Goodness–of–Fit Test for Stochastic Volatility Models

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Abstract

A goodness–of–fit test based on empirical processes is proposed as a model diagnostic check method for continuous time stochastic volatility models. More specifically, as the volatility is not observable, a marked empirical process is constructed from the representation in a state space model form associated to the discretized version of the underlying process. Distributions of these processes are approximated using bootstrap techniques. Some simulation results and an empirical application to an EURIBOR (Euro Interbank Offered Rate) data set are presented for illustration.

Contents

1 Introduction

Understanding and quantifying volatility is one of the main challenges in present–day financial analysis. This is thoroughly justified by its impact in pricing and risk management, among other applications. For instance, it is crucial in pricing and hedging derivatives a good model selection.

However, this is not an easy task to accomplish. The volatility of a process is not directly observed and thus needs to be estimated by some indirect process. In addition, the term *volatility* has different meanings depending on the discipline or field of study, which has given rise to quite a few different definitions of volality throughout the literature (see Ghyles et al. (1996), Shephard (2005) and references therein for a sample of the various approaches and historical background).

Nevertheless, there is a key feature of volatility which somehow unifies the alternative approaches: volatility refers to a measure of variation or oscillation of the observed quantity time series. Intuitively, higher volatility acts as if time would be *running faster* and more information is being added to the observed system (see Shephard (2005) and references therein).

In financial applications, the classical work of Markowitz (1952) connects the volatility directly with investment strategies and risk management. The seminal papers by Black and Scholes (1973) and Merton (1973) in option pricing make use of a powerful simplifying assumption. Namely, that the underlying asset follows a geometric Brownian motion (GBM) with drift

$$dr_t = \mu r_t dt + \sigma r_t dW_t \ , \tag{1}$$

where dW_t refers to the differential of the Wiener process, r_t denotes the asset price and σ and μ are constant values. However, this hypothesis has been deeply scrutinized and questioned in the literature. For instance, even in the case of US stock returns, departures from GBM have been well-documented (see Campbell et al. (1997), Section 9.3.6).

In addition, the unpredictability and evidence of non–stationarity of the volatility in financial time series under different scales has been well documented in the literature and goes back to Mandelbrot (1963) and Officer (1973). This naturally leads to the

proposal of more general models than (1), such as

$$dr_t = m(r_t)dt + \sigma(r_t)dW_t, \tag{2}$$

where the drift m and the volatility σ are now dependent on the underlying asset, r_t. Equation (2) can be analyzed as a parametric model by assuming that

$$dr_t = m(r_t, \theta)dt + \sigma(r_t, \theta)dW_t, \tag{3}$$

where the functional form for m an σ are well–defined within a certain class that depends on an unknown parameter $\theta \in \Theta \subset \mathbb{R}^d$ with d a positive integer. Equation (3) allows for the representation of a fairly broad family of financial models. See for example Andersen and Lund (1997) and Hull and White (1987).

A plethora of volatility definitions and indices arises when volatility models are formulated in a discrete time scale. For instance, a great deal of attention has been paid to models such as the autoregressive conditional heteroskedasticity (ARCH) model by Engle (1982) and its generalizations. The need for continuous models is obvious and crucial for comparisons, model simulations and ultimately pricing and risk-management. As it can be seen in Shephard (2005), there have been efforts in both simulation and inference methods on continuous–time stochastic volatility models. Nevertheless, to the best of our knowledge, the joint use of goodness–of–fit tests and Kalman filtering techniques has not been explored.

In this work, financial models will be considered as continuous in time and described by stochastic differential equations whose coefficients are to be determined parametrically. In particular, the focus will be placed on models for an observed quantity r_t given by the stochastic differential equation

$$\begin{aligned} dr_t &= m_1(r_t, \theta)dt + \sigma_t \upsilon_1(r_t, \theta)dW_{1,t} \\ dg(\sigma_t) &= m_2(g(\sigma_t), \vartheta)dt + \upsilon_2(g(\sigma_t), \vartheta)dW_{2,t} \end{aligned} \tag{4}$$

where g, m_1, υ_1, m_2 and υ_2 are known functions; $\Phi = (\theta, \vartheta) \in \mathbb{R}^d$ is an unknown vector parameter (to be estimated); σ_t^2 is the unobserved volatility and $W_{1,t}$ y $W_{2,t}$ are (possibly correlated) Brownian motions.

As remarked in Campbell et al. (1997) there are many open issues in statistical inference for continuous–time processes with discretely sampled data. For instance, Aït-Sahalia (1993) proposes a nonparametric estimator of the diffusion coefficient (assuming some constraints on the drift). Genon-Catalot et al. (1999) introduce appropriate and explicit functions of the observations to replace either the log-likelihood or the score function. Aït-Sahalia and Kimmel (2007) developed an alternative method that employs maximum likelihood, using closed form approximations to the true (but unknown) likelihood function. Specifically, for Model (3), the goodness–of–fit testing problem has been discussed by Dette and von Lieres und Wilkau (2003), Dette et al. (2006) and Monsalve-Cobis et al. (2011).

For the stochastic volatility model in equation (4), most of the existing methods for goodness–of–fit testing are not directly applicable due the fact that the volatility is not directly observed, but there have been some approaches for testing its components. For example, Lin et al. (2013) propose a goodness–of–fit test for the volatility distribution in (4), based on the deviation between the empirical characteristic function and its parametric counterparts.

In this work, a goodness–of–fit test based on the empirical process is proposed. First, a discretized version of Model (4) is considered. Then, Kalman filtering techniques are applied to obtain the associated state space model. Finally, the ideas described in Monsalve-Cobis et al. (2011) for the construction of some generalized statistical tests are applied to this context. Thus, the goal is to introduce a goodness–of–fit test for the (parametric) drift and volatility functions in those models with a stochastic volatility component. Calibration of the tests is done using bootstrap procedures (see Rodriguez and Ruiz (2012) and Monsalve-Cobis et al. (2011)).

This article is organized as follows: the continuous time stochastic volatility models are presented in Section 2, discussing the corresponding state space structure. In Section 3, the new goodness–of–fit tests for the drift and the volatility is introduced. Section 4 is devoted to the bootstrap strategy used for calibration. Finally, in Section 5, some preliminary simulation results are provided, jointly with a real data application of the tests, dealing with interbank intereset rates in the Eurozone.

2 The Stochastic Volatility Model

Consider the stochastic volatility Model (4), where g, m_1, m_2, υ_1 and υ_2 are known real valued functions satisfying certain regularity conditions in order to ensure the existence and uniqueness of the solution of the underlying stochastic differential equations (see Genon-Catalot et al. (1999) and Lin et al. (2013)). The coefficients in (4) depend on the unknown parameters $\Phi = (\theta, \vartheta) \in \Theta \in \mathbb{R}^d$, and therefore, different models can be generated for stochastic volatility by choosing different parametric forms for the functions g, m_1, m_2, υ_1 and υ_2. The developments presented in this paper will be focused on a widely studied model, which has been used in several financial applications: the *CKLS* model proposed by Andersen and Lund (1997). This model incorporates the volatility as a non observable stochastic factor, being an extension of the *CKLS* model introduced by Chan et al. (1992). The specification proposed by Andersen and Lund (1997) assumes mean reversion both at the level of the interest rate and at the volatility (in log scale). More concretely:

$$\begin{aligned} dr_t &= \kappa_1(\mu - r_t)dt + \sigma_t r_t^{\gamma} dW_{1,t} \\ d\log(\sigma_t^2) &= \kappa_2(\alpha - \log(\sigma_t^2))dt + \xi dW_{2t}, \end{aligned}$$

where W_{1t} and W_{2t} are independent Brownian motions, and $\alpha, \kappa_1, \kappa_2, \mu, \gamma$ and ξ are the unknown parameters.

It should be also noted that it is not unusual to find in Model (4) a correlation between r_t and σ_t as a consequence of the corresponding Brownian processes. In that case, the following kind of dependence structure can be used,

$$dW_{1t} = \rho dW_{2t} + \sqrt{1-\rho^2} dW_{3t},,$$

with W_{2t} and W_{3t} independent Brownian motions. However, along this paper, ρ will be set to 0.

Although model in equation (4) specifies a proper framework for continuous time financial process analysis, in practice, the phenomena associated to such processes is just observed at discrete time points. Hence, discretized versions of continuous time models must be considered for application in practice. For that purpose, assume that

the process $\{r_t : 0 \leq t \leq T\}$ is observed at discrete equally spaced times $t_i = i\Delta$, $i = 0, 1, \ldots, n$, with a fixed $\Delta > 0$ within an observation window $[0, n\Delta = T]$, which increases as n grows. Then, the discrete time version of Model (4) can be formulated as

$$\begin{aligned} r_{t_{i+1}} - r_{t_i} &= m_1(r_{t_i}, \theta)\Delta + \sigma_{t_i} \upsilon_1(r_{t_i}, \theta) \left(W_{1,t_{i+1}} - W_{1,t_i}\right) \\ g(\sigma_{t_{i+1}}) - g(\sigma_{t_i}) &= m_2(g(\sigma_{t_i}), \vartheta)\Delta + \upsilon_2(g(\sigma_{t_i}), \vartheta) \left(W_{2,t_{i+1}} - W_{2,t_i}\right) \end{aligned}$$

and taking into account the properties of the Brownian motion, the process can be expressed as

$$\begin{aligned} \frac{y_{t_i}}{\Delta} &= m_1(r_{t_i}, \theta) + \sigma_{t_i} \upsilon_1(r_{t_i}, \theta)\Delta^{-1/2}\varepsilon_{1,t_i} \\ g(\sigma_{t_{i+1}}) - g(\sigma_{t_i}) &= m_2(g(\sigma_{t_i}), \vartheta)\Delta + \upsilon_2(g(\sigma_{t_i}), \vartheta)\sqrt{\Delta}\varepsilon_{2,t_i}, \end{aligned} \tag{5}$$

where, $y_{t_i} = r_{t_{i+1}} - r_{t_i}$, and $\{\varepsilon_{1,t_i}, \varepsilon_{2,t_i}\}$ are two independent random variables with distribution $N(0,1)$, for $i = 1, \ldots, n$.

An important issue when analyzing the behaviour of the aforemetioned processes is the large sample scheme, since there is not a unique way of defining it. The most natural approach considered in practice consists in taking Δ (spacing between two consecutive observations) as fixed and let the number of observations n grow (see Kessler (2000) and Iacus (2008), for some examples). However, there are other alternatives, as the one considered by Genon-Catalot et al. (1999), where the sampling distance $\Delta = \Delta_n$ goes to zero whereas the window $n\Delta_n$ goes to infinity. The main goal of the different observation schemes is related to keeping the asymptotic properties of the estimators and to allow the use of statistical inference methods (see Lin et al. (2013)).

2.1 State Space Model

The estimation of stochastic volatility models turns out to be a complex problem, partly motivated by the estimation of the transition density funciton of r_t (the state variable), which is itself a difficult task, even under closed formulations. In addition, the state variables that determine the volatility are not directly observable. Thus, the estimation for such a function just from information of the underlying process in its essence calls

for the use of filtering techniques. With this purpose, Kalman filtering techniques are applied to obtain the state space representation (5) of the model in (4). Taking $x_{t_i} = g(\sigma_{t_i})$, with g strictly monotonic and after some algebraic manipulations,

$$\begin{aligned} \frac{y_{t_i}}{\Delta} &= m_1(r_{t_i},\theta) + g^{-1}(x_{t_i})\upsilon_1(r_{t_i},\theta)\Delta^{-1/2}\varepsilon_{1,t_i} \\ x_{t_{i+1}} &= x_{t_i} + m_2(x_{t_i},\vartheta)\Delta + \upsilon_2(x_{t_i},\vartheta)\sqrt{\Delta}\varepsilon_{2,t_i}. \end{aligned} \tag{6}$$

The main goal of this representation is to capture the dynamics of the observable variables y_{t_i} and r_{t_i}, in terms of the unobservable σ_{t_i}. It is important to stress that, for convenience, the state space model is required to fall within the class of linear state space models. This is achieved considering, for example, $g(y) = \log(y^2)$. Thus, Model (5) with $g(\cdot) = 2\log(\cdot)$ can be written as equation (6):

$$\begin{aligned} \frac{y_{t_i}}{\Delta} &= m_1(r_{t_i},\theta) + \sigma_{t_i}\upsilon_1(r_{t_i},\theta)\Delta^{-1/2}\varepsilon_{1,t_i} \\ \log(\sigma^2_{t_{i+1}}) &= \log(\sigma^2_{t_i}) + m_2(\log(\sigma^2_{t_i}),\vartheta)\Delta + \upsilon_2(\log(\sigma^2_{t_i}),\vartheta)\sqrt{\Delta}\varepsilon_{2,t_i}. \end{aligned}$$

Following the derivation in Harvey et al. (1994), denote by e_{t_i} the error obtained from the equation

$$e_{t_i} = \frac{y_{t_i}}{\Delta} - m_1(r_{t_i},\theta) = \sigma_{t_i}\upsilon_1(r_{t_i},\theta)\Delta^{-1/2}\varepsilon_{1,t_i}\ ,$$

which gives:

$$\log(e^2_{t_i}) = \log(\sigma^2_{t_i}) + 2\log(\upsilon_1(r_{t_i},\theta)) - \log(\Delta) + \log(\varepsilon^2_{1,t_i})$$

Now, taking $u_{t_i} = \log(e^2_{t_i})$, and $x_{t_{i+1}} = \log(\sigma^2_{t_{i+1}})$, the following state space model is obtained:

$$\begin{aligned} u_{t_i} &= x_{t_i} + 2\log(\upsilon_1(r_{t_i},\theta)) + \eta_{t_i} - \kappa \\ x_{t_{i+1}} &= x_{t_i} + m_2(x_{t_i},\vartheta)\Delta + \upsilon_2(x_{t_i},\vartheta)\sqrt{\Delta}\varepsilon_{2,t_i} \end{aligned} \tag{7}$$

with $\eta_{t_i} = -\log(\Delta) + \log(\varepsilon^2_{1,t_i}) + \kappa$ and $\kappa = \log(\Delta) - \mathbb{E}\left[\log(\varepsilon^2_{1,t_i})\right]$. The parameter estimation of $\Phi = (\theta,\vartheta)$ can be obtained by maximum likelihood, computing the likelihood from the innovations $\eta_{t_1},\ldots,\eta_{t_n}$.

In the sequel, the estimation can be obtained using Kalman filters considering a mixture of Gaussian variables to approximate the non–Gaussian errors, but other alternatives are also possible. With respect to this issue, note that innovation errors in the previous state space model are not Gaussian. If ε_{1,t_i}^2 follows a lognormal distribution, then the state space model presents Gaussian errors, and it can be estimated using basic Kalman filter techniques. Unfortunately, under the assumption of normality for ε_{1,t_i}, the variable ε_{1,t_i}^2 has a χ^2 distribution with one degree of freedom, and the density under the logarithmic transformation is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(e^x - x)}, \qquad -\infty < x < \infty,$$

with mean -1.2704 and variance $\pi^2/2$. It is clear the need of applying methodology that allows to obtain an equation involving the non observable variable x_{t_i}, with Gaussian mixture distributions. Therefore, writing the observation equation in Model (7) with $u_{t_i} = \log(e_{t_i}^2)$ as

$$u_{t_i} = x_{t_i} + 2\log(\upsilon_1(r_{t_i}, \theta)) + \eta_{t_i} - \kappa$$

where η_{t_i} is zero-mean noise, the assumption of a normal mixture distribution will be considered. In particular, for a mixture of two distributions:

$$\eta_{t_i} - \kappa = I_{t_i} z_{t_i 0} + (1 - I_{t_i}) z_{t_i 1},$$

where I_{t_i} is an *iid* process such that $\mathbb{P}\{I_{t_i} = 0\} = \pi_0$, $\mathbb{P}\{I_{t_i} = 1\} = \pi_1$, $(\pi_0 + \pi_1 = 1)$, $z_{t_i 0} \sim iid\ N(0, \sigma_0^2)$, and $z_{t_i 1} \sim iid\ N(\mu_1, \sigma_1^2)$. The advantage of such procedure hinges upon the use of normality.

The estimation of the model parameters is performed by maximum likelihood, being the log–likelihood function to optimize:

$$\log L(\Phi) = \sum_{i=1}^{n} \log \left(\sum_{j=0}^{1} \pi_j f_j(t_i | t_i - 1) \right)$$

where the transition density $f_j(t_i|t_i - 1)$ is approximated by a normal or normal mixture density, with parameters given by the filter. For details see, for example, Shumway and Stoffer (2011), §6.8 and §6.9.

An alternative method for the estimation of the stochastic volatility can be found in Aït-Sahalia and Kimmel (2007). In this reference, maximum likelihood is also used but considering numerical approximations of the true likelihood. In order to consider positive correlation between Brownian motions, the methods introduced by §6.7-Shumway and Stoffer (2011) and Nisticò (2007), also based on Kalman filtering techniques, could be considered.

3 GOF-Tests

A generalization of the goodness–of–fit test proposed in Monsalve-Cobis et al. (2011) for the stochastic volatility Model (4) will be presented in this section. The proposal follows the methodology developed by Stute (1997) for the regression context, based on empirical residual processes. The goal in this work is to compare the parametric form of the drift functions and the volatility for the model under consideration, establishing as null hypothesis:

$$\mathscr{H}_{0m} : m_1 \in \{m_1(\cdot,\theta) : \theta \in \Theta\} \tag{8}$$

for the parametric form of the drift function, and

$$\mathscr{H}_{0v} : \upsilon_1 \in \{\upsilon_1(\cdot,\theta) : \theta \in \Theta\} \tag{9}$$

for the parametric form of the volatility. The construction of the test statistic and the testing procedure will be described in the next sections.

3.1 Drift Function Test

Assume that $\hat{\Phi} = (\hat{\theta}, \hat{\vartheta})$ is an appropriate estimator (satisfying a root-n consistency condition) of the true parameter $\Phi = (\theta, \vartheta)$ of the stochastic volatility model. The test statistic for assessing the parametric form of the drift function, under the assumption that $\mathscr{H}_{0v}$ given by (9) holds, is based on the empirical process:

$$D_n(r) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i} \leq r\}} \left\{ \frac{y_{t_i}}{\Delta} - m_1(r_{t_i}, \hat{\theta}) \right\}, \quad \text{with } r \in \mathbb{R},$$

being $\mathbf{1}_{\{\cdot\}}$ the indicator function. For constructing a test statistic, a continuous functional $\Psi(\cdot)$ of the empirical process can be considered. In general, such a test statistic will be defined as $T_n = \Psi(D_n)$ and the null hypothesis $\mathscr{H}_{0m}$ is rejected if $T_n > c_{1-\alpha}$ where $c_{1-\alpha}$ satisfies

$$\mathbb{P}\{T_n > c_{1-\alpha} | \mathscr{H}_{0m}\} = \alpha \, .$$

Two examples of such a test statistic are the following

$$T_n^{KS} = \sup_r |D_n(r)|, \qquad \text{and} \qquad T_n^{CvM} = \int_{\mathbb{R}} D_n(r)^2 F_n(dr)$$

being the first one a Kolmogorov-Smirnov (KS) type test and the second one a Cramér-von Mises (CvM) statistic. In the previous formulation, F_n denotes the empirical distribution of $\{r_{t_i}\}_{i=1}^n$. Along the text, $T_n = T_n^{KS}$ or $T_n = T_n^{CvM}$ will be used to indicate the specific statistics under consideration.

3.2 Volatility function test

Focusing now on the volatility component, and similarly to the ideas presented for the test designed for the drift function, assume that $\hat{\Phi} = (\hat{\theta}, \hat{\vartheta})$ is an appropriate estimator of the true parameter $\Phi = (\theta, \vartheta)$ in the volatility model. The goodness–of–fit test for the parametric form of the volatility function under the assumption that $\mathscr{H}_{0m}$ given by (8) holds, is based on the empirical process:

$$V_n(r,x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i} \le r, \hat{\sigma}_{t_i}^2 \le x\}} \left\{ \left(\frac{y_{t_i}}{\Delta} - m_1(r_{t_i}, \hat{\theta}) \right)^2 - \frac{\hat{\sigma}_{t_i}^2 \upsilon_1^2(r_{t_i}, \hat{\theta})}{\Delta} \right\}, \quad \text{with } r,x \in \mathbb{R}.$$

with $\hat{\sigma}_t^2$ an estimate of the volatility. As before, a continuous functional $\Psi(\cdot)$ of the empirical process can be considered to define, in general, the test statistics $U_n = \Psi(V_n)$. Similarly, the null hypothesis $\mathscr{H}_{0\upsilon}$ is rejected if $U_n > c_{1-\alpha}$ where $c_{1-\alpha}$ is the critical value for the α–level test:

$$\mathbb{P}\{U_n > c_{1-\alpha} | \mathscr{H}_{0\upsilon}\} = \alpha \, ,$$

Again, Kolmogorov-Smirnov (KS) and Cramér-von Mises (CvM) statistics can be expressed as

$$U_n^{KS} = \sup_{r,x} |V_n(r,x)|, \qquad \text{and} \qquad U_n^{CvM} = \int\int_{\mathbb{R}^2} (V_n(r,x))^2 F_n(dr,dx)$$

where F_n is the empirical distribution of $\{r_{t_i}, \hat{\sigma}_{t_i}^2\}$. $U_n = U_n^{KS}$ or $U_n = U_n^{CvM}$ will be used to denote the corresponding statistics.

In the definition of $V_n(r,x)$, a parametric model is assumed for the drift m_1. If such a model in $\mathscr{H}_{0m}$ is not specified, a nonparametric estimator for m_1 must be used. In that case, the problem is that: $\mathbb{E}[y_{t_i}/\Delta | r_{t_i}] = m_1(r_{t_i},\theta) + \upsilon_1(r_{t_i},\theta)\Delta^{-1/2}\mathbb{E}\left[g^{-1}(x_{t_i})\varepsilon_{1,t_i}|r_{t_i}\right]$ and additional assumptions on the stochastic volatility would be necessary to obtain a consistent estimator of m_1 (for example, using a kernel estimation). Clearly, this is still an open problem and more research is needed in this direction.

Both for the drift and the volatility tests, the critical values under the null hypothesis, denoted by $c_{1-\alpha}$, must be determined. For that purpose, the distribution of the processes T_n and U_n must be specified, which turns out to be difficult in general. Alternatively, approximations of such critical values by means of bootstrap techniques can be considered for testing purposes. A bootstrap approximation will be introduced in the next section.

4 Bootstrap Approximations

A bootstrap algorithm will be presented for approximating the critical values of the proposed test statistics. The procedure is based on the generation of an artificial sample with the same characteristics of the initial one. From such a sample, critical values are estimated as follows.

First, let $\{(r_t^*)\}$ be an artificial process (to be defined later in detail) and let $\hat{\Phi}^* = (\hat{\theta}^*, \hat{\vartheta}^*)$ be a parameter estimator obtained from such process. Then, the bootstrap versions D_n and V_n are given by:

$$D_n^*(r) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i}^* \le r\}} \left\{ \frac{y_{t_i}^*}{\Delta} - m_1(r_{t_i}^*, \hat{\theta}^*) \right\}, \quad \text{with } r \in \mathbb{R}.$$

$$V_n^*(r,x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}_{\{r_{t_i}^* \le r, \hat{\sigma}_{t_i}^{*2} \le x\}} \left\{ \left(\frac{y_{t_i}^*}{\Delta} - m_1(r_{t_i}^*, \hat{\theta}^*) \right)^2 - \frac{\hat{\sigma}_{t_i}^{*2} \upsilon_1^2(r_{t_i}^*, \hat{\theta}^*)}{\Delta} \right\}, \quad \text{with } r,x \in \mathbb{R}.$$

The critical value $c_{1-\alpha}$ will be approximated by its bootstrap counterpart, $c_{1-\alpha}^*$, so that

$$\mathbb{P}^*\{T_n^* > c_{1-\alpha}^*\} = \alpha, \quad \mathbb{P}^*\{U_n^* > c_{1-\alpha}^*\} = \alpha,$$

where $\mathbb{P}^*$ denotes the probability measure associated to the bootstrap with

$$T_n^{*KS} = \sup_r |D_n^*(r)|, \quad \text{or} \quad T_n^{*CvM} = \int_{\mathbb{R}} D_n^*(r)^2 F_n(dr)$$

and

$$U_n^{*KS} = \sup_{r,x} |V_n^*(r,x)|, \quad \text{or} \quad U_n^{*CvM} = \int\int_{\mathbb{R}^2} V_n^*(r,x)^2 F_n(dr,dx)$$

In practice,

$$c_{1-\alpha}^* = T_n^{*\lceil B(1-\alpha) \rceil}, \quad \text{or} \quad c_{1-\alpha}^* = U_n^{*\lceil B(1-\alpha) \rceil}$$

that is, the $\lceil B(1-\alpha) \rceil$-th order statistic calculated on the B bootstrap replicates $T_n^{*j} = T_n^*$ ($U_n^{*j} = U_n^*$), $1 \le j \le B$. The empirical p-value for the bootstrap sample can be calculated as

$$\frac{\sharp\{T_n^{*j} > T_n\}}{B} \quad \text{or} \quad \frac{\sharp\{U_n^{*j} > U_n\}}{B},$$

that is to say, the p-value is taken as the fraction of values from the bootstrap versions T_n^* (U_n^*) exceeding the value of T_n (U_n). It seems clear that appropriate (consistent) parametric estimates for the stochastic volatility model parameters is crucial for the procedure. In the following section, some aspects concerning the characteristics of the artificial sample used in the bootstrap procedure implementation will be described.

4.1 Bootstrap Resampling

For the construction of the bootstrap sample, the state space model structure must be taken into account. This feature will be illustrated for Model (7). A crucial condition is that such model presents Gaussian errors, or at least, that such errors are approximately Gaussian distributed (which must be checked using some statistical procedure). Under these premises, the bootstrap sample can be generated as follows:

1. Let $\hat{\Phi} = (\hat{\theta}, \hat{\vartheta})$ be the estimator of the true parameter Φ, obtained by maximum likelihood. That is:
$$\hat{\Phi} = \arg\max_{\Phi} L(\Phi),$$
assuming that the errors follow Gaussian mixture distribution.

2. Following Shumway and Stoffer (2011) §6.7, apply the Kalman filter equations to obtain a bootstrap resample $\left\{\left(u^*_{t_i}, x^*_{t_i}, r_{t_i}\right)\right\}$ where the $\{r_{t_i}\}$ remain fixed.

3. Once the bootstrap resample is obtained, estimate the corresponding parameters $\hat{\Phi}^* = (\hat{\theta}^*, \hat{\vartheta}^*)$ associated to the state space model by maximum likelihood, based on $L^*(\Phi)$.

4. Get the bootstrap versions of the aforementioned processes $D^*_n(r,x)$ and $V^*_n(r,x)$, for $x, r \in \mathbb{R}$, and T^*_n, (U^*_n).

5. Repeat Steps 2–4 B times and get copies T^{*j}_n, (U^{*j}_n), for $j = 1, 2, \ldots, B$,

6. Finally, compute the bootstrap approximations of the critical values
$$\hat{c}^*_{1-\alpha} = T^{*\lceil B(1-\alpha)\rceil}_n \quad \text{or} \quad \hat{c}^*_{1-\alpha} = U^{*\lceil B(1-\alpha)\rceil}_n.$$

5 Some Applications

The performance of the testing procedure introduced in this work is illustrated in this section. First, some preliminary simulation results are shown, considering a previously studied model from the financial literature and showing the procedure perfomance. A real data example is also provided. The dataset gathers interest rate curves at the European markets, the EURIBOR®-(Euro Interbank Offered Rate).

As an example of artificial data, consider Model (1) given in (Monsalve-Cobis et al., 2011, equation (35)):

$$dr_t = (0.0408 - 0.5921 r_t)dt + \sigma_t r_t^{1.4999} dW_{1,t}$$

where the deterministic value $\sigma_t = \sqrt{1.6704}$ in the previous reference is replaced by a stochastic volatility model with $d(\log \sigma_t^2) = \omega dW_{2,t}$ being ω an unknown parameter.

Table 1: Empirical power for the null ($\gamma = 1.4999$) and for the two alternatives ($\gamma = 1.25, 1.0$).

	$\gamma = 1.4999$	$\gamma = 1.25$	$\gamma = 1.00$
$\alpha = 0.10$	0.11	0.39	0.89
$\alpha = 0.05$	0.08	0.24	0.77

Equation (7) adapted for this model is given by:

$$\begin{aligned} u_{t_i} &= x_{t_i} + \gamma_0 \log(r_{t_i}^2) - \log\Delta - 1.2704 + \eta_{t_i} \\ x_{t_{i+1}} &= x_{t_i} + \omega\sqrt{\Delta}\varepsilon_{2,t_i} \end{aligned}$$

with $\gamma_0 = 1.4999$, η_{t_i} a random variable following the centered density given in Section 2.1 and ε_{2,t_i} distributed as a standard normal.

The null hypothesis $\mathscr{H}_{0v} : \upsilon_1(r_t, \theta) = r_t^{1.4999}$ was tested under the assumption that the drift is completely known and with $\omega = 0.0046$ in the simulated model. For simulations, $\{r_t : 0 \le t \le T\}$ was observed at discrete equally spaced times $t_i = i\Delta, i = 0, 1, \ldots, n = 300$ with $\Delta = 1/52$. The Kolmogorov–Smirnov U_n^{KS} statistic was applied for testing in 100 trials. The distribution of the test under the null was calibrated by the suggested bootstrap resampling with $B = 1000$. For the resampling, the density of η_{t_i} was simulated using a mixture of seven normal densities as described in Kim et al. (1998). Table 1 shows the empirical power for the levels $\alpha = 0.1, 0.05$ obtained for the null and for the alternatives $\upsilon_1(r_t, \theta) = r_t^{\gamma}$ with $\gamma = 1.4999$ (the null) and $\gamma = 1.25$ and $\gamma = 1.0$.

Needless to say that this is a very simple example and more research is necessary about the theoretical and practical behaviour of the different tests and the simulation in more complex models. Even in this simple case, the optimization procedure of the Kalman filter is quite demanding in computing time. That effect is multiplied here by the number of bootstrap replicates. So, a revision of the (possibly high time consuming) steps involved in the procedure (design of Kalman filter, optimization techniques, constraints for the parameters, ...) is required in order to get better calibration levels in a more extensive study.

As far as the real data set is concerned, the interest rate curves of EURIBOR, representing the rates at which different interbank Euro denominated deposits, with distinct maturities, are offered within Eurozone. Such maturities for the *EURIBOR* time series

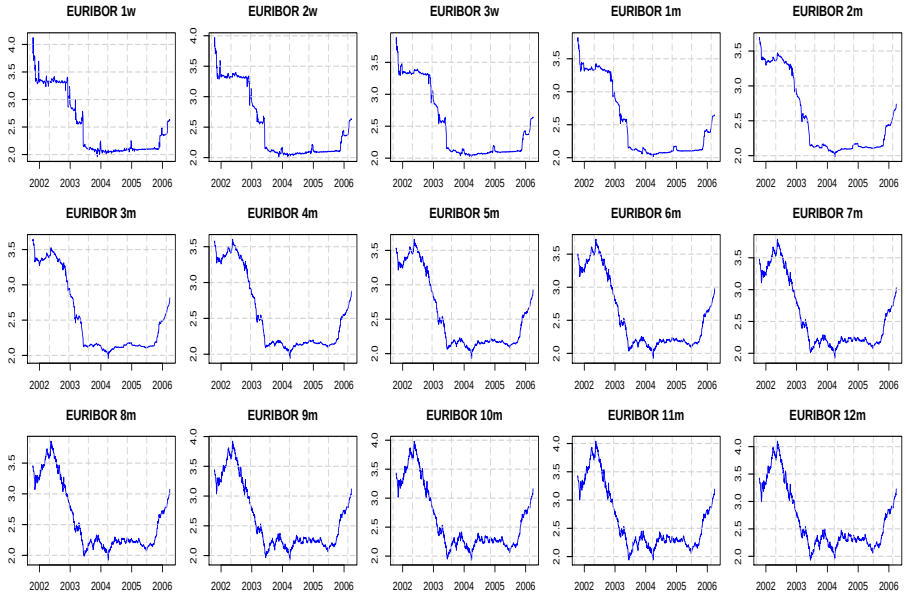

Figure 1: Time series for the interbank deposits in the Eurozone for the period 2001-2006 with maturities of $1,2$, and 3 weeks; and $1,2,\ldots,12$ months.

are $1,2$, and 3 weeks, and $1,2,\ldots,12$ months, being the EURIBOR time frequency a daily scale. For the analysis, the data is divided in two observed time series:

- Previously to the crisis: From October 15th, 2001 till March 31st, 2006
- During the crisis: From January 2nd, 2008 till November 30th, 2011.

Figures 1 and 2 display the graphical evolution of the EURIBOR series during the above mentioned periods. As null hypotheses, for the goodness–of–fit tests for the drift and the volatility, a *CKLS* formulation incorporating the stochastic volatility model proposed in Andersen and Lund (1997) is considered:

$$\begin{aligned} dr_t &= \kappa_1(\mu - r_t)dt + \sigma_t r_t^{\gamma} dW_{1,t} \\ d\log(\sigma_t^2) &= \kappa_2(\alpha - \log(\sigma_t^2))dt + \xi dW_{2t} \end{aligned}$$

where W_{1t} and W_{2t} are independent Brownian motions. The Euler scheme is applied to discretize the model and to obtain the first order approximation

$$\begin{aligned} r_{t_i+1} - r_{t_i} &= \kappa_1(\mu - r_{t_i})\Delta + \sigma_{t_i} r_{t_i}^{\gamma}\sqrt{\Delta}\varepsilon_{1,t_i} \\ \log(\sigma_{t_i+1}^2) - \log(\sigma_{t_i}^2) &= \kappa_2(\alpha - \log(\sigma_{t_i}^2))\Delta + \xi\sqrt{\Delta}\varepsilon_{2,t_i} \end{aligned}$$

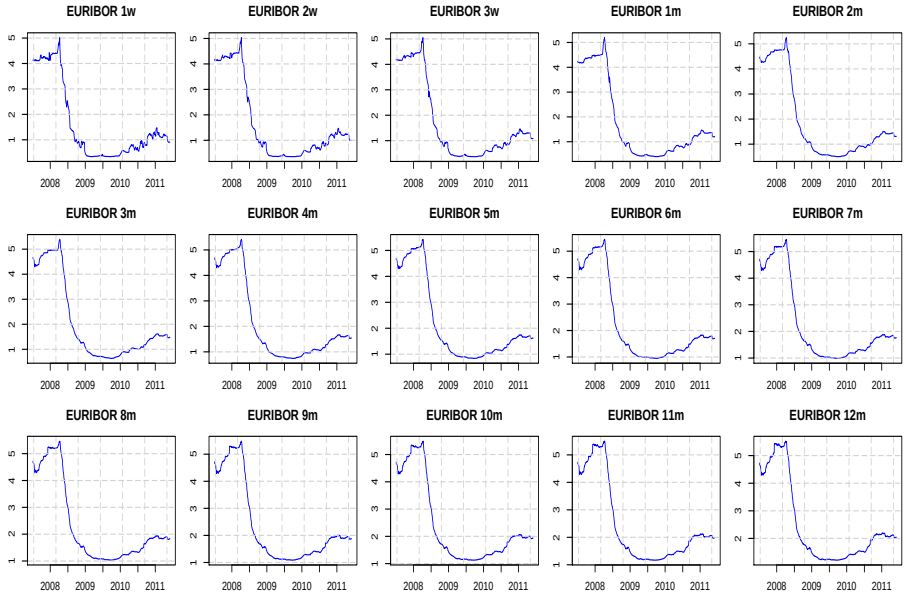

Figure 2: Time series for the interbank deposits in the Eurozone for the period 2008-2011 with maturities of $1,2$, and 3 weeks; and $1,2,\ldots,12$ months.

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independent normal random variables $N(0,1)$ with fixed Δ and weekly frequency (hence, $\Delta = 1/52$). The corresponding general Model (7) is, in this case:

$$\begin{aligned} u_t &= x_t + 2\gamma\log(r_t) - 1.27 + \zeta_t - \log(\Delta) \\ x_t &= \phi_0 + \phi_1 x_{t-1} + \xi\sqrt{\Delta}\varepsilon_{2,t} \end{aligned}$$

where

- $\phi_0 = (1-\kappa_2\Delta)$, $\phi_1 = \kappa_2\alpha\Delta$
- $\zeta_t = \log(\varepsilon_{1,t}^2) + 1.27$
- $u_t = \log(e_t^2)$ and $e_t = Y_t/\Delta - m_1(r_t,\theta) = Y_t/\Delta - \kappa_1(\mu - r_t)$
- $x_t = \log(\sigma_t^2)$.

Based on the above state space model, the proposed tests are applied. Model parameters are estimated by maximum likelihood and Kalman filtering procedures are applied to obtain the non observable variable x_t, required for performing the tests. $B = 500$

Table 2: p-values associated to the goodness–of–fit test for the drift and volatility functions of the stochastic volatility model adjusted to the EURIBOR series *before* the crisis.

	GOF-Drift		GOF-Volatility	
Maturity	$\hat{p}_{value}$	$D_n(r,x)$	$\hat{p}_{value}$	$V_n(r,x)$
1 week	0.036	28.634	0.126	3.717
2 week	0.660	21.603	0.098	2.586
3 week	0.754	27.816	0.984	0.017
1 month	0.752	37.673	0.998	0.003
2 month	0.194	57.085	0.996	0.002
3 month	0.054	51.173	0.990	0.003
4 month	0.186	38.473	0.990	0.004
5 month	0.246	38.926	0.982	0.007
6 month	0.442	33.302	0.990	0.009
7 month	0.238	35.567	0.948	0.013
8 month	0.132	38.118	0.870	0.019
9 month	0.088	38.478	0.766	0.026
10 month	0.050	39.023	0.652	0.033
11 month	0.076	39.375	0.528	0.037
12 month	0.034	42.505	0.440	0.050

bootstrap copies are generated to approximate the distribution of the corresponding processes involved in the tests contruction and estimate the empirical p-values. The results collected in Table 2 and Table 3 were obtained applying the resampling scheme described in Section 4. It can be noted that the p-values associated to the tests for the drift and the volatility of the EURIBOR series, except for a few cases of maturity, *do not* reject the null hypothesis for the periods before the crisis and during the crisis. Therefore, the *CKLS* model, incorporating the volatility factor, is capable of characterizing such series. It is important to emphasize that the *CKLS* model considered in Monsalve-Cobis et al. (2011), without taking into account the stochastic volatility, was rejected in a conclusive way for the volatility component.

In the light of the results, incorporating a stochastic model for the volatility function in a more flexible way seems to allow for a more effective characterization of the EURIBOR series.

Table 3: p-values associated to the goodness–of–fit test for the drift and volatility functions of the stochastic volatility model adjusted to the EURIBOR series *during* the crisis.

	GOF-Drift		GOF-Volatility	
Maturity	$\hat{p}_{value}$	$D_n(r,x)$	$\hat{p}_{value}$	$V_n(r,x)$
1 week	0.222	25.947	0.460	0.198
2 week	0.898	19.519	0.616	0.159
3 week	0.748	23.504	0.402	0.171
1 month	0.996	16.458	0.870	0.129
2 month	0.828	20.524	0.544	0.177
3 month	0.794	19.166	0.596	0.258
4 month	0.476	21.579	0.336	0.310
5 month	0.040	25.021	0.072	0.418
6 month	0.140	27.131	0.096	0.468
7 month	0.436	23.855	0.976	0.013
8 month	0.094	29.190	0.104	0.506
9 month	0.064	28.640	0.066	0.869
10 month	0.056	27.945	0.118	0.656
11 month	0.006	28.735	0.156	0.646
12 month	0.166	26.243	0.098	0.767

References

Aït-Sahalia, Y., 1993. Nonparametric Functional Estimation with Applications to Financial Models. Ph.D. thesis. Department of Economics Massachusetts Institute of Technology.

Aït-Sahalia, Y., Kimmel, R., 2007. Maximum likelihood estimation of stochastic volatility models. Journal of Financial Economics 83, 413–452.

Andersen, T.G., Lund, J., 1997. Estimating continuous-time stochastic volatility models of the short-term interest rate. Journal of Econometrics 77, 343–377.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–54.

Campbell, J.Y., Lo, A.W., MacKinlay, A.C., 1997. The Econometrics of Financial Markets. Princeton University Press.

Chan, K.C., Karolyi, G.A., Longstaff, F.A., Sanders, A.B., 1992. An empirical comparison of alternative models of the short-term interest rate. Journal of Finance 47, 1209–1227.

Dette, H., Podolskij, M., Vetter, M., 2006. Estimation on integrated volatility in continuous-time financial models with applications to goodness-of-fit testing. Scandinavian Journal of Statistics 33, 259–278.

Dette, H., von Lieres und Wilkau, C., 2003. On a test for a parametric form of volatility in continuous time financial models. Finance and Stochastics 7, 363–384.

Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflations. Econometrica 50, 987–1007.

Genon-Catalot, V., Jeantheau, T., Laredo, C., 1999. Parameter estimation for discretely observed stochastic volatility models. Bernoulli 5, 855–872.

Ghyles, E., Harvey, A.C., Renault, E., 1996. Stochastic volatility, in: Statistical Methods in Finance. North Holland, pp. 119–191.

Harvey, A., Ruiz, E., Shephard, N., 1994. Multivariate stochastic variance models. Review of Economic Studies 61, 247–264.

Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatilities. Journal of Finance 42, 281–300.

Iacus, S.M., 2008. Simulation and Inference for stochastics Differencial Equations. Springer.

Kessler, M., 2000. Simple and explicit estimating functions for a discretely observed diffusion process. Scandinavian Journal of Statistics 27, 65–82.

Kim, S., Shephard, N., Chib, S., 1998. Stochastic volatility: likelihood inference and comparison with ARCH models. The Review of Economic Studies 65, 361–393.

Lin, L.C., Lee, S., Guo, M., 2013. Goodness-of-fit test for stochastic volatility models. Journal of Multivariate Analysis 116, 473–498.

Mandelbrot, B., 1963. The variation of certain speculative prices. Journal of Business 36, 394–419.

Markowitz, H., 1952. Portfolio selection. The Journal of Finance 7, 77–91.

Merton, R., 1973. The theory of rational option pricing. Bell Journal of Economics 4, 141–183.

Monsalve-Cobis, A., González-Manteiga, W., Febrero-Bande, M., 2011. Goodness of fit test for interest rates models: an approach based on empirical processes. Computational Statistics & Data Analysis 25, 3073–3092.

Nisticò, S., 2007. Measurement Errors and the Kalman Filter: A Unified Exposition. LL EE Working Document 45. Luiss Lab of European Economics.

Officer, R.R., 1973. The variability of the market factor of the New York Stock Exchange. The Journal of Business 46, 434–453.

Rodriguez, A., Ruiz, E., 2012. Bootstrap prediction mean squared errors of unobserved states based on the Kalman filter with estimated parameters. Computational Statistics & Data Analysis 56, 62 – 74.

Shephard, N., 2005. Stochastic volatility. Working Paper. Nuffield College (University of Oxford).

Shumway, R., Stoffer, D., 2011. Time Series Analysis and Its Applications: With R Examples. Springer Texts in Statistics. 3rd edition.

Stute, W., 1997. Nonparametric model checks for regression. Annals of Statistics 25, 613–641.

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