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Constant Mean Curvature Surfaces in Homogeneous Spaces

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Se um dia tiver que escolher entre o mundo e o amor lembre-se: se escolher o mundo ficará sem o amor, mas se escolher o amor com ele você conquistará o mundo.

Albert Einstein.

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There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics, rediscovering old mathematics, learning a way of thinking from a person or text, or finding a new way to explain or to view an old mathematical structure.

William Thurston.

Abstract

This these will be focused on the study of constant mean curvature surfaces in the homogeneous spaces $\mathbb{E}(\kappa,\tau)$.

Uwe Abresch and Harold Rosenberg discovered a holomorphic quadratic differential defined on any constant mean curvature surface in $\mathbb{E}(\kappa,\tau)$. However, there were no a Codazzi pair associated to this differential when $\tau \neq 0$, that is, a fundamental pair that satisfies the Codazzi equation. The importance of these kind of pairs for classifying constant mean curvature surfaces in product spaces was showed by J. A. Aledo, J. M. Espinar and J. A. Gálvez, among others.

This work is divided in three parts; in the first part, we will define a geometric Codazzi pair (I, II_{AR}), where II_{AR} is a symmetric $(2,0)$-tensor, that we call the Abresch-Rosenberg fundamental form, on any constant mean curvature surface in $\mathbb{E}(\kappa,\tau)$ whose $(2,0)$-part with respect to the conformal structure induced by I is the Abresch-Rosenberg differential. Moreover, we will exhibit some geometric properties of this pair.

In the second part, we will compute a Simons' type formula for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ using the traceless self-adjoint operator S associated to II_{AR}. As applications of this Simons' formula, first, we study the behavior of complete constant mean curvature surfaces Σ with finite Abresch-Rosenberg total curvature immersed in $\mathbb{E}(\kappa,\tau)$, i.e., those that the L^2-norm of S is finite. Observe that complete $H-$surfaces $\Sigma \subset \mathbb{R}^3$ of finite total curvature, that is, those that the L^2-norm of its traceless second fundamental form is finite, are of capital importance on the comprehension of $H-$surfaces. In the case $H = 0$, Osserman's Theorem gives an impressive description of them. If Σ has constant nonzero mean curvature and finite total curvature, then it must be compact. In our case, if H is greater than a constant depending only on κ and τ, we extend the latter result. We also estimate the first eigenvalue of any Schrödinger Operator $L = \Delta + V$, V continuous, defined on $H-$surfaces with finite Abresch-Rosenberg total curvature. Finally, together with the Omori-Yau's Maximum Principle, we classify complete $H-$surfaces (not necessary with finite Abresch-Rosenberg total curvature) in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$.

Finally, in the third part, we will use the Codazzi pair (I, II_{AR}) to classify constant mean curvature immersed compact disks in $\mathbb{E}(\kappa,\tau)$. In particular, we will classify immersed compact disks that meets transversally an Abresch-Rosenberg surface in $\mathbb{E}(\kappa,\tau)$ along the boundary with constant angle. First, we will work when the boundary is a regular curve, after, we will study the case with piece-wise smooth boundary curve.

Keywords: Constant Mean Curvature Surfaces, Homogeneous Space, Codazzi Pair, Fi-

nite Total Curvature, Simons' Formula, Eigenvalue Estimate, Pinching Theorem, Immersed Compact Disk.

Resumo

Esta tese tem o propósito de estudar superfícies de curvatura media constante nos espaços homogêneos $\mathbb{E}(\kappa,\tau)$.

Uwe Abresch e Harold Rosenberg descobriram um diferencial quadrático holomorfo definido em qualquer superficie de curvatura de média constante em $\mathbb{E}(\kappa,\tau)$. Contudo, não existem pares de Codazzi associados a este diferencial quando $\tau \neq 0$, isto é, pares fundamentais de formas quadráticas que satisfazem a equação de Codazzi. A importância destes tipo de pares para classificar superfícies de curvatura média constante em espaços produtos foi mostrada principalmente por J.A. Aledo, J. M. Espinar e J. A. Gálvez.

Este trabalho está divido em três partes; na primeira parte, nós definiremos um par de Codazzi geométrico (I, II_{AR}), onde II_{AR} é um tensor, que nós chamaremos de forma fundamental de Abresch-Rosenberg. Ela estará definida sobre qualquer superfície de curvatura média constante em $\mathbb{E}(\kappa,\tau)$, tal que a parte $(2,0)$ do II_{AR} com respeito ao parâmetro conforme induzido pelo I é o diferencial de Abresch-Rosenberg. Além disso, nós mostraremos algumas propriedades geométricas importantes de este par.

Na segunda parte, nós calcularemos uma equação tipo Simons para superfícies de curvatura média constante em $\mathbb{E}(\kappa,\tau)$ usando o operador auto-adjunto sem traço S associado a II_{AR}. Como aplicações da formula de Simons, em primeiro lugar, estudaremos o comportamento de superfícies de curvatura média constante com curvatura total de Abresch-Rosenberg finita em $\mathbb{E}(\kappa,\tau)$; isto é, as superfícies em que a norma L^2 de S é finita. Observamos que as superfícies de curvatura total finita e com curvatura média constante, isto é, as superfícies que a norma L^2 da sua segunda forma fundamental sem traço é finita, são de muita importância para o entendimento das superfícies de curvatura média constante. No caso mínimo, o Teorema de Osserman dá uma descrição a elas. Se Σ tem curvatura média constante H não zero e tem curvatura total finita, então a superfície é compacta. Em nosso caso, se H é maior que uma constante dependendo só de κ e τ, nós estenderemos o resultado anterior. Além disso, nós estimaremos o primeiro auto-valor de qualquer operador de Schrödinger $L = \Delta + V$, V contínuo, definido em uma superfície de curvatura média constante com curvatura total de Abresch-Rosenberg finita. Finalmente, junto com o princípio do máximo de Omori-Yau, nós classificaremos superficies completas com curvatura média constante (não necessariamente com curvatura de Abresch-Rosenberg finita) em $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$.

Finalmente, na terceira parte, nós usaremos o par de Codazzi (I, II_{AR}) para classificar discos compactos com curvatura média constante imersos no $\mathbb{E}(\kappa,\tau)$. Em particular, nós

classificaremos discos compactos imersos que encontram transversalmente uma superfície de Abresch-Rosenberg do $\mathbb{E}(\kappa,\tau)$ ao longo do bordo com ângulo constante. Primeiro, nós faremos isto quando a fronteira é uma curva suave e depois nós estudaremos o caso quando a fronteira é uma curva suave por partes.

Palavras Chaves:: Superfícies de Curvatura Média Constante, Espaços Homogêneos, Par de Codazzi, Curvatura Total Finita, Fórmula de Simons, Estimativa de Auto-valor, Teorema tipo Pinching, Discos Compactos Imersos.

Contents

Introduction

The study of constant mean curvature surfaces (CMC) in $\mathbb{R}^3$ has been an important research field in differential geometry from the XIX century until today. In particular, the study of minimal surfaces, i.e., those of zero mean curvature, constitutes itself a fundamental area in the theory of surfaces of $\mathbb{R}^3$. Indeed, the second half of the XIX century is considered the first golden period of minimal surface theory in differential geometry due, fundamentally, to the remarkable works of Deluanay [21], Lagrange [36], Legendre [37], Riemann [54], Scherk [57] or Schwarz among others. These authors constructed a variety of different minimal and constant mean curvature surfaces in $\mathbb{R}^3$ however, until the first half of the XX century, the sphere was the only known example of a closed and embedded non zero constant mean curvature surface in $\mathbb{R}^3$.

Around 1950, A.D. Alexandrov [4] and Heinz Hopf [34] gave the most important classification results of CMC surfaces in $\mathbb{R}^3$. Namely:

1. **Alexandrov:** The round sphere is the only compact, closed, embedded CMC surface in $\mathbb{R}^3$.

2. **Hopf:** The round sphere is the only closed, genus zero, immersed CMC surface in $\mathbb{R}^3$.

Eberhard Hopf (cf. [52]) extended the Maximum Principle for harmonic functions, that is, a harmonic function can not have an interior maximum unless it is constant, to more general elliptic partial differential equations. This was a fundamental observation that led A.D. Alexandrov to prove the above theorem. Alexandrov's idea was to compare the original surface with its reflection through planes, today, such a method is known as the *Alexandrov Reflection Method*. Recall that reflections with respect to totally geodesic planes are isometries in $\mathbb{R}^3$ and therefore, the reflection preserves the constancy of the mean curvature.

Heinz Hopf in [34] gave two proofs of his theorem, and both used the fact that any surface admits (locally) a conformal parametrization. In the first proof, Hopf defined a quadratic differential, today known as the Hopf differential, with respect to the conformal parametrization. Such differential encodes important geometric information, the most remarkable is that the zeroes of the Hopf differential coincide with the umbilical points of the surface; the Hopf differential is the $(2,0)$-part of the second fundamental form with respect to the conformal structure given by the first fundamental form. Then, he observed that the Hopf differential is holomorphic if, and only if, the mean curvature of the surface is constant; this equivalence is given by the Codazzi equations that must satisfy any immersed surface in $\mathbb{R}^3$. Therefore, taking into account that in any Riemann surface with genus zero the only holomorphic quadratic differential is the trivial one, the Hopf differential must vanish identically and hence, all points of the surface are umbilical. The classification of umbilical surfaces in $\mathbb{R}^3$ and the hypothesis that the mean curvature is non zero yield the result. The second proof is based on the existence of line fields on any CMC surface with isolated singularities, such singularities coincide with the umbilical points, whose index is negative. The line fields are defined using the Hopf differential. Thus, the Poincaré-Hopf Index Theorem asserts that such field can not exists on a topological sphere, which implies that the Hopf differential must vanish identically.

The existence of a holomorphic quadratic differential on a CMC surface is a powerful tool. For example, this was used by J.C.C Nistche [47] to study free boundary immersed compact CMC disks in the Euclidean unit ball. He showed that the boundary condition implies that the imaginary part of the Hopf differential vanishes along the boundary. Geometrically, this means that the boundary curve of the CMC disk is a line of curvature. Then, we can use complex analysis techniques to infer that the Hopf differential must vanish identically on the CMC disk, therefore, it must be a totally umbilical disk.

After, J. Choe [13] extended Nistche's Theorem when the boundary is smooth except for a finite number of points, the vertices, and conditions on the angles of the vertices. Other generalization has been made by R. Schoen and A. Fraser [30], they extended the Nitsche Theorem for free boundary immersed two-dimensional disks in $\mathbb{B}^n$ defining a complex quartic differential obtained by squaring the Hopf differential.

The underlying idea for the construction of the Hopf differential on surfaces relies on an abstract structure, the *Codazzi pairs*, that is, pairs of quadratic symmetric forms that satisfy the Codazzi equation. For example, in the above situation for CMC surfaces in $\mathbb{R}^3$, the Codazzi pair consists on the first and second fundamental forms of the surface. Under some geometrical conditions, a Codazzi pair allows us to define a holomorphic quadratic differential on the surface that can be used to classify those surfaces under some topological condition. Observe that in any Space Form, $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, the Codazzi equation has the same structure and we can extend the Hopf Theorem to constant mean curvature topological spheres in any Space Form.

The Codazzi equation for an immersed surface Σ in $\mathbb{R}^3$ is given by

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = 0, \qquad X,Y \in \mathfrak{X}(\Sigma), \tag{1}$$

where ∇ is the Levi-Civita connection of the first fundamental form I of Σ, A is the shape operator associated to second fundamental form II, defined by $II(X,Y) = I(A(X),Y)$. This Codazzi equation is, together with the Gauss equation, one of the two classical integrability conditions for surfaces in $\mathbb{R}^3$, and it remains true if we substitute the ambient space $\mathbb{R}^3$ by any other Space Form.

So, the above facts about the Codazzi equation led mathematicians to question which results on surface theory are valid without the Gauss equation; that is, if Σ is a Riemannian surface and we take a pair of quadratic symmetric forms (I,II), with shape operator S, i.e., S is the self-adjoint operator such that $II(X,Y) = I(SX,Y)$, where I is definite positive on

Σ and we suppose that S satisfies the Codazzi equation (1). The question is then, can we classify the pair (I, II) under certain topological conditions on Σ and geometric conditions on the pair? This is how a new research line in differential geometry started, called the theory of Codazzi pairs.

The theory of Codazzi pairs has yielded important results and interesting consequences that generalize theorems from surface theory and apply to others branches of mathematics such as Partial Differential Equations. T.K. Milnor (cf. [40] to [46]), V. Oliker [48] and U. Simon [58] developed this theory and generalized many results that depend only on the Codazzi equation. For example, Liebmann's Theorem, i.e., the only complete surface in $\mathbb{R}^3$ with constant positive Gaussian curvature is the round sphere, and Hopf's Theorem, among others, can be extended to the abstract setting of Codazzi pairs.

In 1968, J. Simons [59] computed the Laplacian of the norm squared of the shape operator A of a minimal surface Σ in $\mathbb{S}^3$;

$$\frac{1}{2}\Delta_\Sigma(|A|^2) = |\nabla A|^2 + |A|^2\,(2 - |A|^2).$$

The above formula is known as a **Simons" formula** of A and it is a useful tool to obtain classification results for minimal surfaces in the sphere. By its importance in the theory of surfaces in $\mathbb{S}^3$, this formula has been extensively generalized by many authors in different situations (cf. [7, 15, 16, 28]).

An important issue to compute the Simons' formula of A is that the fundamental pair composed by the first and the second fundamental form of the minimal surface is a Codazzi pair. In fact; Shiu Cheng and Shing-Tung Yau [14] computed an abstract version of the Simons' formula, that is, they considered a Codazzi pair (I, II) defined on Riemannian surface and computed the Laplacian of the norm squared of II:

$$\frac{1}{2}\Delta(|II|^2) = \sum_{i,j,k}(\phi_{ij,k})^2 + \sum_i \lambda_i(\mathrm{tr}(II))_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2,$$

where $II = \sum_{i,j}\phi_{ij}\,\omega_i \otimes \omega_j$ is the local expression of II in a coframe base $\{\omega_1, \ldots, \omega_n\}$ associated to a local orthonormal frame $\{e_1, \ldots, e_n\}$ of I, R_{ijij} is the Curvature Tensor associated to the metric I, $\mathrm{tr}(II)$ is the trace of II respect to I and λ_i, $i = 1, \ldots, n$ are the eigenvalues of II respect to the orthonormal frame.

As seen so far, the existence of a holomorphic quadratic differential on CMC surfaces in $\mathbb{R}^3$ (or any other Space Form) allows us to understand the geometry of such surfaces. Another underlying geometric property about the ambient space for classifying constant mean curvature surfaces is its isometry group. A Riemannian manifold whose isometry group acts transitively on its points is called *Homogeneous*, for example, the simply connected Space Forms are examples of these manifolds.

W. P. Thurston proved that the building blocks of the Geometrization Conjecture are given by eight maximal model geometries, in other words, a maximal model geometry is a simply connected smooth manifold $\mathscr{M}$ together with a transitive action of the Lie group G on $\mathscr{M}$, which is maximal among groups acting smoothly and transitively on $\mathscr{M}$, with compact stabilizers. Such maximal model geometries can be classified according to the dimension of its isometry group. If the dimension is 6, they correspond to the Space Forms $\mathbb{M}^3(\kappa)$. When the dimension is 3, the manifold has the geometry of the Lie group Sol_3, and when the dimension is 4, they correspond to a 2-parameter family $\kappa, \tau \in \mathbb{R}$, $\kappa - 4\tau^2 \neq 0$, of manifolds denoted by $\mathbb{E}(\kappa, \tau)$. These manifolds correspond to the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$, when $\kappa \neq 0, \tau = 0$, where $\mathbb{M}^2(\kappa)$ is the simply connected surface of constant curvature κ. The Heisenberg space Nil_3, when $\kappa = 0, \tau \neq 0$. The covering space of the linear group $\mathrm{PSl}_2(\mathbb{R})$, when $\kappa < 0, \tau \neq 0$. The Berger sphere $\mathbb{S}^3_B(\kappa, \tau)$, when $\kappa > 0, \tau \neq 0$. It is known that $\mathbb{E}(\kappa, \tau)$ is a Riemannian submersion over $\mathbb{M}^2(\kappa)$ with fiber bundle curvature τ and the fibers are integral curves of a unit Killing field defined in $\mathbb{E}(\kappa, \tau)$.

Constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ has been an active field in differential geometry in the last years. U. Abresch and H. Rosenberg [1, 2] showed the existence of a holomorphic quadratic differential, the Abresch-Rosenberg differential $\mathcal{Q}^{AR}$, on any constant mean curvature surface. Then, they extended Hopf's Theorem and classified the topological spheres with constant mean curvature as the rotationally symmetric surfaces in these spaces. Furthermore, they classified the complete CMC surfaces in $\mathbb{E}(\kappa, \tau)$ such that the Abresch-Rosenberg differential vanishes identically, called Abresch-Rosenberg surfaces, in particular, they are invariant surfaces by a one parameter group of isometries of $\mathbb{E}(\kappa, \tau)$ (cf. [24]).

As we pointed out above, when a surface is immersed in a Space Form, it is well-known that its second fundamental form defines a bilinear symmetric tensor that satisfies the Codazzi equation and the Codazzi equation is fundamental to ensure that the usual Hopf differential is holomorphic on any CMC surface in a Space Form; nevertheless, the above fact is no longer true when the surface is isometrically immersed in a homogeneous 3-manifold $\mathbb{E}(\kappa, \tau)$. However, J.A. Aledo, J.M. Espinar and J.A. Gálvez [3] obtained a geometric Codazzi pair (I, II_S) for any CMC surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ so that the $(2,0)$-part of II_S with respect to a conformal parameter given by the first fundamental I, is the Abresch-Rosenberg differential. As it was observed by S. Cheng and S.T. Yau, the properties of being a Codazzi pair allows us to compute a Simons' type formula of the shape operator S associated to II_S. In fact; this property of (I, II_S) was used by M. Batista [7] to compute the Simons' formula for CMC surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ and, in this way, he classified certain complete CMC surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Moreover, M. Do Carmo and I. Fernández [19] used the properties of the Codazzi pair (I, II_S) to classify certain immersed compact disks in product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

But, despite the existence of a holomorphic quadratic differential on any CMC surface in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, there was no geometric Codazzi pair (I, II_S) defined on the surface such that the $(2,0)$-part of II_S with respect to a conformal parameter induced by I, were the Abresch-Rosenberg differential.

Now, we proceed to describe the contents of this work.

In Chapter 1, we introduce the basic concepts on Riemannian Manifolds and Hypersurface Theory. Next, we continue by defining the most important concepts of the abstract theory of Codazzi pairs on Riemannian surfaces. We finalize this chapter by describing the geometric properties of the Riemaniann homogeneous manifolds $\mathbb{E}(\kappa,\tau)$, and we list the structure equations that must satisfy any constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$.

In Chapter 2, we will study constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$, for this, we begin by defining the Abresch-Rosenberg differential on any constant mean curvature surface, this definition will take us to the Codazzi pair interpretation of the Abresch-Rosenberg differential and its geometric properties. First, we discuss the known case of constant mean curvature surfaces in a product space $\mathbb{M}^2(\kappa)\times\mathbb{R}$. Later, we obtain a geometric Codazzi pair associated to the Abresch-Rosenberg differential on any constant mean curvature surface immersed in $\mathbb{E}(\kappa,\tau)$. Specifically, Lemma 2.2 says

> **Lema 2.2.** *Given a $H-$surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$, consider the symmetric $(2,0)-$tensor given by*
>
> $$II_{AR}(X,Y)=II(X,Y)-\alpha\langle\mathbf{T}_\theta,X\rangle\langle\mathbf{T}_\theta,Y\rangle+\frac{\alpha\,|\mathbf{T}|^2}{2}\langle X,Y\rangle,$$
>
> *where*
>
> - $\alpha=\dfrac{\kappa-4\tau^2}{2\sqrt{H^2+\tau^2}}$,
> - $e^{2i\theta}=\dfrac{H-i\tau}{\sqrt{H^2+\tau^2}}$ *and*
> - $\mathbf{T}_\theta=\cos\theta\,\mathbf{T}+\sin\theta\,J\mathbf{T}$.
>
> *Then, (I,II_{AR}) is a Codazzi Pair with constant mean curvature H. Moreover, the $(2,0)-$part of II_{AR} with respect to the conformal structure given by I agrees (up to a constant) with the Abresch-Rosenberg differential.*

We will continue this chapter with some classification results of constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ and the remainder of this chapter will be devoted to the Abresch-Rosenberg surfaces in $\mathbb{E}(\kappa,\tau)$, that is, the constant mean curvature surfaces whose the Abresch-Rosenberg differential vanishes identically.

In Chapter 3, we compute the Simons' formula for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$. To do so, we use the Codazzi pair (I,II_{AR}) defined in Chapter 2. Hence, we will obtain a formula for the Laplacian of the square norm of the Abresch-Rosenberg fundamental form:

> **Theorem 3.2.** *Let Σ be a $H-$surface in $\mathbb{E}(\kappa,\tau)$. Then, the traceless Abresch-Rosenberg shape operator satisfies*
>
> $$\frac{1}{2}\Delta|S|^2=|\nabla S|^2+2K|S|^2,$$

or, equivalently, away from the zeroes of $|S|$,

$$|S|\,\Delta|S| - 2K\,|S|^2 = |\nabla\,|S||^2 .$$

We are now in position to do some applications to the theory of constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$. First, we will study constant mean surfaces in $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$, with finite Abresch-Rosenberg total curvature, i.e.,

$$\int_\Sigma |S|^2\,dv_g < \infty.$$

We must point out here that the family of complete constant mean curvature surfaces with finite Abresch-Rosenberg total curvature is large. We focus on $H=1/2$ surfaces in $\mathbb{H}^2\times\mathbb{R}$ to show this fact. Recall the following result of Fernández-Mira:

> **Theorem [26, Theorem 16].** *Any holomorphic quadratic differential on an open simply connected Riemann surface is the Abresch-Rosenberg differential of some complete surface* Σ *with* $H=1/2$ *in* $\mathbb{H}^2\times\mathbb{R}$*. Moreover, the space of noncongruent complete mean curvature one half surfaces in* $\mathbb{H}^2\times\mathbb{R}$ *with the same Abresch-Rosenberg differential is generically infinite.*

We will see that, if we take the disk $\mathbb{D}$ as our open Riemann surface and a holomorphic quadratic differential on $\mathbb{D}$ that extends continuously to the boundary, then the $H=1/2$ surface Σ constructed in [26, Theorem 16] has finite Abresch-Rosenberg total curvature.

Then using the Simons' formula and the Sobolev inequality, we can show:

> **Theorem 3.3.** *Let* $\Sigma\subset\mathbb{E}(\kappa,\tau)$ *be a complete* $H-$*surface such that* $H^2+\tau^2\neq 0$*. If* Σ *has finite Abresch-Rosenberg total curvature, that is,*
>
> $$\int_\Sigma |S|^2\,dv_g < +\infty,$$
>
> *then* $|S|$ *goes to zero uniformly at infinity.*

Despite what happens in $\mathbb{R}^3$, a $H-$surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$ with finite Abresch-Rosenberg total curvature is not necessarily conformally equivalent to a compact surface minus a finite number of points, in particular, Σ is not necessarily parabolic. The simplest case is a slice $\mathbb{H}^2$ in $\mathbb{H}^2\times\mathbb{R}$. However, we can obtain:

> **Theorem 3.4.** *Let* Σ *be a complete surface in* $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$*, with finite Abresch-Rosenberg total curvature. Suppose one of the following conditions holds*
>
> 1. $\kappa-4\tau^2>0$ *and* $H^2+\tau^2>\frac{\kappa-4\tau^2}{4}$.
> 2. $\kappa-4\tau^2<0$ *and* $H^2+\tau^2>-\frac{(\sqrt{5}+2)}{4}(\kappa-4\tau^2)$.

Then, Σ must be compact.

As a second application of the Simons' formula, we extend the Simons' first stability eigenvalue estimate given in [59] for compact minimal surfaces in $\mathbb{S}^3$ to Schrödinger Operators $L = \Delta + V$ defined on a complete $H-$surface with finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 \neq 0$ immersed in $\mathbb{E}(\kappa, \tau)$,

Theorem 3.5. *Let Σ be a complete H-surface in $\mathbb{E}(\kappa, \tau)$ with finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 \neq 0$. Let $\lambda_1(L)$ be the first eigenvalue associated to the Schrödinger operator $L := \Delta + V$, $V \in C^0(\Sigma)$. Then, Σ is either invariant by a one a parameter group of isometries of $\mathbb{E}(\kappa, \tau)$, or a Hopf cylinder or*

$$\lambda_1(L) < -\inf_\Sigma \{V + 2K\},$$

where K is the Gaussian curvature of Σ.

Remind that $\mathbb{E}(\kappa, \tau)$ is a Riemannian submersion $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$. Given γ a regular curve in $\mathbb{M}^2(\kappa)$, $\pi^{-1}(\gamma)$ is a surface in $\mathbb{E}(\kappa, \tau)$ that has ξ as a tangent vector field, in this case $\nu = 0$. So, ξ is a parallel vector field along $\pi^{-1}(\gamma)$ and hence $\pi^{-1}(\gamma)$ is a flat surface and its mean curvature is given by $2H = k_g$, where k_g is the geodesic curvature of γ in $\mathbb{M}^2(\kappa)$ (cf. [25]). We will call $\pi^{-1}(\gamma)$ a **Hopf cylinder** of $\mathbb{E}(\kappa, \tau)$ over curve γ. If γ is a closed curve, $\pi^{-1}(\gamma)$ is a flat **Hopf cylinder** and additionally, if π is a circle Riemannian submersion, $\pi^{-1}(\gamma)$ is a **Hopf torus**.

In particular, when L is the Stability (or Jacobi) operator defined on a complete $H-$surface with finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 \neq 0$ immersed in $\mathbb{E}(\kappa, \tau)$, i.e.,

$$L = \Delta + (|A|^2 + \mathrm{Ric}(N)),$$

where, $\mathrm{Ric}(N)$ is the Ricci curvature of the ambient manifold in the normal direction. We obtain the following:

Theorem 3.6. *Let Σ be a complete two sided $H-$surface with finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 \neq 0$ in $\mathbb{E}(\kappa, \tau)$.*

- *If $\kappa - 4\tau^2 > 0$. Then, Σ is either invariant by a one parameter group of isometries of $\mathbb{E}(\kappa, \tau)$, or a Hopf cylinder, or*

$$\lambda_1 < -(4H^2 + \kappa).$$

- *If $\kappa - 4\tau^2 < 0$. Then, Σ is either invariant by a one parameter group of isometries of $\mathbb{E}(\kappa, \tau)$, or a Hopf cylinder, or*

$$\lambda_1 < -(4H^2 + \kappa) - (\kappa - 4\tau^2).$$

Finally, we apply Simons' formula for classifying complete constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$ under natural geometric conditions using the Omori-Yau's Maximum Principle. We can summarize Theorem 3.8 and Theorem 3.9 as follows:

Theorems 3.8 and 3.9.

Let Σ be a complete immersed $H-$surface in $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2 \neq 0$.

- *If $\kappa - 4\tau^2 > 0$, assume that $4(H^2+\tau^2) > \kappa - 4\tau^2$ and*

$$\sup_{\Sigma} |S| < \frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}}.$$

where S is the traceless Abresch-Rosenberg shape operator. Then, Σ is an Abresch-Rosenberg surface in $\mathbb{E}(\kappa,\tau)$. Moreover, if

$$\sup_{\Sigma} |S| = \frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}}.$$

and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S|$, then Σ is a Hopf cylinder.

- *If $\kappa - 4\tau^2 < 0$, assume that $(H^2+\tau^2) > \left|\kappa - 4\tau^2\right|$ and*

$$\sup_{\Sigma} |S| < \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)}$$

where S is the traceless Abresch-Rosenberg shape operator. Then, Σ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa,\tau)$.

Moreover, if

$$\sup_{\Sigma} |S| = \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)}$$

and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S|$, then Σ is a Hopf cylinder.

In Chapter 4, we will study constant mean curvature immersed compact disks ($H-$disks) in $\mathbb{E}(\kappa,\tau)$ using the Codazzi pair (I, II_{AR}). We first study AR-lines of curvature of the $H-$surface Σ in $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2 \neq 0$, that is, curves $\Gamma = \gamma(-\varepsilon,\varepsilon)$ on Σ that satisfies the following condition:

$$S_{AR}(\gamma'(t)) = \lambda(t)\gamma'(t) \text{ for some smooth function } \lambda : (-\varepsilon,\varepsilon) \to \mathbb{R}.$$

Then, as our main result in this chapter is a Joachimstahl's type Theorem for constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$:

Lemma 4.1. *Let $\Sigma_i \subset \mathbb{E}(\kappa,\tau)$, $i=1,2$, be H_i-surfaces so that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Let $\Gamma \subset \Sigma_1 \cap \Sigma_2$ be a regular curve of transversal intersection. Assume that along Γ one has*

a) $\langle N_1, N_2 \rangle$ is constant and *b)* $\sqrt{H_1^2+\tau^2}\langle \mathbf{T}^2_{\theta_2}, N_1\rangle\langle J_2\mathbf{T}^2_{\theta_2}, N_1\rangle = \sqrt{H_2^2+\tau^2}\langle \mathbf{T}^1_{\theta_1}, N_2\rangle\langle J_1\mathbf{T}^1_{\theta_1}, N_2\rangle,$

where $\alpha_i = \frac{\kappa-4\tau^2}{2\sqrt{H_i^2+\tau^2}}$, $\mathbf{T}^i_{\theta_i} = \cos\theta_i\mathbf{T}_i + \sin\theta_i J_i\mathbf{T}_i$ *and* $J_iX = N_i \wedge X$ *for* $i = 1,2$. *Then,* Γ *is an AR-line of curvature for* Σ_1 *if, and only if,* Γ *is an AR-line of curvature for* Σ_2.

With this preliminary result, we can now classify immersed compact disks in $\mathbb{E}(\kappa,\tau)$ with regular boundary. For immersions in product spaces, we have:

Theorem 4.1. *Let* $\phi : \overline{\mathbb{D}} \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ *be a non-minimal* H_1-*disk with regular boundary* Γ. *Suppose that* ϕ *meets transversally an Abresch-Rosenberg* H_2*-surface* Ω *along* Γ *at a constant angle. Assume also that* Γ *is of one of the following types:*

1. Γ *is an horizontal or vertical curve of* Ω
2. *If* $H_1 = H_2$*, the angle function* ν_1 *is opposite to the angle function* ν_2.

Then, $\phi(\overline{\mathbb{D}})$ *is a part of an Abresch-Rosenberg surface in* $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

We continue by obtaining a classification result for immersed compact disks in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$.

Theorem 4.2. *Let* $\phi : \mathbb{D} \to \mathbb{E}(\kappa,\tau)$*,* $\tau \neq 0$*, be a* H_1-*disk with regular boundary, suppose the boundary is parametrized by a regular curve* γ *and it is of one of the following types*

1. γ *is the tangent intersection of the immersion* ϕ *with an Abresch-Rosenberg surface* Ω *with the same mean curvature vector.*
2. γ *is the transverse intersection with constant angle of the immersion* ϕ *with an Abresch-Rosenberg surface* Ω *with the same mean curvature and whose angle function is opposite to the angle function of the immersion* ϕ *along* γ.

Then, $\phi(\mathbb{D})$ *is a part of an Abresch-Rosenberg surface in* $\mathbb{E}(\kappa,\tau)$.

We remark that the above theorem extends a previous result given by M. Do Carmo and I. Fernández in [19]. Finally, we extend Theorem 4.1 and Theorem 4.2, when the boundary is a piece-wise regular curve. First, we will do this for H- disks in product spaces.

Corollary 4.3. *Let* $\phi : \mathbb{D} \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ *be a* H_1- *disk, with* $H_1 \neq 0$ *and piece-wise differentiable boundary* Γ. *Assume also that the following conditions are satisfied:*

1. $\phi(\mathbb{D})$ *is contained as an interior set in a smooth* H_1- *surface* $\hat{\Sigma}$ *in* $\mathbb{E}(\kappa,\tau)$ *without boundary.*

2. The number of vertices in Γ with angle $< \pi$ is less than or equal to 3.

3. Every regular component γ of Γ is a one of the following types:

- *γ is contained in a horizontal slice.*
- *γ is a transverse intersection with constant angle of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω of constant mean curvature $H_2 \neq 0$.*

Then, $\phi(\mathbb{D})$ is a part of an Abresch-Rosenberg surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

Finally, we consider the case $\tau \neq 0$.

Corollary 4.4. *Let $\phi : \mathbb{D} \to \mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, be a H_1- disk with piece-wise differentiable boundary Γ. Assume also that the following conditions are satisfied:*

1. $\phi(\mathbb{D})$ is contained as an interior set in a smooth H_1-surface $\hat{\Sigma}$ on $\mathbb{E}(\kappa, \tau)$ without boundary.

2. The number of vertices in Γ with angle $< \pi$ is less than or equal to 3.

3. Every regular component γ of Γ is one of the following types:

- *γ is a tangent intersection of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω with the same mean curvature vector.*
- *γ is a transverse intersection with constant angle of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω with the same constant mean curvature and whose angle function is opposite to the angle function of $\phi(\mathbb{D})$ along γ.*

Then, $\phi(\mathbb{D})$ is a part of an Abrech-Rosenberg surface in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$.

Chapter 1

Preliminaries

In this chapter we fix the notation, give the definitions and state some results which will be use through this work. In Section 1.1, we will list the usual definitions of Riemannian geometry and the theory of hypersurfaces in Riemannian manifolds. In Section 1.2, we recover the abstract theory of Codazzi pairs on surfaces and we state the abstract version of Hopf Theorem. In Section 1.3, we will resume the basic facts of homogeneous 3-manifolds with isometry group of dimension 4, $\mathbb{E}(\kappa,\tau)$, and some results about immersed surfaces in these manifolds (see [18, 20, 40, 56, 61]).

1.1 Basics on Riemannian Geometry

Let $\mathscr{M}$ be a smooth oriented manifold. Now, we assume that the manifold $\mathscr{M}$ is provided with a metric, this means a symmetric and positive definite $(2,0)$-tensor, denoted by g. So, for each $p \in \mathscr{M}$, one has $g_p : T_p\mathscr{M} \times T_p\mathscr{M} \to \mathbb{R}$. For an arbitrary local chart (U,φ) at $p \in U$, g_p can be written as

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j,$$

where $g_{ij} \in C^\infty(U)$ so that $g_{ij} = g_{ji}$ and $\otimes$ is the tensorial product. Hence, the pair $(\mathscr{M},g)$, a manifold $\mathscr{M}$ provided with a metric g, is called a **Riemannian manifold**.

The functions g_{ij}, $i,j = 1,\dots,n$, are the coefficients of the metric g in the local chart (U,φ).

For a Riemannian manifold $(\mathscr{M},g)$, it is not necessary to distinguish between $\mathfrak{X}(\mathscr{M})$ and $\mathfrak{X}(\mathscr{M})^*$. In fact, we can identify each element $X \in \mathfrak{X}(\mathscr{M})$ with an unique 1-form $\omega \in \mathfrak{X}(\mathscr{M})^*$ using the equality

$$\omega(Y) = g(Y,X),\ \forall Y \in \mathfrak{X}(\mathscr{M})\,. \tag{1.1}$$

We will use the word **tensor** when valued in $\mathfrak{X}(\mathscr{M})$ and **form** when valued in $C^\infty(\mathscr{M})$. We will denote the **Lie bracket** of the vector fields in $\mathfrak{X}(\mathscr{M})$ by $[\ ,\ ]$, that is,

$$[X,Y] = XY - YX \text{ for all } X,Y \in \mathfrak{X}(\mathscr{M}).$$

Given a Riemannian manifold $(\mathscr{M},g)$, there exists a unique affine connection $\overline{\nabla}$ such that

(i) $\overline{\nabla}$ is symmetric, i.e.,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y];$$

(ii) $\overline{\nabla}$ is compatible with g, i.e.,

$$Xg(Y,Z) = g(\overline{\nabla}_X Y,Z) + g(Y,\overline{\nabla}_X Z),$$

for all $X,Y,Z \in \mathfrak{X}(\mathscr{M})$. This leads us to define such unique connection $\overline{\nabla}$ as the **Levi-Civita connection** on $\mathscr{M}$ associated to g.

Let $\overline{\nabla}$ be the Levi-Civita connection associated to a Riemannian metric g and $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ be the basis associated to a local chart $(U,\varphi \equiv (x_1,\ldots,x_n))$. Consider the functions $\Gamma_{ij}^k \in C^\infty(U)$, $i,j,k = 1,,\ldots,n$, given by the relations

$$\overline{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tag{1.2}$$

then, the coefficients Γ_{ij}^k are called the **Christoffel symbols** of the connection $\overline{\nabla}$ on U associated to the metric g.

Associated to the Levi-Civita connection $\overline{\nabla}$ on a Riemannian manifold $(\mathscr{M},g)$, we introduce the **Curvature Tensor** $\overline{R}$ defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]} Z. \tag{1.3}$$

It is well-known that $\overline{R}$ is $C^\infty(\Sigma)$-linear with respect to X,Y,Z and skew-symmetric with respect to X,Y.

Definition 1.1. *Let $(\mathscr{M},g)$ be a Riemannian manifold with Curvature Tensor $\overline{R}$. Given $X_p,Y_p \in T_p\mathscr{M}$ linearly independent, we define the* ***sectional curvature****, $\overline{K}_p(X_p,Y_p)$, related to g at $p \in \mathscr{M}$ for the plane generated by $\left\{X_p,Y_p\right\}$ is given by*

$$\overline{K}_p(X_p,Y_p) = \frac{g(\overline{R}(X_p,Y_p)X_p,Y_p)}{\|X_p \wedge Y_p\|^2}, \tag{1.4}$$

where

$$\|X_p \wedge Y_p\| = \sqrt{\|X_p\|^2\|Y_p\|^2 - g(X_p,Y_p)^2}\,.$$

The definition of $\overline{K}_p(X_p,Y_p)$ does not depend on the choice of the vectors X_p, Y_p, just on the plane generated by them. Moreover, the curvature tensor $\overline{R}$ is completely determined by the sectional curvature when $\overline{K}$ is constant at every point and any plane, and we can recovered it as

$$\overline{R}(X,Y)Z = \overline{K}\left(g(X,Z)Y - g(Y,Z)X\right). \tag{1.5}$$

We will see now other curvature tensors one can define in a Riemanniann manifold $(\mathscr{M},g)$.

Ricci and Scalar curvature

Let $\{e_i\} \in \mathfrak{X}(U)$, $U \subset \mathscr{M}$ open and connected, be a local orthonormal frame of the tangent bundle $TU \subset T\mathscr{M}$. Let us establish our definition for the **Ricci Curvature** and **Scalar Curvature** in $\mathscr{M}$, i.e,

$$\begin{aligned}\overline{\mathrm{Ric}}_g(X,X) &= \sum_{i=1}^{n} \overline{R}(X,e_i,X,e_i),\\ \overline{R}(g) &= \sum_{i=1}^{n} \overline{\mathrm{Ric}}_g(e_i,e_i),\end{aligned}$$

respectively, here $X \in \mathfrak{X}(\mathscr{M})$.

1.1.1 Hypersurfaces Theory

Here, we will remind the most important concepts on hypersurface theory. Along these section we denote by $(\mathscr{M},\langle,\rangle)$ a $(m+1)$-dimensional connected Riemannian manifold, and let $\Sigma \subset \mathscr{M}$ be an immersed, two-sided hypersurface in $\mathscr{M}$. Let us denote by N the unit normal vector field along Σ. Moreover, $\langle,\rangle$ is the metric on $\mathscr{M}$ and g the **first fundamental form of** Σ, that is, the induced metric on Σ by $\langle,\rangle$. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection associated to $\langle,\rangle$ and g, respectively. Denote by $\mathfrak{X}(\Sigma)$ and $\mathfrak{X}(\mathscr{M})$ the linear spaces of smooth vector fields along Σ and $\mathscr{M}$ respectively. We also denote by I the induced metric on Σ, that is, $I \equiv g$.

Remark 1.1. *We will identify I, g and $\langle,\rangle$ when no confusion occurs.*

From (1.3), set

$$\overline{R}(X,Y)Z := \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_X\overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]}Z,\ X,Y,Z \in \mathfrak{X}(\mathscr{M}),$$

the Riemann Curvature Tensor of $\mathscr{M}$. Let $\{e_i\}_1^{m+1} \in \mathfrak{X}(U)$, $U \subset \mathscr{M}$ open and connected, be a local orthonormal frame of the tangent bundle $TU \subset T\mathscr{M}$, then we denote

$$\overline{R}_{ijkl} = \langle \overline{R}(e_i,e_j)e_k,e_l\rangle$$

and, from Definition 1.1, the sectional curvatures in $\mathscr{M}$ are given by

$$\overline{K}_{ij} := \langle \overline{R}(e_i,e_j)e_i,e_j\rangle = \overline{R}_{ijij}.$$

The **Gauss Formula** (see [18]) of a hypersurface Σ is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle A(X),Y\rangle N \text{ for all } X,Y \in \mathfrak{X}(\Sigma),$$

where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the **Weingarten (or Shape) operator** defined as

$$A(X) := -(\overline{\nabla}_X N)^T,$$

that is, $A(X)$ is the tangential component of $-\overline{\nabla}_X\vec{N}$. In fact, we do not need to take the tangential part in the above definition when we are dealing with orientable hypersurfaces in orientable manifolds, but we use the general definition for the sake of completeness.

Since $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a self-adjoint endomorphism, we denote the **mean curvature** and **extrinsic curvature** as

$$H = \frac{1}{m}\mathrm{Tr}(A) \quad \text{and} \quad K_e = \det(A),$$

where Tr and det denote the trace and determinant respectively.

Let $\mathfrak{X}(\Sigma)^\perp$ be the orthogonal complement of $\mathfrak{X}(\Sigma)$ in $\mathfrak{X}(\mathscr{M})$. Let us denote $B : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)^\perp$ the **Vector second fundamental form** of Σ, that is,

$$B(X,Y) := (\overline{\nabla}_X Y)^\perp,\ X,Y \in \mathfrak{X}(\Sigma),$$

here $(\cdot)^\perp$ means the normal part. Therefore, B induces the self-adjoint endomorphism A on Σ, that is,

$$\langle B(X,Y),N\rangle = \langle A(X),Y\rangle,\ X,Y \in \mathfrak{X}(\Sigma),$$

which is called the **second fundamental form**, and we also write it as

$$II(X,Y) = I\,(A(X),Y)\,,\ X,Y \in X.$$

The **mean curvature vector** of Σ is given by

$$m\vec{H}_p = \mathrm{Tr}\,(B_p) = \sum_{i=1}^{m} B_p(v_i,v_i),$$

where $\{v_1,\dots,v_m\}$ is a orthonormal basis of $T_p\Sigma$.

Since $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is self-adjoint, it is diagonalizable and hence let $\{e_1,\dots,e_m\}$ be **principal directions**, i.e.,

$$A(e_i) = -\overline{\nabla}_{e_i}N = \kappa_i e_i,$$

where κ_i are the **principal curvatures**, $i = 1,\dots,m$, in other words, $\{e_1,\dots,e_m\}$ are the eigendirections of S and κ_i, $i = 1,\dots,m$, its eigenvalues.

We say that a point $p \in \Sigma$ is an **umbilic point** if $\kappa_1(p) = \ldots = \kappa_m(p)$, which is equivalent to say that II is proportional to I at p.

Let $\overline{R}$ and R denote the Riemann Curvature tensors of $\mathscr{M}$ and Σ respectively. Then, by the **Gauss Equation** we can relate $\overline{R}$ and R as

$$\overline{R}(X,Y)Z = R(X,Y)Z + \langle A(Y),Z\rangle A(X) - \langle A(X),Z\rangle A(Y) \text{ for all } X,Y,Z,W \in \mathfrak{X}(\Sigma). \quad (1.6)$$

There is another important equation that $\Sigma \subset \mathscr{M}$ must verify, the Codazzi Equation. Given $X,Y \in \mathfrak{X}(\Sigma)$, recall that $AX = -\overline{\nabla}_X N \in \mathfrak{X}(\Sigma)$, the Gauss formula yields

$$\begin{aligned}
\overline{R}(X,Y)N &= \overline{\nabla}_Y\overline{\nabla}_X N - \overline{\nabla}_Y\overline{\nabla}_X N + \overline{\nabla}_{[X,Y]}N \\
&= \overline{\nabla}_X AY - \overline{\nabla}_Y AX - A[X,Y] \\
&= \nabla_X AY - \nabla_Y AX - A[X,Y] - \langle AX,AY\rangle N + \langle AY,AX\rangle N \\
&= \nabla_X AY - \nabla_Y AX - A[X,Y],
\end{aligned}$$

that is, $\overline{R}(X,Y)N \in \mathfrak{X}(\Sigma)$ and the following **Codazzi Equation** holds

$$\overline{R}(X,Y)\vec{N} = \nabla_X AY - \nabla_Y AX - A[X,Y],\ X,Y \in \mathfrak{X}(\Sigma). \tag{1.7}$$

Assume that the ambient manifold is a Space Form $\mathscr{M} = \mathbb{M}^{m+1}(\kappa)$, $\kappa \in \mathbb{R}$, that is, a complete simply connected $(m+1)-$manifold of constant sectional curvature κ for every point $p \in \mathbb{M}^{m+1}(\kappa)$ and any tangent plane. So, by Cartan Theorem (see [18]), we have

$$\mathbb{M}^{m+1}(\kappa) = \begin{cases} \mathbb{S}^{m+1}(\kappa) & \text{if} \quad \kappa > 0, \\ \mathbb{R}^{m+1} & \text{if} \quad \kappa = 0, \\ \mathbb{H}^{m+1}(\kappa) & \text{if} \quad \kappa < 0. \end{cases}$$

Hence, from (1.5), the Riemann curvature tensor $\overline{R}$ can be recovered as

$$\overline{R}(X,Y)Z = \kappa(g(X,Z)Y - g(Y,Z)X), X,Y,Z \in \mathfrak{X}(\mathbb{M}^{m+1}(\kappa)).$$

Therefore, for surfaces in Σ in the Spaces Forms of dimension 3, the Gauss Equation becomes:

$$K = K_e + \kappa, \tag{1.8}$$

where K_e is the extrinsic curvature and the Codazzi equation remind as

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = 0,\ X,Y \in \mathfrak{X}(\Sigma), \tag{1.9}$$

respectively.

1.2 Codazzi Pairs

One important tool in this work is the theory of Codazzi pairs. In this section, we resume the basic concepts of this theory, we follow [3, 40] and references therein. We shall denote by Σ an orientable (and oriented) smooth surface. Otherwise we work with its oriented two-sheeted covering.

Definition 1.2 ([3, 40]). *A fundamental pair on Σ is a pair of real symmetric quadratic forms (I,II) on Σ, where I is a Riemannian metric.*

Associated with a fundamental pair (I,II) we define the shape operator S of the pair as:

$$II(X,Y) = I(S(X),Y) \text{ for any } X,Y \in \mathfrak{X}(\Sigma). \tag{1.10}$$

Conversely, it is clear from (1.10) that the quadratic form II is totally determined by I and S. In other words, to give a fundamental pair on Σ is equivalent to give a Riemannian metric on Σ together with a self-adjoint endomorphism S.

We define the **mean curvature**, the **extrinsic curvature** and the **principal curvatures** of (I,II) as one half of the trace, the determinant and the eigenvalues of the endomorphism S, respectively.

In particular, given local parameters (x,y) on Σ such that

$$I = E\,dx^2 + 2F\,dxdy + G\,dy^2, \qquad II = e\,dx^2 + 2f\,dxdy + g\,dy^2,$$

the mean curvature and the extrinsic curvature of the pair are given, respectively, by

$$H(I,II) = \frac{Eg+Ge-2Ff}{2(EG-F^2)}, \qquad K_e(I,II) = \frac{eg-f^2}{EG-F^2},$$

moreover, the principal curvatures of the pair are $H(I,II) \pm \sqrt{H(I,II)^2 - K_e(I,II)}$.

We shall say that the pair (I,II) is **umbilical** at $p \in \Sigma$ if II is proportional to I at p, or equivalently:

- If both principal curvatures coincide at p, or
- if S is proportional to the identity map on the tangent plane at p, or
- if $H(I,II)^2 - K_e(I,II) = 0$ at p.

We define the **Hopf differential** of the fundamental pair (I,II) as the (2,0)-part of II for the Riemannian metric I. In other words, if we consider Σ as a Riemann surface with respect to the metric I and take a local conformal parameter z, then we can write

$$\begin{aligned} I &= 2\lambda\,|dz|^2, \\ II &= Q\,dz^2 + 2\lambda\,H\,|dz|^2 + \overline{Q}\,d\bar{z}^2. \end{aligned} \tag{1.11}$$

The quadratic form $Q\,dz^2$, which does not depend on the chosen parameter z, is known as the *Hopf differential* of the pair (I,II). We note that (I,II) is umbilical at $p \in \Sigma$ if, and only if, $Q(p) = 0$.

Remark 1.2. *All the above definitions can be understood as natural extensions of the corresponding ones for isometric immersions of a Riemannian surface in a $3-$dimensional ambient space, where I plays the role of the induced metric and II the role of its second fundamental form.*

A specially interesting case happens when the fundamental pair satisfies the Codazzi equation, that is,

Definition 1.3 ([3, 40])**.** *A fundamental pair (I,II) on Σ, with shape operator S, is a* **Codazzi pair** *if*

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = 0, \qquad X,Y \in \mathfrak{X}(\Sigma), \tag{1.12}$$

where ∇ stands for the Levi-Civita connection associated with the Riemannian metric I and $\mathfrak{X}(\Sigma)$ is the set of smooth vector fields on Σ.

Let us also observe that, by (1.11) and (1.12), a fundamental pair (I,II) is a Codazzi pair if and only if

$$Q_{\bar{z}} = \lambda H_z. \tag{1.13}$$

Thus, the equation (1.13) shows the following

Lema 1.1 ([40], **Lemma 6**). *Let (I,II) be a fundamental pair on Σ. Then, any two of the conditions* (i), (ii), (iii) *imply the third:*

(i) *(I,II) is a Codazzi pair.*

(ii) *$H(I,II)$ is constant.*

(iii) *The Hopf differential of the pair is holomorphic.*

Remark 1.3. *We observe that Hopf Theorem [34, p. 138], on the uniqueness of round spheres among immersed constant mean curvature spheres in Euclidean 3-space, can be easily obtained from this result.*

1.3 Homogeneous Spaces $\mathbb{E}(\kappa,\tau)$

We consider simply connected homogeneous Riemannian 3-manifolds whose isometry group has dimension 4.According to standard notation, we embrace these spaces as $\mathbb{E}(\kappa,\tau)$, where κ and τ are constants so that $\kappa - 4\tau^2 \neq 0$ (see [27]).

They can be classified as $\mathbb{M}^2(\kappa)\times\mathbb{R}$ if $\tau = 0$, with $\mathbb{M}^2(\kappa) = \mathbb{S}^2(\kappa)$ if $\kappa > 0$ ($\mathbb{S}^2(\kappa)$ the sphere of curvature κ), and $\mathbb{M}^2(\kappa) = \mathbb{H}^2(\kappa)$ if $\kappa < 0$ ($\mathbb{H}^2(\kappa)$ the hyperbolic plane of curvature κ). If $\tau \neq 0$, $\mathbb{E}(\kappa,\tau)$ is a Berger sphere if $\kappa > 0$, a Heisenberg space if $\kappa = 0$ (of bundle curvature τ), and the universal cover of $\mathrm{PSL}(2,\mathbb{R})$ if $\kappa < 0$. Henceforth we will suppose $\kappa \in \{-1,0,1\}$ (see [27]).

The homogeneous space $\mathbb{E}(\kappa,\tau)$ is a Riemannian submersion $\pi : \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ over a simply connected surface of constant sectional curvature κ. The **fibers** are the inverse image of a point at $\mathbb{M}^2(\kappa)$ by π. The fibers are the trajectories of a unitary Killing field ξ, called the **vertical vector field**.

Denote by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{E}(\kappa,\tau)$, then for all $X \in \mathfrak{X}(\mathbb{E}(\kappa,\tau))$, the following equation holds [56]:

$$\overline{\nabla}_X \xi = \tau X \wedge \xi,$$

where τ is the **bundle curvature**. Note that $\tau = 0$ implies that $\mathbb{E}(\kappa,\tau)$ is a product space (see [27]).

Let $\overline{R}$ the Riemann curvature tensor of $\mathbb{E}(\kappa,\tau)$ associated to connection $\overline{\nabla}$. Then,

Lema 1.2 ([20]). *Let $\mathbb{E}(\kappa,\tau)$ be a homogeneous space with unit Killing field ξ. For all vector fields $X,Y,Z,W \in \mathfrak{X}(\mathbb{E}(\kappa,\tau))$, we have:*

$$
\begin{aligned}
\langle \overline{R}(X,Y)Z,W\rangle = (\kappa-3\tau^2)\,(\langle X,Z\rangle Y - \langle Y,Z\rangle X)& \\
+\,(\kappa-4\tau^2)\,(\langle \xi,Y\rangle\langle \xi,Z\rangle X - \langle \xi,X\rangle\langle \xi,Z\rangle Y)& \\
-\,(\kappa-4\tau^2)\,(\langle Z,Y\rangle\langle \xi,X\rangle \xi + \langle Z,X\rangle\langle \xi,Y\rangle \xi)&
\end{aligned}
\tag{1.14}
$$

1.3.1 Immersed surfaces in $\mathbb{E}(\kappa,\tau)$

Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be an oriented immersed connected surface. We endow Σ with the induced metric of $\mathbb{E}(\kappa,\tau)$, the first fundamental form, which we still denote by $\langle , \rangle$ and N the unit normal vector field along Σ.

Denote by J the **oriented rotation of angle** $\frac{\pi}{2}$ on $T\Sigma$,

$$JX = N \wedge X \text{ for all } X \in \mathfrak{X}(\Sigma).$$

Set $\nu = \langle N,\xi\rangle$ and $\mathbf{T} = \xi - \nu N$, that is, ν is the normal component of the vertical vector field ξ, called the **angle function**, and $\mathbf{T}$ is the **tangent component of the vertical vector field** ξ.

Note that, in a product space $\mathbb{M}^2(\kappa)\times\mathbb{R}$, we have a natural projection onto the fiber $\sigma : \mathbb{M}^2(\kappa)\times\mathbb{R} \to \mathbb{R}$, hence we can define the restriction of σ to the surface Σ, that is, $h : \Sigma \to \mathbb{R}$, $h = \sigma_\Sigma$. The function h is called the **height function** of Σ. So, in $\mathbb{M}^2(\kappa)\times\mathbb{R}$, one can easily observes that $\overline{\nabla}\sigma = \xi$ and hence, $\mathbf{T}$ is the projection of $\overline{\nabla}\sigma$ onto the tangent plane, $\nabla h = \mathbf{T}$. Next, we list the structure equations satisfied by any immersed surface Σ in $\mathbb{E}(\kappa,\tau)$.

Lema 1.3 ([20]). *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be an immersed surface with unit normal vector field N and shape operator A. Let $\mathbf{T}$ and ν the tangent component of the vertical vector field and the angle function respectively. Then, given $X,Y \in \mathfrak{X}(\Sigma)$, the following equations hold:*

$$
\begin{aligned}
K &= K_e + \tau^2 + (\kappa-4\tau^2)\nu^2, && (1.15)\\
T_S(X,Y) &= (\kappa-4\tau^2)\nu(\langle Y,\mathbf{T}\rangle X - \langle X,\mathbf{T}\rangle Y), && (1.16)\\
\nabla_X\mathbf{T} &= \nu(AX - \tau JX), && (1.17)\\
d\nu(X) &= \langle \tau JX - AX, \mathbf{T}\rangle, && (1.18)\\
\|\mathbf{T}\|^2 &+ \nu^2 = 1, && (1.19)
\end{aligned}
$$

where K denotes the Gauss curvature of Σ, K_e the extrinsic curvature and T_S is given by:

$$T_S(X,Y) = \nabla_X AY - \nabla_Y AX - A([X,Y]),$$

for all $X,Y \in \mathfrak{X}(\Sigma)$.

Chapter 2

Constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$

In this chapter, we will define a Codazzi pair (I, II_{AR}) on any $H-$surface in $\mathbb{E}(\kappa,\tau)$ such that the $(2,0)$-part of II_{AR} with respect to a conformal parameter induced by I is the Abresch-Rosenberg differential (up to a constant in the case $\tau \neq 0$), and we will study the geometric properties of this Codazzi pair. Moreover, we will resume the classification results of $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ with constant Abresch-Rosenberg function and finally, we show an interesting Lemma which help us to get important results about $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ in Chapter 3.

This chapter is organized as follows; in Section 2.1, we will recall the structure equations that satisfies any constant mean curvature surface in $\mathbb{E}(\kappa,\tau)$ in terms of a local conformal parameter induced by the first fundamental form. In Section 2.2, we will define the Abresch-Rosenberg differential on any $H-$surface and the Abresch-Rosenberg map. In Section 2.3, we will define the Abresch-Rosenberg shape operator, first we will recover the case $\tau = 0$, second, we will define an Abresch-Rosenberg shape operator when $\tau \neq 0$. Then, we will obtain a fundamental pair (I, II_{AR}) and we study geometric properties of this pair, as the mean curvature or the extrinsic curvature. Moreover, we will compute the square norm of the shape operator of the second fundamental form in terms of the square norm of the Abresch-Rosenberg shape operator. Also, we will show the pair (I, II_{AR}) is a Codazzi pair on Σ whose $(2,0)-$part (with respect to a conformal parameter induced by I) is the Abresch-Rosenberg differential.

In Section 2.4, we will state the classification results for $H-$surfaces on $\mathbb{E}(\kappa,\tau)$ with constant Abresch-Rosenberg map q^{AR}. First, we will study the case $q^{AR} \equiv 0$ on Σ. Next, we will give an alternative proof for the classification result when $q^{AR} \equiv c \neq 0$. In fact, the above result will follow since we are able to show that if two special vector fields defined on any $H-$surface Σ are the principal directions of the traceless shape operator associated to II_{AR}, then Σ is invariant by a one parameter group of isometries of $\mathbb{E}(\kappa,\tau)$. Finally, in Section 2.5, we will define the Abresch-Rosenberg surfaces and we will describe constant mean curvature surfaces whose Abresch-Rosenberg differential vanishes.

2.1 Structure Equations

Let Σ be an orientable, connected, complete H- surface in $\mathbb{E}(\kappa,\tau)$ and let N be a unit normal to Σ. We assume Σ orientable, otherwise we pass to the double cover. In terms of a local conformal parameter z, the first fundamental form $I=\langle,\rangle$ and the second fundamental form are given by

$$\begin{aligned} I &= 2\lambda|dz|^2 && (2.1)\\ II &= Qdz^2+2\lambda H|dz|^2+\overline{Q}d\bar{z}^2, && (2.2)\end{aligned}$$

where $Qdz^2=-\langle\overline{\nabla}_{\partial_z}N,\partial_z\rangle dz^2$ is the usual **Hopf differential** of Σ. Hence, in this conformal coordinate, Lemma 1.3 reads as:

Lema 2.1 ([23, 24])**.** *Given an immersed H- surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$, the following equations are satisfied:*

$$\begin{aligned} K &= K_e+\tau^2+(\kappa-4\tau^2)\nu^2 && (2.3)\\ Q_{\bar{z}} &= \lambda(\kappa-4\tau^2)\nu\mathbf{t} && (2.4)\\ \mathbf{t}_z &= \frac{\lambda_z}{\lambda}\mathbf{t}+Q\nu && (2.5)\\ \mathbf{t}_{\bar{z}} &= \lambda(H+i\tau)\nu && (2.6)\\ \nu_z &= -(H-i\tau)\mathbf{t}-\frac{Q}{\lambda}\bar{\mathbf{t}} && (2.7)\\ |\mathbf{t}|^2 &= \frac{1}{2}\lambda(1-\nu^2), && (2.8)\end{aligned}$$

where $\mathbf{t}=\langle\mathbf{T},\partial_z\rangle$, $\bar{\mathbf{t}}=\langle\mathbf{T},\partial_{\bar{z}}\rangle$, K_e is the extrinsic curvature and K is the Gaussian curvature of I.

2.2 The Abresch-Rosenberg Differential

One of the main points in the work of Abresch-Rosenberg [1, 2] is to prove that certain quadratic differential $\mathcal{Q}^{AR}$ is holomorphic when Σ has constant mean curvature. From being holomorphic, and the Poincaré-Hopf Index Theorem, it is easy to see that such quadratic differential must vanish on a topological sphere. we proceed recalling the definition of this differential.

For an immersed surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$, there is a globally defined quadratic differential, called the **Abresch-Rosenberg differential**.

Definition 2.1 ([1, 2])**.** *Given a local conformal parameter z for I, the **Abresch-Rosenberg differential** is defined by:*

$$\mathcal{Q}^{AR}=Q^{AR}dz^2=(2(H+i\tau)Q-(\kappa-4\tau^2)\mathbf{t}^2)dz^2,$$

moreover, associated to the Abresch-Rosenberg differential we define the ***Abresch-Rosenberg map*** $q^{AR} : \Sigma \to [0, +\infty)$ *given by:*

$$q^{AR} = \frac{4|Q^{AR}|^2}{\lambda^2}.$$

Note that $\mathcal{Q}^{AR}$ *and* q^{AR} *do not depend on the conformal parameter* z*, hence* $\mathcal{Q}^{AR}$ *and* q^{AR} *are globally defined on* Σ*.*

Then, we can show the following

Theorem 2.1 ([1, 2]). *Let* Σ *be a* H*- surface in* $\mathbb{E}(\kappa, \tau)$*, then the Abresch-Rosenberg differential* $\mathcal{Q}^{AR}$ *is holomorphic for the conformal structure induced by the first fundamental form.*

Proof. Using Lemma 2.1, we have the following

$$\begin{aligned} Q^{AR}_{\bar{z}} &= 2(H + i\tau)Q_{\bar{z}} - 2(k - 4\tau^2)\mathbf{t}\mathbf{t}_{\bar{z}} \\ &= 2(H + i\tau)\lambda(k - 4\tau^2)\nu\mathbf{t} - 2(k - 4\tau^2)\mathbf{t}\lambda(H + i\tau)\nu \\ &= 0. \end{aligned}$$

Then $Q^{AR}dz^2$ is a holomorphic quadratic differential on Σ. □

2.3 The Abresch-Rosenberg shape operator

Lemma 1.1 tells us that, from the existence of a holomorphic quadratic differential, we should be able to find a Codazzi pair on any $H-$surface in $\mathbb{E}(\kappa, \tau)$. The pair of real quadratic forms that satisfies the Codazzi condition in the case that $\tau = 0$, i.e., when $\mathbb{E}(\kappa, \tau)$ is a product manifold, was found a long time ago (cf. [3]). Our goal here is to obtain a pair of real quadratic forms on any $H-$surface that satisfies the Codazzi condition and the Abresch-Rosenberg differential appears as its Hopf differential.

2.3.1 $H-$surfaces in $\mathbb{E}(\kappa, \tau)$ with $\tau = 0$

Consider a complete immersed $H-$surface $\Sigma \subset \mathbb{M}^2(\kappa) \times \mathbb{R}$. According to the notation introduced above, in [3] and [7] were defined a self-adjoint endomorphism S along Σ given by

$$SX = 2HAX - \kappa\langle X, \mathbf{T}\rangle\mathbf{T} + \frac{\kappa}{2}|\mathbf{T}|^2 X - 2H^2 X, \; X \in \mathfrak{X}(\Sigma). \tag{2.9}$$

Consider the bilinear symmetric form II_S associated to S given by (2.9). In [3], it was shown that (I, II_S) is a Codazzi pair on Σ when H is constant. Moreover, it is traceless, i.e., $\mathrm{tr}(S) = 0 = H(I, II_S)$, and the Hopf differential associated to (I, II_S) is the Abresch-Rosenberg differential $\mathcal{Q}^{AR}$ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

2.3.2 $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ with $\tau\neq 0$

The main point in this section is to see that the Abresch-Rosenberg differential has an interpretation in terms of Codazzi pairs on any $H-$surface in $\mathbb{E}(\kappa,\tau)$ when $\tau\neq 0$. In this case, we have that $H^2+\tau^2>0$. Define $\theta\in[0,2\pi)$ by

$$e^{2i\theta}=\frac{H-i\tau}{\sqrt{H^2+\tau^2}}.$$

Let $\Sigma\subset\mathbb{E}(\kappa,\tau)$ be a $H-$surface and z be a local conformal parameter. Then, up to the complex constant $H+i\tau$, we can re-define the Abresch-Rosenberg differential as:

$$\begin{aligned} Q^{AR}dz^2 &= \left(Q-\frac{\kappa-4\tau^2}{2(H+i\tau)}\mathbf{t}^2\right)dz^2 \\ &= \left(Q-\frac{\kappa-4\tau^2}{2\sqrt{H^2+\tau^2}}(e^{i\theta}\mathbf{t})^2\right)dz^2. \end{aligned} \tag{2.10}$$

Given the tangential vector field $\mathbf{T}$, define $\mathbf{T}_\theta=\cos\theta\mathbf{T}+\sin\theta J\mathbf{T}$, then

$$\langle\mathbf{T}_\theta,\partial_z\rangle=e^{i\theta}\mathbf{t},$$

hence,

$$Q^{AR}dz^2=\left(\langle A\partial_z,\partial_z\rangle-\alpha\langle\mathbf{T}_\theta,\partial_z\rangle^2\right)dz^2,$$

where $\alpha=\dfrac{\kappa-4\tau^2}{2\sqrt{H^2+\tau^2}}$, and A is the shape operator asociated to N, that is, $II(X,Y)=\langle AX,Y\rangle$, $X,Y\in\mathfrak{X}(\Sigma)$. This leads us to the following definition:

Definition 2.2. *Given a $H-$surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$, the* **Abresch-Rosenberg quadratic form** *II_{AR} is defined as:*

$$II_{AR}(X,Y)=II(X,Y)-\alpha\langle\mathbf{T}_\theta,X\rangle\langle\mathbf{T}_\theta,Y\rangle+\frac{\alpha\,|\mathbf{T}|^2}{2}\langle X,Y\rangle, \tag{2.11}$$

or equivalently, the **Abresch-Rosenberg shape operator** *S_{AR} is defined by:*

$$S_{AR}X=A(X)-\alpha\langle\mathbf{T}_\theta,X\rangle\mathbf{T}_\theta+\frac{\alpha\,|\mathbf{T}|^2}{2}X, \tag{2.12}$$

in particular, the **traceless Abresch-Rosenberg shape operator** *S is given by:*

$$SX=S_{AR}X-HX=A(X)-\alpha\langle\mathbf{T}_\theta,X\rangle\mathbf{T}_\theta+\frac{\alpha\,|\mathbf{T}|^2}{2}X-HX, \tag{2.13}$$

where $X,Y\in\mathfrak{X}(\Sigma)$.

First, we shall study the geometric properties of the above quadratic form and its relationship with the Abresch-Rosenberg differential:

Proposition 2.1. *The following equations hold for the fundamental pair* (I, II_{AR})*:*

(1.) $II_{AR}(\partial_z, \partial_z)dz^2 = Q^{AR}dz^2$, *where z is a local conformal parameter for I.*

(2.) $H(I, II_{AR}) = H$.

(3.) $|S|^2 = 2q_{AR}$.

(4.) $K_e(I, II_{AR}) = K_e(I, II) + \alpha \langle S\mathbf{T}_\theta, \mathbf{T}_\theta \rangle + \dfrac{\alpha^2 |\mathbf{T}|^4}{4}$.

Moreover, the square norm of the shape operator $|A|^2$ and the square norm of the traceless Abresch-Rosenberg shape operator $|S|^2$ are related by

$$|A|^2 = |S|^2 + 2\alpha \langle S\mathbf{T}_\theta, \mathbf{T}_\theta \rangle + \frac{\alpha^2}{2} |\mathbf{T}|^4 + 2H^2, \tag{2.14}$$

and, it holds

$$\frac{|\mathbf{T}|^4}{2} - \frac{\langle S\mathbf{T}_\theta, \mathbf{T}_\theta \rangle^2}{|S|^2} = \frac{\langle S\mathbf{T}_\theta, J\mathbf{T}_\theta \rangle^2}{|S|^2}. \tag{2.15}$$

Proof. Take a local conformal parameter z for $\langle , \rangle$. Then, from the definition of II_{AR} and $\langle \partial_z, \partial_z \rangle = 0$, we have that

$$\begin{aligned} II_{AR}(\partial_z, \partial_z) &= II(\partial_z, \partial_z) - \alpha \langle \mathbf{T}_\theta, \partial_z \rangle^2 + \frac{\alpha |\mathbf{T}|^2}{2} \langle \partial_z, \partial_z \rangle \\ &= II(\partial_z, \partial_z) - \alpha \langle \mathbf{T}_\theta, \partial_z \rangle^2 \\ &= Q^{AR}. \end{aligned}$$

Hence $II_{AR}(\partial_z, \partial_z)dz^2 = Q^{AR}dz^2$ and this shows (1.).

To show (2.), let $p \in \Sigma$ a fixed point and $\{e_1, e_2\}$ an orthonormal base of $T_p\Sigma$, therefore using that $|\mathbf{T}_\theta| = |\mathbf{T}|$, we have the following

$$\begin{aligned} H(I, II_{AR}) &= \frac{1}{2} \mathrm{tra}(S_{AR}) \\ &= \frac{1}{2} \sum_{i=1}^{2} \langle S_{AR}(e_i), e_i \rangle \\ &= \frac{1}{2} \sum_{i=1}^{2} II(e_i, e_i) - \alpha \langle \mathbf{T}_\theta, e_i \rangle^2 + \frac{\alpha |\mathbf{T}|^2}{2} \langle e_i, e_i \rangle \\ &= H - \frac{\alpha}{2} |\mathbf{T}_\theta|^2 + \frac{\alpha}{2} |\mathbf{T}|^2 \\ &= H. \end{aligned} \tag{2.16}$$

To show (3.), Take a local conformal parameter $z = u + iv$ for metric $\langle,\rangle$, that is $\langle,\rangle = 2\lambda\,|dz|^2$. On the one hand, from (1.) and the equation $\langle\partial_z,\partial_z\rangle = 0$ we have:

$$Q^{AR} = \langle S\partial_z,\partial_z\rangle = \frac{\hat{e}-\hat{g}}{4} - i\frac{\hat{f}}{2},$$

where $\hat{e} = \langle S\partial_u,\partial_u\rangle$, $\hat{g} = \langle S\partial_v,\partial_v\rangle$ and $\hat{f} = \langle S\partial_u,\partial_v\rangle$, observe that $\langle S\partial_u,\partial_u\rangle = -\langle S\partial_v,\partial_v\rangle$ since S is a traceless operator. Therefore

$$\left|Q^{AR}\right|^2 = \left(\frac{\hat{e}-\hat{g}}{4}\right)^2 + \left(\frac{\hat{f}}{2}\right)^2 = \frac{\hat{e}^2}{4} + \frac{\hat{f}^2}{4}. \tag{2.17}$$

On the other hand, consider the orthonormal base $\{\frac{1}{\sqrt{\lambda}}\partial_u, \frac{1}{\sqrt{\lambda}}\partial_v\}$, then we compute $|S|^2$ in the above base

$$|S|^2 = \frac{1}{\lambda^2}\left(\langle S\partial_u,\partial_u\rangle^2 + \langle S\partial_v,\partial_v\rangle^2 + 2\langle S\partial_u,\partial_v\rangle^2\right) = \frac{2}{\lambda^2}\left(\hat{e}^2 + \hat{f}^2\right). \tag{2.18}$$

Hence, using the equations (2.17) and (2.18), we get

$$|S|^2 = \frac{8\left|Q^{AR}\right|^2}{\lambda^2} = 2q_{AR}.$$

To show (4.), we use the Abresch-Rosenberg quadratic form (2.11). It is clear that $|\mathbf{T}_\theta| = |J\mathbf{T}_\theta| = |\mathbf{T}|$, then;

$$\begin{aligned} II_{AR}(\mathbf{T}_\theta,\mathbf{T}_\theta) &= II(\mathbf{T}_\theta,\mathbf{T}_\theta) - \frac{\alpha\,|\mathbf{T}|^4}{2}. \\ II_{AR}(\mathbf{T}_\theta,J\mathbf{T}_\theta) &= II(\mathbf{T}_\theta,J\mathbf{T}_\theta). \\ II_{AR}(J\mathbf{T}_\theta,J\mathbf{T}_\theta) &= II(J\mathbf{T}_\theta,J\mathbf{T}_\theta) + \frac{\alpha\,|\mathbf{T}|^4}{2}. \end{aligned} \tag{2.19}$$

Thus, we have

$$II_{AR} = II \text{ on the set } \mathscr{U} = \{p \in \Sigma : |\mathbf{T}|^2(p) = 0\}.$$

Then, take $p \in \Sigma \setminus \mathscr{U}$ and consider the orthonormal basis in $T_p\Sigma$ defined by:

$$e_1 = \frac{\mathbf{T}_\theta}{|\mathbf{T}|} \text{ and } e_2 = \frac{J\mathbf{T}_\theta}{|\mathbf{T}|}. \tag{2.20}$$

From (2.19), we obtain:

$$II_{AR}(e_1,e_1) - II_{AR}(e_2,e_2) = II(e_1,e_1) - II(e_2,e_2) - \alpha\,|\mathbf{T}|^2, \tag{2.21}$$

and, since $\{e_1, e_2\}$ is orthonormal at p and the equations (2.19), we have

$$\begin{aligned} K_e(I, II_{AR}) &= II_{AR}(e_1,e_1)II_{AR}(e_2,e_2) - II_{AR}(e_1,e_2)^2 \\ &= II(e_1,e_1)II(e_2,e_2) - II(e_1,e_2)^2 \\ &\qquad + \frac{\alpha}{2}(II(e_1,e_1) - II(e_2,e_2))\,|\mathbf{T}|^2 - \frac{\alpha^2}{4}\,|\mathbf{T}|^4 . \end{aligned}$$

On the one hand, replacing the formula (2.21) in the last formula for $K_e(I, II_{AR})$ and simplifying terms, we get at p:

$$K_e(I, II_{AR}) = K_e(I, II) + \frac{\alpha}{2}\left(II_{AR}(\mathbf{T}_\theta, \mathbf{T}_\theta) - II_{AR}(J\mathbf{T}_\theta, J\mathbf{T}_\theta)\right) + \frac{\alpha^2\,|\mathbf{T}|^4}{4} \tag{2.22}$$

On the other hand, recall that S is traceless and hence at a point $p \in \Sigma$, we can consider an orthonormal basis $\{E_1, E_2\}$ of principal directions for S, i.e,

$$SE_1 = \lambda E_1,\, SE_2 = -\lambda E_2 \text{ and } |S|^2 = 2\lambda^2.$$

Then, there exists $\beta \in [0, 2\pi)$ such that

$$\mathbf{T}_\theta = |\mathbf{T}|(\cos\beta\, E_1 + \sin\beta E_2),$$

and hence, one can easily check

$$\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle = -\langle SJ\mathbf{T}_\theta, J\mathbf{T}_\theta\rangle, \tag{2.23}$$

hence from (2.13) and (2.23), we have

$$\begin{aligned} II_{AR}(\mathbf{T}_\theta, \mathbf{T}_\theta) - II_{AR}(J\mathbf{T}_\theta, J\mathbf{T}_\theta) &= \langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle - \langle SJ\mathbf{T}_\theta, J\mathbf{T}_\theta\rangle \\ &= 2\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle, \end{aligned} \tag{2.24}$$

thus, joining the equations (2.22) and (2.24) we obtain the expression for $K_e(I, II_{AR})$.

The equation (2.23) and a straightforward shows that

$$\frac{|\mathbf{T}|^4}{2} - \frac{\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle^2}{|S|^2} = \frac{\langle S\mathbf{T}_\theta, J\mathbf{T}_\theta\rangle^2}{|S|^2},$$

which shows (2.15).

Finally, the equation (2.14) can be easily obtained observing that $|A|^2 = 4H^2 - 2K_e$ and $|S|^2 = 2q_{AR} = 2(H^2 - K_e(I, II_{AR}))$. $\square$

Hence, Lemma 1.1 implies:

Theorem 2.2. *Given any H-surface in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, it holds:*

$\mathcal{Q}^{AR}$ is holomorphic if, and only if, (I, II_{AR}) is a Codazzi pair.

Proof. This follows from properties (1.) and (2.) of pair (I, II_{AR}) in Proposition 2.1, Theorem 2.1 and Lemma 1.1. □

So, we can summarize the above in the following

Lema 2.2. *Given a $H-$surface $\Sigma \subset \mathbb{E}(\kappa,\tau)$, $H^2+\tau^2 \neq 0$, consider the symmetric two tensor given by*

$$II_{AR}(X,Y) = II(X,Y) - \alpha\langle \mathbf{T}_\theta, X\rangle\langle \mathbf{T}_\theta, Y\rangle + \frac{\alpha\,|\mathbf{T}|^2}{2}\langle X,Y\rangle,$$

where

- $\alpha = \dfrac{\kappa - 4\tau^2}{2\sqrt{H^2+\tau^2}}$,
- $e^{2i\theta} = \frac{H-i\tau}{\sqrt{H^2+\tau^2}}$ *and*
- $\mathbf{T}_\theta = \cos\theta\mathbf{T} + \sin\theta J\mathbf{T}$.

Then, (I, II_{AR}) is a Codazzi Pair with constant mean curvature H. Moreover, the $(2,0)-$part of II_{AR} with respect to the conformal structure given by I agrees (up to a constant) with the Abresh-Rosenberg differential.

Having disposed the equation (2.14), we have the following

Corollary 2.1. *Let Σ be a $H-$surface on $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2 \neq 0$ and S the traceless Abresch-Rosenberg shape operator defined on Σ. Then, it is equivalent*

- *$|S|$ is bounded,*
- *$|A|$ is bounded,*
- *$|K|$ is bounded.*

Proof. On the one hand, from equation (2.14) is clear that $|A|$ is bounded if, and only if, $|S|$ is bounded on Σ. On the other hand, $4H^2 - 2K_e = |A|^2$ and the Gauss equation, then $|K|$ is bounded if, and only if, $|A|$ is bounded. □

2.4 Classification results for $H-$surfaces in $\mathbb{E}(\kappa,\tau)$

The purpose of this section is to give the principal classification result for complete $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ when the Abresch-Rosenberg map q^{AR} is constant. First, we recall the classification theorem when q^{AR} vanishes.

Theorem 2.3 ([1, 2, 24])**.** *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete $H-$ surface whose Abresch-Rosenberg differential vanishes. Then Σ is invariant by a one parameter group of isometries of $\mathbb{E}(\kappa,\tau)$. Moreover, Σ is either a slice in $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$ if $H=0=\tau$ and in the case $H^2+\tau^2\neq 0$ the Gauss curvature K of these examples satisfies:*

$$16(H^2+\tau^2)K = 16(H^2+\tau^2)^2+16(H^2+\tau^2)(\kappa-4\tau^2)\nu^2-(\kappa-4\tau^2)^2(1-\nu^2)^2$$

and it holds:

- *If either $4(H^2+\tau^2)>\kappa-4\tau^2$ when $\kappa-4\tau^2>0$ or $H^2+\tau^2>-(\kappa-4\tau^2)$ when $\kappa-4\tau^2<0$, then $K>0$, i.e, Σ is a rotationally invariant sphere. In particular, $4H^2+\kappa>0$.*
- *If $4H^2+\kappa=0$ and $\nu=0$, then $K=0$, i.e Σ is either a vertical plane in* Nil_3 *or a vertical cylinder over a horocycle in $\mathbb{H}^2\times\mathbb{R}$ or* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.
- *There exists a point with negative Gauss curvature in the rest of examples.*

Remind that $\mathbb{E}(\kappa,\tau)$ is a Riemannian submersion $\pi:\mathbb{E}(\kappa,\tau)\to\mathbb{M}^2(\kappa)$. Given γ a regular curve in $\mathbb{M}^2(\kappa)$, $\pi^{-1}(\gamma)$ is a surface in $\mathbb{E}(\kappa,\tau)$ that has ξ as a tangent vector field, in this case $\nu=0$. So, ξ is a parallel vector field along $\pi^{-1}(\gamma)$ and hence $\pi^{-1}(\gamma)$ is a flat surface and its mean curvature is given by $2H=k_g$, where k_g is the geodesic curvature of γ in $\mathbb{M}^2(\kappa)$ (cf. [25, Proposition 2.10]). We will call $\pi^{-1}(\gamma)$ a **Hopf cylinder** of $\mathbb{E}(\kappa,\tau)$ over curve γ. If γ is a closed curve, $\pi^{-1}(\gamma)$ is a flat **Hopf cylinder** and additionally, if π is a circle Riemannian submersion, $\pi^{-1}(\gamma)$ is a **Hopf torus**. The latter case occurs when $\mathbb{E}(\kappa,\tau)$ is a Berger sphere (see, [64, Theorem 1]), i.e., $\kappa=1$ and $\tau\neq 0$. If k_g is constant, we call it Hopf $H-$torus.

Now, with the definition of Hopf cylinders, we classify H-surfaces in $\mathbb{E}(\kappa,\tau)$ with constant non-zero Abresch-Rosenberg map q^{AR}.

Theorem 2.4. *Let $\Sigma\subset\mathbb{E}(\kappa,\tau)$ be a complete $H-$surface and suppose q^{AR} is a positive constant map on Σ, then Σ is a Hopf cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

Proof. We can assume, without loss of generality, that Σ is simply-connected by passing to the universal cover.

Since q_{AR} is a positive constant, by the one hand from (cf. [24, Lemma 2.2]) we have $0=\Delta\ln q^{AR}=4K$, that is, the Gaussian curvature vanishes identically on Σ. Then, using Gauss equation for Σ, we have:

$$K_e=-\tau^2-(\kappa-4\tau^2)\nu^2. \tag{2.25}$$

By the other hand, we can work as in the proof of [24, Theorem 2.3] to consider a conformal parameter z for $\langle,\rangle$, such that $\langle,\rangle=|dz|^2$ and then the differential $\mathcal{Q}^{AR}$ is holomorphic with constant norm, thus we conclude $Q^{AR} = c$, where $c \in \mathbb{R}$ is a real constant such that

$$q^{AR} = H^2 - K_e(I, II_{AR}) = 4\left|Q^{AR}\right|^2 = 4c^2,$$

and we obtain that

$$K_e(I, II_{AR}) = H^2 - 4c^2 \text{ is constant on } \Sigma. \tag{2.26}$$

Moreover, since Q^{AR} is a real constant along Σ, we obtain

$$\begin{aligned} 4c = 4Q^{AR} &= II_{AR}\left(\frac{\mathbf{T}_\theta}{|\mathbf{T}|}, \frac{\mathbf{T}_\theta}{|\mathbf{T}|}\right) - II_{AR}\left(\frac{J\mathbf{T}_\theta}{|\mathbf{T}|}, \frac{J\mathbf{T}_\theta}{|\mathbf{T}|}\right) \\ &= 2\frac{\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle}{|\mathbf{T}|^2}, \end{aligned}$$

or, equivalently

$$2c\,|\mathbf{T}|^2 = \langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle. \tag{2.27}$$

Then, from property (4.) of Proposition 2.1, equation (2.25), equation (2.26) and equation (2.27), we obtain

$$H^2 - 4c^2 = -\tau^2 - (\kappa - 4\tau^2)(1 - |\mathbf{T}|^2) + \frac{\alpha^2\,|\mathbf{T}|^4}{4} + 2\alpha c\,|\mathbf{T}|^2,$$

that is,

$$A\,|\mathbf{T}|^4 + B\,|\mathbf{T}|^2 + C = 0, \tag{2.28}$$

where

$$A = \frac{\alpha^2}{4},\ B = 2\alpha c + \kappa - 4\tau^2 \text{ and } C = 4c^2 - H^2 - \tau - (\kappa - 4\tau^2).$$

Since $A \neq 0$, (2.28) tells us that $|\mathbf{T}|$ is constant along Σ and hence ν is constant along Σ, which implies that Σ is a vertical cylinder (cf. [23, Theorem 2.2]). □

2.5 Abresch-Rosenberg surfaces

In this section, we define the Abresch-Rosenberg surfaces, next, we describe them.

Definition 2.3. *A constant mean curvature surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$ is an* **Abresch-Rosenberg surface** *if the Abresch-Rosenberg differential $\mathcal{Q}^{AR}$ vanishes on Σ. In particular by Theorem 2.3, Σ must be invariant by a one parameter group of isometries of $\mathbb{E}(\kappa, \tau)$.*

Now, we give some examples of Abresch-Rosenberg surfaces in $\mathbb{E}(\kappa, \tau)$, as we know from [1, 2], we have a complete description of the Abresch-Rosenberg surfaces in product spaces. Despite that in the case $\tau \neq 0$, we do not have a complete description of the Abresch-Rosenberg surfaces in $\mathbb{E}(\kappa, \tau)$, here we give some examples of these surfaces. We follow the references [1, 2, 29, 35, 49, 50, 63] and the references therein.

- $\mathbb{H}^2 \times \mathbb{R}$.

  Consider $\mathbb{H}^2 \times \mathbb{R}$ as submanifold of Lorentzian 4-space $\mathbb{L}^4$, given by

$$\mathbb{H}^2 \times \mathbb{R} = \{(x_1, x_2, x_3, x_4) : -x_1^2 + x_2^2 + x_3^2 = 1, x_1 > 0\}.$$

  1. **Rotationally invariant spheres [1, 2].** One has embedded constant mean curvature spheres that are rotationally invariant. These examples are parametrized as follows: set $H > \frac{1}{2}$ and $\mathscr{I} = [-1, 1]$. Consider

$$\begin{aligned} r(u) &= 2\operatorname{arcsinh}\left(\sqrt{\frac{1-u^2}{4H^2-1}}\right) \\ h(u) &= \frac{4H}{\sqrt{4H^2-1}} \arcsin\left(\frac{u}{2H}\right) \end{aligned}$$

     and the curve $\alpha : \mathscr{I} \to \mathscr{P}$ given by

$$\alpha(u) = (\cosh r(u), \sinh r(u), 0, h(u)),$$

     where $\mathscr{P} = \{(x_1, x_2, x_3, x_4) \in \mathbb{H}^2 \times \mathbb{R} : x_2 \geq 0, x_3 = 0\}$. Then, the rotationally invariant surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ associated to curve α, can be parametrized as follows:

$$\psi(u, v) = (\cosh r(u), \sinh r(u) \cos v, \sinh r(u) \sin v, h(u)),$$

     for $(u, v) \in \mathscr{I} \times \frac{\mathbb{R}}{2\pi}$. This surface is rotationally invariant sphere with constant mean curvature H.
  2. Convex **rotationally invariant graphs** D_H^2 over horizontal leaves $\mathbb{H}^2 \times \{\xi_0\}$ (cf. [1, 2]), which are asymptotically conical whenever $4H^2 - 1 < 0$.
  3. Rotationally invariant **embedded annulus** C_H^2 with two asymptotically conical ends (cf. [1, 2]). It is generated by rotating a strictly concave curve with asymptotic slopes $\pm \tan(\arccos 2H)$.
  4. **Embedded constant mean curvature surfaces** which are orbits under a two dimensional solvable subgroup of $\operatorname{Iso}(\mathbb{H}^2 \times \mathbb{R})$ [1, 2].

- $\mathbb{S}^2 \times \mathbb{R}$.

  Consider $\mathbb{S}^2 \times \mathbb{R}$ as submanifold of $\mathbb{R}^4$, defined as

$$\mathbb{S}^2 \times \mathbb{R} = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

  Here $\mathbb{R}^4$ denotes the usual 4-dimensional Euclidean Space.

1. **Rotationally invariant spheres [1, 2].** There exists embedded constant mean curvature spheres that are rotationally invariant. These examples are parametrized as follows: Set $H > \frac{1}{2}$ and $\mathscr{I} = [-1,1]$. Consider

$$\begin{aligned} r(u) &= 2\arcsin\left(\frac{4H^2-1+2u^2}{4H^2+1}\right) \\ h(u) &= \frac{4H}{\sqrt{4H^2+1}}\operatorname{arcsinh}\left(\frac{u}{2H}\right). \end{aligned}$$

and the curve $\alpha : \mathscr{I} \to \mathscr{P}$ given by:

$$\alpha(u) = (\cos r(u), \sin r(u), 0, h(u)),$$

where $\mathscr{P} = \{(x_1,x_2,x_3,x_4) \in \mathbb{S}^2 \times \mathbb{R} : x_2 \geq 0, x_3 = 0\}$. Then, the rotationally invariant surface $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ associated to curve α can be parametrized as:

$$\psi(u,v) = (\cos r(u), \sin r(u)\cos v, \sin r(u)\sin v, h(u)),$$

for $(u,v) \in \mathscr{I} \times \frac{\mathbb{R}}{2\pi}$. This sphere has constant mean curvature H.

- **Heisenberg space** Nil_3**.**

Set $k = 0$ and $\tau = 1/2$, then $\mathrm{Nil}_3 = \mathbb{E}(0,1/2)$ ts the Heisenberg space and we can describe it as $\mathrm{Nil}_3 = (\mathbb{R}^3, ds^2)$, where

$$ds^2 = dx^2 + dy^2 + \big(\frac{y}{2}dx - \frac{x}{2}dy + dz\big). \tag{2.29}$$

Now, if we introduce cylindrical coordinates in Nil_3, that is, we parametrize

$$\mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 : z \in \mathbb{R}\}$$

by parameters $r > 0$ and $\theta \in \mathbb{R}/2\pi$. We change coordinates as $x = r\cos\theta$, $y = r\sin\theta$ and the z-coordinate. Then, the metric ds^2 in equation (2.29) takes the form

$$ds^2 = dr^2 + \big(r^2 + \frac{r^4}{4}\big)d\theta^2 + dz^2 - r^2 d\theta dz.$$

1. **Rotationally invariant spheres [29].**
Consider the curve in Nil_3, $\alpha : \mathscr{I} \subset \mathbb{R} \to \mathscr{P}$, where
$\mathscr{P} = \{(r,\theta,z) \in \mathrm{Nil}_3 : r > 0, \theta = 0\}$, given by:

$$\alpha(u) = (r(u), 0, z(u)).$$

Fix $H > 0$, r as a parameter and define:

$$z(r) = \frac{1}{4H}\sqrt{(4+r^2)(4-H^2r^2)} + \frac{1+H^2}{H^2}\arcsin\frac{\sqrt{4-H^2r^2}}{2\sqrt{1+H^2}}$$

here $r \in (0, \frac{2}{H})$. Then, the rotationally invariant surface $\Sigma \subset \mathrm{Nil}_3$ associated to α can be parametrized as:

$$\psi(u,v) = (r(u)\cos\theta, r(u)\sin\theta\cos v, \sin\theta, z(u)),$$

for $(u,\theta) \in \mathscr{I} \times \frac{\mathbb{R}}{2\pi}$ (see [29]). This gives us the rotationally invariant sphere with constant mean curvature H in Nil_3.

2. **Surfaces invariant under translations [29].**

   The $H-$surfaces in Nil_3 invariant under translations are:

   a. The minimal vertical planes.

   b. The minimal surfaces given by the graph of

   $$z = \frac{xy}{2} - c\left(\frac{y\sqrt{1+y^2}}{2} + \frac{1}{2}\ln(y+\sqrt{1+y^2})\right) \quad c \in \mathbb{R}.$$

   c. The H-surfaces given by the graph of

   $$z = \frac{xy}{2} \pm \frac{1}{2H}\left(\sqrt{1+y^2}\sqrt{1-H^2y^2} + \frac{1+H^2}{H}\arcsin\sqrt{\frac{1-H^2y^2}{1+H^2}}\right),$$

   where $-\frac{1}{H} \leq y \leq \frac{1}{H}$.

3. **Catenoids and Horizontal Umbrellas [2]**: These are minimal surfaces in Nil(3) that are invariant under the group of rotations around some vertical axis.

- **Berger Spheres**.

  A Berger sphere, denoted by $\mathbb{S}^3_B(k,\tau)$ is the usual three dimensional sphere in $\mathbb{C}^2$, $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, endowed with the metric

  $$\langle X,Y\rangle_{(k,\tau)} = \frac{4}{\tau}\big(\langle X,Y\rangle + \big(\frac{4\tau^2}{k} - 1\big)\langle X,V\rangle\langle Y,V\rangle\big).$$

  Here $\langle,\rangle$ denotes the standard round metric on $\mathbb{S}^3$, $V : \mathbb{S}^3 \to \mathbb{S}^3$ is given $V(z,w) = (iz, iw)$ and $k > 0$ and $\tau \neq 0$ are constants. $\mathbb{S}^3_B(k,\tau)$ is a model for the homogeneous space $\mathbb{E}(\kappa,\tau)$ when $k > 0$ and $\tau \neq 0$.

  1. **Rotationally invariant spheres [63].** Let $\alpha : \mathscr{I} \subset \mathbb{R} \to \mathbb{S}^2$ be the curve given by:

     $$\alpha(u) = (\cos x(u)e^{iy(u)}, \sin x(u)), \quad \cos x(u) > 0$$

     Let $\Sigma \subset \mathbb{S}^3_B(k,\tau)$ be a rotationally invariant surface associated to α which is parametrized as (see [63]):

     $$\psi(u,v) = (\cos x(u)e^{iy(u)}, \sin x(u)e^{iv}).$$

If $H = 0$, then Σ is a the great 2-sphere given (up to isometries) by:

$$\Sigma = \{(z,w) \in \mathbb{C}^2 : \mathrm{Im}(z) = 0\}.$$

If $H > 0$, Consider $x \in (0, \arctan \frac{\sqrt{k}}{2H})$ as a parameter, the function:

$$y(x) = -\arctan\left(\frac{\tau}{H}\lambda(x)\right) - \delta \frac{H\sqrt{|k-4\tau^2|}}{\tau\sqrt{4H^2+k}} \arctan\left(\frac{|k-4\tau^2|}{\sqrt{4H^2+k}}\lambda(x)\right),$$

where $\delta = sign(k - 4\tau^2)$ and

$$\lambda(x) = \frac{\sqrt{1 - \frac{4H^2}{\tau}\tan^2(x)}}{\sqrt{1 + \frac{4\tau^2}{k}\tan^2(x)}}.$$

The aboves formulas gives us a parametrization of a half rotationally invariant sphere in $\mathbb{S}^3_B$. The other half is obtained by reflecting the solution along the line $x = 0$.

- $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$.

  Set
  $$\widetilde{\mathrm{PSL}(2,\mathbb{R})} : \{(z,w) \in \mathbb{C}^2 : |z|^2 - |w|^2 = 1\}$$
  endowed with the metric
  $$g(E_i,E_j) = -\frac{4}{k}\delta_{ij}, g(V,V) = \frac{16\tau^2}{k^2}, g(V,E_i) = 0, i,j = 1,2.$$
  Where k and τ are real numbers such that $k < 0$ and $\tau \neq 0$ and $\{E_1,E_2,V\}$ is a global frame on $\mathfrak{X}(\widetilde{\mathrm{PSL}(2,\mathbb{R})})$ defined by
  $$E_1(z,w) = (\overline{w},\overline{z}), E_2(z,w) = (i\overline{w}, i\overline{z}), V(z,w) = (iz, iw).$$
  Then $(\widetilde{\mathrm{PSL}(2,\mathbb{R})}, g)$ is a model for an homogeneous space $\mathbb{E}(\kappa,\tau)$ with $k < 0$. $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is a fibration over $\mathbb{H}^2$ with fibers generated by unit Killing vector field $\xi = -\frac{k}{4\tau}V$.

  1. **Rotationally invariant spheres [63].** Let $\alpha : \mathscr{I} \to \mathscr{P}$ be the curve given by:
  $$\alpha(u) = (\cosh x(u) e^{iy(u)}, \sinh x(u)),$$
  where $\mathscr{P} = \{(z,a) \in \mathbb{C} \times \mathbb{R} : |z|^2 - a^2 = 1\}$. Let $\Sigma \subset \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ be a rotationally invariant surface associated to α, which is parametrized as (see [63]):
  $$\psi(u,v) = (\cosh x(u) e^{iy(u)}, \sinh x(u) e^{iv}).$$

Fix $H > 0$, so that $4H^2 + k > 0$ and consider $x \in \left(-\operatorname{arctanh}\frac{\sqrt{-k}}{2H}, 0 \right)$ as a parameter, define the function:

$$y(x) = -\arctan\left(\frac{\tau}{H}\lambda(x)\right) - \frac{H\sqrt{|4\tau^2 - k|}}{\tau\sqrt{4H^2+k}} \arctan\left(\frac{4\tau^2 - k}{\sqrt{4H^2+k}}\lambda(x)\right),$$

where

$$\lambda(x) = \frac{\sqrt{1 + \frac{4H^2}{\tau}\tanh^2(x)}}{\sqrt{1 - \frac{4\tau^2}{k}\tanh^2(x)}}.$$

The above formulas gives us a parametrization of a half rotationally invariant sphere in $\mathrm{PSL}(2,\mathbb{R})$. The other half is obtained by reflecting the solution along the line $x = 0$.
When $\tau = -1$ and $1 - 4H^2 > 0$ in the model of disk for $\mathbb{H}^2$ we can construct invariant surfaces in $\mathrm{PSL}(2,\mathbb{R})$. In this model the surface has the parametrization

$$\varphi(\rho,\theta) = (\tanh\frac{\rho}{2}\cos\theta, \tanh\frac{\rho}{2}\sin\theta, u(\rho)),$$

where ρ is the hyperbolic distance measure from the origin to $\mathbb{D}^2$, $\theta \in [0, 2\pi]$ and

$$u(\rho) = \frac{4\sqrt{2}H}{\sqrt{1-4H^2}}\ln\left(\sqrt{\cosh\rho} + \sqrt{\frac{1+4H^2}{1-4H^2} + \cosh\rho}\right)$$
$$+ 2\arctan\left(-\sqrt{\frac{8H^2}{1-4H^2}}\frac{\sqrt{\cosh\rho}}{\sqrt{\frac{1+4H^2}{1-4H^2}\cosh\rho}}\right).$$

2. **Minimal rotational invariant graphs [50] .** Consider the surface given by the parametrization

$$\varphi(\rho,\theta) = (\tanh\frac{\rho}{2}\cos\theta, \tanh\frac{\rho}{2}\sin\theta, u(\rho))$$

where ρ is the hyperbolic distance measure from the origin to $\mathbb{D}^2$, $\theta \in [0, 2\pi]$ and

$$u(\rho) = \int_{\sinh^{-1} d}^{\rho} \frac{d\sqrt{1 + 4\tau^2\tanh^2\frac{r}{2}}}{\sqrt{\sinh^2 r - d^2}}dr.$$

Then, if $d = 0$ the surface is the **slice** in $\mathrm{PSL}(2,\mathbb{R})$. Now, if $d > 0$ the parametrization φ describes a rotational surface called the **catenoid**, which is embedded and homeomorphic to an annulus.

3. ***H*- surfaces invariant by parabolic isometries of** $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ **[50].** We will take the half-plane model for the hyperbolic space $\mathbb{H}^2$. Consider a curve $\alpha(y) = (0, y, u(y))$ in this model and apply to α a one-parameter group of parabolic isometries Γ. Denote as $\Sigma = \Gamma(\alpha)$ the invariant H- surface by Γ generated by α. Then Σ has a parametrization as

$$\varphi(x,y) = (x,y,u(y)) \ y > 0.$$

where, if $H = 0$, then

$$u(y) = \sqrt{1+4\tau^2}\arcsin(dy).$$

If $H = \frac{1}{2}$, then

$$u(y) = \sqrt{1+4\tau^2}\arcsin(dy-1) + \frac{2\sqrt{1+4\tau^2}}{\tan(\frac{\arcsin(dy-1)}{2})+1}.$$

If $H > \frac{1}{2}$, then

$$\begin{aligned} u(y) &= \sqrt{1+4\tau^2}\arcsin(dy-2H) \\ &\quad - \frac{4H\sqrt{1+4\tau^2}}{\sqrt{4H^2-1}} \arctan\left(\frac{2H\tan(\frac{\arcsin(dy-2H)}{2}+1)}{\sqrt{4H^2-1}}\right). \end{aligned}$$

Chapter 3

Simons' Type Formula in $\mathbb{E}(\kappa, \tau)$ and applications

In this chapter, we obtain a Simons' type formula for the traceless Abresch-Rosenberg shape operator S given on a $H-$surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, defined by (2.13). Using this Simons' formula, we will study the behaviour of complete H- surfaces with finite Abresch-Rosenberg total curvature, we also get an estimate for the first eigenvalue of a Schrödinger operator in any complete H- surface with finite Abresch-Rosenberg total curvature. Finally, we use the Simons' formula together with the Omori-Yau Maximum principle to obtain pinching theorems for complete H-surfaces immersed in $\mathbb{E}(\kappa, \tau)$.

This chapter is organized as follows; in Section 3.1, we will obtain a Simons' Type Formula on any $H-$surface in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, using the Abresch-Rosenberg shape operator defined in Chapter 2. In Section 3.2, we will focus on complete $H-$surfaces Σ with finite Abresch-Rosenberg total curvature immersed in $\mathbb{E}(\kappa, \tau)$ and we study its behavior. In Section 3.3, we will estimate the first eigenvalue of any Schrödinger Operator $L = \Delta + V$, V continuous, defined on $H-$surfaces with finite Abresch-Rosenberg total curvature. Finally, in Section 3.4, using the Simons' Type Formula together with the Omori-Yau's Maximum Principle, we will classify complete $H-$surfaces (not necessary with finite Abresch-Rosenberg total curvature) in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$.

3.1 Simons' Type Formula in $\mathbb{E}(\kappa, \tau)$

Let $\omega_1, \ldots, \omega_n$ be a local dual frame associated to a frame field defined in a neighborhood of a point p on a manifold Σ and ϕ be a symmetric bilinear $(2,0)$-tensor defined on $\mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma)$, so ϕ can be expressed like $\phi = \sum_{i,j} \phi_{ij} \omega_i \times \omega_j$ in this neighborhood.

The structure equations of Σ are given by

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij}+\omega_{ji}=0 \\ d\omega_{ij} &= \sum_k \omega_{ik}\wedge\omega_{kj}+\Omega_{ij}. \end{aligned} \tag{3.1}$$

Where

$$\Omega_{ij} = -\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l.$$

The first covariant derivative of ϕ_{ij} is defined by

$$\sum_k \phi_{ijk}\omega_k = d\phi_{ij}+\sum_k \phi_{kj}\omega_{ki}+\sum_k \phi_{ik}\omega_{kj}. \tag{3.2}$$

The second covariant derivative of ϕ_{ij} is defined by

$$\sum_l \phi_{ijkl}\omega_l = d\phi_{ijk}+\sum_m \phi_{mjk}\omega_{mi}+\sum_m \phi_{imk}\omega_{mj}+\sum_m \phi_{ijm}\omega_{mk}. \tag{3.3}$$

S. Y. Cheng and S. T. Yau used the structure equations of Σ and the covariant derivatives of a tensor to compute the Laplacian of the norm squared of any symmetric tensor that satisfies the Codazzi equation.

Theorem 3.1 ([65]). *Let be a symmetric bilinear $(2,0)$- tensor defined on a manifold Σ, suppose that ϕ can be expressed locally around of one point $p\in\Sigma$ like $\sum_{i,j}\phi_{ij}\omega_i\times\omega_j$ and satisfies a Codazzi equation $\phi_{ij,k}=\phi_{ik,j}$. Then in the point p, we have*

$$\frac{1}{2}\Delta(|\phi|^2) = \sum_{i,j,k}(\phi_{ij,k})^2+\sum_i \lambda_i(\mathrm{tr}(\phi))_{ii}+\frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2, \tag{3.4}$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of tensor ϕ in the point p.

Proof. The exterior differential of the first covariant derivative to be:

$$\sum_{l,k}\phi_{ijkl}\omega_l\wedge\omega_k = \sum_k \phi_{kj}\Omega_{ki}+\sum_k \phi_{ik}\Omega_{kj}, \tag{3.5}$$

Hence the structure equations combined with equation (3.5) gives the relationship:

$$\phi_{ijkl}-\phi_{ijlk} = -\sum_m \phi_{mj}R_{milk}-\sum_m \phi_{im}R_{mjlk}. \tag{3.6}$$

The Laplacian of ϕ_{ij} is defined like $\sum_k \phi_{ijkk}$ and so using equation (3.6), we have:

$$\begin{aligned} \Delta\phi_{ij} &= \sum_k \phi_{ijkk} \\ &= \sum_k(\phi_{ijkk}-\phi_{ikjk})+\sum_k(\phi_{ikjk}-\phi_{ikkj})+\sum_k(\phi_{ikkj}-\phi_{kkij})+\sum_k \phi_{kkij} \\ &= \sum_k(\phi_{ijkk}-\phi_{ikjk})+\sum_k(\phi_{ikkj}-\phi_{kkij})+\sum_k \phi_{kkij}-\sum_{m,k}\phi_{mk}R_{mijk}-\sum_{m,k}\phi_{im}R_{mkkj}, \end{aligned} \tag{3.7}$$

from hypothesis $\phi_{ij,k} = \phi_{ik,j}$, consequently we can reduces the above formula of Laplancian as

$$\Delta\phi_{ij} = \sum_k \phi_{kkij} - \sum_{m,k} \phi_{mk}R_{mijk} - \sum_{m,k} \phi_{im}R_{mkkj}. \tag{3.8}$$

Now $|\phi|^2 = \sum_{i,j}\phi_{ij}^2$, then the Laplacian of $|\phi|^2$ is given by:

$$\Delta|\phi|^2 = \sum_k \big(\sum_{i,j}\phi_{ij}^2\big)_{kk} = 2\big[\sum_{i,j,k}\phi_{ij,k}^2 + \sum_{i,j}\phi_{ij}\Delta\phi_{ij}\big] \tag{3.9}$$

Denotes $\mathrm{tra}(\phi)$ as trace of ϕ, then replaces equation (3.8) in equation (3.9), we get

$$\frac{1}{2}\Delta|\phi|^2 = \sum_{i,j,k}(\phi_{ij,k})^2 + \sum_{i,j}\phi_{ij}(\mathrm{tra}(\phi))_{ij} - \sum_{i,j,k,m}\phi_{ij}\phi_{mk}R_{mijk} - \sum_{i,j,k,m}\phi_{ij}\phi_{im}R_{mkkj}. \tag{3.10}$$

So the equation (3.10) in the point p we obtain the equation (3.4). $\square$

Corollary 3.1. *Let (I,A) be a Codazzi pair on a manifold Σ and $II(X,Y) = I(AX,Y)$ the bilinear symmetric $(2,0)$- tensor associated to pair (I,A). Suppose that II can be expressed around of one point p as $\sum_{i,j}\phi_{ij}\omega_i \times \omega_j$, then in the point p, we have:*

$$\frac{1}{2}\Delta(|A|^2) = |\nabla A|^2 + \sum_i \lambda_i(\mathrm{tra}(A))_{ii} + \frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i - \lambda_j)^2, \tag{3.11}$$

Where e $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A in the point p.

Proof. Choose around of point p a geodesic neighbourhood $\{e_1,\dots,e_n\}$, then in the point p, we have:

$$(\phi_{ij})_k(p) = e_k(I(A(e_i),e_j))(p) = I(\nabla_{e_k}A(e_i)(p),e_j(p)) + I(A(e_i)(p),\nabla_{e_k}e_j(p)),$$

therefore $\sum_{i,j,k}((\phi_{ij})_k)^2(p) = |\nabla A|^2(p)$, hence we get the equation (3.11). $\square$

Now, we will obtain a Simons' type formula for the traceless Abresch-Rosenberg shape operator S defined on a $H-$surface $\Sigma \subset \mathbb{E}(\kappa,\tau)$, $\tau \neq 0$.

Theorem 3.2. *Let Σ be a $H-$surface in $\mathbb{E}(\kappa,\tau)$. Then, the traceless Abresch-Rosenberg shape operator satisfies*

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2K|S|^2, \tag{3.12}$$

or, equivalently, away from the zeroes of $|S|$,

$$|S|\,\Delta|S| - 2K\,|S|^2 = |\nabla\,|S||^2. \tag{3.13}$$

Proof. Since (I, II_{AR}) is a Codazzi pair on Σ, from Lemma 1.1 we get that $(I, II_{AR} - HI)$ is also a Codazzi pair on Σ, observe that $II_{AR} - HI$ is nothing but the traceless Abresch-Rosenberg fundamental form. Hence, from equation (3.11), we obtain

$$\frac{1}{2}\Delta |S|^2 = |\nabla S|^2 + 2K |S|^2 ,$$

as claimed. Here, we have used that $R_{ijij} = K$ since we are working in dimension two. Also, since S is traceless, the two eigenvalues, λ_i, are opposite signs so $(\lambda_i - \lambda_j)^2 = 4\lambda_i^2 = 2|S|^2$.

To obtain (3.13), using $\Delta |S|^2 = 2|S|\Delta |S| + 2|\nabla |S||^2$, we get

$$|S|\Delta |S| + |\nabla |S||^2 = |\nabla S|^2 + 2K|S|^2$$

Thus, since Σ has dimension two and S is traceless and Codazzi, it holds (cf. [10])

$$|\nabla S|^2 = 2|\nabla |S||^2 ,$$

and we finally obtain (3.13). □

3.2 Finite Abresch-Rosenberg Total Curvature

Complete $H-$surfaces $\Sigma \subset \mathbb{R}^3$ of finite total curvature, that is, those that the L^2-norm of its traceless second fundamental form is finite, are of capital importance on the comprehension of $H-$surfaces. If Σ has constant nonzero mean curvature and finite total curvature, then it must be compact. In the case $H = 0$, Osserman's Theorem gives an impressive description of them.

When we move to $H-$surfaces in $\mathbb{E}(\kappa, \tau)$, the traceless part of the second fundamental form encodes less information about the surface. In this case, the traceless part of the Abresch-Rosenberg form gives a better analog to finite total curvature.

Definition 3.1. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H-$surface, $H^2 + \tau^2 \neq 0$. We say that Σ has finite Abresch-Rosenberg total curvature if the L^2-norm of the traceless Abresch-Rosenberg form is finite, i.e,*

$$\int_\Sigma |S|^2 \, dv_g < +\infty.$$

where dv_g is the volume element of Σ.

We must point out here that the family of complete constant mean curvature surfaces with finite Abresch-Rosenberg total curvature is large, as a first observation, any Abresch-Rosenberg surface has finite Abresch-Rosenberg total curvature. We focus on $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ to show this fact. Recall the following result of Fernández-Mira:

> **Theorem [26, Theorem 16].** *Any holomorphic quadratic differential on an open simply connected Riemann surface is the Abresch-Rosenberg differential of some complete surface Σ with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the space of noncongruent complete mean curvature one half surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with the same Abresch-Rosenberg differential is generically infinite.*

Then, we take the disk $\mathbb{D}$ as our open Riemann surface and a holomorphic quadratic differential $\mathcal{Q}$ on $\mathbb{D}$ that extends continuously to the boundary. Let Σ be the $H = 1/2$ surface constructed in [26, Theorem 16]. Now, we will see that Σ has finite Abresch-Rosenberg total curvature.

For a conformal parameter $z \in \mathbb{D}$, we have $I = 2\lambda\,|dz|^2$ and hence $\mathcal{Q} = f(z)dz^2$, for a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ that extends continuously to the boundary. Then, the square norm of the traceless Abresch-Rosenberg operator is given by $|S|^2 = \frac{4|f|^2}{\lambda^2}$ and $dv_g = \lambda^2\,|dz|^2$. Thus, we have

$$\int_\Sigma |S|^2\, dv_g = 4\int_{\mathbb{D}} |f(z)|^2\,|dz|^2 < +\infty,$$

as claimed.

The study of constant mean curvature surfaces with finite Abresch-Rosenberg total curvature is complementary to study of surfaces with finite total curvature in [32].

Note that we are assuming $H^2 + \tau^2 \neq 0$, otherwise we consider the usual Abresch-Rosenberg operator given by (2.9). In [8], the authors studied complete $H-$surfaces of finite Abresch-Rosenberg total curvature in product spaces $\mathbb{M}^2(\kappa)\times\mathbb{R}$. The fundamental tool in [8] is the Simons' Type Formula for $|S|$ developed in [7] when $\tau = 0$. Hence, using our Simons' Type Formula (Theorem 3.2), we can extend to the case $\tau \neq 0$. We begin with:

Proposition 3.1. *Let Σ be an immersed H-surface, $H^2+\tau^2 \neq 0$ in $\mathbb{E}(\kappa,\tau)$ and let $u = |S|$, where S is the traceless Abresch-Rosenberg form in* (2.13). *Then*

$$-\Delta u \leq au^3 + bu, \tag{3.14}$$

where a, b are constants depending on $\kappa - 4\tau^2$ and H.

Proof. First, from (2.14) and $4H^2 - |A|^2 = 2K_e$, we have

$$K_e = H^2 - \frac{1}{2}|S|^2 - \alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle - \alpha^2\frac{|\mathbf{T}|^4}{4}. \tag{3.15}$$

Using the Gauss equation $K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2$ in (3.15), we can rewrite the Gaussian curvature K as follows

$$K = (\kappa-4\tau^2)(1-|\mathbf{T}|^2) + \tau^2 + H^2 - \frac{1}{2}|S|^2 - \alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle - \alpha^2\frac{|\mathbf{T}|^4}{4}. \tag{3.16}$$

Next, replacing (3.16) into (3.13), we obtain:

$$\begin{aligned}\Delta|S| &\geq 2|S|\left((\kappa-4\tau^2)\nu^2 + \tau^2 + H^2 - \frac{1}{2}|S|^2 - \alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle - \alpha^2\frac{|\mathbf{T}|^4}{4}\right)\\ &\geq -|S|^3 - |S|\left(-2\min\{0,\kappa-4\tau^2\} - 2\tau^2 - 2H^2 + \frac{1}{2}\alpha^2 + 2\alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle\right).\end{aligned} \tag{3.17}$$

Since S is a traceless operator, we have that $|S\mathbf{T}_\theta| = \frac{1}{\sqrt{2}}|S|\,|\mathbf{T}_\theta|$, and using the Schwarz inequality $|\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle| \leq |\mathbf{T}_\theta|\,|S\mathbf{T}_\theta|$, we see that

$$-|S|\,(2\alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle) \geq -\frac{2}{\sqrt{2}}|\alpha|\,|S|^2 \geq -\frac{|\alpha|}{\sqrt{2}}|S|^3 - \frac{|\alpha|}{\sqrt{2}}|S|\,. \tag{3.18}$$

Finally, we replace (3.18) into (3.17) and this yields

$$\Delta|S| \geq -\left(1+\frac{|\alpha|}{\sqrt{2}}\right)|S|^3 - \left(-2\min\{0,\kappa-4\tau^2\} - 2\tau^2 - 2H^2 + \frac{1}{2}\alpha^2 + \frac{|\alpha|}{\sqrt{2}}\right)|S|\,,$$

and this shows (3.14).

□

Any constant mean curvature surface Σ in $\mathbb{E}(\kappa,\tau)$ satisfies a Sobolev inequality of the form

$$|f|_2 \leq C_0\,|\nabla f|_1 + C_1\,|f|_1\,, \tag{3.19}$$

for each $f \in C_0(\Sigma)$, where $|f|_p$ denotes the $L^p(\Sigma)$-norm of f and C_0, C_1 are constants that depends only on the mean curvature H of surface (cf. [33]).

Now, let p be a fixed point of Σ. Consider the intrinsic distance function $d(x,p)$ to p and define the open sets

$$B(R) = \{x\in\Sigma : d(p,x) < R\} \quad \textit{and} \quad E(R) = \{x\in\Sigma : d(x,p) > R\},$$

then with the above notations, we can show the following:

Theorem 3.3. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete $H-$surface such that $H^2+\tau^2 \neq 0$. If Σ has finite Abresch-Rosenberg total curvature, that is,*

$$\int_\Sigma |S|^2\,dv_g < +\infty,$$

then $|S|$ goes to zero uniformly at infinity. More precisely, there exist positive constants A, B and a positive radius R_Σ determined by condition $B\int_{E(R_\Sigma)}|S|^2 \leq 1$ such that for $u=|S|$ and for all $R \geq R_\Sigma$,

$$\|u\|_{\infty,E(2R)} = \sup_{x\in E(2R)} u(x) \leq A\Big(\int_{E(R)}|S|^2 dv_g\Big)^{\frac{1}{2}} \tag{3.20}$$

and, there exist positive constants D and E such that the inequality $\int_\Sigma |S|^2\,dv_g \leq D$ implies

$$\|u\|_\infty = \sup_{x\in\Sigma} u(x) \leq E\int_\Sigma |S|^2\,dv_g.$$

Proof. Since the function $u=|S|$ satisfies the Sobolev inequality (3.19) and the inequality (3.14), we can now work as in the proof of [9, Theorem 4.1] to show that u satisfies the inequality (3.20), letting R to infinity shows that $|S|$ goes to zero uniformly to infinity. □

Next, we study $H-$surfaces Σ on $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$ with finite Abresch-Rosenberg total curvature. Despite what happens in $\mathbb{R}^3$, a $H-$surface $\Sigma\subset\mathbb{E}(\kappa,\tau)$ with finite Abresch-Rosenberg total curvature is not necessarily conformally equivalent to a compact surface minus a finite number of points, in particular, Σ is not necessarily parabolic. However, we obtain:

Theorem 3.4. *Let Σ be a complete surface on $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$, with finite Abresch-Rosenberg total curvature. Suppose one of the following conditions holds*

1. $\kappa-4\tau^2>0$ *and* $H^2+\tau^2>\frac{\kappa-4\tau^2}{4}$.
2. $\kappa-4\tau^2<0$ *and* $H^2+\tau^2>-\frac{(\sqrt{5}+2)}{4}(\kappa-4\tau^2)$.

Then, Σ must be compact.

Proof. From (3.16), the Gaussian curvature can be written as

$$K=(\kappa-4\tau^2)(1-|\mathbf{T}|^2)+\tau^2+H^2-\frac{1}{2}|S|^2-\alpha\langle S\mathbf{T}_\theta,\mathbf{T}_\theta\rangle-\alpha^2\frac{|\mathbf{T}|^4}{4}.$$

Now, $|\mathbf{T}_\theta|\leq 1$, S traceless and the Schwarz inequality imply

$$-\alpha\langle S\mathbf{T}_\theta,\mathbf{T}_\theta\rangle\geq-|\alpha|\frac{|S|\,|\mathbf{T}_\theta|}{\sqrt{2}}\geq-\frac{|\alpha|\,|S|}{\sqrt{2}}.$$

Therefore

$$K\geq(\kappa-4\tau^2)\nu^2+(H^2+\tau^2)-\frac{1}{2}|S|^2-|\alpha|\frac{|S|}{\sqrt{2}}-\alpha^2\frac{|\mathbf{T}|^4}{4}.$$

If $\kappa-4\tau^2>0$, then

$$K\geq(H^2+\tau^2)-\frac{\alpha^2}{4}-\frac{1}{2}|S|^2-\alpha\frac{|S|}{\sqrt{2}}.$$

If $\kappa-4\tau^2<0$, then

$$K\geq(\kappa-4\tau^2)+(H^2+\tau^2)-\frac{\alpha^2}{4}-\frac{1}{2}|S|^2+\alpha\frac{|S|}{\sqrt{2}}.$$

In both cases, the hypothesis and the fact that $|S|$ goes to zero uniformly says that there exist a compact set $\overline{\Omega}$ and $\varepsilon>0$ (depending on the compact set) such that the Gaussian curvature satisfies

$$K(p)\geq\varepsilon>0 \text{ for all } p\in\Sigma\setminus\overline{\Omega}.$$

Therefore, Bonnet Theorem implies that $d(p,\partial\Sigma\setminus\overline{\Omega})$ is uniformly bounded for all $p\in\Sigma\setminus\overline{\Omega}$. Thus, Σ must be compact.

□

3.3 First Eigenvalue of Schröndinger Operators

We will use the Simons' Type Formula for the traceless Abresch-Rosenberg shape operator S (3.13) to estimate the first eigenvalue $\lambda_1(L)$ of a Schrödinger operator L defined over a complete $H-$surface Σ immersed in $\mathbb{E}(\kappa,\tau)$ under the assumption that Σ has finite Abresch-Rosenberg total curvature.

Set $V \in C^0(\Sigma)$ and consider the differential linear operator, called **Schrödinger operator**, given by

$$\begin{array}{rccc} L: & C_0^\infty(\Sigma) & \to & C_0^\infty(\Sigma) \\ & f & \to & Lf := \Delta f + Vf, \end{array}$$

where Δ is the Laplacian with respect to the induced Riemannian metric on Σ and $C_0^\infty(\Sigma)$, as always, stands for the linear space of compactly supported piece-wise smooth functions on Σ.

Given a relatively compact domain $\Omega \subset \Sigma$, it is well-known (cf. [12] and [31, Theorem 8.38]), that there exists a positive function $\rho : \Sigma \to \mathbb{R}$ such that

$$\begin{cases} -\Delta\rho &= (V+\lambda_1(L,\Omega))\rho \quad \text{in} \quad \Omega, \\ \rho &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\lambda_1(L,\Omega) = \inf\left\{ \frac{\int_\Omega (|\nabla f|^2 - Vf^2)dv_g}{\int_\Omega f^2\, dv_g} \; : \; f \in C_0^\infty(\Omega) \right\},$$

that is, $\lambda_1(L,\Omega)$ and ρ are the **first eigenvalue** and a **first eigenfunction**, respectively, associated to the Schrödinger operator L on $\Omega \subset \Sigma$.

Now, we can consider the infimum over all the relatively compact domain in Σ and we can define the infimum of the spectrum of L as

$$\lambda_1(L) := \inf\{\lambda_1(L,\Omega) \; : \; \Omega \subset \Sigma \text{ relatively compact}\},$$

in particular,

$$\lambda_1(L) := \liminf_{i\to+\infty}\lambda_1(L,\Omega_i),$$

for any compact exhaustion $\{\Omega_i\}$ of Σ.

Remark 3.1. *It is standard that the regularity conditions above can be relaxed, but this is not important in our arguments.*

First, we will need an important Lemma that relates the Simons' Formula with the first eigenvalue of any Schrödinger operator.

Lema 3.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete $H-$surface such that $H^2+\tau^2 \neq 0$, and let $\Omega \subset \Sigma$ be a relatively compact domain. Let $\lambda_1(L,\Omega)$ and ρ_Ω be the first eigenvalue and a first eigenfunction, respectively, associated to the Schrödinger operator $L := \Delta + V$ on Ω, $V \in C^0(\Omega)$. Set $C_\Omega = |S|(V+\lambda_1(L,\Omega)) + \Delta|S|$, where S is the traceless Abresch-Rosenberg shape operator on Σ and consider $\phi \in C_0^\infty(\Omega')$, where $\overline{\Omega'} \subset \Omega$, then*

$$\int_\Omega \phi^2\,|S|\,C_\Omega \leq \int_\Omega |S|^2\,|\nabla\phi|^2. \tag{3.21}$$

Proof. Denote $\rho_\Omega = \rho$ and $\lambda_1(L,\Omega) = \lambda_1$, by the Maximum Principle we assume $\rho > 0$ in Ω. Set $w := \ln\rho$ in Ω' and note that it is well defined. Moreover, it holds

$$\Delta w = -(V+\lambda_1) - |\nabla w|^2 .$$

Set $\psi = \phi\,|S|$. then, by Stokes' Theorem

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}\left(\psi^2 \nabla w\right) \\ &= \int_\Omega \psi^2 \Delta w + \int_\Omega 2\psi\langle\nabla\psi,\nabla w\rangle \\ &= -\int_\Omega \psi^2 (V+\lambda_1) - \int_\Omega \psi^2 |\nabla w|^2 + \int_\Omega 2\psi\langle\nabla\psi,\nabla w\rangle \\ &\leq -\int_\Omega \psi^2 (V+\lambda_1) + \int_\Omega |\nabla\psi|^2 , \end{aligned}$$

where we have used $-\psi^2|\nabla w|^2 + 2\psi\langle\nabla\psi,\nabla w\rangle \leq |\nabla\psi|^2$. In other words, we have

$$\int_\Omega \psi^2 (V+\lambda_1) \leq \int_\Omega |\nabla\psi|^2 . \tag{3.22}$$

On the other hand, by definition of ψ

$$|\nabla\psi|^2 = \phi^2 |\nabla|S||^2 + 2\phi\,|S|\,\langle\nabla\phi,\nabla|S|\rangle + |S|^2 |\nabla\phi|^2 ,$$

and, since

$$\frac{1}{2}\operatorname{div}\left(\phi^2\nabla|S|^2\right) = \phi^2|S|\Delta|S| + \phi^2|\nabla|S||^2 + 2\phi\,|S|\,\langle\nabla\phi,\nabla|S|\rangle,$$

it yields that

$$|\nabla\psi|^2 = \frac{1}{2}\operatorname{div}\left(\phi^2\nabla|S|^2\right) - \phi^2|S|\Delta|S| + |S|^2|\nabla\phi|^2 ,$$

from where, taking integrals and using again Stokes' Theorem, we obtain

$$\int_\Omega |\nabla\psi|^2 = -\int_\Omega \phi^2|S|\Delta|S| + \int_\Omega |S|^2|\nabla\phi|^2 . \tag{3.23}$$

Thus, joining the equation (3.23) to the inequality (3.22), we get (3.21). □

Now we can use the Lemma 3.1 to estimate the first eigenvalue $\lambda_1(L)$ of the Schrödinger operator $L = \Delta + V$ defined over a complete $H-$surface Σ with finite Abresch-Rosenberg total curvature.

Theorem 3.5. *Let Σ be a complete two-sided $H-$surface in $\mathbb{E}(\kappa,\tau)$ of finite Abresch-Rosenberg total curvature and $H^2+\tau^2 \neq 0$. Denote by $\lambda_1(L)$ the first eigenvalue associated to the Schrödinger operator $L := \Delta + V$, $V \in C^0(\Sigma)$. Then, Σ is either an Abresch-Rosenberg surface, a Hopf cylinder or*

$$\lambda_1(L) < -\inf_\Sigma\{V+2K\} . \tag{3.24}$$

Proof. Assume that $|S|$ is not identically constant on Σ, otherwise Σ is either an Abresch-Rosenberg $H-$surface or a Hopf cylinder. Then, from (3.13), we get

$$C = |S|\,(V + \lambda_1(L)) + \Delta|S| \geq |S|\,(V + \lambda_1(L) + 2K).$$

Note that $C > |S|\,(V + \lambda_1(L) + 2K)$ at some point since, otherwise, it would implies that $|S|$ is constant, which is a contradiction.

Suppose $\lambda_1(L) \geq -\inf_\Sigma\{V + 2K\}$, then $C \geq |S|\,(V + \lambda_1(L) + 2K) \geq 0$. Now, take $p \in \Sigma$ a fixed point and $R > 0$. Denote by $r(x) = d(x,p)$ the distance function from p and $B(p,R)$ the geodesic ball of radius R. Choose $R' < R$ and define ϕ as follows

$$\phi(x) = \begin{cases} 1 & \text{For } x \text{ such that } 0 \leq r(x) \leq \frac{R'}{2}. \\ 2 - \frac{2}{R'}r(x) & \text{For } x \text{ such that } \frac{R'}{2} < r(x) \leq R'. \\ 0 & \text{For } x \text{ such that } R' < r(x) \leq R. \end{cases}$$

Observes that $\phi \in C^1_0(B(p,R'))$, $\overline{B(p,R')} \subset B(p,R)$ and $B(p,R)$ is a relatively compact set on Σ, then Lemma 3.1 implies

$$\int_{B(p,R)} \phi^2\,|S|\,C_{B(p,R)} \leq \int_{B(p,R)} |S|^2\,|\nabla\phi|^2\,,$$

and, since $\phi = 1$ on $B(p, \frac{R'}{2})$ and $C \geq 0$ on Σ, we get

$$\int_{B(p,\frac{R'}{2})} |S|\,C_{B(p,R)} \leq \int_{B(p,R)} \phi^2\,|S|\,C_{B(p,R)} \leq \frac{4}{(R')^2}\int_{\{x\in\Sigma:\frac{R'}{2}<r(x)\leq R'\}} |S|^2.$$

Hence, from the hypothesis that Σ has finite Abresch-Rosenberg total curvature and letting $R' \to \infty$ we obtain

$$\int_\Sigma |S|C \leq 0.$$

The above equation implies that $C \equiv 0$ on Σ, hence $|S|$ must be constant on Σ, which is a contradiction since we are assuming that $|S|$ is not constant. Therefore $\lambda_1(L)$ must satisfy

$$\lambda_1(L) < -\inf_\Sigma\{V + 2K\}.$$

□

When Σ is a closed $H-$ surface immersed in $\mathbb{E}(\kappa,\tau)$, we have that the function $|S|^2$ is integrable on Σ, since $|S|^2$ is a continuous function on Σ. Therefore any closed $H-$surface in $\mathbb{E}(\kappa,\tau)$ has finite Abresch-Rosenberg total curvature. Consequently, from [64, Theorem 1] and Theorem 3.5 we get an estimate for $\lambda_1(L)$ on any closed surface of $\mathbb{E}(\kappa,\tau)$, $H^2 + \tau^2 \neq 0$.

Corollary 3.2. *Let Σ be a closed $H-$surface in $\mathbb{E}(\kappa,\tau)$, $H^2 + \tau^2 \neq 0$. Denote by $\lambda_1(L)$ the first eigenvalue associated to the Schrödinger Operator $L := \Delta + V$, $V \in C^0(\Sigma)$. Then, Σ is either a rotationally symmetric $H-$sphere, a Hopf $H-$tori or*

$$\lambda_1(L) < -\inf_\Sigma\{V + 2K\}. \tag{3.25}$$

3.3.1 Stability Operator

Now, we obtain estimates for the most natural Schrödinger operator of a complete $H-$surface in $\mathbb{E}(\kappa,\tau)$, the Stability (or Jacobi) operator

$$J = \Delta + (|A|^2 + \mathrm{Ric}(N)),$$

where $\mathrm{Ric}(N)$ is the Ricci curvature of the ambient manifold in the normal direction. Hence, in this case, $V \equiv |A|^2 + \mathrm{Ric}(N)$.

Note that, since $\mathrm{Ric}(N) = (\kappa - 4\tau^2)|\mathbf{T}|^2 + 2\tau^2$ and the Gauss equation (1.15), we have

$$\begin{aligned} V + 2K &= 4H^2 + 2(K - K_e) + (\kappa - 4\tau^2)|\mathbf{T}|^2 + 2\tau^2 \\ &= 4H^2 + \kappa + (\kappa - 4\tau^2)\nu^2. \end{aligned}$$

Hence, Theorem 3.5 and the above equality gives:

Theorem 3.6. *Let Σ be a complete two sided $H-$surface of finite Abresch-Rosenberg total curvature in $\mathbb{E}(\kappa,\tau)$, $H^2 + \tau^2 \neq 0$.*

- *If $\kappa - 4\tau^2 > 0$. Then, Σ is either an Abresch-Rosenberg $H-$surface, a Hopf cylinder, or*
$$\lambda_1 < -(4H^2 + \kappa).$$
- *If $\kappa - 4\tau^2 < 0$. Then, Σ is either an Abresch-Rosenberg $H-$surface, or*
$$\lambda_1 < -(4H^2 + \kappa) - (\kappa - 4\tau^2).$$

Remark 3.2. *These estimates were obtained by Alías-Meroño-Ortíz [5] for closed surfaces in $\mathbb{E}(\kappa,\tau)$.*

3.4 Pinching Theorems for $H-$surfaces in $\mathbb{E}(\kappa,\tau)$

In this section we will use the Simons' Type Formula for the traceless Abresch-Rosenberg shape operator (3.12) together with the Omori-Yau Maximum Principle to classify complete $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ satisfying a pinching condition on its Abresch-Rosenberg fundamental form. First, we recall the Omori-Yau Maximum Principle for the reader convenience.

Theorem 3.7 ([65])**.** *Let $\mathcal{M}$ be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(\mathcal{M})$ is bounded from above, then there exits a sequence of points $\{p_j\}_{j\in\mathbb{N}} \in \mathcal{M}$ such that:*

1. $\lim_{j\to\infty} u(p_j) = \sup_{\mathcal{M}} u$.
2. $|\nabla u|(p_j) < \frac{1}{j}$.

3. $\Delta u(p_j) < \frac{1}{j}$.

Second, we study the Simons' Type Formula (3.12) in the set of non-umbilical points of S.

Proposition 3.2. *Let Σ be a $H-$surface in $\mathbb{E}(\kappa,\tau)$. Then, away from the umbilic points of S, it holds*

$$\frac{1}{2}\Delta|S|^2 \geq |\nabla S|^2 + |S|^2 F(|S|), \tag{3.26}$$

where $F(x) = -x^2 + bx + a$ is the second degree polynomial given by

$$\begin{aligned} a &= 2(\kappa - 4\tau^2) + 2(H^2+\tau^2) - 2(\kappa-4\tau^2)|\mathbf{T}|^2 - \alpha^2\frac{|\mathbf{T}|^4}{2}, \\ b &= -\sqrt{2}|\alpha|\,|\mathbf{T}|^2, \end{aligned}$$

where $\alpha = \frac{\kappa-4\tau^2}{2\sqrt{H^2+\tau^2}}$.

Proof. From (3.16), the Gaussian curvature can be written as

$$K = (\kappa-4\tau^2)(1-|\mathbf{T}|^2) + \tau^2 + H^2 - \alpha^2\frac{|\mathbf{T}|^4}{4} - \frac{1}{2}|S|^2 - \alpha\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle.$$

Since $|\langle S\mathbf{T}_\theta, \mathbf{T}_\theta\rangle| \leq |S\mathbf{T}_\theta|\,|\mathbf{T}_\theta| = \frac{1}{\sqrt{2}}|S|\,|\mathbf{T}_\theta|^2$, substituting the above formulas into (3.12) yields

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 \geq |\nabla S|^2 + |S|^2\big(&2(\kappa-4\tau^2)(1-|\mathbf{T}|^2) + 2(\tau^2+H^2) - \alpha^2\frac{|\mathbf{T}|^4}{2} \\ &- |S|^2 - \sqrt{2}|\alpha|\,|S|\,|\mathbf{T}|^2\big), \end{aligned}$$

as claimed. □

So, our next step is to study the first positive root $\bar{x} \in \mathbb{R}^+$ of $F(x)$ so that $F(x) > 0$ for all $x \in (0,\bar{x})$. To do so, we set $t = |\mathbf{T}|^2 \in [0,1]$ and hence, we can rewrite:

$$\begin{aligned} a(t) &= 2(\kappa-4\tau^2) + 2(H^2+\tau^2) - 2(\kappa-4\tau^2)t - \frac{\alpha^2}{2}t^2. \\ b(t) &= -\sqrt{2}|\alpha|t. \end{aligned}$$

In order to obtain a positive real root, the coefficients of F must hold $h(t) = b(t)^2 + 4a(t) > 0$ and $a(t) > 0$ for all $t \in [0,1]$. This means that

$$(H^2+\tau^2) + (\kappa-4\tau^2)(1-t) > 0 \text{ for all } t \in [0,1],$$

and

$$2(H^2+\tau^2) + 2(\kappa-4\tau^2)(1-t) > 0 \text{ for all } t \in [0,1].$$

Proposition 3.3. *Define* $G:[0,1]\to\mathbb{R}$ *as*

$$G(t)=\frac{b(t)+\sqrt{b(t)^2+4a(t)}}{2}=\frac{-\left|\kappa-4\tau^2\right|t+4\sqrt{H^2+\tau^2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)(1-t)}}{2\sqrt{2}\sqrt{H^2+\tau^2}}, \tag{3.27}$$

then, assuming $h(t)>0$ *and* $a(t)>0$ *for all* $t\in[0,1]$, *the function* $G(t)$ *satisfies:*

$$\min_{t\in[0,1]}G(t)=\min\{G(0),G(1)\} \tag{3.28}$$

Proof. We compute the interior critical points of $G(t)$. To do so, we compute the derivative of G at an interior point

$$\begin{aligned}G'(t)&=\frac{b'(t)}{2}\left(1+\frac{b(t)}{\sqrt{b(t)^2+4a(t)}}\right)+\frac{a'(t)}{\sqrt{(b(t))^2+4a(t)}}\\&=-|\alpha|\left(\frac{b(t)+\sqrt{b(t)^2+4a(t)}}{\sqrt{b(t)^2+4a(t)}}\right)-\frac{\alpha^2t+2(\kappa-4\tau^2)}{\sqrt{(b(t))^2+4a(t)}}\\&=\frac{1}{\sqrt{b(t)^2+4a(t)}}\left(-2|\alpha|G(t)-\alpha^2t-2(\kappa-4\tau^2)\right)\end{aligned}$$

Assume that there exists $\bar{t}\in(0,1)$ so that $G'(\bar{t})=0$, then the above equation implies that the function $\Psi(t):=-2|\alpha|G(t)-\alpha^2t-2(\kappa-4\tau^2)$ satisfies $\Psi(\bar{t})=0$ and $\Psi'(\bar{t})=-\alpha^2\bar{t}$. Moreover, observe that

$$G''(t)=R(t)\Psi(t)+\frac{\Psi'(t)}{\sqrt{b(t)^2+4a(t)}},$$

for some smooth function $R:(0,1)\to\mathbb{R}^+$. Hence

$$G''(\bar{t})=\frac{\Psi(\bar{t})}{\sqrt{b(\bar{t})^2+4a(\bar{t})}}=-\frac{\alpha^2\bar{t}}{\sqrt{b(t)^2+4a(t)}}<0.$$

Therefore, G does not have an interior minimum. Therefore, in any case, $\min_{t\in[0,1]}G(t)=\min\{G(0),G(1)\}$. ☐

Next, we compute the minimum of G. To do so, we will distinguish two cases depending on the sign of $\kappa-4\tau^2$.

3.4.1 Case $\kappa-4\tau^2>0$

In this case,

$$(H^2+\tau^2)+(\kappa-4\tau^2)(1-t)\geq 0 \text{ for all } t\in[0,1],$$

and the function $a(t)$ is positive on $[0,1]$, when $4(H^2+\tau^2)>\kappa-4\tau^2$, since

$$a(t)\geq 2(H^2+\tau^2)-\frac{\alpha^2}{2}=\frac{16(H^2+\tau^2)^2-(\kappa-4\tau^2)^2}{8(H^2+\tau^2)}>0.$$

We start by computing the minimum of G:

Proposition 3.4. *If $\kappa-4\tau^2>0$, then*

$$\min_{t\in[0,1]} G(t)=\frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}}.$$

Proof. We compute the derivative of $G(t)$:

$$G'(t)=\frac{1}{2\sqrt{2}\sqrt{H^2+\tau^2}}\left(-(\kappa-4\tau^2)-\frac{2\sqrt{(H^2+\tau^2)}(\kappa-4\tau^2)}{\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)(1-t)}}\right),$$

then $G(t)$ is a decreasing function on the interval $[0,1]$, so $G(1)$ is the minimum value of $G(t)$. $\square$

Now, we are ready to show a pinching theorem for complete $H-$ surfaces in $\mathbb{E}(\kappa,\tau)$, when $\kappa-4\tau^2>0$.

Theorem 3.8. *Let Σ be a complete immersed H-surface in $\mathbb{E}(\kappa,\tau)$, with $\kappa-4\tau^2>0$. Assume that $4(H^2+\tau^2)>\kappa-4\tau^2$ and*

$$\sup_{\Sigma}|S|<\frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}},$$

where S is the traceless Abresch-Rosenberg shape operator defined on Σ. Then, Σ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa,\tau)$. Moreover, if there exists one point $p\in\Sigma$ such that $|S(p)|=\frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}}$, then Σ is a Hopf cylinder.

Proof. Denote $A_2=\frac{4(H^2+\tau^2)-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}}$. First, we show:

> **Claim 1.** *There exists a positive constant d so that $\Delta|S|^2(p)\geq d|S|^2(p)$ for each non umbilical point p of S.*

Proof of Claim 1. From Proposition 3.2, away from the umbilic points of S, we have

$$\frac{1}{2}\Delta|S|^2\geq|\nabla S|^2+|S|^2F(|S|), \tag{3.29}$$

where $F(x)=-x^2+b(t)x+a(t)$. The hypothesis says that $F(0)=a(t)$ is positive for all $t\in[0,1]$, So for p non umbilical point of S, we define

$$F_p(x)=-x^2+b(t(p))x+a(t(p)).$$

Let p such that $b(t(p))<2A_2$. In this case $F_p(\frac{1}{2}b(t(p)))$ or $F_p(0)=a(t(p))$ is the maximum value of $F_p(x)$ on the interval $[0,A_2]$. If $F_p(0)=a(t(p))$ is the maximum value, then $F_p(x)$ is a decreasing polynomial on $[0,A_2]$. Hence,

$$F_p(|S(p)|)\geq F_p(\sup_{\Sigma}|S|)=\frac{d_1}{2}>F_p(A_2)\geq F_p(G(t(p)))=0.$$

Now, suppose $F_p(\frac{1}{2}b(t(p)))$ is the maximum value of $F_p(x)$ on the interval $[0,A_2]$. On the one hand, the polynomial $F_p(x)$ is decreasing on the interval $[\frac{b(t(p))}{2},A_2]$, consequently if $|S(p)| \in [\frac{b(t(p))}{2},A_2]$ we have

$$F_p(|S(p)|) \geq F_p(\sup_\Sigma |S|) = \frac{d_1}{2} > F_p(A_2) \geq F_p(G(t(p))) = 0.$$

On the other hand, on the interval $[0,\frac{b(t(p))}{2}]$ the function $F_p(x)$ is an increasing polynomial. So, If $|S(p)| \in [0,\frac{b(t(p))}{2}]$, we get

$$F_p(|S(p)|) \geq F_p(0) = a(t(p)) > \frac{d_2}{2} = \frac{16(H^2+\tau^2)^2-(\kappa-4\tau^2)^2}{8(H^2+\tau^2)} > 0.$$

Finally, let p such that $b(t(p)) \geq 2A_2$, then is clear that

$$F_p(|S(p)|) \geq F_p(0) = \frac{d_2}{2} > 0,$$

since $F_p(x)$ on the interval $[0,\frac{b(t(p))}{2}]$ is an increasing function. Therefore, the above shows that

$$F_p(|S(p)|) \geq \frac{d}{2}, \tag{3.30}$$

for each p non umbilical point of S, where $d = \min\{d_1,d_2\}$. So, the inequality (3.30) in the equation (3.29) shows that $\Delta|S|^2 \geq d|S|^2$ away from the umbilic points of S. □

Thus, since $|S|$ is bounded by hypothesis, from Corollary 2.1 the Ricci curvature of Σ is bounded from below, then we can apply the Omori-Yau Maximum Principle to the function $|S|^2$. So, there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$ in Σ, such that:

$$\lim_{j\to\infty} |S|^2(p_j) = \sup_\Sigma |S|^2 \ \ \textit{and} \qquad \lim_{j\to\infty} \Delta|S|^2(p_j) \leq 0.$$

The above Claim 1 implies that $\sup_\Sigma |S|^2 = 0$, hence, $|S| = 0$ on Σ and Σ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa,\tau)$.

Moreover, suppose that there exists one point $p \in \Sigma$, such that $|S(p)| = A_2$. So from Claim 1, in a neighbourhood Ω of p we have that

$$\Delta|S| \geq 0.$$

Then the Interior Maximum Principle implies that $|S| \equiv A_2$ on Ω, and hence $|S| \equiv A_2$ on Σ. Finally, $|S|^2 = 2q^{AR}$, thereby, q^{AR} is a positive constant function on Σ and from Lemma 2.4, we conclude that Σ is a Hopf cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$. □

3.4.2 Case $\kappa - 4\tau^2 < 0$

In this case, if $H^2+\tau^2 \geq -(\kappa-4\tau^2)$, then

$$(H^2+\tau^2)+(\kappa-4\tau^2)(1-t) \geq 0 \text{ for all t} \in [0,1]$$

and the function $a(t)$ is positive on $[0,1]$, when $H^2+\tau^2 > -(\kappa-4\tau^2)$

$$a(t) > -2(\kappa-4\tau^2)t - \frac{\alpha^2}{2}t^2 = \frac{-16(\kappa-4\tau^2)(H^2+\tau^2)t-(\kappa-4\tau^2)^2t^2}{8(H^2+\tau^2)},$$

now from inequality $H^2+\tau^2 > -\frac{(\kappa-4\tau^2)}{16}$, we obtain

$$8(H^2+\tau^2)a(t) > (\kappa-4\tau^2)^2t-(\kappa-4\tau^2)^2t^2 = (\kappa-4\tau^2)^2t(1-t) \geq 0.$$

For each $t \in [0,1]$.

Next, we describe the minimum of $G(s,t)$ in the case $\kappa-4\tau^2 < 0$.

Proposition 3.5. *If $\kappa-4\tau^2 < 0$ and $H^2+\tau^2 > -(\kappa-4\tau^2)$, then*

$$\min_{(s,t)\in\overline{\mathscr{A}}} G(s,t) = \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)}.$$

Proof. We compute the derivative of $G(t)$:

$$G'(t) = \frac{1}{2\sqrt{2}\sqrt{H^2+\tau^2}}\left((\kappa-4\tau^2) - \frac{2(\kappa-4\tau^2)\sqrt{(H^2+\tau^2)}}{\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)(1-t)}}\right).$$

For $\kappa-4\tau^2<0$, we have that

$$-2(\kappa-4\tau^2) \leq \frac{-2(\kappa-4\tau^2)\sqrt{(H^2+\tau^2)}}{\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)(1-t)}}, \text{ for all } t\in[0,1],$$

then $G'(t) \geq \frac{-(\kappa-4\tau^2)}{2\sqrt{2}\sqrt{H^2+\tau^2}} \geq 0$ on the interval $[0,1]$, consequently $G(t)$ is an increasing function on $[0,1]$, therefore $G(0)$ is the minimum value of $G(t)$ in $[0,1]$. $\square$

Now, we are ready to announce a pinching theorem for complete $H-$surfaces in $\mathbb{E}(\kappa,\tau)$, with $\kappa-4\tau^2<0$.

Theorem 3.9. *Let Σ be a complete immersed H-surface in $\mathbb{E}(\kappa,\tau)$, with $\kappa-4\tau^2<0$. Assume that $(H^2+\tau^2) > -(\kappa-4\tau^2)$ and*

$$\sup_{\Sigma}|S| < \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)},$$

where S is the traceless Abresch-Rosenberg shape operator defined on Σ. Then, Σ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa,\tau)$. Moreover, if there exists one point $p\in\Sigma$ such that $|S(p)| = \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)}$, then Σ is a Hopf cylinder.

Proof. Denote $A_0 = \sqrt{2}\sqrt{(H^2+\tau^2)+(\kappa-4\tau^2)}$. As above, we show:

Claim 2. *There exists a positive constant d so that $\Delta|S|^2(p) \geq d|S|^2(p)$ for each non umbilical point p of S.*

Proof of Claim 2. From Proposition 3.2, the Simons' formula for $|S|^2$ away from the umbilic points of S is given by

$$\frac{1}{2}\Delta|S|^2 \geq |\nabla S|^2 + |S|^2 F(|S|), \tag{3.31}$$

where $F(x) = -x^2 + b(t)x + a(t)$, the hypothesis says that $F(0) = a(t)$ is positive for all $t \in [0,1]$, so for p non umbilical point of S, we define

$$F_p(x) = -x^2 + b(t(p))x + a(t(p)).$$

Let p such that $b(t(p)) < 2A_0$, therefore $F_p(\frac{1}{2}b(t(p)))$ or $F_p(0)$ is the maximum value of $F_p(x)$ on the interval $[0,A_0]$. If $F_p(0)$ is the maximum value of $F_p(x)$ on the interval $[0,A_0]$, we have that $F_p(x)$ is a positive decreasing function on $[0,A_0]$, then

$$F_p(|S(p)|) \geq F_p(\sup_{\Sigma}|S|) = \frac{d_1}{2} > F_p(A_0) \geq F_p(G(t(p))) = 0.$$

Now, suppose $F_p(\frac{1}{2}b(t(p)))$ is the maximum value of $F_p(x)$ on the interval $[0,A_0]$. On the one hand, $F_p(x)$ is an increasing polynomial on $[0,\frac{b(t(p))}{2}]$, hence, if $|S(p)| \in [0,\frac{b(t(p))}{2}]$, we obtain

$$F_p(|S(p)|) \geq F_p(0) = a(t(p)) \geq \frac{d_2}{2} > 0,$$

where $d_2 = 2(H^2+\tau^2)+2(\kappa-4\tau^2)$. On the other hand, on the interval $[\frac{b(t(p))}{2},A_0]$ the function $F_p(x)$ is decreasing, so, if $|S(p)| \in [\frac{b(t(p))}{2},A_0]$, we get

$$F_p(|S(p)|) \geq F_p(\sup_{\Sigma}|S|) = \frac{d_1}{2} > F_p(A_0) \geq F_p(G(t(p))) = 0.$$

Finally, given p such that $b(t(p)) \geq 2A_0$, then

$$F_p(|S(p)|) \geq F_p(0) \geq \frac{d_2}{2} > 0,$$

since $F_p(x)$ on interval $[0,\frac{b(t(p))}{2}]$ is an increasing function. Therefore

$$F_p(|S(p)|) \geq \frac{d}{2}, \tag{3.32}$$

for any non umbilical point p of S, where $d = \min\{d_1,d_2\}$. So, the inequality (3.32) in the equation (3.31) shows that $\Delta|S|^2 \geq d|S|^2$ away from the umbilic points of S.

□

Thus, since $|S|$ is bounded by hypothesis, from Corollary 2.1 the Ricci curvature of Σ is bounded from below, then we can apply the Omori-Yau Maximum Principle to $|S|^2$. So, there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$ in Σ, such that:

$$\lim_{j\to\infty}|S|^2(p_j) = \sup_{\Sigma}|S|^2 \ \ \textit{and} \qquad \lim_{j\to\infty}\Delta|S|^2(p_j) \leq 0.$$

The above Claim 2 implies that $\sup_{\Sigma}|S|^2 = 0$, hence, $|S| = 0$ on Σ and Σ is invariant by one parameter group of isometries of $\mathbb{E}(\kappa,\tau)$.

Moreover, suppose that there exists one point $p \in \Sigma$, such that $|S(p)| = A_2$. So from Claim 2, in a neighbourhood Ω of p, we have that

$$\Delta|S| \geq 0.$$

Then the Interior Maximum Principle implies that $|S| \equiv A_2$ on Ω, and hence $|S| \equiv A_2$ on Σ. Finally, $|S|^2 = 2q^{AR}$, thereby, q^{AR} is a positive constant function on Σ and from Lemma 2.4, we conclude that Σ is a Hopf cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$. $\square$

Chapter 4

Immersed Compact Disks

In this chapter, we classify constant mean curvature compact disks immersed in $\mathbb{E}(\kappa,\tau)$ under certain geometric conditions along the boundary. First, we will classify immersed compact disks with regular boundary. In particular, these results generalize previous classifications given in [17, 19]. Second, assuming certain differentiability conditions, we will classify immersed compact disks whose boundary is piece-wise smooth.

This chapter is organized as follows; in Section 4.1, we will define the Abresch-Rosenberg lines of curvature and we will exhibit the relationship between these lines and the Abresch-Rosenberg differential. Later, we will show the Key Lemma of this chapter, such result is a Joachimstalh's type Theorem for $H-$surfaces in $\mathbb{E}(\kappa,\tau)$. In Section 4.2, we will classify constant mean curvature compact disks immersed in $\mathbb{E}(\kappa,\tau)$ with geometric hypothesis along the regular boundary. First, we will study the case of compact disks in product spaces $\mathbb{M}^2(\kappa)\times\mathbb{R}$. After, we will study compact disks in $\mathbb{E}(\kappa,\tau)$ for $\tau\neq 0$. In Section 4.3, we will study immersed compact disk with piece-wise smooth boundary assuming that the image of the immersion is contained in the interior of a smooth surface in $\mathbb{E}(\kappa,\tau)$ and the number of vertices with interior angle $<\pi$ at the boundary is less than or equal to 3. Under the above hypothesis, we will extend the previous results of regular case to immersed compact disks with piece-wise smooth boundary.

4.1 Abresch-Rosenberg Lines of curvature

We begin this chapter by defining the concept of line of curvature with respect to the Abresch-Rosenberg shape operator of a $H-$surface Σ in $\mathbb{E}(\kappa,\tau)$ and we characterize these curves in terms of Abresch-Rosenberg differential.

Definition 4.1. *Let Σ be a $H-$surface in $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$ and Γ a regular curve parametrized by a map $\gamma:(-\varepsilon,\varepsilon)\to\Sigma$. We say that $\Gamma=\gamma(-\varepsilon,\varepsilon)$ is a* **line of curvature for the Abresch-Rosenberg shape operator** *S_{AR} if there exists a smooth function $\lambda:(-\varepsilon,\varepsilon)\to\mathbb{R}$ such that $S_{AR}(\gamma'(t))=\lambda(t)\gamma'(t)$. In such case, we call Γ an* **Abresch-Rosenberg line of curvature***, in short, an* **AR-line of curvature***.*

Definition 4.1 says that the tangent vector along the curve γ in Σ is an eigenvector of S_{AR} along γ. So, the definition is nothing but the natural extension of a line of curvature with respect to the shape operator induced by the second fundamental form of a surface in $\mathbb{R}^3$ to the case of the Abresch-Rosenberg shape operator S_{AR} for a $H-$surface in $\mathbb{E}(\kappa,\tau)$. In analogy with the situation in $\mathbb{R}^3$ (see [13]), there exists a link between lines of curvatures with respect to S_{AR} and the Abresch-Rosenberg differential $Q^{AR}dz^2$.

Proposition 4.1. *Let Σ be a $H-$surface in $\mathbb{E}(\kappa,\tau)$, $H^2+\tau^2\neq 0$. Then, $\gamma:(-\varepsilon,\varepsilon)\to\Sigma$ is a line of curvature for S_{AR} if, and only if, the imaginary part of $Q^{AR}dz^2$ is zero along γ.*

Proof. Let $z=u+iv$ be a local conformal parameter of Σ with respect to the first fundamental form I and set

$$\begin{aligned} II_{AR}(\partial_u,\partial_u) &= I(S_{AR}(\partial_u),\partial_u)=L,\\ II_{AR}(\partial_u,\partial_v) &= I(S_{AR}(\partial_u),\partial_v)=M,\\ II_{AR}(\partial_v,\partial_v) &= I(S_{AR}(\partial_v),\partial_v)=N. \end{aligned}$$

The curve $\gamma(t)=(u(t),v(t))$ is a line of curvature with respect to S_{AR} if $S_{AR}(\gamma'(t))=\lambda(t)\gamma'(t)$. In the local coordinates (u,v) this means

$$\begin{bmatrix} L(\gamma(t)) & M(\gamma(t))\\ M(\gamma(t)) & N(\gamma(t))\end{bmatrix}\begin{bmatrix}u'(t)\\ v'(t)\end{bmatrix}=\lambda(t)\begin{bmatrix}u'(t)\\ v'(t)\end{bmatrix}. \tag{4.1}$$

Hence from (4.1), we get the following linear system

$$\begin{aligned} L(\gamma(t))u'(t)+M(\gamma(t))v'(t) &= \lambda(t)u'(t),\\ M(\gamma(t))u'(t)+N(\gamma(t))v'(t) &= \lambda(t)v'(t). \end{aligned} \tag{4.2}$$

On the one hand, from (4.2), we obtain

$$M(\gamma(t))(v'(t))^2+(L(\gamma(t))-N(\gamma(t)))u'(t)v'(t)-M(\gamma(t))(u'(t))^2=0. \tag{4.3}$$

On other hand, from the definition of the Abresch-Rosenberg differential

$$2Q^{AR}(\gamma(t))dz(\gamma(t))^2=(L(\gamma(t))-N(\gamma(t))-2iM(\gamma(t)))dz(\gamma(t))^2, \tag{4.4}$$

then, straightforward computation shows that the imaginary part is given by

$$\begin{aligned} 2\mathrm{Im}(Q^{AR}(\gamma(t))dz(\gamma(t))^2) &= M(\gamma(t))(v'(t))^2+(L(\gamma(t))-N(\gamma(t)))u'(t)v'(t)\\ &\quad -M(\gamma(t))(u'(t))^2, \end{aligned} \tag{4.5}$$

hence, (4.5) is nothing but the left part of (4.3). This shows that the imaginary part of (4.4) vanishes when $\gamma(t)$ is line of curvature with respect to S_{AR}.

Reciprocally, assume that the imaginary part of $Q^{AR}(\gamma(t))dz(\gamma(t))^2$ is zero, this condition is given by (4.5). Then, for all $t\in(-\varepsilon,\varepsilon)$ so that $u'(t)\neq 0$ and $v'(t)\neq 0$ we have

$$\frac{M(\gamma(t))v'(t)+L(\gamma(t))u'(t)}{u'(t)}=\frac{N(\gamma(t))v'(t)+M(\gamma(t)u'(t))}{v'(t)}. \tag{4.6}$$

Now, define the function $\lambda : (-\varepsilon,\varepsilon) \to \mathbb{R}$ as follows

$$\lambda(t) = \begin{cases} \frac{M(\gamma(t))v'(t)+L(\gamma(t))u'(t)}{u'(t)} & \text{, for } t \text{ such that } u'(t) \neq 0 \text{ and } v'(t) \neq 0, \\ L(\gamma(t)) & \text{, for } t \text{ such that } u'(t) \neq 0 \text{ and } v'(t) = 0, \\ N(\gamma(t)) & \text{, for } t \text{ such that } u'(t) = 0 \text{ and } v'(t) \neq 0. \end{cases}$$

Therefore, (4.2) and (4.6) imply $S_{AR}(\gamma'(t)) = \lambda(t)\gamma'(t)$ for all $t \in (-\varepsilon,\varepsilon)$, this shows $\gamma'(t)$ is a line of curvature with respect to S_{AR}. □

Next, we will establish a Joachimstahl's Type Theorem (see [61]) for $H-$surfaces in $\mathbb{E}(\kappa,\tau)$. This is a key step in this chapter.

Lema 4.1 (Key Lemma). *Let $\Sigma_i \subset \mathbb{E}(\kappa,\tau)$,, $i = 1,2$, be H_i-surfaces so that $H_i^2+\tau^2 \neq 0$ and $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ is transversal. Let $\Gamma \subset \Sigma_1 \cap \Sigma_2$ be a regular curve. Assume that along Γ one has*

a) $\langle N_1, N_2\rangle$ is constant and

b) $\sqrt{H_1^2+\tau^2}\langle \mathbf{T}^2_{\theta_2}, N_1\rangle\langle J_2\mathbf{T}^2_{\theta_2}, N_1\rangle = \sqrt{H_2^2+\tau^2}\langle \mathbf{T}^1_{\theta_1}, N_2\rangle\langle J_1\mathbf{T}^1_{\theta_1}, N_2\rangle,$

where $\alpha_i = \frac{\kappa-4\tau^2}{2\sqrt{H_i^2+\tau^2}}$, $\mathbf{T}^i_{\theta_i} = \cos\theta_i\mathbf{T}_i + \sin\theta_i J_i\mathbf{T}_i$ and $J_iX = N_i \wedge X$ for $i = 1,2$. Then, Γ is an AR-line of curvature for Σ_1 if, and only if, Γ is an AR-line of curvature for Σ_2.

Proof. We assume that Γ is an AR-line of curvature for Σ_2, the other case is completely analogous. First, since $\langle N_1(\gamma(t)), N_2(\gamma(t))\rangle = d$ is constant along $\Gamma = \gamma(-\varepsilon,\varepsilon)$, where γ is parametrized by arc-length, then

$$\langle A_1(\gamma'(t)), N_2(\gamma(t))\rangle + \langle N_1(\gamma(t)), A_2(\gamma'(t))\rangle = 0, \tag{4.7}$$

where A_1 and A_2 are the shape operators of Σ_1 and Σ_2 respectively. Now, relating A_1 and A_2 with S^1_{AR} and S^2_{AR} respectively and using that $\gamma(t)$ is a line of curvature for S^2_{AR}, we can rewrite (4.7) as:

$$-\alpha_1\langle \mathbf{T}^1_{\theta_1}, \gamma'(t)\rangle\langle \mathbf{T}^1_{\theta_1}, N_2\rangle - \langle S^1_{AR}(\gamma'(t)), N_2\rangle - \alpha_2\langle \mathbf{T}^2_{\theta_2}, \gamma'(t)\rangle\langle \mathbf{T}^2_{\theta_2}, N_1\rangle = 0, \tag{4.8}$$

where $\alpha_i = \frac{\kappa-4\tau^2}{2\sqrt{H_i^2+\tau^2}}$ and $\mathbf{T}^i_{\theta_i} = \cos\theta_i\mathbf{T}_i + \sin\theta_i J\mathbf{T}_i$, $i = 1,2$.

We can orient γ so that $(1-d^2)N_1 \wedge N_2 = \gamma'(t)$, where d is the contact constant angle between Σ_1 and Σ_2.

Since the intersection is transversal, $\{N_1, N_2, \gamma'(t)\}$ is an oriented basis of $T_{\gamma(t)}\mathbb{E}(\kappa,\tau)$ for each t were the intersection is transversal. Then, the following equations hold:

$$\begin{aligned} \langle \mathbf{T}^1_{\theta_1}, \gamma'(t)\rangle &= (1-d^2)\langle \mathbf{T}^1_{\theta_1}, N_1 \wedge N_2\rangle = -(1-d^2)\langle J_1\mathbf{T}^1_{\theta_1}, N_2\rangle, \\ \langle \mathbf{T}^2_{\theta_2}, \gamma'(t)\rangle &= (1-d^2)\langle \mathbf{T}^2_{\theta_2}, N_1 \wedge N_2\rangle = (1-d^2)\langle J_2\mathbf{T}^2_{\theta_2}, N_1\rangle, \end{aligned} \tag{4.9}$$

where $J_i\mathbf{T}^i_{\theta_i} = N_i \wedge \mathbf{T}^i_{\theta_i}$ for $i = 1,2$. Therefore, (4.8) and (4.9) imply that

$$\begin{aligned}\langle S^1_{AR}(\gamma'(t)), N_2\rangle &= \frac{(1-d^2)(\kappa-4\tau^2)}{\sqrt{H_1^2+\tau^2}}\langle \mathbf{T}^1_{\theta_1}, N_2\rangle\langle J_1\mathbf{T}^1_{\theta_1}, N_2\rangle \\ &\quad - \frac{(1-d^2)(\kappa-4\tau^2)}{\sqrt{H_2^2+\tau^2}}\langle \mathbf{T}^2_{\theta_2}, N_1\rangle\langle J_2\mathbf{T}^2_{\theta_2}, N_1\rangle \\ &= 0,\end{aligned}$$

where we have used item b). Thus, $\langle S^1_{AR}(\gamma'(t)), N_2\rangle = 0$ along Γ. Therefore, since $\langle S^1_{AR}(\gamma'(t)), N_1\rangle = 0$ along Γ, we obtain that $S^1_{AR}(\gamma'(t)) = \lambda(t)\gamma'(t)$, that is, Γ is an AR-line of curvature for Σ_1. □

Observe that we can see item b) above geometrically as follows; Let X be a unitary vector field along Σ, not necessarily tangential. Then, $\{\mathbf{T}_\theta, J\mathbf{T}_\theta\}$ is an orthogonal frame along Σ away from the points where $|\mathbf{T}| = 0$. Let ω be the (oriented) angle between $\mathbf{T}_\theta$ and X, that is,

$$\langle \mathbf{T}_\theta, X\rangle = |\mathbf{T}|\cos\omega,$$

and hence,

$$2\langle \mathbf{T}_\theta, X\rangle\langle J\mathbf{T}_\theta, X\rangle = |\mathbf{T}|^2\sin(2\omega).$$

So, coming back to the situation on the Key Lemma, let ω_i denote the (oriented) angle between $\mathbf{T}^i_{\theta_i}$ and N_j, for $i, j = 1,2$ and $i \neq j$. Hence, item b) can be re-written as

$$\frac{|\mathbf{T}_1|^2}{\sqrt{H_1^2+\tau^2}}\sin(2\omega_1) = \frac{|\mathbf{T}_2|^2}{\sqrt{H_2^2+\tau^2}}\sin(2\omega_2).$$

The remainder of this section will be devoted to study some particular cases of Lemma 4.1.

A curve Γ on a surface Σ in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is **horizontal** if Γ is contained in a horizontal slice $\mathbb{M}^2(\kappa)\times\{\xi_0\}$, for some $\xi_0 \in \mathbb{R}$. On the other hand, the curve Γ in Σ is said to be **vertical** if it is an integral curve of the vector field $\mathbf{T}$.

The Key Lemma gives us general conditions for Γ being an AR-line of curvature of both surfaces Σ_1 and Σ_2. Nevertheless, we will see that certain geometric configurations in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ imply the conditions on the Key Lemma using the vertical and horizontal curves.

Corollary 4.1. *Let Σ_1 and Σ_2 two constant mean curvature surfaces in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ with mean curvatures H_1 and H_2, normal vectors N_1 and N_2 and angle functions ν_1 and ν_2, respectively. Let $\Gamma \subset \Sigma_1\cap\Sigma_2$ be a regular curve parametrized by γ. Suppose that Σ_1 and Σ_2 intersects transversally along Γ at a constant angle and Γ is a AR-line of curvature for Σ_1. Assume along Γ one of the following conditions holds:*

1. *Γ is an horizontal curve of Σ_1.*

2. *Γ is a vertical curve of Σ_1 and Σ_2.*

3. *If $H_1 = H_2 \neq 0$, the angle function ν_1 is opposite to the angle function ν_2.*

Then Γ is an AR-line of curvature for Σ_2.

Proof. Assume that $\Gamma = \gamma(-\varepsilon, \varepsilon)$ where γ is parametrized by arc-length. In $\mathbb{M}^2(\kappa) \times \mathbb{R}$, we have $\mathbf{T}^1_{\theta_1} = \mathbf{T}_1$ and $\mathbf{T}^2_{\theta_2} = \mathbf{T}_2$. In the first case, if γ is horizontal, we obtain

$$\begin{aligned} (1-d^2)\langle J_1\mathbf{T}^1_{\theta_1}, N_2\rangle &= \langle \mathbf{T}^1_{\theta_1}, \gamma'(t)\rangle = \langle \xi, \gamma'(t)\rangle = 0, \\ (1-d^2)\langle J_2\mathbf{T}^2_{\theta_2}, N_1\rangle &= \langle \mathbf{T}^2_{\theta_2}, \gamma'(t)\rangle = \langle \xi, \gamma'(t)\rangle = 0. \end{aligned} \tag{4.10}$$

From (4.10) and Lemma 4.1, $\gamma(t)$ is AR-line of curvature of Σ_2.

In the second case, $\mathbf{T}_2 = \mathbf{T}_1 = \gamma'(t)$ for each t, then $\langle \mathbf{T}_1, N_2\rangle = \langle \mathbf{T}_2, N_1\rangle = 0$ so the hypothesis of Lemma 4.1 holds clearly.

In the third case, suppose $H_1 = H_2 = H$ and $\nu_1 = -\nu_2$, therefore

$$\begin{aligned} H\langle \mathbf{T}_2, N_1\rangle\langle \mathbf{T}_2, \gamma'(t)\rangle &= H\langle \mathbf{T}_2, N_1\rangle\langle \xi, \gamma'(t)\rangle \\ &= H(\nu_2 - \nu_1 d)\langle \mathbf{T}_1 + \nu_1 N_1, \gamma'(t)\rangle \\ &= -H(\nu_1 - \nu_2 d)\langle \mathbf{T}_1, \gamma'(t)\rangle \\ &= -H\langle \mathbf{T}_1, N_2\rangle\langle \mathbf{T}_1, \gamma'(t)\rangle. \end{aligned} \tag{4.11}$$

Hence, from (4.9) and (4.11), we can see again that the hypothesis of Lemma 4.1 hold. Then, in any case, Γ is a AR-line of curvature of Σ_2. $\square$

Next, we give certain geometric configurations for $H-$ surfaces in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, that imply the Key Lemma.

Corollary 4.2. *Let Σ_1 and Σ_2 two $H-$surfaces in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, with normal vectors N_1 and N_2 respectively. Let $\Gamma \subset \Sigma_1 \cap \Sigma_2$ be a regular curve. Suposse that Σ_1 and Σ_2 intersect along Γ at a constant angle. Assume also that:*

1. *If both surfaces are tangent along Γ, then $N_1 = N_2$ along Γ.*

2. *If the intersection between the surfaces is transversal along Γ, then their respective angle functions satisfy $\langle \xi, N_1\rangle = -\langle \xi, N_2\rangle$ along Γ.*

Then, Γ is an AR-line of curvature for Σ_1 if, and only if, Γ is an AR-line of curvature for Σ_2.

Proof. Let $S^i_{AR}X = A_i(X) - \alpha\langle \mathbf{T}^i_{\theta_i}, X\rangle \mathbf{T}^i_{\theta_i} + \frac{\alpha|\mathbf{T}_i|^2}{2}X$ be the Abresch-Rosenberg shape operator of Σ_i, $i = 1, 2$ and J_1, J_2 be the rotations on the tangent bundles of Σ_1 and Σ_2 respectively.

In the first case, we have that $\mathbf{T}^1_{\theta_1} \equiv \mathbf{T}^2_{\theta_2}$ along Γ since $\mathbf{T}_1 \equiv \mathbf{T}_2$ and the surfaces has the same mean curvature. Moreover, if $\Gamma = \gamma(-\varepsilon, \varepsilon)$, then $J_1\gamma' = J_2\gamma'$ and so $II^1_{AR}(\gamma', J_1\gamma') =$

$II^2_{AR}(\gamma', J_2\gamma')$.

Suppose now that we are in case 2. Hence, $\theta_1 = \theta_2 = \theta$ and this implies $\alpha_1 = \alpha_2 = \alpha$, so

$$\begin{aligned}\alpha\langle \mathbf{T}^1_\theta, \gamma'(t)\rangle\langle \mathbf{T}^1_\theta, N_2\rangle = \alpha\langle \mathbf{T}_1, \gamma'(t)\rangle\langle \mathbf{T}_1, N_2\rangle \cos^2\theta \\ + \alpha\cos\theta\sin\theta\langle \mathbf{T}_1, \gamma'(t)\rangle\langle J\mathbf{T}_1, N_2\rangle \\ + \alpha\langle J\mathbf{T}_1, \gamma'(t)\rangle\langle \mathbf{T}_1, N_2\rangle \sin\theta\cos\theta \\ + \alpha\langle J\mathbf{T}_1, \gamma'(t)\rangle\langle J\mathbf{T}_1, N_2\rangle \sin^2\theta.\end{aligned} \tag{4.12}$$

We oriented γ such that $\{N_1, N_2, \gamma'(t)\}$ is an oriented basis of $T_{\gamma(t)}\mathbb{E}(\kappa,\tau)$ for each t were the intersection is transversal, then the following equations holds

$$\begin{aligned}\langle J\mathbf{T}_1, \gamma'(t)\rangle = \langle N_1 \wedge \mathbf{T}_1, \gamma'(t)\rangle = \langle N_2, \mathbf{T}_1\rangle, \\ \langle J\mathbf{T}_1, N_2\rangle = \langle N_1 \wedge \mathbf{T}_1, N_2\rangle = -\langle \gamma'(t), \mathbf{T}_1\rangle.\end{aligned} \tag{4.13}$$

Hence, substituting (4.13) into (4.12), we obtain

$$\begin{aligned}\alpha\langle \mathbf{T}^1_\theta, \gamma'(t)\rangle\langle \mathbf{T}^1_\theta, N_2\rangle = \alpha\langle \mathbf{T}_1, \gamma'(t)\rangle\langle \mathbf{T}_1, N_2\rangle \cos^2\theta \\ - \alpha\langle \mathbf{T}_1, \gamma'(t)\rangle^2 \cos\theta\sin\theta \\ + \alpha\langle \mathbf{T}_1, N_2\rangle^2 \cos\theta\sin\theta \\ - \alpha\langle N_2, \mathbf{T}_1\rangle\langle \gamma'(t), \mathbf{T}_1\rangle \sin^2\theta.\end{aligned} \tag{4.14}$$

Analogously,

$$\begin{aligned}\alpha\langle \mathbf{T}^2_\theta, \gamma'(t)\rangle\langle \mathbf{T}^2_\theta, N_1\rangle = \alpha\langle \mathbf{T}_2, \gamma'(t)\rangle\langle \mathbf{T}_2, N_1\rangle \cos^2\theta \\ + \alpha\cos\theta\sin\theta\langle \mathbf{T}_2, \gamma'(t)\rangle\langle J\mathbf{T}_2, N_1\rangle \\ + \alpha\langle J\mathbf{T}_2, \gamma'(t)\rangle\langle \mathbf{T}_2, N_1\rangle \sin\theta\cos\theta \\ + \alpha\langle J\mathbf{T}_2, \gamma'(t)\rangle\langle J\mathbf{T}_2, N_1\rangle \sin^2\theta.\end{aligned} \tag{4.15}$$

Taking account the orientation $\{N_1, N_2, \gamma'(t)\}$ of $T_{\gamma(t)}\mathbb{E}(\kappa,\tau)$, we get

$$\begin{aligned}\langle J\mathbf{T}_2, N_1\rangle = \langle N_2 \wedge \mathbf{T}_2, N_1\rangle = \langle \gamma'(t), \mathbf{T}_2\rangle, \\ \langle J\mathbf{T}_2, \gamma'(t)\rangle = \langle N_2 \wedge \mathbf{T}_2, \gamma'(t)\rangle = -\langle N_1, \mathbf{T}_2\rangle.\end{aligned} \tag{4.16}$$

So, we can rewrite (4.15) as:

$$\begin{aligned}\alpha\langle \mathbf{T}^2_\theta, \gamma'(t)\rangle\langle \mathbf{T}^2_\theta, N_1\rangle = \alpha\langle \mathbf{T}_2, \gamma'(t)\rangle\langle \mathbf{T}_2, N_1\rangle \cos^2\theta \\ + \alpha\langle \mathbf{T}_2, \gamma'(t)\rangle^2 \cos\theta\sin\theta \\ - \alpha\langle \mathbf{T}_2, N_1\rangle^2 \cos\theta\sin\theta \\ - \alpha\langle N_1, \mathbf{T}_2\rangle\langle \gamma'(t), \mathbf{T}_2\rangle \sin^2\theta.\end{aligned} \tag{4.17}$$

Now, the hypothesis imply that

$$\begin{aligned}\langle \mathbf{T}_1, N_2\rangle &= \nu_2 - \nu_1 d = -\langle \mathbf{T}_1, N_2\rangle,\\ \langle \mathbf{T}_1, \gamma'(t)\rangle &= \langle \mathbf{T}_2, \gamma'(t)\rangle = \langle \xi, \gamma'(t)\rangle.\end{aligned} \tag{4.18}$$

Finally, substituting (4.18) into (4.14) and (4.17), we obtain

$$\sqrt{H^2+\tau^2}\langle \mathbf{T}^2_\theta, N_1\rangle\langle J_2\mathbf{T}^2_\theta, N_1\rangle = \sqrt{H^2+\tau^2}\langle \mathbf{T}^1_\theta, N_2\rangle\langle J_1\mathbf{T}^1_\theta, N_2\rangle.$$

Therefore, the Key Lemma (Lemma 4.1) shows that Γ is an AR-line of curvature of Σ_1 if, and only if Γ is an AR-line of curvature of Σ_2. □

Remark 4.1. *The case $\tau = 0$ was considered by do Carmo-Fernández [19], they consider the usual Abresch-Rosenberg shape operator and studied lines of curvature with respect to this operator. Hence, Corollary 4.2 is an extension of those results when $\tau \neq 0$.*

4.2 Immersed compact disks

Throughout this section, we will denote by $\phi : \mathbb{D} \to \mathbb{E}(\kappa,\tau)$ an immersion from the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\mathbb{E}(\kappa,\tau)$ of constant mean curvature, we call it $H-$disk. Moreover, we will assume that the boundary Γ is a smooth curve.

4.2.1 Immersed compact disks in $\mathbb{E}(\kappa,\tau)$, $\tau = 0$

The classification of immersed compact disks with constant mean curvature in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ was studied by M. Do Carmo and I. Fernández in [19], under certain conditions on the curve Γ, they showed that $\phi(\mathbb{D})$ is a part of an Abresch-Rosenberg surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. In this section, we will classified immersed compact disks, Assuming geometric conditions about Γ more general than the conditions given in [19].

Let Ω be an Abresch-Rosenberg surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. In the following theorem, we denote by $\nu_1 = \langle \xi, N_1\rangle$ as the angle function defined along the immersion ϕ, where N_1 is the unit normal vector field defined along $\phi(\overline{\mathbb{D}})$ and by $\nu_2 = \langle \xi, N_2\rangle$ the angle function defined along Ω, where N_2 is the unit normal vector field defined along surface Ω.

Theorem 4.1. *Let $\phi : \overline{\mathbb{D}} \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ be a non minimal H_1-disk with regular boundary Γ, Suppose that ϕ meets transversally an Abresch-Rosenberg H_2-surface Ω along Γ at a constant angle. Assume also that Γ is of one of the following types:*

1. *Γ is an horizontal or vertical curve of Ω.*
2. *If $H_1 = H_2$, the angle function ν_1 is opposite to the angle function ν_2.*

Then, $\phi(\overline{\mathbb{D}})$ is a part of an Abresch-Rosenberg surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

Proof. Set $\phi(\mathbb{S}^1) = \Gamma$. Now, Γ is an AR-line of curvature of the Abresch-Rosenberg surface Ω, then from hypothesis and Corollary 4.1, γ is an AR-line of curvature of the immersion ϕ. So, from Propostion 4.1, the imaginary part of the Abresch-Rosenberg differential must be zero along γ. Finally, the Schwarz Reflection Lemma implies that the Abresch-Rosenberg differential must be zero on $\phi(\overline{\mathbb{D}})$ and Lemma 2.3 gives the result.

□

In Corollary 4.1, we omitted two interesting cases:

1. When Ω is a slice $\mathbb{M}^2(\kappa) \times \{\xi_0\}$, this case was treated in [17, Theorem 9].
2. When the immersion ϕ is minimal and Γ is horizontal, one can solve this case using the Maximum principle, comparing ϕ with a slice.

Remark 4.2. *Theorem 4.1 generalizes the classification result given by Do Carmo and Fernandez in [19, Corollary 4.1] for immersed compact disks with constant mean curvature. We only assume that Γ is horizontal, without assuming that Γ is a line of curvature of the second fundamental form of the immersion.*

4.2.2 Immersed compact disks in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$

Now, we deal with H- disks in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$. We remember that for this class of immersions, we consider the Abresch-Rosenberg shape operator on $\phi(\mathbb{D})$ defined as

$$S_{AR}X = A(X) - \alpha\langle \mathbf{T}_\theta, X\rangle \mathbf{T}_\theta + \frac{\alpha |\mathbf{T}|^2}{2} X,$$

where $\mathbf{T}_\theta = \cos\theta\mathbf{T} + \sin\theta J\mathbf{T}$ and $\alpha = \frac{\kappa - 4\tau^2}{2\sqrt{H^2+\tau^2}}$. Using Corollary 4.2, we extend the above classification result for the case $\tau \neq 0$.

Theorem 4.2. *Let $\phi : \mathbb{D} \to \mathbb{E}(\kappa,\tau)$, $\tau \neq 0$, be a H_1-disk with regular boundary, suppose the boundary is parametrized by a regular curve γ and it is of one of the following types*

1. *γ is the tangent intersection of the immersion ϕ with an Abresch-Rosenberg surface Ω with the same mean curvature vector.*
2. *γ is the transverse intersection with constant angle of the immersion ϕ with an Abresch-Rosenberg surface Ω with the same mean curvature and whose angle function is opposite to the angle function of the immersion ϕ along γ.*

Then, $\phi(\mathbb{D})$ is a part of an Abresch-Rosenberg surface in $\mathbb{E}(\kappa,\tau)$.

4.3 Immersed compact disks with non-regular boundary

In this section, we will study H_1-disks $\phi : \mathbb{D} \to \mathbb{E}(\kappa, \tau)$ with piece-wise regular boundary. Indeed, we suppose that $\phi(\mathbb{D})$ is contained in the interior set of a differentiable surface without boundary in $\mathbb{E}(\kappa, \tau)$.

First, we will recall a result that gives conditions for a disk type surface to be umbilical with respect to a Codazzi pair with constant mean curvature. For sake of completeness, we will include the proof of this result.

Theorem 4.3 ([22]). *Let Σ be a compact disk with piece-wise smooth boundary. We will call the vertices of the surface to the finite set of non-regular boundary points. Assume that Σ is contained as an interior set in a differentiable surface $\hat{\Sigma}$ without boundary.*

Let (I,II) be a Codazzi pair with constant mean curvature $H(I,II)$ on $\hat{\Sigma}$. Assume also that the following conditions holds:

1. *The number of vertices in $\partial\Sigma$ with an angle $< \pi$ (measured with respect to the metric I) is less than 3.*
2. *The regular curves in $\partial\Sigma$ are lines of curvature for the pair (I,II)*

Then Σ is totally umbilical for the pair (I,II).

Proof. Consider on Σ the Riemannian metric given by I, let z be a conformal parameter. Set $\hat{Q}$ the Hopf differential of the fundamental pair (I,II) and $Q = \hat{Q}|_{\Sigma}$. Assume, that Σ is not totally umbilical respect to pair (I,II), that is, Q does not vanish identically on Σ.

At every non umbilical point of $\hat{\Sigma}$ there exist two orthogonal lines of curvature. whereas at an umbilical point the lines of curvature bend sharply. Since the imaginary part of $\hat{Q}$ is zero on these curves, if we write $\hat{Q} = f(z)dz^2$ in a neighbourhood of a point z_0, the rotation index at an umbilic point z_0 (see [13, 34]) is given by

$$I(z_0) = \frac{-1}{4\pi}\delta \arg f,$$

where $\delta \arg f$ is the variation of the argument of f as we wind once around the singular point.

At an interior umbilic point of Σ, the rotation index of the lines of curvature of Σ coincides with the one of $\hat{\Sigma}$. At a point $z_0 \in \partial\Sigma$ the rotation index of the lines of curvature of Σ is defined as follows. Consider $\varphi : \mathbb{D}_+ \to \Sigma$ an immersion of $\mathbb{D}_+ = \{\xi \in \mathbb{C} : |\xi| < 1, \mathrm{Im}(\xi) \geq 0\}$ into Σ, mapping the diameter of the half disk into $\partial\Sigma$. The lines of curvature can be pulled back to a line field in $\mathbb{D}_+$. Moreover, since the regular curves of $\partial\Sigma$ are lines of curvature, they can be extended by reflection to a continuous line field with singularities on the whole disk. Thus, we define the rotation index $I(z_0)$ of Q at $z_0 \in \partial\Sigma$ to be half of the rotation index of the extended lines of curvature.

If all singularities are isolated, the Poincaré-Hopf Theorem gives that

$$\sum_{z \in \Sigma} I(z) = 1.$$

Now, since $H(I,II)$ is constant, Lemma 1.1 imply that the differential $\hat{Q}$ is holomorphic and then the zeroes of $\hat{Q}$ in $\hat{\Sigma}$ are isolated. In particular, the zeroes of Q are isolated in Σ. Moreover, locally around a zero $z_0 \in \hat{\Sigma}$ of $\hat{Q}$ we have that

$$\hat{Q}(z) = (z - z_0)^k g(z) dz^2, \tag{4.19}$$

where $k \in \mathbb{N}$ and $g(z)$ is a non-vanishing continuous function. Therefore, the rotation index is $-k/2 \leq -1/2$, in particular, it is always negative.

> **Claim:** The boundary singular points are isolated. Moreover, let $z_0 \in \partial\Sigma$ be a singular point, then
>
> 1. if z_0 is not a vertex, its rotation index is $I(z_0) < 0$,
> 2. if z_0 is a vertex of angle $> \pi$, then $I(z_0) < 0$,
> 3. if z_0 is a vertex of angle $< \pi$, then $I(z_0) \leq 1/4$.

Proof. Consider $\varphi : \mathbb{D}_+ \to \Sigma$ a conformal immersion as before, with $\varphi(0) = z_0$. Since $\mathrm{Im}(Q) = 0$ on $\partial\Sigma$, its pull-back can be reflected through the diameter to a continuous quadratic form on the whole unit disk $\mathbb{D}$, that will be denote by Q^*. Notice that when z_0 is a vertex φ' could be zero or infinite.

Let ω be the angle of $\partial\Sigma$ at z_0. Then φ' grows as $|\xi|^{\frac{\omega}{\pi}-1}$ at the origin. Around z_0, $\hat{Q}$ is given by (4.19), although in this case k could be zero. Since, when $\omega = \pi/2$, z_0 is not necessarily a zero of $\hat{Q}$. In particular, z_0 is an isolated singularity. Moreover there are $2(k+2)$ lines of curvature in $\hat{\Sigma}$ emanating from z_0, and meeting at an equal-angle system of angle $\pi/(k+2)$. In particular, since the curves in $\partial\Sigma$ are lines of curvature, ω must be a multiple of $\pi/(k+2)$.

If we write $Q^* = f(\xi)d\xi^2$ for $\xi \in \mathbb{D}$, then

$$f(\xi) = (\varphi(\xi) - \varphi(0))^k (\varphi'(\xi))^2 g(\varphi(\xi)), \quad \xi \in \mathbb{D}_+.$$

Then the variation of the argument of $f(\xi)$ as we wind once around the origin is $2\theta(k+2) - 4\pi$ and the rotation index is

$$I^* = 1 - \frac{\omega}{2\pi}(k+2).$$

In particular, $if \omega \geq \pi$, then $I^* \leq -k/2 < 0$, whereas for $\omega < \pi$ we have $I^* \leq 1/2$ (as $I^* < 1$, and $2I^*$ must be an integer). Since $I(z_0) = I^*/2$, the claim is proved.

□

Finally taking account the above claim and since the number of vertices of angle $< \pi$ is less than or equal to 3, we can conclude that

$$\sum_{z \in \Sigma} I(z) \leq 3/4 < 1,$$

which contradicts the Poincaré-Hopf Theorem and shows that Σ must be totally umbilical.

□

If Σ is a constant mean curvature surface in $\mathbb{E}(\kappa,\tau)$, therefore, the fundamental pair (I, II_{AR}) defined on Σ is a Codazzi pair, such that its mean curvature $H(I, II_{AR})$ is constant. Then, using Theorem 4.3 we obtain the following:

Theorem 4.4. *Let Σ be a $H-$ disk in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$, with piece-wise differentiable boundary. Assume also that the following conditions are satisfied:*

1. *Σ is contained as an interior set in a smooth H-surface $\hat{\Sigma}$ in $\mathbb{E}(\kappa,\tau)$ without boundary.*
2. *The number of vertices in $\partial\Sigma$ with angle $< \pi$ is less than or equal to 3.*
3. *The regular curves in $\partial\Sigma$ are AR-lines of curvatures of Σ.*

Then, Σ is a part of an Abresch-Rosenberg surface in $\mathbb{E}(\kappa,\tau)$.

Theorem 4.1, shows that if a $H-$ disk in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ with horizontal curve Γ as boundary meets an Abresch-Rosenberg surface at a constant angle along Γ, then Γ is an AR- line of curvature of the immersion, then using this fact with Theorem 4.4, we get the following corollary:

Corollary 4.3. *Let $\phi : \mathbb{D} \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ be a H_1- disk, with $H_1 \neq 0$ and piece-wise differentiable boundary Γ. Assume also that the following conditions are satisfied:*

1. *$\phi(\mathbb{D})$ is contained as an interior set in a smooth H_1-surface $\hat{\Sigma}$ in $\mathbb{E}(\kappa,\tau)$ without boundary.*
2. *The number of vertices in Γ with angle $< \pi$ is less than or equal to 3.*
3. *Every regular component γ of Γ is a one of following types:*
   - *γ is contained in a horizontal slice*
   - *γ is a transverse intersection with constant angle of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω of constant mean curvature $H_2 \neq 0$*

Then, $\phi(\mathbb{D})$ is a part of an Abresch-Rosenberg surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

Theorem 4.2, shows that if a $H-$ disk in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$, has regular boundary, then under certain conditions over Γ, we conclude that Γ is an AR- line of curvature of the immersion. So, using the above fact with the Theorem 4.4, we obtain:

Corollary 4.4. *Let $\phi : \mathbb{D} \to \mathbb{E}(\kappa,\tau)$, $\tau \neq 0$, be a H_1-disk with piece-wise differentiable boundary Γ. Assume also that the following conditions are satisfied:*

1. *$\phi(\mathbb{D})$ is contained as an interior set in a smooth H-surface $\hat{\Sigma}$ in $\mathbb{E}(\kappa,\tau)$ without boundary.*
2. *The number of vertices in Γ with angle $< \pi$ is less than or equal to 3.*

3. Every regular component γ of Γ is one of the following types:

- *γ is a tangent intersection of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω with the same mean curvature vector.*
- *γ is a transverse intersection with constant angle of $\phi(\mathbb{D})$ with an Abresch-Rosenberg surface Ω with the same constant mean curvature and whose angle function is opposite to the angle function $\phi(\mathbb{D})$ along γ.*

Then, $\phi(\mathbb{D})$ is a part of an Abresch-Rosenberg surface in $\mathbb{E}(\kappa,\tau)$, $\tau \neq 0$.

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