Analysis of regularization by conjugation for bounded linear operators

Álvarez, A. C.[1], Chapiro, G.[2], García, G. C.[3], and Moreira, C. G. T. A.[1]

[1] IMPA, Rio de Janeiro, RJ 22460-320, Brazil.

[2] Departamento de Matemática, Universidade Federal de Juiz de Fora, Juiz de Fora, MG 36036-900, Brazil.

[3] Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile.

E-mail: meissa98@impa.br,grigori@ice.ufjf.br,galina.garcia@usach.cl, gugu@impa.br

Abstract. The solutions of the linear ill-posed problems are frequently obtained by approximation using the regularization functions. In the present work, bounds of the noise-free part of the regularization error of a certain class of functions are obtained through the norm bound functions that decreasing to zero along the regularization parameter, the so called profile functions. Using the properties of such class and assuming apriori smoothness of the true solution, in terms of sourcewise representations, the profile functions are obtained. We verify that qualification property of the regularization class is a consequence of the properties of such functions. We present a method of construction of a regularization class through conjugation technique by using Julia's functional equation. The conjugation procedure allows incorporating the properties of the operator into the regularization class. Some properties of the Julia's equation solution are obtained which are useful to construct filters for the discrete regularization. Other examples of application consist of generating the classical Tikhonov-Phillips regularization and the non-smooth regularization methods as Landweber and spectral cut-off are embedding in a regularization class by using a mollification process. Numerical examples are presented showing the robustness of the regularization by conjugation developed here.

PACS numbers: 02.30.Zz, 02.30.Sa.

Keywords: Linear ill-posed problems, regularization, functional equation, Julia's equation.

Submitted to: *Inverse Problems*

1. Introduction

We consider the ill-posed problem

$$Kx = y, \tag{1}$$

where K is an injective bounded linear operator between Hilbert spaces X and Y with norm $||\cdot||$ and the range $R(K)$ is not a closed in Y. Then the linear operator equation $Kx = y$ has unique solution $x^\dagger \in \{x \in X : Kx = y\}$ for each $y \in R(K)$ with the property $||x^\dagger|| = \inf\{||x|| : Kx = y\}$, but notice that since $R(K)$ is not closed, then the formal inverse mapping $y \to x = K^{-1}y$ exists for each $y \in R(K)$ and it is discontinuous. This problem is solved by using the Moore-Penrose (generalized) inverse operator $K^\dagger$, see [7] for details. For the bounded linear operator $K : X \to Y$, $K^\dagger$ is defined as the unique linear extension of $\widehat{K}^{-1}$ with the domain $D(K^\dagger) := R(K) + R(K)^\perp$ and kernel $N(K^\dagger) = R(K)^\perp$, where $\widehat{K} := K|_{N(K)^\perp} : N(K)^\perp \to R(K)$, and $N(K) = \{x \in D(K) : Kx = 0\}$. In [7] the uniqueness of the best-approximate solution $x^\dagger = K^\dagger y$ is proved.

Assuming that the right side of Eq. (1) is contaminated with error, we consider $y^\delta \in Y$ instead of y, such that $||y - y^\delta|| \leq \delta$, where the noise level δ is known. In order to obtain approximated solution depending continuously on y^δ it is necessary to apply some sort of regularization. The goal of this paper consists in recovering $x^\dagger$, which is related to the data y^δ by $y^\delta = Kx^\dagger + \delta\xi$, for $||\xi|| \leq 1$ by using an alternative regularization schemes.

The general linear regularization schemes are usually based on a family of piecewise continuous functions $s_\alpha(\tau)$ depending on a parameter $\alpha \in (0, \bar{\alpha}]$, $\tau \in [0, a]$ and $a = ||K^*K||$. Once the regularization s_α is chosen, the regularization method is defined and the regularized solution is obtained by using $x_\alpha^\delta = s_\alpha(K^*K)K^*y^\delta$, where K^* is the adjoint operator of K. The error bound $e(x^\dagger, \alpha, \delta)$ for this approximation satisfies

$$e(x^\dagger, \alpha, \delta) = ||x_\alpha^\delta - x^\dagger|| \leq ||x_\alpha - x^\dagger|| + \delta||s_\alpha(K^*K)K^*||, \tag{2}$$

where $x_\alpha = s_\alpha(K^*K)K^*y$. In the present text, the same framework is used but we assume that $s_\alpha(\tau)$ are of class $C^n[0, a]$ with $n >= 1$. This assumption permits to use the functional equation solutions to built a broad class of regularization schemes. Also, we can show in this case the regularization can be obtained by a conjugation process that improving the behavior of the spectrum of the operator K^*K which is hidden in the analysis of the ill-posed problems.

In [7], a broad theoretical exposition of regularization methods, as well as numerous examples of practical utility are presented. At the same time leaves open the possibility of finding new regularization functions with the properties exposed in such work and that was tackled in several works, see for example [15, 29, 15, 14, 17, 12]. It is because the purposes of our work is to find other alternatives of regularization functions that with certain adjustments include or reproduce the most popular such as Tikhonov-Phillips, Landweber and cut-off regularization methods. In turn, provide us with a constructive method to obtain another kind of these regularization functions. In its turn, the formalism exposed by the new regularization parameter and the property of conjugation leaves open the possibility of comparing each other and classifying them which is open for future work.

In order to simplify the calculations without of generality, the regularization family s_α is defined as $s_\alpha := 1/g_\alpha$ where $g_\alpha \neq 0$). In this notation the residual of the regularization method g_α is denoted by

$$r_{g_\alpha}(\tau) = 1 - \tau s_\alpha(\tau), \tag{3}$$

with $s_\alpha(\tau) = 1/g_\alpha(\tau)$ for $\tau \in [0, a]$, where $\lim_{\tau \to 0}(\tau/g_\alpha(\tau)) < \infty$. Using the spectral decomposition of K^*K it is possible to check that

$$||x_\alpha - x^\dagger|| = ||r_{g_\alpha}(K^*K)x^\dagger||. \tag{4}$$

In order to guarantee the convergence of the approximate solution, the regularization g_α needs to be chosen to control the norm bound of the residual r_α.

Without loss of generality, we substitute the regularization parameter α for an arbitrary index $i \in I$. The convergence of the regularized solution

$$x^\delta_{g_i} = r_{g_i}(K^*K)K^*y^\delta \tag{5}$$

is studied for a sequence of functions $\{g_i\}$ such that $\epsilon_{g_i} \to 0$, where

$$\epsilon_{g_i} = \sqrt{(g_i(0))^2 + (g_i^{'}(0) - 1)^2 + (g_i^{''}(0))^2 + \ldots + (g_i^{(n)}(0))^2}, \tag{6}$$

and $g_i^{(j)}$ denotes the jth derivative of g_i.

Now ϵ_{g_i} plays the role of α from the classical regularization formalism, see [24, 26, 30]. The difference between both formulations consists in that parameter α in the later is used in the construction of the family of regularization functions although ϵ_{g_i} depends on the properties of the regularization family at the origin. One of the advantages of such choice is that it allows to measure different regularization functions.

The properties of regularization class exposed here are similar those exposes in Definition of regularization in [16], but the particular inequality $g_\alpha(\tau) > \gamma_* \sqrt{\alpha}\sqrt{\tau}$ is replaced for a more general condition, i.e. $g_\alpha(\tau) \geq \Psi(\tau)\Theta(\epsilon_{g_\alpha})$ where Ψ and Θ are index functions, i.e. continuous, strictly increasing function that tends to zero when the argument go to zero. These bound index functions allow us to construct some profile functions and to choose an appropriate smoothness for the true solution, which we shall describe in terms of sourcewise representations. Also, this inequality lead to a more general definition of qualification property and to include other examples of regularization which are not studied previously.

In Section 2 the Definition of the regularization class is proposed and it includes classical Tickonov's regularization [30]. The convergence and convergence rate of regularized solutions to the best approximate solution $x^\dagger$ are verified, assuming that $x^\dagger$ satisfy apriori smoother condition similar to ones presented in [7, 16]. The qualification property of this regularized method is also addressed. The convergence of such method is optimal and based on the topological properties on the functions of the regularization class.

The continuously differentiable hypothesis in the Definition of regularization class can be relaxed for embedding some kind of piece continuous regularization functions into a regularization class by means a mollification process, which is studied in Section 3.

Section 4 is developed a method for the construction of the presented class of regularization functions through conjugation using Julia's functional equation [18, 19]. This type of approach reminds the Dynamical System Method (DSM) explained in [27], which proposes to solve the ill-posed problem by rewriting it in the equivalent form that can be studied by using dynamical systems techniques.

Although there is a notable relation between the behavior of the eigenvalues close to the accumulation point and dynamical systems there is a lack of literature addressing this issue applied to the functional equations with ill-posed problems. The present article addresses this problem relating the functional equation theory and the regularized solutions of the operational equation (1). To do so, we study a family of equivalent operational equation $g(K)x = y$, where g is a function belonging to a regularization class, such that the corresponding solutions converge to the best approximate solution when the function g approximates identity. The function g is obtained though a process of conjugation with a diffemeorphism. This approach was used a long time ago (see [1, 5]) in similar problems to understand the Newton's method using conjugation (see [2]) by studying the associated discrete dynamical system in new coordinates. It was also used to study the normal form and spectrum of matrices [6] and more recently in [20].

In Section 5 a numerical examples is presented. We show how to use the regularization class to solve a discrete version of a linear compact operator equation. Although the proof that solution g of the Julia's equation $g(D(\tau)) = D'(\tau)g(\tau)$ for a given D with some properties constitutes a regularization class is quite long and technical, the numerical calculation itself is simple and depends only on the selection of a function D satisfying a set of properties easily verifiable. Therefore, regularization by conjugation offers a practical and robust method to obtain regularization functions.

2. Regularization class

Using the presented results, we prove the main tool of this work that enables to obtain a regularized solution to the operational equation with a bounded linear operator. As in [16], we use the following definition of index function.

Definition 1. *A real function $\phi(t)$, with $(0 < t \leq \bar{t}\,)$ is called an index function if it is continuous, strictly increasing and satisfies the condition* $\lim_{t\to 0+} \phi(t) = 0$.

Definition 2. *Define the family of functions $\mathcal{F}$ as a union of the null function with the subset of index functions $\{\Phi : [0, \bar{\epsilon}] \to \mathbb{R}\}$ such that $\epsilon/\Phi(\epsilon)$ is an index function and* *there exist constants $\beta_1 \leq 1$, $\beta_2 \geq 1$, $r_1 \geq 0$, $r_2 \geq 0$, $\bar{\epsilon} > 0$ such that $r_1\epsilon^{\beta_1} \leq \Phi(\epsilon) \leq r_2\epsilon^{\beta_2}$ for $\epsilon \in [0, \bar{\epsilon}]$.*

The following set of auxiliary functions is defined in order to include different types of regularization classes in a single formalism.

Definition 3. *Define the auxiliary set of functions $\mathcal{G}_a = \{g \in C^n([0,a], \mathbb{R})\}$ with ϵ_g defined in Eq. (6) satisfying the properties*

(a_1) *$\exists\ Q > 0$ and $n \geq 1$, such that $\forall g \in \mathcal{G}_a$, $|g^{(n)}(\tau)| \leq Q\epsilon_g$, $\forall \tau \in [0,a]$; and there exists at least one derivative $g^{(m)}(0) \neq 0$ for $1 \leq m \leq n$.*

(a_2) *$\exists R_1$ constant, such that $\epsilon_g \leq R_1$ for all $g \in \mathcal{G}_a$.*

(a_3) *$g(\tau) > 0$ for $\tau \neq 0$ and $g(\tau)$ is a non-decreasing function.*

(a_4) *$\exists R_2$ constant, such that $|\tau/g(\tau)| < R_2$, $\forall \tau \in [0,a]$, $\forall g \in \mathcal{G}_a$ and $\lim_{\tau\to 0}(\tau/g(\tau)) < \infty$.*

The main regularization class is defined using previous set of functions as follows.

Definition 4. *Given $a \in \mathbb{R}$ and ϵ_g defined in Eq. (6) the regularization class $\mathbb{G}_a$ is defined as $\mathbb{G}_a = \{\bar{g}(\tau) \in C^n([0,a], \mathbb{R});\ \bar{g}(0) \neq 0;\ \bar{g}(\tau) = g(\tau) + \Phi(\epsilon_g);\ g \in \mathcal{G}_a;\ \Phi \in \mathcal{F}\}$, satisfying the property*

i) *There exist index functions Ψ and Θ, such that $\epsilon/\Theta(\epsilon)$ is also an index function for $\epsilon \in (0, \bar{\epsilon}]$ and $\bar{g}(\tau) \geq \Psi(\tau)\Theta(\epsilon_g)$, $\tau \in (0,a]$.*

The next proposition follows immediately from the definition.

Proposition 2.1. *The functions $\bar{g}$ in the regularization class $\mathbb{G}_a$ satisfy*

ii) *$\exists\ Q > 0$ and $n \geq 1$, such that $\forall \bar{g} \in \mathbb{G}_a$, $|\bar{g}^{(n)}(\tau)| \leq Q\epsilon_g$ and there exists at least one derivative $\bar{g}^{(m)}(0) \neq 0$ for $1 \leq m \leq n$.*

iii) *$\exists R_1$ constant, such that $\epsilon_g \leq R_1$ for all $\bar{g} \in \mathcal{G}_a$.*

iv) *$\bar{g}(\tau) > 0$ for $\tau \neq 0$ and $\bar{g}(\tau)$ is non-decreasing function.*

v) *$\exists R_2$ constant, such that $|\tau/\bar{g}(\tau)| < R_2$ for all $\bar{g} \in \mathbb{G}_a$ and $\lim_{\tau\to 0}(\tau/\bar{g}(\tau)) < \infty$.*

Remark 1. *The application $\epsilon_g : \mathcal{G}_a \to [0, \bar{\epsilon}]$ defined by Eq. (6) is surjective. Consider some increasing continuous differentiable function $\upsilon(\alpha)$ going onto the interval $[0, \bar{\epsilon}]$. Notice that the function $h_\alpha(\tau) = \tau + \upsilon(\alpha)$ is of class $\mathbb{G}$ and ϵ_h assumes all values in the interval $[0, \bar{\epsilon}]$.*

The following lemma is required to prove convergence of the regularized solutions.

Lemma 2.2. *Let K be a bounded linear operator. Consider $n \in \mathbb{N}$, $n \geq 1$, $a = ||K^*K||$ and the function $\bar{g} \in \mathbb{G}_a$, with $\bar{g} = g + \Phi$. The error $r_{\bar{g}}$ in (3) satisfies the following conditions*

(a) If Φ is not a null function then there exists a constant M_1 such that

$$|r_{\bar{g}}(\tau)| \leq M_1 \frac{\epsilon_g}{\Phi(\epsilon_g)}, \quad \textit{for}\ \ \tau \in (0,a]. \tag{7}$$

(b) The error $r_{\bar{g}}$ is uniformly bounded in $\mathbb{G}_a$.

(c) For Θ and Ψ from Def. 4 there exists a constant M_2 such that

$$|r_{\bar{g}}(\tau)| \leq M_2 \frac{\epsilon_g}{\Theta(\epsilon_g)} \frac{1}{\Psi(\tau)}, \quad \textit{for } \tau \in (0, a]. \tag{8}$$

Proof. Let $\bar{g} = g + \Phi(\epsilon_g) \in \mathbb{G}_a$. Notice that

$$|r_{\bar{g}}(\tau)| = \left|1 - \frac{\tau}{\bar{g}(\tau)}\right| = \left|\frac{\bar{g}(\tau) - \tau}{\bar{g}(\tau)}\right|. \tag{9}$$

Using Taylor's formula it follows that

$$g(\tau) = g(0) + \frac{g^{'}(0)}{1!}\tau + \frac{g^{''}(0)}{2!}\tau^2 + \ldots + \frac{g^{(n-1)}(0)}{(n-1)!}\tau^{n-1} + \frac{g^{(n)}(\xi)}{(n)!}\tau^n, \tag{10}$$

for some $\xi \in (0, a)$. Substituting (10) into (9) yields

$$\begin{aligned} |r_{\bar{g}}(\tau)| = \frac{1}{|g(\tau) + \Phi(\epsilon_g)|} \times \\ \left|\Phi(\epsilon_g) + g(0) + (g^{'}(0) - 1)\tau + \frac{g^{''}(0)}{2!}\tau^2 + \ldots + \frac{g^{(n-1)}(0)}{(n-1)!}\tau^{n-1} + \frac{g^{(n)}(\xi)}{(n)!}\tau^n\right|. \end{aligned} \tag{11}$$

From (ii) and (iv) of Proposition 2.1 and $|g(0)| \leq \epsilon_g$, $|g'(0) - 1| \leq \epsilon_g$, $|g^{(j)}(0)| \leq \epsilon_g$, with $j = 2, \ldots, n-1$ for $\tau \leq a$, it follows that

$$|r_{\bar{g}}(\tau)| \leq \frac{1}{\Phi(\epsilon_g)} \left(|\Phi(\epsilon_g)| + \epsilon_g \left(1 + a + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!} + Q\frac{a^n}{(n)!}\right)\right). \tag{12}$$

Using from Def. 2 that $\Phi(\tau) \leq r_2\tau^{\beta_2}$ (i.e. $\Phi \in \mathcal{F}$) inequality (12) can be rewritten as

$$|r_{\bar{g}}(\tau)| \leq \frac{\epsilon_g}{|\Phi(\epsilon_g)|} \left(r_2\epsilon_g^{\beta_2 - 1} + 1 + a + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!} + Q\frac{a^n}{(n)!}\right). \tag{13}$$

Taking $M_1 = (r_2 R_1^{\beta_2 - 1} + 1 + a + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!} + Q\frac{a^n}{(n)!})$ we obtain (a). Item (b) follows from the item (v) of Prop. 2.1. In order to prove item (c) substitute (i) from Def. 4 into (11) obtaining

$$|r_h(\tau)| \leq \frac{1}{|\Psi(\tau)\Theta(\epsilon_g)|} \left(|\Phi(\epsilon_g)| + \epsilon_g \left(1 + a + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!} + Q\frac{a^n}{(n)!}\right)\right). \tag{14}$$

Using Def. 2 again, Eq. (8) follows. □

Remark 2. *It is possible to check that Lemma 2.2 still valid if instead of to use the condition $|g^{(n)}(\tau)| \leq Q\epsilon_g$, $\forall \tau \in [0, a]$ we use that the following condition: there exist a constant $Q_1 > 0$ such that $|(g^{(n)}(\xi)\tau^n)|/(n!) \leq Q_1\epsilon_g$ holds, $\forall \xi < \tau$ and $\tau \in (0, a]$. This condition will be used when we study the mollification of non-smooth regularization functions in Section 3.*

In order to prove the convergence of the regularized solutions to $x^\dagger$ and to evaluate the convergence rate it is necessary to introduce the concepts of qualification and smoothness. In this work it is used a more general concept of qualification than that found in the literature, see [7, 16, 21, 22, 23].

Definition 5. *Let Ψ and Π be two index functions on the interval $(0,a]$. A function $\bar{g} = g + \Phi(\epsilon_g)$ of the regularization class $\mathbb{G}_a$ has* **qualification** (Ψ, Π) *with constant $\gamma > 0$ if the error $r_{\bar{g}}$ defined in Eq.* (3) *satisfies*

$$\sup_{0<\tau\leq a} |r_{\bar{g}}(\tau)|\Psi(\tau) \leq \gamma\Pi(\epsilon_g). \tag{15}$$

Notice that from the items (a) and (c) of Lemma 2.2 follows that functions of class $\mathbb{G}_a$ have qualifications $(\tau^\beta, \epsilon_g/\Phi(\epsilon_g))$ with $0 < \beta < 1$ and $(\Psi, \epsilon_g/\Theta(\epsilon_g))$. Substituting ϵ_g for the parameter α and considering $\Pi = \Psi$ the above definition is equivalent to one presented in [16]. Following [16] we introduce the definitions:

Definition 6. *Given constant $c > 0$, bounded linear operator between Hilbert spaces $K : X \to Y$ and index function Λ the regularized solution possesses* **smoothness property** *when $x^\dagger \in H_\Lambda(c) = \{x \in X : x = \Lambda(K^*K)w, ||w|| \leq c\}$.*

For $M \subset H_\Lambda(c)$ and $\bar{g} \in \mathbb{G}_a$ an index function $F : (0, c] \to (0,\infty)$ (see [16] for details) is called **profile function** *for $(M, \bar{g})$ when*

$$\sup_{x\in M} ||r_{\bar{g}}(K^*K)x|| \leq F(\epsilon_g). \tag{16}$$

Lemma 2.3. *Let $K : X \to Y$ be a bounded linear operator between Hilbert spaces X and Y. If $\bar{g} \in \mathbb{G}_a$ with $a = ||K^*K||$ then $||\bar{s}(K^*K)K^*|| \leq \mu/\Psi(\epsilon_g)$ with $\bar{s} := 1/\bar{g}$, for some constant μ and some index function Ψ.*

Proof. By the spectral theory, see [7], we have

$$||\bar{s}(K^*K)K^*y||^2 = \int_0^a (\tau/(g(\tau) + \Phi(\epsilon_g)))^2 d||E_\lambda y||^2, \tag{17}$$

where $y \in R(K)$. Using (i) from Def. 4 we obtain

$$||\bar{s}(K^*K)K^*y||^2 \leq \sup_{\tau\in(0,a]} \{(\tau/\Theta(\tau))^2\}(1/\Psi(\epsilon_g))^2||y||^2. \tag{18}$$

Taking $\mu = a\sqrt{\sup_{\tau\in(0,a]}\{(\tau/\Theta(\tau))^2\}}$, that is limited because of Def. 4 (i), the result follows. □

Next we prove the main result about the convergence of the regularized solutions.

Lemma 2.4. *Let $K : X \to Y$ be a bounded linear operator between Hilbert spaces X and Y. Let $\bar{g} \in \mathbb{G}_a$, with $a = ||K^*K||$. If F is a profile function for $(M, \bar{g})$, where M is some compact subset of $H_\Lambda(c)$ for some constant c. Then the error (defined in Eq. (2)) satisfies*

$$e(x^\dagger, \epsilon_{g^*}, \delta) \leq (1 + \mu)F(\epsilon_{g^*}) \tag{19}$$

for certain $\bar{g}^* \in \mathbb{G}_a$. This implies that it is possible to construct the sequence of regularized solutions converging to $x^\dagger$ as in Eq. (5).

Proof. The proof proceeds in a similar way as in [16]. From Lemma 2.3 it follows

$$e(x^\dagger, \epsilon_g, \delta) \leq F(\epsilon_g) + \delta\mu(1/\Psi(\epsilon_g)). \tag{20}$$

Take the index function $\Omega(\tau) = \Psi(\tau)F(\tau)$ and choose the function $g^* \in \mathbb{G}_a$ such that $\epsilon_{g^*} = \Omega^{-1}(\delta)$. The existence of such $\bar{g}^*$ is guaranteed by Remark 1. Substituting δ into Eq. (20) yields Eq. (19). The result about convergence follows. □

One sequence of regularized solutions converging to $x^\dagger$ is given in

Theorem 2.5. *Consider a bounded linear operator $K : X \to Y$ between Hilbert spaces and the sequence $(\bar{g}_i)$ in the regularization class $\mathbb{G}_a$ from Def. 4. Here $a = ||K^*K||$, $i \in I$, where I denotes some set of indexes. Let $y \in R(K)$ and $\bar{s}_i := 1/\bar{g}_i$, $x_{\bar{g}_i} = (\bar{s}_i(K^*K)K^*)y$, $x^\delta_{\bar{g}_i} = (\bar{s}_i(K^*K)K^*)y^\delta$, with $||y - y^\delta|| < \delta$. Let $x^\dagger \in H_\Psi(R)$ as in Def. 6, where the index function Ψ is given in Def. 4(i) and ϵ_{g_i} defined in Eq. (6). Then there exists an index function F such that*

$$||x_{\bar{g}_i} - x^\dagger|| < F(\epsilon_{g_i}), \tag{21}$$

with $e(x^\dagger, \epsilon_{g_i}, \delta)$ from Eq. (2) satisfying

$$e(x^\dagger, \epsilon_{g_i^*}, \delta) \leq (1 + \mu)F(\epsilon_{g_i^*}), \tag{22}$$

where $\epsilon_{g_i^*} = \Omega^{-1}(\delta)$ for some $\bar{g}_i^* \in \mathbb{G}_a$ with $\Omega(\tau) = \Psi(\tau)F(\tau)$.

Proof. Notice that $x_{\bar{g}_i} - x^\dagger = (\bar{s}_i(K^*K)K^*)y - x^\dagger = (\bar{s}_i(K^*K)K^*K)x^\dagger - x^\dagger = (\bar{s}_i(K^*K)K^*K - I)x^\dagger = (\bar{s}_i(K^*K)K^*K - I)\Psi(K^*K)w$, with $||w|| \leq R$. Using spectral theory, see [7], we obtain

$$||x_{\bar{g}_i} - x^\dagger||^2 = \int_0^{||K||^2} \left(1 - \frac{\lambda}{\bar{g}_i(\lambda)}\right)^2 \Psi^2(\lambda) d||E_\lambda w||^2 = \int_0^{||K||^2} r^2_{\bar{g}_i}(\lambda)\Psi^2(\lambda) d||E_\lambda w||^2.$$

From inequality (8) in Lemma 2.2, we have

$$||x_{\bar{g}_i} - x^\dagger|| \leq M_2(\epsilon_{g_i}/\Theta(\epsilon_{g_i})) \int_0^{||K||^2} d||E_\lambda w||^2 := F(\epsilon_{g_i}). \tag{23}$$

Notice that (from Def. 4-i) $\epsilon_{g_i}/\Theta(\epsilon_{g_i})$ is an index function following that $F(\epsilon_{g_i})$ defined in Eq. (23) is a profile function. Finally, by using (21) in Lemma 2.4 follows (22), where $\epsilon_{g_i^*} = \Omega^{-1}(\delta)$ for some $\bar{g}_i^* \in \mathbb{G}_a$ with $\Omega(\tau) = \Psi(\tau)F(\tau)$. □

Notice that Eq. (22) guarantees that the error is bounded by $F(\Omega^{-1}(\delta))$ leading to the optimal rate of convergence in the sense of [9], i.e. the regularization functions $x^\delta_{\bar{g}_i}$ to $x^\dagger$ as $\delta \to 0$.

2.1. Properties of the regularization class $\mathbb{G}_a$

From Definition 5 and inequality (8) in Lemma 2.2 it follows that the family $\bar{g}_i \in \mathbb{G}_a$ $(a = ||K^*K||)$ possesses qualification (Ψ, F) with Ψ given in Def. 4-(i) and F defined in Eq. (23).

It is possible to prove Theorem 2.5 using (7) instead of (8) and different smoothness condition $x^\dagger = (K^*K)^\beta w$, with $||w|| \leq R$ for the parameter $0 < \beta < 1$. This is equivalent to consider the function $\Psi(\tau) = \tau^\beta$ for $\tau \in [0, a]$, with $0 < \beta < 1$ in the proof of Theorem 2.5. The corresponding profile function will be $F(\epsilon_g) = M_3\epsilon_g/\Phi(\epsilon_g)$ resulting in different convergence rate.

Another important property of the regularization class is *compactness* in the following sense: The set of continuous differentiable functions satisfying condition (a1), (a2) and (a3) of Definition 3 constitute a compact subset of $C^n([0, a], \mathbb{R})$ with the infinite norm. This is a consequence of the Helly's Theorem (see [13], page 45). Also, the limit of sequence in $\mathbb{G}_a$ satisfy condition (i) of Definition 4 and (a4) of Definition 3.

The class $\mathbb{G}_a$ $(a = ||K^*K||)$ is not empty. One possible construction is presented in Section 4 by using a solution of a Julia's functional equation.

3. About the hypotheses of continuous differentiable

Def. 4 assumed that the regularization functions are continuously differentiable up to order $n \geq 1$. In this section, the regularization class will be enriched with some piecewise continuous regularization functions by means of mollification process. This procedure allows to include the regularization functions described in the literature, see [7, 22]. Following [8] the following $C^\infty(\mathbb{R})$ mollifiers are defined:

$$\eta(x) = \begin{cases} C \exp\left\{\dfrac{1}{x^2 - 1}\right\} & |x| < 1; \\ 0 & |x| \geq 1, \end{cases} \qquad \eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right), \tag{24}$$

where the constant C is such that $\int_{-\infty}^{+\infty} \eta(x)dx = \int_{-\infty}^{+\infty} \eta_\epsilon(x)dx = 1$ and the support of η_ϵ is the closed ball $B[0, \epsilon] = \{x \in \mathbb{R} : |x| \leq \epsilon\}$.

Definition 7. *For $\epsilon > 0$ the* **ϵ-mollification** *of a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ is the convolution $f^\epsilon = \eta_\epsilon * f$ on $\mathbb{R}$ given by $f^\epsilon(x) = \int_{-\infty}^{\infty} \eta_\epsilon(x - s)f(s)ds$, for all $x \in \mathbb{R}$.*

Theorem 3.1 (See [8])**.** *(a) For all $\epsilon > 0$, the ϵ-mollification f^ϵ is $C^\infty(\mathbb{R})$ and its nth derivative is given by $(f^\epsilon)^{(n)}(x) = \int_{-\infty}^{\infty} \eta_\epsilon^{(n)}(x - s)f(s)ds$, for all $x \in \mathbb{R}$.*
(b) If f is $C^0(\mathbb{R})$, then f^ϵ converges to f uniformly on compact sets for $\epsilon \to 0^+$.

The next definition of the regularization functions is similar to the one used in [7, 16], it is adjusted to the presented formalism by working only with positive functions.

Definition 8. *A family of positive non-decreasing $g_\alpha(\tau)$ $(0 \leq \tau \leq a)$, defined for parameters $0 \leq \alpha \leq \bar{\alpha}$, is called* **regularization** *if they are piecewise continuous in α and the following properties are valid:*

(i) For each $\tau \in [0, a]$, $|r_\alpha(\tau)| \to 0$ *as* $\alpha \to 0$, *where* $r_\alpha(\tau) = 1 - \tau/g_\alpha(\tau)$.
(ii) There exist constants γ_1 *and* γ_2, *such that* $|r_\alpha(\tau)| \leq \gamma_1$ *and* $|g_\alpha(\tau)| \leq \gamma_2$ *for all* $\alpha \in [0, \bar{\alpha}]$ *and for all* $\tau \in [0, a]$.
(iii) There exists a constant γ_*, *such that* $g_\alpha(\tau) \geq \gamma_* \sqrt{\alpha}\sqrt{\tau}$ *for all* $\alpha \in [0, \bar{\alpha}]$.

The following Lemma (proved in Section 3.1) summarize the properties of the mollifier of the regularization function in Def. 8.

Lemma 3.2. *Let* g_α *be a regularization as in Def. 8, such that* $g_\alpha(\tau) \to \tau$ *when* $\alpha \to 0$ *and*

$$\bar{g}_\alpha(\tau) = \begin{cases} 0 & \tau > a; \\ g_\alpha(\tau) & \tau \in [0, a]; \\ -g_\alpha(-\tau) & \tau \in [-a, 0); \\ 0 & \tau < -a. \end{cases} \tag{25}$$

Then the functions given by

$$f_\alpha^\epsilon(\tau) = \eta_\epsilon * \bar{g}_\alpha(\tau) = \int_{-\epsilon}^{\epsilon} \eta_\epsilon(\tau - s)\bar{g}_\alpha(s)ds, \tag{26}$$

for each fixed $\epsilon > 0$ *satisfy*
i) $\lim_{\alpha\to 0} f_\alpha^\epsilon(\tau) = \tau$, *ii)* $\lim_{\alpha \to 0}(f_\alpha^\epsilon)'(\tau) = \int_{-\epsilon}^{+\epsilon} \eta'_\epsilon(\tau - s)sds = 1$,
iii) For $n \geq 2$, $\lim_{\alpha\to 0}(f_\alpha^\epsilon)^{(n)}(\tau) = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(n)}(\tau - s)sds = 0$.

Remark 3. *From Lemma 3.2 it is possible to deduce that* $\epsilon_{f_\alpha^\epsilon}$, *defined in equation (6), tends to zero when* $\alpha \to 0$.

The following Lemma is used to find mollification parameters.

Lemma 3.3. *Consider the function* $f \in C^1[0, \bar{\epsilon}]$ *such that,* $f(0) = 0$ *and* $|f'| < N < 1$ *for some constant* N. *Let* $f_\alpha^\epsilon(\tau)$ *as in (26),* g_α *from Def. 8,* $\bar{g}_\alpha(\tau)$ *as in (25) such that,*

$$ess \sup_{z \in [0,a]} |\bar{g}_\alpha(-\epsilon_1 z) - \bar{g}_\alpha(-\epsilon_2 z)| \leq L_\alpha |\epsilon_1 - \epsilon_2|, \tag{27}$$

for any $\epsilon_1, \epsilon_2 \in (0, \bar{\epsilon}]$ *and some positive constant* L_α. *Let* $\gamma = (1 + L_\alpha)^{-1}$. *Then the equation*

$$\sigma(\alpha, \epsilon) := \gamma f f_\alpha^\epsilon(0) = \epsilon, \tag{28}$$

possesses a unique fixed point $\epsilon^* = \epsilon^*(\alpha)$ *for each* α, *i.e.* $\sigma(\alpha, \epsilon^*) = \epsilon^*$.

Lemma 3.4. *Let* $f_\alpha^\epsilon(\tau)$ *as in (26),* g_α *from Def. 8,* $\bar{g}_\alpha(\tau)$ *as in (25) such that,*

$$ess \sup_{\tau \in [0,a]} |\bar{g}_{\alpha_1}(\tau) - \bar{g}_{\alpha_2}(\tau)| \leq L|\alpha_1 - \alpha_2|, \tag{29}$$

for any $\alpha_1, \alpha_2 \in (0, \bar{\alpha}]$ *and some positive constant* L. *Given* ϵ, *let us define*

$$\chi_j = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(j)}(-s)ds, \quad L_\epsilon = 2L|\chi_1| + 2L\gamma_2 \sum_{j=0}^{n} |\chi_j|^2, \quad \gamma_\epsilon = (1 + L_\epsilon)^{-1}. \tag{30}$$

Then, for each ϵ, the equation

$$\gamma_\epsilon(\epsilon_{f_\alpha^\epsilon})^2 = \alpha, \tag{31}$$

possesses a unique fixed point $\alpha^* = \alpha^*(\epsilon)$ with $\epsilon_{f_\alpha^\epsilon}$ defined in (6).

Next theorem proves that it is possible to construct a regularization class from regularization functions given in Def. 8.

Theorem 3.5. *Under the hypotheses of Lemmas 3.3 and 3.4 there exists $\epsilon(\alpha)$, such that the family of functions $f_\alpha = \eta_{\epsilon(\alpha)} * \bar{g}_\alpha$ belongs to the regularization class $\mathbb{G}_a$ in the sense that it satisfies conditions (i) in Def. 4 and $(a1) - (a4)$ in Def. 3.*

3.1. Technical proofs

Proof. (Lemma 3.2) For fixed $\epsilon > 0$, using Dominated Convergence Theorem, follows that

$$\lim_{\alpha\to 0} f_\alpha^\epsilon(\tau) = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon(\tau - s)sds = \int_{-1}^{+1} \eta_1(z)(\tau - \epsilon z)dz. \tag{32}$$

Since $\eta_1(z)z$ is an odd function yields $\lim_{\alpha\to 0} f_\alpha^\epsilon(\tau) = \tau$. In a similar way is obtained

$$\lim_{\alpha\to 0}(f_\alpha^\epsilon)'(\tau) = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon'(\tau - s)sds = 1, \tag{33}$$

and for $n \geq 2$, it follows that

$$\lim_{\alpha\to 0}(f_\alpha^\epsilon)^{(n)}(\tau) = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(n)}(\tau - s)g_\alpha(s)ds = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(n)}(\tau - s)sds = 0. \tag{34}$$

□

Proof. (Lemma 3.3) Changing variables $z = (\tau - s)/\epsilon$, Eq. (26) can be rewritten as

$$f_\alpha^\epsilon(\tau) = \eta_\epsilon * \bar{g}_\alpha(\tau) = \int_{-1}^{+1} \eta(z)\bar{g}_\alpha(\tau - \epsilon z)dz, \tag{35}$$

The function $\sigma(\alpha, \epsilon)$ is a contraction for each α. Indeed, consider ϵ_1, ϵ_2 in $[0, \bar{\epsilon}]$, then using (35), (28) and applying the Mean Value Theorem to f results in

$$|\sigma(\alpha, \epsilon_2) - \sigma(\alpha, \epsilon_1)| < \gamma N \int_{-1}^{+1} \eta(z)|\bar{g}_\alpha(-\epsilon_2 z) - \bar{g}_\alpha(-\epsilon_1 z)|dz. \tag{36}$$

Substituting Eq. (27) into (36) and using $\int_{-1}^{+1} \eta(s)ds = 1$ follows that

$$|\sigma(\alpha, \epsilon_2) - \sigma(\alpha, \epsilon_1)| \leq (L_\alpha \gamma |f'|)\, |\epsilon_2 - \epsilon_1|. \tag{37}$$

Substituting γ yields that σ is a contraction and the result follows from the Fixed Point Theorem (see [32]). □

Proof. (Lemma 3.4) Consider $j = 1, \ldots, n-1$. Since $\bar{g}_\alpha$ are uniformly bounded almost everywhere, applying the Dominated Convergence Theorem on the integrals

$$f_\alpha^\epsilon(0) = \eta_\epsilon * \bar{g}_\alpha(0) = \int_{-1}^{+1} \eta(z)\bar{g}_\alpha(-\epsilon z)dz, \tag{38}$$

and

$$(f_\alpha^\epsilon)^{(j)}(0) = \eta_\epsilon^{(j)} * \bar{g}_\alpha(0) = \int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(j)}(-s)\bar{g}_\alpha(s)ds, \tag{39}$$

we obtain that $f_\alpha^\epsilon(0)$ and $(f_\alpha^\epsilon)^{(j)}(0)$ are continuous functions in α. Let us define the functions $\varphi_j : [0, \bar{\alpha}] \to \mathbb{R}$ as $\varphi_0(\alpha) := f_\alpha^\epsilon(0)$, $\varphi_1(\alpha) := (f_\alpha^\epsilon)'(0) - 1$ and $\varphi_j(\alpha) := (f_\alpha^\epsilon)^{(j)}(0)$ for $j > 1$. It follows that

$$(\epsilon_{f_\alpha^\epsilon})^2 = \sum_{j=0}^{n} \varphi_j^2(\alpha). \tag{40}$$

Notice that for $j \neq 1$ $|\varphi_j(\alpha_1) + \varphi_j(\alpha_2)| \leq 2|\chi_j|\gamma_2$, where χ_j is given in (30) and $|\varphi_1(\alpha_1) + \varphi_1(\alpha_2)| \leq (2 + 2|\chi_1|\gamma_2)$. Using (29) and (38)–(39) yields

$$|\varphi_j(\alpha_1) - \varphi_j(\alpha_2)| \leq |\chi_j| \operatorname*{ess\,sup}_{\tau \in [0,a]} |\bar{g}_{\alpha_1}(\tau) - \bar{g}_{\alpha_2}(\tau)| \leq L|\chi_j||\alpha_1 - \alpha_2|. \tag{41}$$

It follows that $|\varphi_j^2(\alpha_1) - \varphi_j^2(\alpha_2)| \leq |\varphi_j(\alpha_1) + \varphi_j(\alpha_2)||\varphi_j(\alpha_1) - \varphi_j(\alpha_2)|$ yielding

$$|(\epsilon_{f_{\alpha_1}^\epsilon})^2 - (\epsilon_{f_{\alpha_2}^\epsilon})^2| \leq \left(2L|\chi_1| + 2L\gamma_2 \sum_{j=0}^{n} |\chi_j|^2\right) |\alpha_1 - \alpha_2| \leq L_\epsilon \alpha_1 - \alpha_2|, \tag{42}$$

where L_ϵ was given in (30). Taking $\gamma_\epsilon = (1 + L_\epsilon)^{-1}$ we obtain from (42) that $\gamma_\epsilon(\epsilon_{f_\alpha^\epsilon})^2$ is a contraction and the result follows from the Fixed Point Theorem. □

Next we prove that it is possible to construct the regularization class from regularization functions given in Def. 8.

Proof. (Theorem 3.5) **[Def. 4 (i)]** Consider $\epsilon = \epsilon(\alpha) > 0$ and $\bar{g}_\alpha$ as in (25). After changing variable $t = -s$ it follows that

$$f_\alpha(\tau) = \int_\epsilon^0 \eta_\epsilon(\tau + t)g_\alpha(t)dt + \int_0^\epsilon \eta_\epsilon(\tau - s)g_\alpha(s)ds \geq \gamma^* \sqrt{\alpha} H(\tau, \epsilon), \tag{43}$$

where (iii) from Def. 8 was used in the last inequality and the function $H(\tau, \epsilon) = \int_{+\epsilon}^0 \eta_\epsilon(\tau + t)\sqrt{t}dt + \int_0^{+\epsilon} \eta_\epsilon(\tau - s)\sqrt{s}ds$ is an index function. Notice that $H(0, \epsilon) = 0$, and

$$\frac{\partial H(\tau, \epsilon)}{\partial \tau} = -\int_0^{+\epsilon} \frac{\partial}{\partial \tau}(\eta_\epsilon(\tau + s) - \eta_\epsilon(\tau - s))\sqrt{s}ds = -\int_0^{+\epsilon} \frac{\partial^2 \eta_\epsilon}{\partial \tau^2}(\theta)\, 2s\sqrt{s}ds, \tag{44}$$

where the Mean Value Theorem (see [28]) was used with $\theta \in [\tau - s, \tau + s]$. From (24) it follows that $\partial_\tau^2 \eta(\theta) < 0$, then $\partial_\tau^2 H(\tau, \epsilon) > 0$ and $H(\tau, \epsilon)$ is an index function. Using (31) in (43) yields

$$f_\alpha(\tau) > \gamma^* \sqrt{\gamma_\epsilon}(\epsilon_{f_\alpha^\epsilon}) H(\tau, \epsilon). \tag{45}$$

Finally, taking the index functions $\Theta(\epsilon) = \gamma^* \sqrt{\gamma_\epsilon}\epsilon$ and $\Psi(\tau) = H(\tau, \epsilon)$ the conditions (i) of Def. 4 follows.

[Def. 3 (a1)] Following Remark 2 we prove condition $|f_\alpha^{(n)}(\xi)\tau^n/(n)!| \leq Q_1(\epsilon_{f_\alpha^\epsilon})^2$. Applying Taylor formula to f_α results in

$$\frac{{f_\alpha}^{(n)}(\xi)}{(n)!}\tau^n = (f_\alpha(\tau) - \tau) - f_\alpha(0) + (1 - {f_\alpha}'(0))\tau - \frac{{f_\alpha}''(0)}{2!}\tau^2 - \ldots - \frac{{f_\alpha}^{n-1}(0)}{(n-1)!}\tau^{n-1}. \quad (46)$$

From Lemma 3.2 we have $f_\alpha(\tau) \to \tau$, when $\alpha \to 0$. Then $|f_\alpha(\tau) - \tau| < \epsilon(\alpha)$ for small α, where $\epsilon(\alpha)$ is the solution of Equation (28). From Lemma 3.3 it is easy to check that $\epsilon^*(\alpha) < f_\alpha^\epsilon(0)$. Using that $|f_\alpha(0)| \leq \epsilon_{f_\alpha}$, $|1 - {f_\alpha}'(0)| \leq \epsilon_{f_\alpha}$ and $|{f_\alpha}^{(j)}(0)| \leq \epsilon_{f_\alpha}$ $(j = 2, \ldots, n-1)$, where ϵ_{f_α} is defined in (6), from (46) follows

$$\left|\frac{{f_\alpha}^{(n)}(\xi)}{(n)!}\tau^n\right| \leq Q_1 \epsilon_{f_\alpha}, \quad (47)$$

where $Q_1 = 3 + (1/2!) + \ldots + (1/(n-1)!)$.

[Def. 3 (a2)] From Def. 8 (ii) g_α and ${f_\alpha}^{(n)}(0)$ are uniformly bounded by γ_2 since

$$|{f_\alpha}^{(n)}(0)| = \left|\int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(n)}(-s)\bar{g}_\alpha(s)ds\right| \leq \left|\int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(n)}(-s)ds\right| \sup_{s\in[0,a]} |\bar{g}_\alpha(s)|. \quad (48)$$

It follows that $\epsilon_{f_\alpha} \leq R_1$, where

$$R_1 = \sup_{1\leq j\leq n} \left|\int_{-\epsilon}^{+\epsilon} \eta_\epsilon^{(j)}(-s)ds\right| \gamma_2.$$

[Def. 3 (a3)] Notice that $f_\alpha > 0$. From (43) if $\tau_1 > \tau_2$, then $\bar{g}_\alpha(\tau_1 - \epsilon z) > \bar{g}_\alpha(\tau_2 - \epsilon z)$ implying $f_\alpha(\tau_1) > f_\alpha(\tau_2)$. So $f_\alpha(\tau)$ is an increasing function for $\tau \in [0, a]$.

[Def. 3 (a4)] From Def. 8, $|\tau/g_\alpha(\tau)| < R_2$ uniformly for $\alpha \in (0, \bar{\alpha})$ and $\lim_{\tau\to 0}(\tau/g_\alpha(\tau)) < \infty$. Using (43) f_α is also uniformly bounded for $\alpha \in (0, \bar{\alpha})$. □

3.2. Examples of the regularization classes obtained by mollification

Well known regularization functions presented in the form of Def. 8 can be written in the form of the proposed formalism. Although they are not continuously differentiable everywhere, it is possible to verify that these functions satisfy the conditions of the Def. 8 and hypothesis of Lemma 3.3.

Example 1. The regularization function known as spectral cut-off, see [16], is

$$g_\alpha(\tau) = \begin{cases} 0 & if \quad 0 < x < \alpha, \\ \tau & if \quad \alpha \leq \tau \leq a. \end{cases} \quad (49)$$

This function is not differentiable at $\tau = \alpha$, however, it satisfies the properties in Def. 8, where the upper bound $\bar{\alpha}$ can be selected arbitrarily.

Example 2. The Landweber iteration (see [7, 16]) produces the regularization method $g_\alpha = h_\alpha + \Phi$ with $\Phi \equiv 0$ and $h_\alpha(0) = 0$ and $h_\alpha(\tau) = \tau/(1-(1-\mu\tau)^{1/\alpha})$ for $\tau \in (0, a]$ with $a = ||K^*K||$. It is not differentiable at $\tau = 0$ and $g_\alpha(\tau) \geq (1/\sqrt{\mu})\sqrt{\tau}\sqrt{\alpha}$ for $\tau \in [0, a]$ with $\mu < 1/a$.

Further, an exhaustive construction of a regularization class is given

4. Construction of a regularization class

The regularization by conjugation formalism presented in Theorem 2.5 is based on the existence of the family $\mathbb{G}_a$ of functions satisfying hypotheses of Definition 4. In this section one form of constructing is presented. The construction is based on the conjugation technique widely used in Dynamical Systems, see [2].

Definition 9. *Auxiliary set of real valued functions* $\mathbb{D}_a$ *is defined. Consider* $a > 0$ *and* $n \geq 1$, *then* $\mathbb{D}_a = \{D \in C^{n+1}([0, a])\}$, *such that*

(a) $D(0) = 0$; $0 < \lambda := D'(0) < 1$; $|D(\tau)| < |\tau|$, $\forall \tau \in [0, a]\backslash\{0\}$.

(b) $((D'(\tau))^2\tau - D(\tau)(D'(\tau)\tau))' > 0$, $\forall \tau \in [0, a]\backslash\{0\}$.

Using the set of auxiliary functions D, the class $\mathbb{G}_a$ can be build as follows.

Theorem 4.1. *Let* $D \in \mathbb{D}_a$ *such that* $D''(0) < 0$ *and* $\epsilon_D < R_d$ *uniformly, where* ϵ_D *is defined as*

$$\epsilon_{D_i} = \sqrt{(D_i''(0))^2 + (D_i'''(0))^2 + (D_i^{(iv)}(0))^2 + \ldots + (D_i^{(n+1)}(0))^2}. \tag{50}$$

Then the solutions $g : [0, a] \to \mathbb{R}$ *of Julia's equation*

$$g(D(\tau)) = D'(\tau)g(\tau) \tag{51}$$

satisfy the conditions (a1)-(a4) of the auxiliary class $\mathcal{G}_a$ *of Def. 3.*

The proof of Theorem 4.1 is given in Section 4.6. Although some properties of Julia's equation are known (see [3, 18, 19] and references therein), other are obtained here to show its relation with the theory of regularization.

Theorem 4.2. *Let* g *be the solution of* (51) *with* $D \in \mathbb{D}_a$. *Assuming*

(a) There exist constants k_1 *and* $0 < \gamma \leq 2$ *such that*

$$g(\tau) \geq k_1\tau^\gamma. \tag{52}$$

(b) There exists an index function Θ, *such that* $\epsilon/\sqrt{\Theta(\epsilon)}$ *is also an index function for* $\epsilon \in (0, \bar{\epsilon}]$.

Then the family $\bar{g} = g + \Theta(\epsilon_g)$, *with* ϵ_g *from* (6) *defines a regularization class* $\mathbb{G}_a$.

Proof. From Theorem 4.1 function $g \in \mathcal{G}_a$ as in Def. 3. We need to prove that $\bar{g} = g(\tau) + \Theta(\epsilon_g)$ satisfies condition (i) of Def. 4 for $\tau \in [0, a]$. Using $(g(\tau) + \Theta(\epsilon_g)) \geq \sqrt{2}(g(\tau))^{1/2}(\Theta(\epsilon_g))^{1/2}$, together with (a) and (b) yields

$$(g(\tau) + \Theta(\epsilon_g)) \geq (\sqrt{2k_1})\tau^{\gamma/2}(\Theta(\epsilon_g))^{1/2},$$

which is the condition (i) from Def. 4, because $\epsilon_g/\sqrt{\Theta(\epsilon_g)}$ is an index function. □

4.1. Examples of regularization class

Two examples are presented next.

Example 1. A relevant particular case corresponds to the family $\mathbb{D}_a$ of functions $D(\tau) = a\tau + \alpha(a-1)$, with $a < 1$ and $\alpha \in (0, \bar{\alpha})$. It satisfies Def. 9 for small α and the set $\mathbb{G}_a$ of corresponding solution of Julia's equation is the Tikhonov's regularization function $g(\tau) = \tau + \alpha$.

Example 2. Consider the monotone decreasing sequence $\{\lambda_n\}_{n=1,2,...}$ of the eigenvalues of a self-adjoint operator K. For the moment assume $\lambda_1 < 1$. Define a real smooth interpolation function $\theta : [0, ||K||] \to \mathbb{R}$, such that $\theta(\tau_n) = \lambda_n$, $n \in \mathbb{N}$, where $\tau_n = (1 - 1/n)||K||$. Let us define the family $\mathbb{D}_a$ of functions $D : [0, ||K||] \to \mathbb{R}$ as

$$D(\tau) = \int_0^\tau \theta(\tau) d\tau. \tag{53}$$

In the case when $\lambda_1 \geq 1$, instead of the sequence $\{\lambda_n\}$, one can use the sequence $\{\lambda_n/(1 + \lambda_1)\}$ to define function θ. Notice that if θ is $C^n[0, ||K||]$ then the function D is $C^{n+1}[0, ||K||]$ and always satisfies the item (i) in Def. 9. In case it satisfies the item (ii) then the presented theory is applicable.

In this example Theorem 2.5 guarantees the bounds for the error based on the proposed function D, which includes information about the eigenvalues of the operator. In Corollary 4.5 this idea is shown with more details.

4.2. Technical proofs

We use some of the results presented in [3, 19] about the solution of Eq. (51), however, we obtain other properties needed to construct the regularization class addressed in this text.

Definition 10. *A local diffeomorphism $h : (-a, a) \to (-c, d)$, with $a, c, d > 0$, such that $D(\tau) = h^{-1}(\lambda h(\tau))$, $\forall \tau \in (-a, a)$ is called a* **conjugation** *between function D and its linear part $L(\tau) = \lambda\tau$, where λ is a positive parameter.*

Next proposition proves the existence of conjugation diffeomorphism for certain class of functions D.

Proposition 4.3. *Assume $a > 0$ and $D : (-a, a) \to \mathbb{R}$, $D \in C^{1+\epsilon}$ for some $\epsilon > 0$ such that (a) $D(0) = 0$, (b) $0 < \lambda := D'(0) < 1$ and (c) $|D(\tau)| < |\tau|$, $\forall \tau \in (-a, a)\backslash\{0\}$. Then there is a unique conjugation h as in Def. 10, such that $h'(0) = 1$. Such diffeomorphism h is of class $C^{1+\epsilon}$ and its derivative is given by*

$$h'(\tau) = \prod_{i=0}^{\infty} \frac{D'(D^i(\tau))}{\lambda}, \tag{54}$$

where $D^0(\tau) = x$ and $D^{i+1}(\tau) = D(D^i(\tau))$, $\forall i \geq 0$.

Proof. Since the function h satisfies $h(D(\tau)) = \lambda h(\tau)$ and, for $\tau = 0$, $h(0) = \lambda h(0)$, it follows that $h(0) = 0$. Moreover, $h(D^n(\tau)) = \lambda^n h(\tau)$, $\forall n \in \mathbb{N}$, yielding $h'(D^n(\tau))(D^n)'(\tau) = \lambda^n h'(\tau)$, and thus

$$h'(\tau) = h'(D^n(\tau))\frac{(D^n)'(\tau)}{\lambda^n} = h'(D^n(\tau))\prod_{j=0}^{n-1}\frac{D'(D^j(\tau))}{\lambda}. \tag{55}$$

Using Remark 4 we get $D^n(\tau) \leq b^n|\tau| \to 0$, when $n \to \infty$. Since $h'(0) = 1$ and $h \in C^1$ it follows that $h'(\tau)$ satisfies equation (54). This implies the uniqueness of h.

Next we prove that the product in equation (54) converges, it is equivalent to prove that

$$f(\tau) = \sum_{j=0}^{\infty}(\log D'(D^j(\tau)) - \log\lambda) \tag{56}$$

converges and so the function f is well defined. Since $D' \in C^\epsilon$, it follows that $\log D' \in C^\epsilon$ thus there exists $k > 0$ such that $|\log D'(z) - \log D'(y)| \leq k|z-y|^\epsilon$, $\forall y, z \in (-a, a)$. In particular, $|\log D'(z) - \log\lambda| \leq k|z|^\epsilon$, $\forall z \in (-a, a)$.

Since $|D^j(\tau)| \leq b^j|\tau| \leq a \cdot b^j$, $\forall \tau \in (-a, a)$, $|\log D'(D^j(\tau)) - \log\lambda| \leq k|D^j(\tau)|^\epsilon \leq k \cdot a^\epsilon \cdot b^{j\epsilon}$, $\forall j \geq 0$, yielding the absolute and uniform convergence of the series in equation (56). Moreover, since $|D^j(x) - D^j(y)| = |(D^j)'(\xi)||x-y|$, for some $\xi \in (x, y)$, and $|(D^j)'(\xi)| = \prod_{j=0}^{j-1}|D'(D^j(\xi))| \leq b^j$, we have, for every $x, y \in (-a, a)$,

$$|f(y) - f(x)| \leq \sum_{j=0}^{\infty}|\log D'(D^j(y)) - \log D'(D^j(x))| \tag{57}$$

$$\leq k\sum_{j=0}^{\infty}|D^j(y) - D^j(x)|^\epsilon \tag{58}$$

$$\leq k\sum_{j=0}^{\infty} b^{j\epsilon}|x-y|^\epsilon = \frac{k}{1-b^\epsilon}|x-y|^\epsilon, \tag{59}$$

yielding $f \in C^\epsilon$. Thus,

$$e^{f(\tau)} = \prod_{j=0}^{\infty}\frac{D'(D^j(\tau))}{\lambda} \tag{60}$$

is also of class C^ϵ. Notice that

$$\begin{aligned} e^{f(D(\tau))} &= \prod_{j=0}^{\infty}\frac{D'(D^j(D(\tau)))}{\lambda} = \prod_{j=1}^{\infty}\frac{D'(D^j(\tau))}{\lambda} = \frac{\lambda}{D'(\tau)}\prod_{j=0}^{\infty}\frac{D'(D^j(\tau))}{\lambda} \\ &= \frac{\lambda}{D'(\tau)}e^{f(\tau)}. \end{aligned} \tag{61}$$

Defining

$$h(\tau) = \int_0^\tau e^{f(t)}dt, \tag{62}$$

we have $h(0) = 0$, $h'(\tau) = e^{f(\tau)}$, and thus $(h(D(\tau)))' = h'(D(\tau))D'(\tau) = e^{f(D(\tau))}D'(\tau) = \lambda e^{f(\tau)} = \lambda h'(\tau)$, $\forall \tau \in (-a, a)$. Since $h(D(0)) = h(0) = 0 = \lambda h(0)$, we have $h(D(\tau)) = \lambda h(\tau), \forall \tau \in (-a, a)$. Since $h'(\tau) = e^{f(\tau)} > 0$, $\forall \tau \in (-a, a)$ then h is a diffeomorphism over its image $(-c, d)$. From $h'(\tau) = e^{f(\tau)} \in C^\epsilon$ it follows that $h \in C^{1+\epsilon}$. Since $h(D(\tau)) = \lambda h(\tau)$, we have $D(\tau) = h^{-1}(\lambda h(\tau))$, $\forall \tau \in (-a, a)$. Finally, notice that

$$f(0) = \sum_{j=0}^{\infty} (\log(D'(D^j(0))) - \log \lambda) = 0, \tag{63}$$

yielding $h'(0) = e^{f(0)} = 1$. □

Remark 4. *Under the hypotheses of the proposition above, taking smaller* a*, if necessary, there are* $\bar{b}$ *and* b *such that* $0 < \bar{b} < D'(\tau) < b < 1$, $\forall \tau \in (-a, a)$ *and* $|D(\tau)| \leq b|\tau| < |\tau|$, $\forall \tau \in (-a, a) \backslash \{0\}$.

Remark 5. *If in the hypothesis of the Proposition above, we choose* D *defined on the interval* $[0, M]$*, the proposition is still valid in the case when function* D *is differentiable at zero.*

Remark 6. • *If* $b \neq 0$*, the unique conjugation* $\tilde{h}$ *of class* C^1 *between* D *and its linear part with* $\tilde{h}'(0) = b$ *is defined by* $\tilde{h}(\tau) = bh(\tau)$.

• *There exists the function* D *of the class* C^1*, such that the product in equation (54) does not converge,* $\forall \tau \neq 0$*, and a conjugation* h *of class* C^1 *in the sense of Definition 10 does not exist.*

Definition 11. *Two functions* $f(w)$ *and* $\tilde{f}(w)$ *are* **similar** *(denoted by* $f(w) \sim \tilde{f}(w)$*) when*

$$\lim_{w \to 0} \frac{f(w)}{\tilde{f}(w)} = 1. \tag{64}$$

Remark 7. *The function* h *of the Proposition 4.3 is such that* $\forall n \in \mathbb{N}$, $D^n(\tau) = h^{-1}(\lambda^n h(\tau))$*. From* $h'(0) = 1$ *it follows that* $h^{-1}(y) \sim y$*, yielding* $D^n(\tau) \sim \lambda^n h(\tau)$*. Thus* $h(\tau)$ *can be obtained through the expression*

$$h(\tau) = \lim_{n \to \infty} \lambda^{-n} D^n(\tau). \tag{65}$$

Given the functions D and h as in Proposition 4.3 it will be checked that the function $g := h/h'$ satisfies Julia's equation studied in [19]:

$$g(D(\tau)) = D'(\tau) g(\tau). \tag{66}$$

If the function D satisfies Def. 9 (a) and the derivatives of D possess uniform limitation than the solution g_D of the functional equation (66) satisfies the hypotheses of Theorem 2.5.

Let us assume that $g : (-a, a) \to \mathbb{R}$ possesses derivative at 0. First, notice that $\forall n \geq 1$, $g(D^n(\tau)) = (D^n)'(\tau) g(\tau)$. We prove it using induction. For $n = 1$, this is the initial functional equation. Assuming the equation valid for n, we have $g(D^{n+1}(\tau)) = g(D(D^n(\tau))) = D'(D^n(\tau)) g(D^n(\tau)) = D'(D^n(\tau))(D^n)'(\tau) g(\tau) = (D \circ D^n)'(\tau) g(\tau) = (D^{n+1})'(\tau) g(\tau)$.

From $\lim_{n\to\infty} D^n(\tau) = 0$ it follows that $g(D^n(\tau)) \sim g'(0)D^n(\tau)$. From $D(\tau) = h^{-1}(\lambda h(\tau))$ it follows that $D^n(\tau) = h^{-1}(\lambda^n h(\tau))$ and $(D^n)'(\tau) = (h^{-1})'(\lambda^n h(\tau))\lambda^n h'(\tau) \sim \lambda^n h'(\tau)$ (since $h'(0) = 1$), so $g(D^n(\tau)) \sim g'(0)D^n(\tau) = g'(0)h^{-1}(\lambda^n h(\tau)) \sim g'(0)\lambda^n h(\tau)$, and thus $g'(0)\lambda^n h(\tau) \sim (D^n)'(\tau)g(\tau) \sim \lambda^n h'(\tau)g(\tau)$. This yields

$$g(\tau) = g'(0)\frac{h(\tau)}{h'(\tau)}. \tag{67}$$

Next we prove that for all $\tilde{b} \in \mathbb{R}$, the function $g(\tau) = \tilde{b}h(\tau)/h'(\tau)$ is a solution of the functional equation (66) even in the case when g does not admit derivative in 0. We have

$$g(D(\tau)) = \frac{\tilde{b}h(D(\tau))}{h'(D(\tau))}. \tag{68}$$

From $h(D(\tau)) = \lambda h(\tau)$ follows $h'(D(\tau))D'(\tau) = \lambda h'(\tau)$ and

$$\frac{h(D(\tau))}{h'(D(\tau))} = \frac{\lambda h(\tau)D'(\tau)}{\lambda h'(\tau)} = \frac{h(\tau)}{h'(\tau)}D'(\tau). \tag{69}$$

Substituting (69) into (68) yields $g(D(\tau)) = \tilde{b}h(\tau)D'(\tau)/h'(\tau) = g(\tau)D'(\tau)$. Thus g in (67) satisfies the functional equation (66).

In order to define the regularization class it is necessary to study the regularity of the class of functions g depending on the regularity of functions D. When $D \in C^{1+\epsilon}$ for some $\epsilon > 0$ this construction determines all continuous solutions g admitting derivatives in 0 (there are other solutions that are only continuous). If h is of class C^k, then h' is of class C^{k-1} and thus g is of class C^{k-1}. Since $h \in C^1$ it is easy to prove (by induction in s) from the expression $g(\tau) = g'(0)h(\tau)/h'(\tau)$ that if $g \neq 0$ and $g \in C^s$ then $h \in C^{s+1}$.

Proposition 4.4. *Consider h satisfying* (65)*, $D \in C^k$, $k \geq 2$, if and only if $h \in C^k$.*

Proof. If $h \in C^k$ then $D(\tau) = h^{-1}(\lambda h(\tau))$ is a composition of functions of class C^k and thus $D \in C^k$.

Reciprocally, if $D \in C^k$ with $k \geq 2$, we may assume, by reducing the interval diameter a if necessary, that $D^{(j)}$ is bounded in $(-a, a)$ for $1 \leq j \leq k$. We have $\log(h'(\tau)) = f(\tau)$ defined by equation (56) and $h \in C^k$, if and only if, $\log h' \in C^{k-1}$, if and only if, $h''/h' = (\log h')' \in C^{k-2}$. We have

$$f'(\tau) = \sum_{j=0}^{\infty}(\log D'(D^j(\tau)) - \log\lambda)' = \sum_{j=0}^{\infty}\frac{D''(D^j(\tau))}{D'(D^j(\tau))}(D^j)'(\tau), \tag{70}$$

where this series of continuous functions converges uniformly in $(-a, a)$, because $D^j(\tau) = h^{-1}(\lambda^j h(\tau))$, and so $(D^j)'(\tau) = (h^{-1})'(\lambda^j h(\tau)) \cdot \lambda^j \cdot h'(\tau)$, thus

$$\sum_{j=0}^{\infty}\frac{D''(D^j(\tau))}{D'(D^j(\tau))}(D^j)'(\tau) = \sum_{j=0}^{\infty}\frac{D''(D^j(\tau))}{D'(D^j(\tau))} \cdot \lambda^j \cdot (h^{-1})'(\lambda^j h(\tau)) \cdot h'(\tau). \tag{71}$$

It follows ‡ that $h \in C^2$.

We will show by induction on r, for $2 \leq r \leq k$ that $h \in C^r$. Indeed, we will prove that $\sum_{j=0}^{\infty} (\log D'(D^j(\tau)) - \log \lambda)^{(r-1)}$ can be written as

$$\sum_{j=0}^{\infty} \frac{\lambda^j P_r((D^{(s)}(D^j(\tau)))_{1 \leq s \leq r}, (h^{(s)}(\tau))_{1 \leq s \leq r-1}, ((h^{-1})^{(s)}(\lambda^j h(\tau)))_{1 \leq s \leq r-1}, \lambda^j)}{(D'(D^j(\tau)))^{r-1}}, \tag{72}$$

where P_r is a polynomial in 4 variables (which depends on r).

By the induction hypothesis according to which $h \in C^{k-1}$, the functions $D^{(s)}(D^j(\tau))$, $1 \leq s \leq r$, $h^{(s)}(x)$, $1 \leq s \leq r-1$ and $(h^{-1})^{(s)}(\lambda^j h(\tau)), 1 \leq s \leq r-1$ are continuous and uniformly bounded in $(-a, a)$. It follows that the series in equation (72) converges uniformly to $(\log h')^{(r-1)}$, which is a continuous function (since it is given by a series of continuous functions which converges uniformly). The claim follows by induction using the following formula

$$\begin{aligned}
(D^{(s)}(D^j(\tau)))' &= D^{(s+1)}(D^j(\tau)) \cdot (D^j)'(\tau) = \\
&= D^{(s+1)}(D^j(x)) \cdot (h^{-1})'(\lambda^j h(\tau)) \cdot \lambda^j \cdot h'(\tau), && (73)\\
(h^s(\tau))' &= h^{s+1}(\tau), && (74)\\
((h^{-1})^{(s)}(\lambda^j h(\tau)))' &= (h^{-1})^{(s+1)}(\lambda^j h(\tau)) \cdot \lambda^j \cdot h'(\tau), && (75)\\
(\lambda^j)' &= 0 && (76)
\end{aligned}$$

and

$$\begin{aligned}
&(D'(D^j(\tau))^{r-1})' = && (77)\\
&= (r-1) D'(D^j(\tau))^{r-2} \cdot D''(D^j(\tau)) \cdot (D^j)'(\tau) && (78)\\
&= (r-1) D'(D^j(\tau))^{r-2} \cdot D''(D^j(\tau)) \cdot (h^{-1})'(\lambda^j h(\tau)) \cdot \lambda^j \cdot h'(\tau). && (79)
\end{aligned}$$

□

Remark 8. *From Proposition 4.4 it follows that $g \in C^1$, if and only if, $D \in C^2$. More generally, $g \in C^k$, $k \geq 1$, if and only if, $D \in C^{k+1}$. It is possible to prove that $g \not\equiv 0$ is derivable at 0, if and only if, $\exists D''(0)$.*

Remark 9. *If D is a real analytic function, then h is also a real analytic function since D can be extended analytically to some disk $B \subset \mathbb{C}$ with center at the origin, where*

$$\prod_{j=1}^{\infty} \frac{D'(D^j(\tau))}{\lambda} \tag{80}$$

is the limit of a sequence of analytic functions in B which converges uniformly. In this case, if g possesses derivative at 0 then g is a real analytic function. The analyticity is important for the investigation of stability properties of the function g depending of the function D, see [3].

‡ Since $D''(D^j(\tau))$, $1/(D'(D^j(\tau)))$, $(h^{-1})'(\lambda^j h(\tau))$ and $h'(\tau)$ are continuous functions which are uniformly bounded in $(-a, a)$, and the series $\sum_{j=0}^{\infty} \lambda^j$ converges absolutely.

It is proved next that the uniform bound of the parameters ϵ_{g_i} depends on the uniform bound of the parameters ϵ_{D_i}.

Corollary 4.5. *Consider the sequence $D_i \in \mathbb{D}_a$, with $i \in I$, and the corresponding sequence of solutions g_{D_i} of Julia's equation* (66)*. If ϵ_{D_i} defined in Eq.* (50) *converges to zero with $i \to \infty$ then $\epsilon_{g_{D_i}} = [(g''_{D_i}(0))^2 + (g'''_{D_i}(0))^2 + (g^{iv}_{D_i}(0))^2 + \ldots + (g^{(n)}_{D_i}(0))^2]^{1/2}$ converges to zero. Moreover there is a constant $M > 0$ such that $\epsilon_{g_i} < M\epsilon_{D_i}$.*

Proof. This proof is a consequence of the proof of Proposition 4.3. Notice that the solution of the functional equation (66) is given by $g_{D_i}(\tau) = h_{D_i}(\tau)/h'_{D_i}(\tau)$, where $h'_{D_i}(\tau) = \exp(f_{D_i}(\tau))$, with f_{D_i} defined in (56). It is possible to check that

$$g'_{D_i}(\tau) = 1 - f'_{D_i}(\tau)g_{D_i}(\tau). \tag{81}$$

Deriving equation (81) and using (81) again results in

$$g''_{D_i}(\tau) = -f'_{D_i}(\tau) + ((f'_{D_i}(\tau))^2 - f''_{D_i}(\tau))g_{D_i}(\tau). \tag{82}$$

Since $g_{D_i}(0) = 0$ then $g''_{D_i}(0) = -f'_{D_i}(0)$ and

$$f'_{D_i}(0) = \sum_{j=0}^{\infty} D''_i(0)(D'_i(0))^{j-1} = D''_i(0)/\lambda(1-\lambda), \tag{83}$$

implying that when $D''_i(0)$ tends to zero with $i \to \infty$ so does $g''_{D_i}(0)$. In general, deriving Eq. (82) $(k-2)$ times and using Eq. (81) on the right side results in

$$g^{(k)}_{D_i}(\tau) = P_k(f', \ldots, f^{(k)})(\tau) + Q_k(f', \ldots, f^{(k)})(\tau)g_{D_i}(\tau), \tag{84}$$

where P_k and Q_k are polynomials depending on derivatives $f', \ldots, f^{(k)}$, $k = 2, \ldots, n$. Notice that $g^{(k)}_{D_i}(0) = P_n(f', \ldots, f^{(k)})(0)$, thus $g^{(k)}_{D_i}(0)$ depends on $f^{(j)}(0)$ for $j = 1 \ldots, k$. It is possible to verify from (56) that the derivative $f^{(j)}(0)$, with $j = 1, \ldots, n$ is the the polynomial function of $(\lambda, D''(0), \ldots, D^{(n+1)}(0))$. Therefore there is a constant $M > 0$ such that $\epsilon_{g_i} < M\epsilon_{D_i}$. The Corollary follows. □

Also, we have

Corollary 4.6. *Consider the sequence $D_i \in \mathbb{D}_a$ and satisfying conditions of Remark 4. The corresponding sequence of solutions g_{D_i} of Julia's equation* (66) *are uniformly bounded in the following sense. There exists a constant C such that $||g_{D_i}||_\infty \leq C$, $\forall i \in I$. Moreover $g_{D_i}(0) = 0$ and $g'_{D_i}(0) = 1$.*

Proof. This proof is a consequence of the proof of Proposition 4.3. □

The following Proposition justifies the property (iv) in Definition 4.

Proposition 4.7. *Consider the polynomials P_j, Q_j defined in the proof of Corollary 4.5 and the same sequence $D_i \in \mathbb{D}_a$ satisfying two more hypotheses:*
a) There exists $m \geq 2$ and $m < n$ such that $P_m(f', \dots, f^{(k)})(0) \neq 0$.
b) The sequences of polynomials are uniformly bounded, i.e., there are constants C_1 and C_2 such that $||P_n(f', \dots, f^{(k)})||_\infty \leq C_1$ and $||Q_n(f', \dots, f^{(k)})||_\infty \leq C_2$, $\forall n > 2$.
Then there exist a constant C_4 such that $||g_{D_i}^{(n)}||_\infty \leq C_5 |g_{D_i}^m(0)|$.

Proof. From (a) it follows that there is a constant C_3 such that $|P_m(f', \dots, f^{(k)})(0)| \geq C_3$. From (84) and (a) it follows that $g_{D_i}^{(m)}(0) = P_m(f', \dots, f^{(k)})(0) \neq 0$ for $2 \leq m < n$ and

$$\left| \frac{g_{D_i}^{(n)}(\tau)}{g_{D_i}^{(m)}(0)} \right| \leq \frac{1}{|g_{D_i}^{(m)}(0)|} \big(P_k(f', \dots, f^{(k)})(\tau) + Q_k(f', \dots, f^{(k)})(\tau) g_{D_i}(\tau) \big). \tag{85}$$

Using in (85) the hypotheses and Corollary 4.6 we obtain

$$|g_{D_i}^{(n)}(\tau)/g_{D_i}^{(m)}(0)| \leq (C_1 + C_2\, C)/C_3 =: C_4, \tag{86}$$

where C is a constant from Corollary 4.5. □

4.3. Explicit solution of the Julia's equation

Consider the solution g of the Julia's equation (66), with $g = h/h^{'}$, where h is the solution of Abel's equation $h(D(\tau)) = D^{'}(0)h(\tau)$, for similar results, see [2, 3, 19]. From equation (55) and using that $h^{'}(0) = 1$ we obtain

$$h^{'}(\tau) = \lim_{n\to\infty} \prod_{j=0}^{n-1} \frac{D^{'}(D^j(\tau))}{\lambda}. \tag{87}$$

Combining equations (65) and (87) we obtain

$$g(\tau) = \lim_{n\to\infty} \frac{D^n(\tau)}{\prod_{j=0}^{n-1} D^{'}(D^j(\tau))}. \tag{88}$$

The formula (88) guarantees $g(0) = 0$, it can be rewritten as an infinite product, which is very useful for analysis and numerical calculations. Let us define

$$R_n = \frac{\tau_n}{\prod_{k=0}^{n-1} D^{'}(\tau_k)}, \quad \rho_n = \frac{D(\tau_n)}{D^{'}(\tau_n)\tau_n}, \quad \tau_n = D^n(\tau)\cdot \tag{89}$$

It follows that

$$R_n = \frac{D(\tau_{n-1})\tau_{n-1}}{D^{'}(\tau_{n-1})\tau_{n-1} \prod_{k=0}^{n-2} D^{'}(\tau_k)} = \frac{D(\tau_{n-1})}{D^{'}(\tau_{n-1})\tau_{n-1}} R_{n-1} = \rho_{n-1} R_{n-1} \tag{90}$$

and

$$g(\tau) = \omega \prod_{n=0}^{\infty} \frac{D(\tau_n)}{D^{'}(\tau_n)\tau_n}. \tag{91}$$

The solution (91) of the Julia's equation (66) is unique except for a constant ω (see [18]), which is chosen in such way that $g'(0) = 1$.

4.4. Sufficient conditions for monotonicity

Next, the sufficient conditions that ensure the monotonicity of the solution of the functional equation (66) is established implying the property (a3) of Def. 3.

Lemma 4.8. *Assume that the hypotheses of Corollary 4.5 are satisfied and let D be such that*

$$((D'(\tau))^2\tau - D(\tau)(D'(\tau)\tau))' > 0 \tag{92}$$

holds for all $\tau \in (0, a]$. Then the solution g of the functional equation (66) is monotone increasing in $(0, a]$.

Proof. We set

$$G(\tau) = D(\tau)/D'(\tau)\tau. \tag{93}$$

Consider $\xi, \tau \in (0, a]$ such as $\tau < \xi$. Let g be the solution of (66). Notice that from (91) we have

$$g(\tau)/g(\xi) = \prod_{j=1}^{\infty}(G(\tau_j)/G(\xi_j)), \tag{94}$$

where τ_j and ξ_j are defined in (89c). Notice that $G'(\tau)$ is the fraction with the numerator equal to the left side of inequality (92) and the denominator equal to $(D'(\tau)\tau)^2$. Then the inequality (92) yields $G' > 0$ for all $\tau \in (0, a]$.

Using that $\tau_j < \xi_j$ (because function D is monotone increasing) and that the function G in (93) is monotone increasing, we have $G(\tau_j) < G(\xi_j)$ for $j = 1, 2 \ldots$ and as a consequence $g(\tau) < g(\xi)$ and g is monotone increasing in the interval $(0, a]$. □

4.5. Sufficient conditions for superlinearity

The sufficient conditions showing that the solution g of Eq. (66) satisfies the property (a4) of Def. 3.

Lemma 4.9. *Assume that the hypotheses of Corollary 4.5 are satisfied and let D such that*

$$D''(0) < 0. \tag{95}$$

Then the solution g of the functional equation (66) satisfy $g(\tau) \geq \tau$ for all $\tau \in [0, a]$.

Proof. To prove this inequality it is sufficient to check that the function $s(\tau) = g(\tau) - \tau$ possesses a local minimum at $\tau = 0$. Since we set $g'(0) = 1$, then $s'(0) = 0$. Notice that $s''(0) = g''(0)$ and $g(0) = 0$. Using $g = h/h'$ and formulas (56) and (62) it is possible to check

$$g''(0) = \frac{D''(0)}{D'(0)} \lim_{n\to\infty} \prod_{j=0}^{n-1} D'(D^j(0)). \tag{96}$$

Since $D^{'}(0)$ is positive and assumption (95) then $g''(0) \leq 0$ and the lemma follows. □

Remark 10. *It is possible verify that, if $D''(0) = 0$ then Lemma 4.9 still valid assuming that first derivative such that $D^{(m)}(0) \neq 0$ is less than zero.*

4.6. Proof of Theorem 4.1

Remark 11 (Property (a2) in Def. 3.)**.** *To guarantee that the set of* $\epsilon_{g_{D_i}}$ *corresponding to the set of solutions* g_{D_i} *of the functional equation (66) is uniformly bounded, we can choose the family of functions* D_i *such that the set* ϵ_{D_i} *defined in Eq.* (50) *is uniformly bounded, see Corollary 4.5.*

Proof. Let us summarize the properties of the regularization class $\mathbb{G}_a$ from Def. 3. Consider g the solution of Julia's equation in (51). Using that ϵ_D are uniformly bounded and Proposition 4.7, Remark 11 and Lemma 4.8 we obtain properties $(a1)$ and $(a2)$ in Def. 3. Also using Lemma 4.8 we obtain that the solution g is monotone increasing function, so we have $(a3)$. The conditions (a4) of Def. 3 is satisfied because $\tau/\bar{g}$ are uniformly bounded. It is easy to see it rewriting $\tau/\bar{g} = \tau/(g+\Phi) \leq (\tau - g)/g + 1$ and using Taylor's formula from Eq. (10) with g in the numerator together with the uniform bound of $\epsilon_{g_{D_i}}$ by Remark 11 and Corollary 4.5.

Also from Eqs. (88) and (91) we have that $g(0) = 0$ and $g^{'}(0) = 1$ thus $\lim_{\tau\to 0}(\tau/g(\tau)) < \infty$ holds. Finally, since $D \in C^{n+1}([0,a])$ and Remark 8 we have that $g \in C^n([0,a])$. □

Hypothesis of Theorem 4.1 on function D provides sufficient conditions to obtain solution g satisfying Def. 3. Notice that the existence of other classes of functions D generating same $\mathbb{G}_a$ is an open problem. Given that, from the class of functions $\mathbb{G}_a$ in Theorem 2.5 we can obtained other regularized solutions as can be observed in Examples 1, 2 in Section 4.1 and Example 3 below. That is why we propose to call $\mathbb{D}_a$ as **generating regularized class**.

Example 3. If we consider the family $D(\tau) = (\lambda/l)(\exp(l\tau) - 1))$, with $\lambda < 1$ and $l < 1$, the regularization class can be obtained by using Equations (56), (62) and (67) resulting in

$$g_D(\tau) = \frac{1}{\exp(f(\tau))}\int_0^\tau \exp(f(s))ds, \tag{97}$$

with $f(\tau) = \sum_{j=1}^{\infty} D^j(\tau)$ and $D^j(\tau) = D(D^{j-1}(\tau))$.

The possibility of creating regularization functions that are easy to implement numerically is one of the advantages of the formulation proposed in the present article.

5. Numerical results

In this section numerical examples are shown illustrating the conjugation regularization technique proposed in the present paper. We present this algorithm by way of example, however a work dedicated solely to the implementation of the numerical details of the regularization functions obtained here will be addressed in future works. Nevertheless we are the proof of how filters can be constructed that are methods of regularization.

Consider the operational equation $Kx = y$, where K is some bounded linear operator. Consider the discrete version of this operational equation given by

$$Ax = b, \tag{98}$$

where A is a $m \times n$ non singular matrix with $m \geq n$. We use a similar method for solving the normal equation $A^T Ax = A^T b$ with a MATLAB toolbox described in [10].

Let (U, V, Σ) be the singular value decomposition (SVD) of the matrix A, i.e.,

$$A = U\Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T, \tag{99}$$

where $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)$ are matrices with orthogonal columns and $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$ has non-negative singular values appearing in non-increasing order $(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n)$. Let us consider the regularized normal equation of (98)

$$\bar{g}(A^T A)x = A^T b^\delta, \tag{100}$$

with $||b - b^\delta|| < \delta$ and $\bar{g}(\sigma_i)$ is defined as in Definition 4 and Theorem 4.2 using $g(\sigma_i)$, which is the solution of (66) for given D. The discrete regularized solution of (100) is given by

$$x_{\bar{g}}^\dagger = \sum_{i=1}^{n} \frac{\sigma_i}{\bar{g}(\sigma_i)} \frac{u_i^T b^\delta}{\sigma_i} v_i, \tag{101}$$

We use Eq. (91) to obtain the function $g = g_D$. Details of how to implement numerically this equation can be found in [4].

Notice that the term $f_i(\sigma_i) = \sigma_i/\bar{g}(\sigma_i)$ in (101) plays the role of the filter, see [10, 17]. More exactly, taking ϵ_g as regularization parameters and using the fact that $\bar{g}$ (with $\Theta(\epsilon) = \epsilon$) satisfy the Hypothesis of Theorem 4.2, we obtain that the function f_i satisfy the condition (4a)-(4c) given in [17] which guarantee that the filter-based method is a regularization method. Also, it is possible to obtain de condition (5a)-(5b) which assured the order optimality. Conditions (4b) and (4c) are easily verifiable. We have that (4a) and (5a) follow of

$$f_i(\sigma_i) \leq (1/\sqrt{2k_1})\sigma_i^{1-\gamma/2}(\epsilon_g)^{-1/2}, \tag{102}$$

with $0 < \gamma < 2$. The condition (5b) is a consequence of the qualification properties which can be deduced of Lemma 2.2.

Thus the procedure described in Section 4 also serves for generate filters that can be directly used to solve the discrete version of the operational equation. Since we have that $\bar{g} = g + \Theta(\epsilon_{g^*})$, where Θ is some index function, in our numerical test we take $\Theta(\epsilon) = \epsilon$ and g given in Eq. (91). The regularization parameter ϵ_{g^*} is obtained solving the equation

$$\Omega^{-1}(\delta) = \epsilon_{g^*}, \tag{103}$$

with Ω is given in Theorem 2.5. In this case $\Omega = k_1\tau^{(\gamma/2)+(3/4)}$. In order to test the algorithm described above the following benchmark problem is used (see [31])

Problem. Obtain the solution $q(x)$ of the equation

$$\int_0^{2\pi} K(x,y)q(x)dx = w(y), \quad \text{with} \quad 0 \leq y \leq 2\pi, \tag{104}$$

where $K(x,y) = \exp(y\cos(x))$, the exact solution is given by $q(x) = \cos(x)$ and $w(y)$ is evaluated numerically.

Next, three cases are shown with different choices of the function D and the corresponding solution g_D of Julia's equation working as the regularization class.

5.1. *Case 1*

In the first experiment, we select the function D which derivative $D'(\tau) = c_1\cos(\tau/c_2)$ where the constants $c_1 = 0.3 \times 10^5$ and $c_2 = 1500$. It is possible to verify that this function D satisfies the sufficient conditions ensuring that the solution g_D of Julia's equation belongs to the regularization class, i.e., satisfying hypothesis of Theorem 4.2.

In Figure 1, we compare the regularized solution obtained by the Tikhonov's filter with parameter α determined by L-curve method [7], the regularized solution obtained by the present method and the exact solution. Tikhonov's filter presented a relative error of 0.03 using l_2 norm. The regularized solution by conjugation resulted in the relative error of 0.01.

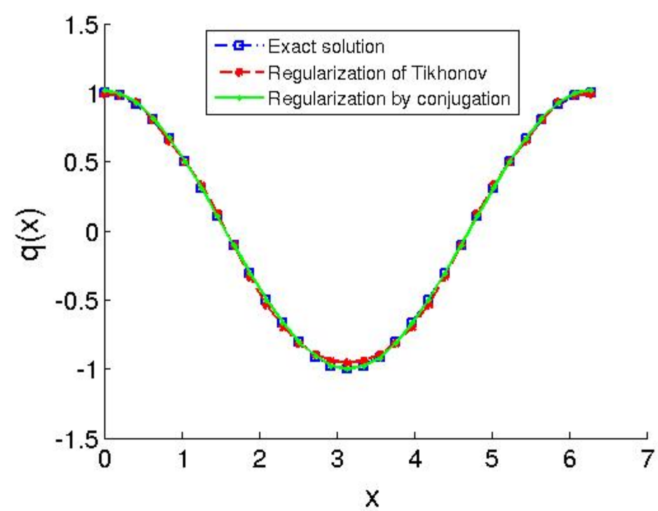

Figure 1. Regularized solutions are compared to the exact solution. Tikhonov's regularization implemented using the method described in [10].

5.2. *Case 2*

In this example the same problem was addressed with function D such that its derivative is $D'(\tau) = 0.1c_1\sin((\tau/850)(\pi/2)) + c_1\cos((\tau/2800)(\pi/2))$, with $c_1 = 0.3 \times 10^5$. Again the function D satisfies the sufficient conditions that ensuring that the solution g_D of Julia's equation belongs to the regularization class. For this parameter value the solution g of Eq. (66) satisfies Definition 4 with the condition $g(\tau) > 0.5 \times 10^{-7}\tau^{1/2}$ (hypothesis (52) in Theorem 4.2). In this case the regularization by conjugation converges with relative error of 0.003.

5.3. Case 3

The last example discussed here considers the function D constructed as described in Example 2 in Section 4.1. In Figure 2 we show the normalized eigenvalues of the discretization of the operator K in (104). For this example the function D satisfies the sufficient conditions that ensuring that the solution g_D of Julia's equation belongs to the regularization class. We have convergence of regularization by conjugation with relative error of 0.01.

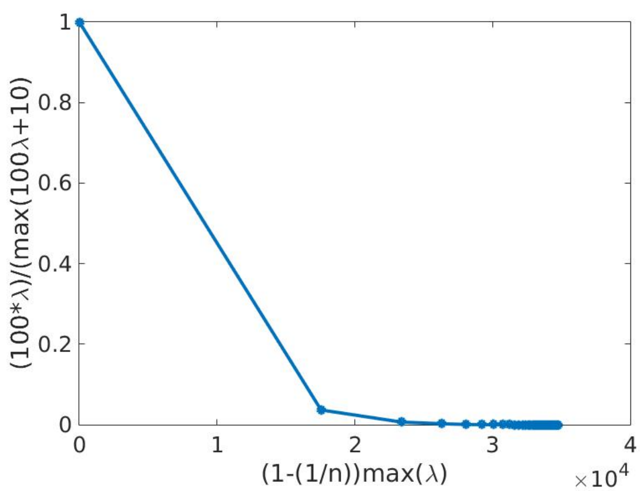

Figure 2. Normalized eigenvalues of the operator in the operational equation (104). The function D' is a spline approximation of the profile of the normalized eigenvalues.

This experiment was also done for the case when the right side of the equation (98) was contaminated with an error $b_\delta = b + \delta$, where $\delta = 1.0 \times 10^{-5}$, resulting in visually indistinguishable result.

This example is interesting because the selection of the function D highlights some known aspects of operational equation with compact linear operator. It is well known that when the eigenvalues of the compact operator tend to zero fast it is more difficult to obtain the regularized solution.

In this case D' represents the decay of the eigenvalues and the convergence of the regularized solution. It depends on the uniform limitation of the higher order derivatives of the function D. In the case of rapid decay of the eigenvalues these limiting constants are larger inducing slower convergence rate when searching for the regularized solution.

6. Discussions and Conclusions

In this paper, we have shown the alternative class of regularization functions that solve an operational equation with bounded linear operator. The presented regularization depends on a set of topological properties of these functions and can be used to obtain different regularization techniques. We study the properties of the regularization class such as smoothness, profile function, qualification and optimal rate of convergence. We believe that this formalism allows to address other properties like saturation introduced in [21].

Another result shown in this paper consists in constructing such a class of regularization functions using Julia's functional equation. It is an elegant procedure that uses Dynamical System techniques. It provides an equivalence between class of functions $\mathbb{D}_a$ (Def. 9), which is easy to construct, and regularization class $\mathbb{G}_a$ (Def. 4). Such approach has a potential to classify the regularization techniques and to search for the optimal regularization technique for a given operator. Some new properties of the solutions of Julia's equation were also presented. For example, Eq. (91), proposed here, is easy to implement numerically and allows to calculate the solutions of Julia's equation efficiently which is used to obtain filters for the discrete regularization method (see an example in numerical Section 5).

This procedure allows to incorporate, through the conjugation transformation, the properties of the operator into the regularization class. Thus it can be used to improve the behavior of the eigenvalues of the bounded operator in order to build a robust regularization method. In other words, if the behavior of the eigenvalues λ_k of the operator is analyzed as a discrete dynamic system (DDS), using conjugation, it becomes a new DDS of eigenvalues $g(\lambda_k)$ with better behavior in the regularization process. In fact, this mechanism is hidden inside Tikhonov's regularization.

In Def. 4 the regularization functions were considered of class $C^n(\mathbb{R})$. However, in the literature (See [7, 16]) it is common to consider regularization functions continuous or even piecewise continuous. The formalism presented in Section 2 is generalized in Section 3 to such cases applying mollification procedure similar to the classical one presented in [8, 25, 11]. Thus the present method also embedding must of the non-smooth regularization methods as Landweber and spectral cut-off.

The advantage of considering the kind of the regularization class described here is that it facilitates the analysis of convergence and rate of convergence of the regularization methods, since we gain as an immediate consequence the qualification and smoothness properties from the characteristics of these functions. Also, in Lemmas 3.2 and 3.3 are given sufficient conditions to include non-smooth regularization functions in the regularization class, however, we believe that using mechanisms similar to process mollification we can include a large number of functions in this context facilitating the analysis.

Acknowledgments

The authors thank Prof. Dan Marchesin for introducing us to the topic. The first author acknowledges Eng. Ely Maranhão for helpful comments and motivation of the results, also thanks to the collaboration of Luzia Maranhão, Lya Vinhas de Quiroga and Hector Quiroga Zambrana.

The first author's work was supported in part by IMPA/CAPES. The second author was supported in part by FAPEMIG under Grant APQ 01377/15. The third author was supported in part by DICYT grant 041633GM and MathAmsud 15MATH-02-SOCDE project.

References

[1] J. Aczél. *On Applications and Theory of Functional Equations.* Elsevier Inc, 1969.

[2] D. S. Alexander. *A History of Complex Dynamics: From Schroder to Fatou and Julia.* Springer Science and Business Median, 2013.

[3] A. C. Alvarez, G. Hime, J. D. Silva, and D. Marchesin. Analytic regularization of an inverse filtration problem in porous media. *Inverse Problems*, 29(2):025006 (20p), 2013.

[4] A.C. Alvarez, P.G. Bedrikovetsky, G. Hime, A.O. Marchesin, D. Marchesin, and J.R. Rodrigues. A fast inverse solver for the filtration function for flow of water with particles in porous media. *Inverse problems*, 22(1):69, 2005.

[5] Boris Hasselblatt Anatole Katok. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications 54. Cambridge University Press, 1995.

[6] V. I. Arnold. *Geometrical methods in the theory of ordinary differential equations*, volume 250. Springer Science & Business Media, 2012.

[7] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375. Springer Science & Business Media, 1996.

[8] Lawrence C Evans. Partial differential equations and Monge-Kantorovich mass transfer. *Current developments in mathematics*, pages 65–126, 1997.

[9] C. W. Groetsch. *The theory of Tikhonov Regularization for Fredholm Equations.* 1984.

[10] P.C. Hasen. Regularization tools: a matlab package for analysis and solution of discrete illposed problems. *Numer. Algo.*, 6:1–35, 1994.

[11] Dinh Nho H\ao. A mollification method for ill-posed problems. *Numerische Mathematik*, 68:469–506, 1994.

[12] Markus Hegland. An optimal order regularization method which does not use additional smoothness assumptions. *SIAM journal on numerical analysis*, 29(5):1446–1461, 1992.

[13] T. H. Hildebrandt. *Introduction to the Theory of Integration.* Academic Press New York and London, 1963.

[14] Michiel E Hochstenbach, Silvia Noschese, and Lothar Reichel. Fractional regularization matrices for linear discrete ill-posed problems. *Journal of Engineering Mathematics*, 93(1):113–129, 2015.

[15] Michiel E Hochstenbach and Lothar Reichel. Fractional Tikhonov regularization for linear discrete ill-posed problems. *BIT Numerical Mathematics*, 51(1):197–215, 2011.

[16] B. Hofmann and P. Mathé. Analysis of profile functions for general linear regularization methods. *SIAM Journal on Numerical Analysis*, 45(3):1122–1141, 2007.

[17] Esther Klann and Ronny Ramlau. Regularization by fractional filter methods and data smoothing. *Inverse Problems*, 24(2):025018, 2008.

[18] M Kuczma. *Functional Equations in a Single Variable.* Polish Scientific Publishers,Warszawa, 1968.

[19] M Kuczma, B Choczewski, and R Ger. *Iterative Functional Equations*, volume 32. Cambridge University Press, 1990.

[20] A. A. Mailybaev. Transformation to versal deformations of matrices. *Linear Algebra and its Applications*, 337(1-3):87–108, 2001.

[21] P. Mathé. Saturation of regularization methods for linear ill-posed problems in Hilbert spaces. *SIAM journal on numerical analysis*, 42(3):968–973, 2004.

[22] P. Mathé and S. V. Pereverzev. Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems*, 19(3):789–803, 2003.

[23] P. Mathé and S. V. Pereverzev. Regularization of some linear ill-posed problems with discretized random noisy data. *Mathematics of Computation*, 75(256):1913–1929, 2006.

[24] V. A. Morozov and M. Stessin. *Regularization methods for ill-posed problems.* Crc Press Boca Raton, FL:, 1993.

[25] Diego A Murio. *The mollification method and the numerical solution of ill-posed problems.* John Wiley & Sons, 2011.

[26] D. L. Phillips. A technique for the numerical solution of certain integral equations of the first kind. *J. Assoc. Comput. Mach.*, 9(1):84–97, 1962.

[27] A G Ramm. *Inverse problems: Mathematical and analytical techniques with applications to engineering.* Springer, 2005.

[28] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-Hill New York, 1964.

[29] X. Sun, J. Liu, X. Han, C. Jiang, and R. Chen. A new improved regularization method for dynamic load identification. *Inverse Problems in Science and Engineering*, 22(7):1062–1076, 2014.

[30] A. N. Tikhonov and V. Y. Arsenin. *Solutions of Ill-Posed Problems.* Wiley, New York, 1977.

[31] Groetsch C. W. *The theory of Tikhonov regularization for Fredholm equations of the first kind.* Boston, MA: Pitman, 1984.

[32] Eberhard Zeidler. Applied functional analysis: main principles and their applications. *Applied mathematical sciences*, 109, 1995.