)(E+F) \\ &= \mathcal{W}'_{\mathcal{V}(-E)} + (r+1-dim_{\mathbb{C}}(V\big|_E))E - dim_{\mathbb{C}}(V\big|_E)F \qquad (1.3.2) \\ &= \mathcal{W}'_{\mathcal{V}(-E)} + dim_{\mathbb{C}} V\big|_F(-P_1 - \ldots - P_n)E - dim_{\mathbb{C}}(V\big|_E)F\end{aligned}$$

In addition, if D is an effective divisor of $\mathcal{C}$ such that D and E have no common components, then $V(-D)\big|_E \subseteq V\big|_E(-D \cdot E)$.

For the convenience of the reader we include a collection of results we will need, without their proofs.

Proposition 1.3.1. *Let C be a nodal union of two smooth curves X and Y, identifying the point $A \in X$ with the point $B \in Y$. Let $p : \mathcal{C} \to \Sigma$ be a smoothing of C, $\mathcal{L}$ an invertible sheaf over $\mathcal{C}$ and $\mathcal{V} \subseteq p_*\mathcal{L}$ a relative linear system of rank $r+1$. Let W' be the degeneracy scheme and W the ramification divisor of $\mathcal{V}$. Suppose that for every $i > 0$ the following is satisfied*

$$\begin{aligned}\dim_{\mathbb{C}}(V\big|_X(-iA)) + \dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \leq r+1 \\ \dim_{\mathbb{C}}(V\big|_Y(-iB)) + \dim_{\mathbb{C}}(V(-iX)\big|_X(-A)) \leq r+1\end{aligned}$$

Then

$$W' = W + Tw_{V|_X}(A)Y + Tw_{V|_Y}(B)X$$

Proof. See [A], Lemma 5.2.2. □

Proposition 1.3.2. *Let C be a smooth curve, $\mathcal{L}$ an invertible sheaf and $V \subseteq H^0(\mathcal{L})$ a linear system of dimension $r+1$. Let $V' = V(-P) \subseteq H^0(\mathcal{L}(-P))$ for some point $P \in C$ such that $V' \neq V$. Suppose V and V' do not have special ramification points on $C - \{P\}$. Then V and V' do not have ramification points in common on $C - \{P\}$.*

Proof. See [A], Lemma 5.2.5. □

Proposition 1.3.3. *Let (C, A) be a general pointed smooth curve of genus g. Then, for every $0 \leq a \leq g-1$, the complete linear system $H^0(\omega_C(-aA))$ does not have special ramification points.*

Proof. See [A], Proposition 5.3.3. □

Proposition 1.3.4. *Let (C, A) be a general pointed smooth curve of genus $g \geq 1$, and i a positive integer. Then the complete linear system $H^0(\omega_C(iA))$ has only simple ramification points distinct from A.*

Proof. See [CEG2], Proposition 3.1. □

Proposition 1.3.5. *Let i_0 be a fixed positive integer. Then for a general curve C of genus g and a general point $R \in C$,*

$$h^0(\omega_C((1+i)R - (a+1)P - (b+1)Q)) = 0$$

for every $P, Q \in C$, every $i = 0, \ldots, i_0$ and every nonnegative integers a and b with $a + b = g + i$.

Proof. See [A], Proposition 5.3.4. □

Proposition 1.3.6. *Let i_0 be a fixed positive integer. Then for a general curve C of genus g and a general point $R \in C$,*

$$h^0(\omega_C((1+i)R - (a-1)P - (b-1)Q)) = 2$$

for every $P, Q \in C - \{R\}$, every $i = 0, \ldots, i_0$ and every positive integers a and b with $a + b = g + i$.

Proof. See [A], Proposition 5.3.6. □

Chapter 2

The moduli space of stable curves and intersection theory

2.1 Construction of $\overline{M_g}$

Let $g \geq 2$ be an integer. Let $\overline{M_g}$ denote the coarse moduli space of stable curves. We will recall how $\overline{M_g}$ is constructed. Given a Deligne-Mumford stable curve X, we have that $\omega_X^{\otimes n}$ is very ample for each $n \geq 3$. Then, we may view X as a closed subscheme of degree $2n(g-1)$ of $\mathbb{P}^N$, where $N = (2n-1)(g-1)-1$, as by Riemann-Roch $h^0(X, \omega_X^{\otimes n}) = (2n-1)(g-1)$ for each $n \geq 2$.

We have that $\omega_X^{\otimes n} \cong \mathcal{O}_X(1)$; we call such a stable curve n-canonically embedded. Let H be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^N$ with Hilbert polynomial $2n(g-1)T + 1 - g$, and $\mathcal{U} \subseteq \mathbb{P}^N \times H$ the universal closed subscheme. There is a locally closed subscheme $K \subseteq H$ parametrizing n-canonically embedded stable curves of genus g. We can get K as follows:

Let $H' \subseteq H$ be the open subscheme parametrizing nodal curves. Let $\mathcal{U}_{H'} \subseteq \mathbb{P}^N \times H'$ be the induced subfamily over H'. This family $\mathcal{U}_{H'}$ admits a Picard algebraic space $\mathrm{Pic}_{\mathcal{U}_{H'}/H'}$ over H'. Furthermore, the sheaves $\omega_{\mathcal{U}_{H'}/H'}^{\otimes n}$ and $\mathcal{O}_{\mathcal{U}_{H'}}(1)$ induce a map

$$H' \to Pic_{\mathcal{U}_{H'}/H'} \times_{H'} Pic_{\mathcal{U}_{H'}/H'}.$$

Then K is the preimage of the diagonal under this map.

We have that K is locally closed in H', because so is the diagonal in $\mathrm{Pic}_{\mathcal{U}_{H'}/H'} \times_{H'} \mathrm{Pic}_{\mathcal{U}_{H'}/H'}$. Let $\mathcal{V} := \mathcal{U}_K \subseteq \mathbb{P}^N \times K$ be the induced subscheme and $v : \mathcal{V} \to K$ the family induced by the second projection $\mathbb{P}^N \times K \to K$.

We have that K is smooth (see [HMo], lemma 3.35).

The family $v : \mathcal{V} \to K$ is versal. To see this, let $\pi : \mathcal{C} \to S$ be a family of genus g stable curves. The pushforward $\pi_*(\omega_\pi^{\otimes n})$ is locally free of rank $N+1$. Thus, for each point $s \in S$, there is an open neighborhood U_s of s and an isomorphism $\mathcal{O}_{U_s}^{\oplus N+1} \to \pi_*(\omega_\pi^{\otimes n})\big|_{U_s}$. We have an induced map $\mathcal{C}_s := \pi^{-1}(U_s) \to \mathbb{P}^N \times U_s$, and this map is an embedding for $n \geq 3$, as the fibers of π are stable. This is a n canonical embedding, so we get a map $U_s \to K$ and by the universal property of the Hilbert scheme, we get a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_s & \longrightarrow & \mathcal{V} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle v} \\ U_s & \longrightarrow & K \end{array}$$

Therefore, v is versal.

The group of automorphisms $PGL(N)$ of $\mathbb{P}^N$ acts naturally on H. Then, there is an induced action $PGL(N) \times K \to K$. Gieseker [G] constructs $\overline{M_g}$ as a geometric GIT quotient of K under this action for any n sufficiently large. The quotient map, $\Phi : K \to \overline{M_g}$, is also the map induced by the family $v : \mathcal{V} \to K$.

2.2 The Picard group of $\overline{M_g}$

Let $\overline{M_g}$ be the coarse moduli space parametrizing stable curves. Let $A^1(\overline{M_g})$ be its Chow group of codimension-1 cycle classes and $\mathrm{Pic}(\overline{M_g})$ its Picard group. Since $\overline{M_g}$ has only finite quotient singularities, every codimension-1 subvariety Y of $\overline{M_g}$ is $\mathbb{Q}-$Cartier, i.e. there is a Cartier divisor D of $\overline{M_g}$ such that $[D] = d[Y]$ for some integer $d > 0$. So we have an isomorphism

$$A^1(\overline{M_g}) \otimes \mathbb{Q} \to Pic(\overline{M_g}) \otimes \mathbb{Q}$$

Now, we will define the Picard group of the moduli functor $\mathrm{Pic}_{fun}(\overline{M_g})$:

Definition 2.2.1. *An element $\gamma \in Pic_{fun}(\overline{M_g}) \otimes \mathbb{Q}$ is a collection of classes $\gamma_\pi \in Pic(S) \otimes \mathbb{Q}$ for each family of stable curves $\pi : \mathcal{C} \to S$, such that for each Cartesian diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{f} & S \end{array}$$

we have $\gamma_{\pi'} \cong f^*(\gamma_\pi)$.

We have an isomorphism (see [HMo], Proposition 3.88)

$$Pic(\overline{M_g}) \otimes \mathbb{Q} \to Pic_{fun}(\overline{M_g}) \otimes \mathbb{Q}$$

It is easy to explain how the isomorphism works: Consider an element $\Gamma \in Pic(\overline{M_g})$. For each family of stable curves $\pi : \mathcal{C} \to S$, we have an induced map $\zeta : S \to \overline{M_g}$. Now, define $\gamma_\pi := \zeta^*(\Gamma)$.

Also, there is an isomorphism (see [HMu], p.50)

$$Pic_{fun}(\overline{M_g}) \to Pic(K)^{PGL(N)},$$

where $\mathrm{Pic}(K)^{PGL(N)} \subseteq \mathrm{Pic}(K)$ is the invariant subgroup under the action of $PGL(N)$. The isomorphism carries an element $\gamma \in \mathrm{Pic}_{fun}(\overline{M_g})$ to γ_v.

2.3 Tautological and boundary classes

There is a natural element $\lambda \in \mathrm{Pic}_{fun}(\overline{M_g})$, which is called a *tautological class*. Given a family $\pi : \mathcal{C} \to S$ of stable curves, define $\lambda_\pi :=\det(\pi_*(\omega_\pi))$, where ω_π is the dualizing sheaf of π.

To define the boundary classes, we need some terminology. Given a connected nodal curve X, a node $P \in X$ is called a *disconnecting node* if $X - \{P\}$ is not connected. Otherwise, P is called a *connecting node*.

For each $i = 0, \ldots, [g/2]$, we define the subsets $\Delta_i' \subseteq K$ as follows: Δ_0' is the set of points $s \in K$ such that the fiber $\mathcal{V}_s$ has a connecting node, and Δ_i', for $i \geq 1$ is the set of points $s \in K$ such that the fiber $\mathcal{V}_s$ has a disconnecting node P, and the closure in $\mathcal{V}_s$ of one of the connected components of $\mathcal{V}_s - \{P\}$ has arithmetic genus i. The subsets $\Delta_i' \subseteq K$ are closed subsets of K of codimension 1. We give them their reduced induced scheme structures. Thus, they are Cartier divisors, because K is smooth. The invertible sheaves associated to the Δ_i' are invariant under the action of $PGL(N)$. Let $\delta_0, \ldots, \delta_{[g/2]}$ denote the corresponding elements of $\mathrm{Pic}_{fun}(\overline{M_g})$. These elements are called *boundary classes*. We can also view λ and the δ_i as elements of $\mathrm{Pic}(\overline{M_g}) \otimes \mathbb{Q}$.

The group $\mathrm{Pic}_{fun}(\overline{M_g})$ is freely generated by λ and the δ_i for $g \geq 3$ (see [AC]). If $g = 2$, then δ_0 and δ_1 form a basis for $\mathrm{Pic}(\overline{M_g}) \otimes \mathbb{Q}$ (see [M]), and we have Mumford's relation:

$$10\lambda = \delta_0 + 2\delta_1$$

For our calculations, it is important the fact that a class $\gamma \in Pic_{fun}(\overline{M_g}) \otimes \mathbb{Q}$ is defined by its value $\gamma_\pi \in Pic(S) \otimes \mathbb{Q}$ on 1-parameter families $\pi : \mathcal{C} \to S$, where $\mathcal{C}$ is smooth. Moreover, it is enough to consider just a sufficiently general family.

2.4 Intersection theory

All definitions and theorems of this section can be found in [F].

Let X be a scheme, and $A_k X$ the Chow group of its $k-$cycles modulo rational equivalence. Let E be a vector bundle over X of rank r. For $i = 0, 1, \ldots$, the $i - th$ *Chern class* $c_i(E)$ is a map

$$c_i(E) \cap \_ : A_k X \to A_{k-i} X$$

defined for all k by the following properties:

1. $c_0(E) = 1$.

2. If $f : X' \to X$ is a flat morphism, then

$$c_i(f^* E) \cap f^* \alpha = f^*(c_i(E) \cap \alpha)$$

for all cycles α on X and all i.

3. (Whitney sum) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles on X, we have

$$c_k(E) = \sum_{i+j=k} c_i(E') c_j(E'').$$

4. (Normalization) If E is a line bundle, and D is a Cartier divisor on X with $\mathcal{O}_X(D) \cong E$, then

$$c_1(E) \cap [X] = [D].$$

5. (Projection formula) If $f : X' \to X$ is a proper morphism, then

$$f_*(c_i(f^* E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$

for all cycles α on X' and all i.

6. (Vanishing) For all $i > r$, we have that

$$c_i(E) = 0.$$

Define the *Chern polynomial* $c_t(E)$ by

$$c_t(E) := 1 + c_1(E)t + \ldots + c_r(E)t^r.$$

Factor $c_t(E) = \prod_{i=1}^{r}(1 + \alpha_i t)$ in a formal way, i.e., the Chern classes of E are the elementary symmetric functions of $\alpha_1, \ldots, \alpha_r$. The α_i are called *Chern roots* of E.

The Chern classes of the dual bundle $E^\vee$ are given by the formula

$$c_i(E^\vee) = (-1)^i c_i(E).$$

For a line bundle L, we have the following formula for the top Chern class of the tensor product $E \otimes L$

$$c_r(E \otimes L) = \sum_{i=0}^{r} c_1(L)^i c_{r-i}(E).$$

Keeping the same notation of Section 1.2, we have the following proposition:

Proposition 2.4.1. $c_1(J^i_\pi(\mathcal{L})) = \binom{i+1}{2} c_1(\omega_\pi) + (i+1)c_1(\mathcal{L})$ *for every* $i \geq 0$.

Proof. By using the truncation exact sequence

$$0 \to \omega_\pi \otimes \mathcal{L} \to J^1_\pi(\mathcal{L}) \to J^0_\pi(\mathcal{L}) \cong \mathcal{L} \to 0$$

we obtain by the Whitney formula

$$c_1(J^1_\pi(\mathcal{L})) = c_1(\omega_\pi \otimes \mathcal{L}) + c_1(\mathcal{L}) = c_1(\omega_\pi) + c_1(\mathcal{L}) + c_1(\mathcal{L}) = c_1(\omega_\pi) + 2c_1(\mathcal{L})$$

More generally, by using the truncation exact sequence

$$0 \to \omega_\pi^{\otimes i} \otimes \mathcal{L} \to J^i_\pi(\mathcal{L}) \to J^{i-1}_\pi(\mathcal{L}) \to 0$$

we obtain by the Whitney formula for every $i \geq 1$

$$c_1(J^i_\pi(\mathcal{L})) = c_1(\omega^i_\pi \otimes \mathcal{L}) + c_1(J^{i-1}_\pi(\mathcal{L})) = ic_1(\omega_\pi) + c_1(\mathcal{L}) + c_1(J^{i-1}_\pi(\mathcal{L}))$$

Therefore, by induction

$$c_1(J^i_\pi(\mathcal{L})) = (1+2+...+i)c_1(\omega_\pi)+(i+1)c_1(\mathcal{L}) = \binom{i+1}{2}c_1(\omega_\pi)+(i+1)c_1(\mathcal{L}).$$

□

The *Chern character* $ch(E)$ of a vector bundle E of rank r is defined by the formula

$$ch(E) = \sum_{i=1}^{r} exp(\alpha_i),$$

where $exp(x) = e^x = \sum_{i=0}^{\infty} x^n/n!$, and $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E. The first terms are

$$ch(E) = r+c_1(E)+\frac{1}{2}(c_1(E)^2-2c_2(E))+\frac{1}{6}(c_1(E)^3-3c_1(E)c_2(E)+3c_3(E))+\ldots$$

The *Todd class* $td(E)$ of a vector bundle E of rank r is defined by the formula

$$td(E) = \prod_{i=1}^{r} Q(\alpha_i),$$

where

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k},$$

where the B_k are the Bernoulli numbers and $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E. The first terms are

$$td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \frac{1}{24}c_1(E)c_2(E) + \ldots$$

When X is non-singular, we write simply $c_i(E)$ in place of $c_i(E) \cap [X]$. Furthermore, every coherent sheaf $\mathcal{F}$ on a non-singular X has a finite resolution by locally free sheaves

$$0 \to E_n \to E_{n-1} \to \ldots \to E_1 \to E_0 \to \mathcal{F} \to 0,$$

so we can extend the definition of Chern classes to coherent sheaves; in fact, just use the Whitney sum to define

$$c_t(\mathcal{F}) := \prod_{i=0}^{n} c_t(E_i)^{(-1)^i}.$$

Given a proper morphism $f : X \to Y$ and a coherent sheaf E over X, recall that the *shriek* of E by f, denoted by $f_!(E)$, is an element of the

Grothendieck group of coherent sheaves on Y (for details and definitions, see [F], p.281). In order to state the Grothendieck-Riemann-Roch Theorem, we just need the following definition:

Definition 2.4.1. *Let $f : X \to Y$ be a proper morphism and E a coherent sheaf over X. Define*

$$ch(f_!(E)) := \sum_{i=0} (-1)^i ch(R^i f_*(E)).$$

Now, we state the Grothendieck-Riemann-Roch Theorem; this theorem will be useful for us, as by using it, we will be able to compute the first Chern class of the pushforward of an invertible sheaf, and this will be important to compute classes of degeneracy loci of evaluation maps (see Section 1.2).

Theorem 2.4.2. *(Grothendieck-Riemann-Roch) Let $f : X \to Y$ be a proper morphism between smooth connected schemes. Then*

$$ch(f_!(E)) = f_*(ch(E) \cdot td(T_{X/Y})),$$

for all coherent sheaf E over X, where $T_{X/Y}$ is the relative tangent sheaf.

If $\pi : \mathcal{C} \to S$ is a family of stable curves, where both $\mathcal{C}$ and S are smooth, then the Grothendieck-Riemann-Roch Theorem can be used to prove the following formulas:

(1) $td_1(T_{\mathcal{C}/S}) = -\frac{1}{2} c_1(\omega_\pi)$.
(2) $\pi_*(td_2(T_{\mathcal{C}/S})) = \lambda_\pi$.
(3) $\pi_*(c_1(\omega_\pi)^2) = 12\lambda - \delta$, where $\delta = \delta_0 + \ldots + \delta_{[g/2]}$.

Finally, we will state the Thom-Porteous Formula, which we will use together with the Grothendieck-Riemann-Roch Theorem to compute classes of degeneracy loci of evaluation maps.

Theorem 2.4.3. *Let X be a smooth connected scheme, $u : E \to F$ a morphism of vector bundles of ranks e and f, and $k \leq min\{e, f\}$. Define $D_k(u)$ as the locus where the map has rank $\leq k$. If $D_k(u)$ has the expected codimension $(e-k)(f-k)$, then*

$$[D_k(u)] = \Delta_{f-k}^{(e-k)}(c(F-E)) \cap [X],$$

where $\Delta_l^{(d)}(c(F-E)) = det((c_{l+j-i}(F-E))_{i,j=1,\ldots,d})$, and for each i, j, $c_{l+j-i}(F-E)$ is the coefficient of t^{l+j-i} in the formal series $c_t(F)/c_t(E)$.

In particular, if $e = f$ *and* $k = e - 1$, *then*

$$D_k(u) = c_1(F) - c_1(E)$$

and if $k = 0$ *and* $E = \mathcal{O}_X$ *then*

$$[D_k(u)] = c_f(F).$$

Chapter 3

Linear systems on rational and elliptic curves

3.1 Rational curves

Proposition 3.1.1. *Let $R_1, \dots, R_n$ be distinct points on $\mathbb{P}^1$, and $a_1, \dots, a_n$ positive integers. Define the linear system*

$$V := H^0(\omega_{\mathbb{P}^1}((a_1+1)R_1)) + \ldots + H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \\ \subseteq H^0(\omega_{\mathbb{P}^1}((a_1+1)R_1 + \ldots + (a_n+1)R_n))$$

Then V is $(a_1 + \ldots + a_n)$-dimensional and has no ramification points on $\mathbb{P}^1 - \{R_1, \dots, R_n\}$. Furthermore, for each i, the orders of vanishing at R_i of the sections in V are

$$0, \dots, a_i - 1, a_i + 1, \dots, a_1 + \ldots + a_n$$

$$\text{and } wt_V(R_i) = \textstyle\sum_{j \neq i} a_j.$$

Proof. Let $\mathcal{L} := \omega_{\mathbb{P}^1}((a_1+1)R_1 + \ldots + (a_n+1)R_n)$. Since for each i

$$U_i := H^0(\omega_{\mathbb{P}^1}((a_i+1)R_i)) \cap \sum_{j \neq i} H^0(\omega_{\mathbb{P}^1}((a_j+1)R_j))$$

is contained in $H^0(\mathcal{L}(-\sum_{j \neq i}(a_j+1)R_j))$ and $H^0(\mathcal{L}(-(a_i+1)R_i))$, we get

$$U_i \subseteq H^0(\mathcal{L}(-(a_1+1)R_1 - \ldots - (a_n+1)R_n)) = H^0(\omega_{\mathbb{P}^1}) = 0$$

for every i, so the dimension of V is $a_1 + \ldots + a_n$. On the other hand, all complete linear systems on $\mathbb{P}^1$ have no ramification points, so the statement of the proposition is true if $n = 1$. Suppose $n \geq 2$ and let us argue by induction on n. For every $0 \leq m \leq a_1 + 1$

$$\begin{aligned} V(-mR_1) = H^0(\omega_{\mathbb{P}^1}((a_1 + 1 - m)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_2 + 1)R_2)) \oplus \dots \\ \dots \oplus H^0(\omega_{\mathbb{P}^1}((a_n + 1)R_n)) \end{aligned}$$

Then $\dim_{\mathbb{C}} V(-mR_1) = a_1 - m + a_2 + \dots + a_n$ for every $0 \leq m \leq a_1$ and $V(-a_1R_1) = V(-(a_1 + 1)R_1)$.

Now, consider the linear system

$$\begin{aligned} V' = H^0(\omega_{\mathbb{P}^1}((a_2 + 1)R_2)) \oplus \dots \oplus H^0(\omega_{\mathbb{P}^1}((a_n + 1)R_n)) \\ \subseteq H^0(\omega_{\mathbb{P}^1}((a_2 + 1)R_2 + \dots + (a_n + 1)R_n)) \end{aligned}$$

Since by induction V' has no ramifications points on $\mathbb{P}^1 - \{R_2, \dots, R_n\}$, and since $\dim_{\mathbb{C}} V' = a_2 + \dots + a_n$ and $V'(-\alpha R_1) = V(-(a_1 + 1 + \alpha)R_1)$ for every integer $\alpha \geq 0$, it follows that $V(-(a_1 + 1 + a_2 + \dots + a_n)R_1) = 0$. Thus, the orders of vanishing at R_1 of the sections in V are

$$0, \dots, a_1 - 1, a_1 + 1, \dots, a_1 + \dots + a_n,$$

whence $wt_V(R_1) = a_2 + \dots + a_n$. Analogously, for each i, the orders of vanishing at R_i of the sections in V are

$$0, \dots, a_i - 1, a_i + 1, \dots, a_1 + \dots + a_n,$$

whence $wt_V(R_i) = \sum_{j \neq i} a_j$. Then

$$wt_V(R_1) + \dots + wt_V(R_n) = (n - 1)(a_1 + \dots + a_n).$$

On the other hand, since

$$\deg(\mathcal{L}) = a_1 + \dots + a_n + n - 2 \text{ and } \dim_{\mathbb{C}} V = a_1 + \dots + a_n,$$

we have by Plücker formula

$$\deg(R_V) = (n - 1)(a_1 + \dots + a_n).$$

Therefore, we have no other ramification points. □

Proposition 3.1.2. *Let $R_1, \dots, R_n$ be distinct points on $\mathbb{P}^1$, and $a_1, \dots, a_n$ positive integers.*

Let $\mathcal{L} := \omega_{\mathbb{P}^1}((a_1 + 1)R_1 + \dots + (a_n + 1)R_n))$. Define the linear system

$$V := H^0(\omega_{\mathbb{P}^1}((a_1 + 1)R_1)) \oplus \dots \oplus H^0(\omega_{\mathbb{P}^1}((a_n + 1)R_n)) \subseteq H^0(\mathcal{L})$$

Let $V_1 \subseteq H^0(\mathcal{L})$ *be a linear system of dimension* $a_1 + \ldots + a_n - 1$ *contained in* V *and containing*

$$H^0(\omega_{\mathbb{P}^1}((a_1-1)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n))$$

Then either V_1 *has no ramification points on* $\mathbb{P}^1 - \{R_1, \ldots, R_n\}$ *or* V_1 *has exactly one ramification point there and the ramification is simple. Furthermore,*

$$wt_{V_1}(R_1) = (\textstyle\sum_{j\neq 1} a_j) + a_1 + \ldots + a_n - 2 + \epsilon_1, \text{ where } \epsilon_1 \in \{0,1\},$$

and for each $i \neq 1$

$$wt_{V_1}(R_i) = (\textstyle\sum_{j\neq i} a_j) - 1 + \epsilon_i, \text{ where } \epsilon_i \in \{0,1\}.$$

Proof. If $a_1 = 1$, then by dimension considerations

$$V_1 = H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)).$$

Then, it follows from Proposition 3.1.1 that

$$wt_{V_1}(R_i) = \textstyle\sum_{j\neq 1,i} a_j = (\sum_{j\neq i} a_j) - 1 + \epsilon_i,$$

where $\epsilon_i = 0$ for every $i \neq 1$, and

$$\begin{aligned} wt_{V_1}(R_1) &= 2(a_2 + \ldots + a_n) \\ &= (\sum_{j\neq 1} a_j) + a_1 + \ldots + a_n - 2 + \epsilon_1, \end{aligned}$$

where $\epsilon_1 = 1$.

Now, assume $a_1 \geq 2$. As the orders of vanishing at R_1 of the sections in V are

$$0, \ldots, a_1 - 1, a_1 + 1, \ldots, a_1 + \ldots + a_n,$$

it follows that the orders of vanishing at R_1 of the sections in V_1 are of the form $\{0, \ldots, a_1 - 1, a_1 + 1, \ldots, a_1 + \ldots + a_n\} - \{l\}$, for some integer $l \in \{0, \ldots, a_1 - 1, a_1 + 1, \ldots, a_1 + \ldots + a_n\}$. Notice that

$$\begin{gathered} V(-2R_1) = H^0(\omega_{\mathbb{P}^1}((a_1-1)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \\ \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \subseteq V_1 \end{gathered}$$

and since $V_1 \subseteq V$, we have $V_1(-2R_1) = V_1 \cap V(-2R_1) = V(-2R_1)$. Then $\dim_{\mathbb{C}} V_1(-2R_1) = a_1 + \ldots + a_n - 2$ and hence $l \leq 1$. Therefore

$$\begin{aligned} wt_{V_1}(R_1) &= wt_V(R_1) + a_1 + \ldots + a_n - 1 - l \\ &= (\sum_{j \neq 1} a_j) + a_1 + \ldots + a_n - 2 + \epsilon_1, \end{aligned}$$

where $\epsilon_1 = 1 - l \in \{0, 1\}$.

To show the equalities $wt_{V_1}(R_i) = (\sum_{j \neq i} a_j) - 1 + \epsilon_i$, where $\epsilon_i \in \{0, 1\}$ and $i \neq 1$, it is enough to consider the case $i = 2$. Notice that

$$\begin{aligned} V(-(a_2+1)R_2) = H^0(\omega_{\mathbb{P}^1}((a_1+1)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_3+1)R_3)) \oplus \ldots \\ \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \end{aligned}$$

Now, consider the linear system

$$\begin{aligned} V' &= H^0(\omega_{\mathbb{P}^1}((a_1+1)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_3+1)R_3)) \oplus \ldots \\ &\ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \\ &\subseteq H^0(\omega_{\mathbb{P}^1}((a_1+1)R_1 + (a_3+1)R_3 \ldots + (a_n+1)R_n)) \end{aligned}$$

Since

$$\begin{aligned} V'(-2R_1) &= H^0(\omega_{\mathbb{P}^1}((a_1-1)R_1)) \oplus H^0(\omega_{\mathbb{P}^1}((a_3+1)R_3)) \oplus \ldots \\ &\ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \\ &\subseteq H^0(\omega_{\mathbb{P}^1}((a_1-1)R_1 + (a_3+1)R_3 \ldots + (a_n+1)R_n)) \end{aligned}$$

has no ramification points on $\mathbb{P}^1 - \{R_1, R_3, \ldots, R_n\}$ and $\dim_{\mathbb{C}} V'(-2R_1) = a_1 + a_3 + \ldots + a_n - 2$, we get $V'(-2R_1 - (a_1 + a_3 + \ldots + a_n - 2)R_2) = 0$ and hence $V(-2R_1 - (a_1 + \ldots + a_n - 1)R_2) = 0$. As we saw in Proposition 3.1.1, we have $\dim_{\mathbb{C}} V(-(a_1 + \ldots + a_n - 1)R_2) = 2$; then, by dimension considerations

$$V = V(-2R_1) \oplus V(-(a_1 + \ldots + a_n - 1)R_2)$$

Therefore $V(-(a_1 + \ldots + a_n - 1)R_2) \nsubseteq V_1$ and we get

$$\dim_{\mathbb{C}} V_1(-(a_1 + \ldots + a_n - 1)R_2) = 1$$

On the other hand, as we saw in Proposition 3.1.1, the orders of vanishing at R_2 of the sections in V are

$$0, \ldots, a_2 - 1, a_2 + 1, \ldots, a_1 + \ldots + a_n$$

So the orders of vanishing at R_2 of the sections in V_1 are of the form

$$\{0, \dots, a_2 - 1, a_2 + 1, \dots, a_1 + \dots + a_n\} - \{l\},$$

for some integer $l \in \{0, \dots, a_2 - 1, a_2 + 1, \dots, a_1 + \dots + a_n\}$. Since we have $\dim_{\mathbb{C}} V_1(-(a_1 + \dots + a_n - 1)R_2) = 1$, it follows that $l \geq a_1 + \dots + a_n - 1$. Thus

$$\begin{aligned} wt_{V_1}(R_2) &= wt_V(R_2) + a_1 + \dots + a_n - 1 - l \\ &= a_1 + a_3 + \dots + a_n + a_1 + \dots + a_n - 1 - l \\ &= a_1 + a_3 + \dots + a_n - 1 + a_1 + \dots + a_n - l \\ &= a_1 + a_3 + \dots + a_n - 1 + \epsilon_2, \end{aligned}$$

where $\epsilon_2 := a_1 + \dots + a_n - l \in \{0, 1\}$.

Now, we will prove the first statement of the proposition. Using the equalities we have shown, we get

$$\sum wt_{V_1}(R_i) = (a_1 + \dots + a_n - 1)n - 1 + \sum \epsilon_i$$

On the other hand, by Plücker formula $\deg(R_{V_1}) = (a_1 + \dots + a_n - 1)n$. Then $0 \leq \sum \epsilon_i \leq 1$ and V_1 has $1 - \sum \epsilon_i$ ramification points on $\mathbb{P}^1 - \{R_1, \dots, R_n\}$, counted with their respective weights. This proves the first statement of the proposition. □

Proposition 3.1.3. *Let $R_1, \dots, R_n$ be distinct points on $\mathbb{P}^1$, and $a_1, \dots, a_n$ positive integers. Define the linear system*

$$\begin{aligned} V := H^0(\omega_{\mathbb{P}^1}((a_1 + 1)R_1)) \oplus \dots \oplus H^0(\omega_{\mathbb{P}^1}((a_n + 1)R_n)) \\ \subseteq H^0(\omega_{\mathbb{P}^1}((a_1 + 1)R_1 + \dots + (a_n + 1)R_n)) \end{aligned}$$

Consider the linear system

$$V_1 := V(-P) \subseteq H^0(\omega_{\mathbb{P}^1}(-P + (a_1 + 1)R_1 + \dots + (a_n + 1)R_n))$$

Then either V_1 has no ramification points on $\mathbb{P}^1 - \{R_1, \dots, R_n\}$ or V_1 has exactly one ramification point there and the ramification is simple. Furthermore, for each i

$$wt_{V_1}(R_i) = (\textstyle\sum_{j \neq i} a_j) - 1 + \epsilon_i, \text{ where } \epsilon_i \in \{0, 1\}.$$

Proof. All complete linear systems on $\mathbb{P}^1$ have no ramification points, so the first statement of the proposition is true if $n = 1$. Suppose $n \geq 2$ and let us argue by induction on n. We have

$$\begin{aligned} V(-a_1R_1) &= V(-(a_1+1)R_1) \\ &= H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)) \end{aligned}$$

Then

$$\begin{aligned} V_1(-a_1R_1) &= V_1(-(a_1+1)R_1) \\ &= (H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)))(-P) \\ &\subseteq H^0(\omega_{\mathbb{P}^1}(-P+(a_2+1)R_2+\ldots+(a_n+1)R_n)) \end{aligned}$$

Now, consider the linear system

$$\begin{aligned} V' &= (H^0(\omega_{\mathbb{P}^1}((a_2+1)R_2)) \oplus \ldots \oplus H^0(\omega_{\mathbb{P}^1}((a_n+1)R_n)))(-P) \\ &\subseteq H^0(\omega_{\mathbb{P}^1}(-P+(a_2+1)R_2+\ldots+(a_n+1)R_n)) \end{aligned}$$

We have $\dim_{\mathbb{C}} V' = a_2 + \ldots + a_n - 1$ (Proposition 3.1.1) and by induction R_1 is at most a simple ramification point of V'. Therefore, the orders of vanishing at R_1 of the sections in V_1 are

$$\begin{gathered} 0, \ldots, a_1-1, a_1+1, \ldots, a_1+\ldots+a_n-1 \text{ or} \\ 0, \ldots, a_1-1, a_1+1, \ldots, a_1+\ldots+a_n-2, a_1+\ldots+a_n, \end{gathered}$$

i.e., $\{0, \ldots, a_1-1, a_1+1, \ldots, a_1+\ldots+a_n\} - \{l\}$, where $l = a_1+\ldots+a_n-1$ or $l = a_1+\ldots+a_n$. Then

$$\begin{aligned} wt_{V_1}(R_1) &= wt_V(R_1) + a_1 + \ldots + a_n - 1 - l \\ &= (\sum_{j\neq 1} a_j) - 1 + a_1 + \ldots + a_n - l \\ &= (\sum_{j\neq 1} a_j) - 1 + \epsilon_1, \end{aligned}$$

where $\epsilon_1 = a_1 + \ldots + a_n - l \in \{0, 1\}$. Analogously, we have for each i

$$wt_{V_1}(R_i) = (\textstyle\sum_{j\neq i} a_j) - 1 + \epsilon_i, \text{ where } \epsilon_i \in \{0,1\}.$$

Using the equalities we have shown, we get

$$\sum wt_{V_1}(R_i) = (a_1+\ldots+a_n-1)(n-1) - 1 + \sum \epsilon_i$$

On the other hand, by Plücker formula $\deg(R_{V_1}) = (a_1 + \ldots + a_n - 1)(n - 1)$. Then $0 \leq \sum \epsilon_i \leq 1$ and V_1 has $1 - \sum \epsilon_i$ ramification points on $\mathbb{P}^1 - \{R_1, \ldots, R_n\}$, counted with their respective weights. This proves the proposition. □

Proposition 3.1.4. *Let $R_1, \ldots, R_n$ be distinct points on $\mathbb{P}^1$ and $l_1, \ldots, l_n$ nonnegative integers.*

Let $\mathcal{L}$ be an invertible sheaf and $\mathcal{L}' := \mathcal{L}(-l_1R_1 - \ldots - l_nR_n)$.

Let $V \subseteq H^0(\mathcal{L})$ be a linear system such that $V \subseteq H^0(\mathcal{L}')$. Let V' denote the linear system V inside $H^0(\mathcal{L}')$ and let $r+1 :=dim_{\mathbb{C}}V$. Then, for each i, we have that $b_0 + l_i, \ldots, b_r + l_i$ are the orders of vanishing at R_i of the sections in V, where $b_0, \ldots, b_r$ are the orders of vanishing at R_i of the sections in V'. Hence, for each i

$$wt_V(R_i) = wt_{V'}(R_i) + l_i(r+1)$$

Proof. To show the equalities $wt_V(R_i) = wt_{V'}(R_i) + l_i dim_{\mathbb{C}}V$, it is enough to consider the case $i = 1$. Since $\mathcal{L}' = \mathcal{L}(-l_1R_1 - \ldots - l_nR_n)$, we have for each $\beta \geq 0$

$$H^0(\mathcal{L}'(-\beta R_1)) = H^0(\mathcal{L}(-(l_1+\beta)R_1 - l_2R_2 - \ldots - l_nR_n)),$$

and since $V \subseteq H^0(\mathcal{L}(-l_2R_2 - \ldots - l_nR_n))$, we have

$$\begin{aligned} H^0(\mathcal{L}'(-\beta R_1)) \cap V &= H^0(\mathcal{L}(-(l_1+\beta)R_1 - l_2R_2 - \ldots - l_nR_n)) \cap V \\ &= H^0(\mathcal{L}(-(l_1+\beta)R_1)) \cap V \end{aligned}$$

Therefore $H^0(\mathcal{L}'(-\beta R_1)) \cap V' = H^0(\mathcal{L}(-(l_1+\beta)R_1)) \cap V$. Thus, if $b_0, \ldots, b_r$ are the orders of vanishing at R_1 of the sections in V', then $b_0+l_1, \ldots, b_r+l_1$ are the orders of vanishing at R_1 of the sections in V. This proves the statement of the proposition. □

3.2 Elliptic curves

Proposition 3.2.1. *Let E be a smooth elliptic curve, A a point of E and g an odd positive integer. Let $\mathcal{L} := \mathcal{O}_E((2g-2)A)$. Consider the linear system*

$$V := H^0(\mathcal{O}_E(gA)) \subseteq H^0(\mathcal{L})$$

Let $V_1 \subseteq H^0(\mathcal{L})$ be a linear system of dimension $g-1$ such that

$$H^0(\mathcal{O}_E((g-2)A)) \subseteq V_1 \subseteq V$$

Then $wt_{V_1}(Q) \leq 2$ for every $Q \in E - \{A\}$ and $wt_{V_1}(A) = (g-1)^2 + \epsilon$, where $\epsilon \in \{0, 1\}$.

Proof. Let $Q \in E - \{A\}$. Notice that $V(-gA) = H^0(\mathcal{O}_E((g-2)A))$ is contained in V_1. As we have

$$V(-gA) \cap V(-(g-3)Q) = H^0(\mathcal{O}_E((g-2)A - (g-3)Q))$$

as subspaces of $H^0(\mathcal{O}_E((2g-2)A))$, it follows that $V(-gA) \cap V(-(g-3)Q)$ has dimension 1. Then, by dimension considerations

$$V = V(-gA) + V(-(g-3)Q)$$

Therefore $V(-(g-3)Q) \not\subseteq V_1$ and $\dim_{\mathbb{C}} V_1(-(g-3)Q) = 2$. On the other hand, the orders of vanishing at Q of the sections in V are of the form $0, \ldots, g-2, a_{g-1}$, where $g-1 \leq a_{g-1} \leq g$, and hence the orders of vanishing at Q of the sections in V_1 are of the form $\{0, \ldots, g-2, a_{g-1}\} - \{l\}$, where $l \in \{0, \ldots, g-2, a_{g-1}\}$. As $\dim_{\mathbb{C}} V_1(-(g-3)Q) = 2$, it follows that $l \geq g-3$. Thus $wt_{V_1}(Q) = wt_V(Q) + g - 1 - l \leq 2$, if Q is an ordinary point of V.

Now, assume Q is an ordinary point of $V(-gA) = H^0(\mathcal{O}_E((g-2)A))$. Then, by dimension considerations

$$V = V(-gA) \oplus V(-(g-2)Q).$$

Therefore $V(-(g-2)Q) \not\subseteq V_1$ and $\dim_{\mathbb{C}} V_1(-(g-2)Q) = 1$. It follows that $l \geq g-2$ and hence $wt_{V_1}(Q) = wt_V(Q) + g - 1 - l \leq 2$.

Now, we will prove that $H^0(\mathcal{O}_E(gA))$ and $H^0(\mathcal{O}_E((g-2)A))$ do not have ramification points in common on $E - \{A\}$, when g is an odd positive integer. Suppose by contradiction that there exists $Q \in E - \{A\}$ which is a ramification point in common of both $H^0(\mathcal{O}_E(gA))$ and $H^0(\mathcal{O}_E((g-2)A))$. Then $h^0(\mathcal{O}_E(gA - gQ)) = 1$ and $h^0(\mathcal{O}_E((g-2)A - (g-2)Q)) = 1$. Thus gA and gQ are linearly equivalent divisors and the same property is true for $(g-2)A$ and $(g-2)Q$. Therefore, $2A$ and $2Q$ are linearly equivalent divisors. Now let $g = 2n+1$; since $2A$ and $2Q$ are linearly equivalent divisors, we have that $2nA$ and $2nQ$ are linearly equivalent divisors. As

$(2n+1)A$ and $(2n+1)Q$ are linearly equivalent divisors, it follows that A and Q are linearly equivalent divisors and hence $Q = A$, a contradiction.

Finally, we will compute $wt_{V_1}(A)$. Since the orders of vanishing at A of the sections in V are $g-2, \ldots, 2g-4, 2g-2$, we have that the orders of vanishing at A of the sections in V_1 are of the form

$$\{g-2, \ldots, 2g-4, 2g-2\} - \{l\}$$

where $l \in \{g-2, \ldots, 2g-4, 2g-2\}$. Since $V(-gA) \subseteq V_1 \subseteq V$, we have that $V_1(-gA) = V(-gA)$. Then $\dim_{\mathbb{C}} V_1(-gA) = g-2$ and $g-2 \leq l \leq g-1$. Therefore

$$wt_{V_1}(A) = wt_V(A) + g - 1 - l = (g-1)^2 + \epsilon$$

where $\epsilon = g - 1 - l \in \{0, 1\}$. $\square$

Chapter 4

The divisor

4.1 Introduction

Our aim is to compute the class of the divisor $\overline{S^2W}$ in $\mathrm{Pic}_{fun}(\overline{M_g})$, defined as the closure of the locus of smooth curves C with a pair of points (P, Q) satisfying that Q is a ramification point of the linear system $H^0(\omega_C(-P))$ with ramification weight at least 3.

Write the class of the divisor we want to compute as

$$\overline{S^2W} := a\lambda - a_0\delta_0 - a_1\delta_1 - \ldots - a_{[g/2]}\delta_{[g/2]}$$

First, we will compute the coefficient a. Let $\pi : \mathcal{X} \to T$ be a family of smooth curves over a smooth curve T. Consider the double product $\mathcal{Y} = \mathcal{X} \times_T \mathcal{X}$ as a family of curves via the first projection $p_1 : \mathcal{Y} \to \mathcal{X}$. Let W be the ramification divisor of the invertible sheaf $\mathcal{L} = \omega_{p_1}(-\Delta)$ with respect to p_1. Notice that $h^0(\mathcal{L}\big|_{\mathcal{Y}_P}) = h^0(\omega_{\mathcal{X}_{\pi(P)}}(-P)) = g-1$ for every $P \in \mathcal{X}$. Then $p_{1*}(\mathcal{L})$ is locally free of rank $g-1$.

Now, we will compute $\pi_* p_{1*}([S^2W])$. By the Thom-Porteous Formula:

$$[W] = c_1(J^{g-2}_{p_1}(\mathcal{L})) - c_1(p_1^* p_{1*}(\mathcal{L}))$$

By using the truncation exact sequences, we obtain (Proposition 2.4.1)

$$c_1(J^{g-2}_{p_1}(\mathcal{L})) = \binom{g-1}{2} c_1(\omega_{p_1}) + (g-1)c_1(\mathcal{L})$$

We have to compute $c_1(p_{1*}(\mathcal{L}))$. Notice that by Riemann-Roch we have $h^1(\mathcal{L}\big|_{\mathcal{Y}_P}) = 1$ for every $P \in \mathcal{X}$, as $h^0(\mathcal{L}\big|_{\mathcal{Y}_P}) = g-1$. It follows that $R^1 p_{1*}(\mathcal{L})$ is invertible.

Consider the long exact sequence

$$0 \to p_{1*}(\mathcal{L}) \to p_{1*}(\omega_{p_1}) \to p_{1*}(\omega_{p_1}\big|_{\Delta}) \to$$

$$R^1 p_{1*}(\mathcal{L}) \to R^1 p_{1*}(\omega_{p_1}) \to R^1 p_{1*}(\omega_{p_1}\big|_\Delta) \to 0.$$

Since $R^1 p_{1*}(\omega_{p_1}\big|_\Delta) = 0$, as the restriction of $\omega_{p_1}\big|_\Delta$ to each fiber is supported at a point, we have a surjection $R^1 p_{1*}(\mathcal{L}) \to R^1 p_{1*}(\omega_{p_1})$. As $R^1 p_{1*}(\mathcal{L})$ is an invertible sheaf and $R^1 p_{1*}(\omega_{p_1}) \cong \mathcal{O}_{\mathfrak{X}}$, it follows that $R^1 p_{1*}(\mathcal{L}) \cong R^1 p_{1*}(\omega_{p_1})$. Then we have an exact sequence

$$0 \to p_{1*}(\mathcal{L}) \to p_{1*}(\omega_{p_1}) \to p_{1*}(\omega_{p_1}\big|_\Delta) \to 0$$

Via the Whitney formula, we have

$$c_1(p_{1*}(\mathcal{L})) = c_1(p_{1*}(\omega_{p_1})) - c_1(p_{1*}(\omega_{p_1}\big|_\Delta)).$$

From $\omega_{p_1} = p_2^*\omega_\pi$, we get $p_{1*}(\omega_{p_1}\big|_\Delta) = \omega_\pi$. On the other hand, since $p_{1*}(\omega_{p_1}) = p_{1*}(p_2^*(\omega_\pi)) \cong \pi^*\pi_*\omega_\pi$,

$$c_1(p_{1*}(\omega_{p_1})) = \pi^* c_1(\pi_*\omega_\pi) = \pi^* c_1(\det \pi_*\omega_\pi) = \pi^*\lambda_\pi$$

Therefore

$$c_1(p_{1*}(\mathcal{L})) = \pi^*\lambda - K_\pi$$

where $\lambda := \lambda_\pi$ and $K_\pi = c_1(\omega_\pi)$.

Let $K_{p_1} = p_2^* K_\pi$ and $K_{p_2} = p_1^* K_\pi$. Then

$$\begin{aligned} [W] &= \binom{g-1}{2} K_{p_1} + (g-1)c_1(\mathcal{L}) - p_1^*(\pi^*\lambda - K_\pi) \\ &= \binom{g-1}{2} K_{p_1} + (g-1)(K_{p_1} - \Delta) - p_1^*\pi^*\lambda + K_{p_2} \\ &= \binom{g}{2} K_{p_1} + K_{p_2} - (g-1)\Delta - p_1^*\pi^*\lambda \end{aligned}$$

By the Thom-Porteous Formula:

$$[S^2W] = c_3(J^2_{p_1}(\mathcal{O}_{\mathcal{Y}}(W)))$$

Using the truncation exact sequence

$$0 \to \omega_{p_1}^{\otimes 2} \otimes \mathcal{O}_{\mathcal{Y}}(W) \to J^2_{p_1}(\mathcal{O}_{\mathcal{Y}}(W)) \to J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W)) \to 0$$

and recalling that $J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W))$ is locally free of rank 2, we get

$$c_3(J^2_{p_1}(\mathcal{O}_{\mathcal{Y}}(W))) = c_2(J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W)))c_1(\omega_{p_1}^{\otimes 2} \otimes \mathcal{O}_{\mathcal{Y}}(W)),$$

and using the truncation exact sequence

$$0 \to \omega_{p_1} \otimes \mathcal{O}_{\mathcal{Y}}(W) \to J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W)) \to \mathcal{O}_{\mathcal{Y}}(W) \to 0$$

we get $c_2(J^1_{p_1}(\mathcal{O}_{\mathcal{Y}}(W))) = c_1(\mathcal{O}_{\mathcal{Y}}(W))c_1(\omega_{p_1} \otimes \mathcal{O}_{\mathcal{Y}}(W))$.

Therefore

$$[S^2W] = c_3(J^2_{p_1}(\mathcal{O}_{\mathcal{Y}}(W))) = [W](K_{p_1} + [W])(2K_{p_1} + [W])$$

On the other hand, since $\mathcal{O}(-\Delta)\big|_\Delta \cong \omega_\pi$ (identifying Δ with $\mathcal{X}$), we have $\mathcal{O}(-\Delta)\big|_\Delta = (p_2^*\omega_\pi)\big|_\Delta = \omega_{p_1}\big|_\Delta$ and $\mathcal{O}(-\Delta)\big|_\Delta = (p_1^*\omega_\pi)\big|_\Delta = \omega_{p_2}\big|_\Delta$. Then $\Delta^2 = -K_{p_1} \cdot \Delta = -K_{p_2} \cdot \Delta$. Using the projection formula and the following formulas

(1)$K^3_{p_1} = 0$, $K^3_{p_2} = 0$ and $(p_1^*\pi^*\lambda)^2 = 0$

(2)$p_{1*}(K_{p_1} \cdot \Delta) = K_\pi$, $\Delta^2 = -K_{p_1} \cdot \Delta = -K_{p_2} \cdot \Delta$

(3)$\pi_*(K^2_\pi) = 12\lambda$, $\pi_*(K_\pi) = 2g - 2$

(4)$\pi^*\pi_*(\alpha) = p_{1*}p_2^*(\alpha)$ for every cycle α on $\mathcal{X}$, we get:

$$\pi_* p_{1*}([S^2W]) = (9g^5 - 51g^4 + 129g^3 - 207g^2 + 174g - 54)\lambda$$

Therefore,

$$a = 9g^5 - 51g^4 + 129g^3 - 207g^2 + 174g - 54.$$

For g odd and $g \geq 5$, we will obtain the coefficient a_i in terms of the coefficient a_1 for every $i > 1$, in Chapter 5, by using the method of test curves. We will use $[g/2] - 1$ test curves, which are induced by families of flag stable curves over $\mathbb{P}^1$. Of crucial importance in the use of the test curves is Proposition 5.2.1, which is similar to [HMo], Thm 6.65, (2); in fact, Proposition 5.2.1 implies that result. To be able to apply our Proposition 5.2.1, we will use Proposition 5.1.1, which is a general result about flag curves. We end up with (see Chapter 5 for more details)

$$a_i = (i(g-i)/(g-1))a_1, \text{ for each } 2 \leq i \leq [g/2].$$

In Chapter 6, we will obtain a lower bound for the coefficient a_i, for every $i \geq 1$. To do it, we will consider a general family $\pi : \mathcal{X} \to T$ of stable curves over a smooth projective curve T. Since the family is general, the singular curves we have in our family have only one node, and these singular curves are not in the support of the divisor we want to compute. We will restrict ourselves to a neighborhood in T of some point t_0, such that $\mathcal{X}_{t_0}$

is a singular fiber and the other fibers are nonsingular. Assume that the singular fiber is reducible.

First, we consider $\mathcal{Y}' = \mathcal{X} \times_T \mathcal{X}$ and blow up to solve the singularities of $\mathcal{Y}'$. Let $\mathcal{B}'$ be this blow up. Using the first projection of $\mathcal{Y}'$ to $\mathcal{X}$, we obtain a map $\rho_1' : \mathcal{B}' \to \mathcal{X}$. We consider ρ_1' as a family of curves over $\mathcal{X}$.

Let ω be the relative dualizing sheaf of $\mathcal{B}'/\mathcal{X}$ and $\mathcal{L}' := \omega(-\widetilde{\Delta})$, where $\widetilde{\Delta}$ is the strict transform of Δ in $\mathcal{B}'$. It will be necessary to do some modifications in order to obtain $h^0(\mathcal{L}'\big|_F) = g-1$ for every fiber F of ρ_1'. After the modifications, we get a family $\rho_1 : \mathcal{B} \to \widetilde{\mathcal{X}}$ of nodal curves over $\widetilde{\mathcal{X}}$ such that $h^0(\mathcal{L}\big|_F) = g-1$ for every fiber F, where $\widetilde{\mathcal{X}}$ is a suitable blow up of $\mathcal{X}$, $\mathcal{Y} := \widetilde{\mathcal{X}} \times_T \mathcal{X}$ and $\mathcal{B}$ is a blow up of $\mathcal{Y}$ which solves its singularities. Then, $\rho_{1*}\mathcal{L}$ is locally free of rank $g-1$ (Proposition 6.2.1). Now consider the natural map $u : \rho_1^*\rho_{1*}\mathcal{L} \to J^{g-2}_{\rho_1}(\mathcal{L})$ and substract excess components of the degeneracy scheme W' of u. Then, we get a divisor W intersecting each fiber in finitely many points (Propositions 6.3.1 and 6.4.2). If W intersects each fiber away from the nodes, with multiplicity at most 2 at each point, then we have $\widetilde{\pi}_*\rho_{1*}(c_3(J^2_{\rho_1}(\mathcal{O}_B(W)))) = [\pi]^*(\overline{S^2W})$, where $[\pi] : T \to \overline{M_g}$ is the map which is induced by π, and $\widetilde{\pi} : \widetilde{\mathcal{X}} \to T$ is the morphism induced by $\pi : \mathcal{X} \to T$. However, we have only proved that S^2W has only finitely many points in the singular fibers (see Proposition 6.4.3 and Hypothesis $(*)$ before that proposition). We must then compute their multiplicities and substract them from $c_3(J^2_{\rho_1}(\mathcal{O}_\mathcal{B}(W)))$ to get $[\pi]^*(\overline{S^2W})$. It turns out that we can obtain a lower bound for the coefficient a_i, for every $i \geq 1$. We end up with

$$-b_i \leq a_i \text{ for every } 1 \leq i \leq [g/2],$$

where

$$\begin{aligned} b_i := {} & 6i^4g^2 - 6i^4g + 12i^4 - 6i^3g^3 - 3i^3g^2 - 3i^3g - 18i^3 + 3i^2g^4 \\ & + 3i^2g^2 + 12i^2g + 6i^2 - 3ig^5 + 12ig^4 - 21ig^3 + 21ig^2 - 21ig + 6i. \end{aligned}$$

Our main theorem is:

Theorem 4.1.1. *Let $\overline{S^2W} \subseteq \overline{M_g}$ be the effective divisor which is defined as the closure of the locus of smooth curves C with a pair of points (P, Q) satisfying that Q is a ramification point of the linear system $H^0(\omega_C(-P))$ with ramification weight at least* 3.

Write the class of $\overline{S^2W}$ in $Pic_{fun}(\overline{M_g}) \otimes \mathbb{Q}$ in the form

$$\overline{S^2W} = a\lambda - a_0\delta_0 - a_1\delta_1 - \ldots - a_{[g/2]}\delta_{[g/2]}$$

Then

$$a = 9g^5 - 51g^4 + 129g^3 - 207g^2 + 174g - 54, \text{ and}$$

(1) If g is an odd integer such that $g \geq 5$, then

$$a_i = (i(g-i)/(g-1))a_1 \text{ for every } 2 \leq i \leq [g/2].$$

(2) Assume the following hypothesis holds:

($*$) *If (X, A) is a general pointed smooth curve, then for every ramification point $P \in X$ of the complete linear system $H^0(\omega_X(-(g_X - 1)A))$ and for every $i \geq 1$, the complete linear system $H^0(\omega_X((i+1)A - P))$ does not have ramification points on $X - \{A\}$ having ramification weight at least* 3.

Then, for every g, we have the following inequalities

$$-b_i \leq a_i \text{ for every } 1 \leq i \leq [g/2],$$

where

$$b_i := 6i^4g^2 - 6i^4g + 12i^4 - 6i^3g^3 - 3i^3g^2 - 3i^3g - 18i^3 + 3i^2g^4 + 3i^2g^2 + 12i^2g + 6i^2 - 3ig^5 + 12ig^4 - 21ig^3 + 21ig^2 - 21ig + 6i.$$

We actually have equalities in Theorem 4.1.1, item (2), without using the hypothesis ($*$), for $g = 3$ and $i = 1$, and for $g = 4$ and $i = 2$. (See Propositions 6.4.4 and 6.4.5.)

4.2 The irreducible case

In this section, we just present a few results which can possibly be useful to compute the coefficient of δ_0 in the expression for $\overline{S^2W}$.

Proposition 4.2.1. *Let X be a nodal curve which is the union of a smooth curve C of genus $g-1$ and a chain of rational smooth curves $E_1, \ldots, E_{n-1}$. Suppose C intersects only E_1 and E_{n-1}. Let $A \in C \cap E_1$ and $B \in C \cap E_{n-1}$ be the unique points of intersection. Assume that (C, A, B) is a two-pointed general smooth curve. Let $p : \mathcal{C} \to \Sigma$ be a smoothing of X. Fix an integer $1 \leq i \leq n-1$ and a section Γ of p intersecting X at a point $P \in E_i$, where P is not a node of X. Let $\mathcal{L} := \omega_p(-\Gamma)$ and let W_0 be the limit ramification divisor of $\mathcal{L}$. Then*

$$mult_Q(W_0) \leq 1, \text{ if } Q \in X \text{ is not a node.}$$

Furthermore, if $n = q(g-1)$, where $q > 0$ is an even integer, and $i = n/2$, then W_0 is reduced and contains no node of X.

Proof. We have

$$\mathcal{L}\Big|_C = \omega_C(A+B), \mathcal{L}\Big|_{E_i} = \mathcal{O}_{E_i}(-1), \mathcal{L}\Big|_{E_j} = \mathcal{O}_{E_j}; j \neq i. \tag{4.2.1}$$

For each $l = 1, \dots, n-1$, set

$$\mathcal{L}_l := \mathcal{L}(-\sum_{r=1}^{n-1} a_{l,r} E_r),$$

where

$$a_{l,r} = (n-r)a_{l,n-1} - \max\{i-r, 0\} - \max\{l-r, 0\}(g-2) - \max\{n-d-r, 0\},$$
$$i + l(g-2) = na_{l,n-1} - (n-d),\ 1 \leq d \leq n,\ a_{l,n-1} \geq 1.$$

If $i + l(g-2)$ is a multiple of n, then $a_{l,n-1} < g-1$(as $i < n$ and $l < n$) and $a_{l,1} + a_{l,n-1} = g-1$, and hence $a_{l,1} \geq 1$. Analogously, if $i + l(g-2)$ is not a multiple of n, then $a_{l,n-1} < g$ and $a_{l,1} + a_{l,n-1} = g$, and hence $a_{l,1} \geq 1$.

If $i + l(g-2)$ is a multiple of n, then we have

$$\mathcal{L}_l\Big|_C = \omega_C(-(a_{l,1}-1)A - (a_{l,n-1}-1)B),$$
$$\mathcal{L}_l\Big|_{E_l} = \mathcal{O}_{E_l}(g-2),$$
$$\mathcal{L}_l\Big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j \neq l.$$

If $i + l(g-2)$ is not a multiple of n and $n - d \neq l$, then

$$\mathcal{L}_l\Big|_C = \omega_C(-(a_{l,1}-1)A - (a_{l,n-1}-1)B),$$
$$\mathcal{L}_l\Big|_{E_l} = \mathcal{O}_{E_l}(g-2),$$
$$\mathcal{L}_l\Big|_{E_{n-d}} = \mathcal{O}_{E_{n-d}}(1),$$
$$\mathcal{L}_l\Big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j \neq l, n-d.$$

If $i + l(g-2)$ is not a multiple of n and $n - d = l$, then

$$\mathcal{L}_l\Big|_C = \omega_C(-(a_{l,1}-1)A - (a_{l,n-1}-1)B),$$
$$\mathcal{L}_l\Big|_{E_l} = \mathcal{O}_{E_l}(g-1),$$
$$\mathcal{L}_l\Big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j \neq l.$$

Thus, $\mathcal{L}_l$ has focus on E_l. Let V_l be the limit linear system of $\mathcal{L}$ on E_l. When $i + l(g-2)$ is a multiple of n or $i + l(g-2)$ is not a multiple of n and $n - d \neq l$, by dimension considerations, $V_l = H^0(\mathcal{O}_{E_l}(g-2))$ and hence $V_l \subseteq H^0(\mathcal{O}_{E_l}(g-2))$ has no ramification point on E_l. Now, assume $i+l(g-2)$ is not a multiple of n and $n-d = l$. Let $Z := \overline{X - C}$ and let A_l and B_l be the nodes of X lying on E_l. By considering the exact sequence

$$0 \to \mathcal{L}_l(-C)\big|_Z \to \mathcal{L}_l\big|_X \to \mathcal{L}_l\big|_C \to 0,$$

we get

$$\begin{aligned} h^0(\mathcal{L}_l\big|_X) &\leq h^0(\mathcal{O}_{E_l}(g-1)(-A-B)) \\ &\quad + h^0(\omega_C(-(a_{l,1}-1)A - (a_{l,n-1}-1)B)) \\ &= g-1, \end{aligned}$$

as A and B are general points of C, and since by semicontinuity we have $h^0(\mathcal{L}_l\big|_X) \geq g-1$, we conclude $h^0(\mathcal{L}_l\big|_X) = g-1$. By the base change theorem we have a surjection $H^0(\mathcal{L}_l) \to H^0(\mathcal{L}_l\big|_X)$. Thus, we have $H^0(\mathcal{O}_{E_l}(g-1)(-A_l-B_l)) \subseteq H^0(\mathcal{L}_l\big|_X) \cong V_l \subseteq H^0(\mathcal{O}_{E_l}(g-1))$ and hence, for $Q \in E_l - \{A_l, B_l\}$, the vanishing sequence of V_l at Q starts with $0, \ldots, g-3$, and since $V_l(-gQ) \subseteq H^0(\mathcal{O}_{E_l}(g-1)(-gQ)) = 0$, we conclude V_l has only simple ramification points on $E_l - \{A_l, B_l\}$.

On the other hand, as $h^0(\mathcal{L}_l\big|_C(-A)) = h^0(\mathcal{L}_l\big|_C(-B)) = 0$, it follows that $V_l(-A_l) = V_l(-B_l) = V_l(-A_l - B_l)$. Then

$$V_l(-(g-1)A_l) = V_l(-(g-1)A_l - B_l) \subseteq H^0(\mathcal{O}_{E_l}(g-1)(-(g-1)A_l - B_l)) = 0$$

and hence A_l(analogously, B_l) is not a ramification point of V_l.

Now, we are going to see what happens on C. It follows from equation 4.2.1 that $\mathcal{L}$ has focus on C and the limit linear system of $\mathcal{L}$ on C is $V_C = H^0(\omega_C)$. Since C is a general smooth curve, $V_C \subseteq H^0(\omega_C(A+B))$ has only simple ramification points on $C - \{A, B\}$. This proves the first statement of the proposition.

We will prove the last statement of the proposition. Assume $n = q(g-1)$, where $q > 0$ is an even integer, and $i = n/2$. Since the limit linear system of $\mathcal{L}$ on C is $V_C = H^0(\omega_C) \subseteq H^0(\omega_C(A+B))$, $wt_{V_C}(A) = wt_{V_C}(B) = g-1$. On the other hand, by Plücker formula, we get

$$\begin{aligned} \deg(W_0) &= (g-1)(g^2-g-1) \text{ and} \\ \deg(R_{V_C}) &= (g-1)(g^2-g-1) - (g-3)(g-1). \end{aligned}$$

Therefore, we have $(g-1)(g^2-g-1)-(g-1)^2$ limit ramification points on $C-\{A,B\}$. Since A_l and B_l are not ramification points of V_l and $\deg(R_{V_l}) = g-1$ for every l such that $i+l(g-2)$ is not a multiple of n and $n-d=l$, it is enough to show that there are exactly $g-1$ integers l satisfying the condition: $1 \le l \le n-1$, $i+l(g-2)$ is not a multiple of n and $n-d=l$. This condition is equivalent to: $1 \le l \le n-1$ and $i+l(g-1)$ is a multiple of n. We have $i+l(g-1)$ is a multiple of n if and only if $l+q/2$ is a multiple of q. Therefore, the condition $1 \le l \le n-1$ and $i+l(g-1)$ is a multiple of n is equivalent to $l = q-q/2, 2q-q/2, \ldots, (g-1)q-q/2$; thus there are exactly $g-1$ integers l satisfying that condition. □

Proposition 4.2.2. *Let X be a nodal curve which is the union of a smooth curve C of genus $g-1$ and a chain of rational smooth curves $E_1, \ldots, E_{n-1}$. Suppose C intersects only E_1 and E_{n-1}. Let $A \in C \cap E_1$ and $B \in C \cap E_{n-1}$ be the unique points of intersection. Assume that (C, A, B) is a two-pointed general smooth curve. Let $p : \mathcal{C} \to \Sigma$ be a smoothing of X. Fix a section Γ of p intersecting X at a point $P \in C$, where P is not a node of X and*

$$h^0(\omega_C(-aA-bB-P)) = 0,$$

for every nonnegative integers a and b with $a+b=g-2$. Let $\mathcal{L} := \omega_p(-\Gamma)$ and let W_0 be the limit ramification divisor of $\mathcal{L}$. Assume that $n = q(g-1)$, where $q > 0$. Then W_0 contains no node of X.

Proof. We have

$$\mathcal{L}\big|_C = \omega_C(A+B-P), \mathcal{L}\big|_{E_j} = \mathcal{O}_{E_j}. \tag{4.2.2}$$

For each $l = 1, \ldots, n-1$, set

$$\mathcal{L}_l := \mathcal{L}(-\sum_{r=1}^{n-1} a_{l,r} E_r),$$

where

$$a_{l,r} = (n-r)a_{l,n-1} - \max\{l-r, 0\}(g-1) - \max\{n-d-r, 0\},$$
$$l(g-1) = na_{l,n-1} - (n-d),\ 1 \le d \le n,\ a_{l,n-1} \ge 1.$$

If $l(g-1)$ is a multiple of n, then $a_{l,n-1} < g-1$(as $i < n$ and $l < n$) and $a_{l,1} + a_{l,n-1} = g-1$, and hence $a_{l,1} \ge 1$. Analogously, if $l(g-1)$ is not a multiple of n, then $a_{l,n-1} < g$ and $a_{l,1} + a_{l,n-1} = g$, and hence $a_{l,1} \ge 1$.

If $l(g-1)$ is a multiple of n, then we have

$$\mathcal{L}_l\big|_C = \omega_C(-(a_{l,1}-1)A-(a_{l,n-1}-1)B-P),$$
$$\mathcal{L}_l\big|_{E_l} = \mathcal{O}_{E_l}(g-1),$$
$$\mathcal{L}_l\big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j\neq l.$$

If $l(g-1)$ is not a multiple of n and $n-d\neq l$, then

$$\mathcal{L}_l\big|_C = \omega_C(-(a_{l,1}-1)A-(a_{l,n-1}-1)B-P),$$
$$\mathcal{L}_l\big|_{E_l} = \mathcal{O}_{E_l}(g-1),$$
$$\mathcal{L}_l\big|_{E_{n-d}} = \mathcal{O}_{E_{n-d}}(1),$$
$$\mathcal{L}_l\big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j\neq l, n-d.$$

If $l(g-1)$ is not a multiple of n and $n-d=l$, then

$$\mathcal{L}_l\big|_C = \omega_C(-(a_{l,1}-1)A-(a_{l,n-1}-1)B-P),$$
$$\mathcal{L}_l\big|_{E_l} = \mathcal{O}_{E_l}(g),$$
$$\mathcal{L}_l\big|_{E_j} = \mathcal{O}_{E_j}, \text{ if } j\neq l.$$

Thus, $\mathcal{L}_l$ has focus on E_l. Let V_l be the limit linear system of $\mathcal{L}$ on E_l. Let A_l and B_l be the nodes of X lying on E_l, with $B_l = A_{l+1}$ for every $l=1,\dots,n-2$. When $l(g-1)$ is a multiple of n, we have

$$h^0(\mathcal{L}_l\big|_C(-A)) = h^0(\mathcal{L}_l\big|_C(-B)) = 0,$$

by the hypothesis of the proposition, which implies

$$V_l(-A_l) = V_l(-B_l) = V_l(-A_l-B_l).$$

Then

$$V_l(-(g-1)A_l) = V_l(-(g-1)A_l-B_l) \subseteq H^0(\mathcal{O}_{E_l}(g-1)(-(g-1)A_l-B_l)) = 0,$$

and hence A_l(analogously, B_l) is not a ramification point of V_l. Now assume that $l(g-1)$ is not a multiple of n. Then $h^0(\mathcal{L}_l\big|_C)=0$, by the hypothesis of the proposition. Thus,

$$V_l = H^0(\mathcal{O}_{E_l}(g-1)(-B_l)), \text{ if } n-d<l,$$
$$V_l = H^0(\mathcal{O}_{E_l}(g-1)(-A_l)), \text{ if } n-d>l, \text{ and}$$
$$V_l = H^0(\mathcal{O}_{E_l}(g)(-A_l-B_l)), \text{ if } n-d=l.$$

Now, we are going to see what happens on C. It follows from equation 4.2.2 that $\mathcal{L}$ has focus on C and the limit linear system of $\mathcal{L}$ on C is $V_C = H^0(\omega_C(A+B-P))$. Notice that A and B are not ramification points of V_C, by the hypothesis of the proposition. On the other hand, by Plücker formula, we get

$$\deg(W_0) = (g-1)(g^2-g-1) \text{ and}$$
$$\deg(R_{V_C}) = (g-1)(g^2-g-1) - (g-2)(g-1).$$

Therefore, we have $(g-1)(g^2-g-1) - (g-2)(g-1)$ limit ramification points on $C - \{A, B\}$. Since A_l and B_l are not ramification points of V_l and $\deg(R_{V_l}) = g - 1$ for every l such that $l(g-1)$ is a multiple of n, it is enough to show that there are exactly $g-2$ integers l satisfying the condition: $1 \leq l \leq n-1$ and $l(g-1)$ is a multiple of n. We have $l(g-1)$ is a multiple of n if and only if l is a multiple of q. Therefore, the condition $1 \leq l \leq n-1$ and $l(g-1)$ is a multiple of n is equivalent to $l = q, 2q, \ldots, (g-2)q$; thus there are exactly $g-2$ integers l satisfying that condition. $\square$

Chapter 5

Flag curves

5.1 A result on flag curves

A *flag curve* is a nodal curve X satisfying the following properties:

(1) It is of *compact type*, i.e., the number of nodes of X is smaller (by one) than the number of components.

(2) Each component of X is either $\mathbb{P}^1$ or an elliptic curve.

(3) Each elliptic component of X contains exactly one node of X.

(4) Each $\mathbb{P}^1$ contains at least 2 nodes of X.

Proposition 5.1.1. *Let X be a flag curve of genus g. Assume g is an odd integer and let $p : \mathcal{C} \to \Sigma := Spec\,\mathbb{C}[[t]]$ be a smoothing of X. Let $\mathcal{C}_*$ be the generic fiber of p and $\overline{\mathcal{C}}_*$ the geometric generic fiber. Then $\overline{\mathcal{C}}_*$ satisfies the following condition:*

for each $P_* \in \overline{\mathcal{C}}_*$, the ramification points of the complete linear system $H^0(\omega_{\overline{\mathcal{C}}_*}(-P_*))$ have ramification weight at most 2.

Proof. Let $P_* \in \overline{\mathcal{C}}_*$. After base change, we may assume that P_* is a rational point of $\mathcal{C}_*$, and thus there is a section Γ of p intersecting $\mathcal{C}_*$ at P_*. After base changes and a sequence of blowups at the singular points of the special fiber $\mathcal{C}_0$, we may assume that $\mathcal{C}$ is regular and that Γ intersects the special fiber at a point P which is not a node of $\mathcal{C}_0$. After all the base changes and the sequence of blowups, each node is replaced by a chain of rational smooth curves and $\mathcal{C}_0$ is still a flag curve.

Let $\mathcal{L} := \omega_p(-\Gamma)$ and let W_0 be the limit ramification divisor of $\mathcal{L}$. To prove the statement of the proposition, it is enough to show that $\mathrm{mult}_Q(W_0) \leq 2$ for every $Q \in \mathcal{C}_0$. There are two cases to consider.

Case (1): P lies on a rational component Y of $\mathcal{C}_0$.

We will show that $\mathrm{mult}_Q(W_0) \leq 2$ for every $Q \in \mathcal{C}_0$. To prove this, we will show that the multiplicity of W_0 at each node of $\mathcal{C}_0$ is 0, and $\mathrm{mult}_Q(W_0) \leq 2$ if $Q \in \mathcal{C}_0$ is not a node.

The limit linear system of ω_p on Y is of the form (see [$EH2$])

$$\begin{aligned} V := H^0(\omega_Y((a_1+1)R_1)) \oplus \ldots \oplus H^0(\omega_Y((a_n+1)R_n)) \\ \subseteq H^0(\omega_Y(2a_1R_1 + \ldots + 2a_nR_n)), \end{aligned}$$

where n is the number of connected components of $\mathcal{C}_0 - Y$, the integers a_j are the genera of the closures of the connected components of $\mathcal{C}_0 - Y$, and each R_j is the point of intersection of Y and the connected component of $\mathcal{C}_0 - Y$ of genus a_j. Notice that if $\omega_p(D_Z)$ has degree $2g-2$ on a component Z of $\mathcal{C}_0$ and degree 0 on the other components of $\mathcal{C}_0$, where $D_Z \subseteq \mathcal{C}_0$ is a divisor, then $\mathcal{L}(D_Z)$ has focus on Z. In this way, we can get a limit linear system V_Z of $\mathcal{L}$ on each component Z of $\mathcal{C}_0$, and the connecting number between $\mathcal{L}(D_{Z_1})$ and $\mathcal{L}(D_{Z_2})$ corresponding to components $Z_1 \neq Z_2$ of $\mathcal{C}_0$ is equal to the connecting number between $\omega_p(D_{Z_1})$ and $\omega_p(D_{Z_2})$ corresponding to Z_1 and Z_2. The limit linear system of $\mathcal{L}$ on Y is

$$V_Y = V(-P) \subseteq H^0(\omega_Y(-P + 2a_1R_1 + \ldots + 2a_nR_n)).$$

It follows from Proposition 3.1.3 that $wt_{V_Y}(Q) \leq 1$ if $Q \in Y - \{R_1, \ldots, R_n\}$, whence $\mathrm{mult}_Q(W_0) \leq 2$ if $Q \in Y$ is not a node of $\mathcal{C}_0$. (In fact, if $Q \in Y$ is not a node of $\mathcal{C}_0$, then $\mathrm{mult}_Q(W_0) \leq 1$.) Now, we have to prove that the multiplicity of W_0 at each point R_j is 0. Assume that $j = 1$ and R_1 is the point of intersection of Y and a rational component Y_1 of $\mathcal{C}_0$. By Propositions 3.1.3 and 3.1.4, we have

$$\begin{aligned} wt_{V_Y}(R_1) &= (\sum_{j\neq 1} a_j) - 1 + \epsilon_1 + (a_1 - 1)(g-1) \\ &= g - a_1 - 1 + \epsilon_1 + (a_1-1)(g-1) \end{aligned}$$

where $\epsilon_1 \in \{0, 1\}$. Let $R'_1, \ldots, R'_m$ be the nodes of $\mathcal{C}_0$ lying on Y_1. Assume $R'_m = R_1$. Let V_{Y_1} be the limit linear system of $\mathcal{L}$ on Y_1. We have

$$\begin{aligned} mult_{R_1}(W_0) &= wt_{V_Y}(R_1) + wt_{V_{Y_1}}(R'_m) + (g-1)(g-2-(2g-2)) \\ &= wt_{V_{Y_1}}(R'_m) + g - a_1 - 1 + \epsilon_1 + (a_1-1)(g-1) - g(g-1) \\ &= wt_{V_{Y_1}}(R'_m) + (g-1)(a_1 - g) - a_1 + \epsilon_1. \end{aligned}$$

On the other hand, consider the limit linear system of ω_p on Y_1

$$V' := H^0(\omega_{Y_1}((a'_1+1)R'_1)) \oplus \ldots \oplus H^0(\omega_{Y_1}((a'_m+1)R'_m)) \\ \subseteq H^0(\omega_{Y_1}(2a'_1R'_1+\ldots+2a'_mR'_m)).$$

We have that $V_{Y_1} \subseteq V'$. By Propositions 3.1.1 and 3.1.4 we have that the orders of vanishing at R'_m of the sections in V' are

$$0+(a'_m-1),\ldots,a'_m-1+(a'_m-1),a'_m+1+(a'_m-1),\ldots,a'_1+\ldots+a'_m+(a'_m-1)$$

and $wt_{V'}(R'_m) = (\sum_{j\neq m} a'_j) + (a'_m-1)g$. Thus, the orders of vanishing at R'_m of the sections in V_{Y_1} are of the form

$$\{a'_m-1,\ldots,2(a'_m-1),a'_m+1+(a'_m-1),\ldots,a'_1+\ldots+a'_m+(a'_m-1)\}-\{l\},$$

for some l. Thus

$$\begin{aligned} wt_{V_{Y_1}}(R'_m) &= wt_{V'}(R'_m)+g-1-l \\ &= (\sum_{j\neq m} a'_j)+(a'_m-1)g+g-1-l \\ &= g-a'_m+(a'_m-1)g+g-1-l. \end{aligned}$$

Therefore, since $a_1+a'_m=g$, we have

$$\begin{aligned} mult_{R_1}(W_0) &= g-a'_m+(a'_m-1)g+g-1-l+(g-1)(a_1-g)-a_1+\epsilon_1 \\ &= (g-1)(a'_m+a_1-g)+(g-a_1-a'_m)+(a'_m-1-l)+\epsilon_1 \\ &= (a'_m-1-l)+\epsilon_1 \leq \epsilon_1 \leq 1. \end{aligned}$$

Since the intersection multiplicity of the ramification divisor of $\mathcal{L}$ and the special fiber at the node R_1 cannot be 1, we have $\mathrm{mult}_{R_1}(W_0)=0$. (Notice that, the only important information about V_Y we have used in the reasoning above is the ramification weight of V_Y at the point R_1.)

Now, we are going to see what happens on Y_1. We have to prove that the multiplicity of W_0 at each point R'_j is 0, and $\mathrm{mult}_Q(W_0)\leq 2$ if $Q\in Y_1$ is not a node of $\mathcal{C}_0$. Since $\mathrm{mult}_{R_1}(W_0)=0$, $a'_m-1-l=-\epsilon_1$. This implies that $l=a'_m-1$ or $l=a'_m$. Then $\dim_{\mathbb{C}} V_{Y_1}(-(a'_m+1)R'_m)=g-2$ and hence $V_{Y_1}\supseteq V'(-(a'_m+1)R'_m)=H^0(\omega_{Y_1}((a'_1+1)R'_1))\oplus\ldots\oplus H^0(\omega_{Y_1}((a'_m-1)R'_m))$.

Now, using the last Formula in Proposition 3.1.2 for $V_1=V_{Y_1}$, and using Proposition 3.1.4, we are able to use the same reasoning above to conclude that $\mathrm{mult}_{R'_j}(W_0)=0$ if R'_j is the point of intersection of Y_1 and a rational component of $\mathcal{C}_0$. Also, using Proposition 3.1.2, we get that $\mathrm{mult}_Q(W_0)\leq 2$ if $Q\in Y_1$ is not a node of $\mathcal{C}_0$. Notice that, we can use the same reasoning above, repeatedly, for each rational component in $\mathcal{C}_0$.

It remains to prove that, if $\bar{Y}$ is a rational component of $\mathcal{C}_0$ intersecting an elliptic component E of $\mathcal{C}_0$, then the point of intersection of $\bar{Y}$ and E does not appear in W_0 and $\text{mult}_Q(W_0) \leq 2$ if $Q \in E$ is not a node of $\mathcal{C}_0$. Let $\bar{R}_1, \dots, \bar{R}_k$ be the nodes of $\mathcal{C}_0$ lying on $\bar{Y}$ and C the node of $\mathcal{C}_0$ lying on E. Assume $\bar{R}_k = C$ and let $V_{\bar{Y}}, V_E$ be the limit linear systems of $\mathcal{L}$ on $\bar{Y}$ and E respectively. We have an equality of the form

$$wt_{V_{\bar{Y}}}(\bar{R}_k) = (\textstyle\sum_{j \neq k} \bar{a}_j) - 1 + \epsilon_k + (\bar{a}_k - 1)(g-1),$$

where $\epsilon_k \in \{0, 1\}$, the integers $\bar{a}_j$ are the genera of the closures of the connected components of $\mathcal{C}_0 - \bar{Y}$, and each $\bar{R}_j$ is the point of intersection of $\bar{Y}$ and the connected component of $\mathcal{C}_0 - \bar{Y}$ of genus $\bar{a}_j$. Since $\bar{a}_k = 1$, $wt_{V_{\bar{Y}}}(\bar{R}_k) = g - 2 + \epsilon_k$. Then

$$\begin{aligned} mult_{\bar{R}_k}(W_0) &= wt_{V_{\bar{Y}}}(\bar{R}_k) + wt_{V_E}(C) + (g-1)(g-2-(2g-2)) \\ &= g - 2 + \epsilon_k + wt_{V_E}(C) - g(g-1) \\ &= wt_{V_E}(C) + g - 2 - g(g-1) + \epsilon_k. \end{aligned}$$

On the other hand, as the limit linear system of ω_p on E is

$$V' := H^0(\omega_E(gC)) \subseteq H^0(\omega_E(2(g-1)C)),$$

it follows that $V_E \subseteq V' = H^0(\omega_E(gC))$. The orders of vanishing at C of the sections in V' are

$$g-2, \dots, 2g-4, 2g-2$$

and $wt_{V'}(C) = g^2 - 2g + 1$. Thus, the orders of vanishing at C of the sections in V_E are of the form

$$\{g-2, \dots, 2g-4, 2g-2\} - \{l\},$$

where $l \in \{g-2, \dots, 2g-4, 2g-2\}$. Then

$$\begin{aligned} wt_{V_E}(C) &= wt_{V'}(C) + g - 1 - l \\ &= g^2 - 2g + 1 + g - 1 - l \\ &= g^2 - g - l. \end{aligned}$$

Therefore

$$\begin{aligned} mult_{\bar{R}_k}(W_0) &= g^2 - g - l + g - 2 - g(g-1) + \epsilon_k \\ &= (g - 2 - l) + \epsilon_k \leq \epsilon_k \leq 1. \end{aligned}$$

It follows that $\mathrm{mult}_{\bar{R}_k}(W_0) = 0$ and hence $\bar{R}_k$ is not a limit ramification point, and $l = g-2$ or $l = g-1$, which implies that $\dim_{\mathbb{C}} V_E(-gC) = g-2$ and hence $V_E \supseteq V'(-gC) = H^0(\omega_E((g-2)C))$.

By using Proposition 3.2.1, we get $\mathrm{mult}_Q(W_0) \leq 2$ if $Q \in E$ is not a node of $\mathcal{C}_0$. This proves the case (1).

Case (2): P lies on an elliptic component E of $\mathcal{C}_0$.

We will show that $\mathrm{mult}_Q(W_0) \leq 2$ for every $Q \in \mathcal{C}_0$. To prove this, we will show that the multiplicity of W_0 at each node of $\mathcal{C}_0$ is 0, and $\mathrm{mult}_Q(W_0) \leq 2$ if $Q \in \mathcal{C}_0$ is not a node.

Let C be the node of $\mathcal{C}_0$ lying on E. Since the limit linear system of ω_p on E is

$$V = H^0(\omega_E(gC)) \subseteq H^0(\omega_E(2(g-1)C)),$$

the limit linear system of $\mathcal{L}$ on E is

$$V_E = V(-P) = H^0(\omega_E(gC - P)) \subseteq H^0(\omega_E(2(g-1)C - P)).$$

Notice that V_E has at most simple ramification points on $E - \{C\}$, whence $\mathrm{mult}_Q(W_0) \leq 2$ if $Q \in E$ is not a node of $\mathcal{C}_0$. Now, we have to prove that the multiplicity of W_0 at the point C is 0. We have that $V_E(-nC) = V_E$ for every $0 \leq n \leq g-2$. For every $n \geq g-2$

$$\begin{aligned} V_E(-nC) &= H^0(\omega_E(gC - P)) \cap H^0(\omega_E(2(g-1)C - P - nC)) \\ &= H^0(\omega_E(2(g-1)C - P - nC)). \end{aligned}$$

Then $\dim_{\mathbb{C}} V_E(-nC) = 2(g-1) - n - 1$ for every $g-2 \leq n \leq 2g-4$. Thus, since $V_E(-(2g-3)C) = H^0(\omega_E(C-P)) = 0$, the orders of vanishing at C of the sections in V_E are $g-2, \ldots, 2g-4$. Hence $wt_{V_E}(C) = (g-1)(g-2)$.

Assume that C is the point of intersection of E and a rational component Y of $\mathcal{C}_0$. Let $R_1, \ldots, R_n$ be the nodes of $\mathcal{C}_0$ lying on Y. Assume $R_n = C$ and let V_Y be the limit linear system of $\mathcal{L}$ on Y. As the limit linear system of ω_p on Y is of the form

$$\begin{aligned} V' := H^0(\omega_Y((a_1+1)R_1)) \oplus \ldots \oplus H^0(\omega_Y((a_n+1)R_n)) \\ \subseteq H^0(\omega_Y(2a_1R_1 + \ldots + 2a_nR_n)), \end{aligned}$$

it follows that $V_Y \subseteq V'$. By Propositions 3.1.1 and 3.1.4 we have that the orders of vanishing at R_n of the sections in V' are

$$0+(a_n-1), \ldots, a_n-1+(a_n-1), a_n+1+(a_n-1), \ldots, a_1+\ldots+a_n+(a_n-1)$$

and $wt_{V'}(R_n) = (\sum_{j\neq n} a_j) + (a_n - 1)g$. Thus, the orders of vanishing at R_n of the sections in V_Y are of the form

$$\{a_n - 1, \ldots, 2(a_n - 1), a_n + 1 + (a_n - 1), \ldots, a_1 + \ldots + a_n + (a_n - 1)\} - \{l\},$$

for some l. Thus

$$\begin{aligned} wt_{V_Y}(R_n) &= wt_{V'}(R_n) + g - 1 - l \\ &= (\sum_{j\neq n} a_j) + (a_n - 1)g + g - 1 - l \\ &= g - a_n + (a_n - 1)g + g - 1 - l. \end{aligned}$$

Since $a_n = 1$, $wt_{V_Y}(R_n) = 2(g-1) - l$. Therefore

$$\begin{aligned} mult_C(W_0) &= wt_{V_Y}(R_n) + wt_{V_E}(C) - g(g-1) \\ &= 2(g-1) - l + (g-1)(g-2) - g(g-1) \\ &= -l. \end{aligned}$$

It follows that $\mathrm{mult}_C(W_0) = 0$ and hence C is not a limit ramification point, and $l = 0$, which implies that

$$V_Y = H^0(\omega_Y((a_1+1)R_1)) \oplus \ldots \oplus H^0(\omega_Y((a_{n-1}+1)R_{n-1})).$$

By Propositions 3.1.1 and 3.1.4, we have for every $k \neq n$

$$\begin{aligned} wt_{V_Y}(R_k) &= (\sum_{j\neq k,n} a_j) + (a_k - 1)(g-1) \\ &= (\sum_{j\neq k} a_j) - 1 + (a_k - 1)(g-1). \end{aligned}$$

Thus, the proof of this case follows as in the case (1). □

5.2 Effective divisors in $\overline{M_g}$

Let g and i be positive integers such that $g \geq 5$ and $2 \leq i \leq [g/2]$. We will define a family of curves over $\mathbb{P}^1$ in the following steps:

Step 1: Fix 3 distinct points R, S and T on $\mathbb{P}^1$, and let $\mathbb{P}^1_{i-1} := \mathbb{P}^1$, $R_{i-1} := R$, $S_{i-1} := S$ and $T_{i-1} := T$. (We use this notation to extend a notation we will see later.) Begin with the fibered product $\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1}$, and then blow up the points $(R_{i-1}, R_{i-1}), (S_{i-1}, S_{i-1})$ and (T_{i-1}, T_{i-1}). Let

$\mathbb{P}^1_{(R_{i-1},R_{i-1})}, \mathbb{P}^1_{(S_{i-1},S_{i-1})}$ and $\mathbb{P}^1_{(T_{i-1},T_{i-1})}$ be the rational curves on the blowup $(\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$ over the points $(R_{i-1}, R_{i-1}), (S_{i-1}, S_{i-1})$ and (T_{i-1}, T_{i-1}) of $\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1}$. The points in the intersections $(\{R_{i-1}\} \times \mathbb{P}^1_{i-1})\tilde{} \cap \mathbb{P}^1_{(R_{i-1},R_{i-1})}$, $(\{S_{i-1}\} \times \mathbb{P}^1_{i-1})\tilde{} \cap \mathbb{P}^1_{(S_{i-1},S_{i-1})}$ and $(\{T_{i-1}\} \times \mathbb{P}^1_{i-1})\tilde{} \cap \mathbb{P}^1_{(T_{i-1},T_{i-1})}$ will be denoted R', S' and T' respectively. Also, abusing notation, we denote the strict transform of each fiber $\{Q\} \times \mathbb{P}^1_{i-1} \subseteq \mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1}$ by $\mathbb{P}^1_{i-1}$.

Step 2: Now, fix g smooth pointed elliptic curves $(E_1, C_1), \ldots, (E_g, C_g)$. Let $\mathcal{Y}$ be the disjoint union of $(\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$, $\mathbb{P}^1 \times E_i$ and $\mathbb{P}^1 \times E_{i+1}$ modulo the identification of the strict transform of the diagonal $\widetilde{\Delta} \subseteq (\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$ with $\mathbb{P}^1 \times \{C_i\} \subseteq \mathbb{P}^1 \times E_i$, and the identification of the strict transform $(\mathbb{P}^1_{i-1} \times \{S_{i-1}\})\tilde{} \subseteq (\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$ with $\mathbb{P}^1 \times \{C_{i+1}\} \subseteq \mathbb{P}^1 \times E_{i+1}$.

Step 3: Assume $i \geq 3$ and consider a chain of $i-2$ three pointed rational curves $(\mathbb{P}^1_1, R_1, S_1, T_1), \ldots, (\mathbb{P}^1_{i-2}, R_{i-2}, S_{i-2}, T_{i-2})$ with $T_j = R_{j+1}$ for every $1 \leq j \leq i-3$. Now, attach the elliptic curves $E_1, \ldots, E_{i-1}$ at the points $R_1, S_1, S_2, \ldots, S_{i-2}$ respectively, identifying the points $C_1, \ldots, C_{i-1}$ with the points $R_1, S_1, S_2, \ldots, S_{i-2}$ respectively, obtaining a nodal curve which we will call X_i. If $i = 2$, we set $X_i := E_1$ and $T_{i-2} := C_1$. Analogously, consider a chain of $g-i-2$ three pointed rational curves $(\mathbb{P}^1_i, R_i, S_i, T_i), \ldots, (\mathbb{P}^1_{g-3}, R_{g-3}, S_{g-3}, T_{g-3})$ such that $T_j = R_{j+1}$ for every $i \leq j \leq g-4$. Now, attach the elliptic curves $E_{i+2}, \ldots, E_g$ at the points $S_i, \ldots, S_{g-3}, T_{g-3}$ respectively, identifying the points $C_{i+2}, \ldots, C_g$ with the points $S_i, \ldots, S_{g-3}, T_{g-3}$ respectively, obtaining a nodal curve which we will call Y_i.

Step 4: Let $\mathcal{X}$ be the disjoint union of $\mathcal{Y}$, $\mathbb{P}^1 \times X_i$ and $\mathbb{P}^1 \times Y_i$ modulo the identification of $(\mathbb{P}^1_{i-1} \times \{R_{i-1}\})\tilde{} \subseteq \mathcal{Y}$ with $\mathbb{P}^1 \times \{T_{i-2}\} \subseteq \mathbb{P}^1 \times X_i$, and the identification of $(\mathbb{P}^1_{i-1} \times \{T_{i-1}\})\tilde{} \subseteq \mathcal{Y}$ with $\mathbb{P}^1 \times \{R_i\} \subseteq \mathbb{P}^1 \times Y_i$. This gives a family $\pi_i : \mathcal{X} \to \mathbb{P}^1$ of stable curves of genus g.

Abusing notation, for each fiber F of π_i, we denote by R_{i-1}, S_{i-1} and T_{i-1} the points in the intersections $F \cap (\mathbb{P}^1_{i-1} \times \{R_{i-1}\})\tilde{}$, $F \cap (\mathbb{P}^1_{i-1} \times \{S_{i-1}\})\tilde{}$ and $F \cap (\mathbb{P}^1_{i-1} \times \{T_{i-1}\})\tilde{}$. Figure 3.1 describes the family given by π_i.

Proposition 5.2.1. *Let $D \subseteq \overline{M_g}$ be an effective divisor, with class*

$$D = a\lambda - a_0\delta_0 - a_1\delta_1 - \ldots - a_{[g/2]}\delta_{[g/2]}$$

If $\pi_i^* D = 0$, for every $2 \leq i \leq [g/2]$, then

$$a_l = (l(g-l)/(g-1))a_1, \textit{ for every } 2 \leq l \leq [g/2].$$

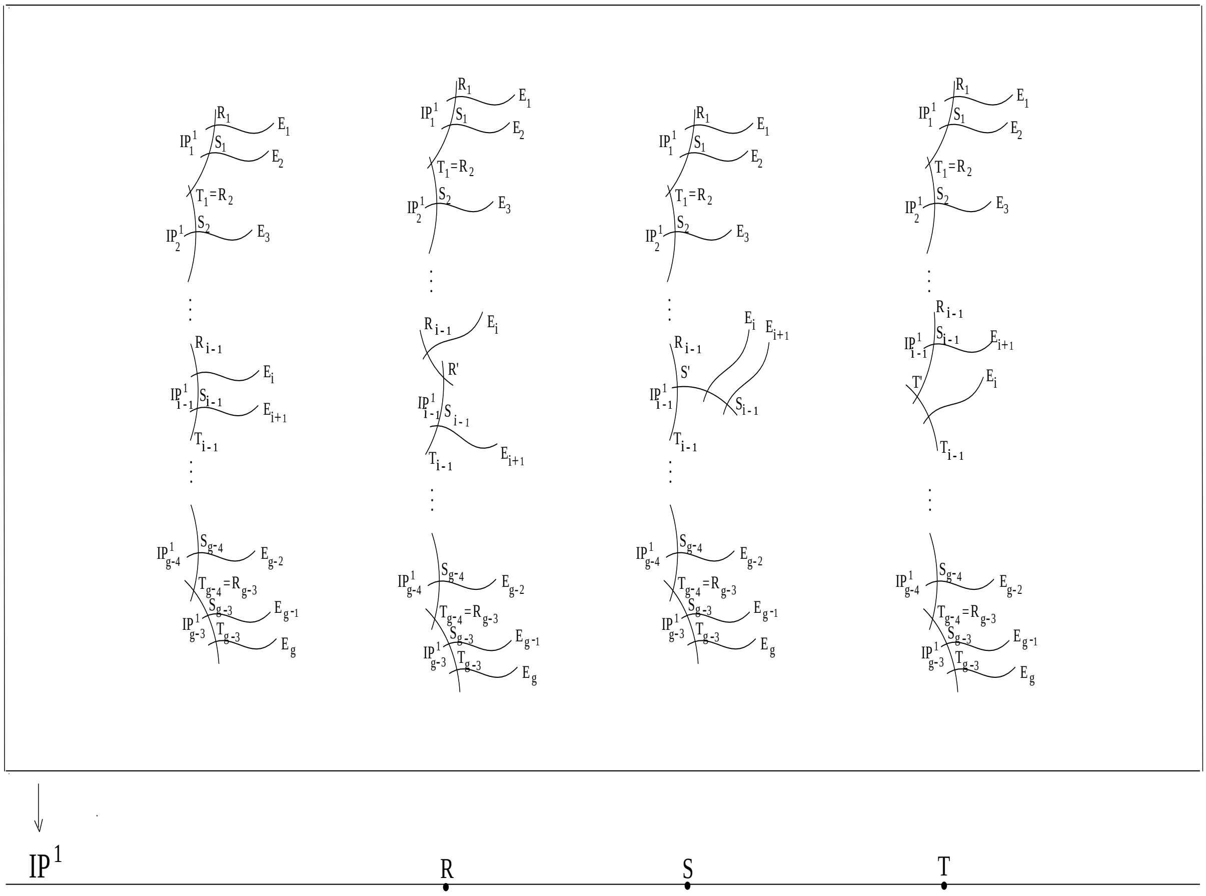

Figure 5.1: The family.

Proof. For every i, the degree of $(\delta_0)_{\pi_i}$ is 0, because each fiber of π_i contains only disconnecting nodes. On the other hand, for every fiber F of π_i, each section of $H^0(\omega_{\pi_i}\big|_F)$ vanishes at each $\mathbb{P}^1$.

Furthermore, we have that $H^0(\omega_{\pi_i}\big|_E) = H^0(\omega_E(C))$ for every elliptic component E of F, where C is the node of F lying on E. The upshot is that

$$H^0(\omega_{\pi_i}\big|_F) = \bigoplus_E H^0(\omega_E),$$

for every fiber F of π_i. Thus, $\pi_{i*}(\omega_{\pi_i})$ is trivial and hence $\deg((\lambda)_{\pi_i}) = 0$.

Assume $i \geq 4$. By the construction of π_i, we have that

$$\begin{aligned} deg((\delta_1)_{\pi_i}) &= \widetilde{\Delta}^2 + ((\mathbb{P}^1_{i-1} \times \{S_{i-1}\})\widetilde{)}^2 \\ &= \Delta^2 - 3 + ((\mathbb{P}^1_{i-1} \times \{S_{i-1}\})^2 - 1) \\ &= 2 - 2(0) - 3 + (0 - 1) = -2, \end{aligned}$$

where $\widetilde{\Delta}$ and $(\mathbb{P}^1_{i-1} \times \{S_{i-1}\})\tilde{}$ are the strict transforms of the diagonal Δ and $\mathbb{P}^1_{i-1} \times \{S_{i-1}\}$ in the blow up $(\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$ of $\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1}$ at the points $(R_{i-1}, R_{i-1}), (S_{i-1}, S_{i-1})$ and (T_{i-1}, T_{i-1}).

On the other hand, we have $\deg((\delta_2)_{\pi_i}) = 1$, as the fiber of π_i over S has a disconnecting node S' such that the closure of one of the connected components of $\pi_i^{-1}(S) - \{S'\}$ has genus 2 and the total space of π_i is smooth at S'. (The total space of π_i is smooth at the point S', as this point can be seen as a point of $(\mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i-1})\tilde{}$, which is a smooth surface.)

For $3 \leq l \leq i-2$, we have $\deg((\delta_l)_{\pi_i}) = 0$, as the family is locally trivial around $\mathbb{P}^1 \times \{Q\} \subseteq \mathcal{X}$ for every Q which is a node of X_i or Y_i.

Now, we will compute $\deg((\delta_{i-1})_{\pi_i})$. Notice that for every fiber F of π_i, the closure of one of the connected components of $F - \{R_{i-1}\}$ has genus $i-1$. If g is even and $i = g/2$, then for each fiber F, the closure of one of the connected components of $F - \{T_{i-1}\}$ has genus $i-1$ and hence

$$\begin{aligned} deg((\delta_{i-1})_{\pi_i}) &= ((\mathbb{P}^1_{i-1} \times \{R_{i-1}\})\tilde{})^2 + ((\mathbb{P}^1_{i-1} \times \{T_{i-1}\})\tilde{})^2 \\ &= ((\mathbb{P}^1_{i-1} \times \{R_{i-1}\})^2 - 1) + ((\mathbb{P}^1_{i-1} \times \{T_{i-1}\})^2 - 1) \\ &= (0-1) + (0-1) = -2 \end{aligned}$$

Otherwise,

$$\begin{aligned} deg((\delta_{i-1})_{\pi_i}) &= ((\mathbb{P}^1_{i-1} \times \{R_{i-1}\})\tilde{})^2 \\ &= (\mathbb{P}^1_{i-1} \times \{R_{i-1}\})^2 - 1 \\ &= 0 - 1 = -1 \end{aligned}$$

To compute $\deg((\delta_i)_{\pi_i})$, first notice that the fiber of π_i over R has a disco– nnecting node R' such that the closure of one of the connected components of $\pi_i^{-1}(R) - \{R'\}$ has genus i and the total space of π_i is smooth at R', and the same holds for the fiber of π_i at T. Now, if g is odd and $i = (g-1)/2$, then for each fiber F, the closure of one of the connected components of $F - \{T_{i-1}\}$ has genus i and hence

$$\begin{aligned} deg((\delta_i)_{\pi_i}) &= 2 + ((\mathbb{P}^1_{i-1} \times \{T_{i-1}\})\tilde{})^2 \\ &= 2 + ((\mathbb{P}^1_{i-1} \times \{T_{i-1}\})^2 - 1) \\ &= 2 + (0-1) = 1 \end{aligned}$$

Otherwise, $\deg((\delta_i)_{\pi_i}) = 2$. Finally, if $i \leq [g/2] - 1$, then

$$\begin{aligned} deg((\delta_{i+1})_{\pi_i}) &= ((\mathbb{P}^1_{i-1} \times \widetilde{\{T_{i-1}\}}))^2 \\ &= ((\mathbb{P}^1_{i-1} \times \{T_{i-1}\})^2 - 1) \\ &= 0 - 1 = -1 \end{aligned}$$

and $\deg((\delta_l)_{\pi_i}) = 0$, if $i + 2 \leq l \leq [g/2]$. Now, as $\pi_i^* D = 0$, for every $2 \leq i \leq [g/2]$, it follows that

$$2a_1 - a_2 + a_{i-1} - 2a_i + a_{i+1} = 0 \text{ for every } 2 \leq i < [g/2],$$

$$2a_1 - a_2 + 2a_{i-1} - 2a_i = 0, \text{ if } g \text{ is even and } i = g/2, \text{ and}$$

$$2a_1 - a_2 + a_{i-1} - a_i = 0, \text{ if } g \text{ is odd and } i = (g-1)/2.$$

For $i = 2, 3$, analogously, we get the same equations. Now, solving the system of $[g/2] - 1$ equations, we get that

$$a_l = (l(g-l)/(g-1))a_1, \text{ for every } 2 \leq l \leq [g/2].$$

□

Corollary 5.2.2. *Let g be an odd positive integer such that $g \geq 5$. Let $\overline{S^2W} \subseteq \overline{M_g}$ be the effective divisor which is defined as the closure of the locus of smooth curves C with a pair of points (P, Q) satisfying that Q is a ramification point of the linear system $H^0(\omega_C(-P))$ with ramification weight at least 3.*

Write the class of $\overline{S^2W}$ in $Pic_{fun}(\overline{M_g}) \otimes \mathbb{Q}$ in the form

$$\overline{S^2W} = a\lambda - a_0\delta_0 - a_1\delta_1 - \ldots - a_{[g/2]}\delta_{[g/2]}$$

Then

$$a_l = (l(g-l)/(g-1))a_1, \text{ for every } 2 \leq l \leq [g/2].$$

Proof. Just combine Propositions 5.1.1 and 5.2.1. □

Chapter 6

The reducible case

6.1 The family

Consider a general family $\pi : \mathcal{X} \to T$ of stable curves over a smooth projective curve T. As the family is general, the singular curves we have in our family have only one node and these curves are not in the divisor $\overline{S^2W}$ we want to compute. We will restrict ourselves to a neighborhood in T of some point t_0, such that $\mathcal{X}_{t_0}$ is a singular fiber and the other fibers are nonsingular. Assume that the singular fiber is reducible.

The special fiber is a nodal union of two general smooth pointed curves (X, A) and (Y, B), identifying A with B. Suppose $g_Y \leq g_X$.

Let $\widetilde{\mathcal{X}}$ be the blowup of $\mathcal{X}$ at the ramification points of the complete linear systems $H^0(\omega_X(-(g_X - 1)A))$ and $H^0(\omega_Y(-(g_Y - 1)B))$. Notice that $\widetilde{\mathcal{X}}$ is the blowup of $\mathcal{X}$ at the supports of the unique effective divisors of X and Y which are linearly equivalent to $K_X - (g_X - 1)A$ and $K_Y - (g_Y - 1)B$, respectively. Notice that the points A and B are not ramification points of the linear systems $H^0(\omega_X(-(g_X - 1)A))$ and $H^0(\omega_Y(-(g_Y - 1)B))$, respectively, as A and B are general points of X and Y respectively. Abusing notation, we denote by X and Y the strict transform of X and Y in $\widetilde{\mathcal{X}}$. Also if $P \in \mathcal{X}$ is one of the blown up points, we denote by $\mathbb{P}^1_P$ the component of the exceptional divisor on $\widetilde{\mathcal{X}}$ corresponding to P. Abusing notation, we denote by $P \in \widetilde{\mathcal{X}}$ the point of intersection of $X \cup Y$ and $\mathbb{P}^1_P$. These points are nodes of the singular fiber of the family $\widetilde{\pi} : \widetilde{\mathcal{X}} \to T$; also this singular fiber has the point $A \in \widetilde{\mathcal{X}}$ as a node.

Let $\mathcal{Y} = \widetilde{\mathcal{X}} \times_T \mathcal{X}$. The singularities of $\mathcal{Y}$ are the points (A, A) and (P, A), where the points $P \in \widetilde{\mathcal{X}} - \{A\}$ are nodes of the singular fiber of $\widetilde{\pi}$.

To solve the singularities of $\mathcal{Y}$, we blow up $X \times X$ and $Y \times Y$; let $\mathcal{B}$ be this blowup. We obtain a $\mathbb{P}^1$ over each singularity of $\mathcal{Y}$. We denote by $\mathbb{P}^1_{(P,A)}$ the rational curve on $\mathcal{B}$ over the point $(P, A) \in \mathcal{Y}$ and $\mathbb{P}^1_{(A,A)}$ the rational curve on $\mathcal{B}$ over the point $(A, A) \in \mathcal{Y}$. Let $\widetilde{\Delta}$ be the strict transform in $\mathcal{B}$ of the inverse image of Δ via the natural morphism $\mathcal{Y} \to \mathcal{X} \times_T \mathcal{X}$. Let Z_{11}, Z_{12}, Z_{21} and Z_{22} be the strict transforms of $X \times X, X \times Y, Y \times X$ and $Y \times Y$ respectively. Let $(\mathbb{P}^1_P \times X)\tilde{}$ and $(\mathbb{P}^1_P \times Y)\tilde{}$ be the strict transforms of $\mathbb{P}^1_P \times X$ and $\mathbb{P}^1_P \times Y$ respectively. A local analysis shows that $\widetilde{\Delta}$ intersects $\mathbb{P}^1_{(A,A)}$ transversally. Also, $\mathbb{P}^1_{(A,A)} = Z_{11} \cap Z_{22}$ and Z_{12}, Z_{21} do not contain $\mathbb{P}^1_{(A,A)}$. If $P \in X - \{A\}$ is a node of the singular fiber of $\widetilde{\pi}$, then $\mathbb{P}^1_{(P,A)} = Z_{11} \cap (\mathbb{P}^1_P \times Y)\tilde{}$ and $Z_{12}, (\mathbb{P}^1_P \times X)\tilde{}$ do not contain $\mathbb{P}^1_{(P,A)}$. Also, if $P \in Y - \{B\}$ is a node of the singular fiber of $\widetilde{\pi}$, then $\mathbb{P}^1_{(P,A)} = Z_{22} \cap (\mathbb{P}^1_P \times X)\tilde{}$ and $Z_{21}, (\mathbb{P}^1_P \times Y)\tilde{}$ do not contain $\mathbb{P}^1_{(P,A)}$. See figure 6.1.

Let $\bar{p}_1, \bar{p}_2$ be the projection maps of $\mathcal{Y}$ and $b : \mathcal{B} \to \mathcal{Y}$ the blowup. Set $\rho_i = \bar{p}_i \circ b$ for $i = 1, 2$, and $\mathcal{L} = \omega_{\rho_1}(-\widetilde{\Delta} - Z_{11})$.

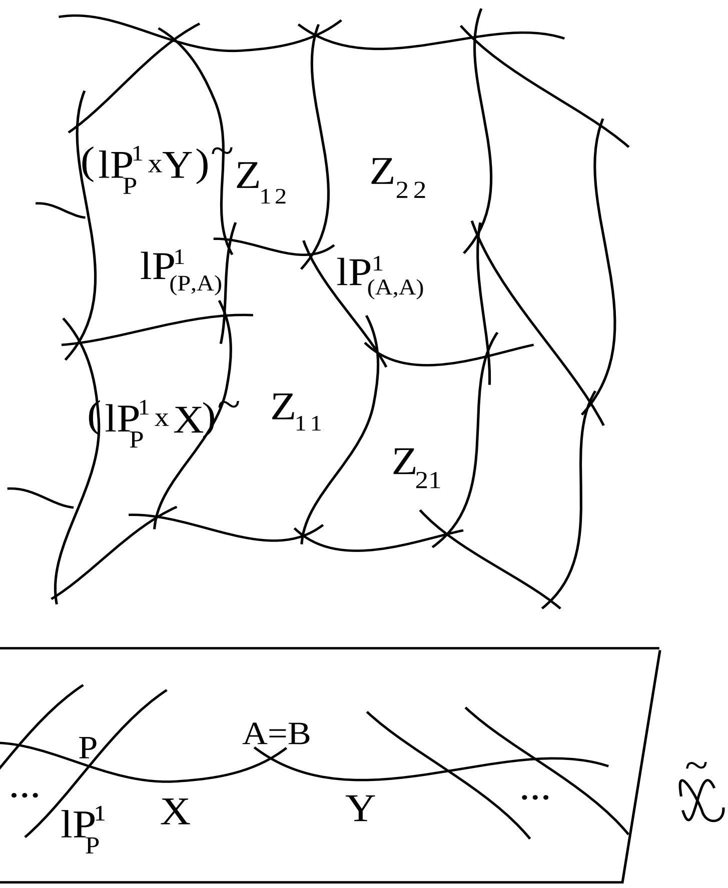

Figure 6.1: The family over $\widetilde{\mathcal{X}}$.

6.2 Direct images

Proposition 6.2.1. *$\rho_{1*}(\mathcal{L})$ is locally free of rank $g-1$.*

Proof. It is enough to show that $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$ for every $P \in \widetilde{\mathcal{X}}$. There are 8 cases to consider.

Case (1): $\mathcal{B}_P$ is a smooth curve. Then

$$H^0(\mathcal{L}\big|_{\mathcal{B}_P}) = H^0(\omega_{\mathcal{X}_{\pi(P)}}(-P))$$

it follows that $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (2): $P \in X - \{A\}$ and P is not a node of the singular fiber of $\widetilde{\pi}$. Then

$$\mathcal{L}\big|_X = \omega_X(2A-P),\ \mathcal{L}\big|_Y = \omega_Y$$

and thus $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (3): $P \in X - \{A\}$ is a node of the singular fiber of $\widetilde{\pi}$. Then

$$\mathcal{L}\big|_X = \omega_X(A-P),\ \mathcal{L}\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1),\ \mathcal{L}\big|_Y = \omega_Y$$

therefore $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (4): $P = A$. Then

$$\mathcal{L}\big|_X = \omega_X(A),\ \mathcal{L}\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},\ \mathcal{L}\big|_Y = \omega_Y$$

it follows that $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (5): $P \in Y - \{B\}$ and P is not a node of the singular fiber of $\widetilde{\pi}$. Then

$$\mathcal{L}\big|_X = \omega_X(A),\ \mathcal{L}\big|_Y = \omega_Y(B-P)$$

Then we have that $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (6): $P \in Y - \{B\}$ is a node of the singular fiber of $\widetilde{\pi}$. Then

$$\mathcal{L}\big|_X = \omega_X(A),\ \mathcal{L}\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},\ \mathcal{L}\big|_Y = \omega_Y(B-P)$$

Therefore $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$.

Case (7): Let $\mathcal{B}_Q$ be the fiber of $\mathcal{B}$ over a point $Q \in \mathbb{P}^1_P - \{P\}$, for some $\mathbb{P}^1_P$ intersecting X. Then

$$\mathcal{L}\big|_X = \omega_X(A-P),\ \mathcal{L}\big|_Y = \omega_Y(B)$$

Then we obtain that $h^0(\mathcal{L}\big|_{\mathcal{B}_Q}) = g-1$.

Case (8): Let $\mathcal{B}_Q$ be the fiber of $\mathcal{B}$ over a point $Q \in \mathbb{P}^1_P - \{P\}$, for some $\mathbb{P}^1_P$ intersecting Y. Then

$$\mathcal{L}\big|_X = \omega_X(A),\ \mathcal{L}\big|_Y = \omega_Y(B-P)$$

and it follows that $h^0(\mathcal{L}\big|_{\mathcal{B}_Q}) = g-1$. □

Proposition 6.2.2. $R^1\rho_{1*}(\mathcal{L}) \cong \mathcal{O}_{\widetilde{\mathcal{X}}}$

Proof. Notice that by Riemann-Roch we have $h^1(\mathcal{L}\big|_{\mathcal{B}_P}) = 1$ for every $P \in \widetilde{\mathcal{X}}$, as $h^0(\mathcal{L}\big|_{\mathcal{B}_P}) = g-1$. It follows that $R^1\rho_{1*}(\mathcal{L})$ is invertible. Let $D = \widetilde{\Delta} + Z_{11}$ and consider the long exact sequence

$$0 \to \rho_{1*}(\mathcal{L}) \to \rho_{1*}(\omega_{\rho_1}) \to \rho_{1*}(\omega_{\rho_1}\big|_D) \to$$

$$R^1\rho_{1*}(\mathcal{L}) \to R^1\rho_{1*}(\omega_{\rho_1}) \to R^1\rho_{1*}(\omega_{\rho_1}\big|_D) \to 0$$

Now, we will show that $R^1\rho_{1*}(\omega_{\rho_1}\big|_D) = 0$ in codimension 2. Indeed, consider the exact sequence

$$0 \to \omega_{\rho_1}(-Z_{11})\big|_{\widetilde{\Delta}} \to \omega_{\rho_1}\big|_D \to \omega_{\rho_1}\big|_{Z_{11}} \to 0$$

Since $R^1\rho_{1*}(\omega_{\rho_1}(-Z_{11})\big|_{\widetilde{\Delta}}) = 0$, as the restriction of $\omega_{\rho_1}(-Z_{11})\big|_{\widetilde{\Delta}}$ to each fiber is supported at a point, we have $R^1\rho_{1*}(\omega_{\rho_1}\big|_D) \cong R^1\rho_{1*}(\omega_{\rho_1}\big|_{Z_{11}})$. To show that $R^1\rho_{1*}(\omega_{\rho_1}\big|_{Z_{11}}) = 0$ in codimension 2, it is enough to show that $h^1(\mathcal{B}_P, (\omega_{\rho_1}\big|_{Z_{11}})\big|_{\mathcal{B}_P}) = 0$ for every $P \in \widetilde{\mathcal{X}}$ away from a codimension- 2 locus. If $P \notin X$, then this is true. Now, let $P \in X$ such that P is not a node of $\widetilde{\mathcal{X}}_{t_0}$. Consider the exact sequence

$$0 \to \omega_{\rho_1}(-Z_{11}) \to \omega_{\rho_1} \to \omega_{\rho_1}\big|_{Z_{11}} \to 0$$

then we have an exact sequence

$$\omega_{\rho_1}(-Z_{11})\big|_{\mathcal{B}_P} \to \omega_{\mathcal{B}_P} \to (\omega_{\rho_1}\big|_{Z_{11}})\big|_{\mathcal{B}_P} \to 0$$

Writing $\mathcal{B}_P = X \cup Y$, we see that the image of the first map vanishes over X, as $X \subseteq Z_{11}$. Thus, this image is $\omega_{\rho_1}(-Z_{11})\big|_Y = \omega_Y$. Hence, we have the exact sequence

$$0 \to \omega_Y \to \omega_{\mathcal{B}_P} \to (\omega_{\rho_1}\big|_{Z_{11}})\big|_{\mathcal{B}_P} \to 0$$

Taking the long exact sequence in cohomology, we get

$$H^1(Y, \omega_Y) \to H^1(\mathcal{B}_P, \omega_{\mathcal{B}_P}) \to H^1(\mathcal{B}_P, (\omega_{\rho_1}\big|_{Z_{11}})\big|_{\mathcal{B}_P}) \to 0$$

By duality, the map $H^1(Y, \omega_Y) \to H^1(\mathcal{B}_P, \omega_{\mathcal{B}_P})$ is a surjection if and only if the map $H^0(\mathcal{B}_P, \mathcal{O}_{\mathcal{B}_P}) \to H^0(Y, \mathcal{O}_Y)$ is injective, which is true. Thus, $R^1\rho_{1*}(\omega_{\rho_1}\big|_D) = 0$ in codimension 2 and hence we have a surjection $R^1\rho_{1*}(\mathcal{L}) \to R^1\rho_{1*}(\omega_{\rho_1})$ in codimension 2. As $R^1\rho_{1*}(\mathcal{L})$ is an invertible sheaf and $R^1\rho_{1*}(\omega_{\rho_1}) \cong \mathcal{O}_{\widetilde{\mathcal{X}}}$, it follows that $R^1\rho_{1*}(\mathcal{L}) \cong \mathcal{O}_{\widetilde{\mathcal{X}}}$. □

6.3 Classes of the degeneracy scheme and the ramification divisor

Let W' be the degeneracy locus of the evaluation map $\rho_1^*\rho_{1*}(\mathcal{L}) \to J^{g-2}_{\rho_1}(\mathcal{L})$ and W the closure of $W' \cap \mathcal{B}_{ns}$.

Proposition 6.3.1.

$$\begin{aligned} W = W' &- \binom{g_Y}{2} Z_{11} - \left(\binom{g_X+1}{2} - 1\right) Z_{12} - \binom{g_Y}{2} Z_{21} - \binom{g_X+1}{2} Z_{22} \\ &- \binom{g_Y+1}{2} \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times X)\tilde{} - \left(\binom{g_X}{2} + 1\right) \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times Y)\tilde{} \\ &- \left(\binom{g_Y}{2} + 1\right) \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times X)\tilde{} - \binom{g_X+1}{2} \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times Y)\tilde{} \end{aligned}$$

Proof. Consider a slice Σ on $\widetilde{\mathcal{X}}$ intersecting $\widetilde{\mathcal{X}}_{t_0}$ transversally at a point which is not a node. Let $\mathcal{S}$ be the fibered product

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow{\scriptstyle \rho_1} \\ \Sigma & \longrightarrow & \widetilde{\mathcal{X}} \end{array}$$

The family of curves $\mathcal{S} \to \Sigma$ has smooth generic fiber. Since the formation of the degeneracy scheme commutes with base change, we have that $f^*(W')$ is the degeneracy scheme of the invertible sheaf $f^*(\mathcal{L})$.

Thus, it is enough to see that the pull back via f of the right side of the equality we want to prove is effective and does not have vertical components for every point $P \in \widetilde{\mathfrak{X}}_{t_0}$ which is not a node. For this, we use Proposition 1.3.1. Abusing notation, we denote by $\mathcal{L}$ the invertible sheaf $f^*(\mathcal{L})$ on $\mathcal{S}$, and let $V = H^0(\mathcal{L})$. There are 4 cases to consider.

Case (1): Let $P \in X - \{A\}$ such that P is not a node of $\widetilde{\mathfrak{X}}_{t_0}$.

We have

$$\mathcal{L}\big|_X = \omega_X(2A - P),\ \mathcal{L}\big|_Y = \omega_Y.$$

Consider the exact sequence

$$0 \to H^0(\omega_Y(-B)) \to H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_X(2A - P)).$$

It follows that $h^0(\mathcal{L}\big|_{X\cup Y}) \le h^0(\omega_Y(-B)) + h^0(\omega_X(2A-P)) = g_Y - 1 + g_X = g - 1$ and since by semicontinuity $h^0(\mathcal{L}\big|_{X\cup Y}) \ge g-1$, then $h^0(\mathcal{L}\big|_{X\cup Y}) = g-1$ and we have a surjection $H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_X(2A-P))$. By the base change theorem we have a surjection $H^0(\mathcal{L}) \to H^0(\mathcal{L}\big|_{X\cup Y})$ and hence $V\big|_X = H^0(\omega_X(2A-P))$.

We will show that

$$\dim_{\mathbb{C}}(V\big|_X(-iA)) + \dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \le g-1 \text{ for every } i \ge 1.$$

We have the exact sequence

$$0 \to V(-iY)\big|_Y(-B) \to V(-iY)\big|_{X\cup Y} \to V(-iY)\big|_X \to 0.$$

Then $\dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \le \dim_{\mathbb{C}}(V(-iY)\big|_{X\cup Y}) = g-1$ and as

$$V\big|_X(-iA) = H^0(\omega_X(-(i-2)A - P)),$$

it follows that $\dim_{\mathbb{C}}(V\big|_X(-iA)) + \dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \le g-1$ for every $i \ge g_X + 2$. On the other hand for $2 \le i \le g_X + 1$ we have

$$V(-iY)\big|_Y(-B) \subseteq H^0(\omega_Y((i-1)B)),$$

which implies that $\dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \le h^0(\omega_Y((i-1)B)) = g_Y + i - 2$; also we have $\dim_{\mathbb{C}}(V\big|_X(-iA)) = h^0(\omega_X(-(i-2)A-P)) = g_X - i + 1$, as P is an ordinary point of the complete linear system $H^0(\omega_X(-(g_X-1)A))$. Finally, for $i=1$ we have $\dim_{\mathbb{C}} V\big|_X(-A) = h^0(\omega_X(A-P)) = g_X - 1$ and $\dim_{\mathbb{C}}(V(-Y)\big|_Y(-B)) \le h^0(\omega_Y) = g_Y$.

Now, we will show that

$$\dim_{\mathbb{C}}(V\big|_Y(-iB)) + \dim_{\mathbb{C}}(V(-iX)\big|_X(-A)) \le g-1 \text{ for every } i \ge 1.$$

Consider the exact sequence

$$0 \to H^0(\omega_X(A-P)) \to H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_Y).$$

Since $h^0(\mathcal{L}\big|_{X\cup Y}) = g-1 = h^0(\omega_X(A-P)) + h^0(\omega_Y)$, we have a surjection $H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_Y)$, and as we have a surjection $H^0(\mathcal{L}) \to H^0(\mathcal{L}\big|_{X\cup Y})$, it follows that $V\big|_Y = H^0(\omega_Y)$. Thus, we get $V\big|_Y(-iB) = H^0(\omega_Y(-iB)) = 0$ for every $i \geq g_Y$. On the other hand, as we have the exact sequence

$$0 \to V(-iX)\big|_X(-A) \to V(-iX)\big|_{X\cup Y} \to V(-iX)\big|_Y \to 0,$$

it follows that $\dim_{\mathbb{C}} V(-iX)\big|_X(-A) \leq \dim_{\mathbb{C}}(V(-iX)\big|_{X\cup Y}) = g-1$. Then $\dim_{\mathbb{C}}(V\big|_Y(-iB)) + \dim_{\mathbb{C}}(V(-iX)\big|_X(-A)) \leq g-1$ for every $i \geq g_Y$. Now, for $i \leq g_Y - 1$ we have

$$V(-iX)\big|_X(-A) \subseteq H^0(\omega_X((i+1)A-P)),$$

which implies that $\dim_{\mathbb{C}} V(-iX)\big|_X(-A) \leq g_X - 1 + i$. Also $\dim_{\mathbb{C}} V\big|_Y(-iB) = h^0(\omega_Y(-iB)) = g_Y - i$.

Thus, the hypothesis of Proposition 1.3.1 are satisfied in this case. Since $V\big|_Y = H^0(\omega_Y)$, we get $Tw_{V|_Y}(B) = 0 + 1 + \ldots + g_Y - 1 = \binom{g_Y}{2}$. Also since $V\big|_X = H^0(\omega_X(2A-P))$, we have $Tw_{V|_X}(A) = 0 + 2 + \ldots + g_X = \binom{g_X+1}{2} - 1$. Hence the multiplicities of X and Y in the degeneracy scheme are what we stated.

Case (2): Let $P \in Y - \{B\}$ such that P is not a node of $\widetilde{\mathfrak{X}}_{t_0}$.

We have

$$\mathcal{L}\big|_X = \omega_X(A),\ \mathcal{L}\big|_Y = \omega_Y(B-P).$$

Considering the exact sequence

$$0 \to H^0(\omega_Y(-P)) \to H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_X(A)),$$

as in the first case, we get $V\big|_X = H^0(\omega_X(A))$. We will show that

$$\dim_{\mathbb{C}}(V\big|_X(-iA)) + \dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \leq g-1 \text{ for every } i \geq 1.$$

We have $V\big|_X(-iA) = H^0(\omega_X(-(i-1)A)) = 0$ for every $i \geq g_X + 1$. On the other hand, for $i \leq g_X$ we have $\dim_{\mathbb{C}}(V\big|_X(-iA)) = h^0(\omega_X(-(i-1)A)) = g_X - i + 1$, and since $V(-iY)\big|_Y(-B) \subseteq H^0(\omega_Y(iB-P))$, we have that $\dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \leq h^0(\omega_Y(iB-P)) = g_Y + i - 2$.

Now, we will show that

$$\dim_{\mathbb{C}}(V\big|_Y(-iB))+\dim_{\mathbb{C}}(V(-iX)\big|_X(-A)) \leq g-1 \text{ for every } i \geq 1.$$

Considering the exact sequence

$$0 \to H^0(\omega_X) \to H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_Y(B-P)),$$

we get $V\big|_Y = H^0(\omega_Y(B-P))$. We have $V\big|_Y(-iB) = H^0(\omega_Y(-(i-1)B-P)) = 0$ for every $i \geq g_Y+1$. On the other hand, for $i \leq g_Y$ we have

$$V(-iX)\big|_X(-A) \subseteq H^0(\omega_X(iA)),$$

which implies that $\dim_{\mathbb{C}}V(-iX)\big|_X(-A) \leq g_X-1+i$. Also $\dim_{\mathbb{C}}V\big|_Y(-iB) = h^0(\omega_Y(-(i-1)B-P)) = g_Y-i$, as P is an ordinary point of the complete linear system $H^0(\omega_Y(-(g_Y-1)B))$.

Thus, the hypothesis of Proposition 1.3.1 are satified in this case. Since $V\big|_Y = H^0(\omega_Y(B-P))$, we get $Tw_{V|_Y}(B) = 1+\ldots+g_Y-1 = \binom{g_Y}{2}$; also since $V\big|_X = H^0(\omega_X(A))$, we have $Tw_{V|_X}(A) = 1+\ldots+g_X = \binom{g_X+1}{2}$. Hence the multiplicities of X and Y in the degeneracy scheme are what we stated.

Case (3): Consider $Q \in \mathbb{P}^1_P - \{P\}$ such that $\mathbb{P}^1_P$ intersects X.

We have

$$\mathcal{L}\big|_X = \omega_X(A-P),\ \mathcal{L}\big|_Y = \omega_Y(B).$$

Considering the exact sequence

$$0 \to H^0(\omega_Y) \to H^0(\mathcal{L}\big|_{X\cup Y}) \to H^0(\omega_X(A-P)),$$

we get $V\big|_X = H^0(\omega_X(A-P))$. We will show that

$$\dim_{\mathbb{C}}(V\big|_X(-iA))+\dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \leq g-1 \text{ for every } i \geq 1.$$

We have $V\big|_X(-iA) = H^0(\omega_X(-(i-1)A-P)) = 0$ for every $i \geq g_X+1$; on the other hand, for $i \leq g_X-1$, since P is a ramification point of the complete linear system $H^0(\omega_X(-(g_X-1)A))$, we have (Propositions 1.3.3 and 1.3.2) $\dim_{\mathbb{C}}(V\big|_X(-iA)) = h^0(\omega_X(-(i-1)A-P)) = g_X-i$. As $V(-iY)\big|_Y(-B) \subseteq H^0(\omega_Y(iB))$, then $\dim_{\mathbb{C}}(V(-iY)\big|_Y(-B)) \leq h^0(\omega_Y(iB)) = g_Y+i-1$.

Finally, for $i = g_X$, we have $\dim_{\mathbb{C}}V\big|_X(-g_XA) = h^0(\omega_X(-(g_X-1)A-P)) = 1$, as P is a ramification point of the complete linear system $H^0(\omega_X(-(g_X-1)A))$. Now, we will show that $\dim_{\mathbb{C}}(V(-g_XY)\big|_Y(-B)) \leq g-2$. As

$$\mathcal{L}(-g_XY)\big|_X = \omega_X(-(g_X-1)A-P),\ \mathcal{L}(-g_XY)\big|_Y = \omega_Y((g_X+1)B),$$

then $\mathcal{L}(-g_XY)$ has focus on Y. Let $V_Y = V(-g_XY)\big|_Y$ be the limit linear system on Y. Then $\dim_{\mathbb{C}} V_Y = g-1$. Since B is a general point of Y, the orders of vanishing at B of the sections of $\omega_Y((g_X+1)B)$ are $\{0,1,\ldots,g_X-1,g_X+1,\ldots,g\}$; and since V_Y has codimension 1 in $H^0(\omega_Y((g_X+1)B))$, the orders of vanishing at B of the sections of $\omega_Y((g_X+1)B)$ in V_Y are of the form $\{0,1,\ldots,g_X-1,g_X+1,\ldots,g\}-\{l\}$, for some $l\in\{0,1,\ldots,g_X-1,g_X+1,\ldots,g\}$. Then we have that $wt_{V_Y}(B) = g_Y+g-1-l$. On the other hand, notice that

$$\mathcal{L}(-g_YX)\big|_X = \omega_X((g_Y+1)A-P),\ \mathcal{L}(-g_YX)\big|_Y = \omega_Y(-(g_Y-1)B),$$

then $\mathcal{L}(-g_YX)$ has focus on X. Let $V_X = V(-g_YX)\big|_X$ be the limit linear system on X. As $\dim_{\mathbb{C}} V_X = g-1 = h^0(\omega_X((g_Y+1)A-P))$, it follows that $V_X = H^0(\omega_X((g_Y+1)A-P))$. Then the orders of vanishing at A of the sections in V_X are $\{0,1,\ldots,g_Y-1,g_Y+1,\ldots,g-2,g\}$ and hence $wt_{V_X}(A) = g_X$. Now, as $\mathcal{L}(-g_YX)$ has focus on X and $\mathcal{L}(-g_XY)$ has focus on Y, it follows that the connecting number between these sheaves with respect to X and Y is $l_{XY} = 0-(-g_Y)+0-(-g_X) = g$. Therefore, we have

$$\begin{aligned} wt_{V_X}(A) + wt_{V_Y}(B) + (g-1)(g-2-l_{XY}) &\geq 0, \text{ i.e.,} \\ g_X+g_Y+g-1-l+(g-1)(g-2-g) &\geq 0. \end{aligned}$$

It follows that $l\leq 1$. If $l=0$, the intersection multiplicity of the ramification divisor and the special fiber at the node would be 1, which is impossible. Thus, we get that $l=1$. It follows that $\dim_{\mathbb{C}} V_Y(-B) = g-2$, that is, $\dim_{\mathbb{C}} V(-g_XY)\big|_Y(-B) = g-2$.

To show the inequality $\dim_{\mathbb{C}} V\big|_Y(-iB) + \dim_{\mathbb{C}} V(-iX)\big|_X(-A) \leq g-1$ is similar to the begining of Case (2), exchanging X with Y and A with B. Thus, we get $V\big|_Y = H^0(\omega_Y(B))$. Now, since $V\big|_X = H^0(\omega_X(A-P))$, $Tw_{V|_X}(A) = 1+\ldots+g_X-2+g_X = \binom{g_X}{2}+1$. Also since $V\big|_Y = H^0(\omega_Y(B))$, $Tw_{V|_Y}(B) = 1+\ldots+g_Y = \binom{g_Y+1}{2}$. Hence the multiplicities of X and Y in the degeneracy scheme are what we stated.

Case (4): Consider $Q\in\mathbb{P}^1_P-\{P\}$ such that $\mathbb{P}^1_P$ intersects Y.

This case is similar to Case (3) exchanging X with Y and A with B. Thus, we get that the multiplicities of X and Y in the degeneracy scheme are what we stated. $\square$

Lemma 6.3.2. *Let* $K_\pi := c_1(\omega_\pi)$, $K_{\rho_1} := c_1(\omega_{\rho_1})$ *and* $K_{\rho_2} := c_1(\omega_{\rho_2})$. *Let* $b' : \widetilde{\mathcal{X}} \to \mathcal{X}$ *be the blow-up. Then*

$$\widetilde{\Delta}^2 = -K_{\rho_1} \cdot \widetilde{\Delta}$$
$$\rho_{1*}(\widetilde{\Delta}^2) = -b'^*(K_\pi)$$

Proof. Let $g : \mathcal{Y} \to \mathcal{X} \times_T \mathcal{X}$ be the natural morphism and $b : \mathcal{B} \to \mathcal{Y}$ the blow-up. Let $f : \widetilde{\Delta} \to \Delta$ be the morphism induced by $g \circ b$, and let $\bar{p}_1, \bar{p}_2$ be the projection maps of $\mathcal{Y}$. Since $(g \circ b)^*(\mathcal{I}_\Delta)$ modulo torsion is $\mathcal{I}_{\widetilde{\Delta}} \cdot \mathcal{I}_{\mathbb{P}^1_{(A,A)}}$, we have that $(g \circ b)^*(\mathcal{I}_\Delta)\big|_{\widetilde{\Delta}}$ modulo torsion is $\mathcal{I}_{\widetilde{\Delta}}\big|_{\widetilde{\Delta}} \cdot \mathcal{I}_{\mathbb{P}^1_{(A,A)} \cdot \widetilde{\Delta}}$. On the other hand, $(g \circ b)^*(\mathcal{I}_\Delta)\big|_{\widetilde{\Delta}} = f^*(\mathcal{I}_\Delta\big|_\Delta)$ and $\mathcal{I}_\Delta\big|_\Delta \cong \Omega^1_\pi = \omega_\pi \otimes \mathcal{J}$ (identifying Δ with $\mathcal{X}$), where $\mathcal{J}$ is the ideal sheaf of the nodes. Then we have that $f^*(\omega_\pi) \otimes \mathcal{I}_{\mathbb{P}^1_{(A,A)} \cdot \widetilde{\Delta}} = \mathcal{I}_{\widetilde{\Delta}}\big|_{\widetilde{\Delta}} \cdot \mathcal{I}_{\mathbb{P}^1_{(A,A)} \cdot \widetilde{\Delta}}$ and hence $\mathcal{I}_{\widetilde{\Delta}}\big|_{\widetilde{\Delta}} = f^*(\omega_\pi)$. Thus, denoting by p_1, p_2 the projection maps of $\mathcal{X} \times_T \mathcal{X}$, we have

$$\mathcal{O}_{\mathcal{B}}(-\widetilde{\Delta})\big|_{\widetilde{\Delta}} = f^*(\omega_\pi) = f^*(\omega_{p_1}\big|_\Delta) = (b^* g^*(\omega_{p_1}))\big|_{\widetilde{\Delta}} = (b^*\omega_{\bar{p}_1})\big|_{\widetilde{\Delta}} = \omega_{\rho_1}\big|_{\widetilde{\Delta}}.$$

Therefore $\widetilde{\Delta}^2 = -K_{\rho_1} \cdot \widetilde{\Delta}$ and $\rho_{1*}(\widetilde{\Delta}^2) = -\rho_{1*}(f^*(K_\pi)) = -b'^*(K_\pi)$. □

Proposition 6.3.3.

$$[W'] = \binom{g}{2} K_{\rho_1} + K_{\rho_2} - \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times X)\tilde{} - \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times Y)\tilde{}$$
$$- \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times X)\tilde{} - \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times Y)\tilde{} - (g-1)\widetilde{\Delta}$$
$$- \rho_1^* \widetilde{\pi}^* \lambda_\pi + (g_X - 1)\rho_1^* X - (g-1) Z_{11}$$

Proof. By the Thom-Porteous Formula:

$$[W'] = c_1(J^{g-2}_{\rho_1}(\mathcal{L})) - c_1(\rho_1^* \rho_{1*}(\mathcal{L})).$$

Using the exact sequences of truncation, we get (Proposition 2.4.1)

$$\begin{aligned} c_1(J^{g-2}_{\rho_1}(\mathcal{L})) &= \binom{g-1}{2} c_1(\omega_{\rho_1}) + (g-1)c_1(\mathcal{L}) \\ &= \binom{g-1}{2} K_{\rho_1} + (g-1)(K_{\rho_1} - \widetilde{\Delta} - Z_{11}) \\ &= \binom{g}{2} K_{\rho_1} - (g-1)\widetilde{\Delta} - (g-1) Z_{11} \end{aligned}$$

Now, we will compute $c_1(\rho_{1*}(\mathcal{L}))$. By Grothendieck-Riemann-Roch we have:

$$\begin{aligned}
ch(\rho_{1!}(\mathcal{L})) &= \rho_{1*}(ch(\mathcal{L})\cdot td(\mathcal{T}_{\mathcal{B}/\widetilde{\mathcal{X}}}))\\
&= \rho_{1*}((1+c_1(\mathcal{L})+\frac{c_1(\mathcal{L})^2}{2}+...)\cdot(1-\frac{K_{\rho_1}}{2}+td_2(\mathcal{T}_{\mathcal{B}/\widetilde{\mathcal{X}}})+...))\\
&= \rho_{1*}(1+(c_1(\mathcal{L})-\frac{K_{\rho_1}}{2})\\
&+(\frac{c_1(\mathcal{L})^2}{2}-\frac{K_{\rho_1}c_1(\mathcal{L})}{2}+td_2(\mathcal{T}_{\mathcal{B}/\widetilde{\mathcal{X}}}))+...)\\
&= \rho_{1*}(1+(\frac{K_{\rho_1}}{2}-\widetilde{\Delta}-Z_{11})\\
&+(\frac{K_{\rho_1}^2+\widetilde{\Delta}^2+Z_{11}^2-2K_{\rho_1}\cdot\widetilde{\Delta}-2K_{\rho_1}\cdot Z_{11}+2\widetilde{\Delta}\cdot Z_{11}}{2}\\
&-\frac{K_{\rho_1}^2-K_{\rho_1}\cdot\widetilde{\Delta}-K_{\rho_1}\cdot Z_{11}}{2}+td_2(\mathcal{T}_{\mathcal{B}/\widetilde{\mathcal{X}}}))+...)
\end{aligned}$$

On the other hand, we have the following formulas:

(1)$\rho_{1*}(Z_{11})=0$, as Z_{11} is vertical with respect to ρ_1.

(2)$\rho_{1*}(Z_{11}^2)=-X$, as $Z_{11}^2=-Z_{11}\cdot(\mathcal{B}_{t_0}-Z_{11})$ and the only component of $Z_{11}\cdot(\mathcal{B}_{t_0}-Z_{11})$ which is not contracted by ρ_1 is $(X\times\{A\})\widetilde{}$.

(3)$\rho_{1*}(K_{\rho_1}\cdot Z_{11})=(2g_X-1)X$, as writing $\mathcal{B}_P=X\cup Y$ for each point $P\in X-\{A\}$ which is not a node of $\widetilde{\mathcal{X}}_{t_0}$ we have that $K_{\rho_1}\big|_X=\omega_X(A)$ has degree $2g_X-1$.

Also, recall that $\rho_{1*}(td_2(\mathcal{T}_{\mathcal{B}/\widetilde{\mathcal{X}}}))=\lambda_{\rho_1}=\widetilde{\pi}^*\lambda_\pi$. Using these formulas and the previous lemma, we get

$$\begin{aligned}
ch(\rho_{1!}(\mathcal{L})) &= (g-2)+\frac{-b'^*(K_\pi)-X-2b'^*(K_\pi)-2(2g_X-1)X+2X}{2}\\
&-\frac{-b'^*(K_\pi)-(2g_X-1)X}{2}+\widetilde{\pi}^*\lambda_\pi+\dots\\
&= (g-2)-b'^*(K_\pi)-(g_X-1)X+\widetilde{\pi}^*\lambda_\pi+\dots,
\end{aligned}$$

Since $R^1\rho_{1*}(\mathcal{L})\cong\mathcal{O}_{\widetilde{\mathcal{X}}}$ (Proposition 6.2.2), we have that $ch(\rho_{1!}(\mathcal{L}))=(g-2)+c_1(\rho_{1*}(\mathcal{L}))+....$ Then

$$c_1(\rho_{1*}(\mathcal{L}))=-b'^*(K_\pi)-(g_X-1)X+\widetilde{\pi}^*\lambda_\pi$$

and hence

$$[W']=\binom{g}{2}K_{\rho_1}-(g-1)\widetilde{\Delta}-(g-1)Z_{11}-\rho_1^*(-b'^*(K_\pi)-(g_X-1)X+\widetilde{\pi}^*\lambda_\pi)$$

$$= \binom{g}{2} K_{\rho_1} - (g-1)\widetilde{\Delta} - (g-1)Z_{11} + \rho_1^* b'^*(K_\pi) + (g_X - 1)\rho_1^* X - \rho_1^* \widetilde{\pi}^* \lambda_\pi$$

On the other hand, as $K_{\widetilde{\pi}} = b'^*(K_\pi) + \sum \mathbb{P}^1_P$, it follows that

$$K_{\rho_2} = \rho_1^*(K_{\widetilde{\pi}}) = \rho_1^* b'^*(K_\pi) + \rho_1^*(\sum \mathbb{P}^1_P),$$

i.e., $\rho_1^* b'^*(K_\pi) = K_{\rho_2} - \rho_1^*(\sum \mathbb{P}^1_P)$. This, together with the formula for $[W']$, implies the proposition. □

By Propositions 6.3.1 and 6.3.3, we get the formula for $[W]$:

$$\begin{aligned}
[W] = & \binom{g}{2} K_{\rho_1} + K_{\rho_2} - (g-1)\widetilde{\Delta} - \rho_1^* \widetilde{\pi}^* \lambda_\pi + (g_X - 1)\rho_1^* X - \binom{g_Y}{2} Z_{21} \\
& - (\binom{g_Y}{2} + g - 1) Z_{11} - (\binom{g_X + 1}{2} - 1) Z_{12} - \binom{g_X + 1}{2} Z_{22} \\
& - (\binom{g_Y + 1}{2} + 1) \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times X)\widetilde{} - (\binom{g_X}{2} + 2) \sum_{\mathbb{P}^1_P \cap X \neq \emptyset} (\mathbb{P}^1_P \times Y)\widetilde{} \\
& - (\binom{g_Y}{2} + 2) \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times X)\widetilde{} - (\binom{g_X + 1}{2} + 1) \sum_{\mathbb{P}^1_P \cap Y \neq \emptyset} (\mathbb{P}^1_P \times Y)\widetilde{}
\end{aligned}$$

6.4 Lower bounds for the coefficients

Let j denote the genus of Y, and let $\lambda := \lambda_\pi$. Our next aim is to compute $\widetilde{\pi}_* \rho_{1*}([W] \cdot ([W] + K_{\rho_1}) \cdot ([W] + 2K_{\rho_1}))$. To do it, it is necessary to compute all intersections appearing. We list some of the most representative formulas in the following lemma (see appendix for a list of all intersections).

Lemma 6.4.1. *We have the following formulas:*

(1) *If* $\mathbb{P}^1_P \cap X \neq \emptyset$, *then* $\widetilde{\pi}_* \rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{})^3) = 2\delta_j$
(2) $\widetilde{\pi}_* \rho_{1*}(K^2_{\rho_1} \cdot Z_{11}) = 0$
(3) $\widetilde{\pi}_* \rho_{1*}(Z^2_{11} \cdot K_{\rho_1}) = -(2g_X - 1) g_X \delta_j$
(4) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot K_{\rho_2}) = -12\lambda + \delta_j$
(5) $\widetilde{\pi}_* \rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot Z_{11}) = (2g_X - 1)(3g_X - 2)\delta_j$
(6) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot Z_{11}) = -(2g_X - 1)\delta_j$
(7) $\widetilde{\pi}_* \rho_{1*}(Z^2_{11} \cdot \widetilde{\Delta}) = -g_X \delta_j$
(8) $\widetilde{\pi}_* \rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \rho_1^* X) = (2g - 2)(3g_X - 2)\delta_j$
(9) $\widetilde{\pi}_* \rho_{1*}(Z^2_{11} \cdot K_{\rho_2}) = -(3g_X - 2)\delta_j$
(10) $\widetilde{\pi}_* \rho_{1*}(Z^2_{11} \cdot \rho_1^* \widetilde{\pi}^* \lambda) = 0$

Proof. To show the calculation techniques, we will prove the formulas $(1), (2), (4), (6), (8)$ and (9).

(1) $(\widetilde{(\mathbb{P}^1_P\times X)})^2 = -\widetilde{(\mathbb{P}^1_P\times X)}\cdot(\mathcal{B}_{t_0} - \widetilde{(\mathbb{P}^1_P\times X)}) = -(\mathbb{P}^1_P\times\{A\}+\{P\}\times X)$.

Then $(\widetilde{(\mathbb{P}^1_P\times X)})^3 = -(\mathbb{P}^1_P\times\{A\}+\{P\}\times X)\cdot\widetilde{(\mathbb{P}^1_P\times X)}$. On the other hand, we have $\mathbb{P}^1_P\times\{A\} = \widetilde{(\mathbb{P}^1_P\times Y)}\cdot\widetilde{(\mathbb{P}^1_P\times X)}$, then $\mathbb{P}^1_P\times\{A\}\cdot\widetilde{(\mathbb{P}^1_P\times X)} = \widetilde{(\mathbb{P}^1_P\times Y)}\cdot(\widetilde{(\mathbb{P}^1_P\times X)})^2$ is the self-intersection of $\mathbb{P}^1_P\times\{A\}$ on $\widetilde{(\mathbb{P}^1_P\times Y)}$; this self-intersection is the self-intersection of $\mathbb{P}^1_P\times\{A\}$ on $\mathbb{P}^1_P\times Y$ minus 1, as $\widetilde{(\mathbb{P}^1_P\times Y)}$ is the blow-up of $\mathbb{P}^1_P\times Y$ at the point $(P,A)\in\mathbb{P}^1_P\times\{A\}$. Since the self-intersection of $\mathbb{P}^1_P\times\{A\}$ on $\mathbb{P}^1_P\times Y$ is 0, $\mathbb{P}^1_P\times\{A\}\cdot\widetilde{(\mathbb{P}^1_P\times X)} = -1$. Analogously, $\{P\}\times X\cdot\widetilde{(\mathbb{P}^1_P\times X)} = -1$. This implies Formula (1).

(2) We have that $K^2_{\rho_1}\cdot Z_{11}$ is the self-intersection of $K_{\rho_1}\cdot Z_{11}$ on Z_{11}. Let $i : Z_{11}\hookrightarrow\mathcal{B}$ and $j : X\hookrightarrow\mathfrak{X}$ be the inclusion maps, $q_2 : X\times X\to X$ the second projection and $\varphi : Z_{11}\to X\times X$ the morphism induced by the blow-up $b:\mathcal{B}\to\widetilde{\mathfrak{X}}\times_T\mathfrak{X}$

$$
\begin{array}{ccccc}
\mathcal{B} & \xrightarrow{\;b\;} & \widetilde{\mathfrak{X}}\times_T\mathfrak{X} & \xrightarrow{\;\bar{p}_2\;} & \mathfrak{X}\\
\uparrow{\scriptstyle i} & & \uparrow & & \uparrow{\scriptstyle j}\\
Z_{11} & \xrightarrow{\;\varphi\;} & X\times X & \xrightarrow{\;q_2\;} & X
\end{array}
$$

Then we have

$$K_{\rho_1}\cdot Z_{11} = i^*K_{\rho_1} = i^*\rho_2^*K_\pi = \varphi^*q_2^*j^*K_\pi = \varphi^*q_2^*(K_X+A).$$

Therefore, the self-intersection of $K_{\rho_1}\cdot Z_{11}$ on Z_{11} is 0, and from this Formula (2) follows.

(4) By using the projection formula and Lemma 6.3.2, we get

$$
\begin{aligned}
\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2\cdot K_{\rho_2}) &= \widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2\cdot\rho_1^*K_{\widetilde{\pi}})\\
&= \widetilde{\pi}_*(\rho_{1*}(\widetilde{\Delta}^2)\cdot K_{\widetilde{\pi}})\\
&= \widetilde{\pi}_*(-b'^*(K_\pi)\cdot(b'^*(K_\pi)+\sum\mathbb{P}^1_P))\\
&= \pi_*b'_*(-b'^*(K^2_\pi)) = -\pi_*(K^2_\pi) = -(12\lambda-\delta_j) = -12\lambda+\delta_j
\end{aligned}
$$

(6) We have that $\widetilde{\Delta}^2\cdot Z_{11}$ is the self-intersection of $\widetilde{\Delta}\cdot Z_{11}$ on Z_{11}; this self-intersection is the self-intersection of the diagonal on $X\times X$ minus 1, as Z_{11} is the blow-up of $X\times X$ at the points (A,A) and (P,A), where the points $P\in X-\{A\}$ are the nodes of $\widetilde{\mathfrak{X}}_{t_0}$, and the diagonal of $X\times X$ passes through the point (A,A) and does not pass through the points

(P, A). Since the self-intersection of the diagonal on $X \times X$ is $-(2g_X - 2)$, we have $\widetilde{\Delta}^2 \cdot Z_{11} = -(2g_X - 1)$. Formula (6) follows.

(8) By using the projection formula, we get

$$\begin{aligned}\widetilde{\pi}_* \rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \rho_1^* X) &= \widetilde{\pi}_* \rho_{1*}(K_{\rho_1} \cdot \rho_1^*(K_{\widetilde{\pi}}) \cdot \rho_1^* X) \\ &= \widetilde{\pi}_* \rho_{1*}(K_{\rho_1} \cdot \rho_1^*(K_{\widetilde{\pi}} \cdot X)) \\ &= \widetilde{\pi}_*(\rho_{1*}(K_{\rho_1}) \cdot K_{\widetilde{\pi}} \cdot X) \\ &= \widetilde{\pi}_*((2g-2) K_{\widetilde{\pi}} \cdot X) \\ &= (2g-2)(3g_X - 2)\delta_j\end{aligned}$$

(9) By using the projection formula, we get

$$\begin{aligned}\widetilde{\pi}_* \rho_{1*}(Z_{11}^2 \cdot K_{\rho_2}) &= \widetilde{\pi}_* \rho_{1*}(Z_{11}^2 \cdot \rho_1^*(K_{\widetilde{\pi}})) \\ &= \widetilde{\pi}_*(\rho_{1*}(Z_{11}^2) \cdot K_{\widetilde{\pi}}) \\ &= \widetilde{\pi}_*(-X \cdot K_{\widetilde{\pi}}) \\ &= -(3g_X - 2)\delta_j\end{aligned}$$

□

Using *Singular* [S] for the computations, we obtain
$\widetilde{\pi}_* \rho_{1*}([W] \cdot ([W] + K_{\rho_1}) \cdot ([W] + 2K_{\rho_1})) = a(g)\lambda + b_j(g)\delta_j$, where

$$a(g) = 9g^5 - 51g^4 + 129g^3 - 207g^2 + 174g - 54$$

(as computed in chapter 4) and

$$\begin{aligned}b_j(g) = {} & 6j^4g^2 - 6j^4g + 12j^4 - 6j^3g^3 - 3j^3g^2 - 3j^3g - 18j^3 + 3j^2g^4 \\ & + 3j^2g^2 + 12j^2g + 6j^2 - 3jg^5 + 12jg^4 - 21jg^3 + 21jg^2 - 21jg + 6j.\end{aligned}$$

Proposition 6.4.2. *The divisor W is flat over $\widetilde{\mathfrak{X}}$.*

Proof. It is enough to show that W does not contain any irreducible component of each fiber $\mathcal{B}_P$. Indeed, let $\mathcal{I}_W$ be the ideal sheaf of W and consider the natural morphism $\mathcal{I}_W \hookrightarrow \mathcal{O}_{\mathcal{B}}$. Since $\mathcal{O}_{\mathcal{B}}$ is flat over $\widetilde{\mathfrak{X}}$, we have that if $\mathcal{I}_W\big|_{\mathcal{B}_P} \to \mathcal{O}_{\mathcal{B}_P}$ is injective for every point $P \in \widetilde{\mathfrak{X}}$ (which is true when W does not contain any irreducible component of each singular fiber $\mathcal{B}_P$), then the cokernel of the morphism $\mathcal{I}_W \hookrightarrow \mathcal{O}_{\mathcal{B}}$ is flat over $\widetilde{\mathfrak{X}}$.

There are 4 cases to consider.

Case (1): Let $P = A$ and let $\mathcal{B}_A = X \cup \mathbb{P}^1 \cup Y$ be the fiber of $\mathcal{B}$ over the point A.

Let $\mathcal{L}_1 := \mathcal{L}(-(g_Y - 1)Z_{11} - (g_Y - 1)Z_{21})$. Then, we have

$$\mathcal{L}_1\big|_X = \omega_X(g_Y A),\ \mathcal{L}_1\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},\ \mathcal{L}_1\big|_Y = \omega_Y(-(g_Y-1)B).$$

Then $h^0(\mathcal{L}_1\big|_{\mathcal{B}_A}) = g-1$ and hence $\rho_{1*}(\mathcal{L}_1)$ is locally free of rank $g-1$ in a neighborhood of A.

Consider Σ and $\mathcal{S}$ as in the begining of the proof of Proposition 6.3.1, but assume that Σ is a slice through the point $A \in \widetilde{\mathcal{X}}$ intersecting X and Y transversally at the point A. Since $f^*(\mathcal{L}_1)$ has focus on X, the degeneracy scheme of $f^*(\mathcal{L}_1)$ does not contain X, i.e., the pullback of the degeneracy scheme of $\mathcal{L}_1$ does not contain X. Then W does not contain X.

On the other hand, analogously, by considering the invertible sheaf $\mathcal{L}_2 := \mathcal{L}(-g_X Z_{12} - g_X Z_{22})$ we get that W does not contain Y.

Now, we are going to see what happens on $\mathbb{P}^1$. Abusing notation, we denote by $\mathcal{L}$ the invertible sheaf $f^*(\mathcal{L})$ on $\mathcal{S}$.

Let $\mathcal{L}' := \mathcal{L}(g_X X + (g_Y-1)Y + f^*(\widetilde{\Delta}))$. We have

$$\mathcal{L}'\big|_X = \omega_X(-(g_X-1)A),\ \mathcal{L}'\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(g),\ \mathcal{L}'\big|_Y = \omega_Y(-(g_Y-1)B).$$

Thus $\mathcal{L}'$ has focus on $\mathbb{P}^1$. Let $V' = H^0(\mathcal{L}')$, then $V'\big|_{\mathbb{P}^1} \subseteq H^0(\mathcal{O}_{\mathbb{P}^1}(g))$. As $\mathcal{L}'(-g_X X)\big|_X = \omega_X(A)$, it follows that $V'(-g_X X)\big|_{\mathbb{P}^1}$ has A as a base point. Therefore,

$$V'(-g_X X)\big|_{\mathbb{P}^1} \subseteq H^0(\mathcal{L}'(-g_X X)\big|_{\mathbb{P}^1}(-A)) = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)).$$

On the other hand, we have

$$\begin{aligned}\dim_{\mathbb{C}} V'(-g_X X)\big|_{\mathbb{P}^1} &= g - \dim_{\mathbb{C}} V'(-g_X X)\big|_{X\cup Y}(-A-B)\\ &\geq g - h^0(\omega_X)\\ &= g - g_X = g_Y\end{aligned}$$

Hence $V'(-g_X X)\big|_{\mathbb{P}^1} = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A))$. Analogously, we get $V'(-g_Y Y)\big|_{\mathbb{P}^1} = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_Y+1)B))$, and since

$$H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)) \cap H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_Y+1)B)) = 0$$

as subspaces of $H^0(\mathcal{O}_{\mathbb{P}^1}(g))$, then by dimension considerations,

$$V'\big|_{\mathbb{P}^1} = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_Y+1)B)).$$

Let $R := \mathbb{P}^1 \cap \widetilde{\Delta}$. We have $V'(-f^*(\widetilde{\Delta}))\big|_{\mathbb{P}^1} \subseteq V'\big|_{\mathbb{P}^1}(-R)$. On the other hand, the degree of the ramification divisor of $V'|_{\mathbb{P}^1}$ is g, $wt_{V'|_{\mathbb{P}^1}}(A) = g_Y$ and $wt_{V'|_{\mathbb{P}^1}}(B) = g_X$. Then A and B are the only ramification points of

$V'\big|_{\mathbb{P}^1}$ and hence $\dim_{\mathbb{C}} V'\big|_{\mathbb{P}^1}(-R) = g-1$. Thus, the limit linear system of $\mathcal{L}$ on $\mathbb{P}^1$ is $V'(-f^*(\widetilde{\Delta}))\big|_{\mathbb{P}^1} = V'\big|_{\mathbb{P}^1}(-R)$. Then, the limit linear system of $\mathcal{L}$ on $\mathbb{P}^1$ does not depend on the slice Σ. Notice that, if $N \subseteq W$ is a irreducible curve passing through a point $Q \in \mathbb{P}^1 - \{A, B\}$ such that the generic point of N lies on a smooth fiber of ρ_1, then Q is a limit ramification point on some slice Σ, and hence Q is a ramification point of $V'\big|_{\mathbb{P}^1}(-R)$. We conclude that W does not contain $\mathbb{P}^1$.

Case (2): Consider a fiber $\mathcal{B}_P = X \cup \mathbb{P}^1 \cup Y$, where $\mathbb{P}^1_P \cap X \neq \emptyset$.

By considering the invertible sheaf $\mathcal{L}_1 := \mathcal{L}(-g_Y(\widetilde{\mathbb{P}^1_P \times X}) - (g_Y - 1)Z_{11})$, as in Case (1), we get that W does not contain X.

Now, consider Σ and $\mathcal{S}$ as in the begining of the proof of Proposition 6.3.1, but assume Σ is a slice in $\widetilde{\mathfrak{X}}$ intersecting $\mathbb{P}^1_P$ and X transversally at the point P. Abusing notation, we denote by $\mathcal{L}$ the invertible sheaf $f^*(\mathcal{L})$ on $\mathcal{S}$, and let $V := H^0(\mathcal{L})$. We will see what happens on $\mathbb{P}^1$.

For each $i \geq 0$, let $\mathcal{L}_i := \mathcal{L}(-i\mathbb{P}^1)$ and let W'_i be the degeneracy scheme of $\mathcal{L}_i$. Let $m_{\mathbb{P}^1}(i)$ denote the multiplicity of $\mathbb{P}^1$ in the divisor W'_i. By Equation 1.3.2, we get

$$m_{\mathbb{P}^1}(i) = m_{\mathbb{P}^1}(i+1) + \dim_{\mathbb{C}} V(-i\mathbb{P}^1)\big|_{X\cup Y}(-A-B).$$

Furthermore,

$$\mathcal{L}_i\big|_X = \omega_X(-(i-1)A - P),\ \mathcal{L}_i\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1+2i),\ \mathcal{L}_i\big|_Y = \omega_Y(-iB),$$

which implies that $\mathcal{L}_i$ has focus on $\mathbb{P}^1$ for each $i \geq g_X$, and hence $m_{\mathbb{P}^1}(i) = 0$ for every $i \geq g_X$. It follows that

$$m_{\mathbb{P}^1}(0) = \sum_{i=0}^{g_X-1} dim_{\mathbb{C}} V(-i\mathbb{P}^1)\big|_{X\cup Y}(-A-B).$$

On the other hand, for each $i \geq 0$ we have

$$dim_{\mathbb{C}} V(-i\mathbb{P}^1)\big|_{X\cup Y}(-A-B) \leq h^0(\omega_X(-iA-P)) + h^0(\omega_Y(-(i+1)B)),$$

which implies that

$$\begin{aligned} m_{\mathbb{P}^1}(0) &\leq Tw_{H^0(\omega_X(A-P))}(A) + Tw_{H^0(\omega_Y)}(B) \\ &= \binom{g_X}{2} + 1 + \binom{g_Y}{2}. \end{aligned}$$

This, together with the fact that W' contains Z_{11} and $(\mathbb{P}^1_P \times Y)\tilde{}$ with multiplicities $\binom{g_Y}{2}$ and $\binom{g_X}{2}+1$ respectively (Proposition 6.3.1), implies that W does not contain $\mathbb{P}^1$.

Now, we are going to show that W does not contain Y. Let $\mathcal{L}' := \mathcal{L}(g_X X + (g_Y - 1)Y)$. We have

$$\mathcal{L}'\big|_X = \omega_X(-(g_X-1)A - P),\ \mathcal{L}'\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(g),\ \mathcal{L}'\big|_Y = \omega_Y(-(g_Y-1)B).$$

Thus $\mathcal{L}'$ has focus on $\mathbb{P}^1$. Let $V' = H^0(\mathcal{L}')$, then $V'\big|_{\mathbb{P}^1} \subseteq H^0(\mathcal{O}_{\mathbb{P}^1}(g))$. Since $\mathcal{L}'(-g_X X)\big|_X = \omega_X(A-P)$, we have that $V'(-g_X X)\big|_{\mathbb{P}^1}$ has A as a base point. Then

$$V'(-g_X X)\big|_{\mathbb{P}^1} \subseteq H^0(\mathcal{L}'(-g_X X)\big|_{\mathbb{P}^1}(-A)) = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)).$$

On the other hand, we have

$$\begin{aligned}\dim_{\mathbb{C}} V'(-g_X X)\big|_{\mathbb{P}^1} &= g - 1 - \dim_{\mathbb{C}} V'(-g_X X)\big|_{X\cup Y}(-A-B)\\ &\geq g-1-(g_X-1)\\ &= g - g_X = g_Y\end{aligned}$$

Therefore $V'(-g_X X)\big|_{\mathbb{P}^1} = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A))$. Analogously, we have $V'(-g_Y Y)\big|_{\mathbb{P}^1} \subseteq H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_Y+1)B))$. Now, notice that

$$\begin{aligned}\dim_{\mathbb{C}} V'(-g_Y Y)\big|_{\mathbb{P}^1} &= g - 1 - \dim_{\mathbb{C}} V'(-g_Y Y)\big|_{X\cup Y}(-A-B)\\ &\geq g-1-g_Y\\ &= g_X - 1,\end{aligned}$$

and since

$$H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)) \cap H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_Y+1)B)) = 0$$

as subspaces of $H^0(\mathcal{O}_{\mathbb{P}^1}(g))$, then by dimension considerations,

$$V'\big|_{\mathbb{P}^1} = H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-(g_X+1)A)) \oplus V'(-g_Y Y)\big|_{\mathbb{P}^1}.$$

Let $V_{\mathbb{P}^1} := V'\big|_{\mathbb{P}^1}$. It follows that $V_{\mathbb{P}^1}(-g_X A) = V_{\mathbb{P}^1}(-(g_X+1)A)$. Then, writing the vanishing orders at A of the sections of $V_{\mathbb{P}^1}$,

$$\{0,\dots,g\} - \{l_1, l_2\},$$

where $0 \leq l_1 < l_2 \leq g$, we have $l_1 = g_X$ or $l_2 = g_X$. On the other hand,

$$\mathcal{L}(-g_Y X + (g_Y - 1)Y)\big|_X = \omega_X((g_Y + 1)A - P),$$
$$\mathcal{L}(-g_Y X + (g_Y - 1)Y)\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},$$
$$\mathcal{L}(-g_Y X + (g_Y - 1)Y)\big|_Y = \omega_Y(-(g_Y - 1)B).$$

Then $\mathcal{L}(-g_Y X + (g_Y - 1)Y)$ has focus on X, and by dimension considerations, the limit linear system of $\mathcal{L}$ on X is $V_X = H^0(\omega_X((g_Y+1)A-P))$. As P is a ramification point of $H^0(\omega_X(-(g_X - 1)A))$, it follows the vanishing orders at A of the sections of V_X are $\{0, \dots, g_Y - 1, g_Y + 1, \dots, g - 2, g\}$ and hence $wt_{V_X}(A) = g_X$. Since $\mathcal{L}(-g_Y X + (g_Y - 1)Y)$ has focus on X and $\mathcal{L}(g_X X + (g_Y - 1)Y)$ has focus on $\mathbb{P}^1$, the connecting number between these sheaves with respect to X and $\mathbb{P}^1$ is $l_{X\mathbb{P}^1} = g_Y - 1 - (-g_Y) + g_X - (g_Y - 1) = g$. Therefore, we have

$$wt_{V_X}(A) + wt_{V_{\mathbb{P}^1}}(A) + (g-1)(g-2-l_{X\mathbb{P}^1}) \geq 0, \text{ i.e.,}$$
$$g_X + 2g - 1 - (l_1 + l_2) - 2(g-1) \geq 0.$$

It follows that $l_1 + l_2 \leq g_X + 1$; and since $l_1 = g_X$ or $l_2 = g_X$, we have that $l_1 = 0$ and $l_2 = g_X$, or $l_1 = 1$ and $l_2 = g_X$. But, for $l_1 = 0$ the intersection multiplicity of the ramification divisor and the special fiber at the node is 1, which is impossible. Thus, we get that $l_1 = 1$, $l_2 = g_X$ and hence $wt_{V_{\mathbb{P}^1}}(A) = 2g - 1 - (1 + g_X) = 2(g-1) - g_X$ and the limit ramification divisor does not contain the point A. Also, since $\deg R_{V_{\mathbb{P}^1}} = 2(g-1)$, we have $wt_{V_{\mathbb{P}^1}}(B) \leq g_X$.

On the other hand, we have that

$$V_{\mathbb{P}^1}(-g_Y B) = V_{\mathbb{P}^1}(-(g_Y + 1)B) = V'(-g_Y Y)\big|_{\mathbb{P}^1}$$

has dimension $g_X - 1$. Then, writing the orders of vanishing at B of the sections of $V_{\mathbb{P}^1}$,

$$\{0, \dots, g\} - \{l'_1, l'_2\},$$

where $0 \leq l'_1 < l'_2 \leq g$, we get that $l'_1 = g_Y$. On the other hand,

$$\mathcal{L}(g_X X - (g_X + 1)Y)\big|_X = \omega_X(-(g_X - 1)A - P),$$
$$\mathcal{L}(g_X X - (g_X + 1)Y)\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},\ \mathcal{L}(g_X X - (g_X + 1)Y)\big|_Y = \omega_Y((g_X + 1)B).$$

Then $\mathcal{L}(g_X X - (g_X + 1)Y)$ has focus on Y. Let V_Y be the limit linear system on Y; writing the orders of vanishing at B of the sections of V_Y, $\{0, \dots, g_X - 1, g_X + 1, \dots, g\} - \{l\}$, we get $wt_{V_Y}(B) = g_Y + g - 1 - l$. As $\mathcal{L}(g_X X + (g_Y - 1)Y)$ has focus on $\mathbb{P}^1$ and $\mathcal{L}(g_X X - (g_X + 1)Y)$ has focus on Y, it follows that the connecting number between these sheaves with respect to $\mathbb{P}^1$ and Y is $l_{\mathbb{P}^1 Y} = g_Y - 1 - g_X + g_X - (-(g_X+1)) = g$. Therefore, we have

$$wt_{V_{\mathbb{P}^1}}(B) + wt_{V_Y}(B) + (g-1)(g-2-l_{\mathbb{P}^1 Y}) \geq 0, \text{ i.e.,}$$
$$2g - 1 - (l'_1 + l'_2) + g_Y + g - 1 - l - 2(g-1) \geq 0.$$

It follows that $l'_1 + l'_2 + l \leq g_Y + g$; and hence $l'_2 + l \leq g$. Now, since $wt_{V_{\mathbb{P}^1}}(B) \leq g_X$, we have $2g - 1 - (l'_1 + l'_2) \leq g_X$; then $l'_1 + l'_2 \geq g - 1 + g_Y$, i.e., $l'_2 \geq g - 1$. We conclude that $l = 1$ and $l'_2 = g - 1$, or $l = 0$ and $g - 1 \leq l'_2 \leq g$. But, for $l = 0$ and $l'_2 = g - 1$, the intersection multiplicity of the ramification divisor and the special fiber at the node is 1, which is impossible. The remaining cases, that is, $l = 1$ and $l'_2 = g - 1$, $l = 0$ and $l'_2 = g$, imply the limit ramification divisor does not contain the point B.

Now, notice that when $l = 1$ and $l'_2 = g-1$; we have $wt_{V_{\mathbb{P}^1}}(B) = g_X$, and as $wt_{V_{\mathbb{P}^1}}(A) = 2(g-1) - g_X$, it follows that the limit ramification divisor does not contain any point of $\mathbb{P}^1$.

On the other hand, by using the formula for the class of W (see after proposition 6.3.3), together with the following facts: the intersection multiplicities of all K_{ρ_1}, K_{ρ_2}, $\widetilde{\Delta}$ and $\rho_1^* \widetilde{\pi}^* \lambda_\pi$ with $\mathbb{P}^1$ are zero; and $Z_{11} \cdot \mathbb{P}^1 = -1$, $Z_{12} \cdot \mathbb{P}^1 = 1$, $(\mathbb{P}^1_P \times X)\tilde{} \cdot \mathbb{P}^1 = 1$, $(\mathbb{P}^1_P \times Y)\tilde{} \cdot \mathbb{P}^1 = -1$, we conclude that $W \cdot \mathbb{P}^1 = 1$. Thus, since W does not contain $\mathbb{P}^1$, there is a unique irreducible component of W intersecting $\mathbb{P}^1$, and this intersection is transversal. Let $D := W \cap \mathbb{P}^1$. Then, $D \in \mathbb{P}^1$ is a limit ramification point on the slice Σ. It follows that the case $l = 1$ and $l'_2 = g - 1$ is impossible, and hence $l = 0$ and $l'_2 = g$. Then $V_Y = H^0(\omega_Y(g_X B))$. Thus, the limit linear system on Y does not depend on the slice Σ, and hence W does not contain Y.

Case (3): Consider a fiber $\mathcal{B}_P = X \cup \mathbb{P}^1 \cup Y$, where $\mathbb{P}^1_P \cap Y \neq \emptyset$.

This case is similar to Case (2).

Case (4): $P \in \widetilde{\mathcal{X}}_{t_0}$ is not a node.

By the proof of Proposition 6.3.1, we have that for each slice Σ passing through the point P, $f^*(W)$ does not contain any vertical component on $\mathcal{S}$. □

Now, we want to prove that $S^2 W \cap \mathcal{B}_{t_0}$ is finite. There are some cases which are easy to handle (see Propositions 6.4.4 and 6.4.5), but in the general case, we need to state the following hypothesis:

Hypothesis $(*)$.

If (X, A) is a general pointed smooth curve, then for every ramification point $P \in X$ of the complete linear system $H^0(\omega_X(-(g_X-1)A))$ and for every $i \geq 1$, the complete linear system $H^0(\omega_X((i+1)A-P))$ does not have ramification points on $X - \{A\}$ having ramification weight at least 3.

Proposition 6.4.3. *$S^2W \cap \mathcal{B}_{t_0}$ is finite.*

Proof. Since W is flat, for each singular fiber $\mathcal{B}_P$, we have that $W \cap \mathcal{B}_P$ is the limit ramification divisor over any slice Σ intersecting $\widetilde{\mathcal{X}}_{t_0}$ transversally at the point P. We are going to see what happens on each fiber; keeping the notation as in the begining of the proof of Proposition 6.3.1, we denote by $\mathcal{L}$ the invertible sheaf $f^*(\mathcal{L})$ on $\mathcal{S}$.

Case (1): $P \in X$ is not a node of $\widetilde{\mathcal{X}}_{t_0}$.

We have

$$\mathcal{L}(-(g_X+1)Y)\big|_X = \omega_X(-(g_X-1)A-P),$$
$$\mathcal{L}(-(g_X+1)Y)\big|_Y = \omega_Y((g_X+1)B).$$

Then $\mathcal{L}(-(g_X+1)Y)$ has focus on Y, and since P is not a ramification point of $H^0(\omega_X(-(g_X-1)A))$, by dimension considerations we have that the limit linear system of $\mathcal{L}$ on Y is $V_Y = H^0(\omega_Y(g_X B))$. On the other hand,

$$\mathcal{L}(-(g_Y-1)X)\big|_X = \omega_X((g_Y+1)A-P),$$
$$\mathcal{L}(-(g_Y-1)X)\big|_Y = \omega_Y(-(g_Y-1)B).$$

Then $\mathcal{L}(-(g_Y-1)X)$ has focus on X, and by dimension considerations we have that the limit linear system of $\mathcal{L}$ on X is $V_X = H^0(\omega_X((g_Y+1)A-P))$.

Since B is a general point of Y, the orders of vanishing at B of the sections in V_Y are $\{1, \ldots, g_X-1, g_X+1, \ldots, g\}$; then $wt_{V_Y}(B) = g_Y+g-1$. Also, $\{0, \ldots, g_Y-1, g_Y+1, \ldots, g-1\}$ are the orders of vanishing at A of the sections in V_X; hence $wt_{V_X}(A) = g_X - 1$. Thus, the number of ramification points of V_X and V_Y on $(X - \{A\}) \cup (Y - \{B\})$ is

$$(g-1)((g-2)(g_X-1)+2g_X-2+g_Y)-(g_X-1)$$
$$+(g-1)((g-2)(g_Y-1)+2g_Y-2+g_X+1)-(g_Y+g-1);$$

this sum is $(g-1)(g^2-g-1)$, and since this number is the total number of limit ramification points, we have that W does not contain the node. On the other hand, as $V_Y = H^0(\omega_Y(g_X B))$, it follows from Proposition 1.3.4 that $wt_{V_Y}(Q) = 1$ for every ramification point Q of V_Y on $Y - \{B\}$. Also, for $Q \in X - \{A\}$, it follows from Propositions 1.3.5 and 1.3.6 that

$$h^0(\omega_X((g_Y+1)A-P-(g+1)Q))=0,$$
$$h^0(\omega_X((g_Y+1)A-P-(g-3)Q))=2.$$

Then, the orders of vanishing at Q of the sections in V_X are

$$\{0,\dots,g-4,a_{g-3},a_{g-2}\},\text{ where } a_{g-2}\le g.$$

Thus, $wt_{V_X}(Q)\ge 3$ if and only if $h^0(\omega_X((g_Y+1)A-P-gQ))=1$ and $h^0(\omega_X((g_Y+1)A-P-(g-2)Q))=2$; and in this case $wt_{V_X}(Q)=3$. Since $h^0(\omega_X((g_Y+1)A-P-gQ))=1$ if and only if Q is a ramification point of $H^0(\omega_X((g_Y+1)A))$ and P is a ramification point of $H^0(\omega_X((g_Y+1)A-gQ))$, we get that there only exist a finitely many points $(P,Q)\in Z_{11}$ such that $P\in X$ is not a node of $\widetilde{\mathfrak{X}}_{t_0}$, $Q\in X-\{A\}$ and $wt_{V_X}(Q)=3$, where $V_X=H^0(\omega_X((g_Y+1)A-P))$.

Case (2): $P\in Y$ is not a node of $\widetilde{\mathfrak{X}}_{t_0}$.

This case is similar to Case (1):

$$V_X=H^0(\omega_X(g_YA))\subseteq H^0(\omega_X((g_Y+1)A))\text{ and}$$
$$V_Y=H^0(\omega_Y((g_X+1)B-P))$$

are the limit linear systems on X and Y respectively. Also, W does not contain the node, $wt_{V_X}(Q)=1$ for every ramification point Q of V_X on $X-\{A\}$; for each $Q\in Y-\{B\}$ we have that $wt_{V_Y}(Q)\ge 3$ if and only if $wt_{V_Y}(Q)=3$; and there exists a finitely many points $(P,Q)\in Z_{22}$ such that $P\in Y$ is not a node of $\widetilde{\mathfrak{X}}_{t_0}$, $Q\in Y-\{B\}$ and $wt_{V_Y}(Q)=3$.

Case (3): Consider $Q\in\mathbb{P}^1_P-\{P\}$ such that $\mathbb{P}^1_P$ intersects X.

By the proof of Case (3) of Proposition 6.3.1, we have that the limit linear systems on X and Y satisfy $V_X=H^0(\omega_X((g_Y+1)A-P))$ and $V_Y\subseteq H^0(\omega_Y((g_X+1)B))$. Also, we obtained that $l=1$, where $\{0,1,\dots,g_X-1,g_X+1,\dots,g\}-\{l\}$ are the orders of vanishing at B of the sections of $\omega_Y((g_X+1)B)$ in V_Y. Therefore W does not contain the node B, $V_Y\ne H^0(\omega_Y(g_XB))$ and $V_Y\supseteq H^0(\omega_Y((g_X-1)B))$.

Now, consider a linear system $V\subseteq H^0(\omega_Y((g_X+1)B))$ satisfying

$$V\supseteq H^0(\omega_Y((g_X-1)B))\text{ and }\dim_{\mathbb{C}}V=g-1.$$

We have that $D\in Y-\{B\}$ is a ramification point of V if and only if $V(-(g-1)D)\ne 0$, i.e., $V\supseteq H^0(\omega_Y((g_X+1)B-(g-1)D))$. On the other hand, for every $D\in Y-\{B\}$,

$$H^0(\omega_Y((g_X-1)B))\cap H^0(\omega_Y((g_X+1)B-(g-1)D))=0$$

as subspaces of $H^0(\omega_Y((g_X+1)B))$. Hence, by dimension considerations, $D \in Y - \{B\}$ is a ramification point of V if and only if

$$V = H^0(\omega_Y((g_X-1)B)) \oplus H^0(\omega_Y((g_X+1)B-(g-1)D)).$$

It follows that V_Y and $H^0(\omega_Y(g_X B))$ do not have ramification points in common on $Y-\{B\}$, and since the limit linear system on $Y \subseteq \mathcal{B}_P$ is $H^0(\omega_Y(g_X B))$, we have that for every $D \in Y-\{B\}$, W does not contain $\mathbb{P}^1_P \times \{D\}$.

On the other hand, by using the formula for the class of W (see after Proposition 6.3.3), we conclude that $W \cdot \mathbb{P}^1_P \times \{D\} = 1$ for every point $D \in Y-\{B\}$. Then, for each $D \in Y-\{B\}$, we have W intersects $\mathbb{P}^1_P \times \{D\}$ at a single point; thus, as Q varies in $\mathbb{P}^1_P - \{P\}$, the limit linear system V_Y on $Y \subseteq \mathcal{B}_Q$ varies through distinct subspaces of $H^0(\omega_Y((g_X+1)B))$, and furthermore, those limit linear systems on Y do not have ramification points in common on $Y-\{B\}$. Also, notice that, as Q varies in $\mathbb{P}^1_P$, the limit linear system V_Y on $Y \subseteq \mathcal{B}_Q$ varies through all the subspaces V of $H^0(\omega_Y((g_X+1)B))$ satisfying $V \supseteq H^0(\omega_Y((g_X-1)B))$ and $\dim_{\mathbb{C}} V = g-1$.

Since $H^0(\omega_Y((g_X-1)B)) \subseteq V_Y \subseteq H^0(\omega_Y((g_X+1)B))$, for every point $D \in Y-\{B\}$ we have that $wt_{V_Y}(D) = wt_{H^0(\omega_Y((g_X+1)B))}(D) + g - 1 - l'$, where $\{b_0,\dots,b_{g-1}\} - \{l'\}$ are the orders of vanishing at D of the sections in V_Y, with $\{b_0,\dots,b_{g-1}\}$ the orders of vanishing at D of the sections of $H^0(\omega_Y((g_X+1)B))$. If $D \in Y-\{B\}$ is an ordinary point of the linear system $H^0(\omega_Y((g_X-1)B))$ then $\{0,\dots,g-3\} \subseteq \{b_0,\dots,b_{g-1}\} - \{l'\}$, and hence $l' \geq g-2$; and since $wt_{H^0(\omega_Y((g_X+1)B))}(D) \leq 1$, we have $wt_{V_Y}(D) \leq 2$. On the other hand, as $H^0(\omega_Y((g_X-1)B))$ has only simple ramification points on $Y-\{B\}$, it follows that $\{0,\dots,g-4\} \subseteq \{b_0,\dots,b_{g-1}\} - \{l'\}$. Therefore $l' \geq g-3$, and if $D \in Y-\{B\}$ is an ordinary point of $H^0(\omega_Y((g_X+1)B))$ then $wt_{V_Y}(D) \leq 2$.

Now, consider $D \in Y-\{B\}$ such that D is a ramification point in common of the linear systems $H^0(\omega_Y((g_X-1)B))$ and $H^0(\omega_Y((g_X+1)B))$. Then, we have

$$\{0,\dots,g-4,g-2\} \subseteq \{b_0,\dots,b_{g-1}\} - \{l'\} \text{ and}$$
$$\{b_0,\dots,b_{g-1}\} = \{0,\dots,g-2,g\};$$

which imply $l' = g-3$ or $l' = g$. Therefore, if D is a ramification point of V_Y, then $l' = g-3$ and $wt_{V_Y}(D) = 3$. Then, for $D \in Y-\{B\}$, we have $wt_{V_Y}(D) \geq 3$ if and only if $wt_{V_Y}(D) = 3$. Also, if α is the number of ramification points in common on $Y - \{B\}$ of the linear systems $H^0(\omega_Y((g_X - 1)B))$ and $H^0(\omega_Y((g_X + 1)B))$, then there exist α points $(Q, D) \in (\mathbb{P}^1_P \times Y)\tilde{}$ such that $Q \in \mathbb{P}^1_P - \{P\}$, $D \in Y - \{B\}$ and $wt_{V_Y}(D) = 3$, where V_Y is the limit linear system on $Y \subseteq \mathcal{B}_Q$.

On the other hand, the hypothesis $(*)$ implies that there are no points $(Q, D) \in (\mathbb{P}^1_P \times X)\tilde{}$ such that $Q \in \mathbb{P}^1_P - \{P\}$, $D \in X - \{A\}$ and $wt_{V_X}(D) \geq 3$, where $V_X = H^0(\omega_X((g_Y + 1)A - P))$.

Case (4): Consider $Q \in \mathbb{P}^1_P - \{P\}$ such that $\mathbb{P}^1_P$ intersects Y.

This case is similar to Case (3): We have the limit linear systems on X and Y satisfy

$$V_X \subseteq H^0(\omega_X((g_Y + 1)A)) \text{ and } V_Y = H^0(\omega_Y((g_X + 1)B - P)).$$

Also, W does not contain the node, $V_X \supseteq H^0(\omega_X((g_Y - 1)A))$ and $V_X \neq H^0(\omega_X(g_Y A))$. If β is the number of ramification points in common on $X - \{A\}$ of the linear systems $H^0(\omega_X((g_Y - 1)A))$ and $H^0(\omega_X((g_Y + 1)A))$, then there exist β points $(Q, D) \in (\mathbb{P}^1_P \times X)\tilde{}$ such that $Q \in \mathbb{P}^1_P - \{P\}$, $D \in X - \{A\}$ and $wt_{V_X}(D) = 3$, where V_X is the limit linear system on $X \subseteq \mathcal{B}_Q$. Also, there are no points $(Q, D) \in (\mathbb{P}^1_P \times Y)\tilde{}$ such that $Q \in \mathbb{P}^1_P - \{P\}$, $D \in Y - \{B\}$ and $wt_{V_Y}(D) \geq 3$, where $V_Y = H^0(\omega_Y((g_X + 1)B - P))$.

Case (5): Let $P = A$ and let $\mathcal{B}_A = X \cup \mathbb{P}^1 \cup Y$ be the fiber of $\mathcal{B}$ over the point A.

We have

$$\mathcal{L}(-(g_Y - 1)X + (g_Y - 1)Y)\big|_X = H^0(\omega_X(g_Y A)),$$
$$\mathcal{L}(-(g_Y - 1)X + (g_Y - 1)Y)\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1},$$
$$\mathcal{L}(-(g_Y - 1)X + (g_Y - 1)Y)\big|_Y = H^0(\omega_Y(-(g_Y - 1)B)).$$

Then $\mathcal{L}(-(g_Y - 1)X + (g_Y - 1)Y)$ has focus on X and $V_X = H^0(\omega_X(g_Y A))$. Also,

$$\mathcal{L}(g_X X - g_X Y)\big|_X = H^0(\omega_X(-(g_X - 1)A)),\ \mathcal{L}(g_X X - g_X Y)\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}.$$
$$\mathcal{L}(g_X X - g_X Y)\big|_Y = H^0(\omega_Y(g_X B));$$

Then $\mathcal{L}(g_X X - g_X Y)$ has focus on Y and $V_Y = H^0(\omega_Y(g_X B))$. On the other hand,

$$\mathcal{L}(g_X X + (g_Y - 1)Y)\big|_X = H^0(\omega_X(-(g_X - 1)A)),$$
$$\mathcal{L}(g_X X + (g_Y - 1)Y)\big|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(g)(-R),$$
$$\mathcal{L}(g_X X + (g_Y - 1)Y)\big|_Y = H^0(\omega_Y(-(g_Y - 1)B));$$

where $R = \mathbb{P}^1 \cap \widetilde{\Delta}$. Then $\mathcal{L}(g_X X + (g_Y - 1)Y)$ has focus on $\mathbb{P}^1$.

Let $\{0, \dots, g-1\} - \{l''\}$ be the orders of vanishing at A of the sections in the limit linear system $V_{\mathbb{P}^1}$. It follows that

$$wt_{V_X}(A) + wt_{V_{\mathbb{P}^1}}(A) + (g-1)(g-2-(g-1)) \geq 0, \text{ i.e.,}$$
$$g_X + g - 1 - l'' - (g-1) \geq 0.$$

Then $l'' \leq g_X$. By the proof of Case (1) of Proposition 6.4.2, we have that $V_{\mathbb{P}^1} \supseteq H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-R-(g_X+1)A)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-R-(g_Y+1)B))$, and since the numbers $0, \dots, g_X - 2$ are the orders of vanishing at A of the sections in the linear system

$$H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-R-(g_X+1)A)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-R-(g_Y+1)B)) \subseteq H^0(\mathcal{O}_{\mathbb{P}^1}(g)(-R)),$$

we get $l'' \geq g_X - 1$. Therefore $g_X - 1 \leq l'' \leq g_X$, but for $l'' = g_X - 1$, the intersection multiplicity of the ramification divisor and the special fiber at the point A is 1, which is impossible. Thus, $l'' = g_X$ and W does not contain the point A. Analogously, W does not contain the point B.

On the other hand, the number of ramification points of V_X and V_Y on $(X - \{A\}) \cup (Y - \{B\})$ is

$$(g-1)((g-2)(g_X-1) + 2g_X - 2 + g_Y) - g_X$$
$$+(g-1)((g-2)(g_Y-1) + 2g_Y - 2 + g_X) - g_Y.$$

This sum is $(g-1)(g^2-g-1)-1$, which implies there is a unique ramification point of $V_{\mathbb{P}^1}$ on $\mathbb{P}^1 - \{A, B\}$, and this point has ramification weight 1. Noticing that V_X has only simple ramification points on $X - \{A\}$ and V_Y has only simple ramification points on $Y - \{B\}$, we conclude there are no points lying on this fiber of weight at least 3.

Case (6): Consider the fiber $\mathcal{B}_P = X \cup \mathbb{P}^1 \cup Y$, where $P \in X - \{A\}$ is a node of $\widetilde{\mathcal{X}}_{t_0}$.

By the proof of Proposition 6.4.2, the limit linear systems on X and Y are $V_X = H^0(\omega_X((g_Y+1)A-P))$ and $V_Y = H^0(\omega_Y(g_X B))$ respectively; also, there is a unique limit ramification point on $\mathbb{P}^1 - \{A, B\}$, and this point has ramification weight 1. As in Case (3), we have $V_X = H^0(\omega_X((g_Y+1)A-P))$ does not have ramification points on $X - \{A\}$ having ramification weight at least 3. Also, since $V_Y = H^0(\omega_Y(g_X B))$, we get $wt_{V_Y}(D) = 1$ for every ramification point D of V_Y on $Y - \{B\}$. Therefore, there are no points lying on this fiber of weight at least 3.

Case (7): Consider the fiber $\mathcal{B}_P = X \cup \mathbb{P}^1 \cup Y$, where $P \in Y - \{B\}$ is a node of $\widetilde{\mathcal{X}}_{t_0}$.

This case is similar to Case (6); there are no points lying on this fiber of weight at least 3. □

In the following propositions, we get more information without using the hypothesis $(*)$.

Proposition 6.4.4. *If $g = 3$, then $S^2W \cap \mathcal{B}_{t_0} = \emptyset$.*

Proof. We have $g_X = 2$ and $g_Y = 1$. As in Proposition 6.4.3, we will see what happens on each singular fiber $\mathcal{B}_P$.

Case (1): $P \in X$ is not a node of $\widetilde{\mathcal{X}}_{t_0}$.

It is enough to show that there is no point $Q \in X - \{A\}$ such that $h^0(\omega_X(2A - P - 3Q)) = 1$ and $h^0(\omega_X(2A - P - Q)) = 2$.

Suppose $h^0(\omega_X(2A - P - 3Q)) = 1$ and $h^0(\omega_X(2A - P - Q)) = 2$. As $\deg(\omega_X(2A - P - 3Q)) = 0$ and $h^0(\omega_X(2A - P - 3Q)) = 1$, it follows that $K_X + 2A - P$ and $3Q$ are linearly equivalent divisors; and since we have $h^0(\omega_X(2A - P - Q)) = 2$, we conclude $h^0(\mathcal{O}_X(2Q)) = 2$. Then $h^0(\omega_X(-2Q)) = 1$, and since $\deg(\omega_X(-2Q)) = 0$, we have that K_X and $2Q$ are linearly equivalent. We conclude that $2A$ and $P + Q$ are linearly equivalent; but this is impossible, as $h^0(\mathcal{O}_X(2A)) = 1$.

Case (2): $P \in Y$ is not a node of $\widetilde{\mathcal{X}}_{t_0}$.

Notice that, since $g_Y = 1$, $V_Y = H^0(\omega_Y((g_X + 1)B - P))$ has only simple ramification points.

Case (3): Consider $Q \in \mathbb{P}^1_P - \{P\}$ such that $\mathbb{P}^1_P$ intersects X.

Notice that $H^0(\omega_Y(B))$ has only the point B as a ramification point. Also, as showed in the case (1), there is no point $Q \in X - \{A\}$ such that $h^0(\omega_X(2A - P - 3Q)) = 1$ and $h^0(\omega_X(2A - P - Q)) = 2$.

Finally, notice that we do not have Case (4) of Proposition 6.4.3, as $g_Y = 1$. □

Proposition 6.4.5. *If $g = 4$ and $g_Y = 2$, then $S^2W \cap \mathcal{B}_{t_0} = \emptyset$.*

Proof. We have $g_X = 2$ and $g_Y = 2$. As in Proposition 6.4.3, we will see what happens on each singular fiber $\mathcal{B}_P$.

Case (1): $P \in X$ is not a node of $\widetilde{\mathcal{X}}_{t_0}$.

It is enough to show that there is no point $Q \in X - \{A\}$ such that $h^0(\omega_X(3A - P - 4Q)) = 1$ and $h^0(\omega_X(3A - P - 2Q)) = 2$.

Suppose $h^0(\omega_X(3A-P-4Q))=1$ and $h^0(\omega_X(3A-P-2Q))=2$. As $\deg(\omega_X(3A-P-4Q))=0$ and $h^0(\omega_X(3A-P-4Q))=1$, it follows that K_X+3A-P and $4Q$ are linearly equivalent divisors; and since we have $h^0(\omega_X(3A-P-2Q))=2$, we conclude $h^0(\mathcal{O}_X(2Q))=2$. Then $h^0(\omega_X(-2Q))=1$, and since $\deg(\omega_X(-2Q))=0$, we get that K_X and $2Q$ are linearly equivalent. We conclude that $3A-P$ and $2Q$ are linearly equivalent; which implies that

$$2=h^0(\omega_X)=h^0(\mathcal{O}_X(2Q))=h^0(\mathcal{O}_X(3A-P))$$

and hence $h^0(\omega_X(P-3A))=1$. It follows that $h^0(\omega_X(-3A))=1$, which is impossible.

Case (2): $P\in Y$ is not a node of $\widetilde{\mathfrak{X}}_{t_0}$.

This case is similar to Case (1).

Case (3): Consider $Q\in\mathbb{P}^1_P-\{P\}$ such that $\mathbb{P}^1_P$ intersects X.

As showed in the case (1), there is no point $Q\in X-\{A\}$ such that $h^0(\omega_X(3A-P-4Q))=1$ and $h^0(\omega_X(3A-P-2Q))=2$. Thus, it is enough to show that $H^0(\omega_Y(B))$ and $H^0(\omega_Y(3B))$ do not have ramification points in common on $Y-\{B\}$. Suppose $Q\in Y-\{B\}$ is a ramification point in common of the complete linear systems $H^0(\omega_Y(B))$ and $H^0(\omega_Y(3B))$. We have $h^0(\omega_Y(B-2Q))=1$, i.e., $h^0(\omega_Y(-2Q))=1$. Then K_Y and $2Q$ are linearly equivalent. On the other hand, since $h^0(\omega_Y(3B-4Q))=1$, we have $h^0(\mathcal{O}_Y(3B-2Q))=1$; it follows that there exists a point $R\in Y$ such that $3B-2Q$ and R are linearly equivalent. Thus, we have

$$2=h^0(\omega_Y)=h^0(\mathcal{O}_Y(2Q))=h^0(\mathcal{O}_Y(3B-R))$$

and hence $h^0(\omega_Y(R-3B))=1$. As $B\in Y$ is a general point, it follows that $R\neq B$; therefore $h^0(\omega_Y(-3B))=1$, which is impossible.

Case (4): Consider $Q\in\mathbb{P}^1_P-\{P\}$ such that $\mathbb{P}^1_P$ intersects Y.

This case is similar to Case (3). □

Observe that if we write the class of $\overline{S^2W}$ as

$$\overline{S^2W}:=a\lambda-a_0\delta_0-a_1\delta_1-\ldots-a_{[g/2]}\delta_{[g/2]},$$

then Propositions 6.4.4 and 6.4.5 tell us that $a_1=b_1$ for $g=3$, and $a_2=b_2$ for $g=4$, where $b_j:=b_j(g)$ are the numbers computed before Proposition 6.4.2.

Appendix A

Intersections appearing in the reducible case

A.1 List of intersections

We have the following formulas:

(1) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^3) = 2\delta_j$

(2) $\widetilde{\pi}_*\rho_{1*}(K^2_{\rho_1} \cdot Z_{11}) = 0$

(3) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot K_{\rho_1}) = -(2g_X - 1)g_X\delta_j$

(4) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2 \cdot K_{\rho_2}) = -12\lambda + \delta_j$

(5) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot Z_{11}) = (2g_X - 1)(3g_X - 2)\delta_j$

(6) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2 \cdot Z_{11}) = -(2g_X - 1)\delta_j$

(7) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot \widetilde{\Delta}) = -g_X\delta_j$

(8) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \rho_1^*X) = (2g - 2)(3g_X - 2)\delta_j$

(9) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot K_{\rho_2}) = -(3g_X - 2)\delta_j$

(10) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$

(11) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2 \cdot Z_{22}) = -(2g_Y - 1)\delta_j$

(12) $\widetilde{\pi}_*\rho_{1*}(Z^2_{22} \cdot \widetilde{\Delta}) = -g_Y\delta_j$

(13) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot Z_{22} \cdot \widetilde{\Delta}) = \delta_j$

(14) $\widetilde{\pi}_*\rho_{1*}(Z^3_{11}) = g_X\delta_j$

(15) $\widetilde{\pi}_*\rho_{1*}(Z^3_{22}) = g_Y\delta_j$

(16) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot Z_{22}) = -\delta_j$

(17) $\widetilde{\pi}_*\rho_{1*}(Z^2_{22} \cdot Z_{11}) = -\delta_j$

(18) $\widetilde{\pi}_*\rho_{1*}(Z^2_{11} \cdot \rho_1^*X) = g_X\delta_j$

(19) $\widetilde{\pi}_*\rho_{1*}(Z^2_{22} \cdot \rho_1^*X) = -\delta_j$

(20) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot Z_{22} \cdot \rho_1^*X) = 0$

(21) $\widetilde{\pi}_*\rho_{1*}((\rho_2^*X)^2 \cdot \rho_1^*X) = 0$
(22) $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(23) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot Z_{22} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(24) $\widetilde{\pi}_*\rho_{1*}((\rho_2^*X)^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(25) $\widetilde{\pi}_*\rho_{1*}(Z_{11}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times X)}) = 0$
(26) $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times Y)}) = 0$
(27) If $\mathbb{P}_P^1 \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(Z_{11}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times Y)}) = -\delta_j$
(28) If $\mathbb{P}_P^1 \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(Z_{11}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times Y)}) = 0$
(29) If $\mathbb{P}_P^1 \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times X)}) = -\delta_j$
(30) If $\mathbb{P}_P^1 \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot \widetilde{(\mathbb{P}_P^1 \times X)}) = 0$
(31) $\widetilde{\pi}_*\rho_{1*}(Z_{11}^2 \cdot \rho_2^*X) = g_X\delta_j$
(32) $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot K_{\rho_2}) = -(3g_Y - 2)\delta_j$
(33) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot Z_{22} \cdot K_{\rho_2}) = 0$
(34) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \widetilde{\Delta}) = 12\lambda - \delta_j$
(35) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta}^2) = -12\lambda + \delta_j$
(36) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^3) = 12\lambda - \delta_j$
(37) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \widetilde{\Delta}) = 12\lambda - \delta_j$
(38) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \widetilde{\Delta}) = 12\lambda - (g-1)\delta_j$
(39) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot Z_{11}) = (2g_X - 1)\delta_j$
(40) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot Z_{22}) = (2g_Y - 1)\delta_j$
(41) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot Z_{11}) = (3g_X - 2)\delta_j$
(42) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot Z_{22}) = (3g_Y - 2)\delta_j$
(43) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^3) = 0$
(44) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^3) = 0$
(45) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot K_{\rho_2}) = (2g-2)(12\lambda - \delta_j)$
(45) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot K_{\rho_1}) = (2g-2)(12\lambda - (g-1)\delta_j)$
(46) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta}^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = -(2g-2)\lambda$
(47) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(48) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = (2g-2)^2\lambda$
(49) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(50) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = (2g-2)\lambda$
(51) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = (2g-2)\lambda$
(52) $\widetilde{\pi}_*\rho_{1*}((\rho_1^*X)^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(53) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot \rho_2^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(54) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{11} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$

(55) $\widetilde{\pi}_*\rho_{1*}(Z_{22} \cdot \rho_2^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(56) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot \rho_2^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(57) $\widetilde{\pi}_*\rho_{1*}(Z_{22} \cdot \rho_1^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(58) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot Z_{22} \cdot K_{\rho_1}) = 0$
(59) $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot K_{\rho_1}) = -(2g_Y - 1)g_Y\delta_j$
(60) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(61) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(62) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(63) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(64) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot \widetilde{(\mathbb{P}^1_P \times Y)} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(65) If $P \neq Q$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot \widetilde{(\mathbb{P}^1_Q \times X)} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(66) If $P \neq Q$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times Y)} \cdot \widetilde{(\mathbb{P}^1_Q \times Y)} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(67) If $P \neq Q$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot \widetilde{(\mathbb{P}^1_Q \times Y)} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(68) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot Z_{11} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(69) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot Z_{22} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(70) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times Y)} \cdot Z_{11} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(71) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times Y)} \cdot Z_{22} \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(72) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times X)} \cdot \rho_1^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(73) $\widetilde{\pi}_*\rho_{1*}(\widetilde{(\mathbb{P}^1_P \times Y)} \cdot \rho_1^*X \cdot \rho_1^*\widetilde{\pi}^*\lambda) = 0$
(74) $\widetilde{\pi}_*\rho_{1*}((\rho_1^*\widetilde{\pi}^*\lambda)^3) = 0$
(75) $\widetilde{\pi}_*\rho_{1*}((\rho_1^*X)^3) = 0$
(76) $\widetilde{\pi}_*\rho_{1*}((\rho_2^*X)^3) = 0$
(77) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^3) = \delta_j$
(78) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^3) = \delta_j$
(79) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^3) = 2\delta_j$
(80) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \rho_1^*X) = 0$
(81) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \rho_2^*X) = 0$
(82) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot Z_{22}) = 0$
(83) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(84) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1}^2 \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(85) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \rho_1^*X) = 0$
(86) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \rho_2^*X) = 0$
(87) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot Z_{11}) = 0$
(88) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot Z_{22}) = 0$
(89) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(90) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2}^2 \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$

(91) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(92) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(93) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(94) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(95) $\widetilde{\pi}_*\rho_{1*}(\rho_2^*X \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(96) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(97) $\widetilde{\pi}_*\rho_{1*}(Z_{22} \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(98) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(99) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times Y)\widetilde{\ } \cdot (\rho_1^*\widetilde{\pi}^*\lambda)^2) = 0$
(100) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot (\rho_1^*X)^2) = -(2g-2)g_X\delta_j$
(101) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot (\rho_1^*X)^2) = 0$
(102) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot (\rho_1^*X)^2) = -g_X\delta_j$
(103) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot (\rho_1^*X)^2) = 0$
(104) $\widetilde{\pi}_*\rho_{1*}(Z_{22} \cdot (\rho_1^*X)^2) = 0$
(105) $\widetilde{\pi}_*\rho_{1*}(\rho_2^*X \cdot (\rho_1^*X)^2) = 0$
(106) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\rho_1^*X)^2) = 0$
(107) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times Y)\widetilde{\ } \cdot (\rho_1^*X)^2) = 0$
(108) $\widetilde{\pi}_*\rho_{1*}(Z_{22}^2 \cdot \rho_2^*X) = -g_Y\delta_j$
(109) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot (\rho_2^*X)^2) = 0$
(110) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot (\rho_2^*X)^2) = -(2g-2)\delta_j$
(111) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot (\rho_2^*X)^2) = -\delta_j$
(112) $\widetilde{\pi}_*\rho_{1*}(Z_{11} \cdot (\rho_2^*X)^2) = 0$
(113) $\widetilde{\pi}_*\rho_{1*}(Z_{22} \cdot (\rho_2^*X)^2) = 0$
(114) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\rho_2^*X)^2) = 0$
(115) $\widetilde{\pi}_*\rho_{1*}((\mathbb{P}^1_P \times Y)\widetilde{\ } \cdot (\rho_2^*X)^2) = 0$
(116) $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot K_{\rho_1}) = -(2g_X-1)\delta_j$
(117) $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{)}^2 \cdot K_{\rho_1}) = -(2g_Y-1)\delta_j$
(118) $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot K_{\rho_2}) = \delta_j$
(119) $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{)}^2 \cdot K_{\rho_2}) = \delta_j$
(120) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot \widetilde{\Delta}) = -\delta_j$
(121) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot \widetilde{\Delta}) = 0$
(122) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{)}^2 \cdot \widetilde{\Delta}) = 0$
(123) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{)}^2 \cdot \widetilde{\Delta}) = -\delta_j$
(124) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot \rho_1^*X) = -\delta_j$
(125) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot \rho_1^*X) = 0$
(126) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(((\mathbb{P}^1_P \times X)\widetilde{)}^2 \cdot Z_{11}) = -\delta_j$

(127) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot Z_{11}) = 0$
(128) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot Z_{22}) = 0$
(129) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot Z_{22}) = -\delta_j$
(130) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot Z_{22}) = 0$
(131) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot Z_{22}) = -\delta_j$
(132) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot Z_{11}) = -\delta_j$
(133) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot Z_{11}) = 0$
(134) $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot \rho_2^*X) = \delta_j$
(135) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = -\delta_J$
(136) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times X)})^2 \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(137) $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot \rho_2^*X) = -\delta_j$
(138) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(139) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}((\widetilde{(\mathbb{P}^1_P \times Y)})^2 \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = -\delta_j$
(140) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot Z_{22}) = (2g_Y - 1)(3g_Y - 2)\delta_j$
(141) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \rho_2^*X) = (2g_X - 1)(2g - 2)\delta_j$
(142) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = -(2g_X - 1)\delta_j$
(143) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot K_{\rho_2} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = -(2g_Y - 1)\delta_j$
(144) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot \rho_1^*X) = (2g_X - 1)\delta_j$
(145) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(146) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(147) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \widetilde{\Delta} \cdot \rho_2^*X) = (2g_X - 1)\delta_j$
(148) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_1^*X) = 0$
(149) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{11}) = 0$
(150) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{22}) = 0$
(151) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_2^*X) = 0$
(152) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(153) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(154) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot Z_{11}) = -(2g_X - 1)g_X\delta_j$
(155) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot Z_{22}) = (2g_Y - 1)\delta_j$
(156) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot \rho_2^*X) = 0$
(157) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = (2g_X - 1)\delta_j$
(158) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(159) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = (2g_Y - 1)\delta_j$
(160) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(161) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = (2g_X - 1)\delta_j$
(162) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$

(163) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{11} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(164) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{11} \cdot \rho_2^* X) = 0$
(165) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{22} \cdot \rho_2^* X) = 0$
(166) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(167) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(168) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = (2g_Y - 1)\delta_j$
(169) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(170) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = -(2g_X - 1)\delta_j$
(171) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(172) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_1} \cdot (\mathbb{P}^1_P \times X)\widetilde{} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(173) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot \rho_1^* X) = (3g_X - 2)\delta_j$
(174) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot \rho_2^* X) = (2g_X - 1)\delta_j$
(175) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = -\delta_j$
(176) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(177) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(178) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \widetilde{\Delta} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = -\delta_j$
(179) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_1^* X) = 0$
(180) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{11}) = 0$
(181) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{22}) = 0$
(182) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_2^* X) = 0$
(183) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(184) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(185) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^* X \cdot Z_{11}) = 0$
(186) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^* X \cdot Z_{22}) = 0$
(187) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^* X \cdot \rho_2^* X) = 0$
(188) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(189) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_1^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(190) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{11} \cdot \rho_2^* X) = -(3g_X - 2)\delta_j$
(191) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{11} \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(192) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{11} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(193) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{22} \cdot \rho_2^* X) = (3g_Y - 2)\delta_j$
(194) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = 0$
(195) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = 0$
(196) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{)}) = \delta_j$
(197) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = -\delta_j$
(198) $\widetilde{\pi}_*\rho_{1*}(K_{\rho_2} \cdot (\mathbb{P}^1_P \times X)\widetilde{} \cdot (\mathbb{P}^1_P \times Y)\widetilde{)}) = -\delta_j$

(199) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_1^*X) = 0$
(200) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{11}) = 0$
(201) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{22}) = 0$
(202) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_2^*X) = 0$
(203) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(204) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*\widetilde{\pi}^*\lambda \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(205) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot Z_{11}) = -g_X\delta_j$
(206) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot Z_{22}) = \delta_j$
(207) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot \rho_2^*X) = -\delta_j$
(208) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = \delta_j$
(209) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(210) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_1^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(211) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{11} \cdot \rho_2^*X) = -\delta_j$
(212) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = \delta_j$
(213) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(214) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(215) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{22} \cdot \rho_2^*X) = \delta_j$
(216) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{22} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(217) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{22} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = \delta_j$
(218) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot Z_{22} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(219) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(220) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(221) $\widetilde{\pi}_*\rho_{1*}(\widetilde{\Delta} \cdot \widetilde{(\mathbb{P}^1_P \times X)} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(222) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*\widetilde{\pi}^*\lambda \cdot Z_{11} \cdot Z_{22}) = 0$
(223) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(224) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*\widetilde{\pi}^*\lambda \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(225) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{11} \cdot \rho_2^*X) = g_X\delta_j$
(226) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(227) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{11} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(228) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{22} \cdot \rho_2^*X) = \delta_j$
(229) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{22} \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(230) $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot Z_{22} \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$
(231) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = -\delta_j$
(232) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times X)}) = 0$
(233) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = \delta_j$
(234) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_*\rho_{1*}(\rho_1^*X \cdot \rho_2^*X \cdot \widetilde{(\mathbb{P}^1_P \times Y)}) = 0$

(235) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(\rho_1^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = \delta_j$
(236) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(\rho_1^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(237) $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot Z_{22} \cdot \rho_2^* X) = 0$
(238) $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times X)\widetilde{\ }) = 0$
(239) $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot Z_{22} \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(240) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ }) = -\delta_j$
(241) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ }) = 0$
(242) $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(243) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = \delta_j$
(244) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{11} \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(245) $\widetilde{\pi}_* \rho_{1*}(Z_{22} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ }) = 0$
(246) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{22} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(247) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{22} \cdot \rho_2^* X \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = \delta_j$
(248) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{22} \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(249) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(Z_{22} \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = \delta_j$
(250) $\widetilde{\pi}_* \rho_{1*}(\rho_2^* X \cdot (\mathbb{P}^1_P \times X)\widetilde{\ } \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$
(251) If $\mathbb{P}^1_P \cap X \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{\ })^2 \cdot \rho_1^* X) = -\delta_j$
(252) If $\mathbb{P}^1_P \cap Y \neq \emptyset$, then $\widetilde{\pi}_* \rho_{1*}(((\mathbb{P}^1_P \times Y)\widetilde{\ })^2 \cdot \rho_1^* X) = 0$
(253) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot \rho_1^* X) = -(2g_X - 1)\delta_j$
(254) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot \rho_2^* X) = -(2g_X - 1)\delta_j$
(255) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot (\mathbb{P}^1_P \times X)\widetilde{\ }) = 0$
(256) $\widetilde{\pi}_* \rho_{1*}(\widetilde{\Delta}^2 \cdot (\mathbb{P}^1_P \times Y)\widetilde{\ }) = 0$

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