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# SIMPLE LYAPUNOV SPECTRUM FOR LINEAR COCYCLES OVER CERTAIN PARTIALLY HYPERBOLIC MAPS

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Para mis padres y hermanos.

# Abstract

Criteria for the simplicity of the Lyapunov spectra of linear cocycles have been found by Guivarc'h-Raugi, Gol'dsheid-Margulis and, more recently, Bonatti-Viana and Avila-Viana. In all the cases, the authors consider cocycles over hyperbolic systems, such as shifts or Axiom A diffeomorphism.

In this thesis we propose to extend such criteria to situations where the base map is just partially hyperbolic. This raises a lot of new issues concerning, among others, the recurrence of the holonomy maps and the (lack of) continuity of the disintegrations of $u$-states.

Our first results are stated for partially hyperbolic skew-products whose iterates have bounded derivatives along center leaves. They allow us, in particular, to exhibit non-trivial examples of stable simplicity in the partially hyperbolic setting.

The second result is when we consider $SL(2,\mathbb{R})$ cocycles over partially hyperbolic diffeomorphisms. Under a hypothesis over the behavior of the cocycles over a compact center leaf of the diffeomorphism, we prove that the cocycle is accumulated by open sets where the Lyapunov exponents are non-zero.

**Keywords:** Linear cocycles, Lyapunov exponents, fiber bunched cocycles, partially hyperbolic diffeomorphism, skew-product.

# Contents

# CHAPTER 1

## Introduction

The theory of linear cocycles is a classical and rather developed field of Dynamics and Ergodic Theory, whose origins go back to the works of Furstenberg, Kesten [16, 14] and Oseledets [21]. The simplest examples are the derivative transformations of smooth dynamical systems. However, the notion of linear cocycle is a lot more general and flexible, and arises naturally in many other situations, such as the spectral theory of Schrödinger operators.

Among the outstanding issues is the problem of simplicity: when is it the case that the dimension of all Oseledets subspaces is equal to 1. This was first studied by Guivarc'h-Raugi [18] and Gol'dsheid-Margulis [17], who obtained explicit simplicity criteria for random i.i.d. products of matrices. Bonatti-Viana [9] and Avila-Viana [3] have much extended the theory, to include a much broader class of (Hölder continuous) cocycles over hyperbolic maps.

Our purpose in this work is to initiate the study of simplicity in the context of cocycles over partially hyperbolic maps. The study of partially hyperbolic systems was introduced in the works of Brin-Pesin [11] and Hirsch-Pugh-Shub [19] and has been at the heart of much recent progress in this area. While sharing many of the important features of uniformly hyperbolic systems, partially hyperbolic maps are a lot more flexible and encompass many new interesting phenomena.

As we are going to see, the study of linear cocycles over partially hyperbolic maps introduces a host of new issues. We will deal with these issues in the context when the partially hyperbolic map is given by a partially hyperbolic skew-product.

The case when the map is non-uniformly hyperbolic, that is, when all center Lyapunov exponents are non-zero, is better understood. Indeed, the case of 2-dimensional cocycles is covered by Viana [24] and for general dimension $d \geq 2$ it should be possible to combine a symbolic description as in Sarig [23] with the main result of Avila-Viana [3], that deals with cocycles over countable shifts.

For this reason, we focus on the opposite case: namely, we take the skew-product to be mostly neutral along the center direction, meaning that its iterates are have bounded derivatives along the vertical leaves $\{\hat{x}\} \times K$.

Concerning the linear cocycle, we take it to admit strong-stable and strong-unstable holonomies, in the sense of Bonatti-Gomez-Mont-Viana [8] and Avila-Santamaria-Viana [2]. The simplicity conditions in our main result, that we are going to state next, may be viewed as extensions of the pinching and twisting conditions in Bonatti-Viana [9] and Avila-Viana [3] to the present partially hyperbolic setting.

Firstly, we call the linear cocycle *uniformly pinching* if there exists some fixed (or periodic) vertical leaf $\ell = \{\hat{p}\} \times K$ such that the restriction to $\ell$ of every exterior power of the cocycle admits an invariant dominated decomposition

$$\ell \times \Lambda^k(\mathbb{C}^d) = E^{k,1} \oplus \cdots \oplus E^{k,\dim \Lambda^k(\mathbb{C}^d)}$$

into 1-dimensional subbundles. In particular, this decomposition is continuous and the Lyapunov exponents along the factor subbundles $E^j$ are all distinct.

Secondly, we say that the linear cocycle is *uniformly twisting* if, for any $su$-path $\gamma$ connecting points $(\hat{p}, t) \in \ell$ and $(\hat{p}, s) \in \ell$, the push-forward of the decomposition

$$E^{1,1}(t) \oplus \cdots \oplus E^{1,d}(t)$$

at $(\hat{p}, t)$ under the concatenation of strong-stable and strong-unstable holonomies over $\gamma$ is in general position with respect to the decomposition

$$E^{1,1}(s) \oplus \cdots \oplus E^{1,d}(s)$$

at $(\hat{p}, s)$. By the latter, we mean that the image of any sum of $k$ subspaces $E^{1,i}(t)$ is transverse to the sum of any $d-k$ subspaces $E^{1,j}(s)$, for any $1 \leq k \leq d-1$.

We say that the linear cocycle is *uniformly simple* if it is both uniformly pinching and uniformly twisting.

**Theorem A.** Every uniformly simple linear cocycle over a partially hyperbolic skew-product with mostly neutral center direction has simple Lyapunov spectrum relative to any invariant probability measure with local partial product structure.

The notion of local partial product structure will be recalled in Chapter 3, where we will also give a more precise version of the theorem. Indeed, as we will see, the conclusion holds under weaker (non-uniform) versions of the pinching and twisting conditions. Moreover, already in the form given in Theorem A, our simplicity criterion holds for a subset of Hölder continuous cocycles with non-empty interior.

If we deal with $\mathrm{SL}(2,\mathbb{R})$ cocycles there are more results about genericity and stability of simple or not simple spectrum. In this case simplicity is equivalent to non-zero Lyapunov exponents.

Results about genericity of zero Lyapunov exponents were announced by Mañe and proved by Bochi [7], they proved that $C^0$ generically $\mathrm{SL}(2,\mathbb{R})$ cocycles are uniformly hyperbolic or have zero Lyapunov exponents. Also Avila [5] proved in a very general setting (including the previous case) that there exist a dense set of cocycles with non-zero Lyapunov exponents.

On the other hand, the situation changes radically in the context of (Hölder) fiber bunched cocycles. Viana [24], Avila-Viana [4], Avila-Viana-Santamaria [1] proved that in this class generically the Lyapunov exponents are non-zero. In particular, theorem A [1], gives examples of cocycles with stably non-zero exponents when the base dynamics is partially hyperbolic, volume preserving and accessible.

Here, we provide a new construction that does not require volume preserving nor accessiblility. Under certain conditions we prove, in Chapter 9 that the cocycle is acumulated by open sets where the Lyapunov exponents are non-zero. We prove the next theorem:

**Theorem B.** Let $\mu$ be a $f$-invariant ergodic measure with zero center Lyapunov exponent and projective product structure.

Let $A : M \to SL(2,\mathbb{R})$ be a fiber bunched cocycle, such that the restriction to some periodic center leave of $f$ is non-uniformly hyperbolic and is a weak continuity point of Lyapunov exponent. Then $A$ is accumulated by stably non-zero cocycles.

By non-uniformly hyperbolic we mean that the Lyapunov exponents are non-zero and the concept of weak continuity is a type of continuity of Lyapunov exponents, for non-ergodic measures, that we introduce in Chapter 9.

In the proof of this fact we use an equivalence condition between continuity of Lyapunov exponents and Oseledets decomposition in the case of $SL(2,\mathbb{R})$ cocycles. Actually we prove a much more general version, that is an interesting result by itself.

We prove this equivalence in the case of semi-invertible cocycles of any dimension, this means that the base map is invertible but the matrices may not be invertible.

**Theorem C.** $A$ is a continuity point for the Lyapunov exponents if and only if it is a continuity point for the Oseledets subspaces with respect to the measure $\mu$.

## 1.1 Structure of the work

This work is divided in 3 parts.

- The first one, from Chapter 3 to 7, is devoted to prove Theorem A. Actually we prove a stronger result that gives a condition for simple Lyapunov spectrum for cocycles $A : M \to GL(d, \mathbb{C})$ with $f$ a partially hyperbolic skew-product.
- In the second part, chapter 8, we prove the equivalence between continuity of Lyapunov Exponents and Oseledets decomposition, wich correspond to Theorem C. As mentioned before, a particular case of this result will be used in chapter 9.
- The third part, chapter 9, is devoted to prove Theorem B.

These 3 results are relatively independent, and can be read separate.

In chapter 2 we give the precise definitions and preliminary results that will be used in the whole thesis.

# CHAPTER 2

## Definitions and Preliminary Results

Let $(M, \mathfrak{B}, \mu)$ be a measurable space and $f : M \to M$ an invertible measurable map that preserves $\mu$. Fixed $d \in \mathbb{N}$, every measurable matrix valued map $A : M \to M(d, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) defines a *linear cocycle over* $f$,

$$F_A : M \times \mathbb{K}^d \to M \times \mathbb{K}^d, \quad F_A(x, v) = (f(x), A(x)v).$$

Its iterates are given by

$$F_A^n(x, v) = (f^n(x), A^n(x)v),$$

Were

$$A^n(x) = \begin{cases} A(f^{n-1}(x)) \dots A(f(x))A(x) & \text{if } n > 0 \\ \mathrm{id} & \text{if } n = 0 \end{cases}$$

It was proved in [13] that, if $\int \log^+ \|A\| d\mu < \infty$, for $\mu$-almost every point $x \in M$, there exist measurable functions $\lambda_1(x) > \dots > \lambda_l(x) \geq -\infty$, and a direct sum decomposition $\mathbb{R}^d = E_x^{1,A} \oplus \dots \oplus E_x^{l,A}$ into vector subspaces, such thatfor every $1 \leq i \leq l$,

- $\dim(E_x^{i,A})$ is constant on orbits,
- $A^n(x)E_x^{i,A} \subseteq E_{f^n(x)}^{i,A}$ with equality when $\lambda_i > -\infty$

and

- $\lambda_i(x) = \lim_{n \to +\infty} \dfrac{1}{n} \log \| A^n(x)v \|$ for every non-zero $v \in E_x^{i,A}$.

The $\lambda_i$'s are called Lyapunov exponents and the bundles $x \mapsto E_x^{i,A}$ are called Lyapunv spaces. If $\mu$ is ergodic the $\lambda_i$'s and the dimensions are constant. Denote by

$$\gamma_1(A) \geq \gamma_2(A) \geq \dots \geq \gamma_d(A)$$

the Lyapunov exponents of $(f, A)$ counted with multiplicities.

This result extends the *multiplicative ergodic theorem* of Oseledets [21] in the case where the matrices may not be invertible. We call this cocycles *semi-invertible.*

In the case of maps $A : M \to GL(d, \mathbb{C})$ such that $\log \left\|A^{-1}\right\|$ and $\log \|A\|$ are $\mu$-integrable (Oseledets Theorem), the results are also true for $n < 0$ and $\lambda_l > -\infty$.

By Kingman's subadditive ergodic Theorem $\lambda_1 = \lim \frac{1}{n} \int \log \|A^n\|$.

We say that $A$ has simple Lyapunov spectrum if all Oseledets sub spaces have $\dim E^i(x) = 1$. This gives a much more simple behavior of the dynamic of the cocycle. For example if we call

$$\mathbb{P}F_A : M \times \mathbb{C}P^{d-1} \to M \times \mathbb{C}P^{d-1}, \quad \mathbb{P}F_A(x, [v]) = (f(x), [A(x)v])$$

every $\mathbb{P}F_A$-invariant measure that projects on $\mu$ is concentrated in the bundle $\{E^1_x, \cdots, E^d_x\}$, this is proved in Proposition 8.2.1. By Poincare's Recurrence the complement of this set is contained in the set of wandering points. If $A : M \to SL(2, \mathbb{R})$ we have $\lambda_1 = -\lambda_2$, then simple spectrum is equivalent to non zero Lyapunov exponents.

When $(M, d)$ is a metric space, we define an uniform topology in the space $C^0(M)$ of continuous cocycles $A : M \to M(d, \mathbb{K})$ by the norm

$$\|A\|_\infty = \sup_x \|A(x)\|.$$

An interesting case is when the cocycles have more regularity. In particular we are interested in the $\alpha$-Hölder cocycles, denoted by $H^\alpha(M)$. The norm

$$\|A\|_\alpha = \sup_{x \in M} \|A(x)\| + \sup_{x \neq y} \frac{\|A(x) - A(y)\|}{\operatorname{dist}(x, y)^\alpha}$$

defines a topology in $H^\alpha(M)$ that we call *$\alpha$-Hölder topology.*

A cocycle $A$ is called *stably simple* if, in some topology, there exists an open set $A \in \mathcal{V}$ such that every $B \in \mathcal{V}$ has simple Lyapunov spectrum.

We say that $A \in C^0(M)$ is a *continuity point for the Lyapunov exponents* if for every sequence $\{A_k\}_k \subset C^0(M)$ converging to $A$ we have $\lim_{k\to\infty} \gamma_i(A_k) = \gamma_i(A)$ for every $1 \leq i \leq d$. Observe that in this case for every $k$ sufficiently large we have

$$\gamma_1(A_k) \geq \gamma_{\tilde{d}_1}(A_k) > \gamma_{\tilde{d}_1+1}(A_k) \geq \gamma_{\tilde{d}_2}(A_k) > \ldots > \gamma_{\tilde{d}_{l-1}+1}(A_k) \geq \gamma_d(A_k)$$

where $\tilde{d}_i = \sum_{j=1}^{i} d_j(A)$ for every $1 \leq i \leq l$. In particular, $A_k$ has at least $l$ different Lyapunov exponents and the sum of the dimensions of the Oseledets subspaces associated with $\gamma_{\tilde{d}_{j-1}+1}(A_k), \ldots, \gamma_{\tilde{d}_j}(A_k)$ coincide with the dimension of $E^{j,A}_x$ for every $1 \leq j \leq l$ where $\tilde{d}_0 = 0$. This motivates the following definition.

Given a sequence $\{A_k\}_k \subset C^0(M)$ converging to $A \in C^0(M)$ we say that the *Oseledets subspaces of* $A_k$ *converge to those of* $A$ *with respect to the measure* $\mu$ if for every $k$ sufficiently large there exists a direct sum decomposition $\mathbb{R}^d = F_x^{1,A_k} \oplus \ldots \oplus F_x^{l,A_k}$ into vector subspaces such that the following conditions are satisfied:

i) $F_x^{i,A_k} = E_x^{j,A_k} \oplus E_x^{j+1,A_k} \oplus \ldots \oplus E_x^{j+t,A_k}$ for some $j \in \{1, \ldots, l_k\}$ and $t \geq 0$;

ii) $\dim(F_x^{i,A_k}) = \dim(E_x^{i,A})$ for every $i = 1, \ldots, l$;

iii) for every $\delta > 0$ and $1 \leq i \leq l$ we have

$$\mu\left(\{x \in M; \measuredangle(F_x^{i,A_k}, E_x^{i,A}) > \delta\}\right) \xrightarrow{k\to\infty} 0,$$

where the *angle* $\measuredangle(E, F)$ between two subspaces $E$ and $F$ of $\mathbb{R}^d$ is defined as follows: given $w \in \mathbb{R}^d$ we define

$$\operatorname{dist}(w, E) = \inf_{v\in E} \|w - v\|.$$

More generally, we may consider the distance between $E$ and $F$ given by

$$\operatorname{dist}(E, F) = \sup_{v\in E, w\in F} \left\{\operatorname{dist}\left(\frac{v}{\|v\|}, F\right), \operatorname{dist}\left(\frac{w}{\|w\|}, E\right)\right\}. \tag{2.1}$$

Then, the angle between $E$ and $F$ is just $\measuredangle(E, F) = \sin^{-1}(\operatorname{dist}(E, F))$. A cocycle $A$ is said to be a *continuity point for the Oseledets decomposition with respect to the measure* $\mu$ if the above requirements are satisfied for every sequence $\{A_k\}_k \subset C^0(M)$ converging to $A$.

## 2.1 Partially hyperbolic maps

Given any $x \in M$ and $\varepsilon > 0$, we define the *local stable* and *unstable sets* of $\hat{x}$ with respect to $f$ by

$$W_\epsilon^s(x) := \{y \in M : \operatorname{dist}_M(f^n(x), f^n(y)) \leq \epsilon,\ \forall n \geq 0\},$$
$$W_\epsilon^u(x) := \{y \in M : \operatorname{dist}_M(f^n(x), f^n(y)) \leq \epsilon,\ \forall n \leq 0\},$$

respectively.

**Definition 2.1.1.** We say that a homeomorphism $f : M \to M$ is *hyperbolic* whenever there exist constants $C, \epsilon, \tau > 0$ and $\lambda \in (0, 1)$ such that the following conditions are satisfied:

- $\operatorname{dist}_M(f^n(y_1), f^n(y_2)) \leq C\lambda^n \operatorname{dist}_M(y_1, y_2)$, $\forall x \in M$, $\forall y_1, y_2 \in W_\epsilon^s(x)$, $\forall n \geq 0$;

- $\operatorname{dist}_M(f^{-n}(y_1), f^{-n}(y_2)) \leq C\lambda^n \operatorname{dist}_M(y_1, y_2)$, $\forall x \in M$, $\forall y_1, y_2 \in W^u_\epsilon(x)$, $\forall n \geq 0$;
- If $\operatorname{dist}_M(x, y) \leq \tau$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ intersect in a unique point which is denoted by $[x, y]$ and depends continuously on $x$ and $y$.

A diffeomorphism $f : M \to M$ is called *partially hyperbolic* if there exist a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

invariant under the derivative $Df$, a Riemannian metric $\|\cdot\|$ on $M$, and positive continuous functions $\nu$, $\hat\nu$, $\gamma$, $\hat\gamma$ with $\nu$, $\hat\nu < 1$ and $\nu < \gamma < \hat\gamma^{-1} < \hat\nu^{-1}$ such that, for any unit vector $v \in T_pM$,

$$\|Df(p)v\| < \nu(p) \qquad \text{if } v \in E^s(p), \tag{2.2}$$
$$\gamma(p) < \|Df(p)v\| < \hat\gamma(p)^{-1} \qquad \text{if } v \in E^c(p), \tag{2.3}$$
$$\hat\nu(p)^{-1} < \|Df(p)v\| \qquad \text{if } v \in E^u(p). \tag{2.4}$$

All three sub-bundles $E^s$, $E^c$, $E^u$ are assumed to have positive dimension.

A partially hyperbolic diffeomorphism $f : M \to M$ is called *dynamically coherent* if there exist invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ with smooth leaves tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$, respectively. Intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ one obtains a center foliation $\mathcal{W}^c$ whose leaves are tangent to the center sub-bundle $E^c$ at every point.

The stable and unstable bundles $E^s$ and $E^u$ are uniquely integrable and their integral manifolds form two transverse continuous foliations $W^{ss}$ and $W^{uu}$, whose leaves are immersed submanifolds of the same class of differentiability as $f$. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under $f$, in the sense that

$$f(W^{ss}(x)) = W^{ss}(f(x)) \quad \text{and} \quad f(W^{uu}(x)) = W^{uu}(f(x)),$$

where $W^{ss}(x)$ and $W^{uu}(x)$ denote the leaves of $W^{ss}$ and $W^{uu}$, respectively, passing through any $x \in M$. These foliations are, usually, *not* transversely smooth: the holonomy maps between any pair of cross-sections are not even Lipschitz continuous, in general, although they are always $\alpha$-Hölder continuous for some $\alpha > 0$.

Let $d = \dim M$ and $\mathcal{F}$ be a continuous foliation of $M$ with $k$-dimensional smooth leaves, $0 < k < d$. Let $\mathcal{F}(p)$ be the leaf through a point $p \in M$ and $\mathcal{F}(p, R) \subset \mathcal{F}(p)$ be the neighborhood of radius $R > 0$ around $p$, relative to the distance defined by the Riemannian metric restricted to $\mathcal{F}(p)$. A *foliation box* for $\mathcal{F}$ at $p$ is the image of an embedding

$$\Phi : \mathcal{F}(p, R) \times \mathbb{R}^{d-k} \to M$$

such that $\Phi(\cdot, 0) = \mathrm{id}$, every $\Phi(\cdot, y)$ is a diffeomorphism from $\mathcal{F}(p, R)$ to some subset of a leaf of $\mathcal{F}$ (we call the image a *horizontal slice*), and these diffeomorphisms vary continuously with $y \in \mathbb{R}^{d-k}$. Foliation boxes exist at every $p \in M$, by definition of continuous foliation with smooth leaves. A *cross-section* to $\mathcal{F}$ is a smooth codimension-$k$ disk inside a foliation box that intersects each horizontal slice exactly once, transversely and with angle uniformly bounded from zero.

Then, for any pair of cross-sections $\Gamma$ and $\Gamma'$, there is a well defined *holonomy map* $\Gamma \to \Gamma'$, assigning to each $x \in \Gamma$ the unique point of intersection of $\Gamma'$ with the horizontal slice through $x$.

## 2.2 Linear cocycles with holonomies

Let $A : M \to GL(d, \mathbb{C})$ be an $\alpha$-Hölder continuous map for some $\alpha > 0$.

We say that the cocycle admits *strong-stable holonomies* and *strong-unstable holonomies*, in the sense of [8, 2].

By *strong-stable holonomies* we mean a family of linear transformations $H^{ss}_{p,q} : \mathbb{C}^d \to \mathbb{C}^d$, defined for each $p, q \in M$ with $q \in W^{ss}_{loc}(p)$ and such that, for some constant $L > 0$,

(a) $H^s_{f^j(p), f^j(q)} = A^j(q) \circ H^s_{p,q} \circ A^j(p)^{-1}$ for every $j \geq 1$;

(b) $H^s_{p,p} = \mathrm{id}$ and $H^s_{p,q} = H^s_{z,q} \circ H^s_{p,z}$ for any $z \in W^{ss}_{loc}(p)$;

(c) $\|H^s_{p,q} - \mathrm{id}\| \leq L \,\mathrm{dist}(p, q)^\alpha$.

*Strong unstable holonomies* $H^{uu}_{p,q} : \mathbb{C}^d \to \mathbb{C}^d$ are defined analogously, for the pairs $(p, q)$ with $q \in W^{uu}_{loc}(p)$.

It was shown in [2] that strong-stable holonomies and strong-unstable holonomies do exist, in particular, when the cocycle is *fiber bunched*. By the latter we mean that there exist $C > 0$ and $\theta < 1$ such that

$$\|A^n(p)\|\|A^n(p)^{-1}\| \min\{\hat{\nu}(p), \nu(p)\}^{n\alpha} \leq C\theta^n \quad \text{for every } p \in M \text{ and } n \geq 0,$$

where $\hat{\nu}(p), \nu(p)$ are the hyperbolicity functions for $f$ as in conditions (i)-(ii) above.

## 2.3 Invariance Principle

A more general type of cocycles are the *smooth cocycles* over $f$, let us give the definitions

Let $\pi : E \mapsto M$ be a fiber bundle with smooth fibers modeled on some Riemannian manifold $N$. A smooth cocycle over $f : M \to M$ is a continuous transformation $F : E \mapsto E$ such that $\pi \circ F = f \circ \pi$, every $F_x : E_x \mapsto E_{f(x)}$ is a $C^1$ diffeomorphism depending continuously on $x$, relative to the uniform

$C^1$ distance in the space of $C^1$ diffeomorphisms on the fibers, and the norms of the derivative $DF_x(\xi)$ and its inverse are uniformly bounded.

In particular, the functions

$$(x, \xi) \mapsto \log \|DF_x(\xi)\| \quad \text{and} \quad (x, \xi) \mapsto \log \left\|DF_x(\xi)^{-1}\right\|$$

are bounded. Then (Kingman [20]), given any $F$-invariant probability $m$ on $E$, the extremal Lyapunov exponents of $F$ $\lambda_+ = \lim_{n\to\infty} \frac{1}{n} \log \|DF_x^n(\xi)\|$ and $\lambda_- = \lim_{n\to\infty} \frac{1}{n} \log \left\|DF_x^n(\xi)^{-1}\right\|^{-1}$ are well defined at $m$-almost every $(x, \xi) \in E$

We call *stable holonomies* a family of transformations $h^{ss}_{p,q} : E_p \mapsto E_q$, defined for each $p, q \in M$ with $q \in W^{ss}_{loc}(p)$ and such that, for some constant $L > 0$,

(a) $h^s_{f^j(p),f^j(q)} = F^j_q \circ h^s_{p,q} \circ F^{j^{-1}}_p$ for every $j \geq 1$;

(b) $h^s_{p,p} = \mathrm{id}$ and $h^s_{p,q} = H^s_{z,q} \circ h^s_{p,z}$ for any $z \in W^{ss}_{loc}(p)$;

(c) $(p, q, \xi) \mapsto h^s_{p,q}(\xi)$ varies continuously with respect to pairs $(x, y)$ in the same strong stable leaf;

(d) There are $C > 0$ and $\gamma > 0$ such that $H^s_{p,q}$ is $(C, \gamma)$-Hölder continuous for every $x$ and $y$ in the same local strong stable leaf.

The *unstable holonomies* $h^{uu}_{p,q} : E_p \mapsto E_q$ are defined analogously, for the pairs $(p, q)$ with $q \in W^{uu}_{loc}(p)$.

The linear, strong stable and strong unstable holonomies defined in the previous section are a particular case of holonomies.

Here we recall an important theorem, known as invariance principle, that will be essential in the work, This is Theorem 4.1 of [2].

**Theorem 2.3.1.** *Let $f$ be a $C^1$ partially hyperbolic diffeomorphism, $F$ be a smooth cocycle over $f$, $\mu$ be a $f$-invariant probability, and $m$ be a $F$-invariant probability on $E$ such that $\pi m = \mu$. Then if $F$ admits invariant stable and unstable holonomies and $\lambda = \lambda_+ = 0$ then for any desintegration $m_x : x \in M$ of $m$ into conditional probabilities along the fibers, there exist:*

*($\alpha$) A full $\mu$-measure subset $M^s$ such that $m_z = (h^s_{y,z})m_y$ for every $y, z \in M^s$ in the same strong-stable leaf;*

*($\beta$) A $\mu$-measure subset $M^u$ such that $m_z = (h^u_{y,z})m_y$ for every $y, z \in M^u$ in the same strong-unstable leaf.*

A measure $m$ is called *su-state* if it has the properties of the conclusion of the previous theorem. If $M^s = M$ we say that the measure is *s-invariant*, and if $M^u = M$ we say that if is *u-invariant*.

In the case where $\tilde{f} : \tilde{M} \mapsto \tilde{M}$ is hyperbolic ($E^c_x = \{0\}$) we say that an invariant measure $\mu$ has *local product structure* if there exist $\mu^s$ and $\mu^u$ measures in $W^s_{loc}$ and $W^u_{loc}$ respectively, such that locally $\mu \sim \mu^s \times \mu^u$. We recall an important Result for this type of cocycles, whose proof can be found in [4, Proposition 4.8].

**Proposition 2.3.2.** *Assume $\tilde{F} : \tilde{M} \to \tilde{M}$, the smooth cocycle defined over $\tilde{f}$, admits s-holonomy and u-holonomy. Assume $\tilde{f}$ and $\pi_*\mu$ have local product structure, as described above. If $m$ is an su-state then it admits a disintegration which is s-invariant and u-invariant and whose conditional probabilities $m_x$ vary continuously with $x$ in the support of $\pi_*\mu$.*

# CHAPTER 3

# First statements

Now let us give the precise definitions of the notions involved, as well as a refined version of Theorem A.

## 3.1 Partially hyperbolic skew-products

Let $\hat{\sigma} : \hat{\Sigma} \to \hat{\Sigma}$ be any two-sided finite or countable shift. By this we mean that $\hat{\Sigma}$ is the set of two-sided sequences $(x_n)_{n\in\mathbb{Z}}$ in some set $X \subset \mathbb{N}$ with $\#X > 1$, and the map $\hat{\sigma}$ is given by

$$\hat{\sigma}\left((x_n)_{n\in\mathbb{Z}}\right) = (x_{n+1})_{n\in\mathbb{Z}} .$$

Let $\mathrm{dist} : \hat{\Sigma} \times \hat{\Sigma} \to \mathbb{R}$ be the distance defined by

$$\mathrm{dist}(\hat{x}, \hat{y}) = \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|}} |x_k - y_k| \tag{3.1}$$

where $\hat{x} = (x_k)_{k\in\mathbb{Z}} \in \hat{\Sigma}$ and $\hat{y} = (y_k)_{k\in\mathbb{Z}} \in \hat{\Sigma}$. Then $\hat{\Sigma}$ is a compact metric space. Moreover, $\hat{\sigma}$ is a *hyperbolic homeomorphism*, as we are going to explain.

Given any $\hat{x} \in \hat{\Sigma}$ and $\varepsilon > 0$, we define the *local stable* and *unstable sets* of $\hat{x}$ with respect to $\hat{\sigma}$ by

$$W^s_{loc}(\hat{y}) = \{\hat{x} : x_k = y_k \text{ for every } k \geq 0\} \text{ and}$$
$$W^u_{loc}(\hat{y}) = \{\hat{x} : x_k = y_k \text{ for every } k \leq 0\}.$$

Observe that, taking $\lambda = 1/2$ and $\tau = 1/2$,

(i) $\operatorname{dist}(\hat{\sigma}^n(\hat{y}_1), \hat{\sigma}^n(\hat{y}_2)) \leq \lambda^n \operatorname{dist}(\hat{y}_1, \hat{y}_2)$ for any $\hat{y} \in \hat{\Sigma}$, $\hat{y}_1, \hat{y}_2 \in W^s_{loc}(\hat{y})$ and $n \geq 0$;

(ii) $\operatorname{dist}(\hat{\sigma}^{-n}(\hat{y}_1), \hat{\sigma}^{-n}(\hat{y}_2)) \leq \lambda^n \operatorname{dist}(\hat{y}_1, \hat{y}_2)$ for any $\hat{y} \in \hat{\Sigma}$, $\hat{y}_1, \hat{y}_2 \in W^u_{loc}(\hat{y})$ and $n \geq 0$;

(iii) if $\operatorname{dist}(\hat{x}, \hat{y}) \leq \tau$, then $W^s_{loc}(\hat{x})$ and $W^u_{loc}(\hat{y})$ intersect in a unique point, which is denoted by $[x, y]$ and depends continuously on $x$ and $y$.

By *partially hyperbolic skew-product* over the shift map $\hat{\sigma}$ we mean a homeomorphism $\hat{f} : \hat{\Sigma} \times K \to \hat{\Sigma} \times K$ of the form

$$\hat{f}(\hat{x}, t) = \big(\hat{\sigma}(\hat{x}), \hat{f}_{\hat{x}}(t)\big)$$

where $K$ is a compact Riemann manifold and the maps $\hat{f}_{\hat{x}} : K \to K$ are diffeomorphisms satisfying

$$\lambda \|d\hat{f}_{\hat{x}}(t)\| < 1 \text{ and } \lambda \|d\hat{f}^{-1}_{\hat{x}}(t)\| < 1 \text{ for every } (\hat{x}, t) \in \hat{\Sigma} \times K, \tag{3.2}$$

where $\lambda$ is a constant as in (i) - (ii). We also assume the following Hölder condition: there exist $C > 0$ and $\alpha > 0$ such that the $C^1$-distance between $\hat{f}_{\hat{x}}$ and $\hat{f}_{\hat{y}}$ is bounded by $C \operatorname{dist}(\hat{x}, \hat{y})^\alpha$ for every $\hat{x}, \hat{y} \in \hat{\Sigma}$.

We say that $\hat{f}$ has *mostly neutral center direction* if the family of maps $\hat{f}^n_{\hat{x}} : K \to K$ defined for $n \in \mathbb{Z}$ and $\hat{x} \in \hat{\Sigma}$ by

$$\hat{f}^n_{\hat{x}} = \begin{cases} \hat{f}_{\hat{\sigma}^{n-1}(\hat{x})} \circ \cdots \circ \hat{f}_{\hat{x}} & \text{if } n > 0 \\ \text{id} & \text{if } n = 0 \\ \hat{f}^{-1}_{\hat{\sigma}^n(x)} \circ \cdots \circ \hat{f}^{-1}_{\hat{\sigma}^{-1}(\hat{x})} & \text{if } n < 0. \end{cases}$$

have bounded derivatives, that is, if there exists $C > 0$ such that

$$\left\| D\hat{f}^n_{\hat{x}} \right\| \leq C \text{ for every } \hat{x} \in \hat{\Sigma} \text{ and } n \in \mathbb{Z}.$$

**Remark 3.1.1.** Clearly, this implies that the $\{\hat{f}^n_{\hat{x}} : n \in \mathbb{Z} \text{ and } \hat{x} \in \hat{\Sigma}\}$ is equi-continuous. When the maps $\hat{f}_{\hat{y}}$ are $C^{1+\epsilon}$, equi-continuity alone suffices for all our purposes (see Remark 4.2.5).

A few comments are in order concerning the scope of the notion of partially hyperbolic skew-product. To begin with, the shift $\hat{\sigma} : \hat{\Sigma} \to \hat{\Sigma}$ may be replaced by a sub-shift $\hat{\sigma}_T : \hat{\Sigma}_T \to \hat{\Sigma}_T$ associated to a transition matrix $T = (T_{i,j})_{i,j \in X}$. By this we mean that $T_{i,j} \in \{0, 1\}$ for every $i, j \in X$ and $\hat{\sigma}_T$ is the restriction of the shift map $\hat{\sigma}$ to the subset $\hat{\Sigma}_T$ of sequences $(x_n)_{n \in \mathbb{Z}}$ such that $T_{x_n, x_{n+1}} = 1$ for every $n \in \mathbb{Z}$. One way to reduce the sub-shift case to the full shift case is through inducing. Namely, fix any cylinder $[i] = \{(x_n)_{n \in \mathbb{Z}} \in \hat{\Sigma}_T : x_0 = i\}$ with positive measure and consider the first return map $g : [i] \to [i]$ of $\hat{\sigma}_T$ to $[i]$. This is conjugate to a full countable shift (with the return times as symbols) and it preserves the normalized restriction to the cylinder of the $\hat{\sigma}_T$-invariant measure. All the conditions that follow are not affected by this procedure. Moreover, every linear cocycle $F$ over $\hat{\sigma}_T$ gives rise, also through inducing, to a linear cocycle over $g$ whose Lyapunov spectrum is just a rescaling of the Lyapunov spectrum of $F$. In particular, simplicity may also be read out from the induced cocycle.

Moreover, although we choose to formulate our approach in a symbolic set-up, for skew-products over shifts, it is clear that it extends to other situations that are more geometric in nature. For example, take $g : N \to N$ to be a partially hyperbolic diffeomorphism on a compact 3-dimensional manifold $N$ and assume that there exists an embedded closed curve $\gamma \subset N$ such that $g(\gamma) = \gamma$ and some connected component of $W^s_{loc}(\gamma) \cap W^u_{loc}(\gamma) \setminus \gamma$ is a closed curve. By [10], $g$ is conjugate up to finite covering to a skew-product over a linear Anosov diffeomorphism of the 2-torus. Thus, using a Markov partition for the Anosov map, one can semi-conjugate $g$ to a partially hyperbolic skew-product over a sub-shift of finite type. In this way, the conclusions of this work can be adapted to linear cocycles over such a diffeomorphism.

## 3.2 Stable and unstable holonomies

Property (3.2) is a condition of *domination* (or *normal hyperbolicity*, in the spirit of [19]): it means that any expansion and contraction of $\hat{f}_{\hat{x}}$ along the fibers $\{\hat{x}\} \times K$ are dominated by the hyperbolicity of the base map $\hat{\sigma}$. For our purposes, its main relevance is that it ensures the existence of strong-stable and strong-unstable “foliations” for $\hat{f}$, as we explain next.

Let the product $\hat{M} = \hat{\Sigma} \times K$ be endowed with the distance defined by

$$\mathrm{dist}_{\hat{M}}((\hat{x}_1, t_1), (\hat{x}_2, t_2)) = \mathrm{dist}_{\hat{\Sigma}}(\hat{x}_1, \hat{x}_2) + d(t_1, t_2),$$

where $d$ is the distance induced by the Riemannian metric on $K$ (on the right-hand side, dist denotes the distance (3.1) on $\hat{\Sigma}$).

We consider the *stable holonomies*

$$h^s_{\hat{x},\hat{y}} : K \to K, \quad h^s_{\hat{x},\hat{y}} = \lim_{n\to\infty} \left(\hat{f}^n_{\hat{y}}\right)^{-1} \circ \hat{f}^n_{\hat{x}},$$

defined for every $\hat{x}$ and $\hat{y}$ such that $\hat{x} \in W^s_{loc}(\hat{y})$, and *unstable holonomies*

$$h^u_{\hat{x},\hat{y}} : K \to K, \quad h^u_{\hat{x},\hat{y}} = \lim_{n\to-\infty} \left(\hat{f}^n_{\hat{y}}\right)^{-1} \circ \hat{f}^n_{\hat{x}}$$

defined for every $\hat{x}$ and $\hat{y}$ such that $\hat{x} \in W^u_{loc}(\hat{y})$. That these families of maps exist follows from the assumption (3.2), using arguments from [8]; see for instance [6] which deals with a similar setting.

Then we define the *local strong-stable set* and the *local strong-unstable set* of each $(\hat{x}, t) \in \hat{M}$ to be

$$W^{ss}_{loc}(\hat{x}, t) = \{(\hat{y}, s) \in \hat{M} : \hat{y} \in W^s_{loc}(\hat{x}) \text{ and } s = h^s_{\hat{x},\hat{y}}(t)\} \text{ and}$$
$$W^{uu}_{loc}(\hat{x}, t) = \{(\hat{y}, s) \in \hat{M} : \hat{y} \in W^u_{loc}(\hat{x}) \text{ and } s = h^u_{\hat{x},\hat{y}}(t)\},$$

respectively. It is easy to check that

$$(\hat{y}, s) \in W^{ss}_{loc}(\hat{x}, t) \quad \Rightarrow \quad \lim_{n\to+\infty} \operatorname{dist}_{\hat{M}}(\hat{f}^n(\hat{y}, s), \hat{f}^n(\hat{x}, t)) = 0$$

and analogously on strong-unstable sets for time $n \to -\infty$.

## 3.3 Measures with partial product structure

Throughout, we take $\hat{\mu}$ to be an $\hat{f}$-invariant measure with *partial product structure*, that is, a probability measure of the form $\hat{\mu} = \rho\mu^s \times \mu^u \times \mu^c$ where:

- $\rho : \hat{M} \to (0, +\infty)$ is a continuous function;
- $\mu^s$ is a probability measure supported on $\Sigma^- = X^{\mathbb{Z}_{<0}}$;
- $\mu^u$ is a probability measure supported on $\Sigma^+ = X^{\mathbb{Z}_{\geq 0}}$;
- $\mu^c$ is a probability measure on the manifold $K$.

We also assume the following boundedness condition: there exists $\kappa > 0$ such that

$$\frac{1}{\kappa} \leq \frac{\tilde{\rho}(x^s, x^u)}{\tilde{\rho}(x^s, z^u)} \leq \kappa \quad \text{and} \quad \frac{1}{\kappa} \leq \frac{\tilde{\rho}(x^s, x^u)}{\tilde{\rho}(z^s, x^u)} \leq \kappa \tag{3.3}$$

for every $x^s$, $z^s \in \Sigma^-$ and $x^u$, $z^u \in \Sigma^+$, where $\tilde{\rho} : \Sigma \to \mathbb{R}$ is defined by

$$\tilde{\rho}(x^s, x^u) = \int \rho(x^s, x^u, t) d\mu^c(t).$$

Observe that when $\hat{\Sigma}$ is a finite shift space this is an immediate consequence of compactness and the continuity of $\rho$.

For each $\hat{x} \in \hat{\Sigma}$, let $\hat{\mu}^c_{\hat{x}}$ denote the normalization of $\rho(\hat{x}, \cdot)\mu^c$. The family $\{\hat{\mu}^c_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ is a Rokhlin disintegration of $\hat{\mu}$ along the vertical fibers. The assumption that $\hat{\mu}$ is invariant under $\hat{f}$, together with the fact that $\hat{\mu}^c_{\hat{x}}$ depends continuously on $\hat{x}$, implies that

$$(\hat{f}_{\hat{x}})_* \hat{\mu}^c_{\hat{x}} = \hat{\mu}^c_{\hat{\sigma}(\hat{x})} \quad \text{for every } \hat{x} \in \hat{\Sigma}. \tag{3.4}$$

We will also see in Chapter 4.2 that this disintegration is *holonomy invariant*:

$$\begin{aligned} (h^s_{\hat{x},\hat{y}})_* \hat{\mu}^c_{\hat{x}} &= \hat{\mu}^c_{\hat{y}} \text{ whenever } \hat{y} \in W^s(\hat{x}) \quad \text{and} \\ (h^u_{\hat{x},\hat{y}})_* \hat{\mu}^c_{\hat{x}} &= \hat{\mu}^c_{\hat{y}} \text{ whenever } \hat{y} \in W^u(\hat{x}). \end{aligned} \tag{3.5}$$

**Remark 3.3.1.** In particular, if $\hat{p}$ is a fixed point of the shift map then $\hat{\mu}^c_{\hat{p}}$ is invariant under $\hat{f}_{\hat{p}}$. Clearly, it is equivalent to $\mu^c$. Moreover, if $\hat{z}$ is a homoclinic point of $\hat{p}$, that is, a point in $W^s(\hat{p}) \cap W^u(\hat{p})$, then $(h^s_{\hat{z},\hat{p}} \circ h^u_{\hat{p},\hat{z}})_* \hat{\mu}^c_{\hat{p}} = (h^u_{\hat{z},\hat{p}} \circ h^s_{\hat{p},\hat{z}})_* \hat{\mu}^c_{\hat{p}} = \hat{\mu}^c_{\hat{p}}$.

We assume that our cocycle admits strong stable and strong unstable holonomies as defined in section 2.2

For $1 \le l \le d-1$, let $\mathrm{Sec}(K, \mathrm{Grass}(l,d))$ denote the space of measurable maps $V$ from (some full $\mu^c$-measure subset of) $K$ to the Grassmannian manifold of all $l$-dimensional subspaces of $\mathbb{R}^d$. For each $\hat{x} \in \hat{\Sigma}$, consider the following push-forward maps

$$\mathrm{Sec}(K, \mathrm{Grass}(l,d)) \to \mathrm{Sec}(K, \mathrm{Grass}(l,d)) :$$

(a) $V \mapsto \mathcal{F}_{\hat{x}} V$ given by

$$\mathcal{F}_{\hat{x}} V(t) = \hat{A}(\hat{x}, s) V(s) \text{ with } s = (f_{\hat{x}})^{-1}(t);$$

(b) $V \mapsto \mathcal{H}^s_{\hat{x},\hat{y}} V$ given, for $\hat{y} \in W^s_{loc}(\hat{x})$, by

$$\mathcal{H}^s_{\hat{x},\hat{y}} V(t) = H^s_{(\hat{x},s),(\hat{y},t)} V(s) \text{ with } s = h^s_{\hat{y},\hat{x}}(t);$$

(c) $V \mapsto \mathcal{H}^u_{\hat{x},\hat{y}} V$ given, for $\hat{y} \in W^u_{loc}(\hat{x})$, by

$$\mathcal{H}^u_{\hat{x},\hat{y}} V(t) = H^u_{(\hat{x},s),(\hat{y},t)} V(s) \text{ with } s = h^u_{\hat{y},\hat{x}}(t).$$

## 3.4 Pinching and twisting

Now we state our refined criterion for simplicity of the Lyapunov spectrum.

We call the cocycle $\hat{F}$ *pinching* if there exists some fixed (or periodic) vertical leaf $\ell = \{\hat{p}\} \times K$ such that the restriction to $\ell$ of every exterior power $\Lambda^k \hat{F}$ has simple Lyapunov spectrum, relative to the $\hat{f}_{\hat{p}}$-invariant measure $\hat{\mu}^c_{\hat{p}}$ (Remark 3.3.1). In other words, the Lyapunov exponents $\lambda_1, \cdots, \lambda_d$ are such that, for each $1 \le k \le d-1$ and $\hat{\mu}^c_{\hat{p}}$-almost every $t \in K$, the sums

$$\lambda_{i_1}(\hat{p}, t) + \cdots + \lambda_{i_k}(\hat{p}, t), \qquad 1 \le i_1 < \cdots < i_k \le d$$

are all distinct. It is clear that $\hat{F}$ is pinching if it is uniformly pinching.

Take $\hat{F}$ to be pinching and let $TK = E^1(t) \oplus \cdots \oplus E^d(t)$ denote the Oseledets decomposition of $\hat{F}$ restricted to $\ell$. The maps $t \mapsto E^i(t)$ are defined on a full $\hat{\mu}^c_{\hat{p}}$-measure set. Now let $\hat{z}$ be some homoclinic point of $\hat{p}$ and $\imath \ge 1$, such that

$$\hat{z} \in W^u_{loc}(\hat{p}) \quad \text{and} \quad \hat{\sigma}^{\imath}(\hat{z}) \in W^s_{loc}(\hat{p}).$$

Define

$$V^i = (\mathcal{H}^u_{\hat{z},\hat{p}} \circ \mathcal{F}^{-\imath}_{\hat{z}} \circ \mathcal{H}^s_{\hat{p},\hat{\sigma}^{\imath}(\hat{z})}) E^i \quad \text{for } i = 1, \ldots, d. \tag{3.6}$$

Note that $V^i$ is also defined on a full $\hat{\mu}^c_{\hat{p}}$-measure set since the maps $\hat{f}_{\hat{x}}$ and the holonomies $h^s_{\hat{x},\hat{y}}$ and $h^u_{\hat{x},\hat{y}}$ preserve $\hat{\mu}^c_{\hat{p}}$ (Remark 3.3.1).

Now let $B(t) = (\beta_{i,j}(t))_{i,j}$ be the $d \times d$-matrix defined by

$$V^i(t) = \sum_{j=1}^{d} \beta_{i,j}(t) E^j(t), \quad \text{for } i = 1, \dots, d.$$

We call the cocycle $\hat{F}$ *twisting* if, some choice of the homoclinic point $\hat{z}$, all the algebraic minors $m_{I,J}(t)$ of $B(t)$ decay sub-exponentially along the orbits of $\hat{f}_{\hat{p}}$, meaning that

$$\lim_{n\to\infty} \frac{1}{n} \log |m_{I,J}(\hat{f}^n_{\hat{p}}(t))| = 0 \quad \text{for } \hat{\mu}^c_{\hat{p}}\text{-almost every } t \in K \tag{3.7}$$

and every $I, J \subset \{1, \dots, d\}$ with $\#I = \#J$.

It is clear that this holds if $\hat{F}$ is uniformly twisting, because in this case the algebraic minors are uniformly bounded. More generally (see for instance [25, Corollary 3.11]), the property (3.7) holds whenever the function $\log |m_{I,J}| \circ \hat{f}_{\hat{p}} - \log |m_{I,J}|$ is $\hat{\mu}^c_{\hat{p}}$-integrable.

## 3.5 Main statements and outline of the proof

We say that the cocycle $\hat{F}$ is *simple* if it is both pinching and twisting, in the sense of the previous section. That is the case, in particular, if $\hat{F}$ is uniformly pinching and uniformly twisting. Thus Theorem A is contained in the following result:

**Theorem D.** If $\hat{F} : \hat{M} \times \mathbb{C}^d \to \hat{M} \times \mathbb{C}^d$ is a simple cocycle then its Lyapunov spectrum is simple.

Let $H^\alpha(\hat{M})$ denote the space of all $\alpha$-Hölder continuous maps $A : \hat{M} \to \mathrm{SL}(2, \mathbb{R})$. We are going to prove in Section 7.2 that the uniform pinching and uniform twisting conditions are open with respect to this topology. Thus Theorem A implies

**Theorem E.** There is a non-empty open subset of $A \in H^\alpha(\hat{M})$ such that the associated linear cocycle $\hat{F}$ over $\hat{f}$ has simple Lyapunov spectrum.

In many contexts of linear cocycles over *hyperbolic* systems, simplicity turns out to be a generic condition: it contains an open and dense subset of cocycles (precise statements can be found in Viana [25]). This is related to the fact that, in the hyperbolic setting, the pinching and twisting conditions are just transversality conditions, they clearly hold on the complement of suitable submanifolds with positive codimension. At present, it is unclear how this can be extended to the partially hyperbolic setting. Uniform pinching is surely not a generic condition, in general, but it might be locally generic in some special situations, for instance when $\hat{f}_{\hat{p}}$ is quasi-periodic.

We close this section by outlining the overall strategy of the proof of Theorem 8.1.1. For every $1 \leq \ell < d$, we want to find complementary $\hat{F}-$invariant measurable sections

$$\xi : \hat{M} \to \mathrm{Grass}(l, d) \quad \text{and} \quad \eta : \hat{M} \to \mathrm{Grass}(d - l, d) \tag{3.8}$$

such that the Lyapunov exponents of $\hat{F}$ along $\xi$ are strictly larger than those along $\eta$.

The starting point is to reduce the problem to the case when the maps $\hat{f}_{\hat{x}}$ and the matrices $\hat{A}(\hat{x}, t)$ depend on $\hat{x}$ only through its positive part $x^u$. This we do in Section 4.1, using the stable holonomies to conjugate the original dynamics to others with these properties. Then $\hat{f} : \hat{M} \to \hat{M}$ projects to a transformation $f : M \to M$ on $M = \Sigma^+ \times K$ which is a skew-product over the one-sided shift $\sigma : \Sigma^+ \to \Sigma^+$ and, similarly, the linear cocycle $\hat{F} : \hat{M} \times \mathbb{C}^d \to \hat{M} \times \mathbb{C}^d$ projects to a linear cocycle $F : M \times \mathbb{C}^d \to M \times \mathbb{C}^d$ over the transformation $\hat{f}$.

We also denote by $\hat{F}$ and $F$ the actions

$$\hat{F} : \hat{M} \times \mathrm{Grass}(l, d) \to \hat{M} \times \mathrm{Grass}(l, d) \text{ and}$$
$$F : M \times \mathrm{Grass}(l, d) \to M \times \mathrm{Grass}(l, d)$$

induced by the two linear cocycles on the Grassmannian bundles. Still in Section 4.1, using very classical arguments, we relate the invariant measures of $\hat{f}$ and $\hat{F}$ with those of $f$ and $F$, respectively.

In Section 4.2 we study *u-states*, that is $\hat{F}$-invariant probability measures $\hat{m}$ whose Rokhlin disintegration $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ are invariant under unstable holonomies, as well as the corresponding $F$-invariant probability measures $m$. Here we meet the first important new difficulty arising from the fact that $\hat{f}$ is only partially hyperbolic.

Indeed, in the hyperbolic setting such measures $m$ are known to admit *continuous* Rokhlin disintegration $\{m_x : x \in M\}$ along the fibers $\{x\} \times \mathrm{Grass}(l, d)$ and this fact plays a key part in the arguments of Bonatti-Viana [9] and Avila-Viana [3].

In the partially hyperbolic setting, the situation is far more subtle: the disintegration $\{m_x : x \in \Sigma\}$ along the sets $\{x\} \times K \times \mathrm{Grass}(l, d)$ is still continuous, but there is no reason why this should extend to the disintegration

$$\{m_{x,t} : (x, t) \in M\}$$

along fibers $K \times \mathrm{Grass}(l, d)$, which is what one really needs. The way we make up for this is by proving a kind of $L^1$-continuity: if $(x_i)_i \to x$ in $\Sigma$ then $(m_{x_i,t})_i \to m_{x,t}$ in $L^1(\mu^c)$. See Proposition 4.3.5 for the precise statement.

This also leads to our formulating the arguments in terms of measurable sections $K \to \mathrm{Grass}(l, d)$ of the Grassmannian bundle, which is perhaps another significant novelty in this paper. The properties of such sections are studied in Section 5.1. The key result (Proposition 5.1.1) is that, under pinching and twisting, the graph of every invariant Grassmannian section has zero $m_x$-measure, for every $x \in M$ and any $u$-state $\hat{m}$.

These results build up to Section 5.3, where we prove that every $u$-state $\hat{m}$ has an atomic Rokhlin disintegration. More precisely (Theorem 5.3.1), there exists a measurable section $\xi : \hat{M} \to \mathrm{Grass}(l, d)$ such that, given any $u$-state $\hat{m}$ on $\hat{M} \times \mathrm{Grass}(l, d)$, we have

$$\hat{m}_{\hat{x},t} = \delta_{\xi(\hat{x},t)} \quad \text{for } \hat{\mu}\text{-almost every } (\hat{x}, t) \in \hat{M}. \tag{3.9}$$

Thus we construct the invariant section $\xi : \hat{M} \to \mathrm{Grass}(l, d)$ in (3.8).

To find the complementary invariant section $\eta : \hat{M} \to \mathrm{Grass}(d - l, d)$, in Section 6.1 we apply the same procedure to the adjoint cocycle $\hat{F}^*$, that is, the linear cocycle defined over $\hat{f}^{-1} : \hat{M} \to \hat{M}$ by the function

$$\hat{x} \mapsto \hat{A}^*(x) = \text{ adjoint of } \hat{A}(\hat{f}^{-1}(\hat{x})).$$

We check (Proposition 6.1.1) that this cocycle $\hat{F}^*$ is pinching and twisting if and only if $\hat{F}$ is. So, the previous arguments yield a $\hat{F}^*$-invariant section $\xi^* : \hat{M} \to \mathrm{Grass}(l, d)$ related to the $u$-states of $\hat{F}^*$. Then we just take $\eta = \left(\xi^*\right)^{\perp}$.

Finally, in Chapter 7 we check that the eccentricity, or lack of conformality, of the iterates $\hat{A}^n$ goes to infinity $\hat{\mu}$-almost everywhere (see Proposition 7.1.1 for the precise statement) and we use this fact to deduce that every Lyapunov exponent of $\hat{F}$ along $\xi$ is strictly larger than any Lyapunov exponents of $\hat{F}$ along $\eta$. At this stage the arguments are again very classical. This concludes the proof of Theorem 8.1.1.

Theorem E is proven in Section 7.2. The appendix contain material that seems to be folklore, but for which we could not find explicit references. In Appendix A.1 we check that the Lyapunov spectra of a linear cocycle and its adjoint coincide. In Appendix A.2 we show that continuous maps are dense in the corresponding $L^1$ space, whenever the target space is geodesically convex.

CHAPTER 4

# Projective measures

In this chapter we prove some results about measure $\hat{m}$ on $\hat{M} \times \mathrm{Grass}(l, d)$ that projects on $\hat{\mu}$. We also introduce the concept of $u$-states, that will be very useful in the rest of the text.

## 4.1 Convergence of measures

Recall that $\hat{\Sigma} = \Sigma^- \times \Sigma^+$. Accordingly, we write every $\hat{x} \in \hat{\Sigma}$ as $(x^s, x^u)$. For simplicity, we also write $\Sigma = \Sigma^+$ and $x = x^u$. Let $P : \hat{\Sigma} \to \Sigma$ be the canonical projection $P(\hat{x}) = x$ and $\sigma : \Sigma \to \Sigma$ be the one-sided shift. We also consider $M = \Sigma \times K$.

Our first step is to show that, up to conjugating the cocycle in a suitable way, we may suppose that:

(a) the base dynamics $f_{\hat{x}}$ along the center direction depends only on $x$;

(b) the matrix $\hat{A}(\hat{x}, t)$ depend only on $(x, t)$.

Let us explain how such a conjugacy may be defined using the stable holonomies.

Let $x^s \in \Sigma^-$ be fixed. Por any $\hat{y} \in \hat{\Sigma}$, let $\phi(\hat{y}) = (x^s, y)$ and then define

$$h(\hat{y}, t) = \big(\hat{y}, h^s_{\varphi(\hat{y}),\hat{y}}(t)\big).$$

Then $\tilde{f} = h^{-1} \circ f \circ h$ is given by

$$\tilde{f}(\hat{y}, t) = \big(\hat{\sigma}(\hat{y}), \tilde{f}_{\hat{y}}(t)\big), \quad \text{with } \tilde{f}_{\hat{y}}(t) = h^s_{\hat{\sigma}(\phi(\hat{y})),\phi(\hat{\sigma}(\hat{y}))} f_{\phi(\hat{y})}(t).$$

Notice that $\tilde{f}_{\hat{y}}$ does depend only on $y$ (because $\phi$ does).

Assume that (a) is satisfied. Define $\hat{\phi}(\hat{y},t)=(\phi(\hat{y}),t)$ and then let

$$H(\hat{y},t)=H^s_{\hat{\phi}(\hat{y},t),(\hat{y},t)}.$$

Define $\tilde{A}(\hat{y},t)=H(\hat{f}(\hat{y},t))^{-1}\circ A(\hat{y},t)\circ H(\hat{y},t)$. Then

$$\tilde{A}(\hat{y},t)=H^s_{\hat{f}(\hat{\phi}(\hat{y},t)),\hat{\phi}(\hat{f}(\hat{y},t))}\circ \hat{A}(\hat{\phi}(\hat{y},t)),$$

which only depends on $(y,t)$. Clearly, this procedure does not affect the Lyapunov exponents.

For each $1\le l<d$, the linear cocycle $\hat{F}:\hat{M}\times\mathbb{C}^d\to\hat{M}\times\mathbb{C}^d$ induces a projective cocycle $\hat{F}:\hat{M}\times \mathrm{Grass}(l,d)\to\hat{M}\times\mathrm{Grass}(l,d)$ through

$$\hat{F}(\hat{q},V)=(\hat{f}(\hat{q}),\hat{A}(\hat{q})V). \tag{4.1}$$

From now on, we assume that both (a) and (b) are satisfied. Then, there exists

$$f:M\to M,\quad f(x,t)=(\sigma(x),f_x(t))$$

such that

$$(P\times\mathrm{id}_K)\circ\hat{f}=f\circ(P\times\mathrm{id}_K)$$

and there exists $A:M\to GL(d,\mathbb{C})$ such that $\hat{A}=A\circ(P\times\mathrm{id}_K)$. Then the map

$$F:M\times\mathrm{Grass}(l,d)\to M\times\mathrm{Grass}(l,d),\quad F(p,V)=(f(p),A(p)V)$$

satisfies

$$\left(P\times\mathrm{id}_K\times\mathrm{id}_{\mathrm{Grass}(l,d)}\right)\circ\hat{F}=F\circ\left(P\times\mathrm{id}_K\times\mathrm{id}_{\mathrm{Grass}(l,d)}\right).$$

Define $\pi:\hat{M}\times\mathrm{Grass}(l,d)\to\hat{M}$ to be the canonical projection on the first coordinate. Let $\hat{m}$ be any Borel probability measure on $\hat{M}\times\mathrm{Grass}(l,d)$ that projects down to $\hat{\mu}$ under $\pi$. Denote

$$\mu=(P\times\mathrm{id}_K)_*\hat{\mu}\quad\text{and}\quad m=\left(P\times\mathrm{id}_K\times\mathrm{id}_{\mathrm{Grass}(l,d)}\right)_*\hat{m}.$$

**Proposition 4.1.1.** *Let $(N,\mathfrak{B},\eta)$ be a Lebesgue probability space and $g:N\to N$ be a measurable map that preserves $\mu$. Let $\{\eta_x:x\in N\}$ be the Rokhlin disintegration of $\eta$ with respect to the partition into pre-images $g^{-1}(x)$. Let*

$$G:N\times L\to N\times L,\quad G(x,y)=(g(x),G_x(y))$$

*be a skew-product over $g$ and, given any probability measure $m$ on $N\times L$ that projects down to $\eta$, let $\{m_x:x\in N\}$ be its Rokhlin disintegration with respect to the partition into vertical fibers $\{x\}\times L$. Then $m$ is invariant under $G$ if and only if*

$$m_x=\int(G_z)_*m_z\,d\eta_x(z)\quad \textit{for } \eta\textit{-almost every } x\in N.$$

*Proof.* Let $\mathcal{P} = \{\{x\} \times L, \text{ for } x \in N\}$. Define

$$\tilde{m}_x = \int G_{z*} m_z d\eta_x(z)$$

we have that

$$\tilde{m}_x(\{x\} \times L) = 1$$
$$\int \tilde{m}_x(A) d\mu = \int \int m_z \left(G_z^{-1}(A)\right) d\eta_x(z) d\eta(x).$$

Let $\pi : N \to \mathcal{P}$ be the projection to the partition space, it is easy to see that $\pi(x) = g^{-1}(g(x))$, then we can write $\eta_{\pi(x)} = \eta_{g(x)}$.

We have that for any measurable function $\varphi : N \to \mathbb{C}$,

$$\int \int \varphi(z) d\eta_x(z) \pi_* \mu(x) = \int \varphi(x) d\mu.$$

Then

$$\begin{aligned}
\int \varphi(x) d\eta &= \int \int \varphi(z) d\eta_{\pi(x)}(z) d\eta(x) \\
&= \int \int \varphi(z) d\eta_{g(x)}(z) d\eta(x) \\
&= \int \int \varphi(z) d\eta_x(z) d\eta(x)
\end{aligned}$$

So we have that

$$\int \tilde{m}_x(A) d\eta = \int m_x \left(G_x^{-1}(A)\right) d\eta(x) = m\left(G^{-1}(A)\right)$$

Then, by the uniqueness of the Rokhlin disintegration, $\tilde{m}_x$ is a disintegration of $m$ if and only if $m(G^{-1}(A)) = m(A)$. □

For every $\hat{q} \in \hat{M}$, define $q_n = (P \times \mathrm{id}_K)(\hat{f}^{-n}(\hat{q}))$. Next, we are going to prove the following proposition:

**Proposition 4.1.2.** *For $\hat{\mu}$-almost every $\hat{q} \in \hat{M}$,*

*(a) $A^n(q_n)_* m_{q_n} \to \hat{m}_{\hat{q}}$ in the weak* topology*

*(b) for any $k \geq 1$ and any choice of points $y_{n,k}$ with $f^k(y_{n,k}) = q_n$ and such that $y_{n,k}$ stay in a compact set*

$$\lim_{n\to\infty} A^n(q_n)_* m_{q_n} = \lim_{n\to\infty} A^{n+k}(y_{n,k})_* m_{y_{n,k}}.$$

*Proof.* Let $g : \mathrm{Grass}(l, d) \to \mathbb{C}$ be any continuous function. Define

$$\hat{I}_n : \hat{M} \to \mathbb{C}, \quad \hat{I}_n(\hat{q}) = \int g \circ A^n(q_n) dm_{q_n}$$

and

$$I_n : M \to \mathbb{C}, \quad I_n(q) = \int g \circ A^n(q) dm_q.$$

Note that $\hat{I}_n(\hat{q}) = I_n \circ (P \times \mathrm{id}_K) \circ \hat{f}^{-n}(\hat{q})$.

For each $n \geq 1$, let $\mathfrak{C}_n$ be the $\sigma$-algebra generated by the measurable sets of the form $[-n : A_{-n}, \dots, A_k] \times B$ with $k \geq 0$, $A_{-n}, \dots, A_k \subset X$ and $B \subset K$. Moreover, let $\mathfrak{C}_0$ be the Borel $\sigma$-algebra of $M$. Observe that $I_n$ is $\mathfrak{C}_n$-measurable, since $\mathfrak{C}_n = \hat{f}^n((P \times \mathrm{id}_K)^{-1}(\mathfrak{C}_0))$. Moreover, the Borel $\sigma$-algebra of $\hat{M}$ is generated by $\cup_{n \in \mathbb{N}} \mathfrak{C}_n$.

We need the next Lemma:

**Lemma 4.1.3.** *For each $k \geq 1$, let $\{\mu_q^k : q \in M\}$ be the Rokhlin disintegration of $\mu$ with respect to the partition into pre-images $f^{-k}(q)$. Then $I_n(q) = \int I_{n+k}(z) d\mu_q^k(z)$*

*Proof.* By Proposition 4.1.1

$$\begin{aligned} I_n(q) &= \int g\, A^n(q) d\left(\int A^k(z)_* m_z d\mu_q^k\right) \\ &= \int\int g\, A^n(q) d\left(A^k(z)_* m_z d\mu_q^k\right) \\ &= \int\int g\, A^{n+k}(z) dm_z d\mu_q^k(z) \\ &= \int I_{n+k}(z) d\mu_q^k(z), \end{aligned}$$

which proves the claim. □

Then for any $\psi : \hat{M} \to \mathbb{C}$, $\mathfrak{C}_n$ measurable function, i.e: $\psi = \psi_n \circ (P \times \mathrm{id}_K) \circ \hat{f}$, for some $\psi_n : M \to \mathbb{C}$ measurable

$$\begin{aligned} \int \hat{I}_n(\hat{q}) \psi(\hat{q}) d\hat{\mu}(\hat{q}) &= \int I_n(q) \psi_n(q) d\mu(q) \\ &= \int\int I_{n+k}(z) d\mu_q^k(z) \psi_n(q) d\mu(q) \\ &= \int\int I_{n+k}(z) \psi_n(f^k(z)) d\mu_q^k(z) d\mu(q) \\ &= \int \hat{I}_{n+k}(\hat{q}) \psi(\hat{q}) d\hat{\mu}(\hat{q}). \end{aligned}$$

So $\hat{I}_n$ is a martingale. Then for almost every $\hat{q} \in \hat{M}$ there exist $\lim_{n\to\infty} \hat{I}_n = \hat{I}(g)$.

Define the linear functional

$$I : C\left(\mathrm{Grass}(l,d),\mathbb{C}\right) \to \mathbb{C}, \quad \hat{I}(g) = \lim_{n\to\infty} \hat{I}_n(g)$$

where $C(\mathrm{Grass}(l,d),\mathbb{C})$ is the space of continuous functions from the Grassmanian $\mathrm{Grass}(l,d)$ to $\mathbb{C}$. By the Riesz-Markov theorem there exist a probability measure $\tilde{m}_{\hat{q}}$ in $\mathrm{Grass}(l,d)$ such that $\int g d\tilde{m}_{\hat{q}} = I(g)(\hat{q})$ for every $g$.

To see that $\tilde{m}_{\hat{q}} = \hat{m}_{\hat{q}}$, let $\psi$ be any $\mathfrak{C}_n$-measurable function. Taking the limit as $n \to \infty$, we get that

$$\int \hat{I}(\hat{q})\psi(\hat{q})d\hat{\mu}(\hat{q}) = \int \hat{I}_n(q)\psi(\hat{q})d\hat{\mu}(\hat{x}).$$

This may be rewritten as

$$\int \psi(\hat{q}) \int g(\xi)d\tilde{m}_{\hat{q}}(\xi)d\hat{\mu}(\hat{q}) = \int \psi(\hat{q}) \int g(A^n(q_n)\xi)dm_{q_n}(\xi)d\hat{\mu}(\hat{q}).$$

Using the invariance of $\hat{\mu}$ and the invariance of $\hat{m}$, we get that

$$\int\int \psi(\hat{q})g(\xi)d\tilde{m}_{\hat{q}}(\xi)d\hat{\mu}(\hat{q}) = \int\int \psi(\hat{q})g(\eta)d\hat{m}_{\hat{q}}(\eta)d\hat{\mu}(\hat{q}).$$

These relations extend immediately to linear combinations of functions $\psi\times g$. Since these form a dense subset of all bounded measurable functions on $\hat{M} \times \mathrm{Grass}(l,d)$, this implies that $\tilde{m}_{\hat{q}} = \hat{m}_{\hat{q}}$ for $\hat{\mu}$-almost every $\hat{q}$.

This proves the claim (a) of Proposition 4.1.2.

To prove part (b), observe that

$$\mu_q^k = \sum_{y:f^k(y)=p} \frac{1}{J_\mu f^k(y)}\delta_y.$$

For any $n \geq 0$ and $k \geq 1$, define

$$S_{n,k} = \int\int \left(I_{n+k}(y) - \hat{I}_n(\hat{q})\right)^2 d\mu_q^k(y)\, d\hat{\mu}(\hat{q}),$$

by [9, Lemma 3.4] we have that

$$S_{n,k} = \int \hat{I}_{n+k}(\hat{q})^2\, d\hat{\mu}(\hat{q}) - \int \hat{I}_n(\hat{q})^2\, d\hat{\mu}(\hat{q})$$

then, for every $s \geq 1$

$$\sum_{n=1}^{s} \int\int \left(I_{n+k}(y) - \hat{I}_n(\hat{q})\right)^2 d\mu_q^k(y)\, d\hat{\mu}(\hat{q}) = \sum_{n=1}^{s} S_{n,k} \leq 2k(\sup\|g\|)^2.$$

Consequently,

$$\int \left(I_{n+k}(y) - \hat{I}_n(\hat{q})\right)^2 d\mu_q^k(y) = \sum_{f^k(y)=q_n} \frac{1}{J_\mu f^k(y)}\left(I_{n+k}(y) - \hat{I}_n(\hat{q})\right)^2$$

converges to zero when $n \to \infty$, for $\hat{\mu}$ almost every $\hat{q} \in \hat{M}$. Since $y_{n,k}$ stays in a compact set and the Jacobian is continuous it follows that

$$(I_{n+k}(y_{n,k}) - \hat{I}_n(\hat{q}))^2$$

converges to zero for $\hat{\mu}$ almost every $\hat{q} \in \hat{M}$. In other words, there exist a total measure subset such that

$$|\int A^n(q_n)_* m_{q_n} - A^{n+k}(y_{n,k})_* m_{y_{n,k}}|$$

converges to zero as $n \to \infty$. Claim (b) follows again considering a dense subset of continuous functions. $\square$

We have a similar, but stronger result for the disintegration of $\hat{\mu}$ with respect to the partition in central manifolds. In order to state it, let

$$\hat{\mu}^c_{\hat{x}} = \frac{\rho(\hat{x}, \cdot)}{\int \rho(\hat{x}, t) d\mu^c(t)} \mu^c$$

for $x \in \hat{M}$ and

$$\mu^c_x = \frac{\int \rho(x^s, x, \cdot) d\mu^s(x^s)}{\int\int \rho(x^s, x, t) d\mu^s(x^s) d\mu^c(t)} \mu^c$$

for $x \in M$. It is easy to see that $\{\hat{\mu}^c_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ is a (continuous) disintegration of $\hat{\mu}$ with respect to the partition $\hat{\mathcal{P}} = \{\{\hat{x}\} \times K : \hat{x} \in \hat{\Sigma}\}$ and $\{\mu^c_x : x \in \Sigma\}$ is a continuous disintegration of $\mu$ with respect to the partition $\mathcal{P} = \{\{x\} \times K : x \in \Sigma\}$. It is important to observe that the next statement is for *every* $\hat{x} \in \hat{\Sigma}$.

**Proposition 4.1.4.** *There exist continuous functions $\rho_n : \hat{M} \to \mathbb{R}$ such that*

$$\left(f^n_{P(\hat{\sigma}^{-n}\hat{x})}\right)_* \mu^c_{P(\hat{\sigma}^{-n}\hat{x})} = \rho_n \mu^c \quad \textit{and} \quad \rho_n \to \frac{\rho}{\int \rho(\cdot, t) d\mu^c(t)}$$

*at every point. In particular, for every $\hat{x} \in \hat{\Sigma}$,*

$$\left(f^n_{P(\hat{\sigma}^{-n}\hat{x})}\right)_* \mu^c_{P(\hat{\sigma}^{-n}\hat{x})} \to \hat{\mu}^c_{\hat{x}}.$$

*in the weak* topology.*

*Proof.* Let us define

$$\hat{\varrho}(\hat{x}, \cdot) = \frac{\rho(\hat{x}, \cdot)}{\int \rho(\hat{x}, t) d\mu^c(t)} \quad \text{and} \quad \varrho(x, t) = \int \hat{\varrho}(x^s, x, t) d\mu^s(x^s).$$

Let $n \geq 1$. By the invariance of $\hat{\mu}$,

$$\left(\hat{f}^n_{\hat{x}}\right)_* \hat{\mu}^c_{\hat{x}} = \hat{\mu}^c_{\hat{\sigma}^n(\hat{x})}$$

for $\hat{\mu}$-almost every $\hat{x}$ and, by continuity, the equality extends to every $\hat{x} \in \hat{\Sigma}$. In other words,

$$\int \varphi \circ \hat{f}_{\hat{x}}^{n}(t) \hat{\varrho}(\hat{x}, t) d\mu^{c}(t) = \int \varphi(t) \hat{\varrho}(\hat{\sigma}^{n}(\hat{x}), t) d\mu^{c}(t)$$

for any continuous $\varphi : K \to \mathbb{R}$ and every $\hat{x}$. Also, by a change of variables,

$$\int \varphi \circ \hat{f}_{\hat{x}}^{n}(t) \hat{\varrho}(\hat{x}, t) d\mu^{c}(t) = \int \varphi(s) \hat{\varrho}\left(\hat{x}, \hat{f}_{\hat{\sigma}^{n}(\hat{x})}^{-n}(s)\right) J \hat{f}_{\hat{\sigma}^{n}(\hat{x})}^{-n} d\mu^{c}(s)$$

for any continuous $\varphi : K \to \mathbb{R}$ and every $\hat{x}$. Comparing the right-hand side of these two relations, we conclude that

$$\hat{\varrho}\left(\hat{x}, \hat{f}_{\hat{\sigma}^{n}(x)}^{-n}(s)\right) J \hat{f}_{\hat{\sigma}^{n}(x)}^{-n} = \hat{\varrho}(\hat{\sigma}^{n}(\hat{x}), t) \tag{4.2}$$

on the support of $\mu^{c}$. Analogously,

$$\begin{aligned}
&\int \varphi(t) d\left(f_{P(\hat{\sigma}^{-n}\hat{x})_{*}}^{n} \mu_{P(\hat{\sigma}^{-n}\hat{x})}^{c}\right)(t) \\
&\quad = \int \varphi \circ f_{P(\hat{\sigma}^{-n}\hat{x})}^{n}(t) \varrho(P\left(\hat{\sigma}^{-n}\hat{x}\right), t)\, d\mu^{c}(t) \qquad (4.3) \\
&\quad = \int \varphi(s) \varrho\left(P\left(\hat{\sigma}^{-n}\hat{x}\right), \hat{f}_{\hat{x}}^{-n}(s)\right) J \hat{f}_{\hat{x}}^{-n}(s) d\mu^{c}(s)
\end{aligned}$$

for every continuous $\varphi : K \to \mathbb{R}$ and any $\hat{x}$. Define,

$$\rho_{n}(\hat{x}, s) = \varrho\left(P\left(\hat{\sigma}^{-n}\hat{x}\right), \hat{f}_{\hat{x}}^{-n}(s)\right) J \hat{f}_{\hat{x}}^{-n}(s).$$

The definition of $\varrho$ gives that

$$\rho_{n}(\hat{x}, s) = \int \hat{\varrho}\left(x^{-}, P\left(\hat{\sigma}^{-n}\hat{x}\right), \hat{f}_{\hat{x}}^{-n}(s)\right) J \hat{f}_{\hat{x}}^{-n}(s) d\mu^{s}(x^{-}).$$

Combining this with (4.2), we obtain that

$$\rho_{n}(\hat{x}, s) = \int \hat{\varrho}\left(\hat{\sigma}^{n}\left(x^{-}, P\left(\hat{\sigma}^{-n}\hat{x}\right)\right), s\right) d\mu^{s}(x^{-}).$$

By continuity of $\hat{\varrho}$, the expression $\hat{\varrho}\left(\hat{\sigma}^{n}\left(x^{s}, P\left(\hat{\sigma}^{-n}\hat{x}\right)\right), t\right)$ converges to $\hat{\varrho}\left(\hat{x}, t\right)$ at every point. So, by dominated convergence, the previous identity yields

$$\rho_{n}(\hat{x}, s) \to \hat{\varrho}\left(\hat{x}, s\right).$$

at every point. In view of (4.3), this completes the argument. $\square$

## 4.2 Existence and properties of $u$-states

A probability measure $\hat{m}$ in $\hat{M} \times \text{Grass}(l,d)$ is an *u-state* if there exists a total measure set $\tilde{M} \subset \hat{M}$ such that $\hat{m}_q = H^u_{b,q*}\hat{m}_p$ for every $b, q \in \tilde{M}$ with $b \in W^{uu}_{loc}(p)$.

**Proposition 4.2.1.** *There exist some $\hat{F}$-invariant u-state measure $\hat{m}$ that projects on $\hat{\mu}$.*

This entirely analogous to Proposition 4.2 of [3], and so we only outline the proof. The idea is to fix some $\hat{x} \in \hat{\Sigma}$ and define a homeomorphism between the measures in $\{\hat{x}\} \times W^s_{loc}(\hat{x}) \times K$ that projects to $\mu^s$ and the $u$-states. Since this space is compact we have that the space of $u$-states measures is also compact and $\hat{F}_*$-invariant, so we have that any accumulation point of $n^{-1}\sum_{j=0}^{n-1}\hat{F}^j_*\hat{m}$ is also a $u$-state.

### 4.2.1 Bounded distortion

Let $\pi_1 : \hat{M} \to \hat{\Sigma}$ be the canonical projection $\pi_1(\hat{x},t) = \hat{x}$ and denote $\hat{\nu} = \pi_{1*}\hat{\mu}$. Equivalently,

$$\hat{\nu}(E) = \int_{E\times K} \rho(x^s,x,t)\, d\mu^s(x^s)\, d\mu^u(x)\, d\mu^c(t)$$

for any measurable set $E \subset \hat{\Sigma}$. For each $x \in \Sigma$, define $\hat{\nu}_x$ to be the normalization of

$$\mu^s \int \rho(\cdot,x,t)\, d\mu^c(t).$$

Then $\{\hat{\nu}_x : x \in \Sigma\}$ is a continuous Rokhlin disintegration of $\hat{\nu}$ with respect to the partition into local stable sets $W^s_{loc}(\hat{x})$.

The measure $\hat{\nu}$ satisfies the properties of local product structure, boundedness and continuity in [3, Section 1.2]. In what follows, we recall a few results about this type of measures that we will use later.

For each $x^u \in \Sigma^+$ and $k \geq 1$ let the *backward average* measure $\mu^u_{k,x^u}$ of the map $\sigma$ be defined by

$$\mu^u_{k,x^u} = \sum_{\sigma^k(z)=x^u} \frac{1}{J\sigma^k(z)}\delta_z$$

where $J\sigma^k : \Sigma^+ \to \mathbb{R}$ is the Jacobian of $\mu^u$ with respect to $\sigma^k$.

**Lemma 4.2.2.** *Given any cylinder $I^u = [\iota_0,\dots,\iota_{k-1}] \subset \Sigma^+$ and any $z^u \in I^u$,*

$$\hat{\sigma}^k_*\hat{\nu}_{z^u} = J\sigma^k(z^u)(\hat{\nu}_{\sigma^k(z^u)} \mid I^s)$$

*where $\{\hat{\nu}_{z^u} : z^u \in \Sigma^+\}$ is the disintegration of $\hat{\nu}$ with respect to the partition $\{\Sigma^- \times \{z^u\} : z^u \in \Sigma^+\}$.*

*Proof.* Analogous to [3, Lemma 2.6]. □

**Lemma 4.2.3.** *For every $x^u \in \Sigma^+$ and every cylinder $[J] \subset \Sigma^+$,*

$$\kappa\mu^u([J]) \geq \limsup_n \frac{1}{n}\sum_{k=0}^{n-1}\mu^u_{k,x^u}([J]) \geq \liminf_n \frac{1}{n}\sum_{k=0}^{n-1}\mu^u_{k,x^u}([J]) \geq \frac{1}{\kappa}\mu^u([J])$$

*Proof.* Analogous to [3, Lemma 2.7]. □

As a direct consequence, for every cylinder $[J] \subset \Sigma^+$ and every $x^u \in \Sigma^+$,

$$\limsup_k \mu^u_{k,x^u}([J]) \geq \frac{1}{\kappa}\mu^u([J]) \tag{4.4}$$

### 4.2.2 Estimating the Jacobians

We call the *extremal center Lyapunov exponents* of $\hat{f}$ the limits

$$\lambda^{c+} = \lim_n \frac{1}{n}\log\left\|D\hat{f}^n_{\hat{x}}(t)\right\| \quad \text{and} \quad \lambda^{c-} = \lim_n -\frac{1}{n}\log\left\|D\hat{f}^n_{\hat{x}}(t)^{-1}\right\|.$$

The Oseledets theorem [21] ensures that these numbers are well defined at $\hat{\mu}$-almost every point.

Since we assume that the maps $\hat{f}^n_{\hat{x}}$ have uniformly bounded derivatives, in our case we have

**Lemma 4.2.4.** $\lambda^{c+} = \lambda^{c-} = 0$.

**Remark 4.2.5.** When the maps $\hat{f}^n_{\hat{x}}$ are $C^{1+\epsilon}$, equi-continuity alone suffices to get the conclusion of Lemma 4.2.4. This can be shown using Pesin theory, as follows.

Suppose that $\lambda^{c+} > 0$. Then we have a Pesin unstable manifold defined $\hat{\mu}$-almost everywhere. This implies that there exist $\hat{x} \in \hat{\Sigma}$ and $t \neq s \in K$ such that

$$\operatorname{dist}_K\left(\hat{f}^{-n}_{\hat{x}}(t), \hat{f}^{-n}_{\hat{x}}(s)\right) \to 0.$$

Then, given points $t$ and $s$ in the unstable manifold and given any $\delta > 0$, there exists $n$ such that $\operatorname{dist}_K(\hat{f}^{-n}_{\hat{x}}(t), \hat{f}^{-n}_{\hat{x}}(s)) < \delta$. This implies that the family is not equi-continuous. The proof for $\lambda^{c-}$ is analogous.

Recall that we also assume that $\hat{f} : \hat{M} \to \hat{M}$ admits s-holonomies and u-holonomies and $\hat{\mu}$ has partial product structure. As explained before, this implies that $(\pi_1)_*\hat{\mu}$ has local product structure.

**Lemma 4.2.6.** *The disintegration $\{\mu^c_x : x \in \Sigma\}$ is $f$-invariant, in the sense that $f_{x*}\mu^c_x = \mu_{\sigma(x)}$ for every $x \in \Sigma$.*

*Proof.* For every $x \in \Sigma$ we have

$$\left(f_x^{-1}\right)_* \mu^c_{\sigma(x)} = J(x,\cdot)\mu^c_x + \eta_x$$

where $\eta_x$ is singular respect to $\mu^c_{\sigma(x)}$. Define $h = \int -\log J d\mu$. By [4, Proposition 3.1] we have that zero Lyapunov exponent implies $h = 0$ this implies $J = 1$ at $\mu$-almost every point. Then $\left(f_x^{-1}\right)_* \mu^c_{\sigma(x)} = \mu^c_x$, which is the same as $f_{x*}\mu^c_x = \mu_{\sigma(x)}$, almost everywhere. The continuity of the disintegration implies that the equality is true everywhere. □

**Corollary 4.2.7.** *$\mu^c_x = \hat{\mu^c}_{\hat{x}}$ for every $\hat{x} \in \hat{\Sigma}$, where $x = P(\hat{x})$.*

*Proof.* We have that $f^{-1}_{\sigma(x)_*}\mu^c_{\sigma(x)} = \mu^c_x$, and by Proposition 4.1.4

$$\hat{\mu}^c_{\hat{x}} = \lim f^n_{\sigma^{-n}(\hat{x})}\mu^c_{P(\sigma^{-n}(\hat{x}))}.$$

Then the sequence in the 4.1.4 is constant and equal to $\mu^c_x$. □

We also have

**Lemma 4.2.8.** *The map $\hat{x} \mapsto \hat{\mu}^c_{\hat{x}}$ is continuous. Moreover, the disintegration is both u- and s-invariant:*

*(a) $\left(h^u_{\hat{x},\hat{y}}\right)_* \hat{\mu}^c_{\hat{x}} = \hat{\mu}^c_{\hat{y}}$ for every $\hat{x} \in W^u(\hat{y})$ and*

*(b) $\left(h^s_{\hat{x},\hat{z}}\right)_* \hat{\mu}^c_{\hat{x}} = \hat{\mu}^c_{\hat{z}}$ for every $\hat{x} \in W^u(\hat{z})$.*

*Proof.* By Theorem 2.3.1 and Proposition 2.3.2 there exist some Rokhlin disintegration $\{\tilde{\mu}^c_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ which is continuous, $u$-invariant and $s$-invariant. By essential uniqueness, $\tilde{\mu}^c_{\hat{x}} = \hat{\mu}^c_{\hat{x}}$ for $\hat{\mu}$-almost every $x$. Since both disintegrations are continuous, it follows that they coincide, and so $\{\hat{\mu}^c_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ is continuous, $u$-invariant and $s$-invariant, as claimed. □

**Remark 4.2.9.** If we define $Jf^j_x : K \to \mathbb{R}$, as the Jacobian of $f^j_x$ with respect to $\mu^c$ we have that $Jf^j_x(t) = \frac{\varrho(\sigma^j(x),t)}{\varrho(x,t)}$. This implies that the Jacobians with respect to $\mu^c$ are bounded from above and below. The same argument shows that that the Jacobians of the holonomies $Jh^*_{\hat{x},\hat{y}}$ with respect to $\mu^c$ are bounded.

For every $x \in \Sigma$ let

$$F_x : K \times \mathrm{Grass}(l,d) \to K \times \mathrm{Grass}(l,d), \quad F_x(t,V) = (f_x(t), A(x,t)V)$$

and for every $\hat{x}, \hat{y} \in \hat{\Sigma}$ in the same unstable set let

$$H_{\hat{x},\hat{y}} : K \times \mathrm{Grass}(l,d) \to K \times \mathrm{Grass}(l,d)$$
$$H_{\hat{x},\hat{y}}(t,V) = (h_{\hat{x},\hat{y}}(t), H^u_{(\hat{x},t)(\hat{y},h_{\hat{x},\hat{y}}(t))}V).$$

**Corollary 4.2.10.** *If $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ is a disintegration of an invariant u-state $\hat{m}$ with respect to the partition $\{\hat{x} \times K \times \mathrm{Grass}(l,d) : \hat{x} \in \hat{\Sigma}\}$ then*

$$\hat{m}_{\hat{\sigma}^n(\hat{x})} = F^n_{x\,*}\hat{m}_{\hat{x}}$$

*for every $n \geq 1$, every $x \in \Sigma$, and $\hat{\nu}_x$-almost every $\hat{x} \in W^s_{loc}(x)$.*

*Proof.* Since $\hat{m}$ is $\hat{F}$-invariant, the equality is true for all $n \geq 1$ and $\hat{\nu}$-almost all $\hat{z} \in \hat{\Sigma}$ or, equivalently, for $\hat{\nu}_z$-almost every $\hat{z} \in W^s_{loc}(z)$ and $\nu$-almost every $z \in \Sigma$. Consider an arbitrary point $x \in \Sigma$. Since $\nu$ is positive on open sets, $x$ may be approximated by points $z$ such that

$$\hat{m}_{\hat{\sigma}^n(\hat{z})} = F^n_{z\,*}\hat{m}_{\hat{z}}$$

for every $n \geq 1$ and $\hat{\mu}_z$-almost every $\hat{z} \in W^s_{loc}(z)$. Since the conditional probabilities of $\hat{m}$ are invariant under unstable holonomies, it follows that

$$\hat{m}_{\hat{\sigma}^n(\hat{x})} = (H_{\hat{\sigma}^n(z),\hat{\sigma}^n(x)})_* F^n_{z\,*}\hat{m}_{\hat{z}} = F^n_{x\,*}(H_{\hat{z},\hat{x}})_*\hat{m}_{\hat{z}} = F^n_{x\,*}\hat{m}_{\hat{x}}$$

for $\hat{\mu}_z$-almost every $\hat{z} \in W^s_{loc}(z)$, where $\hat{x}$ is the unique point in $W^s_{loc}(x) \cap W^u_{loc}(\hat{z})$. Since the measures $\hat{\mu}_x$ and $\hat{\mu}_z$ are equivalent, this is the same as saying that the last equality holds for $\hat{\mu}_x$-almost every $\hat{x} \in W^s_{loc}(x)$, as claimed. $\square$

## 4.3 $L^1$-continuity of conditional probabilities

Let $\hat{m}$ be a $u$-state on $\hat{M} \times \mathrm{Grass}(l,d)$ that projects to $\hat{\mu}$ under the canonical projection $\pi : \hat{M} \times \mathrm{Grass}(l,d) \to \hat{M}$. As before, denote

$$\mu = (P \times \mathrm{id}_K)_*\,\hat{\mu} \quad \text{and} \quad m = \left(P \times \mathrm{id}_K \times \mathrm{id}_{\mathrm{Grass}(l,d)}\right)_*\hat{m}.$$

Let $\{m_x : x \in \Sigma\}$ be a Rokhlin disintegration of $m$ with respect to the partition $\{\{x\} \times K \times \mathrm{Grass}(l,d), x \in \Sigma\}$. Thus each $m_x$ is a probability measure on $K \times \mathrm{Grass}(l,d)$.

Also, let $\{m_{x,t} : (x,t) \in M\}$ be a Rokhlin disintegration of $m$ with respect to the partition $\{\{(x,t)\} \times \mathrm{Grass}(l,d), (x,t) \in \Sigma \times K\}$: each $m_{x,t}$ is a probability measure on $\mathrm{Grass}(l,d)$.

It is easy to check that $x \mapsto m_x$ may be chosen to be continuous with respect to the weak* topology, indeed, we will do that in a while. The corresponding statement for $x \mapsto m_{x,t}$ is false, in general. However, the main point of this section is to show that the family $\{m_{x,t} : (x,t) \in M\}$ does have some continuity property:

**Proposition 4.3.1.** *Let $(x_n)_n$ be a sequence of elements of $\Sigma$ converging to some $x \in \Sigma$. Then there exists a sub-sequence $(x_{n_k})_k$ such that*

$$m_{x_{n_k},t} \to m_{x,t} \text{ as } k \to \infty$$

*in the weak* topology, for $\mu^c$-almost every $t \in K$.*

We will deduce this proposition from a somewhat stronger $L^1$-continuity result, whose precise statement will be given in Proposition 4.3.5. The key ingredient in the proofs is a result about maps on geodesically convex metric spaces that we are going to state in Lemma 4.3.3 and which will also be useful at latter stages of our arguments.

**Definition 4.3.2.** We say that a metric space $N$ is *geodesically convex* if there exist $\tau > 0$ such that for every $u, v \in N$ there exist a continuous path $\lambda : [0,1] \to N$, with $\lambda(0) = u$, $\lambda(1) = v$ and

$$\mathrm{dist}_N(\lambda(t), \lambda(s)) \le \tau \, \mathrm{dist}_N(u,v) \text{ for every } s, t \in [0,1].$$

Examples of geodesically convex spaces are: convex subset of a Banach space, path connected compact metric spaces, complete connected Riemannian manifolds. The spaces of maps with values on a geodesically convex space are analyzed in Appendix A.2.

**Lemma 4.3.3.** *Let $L$ be a geodesically convex metric space and $(K, \mathfrak{B}_K, \mu_K)$ be a probability space such that $K$ is a normal topological space, $\mathfrak{B}_K$ is the Borel $\sigma$-algebra of $K$ and the measure $\mu_K$ is regular.*

*Let $H_{j,t} : L \to L$ and $h_j : K \to K$, with $j \in \mathbb{N}$ and $t \in K$, be such that*

$$\big(H_{j,t}(x)\big)_j \to x \quad \textit{and} \quad \big(h_j(t)\big)_j \to t,$$

*uniformly in $t \in K$ and $x \in L$ and, moreover, the Jacobian $Jh_j(t)$ of each $h_j$ with respect to $\mu_K$ is uniformly bounded. Then*

$$\lim_j \int \mathrm{dist}_L \big(\psi(t), H_{j,t} \circ \psi \circ h_j(t)\big) \, d\mu_K(t) = 0$$

*for every bounded measurable map $\psi : K \to L$.*

*Proof.* Take $j \in \mathbb{N}$ to be sufficiently large that $d_L(H_{j,t}(x), x) < \epsilon/4$ for every $t$ and $x$. Then,

$$\begin{aligned}\int \mathrm{dist}_L(\psi, H_{j,t} \circ \psi \circ h_j) \, d\mu_K &\le \int \big( \mathrm{dist}_L(\psi, \psi \circ h_j) \\ &\qquad + \mathrm{dist}_L(\psi \circ h_j, H_{j,t} \circ \psi \circ h_j)\big) \, d\mu_K \\ &\le \int \mathrm{dist}_L(\psi, \psi \circ h_j) \, d\mu_K + \frac{\epsilon}{4}.\end{aligned}$$

Let $C > 1$ be a uniform bound for $Jh_j(t)$. By Theorem A.2.1, given $\epsilon > 0$ there exists a continuous map $\tilde{\psi} : K \to L$ such that

$$\int \mathrm{dist}_L(\tilde{\psi}, \psi) \, d\mu_K < \frac{\epsilon}{4C}.$$

Then, by change of variables,

$$\int \operatorname{dist}_L(\tilde{\psi} \circ h_j, \psi \circ h_j)\, d\mu_K \leq C \int \operatorname{dist}_L(\tilde{\psi}, \psi)\, d\mu_K < \frac{\epsilon}{4}.$$

Hence

$$\begin{aligned}
&\int \operatorname{dist}_L(\psi, \psi \circ h_j)\, d\mu_K \\
&\leq \int \big(\operatorname{dist}_L(\psi, \tilde{\psi}) + \operatorname{dist}_L(\tilde{\psi}, \tilde{\psi} \circ h_j) + \operatorname{dist}_L(\tilde{\psi} \circ h_j, \psi \circ h_j)\big)\, d\mu_K \\
&\leq \int \operatorname{dist}_L(\tilde{\psi}, \tilde{\psi} \circ h_j)\, d\mu_K + \frac{\epsilon}{2}.
\end{aligned}$$

By the continuity of $\tilde{\psi}$, increasing $j$ if necessary,

$$\operatorname{dist}_L(\tilde{\psi}(t), \tilde{\psi} \circ h_j(t)) < \frac{\epsilon}{4} \text{ for every } t \in K.$$

The conclusion follows from these inequalities. □

**Lemma 4.3.4.** *Let $(x_n)_n$ be a sequence of elements of $\Sigma$ converging to some $x \in \Sigma$, also let $f^{j_n}_{x_n}$ converge uniformly to $g : K \to K$, and $\sigma^{j_n}(x_n) \to z$. Then $g$ is absolutely continuous with respect to $\mu^c$ and has bounded Jacobian.*

*Proof.* By Lemma 4.2.6 we have that

$$(f^{j_n}_{x_n})_* \mu^c_{x_n} = \mu^c_{\sigma^{j_n}(x_n)}$$

making $n \to \infty$ we get

$$g_* \mu^c_x = \mu^c_z$$

this implies that $Jg_{\mu^c} = \frac{\varrho(z,t)}{\varrho(x,t)}$ is uniformly bounded. □

**Proposition 4.3.5.** *For every continuous $\varphi : \operatorname{Grass}(l,d) \to \mathbb{R}$, if $x_n \in \Sigma$ is a sequence such that $\lim_{n\to\infty} x_n = x$, also let $f^{j_n}_{x_n}$ converge uniformly to $g : K \to K$. Then $\int \varphi dm_{x_n, f^{j_n}_{x_n}(t)}$ converges in $L^1(\mu^c)$ to $\int \varphi\, dm_{x,g(t)}$.*

*Proof.* Let $\varphi : \operatorname{Grass}(l,d) \to \mathbb{R}$ be a continuous function. For simplicity call $t_n = f^{j_n}_{x_n}(t)$

Fixed $x^s \in \Sigma^-$, let

$$h^u_{n,x^s}(t) = h^u_{(x^s,x_n),(x^s,x)}(t_n)$$

and

$$H^u_{\left(x^s,x,h^u_{n,x^s}(t)\right),(x^s,x_n,t_n)} = H^u_{n,x^s,t}.$$

We have that

$$\hat{m}_{x^s,x_n,t_n} = \left(H^u_{n,x^s,t}\right)_* \hat{m}_{x^s,x,h^u_{n,y}(t_n)}.$$

So applying the lemma 4.3.3 with $L = \mathcal{M}$, the space of Grass$(l,d)$ probability measures (that is a compact metric space), $H_{j,t} = \left(H^u_{j,x^s,t}\right)_*$, $h_j = h^u_{j,x^s}(t_j)$ and

$$\psi : K \to \mathcal{M}, \quad \psi(t) = \hat{m}_{x^s,x,g(t)}$$

we have that $\int \varphi d\hat{m}_{x^s,x_n,f^{j_n}_{x_n}(t))}$ converges in $L^1_{\mu^c}$ to $\int \varphi d\hat{m}_{y,x,g(t)}$.

By Rokhlin disintegration we know that

$$m_{x,t} = \int \hat{m}_{x^s,x,t}\rho(x^s,x,t)d\mu^s(x^s)$$

so

$$\int |\int \varphi(v)dm_{x_n,t_n} - \int \varphi(v)dm_{x,g(t)}|d\mu^c \leq$$
$$\int\int |\int \varphi\rho(x^s,x_n,t_n)d\hat{m}_{x^s,x_n,t_n} - \int \varphi\rho(x^s,x,g(t))d\hat{m}_{x^s,x,g(t)}|d\mu^c d\mu^s$$

as the integrand goes to zero for every $x^s \in \Sigma^-$, by dominated convergence

$$\lim_{n\to\infty} \int |\int \varphi(v)dm_{x_n,t_n} - \int \varphi(v)dm_{x,g(t)}|d\mu^c = 0$$

□

**Corollary 4.3.6.** *The disintegration $m_x$, $x \in \Sigma$, is continuous.*

*Proof.* Let $\varphi : K \times \mathrm{Grass}(l,d) \to \mathbb{R}$ be a continuous function, and $\rho'(z,t) = \int \rho(x^s,z,t)d\mu^s$.

Then if $x_n \to x$ we have that

$$|\int \varphi dm_{x_n} - \int \varphi dm_x| =$$
$$|\int\int \varphi(t,v)dm_{x_n,t}\rho'(x_n,t)d\mu^c - \int\int \varphi(t,v)dm_{x,t}\rho'(x,t)d\mu^c| \leq$$
$$\int |\int \varphi(t,v)\rho'(x_n,t)dm_{x_n,t} - \int \varphi(t,v)\rho'(x_n,t)dm_{x,t}|d\mu^c$$

and a small modification of the argument of the previous proposition shows that

$$\int \varphi(t,v)\rho'(x_n,t)dm_{x_n,t} \to_{L^1_{\mu^c}} \int \varphi(t,v)\rho'(x_n,t)dm_{x,t}.$$

So the conclusion follows integrating in $t$. □

Given $x_n \to x$, for every $\varphi$ continuous, there exist a sub-sequence such that that

$$\int \varphi dm_{x_n,t} \to \int \varphi dm_{x,t} \text{ in } \mu^c \text{ almost every } t.$$

Taking a subset $\{\varphi^j : K \to \mathbb{R}, j \in \mathbb{N}\}$ dense in the space of continuous functions with norm 1 and constructing a diagonal sequence we have a subsequence such that

$$\int \varphi^j dm_{x_{n_k},t} \to \int \varphi^j dm_{x,t} \text{ for every } \varphi^j$$

in a set of total $\mu^c$ measure.

If $\psi : \mathrm{Grass}(l,d) \to \mathbb{R}$ is a continuous function with norm 1, for every $\epsilon > 0$ there exists $\varphi^j$ such that $\left\|\psi - \varphi^j\right\| < \epsilon$, then

$$\begin{aligned} \limsup_{k\to\infty} \int \psi dm_{x_{n_k},t} &\leq \limsup_{k\to\infty} \int \mid \psi - \varphi^j \mid dm_{x_{n_k},t} + \lim \int \varphi^j dm_{x_{n_k},t} \\ &< \epsilon + \int \varphi dm_{x,t}. \end{aligned}$$

Analogously

$$\liminf_{k} \int \psi dm_{x_{n_k},t} > \int \varphi dm_{x,t} - \epsilon.$$

Hence, for every continuous function $\lim_k \int \varphi dm_{x_{n_k},t} = \int \varphi dm_{x,t}$ for $\mu^c$-almost every $t$.

This proves Proposition 4.3.1.

CHAPTER 5

# Finding the Oseledets subspaces

In this chapter we prove that our $u$-states are actually Dirac measures. This will allow us to find the Oseledets subspace more expanded by the cocycle.

## 5.1 Graphs of Grassmanians

Using Rokhlin disintegration we have that $m_x = \int m_{x,t}\varrho(x,t)d\mu^c$. Remember that $m_x$ is a probability measure on $K \times \mathrm{Grass}(l,d)$ and $m_{x,t}$ is a probability measure on $\mathrm{Grass}(l,d)$.

We embed $\mathrm{Grass}(l,d) \hookrightarrow \mathbb{P}\Lambda^l\left(\mathbb{C}^d\right)$, where $\mathbb{P}\Lambda^l\left(\mathbb{C}^d\right)$ is the projectivization of the exterior product $\Lambda^l\left(\mathbb{C}^d\right)$. The image under this embedding is a closed set called *$l$-vectors*.

This set can be seen as the projectivization of

$$\Lambda^l_v(\mathbb{C}^d) = \{w_1 \wedge w_2 \wedge \cdots \wedge w_l \in \Lambda^l\left(\mathbb{C}^d\right), \text{ such that } w_i \in \mathbb{C}^d \text{ for } 1 \leq i \leq l\}.$$

that we denote by $\mathbb{P}\Lambda^l_v(\mathbb{C}^d)$.

Every linear transformation $B : \mathbb{C}^d \to \mathbb{C}^d$ induce an action in $\Lambda^l\left(\mathbb{C}^d\right)$, that we also denote by $B : \Lambda^l\left(\mathbb{C}^d\right) \to \Lambda^l\left(\mathbb{C}^d\right)$, this action preserves $\Lambda^l_v(\mathbb{C}^d)$. For a more clear picture about $l$ vectors see of [3, Section 2].

Given $V \in \mathrm{Grass}(d-l,d)$, a $(d-l)$ subspace, we define the *geometric hyperplane* $\mathfrak{H}V$

$$\mathfrak{H}V = \{W \in \mathrm{Grass}(l,d), \text{ such that } W \cap V \neq \{0\}\}$$

namely this are the $d$-dimensional subspaces that have non-trivial intersection with $V$.

Taking $v \in \Lambda_v^{d-l}(\mathbb{C}^d)$ an $(d-l)$-vector that represents $V$, and for every $W \in \mathrm{Grass}(l,d)$ calling $w \in \Lambda_v^l(\mathbb{C}^d)$ a $l$-vector that represents $W$, it is easy to see that

$$\mathfrak{H}V = \{W \in \mathrm{Grass}(l,d), \text{ such that } w \wedge v = 0\}$$

The space of hyperplanes could be seen as a subset of $\mathrm{Grass}(d-l,d)$ (by $V \to \mathfrak{H}V$). Denote by $\mathfrak{H}\,\mathrm{Sec}$ the space of sections $V : K \to \mathrm{Grass}(d-l,d)$.

If $V \in \mathfrak{H}\,\mathrm{Sec}$ is a measurable section, his

$$\mathrm{graph}\,\mathfrak{H}\,(V) = \{(t,v) \in K \times \mathrm{Grass}(l,d), \text{ with } v \in \mathfrak{H}V(t)\}$$

has measure $m_x\,(\mathrm{graph}\,\mathfrak{H}(V)) = \int m_{x,t}\,(\mathfrak{H}V_t)\,\varrho(x,t)d\mu^c$.

### 5.1.1 Graphs have measure zero

The purpose of this section is to prove the next proposition

**Proposition 5.1.1.** *For every $x \in M$ any measurable $V \in \mathfrak{H}\,\mathrm{Sec}$, of dimension $l < d$, has $m_x(\mathrm{graph}\,\mathfrak{H}(V)) = 0$ measure.*

Fix $x$ and consider the next function

$$G : K \times \mathrm{Grass}(d-l,d) \mapsto \mathbb{R}, \quad G\,(t,V) = m_{x,t}(\mathfrak{H}V).$$

This is a measurable function.

Let

$$g : K \to \mathbb{R}, \quad g(t) = \sup_V G(t,V)$$

this function is also measurable.

Also, it is easy to see that for every fix $t \in K$ there exist some $V_t$ such that $g(t) = G\,(t,V_t)$, and the set $B_t \subset \mathrm{Grass}(d-l,d)$ of $V_t$ such that $g(t) = G\,(t,V_t)$ is compact.

Now we use the next result whose proof can be found in [12]

Let $(X,\mathfrak{B},\mu)$ be a complete probability space and $Y$ be a separable complete metric space. Denote by $\mathfrak{B}(Y)$ the Borel $\sigma-$algebra of $Y$.

**Proposition 5.1.2.** *Let $\kappa(Y)$ be the space of compact subsets of $Y$, with the Hausdorff topology. The following are equivalent:*

- *a map $x \to K_x$ from $X$ to $\kappa(Y)$ is measurable;*
- *its graph $\{(x,y) : y \in K_x\}$ is in $\mathfrak{B} \otimes \mathfrak{B}(Y)$;*
- *$\{x \in X : K_x \cap U \neq \emptyset\} \in \mathfrak{B}$ for any open set $U \subset Y$*

*Moreover, any of these conditions implies that there exist a measurable map $\eta : X \to Y$ such that $\eta(x) \in K_x$ for every $x \in X$*

Applying this proposition we have that there exists some measurable section, $W : K \to \mathrm{Grass}(d-l,d)$ such that $W_t \in B_t$ for every $t \in K$.

So there exists a measurable $W \in \mathrm{Sec}(K, \mathrm{Grass}(l,d))$ such that $g(t) = G(t, W_t)$.

Given $V \in \mathfrak{H}\,\mathrm{Sec}$ we have

$$\begin{aligned} m_x\left(\operatorname{graph}\mathfrak{H}(V)\right) &= \int m_{x,t}\left(\mathfrak{H}V_x\right)\varrho(x,t)d\mu^c \\ &\leq \int \sup_Z m_{x,t}\left(\mathfrak{H}Z\right)\varrho(x,t)d\mu^c \\ &= \int m_{x,t}\left(\mathfrak{H}W_t\right)\varrho(x,t)d\mu^c. \end{aligned}$$

As $x \in \Sigma$ was arbitrary, we proved that for every $x \in \Sigma$ there exists $W^x \in \mathfrak{H}\,\mathrm{Sec}$ measurable such that

$$\sup_{V\in \mathrm{Sec}(K,\mathrm{Grass}(d-l,d))} m_x\left(\operatorname{graph}\mathfrak{H}(V)\right) = m_x\left(\operatorname{graph}\mathfrak{H}(W^x)\right)$$

and the section reaches the supreme, if and only if, $m_{x,t}\left(\mathfrak{H}W_t^x\right) = g(t)$ for $\mu^c$-almost every $t \in K$.

Now let

$$\gamma = \sup_{x\in\Sigma}\sup_{V\in\mathfrak{H}\,\mathrm{Sec}} m_x\left(\operatorname{graph}\mathfrak{H}(V)\right),$$

we are going to prove that for every $x \in \Sigma$ there exists a section $W^x$ that attains the supreme.

**Proposition 5.1.3.** *For every $x \in \Sigma$ there exist some $W^x$ section such that $m_x(\operatorname{graph}\mathfrak{H}(W^x)) = \gamma$.*

*Proof.* Fix a cylinder $[J] \subset \Sigma$ and a positive constant $c < \frac{\mu([J])}{\kappa}$, where $\kappa$ is the constant given in equation (4.4). Let $z \in \Sigma$ and $V \in \mathfrak{H}\,\mathrm{Sec}$ with $m_z(\operatorname{graph}\mathfrak{H}(V)) > 0$. For each $y \in \sigma^{-k}(z)$, let $V^y = \mathcal{F}_{\hat{y}}^{-k}V$. Then

$$\begin{aligned} m_z(\operatorname{graph}\mathfrak{H}(V)) = \int m_y(\operatorname{graph}\mathfrak{H}(V_y)d\mu_{k,z}(y) \\ \leq \mu_{k,z}([J])\sup\{m_y(\operatorname{graph}\mathfrak{H}(V_y) : y \in [J]\} \\ + (1-\mu_{k,z}([J]))\gamma \end{aligned}$$

By (4.4), there exist arbitrary large values of $k$ such that $\mu_{k,z}([J]) \geq c$. Then

$$m_z(\operatorname{graph}\mathfrak{H}(V)) \leq c\sup\{m_y(\operatorname{graph}\mathfrak{H}(V_y)) : y \in [J]\} + (1-c)\gamma$$

Varying the point $z \in \Sigma$ and $V$, we can make the left hand side arbitrarily close to $\gamma$. It follows that

$$\sup\{m_y(\operatorname{graph}\mathfrak{H}(V_y) : y \in [J]\} \geq \gamma, .$$

This proves that the supremum over any cylinder $[J]$ coincides with $\gamma$. Then given any $x \in \Sigma$ we may find a sequence $x_n \to x$ such that $m_{x_n}(\operatorname{graph}\mathfrak{H}(W^{x_n})) \to \gamma$, by proposition 4.3.1 we may assume that there exists a set $X \subset K$ of total $\mu^c$ measure such that $m_{x_n,t} \rightharpoonup^* m_{x,t}$.

$$\begin{aligned}\gamma &= \lim \int m_{x_n,t}\left(\mathfrak{H}W_t^{x_n}\right)\varrho(x_n,t)d\mu^c \\ &\leq \int \limsup m_{x_n,t}\left(\mathfrak{H}W_t^{x_n}\right)\varrho(x_n,t)d\mu^c.\end{aligned}$$

Now for every fixed $t \in X$ we have that there exist a sub-sequence $x_{n_k}$ and $V \in \operatorname{Grass}(d-l,d)$ such that $W_t^{x_{n_k}} \to V$. Taking a neighborhood of $V_\epsilon$ of $V$

$$\limsup m_{x_n,t}\left(\mathfrak{H}W_t^{x_n}\right) \leq m_{x,t}\left(\mathfrak{H}V_\epsilon\right).$$

Then making $\epsilon \to 0$

$$\begin{aligned}\limsup m_{x_n,t}\left(\mathfrak{H}W_t^{x_n}\right) &\leq m_{x,t}\left(\mathfrak{H}V\right) \\ &\leq m_{x,t}\left(\mathfrak{H}W_t^{x}\right).\end{aligned}$$

So

$$\gamma \leq \int m_{x,t}\left(\mathfrak{H}W_t^{x}\right)\varrho(x,t)d\mu^c.$$

□

**Lemma 5.1.4.** *For any $x \in \Sigma$ and any $V \in \mathfrak{H}\operatorname{Sec}$, we have that*

$$m_x(\operatorname{graph}\mathfrak{H}(V)) = \gamma$$

*if and only if $m_y(\operatorname{graph}\mathfrak{H}(\mathcal{F}_{\hat{y}}^{-1}V)) = \gamma$ for every $y \in \sigma^{-1}(x)$.*

*Proof.* By the continuity of the disintegration $m_x$ and the $F$ invariance of $m$ we have that $m_x = \sum_{\sigma(y)=x} \frac{1}{J\sigma(y)} F_{y*} m_y$, with $\sum_{\sigma(y)=x} \frac{1}{J\sigma(y)} = 1$. Then

$$m_x(\operatorname{graph}\mathfrak{H}V) = \sum_{\sigma(y)=x} \frac{1}{J\sigma(y)} m_y(\operatorname{graph}\mathfrak{H}\mathcal{F}_{\hat{y}}^{-1}V). \tag{5.1}$$

Since the maximum is $\gamma$ we get that $m_x(\operatorname{graph}\mathfrak{H}V) = \gamma$ if and only if $m_y(\operatorname{graph}\mathfrak{H}\mathcal{F}_{\hat{y}}^{-1}V) = \gamma$. □

**Lemma 5.1.5.** *For any $x \in \Sigma$ and any $W \in \mathfrak{H}\operatorname{Sec}$ we have that*

$$\hat{m}_{\hat{x}}(\operatorname{graph}\mathfrak{H}(W)) \leq \gamma \text{ for } \hat{\nu}_x \text{ almost every } \hat{x} \in W^s_{loc}(x).$$

*Hence, $m_x(\operatorname{graph}\mathfrak{H}(W)) = \gamma$ if and only if $\hat{m}_{\hat{x}}(\operatorname{graph}\mathfrak{H}(W)) = \gamma$ for $\mu^s$-almost every $\hat{x}$ in $W^s_{loc}(x)$.*

*Proof.* Suppose there is $y \in \Sigma$, any $V \in \mathfrak{H}\,\mathrm{Sec}$, a constant $\gamma_1 > \gamma$, and a positive $\hat{\nu}$-measure subset $X$ of $W^s_{loc}(y)$ such that $\hat{m}_{\hat{y}}(\mathrm{graph}\,\mathfrak{H}(V)) \geq \gamma_1$ for every $\hat{y} \in X$. For each $m < 0$, consider the partition of $W^s_{loc}(y) \approx \Sigma^-$ determined by the cylinders $[I]^s = [\iota_m, \dots, \iota_{-1}]^s$, with $\iota_j \in \mathbb{N}$. Since these partitions generate the $\sigma$-algebra of the local stable set, given any $\varepsilon > 0$ we may find $m$ and $I$ such that

$$\hat{\nu}_y(X \cap [I]^s) \geq (1-\varepsilon)\hat{\nu}_y([I]^s).$$

Observe that $[I]^s \approx [I]^s \times \{y\}$ coincides with $\hat{\sigma}^n(W^s_{loc}(x))$, where $x = \sigma_I^{-n}(y)$. So, using also Lemma 4.2.2,

$$\hat{\nu}_x\big(\hat{\sigma}^{-n}(X) \cap W^s_{loc}(x)\big) = (\hat{\sigma}^n_* \hat{\nu}_x)(X \cap [I]^s) = J_\mu \sigma^n(x)\, \hat{\nu}_y\big(X \cap [I]^s\big).$$

By the previous inequality and Lemma 4.2.2, this is bounded below by

$$(1-\varepsilon)\, J_\mu \sigma^n(x)\, \hat{\nu}_y\big([I]^s\big) = (1-\varepsilon)\, (\hat{\sigma}^n_* \hat{\nu}_x)\big([I]^s\big) = \hat{\nu}_x\big(W^s_{loc}(x)\big) = 1-\varepsilon.$$

In this way we have shown that

$$\hat{\nu}_y\big(\hat{\sigma}^{-n}(X) \cap W^s_{loc}(x)\big) \geq (1-\varepsilon).$$

Fix $\varepsilon > 0$ small enough so that $(1-\varepsilon)\gamma_1 > \gamma_0$. Using Corollary 4.2.10, we find that

$$\hat{m}_{\hat{x}}(\mathrm{graph}\,\mathfrak{H}(\mathcal{F}_x^{-n} V)) = \hat{m}_{\hat{y}}(\mathrm{graph}\,\mathfrak{H}(V)) \geq \gamma_1$$

for $\hat{\nu}_x$-almost every $\hat{x} \in \hat{\sigma}^{-n}(X) \cap W^s_{loc}(x)$. It follows that

$$m_x(\mathrm{graph}\,\mathfrak{H}(\mathcal{F}_x^{-n} V)) = \int \hat{m}_{\hat{x}}(\mathrm{graph}\,\mathfrak{H}(\mathcal{F}_x^{-n} V))\, d\hat{\nu}_x(\hat{x}) \geq (1-\varepsilon)\gamma_1 > \gamma,$$

which contradicts the definition of $\gamma_0$. This contradiction proves the first part of the lemma. The second one is a direct consequence, using the fact that $m_x(\mathrm{graph}\,\mathfrak{H}(V))$ is the $\hat{\nu}_x$-average of all $\hat{m}_{\hat{x}}(\mathrm{graph}\,\mathfrak{H}(V))$. □

## 5.2 Sections over the periodic point

From now on, denote by $p = P(\hat{p})$, and $z = P(\hat{z})$ where $\hat{p}$ is the fixed point and $\hat{z}$ is the homoclinic point given in the definition of pinching and twisting. Recall that $\imath \in \mathbb{N}$ such that $\hat{\sigma}^\imath(\hat{z}) \in W^s_{loc}(\hat{p})$.

By the pinching condition the Oseledets decomposition of the restriction of $F$ to the periodic leaf $\hat{p} \times K$, $\{E^1_t, \dots, E^d_t\}$, gives, at $\mu^c$-almost every $t \in K$, a linear basis of $\mathbb{C}^d$.

This defines a linear base of $\Lambda^{(d-l)}\left(\mathbb{C}^d\right)$ and $\Lambda^l\left(\mathbb{C}^d\right)$ at $\mu^c$-almost every $t \in K$ given by

$$\{E^I_t = E^{i_1}_t \wedge E^{i_2}_t \wedge \dots E^{i_{d-l}}_t, \text{ for } I = \{i_1 < i_2 < \dots < i_{d-l}\}\}$$

and

$$\{E_t^J = E_t^{j_1} \wedge E_t^{j_2} \wedge \dots E_t^{j_l}, \text{ for } J = \{j_1 < j_2 < \dots < j_l\}\}.$$

This basis are invariant by the action of $A$ in $\Lambda^{(d-l)}\left(\mathbb{C}^d\right)$ and $\Lambda^l\left(\mathbb{C}^d\right)$

$$A(p,t)E_t^J \subset \mathbb{C}E_{f_p(t)}^J.$$

**Remark 5.2.1.** If $V \in \sec(K, \mathrm{Grass}(l,d))$ has a base (can be completed to a base of $\mathbb{C}^d$) that is sufficiently far away of the invariant subspaces, then denoting by

$$V = \sum_{I=i_1,i_2,\dots,i_l} v_I(t)E_t^{i_1} \wedge E_t^{i_2} \wedge \dots \wedge E_t^{i_l}$$

for every multi-index $I$, $\lim_{n\to\infty} \frac{1}{n} \log v_I\left(f_p^{qn}(t)\right) = 0$ for $\mu^c$-almost every $t$.

Let $W^p \in \mathfrak{H}\,\mathrm{Sec}$ be the section such that $m_p\left(\mathrm{graph}\,\mathfrak{H}\,W^p\right) = \gamma$.

Let $W^j = \mathcal{F}_p^{-j}W^p$ then also $m_p\left(\mathrm{graph}\,\mathfrak{H}\,W^j\right) = \gamma$.

In the invariant base we can express $W$ by

$$W_t^p = \sum_{I=i_1,i_2,\dots,i_{d-l}} w_I(t)E_t^{i_1} \wedge E_t^{i_2} \wedge \dots E_t^{i_{d-l}}$$

with $|w_I| \leq 1$ and $\sum w_I^2 = 1$.

Then

$$W_t^j = \sum_{I=i_1,i_2,\dots,i_{d-l}} w_I(f_p^j(t))\left(A^j(p,t)\right)^{-1} E_{f_p^j(t)}^{i_1} \wedge \dots \wedge \left(A^j(p,t)\right)^{-1} E_{f_p^j(t)}^{i_{d-l}}$$

and the invariance of the Oseledets subspaces implies that

$$W_t^j = \sum_{I=i_1,i_2,\dots,i_{d-l}} w_I(f_p^j(t))a_{i_1}^j(t)\dots a_{i_{d-l}}^j(t)E_t^{i_1} \wedge E_t^{i_2} \wedge \dots E_t^{i_{d-l}},$$

where

$$a_k^j(t) = |A^j(p,t)^{-1}E_{f_p^j(t)}^k|.$$

We should order the multi-index $I$ such that the sums $\lambda_{i_1}+\lambda_{i_2}+\dots+\lambda_{i_{d-l}}$ are in decreasing order.

Let $w_{\tilde{I}}$ be the first non-zero $w_I$.

Let us assume first that $f_p$ is ergodic, since the non-ergodic case will be reduced to the first one by ergodic decomposition

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} |w_{\tilde{I}}|\left(f_p^j(t)\right) = \int |w_{\tilde{I}}| d\mu_p^c.$$

So for almost every $t$, $\lim \dfrac{1}{n} \sum_{j=0}^{n-1} |w| \left(f_p^j(t)\right) > \delta > 0$, for some $\delta > 0$. Also for almost every $t$, $m_{p,t}\left(W_t^j\right) = \sup_W m_{p,t}(W)$. Then fixing $t$ in that total measure set and using a sub-sequence such that $|w_{\tilde{I}}(f^{j_k q})| > \delta > 0$ we have that

$$W_t^{j_k} \to E_t^{j_1} \wedge E_t^{j_2} \wedge \cdots \wedge E_t^{j_l} = E_t^{\tilde{I}}$$

and $m_{p,t}(E_t^J) = \sup_W m_{p,t}(W)$ for almost every $t$.

So we can assume from the beginning that the section $W^p$ is one of the invariant sections.

Define $W' = \mathcal{F}_z^{-\imath} W^p$. We got that $m_z(\operatorname{graph} \mathfrak{H}\, W') = \gamma$ and by corollary 5.1.5 $\hat{m}_{(z^u,\hat{z})}(\operatorname{graph} \mathfrak{H}\, W') = \gamma$ for $\mu^s$-almost every $(z^u,\hat{z}) \in W^s_{loc}(\hat{z})$.

For each $(x^s,p) \in W^s_{loc}(\hat{p})$, define $W_{(x^s,p)} = \mathcal{H}^u_{(x^s,z),(x^s,p)}(W')$, where $(x^s,z)$ is the unique point in $W^u_{loc}((x^s,p)) \cap W^s_{loc}(\hat{z})$.

Since $\hat{m}$ is an $u$-state, and $h^u_{\hat{z},\hat{p}_*}\mu^c_{\hat{z}} = \mu^c_{\hat{p}}$, this implies that $\hat{m}_\eta(W_{(x^s,p)}) = \gamma$, for $\mu^s$-almost every $\eta \in W^s_{loc}(p)$.

### 5.2.1 Intersection of hyperplane sections

Let us denote by $W^j_{(x^s,p)} = \mathcal{F}^{-j}_{\hat{\sigma}^j(x^s,p)} W_{\hat{\sigma}^j(x^s,p)}$. For $(x^s,p) = \hat{p}$, $W^j_{\hat{p}} = \mathcal{F}^{-j}_{\hat{p}} W_{\hat{p}}$.

We are going to prove that for a large set of $j$'s the $W^j_{\hat{p}}$ have no intersection.

Let $v(t)$ be the $(d-l)$-vector that represents $W_{\hat{p}}$. We can write it

$$v(t) = \sum_I v_I(t) E_t^{i_1} \wedge E_t^{i_2} \wedge \cdots \wedge E_t^{i_{d-l}}$$

for $I = \{i_1 < i_2 < \dots i_{d-l}\}$, $0 < i_k \le l$. Let $J = \{j_1 < j_2 < \cdots < j_{d-l}\}$ be the complement $J = \{1,2,\dots,l\} - I$.

We have that

$$\begin{aligned} \mathcal{F}^{-j}_{\hat{p}} v(t) &= A^j(\hat{p},t)^{-1} v\left(f^j_p(t)\right) \\ &= \sum_I v_I\left(f^j_p(t)\right) a_I^{-j}(t) E_t^I \end{aligned}$$

where $a_I^{-j}(t) = \left(a^j_{i_1}(t) \dots a^j_{i_{d-l}}(t)\right)^{-1}$.

Let $N = dim \Lambda^l\left(\mathbb{C}^d\right)$. Given $m_1 < m_2 < \cdots < m_N$ if $W_t^{m_1} \cap \cdots \cap W_t^{m_N} \neq \emptyset$ there will exists some $\omega : K \to \Lambda^l\left(\mathbb{C}^d\right)$, $\omega(t) \neq 0$ such that

$$\omega(t) \wedge \mathcal{F}^{-m_k *}_{\hat{p}} v(t) = 0 \tag{5.2}$$

Let $\omega(t) = \sum_I \omega_I(t) E_t^J$. Then (5.2) means

$$\sum_I a_J^{-m_k}(t) v_I\left(f^{m_k}(t)\right) \omega_I(t) \varpi_I = 0$$

for every $m_k$. ($\varpi_I$ is 1 or $-1$ accordingly to the sign of $E_t^{i_1} \wedge \cdots \wedge E_t^{i_{d-l}} \wedge E_t^{j_1} \wedge \cdots \wedge E_t^{j_{d-l}}$).

This gives a system of equations

$$B(t)x = 0$$

$$B(t) = \begin{pmatrix} a_{I_1}^{-m_1}(t)v_{I_1}(f^{m_1}(t)) & \dots & a_{I_N}^{-m_1}(t)v_{I_N}(f^{m_1}(t)) \\ \vdots & \vdots & \vdots \\ a_{I_1}^{-m_N}(t)v_{I_1}(f^{m_N}(t)) & \dots & a_{I_N}^{-m_N}(t)v_{I_N}(f^{m_N}(t)) \end{pmatrix}$$

$$x = (\varpi_{I_1}\omega_{I_1}, \dots, \varpi_{I_N}\omega_{I_N})^T$$

So we need to prove that for some large sequence $m_1 < \dots < m_N$, the $\det(B(t)) \neq 0$

**Lemma 5.2.2.** *Let $a_j^n : K \to \mathbb{C}$ for $1 \leq j \leq d$ and $n \in \mathbb{N}$ be measurable functions such that $\lim \frac{1}{n} \log |a_j^n(t)| = \lambda_j$, $\mu^c$-almost everywhere and $\lambda_1 < \lambda_2 < \dots < \lambda_d$.*

*Then for every $M \in \mathbb{N}$ and $\delta > 0$ there exist $n_1 < n_2 < \dots < n_M$ and $\tilde{K} \subset K$ with $\mu(\tilde{K}) > 1 - \delta$ such that for any choice of $n_{k_1} < n_{k_2} < \dots < n_{k_d}$ the matrix $B(t) \in \mathbb{C}^{d \times d}$, $B_{i,j}(t) = \left(a_i^{n_{k_j}}(t)\right)$ has non-zero determinant for every $t \in \tilde{K}$*

*Proof.* Let $m_1 < m_2 < \dots < m_d$ be natural numbers, then we have the determinant

$$\det(B(t)) = \begin{bmatrix} a_1^{m_1}(t) & \dots & a_d^{m_1}(t) \\ \vdots & \dots & \vdots \\ a_1^{m_d}(t) & \dots & a_d^{m_d}(t) \end{bmatrix}$$

if $a_i^{m_1}(t) \neq 0$ we can write

$$\det(B)(t) = a_1^{m_1}(t) \dots a_d^{m_1}(t) \begin{pmatrix} 1 & \dots & 1 \\ \frac{a_1^{m_2}(t)}{a_1^{m_1}(t)} & \dots & \frac{a_d^{m_2}(t)}{a_d^{m_1}(t)} \\ \vdots & \vdots & \vdots \\ \frac{a_1^{m_d}(t)}{a_1^{m_1}(t)} & \dots & \frac{a_d^{m_d}(t)}{a_d^{m_1}(t)} \end{pmatrix}$$

subtracting the first column from on the others columns we end up with

$$\det(B(t)) = a_1^{m_1}(t) \dots a_d^{m_1}(t) \begin{pmatrix} a_2'^{m_2,m1}(t) & \dots & a_d'^{m_2,m1}(t) \\ \vdots & \vdots & \vdots \\ a_2'^{m_d,m1}(t) & \dots & a_d'^{m_d,m1}(t) \end{pmatrix}$$

where

$$a_i'^{k,j}(t) = \frac{a_i^k(t)}{a_i^j(t)} - \frac{a_1^k(t)}{a_1^j(t)}.$$

Doing this again we define inductively in $l$

$$a_j^{(l)k_{l+1},k_l,\ldots,k_1}(t) = \frac{a_j^{(l-1)k_{l+1},k_{l-1},\ldots,k_1}(t)}{a_j^{(l-1)k_l,\ldots,k_1}(t)} - \frac{a_l^{(l-1)k_{l+1},k_{l-1},\ldots,k_1}(t)}{a_l^{(l-1)k_l,\ldots,k_1}(t)}$$

for $j > l$ if $a_j^{(l-1)k_l,\ldots,k_1}(t) \neq 0$ for every $j$.

By induction is easy to see that given $k_1 < k_2 < \ldots k_l$ we have that $\lim \frac{1}{n} \log a_j^{(l)n,k_l,\ldots,k_1}(t) = \lambda_j$.

So let us define $n_1$ such that $a_j^{n_1}(t) \neq 0$ for every $j$ in a set $K_1 \subset K$ with $\mu(K_1) > 1 - \frac{\delta}{M}$. Then define $n_2 > n_1$ such that $a_j^{n_2}(t) \neq 0$ and $a_j'^{n_2,n_1}(t) \neq 0$ in a set $K_2 \subset K_1$ with $\mu(K_2) > 1 - \frac{2\delta}{M}$.
So inductively, given $n_1 < n_2 < \ldots n_l$ and $K_l$ we define $n_{l+1}$ such that $a_j^{(k)n_{l+1},n_{i_k},\ldots n_{i_1}}(t) \neq 0$ for every choice of $\{n_{i_1} < \ldots n_{i_k}\} \subset \{n_1 < n_2 < \cdots < n_l\}$ in a set $K_{l+1} \subset K_l$ with $\mu(K_{l+1}) > 1 - \frac{(l+1)\delta}{M}$. So for every choice of $n_{k_1} < n_{k_2} < \cdots < n_{k_d}$ we have that

$$\det(B(t)) = a_1^{n_{k_1}}(t) \ldots a_d^{(d)n_{k_d},n_{k_{d-1}},\ldots,n_{k_1}}(t)$$

$\square$

Using this Lemma, we prove that the matrix $B(t)$ is invertible, so the only solution is the trivial one.

*Proof of Theorem 5.1.1.* Now to prove the theorem let $2\delta < \gamma$ and $C > 0$ such that $C(\gamma - 2\delta) > 1$ and take the sequence of integers $I = \{n_1, n_2, \ldots, n_{CN}\}$ given by the Lemma 5.2.2. Then we have that any intersection of any $N$ combination of the $W_t^k$, $k \in I$ is empty for every $t \in \tilde{K}$, with $\mu^c(K) > 1 - \delta$.

It is easier to look first the case when $\hat{m}_{\hat{p}}(\mathrm{graph}\,\mathfrak{H} W_{\hat{p}}^j) = \gamma$. For this we have

$$\hat{m}_{\hat{p}}\Big(\bigcup_{j \in I} \mathrm{graph}\,\mathfrak{H}\, W_{\hat{p}}^j\Big) \geq \frac{1}{N} \sum_I \hat{m}_{\hat{p}}\Big(\mathrm{graph}\,\mathfrak{H} W_{\hat{p}}^j|_{\tilde{K}}\Big) \geq C(\gamma - \delta).$$

A contradiction because the measure is a probability.

If not, as almost every $(x^s, p) \in W_{loc}^s(\hat{p})$ have $\hat{m}_{(x^s,p)}(W_\eta^{n_j}) = \gamma$ we can take a sequence of $(x_k^s, p) \to \hat{p}$ with such property.

We also can find a system of equations $B_k(t)$ as above with the coefficients of $W_{(x^s,p)_k}^{n_i}$. Using that $\mathcal{H}_{(x_k^s,z),(x_k^s,p)}^u$ converges uniformly to $\mathcal{H}_{\hat{z},\hat{p}}^u$, and using the lemma 4.3.3 we can find a sub-sequence such that $\lim_{k\to\infty} W_{(x_k^s,p)}^{n_i}(t) = W_{\hat{p}}^{n_i}(t)$ for $\mu^c$-almost every $t \in K$. This proves that the coefficients of $B_k$ converges almost everywhere to $B$.

As $\det(B(t)) \neq 0$ in $\tilde{K}$, taking $\beta > 0$ small enough we can reduce $\tilde{K}$ a little such that $|\det(B(t))| > \beta > 0$. Then reducing $\tilde{K}$ a little more, with measure still $\mu^c(\tilde{K}) > 1 - 2\delta$, such that the convergence of $B_k$ is uniform (Egorof theorem), we have that in $\tilde{K}$ the intersection of $N$ of the $W^{n_i}_{(x^s_k,p)}(t)$ is empty. So we have as before

$$\begin{aligned}\hat{m}_{(x^s_k,p)}\Big(\bigcup_{j\in I}\operatorname{graph}\mathfrak{H}\,W^j_{(x^s_k,p)}\Big) &\geq \frac{1}{N}\sum_I \hat{m}_{(x^s_k,p)}\Big(\operatorname{graph}\mathfrak{H}W^j_{(x^s_k,p)}|_{\tilde{K}}\Big)\\ &\geq C(\gamma - 2\delta).\end{aligned}$$

Again a contradiction because the measure is a probability. □

## 5.3 Convergence to Dirac measures

The goal of this section is to prove the following theorem:

**Theorem 5.3.1.** *There exists a measurable map $\xi : \hat{M} \to \operatorname{Grass}(l,d)$ such that, given any u-state $\hat{m}$ on $\hat{M} \times \operatorname{Grass}(l,d)$, we have*

$$\hat{m}_{\hat{x},t} = \delta_{\xi(\hat{x},t)} \quad \textit{for } \hat{\mu}\textit{-almost every } (\hat{x},t) \in \hat{M}.$$

*In particular, there exists an unique u-state.*

### 5.3.1 Quasi-projective maps

We begin by recalling the notion of *quasi-projective map*, which was introduced by Furstenberg [15] and extended by Gol'dsheid-Margulis [17]. See also see Section 2.3 of [3] for a related discussion.

Let $v \mapsto [v]$ be the canonical projection from $\mathbb{C}^d$ minus the origin to the projective space $\mathbb{C}P^{d-1}$. We call $P_\# : \mathbb{C}P^{d-1} \to \mathbb{C}P^{d-1}$ a *projective map* if there is some $P \in GL(d,\mathbb{C})$ that induces $P_\#$ through $P_\#([v]) = [P(v)]$. The space of projective maps has a natural compactification, the space of *quasi-projective maps*, defined as follows. The quasi-projective map $Q_\#$ induced in $\mathbb{C}P^{d-1}$ by a non-zero, possibly non-invertible, linear map $Q : \mathbb{C}^d \to \mathbb{C}^d$ is given by

$$Q_\#([v]) = [Q(v)].$$

Observe that $Q_\#$ is well defined and continuous on the complement of the *kernel* $\operatorname{Ker} Q_\# = \{[v] : v \in \operatorname{Ker} Q\}$.

More generally, one calls $P_\# : \operatorname{Grass}(l,d) \to \operatorname{Grass}(l,d)$ a *projective map* if there is $P \in GL(d,\mathbb{C})$ that induces $P_\#$ through $P_\#(\xi) = P(\xi)$. Furthermore, the quasi-projective map $Q_\#$ induced in $\operatorname{Grass}(l,d)$ by a non-zero, possibly non-invertible, linear map $Q : \mathbb{C}^d \to \mathbb{C}^d$ is given by

$$Q_\#\xi = Q(\xi).$$

Observe that $Q_\#$ is well defined and continuous on the complement of the *kernel* $\operatorname{Ker} Q_\# = \{\xi \in \operatorname{Grass}(l,d) : \xi \cap \operatorname{Ker} Q \neq \{0\}\}$.

The space of quasi-projective maps inherits a topology from the space of non-zero linear maps, through the natural projection $Q \mapsto Q_{\#}$. Clearly, every quasi-projective map $Q_{\#}$ is induced by some linear map $Q$ such that $\|Q\| = 1$. It follows that the space of quasi-projective maps on any $\mathrm{Grass}(l,d)$ is compact for this topology.

The following two lemmas are borrowed from Section 2.3.

**Lemma 5.3.2.** *The kernel* $\operatorname{Ker} Q_{\#}$ *of any quasi-projective map is contained in some hyperplane of* $\mathrm{Grass}(l,d)$.

**Lemma 5.3.3.** *If* $(P_n)_n$ *is a sequence of projective maps converging to some quasi-projective map* $Q$ *of* $\mathrm{Grass}(l,d)$*, and* $(\nu_n)_n$ *is a sequence of probability measures in* $\mathrm{Grass}(l,d)$ *converging weakly to some probability* $\nu$ *with* $\nu(\operatorname{Ker} Q) = 0$*, then* $(P_n)_*\nu_n$ *converges weakly to* $Q_*\nu$.

### 5.3.2 Convergence

Recall that, given $1 \le l \le d-1$ and $1 \le i_1 < \cdots < i_l \le d$, we write

$$E^{i_1,\dots,i_l}(t) = E^{i_1}(t) \wedge \cdots \wedge E^{i_l}(t) \in \Lambda^l\left(\mathbb{C}^d\right)$$

for every $t \in K$ such that the Oseledets subspaces $E^i_t$ are defined. We give ourselves the right to denote by $E^{i_1,\dots,i_l}_t$ also the direct sum

$$E^{i_1}(t) \oplus \cdots \oplus E^{i_l}(t) \in \mathrm{Grass}(l,d).$$

Thus each $E^{i_1,\dots,i_l}$ is an element of $\mathrm{Sec}(K, \mathrm{Grass}(l,d))$.

Let $\hat p \in \hat\Sigma$ be the fixed point of $\hat\sigma$ and $\hat z \in \hat\Sigma$ be a homoclinic point of $\hat p$ with $\hat z \in W^u_{loc}(\hat p)$. Fix $\imath \in \mathbb{N}$ such that $\hat\sigma^{\imath}(\hat z) \in W^s_{loc}(\hat p)$. For each $k \ge 0$, denote $\hat z_k = \hat\sigma^{-k}(\hat z)$ and $z_k = P(\hat z_k)$. Observe that $\hat f_{\hat z_k} = f_{z_k}$ and, similarly, $\hat A(\hat p, t) = A(p,t)$. We take advantage of this fact to simplify the notations a bit in the arguments that follow.

**Proposition 5.3.4.** *Let* $\eta = \mathcal{H}^u_{\hat p,\hat z} E^{1,\dots,l} \in \mathrm{Sec}(K, \mathrm{Grass}(l,d))$. *For every sequence* $(k_j)_j \to \infty$ *there exists a sub-sequence* $(k'_i)_i$ *such that*

$$\lim_{i\to\infty} A^{k'_i}\big(z_{k'_i}, t_{k'_i}\big)_* m_{z_{k'_i},t_{k'_i}} = \delta_{\eta(t)}, \ \textit{where } t_k = (f^k_{z_k})^{-1}(t),$$

*for* $\mu^c$*-almost every* $t \in K$.

*Proof.* We have that

$$\begin{aligned} f^k_{z_k} &= h^u_{\hat p,\hat z} \circ f^k_p \circ h^u_{\hat z_k,\hat p} \text{ and} \\ A^k(z_k,t_k) &= H^u_{(\hat p, h^u_{\hat z,\hat p}(t)),(\hat z,t)}\, A^k(p, h^u_{\hat z_k,\hat p}(t_k))\, H^u_{(\hat z_k,t_k),(\hat p,h^u_{\hat z_k,\hat p}(t_k))}. \end{aligned}$$

So $\left(A^k_{z_k,t_k}\right)_* m_{z_k,t_k}$ is equal to

$$\big(H^u_{(\hat p, h^u_{\hat z,\hat p}(t)),(\hat z,t)}\, A^k(p, h^u_{\hat z_k,\hat p}(t_k))\big)\big(H^u_{(\hat z_k,t_k),(\hat p,h^u_{\hat z_k,\hat p}(t_k))}\big)_* m_{z_k,t_k}.$$

Note that $H^u_{(\hat{z}_k,t_k),(\hat{p},h^u_{\hat{z}_k,\hat{p}}(t_k))}$ converges uniformly to the identity map id, because $\hat{z}_k$ converges to $\hat{p}$.

Let $K_0 \subset K$ be a full $\mu^c$-measure such that the conclusion of the Oseledets theorem holds at $(\hat{p}, t)$ for every $t \in K_0$. We claim that for any $t \in K_0$ and every sub-sequence of

$$A^k(p, h^u_{\hat{z}_k,\hat{p}}(t_k))$$

that converges, the limit is a quasi-projective transformation $Q_\#$ that maps every point outside $\operatorname{Ker} Q_\#$ to $E^{1,\dots,l}(h^u_{\hat{z}_k,\hat{p}}(t_k)) \in \operatorname{Grass}(l,d)$. This can be seen as follows.

Given $w \in \Lambda^l\left(\mathbb{C}^d\right)$ and $k \geq 1$, we may write

$$w = \bigoplus_{1\leq i_1<\cdots<i_l\leq d} w_k^{i_1,\dots,i_l} E^{i_1,\dots,i_l}\left((f_p^k)^{-1} h_{\hat{z},\hat{p}}(t)\right)$$

with coefficients $w_k^1, \dots, w_k^N \in \mathbb{C}$. It follows from the Oseledets theorem that $k \mapsto w_k^i$ is sub-exponential for every $i = 1, \dots, N$. Recall that $(f_p^k)^{-1}(h(t)) = h_k(t_k)$. Then, the action of $A^k(p, h_k(t_k))$ in the projectivization of the exterior power is given by

$$A^k(p, h_k(t_k))w = \bigoplus_{j=1}^{N} w_k^j \frac{\left\|A^k(p, (f_p^k)^{-1}(h(t)) E^{I_j}_{(f_p^k)^{-1}(h(t)}\right\|}{\left\|A^k(p, (f_p^k)^{-1}(h(t))\right\|} E^{I_j}_{h(t)}.$$

The quotient of the norms converges to zero for any $j > 1$. Thus, either $A^k(p, h_k(t_k))w \to E^{I_1}_{h(t)}$ or $A^k(p, h_k(t_k))w \to 0$. The latter case means that $w$ is in the kernel of the limit. Thus, any limit quasi-projective transformation does map the complement of the kernel to $E^{I_1}_{h(t)}$, as claimed.

As an immediate consequence we get that for any $t \in K_0$ and every sub-sequence of

$$H^u_{(\hat{p},h(t))),(\hat{z},t)}\, A^k(p, h_k(t_k))$$

that converges, the limit is a quasi-projective transformation that maps every point outside the kernel to $H^u_{(\hat{p},h(t))),(\hat{z},t)} E^{I_1}_{h(t)}$.

By Remark 3.1.1, the family $\{f^n_{z_k} : n, k \geq 1\}$ is equicontinuous. Using Arzela-Ascoli, it follows that we can find a sub-sequence of $(k_j)_j$ along which the family $(f^k_{z_k})^{-1}$ converges to some $g : K \mapsto K$. Then, by Proposition 4.3.5, there exists a further subsequence $(k'_i)_i$ and a full $\mu^c$-measure set $K_1 \subset K$ such that

$$m_{z_{k'_i},t_{k'_i}} \to m_{p,g(t)}$$

for every $t \in K_1$.

By Proposition 5.1.1 and Lemma 4.3.4, there exists a full $\mu^c$-measure set $K_2 \subset K$ such that $m_{p,g(t)}$ gives zero weight to every hyperplane of $\operatorname{Grass}(l,d)$ for every $t \in K_2$. Then, by Lemma 5.3.3 and the previous observations,

$$\lim_{k\to\infty} A^k(z_k, t_k)_* m_{z_k,t_k} = \delta_{\eta(t)}$$

along any sub-sequence such that $A^k(z_k, t_k)$ converges. This yields the claim of the proposition. $\square$

**Remark 5.3.5.** Actually, the same argument shows that this proposition is true for every $w \in W^u(\hat{p})$.

It follows from Proposition 4.1.2 that there is a full $(\mu^s \times \mu^u)$-measure subset of points $\hat{x} \in \hat{\Sigma}$ such that

$$\lim_{n\to\infty} A^n(x_n, t_n^{\hat{x}})_* m_{x_n, t_n^{\hat{x}}} = \hat{m}_{\hat{x},t} \tag{5.3}$$

for $\mu^c$-almost every $t \in K$, $x_n = P\big(\hat{\sigma}^{-n}(\hat{x})\big)$ and $t_n^{\hat{x}} = (f_{x_n}^n)^{-1}(t)$. Since the shift is ergodic with respect to the projection of $\hat{\mu}$ on $\hat{\Sigma}$, one may also require that

$$\lim_{j\to\infty} \hat{\sigma}^{-n_j}(\hat{x}) = \hat{z}$$

for some sub-sequence $(n_j)_j \to \infty$.

Fix any $\hat{x} \in \hat{\Sigma}$ such that both conditions hold. Let $k \geq 1$ be fixed, for the time being. Then (5.3) implies that

$$\begin{aligned}
&\lim_{j\to\infty} A^{n_j}(x_{n_j}, t_{n_j}^{\hat{x}})_* m_{x_{n_j}, t_{n_j}^{\hat{x}}} \\
&\qquad = \lim_{j\to\infty} A^{n_j+k}(x_{n_j+k}, t_{n_j+k}^{\hat{x}})_* m_{x_{n_j+k}, t_{n_j+k}^{\hat{x}}} \\
&\qquad = \lim_{j\to\infty} A^{n_j}(x_{n_j}, t_{n_j}^{\hat{x}})_* A^k(x_{n_j+k}, t_{n_j+k}^{\hat{x}})_* m_{x_{n_j+k}, t_{n_j+k}^{\hat{x}}}.
\end{aligned} \tag{5.4}$$

Note also that, by definition,

$$t_{n_j}^{\hat{x}} = f_{x_{n_j}+k}^k(t_{n_j+k}^{\hat{x}}).$$

We use once more the fact that $\{\hat{f}_{\hat{x}}^n : n \in \mathbb{Z} \text{ and } \hat{x} \in \hat{\Sigma}\}$ is equicontinuous (Remark 4.2.5). Using Ascoli-Arzela, it follows that there exists a sequence $(n_j)_j \to \infty$ such that $(f_{x_{n_j}}^{n_j})_j^{-1}$ converges to some $g : K \to K$. Up to further restricting to a sub-sequence if necessary, Proposition 4.3.5 ensures that

$$m_{x_{n_j+k}, t_{n_j+k}^{\hat{x}}} \text{ converges to } m_{z_k, g(t)_k^{\hat{z}}} \text{ for } \mu^c\text{-almost every } t.$$

Fix any $t \in K$ such that the previous claims are fulfilled. Let $(n_i')_i$ be any sub-sequence of $(n_j)_j$ such that $A^{n_i'}\big(x_{n_i'}, t_{n_i'}^x\big)$ converges to some quasi-projective map $Q : \text{Grass}(l, d) \to \text{Grass}(l, d)$. Then (5.4) may be written as

$$Q_* A^k\big(z_k, g(t)_k^{\hat{z}}\big)_* m_{z_k, g(t)_k^{\hat{z}}}$$

If $\eta(g(t)) \notin \operatorname{Ker} Q$ then, making $k \to \infty$, we may use Lemma 5.3.3 and Proposition 5.3.4 to conclude that $\hat{m}_{x,t} = \delta_{Q\,\eta(g(t))}$. This gives the conclusion of Theorem 5.3.1 under this assumption.

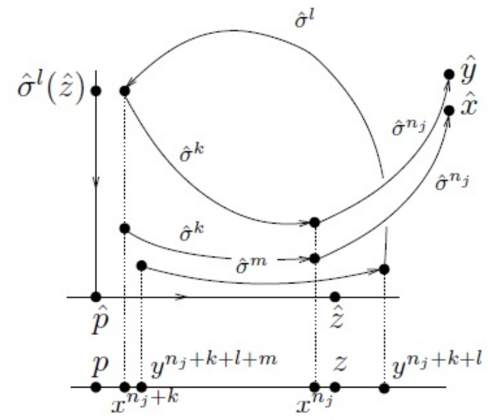


Figure 5.1: Proof of Theorem 5.3.1: avoiding the kernel $\operatorname{Ker} Q$

Let us prove that we can always reduce the proof to this case. Recall that $\imath \in \mathbb{Z}$ was chosen such that $\hat\sigma^\imath(\hat z) \in W^s_{loc}(\hat p)$. Define $\hat y \in \hat\Sigma$ by

$$\hat\sigma^{-n_j-k}(\hat y) \in W^u_{loc}(\hat\sigma^\imath(\hat z)) \cap W^s_{loc}(x_{n_j+k})$$
$$\hat\sigma^\imath(\hat w) \in W^u_{loc}(\hat\sigma^\imath(\hat z)) \cap W^s_{loc}(z_k).$$

Note that $\hat y$ depends on $k$ and $j$ and $\hat w$ just depends on $k$. We denote $y = P(\hat y)$ and $w = P(\hat w)$. Moreover, $y_n = P(\hat\sigma^{-n}(\hat y))$ and $w_n = P(\hat\sigma^{-n}(\hat w))$ for each $n \geq 0$. Let $m \in \mathbb{N}$ be fixed, for the time being. We have that $x_i = y_i$ with $0 \leq i \leq n_j + k$. So,

$$\hat\sigma^{\imath+m}\big(P(\hat\sigma^{-n_j-k-\imath-m}(\hat y))\big) = y_{n_j+k} = x_{n_j+k}.$$

Also $\hat\sigma^{-n_j-k}(y) \to \hat\sigma^\imath(\hat w)$, and so

$$\hat\sigma^{-n_j-k-\imath-m}(y) \to \hat\sigma^{-m}(\hat w) \text{ when } j \to \infty.$$

Therefore, by Proposition 4.1.2 part (b),

$$\begin{aligned}\hat m_{\hat x,t} &= \lim_{j\to\infty} A^{n_j+k}\big(x_{n_j+k}, t^x_{n_j+k}\big)_* m_{x_{n_j+k}, t^x_{n_j+k}} \\ &= \lim_{j\to\infty} A^{m_j}\big(y_{m_j}, t^{\hat y}_{m_j}\big)_* m_{y_{m_j}, t^{\hat y}_{m_j}}\end{aligned}$$

where $m_j = n_j + k + \imath + m$. The last expression may be rewritten as

$$A^{n_j}\big(x_{n_j}, t^{\hat x}_{n_j}\big)_* A^{k+\imath}\big(y_{n_j+k+\imath}, t^{\hat y}_{y_{n_j+k+\imath}}\big)_* A^m\big(y_{m_j}, t^{\hat y}_{m_j}\big)_* m_{y_{m_j}, t^{\hat y}_{m_j}}.$$

Making $j \to \infty$,

$$\begin{aligned}
&\left(f_{y_{n_j+k+\imath}}^{n_j+k+\imath}\right)^{-1} \to \left(f_{\hat{w}}^{k+\imath}\right)^{-1} \circ g \\
&A^{k+\imath}\left(y_{n_j+k+\imath}, t_{y_{n_j+k+\imath}}^{\hat{y}}\right) \to A^{k+\imath}\left(w, \left(f_w^{k+\imath}\right)^{-1} g(t)\right) \\
&A^m\left(y_{m_j}, t_{m_j}^{\hat{y}}\right) \to A^m\left(w_m, \left(f_{w_m}^{k+\imath+m}\right)^{-1} g(t)\right)
\end{aligned}$$

and, restricting to a sub-sequence if necessary,

$$m_{y_{m_j}, t_{m_j}^{\hat{y}}} \to m_{w_m, \left(f_{w_m}^{k+\imath+m}\right)^{-1} g(t)} \text{ for } \mu^c\text{-almost every } t.$$

**Lemma 5.3.6.** *Denote $\tilde{\eta}(s) = H^u_{(\hat{p}, \tilde{h}(s)), (\hat{w}, s)} E^{I_1}_{\tilde{h}(s)}$ with $\tilde{h}(s) = h^u_{\hat{w}, \hat{p}}(s)$. Then there exists a full $\mu^c$-measure set $\tilde{K} \subset K$ such that for every $t \in \tilde{K}$ there exists a sub-sequence of $(n_j)_j(t)$ such that*

$$A^{n_j}\left(x_{n_j}, t_{n_j}^{\hat{x}}\right) \circ A^{k+\imath}\left(y_{n_j+k+\imath}, t_{y_{n_j+k+\imath}}^{\hat{y}}\right)$$

*converges to some quasi-projective transformation $\tilde{Q}$ such that the point $\tilde{\eta}\big((f_w^{k+\imath})^{-1} g(t)\big)$ is not in $\operatorname{Ker} \tilde{Q}$ for any $k = k(t)$ sufficiently large.*

*Proof.* As before denote by $h = h^u_{\hat{z}, \hat{p}}$ and $h_k = h^u_{\hat{z}_k, \hat{p}}$.

Observe that $(f_w^{k+\imath})^{-1} = \left(f_w^{\imath}\right)^{-1}\left(f_{z_k}^k\right)^{-1}$, and $\hat{w} \to \hat{z}$ when $k \to \infty$. First let us take a sub-sequence of $k$ such that $\left(f_{z_k}^k\right)^{-1}$ converges uniformly to some $\phi$, and observe that $(f_p^k)^{-1} h = h_k (f_{z_k}^k)^{-1}$ converges uniformly to some $\phi$, again by lemma 4.3.4 the $\phi$ is absolutely continuous with respect to $\mu^c$.

Remember that

$$\tilde{\eta}\big((f_w^{k+\imath})^{-1} g(t)\big) = H^u_{(\hat{p}, \tilde{h}((f_w^{k+\imath})^{-1})), (\hat{w}, (f_w^{k+\imath})^{-1})} E^{I_1}_{\tilde{h}((f_w^{k+\imath})^{-1})}$$

with $\tilde{h}(s) = h^u_{\hat{w}, \hat{p}}(s)$.

Now, using Lemma 4.3.3, take a sub-sequence and a total $\mu^c$ measure subset by such that

$$\tilde{\eta}\big((f_w^{k+\imath})^{-1} g(t)\big) \text{ converges to } \eta\big(\left(f_z^{\imath}\right)^{-1} \phi g(t)\big)$$

and

$$E((f_p^k)^{-1} h g(t)) \text{ converges to } E(\phi g(t))\big) \tag{5.5}$$

in a total $\mu^c$-measure subset.

Moreover, the twisting condition implies that

$$\mathcal{H}^u_{\hat{z}, \hat{p}} \mathcal{F}_z^{\imath} E_t^{I_1} \cap \left(E_t^{j_{l+1}} + \cdots + E_t^{j_d}\right) = \{0\} \tag{5.6}$$

for any $j_{l+1}, \ldots, j_d \in \{1, \ldots, d\}$ and a full $\mu^c$-measure set of values of $t \in K$. In other words, $A^{\imath}(\eta(h(t))$ does not belong to any of the the hyperplanes of

$\mathrm{Grass}(l,d)$ determined by the Oseledets decomposition at the point $(\hat{p},t)$. Take a total measure subset such that (5.6) is satisfied for $\phi(g(t))$.

Take $\tilde{K}\subset K$ as the intersection of the two full $\mu^c$-measure sets in the previous paragraph.

Fix any $t\in\tilde{K}$ such that in addition $(\hat{p},hg(t))$ satisfies the conclusion of the Oseledets theorem. As before, consider any sub-sequence $(n_i')_i$ of $(n_j)_j$ such that $A^{n_i'}(x_{n_i'},t^{\hat{x}}_{n_i'})$ converges to some quasi-projective transformation $Q$ when $i\to\infty$. Then

$$A^{n_i'}\big(x_{n_i'},t^{\hat{x}}_{n_i'}\big)\circ A^{k+\imath}\big(y_{n_i'+k+\imath},t^{\hat{y}}_{y_{n_i'+k+\imath}}\big)$$

converges to $\tilde{Q}=Q\circ A^{k+\imath}\big(w,\big(f_w^{k+\imath}\big)^{-1}g(t)\big)$ when $i\to\infty$. Moreover,

$$\begin{aligned}\operatorname{Ker}\tilde{Q}&=A^{k+\imath}\big(w,\big(f_w^{k+\imath}\big)^{-1}g(t)\big)^{-1}\operatorname{Ker}Q\\&=A^{\imath}\big(w,\big(f_w^{k+\imath}\big)^{-1}g(t)\big)^{-1}A^{k}\big(z_k,\big(f_{z_k}^{k}\big)^{-1}g(t)\big)^{-1}\operatorname{Ker}Q.\end{aligned}$$

Next, observe that

$$A^{k}\big(z_k,\big(f_{z_k}^{k}\big)^{-1}g(t)\big)^{-1}=\Theta_k\,A^{-k}(p,hg(t))\,\Theta \tag{5.7}$$

where $h=h^u_{\hat{p},\hat{z}}$ and

$$\Theta=H^u_{(\hat{z},g(t)),(\hat{p},hg(t))}\text{ and }\Theta_k=H^u_{(\hat{p},(f_p^k)^{-1}hg(t)),(\hat{z}_k,(f^k_{z_k})^{-1}(g(t))}.$$

By Lemma 5.3.2, the kernel of $Q$ is contained in some hyperplane $\mathfrak{H}v$ of $\mathrm{Grass}(l,d)$. Hence, $\Theta(\operatorname{Ker}Q)$ is contained in the hyperplane $\Theta(\mathfrak{H}v)$, of course. Since we take $t\in K$ to be such that the Oseledets theorem holds at $(\hat{p},t)$, the backward iterates $A^{-k}(p,hg(t))\Theta(\mathfrak{H}v)$ are exponentially asymptotic to some hyperplane section $\mathfrak{H}E$ that is defined by a $(d-l)$-dimensional sum $E$ of Oseledets subspaces. This remains true for $\Theta_kA^{-k}(p,hg(t))\Theta(\mathfrak{H}v)$ because $\Theta_k$ converges exponentially fast to to the identity map, since $\hat{z}_k$ converges to $\hat{p}$. In other words, using (5.7),

$$\mathrm{dist}_{\mathrm{Grass}(l,d)}\left(A^{k}\big(z_k,\big(f_{z_k}^{k}\big)^{-1}g(t)\big)^{-1}\mathfrak{H}v,\mathfrak{H}E((f_p^k)^{-1}hg(t))\right)$$

goes to zero exponentially fast as $k\to\infty$. Then by (5.5) we have that $A^{k}\big(z_k,\big(f_{z_k}^{k}\big)^{-1}g(t)\big)^{-1}\mathfrak{H}v$ converges to $E(\phi g(t))$. Then we have that $\operatorname{Ker}\tilde{Q}\subset A^{\imath}\big(z,(f_z^{\imath})^{-1}\phi g(t)\big)^{-1}\mathfrak{H}E(\phi g(t))$.

Recall (Section 5.2) that $\hat{z}$ was chosen in $W^u_{loc}(\hat{p})$ and $\imath\in\mathbb{N}$ is such that $\hat{\sigma}^{\imath}(\hat{z})\in W^s_{loc}(\hat{p})$. Recall also (from Section 4.1) that in the present setting all the local stable holonomies $h^s$ and $H^s$ are trivial. In particular, (3.6) means that

$$V^i(t')=H^u_{(\hat{z},t_1),(\hat{p},t')}\hat{A}^{-\imath}(\hat{\sigma}^{\imath}(\hat{z}),s)E^i(s)=H^u_{(\hat{z},t_1),(\hat{p},t')}\hat{A}^{\imath}(\hat{z},t_1)^{-1}E^i(s)$$

with $t' = \phi g(t)$, $t_1 = h^u_{\hat{p},\hat{z}}(t')$ and $s = \hat{f}^{\imath}_{\hat{z}}(t_1)$, then by the twisting condition $\oplus_{j\in J} V^j$ and any sum $\oplus_{i\in I} E^i$ with $\#I + \#J = d$ do not intersect. In particular, the distance between

$$H^u_{(\hat{z},t_1),(\hat{p},t')}\hat{A}^{-\imath}(\hat{\sigma}^{\imath}(\hat{z}),s)E(s) \quad \text{and} \quad E^{I_1}(s)$$

is positive. Equivalently, the distance between

$$\hat{A}^{-\imath}(\hat{\sigma}^{\imath}(\hat{z}),s)E(s) \quad \text{and} \quad \eta(t_1) = H^u_{(\hat{p},t'),(\hat{z},t_1)}E^{I_1}(s)$$

is positive. Then $\eta\big(\big(f^{\imath}_{z}\big)^{-1}\phi g(t)\big)$ does not intersect

$$\hat{A}^{-\imath}(\hat{\sigma}^{\imath}(\hat{z}),\big(\phi g(t)\big))E\big(\phi g(t)\big) = A^{\imath}\big(z,(f^{\imath}_{z})^{-1}\phi g(t)\big)^{-1}E(\phi g(t)),$$

which implies that $\eta\big(\big(f^{\imath}_{z}\big)^{-1}\phi g(t)\big) \notin \operatorname{Ker}\tilde{Q}$.

So for every $k$ sufficiently large and the observations at the beginning of the proof the results follows. $\square$

Having established Lemma 5.3.6, we can now use the same argument as previously, to conclude that $\hat{m}_{\hat{x},t} = \delta_{\tilde{Q}\eta}$ at $\mu^c$-almost every point also in this case. To do this, observe that for every $m$ and $k$ fixed there exist a sub-sequence and a subset of total measure, such that

$$m_{y_{m_j},t^{\hat{y}}_{m_j}} \to m_{w_m,\left(f^{k+\imath+m}_{w_m}\right)^{-1}g(t)} \text{ for } \mu^c\text{-almost every } t. \tag{5.8}$$

Then intersecting these sets with the ones given by Lemma 5.3.6 and Proposition 5.3.4 (for every $\hat{w}$ as in the remark) we get a total measure subset $K' \subset K$ and using a diagonal argument a sub-sequence of $j$, such that (5.8) is true for every $t \in K'$, $k$ and $m$, that we fix from now on.

Taking $t \in K'$ and using Lemma 5.3.6 a further sub-sequence $j' \to \infty$ and $k$ we get

$$\hat{m}_{\hat{x},t} = \tilde{Q}_*\big(A^m(w_m,(f^{k+\imath}_{w_m})^{-1}g(t))\big)_* m_{w_m,(f^{k+\imath}_{w_m})^{-1}g(t)}.$$

Then making $m \to \infty$ by Lemma 5.3.3 and Proposition 5.3.4 we get

$$\hat{m}_{\hat{x},t} = \delta_{\xi(\hat{x},t)},$$

where $\xi(\hat{x},t) = \tilde{Q}\tilde{\eta}((f^{k+\imath}_{w})^{-1}g(t)$.

To finish the proof of Theorem 5.3.1, let $\tilde{M} \subset \hat{M}$ be the set of $(\hat{x},t) \in \hat{M}$ such that $\hat{m}_{(\hat{x},t)}$ is a Dirac measure. Observe that $(\hat{x},t) \mapsto \hat{m}_{(\hat{x},t)}$ is measurable and the set of Dirac measures in the weak* topology is closed, then $\tilde{M}$ is measurable.

Thus we proved that $\hat{m}_{\hat{x},t}$ is a Dirac measure for $\hat{\nu}$-almost every $\hat{x} \in \hat{\Sigma}$ and $\hat{\mu}^c_{\hat{x}}$-almost every $t \in K$ . This implies that $\tilde{M}$ has total $\hat{\mu}$ measure. So, we have completed the proof of Theorem 5.3.1.

CHAPTER 6

# Eccentricity

We use the following notation: given $V \in \mathrm{Grass}(l,d)$ and $\epsilon > 0$ we call $C_\epsilon(V)$ the cone of size $\epsilon$ with axis $V$. Given $\delta > 0$ and a hyperplane section $H$ we call $H_\delta \subset \mathrm{Grass}(l,d)$ the $\delta$-neighborhood of $H$.

We will need to use the next lemma, which proof is in [3]:

**Lemma 6.0.1.** *Given $C \geq 1$ and $\delta > 0$ there exists $\epsilon > 0$ such that, for any $V \in \mathrm{Grass}(l,d)$ and any diagonal operator $D$ with eccentricity $E(l,D) \leq C$, one may find a hyperplane section $H$ of $\mathrm{Grass}(l,d)$ such that $D^{-1}(C_\epsilon(V)) \subset H_\delta$.*

**Lemma 6.0.2.** *Let $\mathcal{N}$ be a family of measures with the property that there exists $\delta > 0$ such that, for every hyperplane $H$, the $\delta$-neighborhood $H_\delta$ has $\nu(H_\delta) \leq \frac{1}{2}$ for every $\nu \in \mathcal{N}$. Let $L_n$ be a sequence of linear transformations such that there exist $\nu_n \in \mathcal{N}$ that $L_{n*}\nu_n \to \delta_\xi$, then*

- *The eccentricity of $L_n$ $E(l, L_n) \to \infty$*
- *The image $L_n\xi_n$ of the most expanding direction $\xi_n$ of $L_n$ converges to $\xi$*

*Proof.* First lets see we can assume that the $L_n$ are diagonal operators. Using the polar decomposition we can find $K_n$ and $\tilde{K}_n$ unitary operators such that $L_n = K_n D_n \tilde{K}_n$ where $D_n$ is a diagonal operator.

By definition $E(l, L_n) = E(l, D_n)$. Let $U(l,d)$ be the group of unitary operators, this group is compact. Then it is easy to see that the family $U(l,d)_*\mathcal{N}$ also satisfies the hypothesis, maybe with a smaller $\delta$. So we can replace $\nu_n$ by $\tilde{\nu}_n = (\tilde{K}_n)_*\nu_n$.

Hence $(K_n D_n)_*\tilde{\nu}_n \to \delta_\xi$. So, passing to a sub-sequence if necessary, we can assume that $D_{n*}\tilde{\nu}_n \to \delta_{\tilde{\xi}}$.

So, from now on, we assume that $L_n$ are diagonal operators. For every $\epsilon > 0$

$$\nu_n\big({L_n}^{-1}(C_\epsilon(\xi))\big) = L_{n*}\nu_n(C_\epsilon(\xi)) \to 1,$$

where $C_\epsilon(\xi)$ is the cone of width $\epsilon$ around $\xi$ then there exist an $n$ such that ${L_n}^{-1}(C_\epsilon(\xi))$ is not contained in any $H_\delta$.

By lemma 6.0.1 we have that $E(l, L_n) \to \infty$.

For the second statement, using the first part, we know that the eccentricity goes to $\infty$. Then given any fixed width $\epsilon > 0$ we can find $\beta_n \to \infty$ such that

$$L_n\big(C_{\beta_n}(\xi_n)\big) \subset C_\epsilon(L_n(\xi_n))$$

This implies that the $L_{n*}\nu_n$ mass of the $\epsilon-$neighborhood of $L_n\xi_n$ converges to 1 as $n \to \infty$. Since $L_{n*}\nu_n \to \delta_\xi$, this implies that $L_n(\xi_n) \to \xi$ $\square$

For almost every $(\hat{x}, t) \in \hat{M}$ we know that $m_{P(\hat{x},t)}(H) = 0$ for every hyperplane $H$, then for $(\hat{x}, t)$ there exists $\delta > 0$ such that $m_{x,t}(H_\delta) \leq \frac{1}{2}$. Now let $M_\delta$ be the set of points in $M$ such that the last property is realized by $\delta_0 \geq \delta$. Then $\mu(M_\delta) \to 1$ as $\delta \to 0$. Taking $\hat{M}_\delta = (P \times \mathrm{id}_K)^{-1} M_\delta$ we have the next corollary.

**Corollary 6.0.3.** *For every $0 < c < 1$, there exists a set $M_c \subset \hat{M}$ with $\hat{\mu}(\hat{M}_c) > c$ such that $E\big(l, A^n(\hat{f}^{-n}(\hat{x}, t))\big) \to \infty$, and the image of the most expanded subspace by $A^n(\hat{f}^{-n}(\hat{x}, t))$ converges to $\xi(\hat{x}, t)$, restricted to the iterates $\hat{f}^{-n}(\hat{x}, t) \in M_c$.*

## 6.1 Adjoint cocycle

Let us fix a continuous Hermitian form in $\mathbb{C}^d$ $\langle \cdot, \cdot \rangle_{(\hat{x},t)}$, $(\hat{x}, t) \in \hat{M}$.

Let $\hat{F}_{\hat{A}_*} : M \times \mathbb{C}^d \to M \times \mathbb{C}^d$ be the adjoint cocycle defined over $\hat{f}^{-1} : \hat{M} \to \hat{M}$, by

$$\hat{A}_*^n(\hat{q}) = \begin{cases} \hat{A}(f^{-n}(\hat{x}, t))^* \dots \hat{A}(f^{-1}(\hat{x}, t))^* & \text{if } n > 0 \\ Id & \text{if } n = 0 \\ \hat{A}(f^{n-1}(\hat{x}, t)){^*}^{-1} \dots \hat{A}(\hat{x}, t){^*}^{-1} & \text{if } n < 0 \end{cases}$$

for $(\hat{x}, t) \in \hat{M}$. We have that $W^{ss}_{f^{-1}}(\hat{x}, t) = W^{uu}_f(\hat{x}, t)$ and $W^{uu}_{f^{-1}}(\hat{x}, t) = W^{ss}_f(\hat{x}, t)$. Also, it is easy to see that

$$H^{u,\hat{A}_*}_{(\hat{x},t),(\hat{y},s)} = (H^{s,A}_{(\hat{y},s),(\hat{x},t)})^* \text{ and } H^{s,\hat{A}_*}_{(\hat{x},t),(\hat{z},r)} = (H^{u,A}_{((\hat{z},r)),(\hat{x},t)})^*$$

for every $(\hat{x}, t), (\hat{y}, s)$ in the same $f^{-1}$-unstable set and every $(\hat{x}, t), (\hat{z}, r)$ in the same stable set.

The same hypothesis in $\hat{F}_{\hat{A}}$ are fulfills by $\hat{F}_{\hat{A}_*}$. Indeed:

**Proposition 6.1.1.** *$\hat{A}$ is simple, if and only if, $\hat{A}_*$ is simple.*

*Proof.* By theorem A.1.1 we know that $\hat{A}_*$ is pinching, if and only if, $\hat{A}$ is pinching. Also the Oseledets decomposition of the restriction of $\hat{A}_*$ is given by the orthogonal complements of the Oseledets subspaces for $A$. i.e:

$$E_*^j = \left[E^1 \oplus \cdots \oplus \hat{E^j} \oplus \cdots \oplus E^k\right]^\perp$$

where $\hat{E^j}$ is the subspace which is not in the sum.

Let $\phi_{\hat{p},\hat{z}} = \mathcal{H}^s_{\hat{z},\hat{p}} \circ \mathcal{H}^u_{\hat{p},\hat{z}}$. We have to prove that for every sum of $l$ of the Oseledets invariant subspaces the coefficients of the push forward by $\phi^*_{\hat{p},\hat{z}}$ in the Oseledets basis grows sub-exponentially.

Call $g = f_{\hat{p}}$. Take $e^j(t) \in E^j(t)$ and $e^j_*(t) \in E^j_*$, unitary vectors.

Let us prove for the case of one dimensional Oseledets subspaces, i.e: $l = 1$. Call $h : K \to K$, $h(t) = h^s_{\hat{z},\hat{p}} \circ h^u_{\hat{p},\hat{z}}$, and $H_{h^{-1}(t),t} = H^s_{(\hat{z},h^u_{\hat{p},\hat{z}}\circ h^{-1}(t)),(\hat{p},t)} \circ H^u_{(\hat{p},h^{-1}(t)),(\hat{z},h^u_{\hat{p},\hat{z}}\circ h^{-1}(t))}$, then

$$\phi_{\hat{p},\hat{z}} V(t) = H_{h^{-1}(t),t}\left(V(h^{-1}(t))\right).$$

So we have that

$$\phi^*_{\hat{p},\hat{z}} V(t) = {H_{t,h(t)}}^* \left(V(h(t))\right). \tag{6.1}$$

The twisting condition means that if

$$\phi_{\hat{p},\hat{z}} e^k(t) = \sum_{j=1}^d a_{k,j}(t) e^j(t),$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log|a_{k,j}(g^n(t))| = 0.$$

So we have to prove the same for the adjoint, i.e.

$$\phi^*_{\hat{p},\hat{z}} e^k_*(t) = \sum_{j=1}^d \beta_{k,j}(t) e^j_*(t).$$

So

$$\begin{aligned}
\beta_{k,j}(t)\left\langle e^j_*(t), e^j(t)\right\rangle &= \left\langle \phi^*_{\hat{p},\hat{z}} e^k_*(t), e^j(t)\right\rangle \\
&= \left\langle e^k_*(h(t)), \phi_{\hat{p},\hat{z}} e^j(h(t))\right\rangle \\
&= \overline{a_{j,k}(h(t))}\left\langle e^k_*(h(t)), e^k(h(t))\right\rangle,
\end{aligned}$$

by definition $\left\langle e^i_*(t), e^i(t)\right\rangle = \cos(\alpha^i(x))$ for every $1 \le i \le d$, where

$$\begin{aligned}
\alpha^i(t) &= \measuredangle\left(e^j_*(t), e^j(t)\right) \\
&= \frac{\pi}{2} - \measuredangle\left(e^j_*(t), E^1_t \oplus \cdots \oplus \hat{E^j}_t \oplus \cdots \oplus E^k_t\right)
\end{aligned}$$

so by Oseledets theorem

$$\lim_{n\to\infty}\frac{1}{n}\log|\langle e_*^i(g^n(t)),e^i(g^n(t))\rangle|=0$$

almost everywhere. Then

$$\lim_{n\to\infty}\frac{1}{n}\log|\beta_{k,j}(g^n(t))|=\lim_{n\to\infty}\frac{1}{n}\log|a_{j,k}(g^n(h(t)))|$$

also as $h:K\to K$ preserves $\mu^c_{\hat{p}}$ the equality holds in a total $\mu^c_{\hat{p}}$-measure set.

For $l>1$ the proof is exactly the same, just take the inner product $\Lambda^l\left(\mathbb{C}^d\right)$ induced by $\langle\cdot,\cdot\rangle$ i.e:

$$\langle v_1\wedge\cdots\wedge v_l,w_1\wedge\cdots\wedge w_l\rangle_{\Lambda^l\left(\mathbb{C}^d\right)}=\det\left(\langle v_i,w_j\rangle\right)$$

on $l$ vectors. Then $A$ is pinching if and only if $A_*$ is. □

Applying the corollary 6.0.3 to this cocycle we have that

- There exists a section $\xi^*:\hat{M}\to \mathrm{Grass}(l,d)$ which is invariant under the cocycle $F_{\hat{A}_*}$ and under the unstable holonomies of $\hat{A}_*$
- Given any $c>0$ there exist a $M_c$ of $\hat{\mu}\left(\hat{M}_c\right)>0$ such that restricted to the sub-sequence of iterates $\hat{f}(p)$ in $M_c$, the eccentricity $E\left(l,\hat{A}^n_*\left(\hat{f}^n(p)\right)\right)=E\left(l,A^n\left(p\right)\right)$ goes to infinity and the image $\hat{A}^n_*\left(\hat{f}^n(p)\right)\zeta^a_n\left(\hat{f}^n(p)\right)$ of the $l$-subspace most expanded tends to $\xi^*(p)$ as $n\to\infty$

**Lemma 6.1.2.** *For $\hat{\mu}$-almost every $(\hat{x},t)$, the subspace $\xi(\hat{x},t)$ is transverse to the orthogonal complement of $\xi^*(\hat{x},t)$.*

*Proof.* We may take the stable holonomies of $A$ to be trivial, this means that the unstable holonomies of $\hat{A}_*$ are trivial. So $\xi^*$ is invariant under unstable holonomies means that it is constant on local stable sets of $\hat{f}$.

Hence the same is true about its orthogonal complement, let us call this subspace by $H(\hat{x},t)$, as observed before this only depends on $x,t$ were $x=P(\hat{x})$, so the graph of $H(x,\cdot)$ over $K$ has zero $m_x$-measure.

$$\begin{aligned} m_x(\mathrm{graph}\,\mathfrak{H}\,H_x) &= \int\int\delta_{\xi(\hat{x},t)}(H(x,t))d\mu^c(t)d\mu^s_x(\hat{x})= \\ &= \mu^c\times\mu^s\left(\{\hat{x},t:\xi(\hat{x},t)\in H(x,t)\}\right)= \\ &= 0\end{aligned}$$

for almost every $x$. Then $\hat{\mu}\left(\{\hat{x},t:\xi(\hat{x},t)\in H(x,t)\}\right)=0$. This proves the lemma. □

# CHAPTER 7

## Proof of Theorem A

In this chapter we prove our first result, as mentioned before this will be proved as a Corollary of Theorem D. In this chapter we also prove Theorem E.

### 7.1 Proof of Theorem D

Let $(\hat{x}, t) \in \hat{M}$, and denote by $\eta(\hat{x}, t) \in \mathrm{Grass}(d-l, d)$ the orthogonal complement of $\xi^*(\hat{x}, t)$. $\xi^*$ is invariant under $\hat{A}_*$, which means $\hat{A}_*(\hat{x}, t)\xi^*(\hat{x}, t) = \xi^*\left(\hat{f}^{-1}(\hat{x}, t)\right)$. This implies that $\eta$ is invariant under $A$.

According to Lemma 6.1.2, we have that $\mathbb{C}^d = \xi(\hat{x}, t) \oplus \eta(\hat{x}, t)$ at almost every point $p \in M$. We want to prove that the Lyapunov exponents of $A$ along $\xi$ are strictly bigger than those along $\eta$. So let

$$\xi(\hat{x}, t) = \xi^1(\hat{x}, t) \oplus \cdots \oplus \xi^u(\hat{x}, t) \quad \text{and} \quad \eta(\hat{x}, t) = \eta^s(\hat{x}, t) \oplus \cdots \oplus \eta^1(\hat{x}, t)$$

be the Oseledets decomposition of $A$ restricted to the two invariant sub bundles, where $\xi^u$ corresponds to the smallest Lyapunov exponent among $\xi^i$, and $\eta^s$ is the largest among all $\eta^j$. Denote $d_u = \dim \xi^u$ and $d_s = \dim \eta^s$, and let $\lambda_u$ and $\lambda_s$ be the Lyapunov exponents associated to these two sub bundles, respectively. Define

$$\Delta^n(\hat{x}, t) = \frac{\det\left(A^n(\hat{x}, t), \xi^u(\hat{x}, t)\right)^{\frac{1}{d_u}}}{\det\left(A^n(\hat{x}, t), W(\hat{x}, t)\right)^{\frac{1}{d_u+d_s}}},$$

Where $W(\hat{x}, t) = \xi^u(\hat{x}, t) \oplus \eta^s(\hat{x}, t)$. Oseledets theorem gives that

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta^n(\hat{x}, t) = \frac{d_s}{d_u + d_s}\left(\lambda_u - \lambda_s\right).$$

The proof of the following proposition is identical to the proof of Proposition 7.3 in [3]:

**Proposition 7.1.1.** *For every $0 < c < 1$ there exists a set $\hat{M}_c \subset M$ with $\hat{\mu}^c(\hat{M}_c) > c$ such that for almost every $p \in M$*

$$\lim_{n\to\infty} \Delta^n(\hat{x}, t) = \infty$$

*restricted to the sub-sequence of values $n$ for which $\hat{f}^n(\hat{x}, t) \in \hat{M}_c$.*

So now fix some $0 < c < 1$ and $\hat{M}_c$ given by proposition 7.1.1. Let $g : M_c \to M_c$ be the first return map, defined by

$$g\left(\hat{x}, t\right) = \hat{f}^{r(\hat{x},t)}\left(\hat{x}, t\right).$$

Then we can define the induced cocycle $G : M_c \times \mathbb{C}^d \to M_c \times \mathbb{C}^d$

$$G\left(\left(\hat{x}, t\right), v\right) = \left(g\left(\hat{x}, t\right), D(\hat{x}, t)v\right),$$

where $D(\hat{x}, t) = \hat{A}^{r(\hat{x},t)}\left(\hat{x}, t\right)$.

We also have that the Lyapunov exponents of $G$ with respect to $\frac{\hat{\mu}}{\hat{\mu}(M_c)}$ are the products of the exponents of $\hat{F}_A$ by the average return $\frac{1}{\hat{\mu}(M_c)}$. Thus, to show that $\lambda_u > \lambda_s$, it suffices to prove it for $G$. But defining

$$\tilde{\Delta}^k = \frac{\det\left(D^k(\hat{x}, t), \xi^u(\hat{x}, t)\right)^{\frac{1}{d_u}}}{\det\left(D^k(\hat{x}, t), W(\hat{x}, t)\right)^{\frac{1}{d_u+d_s}}}.$$

Then $\tilde{\Delta}^k(\hat{x}, t)$ is a sub-sequence of $\Delta^n(\hat{x}, t)$ such that $\hat{f}^n(\hat{x}, t) \in \hat{M}_c$. So applying Proposition 7.1.1 we conclude that

$$\lim_{n\to\infty} \sum_{j=0}^{k-1} \log \tilde{\Delta}\left(g^j(\hat{x}, t)\right) = \lim_{n\to\infty} \log \tilde{\Delta}^k\left(p\right) = \infty,$$

for $\hat{\mu}$-almost every $(\hat{x}, t) \in \hat{M}_c$. We need the next well-known Lemma.

**Lemma 7.1.2.** *Let $T : X \to X$ be a measurable transformation preserving a probability measure $\nu$ in $X$, and $\varphi : X \to \mathbb{R}$ be a $\nu$-integrable function such that $\lim_{n\to\infty} \sum_{j=0}^{n-1}\left(\varphi \circ T^j\right) = +\infty$ at $\nu$-almost every point. Then $\int \varphi d\nu > 0$.*

Applying the lemma to $T = g$ and $\varphi = \log \tilde{\Delta}$ we find that

$$\lim_{k\to\infty} \frac{1}{k} \log \tilde{\Delta}^k(\hat{x}, t) = \lim_{k\to\infty} \sum_{j=0}^{k-1} \log \tilde{\Delta}\left(g^j(\hat{x}, t)\right) = \int \log \tilde{\Delta} \frac{d\hat{\mu}}{\hat{\mu}(\hat{M}_c)} > 0$$

at $\hat{\mu}$-almost every point. On the other hand the relation between Lyapunov exponents gives

$$\lim_{k\to\infty}\frac{1}{k}\log\tilde{\Delta}^k(\hat{x},t)=\frac{d_s}{d_u+d_s}\left(\lambda_u-\lambda_s\right)\frac{1}{\hat{\mu}(\hat{M}_c)}.$$

This means that $\lambda_u>\lambda_s$. So there is a gap between the first $l$ Lyapunov exponents and the remaining $d-l$ ones. Since this applies for every $1\leq l\leq d$, we conclude that the Lyapunov spectrum is simple. Proving the Theorem D.

## 7.2 Proof of Theorem E

This section is devoted to prove Theorem E.

**Lemma 7.2.1.** *The set of simple cocycles with strong pinching and twisting is non-empty.*

*Proof.* Take $A\in GL(d,\mathbb{C})$ that has $d$ diferent eigenvalues without resonance. Now take the constant cocycle $\hat{A}:\hat{M}\to GL(d,\mathbb{C})$, $\hat{A}(\hat{x},t)=A$. We have that $H^{*}_{\cdot,\cdot}=\mathrm{id}$ for every $*\in\{s,u\}$.

Let us call $h^u_{\hat{p},\hat{z}}(t)=h^u(t)$ and $h=h^s_{\hat{z},\hat{p}}\circ h^u_{\hat{p},\hat{z}}$, denote by

$$H^{\hat{A}}_t=H^{s,\hat{A}}_{(\hat{z},h^u(t)),(\hat{p},h(t))}H^{u,\hat{A}}_{(\hat{p},t),(\hat{z},h^u(t))},$$

by definition of the holonomies we have that

$$H^{s,\hat{A}}_{(\hat{z},h^u(t)),(\hat{p},h(t))}=\hat{A}(\hat{p},h(t))^{-1}H^{s,\hat{A}}_{\hat{\sigma}(\hat{z},h^u(t)),\hat{\sigma}(\hat{p},h(t))}\hat{A}(\hat{x},h^u(t)).$$

For every $\delta>0$ let $B(\hat{x},\delta)$ be the open ball of radius $\delta$ centered at $\hat{x}$. Fix $r>0$ such that $\hat{\sigma}(B(\hat{z},r))\cap B(\hat{z},r)=\hat{\sigma}^{-1}(B(\hat{z},r))\cap B(\hat{z},r)=\emptyset$, and define $\psi:\hat{M}\to\mathbb{R}$ $C^\infty$ such that

$$\psi(\hat{x})=\begin{cases}1 & \text{if}\quad \hat{x}=\hat{z}\\ 0 & \text{if}\quad \hat{x}\notin B(\hat{z},r).\end{cases}$$

Take a matrix $r\in M(d,\mathbb{C})$ with small norm $\|r\|$ that we will choose later, and such that all the minors of the matrix of $\mathrm{id}+r$ in the base of eigen-values of $A$ are non-zero. This matrix exists because the set of matrices that have all the minors different from zero form an open set in the Zariski topology, then it is also dense in the usual topology.

Now define $\tilde{A}(\hat{x},t)=\hat{A}(\mathrm{id}+\psi(\hat{x})r)$. We have that

$$H^{s,\tilde{A}}_{(\hat{x},h^u(t)),(\hat{p},h(t))}=\hat{A}(\hat{p},h(t))^{-1}H^{s,\hat{A}}_{\hat{\sigma}(\hat{x},h^u(t)),\hat{\sigma}(\hat{p},h(t))}\hat{A}(\hat{x},h^u(t))(\mathrm{id}+r)$$

as the perturbation does not affect $\hat{A}$ outside $B(\hat{z}, r)$ we have

$$H^{s,\tilde{A}}_{(\hat{x},h^u(t)),(\hat{p},h(t))} = A^{-1}\,\mathrm{id}\,A\,(\mathrm{id}+r) = (\mathrm{id}+r)$$

So $H^{\tilde{A}}_t = (\mathrm{id}+r)$.

As the Oseledets subspaces in $l$ are the eigen-spaces of $A$ the strong twisting condition is equivalent to the matrix $H^{\tilde{A}}_t$, written in the eigen-vectors basis, has all his minors different from zero.

Then $\tilde{A}$ is strong twisting and strong pinching. Making the norm of $r$ smaller we can make $\tilde{A}$ arbitrary close to $\hat{A}$.

□

We need to prove that the set of simple cocycles with strong pinching and twisting is open.

**Lemma 7.2.2.** *The set of strong pinching and strong twisting cocycles is open.*

*Proof.* First the dominated decomposition is equivalently to have a family of stable and unstable cone fields $C^s_l(t)$ and $C^u_l(t)$ such that $C^s_l(t)$ is $l$-dimensional cone and $C^u_l(t)$ is $(d-l)$- dimensional cone, for every $1 \leq l < d$.

So if $\hat{A}$ is strongly pinching, let us call $E^s_l(t) = E^1_t + \cdots + E^l_t$ and $E^u_l(t) = E^l_t + \cdots + E^d_t$. We can take the cones

$$C^s_{\epsilon,l}(t) = \{v = v^u_l + v^s_l,\ \frac{\|v^u_l\|}{\|v^s_l\|} \leq \epsilon \text{ where } v^u_l \in E^u_l(t) \text{ and } v^s_l \in E^s_l(t)\}$$

and

$$C^u_{\epsilon,l}(t) = \{v = v^u_l + v^s_l,\ \frac{\|v^s_l\|}{\|v^u_l\|} \leq \epsilon \text{ where } v^u_l \in E^u_l(t) \text{ and } v^s_l \in E^s_l(t)\}$$

which are also stable and unstable cones for $B$ close to $\hat{A}$.

Also this shows, that if $B$ is sufficiently close we can choose $\epsilon$ very small. This implies that the spaces $E^i_{\hat{A}}$ varies continuously with $B$.

Now in the topology $H^{\hat{\alpha}}(\hat{M})$, $\hat{A}_k \to \hat{A}$ implies that $H^{u,s}_{\hat{A}_k} \to H^{u,s}_{\hat{A}}$, this can be seen in [2]. Then if $\hat{A}_k \to \hat{A}$ the spaces $E^i_{\hat{A}_k} \to E^i_{\hat{A}}$ and $H^s_{\hat{A}_k}{}^* H^u_{\hat{A}_k}{}^* \to H^s_{\hat{A}}{}^* H^u_{\hat{A}}{}^*$, so for $k$ sufficiently large the transversality condition, strong twisting, is also verified. □

The last condition to be proved is that for $B$ sufficiently close to $\hat{A}$, the Lyapunov spectrum of $B$ restricted to the periodic $\hat{p} \times K$ is also *simple without resonance*. For this we will prove a stronger property.

**Lemma 7.2.3.** *If $\hat{A}$ is strong pinching, then it is a continuous point of his Lyapunov exponents, i.e: If $\hat{A}_k \to \hat{A}$ then $\lambda^k_i \to \lambda_i$, where $\lambda_i$ is the $i-$th smaller Lyapunov exponent.*

*Proof.* First of all, strong pinching implies simple Lyapunov spectrum. By definition of dominated splitting there exist $C > 0$ and $\theta > 1$ such that

$$\frac{\left\|\hat{A}^n v^{i+1}\right\|}{\left\|\hat{A}^n v^i\right\|} \geq C\theta^n \tag{7.1}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log \left\|\hat{A}^n v^{i+1}\right\| - \lim_{n\to\infty} \frac{1}{n} \log \left\|\hat{A}^n v^i\right\| \geq \log\theta > 0.$$

By the previous lemma, for a fixed $1 \leq i \leq d$, $E^i_{\hat{A}_k} \to E^i_{\hat{A}}$. Also $\lambda^k_i = \int \log \frac{\|\hat{A}_k v\|}{\|v\|} dm^i_k$ where $m^i_k = \int \delta_{E^i_k(t)} \mu^c$.

Then $m^i_k \rightharpoonup m^i$. So

$$\begin{aligned} \lim_{k\to\infty} \lambda^k_i &= \lim_{k\to\infty} \int \log \frac{\left\|\hat{A}_k v\right\|}{\|v\|} dm^i_k = \\ \int \log \frac{\left\|\hat{A} v\right\|}{\|v\|} dm^i &= \lambda_i \end{aligned}$$

□

*Proof of Theorem E.* By Lemma 7.2.2 and 7.2.3 the subset of simple strong pinching and strong twisting cocycles is open.

By Theorem D every simple cocycle has simple Lyapunov spectrum. □

# CHAPTER 8

## Continuity Equivalence

In this chapter we prove an equivalence of continuity of Lyapunov exponents and Oseledets subspaces. This is going to be used in Chapter 9. Actually here we proved a more general version that the one we need in Chapter 9, the reader interested only in that case may read only section 8.3.1 (see remark 8.3.4).

Only in this chapter we consider semi-invertible cocycles, i.e: an invertible ergodic measure preserving dynamical system $f : M \to M$ and a measurable matrix-valued map $A : M \to M(d, \mathbb{R})$.

This part is a joint work with Lucas Backes.

### 8.1 Statements

The main result of this chapter is the next theorem:

**Theorem 8.1.1.** *Let $\{A_k\}_k \subset C^0(M)$ be a sequence converging to $A \in C^0(M)$. Then $\lim_{k\to\infty} \gamma_i(A_k) = \gamma_i(A)$ for every $1 \leq i \leq d$ if and only if the Oseledets subspaces of $A_k$ converge to those of $A$ with respect to the measure $\mu$.*

As a simple consequence of our main theorem we get Theorem C.

It is worth noticing that the proof presented bellow also works with obvious adjustments if we allow the base dynamics $f$ to vary. More precisely, if we consider a sequence of ergodic $\mu$-measure preserving maps $f_k : M \to M$ converging uniformly to $f : M \to M$ and a sequence $\{A_k\}_k \subset C^0(M)$ converging to $A \in C^0(M)$, then a similar statement to the one of Theorem 8.1.1 also works for Lyapunov exponents and Oseledets subspaces of $(A_k, f_k)$ and $(A, f)$. We write the proof in the case when the base dynamics is fixed just to avoid unnecessary notational complications.

## 8.2 Preliminary Results

This section is devoted to present some preliminary results that are going to be used in the proof of Theorem 8.1.1.

### 8.2.1 Semi-projective cocycles

Let $\mathbb{R}P^{d-1}$ denote the real $(d-1)$-dimensional projective space, that is, the space of all one-dimensional subspaces of $\mathbb{R}^d$. Given a continuous map $A : M \to M(d, \mathbb{R})$, we want to define an action on $\mathbb{R}P^{d-1}$ which is, in some sense, induced by $A$. If $(x, [v]) \in M \times \mathbb{R}P^{d-1}$ is such that $A(x)v \neq 0$ then we have a natural action induced by $A$ on $\mathbb{R}P^{d-1}$ which is just given by $A(x)\,[v] = [A(x)v]$. The difficulty appears when $A(x)v = 0$ for some $v \neq 0$. To bypass this issue, let us consider the closed set given by

$$\mathrm{Ker}(A) = \{(x, [v]) \in M \times \mathbb{R}P^{d-1};\ A(x)v = 0\}.$$

If $\mu(\pi(\mathrm{Ker}(A))) = 0$ where $\pi : M \times \mathbb{R}P^{d-1} \to M$ denotes the canonical projection on the first coordinate, then $A(x)$ is invertible for $\mu$-almost every $x \in M$ and hence it naturally induces a map on $\mathbb{R}P^{d-1}$ which is defined $\mu$-almost everywhere and is all we need. Otherwise, if $\mu(\pi(\mathrm{Ker}(A))) > 0$ let us consider the set

$$K(A) = \{(x, [v]) \in M \times \mathbb{R}P^{d-1};\ A^n(x)v = 0 \text{ for some } n > 0\}.$$

Observe that $K(A) \cap \{x\} \times \mathbb{R}P^{d-1} \subset \{x\} \times E_x^{l,A}$ for every *regular* point $x \in M$.

Since $\pi(K(A))$ is an $f$-invariant set and $\mu$ is ergodic it follows that $\mu(\pi(K(A))) = 1$. Thus, we can define a mensurable section $\sigma : M \to \mathbb{R}P^{d-1}$ such that $(x, \sigma(x)) \in K(A)$. Moreover, we can do this in a way such that if $x \in \pi(\mathrm{Ker}(A))$ then $(x, \sigma(x)) \in \mathrm{Ker}(A)$. Fix such a section. We now define the *semi-projective cocycle* associated to $A$ and $f$ as being the map $F_A : M \times \mathbb{R}P^{d-1} \to M \times \mathbb{R}P^{d-1}$ given by

$$F_A(x, [v]) = \begin{cases} (f(x), [A(x)v]) \text{ if } A(x)v \neq 0 \\ (f(x), \sigma(f(x)) \text{ if } A(x)v = 0. \end{cases}$$

This is a measurable function which coincides with the usual projective cocycle outside $\mathrm{Ker}(A)$. In particular, it is continuous outside $\mathrm{Ker}(A)$. From now on, given a non-zero element $v \in \mathbb{R}^d$ we are going to use the same notation to denote its equivalence class in $\mathbb{R}P^{d-1}$.

Given a measure $m$ on $M \times \mathbb{R}P^{d-1}$, observe that if $m(\mathrm{Ker}(A)) = 0$ then $F_{A*}m$ does not depend on the way the section $\sigma$ was chosen. Indeed, if $\psi : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$ is a mensurable function then

$$\int_{M \times \mathbb{R}P^{d-1}} \psi \circ F_A dm = \int_{M \times \mathbb{R}P^{d-1} \setminus \mathrm{Ker}(A)} \psi \circ F_A dm.$$

In the sequel, we will be primarily interested in $F_A$-invariant measures on $M\times\mathbb{R}P^{d-1}$ that *projects* on $\mu$, that is, $\pi_* m=\mu$ and such that $m(\mathrm{Ker}(A))=0$. Our first result states if the cocycle $A$ has two different Lyapunov exponents then any such a measure may be written as a convex combination of measures concentrated on a suitable combination of the Oseledets subspaces. An useful notation that we are going to use through the chapter is the following:

$$E_x^{s_i,A}=E_x^{i+1,A}\oplus\cdots\oplus E_x^{l,A}$$

and

$$E_x^{u_i,A}=E_x^{1,A}\oplus\cdots\oplus E_x^{i,A}$$

which denotes, respectively, the Oseledets slow and fast subspaces of 'order i' associated to $A$ and

$$E^{i,A}=\{(x,v)\in M\times\mathbb{R}P^{d-1};\ v\in E_x^{i,A}\}.$$

**Proposition 8.2.1.** *If $\gamma_i(A)>\gamma_{i+1}(A)$ then every $F_A$-invariant measure projecting to $\mu$ and such that $m(\mathrm{Ker}(A))=0$ is of the form $m=am^{u_i}+bm^{s_i}$ for some $a,b\in[0,1]$ such that $a+b=1$, where $m^*$ is an $F_A$-invariant measure projecting on $\mu$ such that its disintegration $\{m_x^*\}_{x\in M}$ with respect to $\mu$ satisfies $m_x^*(E_x^*)=1$ for $*\in\{s_i,u_i\}$.*

*Proof.* Given $j\in\mathbb{N}$ let us consider the set $B_j$ defined by

$$\left\{(x,v)\in M\times\mathbb{R}P^{d-1};\ |\sin\measuredangle(v,E_x^*)|\geq\frac{1}{j}|\sin\measuredangle(E_x^{u_i},E_x^{s_i})|\text{ for }*=s_i,u_i\right\}.$$

Since $\gamma_i(A)>\gamma_{i+1}(A)$ it follows that for any $(x,v)\in B_j$, the angle between $A^n(x)v$ and $E^{u_i}_{f^n(x)}$ decays exponentially fast when $n$ goes to $+\infty$. Therefore, since by Oseledets' theorem the angle $\measuredangle(E_x^{u_i},E_x^{s_i})$ decays subexponentially it follows that every $(x,v)\in B_j$ leaves $B_j$. Consequently, by Poincaré's recurrence theorem $m(B_j)=0$ for every $j\in\mathbb{N}$. Hence, the measure $m$ is concentrated on $\{(x,E_x^{u_i});\ x\in M\}\cup\{(x,E_x^{s_i});\ x\in M\}$. Let $\{m_x\}_{x\in M}$ be a disintegration of $m$ with respect to $\mu$. It follows then by the previous observations that $m_x(E_x^{s_i})+m_x(E_x^{u_i})=1$ for $\mu$-almost every $x\in M$. Thus, letting $m_x^*$ be the normalized restriction of $m_x$ to $E_x^*$ for $*\in\{s_i,u_i\}$ we get that $m_x=a(x)m_x^{u_i}+b(x)m_x^{s_i}$ where $a(x)=m_x(E_x^{u_i})$ and $b(x)=m_x(E_x^{s_i})$. To conclude the proof, since our measure $\mu$ is ergodic, it only remains to observe that both $a$ and $b$ are invariant functions and consequently constant functions. This follows easily from the invariance of the Oseledets spaces and the fact that, since $m$ is $F_A$-invariant, $m_{f(x)}=A(x)_*m_x$ for $\mu$-almost every $x\in M$. Indeed,

$$\begin{aligned}a(f(x))=m_{f(x)}(E^{u_i}_{f(x)})&=A(x)_*m_x(E^{u_i}_{f(x)})\\&=m_x(E_x^{u_i})=a(x)\end{aligned}$$

as we want. □

Our next result gives the existence of $F_A$-invariant measures concentrated on Oseledets subspaces. This is going to be used in Section 8.4.

**Proposition 8.2.2.** *For every $1 \leq j < l$, there exists an $F_A$-invariant measure $m$ projecting to $\mu$ and concentrated on $E^{j,A} = \{(x,v) \in M \times \mathbb{R}P^{d-1};\ v \in E^{j,A}_x\}$. In particular, it satisfies $m(\mathrm{Ker}(A)) = 0$.*

*Proof.* Let $\mathcal{M}_j$ be the space of all probability measures on $M \times \mathbb{R}P^{d-1}$ such that $m(E^{j,A}) = 1$ and $\pi_* m = \mu$. In particular, $m(\mathrm{Ker}(A)) = 0$ for every $m \in \mathcal{M}_j$.

Let us consider now the map $F_{A*} : \mathcal{M}_j \to \mathcal{M}_j$ given by $F_{A*}m$. From the invariance of $E^{j,A}$ and the definition of $\mathcal{M}_j$ it follows that $F_{A*}$ is well defined and moreover does not depend on the choice of the section $\sigma$ in the definition of the semi-projective cocycle. Furthermore, it is continuous. Indeed, let $\{m_k\}_k \subset \mathcal{M}_j$ be a sequence converging to $m$ in the weak$^*$ topology and $\psi : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$ a continuous map. By Lusin's Theorem, given $\epsilon > 0$ there exists a compact set $K \subset M$ such that $\mu(M \setminus K) < \frac{\epsilon}{4\|\psi\|}$ and $x \to E^{j,A}_x$ is continuous when restricted to $K$. Now, since $\mathrm{Ker}(A) \cap E^{j,A} = \emptyset$ and $\psi \circ F_A$ is continuous outside $\mathrm{Ker}(A)$, it follows from Tietze extension theorem that there exists a continuous function $\hat{\psi} : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$ satisfying $\hat{\psi}(p) = \psi \circ F_A(p)$ for every $p \in \{(x,v) \in K \times \mathbb{R}P^{d-1};\ v \in E^{j,A}_x\}$ and $\left\|\hat{\psi}\right\| \leq \|\psi\|$. Then,

$$\left|\int \psi \circ F_A dm_k - \int \psi \circ F_A dm\right| \leq \left|\int \hat{\psi} dm_k - \int \hat{\psi} dm\right| + \epsilon.$$

Consequently, taking $k$ sufficiently large, $|\int \psi \circ F_A dm_k - \int \psi \circ F_A dm| < 2\epsilon$ as we claimed.

We observe now that $\mathcal{M}_j$ is a closed subset of the set of all probability measures of $M \times \mathbb{R}P^{d-1}$. In fact, let $\{m_k\}_k \subset \mathcal{M}_j$ be a sequence converging to $m$. As before, given $\epsilon > 0$ there exists a compact set $K \subset M$ such that $\mu(M \setminus K) < \epsilon$ and $x \to E^{j,A}_x$ is continuous when restricted to $K$. Thus, since $E^{j,A}_K := \{(x,v) \in K \times \mathbb{R}P^{d-1};\ v \in E^{j,A}_x\}$ is a closed subset of $M \times \mathbb{R}P^{d-1}$, it follows that

$$m(E^{j,A}) \geq m(E^{j,A}_K) \geq \limsup_{k \to \infty} m_k(E^{j,A}_K).$$

Therefore, as $m_k(E^{j,A}) = 1$ and $\mu(M \setminus K) < \epsilon$ we get that $m_k(E^{j,A}_K) > 1 - \epsilon$ for every $k$ and consequently, since $\epsilon > 0$ was arbitrary and $m$ is a probability measure, $m(E^{j,A}) = 1$ and $\mathcal{M}_j$ is closed.

To conclude the proof, it only remains to observe that given any $m \in \mathcal{M}_j$, every accumulation point of $\frac{1}{n}\sum_{k=0}^{n-1} F^k_{A*} m$ gives rise to an $F_A$-invariant measure concentrated on $E^{j,A}$. This follows easily from the previous observations. $\square$

**Remark 8.2.3.** Letting $\varphi_A : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$ be the map given by

$$\varphi_A(x, v) = \log \frac{\| A(x)v \|}{\| v \|},$$

it follows easily from the definition and Birkhoff's ergodic theorem that, for every $F_A$-invariant probability measure $m$ concentrated on $E^{j,A}$ and projecting to $\mu$,

$$\lambda_j(A) = \int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x, v)dm.$$

### 8.2.2 The adjoint cocycle

Given $x \in M$, let $A_*(x) : (\mathbb{R}^d)^* \to (\mathbb{R}^d)^*$ be the adjoint operator of $A(f^{-1}(x))$ defined by

$$(A_*(x)u)v = u(A(f^{-1}(x))v) \text{ for each } u \in (\mathbb{R}^d)^* \text{ and } v \in \mathbb{R}^d. \tag{8.1}$$

Fixing some inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^d$ and identifying the dual space $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$ we get the map $A_* : M \to M(d, \mathbb{R})$ and equation (8.1) becomes

$$\langle A(f^{-1}(x))u, v\rangle = \langle u, A_*(x)v\rangle \text{ for every } u, v \in \mathbb{R}^d.$$

The *adjoint cocycle* of $A$ is then defined as the cocycle generated by the map $A_* : M \to M(d, \mathbb{R})$ over $f^{-1} : M \to M$.

An useful remark is that the Lyapunov exponents counted with multiplicities of the adjoint cocycle are the same as those of the original cocycle. This follows from the fact that a matrix $B$ and its transpose $B^T$ have the same singular values combined with Kingman's sub-additive theorem. Moreover, Oseledets subspaces of the adjoint cocycle are strongly related with the ones of the original cocycle. More precisely,

**Lemma 8.2.4.** $E_x^{s_i,A} = (E_x^{u_i,A_*})^\perp$ *where the right-hand side denotes the orthogonal complement of the space* $E_x^{u_i,A_*}$.

*Proof.* By contradiction, suppose there exist $v \in E_x^{s_i,A}$ and $u \in E_x^{u_i,A_*}$ such that $\langle v, u\rangle \neq 0$. We may assume $i < l$ otherwise the lemma trivially holds. In this case, for each $n \in \mathbb{N}$ the map $A^n(f^{-n}(x)) : E_{f^{-n}(x)}^{s_i,A} \to E_x^{s_i,A}$ is surjective and thus we may find unitary vectors $v_n \in E_{f^{-n}(x)}^{s_i,A}$ such that $A^n(f^{-n}(x))v_n$ are multiples of $v$. By definition,

$$\begin{aligned}\langle A^n(f^{-n}(x))v_n, u\rangle &= \langle v_n, (A^n(f^{-n}(x)))^* u\rangle \\ &= \langle v_n, A_*^n(x)u\rangle.\end{aligned}$$

Now, since $\langle A^n(f^{-n}(x))v_n, u\rangle$ grows at an exponential rate smaller than $\lambda_i(A)$ while $\langle v_n, A_*^n(x)u\rangle$ grows at an exponential rate at least $\lambda_i(A)$ we get a contradiction. Therefore, $E_x^{s_i,A} \subset (E_x^{u_i,A_*})^\perp$. Now, since they have the same dimension the lemma follows.

□

## 8.3 Continuity of Lyapunov exponents implies continuity of Oseledets subspaces

At this section we are going to prove that continuity of Lyapunov exponents implies continuity of Oseledets subspaces. Thus, let $\{A_k\}_k \subset C^0(M)$ be a sequence converging to $A \in C^0(M)$ and suppose $\lim_{k\to\infty} \gamma_i(A_k) = \gamma_i(A)$ for every $1 \leq i \leq d$. We start with an auxiliary lemma.

**Lemma 8.3.1.** *Let $m_k$ be a sequence of $F_{A_k}$-invariant measures concentrated on $E^{1,A_k}$ and suppose they converge to a measure $m$.*

*Then $m(\mathrm{Ker}(A)) = 0$ and moreover $m$ is an $F_A$-invariant measure.*

*Proof.* We start proving that $m(\mathrm{Ker}(A)) = 0$. Suppose by contradiction that $m(\mathrm{Ker}(A)) = 2c > 0$. For each $\delta > 0$ let us consider

$$K_\delta = \left\{(x, v) \in M \times \mathbb{R}P^{d-1};\ \left\| A(x)\frac{v}{\|v\|} \right\| < \delta \right\}.$$

These are open sets such that $\mathrm{Ker}(A) = \cap_{\delta>0} K_\delta$ and $m(K_\delta) \geq m(\mathrm{Ker}(A)) > c > 0$.

Fix $b \in \mathbb{R}$ such that

$$b < \gamma_1(A) - \sup_{k,x,\|v\|=1} \log \|A_k(x)v\|$$

and let $\delta > 0$ be such that $\log y < \frac{b}{c}$ for every $y < 2\delta$. Then, for every $k$ sufficiently large $m_k(K_\delta) > c > 0$ and $\left\| A_k(x)\frac{v}{\|v\|} \right\| < 2\delta$ for every $(x, v) \in K_\delta$ and consequently

$$\gamma_1(A_k) = \int \varphi_{A_k} dm_k < b + \sup_{k,x,\|v\|=1} \log \|A_k(x)v\|$$

contradicting the choice of $b$. Thus, $m(\mathrm{Ker}(A)) = 0$ as we want.

To prove that $m$ is $F_A$-invariant one only has to show that, given a continuous map $\psi : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$,

$$\lim_{k\to\infty} \int \psi \circ F_{A_k} dm_k = \int \psi \circ F_A dm. \tag{8.2}$$

Indeed, if (8.2) is true then, since $m_k$ is $F_{A_k}$-invariant,

$$\int \psi \circ F_A dm = \lim_{k\to\infty} \int \psi \circ F_{A_k} dm_k = \lim_{k\to\infty} \int \psi dm_k = \int \psi dm.$$

In order to prove (8.2) we start noticing that

$$\begin{aligned} \left| \int \psi \circ F_{A_k} dm_k - \int \psi \circ F_A dm \right| \leq & \int |\psi \circ F_{A_k} - \psi \circ F_A| dm_k \\ & + \left| \int \psi \circ F_A dm_k - \int \psi \circ F_A dm \right|. \end{aligned}$$

Now observing that, for every $k$ sufficiently large, $\|A_k(x)v/\|v\|\| > \frac{\delta}{2}$ if $(x,v) \in K_\delta^c$ and recalling the definition of semi-projective cocycle it follows that $\psi \circ F_{A_k}$ converges uniformly to $\psi \circ F_A$ outside $K_\delta$. Given $\varepsilon > 0$ let $\delta > 0$ be such that $m(\overline{K_\delta}) < \frac{\epsilon}{2\|\psi\|}$. Then, taking $k$ sufficiently large such that $|\psi \circ F_{A_k} - \psi \circ F_A| < \epsilon$ outside $K_\delta$ and $m_k(\overline{K_\delta}) < \frac{\epsilon}{2\|\psi\|}$ we get

$$\int |\psi \circ F_{A_k} - \psi \circ F_A| dm_k < 2\epsilon.$$

To bound $\left|\int \psi \circ F_A dm_k - \int \psi \circ F_A dm\right|$, let $\hat{\psi} : M \times \mathbb{R}P^{d-1} \to \mathbb{R}$ be a continuous function which is equal to $\psi \circ F_A$ outside $K_\delta$ and $\left\|\hat{\psi}\right\| \leq \|\psi\|$. Note that the existence of such a map is guaranteed once again by Tietze extension theorem. Then,

$$\left|\int \psi \circ F_A dm_k - \int \psi \circ F_A dm\right| \leq \left|\int \hat{\psi} dm_k - \int \hat{\psi} dm\right| + 2\epsilon.$$

Now, taking $k$ sufficiently large such that $\left|\int \hat{\psi} dm_k - \int \hat{\psi} dm\right| < \epsilon$ it follows that

$$\left|\int \psi \circ F_{A_k} dm_k - \int \psi \circ F_A dm\right| < 5\epsilon$$

proving (8.2) and consequently the lemma.

□

**Remark 8.3.2.** Observe that in the proof of the previous lemma we did not use the full strength of the requirement $\lim_{k\to\infty} \gamma_i(A_k) = \gamma_i(A)$ for every $1 \leq i \leq d$. Indeed, it is enough that $\lim_{k\to\infty} \int \varphi_{A_k} dm_k > -\infty$. This is going to be used in Section 8.4.

### 8.3.1 Continuity of the fastest Oseledets subspace

Our next proposition deals with the case when $d_1(A) = 1$. That is, the case when the dimension of the Oseledets subspace associated with $\lambda_1(A)$ is 1.

**Proposition 8.3.3.** *If $A$ is such that $\gamma_1(A) > \gamma_2(A)$ then $E_x^{1,A_k}$ converges to $E_x^{1,A}$ with respect to the measure $\mu$. More precisely, for every $\delta > 0$*

$$\mu(\{x \in M;\ \measuredangle(E_x^{1,A_k}, E_x^{1,A}) < \delta\}) \xrightarrow{k\to\infty} 1.$$

*Proof.* We start observing that, since $\gamma_j(A_k) \xrightarrow{k\to\infty} \gamma_j(A)$ for every $1 \leq j \leq d$ and $\gamma_1(A) > \gamma_2(A)$, for every $k$ sufficiently large $\gamma_1(A_k) > \gamma_2(A_k)$ and thus $E_x^{1,A_k}$ is also one-dimensional. Let us assume without loss of generality that this is indeed the case for every $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let us consider the measure

$$m_k = \int_M \delta_{(x, E_x^{1,A_k})} d\mu(x)$$

and let $m^u$ be the measure given by

$$m^u = \int_M \delta_{(x, E_x^{1,A})} d\mu(x).$$

Observe that these are, respectively, $F_{A_k}$ and $F_A$-invariant measures on $M \times \mathbb{R}P^{d-1}$ concentrated on $E^{1,A_k}$ and $E^{1,A}$ and projecting to $\mu$. Consequently, it follows from Remark 8.2.3 that

$$\gamma_1(A_k) = \int_{M\times\mathbb{R}P^{d-1}} \varphi_{A_k}(x,v)dm_k \tag{8.3}$$

and

$$\gamma_1(A) = \int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm^u. \tag{8.4}$$

We claim now that $m_k$ converges to $m^u$ in the weak* topology. Indeed, let $\{m_{k_i}\}_{i\in\mathbb{N}}$ be a convergent subsequence of $\{m_k\}_{k\in\mathbb{N}}$ and suppose that it converges to $m$. Since $M \times \mathbb{R}P^{d-1}$ is a compact space it suffices to prove that $m = m^u$. Observing that, for each $i \in \mathbb{N}$ the measure $m_{k_i}$ is a $F_{A_{k_i}}$-invariant measure projecting to $\mu$, it follows from Lemma 8.3.1 that $m$ is a $F_A$-invariant measure projecting to $\mu$ and moreover $m(\mathrm{Ker}(A)) = 0$. Furthermore, since

$$\gamma_1(A_{k_i}) \xrightarrow{i\to+\infty} \gamma_1(A)$$

and

$$\int_{M\times\mathbb{R}P^{d-1}} \varphi_{A_{k_i}}(x,v)dm_{k_i} \xrightarrow{i\to+\infty} \int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm$$

it follows from (8.3) that

$$\gamma_1(A) = \int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm.$$

Thus, from Lemma 8.3.3 we get that $m = m^u$ as claimed. In fact, otherwise we would have $m = am^{u_1} + bm^{s_1}$ where $a, b \in (0,1)$ are such that $a + b = 1$ and $m^{s_1}$ is a $F_A$-invariant measure concentrated on $\{(x, E_x^{s_1});\ x \in M\}$. Therefore,

$$\begin{aligned}
\gamma_1(A) &= \int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm \\
&= a\int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm^{u_1} + b\int_{M\times\mathbb{R}P^{d-1}} \varphi_A(x,v)dm^{s_1} \\
&\le a\gamma_1(A) + b\gamma_2(A) < \gamma_1(A).
\end{aligned}$$

Let us consider now the measurable map $\psi : M \to \mathbb{R}P^{d-1}$ given by

$$\psi(x) = E_x^{1,A}.$$

Note that its graph has full $m^u$-measure. By Lusin's Theorem, given $\varepsilon > 0$ there exists a compact set $K \subset M$ such that the restriction $\psi_K$ of $\psi$ to $K$ is continuous and $\mu(K) > 1 - \varepsilon$. Now, given $\delta > 0$, let $V \subset M \times \mathbb{R}P^{d-1}$ be an open neighborhood of the graph of $\psi_K$ such that

$$V \cap (K \times \mathbb{R}P^{d-1}) \subset V_\delta$$

where

$$V_\delta := \{(x, v) \in K \times \mathbb{R}P^{d-1};\ \measuredangle(v, \psi(x)) < \delta\}.$$

By the choice of the measures $m_k$,

$$m_k(V_\delta) = \mu(\{x \in K;\ \measuredangle(E_x^{1,A_k}, E_x^{1,A}) < \delta\}). \tag{8.5}$$

Now, as $m_k \xrightarrow{k\to\infty} m^u$ it follows that $\liminf m_k(V) \geq m^u(V) > 1 - \varepsilon$. On the other hand, as $m_k(K \times \mathbb{R}P^{d-1}) = \mu(K) > 1 - \varepsilon$ for every $k \in \mathbb{N}$, it follows that

$$m_k(V_\delta) \geq m_k(V \cap (K \times \mathbb{R}P^{d-1})) \geq 1 - 2\varepsilon \tag{8.6}$$

for every $k$ large enough. Thus, combining (8.5) and (8.6), we get that $\mu(\{x \in M;\ \measuredangle(E_x^{1,A_k}, E_x^{1,A}) < \delta\}) \geq 1 - 2\varepsilon$ for every $k$ large enough completing the proof of the proposition.

□

**Remark 8.3.4.** In the case of $SL(2,\mathbb{R})$ cocycles, we have that both Oseledets subspaces are one dimensional or the decomposition is trivial, then the previous proposition gives the Theorem C in the case of $SL(2,\mathbb{R})$ cocycles.

### 8.3.2 Continuity of the Oseledets fast subspace of order $i$

We now prove that the Oseledets fast subspace of order $i$ of $A_k$ converges to the respective Oseledets subspace of $A$. The idea is to consider the cocycle induced by $A$ on a suitable exterior power and then deduce the general case from the previous one.

**Proposition 8.3.5.** *For every $1 \leq i \leq l$ and $\delta > 0$ we have that*

$$\mu(\{x \in M;\ \measuredangle(E_x^{u_i,A_k}, E_x^{u_i,A}) < \delta\}) \xrightarrow{k\to\infty} 1.$$

We take the *Plücker embedding* $\Phi : \mathrm{Grass}(j,d) \to \mathbb{P}\Lambda^j\left(\mathbb{R}^d\right)$. If $\rho(.,.)$ is a distance on $\mathbb{P}(\Lambda^j\left(\mathbb{R}^d\right))$ we may push it back to $\mathrm{Grass}(j,d)$ via $\Phi$. More precisely, the map $\mathrm{dist}_{\Lambda^j(\mathbb{R}^d)} : \mathrm{Grass}(j,d) \times \mathrm{Grass}(j,d) \to \mathbb{R}$ given by

$$\mathrm{dist}_{\Lambda^j(\mathbb{R}^d)}(E_1, E_2) = \rho(\Phi(E_1), \Phi(E_2))$$

is a distance on $\mathrm{Grass}(j,d)$ and moreover, if $\rho$ is a distance given by an inner product in the linear space $\Lambda^j\left(\mathbb{R}^d\right)$ then $\mathrm{dist}_{\Lambda^j\left(\mathbb{R}^d\right)}$ is equivalent to the distance defined in (2.1).

Recall that the Lyapunov exponents of the cocycle induced in the exterior power are given by

$$\{\gamma_{i_1}(A)+\ldots+\gamma_{i_j}(A);\ 1\leq i_1<\ldots<i_j\leq l\}. \tag{8.7}$$

and the Oseledets subspaces by

$$E_x^{1,A}\wedge\ldots\wedge E_x^{i,A}. \tag{8.8}$$

*Proof of Proposition 8.3.5.* Observe that if $i=l$ then there is noting to do since $E_x^{u_l,A_k}=\mathbb{R}^d=E_x^{u_l,A}$ for every $k$ sufficiently large. So, from now on let us assume $i<l$.

Consider $r=d_1(A)+\ldots+d_i(A)$ and let $\Lambda^r A$ and $\Lambda^r A_k$ be the cocycles over $f$ induced by $A$ and $A_k$, respectively, on the $r$th exterior power. Since we are assuming $i<l$ it follows from (8.7) that $\gamma_1(\Lambda^r A)>\gamma_2(\Lambda^r A)$. Thus, from Proposition 8.3.3 we get that, for every $\delta'>0$,

$$\mu(\{x\in M;\ \measuredangle(E_x^{1,\Lambda^r A_k},E_x^{1,\Lambda^r A})<\delta'\})\xrightarrow{k\to\infty}1$$

which from (8.8) is equivalent to

$$\mu(\{x\in M;\ \measuredangle(E_x^{1,A_k}\wedge\ldots\wedge E_x^{i,A_k},E_x^{1,A}\wedge\ldots\wedge E_x^{i,A})<\delta'\})\xrightarrow{k\to\infty}1.$$

Consequently, from the definition of $\mathrm{dist}_{\Lambda^r(\mathbb{R}^d)}$ it follows that

$$\mu(\{x\in M;\ \mathrm{dist}_{\Lambda^r(\mathbb{R}^d)}(E_x^{1,A_k}\oplus\ldots\oplus E_x^{i,A_k},E_x^{1,A}\oplus\ldots\oplus E_x^{i,A})<\delta'\})\xrightarrow{k\to\infty}1.$$

Now, using the fact that the distances $\mathrm{dist}_{\Lambda^r(\mathbb{R}^d)}$ and dist are equivalent it follows that for every $\delta>0$,

$$\mu(\{x\in M;\ \measuredangle(E_x^{1,A_k}\oplus\ldots\oplus E_x^{i,A_k},E_x^{1,A}\oplus\ldots\oplus E_x^{i,A})<\delta\})\xrightarrow{k\to\infty}1$$

as we want. □

As a simple consequence of the previous proposition applied to adjoint cocycles $A_{k_*}$ and $A_*$ combined with Lemma 8.2.4 we get that

**Corollary 8.3.6.** *For every $0\leq i\leq l-1$ and $\delta>0$ we have that*

$$\mu(\{x\in M;\ \measuredangle(E_x^{s_i,A_k},E_x^{s_i,A})<\delta\})\xrightarrow{k\to\infty}1.$$

### 8.3.3 Proof of the direct implication of Theorem 8.1.1

The *cone* of radius $\alpha > 0$ around a subspace $V$ of $\mathbb{R}^d$ is defined as

$$C_\alpha(V) = \left\{ w_1 + w_2 \in V \ \oplus V^\perp;\ \|w_2\| < \alpha \|w_1\| \right\}.$$

Observe that this is equivalent to

$$C_\alpha(V) = \left\{ w \in \mathbb{R}^d;\ \operatorname{dist}\left( \frac{w}{\|w\|}, V \right) < \alpha \right\}$$

where dist is the distance defined in (2.1).

In order to prove the direct implication of our main theorem we are going to need the following auxiliary result.

**Lemma 8.3.7.** *Given $1 \leq i \leq l$, $\epsilon > 0$ and $\delta > 0$ there exist a subset $K = K(\epsilon) \subset M$ with $\mu(K) > 1 - \epsilon$ and $\delta' = \delta'(\epsilon, \delta) > 0$, such that for every $x \in K$,*

$$C_{\delta'}(E_x^{u_i,A}) \cap C_{\delta'}(E_x^{s_{i-1},A}) \subset C_\delta(E_x^{i,A}).$$

*Proof.* For every regular point $x \in M$ we can define an inner product $\langle , \rangle_x$ on $\mathbb{R}^d$ such that $\{E_x^{i,A}\}_{i=1}^l$ are mutually orthogonal. Moreover, this family of inner products may be chosen to be measurable. Let $K \subset M$ be a compact subset of $M$ with $\mu(K) > 1 - \epsilon$ and such that $\langle , \rangle_x$ is continuous when restricted to $K$. Then, there exists $C > 1$ such that $\frac{1}{C}\|v\| \leq \|v\|_x \leq C\|v\|$. Take $\delta' := \frac{\delta}{4C^2} > 0$.

Given $v \in C_{\delta'}(E_x^{u_i,A}) \cap C_{\delta'}(E_x^{s_{i-1},A})$, for every $x \in K$ we can write $v = v_i + v_{u_{i-1}} + v_{u_i}^\perp$ where

$$v_i = \operatorname{Proj}_{E_x^{i,A}}(v),\ \ v_{u_{i-1}} = \operatorname{Proj}_{E_x^{u_{i-1},A}}(v) \ \text{ and } \ v_{u_i}^\perp = \operatorname{Proj}_{(E_x^{u_i,A})^\perp}(v).$$

Analogously $v = v_i + v_{s_i} + v_{s_{i-1}}^\perp$. From the definition of cone we get that $\left\|v_{u_i}^\perp\right\| < \delta'$ and $\left\|v_{s_{i-1}}^\perp\right\| < \delta'$ and consequently,

$$\left\|v_{s_i} - v_{u_{i-1}}\right\| < 2\delta'.$$

Now, from the definition of $v_{s_i}$ and $v_{u_{i-1}}$ and the choice of $C$ it follows that

$$\|v_s\|_x < 2C\delta'.$$

Consequently,

$$\|v_s\| < 2C^2\delta' < \frac{\delta}{2}$$

and thus, if $v = v_i + v_i^\perp$ then

$$\left\|v_i^\perp\right\| \leq \left\|v_{s_i} + v_{s_{i-1}}^\perp\right\| < \delta$$

which implies that $v \in C_\delta(E_x^{i,A})$ as we want. □

Given $\varepsilon > 0$, let $K \subset M$ and $\delta' > 0$ be given by the previous lemma. Proposition 8.3.5 and Corollary 8.3.6 gives us that for every $1 \leq i \leq l$ and $k$ sufficiently large the sets

$$A^{u_i} = \{x \in M;\ \measuredangle(E_x^{u_i,A_k}, E_x^{u_i,A}) \geq \delta'\}$$

and

$$A^{s_{i-1}} = \{x \in M;\ \measuredangle(E_x^{s_{i-1},A_k}, E_x^{s_{i-1},A}) \geq \delta'\}$$

are such that $\mu(A^{u_i}) < \epsilon$ and $\mu(A^{s_{i-1}}) < \epsilon$. Now, observing that, for $x \notin A^{u_i} \cup A^{s_{i-1}}$ and $k$ sufficiently large,

$$E_x^{i,A_k} = E_x^{u_i,A_k} \cap E_x^{s_{i-1},A_k} \subset C_{\delta'}(E_x^{u_i,A}) \cap C_{\delta'}(E_x^{s_{i-1},A})$$

it follows from Lemma 8.3.7 that, for every $x \in K \setminus (A^{u_i} \cup A^{s_{i-1}})$ and $k$ sufficiently large, $E_x^{i,A_k} \subset C_\delta(E_x^{i,A})$. Consequently, $\mu(\{x \in M;\ \measuredangle(E_x^{i,A_k}, E_x^{i,A}) < \delta'\}) \geq 1 - 3\varepsilon$ for every $k$ sufficiently large as we want.

## 8.4 Continuity of Oseledets subspaces implies continuity of Lyapunov exponents

This section is devoted to prove the reverse implication of Theorem 8.1.1. So, let $\{A_k\}_k \subset C^0(M)$ be a sequence converging to $A \in C^0(M)$ and suppose that for every $k$ sufficiently large there exists a direct sum decomposition $\mathbb{R}^d = F_x^{1,A_k} \oplus \ldots \oplus F_x^{l,A_k}$ into vector subspaces such that

i) $F_x^{i,A_k} = E_x^{j,A_k} \oplus E_x^{j+1,A_k} \oplus \ldots \oplus E_x^{j+t,A_k}$ for some $j \in \{1, \ldots, l_k\}$ and $t \geq 0$;

ii) $\dim(F_x^{i,A_k}) = \dim(E_x^{i,A})$ for every $i = 1, \ldots, l$

and moreover that

iii) for every $\delta > 0$ and $1 \leq i \leq l$ we have

$$\mu\left(\{x \in M; \measuredangle(F_x^{i,A_k}, E_x^{i,A}) > \delta\}\right) \xrightarrow{k\to\infty} 0.$$

Given $1 \leq i < l$, we start proving that if

$$\mu\left(\{x \in M,\ \measuredangle(F_x^{i,A_k}, E_x^{i,A}) > \delta\}\right) \to 0$$

for every $\delta > 0$ then $\gamma_j(A_k) \to \gamma_j(A)$ for every $d_0(A)+d_1(A)+\ldots+d_{i-1}(A) < j \leq d_1(A) + \ldots + d_i(A)$ where $d_0(A) = 0$.

For each $k \in \mathbb{N}$, let $m_k$ be a $F_{A_k}$-invariant measure supported on $\{(x, v) \in M \times \mathbb{R}P^{d-1};\ v \in F_x^{i,A_k}\}$ which projects to $\mu$ and such that

$$\gamma_j(A_k) = \int \varphi_{A_k}(x, v) dm_k. \tag{8.9}$$

The existence of such a measure is guaranteed by Proposition 8.2.2 and Remark 8.2.3. Passing to a subsequence we may assume that $m_k$ converges in the weak* topology to some measure $m$. From Lemma 8.3.1 it follows that $m$ is a $F_A$-invariant measure projecting to $\mu$ and moreover that $m(\mathrm{Ker}(A)) = 0$. To conclude the proof it suffices to observe that $m$ is supported on $\{(x,v) \in M \times \mathbb{R}P^{d-1};\ v \in E_x^{i,A}\}$. Indeed, if that is the case then invoking Remark 8.2.3 we get

$$\lim_{k\to\infty} \gamma_j(A_k) = \lim_{k\to\infty} \int \varphi_{A_k} dm_k = \int \varphi_A dm = \gamma_j(A)$$

for every $d_0(A) + d_1(A) + \ldots + d_{i-1}(A) < j \leq d_1(A) + \ldots + d_i(A)$ as we want.

Given $\varepsilon > 0$, let $K \subset M$ be a compact set with $\mu(K) > 1 - \frac{\varepsilon}{2}$ and such that $E_x^{i,A}$ is continuous when restricted to $K$. For each $\delta > 0$ let us consider

$$G_\delta = \{(x,v) \in K \times \mathbb{R}P^{d-1};\ \measuredangle(v, E_x^{i,A}) \leq \delta\}.$$

This is a closed set and thus, by the weak* convergence of the sequence $\{m_k\}_k$,

$$m(G_\delta) \geq \limsup_{k\to\infty} m_k(G_\delta). \tag{8.10}$$

Since $m_k$ projects to $\mu$ it follows by Rokhlin's disintegration theorem that $m_k$ can be written as $m_k = \int m_x^k d\mu(x)$ where $\{m_x^k\}_{x\in M}$ are measures on $\mathbb{R}P^{d-1}$. Moreover, from the choice of $m_k$ it follows that $m_x^k(F_x^{i,A_k}) = 1$ for $\mu$-almost every $x \in M$. Consequently,

$$m_k(G_\delta) = \int m_x^k(G_\delta) d\mu \geq 1 - \mu\left(K^c \cup \{x \in M;\ \measuredangle(F_x^{i,A_k}, E_x^{i,A}) > \delta\}\right). \tag{8.11}$$

Now, let $k_\delta \in \mathbb{N}$ be such that $\mu\Big(\{x \in M;\ \measuredangle(F_x^{i,A_k}, E_x^{i,A}) > \delta\}\Big) < \frac{\varepsilon}{2}$ for every $k \geq k_\delta$. Thus, invoking (8.11) we get that $m_k(G_\delta) \geq 1 - \varepsilon$ for every $\delta > 0$ as far as $k \geq k_\delta$. Hence, it follows from (8.10) that $m(G_\delta) \geq 1 - \varepsilon$ for every $\delta > 0$. Consequently,

$$\begin{aligned} m(\{(x,v) \in M \times \mathbb{R}P^{d-1};\ v \in E_x^{i,A}\}) &\geq m(\{(x,v) \in K \times \mathbb{R}P^{d-1};\ v \in E_x^{i,A}\}) \\ &\geq \lim_{\delta\to 0} m(G_\delta) \geq 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we conclude that $m(\{(x,v) \in M \times \mathbb{R}P^{d-1};\ v \in E_x^{i,A}\}) = 1$ as claimed.

It remains to consider the case when $i = l$. If $\lambda_l(A) > -\infty$, then the previous argument also works for this case. Otherwise, if $\lambda_l(A) = -\infty$ it suffices to prove that $\gamma_j(A_k) \to -\infty$ for $j = d_1(A) + \ldots + d_{l-1}(A) + 1$. Suppose that is not the case, that is, $\limsup_{k\to\infty} \gamma_j(A_k) > -\infty$. Passing to a subsequence, if necessary, we may assume that $\lim_{k\to\infty} \gamma_j(A_k) = a > -\infty$

and moreover that the sequence of measures $\{m_k\}_k$ given as in (8.9) converge to some measure $m$. It follows then from Remark 8.3.2 that $m$ is a $F_A$-invariant measure and $m(\mathrm{Ker}(A)) = 0$. Proceeding as we did in the previous case we conclude that $m(E^{l,A}) = 1$ and

$$-\infty > a = \lim_{k\to\infty} \gamma_j(A_k) = \lim_{k\to\infty} \int \varphi_{A_k} dm_k = \int \varphi_A dm.$$

On the other hand, Birkhoff's ergodic theorem implies that $\int \varphi_A dm = -\infty$ which gives us a contradiction. Therefore, $\gamma_j(A_k) \to -\infty$ for $j = d_1(A) + \ldots + d_{l-1}(A) + 1$ and hence $\gamma_j(A_k) \to -\infty$ for every $j \in \{d_1(A) + \ldots + d_{l-1}(A) + 1, \ldots, d\}$ completing the proof of Theorem 8.1.1.

# CHAPTER 9

## Two dimensional cocycles

In this chapter we prove a stronger result than Theorem D when the cocycles are in $SL(2,\mathbb{R})$. Here we deal with a more general setting for the base dynamics that we are going to define now.

The notations of this chapter does not correspond to the notations of the previous ones.

### 9.1 Definitions and Statements

Let $f : M \to M$ be a partially hyperbolic map, dynamically coherent with compact center leaves, and $\mu$ an ergodic invariant probability measure.

Let $\tilde{M} = M/\mathcal{W}^c$ be the quotient of $M$ by the center foliation, and $\pi : M \to \tilde{M}$ be the quotient map. We say that the center leaves *form a fiber bundle* if for any $\mathcal{W}^c(x) \in \tilde{M}$ there is a neighborhood $V \subset \tilde{M}$ of $\mathcal{W}^c(x)$ and a homeomorphism

$$h_x : V \times \mathcal{W}^c(x) \to \pi^1(V)$$

smooth along the verticals $\{\ell\} \times \mathcal{W}^c(x)$ and mapping each vertical onto the corresponding center leaf $\ell$.

In this chapter we deal with cocycles $\alpha$-H"older cocycles $A : M \to \mathrm{SL}(2,\mathbb{R})$ and the $\alpha$-H"older topology of $H^\alpha(M)$, defined in Chapter 2.

**Definition 9.1.1.** Given an invertible measurable map $g : N \mapsto N$ an invariant measure $\eta$ and an integrable cocycle $A : N \to SL(2,\mathbb{R})$ we say that $A$ is *non-uniformly hyperbolic* if $\lambda^+(x) > 0$ for $\eta$ almost every point.

Here we will need a concept of continuity of Lyapunov exponents for non ergodic measures.

**Definition 9.1.2.** Given an invertible measurale map $g : N \mapsto N$ an invariant measure $\eta$ and an integrable cocycle $A : N \to SL(2,\mathbb{R})$ we say that $A$ is a *weak continuity point of Lyapunov exponents* if for every $A_k$ that converges to $A$ implies that $\lambda^+_{A_k} : N \to \mathbb{R}$ converges in measure to $\lambda^+_A : N \to \mathbb{R}$.

Observe that as $A_k \to A$ this implies that $\sup_k \|A_k\|$ is bounded and in consequence $\lambda^+_{A_k}$ is bounded, also we are dealing with probability measures, then convergence in measure is equivalent to convergence in $L^1_\eta$.

**Definition 9.1.3.** We say that $A \in H^\alpha(M)$ is *stably non-uniformly hyperbolic* if there exist an open set $A \in \mathcal{V} \subset H^\alpha(M)$ such that every $B \in \mathcal{V}$ has $L(B,\mu) > 0$.

Observe that as we assume that $\mu$ is ergodic $L(B,\mu) > 0$ is equivalent to non-uniformly hyperbolic.

The fiber bundle condition gives that the quotient $\tilde{M} = M/\mathcal{W}^c$ is a topological manifold and the induced $\tilde{f} : \tilde{M} \to \tilde{M}$ is a hyperbolic homeomorphism (as in Definition 2.1.1).

We say that $\mu$ has *projective product structure* if locally $\pi_*\mu \sim \mu^s \times \mu^u$ (this measures are equivalent).

As before, we are interested in the case where $f$ is not hyperbolic in the center direction, so in what follows we also assume that the extremal center Lyapunov exponents of $f$ are zero.

The principal result of this chapter is Theorem B, that we recall here

**Theorem 9.1.4.** *Let $\mu$ be a $f$-invariant ergodic measure with zero center Lyapunov exponent and projective product structure.*

*Let $A : M \to SL(2,\mathbb{R})$ be a fiber bunched cocycle, such that the restriction to some periodic center leaf of $f$ is a weak continuity point of Lyapunov exponent and non-uniformly hyperbolic. Then $A$ is accumulated by stably non-uniformly hyperbolic cocycles.*

## 9.2 Holonomies

The key concept, here again, are the holonomies. Let us define the ones that we consider here.

**Definition 9.2.1.** Given $\tilde{x} \in \tilde{M}$ and $\tilde{y} \in \tilde{M}$, such that $\tilde{y} \in W^s(\tilde{x})$, we can define the *stable holonomy* as the holonomy given by the stable foliation, i.e: $h^s_{\tilde{x},\tilde{y}} : \mathcal{W}^c(\tilde{x}) \to \mathcal{W}^c(\tilde{y})$ where $h^s_{\tilde{x},\tilde{y}}(t)$ is the first intersection between $\mathcal{W}^s(t)$ and $\mathcal{W}^c(\tilde{y})$.

Analogously for $\tilde{z} \in W^u(\tilde{x})$ we define the *unstable holonomy* $h^u_{\tilde{x},\tilde{z}} : \mathcal{W}^c(\tilde{x}) \to \mathcal{W}^c(\tilde{z})$ changing the stable by unstable manifolds.

It is well known that there exists $\tilde{f}$ periodic points, that in this case correspond to $f$ periodic center leaves. To simplify notation lets assume that the center leaf $\tilde{p}$ is fixed, $\tilde{f}(\tilde{p}) = \tilde{p}$ (all the arguments and results are not affected by taking an iterate such that $\tilde{p}$ is fixed).

Given a periodic center leaf $\tilde{p} \in \tilde{M}$, let us denote by $K = \mathcal{W}^c(\tilde{p})$, and let $\tilde{z} \in \tilde{M}$ be homoclinic point for $\tilde{p}$, i.e: there exist $\ell > 0$ such that $\tilde{z} \in W^u_{loc}(\tilde{p}) \cap \tilde{f}^{-\ell}(W^s_{loc}(\tilde{p}))$. We can define $h : K \to K$ by

$$h = h^s_{\tilde{z},\tilde{p}} \circ h^u_{\tilde{p},\tilde{z}}$$

As $A$ is fiber bunched we have defined the strong stable and strong unstable holonomies as in section 2.2.

Let $H^A_t : \mathbb{R}^2 \to \mathbb{R}^2$

$$H^A_t = H^{s,A}_{\left(\tilde{z},h^u_{\tilde{p},\tilde{z}}(t)\right),(\tilde{p},h(t))} H^{u,A}_{(\tilde{p},t),\left(\tilde{z},h^u_{\tilde{p},\tilde{z}}(t)\right)}$$

### 9.2.1 Invariance principle

Let $P : M \times \mathbb{R}P^1 \to M, \quad P(x, v) = x$ be the projection to the first coordinate and let $m$ be a measure in $M \times \mathbb{R}P^1$ such that $P_* m = \mu$.

By Rokhlin [22] we can disintegrate the measure $m$ with respect to the partitions $\mathcal{P} = \{\{p\} \times \mathbb{R}P^1, \, p \in M\}$, $\hat{\mathcal{P}} = \{\mathcal{W}^c_p \times \mathbb{R}P^1, \, p \in M\}$. Lets call the first conditionals measures $m_x$, $x \in M$, and the second $\tilde{m}_{\tilde{x}}$ ,$\tilde{x} \in \tilde{M}$.

Also we can disintegrate $\mu$ in the partition $\tilde{\mathcal{P}} = \{\mathcal{W}^c_{\tilde{p}}, \, \tilde{p} \in \tilde{M}\}$, we call this disintegration $\mu^c_{\tilde{x}}$, $\tilde{x} \in \tilde{M}$

It is easy to see that

$$\tilde{m}_{\tilde{x}} = \int m_x d\mu^c_{\tilde{x}}$$

( [26, exercise 5.2.1]).

For $\tilde{y} \in W^u(\tilde{x})$ define $\mathcal{H}^u_{\tilde{x},\tilde{y}} : \mathcal{W}^c(\tilde{x}) \times \mathbb{R}P^1 \to \mathcal{W}^c(\tilde{y}) \times \mathbb{R}P^1$

$$\mathcal{H}^u_{\tilde{x},\tilde{y}}(t, v) = \left(h^u_{\tilde{x},\tilde{y}}(t), H^u_{(\tilde{x},t),(\tilde{p},h^u_{\tilde{x},\tilde{y}}(t))} v\right)$$

Because $\pi_*\mu$ has product structure, by the invariance principle (proposition 2.3.1 and 2.3.2) there exist a continuous disintegration $\mu^c_{\tilde{x}}$ *su* invariant everywhere. i.e: $\tilde{M}^s = \tilde{M}^u = \tilde{M}$. Then we have that $h_*\mu^c_{\tilde{p}} = \mu^c_{\tilde{p}}$ and denoting by

$$f_{\tilde{p}} : K \to K, \quad f_{\tilde{p}} = f \mid_{\mathcal{W}^c(\tilde{p})}$$

then $f_{\tilde{p}*}\mu^c_{\tilde{p}}$.

**Proposition 9.2.2.** *If $L(A, \mu) = 0$, $m$ admits a continuous disintegration $\tilde{m}_{\tilde{x}}, \tilde{x} \in \tilde{M}$ with respect to $\hat{\mathcal{P}}$ that is su-invariant. Moreover, for every $\tilde{x} \in \tilde{M}$ and $\tilde{y} \in \tilde{M}$ in the same unstable leave*

$$H^*_{(\tilde{x},x),(\tilde{y},h^*_{\tilde{x},\tilde{y}}(x))_*} m_x = m_{h^*_{\tilde{x},\tilde{y}}(x)} \textit{ for } \mu^c_{\tilde{x}} \textit{ almost every } x \in \mathcal{W}^c_{\tilde{x}}.$$

*Proof.* Let us prove for $* = u$, the case $* = s$ is analogous. By proposition 2.3.1 there exist a $m_x, x \in M$ Rokhlin disintegration *su invariant*.

Take $\tilde{x} \in \tilde{M}$ and $\tilde{y} \in \tilde{M}$ in the same unstable leave and such that for $\mu^c_{\tilde{x}}$ almost every $x \in \mathcal{W}^c(\tilde{x})$ and for $\mu^c_{\tilde{y}}$ almost every $y \in \mathcal{W}^c(\tilde{y})$ belongs to $M^u$. Then

$$
\begin{aligned}
\mathcal{H}^u_{\tilde{x},\tilde{y}*}\tilde{m}_{\tilde{x}}(B) &= \int m_x\left({H^u_{\tilde{x},\tilde{y}}}^{-1}(B)\right) d\mu^c_{\tilde{x}}(x)\\
&= \int m_x\left({H^u_{(\tilde{x},x),(\tilde{y},h^u_{\tilde{x},\tilde{y}}(x))}}^{-1}(B_{h^u_{\tilde{x},\tilde{y}}(x)})\right) d\mu^c_{\tilde{x}}(x)\\
&= \int {H^u_{(\tilde{x},x),(\tilde{y},h^u_{\tilde{x},\tilde{y}}(x))}}_* m_x\left(B_{h^u_{\tilde{x},\tilde{y}}(x)}\right) d\mu^c_{\tilde{x}}(x)\\
&= \int m_{h^u_{\tilde{x},\tilde{y}}}\left(B_{h^u_{\tilde{x},\tilde{y}}(x)}\right) d\mu^c_{\tilde{x}}(x)\\
&= \int m_y\left(B_y\right) d\left(h^u_{\tilde{x},\tilde{y}*}\mu^c_{\tilde{x}}\right)(y)\\
&= \int m_y\left(B_y\right) d\left(\mu^c_{\tilde{y}}\right)(y) = \tilde{m}_{\tilde{y}}(B)
\end{aligned}
$$

so the measure $\tilde{m}_{\tilde{x}}$ is $u$ invariant, analogously we can find a total measure set such that $\tilde{m}_{\tilde{x}}$ is $s$ invariant. Using Proposition 2.3.2 we conclude that $m$ admits a continuous $\hat{\mathcal{P}}$ disintegration $s$ and $u$ invariant.

The continuity and the $su$ invariance implies that, for every $\tilde{x} \in \tilde{M}$ and $\tilde{y} \in \tilde{M}$ in the same unstable leave

$$
{H^u_{(\tilde{x},x),(\tilde{y},h^u_{\tilde{x},\tilde{y}}(x))}}_* m_x = m_{h^u_{\tilde{x},\tilde{y}}(x)} \text{ for } \mu^c_{\tilde{x}} \text{ almost every } x \in \mathcal{W}^c_{\tilde{x}}.
$$

as claimed. ∎

Then if $L(A,\mu) = 0$ we have that ${H^A_t}_* m_t = m_{h(t)}$ for $\mu^c_{\tilde{p}}$ almost every $t \in K$.

$A\mid_K\colon K \to SL(2,\mathbb{R})$ defines a linear cocycle over $f_{\tilde{p}}$ with invariant measure $\mu_K = \mu^c_{\tilde{p}}$. For almost every $t \in K$ we have defined $\mathbb{R}^2 = E^u_t + E^s_t$, where $E^u_t$ is the Oseledets subspace corresponding to $\lambda^+_{\mu_K}(t)$ and $E^s_t$ the corresponding to $\lambda^-_{\mu_K}(t)$. In the case that $\lambda^+_{\mu_k}(t) = \lambda^-_{\mu_k}(t)$ we have that $E^u_t = E^s_t = \mathbb{R}^2$.

**Definition 9.2.3.** We say that $A \in H^\alpha(M)$ is

- *Weakly pinching* if there exist a periodic center leaf $K = \mathcal{W}^c(p)$ such that $A\mid_K\colon K \to SL(2,\mathbb{R})$ is non-uniformly hyperbolic with respect to $\mu_K$.
- *Weakly twisting* if there exist $\tilde{K} \subset K$ with $\mu_K(\tilde{K}) > 0$ such that $H^A_t\left(\{E^u_t, E^s_t\}\right) \cap \{E^u_{h(t)}, E^s_{h(t)}\} = \emptyset$

**Proposition 9.2.4.** *Suppose $L(A,\mu)=0$, then*

$$\{H_t^A E_t^u, H_t^A E_t^s\} \cap \{E_{h(t)}^u, E_{h(t)}^s\} \neq \emptyset$$

*for $\mu_K$ almost every $t \in K$.*

*Proof.* If $\lambda^+_{\mu_K}(t) = 0$ or $\lambda^+_{\mu_K}(h(t)) = 0$ the result is trivial. If not, by Proposition 8.2.1 (aplied to every ergodic component) the disintegration of the $F$ invariant measures are of the form $m_t = a(t)\delta_{E_t^u} + b(t)\delta_{E_t^s}$ with $a(t)+b(t)=1$, the same replacing $t$ for $h(t)$.

As the cocycle is fixed we denote $H_t = H_t^A$. By proposition 9.2.2 we can take a total measure set such that $H_{t*}m_t = m_{h(t)}$, this means that

$$a(t)\delta_{H_tE_t^u} + b(t)\delta_{H_tE_t^s} = a(h(t))\delta_{E_{h(t)}^u} + b(h(t))\delta_{E_{h(t)}^s}$$

almost every $t \in K$.

So $\operatorname{supp}(m_{h(t)}) = \operatorname{supp}(H_{t*}m_t)$. Then as $\operatorname{supp}(H_{t*}m_t) \subset \{H_tE_t^u, H_tE_t^s\}$ and $\operatorname{supp}(m_{h(t)}) \subset \{E_{h(t)}^u, E_{h(t)}^s\}$ the result follows. □

As a direct corollary we have

**Corollary 9.2.5.** *Given $f: M \to M$ and $\mu$ $f$ invariant with local product structure and let $A \in H^\alpha(M)$ be weakly twisting then $L(A,\mu) > 0$*

## 9.3 Proof of Theorem 9.1.4

**Lemma 9.3.1.** *Assume that $A: K \to SL(2,\mathbb{R})$ is non-uniformly hyperbolic and a weak continuity point of Lyapunov exponents, then it is a continuity point, in measure, of the Oseledets decomposition.*

*Proof.* Take an ergodic decomposition of $\mu_K$, $\{\mu_E, E \in \mathcal{P}\}$, where $\mathcal{P}$ is the partition given by the ergodic decomposition, we have that if $t \in E$ then $\lambda_A^+(t) = \lambda_A^+(E)$.

Suppose by contradiction that there exist $\delta > 0$ and $A_n \to A$ such that $\mu_k\{t \in K, \angle(E_t^{*,A_n}, E_t^{*,A}) > \delta\} > \delta$ for $* = s$ or $u$. Here we use the convention that if $\dim(E) \neq \dim(F)$, $\angle(E,F) = \pi$. Take a subsequence of $\lambda^+_{A_{n_k}}$ that converges $\mu_K$ almost every $t \in K$ to $\lambda_A^+$.

This implies that $\lambda^+_{A_{n_k}}(E) \to \lambda^+_A(E)$ for almost every $E \in \mathcal{P}$, then by Theorem C aplied to every ergodic component we have that $\mu_E\{t \in K, \angle(E_t^{*,A_{n_k}}, E_t^{*,A}) > \delta\} \to 0$, For almost every $E \in \mathcal{P}$.

Then by dominated convergence

$$\mu_K\{t \in K, \angle(E_t^{*,A_{n_k}}, E_t^{*,A}) > \delta\} = \int \mu_E\{t \in K, \angle(E_t^{*,A_{n_k}}, E_t^{*,A}) > \delta\}\, d\mu_K$$

converges zero.

This contradiction proves the Lemma.

□

**Lemma 9.3.2.** *Let $A : M \to SL(2,\mathbb{R})$ be weakly twisting and weakly pinching, such that $A\,|_K \colon K \to SL(2,\mathbb{R})$ is a weak continuity point of Lyapunov exponent then it is stable weakly twisting.*

*Proof.* Reducing $\tilde{K}$, given in definition 9.2.3, we can assume that there exist $\epsilon > 0$ such that

$$\min_{a,b\in\{u,s\}} \measuredangle \left( H_t^A E^a(t), E^b(h(t)) \right) > \epsilon$$

for every $t \in \tilde{K}$ and $\mu^c(\tilde{K}) > 2c > 0$.

Take $0 < \delta < \frac{\epsilon}{6}$ and such that $\measuredangle \left( H_t^A V, H_t^A W \right) < \frac{\epsilon}{6}$ for every $V, W \in \mathbb{R}P^1$ with $\measuredangle \left( H_t^A V, H_t^A W \right) < \delta$.

Now by the continuity of the Oseledets spaces, given by Lemma 9.3.1 , for every $B \in H^\alpha(M)$ sufficiently close to $A$ there exist $\hat{K} \subset K$ with $\mu^c(\hat{K}) > 1 - \frac{c}{3}$ such that $\measuredangle \left( E_B^*(t), E_A^*(t) \right) < \delta$, $* \in \{u, s\}$.

As in $H^\alpha(M)$, $H_t^A$ varies continuously with respect to $A$, we have that for $B$ sufficiently close to $A$

$$\measuredangle \left( H_t^B E_B^*(t), H_t^A E_B^*(t) \right) < \frac{\epsilon}{6}$$

So taking $K' = \hat{K} \cap \tilde{K}$ we have that $\mu^c(K') > \frac{2c}{3}$, then $\mu^c(h(\tilde{K} \cap \hat{K}) \cap \hat{K}) > \frac{c}{3}$.

So for every $t \in h^{-1} \left( h(\tilde{K} \cap \hat{K}) \cap \hat{K} \right)$ we have

$$\measuredangle \left( H_t^B E_B^*(t), H_t^A E_A^*(t) \right) < \frac{\epsilon}{3} \quad \text{and} \quad \measuredangle \left( E_B^*(h(t)), E_A^*(h(t)) \right) < \frac{\epsilon}{3}.$$

Then

$$\min_{a,b\in\{u,s\}} \measuredangle \left( H_t^B E_B^a(t), E_B^b(h(t)) \right) > \frac{\epsilon}{3}.$$

□

**Lemma 9.3.3.** *For every $A$ weakly pinching such that $A\,|_K \colon K \to SL(2,\mathbb{R})$ is a continuity point of Lyapunov exponent there exists $\hat{A} \in H^\alpha(M)$, weakly twisting and weakly pinching, arbitrary close such that $A\,|_K = \hat{A}\,|_K$*

*Proof.* Assume that there exist a total measure set $K'' \subset K$ with the property that for every $t \in K''$, $\{H_t^A E_t^u, H_t^A E_t^s\} \cap \{E_{h(t)}^u, E_{h(t)}^s\} \neq \emptyset$

Lets call $h_{\tilde{p},\tilde{z}}^u(t) = h^u(t)$ then

$$H_t^A = H^s_{(\tilde{z},h^u(t)),(\tilde{p},h(t))} H^u_{(\tilde{p},t),(\tilde{z},h^u(t))}.$$

By definition we have that

$$H^s_{(\tilde{z},h^u(t)),(\tilde{p},h(t))} = A(\tilde{p}, h(t))^{-1} H^s_{f(\tilde{z},h^u(t)),f(\tilde{p},h(t))} A(\tilde{z}, h^u(t))$$

Fix $r > 0$ such that $f(B(\tilde{z}, r)) \cap B(\tilde{z}, r) = f^{-1}(B(\tilde{z}, r)) \cap B(\tilde{z}, r) = \emptyset$, and define a $C^\infty$ function $\psi : \tilde{M} \to \mathbb{R}$ such that

$$\psi(\tilde{x}) = \begin{cases} 1 & \text{if} \quad \tilde{x} = \tilde{z} \\ 0 & \text{if} \quad \tilde{x} \notin B(\tilde{z}, r) \end{cases}$$

Let $\theta > 0$ be small, that we will determinate later, and multiply $A$ by the rotation of $R : M \to SL(2,\mathbb{R})$, $R_{\psi(x)\theta}$. So we have the new cocycle $\hat{A} = AR$, with stable holonomy

$$H^{s,\hat{A}}_{(\tilde{z},h^u(t)),(\tilde{p},h(t))} = A(\tilde{p},h(t))^{-1}H^s_{f(\tilde{z},h^u(t)),f(\tilde{p},h(t))}A(\tilde{z},h^u(t))R_\theta.$$

So the new map $H_t^{\hat{A}}$ corresponding to $\hat{A}$ becomes

$$H_t^{\hat{A}} = A(\tilde{p},h(t))^{-1}H^s_{f(\tilde{z},h^u(t)),f(\tilde{p},h(t))}A(\tilde{z},h^u(t))R_\theta H^u_{(\tilde{p},t),(\tilde{z},h^u(t))}$$

Fix set $\Gamma \subset K''$ with $\mu^c_{\tilde{p}}(\Gamma) > 0$ such that there exist $\gamma > 0$ and $\measuredangle\,(E^u_A(t), E^s_A(t)) > \gamma$ for every $t \in \Gamma$.

Fix $t \in \Gamma$ and denote by

$$B_t = A(\tilde{p},h(t))^{-1}H^s_{f(\tilde{z},h^u(t)),f(\tilde{p},h(t))}A(\tilde{z},h^u(t))$$
$$H'_t = H^u_{(\tilde{p},t),(\tilde{z},h^u(t))}$$

Then

$$B_t R_\theta H'_t E^*_t \in \{E^u_{h(t)}, E^s_{h(t)}\}$$

if and only if

$$R_\theta H'_t E^*_t \in \{B_t^{-1}E^u_{h(t)}, B^{-1}E^s_{h(t)}\}$$

We have two posibilities or $R_\theta H E^*_t \subset \{B^{-1}E^u_{h(t)}, B^{-1}E^s_{h(t)}\}$, for $* = u$ and $s$, or only for $s$ or $u$ (supose $s$) and there exist a positive measure sub-set of $\tilde{\Gamma} \subset \Gamma$ such that $R_\theta H E^u_t \notin \{B_t^{-1}E^u_{h(t)}, B_t^{-1}E^s_{h(t)}\}$.

For the first case, let $h(t) \in \Gamma$ taking $0 < \theta < \gamma'$, where

$$\gamma' = \min_{\measuredangle(U,V)>\gamma} \measuredangle\left(B_t^{-1}U, B_t^{-1}V\right),$$

we have that $R_\theta H'_t E^*(t) \notin \{B_t^{-1}E^u_{h(t)}, B_t^{-1}E^s_{h(t)}\}$.

For the second case, taking $\tilde{\Gamma}$ smaller if we need, there exist $\tilde{\gamma}$ such that

$$\measuredangle\left(R_\theta H'_t E^u_t, B_t^{-1}E^*_{h(t)}\right) > \tilde{\gamma}.$$

So just take $0 < \theta < \min\{\gamma', \tilde{\gamma}\}$.

Then for every $t \in h^{-1}(\tilde{\Gamma})$ we have that

$$\{H_t^{\hat{A}}E^u_t, H_t^{\hat{A}}E^s_t\} \cap \{E^u_{h(t)}, E^s_{h(t)}\} = \emptyset$$

Making $\theta$ smaller we make $\hat{A}$ closer to $A$. As the perturbation do not affect $A\,|_K$ the lemma follows □

Now we can prove Theorem 9.1.4

*Proof of Theorem 9.1.4.* By Lemma 9.3.3 there exist $\hat{A}$, arbitrary close to $A$, with the weakly twisting and weakly pinching property, such that $A\,|_K = \hat{A}\,|_K$. By Lemma 9.3.2 $\hat{A}$ is stable weakly twisting then by corollary 9.2.5 $\hat{A}$ is stably non-uniformly hyperbolic. □

# APPENDIX A

## Appendix

## A.1 Lyapunov exponents of the adjoint cocycle

Let $f : M \to M$ be a dynamical system, $A : M \to GL(d, \mathbb{C})$ and $F_A : M \times \mathbb{C}^d \to M \times \mathbb{C}^d$ the induced linear cocycle.

Let $F_{A_*} : M \times \mathbb{C}^d \to M \times \mathbb{C}^d$ be the adjoint cocycle over $f^{-1} : M \to M$, defined in 6.1

**Theorem A.1.1.** *$F_A$ and $F_{A_*}$ have the same Lyapunov exponents, also if $E^1, \cdots, E^k$ are the Lyapunov spaces corresponding to $\lambda_1, \ldots, \lambda_k$ respectively, then the Lyapunov spaces of the adjoint are*

$$E_*^j = \left[E^1 \oplus \cdots \oplus \hat{E^j} \oplus \cdots \oplus E^k\right]^\perp$$

*where $\hat{E^j}$ is the space that is not in the sum.*

*Proof.* First of all if $e_*^j(x) \in E_*^j(x)$ and $v^j \in E_*^j(f^{-1}(x))^\perp$ then

$$\left\langle v^j, A_*(x)e_*^j(x)\right\rangle = \left\langle A(f^{-1}(x))v^j, e_*^j(x)\right\rangle = 0$$

then $A_*(x)E_*^j(x) = E_*^j(f^{-1}(x))$.

So we have for almost every $x \in M$ an $\hat{A}_*$ invariant decomposition.

Call $e_*^j(x) \in E_*^j(x)$ a unitary vector and $e^j(x) \in E^j(x)$. Then for $k \neq j$

$$\left\langle e^k(f^{-n}(x)), \hat{A}_*^n(x)e_*^j(x)\right\rangle = 0$$

and

$$\left\langle e^j(f^{-n}(x)), \hat{A}_*^n(x)e_*^j(x)\right\rangle = \left\|\hat{A}_*^n(x)e_*^j(x)\right\| \cos(\alpha^j(f^{-n}(x))) \tag{A.1}$$

where

$$\begin{aligned}\alpha^j(x) &= \measuredangle\left(e_*^j(x), e^j(x)\right)\\ &= \frac{\pi}{2} - \measuredangle\left(e_*^j(x), E_x^1 \oplus \cdots \oplus \hat{E^j}_x \oplus \cdots \oplus E_x^k\right)\end{aligned}$$

so by Oseledets theorem $\lim_{n\to\infty} \frac{1}{n} \log|\cos(\alpha^j(f^{-n}(x))| = 0$.

On the other hand we have

$$\left\langle A^n(f^n(x))e^j(f^{-n}(x)), e_*^j(x)\right\rangle = \left\|A^n(f^n(x))e^j(f^{-n}(x))\right\| \cos(\alpha^j(x) \quad \text{(A.2)}$$

Now by the $A$ invariance we have $\frac{A^n(f^n(x))e^j(f^{-n}(x))}{\|A^n(f^n(x))e^j(f^{-n}(x))\|} = e^j(x)$ so

$$\begin{aligned}-\lambda_j &= \lim_{n\to\infty} \frac{1}{n} \log\left\|A^{-n}(x)e^j(x)\right\| \quad \text{(A.3)}\\ &= \lim_{n\to\infty} \frac{1}{n} \log\left\|\frac{e^j(f^{-n}(x))}{\|A^n(f^n(x))e^j(f^{-n}(x))\|}\right\|\\ &= -\lim_{n\to\infty} \frac{1}{n} \log\left\|A^n(f^n(x))e^j(f^{-n}(x))\right\|\end{aligned}$$

So by (A.1), (A.2) and (A.3)

$$\lim_{n\to\infty} \frac{1}{n}\left\|\hat{A}_*^n(x)e_*^j(x)\right\| = \lambda_j$$

□

## A.2 Density of continuous maps in $L^1(M, N)$

Let $M$ be a normal topological space and $N$ be a geodesically convex metric space (Definition 4.3.2). Denote by $\mathcal{F}$ the set of measurable maps $f : M \to N$. Given any regular $\sigma$-finite Borel measure $\mu$ on $M$, fix any point $\hat{0} \in N$ and define

$$L_\mu^1(M, N) = \{f \in \mathcal{F} : \int \operatorname{dist}_N\left(f(x), \hat{0}\right) d\mu(x) < \infty\}.$$

When $\mu$ is a finite measure, the choice of $\hat{0} \in N$ is irrelevant: different choices yield the same space $L_\mu^1(M, N)$.

The function $\operatorname{dist}_{L_\mu^1(M,N)} : L_\mu^1(M, N) \times L_\mu^1(M, N) \to \mathbb{R}$ defined by

$$\operatorname{dist}_{L_\mu^1(M,N)}(f, g) = \int d_N\big(f(x), g(x)\big) d\mu(x)$$

is a distance in $L_\mu^1(M, N)$. In this appendix we prove

**Proposition A.2.1.** *The subset of continuous maps $f : M \to N$ is dense in the space $L_\mu^1(M, N)$.*

We call $s : M \to N$ a *simple map* if there exist points $v_1, \dots, v_k \in N$ pairwise disjoint measurable sets $A_1, \dots, A_k \subset M$ with finite $\mu$-measure such that

$$s(x) = \begin{cases} v_i & \text{if } x \in A_i \\ \hat{0} & \text{if } x \notin \cup_{i=1}^k A_i \end{cases}$$

**Lemma A.2.2.** *The set $\mathcal{S}$ of simple functions is dense in $L^1_\mu(M, N)$.*

*Proof.* Consider any $f \in L^1_\mu(M, N)$. Given $\epsilon > 0$, fix a set $K_0 \subset M$ with finite $\mu$-measure and such that

$$\int_{M \setminus K_0} \operatorname{dist}_N(f(x), \hat{0})\, d\mu(x) < \frac{\epsilon}{4}.$$

Let $\{v_1, \dots, v_i, \dots\}$ be a countable dense subset of $N$. The family

$$\{B\big(v_i, \frac{\epsilon}{\mu(K_0)}\big) : i \in \mathbb{N}\}$$

covers $N$ and, consequently,

$$B_i = B\big(v_i, \frac{\epsilon}{2\mu(K_0)}\big) \setminus \bigcup_{j<i} B\big(v_i, \frac{\epsilon}{2\mu(K_0)}\big), \quad i \in \mathbb{N}$$

is a partition of $N$. Then $A_i = K_0 \cap f^{-1}(B_i)$, $i \in \mathbb{N}$ is a partition of $K_0$ into measurable sets. Fix $k \in \mathbb{N}$ large enough that

$$\int_{K_0 \setminus \bigcup_{i=1}^k A_i} \operatorname{dist}_N(f(x), \hat{0})\, d\mu(x) < \frac{\epsilon}{4}.$$

Now define $s : M \to N$ by

$$s(x) = \begin{cases} v_i & \text{if } x \in A_i \text{ for } i = 1, \dots, k \\ \hat{0} & \text{if } x \notin \cup_{i=1}^k A_i. \end{cases}$$

Then

$$\int_{M \setminus \cup_{i=1}^k A_i} \operatorname{dist}_N(f(x), s(x))\, d\mu(x) = \int_{M \setminus \cup_{i=1}^k A_i} \operatorname{dist}_N(f(x), \hat{0})\, d\mu(x) < \frac{\epsilon}{2}$$

and

$$\int_{\cup_{i=1}^k A_i} \operatorname{dist}_N(f(x), s(x))\, d\mu(x) \leq \mu\big(\cup_{i=1}^k A_i\big) \frac{\epsilon}{\mu(K_0)} < \frac{\epsilon}{2}.$$

Thus $\operatorname{dist}_{L^1_\mu(M,N)}(f, s) < \epsilon$, which proves the lemma. □

**Lemma A.2.3.** *For every $s \in \mathcal{S}$ and $\epsilon > 0$ there exists a continuous map $f : M \to N$ such that $\operatorname{dist}_{L^1_\mu(M,N)}(f, s) < \epsilon$.*

*Proof.* Let $A_i$ and $v_i$, $i = 1, \ldots, k$ be as in the definition of the simple map $s$. For each $i = 1, \ldots, k$, consider a compact set set $K_i \subset A_i$ such that $\mu(A_i \setminus K_i) < \epsilon$. Since the $K_i$ are are pairwise disjoint, and $M$ is assumed to be normal, there exist pairwise disjoint open sets $B_i \supset K_i$, $i = 1, \ldots, k$ with $\mu(B_i \setminus K_i) < \epsilon$. By Urysohn, we can find continuous functions $\psi_i : M \to \mathbb{R}$, $i = 1, \ldots, k$ such that

$$\psi_i(x) = \begin{cases} 1 & \text{if } x \in K_i \\ 0 & \text{if } x \notin B_i. \end{cases} \tag{A.4}$$

Now we use the assumption that $N$ is geodesically convex. For each $i = 1, \ldots, k$, fix $\lambda_i : [0, 1] \to N$ with $\lambda_i(1) = v_i$ and $\lambda_i(0) = \hat{0}$. Then define $f : M \to N$ by

$$f(x) = \begin{cases} \lambda_i(\psi_i(x)) & \text{if } x \in B_i \text{ with } i = 1, \ldots, k \\ \hat{0} & \text{if } x \notin \bigcup_{i=1}^{k} B_i. \end{cases}$$

It is clear that $f$ is continuous, because the $B_i$ are open and pairwise disjoint. Moreover, $f(x) = s(x)$ for every

$$x \notin \bigcup_{i=1}^{k} A_i \setminus K_i \cup B_i \setminus K_i$$

So if $C = \max \tau \operatorname{dist}_N(v_i, \hat{0})$ then $\operatorname{dist}_{L^1_\mu(M,N)}(f, s) < 2Ck\epsilon$. $\square$

Proposition A.2.1 is an immediate consequence of Lemmas A.2.2 and A.2.3.

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