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Determination of 2-dimensional GK foliations on $\mathbb{P}^n$ associated to the affine Lie algebra

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To my lovely wife Maíra.

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O cientista não estuda a natureza porque ela é útil; ele a estuda porque se deleita nela, e se deleita nela porque ela é bela. Se a natureza não fosse bela, não valeria a pena ser conhecida, e se não valesse a pena ser conhecida, a vida não valeria a pena ser vivida.

Henry Poincaré

Eu acredito demais na sorte. E tenho constatado que, quanto mais duro eu trabalho, mais sorte eu tenho.

Coleman Cox

Resumo

Denote por $\mathcal{F}(d,3)$ o espaço das folheações de codimensão 1 e grau d em $\mathbb{P}^3$. Nós exibimos todas as componentes GK $\mathcal{F}(p,q,r;\lambda,d) \subset \mathcal{F}(d,3)$ associadas à álgebra de Lie afim $\mathfrak{aff}(\mathbb{C})$, onde $p > q > r$ são inteiros positivos relativamente primos. Em particular, nós damos uma resposta a um problema que aparece em [2], sobre a existência de componentes GK da forma $\mathcal{F}(p,q,r;\lambda,d)$, onde $p > q > r$ estão fixados.

Em seguida, construímos componentes $\mathcal{F}(p_1,p_2,\ldots,p_n;\lambda,d)$ de $\mathcal{F}_2(d,n)$, o espaço das folheações holomorfas de dimensão 2 e grau d em $\mathbb{P}^n$. Finalmente, nós apresentamos uma caracterização das componentes GK $\mathcal{F}(p_1,p_2,\ldots,p_n;\lambda,d)$, e usamos este resultado para exibir todas as componentes GK $\mathcal{F}(p,q,r,s;\lambda,d) \subset \mathcal{F}_2(d,4)$.

Palavras chaves: Componentes irredutíveis do espaço de folheações. Componentes associadas à álgebra de Lie afim. Folheações GK.

Abstract

Let $\mathcal{F}(d,3)$ denotes the space of foliations of codimension 1 and degree d on $\mathbb{P}^3$. We exhibit all GK components $\mathcal{F}(p,q,r;\lambda,d)\subset\mathcal{F}(d,3)$ associated to the affine Lie Algebra $\mathfrak{aff}(\mathbb{C})$, where $p>q>r$ are relatively prime positive integers. In particular, we give an answer to a problem that appears in [2], about whether there exist GK components of the form $\mathcal{F}(p,q,r;\lambda,d)$, if $p>q>r$ are fixed.

Next we construct components $\mathcal{F}(p_1,p_2,\dots,p_n;\lambda,d)$ of $\mathcal{F}_2(d,n)$, the space of 2-dimensional holomorphic foliations of degree d on $\mathbb{P}^n$. Finally, we present a characterization of the GK components $\mathcal{F}(p_1,p_2,\dots,p_n;\lambda,d)$, and we use this result to exhibit all GK components $\mathcal{F}(p,q,r,s;\lambda,d)\subset\mathcal{F}_2(d,4)$.

Keywords: Irreducible components of the space of foliations. Components associated to the affine Lie algebra. GK foliations.

Contents

Chapter 1

Introduction

In this chapter, we introduce the space of 2-dimensional holomorphic foliations on $\mathbb{P}^n$. Some subvarieties of these spaces which are associated to the affine Lie Algebra are introduced as well. We also present the main results of this work.

1 Codimension 1 and dimension 2 holomorphic foliations on $\mathbb{P}^n$

Let $\mathcal{F}$ be a holomorphic singular foliation of codimension one on $\mathbb{P}^n$. The degree of $\mathcal{F}$ is, by definition, the number of tangencies (counted with multiplicities) of a generic linearly embedded $\mathbb{P}^1$ with $\mathcal{F}$. It is well known that a holomorphic singular foliation $\mathcal{F}$ of codimension one and degree d on $\mathbb{P}^n$ can be defined in homogeneous coordinates by an integrable one-form $\Omega = \sum_{j=0}^{n} A_j(z)dz_j$, where the A_j´s are homogeneous polynomials of degree $d+1$, satisfying the so-called *Euler's* condition

$$\sum_{j=0}^{n} z_j A_j(z) \equiv 0 \tag{1.1}$$

and $cod_{\mathbb{C}}(Sing(\Omega)) \geq 2$, where $Sing(\Omega)$ is the singular set of Ω,

$$Sing(\Omega) = \{z \in \mathbb{C}^{n+1}; A_0(z) = A_1(z) = \cdots = A_n(z) = 0\}.$$

The form Ω is called a homogeneous expression of $\mathcal{F}$. Moreover, if Ω_1 is another form as above which defines $\mathcal{F}$, then $\Omega_1 = \lambda.\Omega$, where $\lambda \in \mathbb{C}^*$.

The singular set of $\mathcal{F}$, $Sing(\mathcal{F})$, is $\Pi_n(Sing(\Omega)) = \Pi_n(Sing(\mathcal{F}^*))$, where $\Pi_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection. Recall that the integrability condition is given by

$$\Omega \wedge d\Omega = 0. \tag{1.2}$$

The leaves of $\mathcal{F}$ are of the form $\Pi_n(L)$, where L is a leaf of $\mathcal{F}^*$, that is, a codimension-1 solution of the differential equation $\Omega = 0$.

The above facts imply that the set of codimension one holomorphic singular foliations of degree d on $\mathbb{P}^n$, denoted by $\mathcal{F}(d,n)$, can be identified to the projectivization of the following space

$$\Big\{\Omega = \textstyle\sum_{j=0}^{n} A_j(z)dz_j;\ A_j \text{ is a homogeneous polynomial of degree } d+1 \text{ on } \mathbb{C}^{n+1};\ \sum_{j=0}^{n} z_j A_j(z) \equiv 0;$$
$$\Omega \wedge d\Omega \equiv 0 \text{ and } cod(Sing(\Omega)) \geq 2\Big\}$$

This means that $\mathcal{F}(d,n)$ can be thought as a Zariski open set of an algebraic set of some projective space (in fact, an intersection of quadrics).

Recall that a holomorphic q-form ω in a complex manifold M of dimension n is said to be locally decomposable outside the singular set (LDS), if for every $p \in M \setminus Sing(\omega)$ there exists a neighbourhood $V_p \ni p$ and a system of 1-forms $\alpha_1, \ldots, \alpha_q$ on V_p such that

$$\omega|_{V_p} = \alpha_1 \wedge \ldots \wedge \alpha_q.$$

We also say that ω is integrable if the system $\{\alpha_1, \ldots, \alpha_q\}$ above can be chosen integrable, that is,

$$d\alpha_j \wedge \omega = 0 \text{ for all } 1 \leq j \leq q.$$

Similarly, a dimension 2 (codimension $n-2$) foliation $\mathcal{F}$ on $\mathbb{P}^n$ of degree d can be given in the following two equivalent ways

(a) In homogeneous coordinates in $\mathbb{C}^{n+1}$ by an homogeneous polynomial integrable $(n-2)$-form Ω of degree $d+1$ satisfying $i_R\Omega = 0$, where R is the radial vector field on $\mathbb{C}^{n+1}$, Ω having singular set of codimension ≥ 2 and coinciding with $\Pi_n^{-1}(Sing(\mathcal{F}))$. Two such $(n-2)$-forms Ω and Ω_1 are related by $\Omega_1 = \lambda.\Omega$, for some $\lambda \in \mathbb{C}^*$;

(b) In affine coordinates $(x_1, \ldots, x_n) \in \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ by an integrable polynomial $(n-2)$-form ω in $\mathbb{C}^n$ with singular set $Sing(\omega) = \{p \in \mathbb{C}^n \mid \omega(p) = 0\}$ of codimension ≥ 2 and $Sing(\mathcal{F}) \cap \mathbb{C}^n = Sing(\omega)$. The $(n-2)$-form ω admits a decomposition $\omega = \omega_0 + \omega_1 + \ldots + \omega_{d+1}$, where ω_i is a homogeneous $(n-2)$-form of degree $i, i = 0, \ldots, d+1, i_R\omega_{d+1} = 0$ and $i_R\omega_d \neq 0$ if $\omega_{d+1} = 0$.

The degree of $\mathcal{F}$ is the degree of the tangency of the foliation with a generic $\mathbb{P}^{n-2}$ linearly embedded in $\mathbb{P}^n$. The projectvization of the set of homogeneous polynomial integrable $(n-2)$-forms Ω of degree $d+1$ which have singular set of codimension equal or greater than 2 satisfying the previous conditions will be denoted by $\mathcal{F}_2(d,n)$, the space of 2-dimensional singular holomorphic foliations on $\mathbb{P}^n$ of degree d. As $\mathcal{F}(d,n)$, $\mathcal{F}_2(d,n)$ is a quasi projective variety and we are interested in its decomposition into irreducible components.

The problem of identify and classify the irreducible components of $\mathcal{F}(d,n)$ seems to have been initiated by Jouanolou in [[12]], where he shows that $\mathcal{F}(0,n)$ has only one irreducible component and $\mathcal{F}(1,n)$ has two irreducible components, $n \geq 3$.

Some irreducible components (that can be described by geometric and dynamic properties of a generic element) of $\mathcal{F}(d,n)$ are known: rational [[11]], logarithmic [[1]], linear pull-back [[3]], generic pull-back [[5]], associated to the affine Lie algebra [[2]] and more recently branched pull-back [[7]].

The classification of $\mathcal{F}(2,n), n \geq 3$, was achieved by Cerveau and Lins Neto in [[6]], where they show that $\mathcal{F}(2,n)$ has six irreducible components, two of rational type, two of logarithm type, one of linear pull-back type and finally one known as the exceptional component. The classification of $\mathcal{F}(d,n), d \geq 3$, is still unknown.

The literature on the irreducible components of $\mathcal{F}_2(d,n)$ is not as extensive in comparison with the literature on the irreducible components of $\mathcal{F}(d,n)$. The classification of $\mathcal{F}_2(0,n)$ was given in [18, theorem 3.8]: a 2-dimensional foliation of degree zero on $\mathbb{P}^n$ is defined by a linear projection from $\mathbb{P}^n$ to $\mathbb{P}^{n-2}$. The classification of the irreducible components of $\mathcal{F}_2(1,n)$ was given in [19, theorem 6.2 and corollary 6.3], where they show that $\mathcal{F}_2(1,n)$ has two irreducible components. Both results are actually about the space of codimension q foliations on $\mathbb{P}^n$, where $q \geq 2$.

The components of $\mathcal{F}(d,3)$ associated to the affine Lie algebra, which we describe next, are the generalization of the exceptional component for higher degrees.

2 Irreducible components associated do the affine Lie algebra

Before stating the main theorems of this work, let us introduce some results related to components associated to the affine Lie algebra $\mathfrak{aff}(\mathbb{C}) := \{e_1, e_2, [e_1, e_2] = e_2\}$. They are given by some special representations of $\mathfrak{aff}(\mathbb{C})$ in the algebra of polynomial vector fields of an affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$.

Let $p > q > r \geq 1$ be relatively prime integers and S be the semi-simple vector field on $\mathbb{C}^3$ defined by

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}.$$

Let X be another polynomial vector field on $\mathbb{C}^3$ such that $[S, X] = \lambda X$, for some $\lambda \in \mathbb{Z}$. By definition, S and X give a representation of the affine Lie algebra in the algebra of polynomial vector fields of $\mathbb{C}^3$ if $\lambda \neq 0$. If we suppose that S and X are linearly independent at generic points, then these vector fields generate an algebraic foliation $\mathcal{F} = \overline{\mathcal{F}}(S, X)$ on $\mathbb{C}^3$, which is given by the integrable 1-form

$$\omega = i_S i_X (dx \wedge dy \wedge dz),$$

where i denotes the interior product. Indeed, the integrability of ω comes from the relation $[S, X] = \lambda X$.

Since ω is a polynomial 1-form, this foliation can be extended to a singular foliation of $\mathbb{P}^3$, which will be denoted by $\mathcal{F}(S, X)$. Observe that S extends to a holomorphic vector field on $\mathbb{P}^3$ and that its trajectories are contained in the leaves of $\mathcal{F}(S, X)$. On the other hand, in general, the vector field X is meromorphic in $\mathbb{P}^3$, but the foliation defined by it on $\mathbb{C}^3$ extends to a foliation on $\mathbb{P}^3$, which will be denoted by $\mathcal{G}_X$, whose leaves are also contained in the leaves of $\mathcal{F}(S, X)$. Remark that the singular set of $\mathcal{F}(S, X)$, denoted by $Sing(\mathcal{F}(S, X))$, is invariant under the flow of S (see proposition 2.1 (b) below).

Set

$$\mathcal{F}(p, q, r; \lambda, d) := \{\mathcal{F} \in \mathcal{F}(d, 3) \mid \mathcal{F} = \overline{\mathcal{F}}(S, X) \text{ in some affine chart}\}.$$

In [[2]] is shown that they are irreducible subvarieties of $\mathcal{F}(d, 3)$.

Next, we see a condition which implies the local stability of the singularities of $\mathcal{F}(S, X)$ by small perturbations of the form defining the foliation.

Definition 1.1. Let ω be an integrable $(n-2)$-form defined in a neighbourhood of $p \in \mathbb{C}^n$. We say that p is a *generalized Kupka* (GK) singularity of ω if $\omega(p) = 0$ and either $d\omega(p) \neq 0$ or p is an isolated singularity of $d\omega$.

We would like to note that this definition depends only on the foliation defined by ω, in the sense that p is a GK singularity of ω if and only if p is a GK singularity of $f.\omega, \forall f \in \mathcal{O}_p^*$.

Let us fix some coordinate system $z = (z_1, \ldots, z_n)$ around p, such that $z(p) = 0$. Then, since $d\omega$ is a $(n-2)$-form, there exists a unique vector field Y such that

$$d\omega = i_Y (dz_1 \wedge \cdots \wedge dz_n),$$

so that 0 is a GK singularity of ω if and only if either $Y(0) \neq 0$ or 0 is an isolated singularity of Y. The vector field Y will be called rotational of ω and denoted by $Y = rot(\omega)$.

Definition 1.2. A two-dimensional holomorphic foliation $\mathcal{F}$ in a complex manifold M of dimension n is GK if all the singularities of $\mathcal{F}$ are GK.

We have the following theorem ([[2]])

Theorem 1.3. *Suppose that $\mathcal{F}(p, q, r; \lambda, d)$ contains some GK foliation, where $\lambda \neq 0$. Then $\overline{\mathcal{F}(p, q, r; \lambda, d)}$ is an irreducible component of $\mathcal{F}(d, 3)$.*

Theorem 1.3 gives rise to the question of determining the families $\mathcal{F}(p,q,r;\lambda,d)$ which contain some GK foliation, and consequently which are irreducible components of the space $\mathcal{F}(d,3)$. In the case $p>q>r\geq 1$ very few of these families are known. One of these examples is given by $\mathcal{F}(d^2+d+1,d+1,1;-1,d+1)$. This case is a generalization of the exceptional component (that corresponds to the case $d=1$) and belong to a family called Klein-Lie foliations of $\mathbb{P}^3$, so we have the following corollary ([[2]], corollary 3 of Theorem 1)

Corollary 1.4. *For any $d\geq 1$, $\overline{\mathcal{F}(d^2+d+1,d+1,1;-1,d+1)}$ is an irreducible component of $\mathcal{F}(d+1,3)$ of dimension $N(d)$, where $N(1)=13$ and $N(d)=14$ if $d>1$. Moreover, this component is the closure of a $\mathbb{P}GL(4,\mathbb{C})$ orbit on $\mathcal{F}(d+1,3)$.*

Remark 1.5. The families of foliations that appear in corollary 1.4 are of the form $\mathcal{F}(p,q,r;\lambda,d+1)$, where $\lambda<0$. As we shall see soon, if $\mathcal{F}$ is GK and lies in one of those families, then all the singularities of $\mathcal{F}$ are Kupka in some affine open set $(E,(x,y,z))$. The opposite is also true, in the following sense: suppose $\mathcal{F}\in\mathcal{F}(p,q,r;\lambda,d+1)$ is GK and all the singularities are Kupka in some affine open set $(E,(x,y,z))$. Then such family is like in corollary 1.4, that is,

$$\mathcal{F}(p,q,r;\lambda,d+1)=\mathcal{F}(d^2+d+1,d+1,1;-1,d+1)$$

for some d. This assertion is contained in corollary 4.2.1 of [[13]].

Corollary 1.4 tell us that for each degree $d\geq 2$ there is at least one irreducible component of $\mathcal{F}(d,3)$ of the form $\mathcal{F}(p,q,r;\lambda,d)$. Recently, in the case $p>q=r$, the following theorem was proved ([[10]])

Theorem 1.6. *If $p>q$, $gcd(p,q)=1, p\geq 3$ and $k\geq 1$, then*

$$\overline{\mathcal{F}(p,q,q;q(kp-1),kp+1)}$$

is an irreducible component of $\mathcal{F}(kp+1,3)$ and

$$\overline{\mathcal{F}(p,q,q;kpq,kp+2)}$$

is an irreducible component of $\mathcal{F}(kp+2,3)$.

Necessary conditions for $\mathcal{F}(p,q,r;\lambda,d+1)$ to contain a GK foliation are given by (see theorem 3 of [[2]] and theorem 4.2 of [[13]])

Theorem 1.7. *Suppose that $\mathcal{F}(p,q,r;\lambda,d+1)$ contains some GK foliation, where $p>q>r$ are relatively prime positive integers. Set $q_1=p-r$, $r_1=p-q$, $\lambda_1=p(d-1)-\lambda, N(d)=d^3+d^2+d+1$. Then*

(a) $m:=\frac{(\lambda+p)(\lambda+q)(\lambda+r)}{pqr}\in\mathbb{Z}_{\geq 0}$;

(b) $m_1:=\frac{(\lambda_1+p)(\lambda_1+q_1)(\lambda_1+r_1)}{pq_1r_1}\in\mathbb{Z}_{\geq 0}$;

(c) $N(d)-1\leq m+m_1\leq N(d)$, *if* $d\geq 2$.

Next result asserts that for fixed $p>q>r\geq 1$, the number of families $\mathcal{F}(p,q,r;\lambda,d+1)$ containing a GK foliation is finite (see [[2]], theorem 3).

Theorem 1.8. *If $p>q>r\geq 1$ are fixed, then the set*

$$\mathcal{P}(p,q,r)=\{(d,\lambda)\mid d\geq 2,\lambda\in\mathbb{Z} \text{ and } \mathcal{F}(p,q,r;\lambda,d+1) \text{ contains a GK foliation } \} \text{ is finite.}$$

The idea of the proof of the theorem 1.8 is to show that for fixed $p>q>r\geq 1$, there are only a finite number of pairs (d,λ) satisfying $m+m_1\leq N(d)$, according to theorem 1.7.

Motivated by theorem 1.3, the following question was posed in [[2]]

Problem 1.9. *Given three positive integers $p>q>r\geq 1$, are there $(\lambda\neq 0,d)$ such that $\mathcal{F}(p,q,r;\lambda,d+1)$ contains a GK foliation?*

3 The present work

Our first result provides, where d is fixed a priori, all GK components of $\mathcal{F}(d+1,3)$ given by theorem 1.3

Theorem 1.10. *Let $p > q > r \geq 1$ be positive integers, where $gcd(p,q,r) = 1$. $\mathcal{F}(p,q,r;\lambda,d+1) \subset \mathcal{F}(d+1,3)$ contains a GK foliation, for some $\lambda \in \mathbb{Z}, d \geq 2$, if and only if either p,q,r,λ,d or $p, q_1 = p-r, r_1 = p-q, \lambda_1 = p(d-1)-\lambda, d$ satisfy one of the following relations*

(a) $p = d > q = r+1 > r, \lambda = dr$;

(b) $p = kd > q = md+k > r = md, \lambda = md^2, gcd(k,m) = 1, k$ divides $d+1$;

(c) $p > q = m(d+1) > r = md, \lambda = md^2, gcd(p,m) = 1, p$ divides either d^2 or d^2+d+1;

(d) $p > q = md > r = m(d-1), \lambda = m(d^2-d), gcd(p,m) = 1, p$ divides either d^2-d, or d^2, or d^2-1.

Remark 1.11. The cases of corollary 1.4 can be obtained from theorem 1.10 by substituting $p = d^2+d+1, m = d$ in (c), since $\mathcal{F}(p,q,r;\lambda,d+1) = \mathcal{F}(p,q_1,r_1;\lambda_1,d+1)$, where

$$q_1 = p-r, r_1 = p-q, \lambda_1 = p(d-1)-\lambda$$

(see corollary 2.12 below).

Corollary 1.12. *If $q \geq 3$, there are no $\lambda \neq 0$ and $d \geq 2$ such that $\mathcal{F}(q+1,q,1;\lambda,d+1)$ contains some GK foliation.*

It follows from corollary 1.12 that the answer to the problem 1.9 is no.
Theorem 1.10 provides several families like those of corollary 1.4.

Corollary 1.13. *For $d \geq 2$, $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ is an irreducible component of $\mathcal{F}(d+1,3)$ for the following values of p,q,r,λ*

p	q	r	λ
d^2+d	$2d+1$	d	d^2
d^2	$d+1$	d	d^2
d^2+d+1	$d+1$	d	d^2
d^2-d	d	$d-1$	d^2-d
d^2	d	$d-1$	d^2-d
d^2-1	d	$d-1$	d^2-d

From theorem 1.10 is immediate to obtain, for instance, the list of all GK components of degree 3, 4 and 5 provided by theorem 1.3.

Corollary 1.14. *There are 6 GK components of the type $\overline{\mathcal{F}(p,q,r;\lambda,3)}$, for the following values of p,q,r,λ*

p	7	7	6	4	4	3
q	6	3	5	3	2	2
r	4	2	2	2	1	1
λ	8	4	4	4	2	2

There are 13 GK components of the type $\overline{\mathcal{F}(p,q,r;\lambda,4)}$, for the following values of p,q,r,λ

p	13	13	13	12	9	9	9	9	8	6	6	4	3
q	12	8	4	7	8	6	4	3	3	5	3	3	2
r	9	6	3	3	6	4	3	2	2	3	2	2	1
λ	27	18	9	9	18	12	9	6	6	9	6	6	3

There are 19 GK components of the type $\overline{\mathcal{F}(p,q,r;\lambda,5)}$, for the following values of p,q,r,λ

p	21	21	21	20	20	20	16	16	16	16	15	15	12	8	8	7	6	5	4
q	20	10	5	17	13	9	15	12	5	4	8	4	4	5	4	5	4	4	2
r	16	8	4	12	8	4	12	9	4	3	6	3	3	4	3	4	3	3	1
λ	64	32	16	48	32	16	48	36	16	12	24	12	12	16	12	16	12	12	4

Next we construct components of $\mathcal{F}_2(d,n), n>3$, associated to the affine Lie algebra.

Let $S=\sum_{j=1}^n p_j z_j \partial/\partial z_j$ be a linear vector field on $\mathbb{C}^n$, where $p_1>p_2>\cdots>p_n$ are relatively prime positive integers, and X another polynomial vector field on $\mathbb{C}^n$ where $[S,X]=\lambda.X$, for some $\lambda\in\mathbb{Z}$. Once again, if S and X are linearly independent at generic points, these vector fields generate an algebraic foliation $\mathcal{F}=\overline{\mathcal{F}}(S,X)$ on $\mathbb{C}^n$, which is defined by $\omega=i_S i_X(dz_1\wedge\cdots\wedge dz_n)$.

Set

$$\mathcal{F}(p_1,\ldots,p_n;\lambda,d):=\{\mathcal{F}\in\mathcal{F}_2(d,n)\mid \mathcal{F}=\overline{\mathcal{F}}(S,X)\text{ in some affine chart}\}.$$

By similar reasons to the case $n=3$, $\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$ is an irreducible subvariety of $\mathcal{F}_2(d,n)$.

Definition 1.15. Let ω be an integrable $(n-2)$-form defined in a neighbourhood of $p\in\mathbb{C}^n, n>3$. We say that p is a *weakly generalized Kupka* (WGK) singularity of ω if $\omega(p)=0$ and $cod_{\mathbb{C}}(Sing(d\omega))\geq 3$. The latter expression refers to the codimension of the singular set of the germ of $d\omega$ at p. By convention $cod_{\mathbb{C}}(\emptyset)=n+1$.

Once again this definition depends only on the foliation defined by ω, in the sense that p is a WGK singularity of ω if and only if p is a WGK singularity of $f.\omega, \forall f\in\mathcal{O}_p^*$.

Definition 1.16. A dimension two holomorphic foliation $\mathcal{F}$ in a complex manifold M of dimension n is WGK if all the singularities of $\mathcal{F}$ are WGK.

If $\mathcal{F}\in\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$, denote by $q(\mathcal{F})$ the point of $\mathbb{P}^n$ that corresponds to $0\in E\cong\mathbb{C}^n$, where $E\subset\mathbb{P}^n$ is the affine open set where $\mathcal{F}$ is defined by $\omega=i_S i_X(dz_1\wedge\cdots\wedge dz_n)$. Then we have the following

Theorem 1.17. *If $\lambda>0$ and $\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$ contains some WGK foliation $\mathcal{F}$, where $q(\mathcal{F})$ is a GK singularity of $\mathcal{F}$, then $\overline{\mathcal{F}(p_1,\ldots,p_n;\lambda,d)}$ is an irreducible component of $\mathcal{F}_2(d,n)$. In particular, if $\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$ contains some GK foliation, where $\lambda\neq 0$, then $\overline{\mathcal{F}(p_1,\ldots,p_n;\lambda,d)}$ is an irreducible component of $\mathcal{F}_2(d,n)$.*

Remark 1.18. If $\mathcal{F}\in\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$ is such that $cod(Sing(\mathcal{F}))\geq 3$, then $\overline{\mathcal{F}(p_1,\ldots,p_n;\lambda,d)}$ is an irreducible component of $\mathcal{F}_2(d,n)$ (see section 5.2 of [[9]]).

Next, we give a characterization of the families $\mathcal{F}(p_1,\ldots,p_n;\lambda,d)$ containing some GK foliation on $\mathbb{P}^n, n\geq 3$. This will be set in terms of one analytic condition and arithmetic relations on some parameters, that we define next.

By convention, set $p_{n+1}=0$, and for $i=1,\ldots,n-1, j=1,\ldots,n$ denote by c_{ij} the relation

$$c_{ij}=\begin{cases} p_j+\lambda=p_{i+1}d, & \text{if } j\leq i\\ p_{j+1}+\lambda=p_{i+1}d, & \text{if } j>i.\end{cases}$$

Set

$$\begin{cases}\tau=\lambda+tr(S)=\lambda+\sum_{k=1}^n p_k,\\ \tau_i=\tau-p_i(n+d), i=2,\ldots,n,\\ \lambda_1=p_1(d-1)-\lambda.\end{cases}\tag{1.3}$$

Finally define

$$W_0 = \{\text{polynomial vector fields } Y \text{ in } \mathbb{C}^n \mid [S,Y] = \lambda Y, div(Y) \equiv 0, deg(Y) \leq d+1, i_R i_S i_{Y_{d+1}} \nu \equiv 0\},$$

where we consider $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ and $\nu = dz_1 \wedge \cdots \wedge dz_n$.

In the definition of W_0, $div(Y)$, $deg(Y)$ and Y_{d+1} denote the divergent, the degree and the term of degree $d+1$ in the expansion of the polynomial vector field Y in homogeneous coordinates, respectively. The radial vector field of $\mathbb{C}^n$ is denoted by R. We point out that W_0 is the ambient space of $Y = rot(\omega)$, where $\omega = i_S i_X \nu$ defines a foliation of $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ in some affine chart.

We have

Theorem 1.19. *The families $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1) \subset \mathcal{F}_2(d+1, n), d \geq 2$, containing some GK foliation, are (precisely) those where*

a) 0 is an isolated singularity of some $Y \in W_0$

and $p_1, \ldots, p_n, \lambda$ satisfy either

b.1)
- $c_{11}, c_{22}, \ldots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \ldots, c_{n-1,n}$, *for some* $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$
- $\tau_j \neq 0, j = 2, 3, \ldots, n$

or

b.2)
- $\lambda = p_i(d-1), c_{11}, c_{22}, \ldots, c_{i-2,i-2}, c_{i,i+1}, c_{i+1,i+2}, \ldots, c_{n-1,n}$, *for some* $2 \leq i \leq \lfloor \frac{n+2}{2} \rfloor$
- $\tau_j \neq 0, j \in \{2, 3, \ldots, n\} \setminus \{i\}$

In particular $\lambda = p_n d$ and p_1 divides $p_k + \lambda$, for some $k \in \{1, \ldots, n\}$.

The GK foliations of theorem 1.19 b.1 have only two singularities that are non-kupka, with exception to the case $i = 0$, which may occur foliations having only one singularity of such type. On the other hand, the GK foliations of theorem 1.19 b.2 have only three singularities that are non-Kupka. The above theorem is a basis for explicitly determining the GK components of type $\overline{\mathcal{F}(p_1, \ldots, p_n; \lambda, d)}$. In particular, we obtain a degree classification of the components in $\mathcal{F}_2(d+1, 4)$ of this type

Theorem 1.20. *Let $p > q > r > s \geq 1$ be positive integers, where $gcd(p,q,r,s) = 1$. $\mathcal{F}(p,q,r,s;\lambda, d+1) \subset \mathcal{F}_2(d+1, 4)$ contains a GK foliation, for some $\lambda \in \mathbb{Z}, d \geq 2$, if and only if either p, q, r, s, λ, d or $p, q_1 = p-s, r_1 = p-q, s_1 = p-q, \lambda_1, d$ satisfy one of the following relations*

(a) $p > q = m(d^2+d+1) > r = m(d^2+d) > s = md^2, \lambda = md^3, gcd(p,m) = 1$, p divides either d^3 or d^3+d^2+d+1;

(b) $p = kd > q = md+k > r = m(d+1) > s = md, \lambda = md^2, gcd(k,m) = 1$, either k divides d, or kd divides $m(d^2+d)+k$ (which implies $k = jd$ where j divides $d+1$), or d divides m and k divides d^2+d+1, or k divides $d+1$ and $gcd(\frac{m(d+1)}{k}, d) = 1$;

(c) $p > q = md^2 > r = m(d^2-1) > s = m(d^2-d), \lambda = m(d^3-d^2), gcd(p,m) = 1$, p divides either d^3-d^2, or d^3, or d^3-1;

(d) $p = kd > q = m(d-1)+k > r = md > s = m(d-1), \lambda = m(d^2-d), gcd(k,m) = 1$, either k divides $d-1$, or k divides d, or d divides m and k divides d^2-1.

Corollary 1.21. *For $d \geq 2$, $\overline{\mathcal{F}(p,q,r,s;\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,4)$ for the following values of p, q, r, s, λ*

p	q	r	s	λ
d^3	d^2+d+1	d^2+d	d^2	d^3
d^3+d^2+d+1	d^2+d+1	$d+1$	1	-1
d^3+d^2+d+1	d^2+d+1	d^2+d	d^2	d^3
d^2	$2d$	$d+1$	d	d^2
d^3+d^2	d^3	d^3-2d-1	d^3-d^2-d	$d^4-d^3-d^2$
d^3+d^2+d	$2d^2+d+1$	d^2+d	d^2	d^3
d^2+d	$2d+1$	$d+1$	d	d^2
d^3-d^2	d^2	d^2-1	d^2-d	d^3-d^2
d^3	d^2	d^2-1	d^2-d	d^3-d^2
d^3-1	d^2	d^2-1	d^2-d	d^3-d^2
d^2-d	$2(d-1)$	d	$d-1$	d^2-d
d^2	$2d-1$	d	$d-1$	d^2-d
d^2+d	d^2+1	d^2	d^2-d	d^3-d^2
d^3-d	$2d^2-d-1$	d^2	d^2-d	d^3-d^2

Corollary 1.22. *There are 10 GK components of the type $\overline{\mathcal{F}(p,q,r,s;\lambda,3)}$, for the following values of p,q,r,s,λ*

p	15	15	14	12	8	8	7	6	6	4
q	14	7	11	8	7	4	4	5	5	3
r	12	6	6	3	6	3	3	4	3	2
s	8	4	4	2	4	2	2	2	2	1
λ	16	8	8	4	8	4	4	4	4	2

There are 22 GK components of the type $\overline{\mathcal{F}(p,q,r,s;\lambda,4)}$, for the following values of p,q,r,s,λ

p	40	40	39	39	27	27	27	27	26	24	24
q	39	13	31	22	26	18	13	9	9	18	14
r	36	12	24	12	24	16	12	8	8	15	9
s	27	9	18	9	18	12	9	6	6	10	6
λ	81	27	54	27	54	36	27	18	18	30	18

p	20	18	18	13	12	12	12	9	9	6	6
q	13	9	9	9	10	7	6	7	6	5	4
r	12	8	4	8	9	4	3	6	4	4	3
s	9	6	3	6	6	3	2	4	3	3	2
λ	27	18	9	18	18	9	6	12	9	9	6

Given $p_1 > p_2 > \cdots > p_n$ positive integers, we set

$$\overline{p}_1 = p_1, \overline{p}_i = p_1 - p_{n-i+2}, i = 2, \ldots, n. \tag{1.4}$$

Note that $\overline{p}_1 > \overline{p}_2 > \cdots > \overline{p}_n$ and $gcd(\overline{p}_1, \cdots, \overline{p}_n) = 1$ whenever $gcd(p_1, \cdots, p_n) = 1$.

The next proposition ensures that the families of corollaries 1.14 and 1.22 are pairwise distinct (see also corollary 2.12).

Proposition 1.23. *Assume that $p_1 > p_2 > \cdots > p_n$ and $l_1 > l_2 > \cdots > l_n$ are two sequences of positive integers, where $gcd(p_1, \ldots, p_n) = gcd(l_1, \ldots, l_n) = 1$. Suppose that $\overline{\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)} = \overline{\mathcal{F}(l_1, \ldots, l_n; \xi, d+1)}$ and one of the families (therefore both) contains a GK foliation. Then, either $l_1 = p_1, \ldots, l_n = p_n, \xi = \lambda$ or $l_1 = \overline{p}_1, \ldots, l_n = \overline{p}_n, \xi = \lambda_1$.*

With respect to irreducible components of $\mathcal{F}_2(d,n), n > 4$, we have a generalization of the Klein-foliations

Corollary 1.24. *Let $p_1 > p_2 > \ldots > p_n$ be positive integers defined by $p_i = \sum_{j=0}^{n-i} d^j, i = 1, \ldots, n$. Then, for every $d \geq 1$, $\overline{\mathcal{F}(p_1, \ldots, p_n; -1, d+1)}$ is an irreducible component of $\mathcal{F}_2(d,n)$. Furthermore, this is the unique GK component provided by theorem 1.17 where the GK foliations belonging to it have only one non-Kupka singularity.*

Chapter 2

Preliminaries

In this chapter, we introduce the machinery needed to develop the main results. Also we obtain some results as consequence of the kind of singularity that appears on GK foliations. The tangent sheaf of such foliations is determined as well.

1 Quasi-homogeneous vector fields

In this section we will adopt the following convention: given a polynomial vector field (resp. form) on $\mathbb{C}^n$, say X (resp. ω), we will write $X = X_0 + X_1 + \cdots + X_k$ (resp. $\omega = \omega_0 + \omega_1 + \cdots + \omega_k$) to denote its decomposition into homogeneous polynomial vector fields (resp. forms) in the variables $(z_1, \ldots, z_n)$.

Also S will stand for the linear vector field $S = \sum_{j=1}^{n} p_j z_j \partial/\partial z_j$ on $\mathbb{C}^n$, where $p_1, \ldots, p_n$ are integers. In addition, if $p_1, \ldots, p_n$ are positive, we say that a holomorphic vector field X on $\mathbb{C}^n$ is *quasi-homogeneous* with respect to S, with weight $\lambda \in \mathbb{Z}$, if

$$[S, X] = \lambda.X.$$

Next proposition is an adapted version, although the same proof holds, of the proposition 4.2.1 of [[13]]. Recall that if 0 is an isolated singularity of a holomorphic vector field $Y = \sum_{i=1}^{n} P_i(z)\partial/\partial z_i$ defined in an open set $0 \in U \subset \mathbb{C}^n$, then the multiplicity of Y at 0 is by definition

$$m(Y, 0) = dim_{\mathbb{C}} \tfrac{\mathcal{O}_n}{\langle P_1, \ldots, P_n \rangle},$$

where $\langle P_1, \ldots, P_n \rangle$ denotes the ideal of $\mathcal{O}_n$ generated by the germs of $P_1, \ldots, P_n$ at 0.

Proposition 2.1. *Let $X \neq 0$ be a holomorphic vector field on $\mathbb{C}^n$, where $[S, X] = \lambda.X$. Suppose that $p_1 \geq p_2 \geq \cdots \geq p_n$. Then*

(a) $\lambda \in \mathbb{Z}$;

(b) $Ld(S, X) := \{z \in \mathbb{C}^n \mid S(z)$ and $X(z)$ are linearly dependent$\}$ is a union of orbits of the action induced by the vector field S, $S_t(z) := exp(t.S).z$;

Additionally, if $p_n \geq 1$ then

(c) $\lambda \geq -p_1$ and X is a polynomial vector field;

(d) If $0 \in \mathbb{C}^n$ is an isolated singularity of X then

$$m(X,0)=\frac{\prod_{j=1}^{n}(p_j+\lambda)}{\prod_{j=1}^{n}p_j}.$$

A closer look at the relation $[S,X]=\lambda.X$ yields

Proposition 2.2. *Let $X=\sum_{j=1}^{n}X_j(z)\partial/\partial z_j$ be a holomorphic vector field on $\mathbb{C}^n$. Then the following are equivalent*

(a) $[S,X]=\lambda.X$*;*

(b) $S(X_j)=(\lambda+p_j).X_j, 1\le j\le n$*;*

(c) $X_j\left(t^{p_1}.z_1,\dots,t^{p_n}.z_n\right)=t^{p_j+\lambda}.X_j(z_1,\dots,z_n), \forall 1\le j\le n, \forall t\in\mathbb{C}$*;*

(d) If $X_j=\sum_{j\sigma}a_{j\sigma}z^{\sigma}, 1\le j\le n$, where $a_{j\sigma}\in\mathbb{C}$ and for $\sigma=(\sigma_1,\dots,\sigma_n), z^{\sigma}=z_1^{\sigma_1}\cdots z_n^{\sigma_n}$, then $a_{j\sigma}\neq 0 \implies \sum_{k=1}^{n}p_k.\sigma_k=p_j+\lambda$.

For example, if $p_j=1, 1\le j\le n$, then S is the radial vector field on $\mathbb{C}^n$ and the equality $[S,X]=\lambda.X$ implies that X is homogeneous of degree $\lambda+1$.

Remark 2.3. Let X be a holomorphic vector field on $\mathbb{C}^n$ satisfying $[S,X]=\lambda.X$. Assume that

$$p_1\ge p_2\ge\cdots\ge p_n\ge 1.$$

If $0\in\mathbb{C}^n$ is an isolated singularity of X then $\lambda\ge 0$ and if $X(0)\neq 0$ then $\lambda<0$. Suppose first that 0 is an isolated singularity of X. By proposition 2.1 (c), we have $X=\sum_{j=1}^{n}P_j(z)\partial/\partial z_j$, where $P_1,\dots,P_n$ are polynomial functions. Suppose, by contradiction, that $\lambda<0$. In this case $\lambda>-p_n$, otherwise $p_n+\lambda\le 0$ would imply from proposition 2.2 (d) that $P_n\equiv 0$. Therefore

$$\{z\in\mathbb{C}^n \mid P_1(z)=P_2(z)=\cdots=P_{n-1}(z)=0\}\subset Sing(X),$$

contradicting our assumption that 0 is an isolated singularity of X. On the other hand,

$$-p_n<\lambda<0 \implies 0<p_j+\lambda<p_j, \forall j\in\{1,2,\dots,n\} \implies \prod_{j=1}^{n}(p_j+\lambda)<\prod_{j=1}^{n}p_j.$$

Once more we get a contradiction since $m(X,0)=\frac{\prod_{j=1}^{n}(p_j+\lambda)}{\prod_{j=1}^{n}p_j}\in\mathbb{Z}$. Now, if $X(0)\neq 0$, by proposition 2.2 (d) there exists $j\in\{1,2,\dots,n\}$ such that $\lambda=-p_j<0$.

From now on, we will consider that the eigenvalues of the linear vector field S satisfy

$$p_1>p_2>\cdots>p_n\ge 1.$$

When $[S,X]=\lambda.X$, we can define the integrable $(n-2)$-form $\omega=i_Si_X\nu$ on $\mathbb{C}^n$ ($\nu=dz_1\wedge\cdots\wedge dz_n$). From proposition 2.1, ω is polynomial and $Sing(\omega)$ is a union of orbits of S. If $\omega\not\equiv 0$ then ω defines a two-dimensional foliation on $\mathbb{C}^n$, denoted as in chapter 1 by $\overline{\mathcal{F}}(S,X)$. Of course, the leaves of the one-dimensional foliations defined by S and X are contained in the leaves of $\overline{\mathcal{F}}(S,X)$. Also as in chapter 1, we denote by $\mathcal{F}(S,X)$ the foliation of $\mathbb{P}^n$ defined by ω in affine chart.

We have

$$d\omega=d(i_Si_X\nu)=L_S(i_X\nu)-i_Sd(i_X\nu)=i_{[S,X]}\nu+i_X(L_S\nu)-div(X).i_S\nu. \tag{2.1}$$

Recall that if $Z=\sum_i Z_i\partial/\partial z_i$ is a holomorphic vector field on $\mathbb{C}^n$, then $div(Z)$ is defined by $d(i_Z\nu)=div(Z).\nu$. Equivalently, $div(Z)=\sum_i\frac{\partial Z_i}{\partial z_i}$.

It follows that $d\omega=i_Y\nu$, where

$$Y=\tau.X-div(X).S, \tag{2.2}$$

and $\tau := \lambda + tr(S) = \lambda + \sum_{i=1}^{n} p_i$. From proposition 2.1 (c) we see that $\tau > 0$. Therefore Y is the rotational of ω and we can say that 0 is an isolated singularity of $d\omega$ if and only if 0 is an isolated singularity of Y.

Using (2.2) one verifies that Y satisfies

$$[S,Y] = \lambda.Y, \omega = \frac{1}{\tau} i_S i_Y \nu, div(Y) = 0.$$

Furthermore, if

$$[S,X] = \lambda.X, \omega = i_S i_X \nu,$$

then X is a scalar multiple of $Y = rot(\omega)$ if and only if $div(X) = 0$. From $\omega = \frac{1}{\tau} i_S i_Y \nu$, we conclude that S and Y also generate the foliation defined by ω on $\mathbb{C}^n$.

Recall that a dimension one singular holomorphic foliation $\mathcal{G}_X$ of degree d on $\mathbb{P}^n$ is given in some affine chart $E \simeq \mathbb{C}^n$ by a polynomial vector field

$$X = X_0 + X_1 + \ldots + X_{d+1},$$

where $X_{d+1} = g_d R$, g_d is a homogeneous polynomial of degree d. If $g_d \equiv 0$ then $X_d \neq 0$ and it is not of the form $X_d = g_{d-1} R$, where g_{d-1} is a homogeneous polynomial of degree $d-1$.

Lemma 2.4. *Suppose that $\mathcal{F} = \mathcal{F}(S,X) \in \mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$. Then X can be chosen satisfying $deg(\mathcal{G}_X) = d$. Reciprocally, let X be a polynomial vector field with $[S,X] = \lambda X$ and suppose that $deg(\mathcal{G}_X) = d$. Then $deg(\mathcal{F}(S,X)) \leq d+1$, and the equality occurs if and only if either $X_{d+1} \neq 0$ or $X_{d+1} = 0$ and X_d is not of the form $f_{d-1} \cdot S + h_{d-1} \cdot R$, for homogeneous polynomials f_{d-1} and h_{d-1} of degree $d-1$.*

Proof. Suppose first that $deg(\mathcal{F}(S,X)) = d+1$, so on the open set where $\omega = i_S i_X \nu$ defines $\mathcal{F}(S,X)$ we write the decomposition

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_{d+2}, i_R \omega_{d+2} = 0.$$

From $d\omega = i_Y \nu$, $\omega = \frac{1}{\tau} i_S i_Y \nu$ we see that

$$Y = Y_0 + Y_1 + \cdots + Y_{d+1}.$$

In addition $i_R i_S i_{Y_{d+1}} \nu = 0$ since $\omega_{d+2} = \frac{1}{\tau} i_S i_{Y_{d+1}} \nu$. As $Ld(R,S)$ is a union of lines, in particular has codimension greater than two, it follows from the parametric De Rham division theorem ([[3]] or [[18]]) and from Hartog's theorem that there exist holomorphic functions f and g on $\mathbb{C}^n$ such that $Y_{d+1} = f.S + h.R$. As Y_{d+1}, R, S are homogeneous, we have $Y_{d+1} = f_d.S + h_d.R$, where f_d and g_d are homogeneous polynomials of degree d. If we define

$$\overline{X} = Y - f_d.S = \sum_{i=0}^{d} Y_i + h_d.R,$$

notice that $\mathcal{F} = \mathcal{F}(S,X) = \mathcal{F}(S,\overline{X})$ and $deg(\mathcal{G}_{\overline{X}}) = d$.

Reciprocally, suppose $deg(\mathcal{G}_X) = d$, so

$$X = X_0 + X_1 + \cdots + X_d + X_{d+1}, X_{d+1} = g_d R.$$

Hence

$$\omega = i_S i_X \nu = \sum_{i=1}^{d+2} \omega_i, \omega_i = i_S i_{X_{i-1}} \nu, i = 1, \ldots, d+2.$$

We have $deg(\mathcal{F}(S,X)) \leq d+1$ since $i_R\omega_{d+2} = 0$. Therefore $deg(\mathcal{F}(S,X)) = d+1$ if and only if

$$\omega_{d+2} = g_d i_S i_R \nu \neq 0$$

or $\omega_{d+2} = 0$ and

$$i_R\omega_{d+1} = i_R i_S i_{X_d}\nu \neq 0,$$

equivalently, $g_d \neq 0$ or $X_d \neq f_{d-1}.S + h_{d-1}.R$, for some homogeneous polynomials f_{d-1}, g_{d-1} of degree $d-1$. □

By the previous lemma, given $\mathcal{F} \in \mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$, we can assume that $\mathcal{F}$ is defined in some affine chart by $\omega = i_S i_X \nu, [S,X] = \lambda.X$ and $deg(\mathcal{G}_X) = d$.

Before stating next result, we recall the parameters $\tau, \tau_i, i = 2,\ldots,n, \lambda_1$ and the numbers $\overline{p}_1,\ldots,\overline{p}_n$, all defined on section 1.3 at (1.3) and (1.4), respectively. Denote

$$\nu_0 = dx_1 \wedge \cdots \wedge dx_n, \nu_1 = du_1 \wedge \cdots \wedge du_n.$$

Proposition 2.5. *Given $\mathcal{F} \in \mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$, there exist affine coordinate systems $(E_0,(x_1,\ldots,x_n))$ and $(E_i,(u_1,\ldots,u_n)), i = 1,\ldots,n$, such that $\mathbb{P}^n = E_0 \cup \cdots \cup E_n$ and*

(a) On E_0, $\mathcal{F}$ is defined by $\omega = i_S i_X \nu_0$, $[S,X] = \lambda.X$, $deg(\mathcal{G}_X) = d$. If $Y = rot(\omega)$, then

$$Y = \tau.X - div(X).S, [S,Y] = \lambda.Y, \omega = \frac{1}{\tau} i_S i_Y \nu_0.$$

(b) On E_1, S is given by $-S_1$, where

$$S_1 = \overline{p}_1 u_1 \partial\ \partial u_1 + \cdots + \overline{p}_n u_n \partial/\partial u_n.$$

If X_1 is the polynomial vector field defining $\mathcal{G}_X$ on E_1, then $[S_1, X_1] = \lambda_1 X_1$, and $\mathcal{F}$ is defined by $\omega_1 = i_{S_1} i_{X_1} \nu_1$ on this chart. Additionally, if $Y_1 = rot(\omega_1)$, then

$$Y_1 = \tau_1.X_1 - div(X_1).S_1, [S_1, Y_1] = \lambda_1.Y_1, \omega_1 = \frac{1}{\tau_1} i_{S_1} i_{Y_1} \nu_1,$$

where $\tau_1 = \lambda_1 + tr(S_1) = \lambda_1 + \sum_{j=1}^n \overline{p}_j$.

(c) On $E_i, i = 2,\ldots,n$, S is given by $S_i = \sum_{j=1}^n \rho_j u_j \partial/\partial u_j$, where

$$\rho_1 = p_1 - p_i > \cdots > \rho_{i-1} = p_{i-1} - p_i > 0 > \rho_i = p_{i+1} - p_i > \cdots > \rho_{n-1} = p_n - p_i > \rho_n = -p_i.$$

If X_i is the polynomial vector field defining $\mathcal{G}_X$ on E_i, then $[S_i, X_i] = \lambda_i X_i$, where $\lambda_i = \lambda - p_i(d-1)$, and $\mathcal{F}$ is defined by $\omega_i = i_{S_i} i_{X_i} \nu_1$ on E_i. Additionally, if $Y_i = rot(\omega_i)$, then

$$Y_i = \tau_i.X_i - div(X_i).S_i, [S_i, Y_i] = \lambda_i.Y_i, \tau_i.\omega_i = i_{S_i} i_{Y_i} \nu_1.$$

(d) The linear vector field S, thought as a holomorphic vector field of $\mathbb{P}^n$, has $n+1$ singularities, which we denote by $q_0,\ldots,q_n$. They are the points of $\mathbb{P}^n$ corresponding to $0 \in E_i \cong \mathbb{C}^n, i = 0,\ldots,n$, respectively.

(e) Denote $(z_0 : z_1 : \cdots : z_n)$ as homogeneous coordinates in $\mathbb{P}^n$. Then, up to a linear automorphism of $\mathbb{P}^n$, we can assume that

$$E_0 = \{(x_1 : \cdots : x_n : 1) | (x_1, \cdots, x_n) \in \mathbb{C}^n\}, E_1 = \{(1 : u_n : u_{n-1} : \cdots : u_1) | (u_1, \cdots, u_n) \in \mathbb{C}^n\},$$
$$E_i = \{(u_1 : \cdots : u_{i-1} : 1 : u_i : \cdots : u_n) | (u_1, \cdots, u_n) \in \mathbb{C}^n\}, i \in \{2,3,\ldots,n\}.$$

Proof. Set $E_0, \dots, E_n$ as in the item (e) above. Clearly it suffices to check items from (a) to (d) in this case. So $\mathcal{F}$ is defined on E_0 by the $(n-2)$-form

$$\omega = i_S i_X \nu_0, [S, X] = \lambda.X.$$

By lemma 2.4, we can assume that $deg(\mathcal{G}_X) = deg(\mathcal{F}) - 1 = d$. From formula (2.2) above it follows (a). Let us look for expressions of $\mathcal{F}, S, X$ in the other charts.

The vector field S is linear and extends to a holomorphic vector field on $\mathbb{P}^n$, which still will be denoted by S. As $S = \sum_{j=1}^n p_j x_j \partial/\partial x_j$ on E_0, we have that S given on E_1 by

$$-S_1 := -p_1 u_1 \partial/\partial u_1 - \sum_{j=2}^n \overline{p}_j u_j \partial/\partial u_j.$$

Recall that $\overline{p}_j = p_1 - p_{n-j+2}, j \in \{2, \dots, n\}$. Note that if $p_1 > p_2 > \dots > p_n$, then $p_1 > \overline{p}_2 > \dots > \overline{p}_n$.

On E_i, $2 \le i \le n$, S is given by

$$S_i := -p_i u_n \partial/\partial u_n + \sum_{j=1}^{i-1} (p_j - p_i) u_j \partial/\partial u_j + \sum_{j=i}^{n-1} (p_{j+1} - p_i) u_j \partial/\partial u_j.$$

The global field S has $n+1$ singularities, they are the points of $\mathbb{P}^n$ denoted by $q_0, q_1, \dots, q_n$. Observe that they correspond to $0 \in E_i, i \in \{0, \dots, n\}$, respectively. It follows (d). The change of coordinates from E_0 to E_1 is given by

$$u_1 = \frac{1}{x_1}, u_2 = \frac{x_n}{x_1}, u_3 = \frac{x_{n-1}}{x_1}, \dots, u_n = \frac{x_2}{x_1}.$$

As $deg(\mathcal{G}_X) = d$, X has a pole of order $d-1$ at $u_1 = 0$ and can be written $X = \frac{X_1}{u_1^{d-1}}$, where X_1 defines $\mathcal{G}_X$ on the chart E_1. The vector field $S_1 = -S$ on E_1 has positive eigenvalues and it will be considered on this chart. We have

$$[S_1, X_1] = [-S, u_1^{d-1}.X] = S_1(u_1^{d-1}).X - u_1^{d-1}.[S, X] = p_1(d-1)u_1^{d-1}.X - u_1^{d-1}.\lambda.X = \lambda_1.X_1,$$

where $\omega_1 = i_{S_1} i_{X_1} \nu_1$ defines $\mathcal{F}$ on E_1 (see proposition 2.11 below). If $Y_1 = rot(\omega_1)$, i.e., $d\omega_1 = i_{Y_1}\nu_1$, it follows from (2.1) that

$$Y_1 = \tau_1.X_1 - div(X_1).S_1, \tag{2.3}$$

where $\tau_1 = \lambda_1 + tr(S_1) = \lambda_1 + (n+1)p_1 - \sum_{i=1}^n p_i$. Note that, just as τ, from proposition 2.1 (c) we have that $\tau_1 > 0$. It follows (b).

By similar reasons, for $i \in \{2, \dots, n\}$, one has $X = \frac{X_i}{u_n^{d-1}}$, where X_i defines $\mathcal{G}_X$ on E_i,

$$[S_i, X_i] = \lambda_i X_i, \lambda_i = \lambda - p_i(d-1),$$

and $\omega_i = i_{S_i} i_{X_i} \nu_1$ defines $\mathcal{F}$ on $E_i, i \in \{2, \dots, n\}$. Set $\tau_i = \lambda_i + tr(S_i) = \sum_{k=1}^n p_k - (n+1)p_i$ and once more it follows from (2.1) that

$$Y_i = \tau_i.X_i - div(X_i).S_i \tag{2.4}$$

for $i \in \{2, \dots, n\}$. From (2.3) and (2.4) we get

$$[S_i, Y_i] = \lambda_i.Y_i, \tag{2.5}$$

$$\tau_i.\omega_i = i_{S_i} i_{Y_i} \nu_1, \tag{2.6}$$

$i \in \{1, 2, \dots, n\}$. It follows (c). □

2 Generalized Kupka and quasi-homogeneous singularities

Throughout this section, we shall consider $\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ in the situation of proposition 2.5. Recall that $p \in \mathbb{C}^n$ is a *generalized Kupka* (GK) singularity of the integrable 1-form ω if $\omega(p) = 0$ and either $d\omega(p) \neq 0$ or p is an isolated singularity of $d\omega$ (see definition 1.1). When $\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ is GK, where $\lambda > 0$, the singularity $q_0 \in E_0$ is of a special type, which we define now.

Definition 2.6. We say that $p \in \mathbb{C}^n$ is a *quasi-homogeneous* singularity of ω if p is an isolated singularity of $Y = rot(\omega)$ and the germ of Y at p is nilpotent (as a derivation in the local ring of formal power series at p).

If we fix some coordinate system $z = (z_1, \ldots, z_n)$ around p, where $z(p) = 0$, equivalently p is a quasi-homogeneous singularity of ω if $DY(0)$ is linear nilpotent. We would like to note that the concepts of definition above are independent of the non-vanishing n-form used to calculate the rotational Y of ω. Indeed, they depend only on the germ of foliation defined by ω, in the sense that

$$0 \text{ is a quasi-homogeneous singularity of } \omega \iff 0 \text{ is a quasi-homogeneous singularity of } f.\omega, \forall f \in \mathcal{O}_n^*.$$

The definition is justified by the following result ([[14]])

Theorem 2.7. *Let $p \in \mathbb{C}^3$ be a quasi-homogeneous singularity of an integrable 1-form ω. Then there exist a local chart $(U, (x, y, z))$ around p such that $x(p) = y(p) = z(p) = 0$ and two germs of holomorphic vector fields S and Z such that*

(a) $\omega = i_S i_X(dx \wedge dy \wedge dz), dw = i_Z(dx \wedge dy \wedge dz)$*;*

(b) $S = \frac{1}{q}T$*, where* $T = p_1.x\partial/\partial x + p_2.y\partial/\partial y + p_3.z\partial/\partial z, q, p_1, p_2, p_3 \in \mathbb{N}$ *and* $tr(S) < 1$*;*

(c) $L_S(\omega) = \omega$ *and* $[S, Z] = (1 - tr(S)).Z$*.*

In particular, the form ω has polynomial coefficients in the coordinate system (x, y, z), which in turn are quasi-homogeneous with respect to T.

When $n = 3$, we are mainly interested in the following corollary of the proof of theorem 2.7

Corollary 2.8. *Assume that $\omega = i_Z i_Y \nu, d\omega = i_Y \nu$, where $\nu = dx \wedge dy \wedge dz$, and $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of Y. Then the eigenvalues of $DZ(0)$ are all positive rational numbers.*

We will use proposition 2.10 below in the proof of theorem 1.10, which is based on the following lemma ([[15]].

Lemma 2.9. *Let A and L be linear vector fields on $\mathbb{C}^n$ such that $[L, A] = \mu.A$, where $\mu \neq 0$. Then A is nilpotent.*

Proof. It is a known fact from linear algebra that if B and C are two linear endomorphisms of $\mathbb{C}^n$, then $B.C$ and $C.B$ have the same characteristic polynomial, consequently $tr(B.C - C.B) = 0$. We show by induction on $m \in \mathbb{N}$ that

$$[L, A^m] = m.\mu.A^m,$$

and by the latter result we get $tr(A^m) = 0$ because $tr([L, A^m]) = 0$. This implies that all eigenvalues of A vanish and that A is nilpotent. In fact, if the eigenvalues of A are $\mu_1, \ldots, \mu_n$ then

$$tr(A^m) = \sum_j \mu_j^m, \forall m \in \mathbb{Z} \implies \sum_j \mu_j^m = 0, \forall m \in \mathbb{N} \implies \mu_1 = \cdots = \mu_n = 0$$

Finally, let us assume by induction that $\left[L, A^{m-1}\right] = (m-1).\mu.A^{m-1}, m \geq 2$.
Then

$$[L, A^m] = A^m.L - L.A^m = A.(A^{m-1}.L - L.A^{m-1}) + (A.L - L.A).A^{m-1} =$$
$$A.\left[L, A^{m-1}\right] + [L, A].A^{m-1} = m.\mu.A^m,$$

by the induction hypothesis. □

Proposition 2.10. *Suppose that $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d+1)$ is GK, where $\lambda \in \mathbb{Z}_{>0}$ and $p > q > r \geq 1$ are positive integers. Then*

(a) The singularity $q_0 \in E_0 \cap Sing(\mathcal{F})$ is quasi-homogeneous;

(b) If $q_2 \in E_2 \cap Sing(\mathcal{F})$ (respectively $q_3 \in E_3 \cap Sing(\mathcal{F})$) is a non-Kupka singularity, then $\lambda = q(d-1)$ (respectively $\lambda = r(d-1)$).

Proof. We use the notation previously established in the case $n = 3$. For (a), note that $\mathcal{F}$ is defined on E_0 by

$$\omega = i_S i_X \nu_0, [S, X] = \lambda.X.$$

As $\lambda > 0$ and $[S, Y] = \lambda.Y$ ($Y = rot(\omega)$), by remark 2.3 it follows that 0 is an isolated singularity of Y. Also from $[S, Y] = \lambda.Y$ we have that

$$[S, DY(0)] = \lambda.DY(0),$$

then the result follows from lemma 2.9 with $L = S, A = DY(0), \mu = \lambda > 0$.

For (b), suppose by contradiction that q_2 is a non-Kupka singularity and $\lambda \neq q(d-1)$ (for $q_3, \lambda \neq r(d-1)$ is analogous). We know that $\mathcal{F}$ is defined on E_2 by

$$\omega_2 = i_{S_2} i_{X_2} \nu_1, [S_2, X_2] = \lambda_2.X_2,$$

where $\lambda_2 = \lambda - q(d-1)$. We also have $[S_2, Y_2] = \lambda_2 Y_2$, which implies

$$[S_2, DY_2(0)] = \lambda_2.DY_2(0),$$

where $Y_2 = rot(\omega_2)$. As $\lambda_2 \neq 0$ we conclude from lemma 2.9 with $L = S_2, A = DY_2(0), \mu = \lambda_2 \neq 0$ that $DY_2(0)$ is nilpotent.

If $\tau_2 = \lambda_2 + tr(S_2) \neq 0$, then

$$\omega_2 = \frac{1}{\tau_2} i_{S_2} i_{Y_2} \nu_1 = i_{\frac{S_2}{\tau_2}} i_{Y_2} \nu_1$$

and from corollary 2.8 we get a contradiction, since the eigenvalues of $\frac{S_2}{\tau_2}$ are

$$\frac{p-q}{\tau_2}, \frac{r-q}{\tau_2}, -\frac{q}{\tau_2},$$

not all positives.

If $\tau_2 = 0$, then

$$0 = \tau_2.\omega_2 = i_{S_2} i_{Y_2} \nu_1.$$

Since q_2 is an isolated of S_2, we can apply the parametric De Rham division theorem to obtain a germ of holomorphic function (indeed polynomial) f at $0 \in \mathbb{C}^3$ such that $Y_2 = f.S_2$. Set $l = f(0)$. If $l = 0$ then the zeros of Y_2 are not isolated since the zeros of f are not, and we obtain a contradiction since $\mathcal{F}$ is GK. If $l \neq 0$ then the eigenvalues of Y_2 are

$$l.(p-q), l.(r-q), l.(-q),$$

once again we obtain a contradiction since $DY_2(0)$ is nilpotent. □

3 Foliations with split tangent sheaf

Let $\mathcal{F}$ be a two dimensional holomorphic foliation on a complex manifold M of dimension $n \geq 3$. The *tangent sheaf* of $\mathcal{F}$, denoted by $\mathcal{TF}$, is the sheaf whose stalk for every $p \in M$ is given by

$$\mathcal{T}_p\mathcal{F} = \{v \in \mathcal{X}_p \mid v \text{ is tangent to } \mathcal{F}\}.$$

In this case, $\mathcal{TF}$ is a coherent sheaf of generic rank two and we say that the tangent sheaf of $\mathcal{F}$ splits if $\mathcal{TF} = \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_1, \mathcal{E}_2$ are subsheafs of rank one of $\mathcal{TF}$. One can show that the tangent sheaf of $\mathcal{F}$ splits if and only if there exist two foliations by curves $\mathcal{G}_1, \mathcal{G}_2$ on M, such that if $p \in M \setminus Sing(\mathcal{F})$ then $p \notin Sing(\mathcal{G}_j), j = 1, 2$, and $T_p\mathcal{F} = T_p\mathcal{G}_1 \oplus T_p\mathcal{G}_2$ ([[13]], remark 4.1.4). In this case we say that the foliations $\mathcal{G}_1$ and $\mathcal{G}_2$ generate $\mathcal{F}$.

Proposition 2.11. *Let $\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ and assume that $\mathcal{F}$ is generated in some affine chart by S and X so that $deg(\mathcal{G}_X) = d$, like in lemma 2.4. Then $\mathcal{G}_S$ and $\mathcal{G}_X$ generate $\mathcal{F}$. In particular, the tangent sheaf of $\mathcal{F}$ splits.*

Proof. Without loss of generality, assume that we are in the situation of proposition 2.5 (e), i.e., $\mathcal{F}$ is defined on E_0 by

$$\omega = i_S i_X \nu_0, [S, X] = \lambda.X.$$

As $deg(\mathcal{G}_X) = d$, we can write $X = P + g.R$, where g is a homogeneous polynomial of degree d and

$$P = \sum_{i=1}^{n} A_i(x_1, \ldots, x_n)\partial/\partial x_i$$

is polynomial of degree d.

The change of coordinates from E_0 to E_1 is given by

$$u_1 = \frac{1}{x_1}, u_2 = \frac{x_n}{x_1}, \ldots, u_n = \frac{x_2}{x_1},$$

and $deg(\mathcal{G}_X) = d$ implies that $X = \frac{X_1}{u_1^{d-1}}$ in $E_0 \cap E_1$, where $X_1 = \sum_{i=1}^{n} P_i\partial/\partial u_i$ is a polynomial vector field representing $\mathcal{G}_X$ in the chart E_1. In fact

$$\begin{cases} P_1(u_1, \ldots, u_n) = -u_1^{d+1}A_1(\frac{1}{u_1}, \frac{u_n}{u_1}, \ldots, \frac{u_2}{u_1}) - g(1, u_n, \ldots, u_2), \\ P_k(u_1, \ldots, u_n) = u_1^d A_{n+2-k}(\frac{1}{u_1}, \frac{u_n}{u_1}, \ldots, \frac{u_2}{u_1}) - u_k u_1^d A_1(\frac{1}{u_1}, \frac{u_n}{u_1}, \ldots, \frac{u_2}{u_1}), 2 \leq k \leq n. \end{cases}$$

In the chart E_1, S is given by

$$-S_1 = -p_1 u_1\partial/\partial u_1 - (p_1 - p_n)u_2\partial/\partial u_2 - \ldots - (p_1 - p_2)u_n\partial/\partial u_n.$$

Observe that S and X generate $\mathcal{F}$ on E_0, so $\mathcal{G}_S$ and $\mathcal{G}_X$ generate $\mathcal{F}$ unless $S_1(p)$ and $X_1(p)$ are linearly dependent at every point of $H \cap E_1$, where $H = \{(z_0 : z_1 : \cdots : z_n) \in \mathbb{P}^n \mid z_n = 0\}$ is the hyperplane at infinity corresponding to E_0. Clearly the last assertion is equivalent to

$$\{(u_1, \ldots, u_n) \in E_1 \mid u_1 = 0\} \subset \{p \in E_1 \mid S_1(p) \wedge X_1(p) = 0\}.$$

Denote by $A_i^{(d)}$ the homogeneous term of degree d of $A_i, 1 \leq i \leq n$. As

$$S_1(0, u_2, \ldots, u_n) = (p_1 - p_n)u_2\partial/\partial u_2 + \ldots + (p_1 - p_2)u_n\partial/\partial u_n,$$

$$X_1(0, u_2, \ldots, u_n) = -g(1, u_n, \ldots, u_2)\partial/\partial u_1 + \sum_{k=2}^{n}(A_{n+2-k}^{(d)}(1, u_n, \ldots, u_2) - u_k A_1^{(d)}(1, u_n, \ldots, u_2))\partial/\partial u_k$$

one has $\{(u_1,\ldots,u_n)\in E_1 \mid u_1=0\}\subset\{p\in E_1 \mid S_1(p)\wedge X_1(p)=0\}$ if and only if

$$\begin{cases} g(1,u_n,\ldots,u_2)\equiv 0 \\ u_k \mid A^{(d)}_{n+2-k}(1,u_n,\ldots,u_2), 2\le k\le n \\ \frac{A^{(d)}_{n+2-i}(1,u_n,\ldots,u_2)-u_iA^{(d)}_1(1,u_n,\ldots,u_2)}{(p_1-p_{n+2-i})u_i}=\frac{A^{(d)}_{n+2-j}(1,u_n,\ldots,u_2)-u_jA^{(d)}_1(1,u_n,\ldots,u_2)}{(p_1-p_{n+2-j})u_j}, i,j=2,\ldots,n. \end{cases}$$

If we go back to the variables $u_1=\frac{1}{x_1}, u_2=\frac{x_n}{x_1},\ldots,u_n=\frac{x_2}{x_1}$, the equations above are equivalent respectively to

$$\begin{cases} g(x_1,\ldots,x_n)\equiv 0 \\ x_k \mid A^{(d)}_k(x_1,\ldots,x_n), 2\le k\le n \\ (p_1-p_i)\frac{A^{(d)}_j(x_1,\ldots,x_n)}{x_j}-(p_1-p_j)\frac{A^{(d)}_i(x_1,\ldots,x_n)}{x_i}=(p_j-p_i)\frac{A^{(d)}_1(x_1,\ldots,x_n)}{x_1}, i,j=2,\ldots,n. \end{cases}$$

Set $X^{(d)}=\sum_{i=1}^n A^{(d)}_i\partial/\partial x_i$, and we claim that the last two set of conditions above are equivalent to $X^{(d)}=f.S+h.R$, for some homogeneous polynomials f and h of degree $d-1$. Indeed, if $X^{(d)}=f.S+h.R$ then $A^{(d)}_k=x_k.(h+p_k.f)$, consequently

$$x_k \mid A^{(d)}_k, k=1,\ldots,n.$$

A simply verification shows that the last set of equalities is also true.

Conversely, set

$$f=\frac{A^{(d)}_2/x_2-A^{(d)}_3/x_3}{p_2-p_3}, h=\frac{p_2A^{(d)}_3/x_3-p_3A^{(d)}_2/x_2}{p_2-p_3}.$$

So f and h are homogeneous polynomials of degree $d-1$, and one verifies that $A^{(d)}_2=x_2(h+p_2f)$ and $A^{(d)}_3=x_3(h+p_3f)$. Making the substitutions

$$i=2, j=3, A^{(d)}_2/x_2=h+p_2f, A^{(d)}_3/x_3=h+p_3f$$

in the relation above we see that $A^{(d)}_1=x_1(h+p_1f)$. For $k\in\{2,\ldots,n\}, k\neq 2,3$, substituting

$$i=k, j=2, A^{(d)}_2/x_2=h+p_2f, A^{(d)}_1/x_1=h+p_1f$$

we see that $A^{(d)}_k=x_k(h+p_kf)$. Thus $X^{(d)}=f.S+h.R$.

Consequently, if $g\equiv 0$ by lemma 2.4 $X^{(d)}$ is not the form $X^{(d)}=f.S+h.R$ for some f,h homogeneous polynomials of degree $d-1$, hence $\mathcal{G}_S$ and $\mathcal{G}_X$ generate $\mathcal{F}$. □

Corollary 2.12. $\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)=\mathcal{F}(\overline{p}_1,\ldots,\overline{p}_n;\lambda_1,d+1)$.

Proof. By symmetry, it is sufficient to show that $\mathcal{F}\in\mathcal{F}(\overline{p}_1,\ldots,\overline{p}_n;\lambda_1,d+1)$ if $\mathcal{F}\in\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$. But it follows from propositions 2.11 and 2.5 (b). □

Corollary 2.13. *If* $\mathcal{F}\in\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$ *then* $\mathcal{TF}=\mathcal{O}\oplus\mathcal{O}(1-d)$.

Proof. As we saw, $\mathcal{G}_S$ and $\mathcal{G}_X$ generate $\mathcal{F}$. S is a global vector field in $\mathbb{P}^n$ with singular set of codimension greater or equal than two, whereas X can be thought as a meromorphic vector field with singular set of codimension greater or equal than two and a polar divisor of order $d-1$. Then the corollary follows. □

Chapter 3

Proof of the results related to the case $n = 3$

1 Proof of theorem 1.10

Theorem 1.10. Let $p > q > r \geq 1$ be positive integers, where $gcd(p, q, r) = 1$. $\mathcal{F}(p, q, r; \lambda, d + 1) \subset \mathcal{F}(d + 1, 3)$ contains a GK foliation, for some $\lambda \in \mathbb{Z}, d \geq 2$, if and only if either p, q, r, λ, d or $p, q_1 = p - r, r_1 = p - q, \lambda_1 = p(d - 1) - \lambda, d$ satisfy one of the following relations

(a) $p = d > q = r + 1 > r, \lambda = dr$;

(b) $p = kd > q = md + k > r = md, \lambda = md^2, gcd(k, m) = 1, k$ divides $d + 1$;

(c) $p > q = m(d + 1) > r = md, \lambda = md^2, gcd(p, m) = 1, p$ divides either d^2 or $d^2 + d + 1$;

(d) $p > q = md > r = m(d - 1), \lambda = m(d^2 - d), gcd(p, m) = 1, p$ divides either $d^2 - d$, or d^2, or $d^2 - 1$.

Proof. The idea of the proof is the following. By proposition 2.10, given a GK foliation $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d+1)$, it follows that either q_2 or q_3 are singularities of Kupka type. So we use this information and proposition 2.2 (d) to obtain necessary conditions on the parameters p, q, r, λ, d. Then, from these necessary conditions and from the information that q_0 is GK we find those conditions where q_0, q_2, q_3 are GK. This will be enough due to the following lemma

Lemma 3.1. *A foliation $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d + 1)$ is GK if and only if the singularities q_0, q_2, q_3 of $\mathcal{F}$ are GK.*

Proof. Of course, if $\mathcal{F}$ is GK then the singularities q_0, q_2, q_3 are GK. Conversely, assume that q_0, q_2, q_3 are GK singularities of $\mathcal{F}$. Suppose, by contradiction, that $\mathcal{F}$ is not GK. This means that there exists $x \in Sing(\mathcal{F})$ that is not GK. In particular, x is a non-Kupka singularity of $\mathcal{F}$.

Suppose first that $x \notin Sing(S) = \{q_0, q_1, q_2, q_3\}$, i.e., $x \neq q_1$. It is not difficult to see that the orbit of the global vector field S passing throughout any point $z \notin Sing(S)$ accumulates at two points of $Sing(S)$. So, there exists $i \in \{0, 2, 3\}$ such that q_i belongs to the closure of the orbit of S passing through x. By proposition 2.1 (b), since

$$[S_i, Y_i] = \lambda_i.Y_i, Y_i = rot(\omega_i),$$

where ω_i defines $\mathcal{F}$ on E_i (by convention $S_0 = S$, $\omega_0 = \omega$, $Y_0 = Y$), it follows that the orbit of S passing through x is contained in $Sing(Y_i)$. We obtain a contradiction, since q_i is GK.

Next, suppose that $x = q_1$. Then, q_1 is not GK implies that there exists a curve $\gamma \subset Sing(Y_1)$ invariant by the flow of S on E_1. The latter is a consequence of proposition 2.1 (b), since $[S_1, Y_1] = \lambda_1.Y_1$, and the fact that q_1 belongs to the closure of every orbit of S on E_1. From the relation

$$\omega_1 = \frac{1}{\tau_1} i_{S_1} i_{Y_1} \nu_1,$$

we see that $Sing(Y_1) \subset Sing(\mathcal{F})$. Then any point $x \in \gamma \setminus \{q_1\}$ is a singularity of $\mathcal{F}$ different from q_0, q_1, q_2, q_3 that is not GK. Once again this contradicts q_0, q_2, q_3 being GK. $\square$

We can assume that the affine coordinate systems of the requested GK foliation $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d+1)$ is like in proposition 2.5 (e). So on E_0, $\mathcal{F}$ is defined by

$$\omega = i_S i_X \nu_0, [S, X] = \lambda.X, deg(\mathcal{G}_X) = d.$$

As $deg(\mathcal{G}_X) = d$, we have $X = (A + gx)\partial/\partial x + (B + gy)\partial/\partial y + (C + gz)\partial/\partial z$, where A, B, C, g are polynomials with $deg(A), deg(B), deg(C) \leq d$ and g is homogeneous of degree d. Write

$$A = \sum_{i+j+k \leq d} a_{ijk} x^i y^j z^k, \qquad B = \sum_{i+j+k \leq d} b_{ijk} x^i y^j z^k,$$
$$C = \sum_{i+j+k \leq d} c_{ijk} x^i y^j z^k, \qquad g = \sum_{i+j+k = d} g_{ijk} x^i y^j z^k.$$

Recall, for example, by proposition 2.2 (d), if $a_{ijk} \neq 0$ then $pi + qj + rk = p + \lambda$. In this proof, given a polynomial vector field Y, in order to avoid some confusion with the rotational vector fields Y_i on E_i, we will write

$$Y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \cdots$$

to denote its decomposition into homogeneous polynomial vector fields.

Next we write the jets of order 1 of Y_2 and Y_3 in terms of the parameters defining X. We have

$$\omega = i_S i_X \nu_0 = [rzB - qyC + (r-q)yzg]dx + [pxC - rzA + (p-r)xzg]dy + [qyA - pxB + (q-p)xyg]dz,$$

so a homogeneous form of $\mathcal{F}$ is given by $\Omega = A_0 dx + A_1 dy + A_2 dz + A_3 dw$, where

$$\begin{cases} A_0 = rzw\tilde{B} - qyw\tilde{C} + (r-q)yzg \\ A_1 = pxw\tilde{C} - rzw\tilde{A} + (p-r)xzg \\ A_2 = qyw\tilde{A} - pxw\tilde{B} + (q-p)xyg \\ A_3 = (r-q)yz\tilde{A} + (p-r)xz\tilde{B} + (q-p)xy\tilde{C} \end{cases}$$

and $\tilde{A} = w^d A(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}), \tilde{B} = w^d B(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}), \tilde{C} = w^d C(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$.

On the chart E_2,

$$\omega_2 = \Omega|_{E_2} = A_0(u, 1, v, w)du + A_2(u, 1, v, w)dv + A_3(u, 1, v, w)dw.$$

From $d\omega_2 = i_{Y_2}\nu_1$ it follows that

$$Y_2 = \left(\frac{\partial}{\partial v}A_3 - \frac{\partial}{\partial w}A_2\right)\partial/\partial u + \left(\frac{\partial}{\partial w}A_0 - \frac{\partial}{\partial u}A_3\right)\partial/\partial v + \left(\frac{\partial}{\partial u}A_2 - \frac{\partial}{\partial v}A_0\right)\partial/\partial w,$$

so

$$\begin{cases} Y_2^{(0)} = (r - 2q)a_{0d0}\partial/\partial u + (p - 2q)c_{0d0}\partial/\partial v + (2q - p - r)g_{0d0}\partial/\partial w, \\ Y_2^{(1)} = Y_2^{(1)}(u)\partial/\partial u + Y_2^{(1)}(v)\partial/\partial v + Y_2^{(1)}(w)\partial/\partial w, \end{cases}$$

where

$$
\begin{cases}
Y_2^{(1)}(u) = [(r-2q)a_{1,d-1,0} + (2p-r)b_{0d0} + (q-p)c_{0,d-1,1}]u + (2r-3q)a_{0,d-1,1}v + \\
(r-3q)a_{0,d-1,0}w, \\
Y_2^{(1)}(v) = (2p-3q)c_{1,d-1,0}u + [(q-r)a_{1,d-1,0} + (2r-p)b_{0d0} + (p-2q)c_{0,d-1,1}]v + \\
(p-3q)c_{0,d-1,0}w, \\
Y_2^{(1)}(w) = (3q-2p-r)g_{1,d-1,0}u + (3q-p-2r)g_{0,d-1,1}v + [qa_{1,d-1,0} - (p+r)b_{0,d,0} + qc_{0,d-1,1}]w.
\end{cases}
$$

Analogously, on the chart E_3

$$
\omega_3 = \Omega|_{E_3} = A_0(u,v,1,w)du + A_1(u,v,1,w)dv + A_3(u,v,1,w)dw
$$

and $Y_3^{(0)} = (2r-q)a_{00d}\partial/\partial u + (2r-p)b_{00d}\partial/\partial v + (p+q-2r)g_{00d}\partial/\partial w$.

Observe that

$$
\begin{cases}
(r-2q)a_{0d0} \neq 0 \implies a_{0d0} \neq 0 \implies p+\lambda = qd & \text{(I)} \\
(p-2q)c_{0d0} \neq 0 \implies c_{0d0} \neq 0 \implies r+\lambda = qd & \text{(II)} \\
(2q-p-r)g_{0d0} \neq 0 \implies g_{0d0} \neq 0 \implies \lambda = qd & \text{(III)} \\
(2r-q)a_{00d} \neq 0 \implies a_{00d} \neq 0 \implies p+\lambda = rd & \text{(IV)} \\
(2r-p)b_{00d} \neq 0 \implies b_{00d} \neq 0 \implies q+\lambda = rd & \text{(V)} \\
(p+q-2r)g_{00d} \neq 0 \implies g_{00d} \neq 0 \implies \lambda = rd & \text{(VI)}
\end{cases}
$$

So if $Y_2(q_2) \neq 0$, then we have either (I), or (II), or (III). Similarly, if $Y_3(q_3) \neq 0$, we have either (IV), or (V), or (VI).

If $\mathcal{F}$ is GK, we have four possibilities

a) q_2 and q_3 are Kupka singularities, which means $Y_2(q_2) \neq 0$ and $Y_3(q_3) \neq 0$;

   One can check that among the six conditions above there are only three pairs which can occur simultaneously: (I) and (VI), (II) and (VI), (I) and (V) (for example, we cannot have (II) and (IV) at the same time because it would imply $qd = r + \lambda < p + \lambda = rd$, which is a contradiction since $q > r$). So it is necessary that one of the three conditions occur

   a.1) $p + \lambda = qd$ and $\lambda = rd$;

   a.2) $r + \lambda = qd$ and $\lambda = rd$;

   a.3) $p + \lambda = qd$ and $q + \lambda = rd$.

   A simple verification shows the equivalences

$$
p + \lambda = qd \iff \lambda_1 = r_1 d, q + \lambda = rd \iff r_1 + \lambda_1 = q_1 d.
$$

   Since $\mathcal{F}(p,q,r;\lambda,d+1) = \mathcal{F}(p,q_1,r_1;\lambda_1,d+1)$, the families $\mathcal{F}(p,q,r,\lambda,d+1)$ containing some GK foliation satisfying a.3 coincide with those satisfying a.2, thus we can treat only the cases a.1 and a.2.

b) q_2 is a non-Kupka singularity and q_3 is a Kupka singularity;

   By proposition 2.10, we have that $\lambda = q(d-1)$. In addition, we must have (IV), (V) or (VI) above. It follows that $\lambda = q(d-1) = rd$ (for example, $\lambda = q(d-1)$ implies $q + \lambda = qd$, so we cannot have $p + \lambda = rd$ since $p + \lambda > q + \lambda$ and $rd < qd$).

c) q_2 is a Kupka singularity and q_3 is a non-Kupka singularity;

By proposition 2.10, we have that $\lambda = r(d-1)$. In addition, we must have (I), (II) or (III) above. Proceeding in a similar way to the previous item, it follows that $\lambda = r(d-1)$ and $p + \lambda = qd$. From the equivalences

$$\lambda = r(d-1) \iff \lambda_1 = q_1(d-1), p + \lambda = qd \iff \lambda_1 = r_1 d,$$

and from $\mathcal{F}(p,q,r;\lambda,d+1) = \mathcal{F}(p,q_1,r_1;\lambda_1,d+1)$, we see that the families $\mathcal{F}(p,q,r,\lambda,d+1)$ containing some GK foliation and satisfying (c) coincide with those satisfying (b), thus we can treat only case (b).

d) q_2 and q_3 are non-Kupka singularities.

By proposition 2.10 (b), this is not possible.

In all cases $\lambda = rd > 0$, then it follows from proposition 2.10 (a) that q_0 must be quasi-homogeneous singularity of $\mathcal{F}$. In particular $Y(q_0) = 0$, where $Y = rot(\omega)$. Let us write

$$Y = A_1 \partial/\partial x + B_1 \partial/\partial y + C_1 \partial/\partial z$$

and note that a term with the monomial x^m must appear in the expansion of either A_1, or B_1, or C_1, otherwise $\{(x,y,z) \in \mathbb{C}^3 \mid y = z = 0\} \subset Sing(Y)$ and this clearly contradicts the fact that 0 is an isolated singularity of Y. So either $p + \lambda = pm$, or $q + \lambda = pm$, or $r + \lambda = pm$ and consequently p divides either $p + \lambda$, or $q + \lambda$ or $r + \lambda$.

Notice that the families $\mathcal{F}(p,q,r;\lambda,d+1)$ containing some GK foliation satisfying a.1 and p dividing $q + \lambda$ coincide with the families satisfying a.1 and p dividing $r + \lambda$. This is due to the equivalences

$$p + \lambda = qd \iff \lambda_1 = r_1 d, \lambda = rd \iff p + \lambda_1 = q_1 d, p \mid q + \lambda \iff p \mid r_1 + \lambda_1$$

and the relation $\mathcal{F}(p,q,r;\lambda,d+1) = \mathcal{F}(p,q_1,r_1;\lambda_1,d+1)$. So only the case a.1 where p divides $r + \lambda$ will be considered.

Let us summarize the necessary relations on p,q,r,λ,d we have found so far. We will obtain from them those relations which are also sufficient in order to $\mathcal{F}(p,q,r;\lambda,d+1)$ contains some GK foliation.

a.1) $p + \lambda = qd, \lambda = rd$, p divides either $p + \lambda$ or $r + \lambda$;

a.2) $r + \lambda = qd, \lambda = rd$, p divides either $p + \lambda$, or $q + \lambda$, or $r + \lambda$;

b) $\lambda = q(d-1) = rd$, p divides either $p + \lambda$, or $q + \lambda$, or $r + \lambda$.

Set

$$\begin{aligned} W_0 = &\{\text{polynomial vector fields } Y \text{ in } E_0 \cong \mathbb{C}^3 \mid [S,Y] = \lambda Y, div(Y) \equiv 0, deg(Y) \leq d+1, \\ & i_R i_S i_{Y^{(d+1)}} \nu \equiv 0\}. \end{aligned}$$

Observe that W_0 is a finite-dimensional vector space over $\mathbb{C}$ (indeed, W_0 is a subspace of the finite dimensional vector space $\{X \mid [S,X] = \lambda X\}$). Given $Y \in W_0$, define

$$\omega_Y = \frac{1}{\tau} i_S i_Y \nu_0.$$

One can check that

$$d\omega_Y = i_Y \nu_0,$$

i.e., W_0 is nothing more than the ambient space of $Y = rot(\omega_Y)$, whenever ω_Y defines a foliation $\mathcal{F} \in \mathcal{F}(p,q,r;\lambda,d+1)$ on affine charts (recall that $\tau = \lambda + p + q + r > 0$). Let V_0 stand for the projectivization $V_0 = \mathbb{P}(W_0)$.

If ω_Y does not define a foliation that extends to a foliation of degree $d+1$ on $\mathbb{P}^3$, there are two possibilities: either $cod(Sing(\omega)) = 1$ (if $\omega_Y \not\equiv 0$) or ω_Y defines a foliation of $\mathbb{P}^3$ denoted by $\mathcal{F}(S,Y)$ and $deg(\mathcal{F}(S,Y)) < d+1$. Set

$$\Gamma_0 = \{[Y] \in V_0 \mid cod(Sing(\omega_Y)) \leq 1 \text{ or } deg(\mathcal{F}(S,Y)) < d+1\}.$$

Also set

$$\Sigma_0 = \{[Y] \in V_0 \mid 0 \in \mathbb{C}^3 \text{ is a non-isolated singularity of } Y\}.$$

Lemma 3.2. *Γ_0 and Σ_0 are algebraic subsets of V_0.*

Proof. First we show that Γ_0 is an algebraic subset of V_0. Let $\mathcal{H}$ be the set of the integrable homogeneous one-forms Ω of degree $d+2$ on $\mathbb{C}^4$ satisfying $i_R\Omega = 0$. Also, given $Y \in W_0$, let Ω_Y be the homogeneous one-form of degree $d+2$ obtained by homogenizing ω_Y (in the same way as ω_Y provides a foliation of degree $d+1$).

One has the linear map $Y \mapsto \Omega_Y$ between the vector spaces W_0 and $\mathcal{H}$. This map is injective. For, if $\Omega_Y = 0$ then

$$\omega_Y = \Omega_Y|_{E_0} = \tfrac{1}{\tau} i_S i_Y \nu_0 = 0.$$

So there exists a polynomial f such that $Y = f.S$, and $S(f) = \lambda.f$ because $[S,Y] = \lambda.Y$. This implies that

$$0 = div(Y) = \tau.f,$$

and we get $f = 0$. Therefore $Y = 0$.

This injective linear map induces a regular map $\pi : V_0 \to \mathbb{P}\mathcal{H}$. Set

$$\mathcal{J} = \{[\Omega] \in \mathbb{P}\mathcal{H} \mid cod(Sing(\Omega)) = 1\}.$$

It is a known fact that $\mathcal{J}$ is a proper algebraic subset of $\mathbb{P}\mathcal{H}$ (indeed $\mathcal{F}(d+1,3) = \mathcal{H}\backslash\mathcal{J}$). We conclude by noting that $\Gamma_0 = \pi^{-1}(\mathcal{J})$.

Next we show that Σ_0 is an algebraic subset of V_0. The fact that 0 is a non-isolated singularity of $Y \in W_0$ means that Y has another singularity which differs from 0. This is because $Sing(Y)$ is invariant by the flow of S and $0 \in \mathbb{C}^3$ belongs to the closure of every orbit of S. Let us write

$$Y = A_1\partial/\partial x + B_1\partial/\partial y + C_1\partial/\partial z.$$

Then by proposition 2.2 (c)

$$\tilde{A}_1 := A_1(x^p, y^q, z^r), \tilde{B}_1 := Y_2(x^p, y^q, z^r), \tilde{C}_1 := C_1(x^p, y^q, z^r)$$

are homogeneous polynomials of degree $p+\lambda, q+\lambda, r+\lambda$ respectively. It is clear that the system of equations

$$A_1 = B_1 = C_1 = 0$$

has a nontrivial solution if and only if the system of equations

$$\tilde{A}_1 = \tilde{B}_1 = \tilde{C}_1 = 0$$

has a nontrivial solution. Let

$$Res_{d_1,d_2,d_3}(F_1, F_2, F_3)$$

denote the multipolynomial resultant for three homogeneous polynomials F_1, F_2, F_3 of degrees d_1, d_2, d_3, respectively ([[8]], chapter 3, §2). The system

$$\tilde{A}_1 = \tilde{B}_1 = \tilde{C}_1 = 0$$

has a nontrivial solution if and only if

$$Res_{p+\lambda,q+\lambda,r+\lambda}(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1) = 0,$$

which is an algebraic equation in the coefficients of $\tilde{A}_1, \tilde{B}_1, \tilde{C}_1$ and consequently in the coefficients of A_1, B_1 and C_1, i.e., in the the coordinates of V_0. Therefore Σ_0 is algebraic. □

Remark 3.3. Although we have written ω in two seemingly different ways, namely $\omega = i_S i_X \nu_0$ and $\omega = \frac{1}{\tau} i_S i_Y \nu_0$, we saw in the proof of lemma 2.4 that there exist homogeneous polynomials f, h of degree d such that $Y^{(d+1)} = f.R + h.S$, and if

$$X = \tfrac{Y - h.S}{\tau},$$

then

$$\omega = i_S i_X \nu_0 = \frac{1}{\tau} i_S i_Y \nu_0$$

and $deg(\mathcal{G}_X) = d$. In particular, $j^d(X) = \frac{1}{\tau}.j^d(Y)$, where j^k denotes the k-th jet of the corresponding vector field. For this reason we maintain

$$a_{ijk}, b_{ijk}, c_{ijk}$$

to represent the coefficients of Y. With respect to the homogeneous term of degree $d+1$ of Y, $Y^{(d+1)}$, from

$$\lambda = rd, S(f) = \lambda.f, S(h) = \lambda.h,$$

we have that $f = f_{00d} z^d$ and $h = h_{00d} z^d$. For instance, if $f = \sum f_{ijk} x^i y^j z^k$, then $S(f) = \lambda.f$ means that

$$f_{ijk} \neq 0 \implies pi + qj + rk = \lambda = rd,$$

and clearly the only possible solution of the latter equation is $i = j = 0, k = d$. Then $f = f_{00d} z^d$, for some $f_{00d} \in \mathbb{C}$. A straightforward calculation shows that the term of degree d of $div(Y)$ is given by

$$((d+3) f_{00d} + \tau h_{00d}) z^d,$$

and from $div(Y) \equiv 0$ we see that there exist a scalar μ such that $f_{00d} = \mu\tau, h_{00d} = -\mu(d+3)$. Thus $Y^{(d+1)} = \mu z^d (\tau.R - (d+3).S)$, for $\mu \in \mathbb{C}$. Thus we consider

$$(a_{ijk} : b_{ijk} : c_{ijk} : \mu)$$

as coordinates of V_0.

Lemma 3.4. *In the above conditions a.1, a.2, b, with exception to condition a.2 where p divides $r+\lambda$, there exists a proper algebraic subset $\Delta_0 \subset V_0$ such that $\mathcal{F}(S, Y) \in \mathcal{F}(p, q, r; \lambda, d+1)$ is GK if $[Y] \in V_0 \backslash \Delta_0$.*

Proof. We begin by showing that in the condition a.2) where p divides $r + \lambda$, a family $\mathcal{F}(p, q, r; \lambda, d+1)$ has no GK foliations. So we are in the situation where

$$r + \lambda = qd, \lambda = rd, p \mid r + \lambda.$$

As $\lambda > 0$, by remark 2.3 and proposition 2.1 (d) it suffices to show that

$$m_1 = \tfrac{(p+\lambda)(q+\lambda)(r+\lambda)}{pqr} \notin \mathbb{Z}.$$

From $r(d+1) = qd$ and $gcd(d, d+1) = 1$, there exists positive integer m such that $q = m(d+1), r = md$. Then

$$gcd(p, q, r) = 1 \iff gcd(p, m) = 1.$$

Thus $p \mid r + \lambda \iff p \mid d(d+1)$, since $\lambda = rd = md^2$.

Certainly $gcd(p, d+1) \neq 1$, otherwise $p \mid d$ which implies $m(d+1) = q < p < d$, which is a contradiction. A straightforward calculation shows that

$$m_1 = \frac{(p + md^2)(d^2 + d + 1)}{p}.$$

Suppose, by contradiction, that $m_1 \in \mathbb{Z}$. Then $gcd(p, m) = 1$ implies that $p \mid d^2(d^2 + d + 1)$. Clearly a prime factor of p and $d+1$ cannot divide neither d^2 nor d^2+d+1, which is a contradiction. So $m_1 \notin \mathbb{Z}$ and $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d+1)$ is never GK.

For all other cases set

$$\alpha_p = \tau - p(d+3), \alpha_q = \tau - q(d+3), \alpha_r = \tau - r(d+3).$$

With this notation, by remark 3.3

$$Y^{(d+1)} = \mu z^d(\tau.R - (d+3).S) = \mu\alpha_p x z^d \partial/\partial x + \mu\alpha_q y z^d \partial/\partial y + \mu\alpha_r z^{d+1}\partial/\partial z.$$

Note that we always have $\alpha_p = q + r - 2p + (r-p)d < 0$ and $\alpha_r = p + q - 2r > 0$.

Next we denote the remaining seven cases by letters $a), b), \ldots, g)$. We show that they correspond to the conditions of theorem 1.10.

In the sequel, keep in mind that given $\mathcal{F} \in \mathcal{F}(p, q, r; \lambda, d+1)$ defined on E_0 by $\omega = \frac{1}{\tau} i_S i_Y \nu_0$, where $Y \in W_0$, q_0 is a GK singularity of $\mathcal{F}$ if and only if $[Y] \in V_0 \setminus (\Gamma_0 \cup \Sigma_0)$. This is due to remark 2.3, since $\lambda = rd > 0$. Similarly, we will see that q_2 and q_3 being GK are given by Zariski-open conditions.

(a) $p + \lambda = qd, \lambda = rd, p \mid p + \lambda$
   As $p = (q-r)d, \lambda = rd$, it follows that

$$p \mid p + \lambda \iff q - r \mid r.$$

   If $q - r \mid r$, then $q - r \mid q = (q-r) + r$. Since $q - r \mid p$ and $gcd(p, q, r) = 1$, we have that $q - r = 1$. Thus $p = (q-r)d = d$ and $q = r + 1$. So we are in the situation of theorem 1.10 (a).

   - $V_0 \setminus \Gamma_0 \neq \emptyset$
     Set

$$Y_{aux} = y^d \partial/\partial x + z^d(\tau R - (d+3)S).$$

     Then $Y_{aux} \in W_0$ and

$$\begin{aligned}\tau\omega &= i_S i_{Y_{aux}} \nu_0 \\ &= \tau(r-q) y z^{d+1} dx + (-r y^d z + \tau(p-r) x z^{d+1}) dy + (q y^{d+1} + \tau(q-p) x y z^d) dz\end{aligned}$$

     is such that $cod(Sing(\omega)) \geq 2$ and $deg(\mathcal{F}(S, Y_{aux})) = d+1$, i.e., $[Y_{aux}] \in V_0 \setminus \Gamma_0$.

   - $V_0 \setminus \Sigma_0 \neq \emptyset$
     Set $l = \frac{p+\lambda}{p} = \frac{qd}{p}$. Then $1 < l < d$. Take

$$Y = (x(\alpha_p z^d + a x^{l-1}) + y^d)\partial/\partial x + y(\alpha_q z^d + b x^{l-1})\partial/\partial y + z(\alpha_r z^d + c x^{l-1})\partial/\partial z.$$

Then $Y \in W_0$ as long as $l.a + b + c = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\alpha_r.a - \alpha_p.c \neq 0, \alpha_q.c - \alpha_r.b \neq 0, a \neq 0, b \neq 0.$$

If we consider the equation $l.a + b + c = 0$ as a hyperplane on $\mathbb{C}^3$ with coordinates (a, b, c), then the hyperplanes

$$\alpha_r.a - \alpha_p.c \neq 0, \alpha_q.c - \alpha_r.b \neq 0, a \neq 0, b \neq 0$$

are all different from $l.a + b + c = 0$, which turns this choice possible. Thus $V_0 \setminus \Sigma_0 \neq \emptyset$.

As $p + \lambda = qd, \lambda = rd$, by proposition 2.2 (d) we have that $c_{0d0} = g_{0d0} = a_{00d} = b_{00d} = 0$. Consequently (see remark 3.3 and the equations involving $Y_2^{(0)}$ and $Y_3^{(0)}$ above)

$$Y_2^{(0)} = \frac{1}{\tau}.(r - 2q)a_{0d0}\partial/\partial u,$$
$$Y_3^{(0)} = (p + q - 2r)\mu\partial/\partial w.$$

As $r - 2q, p + q - 2r \neq 0$ we take $\Delta_0 = \Gamma_0 \cup \Sigma_0 \cup H_1 \cup H_2$, where $H_1, H_2 \subset V_0$ are the hyperplanes

$$H_1 = \{[Y] \in V_0 \mid a_{0d0} = 0\},$$
$$H_2 = \{[Y] \in V_0 \mid \mu = 0\}.$$

If $[Y] \in V_0 \setminus \Delta_0$, then q_0 is a quasi-homogeneous singularity of $\mathcal{F}$ and q_2 and q_3 are Kupka singularities, consequently by lemma 3.1 $\mathcal{F}$ is GK.

(b) $p + \lambda = qd, \lambda = rd, p \mid r + \lambda$
As $p = (q - r)d, \lambda = rd$, it follows that

$$p \mid r + \lambda \iff (q - r)d \mid r(d + 1).$$

Since $gcd(d, d+1) = 1$, it is necessary that $d \mid r$, equivalently, there exists $m \in \mathbb{N}$ such that $r = md$. Set $k = q - r$, then

$$p \mid r + \lambda \iff k \mid m(d + 1).$$

But $gcd(p, q, r) = 1$ implies that $gcd(k, m) = 1$, thus

$$p \mid r + \lambda \iff k \mid d + 1.$$

So $p = kd, q = r + k = md + k, r = md, \lambda = md^2$ and we are in the situation of theorem 1.10 (b).

- $V_0 \setminus \Gamma_0 \neq \emptyset$
  The same proof of the item (a).
- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{r+\lambda}{p}$. Then $l < \frac{p+\lambda}{p} = \frac{qd}{p} < d$. Take

$$Y = (\alpha_p x z^d + y^d)\partial/\partial x + \alpha_q y z^d \partial/\partial y + (\alpha_r z^{d+1} + x^l)\partial/\partial z.$$

  Then $Y \in W_0$ and 0 is an isolated singularity of Y.

The rest of the proof is the same of the item (a).

(c) $r+\lambda = qd, \lambda = rd, p \mid p+\lambda$
As $r(d+1) = qd$ and $gcd(d, d+1) = 1$, there exists positive integer m such that $q = m(d+1), r = md$. Then

$$gcd(p, q, r) = 1 \iff gcd(p, m) = 1.$$

We have that

$$p \mid p+\lambda \iff p \mid d^2,$$

since $\lambda = rd = md^2$. So we are in the situation of theorem 1.10 (c), where $p \mid d^2$.

- $V_0 \setminus \Gamma_0 \neq \emptyset$
Set
$$Y_{aux} = y^d \partial/\partial z + z^d(\tau R - (d+3)S).$$
Then $Y_{aux} \in W_0$ and
$$\begin{aligned}\tau\omega &= i_S i_{Y_{aux}} \nu_0 \\ &= (\tau(r-q)yz^{d+1} - qy^{d+1})dx + (pxy^d + \tau(p-r)xz^{d+1})dy + \tau(q-p)xyz^d dz\end{aligned}$$
is such that $cod(Sing(\omega)) \geq 2$ and $deg(\mathcal{F}(S, Y_{aux})) = d+1$.
- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{p+\lambda}{p}$. Then $1 < l < d$. Take
$$Y = x(\alpha_p z^d + ax^{l-1})\partial/\partial x + y(\alpha_q z^d + bx^{l-1})\partial/\partial y + (z(\alpha_r z^d + cx^{l-1}) + y^d)\partial/\partial z.$$
Then $Y \in W_0$ as long as $l.a + b + c = 0$. Furthermore, 0 is an isolated singularity of Y if and only if
$$\alpha_p.b - \alpha_q.a \neq 0, \alpha_p.c - \alpha_r.a \neq 0, a \neq 0.$$
This choice is possible by a similar reason that happened in the proof that $V_0 \setminus \Sigma_0 \neq \emptyset$ of item (a).

As $r+\lambda = qd, \lambda = rd$, by proposition 2.2 (d) we have that $a_{0d0} = g_{0d0} = a_{00d} = b_{00d} = 0$. Consequently

$$\begin{aligned}Y_2^{(0)} &= \frac{1}{\tau}.(p-2q)c_{0d0}\partial/\partial u, \\ Y_3^{(0)} &= (p+q-2r)\mu\partial/\partial w.\end{aligned}$$

At a first moment we could have $p - 2q = 0$, but we claim that it never happens. In fact, suppose that this is not true; then

$$p = 2q = 2m(d+1)$$

and from $gcd(p, q, r) = 1$ we get $m = 1$. Since $p \mid p+\lambda$, we have $2(d+1) \mid d^2$, which is a contradiction. Then $p - 2q \neq 0$.

We take $\Delta_0 = \Gamma_0 \cup \Sigma_0 \cup H_1 \cup H_2$, where $H_1, H_2 \subset V_0$ are the hyperplanes

$$\begin{aligned}H_1 &= \{[Y] \in V_0 \mid c_{0d0} = 0\}, \\ H_2 &= \{[Y] \in V_0 \mid \mu = 0\}.\end{aligned}$$

Once again, if $[Y] \in V_0 \setminus \Delta_0$, then q_0 is a quasi-homogeneous singularity of $\mathcal{F}$ and q_2 and q_3 are Kupka singularities, consequently by lemma 3.1 $\mathcal{F}$ is GK.

(d) $r+\lambda = qd, \lambda = rd, p \mid q+\lambda$
As in the item (c), we have $p > q = m(d+1) > r = md, \lambda = md^2$ for some positive integer m. The condition $p \mid q+\lambda$ is equivalent to $p \mid d^2+d+1$. So we are in the situation of theorem 1.10 (c), where $p \mid d^2+d+1$.

- $V_0 \setminus \Gamma_0 \neq \emptyset$
  The same proof of item (c).
- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{q+\lambda}{p} < d$. Take

  $$Y = \alpha_p x z^d \partial/\partial x + (\alpha_q y z^d + x^l)\partial/\partial y + (\alpha_r z^{d+1} + y^d)\partial/\partial z,$$

  then $Y \in W_0$. In this case $\alpha_q = p - 2q$ is different from 0. Indeed, if we suppose it is not the case, we can proceed just as at the end of item (c) to conclude that $p = 2(d+1) \mid d^2+d+1$, which is a contradiction. Then 0 is an isolated singularity of Y.

The rest of the proof is the same of the item (c).

(e) $\lambda = rd = q(d-1), p \mid p+\lambda$
As $rd = q(d-1)$ and $gcd(d, d-1) = 1$, there exists positive integer m such that $q = md, r = m(d-1)$. Then

$$gcd(p,q,r) = 1 \iff gcd(p,m) = 1.$$

Thus

$$p \mid p+\lambda \iff p \mid d(d-1),$$

since $\lambda = md(d-1)$, and we are in the situation of theorem 1.10 (d), where $p \mid d^2 - d$.

- $V_0 \setminus \Gamma_0 \neq \emptyset$
  Take

  $$X = xz^d \partial/\partial x + (yz^d + y^d)\partial/\partial y + z^{d+1}\partial/\partial z.$$

  The vector field X satisfies $[S, X] = \lambda.X$. Set

  $$\omega = i_S i_X \nu = yz((r-q)z^d + ry^{d-1})dx + (p-r)xz^{d+1}dy + xy((q-p)z^d - py^{d-1})dz,$$

  so $cod(Sing(\omega)) \geq 2$ and $deg(\mathcal{F}(S,X)) = d+1$. If $Y = rot(\omega)$, then $Y \in V_0 \setminus \Gamma_0$.
- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{p+\lambda}{p}$. Then $1 < l = 1 + \frac{rd}{p} < d+1$. Take

  $$\begin{aligned} Y =& x(\alpha_p z^d + a x^{l-1} + a_1 y^{d-1})\partial/\partial x + y(\alpha_q z^d + b x^{l-1} + b_1 y^{d-1})\partial/\partial y + \\ & z(\alpha_r z^d + c x^{l-1} + c_1 y^{d-1})\partial/\partial z. \end{aligned}$$

  Then $Y \in W_0$ as long as $l.a + b + c = 0$ and $a_1 + d.b_1 + c_1 = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

  $$\begin{vmatrix} \alpha_p & a & a_1 \\ \alpha_q & b & b_1 \\ \alpha_r & c & c_1 \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_p & a \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b_1 \\ \alpha_r & c_1 \end{vmatrix} \neq 0, \begin{vmatrix} a & a_1 \\ b & b_1 \end{vmatrix} \neq 0, a \neq 0, b_1 \neq 0.$$

  where $|\cdot|$ denotes the determinant of the respective matrix. Next we see that such choice is always possible. Consider $\mathbb{C}^4$ with coordinates (a, a_1, b, b_1). After making the substitutions

$c = -l.a - b$ and $c_1 = -d.b_1 - c_1$, we see that the conditions above provided by the 2×2 determinants are given by non-empty Zariski open sets of $\mathbb{C}^4$, the same holds for conditions $b_1 \neq 0, a \neq 0$. Define the polynomial in the variables a, a_1, b, b_1

$$H(a, a_1, b, b_1) = \begin{vmatrix} \alpha_p & a & a_1 \\ \alpha_q & b & b_1 \\ \alpha_r & -l.a - b & -d.b_1 - c_1 \end{vmatrix}.$$

H is a homogeneous polynomial of degree 2 and expanding the determinant above we find $\alpha_p.(1-d)$ as coefficient of the term $b.b_1$. So H does not vanish identically and the first condition is also given by a non-empty Zariski open set of $\mathbb{C}^4$. Consequently $V_0 \setminus \Sigma_0 \neq \emptyset$.

As $\lambda = rd = q(d-1)$, by proposition 2.2 (d) we have that $a_{00d} = b_{00d} = a_{0,d-1,0} = c_{1,d-1,0} = c_{0,d-1,0} = g_{1,d-1,0} = g_{0,d-1,1} = 0$. Consequently

$$\begin{aligned} Y_2^{(1)} &= \frac{1}{\tau}.(L_1 u\partial/\partial u + L_2 v\partial/\partial v + L_3 w\partial/\partial w), \\ Y_3^{(0)} &= (p + q - 2r)\mu\partial/\partial w, \end{aligned}$$

where (see the equation involving $Y_2^{(1)}$ above)

$$\begin{aligned} L_1 &= (r - 2q)a_{1,d-1,0} + (2p - r)b_{0d0} + (q - p)c_{0,d-1,1}, \\ L_2 &= (q - r)a_{1,d-1,0} + (2r - p)b_{0d0} + (p - 2q)c_{0,d-1,1}, \\ L_3 &= qa_{1,d-1,0} - (p + r)b_{0,d,0} + qc_{0,d-1,1}. \end{aligned}$$

Since the coefficient of x^{l-1} in the expansion of $div(Y)$ must be 0, for any $Y \in W_0$, we see that

$$a_{1,d-1,0} + d.b_{0d0} + c_{0,d-1,1} = 0.$$

None of L_1, L_2, L_3 are scalar multiple of the hyperplane

$$a_{1,d-1,0} + d.b_{0d0} + c_{0,d-1,1} = 0,$$

so we take $\Delta_0 = \Gamma_0 \cup \Sigma_0 \cup H_1 \cup H_2 \cup H_3 \cup H_4$, where

$$\begin{aligned} H_1 &= \{[Y] \in V_0 \mid L_1 = 0\}, \\ H_2 &= \{[Y] \in V_0 \mid L_2 = 0\}, \\ H_3 &= \{[Y] \in V_0 \mid L_2 = 0\}, \\ H_4 &= \{[Y] \in V_0 \mid \mu = 0\}. \end{aligned}$$

Recall that $\lambda_2 = \lambda - q(d-1) = 0$ and in this case, if $[Y] \in V_0 \setminus \Delta_0$, then q_2 is an isolated singularity of Y_2 with $m(Y_2, q_2) = 1$, since $det(DY_2(q_2)) = L_1.L_2.L_3 \neq 0$. The singularity q_0 is quasi-homogeneous and q_3 is of Kupka type. By lemma 3.1 the result follows.

(f) $\lambda = rd = q(d-1), p \mid q + \lambda$
As in the item (e), we have $p > q = md > r = m(d-1), \lambda = md(d-1)$. The condition $p \mid q + \lambda$ is equivalent to $p \mid d^2$. So we are in the situation of theorem 1.10 (d), where $p \mid d^2$.

- $V_0 \setminus \Gamma_0 \neq \emptyset$
  The same proof of the item (e).

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{q+\lambda}{p} = \frac{qd}{p} < d$. Take

$$Y = x(\alpha_p z^d + ay^{d-1})\partial/\partial x + (y(\alpha_q z^d + by^{d-1}) + x^l)\partial/\partial y + z(\alpha_r z^d + cy^{d-1})\partial/\partial z,$$

  then $Y \in W_0$ as long as $a + d.b + c = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_r & c \end{vmatrix} \neq 0, a \neq 0, b \neq 0.$$

  By similar reasons of the previous items, we have that $V_0 \setminus \Sigma_0 \neq \emptyset$.

The rest of the proof is the same of the item (e).

(g) $\lambda = rd = q(d-1), p \mid r + \lambda$
As in the previous two items, we have $p > q = md > r = m(d-1), \lambda = md(d-1)$. The condition $p \mid r + \lambda$ is equivalent to $p \mid d^2 - 1$. So we are in the situation of theorem 1.10 (d), where $p \mid d^2 - 1$.

- $V_0 \setminus \Gamma_0 \neq \emptyset$
  The same proof of the item (e).
- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{r+\lambda}{p} = \frac{r(d+1)}{p} < d + 1$. Take

$$Y = x(\alpha_p z^d + ay^{d-1})\partial/\partial x + y(\alpha_q z^d + by^{d-1})\partial/\partial y + (z(\alpha_r z^d + cy^{d-1}) + x^l)\partial/\partial z,$$

  then $Y \in W_0$ as long as $a + d.b + c = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_q & b \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_r & c \end{vmatrix} \neq 0, b \neq 0.$$

  By similar reasons of the previous items, we have that $V_0 \setminus \Sigma_0 \neq \emptyset$.

The rest of the proof is the same of the item (e).

□

□

2 Proof of the corollary 1.12

Corollary 1.12. If $q \geq 3$, there are no $\lambda \neq 0$ and $d \geq 2$ such that $\mathcal{F}(q+1, q, 1; \lambda, d+1)$ contains some GK foliation.

Proof. It is easy to see that $p = q + 1, q, r = 1$ ($q \geq 3$) never satisfy any of the four relations of theorem 1.10. Note that for these values of p, q, r, with the notation of remark 1.11 we have that $q = q_1, r = r_1$. Then by proposition 1.23 our corollary follows. □

3 Proof of the corollary 1.13

Corollary 1.13. For $d \geq 2$, $\overline{\mathcal{F}(p,q,r;\lambda,d+1)}$ is a GK irreducible component of $\mathcal{F}(d+1,3)$ for the following values of p,q,r,λ

p	q	r	λ
d^2+d	$2d+1$	d	d^2
d^2	$d+1$	d	d^2
d^2+d+1	$d+1$	d	d^2
d^2-d	d	$d-1$	d^2-d
d^2	d	$d-1$	d^2-d
d^2-1	d	$d-1$	d^2-d

Proof. In theorem 1.10, just make the substitutions $m=1, k=d+1$ in (b), $m=1, k=d^2$ in (c), $m=1, k=d^2+d+1$ in (c), $m=1, k=d^2-d$ in (d), $m=1, k=d^2$ in (d) and finally $m=1, k=d^2-1$ in (d). $\square$

Chapter 4

The case $n > 3$

In this chapter, we recall a recent result concerning stability of quasi-homogeneous singularity when $n > 3$. We also give the proofs of the remaining theorems.

1 Quasi-Homogeneous singularities

Recall that a singularity $p \in \mathbb{C}^n$ of the germ of a $(n-2)$-form ω at p is quasi-homogeneous if it is an isolated singularity of $Y = rot(\omega)$ and the linear part $DY(0)$ is nilpotent. Recently a result analogous to theorem 2.7 was proved in the case $n > 3$ (see [[15]], theorem 2)

Theorem 4.1. *Assume that $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of ω. Then there exists a holomorphic coordinate system $w = (w_1, \ldots, w_n)$ around $0 \in \mathbb{C}^n$ where ω has polynomial coefficients. More precisely, there exist two polynomial vector fields Z and Y in $\mathbb{C}^n$ such that*

(a) $Z = S + N$, where $S = \sum_{j=1}^{n} p_j w_j \partial/\partial w_j$ is linear semi-simple with eigenvalues $p_1, \ldots, p_n \in \mathbb{Z}_{>0}$, $DN(0)$ is linear nilpotent and $[S, N] = 0$;

(b) $[N, Y] = 0$ and $[S, Y] = \lambda.Y$, where $\lambda \in \mathbb{Z}_{>0}$. In other words, Y is quasi-homogeneous with respect to S with weight λ;

(c) In this coordinate system we have $\omega = \frac{1}{\lambda + tr(S)} i_Z i_Y dw_1 \wedge \ldots \wedge dw_n$ and $L_Y(\omega) = (\lambda + tr(S))\omega$.

Definition 4.2. In the situation of theorem 4.1, $S = \sum_{j=1}^{n} p_j w_j \partial/\partial w_j$ and $L_S(Y) = \lambda.Y$, we say that the quasi-homogeneous singularity is of type $(p_1, \ldots, p_n; \lambda)$.

In the definition 4.2, if we assume that the eigenvalues of S are relatively prime, the type of the singularity is uniquely determined. In other words, if the quasi-homogeneous singularity is of types $(p_1, \ldots, p_n; \lambda)$ and $(l_1, \ldots, l_n; \lambda_1)$ simultaneously, with $gcd(p_1, \ldots, p_n) = gcd(l_1, \ldots, l_n) = 1$, then $p_1 = l_1, \ldots, p_n = l_n, \lambda = \lambda_1$.

One corollary of the proof of theorem 4.1 is the following

Corollary 4.3. *Assume that $\omega = i_Z i_Y \nu, d\omega = i_Y \nu$, where $\nu = dw_1 \wedge \ldots \wedge dw_n$, and $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of Y. Then the eigenvalues of $DZ(0)$ are positive rational numbers.*

We have used corollary 2.8 and lemma 2.9 to show proposition 2.10. Similarly, we can use corollary 4.3 and lemma 2.9 to obtain the analogous of proposition 2.10 for the case $n > 3$

Proposition 4.4. *Let $p_1 > \ldots > p_n \geq 1$ be positive integers, $S = \sum_{i=}^{n} p_i z_i \partial/\partial x_i$ and X polynomials vector fields on $\mathbb{C}^n$ such that $[S, X] = \lambda.X, \lambda \in \mathbb{Z}_{>0}$. Suppose that $\mathcal{F} = \mathcal{F}(S, X) \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ is GK. Then* *(a) The singularity $q_0 \in E_0 \cap Sing(\mathcal{F})$ is quasi-homogeneous;*

(b) If $q_i \in E_i \cap Sing(\mathcal{F})$ is a non-Kupka singularity, then $\lambda = p_i(d-1)$, for $i = 2, 3, \ldots, n$.

In the next result ([[15]], theorem 3) we will consider the problem of deformation of two dimensional foliations with a quasi-homogeneous singularity. Consider a holomorphic family of $(n-2)$-forms, $(\omega_t)_{t\in U}$, defined on a polydisc Q of $\mathbb{C}^n$, where the space of parameters U is an open set of $\mathbb{C}^k$ with $0 \in U$. Let us assume that

- For each $t \in U$ the form ω_t defines a two dimensional foliation $\mathcal{F}_t$ on Q. Let $(Y_t)_{t\in U}$ be the family of holomorphic vector fields on Q such that $d\omega_t = i_{Y_t}\nu, \nu = dz_1 \wedge \ldots \wedge dz_n$;
- $0 \in \mathbb{C}^n$ is a quasi-homogeneous singularity of $\mathcal{F}_0$.

Theorem 4.5. *In the above situation there exist a neighbourhood $0 \in V \subset U$, a polydisc $0 \in P \subset Q$, and a holomorphic map $\mathcal{P} : V \to P \subset \mathbb{C}^n$ such that $\mathcal{P}(0) = 0$ and for any $t \in V$ then $\mathcal{P}(t)$ is the unique quasi-homogeneous singularity of $\mathcal{F}_t$ in P. Moreover, $\mathcal{P}(t)$ is of the same type as $\mathcal{P}(0)$, in the sense that if 0 is a quasi-homogeneous singularity of type $(p_1, \ldots, p_n; \lambda)$ of $\mathcal{F}_0$ then $\mathcal{P}(t)$ is a quasi-homogeneous singularity of type $(p_1, \ldots, p_n; \lambda)$ of $\mathcal{F}_t, \forall t \in V$.*

2 Proof of theorem 1.17

***Theorem 1.17**.* If $\lambda > 0$ and $\overline{\mathcal{F}(p_1, \ldots, p_n; \lambda, d)}$ contains some WGK foliation $\mathcal{F}$, where $q(\mathcal{F})$ is a GK singularity of $\mathcal{F}$, then $\overline{\mathcal{F}(p_1, \ldots, p_n; \lambda, d)}$ is an irreducible component of $\mathcal{F}_2(d, n)$. In particular, if $\mathcal{F}(p_1, \ldots, p_n; \lambda, d)$ contains some GK foliation, where $\lambda \neq 0$, then $\overline{\mathcal{F}(p_1, \ldots, p_n; \lambda, d)}$ is an irreducible component of $\mathcal{F}_2(d, n)$.

Proof. Let $\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ be the required WGK foliation and assume without loss of generality that $\mathcal{F}$ is like in proposition 2.5 (e), so $q_0 = (0 : \cdots : 0 : 1)$ is a GK singularity of $\mathcal{F}$; by proposition 4.4 (a) it is a quasi-homogeneous singularity. Let $(\mathcal{F}_t)_{t\in\Sigma}$ be a holomorphic family of foliations in $\mathcal{F}_2(d+1, n)$, parameterized in a open set $0 \in \Sigma \subset \mathbb{C}$, where $\mathcal{F}_0 = \mathcal{F}$, and $(\Omega_t)_{t\in\Sigma}$ a holomorphic family of respective homogeneous $(n-2)$-form on $\mathbb{C}^{n+1}$ that defines $\mathcal{F}_t$. It suffices to prove that $\mathcal{F}_t \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ for small $|t|$.

First, let us check that $\mathcal{F}_t$ is WGK for small $|t|$. Define $\omega_{i,t} = \Omega_t|_{E_i}, i = 0, \ldots, n$. Set

$$\mathcal{S}_{i,t} = \{[z] \in E_i \mid \omega_{i,t}(z) = 0\}, \mathcal{T}_{i,t} = \{[z] \in E_i \mid d\omega_{i,t}(z) = 0\},$$

and denote by $\mathcal{Q}_{i,t}$ and $\mathcal{R}_{i,t}$ the union of the components of codimension ≥ 3 and the union of the components of codimension ≤ 2 of the analytic set $\mathcal{T}_{i,t}$, respectively. By definition, $\mathcal{F}_t$ is WGK on E_i means that $\mathcal{S}_{i,t} \cap \mathcal{Q}_{i,t} = \emptyset$. For each $p \in \mathbb{P}^n$, take an open set V_p with compact closure such that

$$p \in V_p \subset \overline{V_p} \subset E_i,$$

for some $i = i(p) \in \{0, \ldots, n\}$. As $\mathcal{F}_0$ is WGK, there exists $\epsilon_p > 0$ such that

$$\mathcal{S}_{i,t} \cap \mathcal{Q}_{i,t} \cap \overline{V_p} = \emptyset,$$

if$|t| < \epsilon_p$. From the compactness of $\mathbb{P}^n$, we can assume that there exist a finite number of points $p_1, \ldots, p_m$ such that

$$\mathbb{P}^n = \bigcup_{j=1}^{m} V_{p_m}.$$

Then $\mathcal{F}_t$ is WGK, if $|t| < \epsilon$, where

$$\epsilon = \min_{j \in \{1,\dots,m\}} \epsilon_{p_j}.$$

Now we show that if $\mathcal{F}$ is WGK, then the tangent sheaf $\mathcal{TF}$ of $\mathcal{F}$ is locally free. For, it suffices to show that $\mathcal{T}_p\mathcal{F}$ has two generators for $p \in Sing(\mathcal{F})$. Choose a neighbourhood $V \ni p$ biholomorphic to a polydisc of $\mathbb{C}^n$ and suppose that η defines $\mathcal{F}$ on V. Set $Y = rot(\eta)$, i.e., Y is the holomorphic vector field on V satisfying

$$d\eta = i_Y \mu,$$

where μ is a non-vanishing n-form defined on V. Since $\mathcal{F}$ is WGK it follows that $Y \not\equiv 0$; then the integrability of η is equivalent to $i_Y \eta = 0$ (see [[15]], proposition 1 and remark 1.2). As $\mathcal{F}$ is WGK, we can assume that

$$cod_{\mathbb{C}}(Sing(Y)) \geq 3.$$

It follows from the parametric De Rham division theorem that there exists a holomorphic vector field Z defined on V such that $\eta = i_Y i_Z \mu$. If X is a vector field satisfying $i_X \eta = 0$, using the parametric De Rham division theorem once more, there exist holomorphic functions a and b defined on $V \setminus Sing(\eta)$ such that

$$X = aY + bZ.$$

Since

$$cod_{\mathbb{C}}(Sing(\eta)) \geq 2,$$

it follows from Hartog's Theorem that a and b can be extended holomorphically to V, which proves that $\mathcal{T}_p\mathcal{F}$ has two generators (given by the germs of Y and Z at p).

Thus, if we take Σ small then for any $t \in \Sigma$, $\mathcal{F}_t$ is WGK and $\mathcal{TF}_t$ is locally free. Being locally free, $(\mathcal{TF}_t)_{t\in\Sigma}$ is a holomorphic family of rank two vector bundles over $\mathbb{P}^n$, which can be seen as a deformation of the rank two vector bundle $\mathcal{TF}_0$.

Let $E \to M$ be a holomorphic vector bundle over a compact complex manifold M. The space of deformations of E is isomorphic to $H^1(M, End(E))$, where $End(E)$ is the sheaf of endomorphisms of E ([[19]]). Applying this result to

$$\mathcal{TF}_0 = \mathcal{O} \oplus \mathcal{O}(1-d),$$

we have

$$End(\mathcal{TF}_0) = \mathcal{TF}_0^* \otimes \mathcal{TF}_0,$$

where $\mathcal{TF}_0^* = \mathcal{O} \oplus \mathcal{O}(d-1)$ is the dual bundle of $\mathcal{TF}_0$. Thus

$$H^1(M, End(\mathcal{TF}_0)) = 0,$$

since $End(\mathcal{TF}_0)$ splits as direct sum of line bundles (see [[17]], theorem 2.3.1).

It follows that

$$\mathcal{TF}_t \simeq \mathcal{O} \oplus \mathcal{O}(1-d),$$

for small $|t|$. Thus $\mathcal{F}_t$ is generated by two foliations of dimension one, say $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$, where $\mathcal{G}_1(t)$ corresponds to the factor $\mathcal{O}$ and $\mathcal{G}_2(t)$ to the factor $\mathcal{O}(1-d)$. As a consequence, $\mathcal{G}_1(t)$ is generated by a linear vector field S_t on $\mathbb{P}^n$ and $\mathcal{G}_2(t)$ is generated by a polynomial vector field X_t, where

$$deg(\mathcal{G}_{X_t}) = deg(\mathcal{G}_2(t)) = d.$$

According to theorem 4.5, using a holomorphic family of automorphisms of $\mathbb{P}^n$ if necessary, we can assume that q_0 is a quasi-homogeneous singularity of $\mathcal{F}_t$. As S_0 defines $\mathcal{G}_1(0) = \mathcal{G}_S$ and X_0 defines $\mathcal{G}_2(0) = \mathcal{G}_X$, we can also assume that $S_0 = S$ and $X_0 = X$. Thus $\mathcal{F}_t$ is defined on E_0 by

$$\omega_t = i_{S_t} i_{X_t} \nu, \nu = dx_0 \wedge \cdots \wedge dx_n, S_0 = S, X_0 = X.$$

Set $Y_t = rot(\omega_t)$ on E_0. From $i_{Y_t}\omega_t = 0$, it follows from the division theorem and from Hartog's theorem that

$$Y_t = a_t S_t + b_t X_t,$$

where a_t, b_t are holomorphic functions on E_0. The left side of the equality $i_{S_t} i_{Y_t}\nu = b_t\omega_t$ is polynomial, then b_t must be polynomial. But we have

$$i_{S_0} i_{Y_0}\nu = \tau i_{S_0} i_{X_0} = \tau\omega_0,$$

where $\tau = \lambda + \sum_{k=1}^{n} p_k$. In particular $Sing(i_{S_t} i_{Y_t}\nu)$ has no divisorial components for small $|t|$, then $b_t \in \mathbb{C}^*$.

Applying corollary 4.3 to

$$\omega_t = i_{\tilde{S}_t} i_{Y_t}\nu, d\omega_t = i_{Y_t}\nu,$$

where

$$\tilde{S}_t = \frac{S_t}{b_t},$$

we obtain that the eigenvalues of $\tilde{S}_t$ are positive rational numbers, and consequently they are equal to the eigenvalues of

$$\frac{S_0}{b_0} = \frac{S}{\tau},$$

since the eigenvalues of $\tilde{S}_t$ vary holomorphically with respect to t.

It is a general result that given a holomorphic $(n-2)$-form η and holomorphic vector fields Z, W satisfying

$$\eta = i_Z i_W \nu, d\eta = i_W \nu,$$

then

$$[Z, W] = (1 - div(Z))W.$$

Consider an affine coordinate system $(E, (x_1, \ldots, x_n))$, where

$$\tilde{S}_t = \frac{p_1}{\tau} x_1 \partial/\partial x_1 + \cdots + \frac{p_n}{\tau} x_n \partial/\partial x_n.$$

Then

$$[\tilde{S}_t, Y_t] = (1 - div(\tilde{S}_t))Y_t = \frac{\lambda}{\tau} Y_t,$$

and after multiplying both sides of the latter relation by τ we see that $\mathcal{F}_t \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$. $\square$

3 Proof of theorem 1.19

Theorem 1.19. The families $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1) \subset \mathcal{F}_2(d+1, n), d \geq 2$, containing some GK foliation, are (precisely) the families $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ where

a) 0 is an isolated singularity of some $Y \in W_0$

and $p_1, \ldots, p_n, \lambda$ satisfy either

b.1)
- $c_{11}, c_{22}, \ldots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \ldots, c_{n-1,n}$, for some $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$
- $\tau_j \neq 0, j = 2, 3, \ldots, n$

or

b.2)
- $\lambda = p_i(d-1), c_{11}, c_{22}, \ldots, c_{i-2,i-2}, c_{i,i+1}, c_{i+1,i+2}, \ldots, c_{n-1,n}$, for some $2 \leq i \leq \lfloor \frac{n+2}{2} \rfloor$

- $\tau_j \neq 0, j \in \{2, 3, \ldots, n\} \setminus \{i\}$

In particular $\lambda = p_n d$ and p_1 divides $p_k + \lambda$, for some $k \in \{1, \ldots, n\}$.

Proof. We begin by showing that the conditions of theorem 1.19 are necessary. With respect to the GK foliation

$$\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1),$$

we can assume that we are in the situation of proposition 2.5 (e). By proposition 4.4 (b), since $d \geq 2$ and $p_1, \ldots, p_n$ are pairwise distinct, the singularities $q_2, q_3, \ldots, q_n$ are Kupka, with at most one exception.

Suppose first that the singularities $q_2, \ldots, q_n$ are all of Kupka type. We show that either $p_1, \ldots, p_n, \lambda, d$ or $\overline{p}_1, \ldots, \overline{p}_n, \lambda_1, d$ satisfy the relations of theorem 1.19.b.1. In fact, in this case, for each $i \in \{2, \ldots, n\}$, $Y_i(0) \neq 0$, where $Y_i = rot(\omega_i)$. If the j-th entry of Y_i is not 0, then from

$$[S_i, Y_i] = \lambda_i . Y_i,$$

and from the proposition 2.2 (d), we have that (see the eigenvalues of S_i in proposition 2.5 (c))

$$\begin{cases} \lambda - p_i(d-1) + p_j - p_i = 0, \text{ if } j \leq i-1 \\ \lambda - p_i(d-1) + p_{j+1} - p_i = 0, \text{ if } i \leq j \leq n-1 \\ \lambda - p_i(d-1) - p_i = 0, \text{ if } j = n. \end{cases}$$

Note that in any case it is equivalent to say that condition c_{i-1j} is satisfied. As $q_2, \ldots, q_n$ are all Kupka singularities of $\mathcal{F}$, for each $i \in \{1, \ldots, n-1\}$, c_{ij} must hold for some $j \in \{1, \ldots, n\}$. An easy check shows that if c_{ij} and $c_{i_1 j_1}$ hold simultaneously, then $i_1 > i$ implies that $j_1 > j$. Thus $p_1, \ldots, p_n, \lambda, d$ must satisfy the relations $c_{1,j_1}, c_{2,j_2}, \ldots, c_{n-1,j_{n-1}}$, where $j_k \in \{k, k+1\}$, $k = 1, \ldots, n-1$. Furthermore, if $j_{k_0} = k_0 + 1$ for some $k_0 \in \{1, \ldots, n\}$, then $j_k = k$, for $k < k_0$ and $j_k = k+1$, for $k > k_0$. In other words, $p_1, \ldots, p_n, \lambda, d$ satisfy

$$c_{11}, c_{22}, \ldots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \ldots, c_{n-1,n}, i \in \{0, \ldots, n-1\}.$$

Define the conditions $\overline{c}_{ij}$ as c_{ij} substituting p_k by $\overline{p}_k$ and λ by λ_1, $i = 1, \ldots, n-1, j = 1, \ldots, n$. From

$$c_{ij} \text{ holds } \iff \overline{c}_{n-i,n-j+1} \text{ holds},$$
$$\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1) = \mathcal{F}(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n; \lambda_1, d+1),$$

we can assume that either $p_1, \ldots, p_n, \lambda, d$ or $\overline{p}_1, \ldots, \overline{p}_n, \lambda_1, d$ satisfy

$$c_{11}, c_{22}, \ldots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \ldots, c_{n-1,n}, 0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor.$$

So we are in the situation of theorem 1.19.b.1. In addition, from

$$\tau_i . \omega_i = i_{S_i} i_{Y_i} \nu_1, i = 2, \ldots, n,$$

if $\tau_i = 0$ there exists a polynomial f_i such that $Y_i = f_i . S_i$. In particular $Y_i(q_i) = 0$. Thus, if $q_2, \ldots, q_n$ are Kupka singularities of $\mathcal{F}$, then

$$\tau_i \neq 0, i = 2, \ldots, n.$$

Now we suppose that the singularities $q_2, \ldots, q_n$ are all of Kupka type, with exactly one exception. We show that either $p_1, \ldots, p_n, \lambda, d$ or $\overline{p}_1, \ldots, \overline{p}_n, \lambda_1, d$ satisfy the relations of theorem 1.19.b.2. If the non-Kupka singularity of $\mathcal{F}$ is q_i, $i \in \{2, \ldots, n\}$, then by proposition 4.4 (b) we have $\lambda = p_i(d-1)$.

In this case, an easy check shows that the condition c_{i-1j} is not satisfied, for any $1 \leq j \leq n$. Additionally, if $c_{i_1 j_1}$ holds, for $i_1 > i-1$, from $\lambda = p_i(d-1)$ we have that $i+1 \leq j_1 \leq n$. Also, if $c_{i_1 j_1}$ holds and $i_1 < i-1$, once again from $\lambda = p_i(d-1)$ it follows that $1 \leq j_1 \leq i-2$. Finally, recall that

c_{ij} and $c_{i_1j_1}$ hold simultaneously, then $i_1 > i$ implies that $j_1 > j$. As $q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n$ are Kupka singularities of $\mathcal{F}$, $p_1, \ldots, p_n, \lambda, d$ must satisfy the relations

$$c_{11}, c_{22}, \ldots, c_{i-2,i-2}, \lambda = p_i(d-1), c_{i,i+1}, c_{i+1,i+2}, \ldots, c_{n-1,n}, i = 2, \ldots, n.$$

From

$$\lambda = p_i(d-1) \iff \lambda_1 = \overline{p}_{n+2-i}(d-1),$$

and the other relations of symmetry that we saw in the first part, we can assume that either $p_1, \ldots, p_n, \lambda, d$ or $\overline{p}_1, \ldots, \overline{p}_n, \lambda_1, d$ satisfy

$$c_{11}, c_{22}, \ldots, c_{i-2,i-2}, \lambda = p_i(d-1), c_{i,i+1}, c_{i+1,i+2}, \ldots, c_{n-1,n}, 2 \leq i \leq \lfloor \frac{n+2}{2} \rfloor.$$

So we are in the situation of theorem 1.19.b.2. With the exception to q_i, the remaining singularities $q_2, \ldots, q_n$ are of Kupka type. By a similar reason that we saw in the first part, we have

$$\tau_j \neq 0, j \in \{2, \ldots, n\}, j \neq i.$$

Finally, recall the definition of $Y = rot(\omega) \in W_0$, where ω defines $\mathcal{F}$ on E_0. As $c_{n-1,n}$ holds, we have that $\lambda = p_n d > 0$. Then, by remark 2.3, as q_0 is a GK singularity of $\mathcal{F}$, we have that 0 is an isolated singularity of $Y \in W_0$. Note this implies that $V_0 \setminus \Sigma_0 \neq \emptyset$, where $V_0 = \mathbb{P}(W_0)$. So we have the condition of theorem 1.19.a.

Next we show that under the above conditions, $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ contains some GK foliation. The proof follows immediately from the next two lemmas

Lemma 4.6. *A foliation $\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ is GK if and only if the singularities $q_0, q_2, q_3, \ldots, q_n$ of $\mathcal{F}$ are GK.*

Proof. The proof is essentially the same of lemma 3.1. □

Set

$$V_0 = \mathbb{P}(W_0) = \{[Y] \mid Y \in W_0, Y \neq 0\},$$

and define Γ_0, Σ_0 as in the case $n = 3$. The proof that these sets are algebraic subsets of V_0 is essentially the same of lemma 3.2.

Lemma 4.7. *Under the above conditions, there exists a proper algebraic subset $\Delta_0 \subset V_0$ such that the singularities $q_0, q_2, q_3, \ldots, q_n$ of $\mathcal{F}(S, Y) \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ are GK if $[Y] \in V_0 \setminus \Delta_0$.*

Proof. The idea is to show that under the hypothesis above $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ is not empty and find the 1-jet of $Y_j = rot(\omega_j), j = 2, \ldots, n$. Note that by assumption $V_0 \setminus \Sigma_0 \neq \emptyset$, so the singularity q_0 of $\mathcal{F}(S, Y)$ is quasi-homogeneous if $[Y] \in V_0 \setminus \Sigma_0$.

First we show that $\mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$ is not empty, that means that $V_0 \setminus \Gamma_0 \neq \emptyset$. In both situations of theorem 1.19 we have that $\lambda = p_n d$. In addition, if $n > 3$, with exception to the case $n = 4$, where $i = 3$ in (b), c_{n-2n-1} is satisfied, i.e.,

$$p_n + \lambda = p_{n-1} d.$$

Then

$$X = x_n^d R + x_{n-1}^d \partial/\partial x_n$$

is such that $[S, X] = \lambda.X$. It follows that $Ld(S, X)$ has no divisorial components. In fact, suppose by contradiction that is not true. By looking at the first and second entries of S and X, we conclude that the only possible hypersurfaces contained in $Ld(S, X)$ are

$$\{[x] \in \mathbb{C}^n \mid x_1 = 0\}, \{[x] \in \mathbb{C}^n \mid x_2 = 0\}, \{[x] \in \mathbb{C}^n \mid x_n = 0\}.$$

By looking at the remaining entries of S and X, we can see that none of those hypersurfaces are contained in $Ld(S,X)$. Then $Y = rot(\omega)$ is such that

$$[Y] \in V_0 \setminus \Gamma_0,$$

where

$$\omega = i_S i_X \nu_0,$$

since ω gives rise to a foliation of degree $d+1$ on $\mathbb{P}^n$. On the other hand, in the case $n = 4$, where $i = 3$ in (b),

$$X = x_4^d R + x_2^d \partial/\partial x_1$$

satisfies $[S,X] = \lambda.X$. One can check that $Ld(S,X)$ has no divisorial components and that $\omega = i_S i_X \nu_0$ gives rise to a foliation of degree $d+1$ on $\mathbb{P}^n$. So $Y = rot(\omega)$ is such that

$$[Y] \in V_0 \setminus \Gamma_0.$$

Next, given a holomorphic vector field Y, we will write $Y = Y^{(0)} + Y^{(1)} + Y^{(2)} + \cdots$ to denote its decomposition into homogeneous polynomial vector fields.

Given $Y \in W_0$ and

$$\omega = \frac{1}{\tau} i_S i_Y \nu_0,$$

we will find $Y_j^{(0)}$ and $Y_j^{(1)}$, where $Y_j = rot(\omega_j)$, $j = 2, \ldots, n$. As in the case $n = 3$, it can be proved that

$$Y^{(d+1)} = \mu x_n^d (\tau R - (n+d)S),$$

for some $\mu \in \mathbb{C}$. Next set

$$X = \frac{Y + \mu(n+d)x_n^d S}{\tau},$$

so we have (see also the proof of lemma 2.4 and remark 3.3)

$$\omega = i_S i_X \nu_0 = \frac{1}{\tau} i_S i_Y \nu_0.$$

Let us write

$$Y = P + Y^{(d+1)},$$

where

$$P = \sum_{k=1}^{n} A_k \partial/\partial x_k$$

is polynomial of degree d. Thus

$$X = \frac{P}{\tau} + \mu x_n^d R.$$

The change of coordinates from E_0 to E_j, $j = 2, \ldots, n$, is given by

$$u_1 = \frac{x_1}{x_j}, \ldots, u_{j-1} = \frac{x_{j-1}}{x_j}, u_j = \frac{x_{j+1}}{x_j}, \ldots, u_{n-1} = \frac{x_n}{x_j}, u_n = \frac{1}{x_j}.$$

If $deg(\mathcal{G}_X) = d$ (for instance, if $[Y] \in V_0 \setminus \Gamma_0$), we have $X = \frac{X_j}{u_n^{d-1}}$ in $E_0 \cap E_j$, where

$$X_j = \sum_{k=1}^{n} P_k \partial/\partial u_k \tag{4.1}$$

is a polynomial vector field representing $\mathcal{G}_X$ in the chart E_j. In fact

$$P_k = \begin{cases} \tilde{A}_k - u_k\tilde{A}_j, \text{ if } 1 \le k \le j-1 \\ \tilde{A}_{k+1} - u_k\tilde{A}_j, \text{ if } j \le k \le n-1 \\ -u_n\tilde{A}_j - \mu u_{n-1}^d, \text{ if } k = n, j \ne n \\ -u_n\tilde{A}_j - \mu, \text{ if } k = j = n \end{cases} \tag{4.2}$$

where

$$\tilde{A}_l = \frac{u_n^d}{\tau} A_l\left(\frac{u_1}{u_n}, \dots, \frac{u_{j-1}}{u_n}, \frac{1}{u_n}, \frac{u_j}{u_n}, \dots, \frac{u_{n-1}}{u_n}\right), 1 \le l \le n.$$

Let us write

$$A_k = \sum_{|\sigma| \le d} a_{k,\sigma} z^\sigma, k = 1, \dots, n, \tag{4.3}$$

where $|\sigma| = \sum_{k=1}^n \sigma_k$ for $\sigma = (\sigma_1, \cdots, \sigma_n)$, and denote $\sigma_j = (0, \cdots, 0, d, 0, \cdots, 0)$, where the value d appears in the j-th entry.

From

$$Y_j = \tau_j X_j - div(X_j)S_j, j = 2, \dots, n,$$

we see that

$$Y_j^{(0)} = \tau_j. \sum_{k=1}^n c_k \partial/\partial u_k,$$

where

$$c_k = \begin{cases} \frac{a_{k,\sigma_j}}{\tau}, \text{ if } 1 \le k \le j-1 \\ \frac{a_{k+1,\sigma_j}}{\tau}, \text{ if } j \le k \le n-1 \\ 0, \text{ if } k = n, j \ne n \\ -\mu, \text{ if } k = j = n \end{cases}$$

Suppose first that we are in the situation of theorem 1.19.b.1. Thus, if the following relations are satisfied

$$c_{11}, c_{22}, \dots, c_{ii}, c_{i+1,i+2}, c_{i+2,i+3}, \dots, c_{n-1,n}, 0 \le i \le \lfloor \tfrac{n-1}{2} \rfloor,$$

we have

$$\begin{cases} Y_j^{(0)} = \frac{\tau_j}{\tau} a_{j-1,\sigma_j} \partial/\partial u_{j-1}, \text{ if } 2 \le j \le i+1 \\ Y_j^{(0)} = \frac{\tau_j}{\tau} a_{j+1,\sigma_j} \partial/\partial u_j, \text{ if } i+1 < j \le n-1 \\ Y_n^{(0)} = -\tau_n \mu \partial/\partial u_n \end{cases}$$

Therefore we have that $H_k \ne V_0, 2 \le k \le n$, where

$$\begin{cases} H_j = \{[Y] \in V_0 \mid a_{j-1,\sigma_j} = 0\}, \text{ if } 2 \le j \le i+1 \\ H_j = \{[Y] \in V_0 \mid a_{j+1,\sigma_j} = 0\}, \text{ if } i+1 < j \le n-1 \\ H_n = \{[Y] \in V_0 \mid \mu = 0\} \end{cases}$$

Note that q_k is a Kupka singularity of $\mathcal{F}(S, Y)$ if

$$[Y] \in V_0 \setminus H_k, k = 2, \dots, n.$$

Finally, take $\Delta_0 = \Gamma_0 \cup \Sigma_0 \cup H$, where

$$H = \bigcup_{k=2}^n H_k.$$

Suppose now that we are in the situation of theorem 1.19.b.2. Thus the following are satisfied

$$c_{11}, c_{22}, \ldots, c_{i-2,i-2}, \lambda = p_i(d-1), c_{i,i+1}, c_{i+1,i+2}, \ldots, c_{n-1,n}, 2 \leq i \leq \lfloor \tfrac{n+2}{2} \rfloor$$

and we have that

$$\begin{cases} Y_j^{(0)} &= \frac{\tau_j}{\tau} a_{j-1,\sigma_j} \partial/\partial u_{j-1}, \text{ if } 2 \leq j \leq i-2 \\ Y_j^{(0)} &= \frac{\tau_j}{\tau} a_{j+1,\sigma_j} \partial/\partial u_j, \text{ if } i \leq j \leq n-1 \\ Y_n^{(0)} &= -\tau_n \mu \partial/\partial u_n \end{cases}$$

It follows that $H_k \neq V_0, 2 \leq k \leq n, k \neq i$, where

$$\begin{cases} H_j = \{[Y] \in V_0 \mid a_{j-1,\sigma_j} = 0\}, \text{ if } 2 \leq j \leq i-2 \\ H_j = \{[Y] \in V_0 \mid a_{j+1,\sigma_j} = 0\}, \text{ if } i \leq j \leq n-1 \\ H_n = \{[Y] \in V_0 \mid \mu = 0\} \end{cases}$$

Set

$$H = \bigcup_{\substack{k=2 \\ k \neq i}}^{n} H_k.$$

If

$$[Y] \in V_0 \setminus H,$$

then q_k is a Kupka singularity of $\mathcal{F}(S, Y)$, $k = 2, \ldots, n, k \neq i$. Define

$$L = \{[Y] \in V_0 \mid det(DY_i(q_i)) = 0\}.$$

Observe that if $[Y] \in V_0 \setminus L$, q_i is an isolated singularity of Y_i, since $det(DY_i(q_i)) \neq 0$. Next we show that L is a proper algebraic subset of V_0, so we can take

$$\Delta_0 = \Gamma_0 \cup \Sigma_0 \cup H \cup L.$$

It is clear that $L \subset V_0$ is an algebraic subset, and we show that L is proper as well. For, let us consider first the case where $\tau_i \neq 0$. Define the following vector space over $\mathbb{C}$

$$\begin{aligned} W_1 = \{&\text{polynomial vector fields } Y \text{ in } E_i \cong \mathbb{C}^n \mid [S_i, Y] = \lambda_i Y, div(Y) \equiv 0, deg(Y) \leq d+1, \\ &i_R i_{S_i} i_{Y^{(d+1)}} \nu_1 \equiv 0\}. \end{aligned}$$

W_1 is nothing more than the ambient space of $Y_i = rot(\omega_i)$ on E_i, where $\omega = i_S i_X \nu_0$ defines

$$\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$$

on E_0. Given $Y \in W_0$, set Ω_Y as in the proof of lemma 3.2. The map

$$Y \mapsto Y_i = rot(\Omega_Y|_{E_i})$$

between W_0 and W_1 is an isomorphism of $\mathbb{C}$-vector spaces. Indeed, the proof that it is injective is the same as the proof that the map between the spaces W_0 and $\mathcal{H}$ of lemma 3.2 is injective. In addition, given $Y_1 \in W_1$, set

$$\omega_1 = \frac{1}{\tau_i} i_{S_i} i_{Y_1} \nu_1$$

and let Ω_1 denote the one-form of degree $d+2$ obtained by homogenizing ω_1. So $\omega_0 = \Omega_{Y_1}|_{E_0}$ is such that $Y = rot(\omega_0)$ is the pre-image of Y_1, thus the map is surjective. So we have an induced biregular map between V_0 and V_1, where

$$V_1 = \{[Y] \mid Y \in W_1, Y \neq 0\}.$$

From the latter biregular map, it suffices to exhibit $\overline{Y} \in W_1$ such that $det(D\overline{Y}(0)) \neq 0$, then it follows that L is proper. We can take

$$\overline{Y}_i = \sum_{k=1}^{n} \epsilon_k u_k \partial/\partial u_k,$$

where $\epsilon_1, \ldots, \epsilon_n \in \mathbb{C}^*$ satisfy

$$\sum_{k=1}^{n} \epsilon_k = 0.$$

Note that $[S_i, \overline{Y}] = \lambda_i \overline{Y}$ since $\lambda_i = \lambda - p_i(d-1) = 0$.

Finally, suppose that $\tau_i = 0$. Substituting $j = i$ into (4.1) above, from (4.2) and (4.3) we have that

$$X_i^{(1)} = \sum_{k=1}^{i-1}(a_{k,\sigma_{ki}} - a_{i,\sigma_i})u_k + \sum_{k=i}^{n-1}(a_{k+1,\sigma_{k+1,i}} - a_{i,\sigma_i})u_k - a_{i,\sigma_i}u_n,$$

where $\sigma_{ki} = (\sigma_1, \ldots, \sigma_n)$, with

$$\sigma_l = \begin{cases} 1, \text{ if } l = k \\ (d-1), \text{ if } l = i \\ 0, \text{ if } l \neq k, i \end{cases}$$

Next, write

$$Y_i^{(1)} = \sum_{k=1}^{n} L_k u_k \partial/\partial u_k.$$

From

$$Y_i = \tau_i X_i - div(X_i)S_i = -div(X_i)S_i,$$

we have that

$$L_1 = -(p_1 - p_i)\left(\sum_{k=1}^{i-1} a_{k,\sigma_{ki}} + \sum_{k=i}^{n-1} a_{k+1,\sigma_{k+1,i}} - n a_{i,\sigma_i}\right)$$

and $L_2, \ldots, L_n$ are all a scalar multiple of L_1. But the only restriction to the variables of L_1 that appears in the definition of V_0 is

$$M = \sum_{k=1}^{i-1} a_{k,\sigma_{ki}} + \sum_{k=i}^{n-1} a_{k+1,\sigma_{k+1i}} + (d+1)a_{i,\sigma_i} = 0,$$

since the coefficient of x_i^d in $div(Y) = 0$ must be 0. As M is not a scalar multiple of L_1, we conclude that there exists $[Y] \in V_0 \setminus L$, i.e., L is proper.

$\square$

$\square$

4 Proof of theorem 1.20

Theorem 1.20. Let $p > q > r > s \geq 1$ be positive integers, where $gcd(p, q, r, s) = 1$. $\mathcal{F}(p, q, r, s; \lambda, d+1) \subset \mathcal{F}_2(d+1, 4)$ contains a GK foliation, for some $\lambda \in \mathbb{Z}, d \geq 2$, if and only if either p, q, r, s, λ, d or $p, q_1 = p - s, r_1 = p - q, s_1 = p - q, \lambda_1, d$ satisfy one of the following relations

(a) $p > q = m(d^2+d+1) > r = m(d^2+d) > s = md^2, \lambda = md^3, gcd(p,m) = 1$, p divides either d^3 or d^3+d^2+d+1;

(b) $p = kd > q = md+k > r = m(d+1) > s = md, \lambda = md^2, gcd(k,m) = 1$, either k divides d, or kd divides $m(d^2+d)+k$ (which implies $k = jd$ where j divides $d+1$), or d divides m and k divides d^2+d+1, or k divides $d+1$ and $gcd(\frac{m(d+1)}{k}, d) = 1$;

(c) $p > q = md^2 > r = m(d^2-1) > s = m(d^2-d), \lambda = m(d^3-d^2), gcd(p,m) = 1$, p divides either d^3-d^2, or d^3, or d^3-1;

(d) $p = kd > q = m(d-1)+k > r = md > s = m(d-1), \lambda = m(d^2-d), gcd(k,m) = 1$, either k divides $d-1$, or k divides d, or d divides m and k divides d^2-1.

Proof. In the case $n = 4$, we use the notation $p_1 = p > p_2 = q > p_3 = r > p_4 = s$ in theorem 1.19, and x, y, z, w as the coordinates of $E_0 \cong \mathbb{C}^4$. In theorem 1.19.b.1, we have $0 \le i \le \lfloor \frac{4-1}{2} \rfloor = 1$, and in theorem 1.19.b.2, $2 \le i \le \lfloor \frac{4+2}{2} \rfloor = 3$, for a total of 4 possibilities.

In what follows, items (a), (b), (c) and (d) correspond to theorem 1.19.b.1, $i = 0$, theorem 1.19.b.1, $i = 1$, theorem 1.19.b.2, $i = 2$, theorem 1.19.b.2, $i = 3$, respectively. As we saw, in each case p must divide either $p+\lambda$, or $q+\lambda$, or $r+\lambda$, or $s+\lambda$, which gives rise to four sub-cases. In each sub-case, we must check that $V_0 \setminus \Sigma_0 \neq \emptyset$ and $\tau_l \neq 0$, for some values of l. Set

$$\alpha_p = \tau - p(d+4), \alpha_q = \tau - q(d+4), \alpha_r = \tau - r(d+4), \alpha_s = \tau - s(d+4).$$

If $Y \in W_0$, then

$$Y^{(d+1)} = \mu w^d(\tau.R - (d+4).S) = \mu\alpha_p x w^d \partial/\partial x + \mu\alpha_q y w^d \partial/\partial y + \mu\alpha_r z w^d \partial/\partial z + \mu\alpha_s w^{d+1}\partial/\partial w,$$

for some $\mu \in \mathbb{C}$ (see the proof of theorem 1.19).

We have that $\tau_2 = \alpha_q, \tau_3 = \alpha_r, \tau_4 = \alpha_s$. In any case $\alpha_p = q+r+s-3p+(s-p)d < 0$ and $\tau_4 = \alpha_s = p+q+r-3s > 0$.

(a) $r+\lambda = qd, s+\lambda = rd, \lambda = sd$
This set of conditions is equivalent to

$$p > q = m(d^2+d+1) > r = m(d^2+d) > s = md^2, \lambda = md^3.$$

Hence $gcd(p,q,r,s) = 1 \iff gcd(p,m) = 1$. In addition $\tau_2 = \alpha_q = p+s-3q$ and $\tau_3 = \alpha_r = p+q-3r$.

Next we show that $\tau_2 \neq 0, \tau_3 \neq 0$. Suppose that $\tau_2 = 0$; this implies that $p = m(2d^2+3d+3)$. Since $gcd(p,q,r,s) = 1$ it follows that $m = 1$. Then

$$p = 2d^2+3d+3, q = d^2+d+1, r = d^2+d, s = d^2, \lambda = d^3.$$

Using polynomial division, we get the following identities

$$4(p+\lambda) = p(2d+3)+3-5d, 4(q+\lambda) = p(2d+1)+(1-3d),$$
$$4(r+\lambda) = p(2d+1)-3(d+1), 4(s+\lambda) = p(2d+1)-(7d+3).$$

If $p \mid p+\lambda$, we can use the first identity to obtain a contradiction, since we would have $p = 2d^2+3d+3$ dividing $3-5d$. On the other hand, if $p \mid q+\lambda$, we can use the second identity to obtain a contradiction, and so on. Then $\tau_2 \neq 0$. Suppose now that $\tau_3 = 0$; this implies that $p = m(2d^2+2d-1)$. Since $gcd(p,q,r,s) = 1$ it follows that $m = 1$. Then

$$p = 2d^2+2d-1, q = d^2+d+1, r = d^2+d, s = d^2, \lambda = d^3.$$

Similarly, we can use the following identities to obtain a contradiction

$$2(p+\lambda) = p(d+1) + 3d - 1, 2(q+\lambda) = pd + 3d + 2,$$
$$2(r+\lambda) = pd + 3d, 2(s+\lambda) = pd + d.$$

Then $\tau_3 \neq 0$.

i) $p \mid p+\lambda$
The condition $p \mid p+\lambda$ means that $p \mid d^3$, since $gcd(p,m) = 1$. So we are in the situation of theorem 1.20 (a), where $p \mid d^3$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{p+\lambda}{p} = 1 + \frac{rd}{p}$. We have $1 < l < d+1$. Take

$$Y = x(\alpha_p w^d + a x^{l-1})\partial/\partial x + y(\alpha_q w^d + b x^{l-1})\partial/\partial y + (z(\alpha_r w^d + c x^{l-1}) + y^d)\partial/\partial z + (w(\alpha_s w^d + e x^{l-1}) + z^d)\partial/\partial w.$$

Then $Y \in W_0$ as long as $l.a + b + c + e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_q & b \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_p & a \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_p & a \\ \alpha_s & e \end{vmatrix} \neq 0, a \neq 0.$$

Making the substitution $e = -(l.a + b + c)$, we see that the conditions above is given by a non-empty Zariski open set on $\mathbb{C}^3$ with coordinates (a,b,c), which shows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

ii) $p \mid q+\lambda$
The condition $p \mid q+\lambda$ means that $p \mid d^3 + d^2 + d + 1$, since $gcd(p,m) = 1$. So we are in the situation of theorem 1.20 (a), where $p \mid d^3 + d^2 + d + 1$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{q+\lambda}{p} = \frac{q}{p} + \frac{rd}{p} < 1 + d$. Take

$$Y = \alpha_p x w^d \partial/\partial x + (\alpha_q y w^d + x^l)\partial/\partial y + (\alpha_r z w^d + y^d)\partial/\partial z + (\alpha_s w^{d+1} + z^d)\partial/\partial w.$$

We see that $Y \in W_0$ and 0 is an isolated singularity of Y.

iii) $p \mid r+\lambda$
We show that $\mathcal{F}(p,q,r,s;\lambda,d+1)$ has no GK foliations, if p,q,r,s,λ,d satisfy the above conditions. In fact, as $\lambda > 0$ by proposition 2.1 (d) it suffices to show that

$$m_1 = \frac{(p+\lambda)(q+\lambda)(r+\lambda)(s+\lambda)}{pqrs} \notin \mathbb{Z}.$$

We have

$$p \mid r+\lambda \iff p \mid d^3 + d^2 + d = d(d^2+d+1),$$

since $gcd(p,m) = 1$. Certainly $gcd(p, d^2+d+1) \neq 1$, otherwise $p \mid d$ which implies

$$m(d^2+d+1) = q < p < d,$$

which is a contradiction. A straightforward calculation shows that

$$m_1 = \frac{(p+md^3)(d^3+d^2+d+1)}{p}.$$

Suppose, by contradiction, that $m_1 \in \mathbb{Z}$. Then $gcd(p, m) = 1$ implies that

$$p \mid d^3(d^3 + d^2 + d + 1).$$

Clearly a prime factor of p and $d^2 + d + 1$ cannot divide neither d^3 nor $d^3 + d^2 + d + 1$, which is a contradiction. So $m_1 \notin \mathbb{Z}$.

iv) $p \mid s + \lambda$

Once more we show that $\mathcal{F}(p, q, r, s; \lambda, d + 1)$ has no GK foliations if p, q, r, s, λ, d satisfy the above conditions. The condition $p \mid s + \lambda$ is equivalent to $p \mid d^3 + d^2$, since $gcd(p, m) = 1$. We cannot proceed as in the previous sub-case, because now $m_1 \in \mathbb{Z}$. Let us write $Y = Y_0\partial/\partial x + Y_1\partial/\partial y + Y_2\partial/\partial z + Y_3\partial/\partial w$, $Y \in W_0$. We claim that $Y_0(x, y, z, 0) \equiv 0$ and $Y_1(x, y, z, 0) \equiv 0$, which implies that 0 is a non-isolated singularity of Y. Let us check that $Y_0(x, y, z, 0) \equiv 0$. Suppose, by contradiction, that is not true. Then a term with the monomial $x^a y^b z^c$ must appear in the expansion of Y_0. Thus $p + \lambda = ap + bq + rc$, equivalently

$$p(a - 1) = m(d^3 - b(d^2 + d + 1) - c(d^2 + d)). \tag{4.4}$$

As $p \mid d^3 + d^2$, we can write $p = j_1 j_2$, where $j_1 \mid d^2$ and $j_2 \mid d + 1$. Since j_1 divides the right side of (4.2), $gcd(j_1, m) = 1$, j_1 divides d^2, it follows that

$$\begin{aligned} &j_1 \mid b(d + 1) + cd = d(b + c) + b \implies j_1 \mid d(d(b + c) + b) \implies j_1 \mid bd \implies \\ &j_1 \mid d(b + c) + b - bd = cd + b \implies j_1 \mid b^2 = b(cd + b) - cbd \implies \\ &j_1 \mid d^2 - b^2 = (d - b)(d + b). \end{aligned}$$

Since j_2 divides the right side of (4.2), $gcd(j_2, m) = 1$, j_2 divides $d + 1$, it follows that

$$j_2 \mid d^3 - b = d^3 + 1 - (b + 1) \implies j_2 \mid b + 1 \implies j_2 \mid d - b = (d + 1) - (b + 1).$$

As $gcd(j_1, j_2) = 1$ and both j_1, j_2 divides $d^2 - b^2 = (d-b)(d+b)$, we have that $p = j_1 j_2 \mid d^2 - b^2$. Thus $m(d^2 + d + 1) = q < p \leq d^2$, and we obtain a contradiction. Therefore $Y_0(x, y, z, 0) \equiv 0$. Next, suppose by contradiction that $Y_1(x, y, z, 0) \not\equiv 0$. Then there are natural numbers a, b, c such that $q + \lambda = ap + bq + rc$, equivalently

$$ap = m(d^3 + d^2 + d + 1 - b(d^2 + d + 1) - c(d^2 + d)). \tag{4.5}$$

Write $p = j_1 j_2$ as before. Since j_1 divides the right side of (4.3), $gcd(j_1, m) = 1$, j_1 divides d^2, it follows that

$$\begin{aligned} &j_1 \mid d + 1 - b(d + 1) - cd = d(1 - b - c) + 1 - b \implies j_1 \mid d(d(1 - b - c) + 1 - b) \implies \\ &j_1 \mid d(b - 1) \implies j_1 \mid 1 - b - cd = (d + 1 - b(d + 1) - cd) - d(b - 1) \implies \\ &j_1 \mid (b - 1)^2 = -c(d(b - 1)) - b(1 - b - cd) + (1 - b - cd) \implies j_1 \mid d^2 - (b - 1)^2. \end{aligned}$$

Since j_2 divides the right side of (4.3), $gcd(j_2, m) = 1$, j_2 divides $d + 1$, it follows that

$$j_2 \mid b \implies j_2 \mid d + 1 - b.$$

As $gcd(j_1, j_2) = 1$ and both j_1, j_2 divides $d^2 - (b - 1)^2 = (d + 1 - b)(d + b - 1)$, we have that $p = j_1 j_2 \mid d^2 - (b - 1)^2$. Thus $m(d^2 + d + 1) = q < p \leq d^2$, and we obtain a contradiction. Therefore $Y_1(x, y, z, 0) \equiv 0$.

(b) $p+\lambda = qd, s+\lambda = rd, \lambda = sd$
This set of conditions is equivalent to

$$p = kd > q = md + k > r = m(d+1) > s = md, \lambda = md^2.$$

Hence $gcd(p,q,r,s) = 1 \iff gcd(k,m) = 1$. In addition

$$\tau_2 = p+q+r+s+\lambda - q(d+4) = r+s-3q \neq 0, \tau_3 = p+q-3r.$$

We claim that $\tau_3 \neq 0$. Suppose, by contradiction, that $\tau_3 = 0$; this implies that $k(d+1) = m(2d+3)$. Since

$$gcd(k,m) = gcd(d+1, 2d+3) = 1,$$

it follows that $m = d+1, k = 2d+3$. Then

$$p = 2d^2+3d, q = d^2+3d+3, r = d^2+2d+1, s = d^2+d, \lambda = d^3+d^2.$$

An easy verification shows that in this case $p+\lambda, q+\lambda, r+\lambda, s+\lambda$ are not multiples of p for any d, we obtain a contradiction. Thus in all sub-cases $\tau_3 = \alpha_r \neq 0$.

i) $p \mid p+\lambda$
The condition $p \mid p+\lambda$ means $k \mid d$, since $gcd(k,m) = 1$. So we are in the situation of theorem 1.20 (b), where $k \mid d$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{p+\lambda}{p} = \frac{qd}{p}$. We have $1 < l < d$. Take

$$\begin{aligned} Y =& (x(\alpha_p w^d + ax^{l-1}) + y^d)\partial/\partial x + y(\alpha_q w^d + bx^{l-1})\partial/\partial y + z(\alpha_r w^d + cx^{l-1})\partial/\partial z + \\ & (w(\alpha_s w^d + ex^{l-1}) + z^d)\partial/\partial w. \end{aligned}$$

Then $Y \in W_0$ as long as $l.a+b+c+e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_p & a \\ \alpha_s & e \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_s & e \end{vmatrix} \neq 0, a \neq 0.$$

Making the substitution $e = -(l.a+b+c)$, we see that the conditions above is given by a non-empty Zariski open set on $\mathbb{C}^3$ with coordinates (a,b,c), which shows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

ii) $p \mid q+\lambda$
The condition $p \mid q+\lambda$ means that $kd \mid m(d^2+d)+k$. Clearly this implies that d divides k, i.e., $k = jd$ for some $j \in \mathbb{N}$. Substituting $k = jd$ in $kd \mid m(d^2+d)+k$ we get $j \mid d+1$, since $gcd(j,m) = 1$. So we are in the situation of theorem 1.20 (b), where $kd \mid m(d^2+d)+k$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{q+\lambda}{p} < d$. Take

$$Y = (\alpha_p x w^d + y^d)\partial/\partial x + (\alpha_q y w^d + x^l)\partial/\partial y + \alpha_r z w^d \partial/\partial z + (\alpha_s w^{d+1} + z^d)\partial/\partial w.$$

We see that $Y \in W_0$ and 0 is an isolated singularity of Y.

iii) $p \mid r+\lambda$
The condition $p \mid r+\lambda$ means that $kd \mid m(d^2+d+1)$, which in turn is equivalent to $k \mid d^2+d+1$ and $d \mid m$ since $gcd(k,m) = gcd(d, d^2+d+1) = 1$. So we are in the situation of theorem 1.20 (b), where $k \mid d^2+d+1$ and $d \mid m$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{r+\lambda}{p} < d$. Take
  $$Y = (\alpha_p x w^d + y^d)\partial/\partial x + \alpha_q y w^d \partial/\partial y + (\alpha_r z w^d + x^l)\partial/\partial z + (\alpha_s w^{d+1} + z^d)\partial/\partial w.$$
  We see that $Y \in W_0$ and 0 is an isolated singularity of Y.

iv) $p \mid s+\lambda$
The condition $p \mid s+\lambda$ means that $k \mid d+1$, since $gcd(k,m) = 1$. We claim that $V_0 \setminus \Sigma_0 \neq \emptyset$ if and only if
$$gcd\left(\frac{m(d+1)}{k}, d\right) = 1,$$
so we are in the situation of theorem 1.20 (b), where additionally $k \mid d+1$.
For, let us write
$$Y = Y_0\partial/\partial x + Y_1\partial/\partial y + Y_2\partial/\partial z + Y_3\partial/\partial w, Y \in W_0.$$
We claim that $Y_0(x,0,z,0) \equiv 0$ and $Y_2(x,0,z,0) \equiv 0$. Let us check that $Y_0(x,0,z,0) \equiv 0$. Suppose, by contradiction, that is not true. Then a term with the monomial $x^a z^b$ must appear in the expansion of Y_0. It follows from proposition 2.2 (d) that $p+\lambda = ap+br$, equivalently,
$$kd(a-1) = m(d^2 - b(d+1)).$$
Hence $k \mid d^2 - b(d+1)$, which implies that $k \mid d^2$ since $k \mid d+1$. From $k \mid d+1, k \mid d^2$ we conclude that $k = 1$ and we get
$$p = d < q = md + k,$$
it is a contradiction. By proceeding in a similar way, we have $Y_2(x,0,z,0) \equiv 0$.
Consequently, if 0 is an isolated singularity of $Y \in W_0$ it is necessary that a term with the monomial $x^a z^b$ appears in the expansion of Y_1. In this case, by proposition 2.2 (d) we have $q+\lambda = ap+br$, equivalently,
$$ad - 1 = mj(d-b),$$
where $j = \frac{d+1}{k}$. It follows that $gcd(mj,d) = 1$. On the other hand, if $gcd(mj,d) = 1$, there exists a integer b such that $d \mid mjb - 1$. We can assume that $0 < b < d$. Thus
$$d \mid mjd - (mjb-1) = mj(d-b) + 1.$$
If we set $a = \frac{mj(d-b)+1}{d} \in \mathbb{Z}_{>0}$, then $q+\lambda = ap+br$. Finally we check that $a+b \leq d$. Suppose, by contradiction, that $a+b \geq d+1$. Thus
$$\begin{aligned}
&q+\lambda = ap+br \implies \\
&q+\lambda > r(a+b) \implies \\
&md+k+md^2 > m(d+1)^2 \iff \\
&k > m(d+1) \iff \\
&\frac{1}{m} > \frac{d+1}{k} = j,
\end{aligned}$$
which is a contradiction. Thus $a+b \leq d$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{s+\lambda}{p} < d$. Take
  $$Y = (\alpha_p x w^d + y^d)\partial/\partial x + (\alpha_q y w^d + x^a z^b)\partial/\partial y + \alpha_r z w^d \partial/\partial z + (\alpha_s w^{d+1} + x^l + z^d)\partial/\partial w.$$
  We see that $Y \in W_0$ and 0 is an isolated singularity of Y.

(c) $s+\lambda = rd, \lambda = q(d-1) = sd$
This set of conditions is equivalent to

$$p > q = md^2 > r = m(d^2-1) > s = m(d^2-d), \lambda = m(d^3-d^2).$$

Hence $gcd(p,q,r,s) = 1 \iff gcd(p,m) = 1$.

By theorem 1.19 we must check that $\tau_3 \neq 0$. In this case $\tau_3 = p+q-3r$. Suppose, by contradiction, that $\tau_3 = 0$; this implies that $p = m(2d^2-3)$. Since $gcd(p,q,r,s) = 1$ it follows that $m = 1$. Then

$$p = 2d^2-3, q = d^2, r = d^2-1, s = d^2-d, \lambda = d^3-d^2.$$

As we did in (a) and (b), we can use the following identities to obtain a contradiction

$$\begin{aligned}&2(p+\lambda) = p(d+1)+3(d-1), 2(q+\lambda) = pd+3d,\\ &2(r+\lambda) = pd+3d-2, 2(s+\lambda) = pd+d.\end{aligned}$$

Then $\tau_3 = \alpha_r \neq 0$.

i) $p \mid p+\lambda$
The condition $p \mid p+\lambda$ means that $p \mid d^3-d^2$, since $gcd(p,m) = 1$. So we are in the situation of theorem 1.20 (c), where $p \mid d^3-d^2$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{p+\lambda}{p} = 1 + \frac{rd}{p}$. We have $1 < l < d+1$. Take

$$\begin{aligned}Y =& x(\alpha_p w^d + ax^{l-1} + a_1 y^{d-1})\partial/\partial x + y(\alpha_q w^d + bx^{l-1} + b_1 y^{d-1})\partial/\partial y +\\ & z(\alpha_r w^d + cx^{l-1} + c_1 y^{d-1})\partial/\partial z + (w(\alpha_s w^d + ex^{l-1} + e_1 y^{d-1}) + z^d)\partial/\partial w.\end{aligned}$$

Then $Y \in W_0$ as long as $l.a + b + c + e = 0$ and $a_1 + d.b_1 + c_1 + e_1 = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix}\alpha_p & a & a_1\\ \alpha_q & b & b_1\\ \alpha_r & c & c_1\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_p & a & a_1\\ \alpha_q & b & b_1\\ \alpha_s & e & e_1\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_p & a\\ \alpha_r & c\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_p & a\\ \alpha_s & e\end{vmatrix} \neq 0,$$
$$\begin{vmatrix}\alpha_q & b_1\\ \alpha_r & c_1\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_q & b_1\\ \alpha_s & e_1\end{vmatrix} \neq 0, \begin{vmatrix}a & a_1\\ b & b_1\end{vmatrix} \neq 0, a \neq 0, b_1 \neq 0.$$

Making the substitutions $e = -(l.a + b + c)$ and $e_1 = -(a_1 + d.b_1 + c_1)$, we see that the conditions above is given by a non-empty Zariski open set on $\mathbb{C}^6$ with coordinates (a, a_1, b, b_1, c, c_1), which shows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

ii) $p \mid q+\lambda$
The condition $p \mid q+\lambda$ means that $p \mid d^3$, since $gcd(p,m) = 1$. So we are in the situation of theorem 1.20 (c), where $p \mid d^3$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{q+\lambda}{p} = \frac{qd}{p} < d$. Take

$$\begin{aligned}Y =& x(\alpha_p w^d + ay^{d-1})\partial/\partial x + (y(\alpha_q w^d + by^{d-1}) + x^l)\partial/\partial y + z(\alpha_r w^d + cy^{d-1})\partial/\partial z +\\ & (w(\alpha_s w^d + ey^{d-1}) + z^d)\partial/\partial w.\end{aligned}$$

Then $Y \in W_0$ as long as $a + b.d + c + e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix}\alpha_p & a\\ \alpha_r & c\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_p & a\\ \alpha_s & e\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_q & b\\ \alpha_r & c\end{vmatrix} \neq 0, \begin{vmatrix}\alpha_q & b\\ \alpha_s & e\end{vmatrix} \neq 0, a \neq 0, b \neq 0.$$

From what we have seen, it follows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

iii) $p \mid r + \lambda$
The condition $p \mid r + \lambda$ means that $p \mid d^3 - 1$, since $gcd(p, m) = 1$. So we are in the situation of theorem 1.20 (c), where $p \mid d^3 - 1$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
Set $l = \frac{r+\lambda}{p} < \frac{q+\lambda}{p} < d$. Take

$$\begin{aligned} Y =& x(\alpha_p w^d + ay^{d-1})\partial/\partial x + y(\alpha_q w^d + by^{d-1})\partial/\partial y + (z(\alpha_r w^d + cy^{d-1}) + x^l)\partial/\partial z + \\ & (w(\alpha_s w^d + ey^{d-1}) + z^d)\partial/\partial w. \end{aligned}$$

Then $Y \in W_0$ as long as $a + b.d + c + e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_q & b \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_s & e \end{vmatrix} \neq 0, b \neq 0.$$

From what we have seen, it follows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

iv) $p \mid s + \lambda$
We show that in this case $\mathcal{F}(p, q, r, s; \lambda, d + 1)$ has no GK foliations. By proposition 2.1 (d), it suffices to show that

$$m_1 = \frac{(p + \lambda)(q + \lambda)(r + \lambda)(s + \lambda)}{pqrs} \notin \mathbb{Z}.$$

The condition $p \mid r + \lambda$ means that $p \mid d^3 - d = d(d+1)(d-1)$, since $gcd(p, m) = 1$. So we can write $p = j_1 j_2 j_3$, where $j_1 \mid d, j_2 \mid d + 1, j_3 \mid d - 1$. A straightforward calculation shows that

$$m_1 = \frac{(p + \lambda)(q + \lambda)(r + \lambda)(s + \lambda)}{pqrs} = \frac{(p + md^2(d - 1))d(d^2 + d + 1)}{p}$$

Suppose by contradiction that $m_1 \in \mathbb{Z}$. Then $gcd(p, m) = 1$ implies that $p \mid d^3(d-1)(d^2+d+1)$. We have that

$$gcd(j_1, d - 1) = gcd(j_1, d^2 + d + 1) = gcd(j_2, d) = gcd(j_2, d^2 + d + 1) = gcd(j_3, d) = 1.$$

Let us check that $gcd(j_3, d^2+d+1) = 1$. For, note that a prime factor j of $d-1$ and d^2+d+1 must divide $d(d+2) = (d-1)+(d^2+d+1)$, so $j \mid d+2$, which implies $j \mid 3 = (d+2)-(d-1)$. Hence $j = 3$, and we obtain a contradiction since $d^2 + d + 1$ is never a multiple of 3. Thus we have that

$$p = j_1 j_2 j_3 \mid d(d - 1) \leq d^2 \leq md^2 = q,$$

and we obtain a contradiction. So $m_1 \notin \mathbb{Z}$.

(d) $p + \lambda = qd, \lambda = r(d - 1) = sd$
This set of conditions is equivalent to

$$p = kd > q = m(d - 1) + k > r = md > s = m(d - 1), \lambda = m(d^2 - d).$$

Hence $gcd(p, q, r, s) = 1 \iff gcd(k, m) = 1$. By theorem 1.19, we must check that $\tau_2 \neq 0$, which is true since $\tau_2 = r + s - 3q < 0$.

i) $p \mid p + \lambda$
The condition $p \mid p + \lambda$ means that $k \mid d - 1$, since $gcd(k, m) = 1$. So we are in the situation of theorem 1.20 (d), where $k \mid d$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{p+\lambda}{p} = \frac{qd}{p}$. We have $1 < l < d$. Take

$$\begin{aligned} Y =& (x(\alpha_p w^d + ax^{l-1} + a_1 z^{d-1}) + y^d)\partial/\partial x + y(\alpha_q w^d + bx^{l-1} + b_1 z^{d-1})\partial/\partial y + \\ & z(\alpha_r w^d + cx^{l-1} + c_1 z^{d-1})\partial/\partial z + w(\alpha_s w^d + ex^{l-1} + e_1 z^{d-1})\partial/\partial w. \end{aligned}$$

  Then $Y \in W_0$ as long as $l.a + b + c + e = 0$ and $a_1 + b_1 + d.c_1 + e_1 = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a & a_1 \\ \alpha_r & c & c_1 \\ \alpha_s & e & e_1 \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b & b_1 \\ \alpha_r & c & c_1 \\ \alpha_s & e & e_1 \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_p & a \\ \alpha_s & e \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_s & e \end{vmatrix} \neq 0,$$
$$\begin{vmatrix} \alpha_r & c_1 \\ \alpha_s & e_1 \end{vmatrix} \neq 0, \begin{vmatrix} a & a_1 \\ c & c_1 \end{vmatrix} \neq 0, \begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} \neq 0, b \neq 0, c_1 \neq 0.$$

  Making the substitutions $e = -(l.a + b + c)$ and $e_1 = -(a_1 + d.b_1 + c_1)$, we see that the conditions above is given by a non-empty Zariski open set on $\mathbb{C}^6$ with coordinates (a, a_1, b, b_1, c, c_1), which shows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

ii) $p \mid q + \lambda$
As $p + \lambda_1 = q_1 d, \lambda_1 = r_1(d-1) = s_1 d, p \mid s_1 + \lambda_1$, the families $\mathcal{F}(p, q, r, s; \lambda, d+1)$ containing some GK foliation coincide with those of the subcase iv) below, where $p \mid s + \lambda$.

iii) $p \mid r + \lambda$
The condition $p \mid r + \lambda$ means that $k \mid d$, since $gcd(k, m) = 1$. So we are in the situation of theorem 1.20 (d), where $k \mid d$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{r+\lambda}{p} <= \frac{rd}{d} < d$. Take

$$\begin{aligned} Y =& (x(\alpha_p w^d + az^{d-1}) + y^d)\partial/\partial x + y(\alpha_q w^d + bz^{d-1})\partial/\partial y + (z(\alpha_r w^d + cz^{d-1}) + x^l)\partial/\partial z \\ & + w(\alpha_s w^d + ez^{d-1})\partial/\partial w. \end{aligned}$$

  Then $Y \in W_0$ as long as $a + b + d.c + e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_s & e \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_s & e \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_r & c \\ \alpha_s & e \end{vmatrix} \neq 0, a \neq 0, b \neq 0, c \neq 0.$$

  From what we have seen, it follows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

iv) $p \mid s + \lambda$
The condition $p \mid s + \lambda$ means that $d \mid m$ and $k \mid d^2 - 1$.

- $V_0 \setminus \Sigma_0 \neq \emptyset$
  Set $l = \frac{s+\lambda}{p} < \frac{r+\lambda}{p} < d$. Take

$$\begin{aligned} Y =& (x(\alpha_p w^d + az^{d-1}) + y^d)\partial/\partial x + y(\alpha_q w^d + bz^{d-1})\partial/\partial y + z(\alpha_r w^d + cz^{d-1})\partial/\partial z \\ & + (w(\alpha_s w^d + ez^{d-1}) + x^l)\partial/\partial w. \end{aligned}$$

  Then $Y \in W_0$ as long as $a + b + d.c + e = 0$. Furthermore, 0 is an isolated singularity of Y if and only if

$$\begin{vmatrix} \alpha_p & a \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_q & b \\ \alpha_r & c \end{vmatrix} \neq 0, \begin{vmatrix} \alpha_r & c \\ \alpha_s & e \end{vmatrix} \neq 0, c \neq 0.$$

  From what we have seen, it follows that $V_0 \setminus \Sigma_0 \neq \emptyset$.

$\square$

5 Proof of the corollary 1.21

Corollary 1.21. For $d \geq 2$, $\overline{\mathcal{F}(p,q,r,s;\lambda,d+1)}$ is an irreducible component of $\mathcal{F}_2(d+1,4)$ for the following values of p,q,r,s,λ

p	q	r	s	λ
d^3	d^2+d+1	d^2+d	d^2	d^3
d^3+d^2+d+1	d^2+d+1	$d+1$	1	-1
d^3+d^2+d+1	d^2+d+1	d^2+d	d^2	d^3
d^2	$2d$	$d+1$	d	d^2
d^3+d^2	d^3	d^3-2d-1	d^3-d^2-d	$d^4-d^3-d^2$
d^3+d^2+d	$2d^2+d+1$	d^2+d	d^2	d^3
d^2+d	$2d+1$	$d+1$	d	d^2
d^3-d^2	d^2	d^2-1	d^2-d	d^3-d^2
d^3	d^2	d^2-1	d^2-d	d^3-d^2
d^3-1	d^2	d^2-1	d^2-d	d^3-d^2
d^2-d	$2(d-1)$	d	$d-1$	d^2-d
d^2	$2d-1$	d	$d-1$	d^2-d
d^2+d	d^2+1	d^2	d^2-d	d^3-d^2
d^3-d	$2d^2-d-1$	d^2	d^2-d	d^3-d^2

Proof. In theorem 1.20, we make the following substitutions in (a): $k=d^3, m=1; k=d^3+d^2+d+1, m=d$ (we use here the relation $\mathcal{F}(p,q,r,s;\lambda,d+1)=\mathcal{F}(p,p-s,p-r,p-q;\lambda_1,d+1)$), $k=d^3+d^2+d+1, m=1$. In (b), we make the following substitutions: $p=d, m=1; p=d^2+d, m=d^2-d-1; p=d^2+d+1, m=d; p=d+1, m=1$. In (c), we make the following substitutions: $p=d^3-d^2, m=1; p=d^3, m=1; p=d^3-1, m=1$. In (d), we make the following substitutions: $k=d-1, m=1; k=d, m=1; k=d+1, m=d; k=d^2-1, m=d$. $\square$

6 Proof of the proposition 1.23

Proposition 1.23. Assume that $p_1 > p_2 > \cdots > p_n$ and $l_1 > l_2 > \cdots > l_n$ are two sequences of positive integers, where $gcd(p_1,\ldots,p_n)=gcd(l_1,\ldots,l_n)=1$. Suppose that $\overline{\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)} = \overline{\mathcal{F}(l_1,\ldots,l_n;\xi,d+1)}$ and one of the families (therefore both) contains a GK foliation. Then, either $l_1=p_1,\ldots,l_n=p_n,\xi=\lambda$ or $l_1=\overline{p}_1,\ldots,l_n=\overline{p}_n,\xi=\lambda_1$.

Proof. Let $\mathcal{F}$ be a GK foliation belonging to both families $\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$ and $\mathcal{F}(l_1,\ldots,l_n;\xi,d+1)$. As we know, $\mathcal{F}\in\mathcal{F}(p_1,\ldots,p_n;\lambda,d+1)$ has at least one quasi-homogeneous singularity q, which is either of type $(p_1,\ldots,p_n,\lambda)$ or $(\overline{p}_1,\ldots,\overline{p}_n,\lambda_1)$ (see propositions 4.4 (a) and corollary 2.12). As also $\mathcal{F}\in\mathcal{F}(l_1,\ldots,l_n;\xi,d+1)$, by a similar reason q is either of type $(l_1,\ldots,l_n,\xi)$ or $(l_1,l_1-l_n,\ldots,l_1-l_2,p(d-1)-\xi)$. Since $gcd(p_1,\ldots,p_n)=gcd(l_1,\ldots,l_n)=1$, it follows the conclusion of the proposition (see the observation after definition 4.2). $\square$

7 Proof of the corollary 1.24

Corollary 1.24. Let $p_1 > p_2 > \ldots > p_n$ be positive integers defined by $p_i=\sum_{j=0}^{n-i} d^j, i=1,\ldots,n$. Then, for every $d\geq 1$, $\overline{\mathcal{F}(p_1,\ldots,p_n;-1,d+1)}$ is an irreducible component of $\mathcal{F}_2(d,n)$. Furthermore, this is the unique GK component provided by theorem 1.17 where the GK foliations belonging to it have only one non-Kupka singularity.

Proof. First, note that in theorem 1.19, with exception to the case b.1, where $i = 0$, we have that $\lambda, \lambda_1 \geq 0$. In fact, $\lambda = p_n d > 0$ and in the remaining cases the condition c_{11} holds, i.e.,

$$p_1 + \lambda = p_2 d,$$

which in turn it is equivalent to

$$\lambda_1 = \overline{p}_n d > 0.$$

By remark 2.3 q_0 and q_1 are non-Kupka singularities of a GK foliation

$$\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1).$$

On the other hand, if we are in the situation of theorem 1.19.b.1, where $i = 0$, it follows that the conditions $c_{12}, c_{23}, \ldots, c_{n-1n}$ hold, then

$$p_3 + \lambda = p_2 d, p_4 + \lambda = p_3 d, \ldots, p_n + \lambda = p_{n-1} d, \lambda = p_n d.$$

Next we check when $\lambda_1 < 0$, the only case where q_1 will be a Kupka singularity of the GK foliation $\mathcal{F}$. Define

$$q_i = \sum_{j=i-2}^{n-2} d^j, i = 2, 3, \ldots, n.$$

It can be shown that there exists positive integer m such that

$$p_2 = mq_2, p_3 = mq_3, \ldots, p_n = mq_n, \lambda = md^{n-1}.$$

In this case,

$$gcd(p_1, \ldots, p_n) = 1 \iff gcd(p_1, m) = 1.$$

As $p_1 \geq p_2 + 1$, we have that

$$\lambda_1 = p_1(d-1) - \lambda \geq (p_2 + 1)(d-1) - md^{n-1} = d - (m+1). \tag{4.6}$$

By theorem 1.19, p_1 divides $p_l + \lambda$, for some $l \in \{1, \ldots, n\}$. Assume first that $l = 1$, which means that p_1 divides d^{n-1}. In particular

$$p_1 \leq d^{n-1},$$

and we claim that

$$m \leq d - 1.$$

In fact, if we suppose that $m > d - 1$, we obtain a contradiction since

$$p_1 > mq_2 = p_2.$$

Thus from inequality (4.6) we have that $\lambda_1 \geq 0$. In the same way, if $l \geq 3$ we have that

$$p_1 \leq q_l + d^{n-1},$$

and one more we claim that

$$m \leq d - 1.$$

In fact, if we suppose that $m \geq d$, we obtain a contradiction since

$$p_1 > mq_2 = p_2.$$

Once again from inequality (4.6) we have that $\lambda_1 \geq 0$. Finally, $l = 2$ means that

$$p_1 \mid d^{n-1} + d^{n-2} + \cdots + d + 1.$$

From $p_1 > mq_2$ we conclude that

$$m \leq d.$$

From (4.6), it follows that $\lambda_1 \geq -1$. Moreover, $\lambda_1 = -1$ means that $m = d$ and

$$p_1 = d^{n-1} + d^{n-2} + \cdots + d + 1.$$

In summary, if

$$\mathcal{F} \in \mathcal{F}(p_1, \ldots, p_n; \lambda, d+1)$$

is GK and $\lambda_1 < 0$ it is necessary that

$$p_1 = d^{n-1} + d^{n-2} + \cdots + d + 1 > p_2 = dq_2 > \cdots > p_n = dq_n, \lambda = d^n.$$

Note that in this case

$$\frac{p_2 + \lambda}{p_1} = d.$$

We conclude by showing that the remaining conditions of theorem 1.19 are satisfied. In fact,

$$\tau_j = \tau - p_j(n+d) = \lambda - p_j(n+d) + \sum_{k=1}^{n} p_k, j = 2, 3, \ldots, n.$$

It follows that

$$\tau_j \neq 0, j = 2, 3, \ldots, n,$$

since $p_2, p_3, \ldots, p_n, \lambda$ are multiple of d and p_1 is not. Next set

$$Y = x_n^d \left(\tau R - (n+d)S\right) + \sum_{k=2}^{n} x_{k-1}^d \partial/\partial x_k.$$

Note that

$$\tau R - (n+d)S = \sum_{k=1}^{n} \tau_k x_k \partial/\partial x_k,$$

where we set

$$\tau_1 = \tau - p_1(n+d) < 0.$$

We have that $Y \in W_0$ and 0 is an isolated singularity of Y. Finally we use corollary 2.12 to obtain the family of the corollary 1.24. $\square$

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