Cut-off Phenomenon for Stochastic Small Perturbations of Dynamical Systems

Gerardo Barrera
Advisor: Milton Jara (IMPA, Brazil)
Jury: Marcelo Miranda Viana da Silva (IMPA, Brazil)
Maria Inés Armendariz (Universidad de Buenos Aires, Argentina)
Stefano Olla (Université Paris Dauphine, France)
Claudio Landim (IMPA, Brazil)

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Contents

Chapter 1

Introduction

During the last decades intense research has been devoted to the study of dynamical systems subjected to random perturbations. Considerable effort has been dedicated to investigate exit times and exit locations from given domains and how they relate to the respective deterministic dynamical system. The theory of large deviations provides the usual mathematical framework for tackling these problems in case of Gaussian perturbations, for details see M. Freidling & A. Wentzell [18] and W. Siegert [26]. We will study the relation to the respective deterministic dynamical systems from a different point of view.

We study the so-called *cut-off phenomenon* for a family of stochastic small perturbations of a given dynamical system. We will focus on the semi-flow of a deterministic differential equation which is perturbed by adding to the dynamics a Brownian forcing of small variance. Under suitable hypotheses on the vector field we will prove that the one-parameter family of perturbed stochastic differential equations presents a profile cut-off in the sense of J. Barrera & B. Ycart [14].

The term "cut-off" was introduced by D. Aldous and P. Diaconis in [6] to describe the phenomenon of abrupt convergence of Markov chains introduced as models of shuffling cards. Since the appearance of [6] many families of stochastic processes have been shown to have similiar properties. For a good introduction to the different definitions of cut-off and the evolution of the concept in discrete time, see J. Barrera & B. Ycart [14] and P. Diaconis [19]. In [11], L. Saloff-Coste gives an extensive list of random walks for which the phenomenon occurs.

How to describe the "cut-off" phenomenon? Before a certain "cut-off time" those processes stay far from equilibrium in the sense that the distance in some sense between the distribution at time t and the equilibrium measure is far from 0; after that instant, the distance decays exponentially fast to zero.

Consider a one-parameter family of stochastic processes in continuous time $\{x^\epsilon\}_{\epsilon>0}$ indexed by $\epsilon > 0$, $x^\epsilon := \{x_t^\epsilon\}_{t\geq 0}$, each one converging to a asymptotic distribution μ^ϵ when t goes to infinity. Let us denote by $d^\epsilon(t)$ the distance between the distribution at time t of the ϵ-th processes and its asymptotic distribution, where the "distance" can be taken to being the total variation, separation, Hellinger, relative entropy, Wasserstein, L^p distances, etc. Following J. Barrera & B. Ycart [14], the cut-off phenomenon for $\{x^\epsilon\}_{\epsilon>0}$ can be expressed at three increasingly sharp levels. Let us denoted by M the "maximum of the distance". In general, M could be infinite. In our case, we will focus on the total variation distance so $M = 1$.

Definition 1.1 (Cut-off). *The family $\{x^\epsilon\}_{\epsilon>0}$ has a cut-off at $\{t_\epsilon\}_{\epsilon>0}$ if $t_\epsilon \to +\infty$ as $\epsilon \to 0$ and*

$$\lim_{\epsilon\to 0} d^\epsilon(ct_\epsilon) = \begin{cases} M & if \quad 0 < c < 1, \\ \\ 0 & if \qquad c > 1. \end{cases}$$

Definition 1.2 (Window Cut-off). *The family $\{x^\epsilon\}_{\epsilon>0}$ has a window cut-off at $\{(t_\epsilon, w_\epsilon)\}_{\epsilon>0}$, if $t_\epsilon \to +\infty$ as $\epsilon \to 0$, $w_\epsilon = o(t_\epsilon)$ and*

$$\begin{aligned} \lim_{c\to -\infty} \liminf_{\epsilon\to 0} d^\epsilon(t_\epsilon + cw_\epsilon) &= M, \\ \lim_{c\to +\infty} \limsup_{\epsilon\to 0} d^\epsilon(t_\epsilon + cw_\epsilon) &= 0. \end{aligned}$$

Definition 1.3 (Profile Cut-off). *The family $\{x^\epsilon\}_{\epsilon>0}$ has profile cut-off at $\{(t_\epsilon, w_\epsilon)\}_{\epsilon>0}$ with profile G, if $t_\epsilon \to +\infty$ as $\epsilon \to 0$, $w_\epsilon = o(t_\epsilon)$,*

$$G(c) := \lim_{\epsilon\to 0} d^\epsilon(t_\epsilon + cw_\epsilon)$$

exists for all $c \in \mathbb{R}$ and

$$\begin{aligned} \lim_{c\to -\infty} G(c) &= M, \\ \lim_{c\to +\infty} G(c) &= 0. \end{aligned}$$

Sequences of stochastic processes for which an explicit profile can be determine are scarce. Explicit profiles are usually out of reach, in particular for the total variation distance; in many cases of interest only cut-off or window cut-off has been obtained so far.

Chapter 2

Stochastic Perturbations: One-Dimensional Case

On this chapter, let $x_0 \in \mathbb{R} \setminus \{0\}$ be fixed and let us consider the semi-flow $\{\psi_t\}_{t\geq 0}$ associated to the solution of the following deterministic differential equation

$$dx_t = -V'(x_t)dt \tag{2.1}$$

for $t \geq 0$. The hypothesis made in Theorem 2.1 on the potential V guarantees existence and uniqueness of solutions of (2.1), as well as all the other (stochastic or deterministic) equations defined below.

Our main Theorem in the one-dimensional case is the following:

Theorem 2.1 (General Case)**.** *Let $V : \mathbb{R} \to \mathbb{R}$ be a one-dimensional potential that satisfies the following:*

i) $V \in \mathcal{C}^3$.

ii) $V(0) = 0$.

iii) $V'(x) = 0$ *if only if* $x = 0$.

iv) There exists $\delta > 0$ *such that* $V''(x) \geq \delta$ *for every* $x \in \mathbb{R}$.

Let us consider the family of Markov processes indexed by $\epsilon > 0$, $x^\epsilon = \{x_t^\epsilon\}_{t\geq 0}$ which are given by the the semi-flow of the following stochastic differential equation,

$$\begin{aligned} dx_t^\epsilon &= -V'(x_t^\epsilon)dt + \sqrt{\epsilon}dW_t, \\ x_0^\epsilon &= x_0 \end{aligned}$$

for $t \geq 0$, where x_0 is a deterministic point in $\mathbb{R}\setminus\{0\}$ and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. This family presents profile cut-off in the sense of the Definition 1.3 with respect to the total variation distance when ϵ goes to zero. The profile function $G : \mathbb{R} \to \mathbb{R}$ is given by

$$G(b) := \left\| \mathcal{N}\left(\tilde{c}e^{-b}, 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}},$$

where $\tilde{c}$ is the non-zero constant given by

$$\lim_{t\to+\infty} e^{V''(0)t}\psi_t \quad = \quad \tilde{c}.$$

The cut-off time t_ϵ and window time w_ϵ are given by

$$\begin{aligned} t_\epsilon &:= \frac{1}{2V''(0)}\left(\ln\left({}^1\!/\!_\epsilon\right) + \ln\left(2V''(0)\right)\right), \\ w_\epsilon &:= \frac{1}{V''(0)} + \epsilon^\gamma, \end{aligned}$$

for some $0 < \gamma < {}^1\!/\!_4$.

This Theorem will be proved at the end of this chapter.

2.1 The Linearized Case

Let us take $\mu \in \mathbb{R}$ and $\sigma^2 \in]0, +\infty[$. We denote by $\mathcal{N}(\mu, \sigma^2)$ the Normal distribution with mean μ and variance σ^2. Given two probability measures μ and ν which are defined in the same measurable space $(\Omega, \mathcal{F})$, we denote the total variation distance between μ and ν by $\|\mu - \nu\|_{\mathbb{TV}} := \sup\limits_{A\in\mathcal{F}} |\mu(A) - \nu(A)|$.

Definition 2.2. *We say that V is a regular potential if $V : \mathbb{R} \to \mathbb{R}$ satisfies*

a) V *is* $\mathcal{C}^3$.

b) $V(0) = 0$.

c) $V'(x) = 0$ *iff* $x = 0$.

d) $V''(0) > 0$.

e) $\lim\limits_{|x|\to+\infty} V(x) = +\infty$.

In order to prove Theorem 2.1 we will prove the analogous result for a "linear approximation" of the vector field V.

Theorem 2.3 (The Linearized Case). *Let us consider the one-parameter family of Markov processes indexed by $\epsilon > 0$, $y^\epsilon = \{y^\epsilon_t\}_{t\geq 0}$ which are given by the the solution of the following linear stochastic differential equation,*

$$\begin{aligned} dy^\epsilon_t &= -V''(\psi_t)y^\epsilon_t dt + \sqrt{\epsilon}dW_t, \\ y^\epsilon_0 &= y_0 \end{aligned} \tag{2.2}$$

for $t \geq 0$, where y_0 is a deterministic point in $\mathbb{R}\setminus\{0\}$, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and V is a regular potential. This family presents profile cut-off in the sense of the Definition 1.3 with respect to the total variation distance when ϵ goes to zero. The profile function $G : \mathbb{R} \to \mathbb{R}$ is given by

$$G(b) \;:=\; \left\|\mathcal{N}\left(ce^{-b},1\right) - \mathcal{N}(0,1)\right\|_{\mathbb{TV}},$$

where c is the non-zero constant given by

$$\lim_{t\to+\infty} e^{V''(0)t}\Phi_t \;=\; c,$$

where $\Phi = \{\Phi_t\}_{t\geq 0}$ is the fundamental solution of the non-autonomous system

$$d\Phi_t = -V''(\psi_t)\Phi_t dt$$

for every $t \geq 0$ with initial condition $\Phi_0 = 1$. The cut-off time t_ϵ and window time w_ϵ are given by

$$\begin{aligned} t_\epsilon &:= \frac{1}{2V''(0)}\left(\ln\left(1/\epsilon\right) + \ln\left(2V''(0)y_0^2\right)\right), \\ w_\epsilon &:= \frac{1}{V''(0)}, \end{aligned}$$

respectively.

Notice that choosing $V(x) = \frac{\alpha x^2}{2}$ we see that the Ornstein-Uhlenbeck process presents profile cut-off. In what follows, we call the solutions $\{y^\epsilon\}_{\epsilon>0}$ of (2.2) the "linear approximations".

In order to prove Theorem 2.3, we will find the qualitative behavior of the semi-flow $\psi = \{\psi_t\}_{t\geq 0}$ at infinity.

The following lemma tells us the asymptotic behavior of the expectation and variance of the "linear approximations".

Lemma 2.4. *Leu us assume the hypothesis of Theorem 2.3. Let us assume that there exists a $\mathcal{C}^2$ function $V : \mathbb{R} \to \mathbb{R}$ such that*

a) $V(0) = 0$.

b) $V'(x) = 0$ *iff* $x = 0$.

c) $V''(0) > 0$.

d) $\lim_{|x| \to +\infty} V(x) = +\infty$.

Then it follows that

i) $\lim_{t \to +\infty} \psi_t = 0$.

ii) $\lim_{t \to +\infty} \Phi_t = 0$.

In addition, let us assume that V is a $\mathcal{C}^3$ function. Then it follows that

iii) There exist constants $c \neq 0$ and $\tilde{c} \neq 0$ such that

$$\lim_{t \to +\infty} e^{V''(0)t} \Phi_t = c,$$

$$\lim_{t \to +\infty} e^{V''(0)t} \psi_t = \tilde{c},$$

where $\Phi = \{\Phi_t\}_{t \geq 0}$ is the fundamental solution of the nonautonomous system

$$d\Phi_t = -V''(\psi_t)\Phi_t dt$$

for every $t \geq 0$ with initial condition $\Phi_0 = 1$.

iv)

$$\lim_{t \to +\infty} \Phi_t^2 \int_0^t \left(\frac{1}{\Phi_s} \right)^2 ds = \frac{1}{2V''(0)}.$$

For the proof of this lemma, see Appendix C.

The following lemma characterizes the distribution of the "linear approximations".

Lemma 2.5. *Under the hypothesis of Theorem 2.3, we have*

$$y_t^\epsilon = \Phi_t y_0 + \sqrt{\epsilon}\Phi_t \int_0^t \frac{1}{\Phi(s)} dW_s \tag{2.3}$$

for every $t \geq 0$*, where* $\Phi = \{\Phi_t\}_{t\geq 0}$ *is the fundamental solution of the non-autonomous system*

$$d\Phi_t = -V''(\psi_t)\Phi_t dt$$

for every $t \geq 0$ *with initial condition* $\Phi_0 = 1$*.*

Proof. It follows from Itô's formula. For details check [13] and [20]. □

Using the decomposition (2.3) of the process y^ϵ into a deterministic part and a mean-zero martingale with respect to the natural filtration of the Brownian motion and using Itô's isometry for Wiener's integral, we obtain

$$\begin{aligned} \mathbb{E}\left[y_t^\epsilon\right] &= y_0\Phi_t, \\ \mathbb{V}\left[y_t^\epsilon\right] &= \epsilon\Phi_t^2 \int_0^t \left(\frac{1}{\Phi_s}\right)^2 ds. \end{aligned}$$

By Lemma 2.5 we have that for each $\epsilon > 0$ and $t > 0$ fixed, y_t^ϵ is a random variable with Normal distribution with mean

$$\nu_t^\epsilon \;:=\; \mathbb{E}\left[y_t^\epsilon\right] = \Phi_t y_0$$

and variance

$$\eta_t^\epsilon \;:\; = \mathbb{V}\left[y_t^\epsilon\right] = \epsilon\Phi_t^2 \int_0^t \left(\frac{1}{\Phi_s}\right)^2 ds.$$

Corollary 2.6. *Let us assume the hypothesis of Theorem 2.3 and let* $\epsilon > 0$ *be fixed. Then the random variable* y_t^ϵ *converges in distribution as* $t \to \infty$ *to a Gaussian random variable* $\mathcal{N}^\epsilon$ *with mean zero and variance* $\frac{\epsilon}{2V''(0)}$*.*

Proof. It follows from Lemma 2.4. □

Now we have all the tools in order to prove Theorem 2.3.

Proof of Theorem 2.3. For each $\epsilon > 0$ and $t > 0$, we define

$$\begin{aligned} d^\epsilon(t) &= \left\| \mathcal{N}\left(\nu_t^\epsilon, \eta_t^\epsilon\right) - \mathcal{N}\left(0, \frac{\epsilon}{2V''(0)}\right) \right\|_{\mathbb{TV}}, \\ D^\epsilon(t) &= \left\| \mathcal{N}\left(\sqrt{\frac{2V''(0)}{\epsilon}} y_0 \Phi_t, 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}. \end{aligned}$$

Using triangle's inequality and Lemma A.1, for each $\epsilon > 0$ and $t > 0$ we obtain

$$\begin{aligned} d^\epsilon(t) &\leq D^\epsilon(t) + \left\| \mathcal{N}\left(0, 2V''(0){\Phi_t}^2 I_t\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}, \\ |d^\epsilon(t) - D^\epsilon(t)| &\leq \left\| \mathcal{N}\left(0, 2V''(0){\Phi_t}^2 I_t\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}, \end{aligned}$$

where $I_t = \int\limits_0^t \left(\frac{1}{\Phi_s}\right)^2 ds$. For each $\epsilon > 0$ let us define

$$t_\epsilon := \frac{1}{2V''(0)} \left(\ln\left({}^1\!/\!{}_\epsilon\right) + b_0\right)$$

and

$$w_\epsilon := \frac{1}{V''(0)}$$

with $b_0 := \ln(2V''(0) y_0^2)$. For every $b \in \mathbb{R}$, we define $\tilde{t}_\epsilon(b) = t_\epsilon + b w_\epsilon$. Using Lemma A.4, we obtain

$$\lim_{\epsilon \to 0} |d^\epsilon(\tilde{t}_\epsilon(b)) - D^\epsilon(\tilde{t}_\epsilon(b))| = 0$$

for every $b \in \mathbb{R}$. Let us consider the function $G : \mathbb{R} \to [0,1]$ defined by

$$G(b) := \left\| \mathcal{N}(ce^{-b}, 1) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}},$$

where $c \neq 0$ is the constant of item *iii*) in Lemma 2.4. Observe that

$$D^\epsilon(\tilde{t}_\epsilon(b)) = \left\| \mathcal{N}\left(e^{V''(0)\tilde{t}_\epsilon(b)} \Phi_{\tilde{t}_\epsilon(b)} e^{-b}, 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}$$

for every $b \in \mathbb{R}$. Therefore, by item *iii*) of Lemma 2.4 and by Lemma A.3, we have

$$\lim_{\epsilon \to 0} D^\epsilon(\tilde{t}_\epsilon(b)) = G(b)$$

for every $b \in \mathbb{R}$. By Lemma A.2, we have $\lim_{b\to+\infty} G(b) = 0$ and $\lim_{b\to-\infty} G(b) = 1$. Consequently, the theorem is proved. □

Corollary 2.7 (The First Order Approximation)**.** *Let us consider the Markov processes $y = \{y_t\}_{t\geq 0}$ which is given by the solution of the following linear stochastic differential equation,*

$$\begin{aligned} dy_t &= -V''(\psi_t)y_t dt + dW_t, \\ y_0 &= 0 \end{aligned}$$

for $t \geq 0$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and V is a regular potential. For every $\epsilon > 0$ fixed, let us define $z^{\epsilon}_t = \psi_t + \sqrt{\epsilon} y_t$ for every $t \geq 0$. Then the family $z^{\epsilon} = \{z^{\epsilon}_t\}_{t\geq 0}$ presents profile cut-off in the sense of Definition 1.3 with respect to the total variation distance when ϵ goes to zero. The profile function $G : \mathbb{R} \to \mathbb{R}$ is given by

$$G(b) \;:=\; \left\| \mathcal{N}\left(\tilde{c}e^{-b}, 1\right) - \mathcal{N}(0,1) \right\|_{\mathrm{TV}},$$

where $\tilde{c}$ is the non-zero constant given by

$$\lim_{t\to+\infty} e^{V''(0)t}\psi_t \;=\; \tilde{c}.$$

and the cut-off time t_ϵ and window time w_ϵ are given by

$$t_\epsilon = \frac{1}{2V''(0)}\left(\ln\left(1/\epsilon\right) + \ln\left(2V''(0)\right)\right)$$

and

$$w_\epsilon = \frac{1}{V''(0)}$$

respectively.

The proof of Theorem 2.3 can be adapted in order to prove this corollary in a straightforward way, so we omit it.

Remark 2.8. *The constants c and $\tilde{c}$ obtained in Lemma 2.4 depend on the initial condition of the semi-flow $\psi = \{\psi_t\}_{t\geq 0}$. Theorem 2.3 and Corollary 2.7 remain true if we take as window time $w'_\epsilon = w_\epsilon + \delta_\epsilon$ for each $\epsilon > 0$, where $\{\delta_\epsilon\}_{\epsilon>0}$ is any sequence of positive numbers such that $\lim_{\epsilon\to 0} \delta_\epsilon = 0$.*

2.2 The Gradient Case

From now on and up to the end of this chapter we will use the following notations and names.

Definition 2.9.

a) *The stochastic Markov process* $x^\epsilon := \{x_t^\epsilon\}_{t\geq 0}$ *defined in Theorem 2.1 is called the Itô diffusion.*

a) *The semi-flow* $\psi := \{\psi_t\}_{t\geq 0}$ *defined by the differential equation (2.1) is called the zeroth order approximation of* x^ϵ.

c) *The stochastic Markov process* $z^\epsilon := \{z_t^\epsilon\}_{t\geq 0}$ *defined in Corollary 2.7 is called the first order approximation of* x^ϵ.

The following lemma will give us the existence of a stationary probability measure for the Itô diffusion $x^\epsilon = \{x_t^\epsilon\}_{t\geq 0}$.

Lemma 2.10. *Let* V *be a regular potential and for every* $\epsilon > 0$, *let us consider the Itô diffusion* $x^\epsilon = \{x_t^\epsilon\}_{t\geq 0}$ *which is given by the following stochastic differential equation,*

$$\begin{aligned} dx_t^\epsilon &= -V'(x_t^\epsilon)dt + \sqrt{\epsilon}dW_t, \\ x_0^\epsilon &= x_0 \end{aligned}$$

for $t \geq 0$, *where* x_0 *is a deterministic point in* $\mathbb{R}\setminus\{0\}$ *and* $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion. Let us assume that*

$$\lim_{|x|\to+\infty} |V'(x)| = +\infty.$$

Then for every $\epsilon > 0$ *fixed, when* $t \to \infty$ *the probability distribution of* x_t^ϵ *converges in distribution to the probability* μ^ϵ *given by*

$$\mu^\epsilon(dx) = \frac{e^{-\frac{2}{\epsilon}V(x)}dx}{M^\epsilon},$$

where $M^\epsilon = \int_{\mathbb{R}} e^{-\frac{2}{\epsilon}V(z)}dz$.

Proof. For details see [23] and [26]. □

Now we will restrict our potential to the class of coercive regular potentials.

Definition 2.11 (Coercive Regular Potential)**.** *Let V be a regular potential. We say that V is a coercive regular potential if there exists $\delta > 0$ such that $V''(x) \geq \delta$ for every $x \in \mathbb{R}$.*

In the class of coercive regular potentials, we restrict ourselves to the class of potentials with bounded second and third derivatives.

Definition 2.12 (Smooth Coercive Regular Potential)**.** *Let V be a coercive regular potential. We say that V is a smooth coercive regular potential if*

$$\kappa_2 := \|V''\|_\infty := \sup_{x\in\mathbb{R}} |V''(x)| < \infty,$$

and

$$\kappa_3 := \|V'''\|_\infty := \sup_{x\in\mathbb{R}} |V''(x)| < \infty.$$

The following lemma tells us that the stationary probability measure of the Itô diffusion $\{x_t^\epsilon\}_{t\geq 0}$ is well approximated in total variation distance by the Normal distribution with mean zero and variance $\frac{\epsilon}{2V''(0)}$.

Lemma 2.13. *Let V be a coercive regular potential, then*

$$\lim_{\epsilon\to 0} \|\mu^\epsilon - \mathcal{N}^\epsilon\|_{\mathbb{TV}} = 0,$$

where $\mathcal{N}^\epsilon$ is a normal distribution with mean zero and variance $\frac{\epsilon}{2V''(0)}$.

Proof. Let $0 < \eta < V''(0)$ be fixed. By Lemma 2.10 the $\mu^\epsilon(dx) = \frac{e^{-\frac{2}{\epsilon}V(x)}dx}{M^\epsilon}$ is a well defined probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then

$$\|\mu^\epsilon - \mathcal{N}^\epsilon\|_{\mathbb{TV}} = \frac{1}{2}\int_{\mathbb{R}} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{M^\epsilon} - \frac{e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}}}{N^\epsilon} \right| dx,$$

where $M^\epsilon = \int_{\mathbb{R}} e^{-\frac{2}{\epsilon}V(x)}dx$ and $N^\epsilon = \int_{\mathbb{R}} e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}}dx = \sqrt{\frac{\pi\epsilon}{V''(0)}}$.

By triangle's inequality, we have

$$\begin{aligned}
\|\mu^\epsilon - \mathcal{N}^\epsilon\|_{\mathbb{TV}} &\leq \frac{1}{2}\int\limits_{\mathbb{R}} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{M^\epsilon} - \frac{e^{-\frac{2}{\epsilon}V(x)}}{N^\epsilon} \right| dx + \frac{1}{2}\int\limits_{\mathbb{R}} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{N^\epsilon} - \frac{e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}}}{N^\epsilon} \right| dx \\
&= \frac{|M^\epsilon - N^\epsilon|}{2N^\epsilon} + \frac{1}{2N^\epsilon}\int\limits_{\mathbb{R}} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}} \right| dx \\
&\leq \frac{1}{N^\epsilon}\int\limits_{\mathbb{R}} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}} \right| dx.
\end{aligned}$$

Recall that V is coercive, that is, there exists $\delta > 0$ such that $V''(x) \geq \delta > 0$ for every $x \in \mathbb{R}$. Then, it follows that

$$\lim_{\epsilon \to 0} \frac{1}{N^\epsilon} \int\limits_{\{x:|x|\geq \beta\}} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}} \right| dx = 0$$

for every $\beta > 0$. By the continuity of V'' at zero, there exists $\delta_\eta > 0$ such that

$$|V''(x) - V''(0)| < \eta$$

for every $|x| < \delta_\eta$.

Also, by Taylor's Theorem with Lagrange remainder, we have that $V''(x) = \frac{V''(\xi_x)x^2}{2}$ for every $|x| < \delta_\eta$ where $|\xi_x| < |x|$. Then,

$$\begin{aligned}
\frac{1}{N^\epsilon}\int\limits_{-\delta_\eta}^{\delta_\eta} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}} \right| dx &= \frac{1}{N^\epsilon}\int\limits_{-\delta_\eta}^{\delta_\eta} \left| e^{-\frac{2}{\epsilon}\frac{V''(\xi_x)x^2}{2}} - e^{-\frac{2}{\epsilon}\frac{V''(0)x^2}{2}} \right| dx \\
&\leq \frac{1}{\epsilon N^\epsilon}\int\limits_{-\delta_\eta}^{\delta_\eta} x^2 e^{-\frac{2}{\epsilon}\frac{\lambda x^2}{2}} |V''(\xi_x) - V''(0)|\, dx \\
&\leq \frac{\eta}{\epsilon N^\epsilon}\int\limits_{-\delta_\eta}^{\delta_\eta} x^2 e^{-\frac{2}{\epsilon}\frac{\lambda x^2}{2}} dx \leq \frac{\eta\sqrt{V''(0)}}{\sqrt{\pi}(2\lambda)^{3/2}} \int\limits_{-\delta_\eta\sqrt{\frac{2\lambda}{\epsilon}}}^{\delta_\eta\sqrt{\frac{2\lambda}{\epsilon}}} x^2 e^{-\frac{x^2}{2}} dx \\
&\leq \frac{\eta\sqrt{V''(0)}}{\sqrt{\pi}(2\lambda)^{3/2}} \int\limits_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx,
\end{aligned}$$

where $\lambda := \min\{\delta, V''(0)\} > 0$. Consequently, first taking $\epsilon \to 0$ and then $\eta \to 0$ we obtain the result. □

The following proposition will give us a quantitative estimation of the distance between the Itô diffusion and the zeroth order and first order approximations.

Proposition 2.14 (Zero Order & First Order Approximation)**.** *Let us assume that V is a smooth coercive regular potential. Let us denote $B_t = \sup_{0\leq s\leq t} |W_s|$ for every $t \geq 0$.*

i) For every $t \geq 0$, we have $|x^{\epsilon}_t - \psi_t| \leq \sqrt{\epsilon} B_t(\kappa_2 t + 1)$. We call this estimate the zeroth order estimate.

ii) For every $t \geq 0$, it follows that $|x^{\epsilon}_t - \psi_t - \sqrt{\epsilon} y_t| \leq \epsilon B^2_t \kappa_3(\kappa_2 t + 1)^2 t$. We call this estimate the first order estimate.

Proof. First we prove item $i)$. Let $\epsilon > 0$ and $t \geq 0$ be fixed. It follows that

$$\begin{aligned} x^{\epsilon}_t - \psi_t &= -\int_0^t (V'(x^{\epsilon}_s) - V'(\psi_s))\, ds + \sqrt{\epsilon} W_t \\ &= -\int_0^t V''(\theta^{\epsilon}_s)\, (x^{\epsilon}_s - \psi_s)\, ds + \sqrt{\epsilon} W_t \\ &= -\sqrt{\epsilon} \int_0^t V''(\theta^{\epsilon}_s) W_s e^{-\int_s^t V''(\theta^{\epsilon}_r) dr} ds + \sqrt{\epsilon} W_t, \end{aligned}$$

where the second inequality follows from the Intermediate Value Theorem, θ^{ϵ}_s is between ψ_s and x^{ϵ}_s and the third inequality follows from the variation of parameters method. Therefore, using Gronwall's inequality we obtain $|x^{\epsilon}_t - \psi_t| \leq \sqrt{\epsilon} B_t(\kappa_2 t + 1)$.

Now we prove item *ii*). Let $\epsilon > 0$ and $t \geq 0$ be fixed. It follows that

$$\begin{aligned}
x_t^\epsilon - \psi_t - \sqrt{\epsilon} y_t &= -\int_0^t \left[V'(x_s^\epsilon) - V'(\psi_s) - V''(\psi_s)\sqrt{\epsilon} y_s\right] ds \\
&= -\int_0^t \left[V''(\theta_s^\epsilon)\left(x_s^\epsilon - \psi_s\right) - V''(\psi_s)\sqrt{\epsilon} y_s\right] ds \\
&= -\int_0^t V''(\psi_s)(x_s^\epsilon - \psi_s - \sqrt{\epsilon} y_t) ds - \\
&\quad \int_0^t \left(V''(\theta_s^\epsilon) - V''(\psi_s)\right)(x_s^\epsilon - \psi_s) ds,
\end{aligned}$$

where the second identity comes from the Intermediate Value Theorem and θ_s^ϵ is between ψ_s and x_s^ϵ. Let us define $e_t := \int_0^t \left(V''(\theta_s^\epsilon) - V''(\psi_s)\right)(x_s^\epsilon - \psi_s) ds$. Again using the Intermediate Value Theorem and the zeroth order estimate already proved, we have

$$|e_t| \leq \int_0^t \kappa_3 (x_s^\epsilon - \psi_s)^2 ds \leq \epsilon B_t^2 \kappa_3 (\kappa_2 t + 1)^2 t$$

for every $t \geq 0$. Consequently, using the variation of parameters method and Gronwall's inequality we obtain

$$\left| x_t^\epsilon - \psi_t - \sqrt{\epsilon} y_t \right| \leq \epsilon B_t^2 \kappa_3 (\kappa_2 t + 1)^3 t.$$

□

This proposition will permit us to prove that two first order approximations with random initial conditions that are "near" are close in total variation distance. This statement is made rigorousin the following proposition.

Proposition 2.15 (Linear Coupling). *Let us assume the same hypothesis of Corollary 2.7 and in addition let us assume that V is a smooth coercive regular potential. Take $\{\delta_\epsilon := \epsilon^\gamma\}_{\epsilon > 0}$, where $0 < \gamma < 1$. Let us denote by $z^\epsilon(X) := \{z_t^\epsilon(X)\}_{t \geq 0}$ the first order* *approximation with initial random condition X. Then, for every $b \in \mathbb{R}$ it follows that*

$$\lim_{\epsilon \to 0} \left\| z^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) - z^{\epsilon}_{b\delta_\epsilon}\left(z^{\epsilon}_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} = 0,$$

where for each $\epsilon > 0$, t_ϵ and w_ϵ are defined in Corollary 2.7 and for each $b \in \mathbb{R}$, $\epsilon_b > 0$ is small enough so that $\tilde{t}_\epsilon(b) := t_\epsilon + bw_\epsilon \geq 0$ for every $0 < \epsilon < \epsilon_b$.

Proof. By Itô's formula we obtain

$$z^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) = \Phi_{b\delta_\epsilon} x^{\epsilon}_{\tilde{t}_\epsilon(b)} + \sqrt{\epsilon}\Phi_{b\delta_\epsilon}\int_0^{b\delta_\epsilon} \frac{1}{\Phi(s)} d\big(W_{\tilde{t}_\epsilon(b)+s} - W_{\tilde{t}_\epsilon(b)}\big),$$

$$z^{\epsilon}_{b\delta_\epsilon}\left(z^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) = \Phi_{b\delta_\epsilon} z^{\epsilon}_{\tilde{t}_\epsilon(b)} + \sqrt{\epsilon}\Phi_{b\delta_\epsilon}\int_0^{b\delta_\epsilon} \frac{1}{\Phi(s)} d\big(W_{\tilde{t}_\epsilon(b)+s} - W_{\tilde{t}_\epsilon(b)}\big)$$

for every $0 < \epsilon < \epsilon_b$, where $\Phi = \{\Phi_t\}_{t\geq 0}$ is the fundamental solution of the non-autonomous system

$$d\Phi_t = -V''(\psi_t)\Phi_t dt$$

for every $t \geq 0$ with initial condition $\Phi_0 = 1$. Applying Lemma B.6 with $X := \Phi_{b\delta_\epsilon} x^{\epsilon}_{\tilde{t}_\epsilon(b)}$, $Y := \Phi_{b\delta_\epsilon} z^{\epsilon}_{\tilde{t}_\epsilon(b)}$ and $Z := \sqrt{\epsilon}\Phi_{b\delta_\epsilon}\int_0^{b\delta_\epsilon} \frac{1}{\Phi(s)} d\big(W_{\tilde{t}_\epsilon(b)+s} - W_{\tilde{t}_\epsilon(b)}\big)$, $\mathcal{G} = \sigma(X, Y)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical probability space of the Brownian motion, we obtain

$$\left\| z^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) - z^{\epsilon}_{b\delta_\epsilon}\left(z^{\epsilon}_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} \leq \frac{1}{\sqrt{2\pi\epsilon \int_0^{b\delta_\epsilon} \left(\frac{1}{\Phi(s)}\right)^2 ds}} \mathbb{E}\left[\left| x^{\epsilon}_{\tilde{t}_\epsilon(b)} - z^{\epsilon}_{\tilde{t}_\epsilon(b)}\right|\right].$$

Using Proposition 2.14, we obtain

$$\left\| z^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) - z^{\epsilon}_{b\delta_\epsilon}\left(z^{\epsilon}_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} \leq \sqrt{\frac{\epsilon}{2\pi \int_0^{b\delta_\epsilon} \left(\frac{1}{\Phi(s)}\right)^2 ds}} \times \kappa_3\left(\kappa_2 \tilde{t}_\epsilon(b) + 1\right)^3 \tilde{t}_\epsilon(b)\mathbb{E}\left[B^2_{\tilde{t}_\epsilon(b)}\right].$$

Using the fact that for each $\epsilon > 0$, $\delta_\epsilon = \epsilon^\gamma$ for some $0 < \gamma < 1$, $\Phi_0 = 1$, the Intermediate

Value Theorem for integrals, Lemma D.1 and Lemma D.3 we obtain the result. □

The following proposition will permit us to change the probability measure in a small interval of time in order to compare the total variation distance of the Itô diffusion and the first order approximation with a random initial condition.

Proposition 2.16 (Short Time Change of Measure). *Let us assume the same hypothesis of Proposition 2.15 and also let us follow the same notation. Then for each* $b \in \mathbb{R}$

$$\lim_{\epsilon \to 0} \left\| x^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) - z^{\epsilon}_{b\delta_\epsilon}\left(x^{\epsilon}_{\tilde{t}_\epsilon(b)}\right) \right\|_{\mathbb{TV}} = 0.$$

Proof. We will use Cameron-Martin-Girsanov Theorem and Novikov's Theorem. For the precise statements of these theorems we use here, see [1] and [12]. Let $\epsilon > 0$, $t \geq 0$ and $b \in \mathbb{R}$ be fixed. Let us define $\gamma^{\epsilon}_t := \frac{V'(x^{\epsilon}_t)}{\sqrt{\epsilon}}$ and $\Gamma^{\epsilon}_t := \frac{(V'(\psi_t) - V''(\psi_t)\psi_t + V''(\psi_t) z^{\epsilon}_t)}{\sqrt{\epsilon}}$. Then, for every $\epsilon > 0$ and $t > 0$ it follows that

$$\begin{aligned} (\gamma^{\epsilon}_t)^2 &\leq 2\kappa_2^2 \frac{(x^{\epsilon}_t - \psi_t)^2}{\epsilon} + 2\kappa_2^2 \frac{(\psi_t)^2}{\epsilon} \\ &\leq 4\kappa_2^2 B_t^2 \left(\kappa_2 t^2 + 1\right) + 2\kappa_2^2 \frac{(\psi_t)^2}{\epsilon} \end{aligned}$$

and

$$\begin{aligned} (\Gamma^{\epsilon}_t)^2 &\leq 2\kappa_2^2 (y_t)^2 + 2\kappa_2^2 \frac{(\psi_t)^2}{\epsilon} \\ &\leq 4\kappa_2^2 B_t^2 \left(\kappa_2 t^2 + 1\right) + 2\kappa_2^2 \frac{(\psi_t)^2}{\epsilon}. \end{aligned}$$

Let us define $I^{\epsilon}(b) := \left[\tilde{t}_\epsilon(b), \tilde{t}_\epsilon(b) + b\delta_\epsilon\right]$. Then, for every $\epsilon > 0$ it follows that

$$\int\limits_{I(\epsilon)} (\gamma^{\epsilon}_t)^2 \leq 4b\kappa_2^2 \delta_\epsilon \left(\kappa_2 \left(\tilde{t}_\epsilon(b) + b\delta_\epsilon\right)^2 + 1\right) \sup_{t \in I^{\epsilon}(b)} B_t^2 + 2b\kappa_2^2 \delta_\epsilon \frac{\sup\limits_{t \in I^{\epsilon}(b)} (\psi_t)^2}{\epsilon}.$$

and

$$\int\limits_{I(\epsilon)} (\Gamma_t^\epsilon)^2 \;\leq\; 4b\kappa_2^2\delta_\epsilon \left(\kappa_2 \left(\tilde{t}_\epsilon(b) + b\delta_\epsilon\right)^2 + 1\right) \sup_{t\in I^\epsilon(b)} B_t^2 + 2b\kappa_2^2\delta_\epsilon \frac{\sup\limits_{t\in I^\epsilon(b)} (\psi_t)^2}{\epsilon}.$$

Using Lemma C.3, there exists a constant $c > 0$ such that

$$\int\limits_{I(\epsilon)} (\gamma_t^\epsilon)^2 \;\leq\; 4b\kappa_2^2\delta_\epsilon \left(\kappa_2 \left(\tilde{t}_\epsilon(b) + b\delta_\epsilon\right)^2 + 1\right) \sup_{t\in I^\epsilon(b)} B_t^2 + 2bc\kappa_2^2\delta_\epsilon$$

and

$$\int\limits_{I(\epsilon)} (\Gamma_t^\epsilon)^2 \;\leq\; 4b\kappa_2^2\delta_\epsilon \left(\kappa_2 \left(\tilde{t}_\epsilon(b) + b\delta_\epsilon\right)^2 + 1\right) \sup_{t\in I^\epsilon(b)} B_t^2 + 2bc\kappa_2^2\delta_\epsilon$$

for $\epsilon > 0$ small enough. Consequently, for any constant $\rho > 0$ it follows that

$$\mathbb{E}\left\{\exp\left[\rho \int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} (\gamma_s^\epsilon)^2\, ds\right]\right\} < +\infty$$

and

$$\mathbb{E}\left\{\exp\left[\rho \int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} (\Gamma_s^\epsilon)^2\, ds\right]\right\} < +\infty$$

for $\epsilon > 0$ small enough. From Novikov's Theorem it follows that

$$\begin{aligned}
\frac{d\mathbb{P}^1_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}}{d\mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}} &:= \exp\left\{\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \gamma_s^\epsilon dW_s - \frac{1}{2}\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} (\gamma_s^\epsilon)^2\, ds\right\}, \\
\frac{d\mathbb{P}^2_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}}{d\mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}} &:= \exp\left\{\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \Gamma_s^\epsilon dW_s - \frac{1}{2}\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} (\Gamma_s^\epsilon)^2\, ds\right\},
\end{aligned}$$

are well defined and they define true probability measures $\mathbb{P}^i_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$, $i \in \{1, 2\}$. From now on and up to the end of this proof we will use the notation $\mathbb{P}^i := \mathbb{P}^i_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$, $i \in \{1,2\}$ and $\mathbb{P} := \mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$. Under the probability measure $\mathbb{P}^1$, $W^1_t := W_t - \int_{\tilde{t}_\epsilon(b)}^{t} \gamma^\epsilon_s ds$ is a Brownian motion on the time interval $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$. Also, under the probability measure $\mathbb{P}^2$, $W^2_t := W_t - \int_{\tilde{t}_\epsilon(b)}^{t} \Gamma^\epsilon_s ds$ is a Brownian motion on the time interval $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$. Consequently,

$$
\begin{aligned}
\frac{d\mathbb{P}^1}{d\mathbb{P}^2} &= \frac{\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \gamma^\epsilon_s dW_s - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\gamma^\epsilon_s\right)^2 ds\right\}}{\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \Gamma^\epsilon_s dW_s - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon_s\right)^2 ds\right\}} \\
&= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\gamma^\epsilon_s - \Gamma^\epsilon_s\right) dW_s - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\left(\gamma^\epsilon_s\right)^2 - \left(\Gamma^\epsilon_s\right)^2\right) ds\right\} \\
&= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\gamma^\epsilon_s - \Gamma^\epsilon_s\right) dW^1_s + \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon_s - \gamma^\epsilon_s\right)^2 ds\right\}.
\end{aligned}
$$

By Pinsker's inequality and the mean-zero martingale property of the stochastic integral, we have for every $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$

$$
\begin{aligned}
\left\| x^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right) - z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} &\le \mathbb{E}_{\mathbb{P}^1}\left[\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon_s - \gamma^\epsilon_s\right)^2 ds\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{P}^1}{d\mathbb{P}}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon_s - \gamma^\epsilon_s\right)^2 ds\right].
\end{aligned}
$$

By Cauchy-Schwarz's inequality and the mean-one Doléans exponential martingale property, we have

$$\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{P}^1}{d\mathbb{P}}\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\,ds\right] \le \sqrt{\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\gamma_s^\epsilon)^2\,ds\right\}\left(\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\,ds\right)^2\right]}$$

$$\le \sqrt{\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\gamma_s^\epsilon)^2\,ds\right\}\right]}\times$$

$$\sqrt{\mathbb{E}_{\mathbb{P}}\left[\left(\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\,ds\right)^4\right]}$$

It follows for $\epsilon > 0$ small enough that

$$\exp\left\{\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\gamma_s^\epsilon)^2\,ds\right\} \le \exp\left\{4b\kappa_2^2\delta_\epsilon\left(\kappa_2\left(\tilde{t}_\epsilon(b)+b\delta_\epsilon\right)^2+1\right)\sup_{t\in I^\epsilon(b)}B_t^2+2bc\kappa_2^2\delta_\epsilon\right\},$$

where the last expression is $\mathbb{P}$-integrable for $\epsilon > 0$ small enough. Using the scaling property of Brownian motion and the distribution of the maximum of the Brownian motion in a compact interval, the last inequality implies that

$$\lim_{\epsilon\to 0}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\rho\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\gamma_s^\epsilon)^2\,ds\right\}\right] = 1.$$

for any constant $\rho > 0$. Also, it is true that

$$\left(\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\,ds\right)^4 \le \left(b\delta_\epsilon\sup_{s\in I^\epsilon(b)}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\right)^4$$

$$\le Cb^4\delta_\epsilon^4\left(\sup_{s\in I^\epsilon(b)}\frac{(x_s^\epsilon-\psi_s)^{16}}{\epsilon^4}+\sup_{s\in I^\epsilon(b)}\frac{|x_s^\epsilon-\psi_s-\sqrt{\epsilon}y_t|^8}{\epsilon^4}\right),$$

where $C = C(\kappa_2, \kappa_3) > 0$ is a constant. Using the last inequality and Proposition 2.14, we obtain that

$$\lim_{\epsilon\to 0}\mathbb{E}_{\mathbb{P}}\left[\left(\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}(\Gamma_s^\epsilon-\gamma_s^\epsilon)^2\,ds\right)^4\right] = 0.$$

Consequently,

$$\left\|x^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)-z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} = 0.$$

□

Now we have all the tools in order to prove our result for the class of bounded coercive regular potentials.

Theorem 2.17 (Smooth Coercive Regular Potentials)**.** *Assume the same hypothesis of Proposition 2.15 and also let us follow the same notation. Let us consider the family* $x^\epsilon=\{x^\epsilon_t\}_{t\geq 0}$ *given by the the semi-flow of the following stochastic differential equation,*

$$\begin{aligned} dx^\epsilon_t &= -V'(x^\epsilon_t)dt+\sqrt{\epsilon}dW_t,\\ x^\epsilon_0 &= x_0 \end{aligned}$$

for $t\geq 0$, *where* x_0 *is a deterministic point in* $\mathbb{R}\setminus\{0\}$ *and* $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion. This family presents profile cut-off in the sense of the Definition 1.3 with respect to the total variation distance when* ϵ *goes to zero. The profile function* $G:\mathbb{R}\to\mathbb{R}$ *is given by*

$$G(b) := \left\|\mathcal{N}\left(\tilde{c}e^{-b},1\right)-\mathcal{N}(0,1)\right\|_{\mathbb{TV}},$$

where $\tilde{c}$ *is the non-zero constant given by*

$$\lim_{t\to+\infty} e^{V''(0)t}\psi_t = \tilde{c}.$$

and the cut-off time t_ϵ *and window time* w_ϵ *are given by*

$$t_\epsilon := \frac{1}{2V''(0)}\left(\ln\left({}^1\!/\!_\epsilon\right)+\ln\left(2V''(0)\right)\right)$$

and

$$w_\epsilon \ := \ \frac{1}{V''(0)} + \epsilon^\gamma,$$

where $0 < \gamma < 1/4$

Proof. Let $\epsilon > 0$ and $t > 0$ be fixed. We define

$$D^\epsilon(t) := \|x_t^\epsilon - \mu^\epsilon\|_{\mathbb{TV}}$$

and

$$d^\epsilon(t) := \|z_t^\epsilon - \mathcal{N}^\epsilon\|_{\mathbb{TV}},$$

where μ^ϵ and $\mathcal{N}^\epsilon$ are given by Lemma 2.10 and Lemma 2.13. For each $b \in \mathbb{R}$, take $\epsilon_b > 0$ such that $\hat{t}^\epsilon(b) := t_\epsilon + b(w_\epsilon + \delta_\epsilon) = \tilde{t}^\epsilon(b) + b\delta_\epsilon \geq 0$ for every $0 < \epsilon < \epsilon_b$. By Corollary 2.7 and Remark 2.8 we know that for each $b \in \mathbb{R}$

$$\lim_{\epsilon \to 0} d^\epsilon\left(\hat{t}^\epsilon(b)\right) \ = \ G(b). \tag{2.4}$$

By definition,

$$\begin{aligned} D^\epsilon(\hat{t}^\epsilon(b)) \ &= \ \left\|x^\epsilon_{\hat{t}^\epsilon(b)} - \mu^\epsilon\right\|_{\mathbb{TV}} \\ &\leq \ \left\|x^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right) - z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} + \left\|z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right) - z^\epsilon_{b\delta_\epsilon}\left(z^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|z^\epsilon_{\hat{t}^\epsilon(b)} - \mathcal{N}^\epsilon\right\|_{\mathbb{TV}} + \|\mathcal{N}^\epsilon - \mu^\epsilon\|_{\mathbb{TV}}. \end{aligned}$$

Using Proposition 2.15, Proposition 2.16, Lemma 2.10, the relation (2.4) and the item $i)$ of Lemma D.2, we have $\limsup_{\epsilon \to 0} D^\epsilon(\hat{t}^\epsilon(b)) \leq G(b)$. In order to obtain the converse inequality, we observe that

$$\begin{aligned} d^\epsilon(\hat{t}^\epsilon(b)) \ &= \ \left\|z^\epsilon_{\hat{t}^\epsilon(b)} - \mathcal{N}^\epsilon\right\|_{\mathbb{TV}} \\ &\leq \ \left\|z^\epsilon_{b\delta_\epsilon}\left(z^\epsilon_{\tilde{t}_\epsilon(b)}\right) - z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} + \left\|z^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right) - x^\epsilon_{b\delta_\epsilon}\left(x^\epsilon_{\tilde{t}_\epsilon(b)}\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|x^\epsilon_{\hat{t}^\epsilon(b)} - \mu^\epsilon\right\|_{\mathbb{TV}} + \|\mu^\epsilon - \mathcal{N}^\epsilon\|_{\mathbb{TV}}. \end{aligned}$$

Again using Proposition 2.15, Proposition 2.16, Lemma 2.10, the relation (2.4) and the item $ii)$ of Lemma D.2 we have $\liminf_{\epsilon\to 0} D^\epsilon(\hat{t}^\epsilon(b)) \geq G(b)$. Consequently,

$$\lim_{\epsilon\to 0} D^\epsilon(\hat{t}^\epsilon(b)) = G(b).$$

□

The following proposition will permit us to approximate a coercive regular potential by a smooth coercive regular potential.

Proposition 2.18 (Removing Boundedness for V'' and V''')**.** *Let us assume that V is a coercive regular potential. For every $M \in]0,+\infty[$, there exists a smooth coercive regular potential $V_M(x)$ which is an approximation of V in the following way: $V_M(x) = V(x)$ for every $|x| \leq \sqrt{2}M$.*

Proof. By hypothesis there exists $\delta > 0$ such that $V''(x) \geq \delta$ for every $x \in \mathbb{R}$. Let $g \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ be an increasing function such that $g(u) = 0$ for $u \leq \frac{1}{2}$ and $g(u) = 1$ if $u \geq 1$. Let $M \in [1,\infty[$ be a fixed number. Let $R_M : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$R_M(x) \quad = \quad g\left(\frac{x^2}{4M^2}\right)\delta + \left(1 - g\left(\frac{x^2}{4M^2}\right)\right)V''(x).$$

Since $V \in \mathcal{C}^3(\mathbb{R},\mathbb{R})$ and $g \in \mathcal{C}^\infty(\mathbb{R},[0,1])$, we have $R_M \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$. We also have that $R_M(x) = V''(x)$ for every $|x| \leq \sqrt{2}M$, $R_M(x) = \delta$ for every $|x| \geq 2M$, $R_M(x) \geq \delta$ for every $x \in \mathbb{R}$, $\|R_M\|_\infty < \infty$ and $\|R'_M\|_\infty < \infty$. Let us define $S_M(x) := \int_0^x R_M(y)dy$ for every $x \in \mathbb{R}$ and let us define $V_M(x) := \int_0^x S_M(y)dy$. Then V_M is a smooth δ-coercive regular potential such that $V_M(x) = V(x)$ for every $|x| \leq \sqrt{2}M$. □

The next proposition will tell us that the approximation of the coercive regular potential by a smooth coercive regular potential also implies an approximation in the total variation distance of the invariant measures associated to the potential V and V_M and the total variation distance for the processes at the "cut-off time" associated to the potentials V and V_M.

Proposition 2.19. *Let V be a coercive regular potential and for every $M > 0$ let V_M be the approximation of V obtained from Proposition 2.18. Let $x^{\epsilon,M} = \left\{x_t^{\epsilon,M}\right\}_{t\geq 0}$ be the Itô diffusion associated to the smooth coercive potential V_M and let $\mu^{\epsilon,M}$ be the invariant probability measure associated to the stochastic process $x^{\epsilon,M}$ defined in Lemma 2.10. Let us denote by $x^\epsilon = \{x_t^\epsilon\}_{t\geq 0}$ the Itô diffusion associated to the coercive potential V and let* *us denote by μ^ϵ the invariant probability measure associated to the stochastic process x^ϵ defined in Lemma 2.10.*

It follows that

i) For every $M > 0$

$$\lim_{\epsilon \to 0} \left\| \mu^\epsilon - \mu^{\epsilon,M} \right\|_{\mathbb{TV}} = 0$$

ii) Using the same notation as in Theorem 2.17, we have

$$\lim_{\epsilon \to 0} \left\| x^\epsilon_{t_\epsilon(b)} - x^{\epsilon,M}_{t_\epsilon(b)} \right\|_{\mathbb{TV}} = 0$$

for every $M > |x_0|$ and every $b \in \mathbb{R}$.

Proof. Let us prove item *i*). Notice that $V''_M(0) = V''(0)$. By triangle's inequality and Lemma A.1, we have

$$\left\| \mu^\epsilon - \mu^{\epsilon,M} \right\|_{\mathbb{TV}} \leq \left\| \mu^\epsilon - \mathcal{N}^\epsilon \right\|_{\mathbb{TV}} + \left\| \mathcal{N}^\epsilon - \mu^{\epsilon,M} \right\|_{\mathbb{TV}}.$$

Taking $\epsilon \to 0$ and using Lemma 2.13 we obtain

$$\lim_{\epsilon \to 0} \left\| \mu^\epsilon - \mu^{\epsilon,M} \right\|_{\mathbb{TV}} = 0$$

for every $M > 0$. Now let us prove item *ii*). Let $\epsilon > 0$ and $M > |x_0| > 0$ be fixed. Let us define $\tau^{\epsilon,M} := \inf\left\{ s \geq 0 : \left| x^{\epsilon,M}_s \right| > M \right\}$. By the variational definition of total variation distance in terms of couplings, see (4.12) of [7],

$$\left\| x^\epsilon_{t_\epsilon(b)} - x^{\epsilon,M}_{t_\epsilon(b)} \right\|_{\mathbb{TV}} \leq \mathbb{P}_{x_0}\left(\tau^{\epsilon,M} \leq t_\epsilon(b) \right).$$

Let us define $\sigma^{\epsilon,M} := \inf\left\{ s \geq 0 : |x^{\epsilon,M}_s - \psi^M_s| > M - |x_0| \right\}$, where $\psi^M := \left\{ \psi^M_t \right\}$ is the semi-flow associated to the autonomous differential equation

$$d\psi^M_t = -V'_M\left(\psi^M_t \right)$$

for every $t \geq 0$ and $\psi^M_0 := x_0$. Using the coercivity hypothesis of V_M we see that the semi-flow ψ^M is decreasing in norm, and $|\psi^M_t| \leq |x_0|$ for every $t \geq 0$. In particular, $\sigma^{\epsilon,M} \leq \tau^{\epsilon,M}$. Consequently. $\mathbb{P}_{x_0}\left(\tau^{\epsilon,M} \leq t_\epsilon(b) \right) \leq \mathbb{P}_{x_0}\left(\sigma^{\epsilon,M} \leq t_\epsilon(b) \right)$.

Therefore it is enough to prove that $\lim\limits_{\epsilon\to 0}\mathbb{P}_{x_0}\left(\sigma^{\epsilon,M} > t_\epsilon(b)\right) = 1$. For every $s \geq 0$, let us define $z_s^{\epsilon,M} := \frac{x_s^{\epsilon,M}-\psi_s^M}{\sqrt{\epsilon}}$. Then $\sigma^{\epsilon,M} = \inf\left\{s \geq 0 : |z_s^{\epsilon,M}| > \frac{M-|x_0|}{\sqrt{\epsilon}}\right\}$. We note that

$$\mathbb{P}_{x_0}\left(\sigma^{\epsilon,M} \geq t_\epsilon(b)\right) = \mathbb{P}_{x_0}\left(\sup_{0\leq s\leq t_\epsilon(b)} \left|z_s^{\epsilon,M}\right| \leq \frac{M-|x_0|}{\sqrt{\epsilon}}\right).$$

Let us define $c_M := M - |x_0| > 0$. We have

$$\mathbb{P}_{x_0}\left(\sup_{0\leq s\leq t_\epsilon(b)} \left|z_s^{\epsilon,M}\right| > \frac{c_M}{\sqrt{\epsilon}}\right) = \mathbb{P}_{x_0}\left(\sup_{0\leq s\leq t_\epsilon(b)} \left(z_s^{\epsilon,M}\right)^2 > \frac{c_M^2}{\epsilon}\right).$$

Using Itô's formula and the coercivity of V_M, we have

$$\left(z_t^{\epsilon,M}\right)^2 \leq t + \Pi_t^{\epsilon,M}$$

for every $t \geq 0$, where the process $\Pi_t^{\epsilon,M} := 2\int\limits_0^t z_s^{\epsilon,M} dW_s$ is a martingale. Then

$$\mathbb{E}\left[\left(z_t^{\epsilon,M}\right)^2\right] \leq t$$

for every $t \geq 0$. Using Itô's isometry, we obtain

$$\mathbb{E}\left[\left(\Pi_t^{\epsilon,M}\right)^2\right] \leq 2t^2$$

for every $t \geq 0$. Let us take $\epsilon_{M,b} > 0$ such that for every $0 < \epsilon < \epsilon_{M,b}$, we have $c_M^2 - \epsilon t_\epsilon(b) > 0$. Using Doob's inequality, we have

$$\begin{aligned}
\mathbb{P}_{x_0}\left(\sup_{0\leq s\leq t_\epsilon(b)} \left(z_s^{\epsilon,M}\right)^2 > \frac{c_M^2}{\epsilon}\right) &\leq \mathbb{P}_{x_0}\left(\sup_{0\leq s\leq t_\epsilon(b)} \left|\Pi_s^{\epsilon,M}\right| > \frac{c_M^2 - \epsilon t_\epsilon(b)}{\epsilon}\right) \\
&\leq \frac{\epsilon^2}{\left(c_M^2 - \epsilon t_\epsilon(b)\right)^2}\mathbb{E}\left[\left(\Pi_{t_\epsilon(b)}^{\epsilon,M}\right)^2\right] \\
&\leq \frac{2\epsilon^2\left(t_\epsilon(b)\right)^2}{\left(c_M^2 - \epsilon t_\epsilon(b)\right)^2}.
\end{aligned}$$

Letting $\epsilon \to 0$ we obtain the desired limit. □

Now we are ready to prove Theorem 2.1. To stress the fact that Theorem 2.1 is just a consequence of what we have proved up to here, let us state this as a Lemma:

Lemma 2.20 (From the Smooth Coercive Case to the General Case)**.** *Let V_M be the approximation of V obtained in Proposition 2.18. Profile cut-off for $\{x_t^{\epsilon,M}\}_{t\geq 0}$ implies profile cut-off for $\{x_t^{\epsilon}\}_{t\geq 0}$ with the same cut-off time, cut-off window and profile function.*

Proof. Recall the notation introduced in Proposition 2.15. Let $\epsilon > 0$ and $t > 0$ be fixed. Let us take $M > \max\{|x_0|, \|\psi\|_\infty\}$. We define

$$D^{\epsilon,M}(t) := \left\|x_t^{\epsilon,M} - \mu^{\epsilon,M}\right\|_{\mathbb{TV}}$$

and

$$D^{\epsilon}(t) := \|x_t^{\epsilon} - \mu^{\epsilon}\|_{\mathbb{TV}}.$$

By triangle's inequality we have

$$D^{\epsilon,M}(t) \quad \leq \quad \left\|x_t^{\epsilon,M} - x_t^{\epsilon}\right\|_{\mathbb{TV}} + D^{\epsilon}(t) + \left\|\mu^{\epsilon} - \mu^{\epsilon,M}\right\|_{\mathbb{TV}}.$$

Recall that $t_\epsilon = \frac{1}{2V''(0)}\left(\ln\left({}^{1}/{}_{\epsilon}\right) + \ln\left(2V''(0)\right)\right)$ and $w_\epsilon = \frac{1}{V''(0)} + \delta_\epsilon$ respectively. Let $b \in \mathbb{R}$ be fixed. Recall that $t_\epsilon(b) = t_\epsilon + bw_\epsilon$. Take $\epsilon_b > 0$ such that for every $0 < \epsilon < \epsilon_b$ we have $t_\epsilon(b) > 0$. Consequently,

$$D^{\epsilon,M}(t_\epsilon(b)) \quad \leq \quad \left\|x_{t_\epsilon(b)}^{\epsilon,M} - x_{t_\epsilon(b)}^{\epsilon}\right\|_{\mathbb{TV}} + D^{\epsilon}(t_\epsilon(b)) + \left\|\mu^{\epsilon} - \mu^{\epsilon,M}\right\|_{\mathbb{TV}}.$$

Therefore, using Proposition 2.19 and Lemma D.2 we have

$$\limsup_{\epsilon\to 0} D^{\epsilon,M}(t_\epsilon(b)) \quad \leq \quad \limsup_{\epsilon\to 0} D^{\epsilon}(t_\epsilon(b)).$$

By Theorem 2.17, we know that $\lim\limits_{\epsilon\to 0} D^{\epsilon,M}(t_\epsilon(b)) = G(b)$. Therefore

$$G(b) \quad \leq \quad \limsup_{\epsilon\to 0} D^{\epsilon}(t_\epsilon(b)).$$

It also follows that

$$D^{\epsilon}(t_\epsilon(b)) \quad \leq \quad \left\|x_{t_\epsilon(b)}^{\epsilon} - x_{t_\epsilon(b)}^{\epsilon,M}\right\|_{\mathbb{TV}} + D^{\epsilon,M}(t_\epsilon(b)) + \left\|\mu^{\epsilon,M} - \mu^{\epsilon}\right\|_{\mathbb{TV}}.$$

Therefore, using Lemma D.2, Proposition 2.19 and Theorem 2.17 we have

$$\liminf_{\epsilon \to 0} D^{\epsilon}(t_{\epsilon}(b)) \leq G(b).$$

We conclude that

$$\lim_{\epsilon \to 0} D^{\epsilon}(t_{\epsilon}(b)) = G(b).$$

□

Chapter 3

Stochastic Perturbations: m-Dimensional Case

In this chapter we consider stochastic perturbations of a dynamical system evolving on $\mathbb{R}^m$ with $m \geq 2$. The assumptions and notations we will made on the potential V are the analogous ones made for the one-dimensional case. For the reader's convenience, we repeat them here. Let us consider the semi-flow $\{\psi(t)\}_{t\geq 0}$ associated to the solution of the following deterministic differential equation

$$dx(t) = -\nabla V(x(t))dt \tag{3.1}$$

for $t \geq 0$ and let $x(0) \in \mathbb{R}^m \setminus \{0\}$ be a fixed initial condition. The hypothesis made in Theorem 3.1 on the potential V guarantees existence and uniqueness of solutions of (3.1), as well as all the other (stochastic or deterministic) equations defined below. Our main result for m-dimensional potentials is the following:

Theorem 3.1 (Gradient Case)**.** *Let $V : \mathbb{R}^m \to [0, +\infty[$ be a m-dimensional potential satisfying:*

i) $V \in \mathcal{C}^2$ *and* $V(0) = 0$.

ii) $\nabla V(x) = 0$ *if and only if* $x = 0$.

iii) *There exist* $0 < \delta \leq \Delta$ *such that*

$$\delta\|y\|^2 \leq y^* H_V(x) y \leq \Delta\|y\|^2$$

for every $x, y \in \mathbb{R}^m$, where y^* is the transposed vector of y and H_V is the Hessian matrix of V.

Let us consider the family of processes $x^\epsilon = \{x^\epsilon(t)\}_{t\geq 0}$ which are given by the the semi-flow of the following stochastic differential equation,

$$\begin{aligned} dx^\epsilon(t) &= -\nabla V(x^\epsilon(t))dt + \sqrt{\epsilon}dW(t), \\ x^\epsilon(0) &= x_0 \end{aligned}$$

for $t \geq 0$, where x_0 is a deterministic vector in $\mathbb{R}^m \setminus \{0\}$ and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion. This family presents profile cut-off in the sense of Definition 1.3 with respect to the total variation distance when ϵ goes to zero. Let α_1 be the smallest eigenvalue of $H_V(0)$. For Lebesgue-almost every x_0, the profile function $G_{x_0} : \mathbb{R} \to [0,1]$ is given by

$$G_{x_0}(b) \quad := \quad \left\| \mathcal{G}\left(e^{-b}v(x_0), I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}},$$

where $v(x_0) \in \operatorname{span}(v_1)$ is the unique non-zero vector in $\mathbb{R}^m$ such that

$$\lim_{t\to+\infty} e^{\alpha_1 t}\psi(t) \quad = \quad v(x_0),$$

where v_1 is the eigenvector associated to the eigenvalue α_1 and the cut-off time t_ϵ and window time w_ϵ are given by

$$t_\epsilon = \frac{1}{2\alpha_1} \ln\left({}^1\!/\!_\epsilon\right)$$

and

$$w_\epsilon = \frac{1}{\alpha_1}$$

respectively.

Remark 3.2. *Since the potential V is coercive, we have $\alpha_1 \geq \delta > 0$.*

The assumptions made in Theorem 3.1 on the potential V are the m-dimensional counterpart of what we called smooth coercive potentials. At present time, we can not extend Theorem 3.1 potentials satisfying only the coercive bound $\delta \leq y^* H_V(x)y$ for any $x, y \in \mathbb{R}^m$. The following Theorem explains to which kind of potentials we are able to extend Theorem 3.1.

Theorem 3.3. *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a potential satisfying i), ii) and the lower bound $\delta \leq y^* H_V(x) y$ for any $x, y \in \mathbb{R}^m$ of Theorem 3.1. Let us* **suppose** *that there exists a potential $\tilde{V}$ satisfying i), ii) and iii) of Theorem 3.1 and such that there exists $r > 0$ such that $V(x) = \tilde{V}(x)$ for every $\|x\| \leq r$. Then, profile cut-off for $\{x^{\epsilon,M}(t)\}_{t\geq 0}$ implies profile cut-off for $\{x^\epsilon(t)\}_{t\geq 0}$ with the same cut-off time, cut-off window and profile function, for Lebesgue-almost every initial condition x_0 with $\|x_0\| < r$.*

The proof of this theorem is exactly the same of Lemma 2.20, so we omit it.

3.1 The Symmetric Ornstein-Uhlenbeck Case

For the reader convenience, we state and prove here a simple particular case of Theorem 3.1, namely when the potential V is quadratic.

Let us take $\mu \in \mathbb{R}^m$ and let $\Sigma \in \mathcal{S}_m$ be a symetric and positive definite square m-dimensional matrix. We denote by $\mathcal{G}(\mu, \Sigma)$ the Gaussian distribution with mean μ and covariance matrix Σ.

Proposition 3.4 (Symmetric Ornstein-Uhlenbeck Process)**.** *Let us consider the one-parameter family of processes $x^\epsilon = \{x^\epsilon(t)\}_{t\geq 0}$ which are given by the solution of the following stochastic differential equation,*

$$\begin{aligned} dx^\epsilon(t) &= -\alpha x^\epsilon(t) dt + \sqrt{\epsilon} dW(t), \\ x^\epsilon(0) &= x(0) \end{aligned}$$

for $t \geq 0$, where $x(0)$ is a deterministic point in $\mathbb{R}^m \setminus \{0\}$, α is a constant symmetric matrix with eigenvalues $0 < \alpha_1 \leq \ldots \leq \alpha_m$ and $\{W(t)\}_{t\geq 0}$ is an m-dimensional standard Brownian motion. This family presents profile cut-off in the sense of the Definition 1.3 with respect to the total variation distance when ϵ goes to zero. Let us write $x(0) = \sum\limits_{k=1}^{m} x_k v_k$ where $\{v_1, \ldots, v_m\}$ is an ordered orthonormal basis of $\mathbb{R}^m$ that conjugates the matrix α with the diagonal matrix $diag(\alpha_1, \ldots, \alpha_m)$. The profile function $G : \mathbb{R} \to \mathbb{R}$ is given by

$$G(b) := \left\| \mathcal{G}\left(\sqrt{2} e^{-b} x_\tau \alpha^{\frac{1}{2}} v_\tau, I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}},$$

where $\tau := \min\{i \in \{1, \ldots, m\} : x_i \neq 0\}$, and where the cut-off time t_ϵ and window time w_ϵ are given by

$$t_\epsilon := \frac{1}{2\alpha_\tau} \ln \left({}^1\!/\!_\epsilon\right)$$

and

$$w_\epsilon \;:=\; \frac{1}{\alpha_\tau}.$$

Proof. Since our process is linear, it is Gaussian. We have that its mean vector and covariance matrix are given by

$$\begin{aligned}\mu^\epsilon(t) &:= \mathbb{E}\left[x^\epsilon(t)\right] = e^{-\alpha t}x(0),\\ \Sigma^\epsilon(t) &:= \mathbb{V}\left[x^\epsilon(t)\right] = \epsilon\int\limits_0^t e^{-2\alpha(t-s)}ds = \frac{\epsilon}{2}\alpha^{-1}\left(I_m - e^{-2\alpha t}\right),\end{aligned}$$

respectively. Again, for each $\epsilon > 0$ fixed, when t goes to infinity we obtain that $x^\epsilon(t)$ converges in distribution to a random variable $x^\epsilon(+\infty)$ which has Gaussian distribution with mean vector $\mu^\epsilon := 0$ and variance matrix $\Sigma^\epsilon := \frac{\epsilon}{2}\alpha^{-1}$. For each $t > 0$, we denote by $\mathcal{G}(\mu^\epsilon(t), \Sigma^\epsilon(t))$ the law of the random variable $x^\epsilon(t)$ and by $\mathcal{G}(\mu^\epsilon, \Sigma^\epsilon)$ the law of the random variable $x^\epsilon(+\infty)$. For every $\epsilon > 0$ and $t > 0$, we write $d^\epsilon(t) := \|\mathcal{G}(\mu^\epsilon(t), \Sigma^\epsilon(t)) - \mathcal{G}(\mu^\epsilon, \Sigma^\epsilon)\|_{\mathbb{TV}}$. Using triangle's inequality for the total variation distance and Lemma B.1, for each $\epsilon > 0$ and $t > 0$, we obtain

$$\begin{aligned}d^\epsilon(t) &= \left\|\mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}e^{-\alpha t}x(0), \alpha^{-1}\left(I_m - e^{-2\alpha t}\right)\right) - \mathcal{G}\left(0, \alpha^{-1}\right)\right\|_{\mathbb{TV}}\\ &\leq \left\|\mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}e^{-\alpha t}x(0), \alpha^{-1}\left(I_m - e^{-2\alpha t}\right)\right) - \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}e^{-\alpha t}x(0), \alpha^{-1}\right)\right\|_{\mathbb{TV}}\\ &\quad + \left\|\mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}e^{-\alpha t}x(0), \alpha^{-1}\right) - \mathcal{G}\left(0, \alpha^{-1}\right)\right\|_{\mathbb{TV}}\\ &\leq \left\|\mathcal{G}\left(0, \alpha^{-1}\left(I_m - e^{-2\alpha t}\right)\right) - \mathcal{G}\left(0, \alpha^{-1}\right)\right\|_{\mathbb{TV}}\\ &\quad + \left\|\mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\alpha^{1/2}e^{-\alpha t}x(0), I_m\right) - \mathcal{G}(0, I_m)\right\|_{\mathbb{TV}},\end{aligned}$$

where in the last inequalities we use several times Lemma B.1. For each $\epsilon > 0$ and $t > 0$, let us define

$$D^\epsilon(t) \;:=\; \left\|\mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\alpha^{1/2}e^{-\alpha t}x(0), I_m\right) - \mathcal{G}(0, I_m)\right\|_{\mathbb{TV}}.$$

Let us consider the function $G : \mathbb{R} \to [0,1]$ defined by

$$G(b) := \left\| \mathcal{G}\left(\sqrt{2}e^{-b}c_\tau \alpha^{\frac{1}{2}} v_\tau, I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}},$$

where $c_\tau := \langle v_\tau, x(0)\rangle \neq 0$ since $x_\tau \neq 0$. It follows that that $G(-\infty) = 1$ and $G(+\infty) = 0$ by Lemma B.4 and Lemma B.2, respectively. For each $\epsilon > 0$ let us define $t_\epsilon := \frac{1}{2\alpha_\tau} \ln\left({}^1\!/\!_\epsilon\right)$ and $w_\epsilon := {}^1\!/\!_{\alpha_\tau}$. Note that for any $\epsilon > 0$ and $b \in \mathbb{R}$ we have

$$D^\epsilon(t_\epsilon + bw_\epsilon) \;\; = \;\; \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}} \alpha^{1/2} e^{-\alpha(t_\epsilon + bw_\epsilon)} x(0), I_m \right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}.$$

Using Lemma B.3, we obtain that

$$\lim_{\epsilon \to 0} D^\epsilon(t_\epsilon + bw_\epsilon) \;\; = \;\; G(b).$$

Now we will prove that $\lim\limits_{\epsilon\to 0} |D^\epsilon(\tilde{t}_\epsilon(b)) - d^\epsilon(\tilde{t}_\epsilon(b))| = 0$, where $\tilde{t}_\epsilon(b) = t_\epsilon + bw_\epsilon$. Using triangle's inequality for the total variation distance, we have

$$|D^\epsilon(\tilde{t}_\epsilon(b)) - d^\epsilon(\tilde{t}_\epsilon(b))| \;\; \leq \;\; \left\| \mathcal{G}\left(0, \alpha^{-1}\left(I_m - e^{-2\alpha \tilde{t}_\epsilon(b)}\right)\right) - \mathcal{G}\left(0, \alpha^{-1}\right) \right\|_{\mathbb{TV}}.$$

By the last inequality and Lemma B.5, we conclude that

$$\lim_{\epsilon \to 0} \left(D^\epsilon(\tilde{t}_\epsilon(b)) - d^\epsilon(\tilde{t}_\epsilon(b))\right) \;\; = \;\; 0$$

for any $b \in \mathbb{R}$. Consequently, for any $b \in \mathbb{R}$

$$\lim_{\epsilon \to 0} d^\epsilon(\tilde{t}_\epsilon(b)) = G(b),$$

which is what we wanted to prove. □

3.2 The Linearized Case

Recall the strategy of proof of the one-dimensional case, Theorem 2.1. As an important intermediate step we prove profile cut-off for a family of processes satisfying a linear, non-homogeneous stochastic differential equation, stated in Corollary 2.7. In what follows we prove the m-dimensional version of this Corollary.

This result holds for a more general class of potentials that Theorem 3.1, which we define as follows.

Definition 3.5 (Regular Coercive Potential)**.** *We say that V is a coercive regular potential if $V : \mathbb{R}^m \to \mathbb{R}$ satisfies*

a) $V(0) = 0$ and $V \in \mathcal{C}^2$.

b) $\nabla V(x) = 0$ if and only if $x = 0$.

c) There is $\delta > 0$ such that $y^* H_V(x) y \geq \delta \|y\|^2$ for every $x, y \in \mathbb{R}^m$, where H_V is the Hessian matrix of V.

The following theorem tells us that the "linear approximations" have profile cut-off.

Theorem 3.6 (The Linearized Case)**.** *Let V be a coercive regular potential. Let us consider the family of processes $y^\epsilon = \{y^\epsilon(t) := \psi(t) + \sqrt{\epsilon} y(t)\}_{t\geq 0}$, where $\{y(t)\}_{t\geq 0}$ is the solution of the following linear stochastic differential equation,*

$$\begin{aligned} dy(t) &= -H_V(\psi(t)) y(t) dt + dW(t), \\ y(0) &= 0 \end{aligned}$$

for $t \geq 0$, where $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion, H_V is the Hessian matrix of V and $\{\psi(t)\}_{t\geq 0}$ is the semi-flow associated to (3.1) with initial condition x_0. Let α_1 be the smallest eigenvalue of $H_V(0)$ and let V_1 be its eigenspace. Let $v(x_0)$ be the unique vector in V_1 such that

$$\lim_{t\to+\infty} e^{\alpha_1 t}\psi(t) = v(x_0).$$

Assume that $v(x_0) \neq 0$ and define the cut-off profile $G_{x_0} : \mathbb{R} \to [0, 1]$ as

$$G_{x_0}(b) := \left\| \mathcal{G}\left(\sqrt{2} e^{-b} H_V(0)^{\frac{1}{2}} v(x_0), I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}.$$

Then the family $\{y^\epsilon\}_{\epsilon>0}$ presents profile cut-off in the sense of [14] with respect to the total variation distance when ϵ goes to zero with profile function G_{x_0} and cut-off time t_ϵ and window time w_ϵ given by

$$t_\epsilon = \frac{1}{2\alpha_1} \ln\left(1/\epsilon\right)$$

and

$$w_\epsilon = \frac{1}{\alpha_1}.$$

Remark 3.7. *By item ii) of Lemma 3.8 below, $v(x_0)$ is well defined and nonzero for Lebesgue-almost every x_0. In particular, Theorem 3.6 holds for Lebesgue-almost every initial condition $x_0 \in \mathbb{R}^m \setminus \{0\}$.*

We can see that the Ornstein-Uhlenbeck case is covered by

$$V(x) \;=\; x^* diag(\alpha_1, \ldots, \alpha_m)x,$$

$x \in \mathbb{R}^m$ and $\alpha_k > 0$ for every $k \in \{1, \ldots, m\}$. In order to prove Theorem 3.6 we need to find the qualitative behavior of the semi-flow $\psi = \{\psi(t)\}_{t\geq 0}$ at infinity.

Lemma 3.8. *Under the hypothesis of Theorem 3.6, we have*

i) For any initial condition x_0, $\psi(t)$ goes to zero as t goes to infinity. Moreover, $\|\psi(t)\| \leq \|x_0\| e^{-\delta t}$ for every $t \geq 0$.

ii) For Lebesgue-almost every x_0,

$$\lim_{t\to+\infty} e^{\alpha_1 t}\psi(t) \;=\; v(x_0) \in \mathbb{R}^m \setminus \{0\},$$

where $v(x_0) \in V_1$ and V_1 is the eigenspace associated to the eigenvalue α_1.

iii) Let us consider the following matrix differential equation,

$$\begin{aligned} d\Lambda^\epsilon(t) &= -H_V(0)\Lambda^\epsilon(t) - \Lambda^\epsilon(t)H_V(0) + \epsilon I_m, \\ \Lambda^\epsilon(0) &= M_0, \end{aligned}$$

where M_0 is a square matrix of dimension m. We have

$$\lim_{t\to\infty} \Lambda^\epsilon(t) = \frac{\epsilon}{2}\left(H_V(0)\right)^{-1}.$$

iv) Let us define the covariance matrix $\Delta^\epsilon(t) := \epsilon\mathbb{E}\left[y(t)\left(y(t)\right)^*\right]$. This matrix satisfies the following matrix differential equation,

$$\begin{aligned} d\Delta^\epsilon(t) &= -H_V(\psi(t))\Delta^\epsilon(t) - \Delta^\epsilon(t)H_V(\psi(t)) + \epsilon I_m, \\ \Delta^\epsilon(0) : &= 0. \end{aligned}$$

We have

$$\lim_{t\to\infty} \Delta^{\epsilon}(t) = \frac{\epsilon}{2}\left(H_V(0)\right)^{-1}.$$

For the proof, see Appendix C.

For each $\epsilon > 0$ and $t > 0$ fixed, $y^{\epsilon}(t)$ is a Gaussian random variable so it is characterized by its mean vector and covariance matrix. The mean vector is given by

$$\nu^{\epsilon}(t) \ := \ \mathbb{E}\left[y^{\epsilon}(t)\right] = \psi(t)$$

and the covariance matrix is given by

$$\eta^{\epsilon}(t) := \mathbb{V}\left[y^{\epsilon}(t)\right] = \epsilon\mathbb{V}\left[y(t)\right] = \epsilon\mathbb{E}\left[y(t)\left(y(t)\right)^{*}\right].$$

Corollary 3.9. *Let us assume the hypothesis of Theorem 3.6. Let $\epsilon > 0$ be fixed, then the random variable $y^{\epsilon}(t)$ converges in distribution as t goes to infinity to a Gaussian random variable $y^{\epsilon}(+\infty)$ with mean zero vector and covariance matrix $\frac{\epsilon}{2}(H_V(0))^{-1}$.*

Proof. It follows by item $i)$ and item $iv)$ of Lemma 3.8. □

Now, we have all the tools in order to prove Theorem 3.6.

Proof. Let us call $\alpha := H_V(0)$. For each $\epsilon > 0$ and $t > 0$, we define

$$\begin{aligned} d^{\epsilon}(t) \ &:= \ \left\| \mathcal{G}\left(\nu^{\epsilon}(t), \eta^{\epsilon}(t)\right) - \mathcal{G}\Big(0, \frac{\epsilon}{2}(H_V(0))^{-1}\Big) \right\|_{\mathbb{TV}} \\ &= \ \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\psi(t), \eta(t)\right) - \mathcal{G}\big(0, \alpha^{-1}\big) \right\|_{\mathbb{TV}}, \end{aligned}$$

where $\eta(t) := 2\mathbb{V}\left[y(t)\right]$ and

$$D^{\epsilon}(t) \ := \ \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\alpha^{\frac{1}{2}}\psi_t, I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}.$$

Using triangle's inequality and Lemma B.1, for each $\epsilon > 0$ and $t > 0$, we obtain

$$\begin{aligned} d^{\epsilon}(t) &\leq \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\psi(t), \eta(t)\right) - \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\psi(t), \alpha^{-1}\right) \right\|_{\mathbb{TV}} + \\ &\quad \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\psi(t), \alpha^{-1}\right) - \mathcal{G}\big(0, \alpha^{-1}\big) \right\|_{\mathbb{TV}}, \\ &= \left\| \mathcal{G}\left(0, \eta(t)\right) - \mathcal{G}\big(0, \alpha^{-1}\big) \right\|_{\mathbb{TV}} + \left\| \mathcal{G}\left(\sqrt{\frac{2}{\epsilon}}\alpha^{\frac{1}{2}}\psi(t), I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}. \end{aligned}$$

Therefore,

$$|d^{\epsilon}(t) - D^{\epsilon}(t)| \leq \left\| \mathcal{G}\left(0, \eta(t)\right) - \mathcal{G}\big(0, \alpha^{-1}\big) \right\|_{\mathbb{TV}}.$$

Recall that $0 < \alpha_1 < \ldots < \alpha_m$ denote the eigenvalues of the matrix α. For each $\epsilon > 0$ let us define $t_{\epsilon} := \frac{1}{2\alpha_1} \ln\left({}^{1}\!/_{\epsilon}\right)$ and $w_{\epsilon} := {}^{1}\!/_{\alpha_1}$. For every $b \in \mathbb{R}$, we define $\tilde{t}_{\epsilon}(b) = t_{\epsilon} + b w_{\epsilon}$. Using the last inequality and Lemma B.5, we obtain

$$\lim_{\epsilon \to 0} |d^{\epsilon}(\tilde{t}_{\epsilon}(b)) - D^{\epsilon}(\tilde{t}_{\epsilon}(b))| = 0$$

for every $b \in \mathbb{R}$. By item *ii)* of Lemma 3.8, for Lebesgue-almost every x_0, it follows that

$$\lim_{t \to +\infty} e^{\alpha_1 t}\psi(t) = v(x_0) \in \mathbb{R}^m \setminus \{0\}.$$

Let us consider the function $G_{x_0} : \mathbb{R} \to [0, 1]$ defined by

$$G_{x_0}(b) := \left\| \mathcal{G}(\sqrt{2}e^{-b}\alpha^{\frac{1}{2}}v(x_0), I_m) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}.$$

Observe that $D^{\epsilon}(\tilde{t}_{\epsilon}(b)) = \left\| \mathcal{G}\left(\sqrt{2}\alpha^{\frac{1}{2}} \frac{\psi(\tilde{t}_{\epsilon}(b))}{\sqrt{\epsilon}}, I_m\right) - \mathcal{G}(0, I_m) \right\|_{\mathbb{TV}}$ for every $b \in \mathbb{R}$. Consequently, we have

$$\lim_{\epsilon \to 0} D^{\epsilon}(\tilde{t}_{\epsilon}(b)) = G_{x_0}(b)$$

for every $b \in \mathbb{R}$. It also follows that $\lim_{b \to +\infty} G_{x_0}(b) = 0$ and $\lim_{b \to -\infty} G_{x_0}(b) = 1$ by the same facts usied in the proof of the Ornstein-Uhlenbeck Case. This proves the theorem. □

Remark 3.10. *In Theorem 3.6 we can take as a window time $w'_{\epsilon} > 0$ such that $\lim_{\epsilon \to 0} w'_{\epsilon} = w > 0$.*

3.3 The General Case

Let us fix some notations and names.

Definition 3.11. *a) We call the process $x^\epsilon := \{x^\epsilon(t)\}_{t\geq 0}$ defined in Theorem 3.1 an m-dimensional* Itô's diffusion.

a) *We call the semi-flow $\psi := \{\psi(t)\}_{t\geq 0}$ defined by the differential equation (3.1) the* zeroth-order approximation *of x^ϵ.*

c) *We call the process $y^\epsilon := \{y^\epsilon(t) := \psi(t) + \sqrt{\epsilon}y(t)\}_{t\geq 0}$ defined in Theorem 3.6 the* first order approximation *of x^ϵ.*

The following lemma tells us the existence of a stationary probability measure for the Itô's diffusion $x^\epsilon = \{x^\epsilon(t)\}_{t\geq 0}$.

Lemma 3.12. *Let V a regular coercive potential and for every $\epsilon > 0$ let us consider the Itô's diffusion $x^\epsilon = \{x^\epsilon(t)\}_{t\geq 0}$ given by the following stochastic differential equation,*

$$\begin{aligned} dx^\epsilon(t) &= -\nabla V(x^\epsilon(t))dt + \sqrt{\epsilon}dW(t), \\ x^\epsilon(0) &= x(0) \end{aligned}$$

for $t \geq 0$, where $x(0)$ is a deterministic point in $\mathbb{R}^m \setminus \{0\}$ and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^m$. Then, for every $\epsilon > 0$ fixed, when t goes to infinity the probability distribution of $x^\epsilon(t)$ converges in distribution to the probability μ^ϵ given by

$$\mu^\epsilon(dx) = \frac{e^{-\frac{2}{\epsilon}V(x)}dx}{M^\epsilon},$$

where $M^\epsilon = \int\limits_{\mathbb{R}^m} e^{-\frac{2}{\epsilon}V(z)}dz$.

Proof. For the proof of this lemma and further considerations, see [23] and [26]. □

The following lemma tells us that the stationary probability measure of the Itô's process $\{x_t^\epsilon\}_{t\geq 0}$ is well approximated in total variation distance by the Gaussian distribution with mean zero and covariance matrix $\frac{\epsilon}{2}\left(H_V(0)\right)^{-1}$.

Lemma 3.13. *Let V be a coercive regular potential. Then*

$$\lim_{\epsilon\to 0} \|\mu^\epsilon - \mathcal{G}^\epsilon\|_{\mathbb{TV}} = 0,$$

where $\mathcal{G}^\epsilon$ is a Gaussian distribution with mean zero and covariance matrix $\frac{\epsilon}{2}\left(H_V(0)\right)^{-1}$.

Proof. Let $0 < \eta < 1$ be fixed. By Lemma 3.12, the measure $\mu^\epsilon(dx) = \frac{e^{-\frac{2}{\epsilon}V(x)}dx}{M^\epsilon}$ is a well-defined probability measure. Then

$$\|\mu^\epsilon - \mathcal{G}^\epsilon\|_{\mathbb{TV}} = \frac{1}{2}\int\limits_{\mathbb{R}^m} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{M^\epsilon} - \frac{e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}}}{N^\epsilon} \right| dx,$$

where $M^\epsilon = \int\limits_{\mathbb{R}^m} e^{-\frac{2}{\epsilon}V(x)}dx$ and $N^\epsilon = \int\limits_{\mathbb{R}^m} e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}}dx = (\pi\epsilon)^{\frac{m}{2}}\left(\det\left((H_V(0))^{-1}\right)\right)^{\frac{1}{2}}$. By triangle's inequality, we have

$$\begin{aligned}
\|\mu^\epsilon - \mathcal{G}^\epsilon\|_{\mathbb{TV}} &\leq \frac{1}{2}\int\limits_{\mathbb{R}^m} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{M^\epsilon} - \frac{e^{-\frac{2}{\epsilon}V(x)}}{N^\epsilon} \right| dx + \frac{1}{2}\int\limits_{\mathbb{R}^m} \left| \frac{e^{-\frac{2}{\epsilon}V(x)}}{N^\epsilon} - \frac{e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}}}{N^\epsilon} \right| dx \\
&= \frac{|M^\epsilon - N^\epsilon|}{2N^\epsilon} + \frac{1}{2N^\epsilon}\int\limits_{\mathbb{R}^m} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}} \right| dx \\
&\leq \frac{1}{N^\epsilon}\int\limits_{\mathbb{R}^m} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}} \right| dx.
\end{aligned}$$

By coercivity, we have that there exist $\delta > 0$ such that $V(x) \geq \frac{\delta}{2}\|x\|^2$ for every $x \in \mathbb{R}^m$. Then

$$\lim_{\epsilon \to 0} \frac{1}{N^\epsilon} \int\limits_{\{x \in \mathbb{R}^m : \|x\| > \beta\}} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{x^* H_V(0)x}{2}} \right| dx = 0$$

for every $\beta > 0$. By the second-order Taylor's Theorem for scalar fields, we have that there exists $0 < \vartheta < 1$ such that for every $\|x\| < \vartheta$,

$$V(x) = \frac{x^* H_V(cx)x}{2},$$

where $c = c(x) \in]0,1[$. By continuity, we can take $0 < \delta_\eta < \vartheta$ such that for every

$\|x\| < \vartheta_\eta$, we have $\|H_V(cx) - H_V(0)\| < \eta$. Then,

$$\frac{1}{N^\epsilon} \int\limits_{\{x\in\mathbb{R}^m : \|x\| < \vartheta_\eta\}} \left| e^{-\frac{2}{\epsilon}V(x)} - e^{-\frac{2}{\epsilon}\frac{x^* H_V(0) x}{2}} \right| dx \leq$$

$$\leq \frac{1}{\epsilon N^\epsilon} \int\limits_{\{x\in\mathbb{R}^m : \|x\| < \vartheta_\eta\}} e^{-\frac{1}{\epsilon}\delta\|x\|^2} \|x^* H_V(cx) x - x^* H_V(0) x\| \, dx$$

$$\leq \frac{\eta}{\epsilon N^\epsilon} \int\limits_{\{x\in\mathbb{R}^m : \|x\| < \vartheta_\eta\}} e^{-\frac{1}{\epsilon}\delta\|x\|^2} \|x\|^2 dx \quad \leq C\eta \int\limits_{\left\{x\in\mathbb{R}^m : \|x\| < \vartheta_\eta \sqrt{\frac{1}{\epsilon}}\right\}} e^{-\delta\|x\|^2} \|x\|^2 dx$$

$$\leq C\eta \int\limits_{\mathbb{R}^m} e^{-\delta\|x\|^2} \|x\|^2 dx,$$

where $C > 0$ is an explicit constant independent of ϵ and η. Consequently, first taking $\epsilon \to 0$ and then $\eta \to 0$ we obtain the result. □

The following proposition will give us the zeroth-order and first-order approximations for Itô's diffusion x^ϵ.

Proposition 3.14 (Zeroth-Order and First-Order Approximation)**.** *Let V be a coercive regular potential. Let us write $B(t) := \sup\limits_{0\leq s\leq t} \|W(s)\|$ for $t \geq 0$.*

i) For every $t \geq 0$, we have $\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|^{2n}\right] \leq c_n \epsilon^n t^n$, where $c_n := \prod\limits_{j=0}^{n-1} (m + 2j)$ for every $n \in \mathbb{N}$.

ii) For every $b \in \mathbb{R}$, there exists $\epsilon_0 > 0$ small enough such that for every $0 < \epsilon < \epsilon_0$,

$$\mathbb{E}\left[\exp\left\{\delta_\epsilon \frac{\|x^\epsilon(t_\epsilon + b\delta_\epsilon) - \psi(t_\epsilon + b\delta_\epsilon)\|^2}{\epsilon}\right\}\right] < +\infty,$$

where $\delta_\epsilon = \epsilon^\gamma$, $\gamma > 0$.

iii) For every $b \in \mathbb{R}$ there exists $\epsilon_0 > 0$ small enough such that for every $0 < \epsilon < \epsilon_0$,

$$\mathbb{E}\left[\exp\left\{\delta_\epsilon \frac{\|x^\epsilon(t_\epsilon + b\delta_\epsilon) - \psi(t_\epsilon + b\delta_\epsilon)\|^2}{\epsilon}\right\}\right] \leq e^{\delta_\epsilon(t_\epsilon + b\delta_\epsilon)m},$$

where $\delta_\epsilon = \epsilon^\gamma$,$\gamma > 0$.

iv) For every $r > 0$ there exist a constant $c(r) > 0$ and $\epsilon_0 > 0$ such that

$$\mathbb{P}\left(\sup_{t \leq t_\epsilon + bw_\epsilon} \|x^\epsilon(t) - \psi(t)\|^2 \geq r\right) \leq c(r)\epsilon^2 (t_\epsilon + bw_\epsilon)^2$$

for every $0 < \epsilon < \epsilon_0$.

v) Assume that there exists $K > 0$ such that

$$\|\nabla V(x) - \nabla V(y)\| \leq K\|x - y\|$$

for every $x, y \in \mathbb{R}^m$. Let $b \in \mathbb{R}$ and let us call $t^* := t_\epsilon + b(w_\epsilon + \delta_\epsilon)$, where $\lim_{\epsilon \to 0} \delta_\epsilon = 0$, t_ϵ and w_ϵ are defined in Theorem 3.6. Then, there exists $\epsilon_0 > 0$ such that

$$\mathbb{E}\left[\|x^\epsilon(t^*) - \psi(t^*) - \sqrt{\epsilon}y(t^*)\|^2\right] \leq C\epsilon^{\frac{3}{2}}(t^*)^{\frac{5}{2}}$$

for every $0 < \epsilon < \epsilon_0$, where $C = C(K, b) > 0$ is a fixed constant.

Proof.

i) Let $\epsilon > 0$ and $t \geq 0$ be fixed. We have

$$\begin{aligned} x^\epsilon(t) - \psi(t) &= -\int_0^t [\nabla V(x^\epsilon(s)) - \nabla V(\psi(s))]\, ds + \sqrt{\epsilon}W(t) \\ &= -\int_0^t \left[\int_0^1 H_V(\psi(s) + \theta(x^\epsilon(s) - \psi(s)))d\theta\right](x^\epsilon(s) - \psi(s))\, ds + \\ &\quad \sqrt{\epsilon}W(t) \\ &= -\int_0^t A^\epsilon(s)(x^\epsilon(s) - \psi(s))\, ds + \sqrt{\epsilon}W(t), \end{aligned}$$

where $A^\epsilon(s) := \int_0^1 H_V(\psi(s) + \theta(x^\epsilon(s) - \psi(s)))d\theta$ and where the second identity follows from the Intermediate Value Theorem for vectorial functions. Let us take $f_1(x) = \|x\|^2$. By Itô formula, it follows that

$$\begin{aligned} d\|x^\epsilon(t) - \psi(t)\|^2 &= [-2(x^\epsilon(t) - \psi(t))^* A^\epsilon(t)(x^\epsilon(t) - \psi(t))\, \epsilon m]\, dt + \\ &\quad 2\sqrt{\epsilon}(x^\epsilon(t) - \psi(t))^* dW(t) \end{aligned}$$

for every $t \geq 0$. Using the coercivity hypothesis for V, we obtain

$$d\|x^\epsilon(t) - \psi(t)\|^2 \leq \epsilon m dt + M_t dW(t)$$

for every $t \geq 0$, where $M(t) := 2\sqrt{\epsilon}\left(x^\epsilon(t) - \psi(t)\right)^*$ for every $t \geq 0$. Notice that $\left\{N(t) := \int\limits_0^t M(s)dW(s)\right\}_{t\geq 0}$ is a local martingale. Then there exists a sequence of increasing stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that almost surely $\tau_n \uparrow \infty$ as n goes to infinity and $\{N^n(t) := N\left(\min\{\tau_n, t\}\right)\}_{t\geq 0}$ is a martingale for every $n \in \mathbb{N}$ fixed. Therefore, taking expectation, using the fact that $\{N^n(t)\}_{t\geq 0}$ is a local martingale for every $n \in \mathbb{N}$ fixed and the fact that V is coercive, we obtain

$$\begin{aligned}
\mathbb{E}\left[\|x^\epsilon\left(\min\{\tau_n, t\}\right) - \psi\left(\min\{\tau_n, t\}\right)\|^2\right] &\leq \epsilon m \min\{\tau_n, t\} \\
&\leq \epsilon dt
\end{aligned}$$

for every $t \geq 0$. Consequently, using Fatou's Lemma, we obtain

$$\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|^2\right] \leq \epsilon dt$$

for every $t \geq 0$. Let us consider $f_{n+1}(x) = \|x\|^{2(n+1)}$. By Itô formula, it follows that

$$\begin{aligned}
d\|x^\epsilon(t) - \psi(t)\|^{2(n+1)} &= \left[-2(n+1)\|x^\epsilon(t) - \psi(t)\|^{2n}\left(x^\epsilon(t) - \psi(t)\right)^*\right. \\
&\quad \left. A^\epsilon(t)\left(x^\epsilon(t) - \psi(t)\right)\right] dt \\
&+ \left[\epsilon(m + 2n)(n+1)\|x^\epsilon(t) - \psi(t)\|^{2n}\right] dt + \\
&+ 2(n+1)\sqrt{\epsilon}\|x^\epsilon(t) - \psi(t)\|^{2n}\left(x^\epsilon(t) - \psi(t)\right)^* dW(t)
\end{aligned}$$

for every $t \geq 0$. Using the local martingale property of Itô integral, the coercivity property of V, the induction hypothesis and the Fatou's Lemma, it follows that

$$\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|^{2(n+1)}\right] \leq c_{n+1}\epsilon^{n+1}t^{n+1}$$

for every $t \geq 0$. Consequently, for every $n \in \mathbb{N}$, it follows that

$$\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|^{2n}\right] \leq c_n \epsilon^n t^n$$

for every $t \geq 0$.

ii) Let $b \in \mathbb{R}$ be fixed. By the Monotone Convergence Theorem, it follows that

$$\mathbb{E}\left[e^{\delta_\epsilon \frac{\|x^\epsilon(t_\epsilon+b\delta_\epsilon)-\psi(t_\epsilon+b\delta_\epsilon)\|^2}{\epsilon}}\right] = \sum_{n=0}^{\infty}\mathbb{E}\left[\frac{\delta_\epsilon^n \|x^\epsilon(t_\epsilon+b\delta_\epsilon)-\psi(t_\epsilon+b\delta_\epsilon)\|^{2n}}{\epsilon^n n!}\right],$$

where $\delta_\epsilon = \epsilon^\gamma$ for some $\gamma > 0$. By item *ii)*, we have

$$\sum_{n=0}^{\infty}\mathbb{E}\left[\frac{\delta_\epsilon^n \|x^\epsilon(t_\epsilon+b\delta_\epsilon)-\psi(t_\epsilon+b\delta_\epsilon)\|^{2n}}{\epsilon^n n!}\right] \leq \sum_{n=0}^{\infty}\frac{\delta_\epsilon^n c_n\left(t_\epsilon+b\delta_\epsilon\right)^n}{n!}.$$

Taking $\epsilon_0 > 0$ such that $2\left(t_\epsilon+b\delta_\epsilon\right)\delta_\epsilon < 1$ for $0<\epsilon<\epsilon_0$ and using the ratio test for convergence series, we have that $\sum\limits_{n=0}^{\infty}\frac{c_n\delta_\epsilon^n(t_\epsilon+b\delta_\epsilon)^n}{n!} < +\infty$ for every $0<\epsilon<\epsilon_0$.

iii) We will use the Itô formula for the function $g_\epsilon(x) = e^{\delta_\epsilon\frac{\|x\|^2}{\epsilon}}$. Let $\kappa_\epsilon = \frac{\delta_\epsilon}{\epsilon} = \frac{\epsilon^\gamma}{\epsilon}$. Then,

$$\begin{aligned} de^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2} &= -2\kappa_\epsilon e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\left(x^\epsilon(t)-\psi(t)\right)^* A^\epsilon(t)\left(x^\epsilon(t)-\psi(t)\right)dt + \\ &\quad \epsilon\left(2\kappa_\epsilon^2 e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\|x^\epsilon(t)-\psi(t)\|^2 + \kappa_\epsilon m e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\right)dt + \\ &\quad 2m\sqrt{\epsilon}\kappa_\epsilon e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\left(x^\epsilon(t)-\psi(t)\right)^* dW(t) \end{aligned}$$

for every $t \geq 0$. Using the coercivity property, we obtain

$$\begin{aligned} de^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2} &= -2\kappa_\epsilon\delta e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\|x^\epsilon(t)-\psi(t)\|^2 dt + \\ &\quad \epsilon\left(2\kappa_\epsilon^2 e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\|x^\epsilon(t)-\psi(t)\|^2 + \kappa_\epsilon m e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\right)dt + \\ &\quad 2m\sqrt{\epsilon}\kappa_\epsilon e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\left(x^\epsilon(t)-\psi(t)\right)^* dW(t). \end{aligned}$$

Taking $\epsilon_0 > 0$ such that $2\epsilon^\gamma \leq \delta$ for every $0<\epsilon<\epsilon_0$, we obtain

$$\begin{aligned} de^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2} &\leq -\kappa_\epsilon\delta e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\|x^\epsilon(t)-\psi(t)\|^2 dt + \\ &\quad \epsilon\kappa_\epsilon m e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}dt + \\ &\quad 2m\sqrt{\epsilon}\kappa_\epsilon e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\left(x^\epsilon(t)-\psi(t)\right)^* dW(t). \end{aligned}$$

For $\epsilon > 0$ small enough, by item *i)* and *iii)* the stochastic integral is a true martingale for $t \in [0, t_\epsilon + b\delta_\epsilon]$. Then,

$$d\mathbb{E}\left[e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\right] \leq \epsilon\kappa_\epsilon m\mathbb{E}\left[e^{\kappa_\epsilon\|x^\epsilon(t)-\psi(t)\|^2}\right]dt$$

for every $t \in [0, t_\epsilon + b\delta_\epsilon]$. Now using the Gronwall Inequality we obtain for $\epsilon > 0$ small enough that $\mathbb{E}\left[e^{\delta_\epsilon \frac{\|x^\epsilon(t_\epsilon+b\delta_\epsilon)-\psi(t_\epsilon+b\delta_\epsilon)\|^2}{\epsilon}}\right] \leq e^{\delta_\epsilon(t_\epsilon+b\delta_\epsilon)m}$, where $\delta_\epsilon = \epsilon^\gamma$ for some $\gamma > 0$.

iv) In the same way as in item *i)*, using Itô's formula and coercivity hypothesis, we have

$$\|x^\epsilon(t) - \psi(t)\|^2 \leq \epsilon dt + N(t)$$

for every $t \geq 0$. By item *i)*, we have that $\{N(t)\}_{t\geq 0}$ is a true martingale. Therefore, taking $\epsilon_0 > 0$ such that $\frac{r}{2} \leq r - \epsilon\left(t_\epsilon + bw_\epsilon\right) \leq \frac{3r}{2}$ for every $0 < \epsilon < \epsilon_0$, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\leq t_\epsilon+bw_\epsilon} \|x^\epsilon(t) - \psi(t)\|^2 \geq r\right) &\leq \mathbb{P}\left(\sup_{t\leq t_\epsilon+bw_\epsilon} \|N(t)\| \geq r - \epsilon\left(t_\epsilon + bw_\epsilon\right)\right) \\
&\leq \frac{\mathbb{E}\left[\|N(t)\|^2\right]}{\left(r - \epsilon\left(t_\epsilon + bw_\epsilon\right)\right)^2} \\
&\leq \frac{16\epsilon \int_0^t \mathbb{E}\left[\|x^\epsilon(s) - \psi(s)\|^2\right] ds}{r^2} \\
&\leq \frac{8m\epsilon^2\left(t_\epsilon + bw_\epsilon\right)^2}{r^2},
\end{aligned}$$

where the second inequality follows from Doob's inequality, the third inequality follows from Itô's isometry and the fourth inequality follows by item *i)* of this proposition.

$v)$ Let $\epsilon > 0$ and $t \geq 0$ be fixed. It follows that

$$\begin{aligned}
x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t) &= -\int_0^t \left[\nabla V(x^\epsilon(s)) - \nabla V(\psi(s)) - H_V(\psi(s))\sqrt{\epsilon}y(s)\right] ds \\
&= -\int_0^t \left[A^\epsilon(s)\left(x^\epsilon(s) - \psi(s)\right) - H_V(\psi(s))\sqrt{\epsilon}y(s)\right] ds \\
&= -\int_0^t \left[H_V(\psi(s))(x^\epsilon(s) - \psi(s) - \sqrt{\epsilon}y(s))\right] ds - \\
&\quad \int_0^t \left[\left(A^\epsilon(s) - H_V(\psi(s))\right)\left(x^\epsilon(s) - \psi(s)\right)\right] ds,
\end{aligned}$$

where $A^\epsilon(s) := \int_0^1 H_V(\psi(s) + \theta\left(x^\epsilon(s) - \psi(s)\right))d\theta$ for every $s \geq 0$ and the second equality comes from the Intermediate Value Theorem. Let us define

$$e(t) := \int_0^t \left[\left(A^\epsilon(s) - H_V(\psi(s))\right)\left(x_s^\epsilon - \psi_s\right)\right] ds.$$

It follows that

$$\begin{aligned}
d\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|^2 &= 2\left(x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\right)^* d\left(x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\right) \\
&= -2\left[\left(x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\right)^* H_V(\psi(t))\right. \\
&\quad \left.\left(x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\right)\right] dt - \\
&\quad 2\left[\left(x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\right)^* \left(A^\epsilon(t) - H_V(\psi(t))\right)\right. \\
&\quad \left.\left(x^\epsilon(t) - \psi(t)\right)\right] dt \\
&\leq -2\delta\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|^2 dt + \\
&\quad 2\left[\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|\|A^\epsilon(t) - H_V(\psi(t)\|\right. \\
&\quad \left.\|x^\epsilon(t) - \psi(t)\|\right] dt \\
&\leq 2\left[\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|\|A^\epsilon(t) - H_V(\psi(t)\|\right. \\
&\quad \left.\|x^\epsilon(t) - \psi(t)\|\right] dt \\
&\leq 2\|x^\epsilon(t) - \psi(t)\|^2\|A^\epsilon(t) - H_V(\psi(t)\|dt + \\
&\quad 2\sqrt{\epsilon}\|x^\epsilon(t) - \psi(t)\|\|y(t)\|\|A^\epsilon(t) - H_V(\psi(t)\|dt
\end{aligned}$$

for every $t \geq 0$. In the same way as in item i), using Itô's formula we obtain $\mathbb{E}\left[\|y(t)\|^2\right] \leq dt$ for every $t \geq 0$. Consequently, we obtain

$$\begin{aligned} d\mathbb{E}\left[\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|^2\right] &\leq 2\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|^2\|A^\epsilon(t) - H_V(\psi(t)\|\right] dt + \\ &\quad 2\sqrt{\epsilon}\mathbb{E}\left[\|x^\epsilon(t) - \psi(t)\|\|y(t)\|\|A^\epsilon(t) - H_V(\psi(t)\|\right] dt \\ &\leq 4\sqrt{c_2}\epsilon t\sqrt{\mathbb{E}\left[\|A^\epsilon(t) - H_V(\psi(t)\|^2\right]} \end{aligned}$$

for every $t \geq 0$, where the second inequality follows using several times Cauchy-Schwarz inequality and item i) of this proposition. Therefore,

$$\begin{aligned} \mathbb{E}\left[\|x^\epsilon(t) - \psi(t) - \sqrt{\epsilon}y(t)\|^2\right] &\leq 4\sqrt{c_2}\epsilon \int_0^t s\sqrt{\mathbb{E}\left[\|A^\epsilon(s) - H_V(\psi(s)\|^2\right]}ds \\ &\leq 4\sqrt{c_2}\epsilon t \int_0^t \sqrt{\mathbb{E}\left[\|A^\epsilon(s) - H_V(\psi(s)\|^2\right]}ds \end{aligned} \tag{3.2}$$

for every $t \geq 0$.

Let us estimate the last integral in the following way:

$$\begin{aligned} \|A^\epsilon(t) - H_V(\psi(t)\|^2 &= \left\| \int_0^1 \left[H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\right] d\theta \right\|^2 \\ &\leq \int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta, \end{aligned}$$

for every $t \geq 0$, where the last inequality follows from Jensen's inequality. Let $r > 0$ be fixed and let us define $\Omega(r, \epsilon) := \left\{\omega \in \Omega : \sup_{0 \leq t \leq t_\epsilon + bw_\epsilon} \|x^\epsilon(t) - \psi(t)\| \geq r\right\}$. By item iv) of this proposition we know that $\mathbb{P}\left(\Omega(r, \epsilon)\right) \leq c(r)\epsilon^2 \left(t_\epsilon + bw_\epsilon\right)^2$. Let us define $t^* := t_\epsilon + bw_\epsilon$. Following (3.2), we have

$$\mathbb{E}\left[\|x^\epsilon(t^*) - \psi(t^*) - \sqrt{\epsilon}y(t^*)\|^2\right] \leq$$

$$4\sqrt{c_2}\epsilon t^* \int_0^{t^*} \sqrt{\mathbb{E}\left[\int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta\right]} \, dt \leq$$

$$4\sqrt{c_2}\epsilon (t^*)^{\frac{3}{2}} \sqrt{\int_0^{t^*} \mathbb{E}\left[\int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta\right] dt}.$$

for every $t \geq 0$, where the first inequality follows from the inequality from above and the second inequality follows from the Cauchy-Schwarz inequality. By Tonelli's Theorem, we have

$$\int_0^{t^*} \mathbb{E}\left[\int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta\right] dt =$$

$$\mathbb{E}\left[\int_0^{t^*} \int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta dt\right].$$

We can split the last integral into two parts. The first one is

$$\mathbb{E}\left[\mathbb{1}_{\Omega(r,\epsilon)} \int_0^{t^*} \int_0^1 \|H_V(\psi(t) + \theta\left(x^\epsilon(t) - \psi(t)\right)) - H_V(\psi(t)\|^2 \, d\theta dt\right] \leq$$

$$\mathbb{E}\left[\mathbb{1}_{\Omega(r,\epsilon)} \int_0^{t^*} \hat{C}K^2 dt\right] \leq$$

$$\hat{C}K^2 t^* \mathbb{P}\left(\Omega(r,\epsilon)\right) \leq$$

$$\hat{C}K^2 c(r)\epsilon^2 \left(t_\epsilon + bw_\epsilon\right)^3,$$

where $\hat{C} = 2m > 0$ is a constant. The first inequality comes from the fact that ∇V is a Lipschitz function, which implies that all the eigenvalues of H_V are bounded by K and using the fact that $\|A\|^2$ is equal to the sum of the squares of its eigenvalues for any symmetric matrix A. The second inequality comes from Tonelli's Theorem and the third inequality comes from the item *iv*) of this proposition. The second one is

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\Omega^c(r,\epsilon)}\int_0^{t^*}\int_0^1 \|H_V(\psi(t)+\theta\left(x^\epsilon(t)-\psi(t)\right))-H_V(\psi(t)\|^2\, d\theta dt\right] &\leq \\
\mathbb{E}\left[\mathbb{1}_{\Omega^c(r,\epsilon)}\int_0^{t^*} L^2\|x^\epsilon(t)-\psi(t)\|^2 dt\right] &\leq \\
L^2\int_0^{t^*}\mathbb{E}\left[\|x^\epsilon(t)-\psi(t)\|^2\right]dt &\leq \\
L^2\int_0^{t^*} c_1\epsilon t dt &\leq \\
L^2 c_1\epsilon(t^*)^2, &
\end{aligned}$$

where $L := L\left(r, \|\psi_0\|\right)$ is the Lipschitz constant of the function $g(x,y) = H_V(x+y) - H_V(x)$ on the compact set $\Lambda := \{(x,y) : \|x\| \leq \|\psi(0)\|, \|y\| \leq r\}$, the second inequality follows from Tonelli's Theorem, the third inequality follows from the item $i)$ of this proposition and the fourth inequality is an straightforward calculation. Consequently,

$$\begin{aligned}
\mathbb{E}\left[\|x^\epsilon(t^*)-\psi(t^*)-\sqrt{\epsilon}y(t^*)\|^2\right] &\leq 4\sqrt{c_2}\epsilon(t^*)^{\frac{3}{2}}\sqrt{2K^2c_{2n}\epsilon^{2n}(t^*)^{2n+1}+L^2c_1\epsilon(t^*)^2} \\
&\leq 4\sqrt{c_2}\epsilon(t^*)^{\frac{3}{2}}\times \\
&\quad \left(\sqrt{2K^2c_{2n}\epsilon^{2n}(t^*)^{2n+1}}+\sqrt{L^2c_1\epsilon(t^*)^2}\right) \\
&\leq 4\sqrt{c_2}\epsilon^{\frac{3}{2}}(t^*)^{\frac{5}{2}}\left(K\sqrt{2c_{2n}\epsilon^{2n-1}(t^*)^{2n-1}}+L\sqrt{c_1}\right) \\
&\leq \hat{K}\epsilon^{\frac{3}{2}}(t^*)^{\frac{5}{2}}\left(\sqrt{\epsilon^{2n-1}(t^*)^{2n-1}}+1\right),
\end{aligned}$$

where $\hat{K} := \max\left\{4L\sqrt{c_1c_2}, 4K\sqrt{2c_2c_{2n}}\right\}$. We can observe that there exists and $\epsilon_0 > 0$ such that $\sqrt{\epsilon^{2n-1}(t^*)^{2n-1}} < 1$ for every $0 < \epsilon < \epsilon_0$. Consequently

$$\mathbb{E}\left[\|x^\epsilon(t^*)-\psi(t^*)-\sqrt{\epsilon}y(t^*)\|^2\right] \leq 2\hat{K}\epsilon^{\frac{3}{2}}(t^*)^{\frac{5}{2}}$$

for every $0 < \epsilon < \epsilon_0$.

□

The next proposition will allows us to prove that the total variation distance of two first-order approximations with (random or deterministic) initial conditions that are close enough is negligible. In order to do that, we will need to keep track of the initial condition of the solution of various equations. Let X be a random variable in $\mathbb{R}^m$ and let $T > 0$. Let $\{\psi(t, X)\}_{t\geq 0}$ denote the solution of

$$\begin{aligned} d\psi(t, X) &= -\nabla V(\psi(t, X))dt, \\ \psi(0) &= X. \end{aligned}$$

Let $\{y(t, X, T)\}_{t\geq 0}$ be the solution of the stochastic differential equation

$$\begin{aligned} dy(t, X, T) &= -H_V(\psi(t, X))y(t, X, T)dt + dW(t+T), \\ y(0, X, T) &= 0 \end{aligned}$$

and define $\{y^{\epsilon}(t, X, T)\}_{t\geq 0}$ as $y^{\epsilon}(t, X, T) := \psi_t(X) + \sqrt{\epsilon}y(t, X, T)$. In what follows, we will always take $T = \tilde{t}_{\epsilon}(b) := t_{\epsilon} + bw_{\epsilon}$, so we will omit it from the notation.

Proposition 3.15 (Linear Coupling)**.** *Let us assume the same hypothesis of Theorem 3.6 and in addition let us assume that ∇V is Lipschitz. For $\epsilon > 0$, define $\delta_{\epsilon} = \epsilon^{\gamma}$, where $0 < \gamma < \frac{1}{4}$. Then, for every $b \in \mathbb{R}$ it follows that*

$$\lim_{\epsilon\to 0} \left\| y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}(\tilde{t}_{\epsilon}(b))\right) - y^{\epsilon}\left(b\delta_{\epsilon}, y^{\epsilon}(\tilde{t}_{\epsilon}(b))\right)\right\|_{\mathbb{TV}} = 0,$$

where for each $\epsilon > 0$, t_{ϵ} and w_{ϵ} are defined in Theorem 3.6 and where for each $b \in \mathbb{R}$, $\tilde{t}_{\epsilon}(b) := \max\{t_{\epsilon} + bw_{\epsilon}, 0\}$.

Proof. By Itô's formula, we obtain

$$y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}(\tilde{t}_{\epsilon}(b))\right) = \Phi(b\delta_{\epsilon})x^{\epsilon}(\tilde{t}_{\epsilon}(b)) + \sqrt{\epsilon}\Phi(b\delta_{\epsilon})\int_0^{b\delta_{\epsilon}} \Phi^{-1}(s)d\big(W(\tilde{t}_{\epsilon}(b)+s) - W(\tilde{t}_{\epsilon}(b))\big),$$

$$y^{\epsilon}\left(b\delta_{\epsilon}, y^{\epsilon}(\tilde{t}_{\epsilon}(b))\right) = \Phi(b\delta_{\epsilon})y^{\epsilon}(\tilde{t}_{\epsilon}(b)) + \sqrt{\epsilon}\Phi(b\delta_{\epsilon})\int_0^{b\delta_{\epsilon}} \Phi^{-1}(s)d\big(W(\tilde{t}_{\epsilon}(b)+s) - W(\tilde{t}_{\epsilon}(b))\big),$$

for every ϵ small enough, where $\Phi = \{\Phi(t)\}_{t\geq 0}$ is the fundamental solution of the non-autonomous system

$$d\Phi(t) \quad = \quad -H_V(\psi(t+\tilde{t}_\epsilon(b)))\Phi(t)dt$$

for every $t \geq 0$, with initial condition $\Phi_0 = I_m$. Applying Lemma B.6 with $X = \Phi(b\delta_\epsilon)x^\epsilon(\tilde{t}_\epsilon(b))$, $Y = \Phi(b\delta_\epsilon)y^\epsilon(\tilde{t}_\epsilon(b))$, $Z = \sqrt{\epsilon}\Phi(b\delta_\epsilon)\int\limits_0^{b\delta_\epsilon}\Phi^{-1}(s)dW(s+\tilde{t}_\epsilon(b))$, $\mathcal{G} = \sigma(X,Y)$ and $(\Omega,\mathcal{F},\mathbb{P})$ the canonical probability space of the Brownian motion W, we have

$$\left\|y^\epsilon\left(b\delta_\epsilon, x^\epsilon(\tilde{t}_\epsilon(b))\right) - y^\epsilon\left(b\delta_\epsilon, y^\epsilon(\tilde{t}_\epsilon(b))\right)\right\|_{\mathbb{TV}} \;\leq\; \frac{\hat{C}}{\sqrt{\epsilon\delta_\epsilon}}\mathbb{E}\left[\left\|x^\epsilon(\tilde{t}_\epsilon(b)) - y^\epsilon(\tilde{t}_\epsilon(b))\right\|\right],$$

where $\hat{C} > 0$ is a constant. Now, using Proposition 3.14 item v), we obtain

$$\left\|y^\epsilon\left(b\delta_\epsilon, x^\epsilon(\tilde{t}_\epsilon(b))\right) - y^\epsilon\left(b\delta_\epsilon, y^\epsilon(\tilde{t}_\epsilon(b))\right)\right\|_{\mathbb{TV}} \;\leq\; \sqrt{C}\hat{C}\frac{\epsilon^{\frac{1}{4}}}{\delta_\epsilon}(\tilde{t}_\epsilon(b))^{\frac{5}{4}}$$

for $\epsilon > 0$ small enough, where the constant C is the constant of item v) of Proposition 3.14. Using Lemma D.1, we obtain the result. □

Proposition 3.16 (Short Time Change of Measure). *Let us assume the same hypothesis of Theorem 3.6 and assume that ∇V is Lipschitz. For each $b \in \mathbb{R}$ we have*

$$\lim_{\epsilon\to 0}\left\|x^\epsilon\left(b\delta_\epsilon, x^\epsilon(\tilde{t}_\epsilon(b))\right) - y^\epsilon\left(b\delta_\epsilon, x^\epsilon(\tilde{t}_\epsilon(b))\right)\right\|_{\mathbb{TV}} \;=\; 0.$$

Proof. We will use the Cameron-Martin-Girsanov Theorem and Novikov's Theorem. Let $\epsilon > 0$, $t \geq 0$ and $b \in \mathbb{R}$ be fixed. Let us define $\gamma^\epsilon(t) := \frac{\nabla V(x^\epsilon(t))}{\sqrt{\epsilon}}$ and $\Gamma^\epsilon(t) := \frac{(\nabla V(\psi(t)) - H_V(\psi(t))\psi(t) + H_V(\psi(t))y^\epsilon(t))}{\sqrt{\epsilon}}$. Using the item ii) of Lemma 3.8 and the same facts used in Proposition 2.16, for any $\rho > 0$, we have

$$\mathbb{E}\left\{\rho\exp\left[\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\left\|\gamma^\epsilon(s)\right\|^2 ds\right]\right\} < +\infty$$

and

$$\mathbb{E}\left\{\rho\exp\left[\int\limits_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\left\|\Gamma^\epsilon(s)\right\|^2 ds\right]\right\} < +\infty$$

for $\epsilon > 0$ small enough. From Novikov's Theorem, it follows that

$$\begin{aligned}
\frac{d\mathbb{P}^1_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}}{d\mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}} &:= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \gamma^\epsilon(s)dW(s) - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\gamma^\epsilon(s)\|^2 ds\right\}, \\
\frac{d\mathbb{P}^2_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}}{d\mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}} &:= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \Gamma^\epsilon(s)dW(s) - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma^\epsilon(s)\|^2 ds\right\},
\end{aligned}$$

are well-defined Radon-Nikodym derivatives and they define true probability measures $\mathbb{P}^i_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$, $i \in \{1,2\}$. From now to the end of this proof we will use the notations $\mathbb{P}^i := \mathbb{P}^i_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$, $i \in \{1,2\}$ and $\mathbb{P} := \mathbb{P}_{\tilde{t}_\epsilon(b)+b\delta_\epsilon}$. Under the probability measure $\mathbb{P}^1$, $W^1(t) := W(t) - \int_{\tilde{t}_\epsilon(b)}^{t} \gamma^\epsilon(s)ds$, where $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$ is a Brownian motion. Also, under the probability measure $\mathbb{P}^2$, $W^2(t) := W(t) - \int_{\tilde{t}_\epsilon(b)}^{t} \Gamma^\epsilon(s)ds$, where $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$ is a Brownian motion. Consequently,

$$\begin{aligned}
\frac{d\mathbb{P}^2}{d\mathbb{P}^1} &= \frac{\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \Gamma^\epsilon(s)dW(s) - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma^\epsilon(s)\|^2 ds\right\}}{\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \gamma^\epsilon(s)dW(s) - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\gamma^\epsilon(s)\|^2 ds\right\}} \\
&= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon(s) - \gamma^\epsilon(s)\right) dW(s) - \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\|\Gamma^\epsilon(s)\|^2 - \|\gamma^\epsilon(s)\|^2\right) ds\right\} \\
&= \exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \left(\Gamma^\epsilon(s) - \gamma^\epsilon(s)\right) dW^2(s) + \frac{1}{2}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma^\epsilon(s) - \gamma^\epsilon(s)\|^2 ds\right\}.
\end{aligned}$$

By Pinsker's inequality and the mean-zero martingale property of the stochastic integral, we have for every $\tilde{t}_\epsilon(b) \le t \le \tilde{t}_\epsilon(b) + b\delta_\epsilon$

$$\begin{aligned}
\left\|\mathbb{P}^1 \circ (x^\epsilon(t))^{-1} - \mathbb{P}^2 \circ (x^\epsilon(t))^{-1}\right\|_{\mathbb{TV}}^2 &\le \left\|\mathbb{P}^1 \circ (x^\epsilon)^{-1} - \mathbb{P}^2 \circ (x^\epsilon)^{-1}\right\|_{\mathbb{TV}}^2 \\
&\le \mathbb{E}_{\mathbb{P}^2}\left[\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma^\epsilon(s) - \gamma^\epsilon(s)\|^2 \, ds\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{P}^2}{d\mathbb{P}} \int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma^\epsilon(s) - \gamma^\epsilon(s)\|^2 \, ds\right].
\end{aligned}$$

By Cauchy-Schwarz's inequality and the mean-one Doléans exponential martingale property, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{P}^1}{d\mathbb{P}} \int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma_s^\epsilon - \gamma_s^\epsilon\|^2 \, ds\right] &\le \sqrt{\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\gamma_s^\epsilon\|^2 \, ds\right\}\left(\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma_s^\epsilon - \gamma_s^\epsilon\|^2 \, ds\right)^2\right]} \\
&\le \sqrt{\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\gamma_s^\epsilon\|^2 \, ds\right\}\right]} \times \\
&\quad \sqrt{\mathbb{E}_{\mathbb{P}}\left[\left(\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\Gamma_s^\epsilon - \gamma_s^\epsilon\|^2 \, ds\right)^4\right]}
\end{aligned}$$

Let us define $I^\epsilon(b) := \left[\tilde{t}_\epsilon(b), \tilde{t}_\epsilon(b) + b\delta_\epsilon\right]$. Then, by Jensen's inequality and the Lipschitz condition on the gradient ∇V, we have

$$\begin{aligned}
\exp\left\{2\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \|\gamma^\epsilon(s)\|^2 \, ds\right\} &\le \frac{1}{b\delta_\epsilon}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \exp\left\{2b\delta_\epsilon \|\gamma^\epsilon(s)\|^2\right\} ds \\
&\le \frac{1}{b\delta_\epsilon}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon} \exp\left\{2Kb\delta_\epsilon \frac{\|x^\epsilon(s)\|^2}{\epsilon}\right\} ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\|\gamma^\epsilon(s)\|^2\,ds\right\}\right] &\leq \frac{1}{b\delta_\epsilon}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2Kb\delta_\epsilon\frac{\|x^\epsilon(s)\|^2}{\epsilon}\right\}\right]ds \\
&\leq \frac{1}{b\delta_\epsilon}\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\exp\left\{2Kb\delta_\epsilon ms\right\}ds \\
&\leq \exp\left\{2Kb\delta_\epsilon m\left(\tilde{t}_\epsilon(b)+b\delta_\epsilon\right)\right\}ds,
\end{aligned}$$

where the first inequality comes from Tonelli's Theorem, the second inequality comes from the item $v)$ of Proposition 3.14 and the third inequality is a straightforward calculation. Consequently,

$$\lim_{\epsilon\to 0}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\|\gamma^\epsilon(s)\|^2\,ds\right\}\right] = 1.$$

Now, we will calculate

$$\mathbb{E}_{\mathbb{P}}\left[\left(\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\|\Gamma^\epsilon_s-\gamma^\epsilon_s\|^2\,ds\right)^4\right] = 0.$$

Let us observe that

$$\begin{aligned}
\|\Gamma^\epsilon(s)-\gamma^\epsilon(s)\|^2 \leq\ & \frac{2\|H_V(\psi(s))\|^2\|x^\epsilon(s)-\psi(s)-\sqrt{\epsilon}y(s)\|^2}{\epsilon}+ \\
& \frac{2\|x^\epsilon(s)-\psi(s)\|^2}{\epsilon}\int_0^1\|H_V\left(\psi(s)+\theta\left(x^\epsilon(s)-\psi(s)\right)\right)-H_V\left(\psi(s)\right)\|^2d\theta
\end{aligned}$$

for every $s\geq 0$. Using the last inequality, several times Jensen inequality, several times Cauchy-Schwartz inequality, the item $i)$, item $v)$ of Proposition 3.14; it suffices to prove that

$$\mathbb{E}_{\mathbb{P}}\left[\int_{\tilde{t}_\epsilon(b)}^{\tilde{t}_\epsilon(b)+b\delta_\epsilon}\int_0^1\|H_V\left(\psi(s)+\theta\left(x^\epsilon(s)-\psi(s)\right)\right)-H_V\left(\psi(s)\right)\|^4d\theta ds\right] = o(\epsilon^\gamma)$$

for some $\gamma > 0$. The proof is analogous to the proof of item v) of Proposition 3.14.

$\square$

Theorem 3.17. *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a smooth coercive regular potential. Let us consider the family of processes $x^\epsilon = \{x^\epsilon(t)\}_{t\geq 0}$ which are given by the the semi-flow of the following stochastic differential equation,*

$$\begin{aligned} dx^\epsilon(t) &= -\nabla V(x^\epsilon(t))dt + \sqrt{\epsilon}dW(t), \\ x^\epsilon(0) &= x_0 \end{aligned}$$

for $t \geq 0$, where x_0 is an initial condition in $\mathbb{R}^m \setminus \{0\}$ and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion. Let α_1 be the smallest eigenvalue of $H_V(0)$ and let V_1 be its eigenspace. For each $x_0 \in \mathbb{R}^m \setminus \{0\}$, let $v(x_0) \in V_1$ such that

$$\lim_{t\to+\infty} e^{\alpha_1 t}\psi(t) = v(x_0).$$

Assume that $v(x_0) \neq 0$ and let $G_{x_0} : \mathbb{R} \to [0,1]$ be the profile function given by

$$G_{x_0}(b) := \left\| \mathcal{G}\left(\sqrt{2}e^{-b}H_V(0)^{\frac{1}{2}}v(x_0), I_m\right) - \mathcal{G}(0, I_m)\right\|_{\mathbb{TV}}.$$

Then the family $\{x^\epsilon(t)\}_{t\geq 0}$ presents profile cut-off with profile function G_{x_0}, cut-off time t_ϵ and window time w_ϵ given by

$$t_\epsilon = \frac{1}{2\alpha_1}\ln\left({}^1\!/\!_\epsilon\right)$$

and

$$w_\epsilon = \frac{1}{\alpha_1}.$$

Remark 3.18. *By item ii) of Lemma 3.8 above, $v(x_0)$ is well defined and nonzero for Lebesgue-almost every x_0. In particular, Theorem 3.17 holds for Lebesgue-almost every initial condition $x_0 \in \mathbb{R}^m \setminus \{0\}$.*

Proof of Theorem 3.17. Let $\epsilon > 0$ and $t > 0$ be fixed. We define

$$D^\epsilon(t) := \|x^\epsilon(t) - \mu^\epsilon\|_{\mathbb{TV}}$$

and

$$d^{\epsilon}(t) := \|y^{\epsilon}(t) - \mathcal{G}^{\epsilon}\|_{\mathbb{TV}},$$

where μ^{ϵ} and $\mathcal{G}^{\epsilon}$ are given in Lemma 3.12 and Lemma 3.13. For each $b \in \mathbb{R}$ take $\epsilon_b > 0$ such that $\hat{t}^{\epsilon}(b) := t_{\epsilon} + b(w_{\epsilon} + \delta_{\epsilon}) = \tilde{t}^{\epsilon}(b) + b\delta_{\epsilon} \geq 0$ for every $0 < \epsilon < \epsilon_b$. By Theorem 3.6 and Remark 3.10, we know that for each $b \in \mathbb{R}$

$$\lim_{\epsilon \to 0} d^{\epsilon}\left(\hat{t}^{\epsilon}(b)\right) = G(b). \tag{3.3}$$

By definition

$$\begin{aligned} D^{\epsilon}(\hat{t}^{\epsilon}(b)) &= \left\|x^{\epsilon}\left(\hat{t}^{\epsilon}(b)\right) - \mu^{\epsilon}\right\|_{\mathbb{TV}} \\ &\leq \left\|x^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right) - y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right) - y^{\epsilon}\left(b\delta_{\epsilon}, y^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|y^{\epsilon}\left(\hat{t}^{\epsilon}(b)\right) - \mathcal{G}^{\epsilon}\right\|_{\mathbb{TV}} + \|\mathcal{G}^{\epsilon} - \mu^{\epsilon}\|_{\mathbb{TV}}. \end{aligned}$$

Using Proposition 3.15, Proposition 3.16, Lemma 3.12, the relation (3.3) and the item $i)$ of Lemma D.2, we have $\limsup_{\epsilon \to 0} D^{\epsilon}(\hat{t}^{\epsilon}(b)) \leq G(b)$. In order to obtain the converse inequality we observe that

$$\begin{aligned} d^{\epsilon}(\hat{t}^{\epsilon}(b)) &= \left\|y^{\epsilon}\left(\hat{t}^{\epsilon}(b)\right) - \mathcal{G}^{\epsilon}\right\|_{\mathbb{TV}} \\ &\leq \left\|y^{\epsilon}\left(b\delta_{\epsilon}, y^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right) - y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|y^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right) - x^{\epsilon}\left(b\delta_{\epsilon}, x^{\epsilon}\left(\tilde{t}_{\epsilon}(b)\right)\right)\right\|_{\mathbb{TV}} + \\ &\quad \left\|x^{\epsilon}\left(\hat{t}^{\epsilon}(b)\right) - \mu^{\epsilon}\right\|_{\mathbb{TV}} + \|\mu^{\epsilon} - \mathcal{G}^{\epsilon}\|_{\mathbb{TV}}. \end{aligned}$$

Again, using Proposition 3.15, Proposition 3.16, Lemma 3.12, the relation (3.3) and the item $ii)$ of Lemma D.2 we have $\liminf_{\epsilon \to 0} D^{\epsilon}(\hat{t}^{\epsilon}(b)) \geq G(b)$. Consequently, $\lim_{\epsilon \to 0} D^{\epsilon}(\hat{t}^{\epsilon}(b)) = G(b)$. □

Appendix A

Properties of the Total Variation Distance of Normal Distribution

Let us take $\mu \in \mathbb{R}$ and $\sigma^2 \in]0, +\infty[$. We denote by $\mathcal{N}(\mu, \sigma^2)$ the Normal distribution with mean μ and variance σ^2.

Lemma A.1. *Let $\{\mu, \tilde{\mu}\} \subset \mathbb{R}$ and $\{\sigma^2, \tilde{\sigma}^2\} \subset]0, +\infty[$ be fixed numbers.*

i) For any constant $c \neq 0$ we have

$$\left\| \mathcal{N}\left(c\mu, c^2\sigma^2\right) - \mathcal{N}\left(c\tilde{\mu}, c^2\tilde{\sigma}^2\right) \right\|_{\mathbb{TV}} = \left\| \mathcal{N}\left(\mu, \sigma^2\right) - \mathcal{N}\left(\tilde{\mu}, \tilde{\sigma}^2\right) \right\|_{\mathbb{TV}}.$$

ii)

$$\left\| \mathcal{N}\left(\mu, \sigma^2\right) - \mathcal{N}\left(\tilde{\mu}, \tilde{\sigma}^2\right) \right\|_{\mathbb{TV}} = \left\| \mathcal{N}\left(|\mu - \tilde{\mu}|, \sigma^2\right) - \mathcal{N}\left(0, \tilde{\sigma}^2\right) \right\|_{\mathbb{TV}}.$$

Proof. This is done using the characterization of the total variation distance between two probability measures which are absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and using the Change of Variable Theorem. $\square$

Lemma A.2. *Let $\mu \in \mathbb{R}$ then*

$$\| \mathcal{N}(\mu, 1) - \mathcal{N}(0, 1) \|_{\mathbb{TV}} = \frac{2}{\sqrt{2\pi}} \int_0^{|\mu|/2} e^{-\frac{x^2}{2}} dx \leq \frac{|\mu|}{\sqrt{2\pi}}.$$

Proof. Also, this is done using the characterization of the total variation distance between two probability measures which are absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and an straightforward calculations. $\square$

Lemma A.3. *Let* $\{\mu_\epsilon\}_{\epsilon>0} \subset \mathbb{R}$ *be a sequence such that* $\lim_{\epsilon\to 0} \mu_\epsilon = \mu \in \mathbb{R}$. *Then*

$$\lim_{\epsilon\to 0} \|\mathcal{N}(\mu_\epsilon, 1) - \mathcal{N}(0,1)\|_{\mathbb{TV}} = \|\mathcal{N}(\mu, 1) - \mathcal{N}(0,1)\|_{\mathbb{TV}}.$$

Proof. This is done using triangle inequality, the item *ii*) of Lemma A.1, Lemma A.2 and the Lemma D.2. □

Lemma A.4. *Let* $\{\sigma_\epsilon^2\}_{\epsilon>0} \subset]0,+\infty[$ *be a sequence such that* $\lim_{\epsilon\to 0} \sigma_\epsilon^2 = \sigma^2 \in]0,+\infty[$. *Then*

$$\lim_{\epsilon\to 0} \left\|\mathcal{N}\big(0,\sigma_\epsilon^2\big) - \mathcal{N}\big(0,\sigma^2\big)\right\|_{\mathbb{TV}} \;=\; 0.$$

Proof. This is done using the item *i*) of Lemma A.1, the characterization of the total variation distance between two probability measures which are absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and an straightforward calculations. □

Lemma A.5 (Total Variation Bounded)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\mathcal{G} \subset \mathcal{F}$ *be a sub-sigma algebra of* $\mathcal{F}$. *Let* $X, Y, Z : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *be random variables such that* X *and* Y *are* $\mathcal{G}$ *measurables and* $X, Y, Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. *Let us consider the following random variables* $X^* = X + Z$ *and* $Y^* = Y + Z$. *Let us suppose that* $E[X^* \mid \mathcal{G}]$ *has normal distribution* $\mathcal{N}(X, \sigma^2)$, $E[Y^* \mid \mathcal{G}]$ *has normal distribution* $\mathcal{N}(Y, \sigma^2)$ *and* $Z \perp \mathcal{G}$. *Then,*

$$\|X^* - Y^*\|_{\mathbb{TV}} \;\leq\; \frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[|X-Y|].$$

Proof. Using the the properties of conditional expectation, the item *i*), item *ii*) of Lemma A.1 and Lemma A.2, we have

$$\begin{aligned}
\|X^* - Y^*\|_{\mathbb{TV}} &= \sup_{F\in\mathcal{F}} \left|\mathbb{E}\left[\mathbb{1}_{(X^*\in F)} - \mathbb{1}_{(Y^*\in F)}\right]\right| \\
&\leq \sup_{F\in\mathcal{F}} \mathbb{E}\left[\left|\mathbb{E}\left[\mathbb{1}_{(X^*\in F)} - \mathbb{1}_{(Y^*\in F)} \mid \mathcal{G}\right]\right|\right] \\
&\leq \sup_{F\in\mathcal{F}} \mathbb{E}\left[\left|\mathbb{P}\left(\mathcal{N}(X,\sigma^2) \in F\right) - \mathbb{P}\left(\mathcal{N}(Y,\sigma^2) \in F\right)\right|\right] \\
&\leq \sup_{F\in\mathcal{F}} \mathbb{E}\left[\frac{1}{\sqrt{2\pi}\sigma}\left|X - Y\right|\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[|X-Y|].
\end{aligned}$$

□

Appendix B

Properties of the Total Variation Distance of Gaussian Distribution

Let us take $\mu \in \mathbb{R}^m$ and $\Sigma \in \mathcal{S}_m$ be a symetric and positive definite square m-dimensional matrix. We denoted by $\mathcal{G}(\mu, \Sigma)$ the Gaussian distribution with vector mean μ and covariance matrix Σ.

Lemma B.1. *Let $\{\mu, \tilde{\mu}\} \subset \mathbb{R}^m$ be two fixed vectors and $\{\Sigma, \tilde{\Sigma}\} \subset \mathcal{S}_m$ be two fixed matrices. It follows*

i) For any scalar $c \neq 0$ we have

$$\left\|\mathcal{G}\left(c\mu, c^2\Sigma\right) - \mathcal{G}\left(c\tilde{\mu}, c^2\tilde{\Sigma}\right)\right\|_{\mathbb{TV}} = \left\|\mathcal{G}(\mu, \Sigma) - \mathcal{G}\left(\tilde{\mu}, \tilde{\Sigma}\right)\right\|_{\mathbb{TV}}.$$

ii)

$$\left\|\mathcal{G}(\mu, \Sigma) - \mathcal{G}\left(\tilde{\mu}, \tilde{\Sigma}\right)\right\|_{\mathbb{TV}} = \left\|\mathcal{G}(\mu - \tilde{\mu}, \Sigma) - \mathcal{G}\left(0, \tilde{\Sigma}\right)\right\|_{\mathbb{TV}}.$$

iii)

$$\left\|\mathcal{G}(\mu, \Sigma) - \mathcal{G}(\tilde{\mu}, \Sigma)\right\|_{\mathbb{TV}} = \left\|\mathcal{G}\left(\Sigma^{-\frac{1}{2}}\mu, I_m\right) - \mathcal{G}\left(\Sigma^{-\frac{1}{2}}\tilde{\mu}, I_m\right)\right\|_{\mathbb{TV}}.$$

iv)

$$\left\|\mathcal{G}(0, \Sigma) - \mathcal{G}\left(0, \tilde{\Sigma}\right)\right\|_{\mathbb{TV}} = \left\|\mathcal{G}\left(0, \tilde{\Sigma}^{-1/2}\Sigma\tilde{\Sigma}^{-1/2}\right) - \mathcal{G}(0, I_m)\right\|_{\mathbb{TV}}.$$

v) Let $\mu = (\mu_1, \dots, \mu_m)^*$ *and* $\tilde{\mu} = (\tilde{\mu_1}, \dots, \tilde{\mu_m})^*$. *Let us define* $\mu = (\mu, 0)$ *and* $\tilde{\mu} = (\tilde{\mu}, 0)$. *Then it follows that*

$$\|\mathcal{G}(\mu, I_{m+1}) - \mathcal{G}(\tilde{\mu}, I_{m+1})\|_{\mathbb{TV}} = \|\mathcal{G}(\mu, I_m) - \mathcal{G}(\tilde{\mu}, I_m)\|_{\mathbb{TV}}.$$

Proof. The item *i*), *ii*), *iii*) and *iv*) are done using the characterization of the total variation distance between two probability measures which are absolutely continuous with respect to the Lebesgue measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$, the Change of Variable Theorem and an straightforward calculations. The item *iv*) is done using the characterization of the total variation distance between two probability measures which are absolutely continuous with respect to the Lebesgue measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ and an straightforward calculations. □

Lemma B.2. *Let* $\mu = (\mu_1, \dots, \mu_m)^* \in \mathbb{R}^m$ *then*

$$\|\mathcal{G}(\mu, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} \leq \frac{\sum\limits_{i=1}^{m} |\mu_i|}{\sqrt{2\pi}}.$$

Proof. This is done using the classical coupling technique. We can write

$$\begin{aligned} \mathcal{G}(\mu, I_m) &= \mathcal{N}(\mu_1, 1) \otimes \cdots \otimes \mathcal{N}(\mu_m, 1), \\ \mathcal{G}(0, I_m) &= \underbrace{\mathcal{N}(0, 1) \otimes \cdots \otimes \mathcal{N}(0, 1)}_{m-times}. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathcal{G}(\mu, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} &= \|\mathcal{N}(\mu_1, 1) \otimes \cdots \otimes \mathcal{N}(\mu_m, 1) - \mathcal{N}(0, 1) \otimes \cdots \otimes \mathcal{N}(0, 1)\|_{\mathbb{TV}} \\ &\leq \sum_{k=1}^{m} \|\mathcal{N}(\mu_k, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}} \\ &\leq \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{m} |\mu_k|. \end{aligned}$$

□

Lemma B.3. *Let* $\{\mu_\epsilon\}_{\epsilon>0} \subset \mathbb{R}^m$ *be a sequence such that* $\lim\limits_{\epsilon \to 0} \mu_\epsilon = \mu \in \mathbb{R}^m$. *Then,*

$$\lim_{\epsilon \to 0} \|\mathcal{G}(\mu_\epsilon, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} = \|\mathcal{G}(\mu, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}}.$$

Proof. This is done using triangle inequality, the item *ii*) of Lemma B.1, Lemma B.2 and the Lemma D.2. □

Lemma B.4. *Let* $\{\mu_\epsilon\}_{\epsilon>0} \subset \mathbb{R}^m$ *be a sequence such that* $\lim_{\epsilon\to 0} \|\mu_\epsilon\| = +\infty$. *Then,*

$$\lim_{\epsilon\to 0} \|\mathcal{G}(\mu_\epsilon, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} = 1.$$

Proof. By definition

$$\|\mathcal{G}(\mu_\epsilon, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} = \frac{1}{2(2\pi)^{\frac{m}{2}}} \int\limits_{\mathbb{R}^m} \left| \exp\left\{-\frac{(x-\mu_\epsilon)^*(x-\mu_\epsilon)}{2}\right\} - \exp\left\{-\frac{x^*x}{2}\right\} \right| dx.$$

Let us define $f : \mathbb{R}^m \to]0,\infty[$ by $f(x) = \exp\left\{-\frac{x^*x}{2}\right\}$. Then, we want to compute

$$\|\mathcal{G}(\mu_\epsilon, I_m) - \mathcal{G}(0, I_m)\|_{\mathbb{TV}} = \frac{1}{2(2\pi)^{\frac{m}{2}}} \int\limits_{\mathbb{R}^m} |f(x-\mu_\epsilon) - f(x)|\, dx.$$

By a classical analysis technique that

$$\lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^m} |f(x-\mu_\epsilon) - f(x)|\, dx = \int\limits_{\mathbb{R}^m} |f(x)| dx, \tag{B.1}$$

when $\int\limits_{\mathbb{R}^m} |f(x)|dx < +\infty$. The last statement implies the result. Now, we will prove the relation (B.1). Let us define $M := \int\limits_{\mathbb{R}^m} |f(x)|dx < +\infty$. Let $\eta > 0$ be fixed. Then, there exist $r = r(\eta) > 0$ large enough such that

$$M - \int\limits_{B(0,r)} |f(x)|dx < \frac{\eta}{4}.$$

Therefore,

$$M - \int\limits_{B(\mu_\epsilon,r)} |f(x-\mu_\epsilon)|dx < \frac{\eta}{4}.$$

Due to $\lim_{\epsilon\to 0} \|\mu_\epsilon\| = +\infty$, then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, we have

$B(0,r) \cap B(\mu_\epsilon, r) = \emptyset$. Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^m} |f(x-\mu_\epsilon) - f(x)|\, dx &\geq \int_{B(0,r)} |f(x-\mu_\epsilon) - f(x)|\, dx + \int_{B(\mu_\epsilon,r)} |f(x-\mu_\epsilon) - f(x)|\, dx \\
&\geq \int_{B(0,r)} (|f(x)| - |f(x-\mu_\epsilon)|)\, dx + \int_{B(\mu_\epsilon,r)} |f(x-\mu_\epsilon)| - |f(x)|\, dx \\
&\geq 2M - \eta.
\end{aligned}$$

Consequently, for every $\eta > 0$, we have

$$2M - \eta \leq \int_{\mathbb{R}^m} |f(x-\mu_\epsilon) - f(x)|\, dx \leq 2M.$$

Now, taking $\eta \to 0$, we obtain the statement.

$\square$

Lemma B.5. *Let* $\{\Sigma_\epsilon\}_{\epsilon>0} \subset \mathcal{S}_m$ *be a sequence such that* $\lim\limits_{\epsilon\to 0} \Sigma_\epsilon = \Sigma \in \mathcal{S}_m$. *Then*

$$\lim_{\epsilon\to 0} \|\mathcal{G}(0,\Sigma_\epsilon) - \mathcal{G}(0,\Sigma)\|_{\mathbb{TV}} = 0.$$

Proof. By item *iv*) of Lemma B.1, for every $\epsilon > 0$, we have

$$\|\mathcal{G}(0,\Sigma_\epsilon) - \mathcal{G}(0,\Sigma)\|_{\mathbb{TV}} = \left\|\mathcal{G}\left(0, \Sigma^{-1/2}\Sigma_\epsilon\Sigma^{-1/2}\right) - \mathcal{G}(0, I_m)\right\|_{\mathbb{TV}}.$$

Consequently, it suffices to prove, when $\lim\limits_{\epsilon\to 0} \Sigma_\epsilon = I_m \in \mathcal{S}_m$. By definition, we have

$$\|\mathcal{G}(0,\Sigma_\epsilon) - \mathcal{G}(0,I_m)\|_{\mathbb{TV}} = \frac{1}{2(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \left| \frac{\exp\left\{-\frac{x^*\Sigma_\epsilon^{-1}x}{2}\right\}}{(\det(\Sigma_\epsilon))^{\frac{1}{2}}} - \exp\left\{-\frac{x^*x}{2}\right\} \right| dx.$$

Let us define the function $f_\epsilon : \mathbb{R}^m \to [0,+\infty[$ by $f_\epsilon(x) = \left| \frac{\exp\left\{-\frac{x^*\Sigma_\epsilon^{-1}x}{2}\right\}}{(\det(\Sigma_\epsilon))^{\frac{1}{2}}} - \exp\left\{-\frac{x^*x}{2}\right\} \right|$. For every $x \in \mathbb{R}^m$, we have $\lim\limits_{\epsilon\to 0} f_\epsilon(x) = 0$. Also, for $\epsilon > 0$ small enough, it follows that

$$f_\epsilon(x) \leq C_1 \exp\left\{-C_2\|x\|^2\right\} + \exp\left\{-\frac{\|x\|^2}{2}\right\}$$

for every $x \in \mathbb{R}^m$, where $C_1 > 0$ and $C_2 > 0$ are constants. Consequently, the results follows from the Dominated Convergence Theorem. $\square$

Lemma B.6 (Total Variation Bounded)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ be a sub-sigma algebra of $\mathcal{F}$. Let $X, Y, Z : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables such that X and Y are $\mathcal{G}$ measurables and $X, Y, Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let us consider the following random variables $X^* = X + Z$ and $Y^* = Y + Z$. Let us suppose that $E[X^* \mid \mathcal{G}]$ has Gaussian distribution $\mathcal{G}(X, \Sigma)$, $E[Y^* \mid \mathcal{G}]$ has Gaussian distribution $\mathcal{G}(Y, \Sigma)$ and $Z \perp \mathcal{G}$. Then,*

$$\|X^* - Y^*\|_{\mathbb{TV}} \leq C(m)\|\Sigma^{-\frac{1}{2}}\|\mathbb{E}[\|X - Y\|],$$

where $C(m) > 0$ is a constant which only depends on m.

Proof. Using the the properties of conditional expectation, the item *i*), the item *ii*), the item *iii*) of Lemma B.1 and Lemma B.2, we have

$$\begin{aligned}
\|X^* - Y^*\|_{\mathbb{TV}} &= \sup_{F \in \mathcal{F}} \left|\mathbb{E}\left[\mathbb{1}_{(X^* \in F)} - \mathbb{1}_{(Y^* \in F)}\right]\right| \\
&\leq \sup_{F \in \mathcal{F}} \mathbb{E}\left[\left|\mathbb{E}\left[\mathbb{1}_{(X^* \in F)} - \mathbb{1}_{(Y^* \in F)} \mid \mathcal{G}\right]\right|\right] \\
&\leq \sup_{F \in \mathcal{F}} \mathbb{E}\left[\left|\mathbb{P}\left(\mathcal{G}(X, \sigma^2) \in F\right) - \mathbb{P}\left(\mathcal{G}(Y, \sigma^2) \in F\right)\right|\right] \\
&\leq \sup_{F \in \mathcal{F}} \mathbb{E}\left[\frac{1}{\sqrt{2\pi}} \sum_{k=1}^{m} \left|\left(\Sigma^{-\frac{1}{2}}(X - Y)\right)_k\right|\right] \\
&\leq C(m)\|\Sigma^{-\frac{1}{2}}\|\mathbb{E}[\|X - Y\|],
\end{aligned}$$

where $C(m) > 0$ is a constant. □

Appendix C

Qualitative and Quantitative Behavior

Lemma C.1. *Let* $V : \mathbb{R} \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *function such that*

a) $V(0) = 0$.

b) $V'(x) = 0$ *iff* $x = 0$.

c) $\lim_{|x| \to +\infty} V(x) = +\infty$.

Then $V(x) > 0$ *if* $x \neq 0$.

Proof. It can be shown that $V'(x) < 0$ if $x < 0$ and $V'(x) > 0$ if $x > 0$. □

Lemma C.2. *Leu us assume the hypothesis of Theorem 2.3. Suppose that there exists a* $\mathcal{C}^2$ *function* $V : \mathbb{R} \to \mathbb{R}$ *such that*

a) $V(0) = 0$.

b) $V'(x) = 0$ *iff* $x = 0$.

c) $V''(0) > 0$.

d) $\lim_{|x| \to +\infty} V(x) = +\infty$.

Then its follows

i) $\lim_{t \to +\infty} \psi_t = 0$.

ii) $\lim_{t \to +\infty} \Phi_t = 0$.

iii) Let us assume that V *is a* $\mathcal{C}^3$ *function. Then there exist constants* $c \neq 0$ *and* $\tilde{c} \neq 0$ *such that*

$$\lim_{t \to +\infty} e^{V''(0)t} \Phi_t = c,$$

$$\lim_{t \to +\infty} e^{V''(0)t} \psi_t = \tilde{c},$$

where $\Phi = \{\Phi_t\}_{t \geq 0}$ *is the fundamental solution of the nonautonomous system*

$$d\Phi_t = -V''(\psi_t)\Phi_t dt$$

for every $t \geq 0$ *with initial condition* $\Phi_0 = 1$.

iv) Let us assume that V *is a* $\mathcal{C}^3$ *function, then*

$$\lim_{t \to +\infty} \Phi_t^2 \int_0^t \left(\frac{1}{\Phi_s} \right)^2 ds = \frac{1}{2V''(0)}.$$

Proof.

i) By our assumptioms $V'(0) = 0$, $V''(0) > 0$ and $V'(x) \neq 0$ if $x \neq 0$. Therefore, the unique critical point zero is asymptotically stable, so there exists an open neighboorhood N_0 of zero such that for every $\psi_0 \in N_0$. It follows that ψ_t goes to zero as t goes to infinity. Let us consider that $\psi_0 \notin N_0$ and $K := V^{-1}([0, V(\psi_0)])$. Then $\psi_t \in K$ for every $t \geq 0$. Also, K is a compact set because of $\lim_{|x| \to +\infty} V(x) = +\infty$. Because K is bounded, then there exist $r > 0$ such that $K \subset B(0, r)$ where we denote $B(0, r) := \{x \in \mathbb{R} : |x| < r\}$ and $\overline{B(0, r)} := \{x \in \mathbb{R} : |x| \leq r\}$ so we we can choose N_0 small enough such that $N_0 \subset B(0, r) \subset \overline{B(0, r)}$. Let us call $\hat{K} := \overline{B(0, r)}$ then $\psi_t \in \hat{K}$ for every $t \geq 0$. Let us define $\delta := \inf_{x \in \hat{K} \setminus N_0} (V'(x))^2 > 0$. Let us suppose that $\psi_t \notin N_0$ for every $t \geq 0$, then $dV(\psi_t) = -(V'(\psi_t))^2 \leq -\delta$ for every $t \geq 0$. Therefore, $0 \leq t \leq \frac{V(\psi_0)}{\delta}$ which is a contradiction. Consequently, there exists $\tau > 0$ such that $\psi_\tau \in N_0$ and consequently, ψ_t goes to zero as t goes to infinity.

ii) By our assumptions it follows that $\Phi_t = \frac{V'(\psi_t)}{V'(\psi_0)}$ for every $t \geq 0$, where $\psi_0 = x_0 \neq 0$. So by item *i)* and continuity of V' we have $\lim_{t \to \infty} \Phi_t = \frac{V'(0)}{V'(\psi_0)} = 0$.

iii) Let us define $H(z) = \left(\frac{V''(0)}{V'(z)} - \frac{1}{z}\right) \mathbb{1}_{\{z \neq 0\}} + \left(\frac{-V'''(0)}{2V''(0)}\right) \mathbb{1}_{\{z=0\}}$, where $\mathbb{1}_A$ denotes the indicator function of the set $A \subset \mathbb{R}$. Let us define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := x \exp\left(\int_0^x H(z)dz\right).$$

Since H is everywhere continuous, then it follows that h is well defined. Let us define $\Psi_t := h(\psi_t)$ for every $t \geq 0$, then $d\Psi_t = -V''(0)\Psi_t dt$ for every $t \geq 0$ and $\Psi_0 = h(\psi_0)$. Therefore,

$$\psi_t \exp\left(V''(0)t\right) = h(\psi_0) \exp\left(-\int_0^{\psi_t} H(z)dz\right) \tag{C.1}$$

for every $t \geq 0$. By Intermediate Value Theorem, for every $t \geq 0$ there exists $\xi_t \in]\min\{0, \psi_t\}, \max\{0, \psi_t\}[$ such that $V'(\psi_t) = V''(\xi_t)\psi_t$. Because of relation (C.1), we see that

$$V'(\psi_t) \exp\left(V''(0)t\right) = V''(\xi_t) h(\psi_0) \exp\left(-\int_0^{\psi_t} H(z)dz\right) \tag{C.2}$$

for every $t \geq 0$. Consequently, by the relation (C.2) and item *ii*), we have

$$\lim_{t\to+\infty} V'(\psi_t) \exp\left(V''(0)t\right) = V''(0)h(\psi_0).$$

Because $sgn(h(x)) = sgn(x)$ for every $x \neq 0$, then $V''(0)h(\psi_0) \neq 0$.

iv) By item *ii*), we have

$$\Phi_t^2 \int_0^t \left(\frac{1}{\Phi_s}\right)^2 ds = (V'(\psi_t))^2 \int_0^t \left(\frac{1}{V'(\psi_s)}\right)^2 ds$$

for each $t \geq 0$. By item $iii)$ and for each $0 < \epsilon < c^2$, we have

$$\limsup_{t\to+\infty} (V'(\psi_t))^2 \int_0^t \left(\frac{1}{V'(\psi_s)}\right)^2 ds \leq \left(\frac{c^2+\epsilon}{c^2-\epsilon}\right)\frac{1}{2V''(0)},$$
$$\liminf_{t\to+\infty} (V'(\psi_t))^2 \int_0^t \left(\frac{1}{V'(\psi_s)}\right)^2 ds \geq \left(\frac{c^2-\epsilon}{c^2+\epsilon}\right)\frac{1}{2V''(0)}.$$

Letting $\epsilon \to 0$, we obtain

$$\lim_{t\to+\infty} (V'(\psi_t))^2 \int_0^t \left(\frac{1}{V'(\psi_s)}\right)^2 ds = \frac{1}{2V''(0)}.$$

□

Lemma C.3. *Using the same notation as in the proof of Theorem 2.17. It follows that*

$$\lim_{\epsilon\to 0} \frac{\sup\limits_{\tilde{t}_\epsilon(b)\leq t\leq \hat{t}_\epsilon(b)} |\psi_t|}{\sqrt{\epsilon}} = \rho(b) \in]0,+\infty[.$$

for every $b \in \mathbb{R}$.

Proof. By continuity we have

$$\frac{\sup\limits_{\tilde{t}_\epsilon(b)\leq t\leq \hat{t}_\epsilon(b)} |\psi_t|}{\sqrt{\epsilon}} = \frac{|\psi_{t^*}|}{\sqrt{\epsilon}}$$

for some $t^* \in [\tilde{t}_\epsilon(b), \hat{t}_\epsilon(b)]$. Then, using the following relation and Lemma 2.4, it becomes straightforward.

$$\frac{|\psi_{t^*}|}{\sqrt{\epsilon}} = e^{V''(0)t^*}|\psi_{t^*}|\frac{e^{-V''(0)t^*}}{\sqrt{\epsilon}}.$$

□

Lemma C.4. *Under the hypothesis of Theorem 3.6, we have*

i) For any initial condition $\psi(0)$, it follows that $\psi(t)$ goes to zero as t goes to infinity. Moreover, $\|\psi(t)\| \leq \|\psi(0)\|e^{-\delta t}$ for every $t \geq 0$.

ii) *Lebesgue almost surely for $\psi(0)$, it follows that*

$$\lim_{t\to+\infty} e^{\alpha_1 t}\psi(t) \quad = \quad v(\psi(0)) \in \mathbb{R}^m \setminus \{0\},$$

where $v(\psi(0)) \in span\{v_1\}$ and v_1 is the eigenvector associated to the eigenvalue α_1.

iii) *Let us consider the following matrix differential equation,*

$$\begin{aligned} d\Lambda^\epsilon(t) &= -H_V(0)\Lambda^\epsilon(t) - \Lambda^\epsilon(t)H_V(0) + \epsilon I_m, \\ \Lambda^\epsilon(0) &\in M(m), \end{aligned}$$

where $M(m)$ is an squared matrix of dimension m. It follows that

$$\lim_{t\to\infty} \Lambda^\epsilon(t) = \frac{\epsilon}{2}\left(H_V(0)\right)^{-1}.$$

iv) *Let $\Delta^\epsilon(t) := \epsilon\mathbb{E}\left[y(t)\left(y(t)\right)^*\right]$. It satisfies the following matrix differential equation,*

$$\begin{aligned} d\Delta^\epsilon(t) &= -H_V(\psi(t))\Delta^\epsilon(t) - \Delta^\epsilon(t)H_V(\psi(t)) + \epsilon I_m, \\ \Delta^\epsilon(0) : &= 0 \in M(m), \end{aligned}$$

where $M(m)$ is an squared matrix of dimension m. It follows that

$$\lim_{t\to\infty} \Delta^\epsilon(t) = \frac{\epsilon}{2}\left(H_V(0)\right)^{-1}.$$

Proof.

i) It follows that

$$\begin{aligned} d\|\psi(t)\|^2 &= 2(\psi(t))^* d\psi(t) \\ &= -2(\psi(t))^*\nabla V(\psi(t)) \\ &\leq -2\delta\|\psi(t)\|^2 \end{aligned}$$

for every $t \geq 0$, where the last inequality follows from Lemma D.5. By the Gronwall Inequality, we have $\|\psi(t)\|^2 \leq \|\psi(0)\|^2 e^{-2\delta t}$ for every $t \geq 0$.

ii) Because all the eigenvalues of $-H_V(0)$ are reals and they are bounded for above by $-\delta < 0$. By Hartman-Grobman Theorem there exist neighborhoods $U, \tilde{U}$ of zero such that $h : U \to \tilde{U}$ is an homeomorphism that conjugate the flows of $\{\psi(t)\}_{t\geq 0}$

with initial condition $\psi(0) \in U$ and the linear flow $\{e^{-H_V(0)t}h(\psi(0))\}_{t\geq 0}$. Moreover $h(x) = x + o(\|x\|)$ when $\|x\|$ goes to zero. For details see [17] and [21]. Let $\psi(0) \in \mathbb{R}^m$. There exist $\tau > 0$ such that $\psi(t) \in U$ for every $t \geq \tau$. Therefore $h(\psi(\tau + t)) = e^{-H_V(0)t}h(\psi_\tau)$ for every $t \geq 0$. There exists an orthonormal basis of $\mathbb{R}^m$ for which the linear flow is written in the following way: $\sum_{i=1}^{m} e^{-\alpha_i t} < \psi(\tau), v_i > v_i$, where $0 < \delta \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m$ are the eigenvalues of $H_V(0)$ and $v_1, v_2, \ldots, v_m$ are the corresponding orthonormal eigenvectors. Then

$$\begin{aligned}\psi(\tau + t) &= h^{-1}\left(\sum_{i=1}^{m} e^{-\alpha_i t} < \psi(\tau), v_i > v_i\right) \\ &= \sum_{i=1}^{m} e^{-\alpha_i t} < \psi(\tau), v_i > v_i \\ &\quad + o\left(\|\sum_{i=1}^{m} e^{-\alpha_i t} < \psi(\tau), v_i > v_i\|\right).\end{aligned}$$

Consequently, for Lebesgue almost every initial condition $\psi(0) \in \mathbb{R}^m$ we have

$$\lim_{t\to\infty} e^{\alpha_1(t)}\psi(t) = e^{\alpha_1 \tau} < \psi(\tau), v_1 > v_1.$$

iii) The explicit solution is given by

$$\Lambda^\epsilon(t) = e^{-H_V(0)t}\Lambda^\epsilon(0)e^{-H_V(0)t} + \epsilon \int_0^t e^{-2H_V(0)s}ds$$

for every $t \geq 0$. Now, an straightforward calculation gives the result. For details, see [15].

iv) By item *i)* of this Lemma and using the local Lipschitz condition of H_V at zero with Lipschitz constant $L_0 > 0$, for every $\eta > 0$, we can take $\tau_\eta := \frac{1}{\delta}\ln\left(\frac{\|\psi(0)\|}{\eta}\right)$ such that

$$\|H_V(\psi(t)) - H_V(0)\| \leq L_0\|\psi(t)\| \leq L_0\|\psi(0)\|e^{-\delta t} \leq L_0\eta$$

for every $t \geq \tau_\eta$. Let us call $\tau := \tau_\eta$. Then,

$$d\Lambda^\epsilon(t + \tau) = -\left(H_V\left(\psi(t + \tau)\right)\Lambda^\epsilon(t) + \Lambda^\epsilon(t)H_V\left(\psi(t + \tau)\right)\right)dt + \epsilon I_m$$

for every $t \geq 0$ with initial condition $\Lambda^\epsilon(\tau)$. Let us consider the following matrix differential equation, Then,

$$\begin{aligned} d\Delta^{\epsilon}(t+\tau) &= -\left(H_V\left(0\right)\Delta^{\epsilon}(t)+\Delta^{\epsilon}(t)H_V\left(0\right)\right)dt+\epsilon I_m, \\ \Delta^{\epsilon}(\tau) &= \Lambda^{\epsilon}(\tau) \end{aligned}$$

for every $t \geq 0$. Let us define $\Pi^{\epsilon}(t) := \Lambda^{\epsilon}(t+\tau) - \Delta^{\epsilon}(t+\tau)$ for every $t \geq 0$. Then,

$$\begin{aligned} d\Pi^{\epsilon}(t) &= -\left(H_V\left(\psi(t+\tau)\right)\Pi^{\epsilon}(t)+\Pi^{\epsilon}(t)H_V\left(\psi(t+\tau)\right)\right)dt+ \\ &\quad \left(H_V(0)-H_V\left(\psi(t+\tau)\right)\right)\Delta^{\epsilon}(t+\tau)dt+ \\ &\quad \Delta^{\epsilon}(t+\tau)\left(H_V(0)-H_V\left(\psi(t+\tau)\right)\right)dt, \\ \\ \Pi^{\epsilon}(\tau) &= 0 \end{aligned}$$

for every $t \geq 0$. Therefore,

$$d\|\Pi^{\epsilon}(t)\|^2 = \sum_{i,j=1}^{m} 2\Pi^{\epsilon}_{i,j}(t)\left(d\Pi^{\epsilon}_{i,j}(t)\right)$$

for every $t \geq 0$. For every $i, j \in \{1, \ldots, m\}$, we have

$$\begin{aligned} d\Pi^{\epsilon}_{i,j}(t) &= -\sum_{k=1}^{m} H_V^{i,k}\left(\psi(t+\tau)\right)\Pi^{\epsilon}_{k,j}(t) - \sum_{k=1}^{m}\Pi^{\epsilon}_{i,k}(t)H_V^{k,j}\left(\psi(t+\tau)\right)+ \\ &\quad \sum_{k=1}^{m}\left[H_V^{i,k}(0)-H_V^{i,k}\left(\psi(t+\tau)\right)\right]\Delta^{\epsilon}_{k,j}(t+\tau)+ \\ &\quad \sum_{k=1}^{m}\Delta^{\epsilon}_{i,k}(t+\tau)\left[H_V^{k,j}(0)-H_V^{k,j}\left(\psi(t+\tau)\right)\right] \end{aligned}$$

for every $t \geq 0$. Consequently, using the δ-coercivity of V, we obtain

$$d\|\Pi^{\epsilon}(t)\|^2 \leq -4\delta\|\Pi^{\epsilon}(t)\|^2 + I(t) + J(t)$$

for every $t \geq 0$, where

$$
\begin{aligned}
I(t) &:= \sum_{i,j=1}^{m} 2\Pi^{\epsilon}_{i,j}(t) \sum_{k=1}^{m} \left[H_V^{i,k}(0) - H_V^{i,k}\left(\psi(t+\tau)\right)\right] \Delta^{\epsilon}_{k,j}(t+\tau), \\
J(t) &:= \sum_{i,j=1}^{m} 2\Pi^{\epsilon}_{i,j}(t) \sum_{k=1}^{m} \Delta^{\epsilon}_{i,k}(t+\tau) \left[H_V^{k,j}(0) - H_V^{k,j}\left(\psi(t+\tau)\right)\right]
\end{aligned}
$$

for every $t \geq 0$. Then, using the Lipschitz local condition, the Cauchy-Schwartz inequality and the fact that $|x| \leq x^2 + 1$ for every $x \in \mathbb{R}$, we have

$$
\begin{aligned}
|I(t)| &\leq 2L_0\eta \left(\|\Pi^{\epsilon}(t)\|^2 + m\right) \left(\|\Delta^{\epsilon}(t+\tau)\|^2 + m\right), \\
|J(t)| &\leq 2L_0\eta \left(\|\Pi^{\epsilon}(t)\|^2 + m\right) \left(\|\Delta^{\epsilon}(t+\tau)\|^2 + m\right)
\end{aligned}
$$

for every $t \geq 0$. By item *iii*) of this Lemma, we obtain that there exists $C > 0$ such that $\|\Delta^{\epsilon}(t+\tau)\|^2 \leq C$ for every $t \geq 0$. Consequently,

$$
d\|\Pi^{\epsilon}(t)\|^2 \leq \left(4L_0\kappa\eta - 4\delta\right)\|\Pi^{\epsilon}(t)\|^2 + 4L_0\kappa m\eta
$$

for every $t \geq 0$, where $\kappa := C + m$. *A priori* we can take $0 < \eta < \frac{3\delta}{4L_0\kappa}$, so

$$
d\|\Pi^{\epsilon}(t)\|^2 \leq -\delta\|\Pi^{\epsilon}(t)\|^2 + 4L_0\kappa m\eta
$$

for every $t \geq 0$. Now, using the Gronwall inequality, letting t goes to infinity and then let η goes to zero, we obtain

$$
\lim_{t\to\infty} \|\Pi^{\epsilon}(t)\|^2 = 0.
$$

Using the last fact and the item *iii*) of this Lemma, we obtain the statement.

□

Appendix D

Tools

Lemma D.1. $\lim_{\epsilon \to 0} \epsilon^{\alpha} \left(\ln\left({}^1\!/\!_\epsilon\right)\right)^{\beta} = 0$ *for every* $\alpha > 0$ *and* $\beta > 0$.

Proof. It follows using the L'Hopital Rule of Calculus several times. □

Lemma D.2. *Let* $\{a_\epsilon\}_{\epsilon>0} \subset \mathbb{R}$ *and* $\{b_\epsilon\}_{\epsilon>0} \subset \mathbb{R}$ *be sequences such that* $\lim_{\epsilon\to 0} b_\epsilon = b \in \mathbb{R}$. *Then*

i) $\limsup_{\epsilon\to 0} (a_\epsilon + b_\epsilon) = \limsup_{\epsilon\to 0} a_\epsilon + b$.

ii) $\liminf_{\epsilon\to 0} (a_\epsilon + b_\epsilon) = \liminf_{\epsilon\to 0} a_\epsilon + b$.

Proof. The proof follows from the definition of limsup and liminf. □

Lemma D.3. *Let* $W := \{W_t\}_{t\geq 0}$ *is a Brownian motion and let us consider* $B_t := \sup_{0\leq s\leq t} |W_s|$ *for each* $t \geq 0$. *Then for each* $k \in \mathbb{N}$, $\mathbb{E}\left[B_t^k\right]$ *has growth of the form* t^{α} *for some* $\alpha > 0$.

Proof. By the Donsker Theorem we can compute explicitly the distribution of B_t for every $t \geq 0$ fixed, and the result follows from an straightforward calculations. □

Definition D.4. *Let* $V : \mathbb{R}^m \to \mathbb{R}$ *be a function. We say that* V *is* δ*-coercive or* δ*-strong convex function if there exists* $\delta > 0$ *such that*

$$V(tx + (1-t)y) \leq tV(x) + (1-t)V(y) - \frac{\delta}{2} t(1-t)\|x-y\|^2$$

for every $x, y \in \mathbb{R}^m$.

The following Lemma provides a characterization of coercive functions.

Lemma D.5 (Characterizations Coercivity Functions). *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a C^2-function. The following statements are equivalents:*

i) *V is δ-coercive or δ-strong convex function.*

ii) *$V(y) \geq V(x) + (\nabla V(x))^*(y-x) + \frac{\delta}{2}\|y-x\|^2$ for every $x, y \in \mathbb{R}^m$. The constant δ is called the convexity parameter of function V.*

iii) *$\left(\nabla V(x) - \nabla V(y)\right)^* (x-y) \geq \delta\|x-y\|^2$ for every $x, y \in \mathbb{R}^m$.*

iv) *$y^* H_V(x) y \geq \delta\|y\|^2$ for every $x, y \in \mathbb{R}^m$, where H_V represents the Hessian matrix associated to the scalar function V.*

Proof. For details see [27]. □

Lemma D.6 (Liptchitz Gradient Coercivity Functions). *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a δ-coercive C^2-function such that the gradient ∇V is Lipschitz with Lipschitz constant $\Delta > 0$. Then, $y^* H_V(x) y \leq \Delta\|y\|^2$ for every $x, y \in \mathbb{R}^m$, where H_V represents the Hessian matrix associated to the scalar function V.*

Proof. For details see [27]. □

Lemma D.7 (Jensen Inequality). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable and if φ is a convex function on the real line, then*

$$\varphi\left(\int_\Omega g \, d\mu\right) \leq \int_\Omega \varphi \circ g \, d\mu.$$

Theorem D.8 (Pinsker Inequality). *Let μ and ν be two probability measures define in the measurable space $(\Omega, \mathcal{F})$. Then it follows that*

$$\|\mu - \nu\|^2_{\mathbb{TV}} \leq 2\mathcal{H}\left(\mu \,|\, \nu\right),$$

where $\mathcal{H}\left(\mu \,|\, \nu\right)$ is the Kullback information of μ respecto to ν and it is define as follows: if $\mu \ll \nu$ then take the Radon-Nikodym derivative $f = \frac{d\mu}{d\nu}$ and define $\mathcal{H}\left(\mu \,|\, \nu\right) := \int\limits_\Omega f \ln(f) d\nu$, in the case $\mu \not\ll \nu$ let us define $\mathcal{H}\left(\mu \,|\, \nu\right) := +\infty$.

Proof. For details check [3] or [9]. □

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