GEOMETRIC STRUCTURES ON DEGREE 2 MANIFOLDS

By

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Abstract

This thesis deals with the issue of expressing structures on degree 2 graded manifolds (such as homological vector fields and Poisson brackets) in classical geometrical terms. We take as a starting point the equivalence, established by H. Bursztyn et al in [6], of the category of degree 2 manifolds with the category of certain exact sequences of vector bundles that we call *involutive*. Building on this equivalence, we describe various objects on degree 2 manifolds in terms of the corresponding vector-bundle information.

First, we characterize degree 3 functions as certain pairs of vector-bundle morphisms. Then we give a characterization of Q-structures (degree 1 homological vector fields) on degree 2 manifolds which leads to a geometric object, defined by maps and brackets on certain sequences of vector bundles, that we call a *Lie 2-algebroid*; upon an additional choice of splittings, these correspond to the existing notion of Lie 2-algebroid defined on a graded vector bundle (also known as *2-term L_∞-algebroid*). We also clarify the connection between Lie 2-algebroids, VB-Courant algebroids (due to D. Li-Bland) and exact V-twisted Courant algebroids (introduced by M. Grutzmann and T. Strobl).

We also describe degree -2 Poisson brackets in terms of a (degenerate) metric and a Lie algebroid structure on the vector-bundle side, which intertwine through a metric preserving representation of the Lie algebroid. By means of the double realization functor, we establish an equivalence with a certain category of double vector bundles endowed with an additional structure, we call *involutive*, and describe degree -2 Poisson brackets on degree 2 manifolds in terms of *double linear* Poisson structures on the corresponding double vector bundle, invariant under the involutive structure, which are shown to be equivalent to a metric *VB*-algebroid structure on the dual (as defined by M. Jotz). Finally, we characterize integrable degree 3 functions on a degree 2 Poisson manifold, that is, a degree 3 function θ satisfying $\{\theta, \theta\} = 0$, which appears naturally in the quotient of a Courant algebroid by symmetries, in terms of vector bundle morphisms that relate the Lie 2-algebroid corresponding to the hamiltonian vector field $Q = \{\theta, \cdot\}$ and the *VB*-algebroid corresponding to $\{\cdot, \cdot\}$. The resulting object generalizes Courant algebroids, which are recovered when the Poisson bracket is symplectic.

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blessed Virgin Mary

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Contents

Chapter 1

Introduction

1.1 Poisson, Lie and Courant brackets

Poisson brackets were introduced independently in the 19th century by the French mathematicians, J. L. Lagrange and S. D. Poisson, in their study of mechanics, fluids and elasticity. Afterwards the geometric, local structure of Poisson manifolds was studied by the Norwegian mathematician S. Lie, motivated by his study of continuous symmetries and differential equations.

In 20th century, Poisson brackets were considered by the physicist P. A. Dirac in his effort of quantizing mechanical systems with constraints, and more generally of finding a mathematical formulation of mechanics suitable to perform quantization. This program was followed, among others, by J.-M. Souriau, A. Lichnerowicz, B. Kostant, S. Sternberg and A. Weinstein with several remarkable contributions (see e.g. [43],[70]).

The infinitesimal invariants of a Poisson structure are encoded by a structure which belongs to the differential geometric generalization of a Lie algebra, and is called *Lie algebroid* ([55],[46]). A Lie algebroid generalizes simultaneously Lie algebras, tangent bundles and (infinitesimal) actions of Lie groups on manifolds. It comprises a Lie bracket $[\cdot,\cdot]$ on the sections of a vector bundle $A \longrightarrow M$, and a vector bundle map $\rho : A \longrightarrow TM$ over the identity, such that

$$[X, fY] = f[X,Y] + \rho(X)(f)Y, \qquad \forall X, Y \in \Gamma(A), f \in C^{\infty}(M).$$

For the case of a Poisson structure (M,π), we have the *cotangent Lie algebroid* ([15],[13]), $(T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$, where $\pi^\sharp : T^*M \longrightarrow TM$ is the map induced by the bivector $\pi\Lambda^2 TM$ and the bracket is determined by the condition

$$[df, dg]_\pi = d(\pi(df, dg)) = d\{f, g\}, \qquad \forall f, g \in C^{\infty}(M).$$

Usually in a mechanical system, given by a hamiltonian function on a symplectic manifold, there appear constraints, which restrict the system to a submanifold, that inherits a presymplectic structure; on the other hand a mechanical system usually has symmetries, that leads naturally to work in a quotient manifold which inherits a Poisson structure. Therefore it raises the interest of having a unified setting that comprises both structures. In the late 1980's, T. J. Courant [14], discovered such unifying structure by means of a new bilinear skew-symmetric map on sections of the *generalized tangent bundle* $\mathbb{T}M = TM \oplus T^*M$,

$$[\![(X,\alpha),(Y,\beta)]\!] = \left([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\langle X,\beta\rangle - \langle Y,\alpha\rangle)\right). \tag{1.1}$$

Independently, in the context of two dimensional variational problems, I. Ya. Dorfman introduced a *non skew-symmetric* version of this bracket,

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \iota_Y d\alpha). \tag{1.2}$$

As it is readily seen, the bracket (1.1) is the skew-symmetrization of (1.2). This bracket (or equivalently its skew-symmetrization) is the key concept to encode in a common setting the integrability conditions for Poisson brackets and presymplectic forms, namely, the former is given by $[\pi,\pi] = 0$, where $\pi \in \Gamma(\Lambda^2 TM)$ is the Poisson bivector and $[\cdot,\cdot]$ is the Schouten bracket, and the latter is given by $d\omega = 0$, where $\omega \in \Gamma(\Lambda^2 T^*M)$ is the 2-form, and d is the de Rham exterior differential. The geometrical structure that comprises both cases, called a *Dirac structure*, is a subbundle of $\mathbb{T}M$ *maximally isotropic* with respect to the natural symmetric non-degenerate pairing on $\mathbb{T}M$, whose space of sections is *closed* with respect to the Courant-Dorfman bracket (1.2), and turns out to be a Lie algebroid. Dirac structures interpolate many of the classical geometries besides symplectic and Poisson geometry, for example the geometry of foliations and complex geometry are encompassed by this structure.

Later it was raised the question of what would be the corresponding concept of *double* of a Lie bialgebroid, in analogy with the so-called *Drinfel'd doubles* for Lie bialgebras. This was achieved in a joint work by A. Weinstein, Z.-J. Liu and P. Xu [44], in which they axiomatized the general notion of a *Courant algebroid* (see Def. 5.1 of the present work), which has as a particular case the *standard* Courant algebroid $(\mathbb{T}M, [\![\cdot,\cdot]\!], \rho, \langle\cdot,\cdot\rangle)$, where $\mathbb{T}M = TM \oplus T^*M$, $[\![\cdot,\cdot]\!]$ is the bracket given by Eq. (1.2), $\rho : \mathbb{T}M \longrightarrow TM$ is the projection, and $\langle\cdot,\cdot\rangle$ is the natural duality pairing $\langle (X,\alpha),(Y,\beta)\rangle = \alpha(Y) + \beta(X)$.

1.2 The supergeometric viewpoint

That rich geometric structures can be encoded by very simple objects on graded super manifolds is known since at least back to the works of Y. Kosmann-Scwarzbach, F. Magri [37], M. Rothstein [57] and A. Yu. Vaǐntrob [67]. At a first level we have the correspondence of degree 1 manifolds with vector bundles, which leads to the following further correspondences:

Classical geometric structures	**Geometry on degree 1 manifolds**
Lie algebroid $(A, [\cdot,\cdot], \rho)$	degree 1 homological ($Q^2 = 0$) vector field Q
Lie algebroid $(A^*, [\cdot,\cdot], \rho)$	degree -1 Poisson brackets $\{\cdot,\cdot\}$
Poisson manifold $(M, \{\cdot,\cdot\})$	degree 2 function π satisfying $\{\pi,\pi\} = 0$
Lie bialgebroid on (A, A^*)	degree 1, homological, Poisson vector field Q

This picture becomes even more interesting when one considers degree 2 manifolds. Indeed, D. Roytenberg, following the ideas of A. Y. Vaǐntrob [67] and Pavol Ševera [62],[63], obtained a super-geometric interpretation of Courant algebroids as degree 3 integrable functions on symplectic degree 2 manifolds, bringing fruitful new insights into the geometry of Courant algebroids [59]. So, we can extend the picture above to include the following equivalences between classical and super sides of geometry:

Classical geometric structures	**Geometry on degree 2 manifolds**
Pseudo-euclidean vector bundle $(E^*, \langle\cdot,\cdot\rangle)$	symplectic degree 2 manifold $(\mathcal{M}, \{\cdot,\cdot\})$
Courant algebroid $(E^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$	symplectic degree 2 manifold $(\mathcal{M}, \{\cdot,\cdot\})$ + degree 3 function θ satisfying $\{\theta,\theta\}$

In particular, the super-geometric interpretation of Courant algebroids and their Dirac structures, draws a parallel between these and Poisson manifolds and their coisotropic submanifolds, providing new insights to their study, suggesting a certain parallel between both theories, and it has been indeed the case in some aspects such as reduction, see e.g. [7],[9],[10]. Also in [63] it was pointed out that a class of Courant structures, called *exact Courant algebroids*, should play the role in string theory analogue to the role played by Poisson structures in classical physics.

When a Lie group acts on a symplectic manifold preserving the structure, the corresponding Poisson brackets can be naturally transported to Poisson brackets of functions on the quotient space, which are naturally identified with those functions on the manifold that are invariant under the action of the group. Similarly, we can define the action of a (graded) Lie group on a degree 2 symplectic manifold, preserving the structure. Then, as in the classical case, we can transport the Poisson brackets to the space of functions on the quotient manifold. Moreover, if there is a degree 3, integrable function on the manifold, invariant under the action of the group, then it corresponds to an integrable degree 3 function on the quotient. This motivates the following questions, whose answers are the purpose of the present work.

Question 1. *What is the classical geometric counterpart for degree 2 Poisson manifolds?*

Question 2. *What is the classical geometric counterpart for integrable degree 3 functions on a degree 2 Poisson manifold*?

In the symplectic case, the hamiltonian vector fields generate the module of vector fields. This allows us to understand the geometric structure of a degree 3 function just in terms of functions of degrees 0 and 1, which give rise to the Courant algebroid structure. When the non-degeneracy condition of the Poisson brackets is broken, we need to work directly with the vector fields of negative degree, whose action on a degree 3 function extract all the information about it. Whereby, in order to answer questions *1* and *2*, we need first answer three more questions:

Question 1a. *What is the classical geometric counterpart for degree 2 manifolds (without any additional structure)?*

Question 2a. *What is the classical geometric counterpart for degree 3 functions on a degree 3 manifold?*

Question 2b. *What is the classical geometric counterpart for degree 1, homological vector fields on a degree 2 manifold?*

The answer to question *1a* was given by H. Bursztyn et al in [6]. The answers to the remaining questions are the content of the present work, which we now describe.

After recalling preliminary facts about double vector bundles in **Ch. 2**, we introduce in **Ch. 3** the concept of a degree 2 manifold $\mathcal{M}$, and find an answer for question *1a* by means of a subcategory of double vector bundles, which we call *involutive*, which consists of pairs (D, H), where D is a *double vector bundle* $(D; A, B; M)_C$, and H, called *involutive structure*, is a *double vector bundle morphism*

$$H : D_A \longrightarrow D_B$$

such that $h_A = -h_B^{-1} : A \longrightarrow B$, $h_C = \mathrm{Id}_C$ and $H^4 = \mathrm{Id}$. The morphisms of the category are double vector bundle morphisms which commute with the respective involutive structures. There are some remarkable facts about this structure:

- The isomorphism H induces an automorphism on $C^\infty(D)$ by pull-back, which even though does not preserve the natural bidegree of the subalgebra

$$C^{\cdot,\cdot}(D) \subset C^\infty(D),$$

  it does preserve the *total degree*[1]. Moreover, for functions on $\mathcal{M}$ of degrees 0,1 and 2, we have the following correspondences:

degree 0: these functions are isomorphic to functions on the body M, and we can pull back them to D.

degree 1: these functions correspond to linear functions on the side bundle E, which can be pulled back to D.

degree 2: these functions correspond to sections of the involutive bundle, which are isomorphic to those double linear functions on D invariant under H.

- By means of a sort of transpose of H we obtain a symmetric double vector bundle isomorphism over the identity

$$\Phi : (D_B^*)_{C^*} \longrightarrow (D_B^*)_{C^*}^*,$$

  which establishes an equivalence between involutive structures on a double vector bundle and *linear metrics* on its dual, recovering the equivalence of categories between degree 2 manifolds and metric double vector bundles obtained independently by M. Jotz in [29].

- The degree -1 and -2 vector fields on $\mathcal{M}$ are naturally identified with linear and core sections, respectively, on D_B (or equivalently D_A, but we choose to stick with the horizontal structure). This allows us to obtain a characterization of degree 3 functions, as pairs of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ satisfying certain compatibility conditions. The first one is a morphism between the core bundles of D and D_B^*. The second one is a morphism between the linear bundle of D and the *involutive bundle* of D_B^*, which is the bundle of double-linear functions on D that are H-invariant. Thus we have an answer for question *2a*.

[1] If a function $f \in C^{\cdot,\cdot}(D)$ has bidegree (p, q), then its total degree is $p + q$.

- Linear sections of D_B are isomorphic to linear sections of $(D_B^*)_{C^*}$ and core sections of D_B are isomorphic to linear functions on the base of $(D_B^*)_{C^*}$. This identifications, together with the ones described in the last item, allows to identify the linear metric on $(D_B^*)_{C^*}$ evaluated on two linear sections, with the commutator bracket on $\mathfrak{X}(\mathcal{M})$ applied to the corresponding degree -1 vector fields.

Then, in **Ch. 4**, in order to find the geometric structure encoded by a homological [2] degree 1 vector field, $Q \in \mathfrak{X}(\mathcal{M})$, we apply the so-called *derived brackets method*, and find a bracket-anchor structure on the vector bundles that correspond to degree -1 and -2 vector fields, which are identified with the linear and core bundles of D_B, respectively. We find the compatibility condition these structure must satisfy in order to characterize completely a degree 1 vector field, and call this structure a *preLie 2-algebroid*. Hereupon, we find the integrability equations that are equivalent to the homological condition of Q, and call a structure satisfying such conditions, a *Lie 2-algebroid*, therefore finding an answer to question *2b*. We show that when a splitting is chosen, Lie 2-algebroids correspond exactly to *split* Lie 2-algebroids, which appear for example on [4] and [24], or equivalently, as shown in [29], to *Dorfman 2-representations*.

Under the identifications mentioned above, between the distinguished functions and sections of D_B and $(D_B^*)_{C^*}$, in **Ch. 5** we are able to find an equivalence between Lie 2-algebroids and *VB-Courant algebroids* [41], and also with *exact V-twisted algebroids* [24].

Using the natural correspondences, mentioned above, between functions of degrees 0,1 and 2 on $\mathcal{M}$ and functions of bidegrees $(0,0),(0,1),(1,0)$ and $(1,1)$ on the corresponding involutive double vector bundle, we obtain in **Ch. 6** a straight-forward characterization of -2 Poisson on $\mathcal{M}$ as double linear Poisson brackets on D *invariant under* H, thus obtaining an answer for question *1*. Under the equivalent viewpoint of metric double vector bundles, the double lienar Poisson brackets on D which are H-invariant, correspond to VB-algebroid structures on D_B^* compatible with the metric, in the sense that Φ is Lie algebroid isomorphism, where $(D_B^*)^*_{C^*}$ is endowed with the natural dual VB-algebroid structure associated (the structure that comes from the double linear Poisson structure dualized with respect to fibration over A). Thus, we recover the equivalence between degree 2 Poisson manifolds and metric VB-algebroids in [29]. We also obtain a classification of degree 2, *regular* Poisson manifolds in terms of certain Chevalley cohomology groups naturally associated to them.

Finally, in **Ch. 7**, we put together the four answers found above to questions *1a, 1, 2a* and *2b*, in order to give the answer to question *2* on the geometric characterization of integrable degree 3 functions on a degree 2 Poisson manifold. As we noted above, the answer obtained by D. Roytenberg in the symplectic case exploits the fact that hamiltonian fields generated by degree 0 functions span the $C^\infty(M)$-module of degree -2 vector fields, and hamiltonian fields generated by degree 1 functions span the $C^\infty(M)$-module of degree -1 vector fields, and these suffice in order to determine a degree 3 function θ, and the degree 4 function $\{\theta,\theta\}$, yielding a Courant algebroid $(\langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$, which encode θ, satisfying Jacobi identity, which is encoded in $\{\theta,\theta\} = 0$.

[2] Homological means $Q^2 = 0$, which is equivalent to $[Q,Q] = 0$.

In the possibly degenerate case, we must determine both θ and $\{\theta,\theta\}$ no longer using functions of degrees 0 and 1, but directly the vector fields of degrees -2 and -1, thus yielding the pair of morphisms $(\theta_1^\sharp,\theta_2^\sharp)$ encoding θ, as we explained above, which intertwine the linear bundle of D_B, $\widehat{E}$, whose sections are isomorphic to degree -1 vector fields, and the involutive bundle on D_B^*, $\widetilde{F}^*$, whose sections are isomorphic to degree 2 functions. Next we find expressions for the Lie 2-algebroid structure corresponding to the vector field $Q=\{\theta,\cdot\}$ in terms of the pair $(\theta_1^\sharp,\theta_2^\sharp)$ and of the VB-algebroid structure corresponding to $\{\cdot,\cdot\}$ and finally we encode the equation $\{\theta,\theta\}=0$ in terms of a curvature term measuring the defect of the morphism $\theta_2^\sharp$ to preserve the corresponding brackets on $\widetilde{F}^*$ and on $\widehat{E}$:

$$\frac{1}{2}\theta_2^\sharp(R_\theta(\phi_1,\phi_2))=[\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]_{\widetilde{F}^*}-\theta_2^\sharp([\phi_1,\phi_2]_{\widehat{E}}),\qquad \forall\phi_1,\phi_2\in\Gamma(\widehat{E}),$$

where $R_\theta:\Gamma(\widehat{E})\times\Gamma(\widehat{E})\longrightarrow\Gamma(\widehat{E})$ is defined in terms of the metric VB-algebroid structure of D_B^*. Such a geometric structure, defined for the triple $(\widehat{E},\widetilde{F}^*,(\theta_1^\sharp,\theta_2^\sharp))$ we call *degenerate Courant algebroid*, which is the geometric object we obtain on the quotient when we have a group acting on a Courant algebroid.

We end this introduction with a summary of the equivalences between classical and super geometric structures we found.

Classical geometric structures	**Graded super geometric data**
involutive double vector bundle (D, H) *or equivalently* Metric double vector bundle $(D_B^*, \langle\cdot,\cdot\rangle_{C^*})$	degree 2 manifold $\mathcal{M}$
sections of the core bundle $E^* \subset D_B^*$	degree 1 functions on $\mathcal{M}$
sections of the core bundle $F \subset D$	degree -2 vector fields on $\mathcal{M}$
bidegree $(1,1)$ functions on D invariant under H *or equivalently* sections of the involutive bundle $\widetilde{F}^* \hookrightarrow D_B^*$	degree 2 functions on $\mathcal{M}$
sections of the linear bundle $\widehat{E} \hookrightarrow D_B$	degree -1 vector fields on $\mathcal{M}$
pairs of compatible morphisms $\theta_1^\sharp : F \longrightarrow E^*$, $\theta_2^\sharp : \widehat{E} \longrightarrow \widetilde{F}^*$	degree 3 functions θ on $\mathcal{M}$
Lie 2-algebroids $([\cdot,\cdot]_{\widehat{E}}, \rho, \partial, \Psi, \Theta)$ *or equivalently* VB-Courant algebroids $((D_B^*)_{C^*}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$	homological degree 1 vector field $Q \in \mathfrak{X}(\mathcal{M})_1, \quad Q^2 = 0$
involutive double linear Poisson brackets on D *or equivalently* metric VB-algebroids $(D_B^*, [\cdot,\cdot]_B, \rho_D, \langle\cdot,\cdot\rangle_{C^*})$	degree 2 Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\})$
generalized Courant algebroids	degree 3 function θ on $(\mathcal{M}, \{\cdot,\cdot\})$ satisfying $\{\theta,\theta\} = 0$

Chapter 2

Preliminaries

2.1 Double vector bundles and morphisms

2.1.1 Definitions and examples

The concept of double vector bundle was originally introduced by J. Pradines [56]. The main references for this section are [32],[46] and [23]. For convenience of the reader, we include in App. A all the complementary facts and details about double vector bundles that will be used in the present work.

Definition 2.1. Consider the commutative square

$$\begin{array}{ccc} D & \xrightarrow{q_A} & A \\ {\scriptstyle q_B}\downarrow & & \downarrow{\scriptstyle q^A} \\ B & \xrightarrow{q^B} & M, \end{array} \tag{2.1}$$

where each side is a vector bundle. The diagram (2.1) is called a *double vector bundle*, and denoted by $(D; A, B; M)$, if the structure maps of the vertical vector bundle structures (projections, zero sections, addition, scalar multiplication) are vector bundle maps with respect to the horizontal vector bundle structures. This requirement is equivalent to the following conditions:

(a) $q_A(d_1 \underset{B}{+} d_2) = q_A(d_1) + q_A(d_2)$

(b) $q_B(d_1 \underset{A}{+} d_3) = q_B(d_1) + q_B(d_3)$

(c) $(d_1 \underset{B}{+} d_2) \underset{A}{+} (d_3 \underset{B}{+} d_4) = (d_1 \underset{A}{+} d_3) \underset{B}{+} (d_2 \underset{A}{+} d_4)$,

for quadruples $d_1, \dots, d_4 \in D$ such that $q_B(d_1) = q_B(d_2)$, $q_B(d_3) = q_B(d_4)$, $q_A(d_1) = q_A(d_3)$ and $q_A(d_2) = q_A(d_4)$. We denote the fibration $D \longrightarrow A$ by D_A and the fibration $D \longrightarrow B$ by D_B. The zero sections are denoted by

$$0^A : M \longrightarrow A; \qquad 0_A : A \longrightarrow D_A;$$
$$0^B : M \longrightarrow B; \qquad 0_B : B \longrightarrow D_B.$$

A general principle that we will use frequently is that in order to verify a property involving linearity, it suffices to verify this property only with respect to the addition, because from there it follows that the property holds for the scalar multiplication with integers, consequently with rationals and hence with all real numbers, because of continuity.

It so happens that, because of the compatibility of the two vector bundle structures on a DVB, the vector bundle structures on D_A and D_B coincide on the intersection $C := \ker q_A \cap \ker q_B$. And C is itself a vector bundle over M, with projection $q^C := q^A \circ q_A|_C = q^B \circ q_B|_C$ (see Prop. A.3). C is called the *core bundle* of D. For a double vector bundle $(D; A, B; M)$ we introduce the notation $(D; A, B; M)_C$ to specify that the core bundle is C. When we describe D with a commutative square like (2.1), we will write

$$\begin{tikzcd} D \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd}$$

to specify that the core bundle is $C \longrightarrow M$.

Definition 2.2. A morphism of double vector bundles, say D, D', is a map $\Phi : D \longrightarrow D'$ which is linear with respect to both fibrations. We will use sometimes the abbreviation *DVB morphism*.

Because of the fiber preserving condition, any DVB morphism $\Phi : D \longrightarrow D'$ preserves the core bundle C, moreover, it induces maps $\varphi_A : A \longrightarrow A'$, $\varphi_B : B \longrightarrow B'$, $\varphi_C, \varphi_M : M \longrightarrow M'$ such that each of (Φ, φ_B), (Φ, φ_A), (φ_A, φ_M) (φ_B, φ_M) and (φ_C, φ_M) is a morphism of the relevant vector bundles. See Props. A.1 and A.9.

Remark 2.3. Frequently we will use suffixes on the respective bundles, for example D_A and $D'_{A'}$, to indicate which fibration on D is being mapped to which fibration on D'. When we omit this specification, it is because the context makes it sufficiently clear, so that we chose not to overload the notation.

Example 2.4. The most fundamental example is the *decomposed* DVB: given three vector bundles A, B and C over the same base manifold M, we set

$$D := A \oplus B \oplus C.$$

Define the projections over A and B in the obvious way: $q_A(a, b, c) = a$ and $q_B(a, b, c) = b$, and both addition operations are also obvious to define:

$$(a, b, c) \underset{A}{+} (a, b', c') = (a, b + b', c + c'); \qquad (a, b, c) \underset{B}{+} (a', b, c') = (a + a', b, c + c').$$

It is explained in the appendix (see Cor. C.8) that, up to double vector isomorphisms, this class covers all the examples of double vector bundles.

A double vector bundle morphism between two decomposed double vector bundles

$$\Phi : A \oplus B \oplus C \longrightarrow A' \oplus B' \oplus C',$$

has the form

$$\Phi(a,b,c) = (\varphi_A(a), \varphi_B(b), \varphi_C(c) + \Psi(a,b)),$$

where the mapping

$$\Psi : A \oplus B \longrightarrow C'$$

is bilinear (see the proof in the appendix, Cor. A.24). If Φ is an isomorphism, then its inverse is given by

$$\Phi^{-1}(a',b',c') = (\varphi_A^{-1}(a'), \varphi_B^{-1}(b'), \varphi_C^{-1}(c') - \varphi_C^{-1} \circ \Psi(\varphi_A^{-1}(a'), \varphi_B^{-1}(b'))).$$

Given a vector bundle $A \longrightarrow M$, its tangent bundle is endowed with a double vector bundle structure ([32],[46])

$$\begin{tikzcd} TA \arrow[r, "q_A"] \arrow[d, "q_{TM}"'] & A \arrow[d, "q^A"] \\ TM \arrow[r, "q^{TM}"] & M, \end{tikzcd} \tag{2.2}$$

where $q_{TM} := dq^A$, where d is the differential. Addition with respect to the vertical structure ($TA \xrightarrow{dq^A} TM$) is again given by a differential:

$$d+ : T(A \times A) \cong TA \times TA \longrightarrow TA,$$

where $+ : A \times A \longrightarrow A$ is the addition in $A \xrightarrow{q^A} M$.

Finally, the zero section 0_{TM} is once more given by a differential: $0_{TM} := d(0^A) : TM \longrightarrow TA$.

The cotangent bundle T^*A is also endowed with a natural double vector bundle structure

$$\begin{tikzcd} T^*A \arrow[r, "\pi_A"] \arrow[d, "\pi_{A^*}"'] & A \arrow[d, "q^A"] \\ A^* \arrow[r, "q^{A^*}"] & M. \end{tikzcd} \tag{2.3}$$

Its double vector bundle structure is best understood as a consequence of being the *dual* (see Sec. 2.1.4 below) of the tangent bundle, we give the details in Ap. F. In particular, there it is shown that a decomposition of TA, or equivalently T^*A amounts to *choosing a linear connection* on A. We encourage the reader to constantly consult Ap. F along the reading of each new concept and result on double vector bundles, in order to gain familiarity with the concrete examples studied in that appendix.

Also it is well-known ([14],[48]) that a Poisson structure on a vector bundle A is *linear*, that is, the Poisson brackets of two functions linear on the fibers is again linear, if and only if the induced morphism $\pi^\sharp : T^*A \longrightarrow TA$ is a double vector bundle morphism.

Definition 2.5. Given a double vector bundle D, as in (2.1), the *flip* of D is the double vector bundle (A.1) (see Prop. A.2). That is, the flip of $(D; A, B; M)$ is $(D; B, A; M)$.

2.1.2 Core and linear sections

A *core section* of D_B is a section of the form

$$\iota \circ \alpha \circ q^B \underset{A}{+} 0_B,$$

where $\alpha : M \longrightarrow C$ is a section of C, ι is just the inclusion $C \hookrightarrow D$ and $\tilde{0}_B$ is the zero section of D_B. We denote the core section corresponding to $\alpha \in \Gamma(C)$ by $\tilde{\alpha}$, and the space of core sections by $\Gamma_{\text{core}}(D_B)$.

Analogously, we can define $\Gamma_{\text{core}}(D_A)$. Of course, $\Gamma_{\text{core}}(D_B)$ and $\Gamma_{\text{core}}(D_A)$ are $C^\infty(M)$-modules, and we have the obvious isomorphisms of modules:

$$\Gamma_{\text{core}}(D_B) \cong \Gamma(C) \cong \Gamma_{\text{core}}(D_A).$$

It is easy to see (Prop. A.10) that core sections are preserved by DVB morphisms which are the identity over B. There is another distinguished kind of sections of D_B that are preserved by such morphisms, which are called *linear sections*. A section $\gamma \in \Gamma(D_B)$ is linear if γ is a bundle morphism from B to D_A, which will be necessarily over a section α of A. The space of linear sections will be denoted by $\Gamma_{\text{lin}}(D_B)$.

Remark 2.6. The space of linear sections, $\Gamma_{\text{lin}}(D_B)$, is an $\mathbb{R}$-vector space, and also a $C^\infty(M)$-module in the natural way and it can be shown that actually $\Gamma_{\text{lin}}(D_B) \cong \Gamma(\widehat{A})$, for some vector bundle $\widehat{A}$, called the *linear bundle*. Moreover, $\widehat{A}$ fits in the exact sequence

$$0 \longrightarrow B^* \otimes C \xrightarrow{\iota} \widehat{A} \xrightarrow{p} A \longrightarrow 0,$$

called the *linear sequence* corresponding to D_B. We have an explicit description for the fibers of $\widehat{A}$, given by

$$\widehat{A}_m := \{\sigma \in \text{Hom}(B_m, D_a) : a \in A_m,\ q_B \circ \sigma = \text{Id}_{B_m}\},$$

where B_m, A_m are the fibers of B and A over $m \in M$, respectively, and D_a is the fiber of D_A over $a \in A$. See Prop. C.2 for more details and proofs.

Analogously, the linear bundle corresponding to $\Gamma_{\text{lin}}(D_A)$, denoted by $\widehat{B}$, fits in the exact sequence

$$0 \longrightarrow A^* \otimes C \xrightarrow{\iota} \widehat{B} \xrightarrow{p} B \longrightarrow 0,$$

and its fiber over $m \in M$ is given by

$$\widehat{B}_m = \{\omega \in \text{Hom}(A_m, D_b) : b \in B_m,\ q_A \circ \omega = \text{Id}_{A_m}\}.$$

A DVB morphism $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ whose induced morphism on the base side bundle $\varphi_B : B \longrightarrow B'$ is invertible, induces naturally a morphism between the corresponding linear bundles

$$\widehat{\Phi}_B : \widehat{A} \longrightarrow \widehat{A'},$$

given by $\widehat{\Phi}_B(\sigma) := \Phi \circ \sigma \circ \varphi_B^{-1}$. For more on induced morphisms on the corresponding linear bundles see App. C, Sec. C.3.

2.1.3 Decompositions

Definition 2.7. Let $(D; A, B; M)$ be a double vector bundle, with core bundle C. A *decomposition* of D is a double vector bundle isomorphism between

$$\Theta : D \longrightarrow A \oplus B \oplus C,$$

inducing the identity map on A, B and C.

A *horizontal lift* of the linear sequence corresponding to $\widehat{A}$ above, is a linear map $\psi : A \longrightarrow \widehat{A}$ such that $p \circ \psi = \mathrm{Id}_A$.

To each horizontal lift ψ on $\widehat{A}$ there exists a corresponding horizontal lift, $\overline{\psi}$ on $\widehat{B}$, the relation between them given by

$$\overline{\psi}(b)(a) := \psi(a)(b).$$

It is easy to see that a decomposition Θ of D is equivalent to a projection $q_C : D \longrightarrow C$ which is linear with respect to both structures, D_A and $_B$. The decomposition is given by

$$\Theta = (q_A, q_B, q_C).$$

Moreover, the decomposition can be shown to be equivalent to a horizontal lift of any of its associated linear sequences, and therefore a double vector bundle always admits a decomposition. The relation between the decomposition and the horizontal lift is given by

$$q_C(d) = \left(d \underset{B}{-} \psi(q_A(d))(q_B(b)) \right) \underset{A}{-} 0_B(q_B(d)).$$

See the appendix, Sec. C.2, for more details and proofs.

2.1.4 Duality

Since a double vector bundle has two vector bundle structures, it also has two duals, one for each structure. In principle, since D^*_A has again two duals, we should obtain a third double vector bundle, and repeating the process we would get a fourth double vector bundle and so on. However, it so happens that the third dual is already canonically isomorphic to D^*_B, and so the story ends here. Namely, the two duals are

$$\begin{tikzcd} D^*_B \arrow[r, "\pi_B"] \arrow[d, "\pi_{C^*}"'] \arrow[dr, phantom, "A^*"] & B \arrow[d, "q^B"] \\ C^* \arrow[r, "q^{C^*}"] & M \end{tikzcd} \; ; \qquad \begin{tikzcd} D^*_A \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "B^*"] & C^* \arrow[d, "q^{C^*}"] \\ A \arrow[r, "q^A"] & M \end{tikzcd} \; ,$$

where, for example, the projection $\pi_{C^*} : D^*_B \longrightarrow C^*$ is given by

$$\langle \pi_{C^*}(d), c \rangle = \langle d, 0_B(\pi_B(d)) \underset{A}{+} c \rangle_B.$$

These two duals are in turn dual to each other with respect to the fibration over C^* (see Prop. B.11). The duality pairing is given by

$$(v|w) = \langle v, d \rangle_A - \langle w, d \rangle_B,$$

where $v \in D^*_A$, $w \in D^*_B$ have $\pi_{C^*}(v) = \pi_{C^*}(w) = k$ and d is any element of D with $q_A(d) = \pi_A(v)$ and $q_B(d) = \pi_B(w)$. This pairing induces isomorphisms of double vector bundles (see Prop. B.13)

$$\Upsilon_A : (D^*_A)_{C^*} \longrightarrow (D^*_B)^*_{C^*}, \qquad \langle \Upsilon_A(v), w\rangle_{C^*} = (v|w)$$
$$\Upsilon_B : (D^*_B)_{C^*} \longrightarrow (D^*_A)^*_{C^*}, \qquad \langle \Upsilon_B(w), v\rangle_{C^*} = (v|w),$$

with $(\Upsilon_A)^* = \Upsilon_B$. Both isomorphisms induce the identity on the sides $C^* \longrightarrow C^*$. Υ_A is the identity on the cores $B^* \longrightarrow B^*$, and induces $-\mathrm{Id}$ on the side bundles $A \longrightarrow A$. Υ_B is the identity on the side bundles $B \longrightarrow B$, and induces $-\mathrm{Id}$ on the cores $A^* \longrightarrow A^*$.

A decomposition $\Theta : D \longrightarrow A \oplus B \oplus C$ naturally induces a decomposition on its dual $\widetilde{\Theta}_A : D^*_A \longrightarrow A \oplus B^* \oplus C^*$ (and, of course, also on D^*_B). The relation between them is $\Theta^*_A = \widetilde{\Theta}_A^{-1}$. The corresponding horizontal lift $\widetilde{\psi} : C^* \longrightarrow \widehat{C^*}$ is given by $\widetilde{\psi}(\kappa)(a) = (q_A, q_C)^*(a, \kappa)$ for every $\kappa \in C^*, a \in A$.

The horizontal lift $\psi : A \longrightarrow \widehat{A}$ induces an isomorphism $K : \widehat{A} \longrightarrow A \oplus \mathrm{Hom}(B, C)$, given by $K(\sigma) = (a, q_C \circ \sigma)$, and the corresponding induced horizontal lift $\psi_* := \overline{(\widetilde{\psi})} : A \longrightarrow \widehat{A}_*$, where $\widehat{A}_*$ is the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(C^*, (D^*_A)_{C^*})$, induces the isomorphism $H : \widehat{A}_* \longrightarrow A \oplus \mathrm{Hom}(C^*, B^*)$, given by $H(\omega) = (a, \pi_{B^*} \circ \omega)$. One can see that we obtain a well-defined isomorphism

$$T : \widehat{A} \longrightarrow \widehat{A}_*,$$

given by $T := H^{-1} \circ \Delta \circ K$, where $\Delta : A \oplus \mathrm{Hom}(B, C) \longrightarrow A \oplus \mathrm{Hom}(C^*, B^*)$ is given by $\Delta(a, \sigma_1) = (a, -\sigma_1^*)$, for every $a \in A$, $\sigma_1 \in \mathrm{Hom}(B, C)$. See Prop. C.17.

It is also useful the isomorphism $Z : \widehat{C^*}_A \longrightarrow \widehat{C^*}_B$, given by

$$Z = T^{-1} \circ \widehat{\Upsilon}_A,$$

where $\widehat{C^*}_A$ is the linear bundle corresponding to the linear sections of D^*_A and $\widehat{C^*}_B$ is the linear bundle corresponding to the linear sections of D^*_B. $\widehat{\Upsilon}_A$ is the morphism induced isomorphisms between the linear bundles induced from the DVB isomorphism Υ_A introduced above. Here, the isomorphism T is taken between $\widehat{C^*}_B$ and $\widehat{C^*}_{B_*}$, so that $(D^*_B)^*_{C^*}$ is playing the role of D, and we are identifying D^*_B with $((D^*_B)^*_{C^*})^*_{C^*}$. See App. C, Sec. C.5, for more details.

Lemma 2.8. *A section $\gamma \in \Gamma(D_A)$ is linear if and only if*

a) $\langle \gamma, \phi\rangle_A$ is a fiberwise linear function for every $\phi \in \Gamma_{\mathrm{lin}}(D^*_A)$, and

b) $\langle \gamma, \widetilde{\xi}\rangle_A$ is fiberwise constant for every $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D^*_A)$.

A section $\widetilde{\beta} \in \Gamma(D_A)$ is core if and only if

c) $\langle \widetilde{\beta}, \phi\rangle_A$ is fiberwise constant for every $\phi \in \Gamma_{\mathrm{lin}}(D^*_A)$ and

d) $\langle \widetilde{\beta}, \widetilde{\xi}\rangle_A = 0$ for every $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D^*_A)$.

For the proof, see Lem. C.28.

An important consequence we can draw of the above lemma, is a characterization of linear sections of a DVB in terms of pairs of certain linear functions satisfying a compatibility condition.

Corollary 2.9. *There is a canonical 1:1 correspondence between sections in* $\Gamma_{\mathrm{lin}}(D^*_B) \cong \Gamma(\widehat{C^*}_B)$ *and pairs of linear maps*

$$f_{\widehat{A}} : \widehat{A} \longrightarrow B^* \quad \text{and} \quad f_C : C \longrightarrow \mathbb{R},$$

such that satisfy the following compatibility condition

$$f_{\widehat{A}}(\tau) = f_C \circ \tau, \qquad \forall \tau \in \mathrm{Hom}(B, C) \subset \widehat{A}.$$

For the proof, see Cor. C.29. Notice that, under this identification, the inclusion $\iota : B^* \otimes A^* \longrightarrow \widehat{C^*}_B$ is given by $\iota(\eta) = \eta^* \circ p : \widehat{A} \longrightarrow B^*$, where $p : \widehat{A} \longrightarrow A$ is the projection.

A direct consequence of Cor. 2.9 is a description of the fibers of the dual linear bundle $\widehat{C^*}_B$ (the one that corresponds to $\Gamma_{\mathrm{lin}}(D^*_B)$), in terms of the fibers of the linear bundle $\widehat{A}$ (the one that corresponds to $\Gamma_{\mathrm{lin}}(D_B)$).

$$(\widehat{C^*}_B)_m = \{\mu \in \mathrm{Hom}(\widehat{A}_m, B^*_m) | \exists \kappa \in C^*_m \ s.t. \ \mu(\tau) = \iota_\kappa(\tau), \ \forall \tau \in B^*_m \otimes C_m \subset \widehat{A}_m\}. \tag{2.4}$$

2.1.5 Double realization

An exact sequence of vector bundles over the manifold M,

$$0 \longrightarrow C \longrightarrow \Omega \longrightarrow A \otimes B \longrightarrow 0$$

over the identity Id_M is called a *double vector sequence* or briefly, a *DVB sequence*, and is denoted by $(\Omega \longrightarrow A \otimes B; M)_C$. Our interest in this class of objects relies on the fact that the natural category they form is equivalent to the category of double vector bundles, and has the advantage that its constituent elements are plain vector bundles over the same manifold M. The morphisms between DVB sequences, which we call *DVS morphisms*, are 4-tuples of maps

$$(\Phi; \varphi_A, \varphi_B; \varphi_M) : (\Omega \xrightarrow{p} A \otimes B; M)_C \longrightarrow (\Omega' \xrightarrow{p'} A' \otimes B'; M')_{C'},$$

where $\Phi : \Omega \longrightarrow \Omega'$, $\varphi_A : A \longrightarrow A'$ and $\varphi_B : B \longrightarrow B'$ are vector bundle morphisms over $\varphi_M : M \longrightarrow M'$ such that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{p} & A \otimes B \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_A \otimes \varphi_B} \\ \Omega' & \xrightarrow{p'} & A' \otimes B' \end{array}$$

commutes. Because of this commutativity condition, Φ preserves C, so it induces a fourth vector bundle morphism $\varphi_C : C \longrightarrow C'$ over φ_M.

DVB sequences together with the above morphisms form a category. Given a double vector sequence $(\Omega \xrightarrow{p} A \otimes B; M)_C$, we can form its *double realization*, which is the double vector bundle

$$\begin{tikzcd} D(\Omega) \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M, \end{tikzcd}$$

where $D(\Omega)$ is given by $D(\Omega) = \{(\omega, a, b) \in \Omega \oplus A \oplus B | p(\omega) = a \otimes b\}$. The projections are given by $q_A(\omega, a, b) = a$ and $q_B(\omega, a, b) = b$.

Analogously, to a DVS morphism $(\Phi; \varphi_A, \varphi_B; \varphi_M)$ between two double vector sequences, se can associate a DVB morphism $(D(\Phi), \varphi_A, \varphi_B; \varphi_M)$ between the corresponding double realizations $D(\Omega)$ and $D(\Omega')$, by

$$D(\Phi)(\omega, a, b) = (\Phi(\omega), \varphi_A(a), \varphi_B(b)).$$

This defines a functor from the DVS category to the DVB category, which happens to be an equivalence of categories. See App. D for all the details, where, in particular, the functor from the DVB category to the DVS category that furnishes the equivalence of categories is described.

2.2 VB algebroids and representations up to homotopy

2.2.1 *VB*-algebroids

Definition 2.10. A *VB-algebroid* is a double vector bundle $(D; A, B; M)$ equipped with a Lie algebroid structure on D_B such that the anchor map $\rho_D : D \longrightarrow TB$ is a bundle morphism over $\rho_A : A \longrightarrow TM$ and where the bracket $[\cdot, \cdot]_D$ is such that

1. $[\Gamma_{\text{lin}}(D_B), \Gamma_{\text{lin}}(D_B)]_D \subset \Gamma_{\text{lin}}(D_B)$,
2. $[\Gamma_{\text{lin}}(D_B), \Gamma_{\text{core}}(D_B)]_D \subset \Gamma_{\text{core}}(D_B)$,
3. $[\Gamma_{\text{core}}(D_B), \Gamma_{\text{core}}(D_B)]_D = 0$.

Remark 2.11. Since ρ_D is a vector bundle morphism with respect to the structure over B, it follows that the condition on ρ_D is equivalent to say that it is a double vector bundle morphism from D to TB.

Remark 2.12. The Lie brackets $[\cdot, \cdot]_D$ and the anchor ρ_D are completely determined by their action on linear and core sections, and on linear and basic functions. This can be seen for example by considering the induced Poisson brackets on D_B^*, and recalling that they are completely determined by their action on coordinate functions. Now, an adapted coordinate system $\{x^i, \beta^b, \alpha^a, \kappa^c\}$ in $D_B^* \cong B \oplus A^* \oplus C^*$ is such that the functions on the x^i are basic and β^b are linear. The functions on the fibers κ^c correspond to core sections on $D_B \cong B \oplus C \oplus A$, and α^a correspond to linear sections on D_B.

A *VB*-algebroid structure induces in a natural way Lie algebroid structures on each term of the linear sequence

$$B^* \otimes C \longrightarrow \widehat{A} \longrightarrow A, \tag{2.5}$$

in such a way that it turns into a Lie algebroid exact sequence. See App. E for details. The induced core map, that comes from the restriction of the anchor $\rho_D|_C : C \longrightarrow B \subset TB$ with opposite sign is called the *core-anchor*, denoted by $\partial := -\rho_D|_C$.

2.2.2 Representations up to homotopy

If A is a Lie algebroid, and E a vector bundle over the same base, an *A-connection* on E is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$, $(\alpha, \varepsilon) \longrightarrow \nabla_\alpha \varepsilon$, such that

$$\nabla_{f\alpha}\varepsilon = f\nabla_\alpha\varepsilon,\ \nabla_\alpha f\varepsilon = f\nabla_\alpha\varepsilon + \rho(\alpha)(f)\varepsilon, \qquad \forall f \in C^\infty(M).$$

The *A-curvature* of ∇ is the tensor given by

$$R_\nabla(\alpha, \beta)(\varepsilon) := \nabla_\alpha\nabla_\beta\varepsilon - \nabla_\beta\nabla_\alpha\varepsilon - \nabla_{[\alpha,\beta]}\varepsilon.$$

When the A-connection is *flat*, that is, $R_\nabla = 0$, ∇ is called a *Lie algebroid representation* of A on E. There is a canonical 1:1 correspondence between A-connections on E, ∇ and degree 1 operators d_∇ on $\Omega(A; E)$, the space of E-valued A-differential forms, which satisfy the derivation rule: $d_\nabla(f\omega) = fd_\nabla\omega + d_A f \wedge \omega$, where d_A is the *de-Rham* exterior derivative on $\Gamma(\Lambda^\cdot A^*)$ associated to the Lie algebroid structure on A, and the space $\Omega(A; E)$ is naturally endowed with a $\Omega(A)$-module structure. It can be seen that ∇ is flat, i.e. a representation on E, if and only if $d_\nabla^2 = 0$. See the details in App. E.

Now let $E = \bigoplus_n E^n$ be a graded vector bundle. Then the space of E-valued A-differential forms, $\Omega(A; E)$ is graded by total degree:

$$\Omega(A; E) = \bigoplus_{i+j=p} \Omega^i(A; E^j).$$

Moreover, if E and F is a graded vector bundle, then $\Omega(A; \mathrm{Hom}(E, F))$ is also endowed with the graded structure that comes from the natural grading on $\mathrm{Hom}(E, F)$, the space of degree-preserving morphisms. Also we can extend d_∇ to an operator $\Omega(A; \mathrm{End}(E))$ by demanding the following Leibniz type rule:

$$d_\nabla(T(\varepsilon)) = (d_\nabla T)(\varepsilon) + (-1)^{|T|} T d_\nabla \varepsilon,$$

where, as usual, $|T|$ denotes the degree of T.

A *representation up to homotopy* of A on E is given by an operator, called the structure operator,

$$D : \Omega(A; E) \longrightarrow \Omega(A; E),$$

which increases the total degree by one and satisfies $D^2 = 0$ and the graded derivation rule:

$$D(\omega\eta) = d_A(\omega)\eta + (-1)^k \omega D(\eta),$$

for all $\omega \in \Omega^k(A)$ and $\eta \in \Omega(A; E)$.

A morphism $\Phi : (E, D_E) \longrightarrow (F, D_F)$ between two representations up to homotopy of A is a degree zero linear map $\Phi : \Omega(A; E) \longrightarrow \Omega(A; F)$ which is $\Omega(A)$ linear and commutes with the structure operators D_E and D_F (warning: we use the same notations D, D_E or

D_F to denote double vector bundles; we hope the context will make it clear which concept we refer in each situation).

There is a canonical 1:1 correspondence between representations up to homotopy (E, D) of A concentrated in two consecutive degrees, say 0 and 1 (so that E is zero in all the other degrees), and the following data:

1. Two vector bundles C and B and a vector bundle map $\partial : C \longrightarrow B$

2. A-connections on C and B, ∇^C and ∇^B, compatible with ∂, which means that $\nabla^B\partial = \partial\nabla^C$.

3. A 2-form $K \in \Omega^2(A; \mathrm{Hom}(B, C))$, called the *curvature form* such that

$$R_{\nabla^C} = -K \circ \partial,\ R_{\nabla^B} = -\partial \circ K,$$

   and

$$d_\nabla K = 0,$$

   where we are considering $\nabla := \nabla^C + \nabla^B$ as an A-connection on $E := C \oplus B$, and viewing $\mathrm{Hom}(B, C)$ naturally seated in $\mathrm{End}(E)$, so that we consider the extension d_∇ to $\Omega(A; \mathrm{End}(E))$, and view K as an element in $\Omega^2(A; \mathrm{End}(E))$.

This kind of representations up to homotopy are called *2-term* representations, and we write $D = \partial + \nabla + K$, where $\nabla := (\nabla^C, \nabla^B)$ is the A-connection on $E = C \oplus E[1]$.

If we choose a horizontal lift, $\psi : A \longrightarrow \widehat{A}$, for the Lie algebroid sequence (2.5), we obtain A-connections on C and B, given respectively by

$$\widetilde{\nabla^C_X \mathbf{c}} := [\widehat{X}, \widetilde{\mathbf{c}}]; \qquad \langle \nabla^B_X \mathbf{b}, \beta\rangle := \rho_A(X)(\langle \mathbf{b}, \beta\rangle) - \langle \rho_D(\widehat{X})(\beta), \mathbf{b}\rangle,$$

where, for any $X \in \Gamma(A)$, we denote by $\widehat{X} := \psi(X) \in \Gamma(\widehat{A}) \cong \Gamma_{\mathrm{lin}}(D_B)$ its corresponding horizontal lift, and for any $\mathbf{c} \in \Gamma(C)$, we denote by $\widetilde{\mathbf{c}} \in \Gamma_{\mathrm{core}}(D_B)$ its corresponding core section.

Defining a curvature form $K \in \Omega^2(A; \mathrm{Hom}(B, C))$ by

$$K(X, Y) := \widehat{[X, Y]_A} - [\widehat{X}, \widehat{Y}]_{\widehat{A}}, \qquad \forall X, Y \in \Gamma(A),$$

it can be shown that this 2-form, together with the core-anchor $\partial : C \longrightarrow B$ and the two A-connections ∇^C and ∇^B defined above define a 2-term representation up to homotopy of A on $E = C \oplus B[1]$, and actually it can be proven that every 2-term representation comes from a *VB*-algebroid and the choice of a horizontal lift.

For more details, see App. E.3 and [23].

2.2.3 Duality

From the axioms defining a *VB*-algebroid, it can be seen that its corresponding linear Poisson structure on D_B^* is actually *double-linear*, that is, it is also linear with respect to the vector bundle structure $D_B^* \xrightarrow{\pi_C^*} C^*$. Therefore, dualizing with respect to this structure we obtain a second VB-algebroid structure on $(D_B^*)^*_{C^*} \cong D_A^*$. If we choose a horizontal lift on (2.5), then the induced horizontal lift on the dual linear sequence

$$C^* \otimes B \longrightarrow \widehat{A}_* \longrightarrow A,$$

which is also a Lie algebroid sequence for what we just observed, gives rise to a 2-term representation up to homotopy D^* on $E^* = B^* \oplus C^*$, which happens to coincide with the dual to the representation D on $E = C \oplus B$ (corresponding to the initial VB-algebroid D_B) in the sense that the following product rule is satisfied :

$$d_A(\nu \wedge \eta) = D^*(\nu) \wedge \eta + (-1)^{|\nu|} \nu \wedge D(\eta), \qquad \forall \eta \in \Omega(A; E), \nu \in \Omega(A; E^*),$$

where $\wedge : \Omega(A; E^*) \otimes \Omega(A; E) \longrightarrow \Omega(A)$ is the wedge product formed using the duality pairing between E and E^*. It can be seen that in this case we have $D^* = \partial^* + \nabla^* - K^*$.

For more details, see App. E.3 and [23].

Chapter 3

Degree 2 manifolds and involutivity

Roughly speaking, a graded manifold is, following Ševera [63], a sheaf of graded algebras over a manifold that locally is spanned by a finite number of generators, and only non-negative degrees are allowed, which is why the are called *graded N-manifolds*, where N stems from the natural numbers to mean that negative degrees are not allowed. The *degree* of a graded N manifold is the highest degree of the generators. A degree 1 manifold is seen to correspond exactly to a vector bundle, both categories being equivalent. Therefore, a graded N-manifold can be seen as a non-linear generalization of a vector bundle (an insight explicitly suggested for the first time by T. Voronov [69], up to our knowledge). As explained in the Introduction (Ch. 1), the interest in graded manifolds lies in the fact that we can extend many of the classical geometric structures (e.g. functions, vector fields, differential forms, Poisson brackets, etc.) to the graded setting, and it turns out that these graded structures usually encode in a very simple fashion some classical structures that are not so easy to handle, providing a powerful source of intuition on how those somewhat complicated classical structures should behave.

In Sec. 3.3 we establish the equivalence between the category of degree 2 manifolds and that of *involutive double vector bundles.* This geometric characterization bridges degree 2 manifolds to all the geometric structures we can associate to a double vector bundle. However, it is not always evident how to associate to an object of one category the corresponding one in the other category, often some non-trivial work is needed. Then we show that through a transpose operation, the involutive structure is equivalent to a *linear metric* on the dual, a result obtained by D. Li-Bland [41], and with a different method by M. Jotz [29]. We end this section building from any DVB D, an involutive one, by taking the Whitney sum $D \underset{A}{\oplus} D^*_A$. This procedure in particular applies to TA, providing one of the most fundamental examples of involutive DVB's. Then we describe the geometric counterpart of vector fields on $\mathcal{M}$ of negative degree, that is, of degrees -1 and -2, which allow to give an interpretation of the commutator of two degree -1 vector fields in terms of the metric, and to characterize geometrically functions on $\mathcal{M}$ of degree 3 in terms of a pair of morphisms of certain vector bundles.

3.1 Graded manifolds

In this section we give a very general description of graded manifolds. The main references we used are [63],[59], [67] and [68].

Definition 3.1. A *graded N-manifold*, $\mathcal{M}$ of dimension $(p|q_1|\dots|q_k)$ is a differential manifold M, called the *body* of the manifold, endowed with a structure sheaf $\mathcal{O}_M$ of graded commutative algebras locally isomorphic to

$$C^\infty(U)\otimes\mathbb{S}^{\cdot}(V^*),$$

where $U\subset\mathbb{R}^p$ and $V=\bigoplus_{i>0}V_i$ is a $\mathbb{N}$-graded vector space, with dim $V_i=q_i$ for $i=1,\dots,k$, and $V_i=\{0\}$ for other i. $\mathbb{S}^{\cdot}(V^*)$ denotes the graded symmetric algebra over $V^*=\bigoplus_{i>0}V_i^*$, so that in particular the parity of each vector space V_i is compatible with its degree. The sheaf morphisms are asked to preserve (total) degree. We will denote by $\mathcal{A}^i$ the set of homogeneous functions of degree i.

Remark 3.2. Notice that the local trivializations of the structure sheaf, together with the degree preserving condition, provides us with k vector bundles, such that the space of sections of the i-th vector bundle is $\mathcal{A}^i$. In the following sections, we will describe in detail in the cases of degree 1 and degree 2 manifolds —the ones that are of our interest in this work— how we can recover the structure sheaf from these vector bundles.

Often we omit the label N, and refer to a graded N-manifold simply as a graded manifold, or else we will omit the adjective *graded* and refer simply as degree k manifolds, hopefully without causing any trouble.

Example 3.3 ([59])**.** Let A be a vector bundle over a manifold M. We can attach a grading to the functions on A such that fiberwise constant functions get degree 0 and fiberwise linear functions get degree 1 (see subsec. 3.1.1 below for more details), obtaining in this way the degree 1 manifold we denote by $A[1]$, whose space of degree 1 functions $\mathcal{A}^1$ coincides with the space of sections of the dual $\Gamma(A^*)$. A particular example of these manifolds is obtained by starting with a manifold M and taking $(TM)^*[1]$, which we prefer to denote in the more standard fashion $T^*[1]M$. The space of degree 1 functions $\mathcal{A}^1$ is isomorphic to the space of vector fields $\mathfrak{X}(M)$. Of course we can also obtain the manifold $T[1]M$ whose space of degree 1 functions is isomorphic to the space of 1-forms $\Omega^1(M)$.

We can go a step further to obtain, for a given vector bundle $A\longrightarrow M$, the two degree 2 manifolds $T[1]A[1]$ and $T^*[2]A[1]$ (see [59] for details). From the coordinate description it is clear that the degree 1 functions of $T[1]A[1]$ are sections of $A\oplus T^*M$, whereas the degree 1 functions of $T^*[2]A[1]$ are the sections of $A\oplus A^*$.

In the appendix (see Sec. F) we describe the double vector bundle structures of TA^* and T^*A^*. Now, the algebra of functions on a double vector bundle, D, admits a subalgebra $C^{\cdot,\cdot}(D)$ which comes with a bi-graded structure (see lemma E.34), and actually the adapted coordinate systems belong to this subalgebra. So we obtain, starting from a double vector bundle, a bi-graded manifold. Now, from a bi-graded manifold, that is, a manifold endowed with an atlas of bigraded fuctions, like the atlas of adapted coordinate systems in lemma E.34, we obtain a graded atlas (of degree 2) simply by assigning degree $r+s$ to a function with bi-degree (r,s). Therefore, given a double vector bundle

$(D; A, B; M)$, there is a degree 2 manifold naturally attached, whose structure sheaf is generated locally by an adapted coordinate atlas shifting the coordinates of the side bundles by 1 and the coordinates of the core bundle by 2. The degree 1 submanifold is given by $M_1 = A[1] \oplus B[1]$.

3.1.1 Example: degree 1 manifolds

Let $\mathcal{M}$ be a 1-manifold. Picking coordinates $\theta_1, \ldots, \theta_q$ for V_1, the structure sheaf is locally isomorphic to $\Gamma(\Lambda^{\cdot}\mathbb{R}^q)$. Since morphisms preserve degree, we see that a change of coordinates $(x, \theta) \longrightarrow (\tilde{x}, \tilde{\theta})$ must have the form

$$\tilde{x}_i = f_i(x); \qquad \tilde{\theta}_i = g_i^j(x)\theta_j.$$

This means that we obtain a vector bundle, where the body M of the 1-manifold $\mathcal{M}$ is the base of the vector bundle A, with fiber isomorphic to V_1, so that the degree 1 functions are the fiberwise linear functions of A, and hence sections of A^*. The cocyles for the change of trivializations are given by the maps in the change of odd variables. Under this identification we have

$$\mathcal{O}_M \cong \Gamma_{(\Lambda^{\cdot} A^*)},$$

where the symbol of the right-hand side stands for the sheaf of sections of $\Lambda^{\cdot}A^*$.

Conversely, consider a vector bundle A over M. The fiberwise linear functions give rise to the sheaf of sections of A^*. Attaching degree 1 to these functions we obtain a 1-manifold by taking M for the body, and the structure sheaf $\mathcal{O}_M$ being given by, for U open in M,

$$\mathcal{O}(U) = \Gamma(\Lambda^{\cdot}(A^*|_U)).$$

In this case we say that the degree 1 manifold $\mathcal{M}$ is $A[1]$, meaning that the fiberwise linear functions (which are sections of A^*) are attached degree 1. So there is a canonical 1:1 correspondence between degree 1 manifolds and vector bundles.

Definition 3.4. A *degree k Poisson manifold* is a k-manifold endowed with a *degree $-k$ Poisson bracket*, which is a $\mathbb{R}$-bilinear mapping

$$\{\cdot, \cdot\} : \mathcal{O}_M \times \mathcal{O}_M \longrightarrow \mathcal{O}_M$$

which satisfies the Poisson brackets axioms taking in account the grading, that is, for every $U \subset M$, and $f, g, h \in \mathcal{O}_M(U)$, we have, using the notation $|\cdot|$ to denote the degree of homogeneous functions,

- $|\{f, g\}| = |f| + |g| - k$
- $\{f, g\} = -(-1)^{(|f|+k)(|g|+k)}\{g, f\}$
- $\{f, gh\} = \{f, g\}h + (-1)^{(|f|+k)|g|}g\{f, h\}$
- $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|+k)(|g|+k)}\{g, \{f, h\}\}$.

Example 3.5. Given a degree 1 Poisson manifold $(\mathcal{M}, \{.,.\})$, consider the corresponding vector bundle A, so that $\mathcal{M} \cong A[1]$. Then $\{.,.\}$ induces a Lie algebroid structure on A^* by the following data:

- $\{s, f\} = \rho(s)(f)$, for $f, s \in \Gamma(\Lambda^{\cdot} A^*)$ with $|f| = 0,\ |s| = 1$
- $\{s_1, s_2\} = [s_1, s_2]_{A^*}$, for $s_1, s_2 \in \Gamma(\Lambda^{\cdot} A^*)$ with $|s_1| = |s_2| = 1$.

Conversely, starting with a Lie algebroid $(A^*, \rho, [.,.])$ over M, consider the degree 1 manifold $A[1]$, so that $C^\infty(A[1]) = \Gamma(\Lambda^{\cdot} A^*)$, and set

- $\{f, g\} = 0$, for $f, g \in C^\infty(A[1])$ with $|f| = |g| = 0$
- $\{s, f\} = \rho(s)(f)$, for $f, s \in C^\infty(A[1])$ with $|f| = 0,\ |s| = 1$
- $\{s_1, s_2\} = [s_1, s_2]$, for $s_1, s_2 \in C^\infty(A[1])$ with $|s_1| = |s_2| = 1$.

Extending this product to any $s_1, s_2 \in C^\infty(A[1])$ via linearity and (graded) Leibniz's rule, we obtain a degree 1 Poisson structure on $A[1]$.

The bracket on $\Gamma(\Lambda^{\cdot} A^*)$ that correspond to the Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{M}$ is called *Schouten bracket*.

3.2 The categories of degree 2 manifolds and involutive structures

Now we get into describing the main object of this thesis, namely degree 2 manifolds. In this section we limit ourselves to recall the basic results about the structure of degree 2 manifolds, which are found in [6]. The main result we borrow from that work, is the fundamental equivalence between the category of degree 2 manifolds and a certain category formed by pairs of vector bundles together with a surjective map between one of these vector bundles and the second exterior power of the other, a category that we chose to call "involutive sequence" category (see Thm. 3.12 below).

A *morphism* $\Psi : \mathcal{N} \longrightarrow \mathcal{M}$ between two degree 2 manifolds is a pair $(\psi, \psi^\sharp)$, where $\psi : N \longrightarrow M$ is a smooth map and

$$\psi^\sharp : \mathcal{O}_M \longrightarrow \psi_* \mathcal{O}_N$$

is a morphism of sheaves over M; in particular, for each open subset $U \subset M$,

$$\psi^\sharp_U : \mathcal{O}_M(U) \longrightarrow \mathcal{O}_N(\psi^{-1}(U))$$

is a morphism of graded algebras.

Proposition 3.6 ([6]). *Let $U \subset \mathbb{R}^p$ be an open set, and denote by $\mathcal{U}$ the degree 2 manifold with body U and structure sheaf $C^\infty(U) \otimes \Lambda^{\cdot}\mathbb{R}^{q_1} \otimes S^{\cdot}\mathbb{R}^{q_2}$, described by coordinates $\{x^i, \varepsilon^\mu, \alpha^\nu\}$. If $\mathcal{N}$ is any degree 2 manifold, then any morphism $\mathcal{N} \longrightarrow \mathcal{U}$ is completely determined by the choice of a map $\psi : N \longrightarrow U$ as well as elements $f^\mu, \beta^\nu \in \mathcal{O}_N(N)$, of degrees 1 and 2, respectively; indeed, the conditions*

$$\psi^\sharp(x^i) = x^i \circ \psi,\ \psi^\sharp(\varepsilon^\mu) = f^\mu,\ \psi^\sharp(\alpha^\nu) = \beta^\nu, \tag{3.1}$$

uniquely determine a morphism of sheaves $\psi^\sharp : \mathcal{O}_U \longrightarrow \psi_* \mathcal{O}_N$.

Locally, a degree 2 manifold $\mathcal{M}$ is isomorphic to $\mathcal{U}$, and the structure sheaf is described by coordinates $\{x^i, \varepsilon^\mu, \alpha^\nu\}$. By Prop. 3.6, and since the morphisms preserve degree, a change of coordinates $(x^i, \varepsilon^\mu, \alpha^\nu) \longrightarrow (\tilde{x}^i, \tilde{\varepsilon}^\mu, \tilde{\alpha}^\nu)$ must have the form

$$\tilde{x}^i(x) = \psi^i(x),\ \tilde{\varepsilon}^\mu(x) = a^\mu_\lambda(x)\varepsilon^\lambda,\ \tilde{\alpha}^\nu(x) = b^\nu_\kappa(x)\alpha^\kappa + \frac{1}{2}c^\nu_{\lambda\eta}(x)\varepsilon^\lambda\varepsilon^\eta, \tag{3.2}$$

where the functions $c^\nu_{\lambda\eta}$ are skew-symmetric on the lower indexes, since the coordinates ε^μ are odd.

Proposition 3.7 ([6]). *There is a 1:1 correspondence, up to isomorphisms, between degree 2 manifolds, with body M, and the following data:*

- *A pair of vector bundles $(E, \widetilde{F})$ over M.*
- *A surjective bundle map $p : \widetilde{F} \longrightarrow \Lambda^2 E$ over Id_M.*

So that we obtain an exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow \widetilde{F} \longrightarrow \Lambda^2 E \longrightarrow 0, \tag{3.3}$$

where $F := \ker p$.

Remark 3.8. If follows from the proof of the proposition above (see [6]) that under this correspondence, degree 0 functions $f \in \mathcal{A}^0$ correspond to functions on the base M; degree 1 functions $\varepsilon \in \mathcal{A}^1$ correspond to sections of E^* and degree 2 functions correspond to sections of $\widetilde{F}^*$.

Proposition 3.9 ([6]). *Let $\mathcal{N}, \mathcal{M}$ be two degree 2 manifolds with sheaf structure $\mathcal{O}_N$ and $\mathcal{O}_M$, respectively. Denote by $\mathcal{A}^i_N$ and $\mathcal{A}^i_N$ the sets of homogeneous functions of degree i on $\mathcal{N}$ and $\mathcal{M}$, respectively, and by $(E_N, \widetilde{F}_N)$ and $(E_M, \widetilde{F}_M)$ the pairs of vector bundles corresponding to $\mathcal{N}$ and $\mathcal{M}$, respectively, given by Prop. 3.7, and the corresponding surjective bundle maps $p_N : \widetilde{F}_N \longrightarrow \Lambda^2 E_N$ and $p_M : \widetilde{F}_M \longrightarrow \Lambda^2 E_M$. Given a smooth map $\psi : N \longrightarrow M$, a morphism $\psi^\sharp : \mathcal{O}_M \longrightarrow \psi_*\mathcal{O}_N$ is completely determined by either of the following data:*

a) a pair of morphisms of sheaves of $C^\infty(M)$-modules $\psi^\sharp_i : \mathcal{A}^i_M \longrightarrow \psi_*\mathcal{A}^i_N$, $i = 1, 2$, such that $\psi^\sharp_2(\mathcal{A}^1_M \cdot \mathcal{A}^1_M) = \psi^\sharp_1(\mathcal{A}^1_M) \cdot \psi^\sharp_1(\mathcal{A}^1_M)$.

b) a pair of vector bundle morphisms $\psi_1 : E_N \longrightarrow E_M$, $\psi_2 : \widetilde{F}_N \longrightarrow \widetilde{F}_M$, covering ψ, such that $\Lambda^2\psi_1 \circ p_N = p_M \circ \psi_2$.

Definition 3.10. A *involutive DVB sequence*, or simply an involutive sequence, is given by a pair of vector bundles $(E, \widetilde{F})$ over a manifold M, together with a surjective bundle map $p : \widetilde{F} \longrightarrow \Lambda^2 E$ over Id_M, so that we obtain the exact sequence (3.3), with $F := \widetilde{F}/\Lambda^2 E$.

A morphism of involutive sequences, or simply an *involutive morphism*, is given by a triple of maps (ψ_2, ψ_1, ψ) satisfying item b) of Prop. 3.9.

The *involutive sequence category* consists of involutive sequences as objects, and the corresponding morphisms between them.

Remark 3.11. The reason for the choice of the name "involutive" is that such sequences are completely determined by a natural extension of them to a double vector sequence (this objects are studied in App. D) endowed with an involution. $\widetilde{F}$ is precisely the fixed set point of the involution, and the morphisms are precisely the restriction of those DVS morphisms that commute with the involution. We will see all this in detail next in subsection 3.2.1.

The next theorem summarizes the previous results.

Theorem 3.12 ([6])**.** *The category of degree 2 manifolds is equivalent to the category of involutive sequences.*

3.2.1 The category of extended involutive sequences

Noting the resemblance of involutive sequences, introduced above in Def. 3.10, with DVB-sequences, described in appendix D, Thm. D.8 suggests already that there should be a geometric characterization of degree 2 manifolds in terms of a certain class of double vector bundles. Since we have a functor linking double vector bundles to double vector sequences (Thm. D.8), in order to link the category of involutive sequences to some category of double vector bundles, we need an intermediate, equivalent, category, which is the "double vector sequence equivalent" of the category of double vector bundles we want to find, which will be obtained by extending involutive sequences in the natural way to obtain objects in the double vector sequence category.

Definition 3.13. Let (E, Ω) be two vector bundles and $p : \Omega \longrightarrow E \otimes E$ a projection. A *involutive structure* on the corresponding double vector sequence

$$0 \longrightarrow F \xrightarrow{\iota} \Omega \xrightarrow{p} E \otimes E \longrightarrow 0, \tag{3.4}$$

where $F = \ker p$, is an automorphism (over the identity on M) $\mathcal{I} : \Omega \longrightarrow \Omega$ that is involutive, i.e. $\mathcal{I}^2 = \mathrm{Id}$, and such that the following diagram commutes

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d, "\mathrm{Id}"] & \Omega \arrow[r, "p"] \arrow[d, "\mathcal{I}"] & E \otimes E \arrow[d, "-*"] \\ F \arrow[r, "\iota"] & \Omega \arrow[r, "p"] & E \otimes E \end{tikzcd} \tag{3.5}$$

where $-*$ is the negative of the transpose: $- * (\phi) = -\phi^*$.

The pair $(\Omega \longrightarrow E \otimes E, \mathcal{I})$ of such sequence and involution will be called an *extended involutive sequence*. We will comprise the data of an extended involutive sequence by $(\Omega, E, F; \mathcal{I})$.

A DVS morphism $(\Phi; \varphi_A, \varphi_B; \varphi)$ (see Def. D.2), between two extended involutive sequences

$$(\Omega, E, F; \mathcal{I}) \xrightarrow{\Phi} (\Omega', E', F'; \mathcal{I}'),$$

is called an *extended involution preserving morphism*, or simply an extended involutive morphism, if

$$\Phi \circ \mathcal{I} = \mathcal{I}' \circ \Phi. \tag{3.6}$$

The extended involutive sequences together with the extended involutive morphisms form the *extended involutive sequence category.*

Proposition 3.14. *If* $(\Phi;\varphi_A,\varphi_B;\varphi)$ *is an extended involutive morphism, then* $\varphi_A=\varphi_B$.

Proof. Let $(\Phi;\varphi_A,\varphi_B;\varphi)$ be an extended involutive morphism. Then, for every $a,b\in E$ we have

$$\Phi\circ\mathcal{I}(a\otimes b)=\Phi(-b\otimes a)=-\varphi_A(b)\otimes\varphi_B(a), \tag{3.7}$$

and

$$\mathcal{I}'\circ\Phi(a\otimes b)=\mathcal{I}'(\varphi_A(a)\otimes\varphi_B(b))=-\varphi_B(b)\otimes\varphi_A(a). \tag{3.8}$$

Since $\Phi\circ\mathcal{I}=\mathcal{I}'\circ\Phi$ holds by (3.6), we conclude from Eqs. (3.7) and (3.8) that

$$-\varphi_A(b)\otimes\varphi_B(a)=-\varphi_B(b)\otimes\varphi_A(a),\qquad\forall a,b\in E,$$

hence, in particular, $\varphi_A(b)=\varphi_B(b)$ for every $b\in E$, which means that $\varphi_A=\varphi_B$.

■

An extended involutive morphism will be denoted by (ψ_2,ψ_1,ψ), where

$$\psi_2:\Omega\longrightarrow\Omega';\qquad\psi_1:E\longrightarrow E';\qquad\psi:M\longrightarrow M'.$$

Remark 3.15. Consider the transpose of an extended involutive sequence $(\Omega,E;\mathcal{I})$:

$$0\longrightarrow E^*\otimes E^*\xrightarrow{p^*}\Omega^*\xrightarrow{\iota^*}F^*\longrightarrow 0. \tag{3.9}$$

Then the following diagram commutes

$$\begin{tikzcd} E^*\otimes E^* \arrow[r, "p^*"] \arrow[d, "-*"] & \Omega^* \arrow[r, "\iota^*"] \arrow[d, "\mathcal{I}^t"] & F^* \arrow[d, "\mathrm{Id}"] \\ E^*\otimes E^* \arrow[r, "p^*"] & \Omega^* \arrow[r, "\iota^*"] & F^*\ . \end{tikzcd} \tag{3.10}$$

Indeed, observe first that the dual of $A\otimes B$ is $A^*\otimes B^*$, with the duality pairing given by

$$\langle a\otimes b,\alpha\otimes\beta\rangle=\langle a,\alpha\rangle\langle b,\beta\rangle.$$

Now, the transpose of the map $-*:E\otimes E\longrightarrow E\otimes E$ given in (3.5) is also $-*:E^*\otimes E^*\longrightarrow E^*\otimes E^*$, the negative of the transpose, since, for $\eta=\varepsilon_1\otimes\varepsilon_2\in E^*\otimes E^*$ and $\tau=e_1\otimes e_2\in E\otimes E$ we have

$$\begin{aligned}\langle\eta,\tau^*\rangle&=\langle\varepsilon_1\otimes\varepsilon_2,e_2\otimes e_1\rangle=\langle\varepsilon_1,e_2\rangle\langle\varepsilon_2,e_1\rangle\\&=\langle\varepsilon_2,e_1\rangle\langle\varepsilon_1,e_2\rangle=\langle\varepsilon_2\otimes\varepsilon_1,e_1\otimes e_2\rangle\\&=\langle\eta^*,\tau\rangle,\end{aligned}$$

from which it follows in general $\langle\eta,\tau^*\rangle=\langle\eta^*,\tau\rangle$ for every $\eta\in E^*\otimes E^*$ and $\tau\in E\otimes E$. Therefore,

$$\langle(-*)^t(\eta),\tau\rangle=\langle\eta,(-*)(\tau)\rangle=\langle\eta,-\tau^*\rangle=\langle-\eta^*,\tau\rangle,$$

which means that $(-*)^t(\eta)=-\eta^*$. Thus we have (3.10).

Therefore, the sequence (3.4) admits an involution $\mathcal{I}$ satisfying (3.5) if and only if the transposed sequence (3.9) admits an involution $\mathcal{I}^t$ satisfying (3.10).

Proposition 3.16. *A double vector sequence like (3.4) admits an involution $\mathcal{I}$ satisfying (3.5) if and only if Ω^* comes with a decomposition*

$$\Omega^* \cong S^2E^* \oplus \widetilde{F}^*, \tag{3.11}$$

where $\widetilde{F}^* := \{\vartheta \in \Omega^* | \mathcal{I}^t(\vartheta) = \vartheta\}$.

Proof. We just saw in Rmk. 3.15 that (3.4) with an involution $\mathcal{I}$ satisfying (3.5) is equivalent to (3.9) with an involution $\mathcal{I}^t$ satisfying (3.10). Now notice that, if $\mathcal{I}^t$ is an extended involution, then

$$\overline{P} := \frac{1}{2}(\mathrm{Id} - \mathcal{I}^t)$$

is a projection, that is $\overline{P}^2 = \overline{P}$. Therefore, we obtain the decomposition

$$\Omega = \mathrm{im}\overline{P} \oplus \ker \overline{P},$$

and $\ker \overline{P} = \widetilde{F}^*$. So in order to obtain (3.12) it remains to show that $S^2E^* = \mathrm{im}\overline{P}$. Given $\eta \in S^2E^*$, we have $\overline{P}(\eta) = \eta$, thus

$$S^2E^* \subset \mathrm{im}\overline{P}.$$

On the other hand, for $\vartheta \in \Omega^*$,

$$\iota^*(\overline{P}(\vartheta)) = \frac{1}{2}\iota^*(\vartheta) - \iota^*(\vartheta) = 0,$$

whereby $\overline{P}(\vartheta) \in E^* \otimes E^*$, whence, by (3.10) we have

$$-(\overline{P}(\vartheta))^* = \mathcal{I}^t(\overline{P}(\vartheta)) = \mathcal{I}^t(\vartheta) - \vartheta = -\overline{P}(\vartheta),$$

whence

$$\overline{P}(\vartheta)^* = \overline{P}(\vartheta),$$

that is, $\overline{P}(\vartheta) \in S^2E^*$, which means that

$$\mathrm{im}\overline{P} \subset S^2E^*,$$

thereby, $\mathrm{im}\overline{P} = S^2E^*$, as we needed to obtain the decomposition (3.12).

Conversely, given the decomposition (3.11), we define an extended involution

$$\mathcal{I}^t : \Omega^* \longrightarrow \Omega^*$$

by

$$\mathcal{I}^t(\eta + \zeta) = -\eta + \zeta, \qquad \forall \eta \in S^2E^*, \zeta \in \widetilde{F}^*.$$

Using the decomposition $E^* \otimes E^* = S^2E^* \oplus \Lambda^2E^*$, it follows immediately that $\mathcal{I}^t$ defined this way makes diagram (3.10) commute.

■

Proposition 3.17. *The decomposition (3.11) is equivalent to the decomposition*

$$\Omega \cong A \oplus \operatorname{Ann}(S^2E^*), \tag{3.12}$$

where $A := \{\phi \in \Omega | \mathcal{I}(\phi) = -\phi\}$ *and* $\operatorname{Ann}(S^2E^*)$ *is the annihilator of* $S^2E^* \subset \Omega^*$*, the relation between them given by* $A = \operatorname{Ann}(\widetilde{F}^*)$ *and* $\widetilde{F} = \operatorname{Ann}(S^2E^*)$.

Proof. We already have the decomposition (3.11). Then we will show that this induces the decomposition (3.12), with $A = \operatorname{Ann}(\widetilde{F}^*)$. Indeed, given $\phi \in A, \vartheta \in \widetilde{F}^*$,

$$\langle \phi, \vartheta \rangle = \langle \phi, \mathcal{I}^t(\vartheta) \rangle = \langle \mathcal{I}(\phi), \vartheta \rangle = -\langle \phi, \vartheta \rangle,$$

thus, $\langle \phi, \vartheta \rangle = 0$, which means that

$$A \subset \operatorname{Ann}(\widetilde{F}^*).$$

Conversely, given $\phi \in \operatorname{Ann}(\widetilde{F}^*)$ and any $\vartheta \in \Omega^*$, then there exist unique $\eta \in S^2E^*$ and $\zeta \in \widetilde{F}^*$ such that $\vartheta = \eta + \zeta$, and thus we have

$$\begin{aligned} \langle \mathcal{I}(\phi), \vartheta \rangle &= \langle \mathcal{I}(\phi), \eta + \zeta \rangle = \langle \phi, \mathcal{I}^t(\eta + \zeta) \rangle \\ &= \langle \phi, -\eta \rangle + \langle \phi, \zeta \rangle = \langle -\phi, \eta \rangle \\ &= \langle -\phi, \vartheta \rangle, \end{aligned}$$

which means that $\mathcal{I}(\phi) = -\phi$, that is $\phi \in A$, which implies that

$$\operatorname{Ann}(\widetilde{F}^*) \subset A,$$

from which we conclude that $A = \operatorname{Ann}(\widetilde{F}^*)$.

∎

Corollary 3.18. *In the situation of Prop. 3.16, the subsets* $A \subset \Omega$ *and* $\widetilde{F}^* \subset \Omega^*$ *are vector bundles, and*

$$A \cong \Omega / \operatorname{Ann}(S^2E^*), \qquad \widetilde{F}^* \cong \Omega^* / S^2E^*.$$

We also have the natural isomorphisms

$$A \cong S^2E \quad \text{and} \quad \operatorname{Ann}(S^2E) \cong \widetilde{F}^*.$$

Also we have

$$S^2E^* = \{\vartheta \in \Omega^* | \mathcal{I}^t(\vartheta) = -\vartheta\}$$

and

$$\operatorname{Ann}(S^2E^*) = \{\phi \in \Omega | \mathcal{I}(\phi) = \phi\}.$$

The restriction of $p : \Omega \longrightarrow E \otimes E$ *to* $\widetilde{F}$ *projects onto* $\Lambda^2 E \subset E \otimes E$*:*

The vector bundle $\widetilde{F}$ *is the fixed set point of* $\mathcal{I}$ *and fits in the exact sequence*

$$0 \longrightarrow F \longrightarrow \widetilde{F} \longrightarrow \Lambda^2 E \longrightarrow 0, \tag{3.13}$$

with

$$F := \ker p = \ker p|_{\widetilde{F}},$$

and of course the dual vector bundle $\widetilde{F}^*$ *fits in the transposed exact sequence*

$$0 \longrightarrow \Lambda^2 E^* \longrightarrow \widetilde{F}^* \longrightarrow F^* \longrightarrow 0. \tag{3.14}$$

Definition 3.19. We call $\widetilde{F}^*$ the *linear involutive bundle*, or just the involutive bundle; the sections of $\widetilde{F}^*$ will be called simply *involutive sections*, omitting the adjective "linear".

Observe that the sequence (3.13) is exactly an involutive sequence, as defined in Def. 3.10. Actually, as we already claimed in the beginning of this subsection, the category of involutive sequences is equivalent to the category of extended involutive sequences.

Proposition 3.20. *The involutive DVS category is equivalent to the IS category.*

Proof. We will define a functor

$$\Xi : \begin{Bmatrix} \text{involutive sequences} \\ + \\ \text{involutive morphisms} \end{Bmatrix} \rightsquigarrow \begin{Bmatrix} \text{extended involutive sequences} \\ + \\ \text{extended involutive morphims,} \end{Bmatrix} \tag{3.15}$$

and show that is essentially surjective and fully faithful, thus establishing the desired equivalence of categories.

In order to define Ξ, set

$$\Xi : \{(\widetilde{F} \xrightarrow{p} \Lambda^2 E)\} \rightsquigarrow \begin{Bmatrix} (\Omega := S^2E \oplus \widetilde{F} \xrightarrow{(\mathrm{Id},p)} S^2E \oplus \Lambda^2E = E \otimes E) \\ + \\ \mathcal{I} := (-\mathrm{Id}, \mathrm{Id}) : S^2E \oplus \widetilde{F} \longrightarrow S^2E \oplus \widetilde{F} \\ \qquad\qquad (\ \varsigma\ ,\ \sigma\ ) \longrightarrow (\ -\varsigma\ ,\ \sigma\ ). \end{Bmatrix} \tag{3.16}$$

and

$$\Xi : (\psi_2, \psi_1, \psi) \rightsquigarrow ((S^2\psi_1, \psi_2); \psi_1, \psi_1; \psi).$$

It is evident that $((S^2\psi_1, \psi_2); \psi_1, \psi_1; \psi)$ is a DVS morphism and commutes with the corresponding involutions $\mathcal{I}$ and $\mathcal{I}'$, so that the functor Ξ is well-defined, and it is routine to check that is actually functorial.

Now let's prove that Ξ is essentially surjective. If we have an involutive sequence $(\Omega, E; \mathcal{I})$ then, by Prop. 3.16 we obtain the decomposition (3.12) from which we obtain the involutive DVS $(\widetilde{F} \xrightarrow{p} \Lambda^2 E)$.

Now, given an involution preserving morphism

$$(\psi_2, \psi_1, \psi) : (\Omega \longrightarrow E \otimes E; M; \mathcal{I}) \longrightarrow (\widehat{F'} \longrightarrow (E') \otimes (E'); M'; \mathcal{I}'),$$

then, for $\zeta \in \widetilde{F}$, $\mathcal{I}(\zeta) = \zeta$, whence

$$\mathcal{I}'(\psi_2(\zeta)) = \psi_2(\mathcal{I}(\zeta)) = \psi_2(\zeta),$$

which implies that $\psi_2(\zeta) \in \widetilde{F'}$, that is

$$\psi_2(\widetilde{F}) \subset \widetilde{F'},$$

and therefore we obtain a morphism

$$\widetilde{\psi}_2 := \psi_2|_{\widetilde{F}} : \widetilde{F} \longrightarrow \widetilde{F'}.$$

Moreover, since (ψ_2, ψ_1, ψ) is a DVS morphism, we have $p' \circ \psi_2 = (\psi_1 \otimes \psi_1) \circ p$, which implies

$$p' \circ \widetilde{\psi}_2 = \Lambda^2 \psi_1 \circ p,$$

so that $(\widetilde{\psi}_2, \psi_1, \psi)$ is an involutive morphism.

Therefore, we have found $(\widetilde{F} \xrightarrow{p} \Lambda^2 E)$ and $\widetilde{\psi}_2$ such that $\Xi(\widetilde{F} \xrightarrow{p} \Lambda^2 E) = (\Omega, E; \mathcal{I})$ and $\Xi(\widetilde{\psi}_2, \psi_1, \psi) = (\psi_2, \psi_1, \psi)$, that is, Ξ is essentially surjective.

Fully faithfulness follows immediately from the observation that because of the compatibility condition $p' \circ \psi_2 = (\psi_1 \otimes \psi_1) \circ p$, ψ is completely determined by its restriction $\widetilde{\psi}_2 = \psi_2|_{\widetilde{F}}$.

∎

Corollary 3.21. *An exact sequence of the form*

$$E^* \otimes E^* \longrightarrow \Omega \longrightarrow F^*$$

is involutive if and only if Ω admits a subbundle $\widetilde{F}^* \subset \Omega$ that fits in the exact sub-sequence

$$\Lambda^2 E^* \longrightarrow \widetilde{F}^* \longrightarrow F^*$$

.

3.3 The category of involutive double vector bundles

The functor between double vector bundles and double vector sequences (Thm. D.8) suggests that there should be a structure defined on double vector bundles that is the double realization of an extended involutive structure defined on a double vector sequence. This is what we achieve by means of the notion of a *involutive structure* on a double vector bundle, given by a pair (D, H), where D is a double vector bundle and H is a DVB morphism of a particular kind between D and its *flip*, which is reminiscent of a complex structure, and we show that is equivalent to a *linear metric* on its dual. After describing the morphisms that preserve involutive structures we show that such morphisms, together with involutive DVB's as objects, form a category –the *involutive DVB category*–, which turns out to be equivalent to the category of (extended) involutive sequences, and thereby to the category of degree 2 manifolds.

Definition 3.22. An *involutive double vector bundle* is a pair (D, H), where D is a double vector bundle $(D; A, B; M)_C$ and $H : D_A \longrightarrow D_B$ is a double vector bundle morphism such that

$$h_C = \mathrm{Id}_C, \qquad h_A = -h_B^{-1} : A \longrightarrow B, \qquad \text{and} \qquad H^4 = \mathrm{Id}. \tag{3.17}$$

An *involutive DVB morphism* is a double vector bundle morphism between two involutive double vector bundles $\Phi : (D, H) \longrightarrow (D', H')$ that preserves the respective involutive structures:

$$\Phi \circ H = H' \circ \Phi. \tag{3.18}$$

Remark 3.23. Considering as sets, we have $D_A = D_B$. But if we consider them with their double vector bundle, their are not equal any more. Nonetheless, we have the map $\mathrm{Flip} : D_B \longrightarrow D_A$ (already introduced in Prop. C.34), that is the identity as a map of sets, but the double vector bundle structures get flipped. Then, more precisely, we should write the last condition in (3.17) as $\overline{H}^4 = \mathrm{Id}$, where $\overline{H} = \mathrm{Flip} \circ H : D_A \longrightarrow D_A$.

The sets of involutive DVB's and involutive morphisms form the *involutive DVB category.*

Remark 3.24. Notice that the conditions of H on the induced morphisms on sides and core bundle imply that H is an isomorphism.

Remark 3.25. In terms of a decomposition $\Theta : D \longrightarrow A \oplus B \oplus C$, we have the expression

$$\overline{H}(a, b, c) = (-h_A^{-1}(b), h_A(a), c + \Psi(a, b)). \tag{3.19}$$

By the equality (3.18) we have

$$\varphi_B \circ h_A = h'_{A'} \circ \varphi_A. \tag{3.20}$$

Since $h_A = -h_B^{-1}$ and $h'_{A'} = -h_{B'}^{-1}$, it follows that

$$\varphi_A = (h'_{A'})^{-1} \circ \varphi_B \circ h_A = h'_{B'} \circ \varphi_B \circ h_B^{-1} = -h'_{B'} \circ \varphi_B \circ h_A.$$

Proposition 3.26. *Let D be a DVB, and let $\Theta : D \longrightarrow A \oplus B \oplus C$ be a decomposition. If H is a DVB morphism that has the expression (3.19), then it is a DVB involutive structure, i.e. $H^4 = \mathrm{Id}$ holds, if and only if Ψ is "h_A-symmetric", that is,*

$$\Psi(a, b) = \Psi(h_A^{-1}(b), h_A(a)) \qquad \forall a \in A, b \in B. \tag{3.21}$$

Proof. Let's compute H^4 using the decomposition:

$$\begin{aligned} H^4(a, b, c) &= H^3(-h_A^{-1}(b), h_A(a), c + \Psi(a, b)) \\ &= H^2(-a, -b, c + \Psi(a, b) - \Psi(h_A^{-1}(b), h_A(a))) \\ &= H(h_A^{-1}(b), -h_A(a), c + 2\Psi(a, b) - \Psi(h_A^{-1}(b), h_A(a))) \\ &= (a, b, c + 2\Psi(a, b) - 2\Psi(h_A^{-1}(b), h_A(a))). \end{aligned} \tag{3.22}$$

Hence, $H^4(a, b, c) = (a, b, c)$ if and only if $\Psi(a, b) = \Psi(h_A^{-1}(b), h_A(a))$.

∎

Proposition 3.27. *Let $H : D_A \longrightarrow D_B$ be a DVB morphism such that its induced core morphism is the identity on C. Then H is a DVB involutive structure if and only if*

$$H^2 = (-\mathbf{1}_A) \circ (-\mathbf{1}_B),$$

where $-\mathbf{1}_A$ is the multiplication by -1 using the D_A structure, and $-\mathbf{1}_B$ is the multiplication by -1 using the D_B structure (both operations commute because of the compatibility conditions of a double vector bundle).

Proof. Using a decomposition, we can write

$$H(a,b,c) = (h_B(b), h_A(a), c + \Psi(a,b)),$$

whence

$$H^2(a,b,c) = (h_B(h_A(a)), h_A(h_B(b)), c + \Psi(a,b) + \Psi(h_B(b), h_A(a)).$$

Then, $H^2(a,b,c) = (-a,-b,c)$ if and only if $h_B = -h_A^{-1}$ and $\Psi(a,b) = \Psi(h_A^{-1}(b), h_A(a))$, which is equivalent to be involutive, as follows from the definition and Prop. 3.26. ∎

Remark 3.28. Again, as in the beginning of Rmk. 3.23, we should actually state that $\overline{H}^2(a,b,c) = (-a,-b,c)$, with $\overline{H} = \mathrm{Flip} \circ H : D_A \longrightarrow D_A$.

The corollary above shows that if the core bundle C is zero, then H is equivalent to a complex structure on the fibers of $D \cong A \oplus B$ over M.

3.3.1 Self-conjugate double vector bundles

In this subsection we will show that in the category of involutive double vector bundles, there is a particular subcategory that is actually equivalent, in a similar way that the involutive sequence category was shown to be equivalent to the extended involutive sequence category.

Proposition 3.29. *The set of involutive double vector bundles (D,H) such that the side bundles coincide $A = B$, and $h_A = -\mathrm{Id}$, with their respective involutive morphisms, forms a subcategory that is equivalent to the whole category of involutive double vector bundles.*

Proof. Given any involutive double vector bundle

$$\begin{tikzcd} D \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd},$$

together with the involutive structure H, then we use h_A to obtain the pull-back bundle

$$\begin{tikzcd} h_A^*(D) \arrow[r, "p_A"] \arrow[d, "q^{h_A}"'] \arrow[dr, phantom, "C"] & A \arrow[d, "q^A"] \\ A \arrow[r, "q^A"] & M \end{tikzcd},$$

given by Prop. B.16. Then we have a DVB isomorphism

$$\widetilde{\Phi} := p_2 : h_A^*(D) \longrightarrow D.$$

We claim that $\widetilde{H} := \widetilde{\Phi}^{-1} \circ H \circ \widetilde{\Phi}$ is an involutive structure. Let's compute the induced morphisms on the side and core bundles, which we denote by $\widetilde{h}_v$, $\widetilde{h}_h$ and $\widetilde{h}_C$, where the suffixes v and h mean vertical and horizontal, respectively. By (the proof of) Prop. B.16, we have $\widetilde{\varphi}_v = \mathrm{Id}_A$, $\widetilde{\varphi}_h = h_A$ and $\widetilde{\varphi}_C = \mathrm{Id}_C$, then

$$h_h = \widetilde{\varphi}_v^{-1} \circ h_B \circ \widetilde{\varphi}_h = h_B \circ h_A = -\mathrm{Id},$$

$$\widetilde{h}_v = \widetilde{\varphi}_h^{-1} \circ h_A \circ \widetilde{\varphi}_v = h_A^{-1} \circ h_A = \mathrm{Id},$$

and

$$\widetilde{h}_C = \widetilde{\varphi}_C \circ \mathrm{Id}_C \circ \widetilde{\varphi}_C^{-1} = \mathrm{Id}_C.$$

The condition $\widetilde{H}^4 = \mathrm{Id}$ follows immediately, since

$$\widetilde{H}^4 = \widetilde{\Phi}^{-1} \circ H^4 \circ \widetilde{\Phi} = \widetilde{\Phi}^{-1} \circ \widetilde{\Phi} = \mathrm{Id}.$$

By construction, $\widetilde{\Phi} : h_A^*(D) \longrightarrow D$ is an involutive DVB morphism. Thus, we have showed that the inclusion functor is essentially surjective. Fully faithfullness is obvious.

■

Definition 3.30. The subcategory of Prop. 3.29 will be called *self-conjugate DVB* category, its objects will be called *self-conjugate double vector bundles* and its morphisms *self-conjugate DVB morphisms*.

When a double vector bundle $(D; A, B; M)_C$ is self-conjugate we will denote its common side bundle by E, so that $E := A = B$, and its core bundle by F, and we will comprise the whole data by $(D; E, F; H)$.

Remark 3.31. If $(\Phi, \varphi_A, \varphi_B; \varphi_M) : (D; E, F; H) \longrightarrow (D', E', F'; H')$ is a self-conjugate morphism, then Eq. (3.20) implies that $\varphi_A = \varphi_B$. Thus we will comprise the data of a self-conjugate morphism by $(\Phi, \varphi_E, \varphi)$.

Remark 3.32. If we choose a decomposition of $(D; E, F; H)$, then H has the form

$$H(a, b, c) = (b, -a, c + \Psi(a, b), \qquad a, b \in E, c \in F, \tag{3.23}$$

and $\Psi : E \oplus E \longrightarrow F$ is a symmetric bilinear map.

3.3.2 Metric double vector bundles

In this section we introduce *linear metrics* (already treated in [41] and also in [29]), and provide a nice characterization for them (Prop. 3.34) already stated in [41]. Next we prove that an involutive structure on D is equivalent to a linear metric on $(D_B^*)_{C^*}$.

Definition 3.33. Given a double vector bundle $(D; A, B; M)_C$, a metric $\langle \cdot, \cdot \rangle_{D_A}$ on D_A is *linear* if

1. $\langle \gamma_1, \gamma_2 \rangle_{D_A}$ is a fiberwise linear function on A for every pair $\gamma_1, \gamma_2 \in \Gamma_{\mathrm{lin}}(D_A)$;

2. $\langle \gamma, \widetilde{\beta} \rangle_{D_A}$ is fiberwise constant for every $\gamma \in \Gamma_{\mathrm{lin}}(D_A)$ and $\widetilde{\beta} \in \Gamma_{\mathrm{core}}(D_A)$;

3. $\langle \widetilde{\beta}_1, \widetilde{\beta}_2 \rangle_{D_A} = 0$, $\forall \widetilde{\beta}_1, \widetilde{\beta}_2 \in \Gamma_{\mathrm{core}}(D_A)$.

A double vector bundle endowed with a linear metric is called *metric double vector bundle*.

Proposition 3.34. *Let $(D; A, B; M)_C$ be a double vector bundle. Then a metric on D_A, $\langle \cdot, \cdot \rangle_{D_A}$, is linear if and only if, the induced symmetric isomorphism $\Phi : D_A \longrightarrow D_A^*$, defined by*

$$\langle \Phi(d_1), d_2 \rangle_A = \langle d_1, d_2 \rangle_{D_A}, \tag{3.24}$$

is a DVB morphism.

Proof. Since Φ is a vector bundle morphism over the identity on A, we can apply the characterization of DVB morphisms given by Prop. C.27. So we need to prove that $\langle \cdot, \cdot \rangle_{D_A}$ is linear if and only if Φ defined by Eq. (3.24) satisfies $\Phi(\Gamma_{\text{lin}}(D_A)) \subset \Gamma_{\text{lin}}(D_A^*)$ and $\Phi(\Gamma_{\text{core}}(D_A)) \subset \Gamma_{\text{core}}(D_A^*)$. But this is a direct consequence of properties *1,2* and *3* of Def. 3.33 together with lemma 2.8 above.

■

Proposition 3.35. *Let $(D; A, B; M)_C$ be a double vector bundle endowed with a linear metric $\langle \cdot, \cdot \rangle_{D_A}$. Then the corresponding symmetric isomorphism*

$$\begin{tikzcd} D \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd} \xrightarrow{\quad\Phi\quad} \begin{tikzcd} D_A^* \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "B^*"] & C^* \arrow[d, "q^{C^*}"] \\ A \arrow[r, "q^A"] & M \end{tikzcd} \tag{3.25}$$

induces isomorphisms $\varphi_B : B \longrightarrow C^*$ and $\varphi_C : C \longrightarrow B^*$ such that

$$\varphi_C = (\varphi_B)^*.$$

Proof. We already saw in Prop. A.25 that a DVB isomorphism induces isomorphisms between the corresponding side and core bundles, in particular φ_B and φ_C are isomorphisms.

Considering the restriction

$$\Phi|_{\ker q_A} : B \oplus C \longrightarrow C^* \oplus B^*,$$

the symmetry of Φ implies, for every $b_1, b_2 \in B$ and every $c_1, c_2 \in C$,

$$\begin{aligned} \langle \varphi_B(b_1), c_2 \rangle + \langle \varphi_C(c_1), b_2 \rangle &= \langle \Phi(b_1, c_1), (b_2, c_2) \rangle_A \\ &= \langle \Phi(b_2, c_2), (b_1, c_1) \rangle_A \\ &= \langle \varphi_B(b_2), c_1 \rangle + \langle \varphi_C(c_2), b_1 \rangle, \end{aligned}$$

in particular, taking $(b_1, = c_1) = (b, 0)$ and $(b_2, c_2) = (0, c)$, we get

$$\langle \varphi_B(b), c \rangle = \langle \varphi_C(c), b \rangle,$$

that is, $\varphi_C = (\varphi_B)^*$.

■

Proposition 3.36. *Given a decomposed DVB, $D = A \oplus B \oplus C$, there is a 1:1 correspondence between linear metrics $\langle \cdot, \cdot \rangle_{D_A}$ on a DVB $(D; A, B; M)_C$ and pairs of vector bundle morphisms*

$$\varphi : B \longrightarrow C^*, \quad \text{and} \quad \tau : A \longrightarrow S^2 B^*.$$

Proof. We already have a 1:1 correspondence between linear metrics and symmetric DVB morphisms $\Phi : D \longrightarrow D_A^*$ over the identity. By Cor. A.24,

$$\Phi(a,b,c) = (a, \varphi_B(b), \varphi_C(c) + \Psi(a,b)), \qquad \Psi : A \oplus B \longrightarrow B^*$$

and by Prop. 3.35, $\varphi_C = \varphi_B^*$. Define $\varphi := \varphi_B$ and $\tau(a) := \Psi_a = \Psi(a,\cdot) : B \longrightarrow B^*$, then it remains to show that Φ is symmetric if and only if $\tau(a)$ is symmetric. We compute

$$\begin{aligned} \langle \Phi(a,b_1,c_1),(a,b_2,c_2)\rangle_A &= \langle (a,\varphi(b_1),\varphi^*(c_1)+\tau(a)(b_1)),(a,b_2,c_2)\rangle_A \\ &= \langle \varphi(b_1),c_2\rangle + \langle \varphi^*(c_1),b_2\rangle + \langle \tau(a)(b_1),b_2\rangle. \end{aligned} \tag{3.26}$$

Now Φ is symmetric if and only if

$$\langle \Phi(a,b_1,c_1),(a,b_2,c_2)\rangle_A = \langle \Phi(a,b_2,c_2),(a,b_1,c_1)\rangle_A,$$

which in view of (3.26) turns out to be equivalent to

$$\langle \tau(a)(b_1),b_2\rangle = \langle \tau(a)(b_2),b_1\rangle,$$

hence Φ is symmetric if and only if $\tau(a) : B \longrightarrow B^*$ is symmetric.

■

Remark 3.37. From the proof of Prop. 3.36 it follows immediately that the linear metrics on a double vector bundle are actually sections of a vector bundle over M, denoted by $S^2_{\mathrm{lin}}(D_A^*)$, with the isomorphism

$$S^2_{\mathrm{lin}}(D_A^*) \cong B^* \otimes C^* \oplus A^* \otimes S^2 B^*.$$

Analogously as we did with linear metrics, we can define *linear 2-forms* on a double vector bundle over, say, the horizontal side bundle B, and likewise we also have *linear bivectors*. The linearity property again can be characterized in terms of the induced morphism between a double vector bundle and its dual; for example in the case of linear bivectors, we can characterize them as those elements of $\Pi \in \Gamma(\Lambda^2 D_B)$ such that the induced morphism $\Pi^\sharp : D_B^* \longrightarrow D$ is a double vector bundle morphism. The sets of linear 2-forms and linear bivectors are denoted by $\Gamma_{\mathrm{lin}}(\Lambda^2 D_B^*)$ and $\Gamma_{\mathrm{lin}}(\Lambda^2 D_B)$, respectively, which are sections of vector bundles over M, denoted by $\Omega^2_{\mathrm{lin}}(D_B)$ and $\Lambda^2_{\mathrm{lin}}(D_B)$, respectively, which are isomorphic to

$$\Omega^2_{\mathrm{lin}}(D_B) \cong A^* \otimes C^* \oplus B^* \otimes \Lambda^2 A^*$$

and

$$\Lambda^2_{\mathrm{lin}}(D_B) \cong C \otimes A \oplus B^* \otimes \Lambda^2 C,$$

respectively.

3.3.3 Equivalence between involutive structures and linear metrics on double vector bundles

Proposition 3.38. *Let $(D; A, B; M)_C$ be a double vector bundle. Then there is a 1:1 correspondence between involutive structures on D and linear metrics on the dual $(D^*_B)_{C^*}$, that is, a symmetric double vector bundle isomorphism*

$$\Phi : (D^*_B)_{C^*} \xrightarrow{\cong} (D^*_B)^*_{C^*}. \tag{3.27}$$

Proof. Suppose first that D is involutive, so that we have the involutive structure $H : D_A \longrightarrow D_B$. Introducing a decomposition, we have

$$H(a, b, c) = (h_A(a), -h_A^{-1}(b), c + \Psi(a, b))$$

whereby

$$H^*_A(b, \kappa, \alpha) = (h_A^{-1}(b), \kappa, -(h_A^{-1})^*(\alpha) + \Psi^*_{h_A^{-1}(b)}(\kappa)),$$

where $\Psi_a : B \longrightarrow C$ is given by $\Psi_a(b) = \Psi(a, b)$. Then, we claim that the isomorphism

$$\Phi = \Upsilon_A \circ H^*_A : (D^*_B)_{C^*} \longrightarrow (D^*_B)^*_{C^*} \tag{3.28}$$

is symmetric. Indeed, the decomposed form of Φ is given by

$$\begin{aligned}\Phi(\kappa, b, \alpha) = \Upsilon_A \circ H^*_A(b, \kappa, \alpha) &= \Upsilon_A(h_A^{-1}(b), \kappa, -(h_A^{-1})^*(\alpha) + \Psi^*_{h_A^{-1}(b)}(\kappa)) \\ &= (\kappa, -h_A^{-1}(b), -(h_A^{-1})^*(\alpha) + \Psi^*_{h_A^{-1}(b)}(\kappa)),\end{aligned}$$

whence

$$\Phi^*_{C^*}(\kappa, b, \alpha) = (\kappa, -h_A^{-1}(b), -(h_A^{-1})^*(\alpha) + \widetilde{\Psi}^*_\kappa(b)),$$

where $\widetilde{\Psi}_\kappa : B \longrightarrow B^*$ is given by $\widetilde{\Psi}_\kappa(b) = \Psi^*_{h^{-1}(b)}(\kappa)$, whereby, for every $b, b' \in B$,

$$\begin{aligned}\langle \widetilde{\Psi}^*_\kappa(b), b' \rangle = \langle \widetilde{\Psi}_\kappa(b'), b \rangle &= \langle \Psi^*_{h_A^{-1}(b')}(\kappa), b \rangle \\ &= \langle \Psi_{h_A^{-1}(b')}(b), \kappa \rangle = \langle \Psi(h_A^{-1}(b'), b), \kappa \rangle.\end{aligned}$$

While,

$$\langle \Psi^*_{h_A^{-1}(b)}(\kappa), b' \rangle = \langle \Psi_{h_A^{-1}(b)}(b'), \kappa \rangle = \langle \Psi(h_A^{-1}(b), b'), \kappa \rangle,$$

hence, $\Phi^*_{C^*} = \Phi$ if and only if

$$\Psi(h_A^{-1}(b'), b) = \Psi(h_A^{-1}(b), b'), \qquad \forall b, b' \in B,$$

which, since h_A^{-1} is an isomorphism, is equivalent to

$$\Psi(a, b) = \Psi(h_A^{-1}(b), h_A(a)), \qquad \forall a \in A, b \in B,$$

which, by Prop. 3.26 is equivalent to $H^4 = \mathrm{Id}$.

Conversely, if there exists a symmetric isomorphism (3.27), then we define

$$H : D_A \longrightarrow D_B$$

by the condition that its transpose H^*_A satisfies (3.28), so that

$$H := \Phi^*_A \circ (\Upsilon^*_A)^{-1} : D \longrightarrow D.$$

By what we have done above, if the decomposed form of Φ is given by

$$\Phi(\kappa, b, \alpha) = (\kappa, \varphi_B(b), \varphi_{A^*}(\alpha) + \widetilde{\Psi}(\kappa, b)),$$

then just reversing the steps we conclude that the decomposed form of H is given by

$$H(a, b, c) = (b, -a, c + \Psi(a, b)),$$

with $\langle \Psi(a, b), \kappa \rangle = -\langle \widetilde{\Psi}(\kappa, a), b \rangle$. And we already saw that in this case $H^4 = \mathrm{Id}$ if and only if Φ is symmetric. Therefore, H is a skew-statomorphism, and thereby D is involutive. ■

Remark 3.39. In [29] M. Jotz defines a morphism between two metric double vector bundles

$$(D, \langle \cdot, \cdot \rangle_A) \dashrightarrow (D', \langle \cdot, \cdot \rangle_{A'})$$

as an isotropic relation

$$\Omega \subset \overline{D} \times D'$$

that is dual to a double vector bundle morphism $\omega : D^*_B \longrightarrow (D')^*_{B'}$. With this definition, she proves in [29] that the category of degree 2 manifolds is equivalent to the category of metric double vector bundles.

It can be verified that morphisms of metric double vector bundles defined in this way, correspond exactly to involutive morphisms between the respective involutive double vector bundles given by Prop. 3.38. Hence, we actually have an equivalence of categories.

3.3.4 Equivalence between involutive sequences and involutive DVB's

Now we establish the connection between the category of involutive sequences (already shown to be equivalent to the category of degree 2 manifolds) and the category of involutive double vector bundles.

Theorem 3.40. *The category of involutive sequences and the involutive DVB category are equivalent.*

Proof. In view of propositions 3.29 and 3.20, it is enough to have an equivalence functor

$$\begin{aligned} \mathfrak{A} : \qquad \{\text{self-conjugate DVB's}\} &\rightsquigarrow \{\text{Extended involutive sequences}\} \\ \{\text{self-conjugate morphisms}\} &\rightsquigarrow \{\text{Extended involutive morphisms}\}. \end{aligned} \tag{3.29}$$

For the construction of this functor and the proof that it is indeed an equivalence of categories, we will rely upon the double realization functor and its inverse, given in Thm. D.8.

Given a self-conjugate DVB $(D; E, F; H)$ we define

$$\mathfrak{A}(D; E, F; H) := (\mathfrak{S}(D), E, F; (\mathrm{Flip}^*)^t \circ \mathfrak{S}(H)),$$

where $\mathfrak{S}(D) = C^\infty_{\mathrm{lin}}(D)^* \xrightarrow{p} E \otimes E$ and $\mathfrak{S}(H) = \widehat{H}$ are given in Thm. D.8, and $(\mathrm{Flip}^*)^t$ comes from transposing diagram (C.59)

$$\begin{tikzcd} C \arrow[r, "p^*"] \arrow[d, "\mathrm{Id}"] & C^\infty_{\mathrm{lin}}(D_B)^* \arrow[r, "\iota_B^*"] \arrow[d, "(\mathrm{Flip}^*)^t"] & B \otimes A \arrow[d, "*"] \\ C \arrow[r, "p^*"] & C^\infty_{\mathrm{lin}}(D_A)^* \arrow[r, "\iota_A^*"] & A \otimes B \end{tikzcd}, \tag{3.30}$$

where in our case, the top sequence, for example, is given by

$$F \xrightarrow{p^*} C^\infty_{\mathrm{lin}}(D_h)^* \xrightarrow{\iota_h^*} E \otimes E,$$

the suffix h meaning, as usual, that we are considering the horizontal vector bundle structure.

We claim that $(\mathfrak{S}(D), E, (\mathrm{Flip}^*)^t \circ \mathfrak{S}(H))$ is an extended involutive sequence. Indeed, taking into account Eq. (3.23) and Prop. C.38, it follows that $\widehat{H}$ satisfies diagram (3.5). Moreover, from Rmks. 3.28 and C.39, and again using Prop. C.38, we see that $((\mathrm{Flip}^*)^t \circ \widehat{H})^2 = \mathrm{Id}$.

Given a self-conjugate DVB morphism $(\Phi, \varphi_E, \varphi) : (D; E, F; H) \longrightarrow (D'; E', F'; H')$, we define

$$\mathfrak{A}(\Phi, \varphi_E, \varphi) = \mathfrak{S}(\Phi; \varphi_E, \varphi_E; \varphi),$$

where, again, $\mathfrak{S}(\Phi; \varphi_E, \varphi_E; \varphi) = (\widehat{\Phi}; \varphi_E, \varphi_E; \varphi)$ is given in Thm. D.8. So functoriality of $\mathfrak{A}$ follows from functoriality of $\mathfrak{D}$. Also, It follows from (3.18) and Rmk. C.39 that

$$\widehat{\Phi} \circ \widehat{H} = \widehat{H'} \circ \widehat{\Phi},$$

thus $(\widehat{\Phi}; \varphi_E, \varphi_E; \varphi)$ is indeed an involution preserving morphism, and therefore we have a well-defined functor $\mathfrak{A}$, as we claimed in (3.29).

Now, let's prove that $\mathfrak{A}$ is essentially surjective. Given an extended involutive sequence

$$(\Omega, E, F; \mathcal{I}),$$

consider the pair

$$(\mathfrak{D}(\Omega \longrightarrow E \otimes E; M)_F; \mathfrak{D}(\mathcal{I}; -\mathrm{Id}_E, \mathrm{Id}_E; \mathrm{Id}_M)),$$

where $\mathfrak{D}(\Omega \longrightarrow E \otimes E) = D(\Omega)$ and $\mathfrak{D}(\widetilde{\mathcal{I}}; -\mathrm{Id}_E, \mathrm{Id}_E; \mathrm{Id}_M) = D(\widetilde{\mathcal{I}})$ are given in Thm. D.8. The morphism $\widetilde{\mathcal{I}}$ is given, using the natural isomorphism $\Omega \cong C^\infty_{\mathrm{lin}}(D(\Omega))^*$ and diagram (3.30) upside down:

$$\begin{tikzcd} F \arrow[r, "p^*"] \arrow[d, "\mathrm{Id}"] & C^\infty_{\mathrm{lin}}(D(\Omega)_v)^* \arrow[r, "\iota_v^*"] \arrow[d, "\mathcal{I}"'] \arrow[dd, bend left, "\widetilde{\mathcal{I}}"] & E \otimes E \arrow[d, "-*"] \arrow[dd, bend left, "-\mathrm{Id}_E \otimes \mathrm{Id}_E"] \\ F \arrow[r, "p^*"] \arrow[d, "\mathrm{Id}"] & C^\infty_{\mathrm{lin}}(D(\Omega)_v)^* \arrow[r, "\iota_v^*"] \arrow[d, "(\mathrm{Flip}^*)^t"'] & E \otimes E \arrow[d, "*"] \\ F \arrow[r, "p^*"] & C^\infty_{\mathrm{lin}}(D(\Omega)_h)^* \arrow[r, "\iota_h^*"] & E \otimes E \end{tikzcd}, \tag{3.31}$$

so that the double realization of $\widetilde{\mathcal{I}}$ is a DVB morphism

$$D(\widetilde{\mathcal{I}}) : D(\Omega) \longrightarrow \overline{D(\Omega)}, \tag{3.32}$$

where $\overline{D(\Omega)}$ stands for the flip of $D(\Omega)$.

Now we claim that $(D(\Omega); E, F; D(\widetilde{\mathcal{I}}))$ is self-conjugate DVB. For this it is enough to show that $H := D(\widetilde{\mathcal{I}})$ is an involution structure that satisfies the condition of Prop. 3.29. By construction, we already have that H is a DVB morphism between $D(\Omega)$ and its flip, and that $h_A = -\mathrm{Id}_A$, $h_B = \mathrm{Id}_B$ and $h_C = \mathrm{Id}_C$. So it remains only to show that $\overline{H}^4 = \mathrm{Id}$, where $\overline{H} = \mathrm{Flip} \circ H$ (see Rmk. 3.23). Observe that we have a natural identification Flip in this case is given by

$$\begin{aligned} \mathrm{Flip} : \overline{D(\Omega)} &\xrightarrow{\cong} D(\Omega) \\ (\phi, e_1, e_2) &\longrightarrow (\phi, e_2, e_1). \end{aligned}$$

Using this identification, from the definition of H we have

$$\begin{aligned} \overline{H}^4(\phi, e_1, e_2) &= \overline{H}^3(\widetilde{\mathcal{I}}(\phi), e_2, -e_1) \\ &= \overline{H}^2(\phi, -e_1, -e_2) \\ &= \overline{H}(\widetilde{\mathcal{I}}(\phi), e_2, -e_1) \\ &= (\phi, e_1, e_2), \end{aligned}$$

that is, $\overline{H}^4 = \mathrm{Id}$, as we wanted. Therefore, we have obtained an involutive DVB

$$(D(\Omega), E, F; H),$$

and because of the way it was built, it is evident that $\mathfrak{A}(D(\Omega), E, F; \widetilde{H})$ is canonically isomorphic to $(\Omega \longrightarrow E \otimes E; M; \mathcal{I})$.

Fully faithfulness of $\mathfrak{A}$ follows directly from Thm. D.8 and the functoriality of $\mathfrak{D}$.

■

3.3.5 Some immediate consequences of the equivalence of categories

We have seen in Thm. 3.40 that, given a self-conjugate DVB, there corresponds in a natural way an involutive sequence. In view of Prop. 3.38, we would like to see how the involutive sequence relates to the symmetric $\Phi : D_{F^*} \longrightarrow D^*_{F^*}$, where $D_{F^*} := D^*_h$ is the dual of the self-conjugate DVB D with respect to the horizontal fibration. This is what we do in the next proposition.

Proposition 3.41. *Consider an involutive DVB $(D; E, F; H)$. Consider its corresponding involutive sequence $(\widehat{F^*} \longrightarrow E \otimes E, \mathcal{I})$, where, according to (the proof of) Thm. 3.40,*

$$\widehat{F^*} = C^\infty_{\mathrm{lin}}(D_A)^* \quad \textit{and} \quad \mathcal{I} = (\mathrm{Flip}^*)^t \circ \widehat{H}.$$

Consider the isomorphism

$$\widetilde{Z}_B := Z_B \circ \mathrm{Flip}^* : C^\infty_{\mathrm{lin}}(D_v) \longrightarrow \widehat{C^*}_h, \qquad \textit{(see diagram C.60)},$$

where, as usual, v and h mean, respectively, the vertical and horizontal vector bundle structures (remember that the side bundles are equal).

Then

$$\widetilde{Z}_B \circ \mathcal{I}^t \circ \widetilde{Z}_B^{-1} = T^{-1} \circ \widehat{\Phi}_h, \tag{3.33}$$

where $T : \widehat{C^*}_h \longrightarrow (\widehat{C^*}_h)_*$ is the isomorphism introduced in Prop. C.17, and $\widehat{\Phi}_h : \widehat{C^*}_h \longrightarrow (\widehat{C^*}_h)_*$ is the induced morphism given in Prop. C.18.

Proof. By Prop. C.36 we have

$$T^{-1} = Z \circ \widehat{\Upsilon_A}^{-1},$$

whereby, using Eq. (3.28) and Rmk. C.39,

$$T^{-1} \circ \widehat{\Phi}_h = Z \circ \widehat{\Upsilon_A}^{-1} \circ \widehat{\Upsilon_A} \circ \widehat{H_v^*} = Z \circ \widehat{H_v^*}. \tag{3.34}$$

On the other hand, by Prop. C.40,

$$\begin{tikzcd} C^\infty_{\mathrm{lin}}(D_h) \arrow[r, "H^*"] \arrow[d, "Z_B"] & C^\infty_{\mathrm{lin}}(D_v) \arrow[d, "Z_A"] \\ \widehat{C^*}_h \arrow[r, "\widehat{H_A^*}"] & \widehat{C^*}_v \end{tikzcd} , \tag{3.35}$$

thus, taking into account that $H^* = \widehat{H}^t$, and using (C.60),

$$\begin{aligned} \widetilde{Z}_B \circ \mathcal{I}^t \circ \widetilde{Z}_B^{-1} &= Z_B \circ \mathrm{Flip}^* \circ H^* \circ \mathrm{Flip}^* \circ (\mathrm{Flip}^*)^{-1} \circ Z_B^{-1} \\ &= Z_B \circ \mathrm{Flip}^* \circ Z_A^{-1} \circ Z_A \circ H^* \circ Z_B^{-1} \\ &= Z \circ \widehat{H_v^*} \circ Z_B \circ Z_B^{-1} \\ &= Z \circ \widehat{H_v^*}, \end{aligned}$$

whence, from (3.34), it follows (3.33).

■

Now we want to compute the symmetric morphism Φ in terms of a special kind of decomposition of D_{F^*}, the ones that come from splittings of the corresponding involutive sequence. First we observe that these splittings actually yield decompositions of the corresponding metric double vector bundle. We put this observation as a remark for future reference.

Remark 3.42. If Ω^* is decomposed as in (3.11), so that the exact sequence

$$0 \longrightarrow E^* \otimes E^* \longrightarrow \Omega^* \longrightarrow F^* \longrightarrow 0 \tag{3.36}$$

becomes

$$0 \longrightarrow \Lambda^2 E^* \oplus S^2 E^* \longrightarrow S^2 E^* \oplus \widetilde{F}^* \longrightarrow F^* \longrightarrow 0, \tag{3.37}$$

then any horizontal lift of the involutive subsequence (3.14) yields an horizontal lift of (3.36). Indeed, since, by definition, the projection $p : \widetilde{F}^* \longrightarrow F^*$ is just the restriction of $p : \Omega^* \longrightarrow F^*$, if we compose a horizontal lift ψ of (3.14) with the inclusion $\widetilde{F}^* \hookrightarrow \Omega^*$ we get a horizontal lift of (3.36).

Proposition 3.43. *Let $(D; E, F; H)$ be a self-conjugate DVB, and denote by $(D_{F^*}, E; \Phi)$ its corresponding dual metric DVB. Then*

$$\Phi = \widetilde{\Theta}^{-1} \circ \Theta, \tag{3.38}$$

where Θ and $\widetilde{\Theta}$ are the decompositions of D_{F^*} and $D^*_{F^*}$, respectively, induced by some horizontal lift $\psi : F^* \longrightarrow \widetilde{F}^*$ of (3.14) (see Rmk. 3.42), where $\widetilde{F}^*$ comes from the decomposition (3.11).

Proof. If $\Theta, \widetilde{\Theta}$ are decompositions corresponding to an arbitrary horizontal lift $\psi : F^* \longrightarrow \widehat{F^*}$ of (3.36), then by Cor. A.24

$$\widetilde{\Theta} \circ \Phi \circ \Theta(\xi, e, \varepsilon) = (\xi, e, \varepsilon + \Psi(\xi, e)), \qquad \Psi \in \Gamma(F \otimes E^* \otimes E^*).$$

Therefore, we must show that if the decompositions come from a horizontal lift $\psi : F^* \longrightarrow \widetilde{F}^*$ of (3.14), then $\Psi = 0$, or equivalently,

$$\widetilde{\Theta} \circ \Phi \circ \Theta = \mathrm{Id}_{F^* \oplus E \oplus E^*}.$$

From Eq. (C.51) and diagram (3.10), and since $\widetilde{F}^*$ is the fixed point set of $\mathcal{I}^t$, it follows that a horizontal lift ψ taking values on $\widetilde{F}^*$ yields the following decomposed expression for the left-hand side of (3.33):

$$\widetilde{Z}_B \circ \mathcal{I}^t \circ \widetilde{Z}_B^{-1}(\eta, f) = (\xi, -\eta^*), \qquad \forall \xi \in \Gamma(F), \eta \in E^* \otimes E^*.$$

From Eqs. (C.25) and (3.33) we conclude that, from an horizontal lift taking values on $\widetilde{F}^*$, the decomposed form of $\widehat{\Phi}$ is given by

$$\widehat{\Phi}(\xi, \eta) = (\xi, \eta).$$

Finally, from Eq. (C.29) we conclude that $\Psi = 0$, and therefore the decomposed form of Φ is the identity, as we wanted.

■

3.3.6 Examples

Proposition 3.44. *Let $(D; A, B; M)_C$ be a double vector bundle. D and D^*_A have one side bundle in common, namely A, so that Prop. C.42 applies, so that we obtain the double vector bundle*

$$D \underset{A}{\oplus} D^*_A.$$

Then

$$\begin{tikzcd} (D \underset{A}{\oplus} D^*_A)^*_{B \oplus C^*} \arrow[r] \arrow[d] & B \oplus C^* \arrow[d] \\ B \oplus C^* \arrow[r] & M \end{tikzcd} \tag{3.39}$$

is a self-conjugate double vector bundle, with core bundle A^*.

Proof. Notice that

$$((D \underset{A}{\oplus} D^*_A)^*_{B\oplus C^*})^*_{B\oplus C^*} \cong D \underset{A}{\oplus} D^*_A,$$

and we have a symmetric isomorphism

$$(D \underset{A}{\oplus} D^*_A)^*_A = D^*_A \underset{A}{\oplus} D \cong D \underset{A}{\oplus} D^*_A,$$

which preserves A, $B\oplus C^*$ and $C\oplus B^*$, then it gives a linear metric dual to a self-conjugate structure.

Involutivity also can be seen as a consequence of Cor. C.44.

■

Example Let A be a vector bundle. Consider the double vector bundles

$$\begin{array}{ccc} TA & \xrightarrow{q_{TM}} & TM \\ {\scriptstyle q_A}\big\downarrow & & \big\downarrow{\scriptstyle q^{TM}} \\ A & \xrightarrow{q^A} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T^*A & \xrightarrow{\pi_{A^*}} & A^* \\ {\scriptstyle \pi_A}\big\downarrow & & \big\downarrow{\scriptstyle q^{A^*}} \\ A & \xrightarrow{q^A} & M; \end{array}$$

Then, by Prop. C.42, we obtain the double vector bundle

$$\begin{array}{ccc} TA \underset{A}{\oplus} T^*A & \xrightarrow{(q_{TM},\pi_{A^*})} & TM \oplus A^* \\ {\scriptstyle q_A}\big\downarrow & & \big\downarrow{\scriptstyle q^{TM\oplus A^*}} \\ A & \xrightarrow{q^A} & M \end{array} \tag{3.40}$$

and by Prop. 3.44, dualizing we get the self-conjugate double vector bundle

$$\begin{array}{ccc} (TA \underset{A}{\oplus} T^*A)^*_{TM\oplus A^*} & \longrightarrow & TM \oplus A^* \\ \big\downarrow & & \big\downarrow \\ TM \oplus A^* & \xrightarrow{q^A} & M \end{array} \tag{3.41}$$

with core bundle A^*.

3.4 The dual sequence of an involutive DVB

Since we want to determine a 1-vector field Q without splitting the degree 2 manifold, we need a notion of dual contraction for functions of degrees 1, 2 and 3, so that we can imitate formulas (H.3) to determine the action of Q on functions of degrees 0,1 and 2. The characterization of a degree 2 manifold in terms of certain double vector bundles allows the notion of duality for double vector bundles come to our aid.

Let D be a self-conjugate double vector bundle, so that $D_{F^*} = D_h^*$ (h stems for the *horizontal* bundle structure) is the double vector bundle

$$\begin{tikzcd} D_{F^*} \arrow[r, "q_E"] \arrow[d, "q_{F^*}"'] \arrow[dr, phantom, "E^*"] & E \arrow[d, "q^E"] \\ F^* \arrow[r, "q^{F^*}"] & M \end{tikzcd} , \tag{3.42}$$

with some symmetric isomorphism $\Phi : D_{F^*} \overset{\Phi}{\cong} D^*_{F^*}$. We want to explore the structure of

$$\begin{tikzcd} D \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "F"] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M \end{tikzcd} , \tag{3.43}$$

particularly we want to describe a symmetric pairing on the linear bundle $\widehat{E}$, taking values on F.

Definition 3.45. Let D be a self-conjugate double vector bundle. Then the linear bundle $\widehat{E}$, corresponding to linear sections of D_h fits in the exact sequence

$$E^* \otimes F \xrightarrow{\iota} \widehat{E} \xrightarrow{\pi} E, \tag{3.44}$$

which will be called the *dual sequence of* (3.14).

Remark 3.46. In Cor. 3.53 we give an intrinsic characterization of the dual sequence, in the sense that it is defined just in terms of the involutive sequence data.

Definition 3.47. Let $\widetilde{F}^*$ be the linear involutive bundle corresponding to a self-conjugate DVB D (which is a vector sub-bundle of the linear bundle corresponding to linear sections of the dual D_{F^*} over E), and let $\widehat{E}$ be the linear bundle corresponding to $\Gamma_{\text{lin}}(D)$. Then there is a tensor $T \in \Gamma((\widehat{E})^* \otimes (\widehat{E})^* \otimes \widetilde{F})$, given by

$$\langle T(\phi_1, \phi_2), \gamma\rangle := \langle \overline{\langle \gamma, \phi_1\rangle}, \phi_2\rangle, \tag{3.45}$$

for $\phi_1, \phi_2 \in \Gamma(\widehat{E}) \cong \Gamma_{\text{lin}}(D)$ and $\gamma \in \Gamma(\widetilde{F}^*) \subset \Gamma(\widehat{F^*}) \cong \Gamma_{\text{lin}}((D_{F^*})_E)$, where, for a function ε on E, linear in the fibers (in our case $\langle \gamma, \phi_1\rangle$), we consider the core section $\overline{\varepsilon}$ inside D_{F^*}, thanks to the identification $C^\infty_{\text{lin}}(E) \cong \Gamma(E^*)$. Thus, the duality pairings in (3.45) make sense.

Proposition 3.48. *Let $W \in \Gamma(S^2(\widehat{E})^* \otimes \widetilde{F})$ be the symmetrization of T, that is*

$$W(\phi_1, \phi_2) := T(\phi_1, \phi_2) + T(\phi_2, \phi_1). \tag{3.46}$$

Then W takes values on F which is viewed inside $\widetilde{F}$ (see the exact sequence (3.13). Therefore, $W \in \Gamma(S^2(\widehat{E})^* \otimes F)$.

Actually, if we choose a horizontal lift $\psi : F^* \longrightarrow \widetilde{F}^*$, *we obtain decompositions in* D_{F^*} *and* D, *and we can write*

$$\phi_i = \eta_i + e_i,\ i = 1, 2, \tag{3.47}$$

where $\eta_i \in \Gamma(E^* \otimes F) \cong \Gamma(\mathrm{Hom}(E, F))$ *and* $e_i \in \Gamma(E)$. *Then*

$$W(\phi_1, \phi_2) = \eta_1(e_2) + \eta_2(e_1) \tag{3.48}$$

holds. In particular, the expression on the right-hand side of (3.48) doesn't depend on the horizontal lift ψ.

Proof. Let $\gamma \in \Gamma(\widetilde{F}^*)$. The horizontal lift ψ allows us to write

$$\gamma = \zeta + \lambda,$$

with $\zeta \in \Gamma(F^*)$ and $\lambda \in \Gamma(\Lambda^2 E^*)$. Then

$$\langle \gamma, \phi_1 \rangle = \eta_1^*(\zeta) + \lambda(e_1),$$

where we are viewing $\eta_1 \in \Gamma(\mathrm{Hom}(E, F))$, so that $\eta_1^* \in \Gamma(\mathrm{Hom}(F^*, E^*))$, and $\lambda \in \Gamma(\Lambda^2 E^*) \subset \Gamma(\mathrm{Hom}(E, E^*))$. Then we have

$$\begin{aligned} \langle T(\phi_1, \phi_2), \gamma \rangle = \langle \overline{\langle \gamma, \phi_1 \rangle}, \phi_2 \rangle &= \langle \eta_1^*(\zeta), e_2 \rangle + \langle \lambda, e_1 \wedge e_2 \rangle \\ &= \langle \zeta, \eta_1(e_2) \rangle + \langle \lambda, e_1 \wedge e_2 \rangle. \end{aligned}$$

Likewise

$$\langle T(\phi_2, \phi_1), \gamma \rangle = \langle \zeta, \eta_2(e_1) \rangle + \langle \lambda, e_2 \wedge e_1 \rangle$$

holds. Therefore,

$$\langle W(\phi_1, \phi_2), \gamma \rangle = \langle T(\phi_1, \phi_2) + T(\phi_2, \phi_1), \gamma \rangle = \langle \eta_1(e_2) + \eta_2(e_1), \zeta \rangle, \tag{3.49}$$

that is

$$W(\phi_1, \phi_2) = \eta_1(e_2) + \eta_2(e_1).$$

■

Remark 3.49. In order to get the expression

$$\eta_1(e_2) + \eta_2(e_1) \tag{3.50}$$

invariant it is crucial the involutivity of D_{F^*}, for, in order to arrive to Eq. (3.49), we needed the skew-symmetry of λ.

We can check also directly that (3.50) doesn't depend on the horizontal lift, in order to see more clearly that this independence strongly relies on the involutivity of D_{F^*}. Let ψ', be another horizontal lift, then according to this splitting we can write

$$\phi_i = \eta_i' + e_i, \quad i = 1, 2,$$

and by Prop. (C.15), we get

$$\eta_i' = \eta_i - \Psi_{e_i}^*, \quad i = 1, 2,$$

where $\Psi : E \oplus F^* \longrightarrow E^*$ is given by

$$\Psi(e,\zeta) = (\psi' - \psi)(\zeta)(e),$$

and $\Psi_e(\zeta) := \Psi(e,\zeta)$, so we conclude that

$$\langle \Psi^*_{e_i}(e_j), \zeta\rangle = \langle \Psi(e_i,\zeta)\rangle e_j\rangle = \langle (\psi' - \psi)(\zeta)(e_i), e_j\rangle.$$

Since $(\psi' - \psi)(\zeta) \in \Lambda^2 E^*$ (and here is entering the fact that D_{F^*} is involutive), we get

$$\langle \Psi^*_{e_i}(e_j), \zeta\rangle = -(\psi' - \psi)(\zeta)(e_j), e_i\rangle = -\langle \Psi^*_{e_j}(e_i), \zeta\rangle, \tag{3.51}$$

whence,

$$\eta_1'(e_2) + \eta_2'(e_1) = \eta_1(e_2) - \Psi^*_{e_1}(e_2) + \eta_2(e_1) - \Psi^*_{e_2}(e_1) = \eta_1(e_2) + \eta_2(e_1). \tag{3.52}$$

Corollary 3.50. *Let $(D; E, F; H)$ be a self-conjugate DVB, so that its dual D_{F^*} is endowed with a linear metric Φ, given by Eq. (3.24). Then, after identifying $F \cong C^\infty_{\mathrm{lin}}(F^*)$, we have*

$$\langle \gamma_1, \gamma_2\rangle_{D_{F^*}} = W(Z \circ \widehat{\Phi}_{F^*}(\gamma_1), Z \circ \widehat{\Phi}_{F^*}(\gamma_2)), \qquad \forall \gamma_1, \gamma_2 \in \Gamma_{\mathrm{lin}}(D_{F^*}), \tag{3.53}$$

where $W : S^2 \widehat{E} \longrightarrow F$ is the pairing introduced in Prop. 3.48 and $Z : \widehat{F^*}_* \longrightarrow \widehat{E}$ is the isomorphism given in Prop. C.33, and

$$\langle \gamma, \overline{\varepsilon}\rangle_{D_{F^*}} = \langle \pi \circ Z \circ \widehat{\Phi}_{F^*}(\gamma), \varepsilon\rangle, \qquad \forall \gamma \in \Gamma_{\mathrm{lin}}(D_{F^*}), \overline{\varepsilon} \in \Gamma_{\mathrm{core}}(D_{F^*}), \tag{3.54}$$

where $\pi : \widehat{E} \longrightarrow E$ is the projection.

Proof. Let's denote $\widehat{E}_{F^*}$ the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(D_{F^*})$. To prove (3.53), let's choose a horizontal lift $\psi : F^* \longrightarrow \widetilde{F}^*$, which induces the corresponding lift $\widetilde{\psi} : E \longrightarrow \widehat{E}_{F^*}$, and also a horizontal lift $\widetilde{\psi}_* : E \longrightarrow \widehat{E}$, on the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(D)$, where $D = (D_{F^*})^*_E$ (see Rmk. 3.42). Denote by $\Theta : D_{F^*} \longrightarrow F^* \oplus E \oplus E^*$ the corresponding decomposition. Then, recalling that the induced morphism of Θ on $\widehat{E}_{F^*}$ is given by $K : \widehat{E}_{F^*} \longrightarrow \mathrm{Hom}(F^*, E^*) \oplus E$ (see Prop. C.10), and writing $K(\gamma_i) = (\eta_i, e_i)$ with $\eta_i \in \mathrm{Hom}(F^*, E^*) \cong F \otimes E^* \cong E^* \otimes F$ and $e_i \in E$, we have, denoting by K' : $\widehat{E} \longrightarrow \mathrm{Hom}(E, F) \oplus E$ the morphism induced by $\widetilde{\psi}_*$,

$$K'(Z \circ \widehat{\Phi}_{F^*}(\gamma_i)) = (\eta_i, e_i),$$

which follows from (C.55) and the fact that Φ is a statomorphism. Thus, using Prop. 3.43

$$\begin{aligned}\langle \Phi(\gamma_1), \gamma_2\rangle_{F^*} &= \langle \Theta(\gamma_1), \Theta(\gamma_2)\rangle = \langle K(\gamma_1), K(\gamma_2)\rangle \\ &= \eta_1(e_2) + \eta_2(e_1) \\ &= W(Z \circ \widehat{\Phi}_{F^*}(\gamma_1), Z \circ \widehat{\Phi}_{F^*}(\gamma_2)).\end{aligned}$$

Analogously, to prove (3.54), writing $\gamma = \eta + \widehat{e}$, we have

$$\langle \gamma, \overline{\varepsilon}\rangle_{D_{F^*}} = \langle e, \varepsilon\rangle = \langle \pi \circ Z \circ \widehat{\Phi}_{F^*}(\gamma), \varepsilon\rangle.$$

∎

Remark 3.51. If we consider the induced metric on the dual $D^*_{F^*}$, which is also linear, given by

$$\langle w_1, w_2\rangle_{D^*_{F^*}} := \langle \Phi^{-1}(w_1), w_2\rangle_{F^*},$$

then we have the following formulas:

$$\langle \omega_1, \omega_2\rangle_{D^*_{F^*}} = W(Z(\omega_1), Z(\omega_2)), \qquad \forall \omega_1, \omega_2 \in \Gamma_{\mathrm{lin}}(D^*_{F^*}), \tag{3.55}$$

$$\langle \omega, \overline{\varepsilon}\rangle_{D^*_{F^*}} = \langle Z(\omega), \overline{\varepsilon}\rangle_E = \langle \pi \circ Z(\omega), \varepsilon\rangle, \qquad \forall \omega \in \Gamma_{\mathrm{lin}}(D^*_{F^*}), \overline{\varepsilon} \in \Gamma_{\mathrm{core}}(D^*_{F^*}). \tag{3.56}$$

3.5 The GLA of vector fields on a degree 2 manifold

In this section we describe the GLA (graded Lie algebra) of vector fields on a degree 2 manifold with the (graded) commutator bracket. We will see that core sections of the corresponding involutive DVB D coincide with -2 vector fields on $\mathcal{M}$ and linear sections on D coincide with -1 vector fields.

Consider the graded vector space $A = \mathfrak{X}(\mathcal{M})$ of graded vector fields on a degree 2 manifold $\mathcal{M}$, which are by definition operators on the sheaf $\mathcal{O}_M$ such that, if $X \in \mathfrak{X}(\mathcal{M})$ is a (homogeneous) degree k vector field, then $X(f)$ has degree $k+r$ for every homogeneous degree r element $f \in \mathcal{O}_M$, that is,

$$|X(f)| = |X| + |f|$$

and satisfies graded Leibniz rule

$$X(fg) = X(f)g + (-1)^{|X||f|} f X(g),$$

for every homogeneous functions $f, g \in \mathcal{O}_M$.

Locally, $\mathfrak{X}(\mathcal{M})$ is an $\mathcal{O}(U)$-module spanned by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \varepsilon^\mu}, \frac{\partial}{\partial \alpha^\nu}\}$, where $\{x^i, \varepsilon^\mu, \alpha^\nu\}$ are local coordinates spanning $\mathcal{O}(U)$. The vector fields $\frac{\partial}{\partial \varepsilon^\mu}$ are dual to ε^μ and are assigned degree -1 by the usual convention of graded duality. The vector fields $\frac{\partial}{\partial \alpha^\nu}$ are dual to α^ν and hence are assigned degree -2. Therefore, $\mathfrak{X}(\mathcal{M})$ is a graded vector space

$$\bigoplus_{k=-\infty}^{\infty} \mathfrak{X}(\mathcal{M})_k,$$

which actually is endowed with an $\mathcal{O}_M$-module structure. In particular, since the degrees of $\mathcal{O}_M$ are non-negative, we see that $\mathfrak{X}(\mathcal{M})_{-2}$ is $C^\infty(M)$-spanned by $\{\frac{\partial}{\partial \alpha^\nu}\}$ and $\mathfrak{X}(\mathcal{M})_{-1}$ is $C^\infty(M)$-spanned by $\{\frac{\partial}{\partial \varepsilon^\mu}, \varepsilon^\mu \otimes \frac{\partial}{\partial \alpha^\nu}\}$. Now, from Eq. (3.2), it follows that if we choose another coordinate system $\{\overline{x}^i, \overline{\varepsilon}^\mu, \overline{\alpha}^\nu\}$ for $\mathcal{O}(U)$, then

$$\frac{\partial}{\partial \overline{\alpha}^\nu} = \frac{\partial(b^j_\kappa(x)\alpha^\kappa)}{\partial \alpha^\nu}\frac{\partial}{\partial \alpha^j} = b^j_\nu(x)\frac{\partial}{\partial \alpha^j}, \tag{3.57}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \bar{\varepsilon}^\mu} &= \frac{\partial (a_j^\lambda(x)\varepsilon^j)}{\partial \varepsilon^\mu}\frac{\partial}{\partial \varepsilon^\lambda} + \frac{\partial \left(b_\kappa^\nu(x)\alpha^\kappa + \frac{1}{2}c_{\lambda\eta}^\nu(x)\varepsilon^\lambda\varepsilon^\eta\right)}{\partial \varepsilon^\mu}\frac{\partial}{\partial \alpha^\nu} \\
&= a_\mu^\lambda(x)\frac{\partial}{\partial \varepsilon^\lambda} + \frac{1}{2}c_{\lambda\eta}^\nu(x)\left(\frac{\partial \varepsilon^\lambda)}{\partial \varepsilon^\mu}\varepsilon^\eta - \varepsilon^\lambda\frac{\partial \varepsilon^\eta}{\partial \varepsilon^\mu}\right)\frac{\partial}{\partial \alpha^\nu} \\
&= a_\mu^\lambda(x)\frac{\partial}{\partial \varepsilon^\lambda} + c_{\mu\eta}^\nu(x)\varepsilon^\eta\frac{\partial}{\partial \alpha^\nu},
\end{aligned}
\tag{3.58}
$$

from which we conclude that $\mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma(F)$ (recall that $F = \ker p \cong \widetilde{F}/\Lambda^2 E$, see Eq. (3.3)), and there is a vector bundle G such that $\mathfrak{X}(\mathcal{M})_{-1} \cong \Gamma(G)$, and G fits in the exact sequence

$$0 \longrightarrow E^* \otimes F \longrightarrow G \longrightarrow E \longrightarrow 0. \tag{3.59}$$

This local analysis suggests the following result.

Proposition 3.52. *Consider a degree 2 manifold $\mathcal{M}$, and its corresponding involutive DVB D. Then, there are canonical isomorphisms of $C^\infty(M)$-modules*

$$\mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma_{\text{core}}(D) \cong \Gamma(F) \quad \textit{and} \quad \mathfrak{X}(\mathcal{M})_{-1} \cong \Gamma_{\text{lin}}(D) \cong \Gamma(\widehat{E}).$$

We will denote these isomorphisms by

$$\xi \longrightarrow \iota_\xi \in \mathfrak{X}(\mathcal{M})_{-2} \quad \textit{and} \quad \phi \longrightarrow \iota_\phi \in \mathfrak{X}(\mathcal{M})_{-1}, \tag{3.60}$$

for $\xi \in \Gamma(F)$, and $\phi \in \Gamma(\widehat{E})$.

Proof. We begin by establishing a canonical identification $\mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma_{\text{core}}(D)$. First observe that any section $\widetilde{\xi} \in \Gamma_{\text{core}}(D)$ is canonically identified with a linear functional $f_\xi \in \text{Hom}(\widehat{F^*}, \mathbb{R})$ such that $E^* \otimes E^* \subset \ker f_\xi$. In fact, these functionals are precisely the ones that descend to the quotient F^*, and therefore are canonically identified with the elements in $\Gamma(F)$. Then take any $X \in \mathfrak{X}(\mathcal{M})_{-2}$. Since X has degree -2, it follows that, for any $\zeta \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*)$, $X(\zeta) \in \mathcal{A}^0 \cong C^\infty(M)$, and by Leibniz rule, and again degree reasons, we have, for any $f \in C^\infty(M)$, $X(f\zeta) = fX(\zeta)$. Then, X defines a linear functional f_X on $\widetilde{F}^*$,

$$f_X(\zeta(m)) := X(\zeta)(m),$$

which satisfies, by Leibniz rule, and once more relying on degree reasons, we conclude that

$$f_X(\varepsilon_1 \wedge \varepsilon_2) = f_X(\varepsilon_1)\varepsilon_2 - \varepsilon_1 f_X(\varepsilon_2) = 0, \qquad \forall \varepsilon_1 \wedge \varepsilon_2 \in \Lambda^2 E^*,$$

which implies that $\Lambda^2 E^* \subset \ker f_X$. We can extend f_X to a functional $f_\xi \in \text{Hom}(\widehat{F^*}, \mathbb{R})$ by setting $f_\xi(\varsigma) := 0$ for every $\varsigma \in S^2 E^*$, so that $E^* \otimes E^* \subset \ker f_\xi$, and therefore we can associate to X the corresponding core section to f_ξ. And conversely, given f_ξ, we just define $X(f) = X(\varepsilon) := 0$ and $X(\zeta) := f_\xi(\zeta)$, and then we can extend it by Leiniz rule to the whole space of functions $\mathcal{A}$ obtaining a vector field $X \in \mathfrak{X}(\mathcal{M})_{-2}$. Therefore we have

$$\mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma_{\text{core}}(D) \cong \Gamma(F).$$

As for $\mathfrak{X}(\mathcal{M})_{-1}$, we want to find a canonical correspondence of vector fields $X \in \mathfrak{X}(\mathcal{M})_{-1}$ with linear sections of D. By the characterization of linear sections given in Cor. 2.9, it is enough to associate to X a pair of linear functions

$$f_{\widehat{F^*}} : \widehat{F^*} \longrightarrow E^* \quad \text{and} \quad f_{E^*} : E^* \longrightarrow \mathbb{R},$$

satisfying $f_{\widehat{F^*}}(\tau) = f_{E^*} \circ \tau$, for every $\tau \in E^* \otimes E^* \subset \widehat{F^*}$. So, using the canonical decomposition $\widehat{F^*} = S^2E^* \oplus \widetilde{F}^*$ given by the involutivity structure, and the canonical identifications $\Gamma(E^*) \cong \mathcal{A}^1$ and $\Gamma(\widetilde{F}^*) \cong \mathcal{A}^2$, we set

$$f_{E^*}(\varepsilon) := X(\varepsilon), \qquad \forall \varepsilon \in \Gamma(E^*)$$

and

$$f_{\widehat{F^*}}(\varsigma + \zeta) := f_{E^*} \circ \varsigma + X(\zeta), \qquad \forall \varsigma \in \Gamma(S^2E^*), \zeta \in \Gamma(\widetilde{F}^*).$$

Since X has degree -1, derivation rule implies that f_{E^*} and $f_{\widehat{F^*}}$ are tensorial with respect to $C^\infty(M)$. Also by degree reasons we have that $X(\varepsilon) \in C^\infty(M)$ and $X(\zeta) \in \Gamma(E^*)$. Therefore $f_{E^*} \in \mathrm{Hom}(E^*, \mathbb{R})$ and $f_{\widehat{F^*}} \in \mathrm{Hom}(\widehat{F^*}, E^*)$. It remains to verify the compatibility condition. First observe that, from Leibniz rule we have, for $\varepsilon_{\mathbf{1}} \wedge \varepsilon_{\mathbf{2}} \in \Gamma(\Lambda^2E^*)$,

$$\begin{aligned} X(\varepsilon_{\mathbf{1}} \wedge \varepsilon_{\mathbf{2}}) &= X(\varepsilon_{\mathbf{1}})\varepsilon_{\mathbf{2}} - \varepsilon_{\mathbf{1}} X(\varepsilon_{\mathbf{2}}) \\ &= f_{E^*}(\varepsilon_{\mathbf{1}})\varepsilon_{\mathbf{2}} - \varepsilon_{\mathbf{1}} f_{E^*}(\varepsilon_{\mathbf{2}}) \\ &= f_{E^*} \circ \varepsilon_{\mathbf{1}} \wedge \varepsilon_{\mathbf{2}}. \end{aligned}$$

By $C^\infty(M)$-linearity, it follows that $X(\lambda) = f_{E^*} \circ \lambda$ for every $\lambda \in \Gamma(\Lambda^2E^*)$. Then, take $\tau \in \Gamma(E^* \otimes E^*)$. We can write $\tau = \varsigma + \lambda$, with $\varsigma \in \Gamma(S^2E^*)$ and $\lambda \in \Gamma(\Lambda^2E^*)$

$$f_{\widehat{F^*}}(\tau) = f_{\widehat{F^*}}(\varsigma + \lambda) = f_{E^*} \circ \varsigma + f_{E^*} \circ \lambda = f_{E^*} \circ \tau.$$

Conversely, given such a pair of linear maps $f_{\widehat{F^*}}, f_{E^*}$, we define

$$X(f) := 0, \quad X(\varepsilon) := f_{E^*}(\varepsilon) \quad \text{and} \quad X(\zeta) := f_{\widehat{F^*}}(\zeta),$$

and observe that we can extend X to the whole space of functions (which is generated as a graded algebra by $\mathcal{A}^0, \mathcal{A}^1$ and $\mathcal{A}^2$) by Leibniz rule, observing that the definitions given above are consistent with this rule because of the linearity of $f_{E^*}, f_{\widehat{F^*}}$ and the compatibility condition they satisfy.

■

Corollary 3.53. *Given an involutive sequence $(E, \widetilde{F}, \widehat{F} \xrightarrow{p} \Lambda^2E)$, then the sections of the dual sequence $E^* \otimes F \longrightarrow \widehat{E} \longrightarrow E$, defined in Def. 3.45, are canonically identified with pairs of linear functions*

$$f_{\widetilde{F}^*} : \widetilde{F}^* \longrightarrow E^* \quad \textit{and} \quad f_{E^*} : E^* \longrightarrow \mathbb{R},$$

satisfying $f_{\widetilde{F}^*}(\tau) = f_{E^*} \circ \tau$, for every $\tau \in \Lambda^2E^* \subset \widetilde{F}^*$.

As a consequence, the fibers of the dual bundle $\widehat{E}$ are given by

$$\widehat{E}_m = \{\mu \in \mathrm{Hom}(\widetilde{F}^*_m, E^*_m) | \exists e \in E_m \ s.t. \ \mu(\lambda) = \iota_e\lambda, \ \forall \lambda \in \Lambda^2E_m\}.$$

Corollary 3.54. *With the identifications of Prop. 3.52, we have, for every* $\phi \in \Gamma(\widehat{E}), \xi \in \Gamma(F), \varepsilon \in \Gamma(E^*)$ *and* $\gamma \in \Gamma(\widetilde{F}^*)$*,*

$$\iota_\phi(\gamma) = \langle \phi, \gamma \rangle_E, \qquad \iota_\phi(\varepsilon) = \langle \phi, \bar{\varepsilon} \rangle_E, \quad \text{and} \quad \iota_\xi(\gamma) = \langle \bar{\xi}, \gamma \rangle_E.$$

Remark 3.55. In the degree 1 case, that is, when we are given a 1-manifold $\mathcal{M}$, there is a vector bundle $A \longrightarrow M$ such that $\mathcal{A}^1 \cong \Gamma(A^*)$ and $\mathcal{M} \cong A[1]$. In this case, we have

$$\mathfrak{X}(\mathcal{M})_{-1} \cong \Gamma(A),$$

since, by degree reasons, $X(s) \in C^\infty(M)$, $\forall s \in \Gamma(A^*), X \in \mathfrak{X}(\mathcal{M})_{-1}$, and from Leibniz rule, and again by degree reasons, $X(fs) = fX(s)$ for all $f \in C^\infty(M)$ and $s \in \Gamma(A^*)$, therefore, we canonically associate to any $X \in \mathfrak{X}(\mathcal{M})_{-1}$ an element in $\Gamma(A)$. And conversely, given $\phi \in \Gamma(A)$, we define $X := \iota_\phi \in \mathfrak{X}(\mathcal{M})_{-1}$ by

$$\iota_\phi(f) := 0 \quad \text{and} \quad \iota_\phi(s) := \langle \phi, s \rangle, \quad \forall f \in C^\infty(M), s \in \Gamma(A^*),$$

and extend X to the whole space $\mathcal{A}$ by Leibniz rule, observing that the definition of $X(fs)$ is consistent with this rule because $\langle \phi, fs \rangle = f \langle \phi, s \rangle$.

The above proposition has, as we will see, some powerful consequences. The first one is that it provides a nice interpretation of the symmetric pairing $W : S^2(\widehat{E}) \longrightarrow F$ introduced in Prop. 3.48.

Proposition 3.56. *The tensor* $W \in S^2(\widehat{E})^* \otimes F$ *introduced in Prop. 3.48, is given by the commutator of the corresponding vector fields:*

$$W(\phi_1, \phi_2) = [\iota_{\phi_1}, \iota_{\phi_2}] \in \mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma(F). \tag{3.61}$$

Proof. From Cor. 3.54 and Eq. (3.45), we get

$$\langle T(\phi_1, \phi_2), \gamma \rangle = \iota_{\phi_2} \iota_{\phi_1}(\gamma), \qquad \forall \phi_1, \phi_2 \in \Gamma(\widehat{E}), \gamma \in \widetilde{F}^*,$$

from which, and taking Cor. 3.54 into account once more, Eq. (3.46) casts

$$\begin{aligned} \langle W(\phi_1, \phi_2), \gamma \rangle &= \langle T(\phi_1, \phi_2) + T(\phi_2, \phi_1), \gamma \rangle \\ &= \iota_{\phi_2} \iota_{\phi_1}(\gamma) + \iota_{\phi_1} \iota_{\phi_2}(\gamma) = [\iota_{\phi_1}, \iota_{\phi_2}](\gamma), \end{aligned}$$

where, for the last equality, we also used that the degree of ι_{ϕ_1} and ι_{ϕ_2} is -1.

∎

3.6 Metric sequences

Inspired in the work of M. Grutzmann and T. Strobl [24], we introduce yet another class of geometric objects, which are in 1:1 correspondence (up to ismorphisms) with degree 2 manifolds. This time the structure is placed on the dual sequence (3.44).

Definition 3.57. A *metric sequence* is defined by the following data

- a triplet of vector bundles, which we denote by $(F, \widehat{E}, E)$,
- a non-degenerate surjective symmetric product $(\cdot,\cdot) : S^2\widehat{E} \longrightarrow F$ taking values in F.
- a surjective vector bundle morphism $\pi : \widehat{E} \longrightarrow E$, such that
- $\pi \circ \pi^* = 0$, where $\pi^* : E^* \otimes F \longrightarrow \widehat{E}$ is the adjoint of π with respect to $(\cdot,\cdot)$:

$$(\pi^*(\eta), \phi) := \eta(\pi(\phi)), \qquad \forall \eta \in E^* \otimes F, \phi \in \widehat{E}.$$

We have the following result.

Theorem 3.58. *An exact sequence of the form*

$$E^* \otimes E^* \longrightarrow \Omega \longrightarrow F^*$$

is involutive if and only if the dual sequence

$$E^* \otimes F \longrightarrow \widehat{E} \longrightarrow E$$

is metric.

Proof. If we have an involutive sequence, we obtained a symmetric product $W : S^2\widehat{E} \longrightarrow F$ in Prop. 3.56.

Conversely, let

$$E^* \otimes F \longrightarrow \widehat{E} \longrightarrow E$$

be a metric sequence. Then, by Cor. 2.9, we can obtain the dual sequence

$$E^* \otimes E^* \longrightarrow \Omega \longrightarrow F^*. \tag{3.62}$$

By Cor. 3.21, the sequence (3.62) corresponds to an involutive sequence (in the sense that it comes from an involutive sequence extended by S^2E^*) if and only if we can find a subbundle $\widetilde{F}^* \subset \Omega$ that fits in the exact subsequence of (5.33)

$$0 \longrightarrow \Lambda^2 E^* \xrightarrow{\iota} \widetilde{F}^* \xrightarrow{p} F^* \longrightarrow 0. \tag{3.63}$$

So we will obtain such subbundle $\widetilde{F}^*$. We begin by observing that axiom *1* of Def. 5.20 implies that $E^* \otimes F$, viewed inside $\widehat{E}$ through the injection π^*, is isotropic with respect to $(\cdot,\cdot)$. Next we observe that we can find an isotropic horizontal lift of (5.29) by first choosing any horizontal lift $\psi_0 : E^* \longrightarrow \widehat{E}$, and then defining $\psi : E \longrightarrow \widehat{E}$ by

$$\psi := \psi_0 - \pi^* \circ B_{\psi_0}, \tag{3.64}$$

where $B_{\psi_0} : E \longrightarrow E^* \otimes F$ is defined by

$$B_{\psi_0}(e_1)(e_2) := \frac{1}{2}(\psi_0(e_1), \psi_0(e_2)), \qquad \forall e_1, e_2 \in E.$$

Observe that, for an isotropic lift we have

$$(\phi_1, \phi_2) = (\eta_1 + \widehat{e_1}, \eta_2 + \widehat{e_2}) = \eta_1(e_2) + \eta_2(e_2). \tag{3.65}$$

Now we define

$$\widetilde{F}^* := \widetilde{\psi}(F^*) \oplus \iota(\Lambda^2 E^*) \subset \Omega, \tag{3.66}$$

where $\widetilde{\psi} : F^* \longrightarrow \Omega$ is the dual horizontal lift given by Cor. C.9 corresponding to *any* *isotropic* horizontal lift $\psi : E^* \longrightarrow \widehat{E}$. We must show that $\widetilde{F}^*$ doesn't depend on the isotropic horizontal lift chosen ψ. Indeed, if $\psi' : E \longrightarrow \widehat{E}$ is another *isotropic* horizontal lift, then by Eq. (3.65) we have, for every $\phi_1, \phi_2 \in \widehat{E}$,

$$\begin{aligned}\eta_1(e_2) + \eta_2(e_1) = (\phi_1, \phi_2) &= \eta_1'(e_2) + \eta_2'(e_1)\\ &= \eta_1(e_2) - \Psi_{e_1}(e_2) + \eta_2(e_1) - \Psi_{e_2}(e_1),\end{aligned}$$

which implies that

$$\Psi_{e_1}(e_2) = -\Psi_{e_2}(e_1), \qquad \forall e_1, e_2 \in E,$$

which in turn, by Prop. C.15, implies that

$$\widetilde{\psi}(\zeta) - \widetilde{\psi}'(\zeta) = \Psi_\zeta^* \in \Lambda^2 E^*, \qquad \forall \zeta \in F^*.$$

Thus, $\widetilde{\psi}'(\zeta) = \widetilde{\psi}(\zeta) + \Psi_\zeta^* \in \widetilde{F}^*$, whence,

$$\widetilde{\psi}'(F^*) \oplus \iota(\Lambda^2 E^*) = \widetilde{F}^*,$$

that is, $\widetilde{F}^*$ doesn't depend on the isotropic horizontal lift ψ chosen. Therefore, we have found an involutive sequence $\Lambda^2 E^* \xrightarrow{\iota} \widetilde{F}^*$.

■

3.7 Geometric description of degree 3 functions on a degree 2 manifold

One of our main aims is to describe a Q-structure on a degree 2 manifold, without the choice of a splitting. To achieve this, the notion of duality is instrumental, which enables us to obtain a geometric characterization of degree 3 functions on a degree 2 manifold in terms of a pair of vector bundle morphisms. This can be seen as a by-product of our characterization of degree 2 manifolds as involutive double vector bundles.

Now we go onto describing a degree 3 function on a degree 2 manifold in terms of its corresponding involutive double vector bundle. More precisely, in terms of the pairs of bundles $(\widetilde{F}, E)$ and $(\widehat{E}, F)$, but we use the whole structures of $D_{F^*} = D_h^*$ and of $(D_{F^*})_E^* = D$ as an auxiliary tool, and also as a geometric guide (recall that the suffix h indicates that we perform the dual using the horizontal structure).

We encourage the reader to consult Sec. 6.4.2 in order to see the concrete example when the degree 3 function comes from a Courant algebroid.

Theorem 3.59. *Let $\mathcal{M}$ be a degree 2 manifold. There is a 1:1, canonical, correspondence between degree 3 functions on $\mathcal{M}$, θ, and pairs $(\theta_1^\sharp, \theta_2^\sharp)$ of vector bundle morphisms*

$$\theta_1^\sharp : F \longrightarrow E^*; \qquad \theta_2^\sharp : \widehat{E} \longrightarrow \widetilde{F}^*,$$

which intertwine in the following way:

1. $\langle\theta_1^\sharp(\xi),\phi\rangle = \langle\theta_2^\sharp(\phi),\xi\rangle, \qquad \forall \xi\in F, \phi\in\widehat{E}.$
2. *For* $\eta\in E^*\otimes F$,
$$\theta_2^\sharp\circ\iota(\eta) = (\theta_1^\sharp\circ\eta)^* - \theta_1^\sharp\circ\eta \in \Lambda^2 E^*$$
holds, where $\iota : E^*\otimes F \longrightarrow \widehat{E}$ *is the inclusion.*
3. *The symmetric part of* $\theta_2^\sharp$ *is given by*
$$\langle\theta_2^\sharp(\phi_1),\phi_2\rangle + \langle\theta_2^\sharp(\phi_2),\phi_1\rangle = \theta_1^\sharp(W(\phi_1,\phi_2)),$$
where $W : S^2(\widehat{E}) \longrightarrow F$ *was introduced in Eq. (3.48).*

Proof. We will describe two processes, one inverse of the other, the first one to obtain the pair $(\theta_1^\sharp,\theta_2^\sharp)$ from a degree 3 function θ on $\mathcal{M}$, and the other to obtain a degree 3 function from a pair $(\theta_1^\sharp,\theta_2^\sharp)$ satisfying the three conditions of the theorem.

Let's introduce provisionally a horizontal lift, which also induces a splitting of the degree 2 manifold. Thus, the space of degree 3 functions is decomposed
$$\mathcal{A}^3 \cong \Gamma(E^*\otimes F^*\oplus\Lambda^3E^*).$$
Again because of the horizontal lift we have the inclusion
$$\Gamma(E^*\otimes F^*\oplus\Lambda^3E^*)\subset\Gamma(\Lambda^2(D_{F^*})_E),$$
given through
$$\Gamma(E^*\otimes F^*\oplus\Lambda^3E^*)\subset\Gamma(E^*\otimes F^*\oplus E^*\otimes\Lambda^2E^*)\cong\Gamma_{\mathrm{lin}}(\Lambda^2(D_{F^*})_E),$$
where the set of the right-hand side is the set of *linear* 2-sections on $(D_{F^*})_E$, see Rmk. 3.37. It is important to notice that the space $\Gamma(E^*\otimes F^*\oplus\Lambda^3E^*)$, viewed as a subspace in $\Gamma(\Lambda^2(D_{F^*})_E)$, is not invariant under a change of horizontal lift, because an element in $\Gamma(E^*\otimes F^*)$ will change to an element in $\Gamma(E^*\otimes F^*\oplus E^*\otimes\Lambda^2E^*)\not\subseteq\Gamma(E^*\otimes F^*\oplus\Lambda^3E^*)$. Thus, the embedding
$$\begin{aligned}\mathcal{A}^3 &\subset \Gamma(\Lambda^2(D_{F^*})_E)\\ \theta &\hookrightarrow \quad \widehat{\theta}\end{aligned} \tag{3.67}$$
depends strongly on the horizontal lift. For a function $\theta\in\mathcal{A}^3$, the corresponding 2-section $\widehat{\theta}\in\Lambda^2(D_{F^*})_E$ decomposes as
$$\widehat{\theta} = \widehat{\theta}_1+\widehat{\theta}_2,$$
with $\widehat{\theta}_1\in\Gamma(E^*\otimes F^*)$ and $\widehat{\theta}_2\in\Gamma(\Lambda^3E^*)$. Then we define
$$\theta_1^\sharp(\xi) := \langle\xi,\widehat{\theta}\rangle = \langle\xi,\widehat{\theta}_1\rangle\in\Gamma(E^*), \tag{3.68}$$
where $\xi\in\Gamma(F)$, and the pairing on the right-hand side stands for the insertion operator of the section $\xi\in\Gamma_{\mathrm{core}}((D_{F^*})_E^*) = \Gamma_{\mathrm{core}}(D)\subset\Gamma(D)$ in the 2-section $\widehat{\theta}_1\in\Gamma(\Lambda^2(D_{F^*})_E)$, which is simply the evaluation
$$\langle\xi,\widehat{\theta}_1\rangle := -\widehat{\theta}_1(\xi), \tag{3.69}$$

after identifying $\Gamma(E^* \otimes F^*) \cong \Gamma(\mathrm{Hom}(F, E^*))$. Notice that $\widehat{\theta}_1$ is well-defined, in the sense that it doesn't depend on the horizontal lift we chose. It is only $\widehat{\theta}_2$ that varies with the choice of the horizontal lift.

Now, for $\phi = \eta + e \in \Gamma(\widehat{E}) \cong \Gamma(E^* \otimes F \oplus E)$, we define

$$\theta_2^\sharp(\phi) = \lambda + \zeta \in \Gamma(\widetilde{F^*}) \cong \Gamma(\Lambda^2 E^* \oplus F^*), \tag{3.70}$$

where

$$\zeta := (\theta_1^\sharp)^*(e), \tag{3.71}$$

and

$$\lambda := (\theta_1^\sharp \circ \eta)^* - \theta_1^\sharp \circ \eta - \langle \widehat{e}, \widehat{\theta}_2 \rangle, \tag{3.72}$$

where, again, the pairing in the last term on the right-hand side stands for the insertion operator, provided by duality contraction.

It is worth making explicit here the inclusion $\Lambda^3 E^* \subset \Lambda^2 (D_{F^*})_E$ for an element of the form $\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ in order to understand what's going on. Denoting the inclusion by $\iota : \Lambda^3 E^* \longrightarrow \Lambda^2 (D_{F^*})_E$, we have

$$\iota(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3) = \overline{\varepsilon_1} \wedge \overline{\varepsilon_2} \otimes \varepsilon_3 - \overline{\varepsilon_1} \wedge \overline{\varepsilon_3} \otimes \varepsilon_2 + \overline{\varepsilon_2} \wedge \overline{\varepsilon_3} \otimes \varepsilon_1, \tag{3.73}$$

where, as usual, for a section $\varepsilon \in \Gamma(E^*)$, $\overline{\varepsilon}$ is the corresponding section in $\Gamma_{\mathrm{core}}((D_{F^*})_E)$, so that, for $\theta_2 = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ we have explicitly

$$\begin{aligned} \langle \widehat{e}, \widehat{\theta}_2 \rangle &= \langle e, \varepsilon_1 \rangle \overline{\varepsilon_2} \otimes \varepsilon_3 - \langle e, \varepsilon_2 \rangle \overline{\varepsilon_1} \otimes \varepsilon_3 - \langle e, \varepsilon_1 \rangle \overline{\varepsilon_3} \otimes \varepsilon_2 + \langle e, \varepsilon_3 \rangle \overline{\varepsilon_1} \otimes \varepsilon_2 \\ &\quad + \langle e, \varepsilon_2 \rangle \overline{\varepsilon_3} \otimes \varepsilon_1 - \langle e, \varepsilon_3 \rangle \overline{\varepsilon_2} \otimes \varepsilon_1 \\ &= \langle e, \varepsilon_1 \rangle \varepsilon_2 \wedge \varepsilon_3 - \langle e, \varepsilon_2 \rangle \varepsilon_1 \wedge \varepsilon_3 + \langle e, \varepsilon_3 \rangle \varepsilon_1 \wedge \varepsilon_2. \end{aligned} \tag{3.74}$$

Now we go onto proving the well definition of $\theta_2^\sharp$. Since $\widehat{\theta_1}$ nor e change with the change of horizontal lift, if follows that ζ doesn't depend on the horizontal lift. So we only need to check that λ is well-defined, namely, that it transforms appropriately under a change of horizontal lift.

So, choose a second horizontal lift, and denote the transition map by $\Psi : F^* \longrightarrow \Lambda^2 E^*$. Under this second horizontal lift we have the decompositions

$$\widehat{\theta} = \widehat{\theta}_1 + \widehat{\theta}_2', \;\; \phi = \eta' + e, \;\text{ and } \theta_2^\sharp(\phi) = \lambda' + \zeta,$$

with

$$\eta' = \eta - \Psi_\varepsilon^* \text{ (Prop. C.15)}, \;\; \lambda' = \lambda + \Psi_{(\theta_1^\sharp)^*(e)} \text{ and } \widehat{\theta}_2' = \widehat{\theta}_2 - \widetilde{\Psi_{(\theta_1^\sharp)^*(\cdot)}}, \tag{3.75}$$

where $\widetilde{\Psi_{(\theta_1^\sharp)^*(\cdot)}} \in \Gamma(\Lambda^3 E^*)$ is given by

$$\widetilde{\Psi_{(\theta_1^\sharp)^*(\cdot)}}(e_1, e_2, e_3) = \Psi_{(\theta_1^\sharp)^*(e_1)}(e_2, e_3) - \Psi_{(\theta_1^\sharp)^*(e_2)}(e_1, e_3) - \Psi_{(\theta_1^\sharp)^*(e_3)}(e_2, e_1). \tag{3.76}$$

Now, computing λ' with the corresponding expressions for η' and $\widehat{\theta}_2'$, we get

$$\begin{aligned} \lambda' &= (\theta_1^\sharp \circ \eta')^* - \theta_1^\sharp \circ \eta' - \langle \widehat{e}, \widehat{\theta}_2' \rangle \\ &= \eta^* \circ (\theta_1^\sharp)^* - \Psi_e \circ (\theta_1^\sharp)^* - \theta_1^\sharp \circ \eta + \theta_1^\sharp \circ \Psi_e^* - \langle \widehat{e}, \widehat{\theta}_2 - \widetilde{\Psi_{(\theta_1^\sharp)^*(\cdot)}} \rangle. \end{aligned} \tag{3.77}$$

Computing from the definitions, we have

$$\begin{aligned}(-\Psi_e \circ (\theta_1^\sharp)^* + \theta_1^\sharp \circ \Psi_e^*)(e_1, e_2) &= -\Psi_e((\theta_1^\sharp)^*(e_1), e_2) + \Psi_e((\theta_1^\sharp)^*(e_2), e_1)\\ &= -\Psi_{(\theta_1^\sharp)^*(e_1)}(e, e_2) - \Psi_{(\theta_1^\sharp)^*(e_2)}(e_1, e),\end{aligned}$$

whence we obtain

$$\begin{aligned}\lambda'(e_1, e_2) &= \eta^* \circ (\theta_1^\sharp)^*(e_1, e_2) - \theta_1^\sharp \circ \eta(e_1, e_2) - \langle \widehat{e}, \widehat{\theta_2}\rangle(e_1, e_2) + \Psi_{\theta_1^\sharp(e)}(e_1, e_2)\\ &= \lambda(e_1, e_2) + \Psi_{\theta_1^\sharp(e)}(e_1, e_2),\end{aligned}$$

which is precisely the corresponding value for λ' under the change of horizontal lift (see Eq. (3.75)).

Thus, $\theta_2^\sharp$ is well-defined. It remains to check that the pair $(\theta_1^\sharp, \theta_2^\sharp)$ satisfies the relations *1, 2*, and *3*, from the theorem. The first two properties follow directly from the way we defined $\theta_2^\sharp$. So let's check property *3*, about the symmetric part of $\theta_2^\sharp$.

By the definitions we have the following, for $\phi_i = \eta_i + e_i,\ i = 1, 2$:

$$\begin{aligned}\langle \theta_2^\sharp(\phi_1), \phi_2\rangle + \langle \theta_2^\sharp(\phi_2), \phi_1\rangle &= \langle \widehat{e_2}, \lambda_1\rangle + \eta_2^*(\zeta_1) + \langle \widehat{e_1}, \lambda_2\rangle + \eta_1^*(\zeta_2)\\ &= \theta_1^\sharp \circ \eta_1(e_2) - \eta_1^* \circ (\theta_1^\sharp)^*(e_2) - \langle \widehat{e_2}, \langle \widehat{e_1}, \widehat{\theta_2}\rangle\rangle + \eta_2^* \circ (\theta_1^\sharp)^*(e_1)\\ &\quad + \theta_1^\sharp \circ \eta_2(e_1) - \eta_2^* \circ (\theta_1^\sharp)^*(e_1) - \langle \widehat{e_1}, \langle \widehat{e_2}, \widehat{\theta_2}\rangle\rangle + \eta_1^* \circ (\theta_1^\sharp)^*(e_2)\\ &= \theta_1^\sharp(\eta_1(e_2) + \eta_2(e_1)) = \theta_1^\sharp(W(\phi_1, \phi_2)).\end{aligned}$$

In order to establish the canonical 1:1 correspondence we need now an inverse procedure, that assigns a degree 3 function –which remitting always to the fixed horizontal lift can be written $\theta = \theta_1 + \theta_2 \in \Gamma(E^* \otimes F^* \oplus \Lambda^3 E^*)$– from a given pair $(\theta_1^\sharp, \theta_2^\sharp)$ satisfying the three conditions of the theorem. We take the obvious option:

$$\begin{aligned}&\theta_1 := \theta_1^\sharp, \qquad \text{where we view } \Gamma(\mathrm{Hom}(F, E^*)) \cong \Gamma(E^* \otimes F^*) \subset \mathcal{A}^3 \cong \Gamma(E^* \otimes F^* \oplus \Lambda^3 E^*),\\ &\qquad\qquad\quad \text{namely } \langle \xi, \theta_1\rangle := -\theta_1^\sharp(\xi);\\ &\langle e_1 \wedge e_2, \theta_2\rangle := -\langle \widehat{e_2}, \theta_2^\sharp(\widehat{e_1})\rangle. \end{aligned} \tag{3.78}$$

We need to verify that θ_2 is well-defined, that is, that (3.78) actually defines an element in $\Gamma(\Lambda^3 E^*)$ and that it transforms correctly with a change of horizontal lift, so that the definition of $\theta = \theta_1 + \theta_2$ doesn't depend on the horizontal lift chosen.

In order to see that θ_2 is in $\Gamma(\Lambda^3 E)$, observe that the right-hand side of (3.78) is skew-symmetric in e_1, e_2, as follows directly from property *3* in the statement of the theorem. Indeed, from Eq. (3.48) follows that $W(\widehat{e_1}, \widehat{e_2}) = 0$. On the other hand, since $\theta_2^\sharp$ takes values on $\widetilde{F}^* \cong \Lambda^2 E^* \oplus F^*$, we have

$$\langle \widehat{e_3}, \overline{\langle \widehat{e_2}, \langle \theta_2^\sharp(\widehat{e_1})\rangle\rangle}\rangle = -\langle \widehat{e_2}, \overline{\langle \widehat{e_3}, \langle \theta_2^\sharp(\widehat{e_1})\rangle\rangle}\rangle.$$

Therefore, it follows that the expression

$$\langle \widehat{e_3}, \overline{\langle \widehat{e_2}, \langle \theta_2^\sharp(\widehat{e_1})\rangle\rangle}\rangle$$

is fully skew-symmetric in e_1, e_2, e_3, which means precisely that $\theta_2 \in \Gamma(\Lambda^3 E^*)$.

Now let's see that (3.78) transforms appropriately with a change of horizontal lift. As before, we denote the transition map by $\Psi : F^* \longrightarrow \Lambda^2 E^*$. If we denote by $\widehat{e_i}'$, $i = 1, 2$ the horizontal lift of e_i according to the second choice, so that $\widehat{e_i}' = \widehat{e_i} - \Psi^*_{e_i}$, and if we denote θ_2' the corresponding element of $\Gamma(\Lambda^3 E^*)$ according to the new horizontal lift, then we have

$$\begin{aligned}
\langle e_1 \wedge e_2, \theta_2' \rangle &= -\langle \widehat{e_2}', \theta_2^\sharp(\widehat{e_1}') \rangle = -\langle \widehat{e_2} - \Psi^*_{e_2}, \theta_2^\sharp(\widehat{e_1} - \Psi^*_{e_1}) \rangle \\
&= -\langle \widehat{e_2}, \theta_2^\sharp(\widehat{e_1}) \rangle + \langle \Psi^*_{e_2}, \theta_2^\sharp(\widehat{e_1}) \rangle + \langle \widehat{e_2}, \theta_2^\sharp(\Psi^*_{e_1}) \rangle \\
&= \langle e_1 \wedge e_2, \theta_2 \rangle + \langle \Psi^*_{e_2}, (\theta_1^\sharp)^*(e_1) \rangle + \langle \widehat{e_2}, (\Psi^*_{e_1})^* \circ (\theta_1^\sharp)^* - \theta_1^\sharp \circ \Psi^*_{e_1} \rangle \\
&= \langle e_1 \wedge e_2, \theta_2 \rangle + \langle \Psi^*_{e_2}, (\theta_1^\sharp)^*(e_1) \rangle + \langle \Psi^*_{e_1}(e_2), (\theta_1^\sharp)^*(\cdot) \rangle - \langle \Psi^*_{e_1}, (\theta_1^\sharp)^*(e_2) \rangle \\
&= \langle e_1 \wedge e_2, \theta_2 \rangle - \Psi_{(\theta_1^\sharp)^*(\cdot)}(e_1, e_2) + \Psi_{(\theta_1^\sharp)^*(e_1)}(\cdot, e_2) + \Psi_{(\theta_1^\sharp)^*(e_2)}(e_1, \cdot) \\
&= \left\langle e_1 \wedge e_2, \theta_2 - \widetilde{\Psi_{(\theta_1^\sharp)^*(\cdot)}} \right\rangle,
\end{aligned}$$

which shows that θ_2 transforms according to (3.75), as we wanted.

Thus, we have obtained the two processes we were seeking: $\theta \leftrightsquigarrow (\theta_1^\sharp, \theta_2^\sharp)$. By construction, it is evident that they are mutually inverse.

∎

Remark 3.60. We saw in Prop. C.24 how to obtain a DVB morphism from a pair of vector bundle morphisms, one between the corresponding linear bundles and the other between the corresponding core bundles. In Thm. 3.59 we have a pair of such morphisms, $(\theta_1^\sharp, \theta_2^\sharp)$, however they don't give rise to a DVB morphism $\Theta : D \longrightarrow D_{F^*}$, since the pair $(\theta_1^\sharp, \theta_2^\sharp)$ does not fulfill the compatibility condition (C.37). Indeed, the compatibility condition demands to have

$$\theta_2^\sharp(\eta) = \theta_1^\sharp \circ \eta,$$

however, the intertwining condition *2* tells us that

$$\theta_2^\sharp(\eta) = (\theta_1^\sharp \circ \eta)^* - \theta_1^\sharp \circ \eta,$$

which is somehow the compatibility condition skew-symmetrized. Actually, a pair which is closer to satisfy the compatibility condition is $(\theta_1^\sharp, -\theta_2^\sharp)$, in this case the defect being given by $-(\theta_1^\sharp \circ \eta)^*$.

The two morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ have a very nice –and useful– interpretation in terms of the identifications of Prop. 3.52.

Proposition 3.61. *Let $\theta \in \mathcal{A}^3$ be a degree 3 function on a degree 2 manifold $\mathcal{M}$. Then the corresponding pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ satisfy*

$$\theta_1^\sharp(\xi) = -\iota_\xi(\theta), \quad \text{and} \quad \theta_2^\sharp(\phi) = -\iota_\phi(\theta).$$

Proof. The identities follow from (the proofs of) Prop. 3.52 and Cor. 2.9, together with Eqs. (3.68), (3.70), (3.69), (3.71), (3.72) and (3.74) in the proof of Thm. 3.59.

∎

Chapter 4

Degree 2 NQ-manifolds

Integrable 1-vector fields on an N-manifold were considered by Ševera [63], who called these structures *NQ-manifolds*. In this chapter we address the geometric description of a degree 1 vector field on a degree 2 manifold, and the conditions on these geometric data that characterize the integrability of such vector fields. We arrive in this way to the concept of a *Lie 2-algebroid*, a geometric structure made up of brackets and vector bundle morphisms, which after introducing a splitting unfolds into the so-called *split* Lie 2-algebroids, or 2-term L_∞-algebroids, which are objects already studied, cf. [4], [65], [66] and the references therein, also for the case that the base is a point, [3].

Definition 4.1. An NQ-manifold is a graded manifold $\mathcal{M}$ together with a degree 1 vector field Q satisfying $[Q, Q] = 0$, i.e. a linear operator Q on $\mathcal{O}_M$ that raises the degree by one and satisfies $Q^2 = 0$ and, for each $U \subset M$,

$$Q(fg) = Q(f)g + (-1)^{|f|} fQ(g), \quad \forall f, g \in \mathcal{O}_M(U).$$

We will refer to an NQ-manifold also as an N-manifold endowed with a Q-structure.

Remark 4.2. Recall that a vector field is *homogeneous of degree k* if

$$|Q(f)| = k + r, \qquad \forall f \in \mathcal{O}_M, \text{ with } |f| = r,$$

and

$$Q(fg) = Q(f)g + (-1)^{k|f|} fQ(g).$$

Example 4.3. As we already observed, a 1-manifold is just a vector bundle A, so that $\mathcal{O}_M$ is isomorphic to $\Gamma_{\Lambda^\cdot A^*}$, the sheaf of sections of $\Lambda^\cdot A^*$. It is well-known that a Q-structure on $\mathcal{M}$ is equivalent to a Lie algebroid structure on A, the correspondence being

- $\rho(X)(f) := \langle Q(f), X\rangle$, for $X \in \Gamma A = (\mathcal{A}^1)^*$ and $f \in C^\infty(M) = \mathcal{A}^0$
- $\langle [X, Y], \alpha\rangle := \rho(X)(\langle Y, \alpha\rangle) - \rho(Y)(\langle X, \alpha\rangle) - \langle Q(\alpha), X \wedge Y\rangle$,
  for $X, Y \in \Gamma A$ and $\alpha \in \Gamma A^* = \mathcal{A}^1$.

4.1 PreLie 2-algebroids

We aim to characterize a Q-structure on a degree 2 manifold in terms of vector bundle morphisms and brackets on the sections of the dual sequence (3.44) of the corresponding involutive sequence (3.14), in a spirit similar to the characterization of VB-algebroids given in Prop. E.9.

In order to gain the intuition of what is happening in the general case, we refer the reader to Sec. 6.32 in order to see the particular case of a Q-structure coming from a Courant algebroid and the description of its corresponding Lie 2-algebroid, called the "cotangent" Lie 2-algebroid.

First we characterize geometrically 1-vector fields on degree 2 manifolds, without the integrability condition. Only then, in the next section, we address the characterization of integrability.

Lemma 4.4. *Consider an involutive sequence as the one in Eq. (3.14). Then we have the dual sequence (3.44) (see also Cor. 3.53). Let $\partial : F \longrightarrow E$ and $\Theta : \widehat{E} \longrightarrow \mathbf{CDO}(F)$ be vector bundle morphisms such that*

$$\Theta(\eta) = \eta \circ \partial, \tag{4.1}$$

for $\eta \in E^* \otimes F \subset \widehat{E}$.

Introduce a horizontal lift of (3.14), which induces a horizontal lift of (3.44) too. Define

$$\begin{aligned} \nabla : & E \longrightarrow \mathbf{CDO}(F) \\ & e \longrightarrow \quad \nabla_e \end{aligned}$$

by

$$\nabla_e \xi := \Theta(\widehat{e})(\xi), \tag{4.2}$$

for $e \in E$ and $\xi \in \Gamma(F)$, where $\widehat{e} \in \widehat{E}$ is the horizontal lift of e. Then the operator

$$\delta : \Gamma(F) \longrightarrow \Gamma(\widehat{E})$$

given by

$$\delta(\xi) := \nabla.\xi + \widehat{\partial(\xi)} \tag{4.3}$$

is well-defined, i.e. it doesn't depend on the chosen horizontal lift.

Proof. If we choose another horizontal lift, then

$$\widehat{e}' = \widehat{e} - \Psi_e^*,$$

where, as usual, $\Psi : F^* \longrightarrow \Lambda^2 E^*$ is the transition of map between the two horizontal lifts. Then, according to this second horizontal lift, we have

$$\nabla'_e \xi = \Theta(\widehat{e}')(\xi) = \Theta(\widehat{e} - \Psi_e^*)(\xi) = \nabla_e \xi - \Psi_e^* \circ \partial(\xi),$$

whence, since $\Psi_{e_1}^*(e_2) = -\Psi_{e_2}^*(e_1)$ (notice that here is entering the involutivity structure of D),

$$\nabla' \xi = \nabla \xi + \Psi_{\partial(\xi)}^*.$$

Therefore

$$\begin{aligned}\delta'(\xi) &= \nabla'\xi + \widehat{\partial(\xi)}' \\ &= \nabla\xi + \Psi^*_{\partial(\xi)} + \widehat{\partial(\xi)} - \Psi^*_{\partial(\xi)} \\ &= \delta(\xi).\end{aligned}$$

∎

Lemma 4.5. *Again consider the dual sequence (3.44). Let $\rho : E \longrightarrow TM$ and $\Psi : \widehat{E} \longrightarrow \mathrm{Diff}^1(E^*)$ be vector bundle morphisms such that*

$$\Psi(\phi)(f\varepsilon) = f\Psi(\phi)(\varepsilon) + \widehat{\rho}(\phi)(f)\varepsilon - \rho^*(df)\langle\phi, \overline{\varepsilon}\rangle, \tag{4.4}$$

for $\phi \in \widehat{E}, \varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$, where $\widehat{\rho} : \widehat{E} \longrightarrow TM$ is given by

$$\widehat{\rho} := \rho \circ \pi.$$

Then the equation

$$\langle \Delta_\Psi(\phi, e), \varepsilon\rangle := \widehat{\rho}(\phi)(\langle e, \varepsilon\rangle) - \rho(e)(\langle\phi, \overline{\varepsilon}\rangle) - \langle\Psi(\phi)(\varepsilon), e\rangle, \tag{4.5}$$

for $\phi \in \Gamma(\widehat{E}), \varepsilon \in \Gamma(E^*), e \in \Gamma(E)$, gives a well-defined first-order bidifferential operator

$$\Delta_\Psi : \Gamma(\widehat{E}) \times \Gamma(E) \longrightarrow \Gamma(E),$$

called the dualization *of Ψ. Actually, the adjoint maps $\mathrm{ad}^l_\phi : \Gamma(E) \longrightarrow \Gamma(E)$ and $\mathrm{ad}^r_e : \Gamma(\widehat{E}) \longrightarrow \Gamma(E)$, given by*

$$\mathrm{ad}^l_\phi := \Delta_\Psi(\phi, \cdot), \ \textit{and}\ \mathrm{ad}^r_e := \Delta_\Psi(\cdot, e)$$

are covariant differential operators, in the sense that the corresponding symbol maps are given by

$$\sigma^l_\phi = \widehat{\rho}(\phi) \otimes \mathrm{Id}_E, \qquad \sigma^r_e = \rho(e) \otimes \pi.$$

Proof. In order to see that Eq. (4.5) gives an element $\Delta_\Psi(\phi, e) \in \Gamma(E)$, we need to check that the pairing of the left-hand side is tensorial in ε. So, let $f \in C^\infty(M)$, then

$$\begin{aligned}\langle \Delta_\Psi(\phi, e), f\varepsilon\rangle &= \widehat{\rho}(\phi)(\langle e, f\varepsilon\rangle) - \rho(e)(\langle\phi, \overline{f\varepsilon}\rangle) - \langle\Psi(\phi)(f\varepsilon), e\rangle \\ &= f\langle\Delta_\Psi(\phi, e), \varepsilon\rangle - \widehat{\rho}(\phi)(f)\langle e, \varepsilon\rangle - \rho(e)(f)\langle\phi, \overline{\varepsilon}\rangle \\ &\quad - \langle\widehat{\rho}(\phi)(f)\varepsilon - \rho^*(df)\langle\phi, \overline{\varepsilon}\rangle, e\rangle \\ &= f\langle\Delta_\Psi(\phi, e), \varepsilon\rangle.\end{aligned}$$

Now, it is easy to see that Δ_Ψ is a first-order bidifferential operator if the last assertion of the lemma, about the symbol maps, is true. But this follows immediately from Eq. (4.5) and the derivation property of the anchor maps ρ and $\widehat{\rho}$ acting on functions.

∎

Definition 4.6. Consider an involutive sequence like (3.14)

$$0 \longrightarrow \Lambda^2 E^* \longrightarrow \widetilde{F}^* \longrightarrow F^* \longrightarrow 0 \tag{4.6}$$

and its associated bundle $\widehat{E}$ (see Cor. 3.53), which fits in the exact sequence

$$0 \longrightarrow E^* \otimes F \longrightarrow \widehat{E} \longrightarrow E \longrightarrow 0. \tag{4.7}$$

A *preLie 2-algebroid* consists in the following structure data

- an anchor map $\rho : E \longrightarrow TM$,
- a pseudoalgebra structure $([\cdot,\cdot], \widehat{\rho})$ on $\Gamma(\widehat{E})$,
- a core map $\partial : F \longrightarrow E$,
- two vector bundle morphisms

$$\begin{array}{rcl} \Psi : \widehat{E} & \longrightarrow & \mathrm{Diff}(E^*) \\ \phi & \longrightarrow & \Psi(\phi) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Theta : \widehat{E} & \longrightarrow & \mathbf{CDO}(F) \subset \mathrm{Diff}(F) \\ \phi & \longrightarrow & \Theta(\phi) \end{array}$$

  taking values on the bundle of first-order differential operators of E^* and F, respectively (see Cor. I.5 in the appendix).

These structure data are related by:

1. $\widehat{\rho} = \rho \circ \pi$,
2. $\Theta(\phi)(f\xi) = f\Theta(\phi)(\xi) + \widehat{\rho}(\phi)(f)\xi$,
3. $\Psi(\phi)(f\varepsilon) = f\Psi(\phi)(\varepsilon) + \widehat{\rho}(\phi)(f)\varepsilon - \rho^*(df)\langle \phi, \overline{\varepsilon}\rangle$,
4. $\Theta(\eta) = \eta \circ \partial$ and $\Delta_\Psi(\eta, \cdot) = \partial \circ \eta$, where Δ_Ψ is the dualization of Ψ, introduced in Lemma 4.5,
5. $\pi([\phi_1, \phi_2]) = \Delta_\Psi(\phi_1, \pi(\phi_2))$,
6. $[\phi, \eta] = \Theta(\phi) \circ \eta - \eta \circ \Delta_\Psi(\phi, \cdot)$,
7. $[\phi_1, \phi_2] + [\phi_2, \phi_1] = \delta(W(\phi_1, \phi_2))$, where $W : S^2(\widehat{E}) \longrightarrow F$ was introduced in Prop. 3.48 and δ is the map given in Eq. (4.3).

for all $\phi, \phi_1, \phi_2 \in \Gamma(\widehat{E}), \xi \in \Gamma(F), \varepsilon \in \Gamma(E^*)$ and $\eta \in \Gamma(E^* \otimes F) \cong \Gamma(\mathrm{Hom}(E, F)) \subset \Gamma(\widehat{E})$.

Remark 4.7. Notice that properties *7* and *2* already determine the left generalized anchor map $\varphi : \widehat{E} \longrightarrow TM \otimes \mathrm{End}(\widehat{E})$, for

$$\begin{aligned} [f\phi_1, \phi_2] &= -[\phi_2, f\phi_1] + \delta(W(f\phi_1, \phi_2)) \\ &= -f[\phi_2, \phi_1] - \widehat{\rho}(\phi_2)(f)\phi_1 + f\delta(W(\phi_1, \phi_2)) + \rho^*(df) \otimes W(\phi_1, \phi_2) \\ &= f[\phi_1, \phi_2] - \widehat{\rho}(\phi_2)(f)\phi_1 + \rho^*(df) \otimes W(\phi_1, \phi_2). \end{aligned}$$

Therefore,

$$\varphi(\phi_1)(df)(\phi_2) = \widehat{\rho}(\phi_2)(f)\phi_1 - \rho^*(df) \otimes W(\phi_1, \phi_2). \tag{4.8}$$

Remark 4.8. Property *1* implies that $\widehat{\rho}$ is determined by ρ. More important is the fact that, by property *5*, Ψ is completely determined by $([\cdot,\cdot],\widehat{\rho})$. Indeed, for every $\phi \in \Gamma(\widehat{E}), \varepsilon \in \Gamma(E^*), e \in \Gamma(E)$, we have

$$\begin{aligned}\langle \Psi(\phi)(\varepsilon), e\rangle &= \widehat{\rho}(\phi)(\langle e,\varepsilon\rangle) - \rho(e)(\langle \phi,\overline{\varepsilon}\rangle) - \langle \Delta_\Psi(\phi,e),\varepsilon\rangle \\ &= \widehat{\rho}(\phi)(\langle e,\varepsilon\rangle) - \widehat{\rho}(\widehat{e})(\langle \phi,\overline{\varepsilon}\rangle) - \langle [\phi,\widehat{e}],\overline{\varepsilon}\rangle,\end{aligned}$$

where $\widehat{e} \in \Gamma(\widehat{E})$ is any horizontal lift of e.

The following theorem justifies the preceding definition.

Theorem 4.9. *Consider a degree 2 manifold with associated exact sequence (3.14), so that its dual sequence is given by (3.44).*

There is a canonical 1:1 correspondence between degree 1 vector fields Q (not necessarily homological) on a degree 2 manifold and preLie 2-algebroid structures on (3.44).

Proof. The equivalence between 1 vector fields, Q, and preLie 2-algebroid structures is given through the following items:

- the anchor map $\rho : E^* \longrightarrow TM$ is given by

$$\rho(\varepsilon)(f) := -\langle Q(f),\varepsilon\rangle, \qquad f \in C^\infty(M), \varepsilon \in \Gamma(E^*), \tag{4.9}$$

- the *core map* $\partial : F \longrightarrow E$ is given by

$$\langle \partial(\xi),\varepsilon\rangle := \langle \xi, Q(\varepsilon)\rangle, \qquad \xi \in \Gamma(F) \cong \Gamma_{\text{core}}(D),\ \varepsilon \in \Gamma(E^*), \tag{4.10}$$

- the vector bundle morphism $\Psi : \widehat{E} \longrightarrow \mathrm{Diff}^1(E^*)$ is given by

$$\Psi(\phi)(\varepsilon) := -\langle \phi, Q(\varepsilon)\rangle, \qquad \phi \in \widehat{E},\ \varepsilon \in \Gamma(E^*) \cong \mathcal{A}^1, \text{ so that } Q(\varepsilon) \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*), \tag{4.11}$$

- the vector bundle morphism $\Theta : \widehat{E} \longrightarrow \mathbf{CDO}(F)$ is given by

$$\langle \Theta(\phi)(\xi),\gamma\rangle := \widehat{\rho}(\phi)(\langle \xi,\gamma\rangle) + \langle \partial(\xi), \langle \phi,\gamma\rangle\rangle - \langle \xi, Q(\gamma)^\sharp_2(\phi)\rangle, \tag{4.12}$$

  where $\widehat{\rho} : \widehat{E} \longrightarrow TM$ is given by

$$\widehat{\rho} := \rho \circ \pi. \tag{4.13}$$

  Here $\phi \in \Gamma(\widehat{E}), \xi \in \Gamma(F)$ and $\gamma \in \Gamma(\widetilde{F}) \cong \mathcal{A}^2$, so that $Q(\gamma) \in \mathcal{A}^3$. For the definition of the morphism $Q(\gamma)^\sharp_2 : \widehat{E} \longrightarrow \widetilde{F}$, see Thm. 3.59.

- the (non skew-symmetric) bracket $[\cdot,\cdot]$ on $\widehat{E}$ is given by

$$\begin{aligned}\langle [\phi_1,\phi_2],\gamma\rangle &:= \Psi(\phi_1)(\langle \phi_2,\gamma\rangle) - \Psi(\phi_2)(\langle \phi_1,\gamma\rangle) \\ &\quad + \rho^*(d(\langle T(\phi_2,\phi_1),\gamma\rangle)) - \langle \phi_2, Q(\gamma)^\sharp_2(\phi_1)\rangle, \qquad (4.14)\\ \langle [\phi_1,\phi_2],\overline{\varepsilon}\rangle &:= \widehat{\rho}(\phi_1)(\langle \phi_2,\overline{\varepsilon}\rangle) - \widehat{\rho}(\phi_2)(\langle \phi_1,\overline{\varepsilon}\rangle) + \langle T(\phi_1,\phi_2), Q(\varepsilon)\rangle, \qquad (4.15)\end{aligned}$$

  for $\phi_1,\phi_2 \in \Gamma(\widehat{E}), \gamma \in \Gamma(\widetilde{F})$ and $\varepsilon \in \Gamma(E^*)$, where $\overline{\varepsilon}$ is the corresponding section in $\Gamma_{\text{core}}(D_{F^*})$ (see Def. 3.47), and T was defined in Eq. (3.45) (again, see Def. 3.47).

In order to establish the canonical 1:1 correspondence, we need to show that:

Q $\longrightarrow$ preLie 2-algebroid: if we begin with a 1 vector field, then the structure maps and brackets satisfy the compatibility equations (*1 - 7*) from Def. 4.6.

preLie 2-algebroid $\longrightarrow$ Q: and conversely, if we begin with a preLie 2-algebroid, then $Q(f), Q(\varepsilon)$ and $Q(\gamma)_2^\sharp$ defined in the equations above, actually determine a 1 vector field on the corresponding degree 2 manifold.

Proof of Q $\longrightarrow$ preLie 2-algebroid. The fact that $\rho, \widehat{\rho}, \partial, \Psi$ and Θ are actually vector bundle morphisms together with properties *1, 2* and *3*, follows directly from the definitions and Leibniz property for Q. From the same reasons also follows the well-definition of the brackets, i.e., tensoriality with respect to γ and $\overline{\varepsilon}$ on the left-hand side of (4.14) and (4.15).

In order to check property *4*, observe that for $\phi = \eta$, the first term on the right-hand side of Eq. (4.12) is zero. Recalling property *2* of $Q(\gamma)_2^\sharp$ given in Thm. 3.59, we see that the third term on the right-hand side of Eq. (4.12) is zero too. Also observe that we don't lose generality taking $\eta = \varepsilon_0 \otimes \xi_0,\ \varepsilon_0 \in \Gamma(E^*), \xi_0 \in \Gamma(F)$. With these considerations, we have

$$\begin{aligned}\langle \Theta(\eta)(\xi), \gamma\rangle &= \widehat{\rho}(\eta)\langle \xi, \gamma\rangle + \langle \partial(\xi), \langle \eta, \gamma\rangle\rangle - \langle \xi, Q(\gamma)_2^\sharp(\eta)\rangle \\ &= \langle \xi_0, \gamma\rangle\langle \partial(\xi), \varepsilon_0\rangle \\ &= \langle \eta \circ \partial(\xi), \gamma\rangle,\end{aligned}$$

for all $\xi \in \Gamma(F), \gamma \in \Gamma(\widetilde{F}^*)$. Similarly, for $\phi = \eta$ the first two terms on the right-hand side of Eq. (4.5) are zero, so that

$$\begin{aligned}\langle \Delta_\Psi(\eta, e), \varepsilon\rangle &= \langle e, \langle \varepsilon_0 \otimes \xi_0, Q(\varepsilon)\rangle\rangle \\ &= \langle \xi_0, Q(\varepsilon)\rangle\langle e, \varepsilon_0\rangle = \langle \partial(\xi_0), \varepsilon\rangle\langle e, \varepsilon_0\rangle \\ &= \langle \partial \circ \eta(e), \varepsilon\rangle,\end{aligned}$$

for all $\varepsilon \in \Gamma(E^*), e \in \Gamma(E)$, thus obtaining property *4*.

Next, we need to check compatibility of Eqs. (4.15) and (4.14), when we take $\gamma = \varepsilon_1 \wedge \varepsilon_2 = \overline{\varepsilon_1} \otimes \varepsilon_2 - \overline{\varepsilon_2} \otimes \varepsilon_1$ in (4.14). On one hand we have from Eq. (4.15)

$$\begin{aligned}\langle [\phi_1, \phi_2], \varepsilon_1 \wedge \varepsilon_2\rangle &= \langle [\phi_1, \phi_2], \overline{\varepsilon_1}\rangle\varepsilon_2 - \langle [\phi_1, \phi_2], \overline{\varepsilon_2}\rangle\varepsilon_1 \\ &= \widehat{\rho}(\phi_1)(\langle \phi_2, \overline{\varepsilon_1}\rangle)\varepsilon_2 - \widehat{\rho}(\phi_2)(\langle \phi_1, \overline{\varepsilon_1}\rangle)\varepsilon_2 + \langle T(\phi_1, \phi_2), Q(\varepsilon_1)\rangle\varepsilon_2 \\ &\quad - \widehat{\rho}(\phi_1)(\langle \phi_2, \overline{\varepsilon_2}\rangle)\varepsilon_1 + \widehat{\rho}(\phi_2)(\langle \phi_1, \overline{\varepsilon_2}\rangle)\varepsilon_1 - \langle T(\phi_1, \phi_2), Q(\varepsilon_2)\rangle\varepsilon_1.\end{aligned} \tag{4.16}$$

On the other hand, from Eq. (4.14) we have

$$\begin{aligned}\langle [\phi_1, \phi_2], \varepsilon_1 \wedge \varepsilon_2\rangle =& \Psi(\phi_1)(\langle \phi_2, \varepsilon_1 \wedge \varepsilon_2\rangle) - \Psi(\phi_2)(\langle \phi_1, \varepsilon_1 \wedge \varepsilon_2\rangle) \\ &+ \rho^*(d(\langle T(\phi_2, \phi_1), \varepsilon_1 \wedge \varepsilon_2\rangle)) - \langle \phi_2, Q(\varepsilon_1 \wedge \varepsilon_2)_2^\sharp(\phi_1)\rangle.\end{aligned} \tag{4.17}$$

Now, using property *3*, we have

a)

$$\begin{aligned}\Psi(\phi_1)(\langle\phi_2,\varepsilon_1\wedge\varepsilon_2\rangle) &= \Psi(\phi_1)(\langle\phi_2,\overline{\varepsilon_1}\rangle\varepsilon_2 - \langle\phi_2,\overline{\varepsilon_2}\rangle\varepsilon_1)\\ &= \Psi(\phi_1)(\varepsilon_2)\langle\phi_2,\overline{\varepsilon_1}\rangle + \widehat{\rho}(\phi_1)(\langle\phi_2,\overline{\varepsilon_1}\rangle)\varepsilon_2 - \rho^*(d(\langle\phi_2,\overline{\varepsilon_1}\rangle))\langle\phi_1,\overline{\varepsilon_2}\rangle\\ &\quad - \Psi(\phi_1)(\varepsilon_1)\langle\phi_2,\overline{\varepsilon_2}\rangle - \widehat{\rho}(\phi_1)(\langle\phi_2,\overline{\varepsilon_2}\rangle)\varepsilon_1 + \rho^*(d(\langle\phi_2,\overline{\varepsilon_2}\rangle))\langle\phi_1,\overline{\varepsilon_1}\rangle.\end{aligned}$$

Analogously,

b)

$$\begin{aligned}\Psi(\phi_2)(\langle\phi_1,\varepsilon_1\wedge\varepsilon_2\rangle) &= \Psi(\phi_2)(\varepsilon_2)\langle\phi_1,\overline{\varepsilon_1}\rangle + \widehat{\rho}(\phi_2)(\langle\phi_1,\overline{\varepsilon_1}\rangle)\varepsilon_2 - \rho^*(d(\langle\phi_1,\overline{\varepsilon_1}\rangle))\langle\phi_2,\overline{\varepsilon_2}\rangle\\ &\quad - \Psi(\phi_2)(\varepsilon_1)\langle\phi_1,\overline{\varepsilon_2}\rangle - \widehat{\rho}(\phi_2)(\langle\phi_1,\overline{\varepsilon_2}\rangle)\varepsilon_1 + \rho^*(d(\langle\phi_1,\overline{\varepsilon_2}\rangle))\langle\phi_2,\overline{\varepsilon_1}\rangle.\end{aligned}$$

Next,

c)

$$\begin{aligned}\rho^*(d(\langle T(\phi_2,\phi_1),\varepsilon_1\wedge\varepsilon_2\rangle)) &= \rho^*(d(\langle\phi_2,\overline{\varepsilon_1}\rangle\langle\phi_1,\overline{\varepsilon_2}\rangle - \langle\phi_2,\overline{\varepsilon_2}\rangle\langle\phi_1,\overline{\varepsilon_1}\rangle))\\ &= \rho^*(d(\langle\phi_2,\overline{\varepsilon_1}\rangle))\langle\phi_1,\overline{\varepsilon_2}\rangle + \rho^*(d(\langle\phi_1,\overline{\varepsilon_2}\rangle))\langle\phi_2,\overline{\varepsilon_1}\rangle\\ &\quad - \rho^*(d(\langle\phi_2,\overline{\varepsilon_2}\rangle))\langle\phi_1,\overline{\varepsilon_1}\rangle - \rho^*(d(\langle\phi_1,\overline{\varepsilon_1}\rangle))\langle\phi_2,\overline{\varepsilon_2}\rangle.\end{aligned}$$

Finally, since $Q(\varepsilon_1\wedge\varepsilon_2) = Q(\varepsilon_1)\varepsilon_2 - \varepsilon_1 Q(\varepsilon_2)$,

d)

$$\begin{aligned}\langle\phi_2, Q(\varepsilon_1\wedge\varepsilon_2)^\sharp_2(\phi_1)\rangle &= \langle\phi_2,\overline{\varepsilon_2}\rangle\langle\phi_1,Q(\varepsilon_1)\rangle - \langle\phi_1,\overline{\varepsilon_2}\rangle\langle\phi_2,Q(\varepsilon_1)\rangle - \langle T(\phi_1,\phi_2),Q(\varepsilon_1)\rangle\varepsilon_2\\ &\quad - \langle\phi_2,\overline{\varepsilon_1}\rangle\langle\phi_1,Q(\varepsilon_2)\rangle + \langle\phi_1,\overline{\varepsilon_1}\rangle\langle\phi_2,Q(\varepsilon_2)\rangle + \langle T(\phi_1,\phi_2),Q(\varepsilon_2)\rangle\varepsilon_1.\end{aligned}$$

Putting items a), b) c) and d) into Eq. (4.17), we see that the two ways of computing

$$\langle[\phi_1,\phi_2],\varepsilon_1\wedge\varepsilon_2\rangle,$$

given by (4.16) and (4.17), coincide.

Property *5* follows immediately from Eqs. (4.5) and (4.15).

In order to check property *6*, again we observe that it suffices to prove it for $\eta = \varepsilon\otimes\xi$, since both sides of that equation behave the same way with respect to the product by functions $f \in C^\infty(M)$.

On one hand, taking property *4* into account, we have

$$\begin{aligned}\langle[\phi,\eta],\gamma\rangle &= \Psi(\phi)(\langle\eta,\gamma\rangle) - \Psi(\eta)(\langle\phi,\gamma\rangle) + \rho^*(d(\langle\phi,\overline{\langle\eta,\gamma\rangle}\rangle)) - \langle\eta, Q(\gamma)^\sharp_2(\phi)\rangle\\ &= \Psi(\phi)(\langle\xi,\gamma\rangle\varepsilon) + \langle\partial(\xi),\langle\phi,\gamma\rangle\rangle\varepsilon + \rho^*(d(\langle\xi,\gamma\rangle\langle\phi,\overline{\varepsilon}\rangle)) - \langle\xi, Q(\gamma)^\sharp_2(\phi)\rangle\varepsilon.\end{aligned} \tag{4.18}$$

On the other hand, using property *2*, we can perform the following calculations:

$$\begin{aligned}\langle\Theta(\phi)\circ\eta(e),\gamma\rangle &= \langle\Theta(\phi)(\langle\varepsilon,e\rangle\xi),\gamma\rangle\\ &= \widehat{\rho}(\phi)(\langle\varepsilon,e\rangle)\langle\xi,\gamma\rangle + \langle\Theta(\phi)(\xi),\gamma\rangle\langle\varepsilon,e\rangle,\end{aligned}$$

for every $\varepsilon \in \Gamma(E^*)$, whence

$$\begin{aligned}\langle \Theta(\phi)\circ\eta,\gamma\rangle &= \langle \xi,\gamma\rangle \widehat{\rho}(\phi)(\langle\cdot,\varepsilon\rangle) + \langle \Theta(\phi)(\xi),\gamma\rangle\varepsilon \\ &= \langle \xi,\gamma\rangle \widehat{\rho}(\phi)(\langle\cdot,\varepsilon\rangle) + \widehat{\rho}(\phi)(\langle\xi,\gamma\rangle)\varepsilon + \langle \partial(\xi),\langle\phi,\gamma\rangle\rangle\varepsilon - \langle \xi, Q(\gamma)_2^\sharp(\phi)\rangle\varepsilon.\end{aligned} \tag{4.19}$$

Now, from (4.5) we have

$$\begin{aligned}-\langle \eta\circ\Delta_\Psi(\phi,\cdot),\gamma\rangle &= -\langle\xi,\gamma\rangle\langle\Delta_\Psi(\phi,\cdot),\varepsilon\rangle \\ &= -\langle \xi,\gamma\rangle\widehat{\rho}(\phi)(\langle\cdot,\varepsilon\rangle) + \rho^*(d(\langle\phi,\overline{\varepsilon}\rangle))\langle\xi,\gamma\rangle + \langle\xi,\gamma\rangle\Psi(\phi)(\varepsilon).\end{aligned} \tag{4.20}$$

Putting together (4.19) and (4.20), and comparing with (4.18), using property *3*, we conclude

$$\langle[\phi,\eta],\gamma\rangle = \langle\Theta(\phi)\circ\eta,\gamma\rangle - \langle\eta\circ\Delta_\Psi(\phi,\cdot),\gamma\rangle,$$

which is property *6*.

Finally, we check property *7*. Computing from Eq. (4.14) and using property *3* of $Q(\gamma)_2^\sharp$, given in Thm. 3.59, we obtain

$$\langle[\phi_1,\phi_2]+[\phi_2,\phi_1],\gamma\rangle = \rho^*(d(\langle W(\phi_1,\phi_2),\gamma\rangle)) - Q(\gamma)_1^\sharp(W(\phi_1,\phi_2)). \tag{4.21}$$

Now, let's introduce a horizontal lift. Taking property *1* from Thm. 3.59, we have

$$\begin{aligned}\langle Q(\gamma)_1^\sharp(W(\phi_1,\phi_2)),\widehat{e}\rangle &= \langle W(\phi_1,\phi_2), Q(\gamma)_2^\sharp(\widehat{e})\rangle \\ &= \widehat{\rho}(\widehat{e})(\langle W(\phi_1,\phi_2),\gamma\rangle) + \langle\partial(W(\phi_1,\phi_2)),\langle\widehat{e},\gamma\rangle\rangle \\ &\quad - \langle\nabla_e^F W(\phi_1,\phi_2),\gamma\rangle \\ &= \widehat{\rho}(\widehat{e})(\langle W(\phi_1,\phi_2),\gamma\rangle) - \langle\langle\partial(W(\phi_1,\phi_2)),\gamma\rangle,\widehat{e}\rangle \\ &\quad - \langle\nabla_e^F W(\phi_1,\phi_2),\gamma\rangle,\end{aligned}$$

whence

$$(Q(\gamma)_1^\sharp)^*(W(\phi_1,\phi_2)) = \rho^*(d(\langle W(\phi_1,\phi_2),\gamma\rangle)) - \langle\delta(W(\phi_1,\phi_2),\gamma\rangle. \tag{4.22}$$

Putting together Eqs. (4.21) and (4.22), we obtain property *7*.

Proof of preLie 2-algebroid $\longrightarrow$ Q. First we observe that Eqs. (4.9)-(4.15) actually give elements $Q(f)\in\Gamma(E^*), Q(\varepsilon)\in\Gamma(\widetilde{F}^*)$ and $Q(\gamma)_2^\sharp(\phi)\in\Gamma(\widetilde{F}^*)$ because of tensoriality of the pairings that are defining these elements, and this is so thanks to the fact that $\rho,\partial,\Psi,\Theta$ are vector bundle morphisms, to the derivation property of the anchor map ρ with respect to functions, and to properties *2,3* of Def. 4.6.

Observe that the definition of $Q(\gamma)_1^\sharp$ is missing. To solve this, define

$$\langle Q(\gamma)_1^\sharp(\xi),\phi\rangle := \langle Q(\gamma)_2^\sharp(\phi),\xi\rangle, \qquad \forall\gamma\in\Gamma(\widetilde{F}^*),\xi\in\Gamma(F),\phi\in\Gamma(\widehat{E}). \tag{4.23}$$

In order to check that this definition indeed gives an element $Q(\gamma)_1^\sharp(\xi)$ in $\Gamma(E^*)$, observe that, from Eq. (4.12) and properties *1* and *4* of Def. 4.6, we have

$$\begin{aligned}\langle\xi,Q(\gamma)_2^\sharp(\eta)\rangle &= \widehat{\rho}(\eta)(\langle\xi,\gamma\rangle) + \langle\partial(\xi),\langle\eta,\gamma\rangle\rangle - \langle\Theta(\eta)(\xi),\gamma\rangle \\ &= \langle\partial(\xi),\langle\eta,\gamma\rangle\rangle - \langle\partial(\xi),\langle\eta,\gamma\rangle\rangle = 0,\end{aligned} \tag{4.24}$$

for every $\eta \in \Gamma(E^* \otimes F), \xi \in \Gamma(F)$. This implies that $Q(\gamma)^\sharp_1(\xi) \in \Gamma(E^*)$.

Next, we need to show that the pair $(Q(\gamma)^\sharp_1, Q(\gamma)^\sharp_2)$ satisfies properties *1, 2* and *3* of Thm. 3.59. Since property *1* is already satisfied by definition, it remains to verify only properties *2* and *3* of Thm. 3.59. Let's begin with property *2*. As usual, it is enough to consider η of the form $\varepsilon \otimes \xi$. Also observe that Eq. (4.24) already implies that $Q(\gamma)^\sharp_2(\eta) \in \Gamma(\Lambda^2 E^*)$, therefore, property *2* is equivalent to

$$\langle e_1 \wedge e_2, Q(\gamma)^\sharp_2(\eta)\rangle = \langle \varepsilon_1 \wedge \varepsilon_2, (Q(\gamma)^\sharp_1 \circ \eta)^* - Q(\gamma)^\sharp_1 \circ \eta\rangle. \tag{4.25}$$

On one hand we have

$$\begin{aligned}\langle e_1 \wedge e_2, (Q(\gamma)^\sharp_1 \circ \eta)^* - Q(\gamma)^\sharp_1 \circ \eta\rangle &= \langle Q(\gamma)^\sharp_2(\widehat{e_2}), \xi\rangle\langle e_1, \varepsilon\rangle - \langle Q(\gamma)^\sharp_2(\widehat{e_1}), \xi\rangle\langle e_2, \varepsilon\rangle \\ &= \rho(e_2)(\langle\gamma, \xi\rangle)\langle e_1, \varepsilon\rangle + \langle\partial(\xi), \langle\gamma, \widehat{e_2}\rangle\rangle\langle e_1, \varepsilon\rangle - \langle\nabla^F_{e_2}\xi, \gamma\rangle\langle e_1, \varepsilon\rangle \\ &\quad - \rho(e_1)(\langle\gamma, \xi\rangle)\langle e_2, \varepsilon\rangle - \langle\partial(\xi), \langle\gamma, \widehat{e_1}\rangle\rangle\langle e_2, \varepsilon\rangle + \langle\nabla^F_{e_1}\xi, \gamma\rangle\langle e_2, \varepsilon\rangle.\end{aligned} \tag{4.26}$$

On the other hand,

$$\begin{aligned}\langle e_1 \wedge e_2, Q(\gamma)^\sharp_2(\eta)\rangle &= \langle e_2, \Psi(\eta)(\langle\widehat{e_1}, \gamma\rangle) - \Psi(\widehat{e_1})(\langle\eta, \gamma\rangle) - \langle[\eta, \widehat{e_1}], \gamma\rangle\rangle \\ &= -\langle\partial(\xi), \langle\widehat{e_1}, \gamma\rangle\rangle\langle e_2, \varepsilon\rangle - \langle\xi, \gamma\rangle\Psi(\widehat{e_1})(\varepsilon) - \rho(e_1)(\langle\xi, \gamma\rangle)\langle e_2, \varepsilon\rangle + \rho(e_2)(\langle\xi, \gamma\rangle)\langle e_1, \varepsilon\rangle \\ &\quad - \langle\widehat{e_2}, \overline{\langle\eta \circ \Delta_\Psi(\widehat{e_1}, \cdot) - \nabla^F_{e_1} \circ \eta + \nabla^F_{\cdot}\eta(\varepsilon_1) + \partial(\eta(\varepsilon_1)), \gamma\rangle}\rangle.\end{aligned} \tag{4.27}$$

Developing the last term in the equality above, we have

$$\begin{aligned}\langle\widehat{e_2}, \overline{\langle\eta \circ \Delta_\Psi(\widehat{e_1}, \cdot) - \nabla^F_{e_1} \circ \eta + \nabla^F_{\cdot}\eta(e_1) + \partial(\eta(e_1)), \gamma\rangle}\rangle &= \langle\xi, \gamma\rangle(\rho(e_1)(\langle e_2, \varepsilon\rangle) - \rho(e_2)(\langle e_1, \varepsilon\rangle)) \\ &\quad - \langle e_2, \Psi(\widehat{e_1})(\varepsilon)\rangle) - \rho(e_1)(\langle e_2, \varepsilon\rangle)\langle\xi, \gamma\rangle - \langle e_2, \varepsilon\rangle\langle\nabla^F_{e_1}\xi, \gamma\rangle + \rho(e_2)(\langle e_1, \varepsilon\rangle)\langle\xi, \gamma\rangle \\ &\quad + \langle e_1, \varepsilon\rangle\langle\nabla^F_{e_2}\xi, \gamma\rangle - \langle e_1, \varepsilon\rangle\langle\partial(\xi), \langle e_2, \gamma\rangle\rangle.\end{aligned} \tag{4.28}$$

Putting (4.28) into (4.27) and after some cancellations we get

$$\begin{aligned}\langle e_1 \wedge e_2, Q(\gamma)^\sharp_2(\eta)\rangle &= -\langle\partial(\xi), \langle\widehat{e_1}\rangle\rangle\langle e_2, \varepsilon\rangle + \rho(e_2)(\langle\xi, \gamma\rangle)\langle e_1, \varepsilon\rangle \\ &\quad - \langle e_1, \varepsilon\rangle\langle\nabla^F_{e_2}\xi, \gamma\rangle + \langle e_2, \varepsilon\rangle\langle\nabla^F_{e_1}\xi, \gamma\rangle - \langle e_1, \varepsilon\rangle\langle\partial(\xi), \langle e_2, \gamma\rangle\rangle.\end{aligned} \tag{4.29}$$

Comparing (4.29) with (4.26), we get (4.25), as we wanted.

Finally, property *3* follows from the same calculations done to obtain Eqs. (4.21) and (4.22). Thus, Thm. 3.59 supplies us, for any $\gamma \in \Gamma(\widetilde{F}^*) \cong \mathcal{A}^2$, a well-defined element $Q(\gamma) \in \mathcal{A}^3$ corresponding to the pair $(Q(\gamma)^\sharp_1, Q(\gamma)^\sharp_2)$.

To end the proof, we need to check that the definitions of $Q(f), Q(\varepsilon)$ and $Q(\gamma)$ are compatible with the Leiniz rule, in the sense that the following identities should hold:

a) $Q(f\varepsilon) = fQ(\varepsilon) + Q(f) \wedge e$,

b) $Q(f\gamma) = fQ(\gamma) + Q(f)\gamma$,

c) $Q(\varepsilon_1 \wedge \varepsilon_2) = Q(\varepsilon_1)\varepsilon_2 - \varepsilon_1 Q(\varepsilon_2)$.

Let's prove item a). On one hand we have

$$\langle \xi, Q(f\varepsilon)\rangle = \langle \partial(\xi), f\varepsilon\rangle = f\langle \xi, Q(\varepsilon)\rangle,$$

and

$$\langle \phi, Q(f\varepsilon)\rangle = -\Psi(\phi)(f\varepsilon) = -f\Psi(\phi)(\varepsilon) - \widehat{\rho}(\phi)(f)\varepsilon + \rho^*(df)\langle \phi, \overline{\varepsilon}\rangle.$$

On the other hand

$$\langle \xi, fQ(\varepsilon) + Q(f) \wedge \varepsilon\rangle = f\langle \xi, Q(\varepsilon)\rangle,$$

and

$$\begin{aligned}
\langle \phi, f(Q(\varepsilon) + Q(f) \wedge e\rangle &= f\langle \phi, Q(\varepsilon)\rangle + \langle \phi, Q(f) \wedge \varepsilon\rangle \\
&= -f\Psi(\phi)(\varepsilon) + \langle \phi, \overline{Q(f)}\rangle\varepsilon - Q(f)\langle \phi, \overline{\varepsilon}\rangle \\
&= -f\Psi(\phi)(\varepsilon) - \widehat{\rho}(\phi)(f)\varepsilon + \rho^*(df)\langle \phi, \overline{\varepsilon}\rangle.
\end{aligned}$$

Comparing, we get item a).

For item b) let's begin computing $\langle \xi, Q(f\gamma)^\sharp_2(\phi)\rangle$ and $\langle \phi_2, Q(f\gamma)^\sharp_2(\phi_1)\rangle$.

$$\begin{aligned}
\langle \xi, Q(f\gamma)^\sharp_2(\phi)\rangle &= \widehat{\rho}(\phi)(f\langle \xi, \gamma\rangle) + f\langle \partial(\xi), \langle \phi, \gamma\rangle\rangle - f\langle \Theta(\phi)(\xi), \gamma\rangle \\
&= f\langle \xi, Q(\gamma)^\sharp_2(\phi)\rangle + \widehat{\rho}(\phi)(f)\langle \xi, \gamma\rangle;
\end{aligned}$$

$$\begin{aligned}
\langle \phi_2, Q(f\gamma)^\sharp_2(\phi_1)\rangle &= \Psi(\phi_1)(f\langle \phi_2, \gamma\rangle) - \Psi(\phi_2)(f\langle \phi_1, \gamma\rangle) + \rho^*(d(f\langle T(\phi_2, \phi_1), \gamma\rangle)) \\
&\quad - \langle [\phi_1, \phi_2], f\gamma\rangle \\
&= f\langle \phi_2, Q(\gamma)^\sharp_2(\phi_1)\rangle + \widehat{\rho}(\phi_1)(f)\langle \phi_2, \gamma\rangle - \rho^*(df)\langle T(\phi_2, \phi_1), \gamma\rangle \\
&\quad - \widehat{\rho}(\phi_2)(f)\langle \phi_1, \gamma\rangle + \rho^*(df)\langle T(\phi_1, \phi_2), \gamma\rangle + \rho^*(df)\langle T(\phi_2, \phi_1), \gamma\rangle \\
&= f\langle \phi_2, Q(\gamma)^\sharp_2(\phi_1)\rangle + \widehat{\rho}(\phi_1)(f)\langle \phi_2, \gamma\rangle - \widehat{\rho}(\phi_2)(f)\langle \phi_1, \gamma\rangle \\
&\quad + \rho^*(df)\langle T(\phi_1, \phi_2), \gamma\rangle.
\end{aligned}$$

Now let's compute $\langle \xi, (fQ(\gamma) + Q(f)\gamma)^\sharp_2(\phi)\rangle$ and $\langle \phi_2, (fQ(\gamma) + Q(f)\gamma)^\sharp_2(\phi_1)\rangle$, and then compare with the preceding results.

$$\begin{aligned}
\langle \xi, (fQ(\gamma) + Q(f)\gamma)^\sharp_2(\phi)\rangle &= f\langle \xi, Q(\gamma)^\sharp_2(\phi)\rangle + \langle \phi, Q(f)\rangle\langle \xi, \gamma\rangle \\
&= f\langle \xi, Q(\gamma)^\sharp_2(\phi)\rangle + \widehat{\rho}(\phi)(f)\langle \xi, \gamma\rangle;
\end{aligned}$$

$$\begin{aligned}
\langle \phi_2, (fQ(\gamma) + Q(f)\gamma)^\sharp_2(\phi_1)\rangle &= f\langle \phi_2, Q(\gamma)^\sharp_2(\phi_1)\rangle + \langle \phi_2, \overline{Q(f)}\rangle\langle \phi_1, \gamma\rangle \\
&\quad - \langle \phi_1, \overline{Q(f)}\rangle\langle \phi_2, \gamma\rangle - \langle T(\phi_1, \phi_2), \gamma\rangle Q(f) \\
&= f\langle \phi_2, Q(\gamma)^\sharp_2(\phi_1)\rangle - \widehat{\rho}(\phi_2)(f)\langle \phi_1, \gamma\rangle \\
&\quad + \widehat{\rho}(\phi_1)(f)\langle \phi_2, \gamma\rangle + \rho^*(df)\langle T(\phi_1, \phi_2), \gamma\rangle.
\end{aligned}$$

Comparing with the results above, we get

$$Q(f\gamma)_2^\sharp = (fQ(\gamma) + Q(f)\gamma)_2^\sharp,$$

which is equivalent to the equality of item b).

Finally, let's prove item c), again computing separately both sides of the pretended equality and then comparing.

On one hand we have

$$\begin{aligned}\langle \xi, Q(\varepsilon_1 \wedge \varepsilon_2)_2^\sharp(\phi)\rangle &= \widehat{\rho}(\phi)(\langle \xi, \varepsilon_1 \wedge \varepsilon_2\rangle) + \langle \partial(\xi), \langle \phi, \varepsilon_1 \wedge \varepsilon_2\rangle\rangle - \langle \Theta(\phi)(\xi), \varepsilon_1 \wedge \varepsilon_2\rangle \\ &= \langle \partial(\xi), \varepsilon_2\rangle\langle \phi, \overline{\varepsilon_1}\rangle - \langle \partial(\xi), \varepsilon_1\rangle\langle \phi, \overline{\varepsilon_2}\rangle.\end{aligned}$$

To compute $\langle \phi_2, Q(\varepsilon_1 \wedge \varepsilon_2)_2^\sharp(\phi_1)\rangle$ we use the calculations already performed in the first part of the proof in Eqs. (4.16), (4.17) and items a), b), and c) that follow to these equations. We obtain the following:

$$\begin{aligned}\langle \phi_2, Q(\varepsilon_1 \wedge \varepsilon_2)_2^\sharp(\phi_1)\rangle = {} & \Psi(\phi_1)(\varepsilon_2)\langle \phi_2, \overline{\varepsilon_1}\rangle - \Psi(\phi_1)(\varepsilon_1)\langle \phi_2, \overline{\varepsilon_2}\rangle \\ & - \Psi(\phi_2)(\varepsilon_2)\langle \phi_1, \overline{\varepsilon_1}\rangle + \Psi(\phi_2)(\varepsilon_1)\langle \phi_1, \overline{\varepsilon_2}\rangle \\ & + \langle T(\phi_1, \phi_2), Q(\varepsilon_1)\rangle\varepsilon_2 - \langle T(\phi_1, \phi_2), Q(\varepsilon_2)\rangle\varepsilon_1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \xi, (Q(\varepsilon_1)\varepsilon_2 - \varepsilon_1 Q(\varepsilon_2))_2^\sharp(\phi)\rangle &= -\langle \phi, \overline{\varepsilon_2}\rangle\langle \xi, Q(\varepsilon_1)\rangle + \langle \phi, \overline{\varepsilon_1}\rangle\langle \xi, Q(\varepsilon_2)\rangle \\ &= -\langle \phi, \overline{\varepsilon_2}\rangle\langle \partial(\xi), \varepsilon_1\rangle + \langle \phi, \overline{\varepsilon_1}\rangle\langle \partial(\xi), \varepsilon_2\rangle;\end{aligned}$$

in order to compute $\langle \phi_2, (Q(\varepsilon_1)\varepsilon_2 - \varepsilon_1 Q(\varepsilon_2))_2^\sharp(\phi_1)\rangle$ we use item d) of the calculations done in the (Q $\longrightarrow$ preLie 2-algebroid) part of the proof of this theorem. We get

$$\begin{aligned}\langle \phi_2, (Q(\varepsilon_1)\varepsilon_2 - \varepsilon_1 Q(\varepsilon_2))_2^\sharp(\phi_1)\rangle = {} & \langle \phi_2, \overline{\varepsilon_2}\rangle\langle \phi_1, Q(\varepsilon_1)\rangle - \langle \phi_1, \overline{\varepsilon_2}\rangle\langle \phi_2, Q(\varepsilon_1)\rangle \\ & - \langle T(\phi_1, \phi_2), Q(\varepsilon_1)\rangle\varepsilon_2 \\ & - \langle \phi_2, \overline{\varepsilon_1}\rangle\langle \phi_1, Q(\varepsilon_2)\rangle + \langle \phi_1, \overline{\varepsilon_1}\rangle\langle \phi_2, Q(\varepsilon_2)\rangle \\ & + \langle T(\phi_1, \phi_2), Q(\varepsilon_2)\rangle\varepsilon_1.\end{aligned}$$

Comparing with the calculations above for $Q(\varepsilon_1 \wedge \varepsilon_2)_2^\sharp$ we see that the results are the same, which implies item c), and the proof of the theorem is complete.

∎

4.2 The derived brackets

Now we aim to describe the derived brackets method developed by Y. Kosmann-Schwarzbach in [34] and, following [24], we will show how to apply the method to the setting of NQ degree 2 manifolds in order to obtain a proof of the geometric characterization of these manifolds in terms of Lie 2-algebroids.

To achieve this M. Grutzmann and T. Strobl apply the derived brackets method to the DGLA of vector fields on a degree 2 manifold with the (graded) commutator bracket, and degree 1 differential given by $d = [Q, \cdot]$. From this viewpoint, for a degree 3 function $\theta \in \mathcal{A}^3$, the corresponding pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, given in Thm. 3.59, are, up to sign, nothing but the action of -1 and -2 vector fields on θ. More precisely, if we denote by $\xi \longrightarrow \iota_\xi \in \mathfrak{X}(\mathcal{M})_{-2}$ the correspondence between core sections of D and -2 vector fields on $\mathcal{M}$, and by $\phi \longrightarrow \iota_\phi \in \mathfrak{X}(\mathcal{M})_{-1}$ the correspondence between linear sections of D and -1 vector fields on $\mathcal{M}$, then

$$\theta_1^\sharp(\xi) = -\iota_\xi(\theta) \in \mathcal{A}^1 \cong \Gamma(E^*), \quad \text{and} \quad \theta_2^\sharp(\phi) = -\iota_\phi(\theta) \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*).$$

At last, the Lie 2-algebroid data turns out to be nothing but the brackets and anchor that the derived brackets method provides.

As a by-product of this insight, we will obtain in Sec. 7.1 an alternative formula for the Lie 2-algebroid brackets $[\cdot, \cdot]$ in the case that Q is exact in a Poisson degree 2 manifod, i.e. $Q = \{\theta, \cdot\}$ for some $\theta \in \mathcal{A}^3$, which involves the pair $(\theta_1^\sharp, \theta_2^\sharp)$ and the VB-algebroid structure corresponding to the Poisson degree 2 manifod given by Thm. 6.14. As another application we will characterize, in Se. 7.2, the integrability of a degree 3 function $\theta \in \mathcal{A}^3$ on a Poisson degree 2 manifold, i.e. the equation $\{\theta, \theta\} = 0$, in terms of the pair $(\theta_1^\sharp, \theta_2^\sharp)$, the corresponding Lie 2-algebroid and a Lie algebroid structure on $\widetilde{F}^*$ induced by the Poisson brackets.

We describe the derived bracket method in the appendix, see Sec. G.3 where we describe the method in general before specializing to the degree 1 case, so we refer to that section for the general concepts, and also to get the basic intuition, since the degree 1 case is much simpler, so that the main ideas appear in a more transparent fashion. However, we chose to recall again the essential concepts in the beginning of this section, in order to make the reading more fluent.

Definition 4.10. A graded vector space A endowed with a graded bracket $[\cdot, \cdot]$, of degree $f \in \mathbb{Z}$ is called a *Loday algebra* if its bracket satisfies graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{(f+a)(f+b)}[b, [a, c]]. \tag{4.30}$$

We will refer to a bracket coming from a Loday algebra as a *Loday bracket*. If the bracket satisfies

$$[a, b] = -(-1)^{(f+a)(f+b)}[b, a], \tag{4.31}$$

then it is called a graded *Lie bracket*, and the corresponding Loday algebra is called a graded *Lie algebra*.

Now consider a derivation d of degree $|d|$ of the Loday algebra $(A, [\cdot, \cdot])$, that is

$$d[a, b] = [da, b] + (-1)^{d(f+a)}[a, db], \tag{4.32}$$

where we write the degree exponents without the bars $|\cdot|$ for sake of simplicity

We notice that when we have a Loday bracket, $[\cdot, \cdot]$, then for each $a_0 \in A$, $d_{a_0} := [a_0, \cdot]$ is a degree $|f| + |a_0|$ derivation of $[\cdot, \cdot]$.

Assume that d is of square 0,

$$d^2 = 0, \tag{4.33}$$

then we have a degree $|f| + |d|$ bracket $[\cdot, \cdot]_d$ given by

$$[a, b]_d = (-1)^{d(f+a)}[da, b]. \tag{4.34}$$

We say that d is *odd* if $|d|$ is odd.

Definition 4.11. A graded Lie algebra $(A, [\cdot, \cdot])$ endowed with a derivation d of its Lie bracket, is called a *differential graded Lie algebra*, and denoted by DGLA. The *derived bracket* of a given DGLA, denoted by $[\cdot, \cdot]_d$ is the bracket defined by f_d in Eq. (4.34) above.

Roughly, the derived brackets procedure consists in encoding the action of Q on the set of functions that generate the whole algebra (and thereby determine completely the vector field Q) by a series of anchor maps and derived brackets. The most simple and enlightening case is when we have a degree 1 manifold, to which we associate a vector bundle A, as explained in Ex. 4.3. This case was worked out in Sec. G.3, and we recall here the principal aspects. In this case the vector field Q is completely determined by its action of functions of degrees 0 and 1, and we can encode these actions by the anchor map

$$\rho_Q(\phi)(f), \qquad \forall f \in C^\infty(M) \cong \mathcal{A}^0,\ \phi \in \Gamma(A),$$

and the bracket

$$\langle Q(\alpha), \phi_1 \wedge \phi_2 \rangle = \rho_Q(\phi_1)(\langle \phi_2, \alpha \rangle) - \rho_Q(\phi_2)(\langle \phi_1, \alpha \rangle) - \langle [\phi_1, \phi_2]_Q, \alpha \rangle,$$

for every $\alpha \in \Gamma(A^*) \cong \mathcal{A}^1,\ \phi_1, \phi_2 \in \Gamma(A)$. Now, the key-point is that $\Gamma(A) \cong \mathfrak{X}(\mathcal{M})_{-1}$, and the bracket above is precisely the derived bracket

$$\iota_{[\phi_1,\phi_2]_Q}(\alpha) = [[\iota_{\phi_1}, Q], \iota_{\phi_2}](\alpha), \tag{4.35}$$

where the DGLA is the graded vector space $\mathfrak{X}(\mathcal{M})$, the Lie bracket is the commutator of graded vector fields and d is given by $[Q, \cdot]$, which is a degree 1 derivation, since Q is a degree 1 element (see the observation after Def. 4.10). Observe that the anchor map admits also the expression

$$\rho_Q(\phi)(f) = \iota_\phi(Q(f)). \tag{4.36}$$

Given a graded NQ degree 2 manifold $(\mathcal{M}, Q)$, consider the DGLA where the graded vector space $A = \mathfrak{X}(\mathcal{M})$ consists of the set of graded vector fields on $\mathcal{M}$ endowed with its natural vector space structure, the Lie bracket $[\cdot, \cdot]$ is given by the (graded) commutator, which is actually a degree 0 Lie bracket (satisfies Eq. (4.31)), and d is given by $[Q, \cdot]$, which is a degree 1 derivation as noticed just after Def. 4.10, since Q is a degree 1 element in $\mathfrak{X}(\mathcal{M})$.

For the degree 2 case, since the module is spanned by functions of degrees 0, 1 and 2, we need three anchor maps, and two derived brackets. The anchor maps we define in analogy (up to sign) with (4.36),

$$\begin{aligned} \rho_Q(\phi)(f) &:= -\iota_\phi(Q(f)), & \rho_Q(\xi)(f) &:= \ 0, \\ \rho_Q(\phi)(\varepsilon) &:= -\iota_\phi(Q(\varepsilon)), & \rho_Q(\xi)(\varepsilon) &:= -\iota_\xi(Q(\varepsilon)), \end{aligned} \tag{4.37}$$

where $\phi \in \Gamma(\widehat{E}) \cong \mathfrak{X}(\mathcal{M})_{-1}$, $\xi \in \Gamma(F) \cong \mathfrak{X}(\mathcal{M})_{-2}$, $f \in C^\infty(M) \cong \mathcal{A}^0$ and $\varepsilon \in \Gamma(E^*) \cong \mathcal{A}^1$.

Now, for $\phi_1, \phi_2, \phi \in \Gamma(\widehat{E}) \cong \mathfrak{X}(\mathcal{M})_{-1}$ and $\xi, \xi_1, \xi_2 \in \Gamma(F) \cong \mathfrak{X}(\mathcal{M})_{-2}$, the vector field $[[\iota_{\phi_1}, Q], \iota_{\phi_2}]$ is in $\mathfrak{X}(\mathcal{M})_{-1} \cong \Gamma(\widehat{E})$, the vector fields $[[\iota_\xi, Q], \iota_\phi] = -[[\iota_\phi, Q], \iota_\xi]$ are in $\mathfrak{X}(\mathcal{M})_{-2} \cong \Gamma(F)$ and the vector field $[[\iota_{\xi_1}, Q], \iota_{\xi_2}]$ is in $\mathfrak{X}(\mathcal{M})_{-3} = 0$. Therefore, we have well defined sections

$$[\phi_1, \phi_2]_Q \in \Gamma(\widehat{E}), \quad [\xi, \phi]_Q = -[\phi, \xi]_Q \in \Gamma(F) \quad \text{and} \quad [\xi_1, \xi_2] = 0,$$

such that

$$\begin{aligned} \iota_{[\phi_1,\phi_2]_Q} &:= -[[\iota_{\phi_1}, Q], \iota_{\phi_2}], & \iota_{[\xi,\phi]_Q} &:= -[[\iota_\xi, Q], \iota_\phi], \\ \iota_{[\phi,\xi]_Q} &:= -[[\iota_\phi, Q], \iota_\xi] & \iota_{[\xi_1,\xi_2]_Q} &: -[[\iota_{\xi_1}, Q], \iota_{\xi_2}] = 0, \end{aligned} \tag{4.38}$$

where .

Now we need to find out the relation between $\rho_Q, [\cdot,\cdot]_Q$ and the structure data of the preLie 2-algebroid of Def. 4.6, given by Eqs. (4.9) - (4.15).

Proposition 4.12. *Let Q be a 1-vector field on a degree 2 manifold. Consider its corresponding involutive sequence (3.14), and the respective dual sequence (3.44). Consider the structure data defined by Q through Eqs. (4.9) - (4.15), ρ, $\widehat{\rho}$, ∂, Ψ, Θ, and $[\cdot,\cdot]$. These data are related to $(\rho_Q, [\cdot,\cdot]_Q)$, given in Eqs. (4.37) and (4.38), in the following way:*

1. $\rho_Q(\phi)(f) = \widehat{\rho}(\phi) = \rho(\pi(\phi))(f)$,
2. $\rho_Q(\xi)(\varepsilon) = -\langle \partial(\xi), \varepsilon \rangle$,
3. $\rho_Q(\phi)(\varepsilon) = \Psi(\phi)(\varepsilon)$,
4. $[\phi, \xi]_Q = \Theta(\phi)(\xi) = -[\xi, \phi]_Q$,
5. $[\phi_1, \phi_2]_Q = [\phi_1, \phi_2]$.

Proof. Items *1, 2* and *3* follow immediately from Eq. (4.37) and Eqs. (4.9), (4.13), (4.10) and (4.11). So let's prove item *4*. First observe that, using Cor. 3.54, Prop. 3.61 and item *1* of Thm. 3.59, we have for every $\gamma \in \Gamma(\widetilde{F}^*) \cong \mathcal{A}^2$

$$\begin{aligned} \iota_{[\xi,\phi]_Q}(\gamma) &= -[[\iota_\phi, \iota_\xi], Q](\gamma) - [\iota_\xi, [\iota_\phi, Q]](\gamma) \\ &= -\iota_\phi \iota_\xi Q(\gamma) + \iota_\xi \iota_\phi Q(\gamma) + [[\iota_\phi, Q], \iota_\xi](\gamma) \\ &= \langle \phi, Q(\gamma)_1^\sharp(\xi) \rangle - \langle \xi, Q(\gamma)_2^\sharp(\phi) \rangle - \iota_{[\phi,\xi]_Q}(\gamma) \\ &= -\iota_{[\phi,\xi]_Q}(\gamma). \end{aligned}$$

Therefore, item *4* is equivalent to

$$\iota_{[\phi,\xi]_Q}(\gamma) = \langle \Theta(\phi)(\xi), \gamma \rangle. \tag{4.39}$$

Taking into account Cor. 3.54, Prop. 3.61 and the fact that a degree -1 function on $\mathcal{M}$ is automatically zero, we have, after comparing to Eq. (4.12),

$$\begin{aligned}
\iota_{[\phi,\xi]_Q}(\gamma) &= -[[\iota_\phi, Q], \iota_\xi](\gamma) \\
&= -[\iota_\phi Q + Q\iota_\phi, \iota_\xi](\gamma) \\
&= -\iota_\phi Q \iota_\xi(\gamma) - Q\iota_\phi\iota_\xi(\gamma) + \iota_\xi\iota_\phi Q(\gamma) + \iota_\xi Q\iota_\phi(\gamma) \\
&= \widehat{\rho}(\phi)(\langle \xi, \gamma\rangle) - \langle \xi, Q(\gamma)_2^\sharp(\phi)\rangle + \langle \partial(\xi), \langle \phi, \gamma\rangle\rangle \\
&= \langle \Theta(\phi)(\xi), \gamma\rangle.
\end{aligned}$$

Finally, it remains to prove item *5*, which is equivalent to show the identities

$$\iota_{[\phi_1,\phi_2]_Q}(\varepsilon) = \langle [\phi_1,\phi_2], \overline{\varepsilon}\rangle \quad \text{and} \quad \iota_{[\phi_1,\phi_2]_Q}(\gamma) = \langle [\phi_1,\phi_2], \gamma\rangle. \tag{4.40}$$

As in the preceding item, we compute

$$\begin{aligned}
\iota_{[\phi_1,\phi_2]_Q}(\varepsilon) &= -[[\iota_{\phi_1}, Q], \iota_{\phi_2}](\varepsilon) = -[\iota_{\phi_1} Q + Q\iota_{\phi_1}, \iota_{\phi_2}](\varepsilon) \\
&= -\iota_{\phi_1} Q \iota_{\phi_2}(\varepsilon) - Q\iota_{\phi_1}\iota_{\phi_2}(\varepsilon) + \iota_{\phi_1}\iota_{\phi_2} Q(\varepsilon) + \iota_{\phi_2} Q\iota_{\phi_1}(\varepsilon) \\
&= \widehat{\rho}(\phi_1)(\langle \phi_2, \overline{\varepsilon}\rangle) + \langle T(\phi_1,\phi_2), Q(\varepsilon)\rangle - \widehat{\rho}(\phi_2)(\langle \phi_1, \overline{\varepsilon}\rangle) \\
&= \langle [\phi_1,\phi_2], \overline{\varepsilon}\rangle;
\end{aligned}$$

$$\begin{aligned}
\iota_{[\phi_1,\phi_2]_Q}(\gamma) &= -[[\iota_{\phi_1}, Q], \iota_{\phi_2}](\gamma) = -[\iota_{\phi_1} Q + Q\iota_{\phi_1}, \iota_{\phi_2}](\gamma) \\
&= -\iota_{\phi_1} Q \iota_{\phi_2}(\gamma) - Q\iota_{\phi_1}\iota_{\phi_2}(\gamma) + \iota_{\phi_2}\iota_{\phi_1} Q(\gamma) + \iota_{\phi_2} Q\iota_{\phi_1}(\gamma) \\
&= \Psi(\phi_1)(\langle \phi_2, \gamma\rangle) + \rho^*(d(\langle T(\phi_2,\phi_1), \gamma\rangle)) - \langle \phi_2, Q(\gamma)_2^\sharp(\phi_1)\rangle - \Psi(\phi_2)(\langle \phi_1, \gamma\rangle) \\
&= \langle [\phi_1,\phi_2], \gamma\rangle.
\end{aligned}$$

■

Next we introduce a few identities that will be needed.

Proposition 4.13. *In the setting of Prop. 4.12, the following identities hold*

a) $\widehat{\rho}(\phi)(f) = -[\iota_\phi, Q](f)$,

b) $\langle \partial(\xi), \varepsilon\rangle = [\iota_\xi, Q](\varepsilon)$,

c) $\Psi(\phi)(\varepsilon) = -[\iota_\phi, Q](\varepsilon) - \rho^*(d(\langle \phi, \overline{\varepsilon}\rangle))$.

Proof. Items a), b) and c), are basically another way of writing items *1, 2* and *3* of Prop. 4.12, after noticing that

$$Q\iota_\phi(f) = 0, \qquad Q\iota_\xi(\varepsilon) = 0 \quad \text{and} \quad Q\iota_\phi(\varepsilon) = \rho^*(d(\langle \phi, \overline{\varepsilon}\rangle)).$$

■

4.3 Lie 2-algebroids

In this section we show, following M. Grutzmann and T. Strobl [24], how to apply the derived bracket method to the situation in which we are given a degree 2 NQ manifold. Namely, the derived bracket method, besides providing a more conceptual understanding of the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ and of the structure equations (4.9) - (4.15) defining a (pre)Lie 2-algebroid from a Q structure, enables to prove in a very simple and direct way the equivalence between the integrability of the Q-structure and Jacobi identity of the corresponding derived bracket, which is given in this case by the Lie 2-algebroid bracket $[\cdot,\cdot]$ and the map Θ (cf. the definition of these in Eqs. (4.12), (4.14) and (4.15) in the proof of Thm. 4.9), thus providing a geometric characterization for the equation $Q^2 = 0$. We begin with a few preliminaries, and in particular we introduce the geometric counterpart of degree 2 NQ manifolds, a structure for which we reserve the name *Lie 2-algebroid*.

Lemma 4.14. *Consider a preLie 2-algebroid. Let's introduce a horizontal lift on the corresponding sequence (4.6), which induces a horizontal lift on its dual (3.44), then*

$$Z(\xi_1, \xi_2) := \nabla^F_{\partial(\xi_1)}\xi_2 + \nabla^F_{\partial(\xi_2)}\xi_1 \tag{4.41}$$

gives a well-defined map, i.e. independent of the choice of horizontal lift, $Z : \Gamma(F) \times \Gamma(F) \longrightarrow \Gamma(F)$.

Now suppose that the preLie 2-algebroid structure comes from an integrable vector field Q, i.e. we have $Q^2 = 0$ then (4.41) gives a tensor $Z \in \Gamma(S^2F^* \otimes F)$ that is independent of the horizontal lift.

Proof. Choose another horizontal lift, and denote, as usual, the transition map by Ψ (do not confuse this Ψ with the vector bundle morphism Ψ of Eq. (4.11), sadly we ended using the same letter for those two quite different maps). Then by property *4* of Def. 4.6 we have

$$\begin{aligned}(\nabla^F)'_{\partial\xi_i}\xi_j &= \Theta(\widehat{\partial(\xi_i)}')(\xi_j) = \Theta(\widehat{\partial(\xi_i)} - \Psi^*_{\partial(\xi_i)})(\xi_j)\\ &= \nabla^F_{\partial(\xi_i)}\xi_j - \Psi^*_{\partial(\xi_i)}(\partial(\xi_j)),\end{aligned}$$

so that, taking into account the skew-symmetry $\Psi^*_{e_1}(e_2) = -\Psi^*_{e_2}(e_1)$ due to the involutivity structure,

$$\begin{aligned}Z'(\xi_1,\xi_2) &= (\nabla^F)'_{\partial(\xi_1)}\xi_2 + (\nabla^F)'_{\partial(\xi_2)}\xi_1\\ &= \nabla^F_{\partial(\xi_1)}\xi_2 - \Psi^*_{\partial(\xi_1)}(\partial(\xi_2)) - \nabla^F_{\partial(\xi_2)}\xi_1 - \Psi^*_{\partial(\xi_2)}(\partial(\xi_1))\\ &= \nabla^F_{\partial(\xi_1)}\xi_2 - \Psi^*_{\partial(\xi_1)}(\partial(\xi_2)) - \nabla^F_{\partial(\xi_2)}\xi_1 + \Psi^*_{\partial(\xi_1)}(\partial(\xi_2))\\ &= \nabla^F_{\partial(\xi_1)}\xi_2 - \nabla^F_{\partial(\xi_2)}\xi_1\\ &= Z(\xi_1,\xi_2).\end{aligned}$$

Thus Z doesn't depend on the choice of horizontal lift.

Now, if the preLie 2-algebroid structure comes from an integrable 1-vector field Q, then property *2* of Def. 4.6 implies, for $f \in C^\infty(M)$,

$$\begin{aligned} Z(\xi_1, f\xi_2) &= fZ(\xi_1,\xi_2) + \rho(\partial(\xi_1))(f)\xi_2 \qquad (4.42)\\ &= fZ(\xi_1,\xi_2) - \langle \partial(\xi_1), Q(f)\rangle \xi_2 \\ &= fZ(\xi_1,\xi_2) - \langle \xi_1, Q^2(f)\rangle \xi_2 \\ &= fZ(\xi_1,\xi_2). \qquad (4.43)\end{aligned}$$

By symmetry, we also have $Z(f\xi_1,\xi_2) = fZ(\xi_1,\xi_2)$. Thus, Z is a tensor in $S^2F^* \otimes F$.

∎

Remark 4.15. Actually we will see in the proof of Thm. 4.20 that when the preLie 2-algebroid structure comes from an integrable 1-vector field Q, Z is identically zero. See Cor. 4.21.

Corollary 4.16. *If the preLie 2-algebroid structure comes from an integrable 1-vector field Q, i.e. $Q^2 = 0$, then*

$$\rho \circ \partial = 0.$$

Proof. This follows from the calculation of $Z(\xi_1, f\xi_2)$ above, but anyway we perform the calculation again:

$$\rho(\partial(\xi))(f) = -\langle \partial(\xi), Q(f)\rangle = -\langle \xi, Q^2(f)\rangle = 0.$$

∎

Definition 4.17. A *Lie 2-algebroid* structure on the dual sequence (3.44) is a preLie 2-algebroid structure such that:

1. For every $\phi_1, \phi_2, \phi_3 \in \Gamma(\widehat{E})$,
$$[\phi_1, [\phi_2, \phi_3]] = [[\phi_1, \phi_2], \phi_3] + [\phi_2, [\phi_1, \phi_3]]. \qquad (4.44)$$
2. $\partial \circ \Theta = \Delta_\Psi \circ \partial$, where $(\Delta_\Psi \circ \partial)(\phi, \xi) := \Delta_\Psi(\phi, \partial(\xi)), \qquad \forall \phi \in \Gamma(\widehat{E}), \xi \in \Gamma(F)$.

Remark 4.18. A *Loday algebroid* on a vector bundle $E \longrightarrow M$ (see [40]) is a pseudoalgebra structure $([\cdot,\cdot], \rho, \varphi)$ on $\Gamma(E)$ (see Rmk. I.11 for the meaning of ρ and φ) such that

$$[s_1, [s_2, s_3]] = [[s_1, s_2], s_3] + [s_2, [s_1, s_3]].$$

Therefore, a Lie 2-algebroid is a particular case of a Loday algebroid structure.

There are two main examples of non-trivial Lie 2-algebroids, which we will consider in the next chapter, since they appear naturally as *VB*-Courant algebroids, a structure that we will show to be equivalent to the Lie 2-algebroid structure through an explicit procedure to go from one structure to the other.

Proposition 4.19 (e.g. [40])**.** *In a Lie 2-algebroid the anchor $\widehat{\rho} : \widehat{E} \longrightarrow TM$ preserves brackets:*

$$\widehat{\rho}([\phi_1, \phi_2]) = [\widehat{\rho}(\phi_1), \widehat{\rho}(\phi_2)], \qquad \forall\ \phi_1\phi_2 \in \Gamma(\widehat{E}).$$

Proof. Since $([\cdot,\cdot],\widehat{\rho})$ is a pseudoalgebra structure, we have, for every $\phi_1,\phi_2\in\Gamma(\widehat{E})$ and every $f\in C^\infty(M)$,

$$[\phi_1, f\phi_2] = f[\phi_1,\phi_2] + \widehat{\rho}(\phi_1)(f)\phi_2.$$

Then, using this property repeatedly we have:

$$\begin{aligned}[\phi_1,[\phi_2,f\phi_3]] &= [\phi_1, f[\phi_2,\phi_3]] + [\phi_1, \widehat{\rho}(\phi_2)(f)\phi_3] \\ &= f[\phi_1,[\phi_2,\phi_3]] + \widehat{\rho}(\phi_1)(f)[\phi_2,\phi_3] \\ &\quad + \widehat{\rho}(\phi_2)(f)[\phi_1,\phi_3] + \widehat{\rho}(\phi_1)\widehat{\rho}(\phi_2)(f)\phi_3;\end{aligned} \tag{4.45}$$

$$[[\phi_1,\phi_2], f\phi_3] = f[[\phi_1,\phi_2],\phi_3] + \widehat{\rho}([\phi_1,\phi_2])(f)\phi_3; \tag{4.46}$$

$$\begin{aligned}[\phi_2,[\phi_1,f\phi_3]] &= [\phi_2, f[\phi_1,\phi_3]] + [\phi_2, \widehat{\rho}(\phi_1)(f)\phi_3] \\ &= f[\phi_2,[\phi_1,\phi_3]] + \widehat{\rho}(\phi_2)(f)[\phi_1,\phi_3] \\ &\quad + \widehat{\rho}(\phi_1)(f)[\phi_2,\phi_3] + \widehat{\rho}(\phi_2)\widehat{\rho}(\phi_1)(f)\phi_3.\end{aligned} \tag{4.47}$$

Then, using property (4.44) and Eqs. (4.45), (4.46), (4.47), after cancelling terms we get:

$$\begin{aligned}0 &= [\phi_1,[\phi_2,f\phi_3]] - [[\phi_1,\phi_2],f\phi_3] - [\phi_2,[\phi_1,f\phi_3]] \\ &= f([\phi_1,[\phi_2,\phi_3]] - [[\phi_1,\phi_2],\phi_3] - [\phi_2,[\phi_1,\phi_3]]) \\ &\quad + \widehat{\rho}(\phi_1)\widehat{\rho}(\phi_2)(f)\phi_3 - \widehat{\rho}([\phi_1,\phi_2])(f)\phi_3 - \widehat{\rho}(\phi_2)\widehat{\rho}(\phi_1)(f)\phi_3 \\ &= ([\widehat{\rho}(\phi_1),\widehat{\rho}(\phi_2)] - \widehat{\rho}([\phi_1,\phi_2]))(f)\phi_3,\end{aligned}$$

and since f and ϕ_3 are arbitrary, we conclude that

$$\widehat{\rho}([\phi_1,\phi_2]) = [\widehat{\rho}(\phi_1),\widehat{\rho}(\phi_2)].$$

■

Theorem 4.20. *There is a canonical 1:1 correspondence between degree 2 NQ-manifolds and Lie 2-algebroids.*

Proof. Thm. 4.9 already provides a canonical 1:1 correspondence between 1-vector fields on a degree 2 manifold and preLie 2-algebroid structures on the corresponding dual sequence (3.44). So the proof of the theorem consists on proving the equivalence between the integrability condition $Q^2=0$ and axioms *1* and *2* of Def. 4.17.

Let's prove first that $Q^2=0$ implies

1) $J(\phi_1,\phi_2,\phi_3)=0$, where $\phi_1,\phi_2,\phi_3\in\Gamma(\widehat{E})$ and J is the Jacobiator

$$J(\phi_1,\phi_2,\phi_3) = [\phi_1,[\phi_2,\phi_3]] - [[\phi_1,\phi_2],\phi_3] - [\phi_2,[\phi_1,\phi_3]].$$

2) $\partial\circ\Theta = \Delta_\Psi\circ\partial$, where Δ_Ψ is the duality operator given in Eq. (4.5).

Observe that $Q^2 = 0$ is equivalent to $[Q, Q] = 0$. Now, using Prop. 4.12, we compute

$$\begin{aligned}0 &= [[[Q, Q], \iota_{\phi_1}], \iota_{\phi_2}], \iota_{\phi_3}] = -[[[\iota_{\phi_1}, Q], Q] - [Q, [\iota_{\phi_1}, Q]], \iota_{\phi_2}], \iota_{\phi_3}] \\ &= -[[[\iota_{\phi_2}, [\iota_{\phi_1}, Q]], Q]] - [[\iota_{\phi_1}, Q], [\iota_{\phi_2}, Q]], \iota_{\phi_3}] \\ &\quad + [[[\iota_{\phi_2}, Q], [\iota_{\phi_2}, Q], [\iota_{\phi_1}, Q]] - [Q, [\iota_{\phi_2}, [\iota_{\phi_1}, Q]]], \iota_{\phi_3}] \\ &= -2[[[\iota_{\phi_1}, Q], [\iota_{\phi_2}, Q]] - [[[\iota_{\phi_1}, Q], \iota_{\phi_2}], Q], \iota_{\phi_3}] \\ &= 2[[\iota_{\phi_3}, [\iota_{\phi_1}, Q]], [\iota_{\phi_2}, Q]] + 2[[\iota_{\phi_1}, Q], [\iota_{\phi_3}, [\iota_{\phi_2}, Q]]] \\ &\quad + 2[[[\iota_{\phi_1}, Q], \iota_{\phi_2}], Q], \iota_{\phi_3}] \\ &= 2\left(\iota_{[\phi_2,[\phi_1,\phi_3]]} - \iota_{[\phi_1,[\phi_2,\phi_3]]} + \iota_{[[\phi_1,\phi_2],\phi_3]}\right),\end{aligned}$$

thus obtaining $J(\phi_1, \phi_2, \phi_3) = 0$.

Now let's prove axiom *2*.

$$\begin{aligned}0 &= [[[Q, Q], \iota_\xi], \iota_\phi] = -[[[\iota_\xi, Q], Q] + [Q, [\iota_\xi, Q]], \iota_\phi] \\ &= [[\iota_\phi, [\iota_\xi, Q]], Q] - [[\iota_\xi, Q], [\iota_\phi, Q]] \\ &\quad + [[\iota_\phi, Q], [\iota_\xi, Q]] - [Q, [\iota_\phi, [\iota_\xi, Q]]] \\ &= 2\left([[\iota_\phi, [\iota_\xi, Q]], Q] - [[\iota_\xi, Q], [\iota_\phi, Q]]\right).\end{aligned}$$

Then, using Prop. 4.12 and Prop. 4.13, we get

$$\begin{aligned}0 &= [[[Q, Q], \iota_\xi], \iota_\phi](\varepsilon) = 2\langle \Theta(\phi)(\xi), Q(\varepsilon)\rangle \\ &\quad + 2\langle \partial(\xi), \Psi(\phi)(\varepsilon) + \rho^*(d(\langle \phi, \overline{\varepsilon}\rangle))\rangle - 2\widehat{\rho}(\phi)(\langle \partial(\xi), \varepsilon\rangle) \\ &= 2\langle \partial \circ \Theta(\phi)(\xi) - \Delta_\Psi(\phi, \partial(\xi)), \varepsilon\rangle,\end{aligned}$$

hence $\partial \circ \Theta - \Delta_\Psi \circ \partial = 0$.

In order to prove the converse, that axioms *1* and *2* of Def. 4.17 imply $Q^2 = 0$, we will work locally, so choose a coordinate system $\{x^i, \varepsilon^\mu, \alpha^\nu\}$ spanning $\mathcal{O}(U)$, then, from what was observed in Subsec. 3.5, and since Q^2 has degree 2, it follows that its local expression is

$$\begin{aligned}Q^2 =& (A^j_\nu \alpha^\nu + B^j_{\mu_r\mu_s}\varepsilon^{\mu_r}\varepsilon^{\mu_s})\frac{\partial}{\partial x^j} + (C^\mu_{\nu_m\mu_m}\alpha^{\nu_m}\varepsilon^{\mu_m} + D^\mu_{\mu_u\mu_v\mu_w}\varepsilon^{\mu_t}\varepsilon^{\mu_u}\varepsilon^{\mu_v})\frac{\partial}{\partial\varepsilon^\mu} \\ &+ (E^\nu_{\nu_1\nu_2}\alpha^{\nu_1}\alpha^{\nu_2} + F^\nu_{\nu_a\mu_a\mu_b}\alpha^{\nu_a}\varepsilon^{\mu_a}\varepsilon^{\mu_b} + G^\nu_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\mu_1}\varepsilon^{\mu_2}\varepsilon^{\mu_3}\varepsilon^{\mu_4})\frac{\partial}{\partial\alpha^\nu}. \qquad (4.48)\end{aligned}$$

Our task is to prove that each coefficient of the expression above is zero. Let's compute each coefficient.

$$\begin{aligned}
A^i_\nu &= \left[Q^2, \frac{\partial}{\partial\alpha^\nu}\right](x^i) = \frac{1}{2}\left[[Q,Q], \frac{\partial}{\partial\alpha^\nu}\right](x^i)\\
&= -\frac{1}{2}\left(\left[\left[\frac{\partial}{\partial^\nu}, Q\right], Q\right](x^i) + \left[Q, \left[\frac{\partial}{\partial\alpha^\nu}, Q\right]\right](x^i)\right)\\
&= -\left(\left[\frac{\partial}{\partial\alpha^\nu}, Q\right](Q(x^i)) + Q\left[\frac{\partial}{\partial\alpha^\nu}, Q\right](x^i)\right)\\
&= -\left[\frac{\partial}{\partial\alpha^\nu}, Q\right](Q(x^i))\\
&= -\left\langle \partial\left(\frac{\partial}{\partial\alpha^\nu}\right), Q(x^i)\right\rangle\\
&= \rho\left(\partial\left(\frac{\partial}{\partial\alpha^\nu}\right)\right)(x^i).
\end{aligned}$$

Now, from Eq. (4.42), we have

$$\rho(\partial(\xi_1))(f)\xi_2 = Z(\xi_1, f\xi_2) - fZ(\xi_1,\xi_2), \qquad \forall\ \xi_1,\xi_2 \in \Gamma(F), f \in C^\infty(M),$$

hence $A^i_\nu = 0$ if $Z \equiv 0$. But in the proof of Thm. H.8, namely, in the computation of $J(\eta_1, \widehat{e_2}, \eta_3)$ we show, using only $\partial \circ \Theta = \Delta_\Psi \circ \partial$, that Eq. (H.20),

$$J(\eta_1, \widehat{e_2}, \eta_3) = 0, \qquad \forall\ \eta_1,\eta_3 \in \mathrm{Hom}(E^* \otimes F), e_2 \in \Gamma(E),$$

implies

$$Z(\xi_1,\xi_2) = 0,\ \forall\ \xi_1,\xi_2 \in \Gamma(F). \tag{4.49}$$

Therefore, axioms *1* and *2* of Def. 4.17 imply that $A^i_\nu = 0$.

Let's compute the next coefficient.

$$\begin{aligned}
B^i_{\mu_{r_0}\mu_{s_0}} &= -\left[\left[Q^2, \frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}\right], \frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right](x^i) = -\frac{1}{2}\left[\left[[Q,Q], \frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}\right], \frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right](x^i)\\
&= \frac{1}{2}\left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right], Q\right] - \left[Q, \left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right]\right], \frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right](x^i)\\
&= \frac{1}{2}\left(\left[\left[\frac{\partial}{\partial^{\mu_{s_0}}}, \left[\frac{\partial}{\partial^{\mu_{r_0}}}, Q\right]\right], Q\right](x^i) + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}, Q\right]\right](x^i)\right)\\
&\quad -\frac{1}{2}\left(\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right]\right](x^i) - \left[Q, \left[\frac{\partial}{\partial\varepsilon^{mu_{s_0}}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right]\right]\right](x^i)\right)\\
&= \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}, Q\right]\right](x^i) - \left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, Q\right], \frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right], Q\right](x^i)\\
&= \rho\left(\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}\right)\left(\rho\left(\frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right)(x^i)\right) - \rho\left(\frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right)\left(\rho\left(\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}\right)(x^i)\right)\\
&\quad - \rho\left(\left[\frac{\partial}{\partial\varepsilon^{\mu_{r_0}}}, \frac{\partial}{\partial\varepsilon^{\mu_{s_0}}}\right]\right)(x^i).
\end{aligned}$$

Using the identity $\rho(e) = \widehat{\rho}(\widehat{e})$ and Prop. 4.19, we conclude that $B^i_{\mu_{r_0}\mu_{s_0}} = 0$.

Now is the time to compute $C^{\mu_p}_{\nu_0\mu_0}$.

$$
\begin{aligned}
C^{\mu_p}_{\nu_0\mu_0} &= \left[\left[Q^2, \frac{\partial}{\partial\alpha^{\nu_0}}\right], \frac{\partial}{\partial\varepsilon^{\mu_0}}\right](\varepsilon^{\mu_p}) = \frac{1}{2}\left[\left[[Q,Q], \frac{\partial}{\partial\alpha^{\nu_0}}\right], \frac{\partial}{\partial\varepsilon^{\mu_0}}\right](\varepsilon^{\mu_p}) \\
&= -\frac{1}{2}\left[\left[\left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right], Q\right] + \left[Q, \left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right]\right], \frac{\partial}{\partial\varepsilon^{\mu_0}}\right](\varepsilon^{\mu_p}) \\
&= \frac{1}{2}\left(\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_0}}, \left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right]\right], Q\right](\varepsilon^{\mu_p}) - \left[\left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_0}}, Q\right]\right](\varepsilon^{\mu_p})\right) \\
&\quad + \frac{1}{2}\left(\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_0}}, Q\right], \left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right]\right](\varepsilon^{\mu_p}) - \left[Q, \left[\frac{\partial}{\partial\varepsilon^{\mu_0}}, \left[\frac{\partial}{\partial\alpha^{\nu_0}}, Q\right]\right]\right](\varepsilon^{\mu_p})\right) \\
&= \left\langle \Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_0}}}\right)\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right), Q(\varepsilon^{\mu_p})\right\rangle + \left\langle \Psi\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_0}}}\right)(\varepsilon^{\mu_p}), \partial\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right)\right\rangle \\
&\quad + \rho\left(\partial\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right)\right)\left(\left\langle\frac{\partial}{\partial\varepsilon^{\mu_0}}, \varepsilon^{\mu_p}\right\rangle\right) - \rho\left(\frac{\partial}{\partial\varepsilon^{\mu_0}}\right)\left(\left\langle\partial\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right), \varepsilon^{\mu_p}\right\rangle\right) \\
&= \left\langle \partial\circ\Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_0}}}\right)\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right) - \Delta_\Psi\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_0}}}, \partial\left(\frac{\partial}{\partial\alpha^{\nu_0}}\right)\right), \varepsilon^{\mu_p}\right\rangle = 0.
\end{aligned}
$$

Let's compute now $D^{\mu_0}_{\mu_a\mu_b\mu_c}$.

$$
\begin{aligned}
D^{\mu_0}_{\mu_a\mu_b\mu_c} &= \left[\left[\left[Q^2, \frac{\partial}{\partial\varepsilon^{\mu_a}}\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right](\varepsilon^{\mu_0}) = \frac{1}{2}\left[\left[\left[[Q,Q], \frac{\partial}{\partial\varepsilon^{\mu_a}}\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right](\varepsilon^{\mu_0}) \\
&= -\left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_a}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_b}}, Q\right]\right] - \left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_a}}, Q\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], Q\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right](\varepsilon^{\mu_0}) \\
&= \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_c}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_a}}, Q\right]\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_b}}, Q\right]\right](\varepsilon^{\mu_0}) + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_a}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_c}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_b}}, Q\right]\right]\right](\varepsilon^{\mu_0}) \\
&\quad + \left[\left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_a}}, Q\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], Q\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right](\varepsilon^{\mu_0}) \\
&= \left\langle \left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_b}}}, \left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_a}}}, \widehat{\frac{\partial}{\partial\varepsilon^{\mu_c}}}\right]\right] - \left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_a}}}, \left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_b}}}, \widehat{\frac{\partial}{\partial\varepsilon^{\mu_c}}}\right]\right] + \left[\left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_a}}}, \widehat{\frac{\partial}{\partial\varepsilon^{\mu_b}}}\right], \widehat{\frac{\partial}{\partial\varepsilon^{\mu_c}}}\right], \overline{\varepsilon^{\mu_0}}\right\rangle \\
&= 0,
\end{aligned}
$$

where the brackets on the last line stand for the Lie 2-algebroid structure, while the brackets of the preceding lines of course stand for the commutator of (graded) vector fields.

Next we compute $E^{\nu_0}_{\nu_a\nu_b}$.

$$\begin{aligned}E^{\nu_0}_{\nu_a\nu_b} &= \left[\left[Q^2,\frac{\partial}{\partial\alpha^{\nu_a}}\right],\frac{\partial}{\partial\alpha^{\nu_b}}\right](\alpha^{\nu_0}) = \frac{1}{2}\left[\left[[Q,Q],\frac{\partial}{\partial\alpha^{\nu_a}}\right],\frac{\partial}{\partial\alpha^{\nu_b}}\right](\alpha^{\nu_0})\\ &= -\frac{1}{2}\left[\left[\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right],Q\right],\frac{\partial}{\partial\alpha^{\nu_b}}\right](\alpha^{\nu_0}) - \frac{1}{2}\left[\left[Q,\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right],\frac{\partial}{\partial\alpha^{\nu_b}}\right](\alpha^{\nu_0})\\ &= \frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_b}},\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right],Q\right](\alpha^{\nu_0}) + \frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right],\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right]\right](\alpha^{\nu_0})\\ &\quad + \frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right],\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right](\alpha^{\nu_0}) + \frac{1}{2}\left[Q,\left[\frac{\partial}{\partial\alpha^{\nu_b}},\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right]\right](\alpha^{\nu_0}).\end{aligned}\tag{4.50}$$

Now observe, as we already did in the beginning of proof of Prop. 4.12, and taking into account the identifications of Prop. 3.52, that for any $X\in\mathfrak{X}(\mathcal{M})_{-1}$, $Y\in\mathfrak{X}(\mathcal{M})_{-2}$ and $\zeta\in\mathcal{A}^2$, we have

$$X(Y(Q(\zeta))) - Y(X(Q(\zeta))) = 0.\tag{4.51}$$

Also observe that, simply for degree reasons,

$$X(Y(\zeta)) = Y(X(\zeta)) = 0.\tag{4.52}$$

Hence,

$$\begin{aligned}&\frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_b}},\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right],Q\right](\alpha^{\nu_0}) + \frac{1}{2}\left[Q,\left[\frac{\partial}{\partial\alpha^{\nu_b}},\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right]\right](\alpha^{\nu_0})\\ &= \left[\left[\frac{\partial}{\partial\alpha^{\nu_b}},\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right],Q\right](\alpha^{\nu_0}) = \left[\frac{\partial}{\partial\alpha^{\nu_b}}\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right] - \left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\frac{\partial}{\partial\alpha^{\nu_b}},Q\right](\alpha^{\nu_0})\\ &= \frac{\partial}{\partial\alpha^{\nu_b}}\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]Q(\alpha^{\nu_0}) - \left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\frac{\partial}{\partial\alpha^{\nu_b}}Q(\alpha^{\nu_0}) = 0.\end{aligned}\tag{4.53}$$

Putting (4.53) into (4.50), we obtain

$$\begin{aligned}E^{\nu_0}_{\nu_a\nu_b} &= \frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right],\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right]\right](\alpha^{\nu_0}) + \frac{1}{2}\left[\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right],\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\right](\alpha^{\nu_0})\\ &= \left[\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right],\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right]\right](\alpha^{\nu_0}).\end{aligned}\tag{4.54}$$

Now, using Props. 4.12 and 4.13 and item *1* of Prop. 3.59,

$$\begin{aligned}&\left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\circ\left[\frac{\partial}{\partial\alpha^{\nu_b}},Q\right](\alpha^{\nu_0}) = \left[\frac{\partial}{\partial\alpha^{\nu_a}},Q\right]\left(\iota_{\frac{\partial}{\partial\alpha^{\nu_a}}}Q(\alpha^{\nu_0}) + \rho^*\left(\left\langle\frac{\partial}{\partial\alpha^{\nu_b}},\alpha^{\nu_0}\right\rangle\right)\right)\\ &\qquad = \left\langle\partial\left(\frac{\partial}{\partial\alpha^{\nu_a}}\right),-Q(\alpha^{\nu_0})^\sharp_1\left(\frac{\partial}{\partial\alpha^{\nu_a}}\right)\right\rangle + \rho\left(\partial\left(\frac{\partial}{\partial\alpha^{\nu_a}}\right)\right)\left(\left\langle\frac{\partial}{\partial\alpha^{\nu_b}},\alpha^{\nu_0}\right\rangle\right)\\ &\qquad = -\left\langle\frac{\partial}{\partial\alpha^{\nu_b}},Q(\alpha^{\nu_0})^\sharp_2\left(\partial\left(\widehat{\frac{\partial}{\partial\alpha^{\nu_a}}}\right)\right)\right\rangle + \widehat{\rho}\left(\partial\left(\widehat{\frac{\partial}{\partial\alpha^{\nu_a}}}\right)\right)\left(\left\langle\frac{\partial}{\partial\alpha^{\nu_b}},\alpha^{\nu_0}\right\rangle\right)\\ &\qquad = \left\langle\Theta\left(\partial\left(\widehat{\frac{\partial}{\partial\alpha^{\nu_a}}}\right)\right)\left(\frac{\partial}{\partial\alpha^{\nu_b}}\right),\alpha^{\nu_0}\right\rangle = \left\langle\nabla_{\partial(\frac{\partial}{\partial\alpha^{\nu_a}})}\frac{\partial}{\partial\alpha^{\nu_b}},\alpha^{\nu_0}\right\rangle.\end{aligned}\tag{4.55}$$

Putting, (4.55) into (4.54), and taking into account Eqs. (4.41) and (4.49), we obtain

$$E^{\nu_0}_{\nu_a\nu_b} = \left\langle Z\left(\frac{\partial}{\partial\alpha^{\nu_a}}, \frac{\partial}{\partial\alpha^{\nu_b}}\right), \alpha^{\nu_0}\right\rangle = 0.$$

Now let's compute $F^{\nu_0}_{\nu_k\mu_k\mu_l}$, using what we already calculated in the computation of $C^{\mu_p}_{\nu_0\mu_0}$.

$$\begin{aligned}
F^{\nu_0}_{\nu_k\mu_k\mu_l} &= \left[\left[\left[Q^2, \frac{\partial}{\partial\alpha^{\nu_k}}\right], \frac{\partial}{\partial\varepsilon^{\mu_k}}\right], \frac{\partial}{\partial\varepsilon^{\mu_l}}\right](\alpha^{\nu_0}) = \frac{1}{2}\left[\left[\left[[Q,Q], \frac{\partial}{\partial\alpha^{\nu_k}}\right], \frac{\partial}{\partial\varepsilon^{\mu_k}}\right], \frac{\partial}{\partial\varepsilon^{\mu_l}}\right](\alpha^{\nu_0})\\
&= \left[\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right], Q\right] - \left[\left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, Q\right]\right], \frac{\partial}{\partial\varepsilon^{\mu_l}}\right](\alpha^{\nu_0})\\
&= -\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right](\alpha^{\nu_0}) + \left[Q, \left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right]\right](\alpha^{\nu_0})\\
&\quad + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, Q\right]\right], \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right](\alpha^{\nu_0}) + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right](\alpha^{\nu_0}).
\end{aligned}$$

Using Eq. (4.51), similarly as we did in (4.53), and using also item *4* of Prop. 4.12, we have

$$\left[Q, \left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right]\right](\alpha^{\nu_0}) = -\left[Q, \left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, \frac{\partial}{\partial\varepsilon^{\mu_k}}\right]_Q\right]\right](\alpha^{\nu_0}) = 0.$$

Hence, using items *4* and *5* of Prop. 4.12, we get

$$\begin{aligned}
F^{\nu_0}_{\nu_k\mu_k\mu_l} &= -\left[\left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right](\alpha^{\nu_0}) + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, Q\right]\right], \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right](\alpha^{\nu_0})\\
&\quad + \left[\left[\frac{\partial}{\partial\varepsilon^{\mu_k}}, Q\right], \left[\frac{\partial}{\partial\varepsilon^{\mu_l}}, \left[\frac{\partial}{\partial\alpha^{\nu_k}}, Q\right]\right]\right](\alpha^{\nu_0})\\
&= \left\langle \Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_l}}}\right)\left(\Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_k}}}\right)\left(\frac{\partial}{\partial\alpha^{\nu_k}}\right)\right), \alpha^{\nu_0}\right\rangle + \left\langle \Theta\left(\left[\widehat{\frac{\partial}{\partial\varepsilon^{\mu_k}}}, \widehat{\frac{\partial}{\partial\varepsilon^{\mu_l}}}\right]\right)\left(\frac{\partial}{\partial\alpha^{\nu_k}}\right), \alpha^{\nu_0}\right\rangle\\
&\quad - \left\langle \Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_k}}}\right)\left(\Theta\left(\widehat{\frac{\partial}{\partial\varepsilon^{\mu_l}}}\right)\left(\frac{\partial}{\partial\alpha^{\nu_k}}\right)\right), \alpha^{\nu_0}\right\rangle\\
&= 0.
\end{aligned}$$

Finally, we compute $G^{\nu_0}_{\mu_a\mu_b\mu_c\mu_d}$. Using the calculations we done to compute $D^{\mu_0}_{\mu_a\mu_b\mu_c}$, and the fact that $\frac{\partial}{\partial\varepsilon^{\mu}}(\alpha^\nu) = 0$, we have

$$\begin{aligned}
G^{\nu_0}_{\mu_a\mu_b\mu_c\mu_d} &= \left[\left[\left[\left[Q^2, \frac{\partial}{\partial\varepsilon^{\mu_a}}\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right], \frac{\partial}{\partial\varepsilon^{\mu_d}}\right](\alpha^{\nu_0})\\
&= \frac{1}{2}\left[\left[\left[\left[[Q,Q], \frac{\partial}{\partial\varepsilon^{\mu_a}}\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right], \frac{\partial}{\partial\varepsilon^{\mu_d}}\right](\alpha^{\nu_0})\\
&= \frac{\partial}{\partial\varepsilon^{\mu_d}} \circ \left[\left[\left[[Q,Q], \frac{\partial}{\partial\varepsilon^{\mu_a}}\right], \frac{\partial}{\partial\varepsilon^{\mu_b}}\right], \frac{\partial}{\partial\varepsilon^{\mu_c}}\right](\alpha^{\nu_0})\\
&= -\frac{\partial}{\partial\varepsilon^{\mu_d}} \circ J_Q\left(\frac{\partial}{\partial\varepsilon^{\mu_a}}, \frac{\partial}{\partial\varepsilon^{\mu_b}}, \frac{\partial}{\partial\varepsilon^{\mu_c}}\right)(\alpha^{\nu_0}) = 0,
\end{aligned}$$

where J_Q stands for the Jacobiator of the brackets $[\cdot,\cdot]_Q$, defined in (4.38).

Therefore, we have showed that each coefficient of Eq. (4.48) is zero. Thus we conclude that axioms *1* and *2* of Def. 4.17 imply $Q^2 = 0$, as we wanted.

■

Corollary 4.21. *Consider a Lie 2-algebroid. Then the map Z introduced in Lemma 4.14 is zero.*

Proof. This is what we obtained in Eq. (4.49).

■

Remark 4.22. The corollary above is precisely what Eq. *4* of Prop. H.4 tells us. We can use that equation once we have the equivalence between the integrability equations of a Lie 2-algebroid and those of a *split* Lie 2-algebroid. This equivalence is proved in Thm. H.8.

Corollary 4.23. *Consider a Lie 2-algebra, namely, a Lie 2-algebroid such that the base manifold is a point $M = \{0\}$. Then we have the following structure, which characterizes it:*

- *an involutive sequence of vector spaces*

$$0 \longrightarrow \Lambda^2 E^* \longrightarrow \widetilde{F}^* \longrightarrow F^* \longrightarrow 0,$$

  so that its dual is

$$0 \longrightarrow E^* \otimes F \longrightarrow \widehat{E} \longrightarrow E \longrightarrow 0,$$

- *linear maps*

$$\begin{aligned} \partial &: F \longrightarrow E, \\ \Psi &: \widehat{E} \longrightarrow \mathrm{End}(E^*), \\ \Theta &: \widehat{E} \longrightarrow \mathrm{End}(F), \end{aligned}$$

- *a bilinear map $[\cdot,\cdot] : \widehat{E} \times \widehat{E} \longrightarrow \widehat{E}$,*

such that, considering the skew-adjoint $\Psi^* : \widehat{E} \longrightarrow \mathrm{End}(E)$ given by

$$\langle \Psi^*(\phi)(e), \varepsilon \rangle := -\langle \Psi(\phi)(\varepsilon), e \rangle,$$

the following properties hold:

1. $\Theta(\eta) = \eta \circ \partial, \qquad \Psi^*(\eta) = \partial \circ \eta, \qquad \forall \eta \in E^* \otimes F.$
2. $\pi([\phi_1, \phi_2]) = \Psi^*(\phi_1)(\pi(\phi_2)), \qquad \forall \phi_1, \phi_2 \in \widehat{E}.$
   In particular $\Psi^*(\phi)(e) = \pi([\phi, \widehat{e}])$, for any horizontal lift $\widehat{e}$ of $e \in E$.
3. $[\phi, \eta] = \Theta(\phi) \circ \eta - \eta \circ \Psi^*(\phi),$
4. $[\phi_1, \phi_2] + [\phi_2, \phi_1] = \delta(W(\phi_1, \phi_2))$, *where $\delta : F \longrightarrow \widehat{E}$ is the linear map defined by Eq. (4.3).*

5. $\partial \circ \Theta = \Psi^* \circ \partial$*,*

6. The bilinear map $[\cdot,\cdot]$ *satisfies Jacobi identity, in the sense that*

$$[\phi_1, [\phi_2, \phi_3]] = [[\phi_1, \phi_2], \phi_3] + [\phi_2, [\phi_1, \phi_3]].$$

Moreover, for every Lie 2-algebra, the maps Θ *and* Ψ *are representations, namely*

$$\Theta([\phi_1, \phi_2]) = [\Theta(\phi_1), \Theta(\phi_2)], \tag{4.56}$$

$$\Psi([\phi_1, \phi_2] = [\Psi(\phi_1), \Psi(\phi_2)], \tag{4.57}$$

where the brackets on the right-hand side of the two equations above stand for the commutator bracket of endomorphisms.

Proof. Except Eq. (4.57), everything in this corollary consists simply on spelling out what we already have obtained in the general case of Lie 2-algebroids, taking into account that in this case we have $\rho \equiv 0$. So, we only need to prove Eq. (4.57).

By Thm. 4.9, Eq. (4.11) we have

$$\Psi(\phi)(\varepsilon) = -\langle \phi, Q(\varepsilon) \rangle,$$

where Q is the 1 vector field that corresponds to the Lie 2-algebra structure, and by Thm. 4.20 we have $Q^2 = 0$. Then, using also Eq. (4.14) of Thm. 4.9, and taking into account that $\rho = 0$,

$$\begin{aligned}
\Psi([\phi_1, \phi_2])(\varepsilon) &= -\langle [\phi_1, \phi_2], Q(\varepsilon) \rangle \\
&= -\Psi(\phi_1)(\langle \phi_2, Q(\varepsilon) \rangle) + \Psi(\phi_2)(\langle \phi_1, Q(\varepsilon) \rangle) + \langle \phi_2, Q^2(\varepsilon)_2^\sharp(\phi_1) \rangle \\
&= \Psi(\phi_1)(\Psi(\phi_2)(\varepsilon)) - \Psi(\phi_2)(\Psi(\phi_1)(\varepsilon)).
\end{aligned}$$

Therefore, $\Psi([\phi_1, \phi_2]) = [\Psi(\phi_1), \Psi(\phi_2)]$.

■

In the general case, when the base manifold is not a point, the same calculations as above enables us to obtain the following formula for $\Psi([\phi_1, \phi_2])$.

Corollary 4.24. *In a Lie 2-algebroid, the following identity holds:*

$$\Psi([\phi_1, \phi_2]) = [\Psi(\phi_1), \Psi(\phi_2)] + \rho^*(d(\langle \phi_1, \overline{\Psi(\phi_2)} \rangle)), \tag{4.58}$$

where the last term is the operator on $\Gamma(E^*)$ *defined by*

$$\rho^*(d(\langle \phi_1, \overline{\Psi(\phi_2)} \rangle))(\varepsilon) := \rho^*(d(\langle \phi_1, \overline{\Psi(\phi_2)(\varepsilon)} \rangle)). \tag{4.59}$$

Proof. Indeed, from the equivalence given by Thm. 4.20 of Lie 2-algebroids with degree 2 NQ manifolds, we can use formulas (4.11) and (4.14), and the fact that $Q^2 = 0$, to obtain the following:

$$\begin{aligned}
\Psi([\phi_1, \phi_2])(\varepsilon) &= -\langle [\phi_1, \phi_2], Q(\varepsilon) \rangle \\
&= -\Psi(\phi_1)(\langle \phi_2, Q(\varepsilon) \rangle) + \Psi(\phi_2)(\langle \phi_1, Q(\varepsilon) \rangle) \\
&\quad - \rho^*(d(\langle \phi_1, \overline{\langle \phi_2, Q(\varepsilon) \rangle} \rangle)) + \langle \phi_2, Q^2(\varepsilon)_2^\sharp(\phi_1) \rangle \\
&= \Psi(\phi_1)(\Psi(\phi_2)(\varepsilon)) - \Psi(\phi_2)(\Psi(\phi_1)(\varepsilon)) + \rho^*(d(\langle \phi_1, \overline{\Psi(\phi_2)(\varepsilon)} \rangle)).
\end{aligned}$$

■

Chapter 5

Other viewpoints to degree 2 NQ manifolds

In this chapter we recall the concept of VB-Courant algebroids, a structure introduced by D. Li-Bland in his PhD thesis [41], and we study the relation of this structure to Lie 2-algebroids and consequently to degree 2 NQ-manifolds. These relations were treated independently by M. Jotz using splittings.

In last chapter we were able to characterize geometrically an NQ structure on a degree 2 manifold in terms of brackets and maps defined on the linear sequence of the involutive DVB D. However, we are not able any more to extend the structure maps and brackets to define a structure on the whole double vector bundle D, for if it were the case, the least thing we must ask is that the brackets $[\cdot,\cdot]$ satisfy Leibniz rule with respect to the second entry when we multiply by a fiberwise linear function, as it is the case when we multiply by a fiberwise constant function. More precisely, we should have then

$$[\phi, \varepsilon \otimes \xi] = \Psi(\phi)(\varepsilon) \otimes \xi + \varepsilon \otimes \Theta(\phi)(\xi),$$

but, while on the left-hand side, taking $f\varepsilon \in \Gamma(E^*)$ in the place of ε, we have

$$[\phi, f\varepsilon \otimes \xi] = f[\phi, \varepsilon \otimes \xi] + \widehat{\rho}(\phi)(f)\varepsilon \otimes \xi, \tag{5.1}$$

on the right-hand, for $f\varepsilon$ instead of ε, we have

$$\begin{aligned}\Psi(\phi)(f\varepsilon) \otimes \xi + f\varepsilon \otimes \Theta(\phi)(\xi) &= f\Psi(\phi)(\varepsilon) \otimes \xi + f\varepsilon \otimes \Theta(\phi)(\xi) + \widehat{\rho}(\phi)(f)\varepsilon \otimes \xi \\ &\quad - \rho^*(df)\langle \phi, \overline{\varepsilon}\rangle \otimes \xi \\ &= f[\phi, \varepsilon \otimes \xi] + \widehat{\rho}(\phi)(f)\varepsilon \otimes \xi - \rho^*(df)\langle \phi, \overline{\varepsilon}\rangle \xi,\end{aligned}$$

a result which does not coincide with (5.1), because of the extra term $-\rho^*(df)\langle \phi, \overline{\varepsilon}\rangle \xi$. Therefore, we have to content ourselves to define the structure just on the dual exact sequence (3.44), with all the structure data of Def. 4.6, with no hope to extend it to D.[1]

[1]This situation is similar to the fact, explained in Rmk. 3.60, that the geometric counterpart of a degree 3 function is a pair of vector bundle morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ which don't feet to give a DVB morphism $\Theta : D \longrightarrow D_{F^*}$.

This drawback may be seen as a manifestation of the fact that, unlike the VB-algebroid case where one of the duals has an induced, dual, Lie algebroid structure (see [23], and Prop. E.24 of the present work), *VB*-Courant algebroids do not share this property. This turns out to be a fruitful insight, since it suggests that the somewhat awkward Lie 2-algebroid structure defined on the sequence (3.44) could be transported to the dual double vector bundle $D^* = D_{F^*} \longrightarrow F^*$ to fit into a much nicer *VB*-Courant algebroid structure. We will show that this is indeed the case.

In the end we recall the concept of exact V-twisted Courant algebroids, studied by M. Grutzmann and T. Strobl [24] and explain how they are related to *VB*-Courant algebroids (and hence with our Lie 2-algebroids).

5.1 VB-Courant algebroids

We begin recalling the general definition of Courant algebroids, introduced by Z.–J. Liu, A. Weinstein and P. Xu [44]. Actually, we will work with an equivalent definition, with a *non-skew bracket*, discovered by D. Roytenberg [58]. Then we give the definition of Li-Bland's *VB*-Courant algebroids and describe the main example, namely the *tangent Courant algebroid* $T\mathbb{E}$ of a Courant algebroid.

Definition 5.1. A *Courant algebroid* is a pseudo-Euclidean vector bundle $(E^*, \langle\cdot,\cdot\rangle)$ over a manifold M, together with a (non skew-symmetric) bracket $[\![\cdot,\cdot]\!]$ and a bundle map $a : E \longrightarrow TM$ (the anchor), satisfying the following properties:

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!], \qquad \forall e_1, e_2, e_3 \in \Gamma(E)$,
2. $a([\![e_1, e_2]\!]) = [a(e_1), a(e_2)] \qquad \forall e_1, e_2 \in \Gamma(E)$,
3. $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + a(e_1)(f)e_2 \qquad \forall e_1, e_2 \in \Gamma(E),\ f \in C^\infty(M)$,
4. $\langle e, [\![e_1, e_2]\!] + [\![e_2, e_1]\!]\rangle = a(e)(\langle e_1, e_2\rangle) \qquad \forall e, e_1, e_2 \in \Gamma(E)$,
5. $a(e)(\langle e_1, e_2\rangle) = \langle [\![e, e_1]\!], e_2\rangle + \langle e_1, [\![e, e_2]\!]\rangle \qquad \forall e, e_1, e_2 \in \Gamma(E)$.

Remark 5.2. It can be shown by a standard argument (cf. for example [40] or Prop. 4.19 above) that properties 1 and 3 of the definition above already imply property 2[2]. Also, Y. Kosmann-Schwarzbach [36] showed that a Courant algebroid can be characterized just with properties 1, 4 and 5 of Def. 5.1.

Example 5.3. The most basic example, introduced in T. Courant PhD thesis [14] is the so-called *standard* Courant algebroid

$$(\mathbb{T}M, [\![\cdot,\cdot]\!], a),$$

where $\mathbb{T}M = TM \oplus T^*M$, the anchor map is $a(X, \alpha) = X$, the projection over TM, and the brackets are defined by the formula

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], \mathcal{L}_X\beta - \iota_Y d\alpha). \tag{5.2}$$

[2] The argument consists in replacing e_3 by fe_3 in property 1 of Def. 5.1, and then compute using property 3. Some terms cancel out and in the end we get $a([\![e_1, e_2]\!])(f)e_3 = [a(e_1), a(e_2)](f)e_3$. See Prop. 4.19.

Actually T. Courant introduced the *skew-symmetric* version of the brackets above. This non-skew symmetric version was first introduced by I. Dorfman [16]. The example above was generalized by Z.–J. Liu, A. Weinstein and P. Xu [44] to the case $E = A \oplus A^*$, where (A, A^*) is a Lie bialgebroid (see Def. G.2):

$$[\![(X,\alpha),(Y\beta)]\!] = ([X,Y]_A + (\mathcal{L}_{A^*})_\alpha Y - \iota_\beta d_{A^*}X, [\alpha,\beta]_{A^*} + (\mathcal{L}_A)_X\beta - \iota_Y d_A\alpha).$$

Other examples of Courant algebroids include *quadratic* Lie algebras, i.e. Lie algebras endowed with a symmetric bilinear form invariant under the adjoint representation, *exact* Courant algebroids, introduced by P. Ševera [62],[64], which is a twisted version of the standard Courant algebroid by a *closed* 3-form $H \in \Omega^3(M)$:

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \iota_Y d\alpha + \iota_Y\iota_X H),$$

and the Courant algebroids coming from *proto-bialgebroids* [58].

Now we introduce the central concept of this chapter, D. Li-Bland's *VB*-Courant algebroids [41]. It is completely parallel to the concept of *VB*-algebroids of A. Gracia-Saz and R. Mehta [23], see Def. 2.10, changing the Lie algebroid structure by a Courant algebroid one, with the only difference that, for convenience, the structure in this case is *flipped*, in the sense that the Courant algebroid is on the vertical bundle structure, rather than the horizontal one.

Definition 5.4. A *VB-Courant algebroid* is a double vector bundle $(D; F^*, B; M)_{E^*}$ equipped with a Courant algebroid structure on D_{F^*} such that the anchor map $a_D : D_{F^*} \longrightarrow TF^*$ is a bundle morphism over $a_B : B \longrightarrow TM$, the metric $\langle\cdot,\cdot\rangle_{D_{F^*}}$ is linear, and where the bracket $[\![\cdot,\cdot]\!]_D$ is such that

1. $[\![\Gamma_{\mathrm{lin}}(D_{F^*}), \Gamma_{\mathrm{lin}}(D_{F^*})]\!]_D \subset \Gamma_{\mathrm{lin}}(D_{F^*})$,
2. $[\![\Gamma_{\mathrm{lin}}(D_{F^*}), \Gamma_{\mathrm{core}}(D_{F^*})]\!]_D \subset \Gamma_{\mathrm{core}}(D_{F^*})$,
3. $[\![\Gamma_{\mathrm{core}}(D_{F^*}), \Gamma_{\mathrm{core}}(D_{F^*})]\!]_D = 0$.

Remark 5.5. Actually D. Li-Bland gives a different definition, requiring the graph of the addition operation $\underset{B}{+}$,

$$\mathrm{gr}(\underset{B}{+}) : D \times D \dashrightarrow D,$$

is a Courant relation. However, he proves the equivalence with the definition we present here, which seems much more natural because of its parallel with the definition of *VB*-algebroids, and also more convenient, since it is easier to handle (at least for us).

Remark 5.6. As in the case of *VB*-algebroids (Rmk. 2.11), since a_D is a vector bundle morphism with respect to the structure over F^*, it follows that the condition on a_D is equivalent to say that it is a DVB morphism from D to TF^*.

Remark 5.7. Again as in the case of *VB*-algebroids (Rmk. 2.12), the Courant brackets $[\![\cdot,\cdot]\!]_D$ and the anchor a_D are completely determined by their action on linear and basic functions and on linear and core sections, which span the whole ring of functions $C^\infty(F^*)$ and the $C^\infty(F^*)$-module of sections $\Gamma(D_{F^*})$, respectively. However, the argument is more subtle in this case, since the lack of skew-symmetry prevents us from dualizing a Courant structure to obtain Poisson-like brackets on the dual bundle, as it is the case for Lie algebroids. Anyway, it is possible to go around this difficulty and still determine the Courant algebroid structure just by knowing its action on linear and core sections and on linear and basic functions. The details are worked out in appendix A of D. Li-bland's thesis, to which we refer the reader. We limit ourselves to present the precise statement for later reference.

Proposition 5.8 ([41]). *Let $\mathbb{E} \longrightarrow M$ be a vector bundle, $a : \mathbb{E} \longrightarrow TM$ be a bundle map, $\langle \cdot, \cdot \rangle$ a metric on $\mathbb{E}$, and let $W \subset \Gamma(\mathbb{E})$ be a subspace of sections which generates $\Gamma(\mathbb{E})$ as a $C^\infty(M)$-module. Suppose that $[\![\cdot, \cdot]\!] : W \times W \longrightarrow W$ is a bracket which satisfies properties 1, 2, 4 and 5 of Def. 5.1 for every $e, e_1, e_2, e_3 \in W$, and that $a \circ a^* = 0$, where $a^* := \flat \circ a^t : T^*M \longrightarrow \mathbb{E}$, and $a^t : T^*M \longrightarrow \mathbb{E}^*$ is the transposed map of a. Then there is a unique extension of $[\![\cdot, \cdot]\!]$ to a Courant bracket on all of $\Gamma(\mathbb{E})$.*

Proof. It is Prop. A.0.3 in appendix A of [41].

∎

Corollary 5.9. *Let $(D; F^*, B; M)_{E^*}$ be a double vector bundle. Suppose that we can define a bracket operation $[\![\cdot, \cdot]\!]$ on $\Gamma_{\text{lin}}(D_{F^*})$ and $\Gamma_{\text{core}}(D_{F^*})$ satisfying properties 1,2 and 3 of Def. 5.4. Suppose also that we can define a bilinear, non-degenerate symmetric pairing $\langle \cdot, \cdot \rangle_{D_{F^*}}$ on $\Gamma_{\text{lin}}(D_{F^*}) \oplus \Gamma_{\text{core}}(D_{F^*})$ satisfying properties 1,2 and 3 of Def. 3.33, so that it extends by bilinearity to a unique linear metric on D_{F^*}. Suppose, moreover, that we can define two maps*

$$\widehat{a} : \widehat{B} \longrightarrow \mathbf{CDO}(F), \qquad a_{E^*} : E^* \longrightarrow F^*,$$

where $\Gamma(\widehat{B}) \cong \Gamma_{\text{lin}}(D_{F^*})$, $\Gamma(\mathbf{CDO}(F)) \cong \Gamma(\mathbf{CDO}(F^*)) \cong \Gamma_{\text{lin}}(TF^*)$ (see Sec. F.2), which satisfy

$$\widehat{a}(\tau) = a_{E^*} \circ \tau \in F \otimes F^* \cong F^* \otimes F, \qquad \forall\, \tau \in \text{Hom}(F^*, E^*) \subset \widehat{B},$$

then we obtain a unique extension to a DVB morphism $a : D_{F^*} \longrightarrow TF^*$ by Prop. C.24.

If $a \circ a^* = 0$ and $([\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle_{D_{F^*}}, a)$ satisfy properties 1,2,4 and 5 of Def. 5.1, then there is a unique extension of a to a DVB morphism on D, a unique extension of $[\![\cdot, \cdot]\!]$ to a Courant bracket on $\Gamma(D_{F^*})$.

Proof. It follows from Prop. 5.8, with $W = \Gamma_{\text{lin}}(D_{F^*}) \oplus \Gamma_{\text{core}}(D_{F^*})$, that the structure defined is a Courant algebroid, and we defined the bracket to satisfy the *VB*-algebroid conditions.

∎

Remark 5.10. Notice that because of Lem. C.26 it suffices to verify $a \circ a^* = 0$ only for linear and core sections.

The above corollary enables to describe easily, for a given Courant algebroid $([\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle, a)$ the *tangent prolonged VB*-Courant algebroid on $T\mathbb{E} \longrightarrow TM$. This structure was described first in [5] by Boumaiza and Zaalani (see also [11] where the closely related jet bundle Courant algebroid is described). We follow Li-Bland's thesis [41] (see also [29]).

Proposition 5.11 ([41])**.** *The tangent bundle $T\mathbb{E}$ of a Courant algebroid $\mathbb{E}\longrightarrow M$,*

$$\begin{tikzcd} T\mathbb{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbb{E}"] & \mathbb{E} \arrow[d] \\ TM \arrow[r] & M \end{tikzcd}$$

carries a unique VB-*Courant algebroid structure over TM such that the pairing and the bracket satisfy*

$$\langle \mathcal{T}e_1, \mathcal{T}e_2\rangle_{T\mathbb{E}} = d\langle e_1, e_2\rangle, \quad \langle \mathcal{T}e_1, \overline{e_2}\rangle_{T\mathbb{E}} = \langle e_1, e_2\rangle, \quad \langle \overline{e_1}, \mathcal{T}e_2\rangle_{T\mathbb{E}} = \langle e_1, e_2\rangle, \quad \langle \overline{e_1}, \overline{e_2}\rangle_{T\mathbb{E}} = 0,$$
$$[\![\mathcal{T}e_1, \mathcal{T}e_2]\!]_{T\mathbb{E}} = \mathcal{T}[\![e_1, e_2]\!], \quad [\![\mathcal{T}e_1, \overline{e_2}]\!]_{T\mathbb{E}} = \overline{[\![e_1, e_2]\!]}, \quad [\![\overline{e_1}, \mathcal{T}e_2]\!]_{T\mathbb{E}} = \overline{[\![e_1, e_2]\!]}, \quad [\![\overline{e_1}, \overline{e_2}]\!]_{T\mathbb{E}} = 0,$$

and the anchor map satisfies

$$a_{T\mathbb{E}}(\mathcal{T}e) = \mathcal{T}a(e) \qquad a_{T\mathbb{E}}(\overline{e}) = \overline{a(e)},$$

for any sections $e, e_1, e_2 \in \Gamma(\mathbb{E})$.

Proof. We define the Courant algebroid structure on sections of the form $\overline{e}$ and $\mathcal{T}e, e \in \Gamma(\mathbb{E})$ by the equations of the statement. We need to verify $a_{T\mathbb{E}}(\tau) = a_{T\mathbb{E}} \circ \tau$, for every $\tau \in T^*M \otimes \mathbb{E}$. So let's take arbitrary $f \in C^\infty(M)$ and $e \in \Gamma(\mathbb{E})$, then

$$\begin{aligned} a_{T\mathbb{E}}(df \otimes \overline{e}) &= a_{T\mathbb{E}}(\mathcal{T}(fe) - f\mathcal{T}e) = \mathcal{T}a(fe) - f\mathcal{T}a(e) \\ &= df \otimes \overline{a(e)} + f\mathcal{T}a(e) - f\mathcal{T}a(e) = df \otimes \overline{a(e)} \\ &= a_{T\mathbb{E}} \circ (df \otimes \overline{e}). \end{aligned}$$

Properties $a_{T\mathbb{E}} \circ a^*_{T\mathbb{E}} = 0$, *1,2,4* and *5* of Def. 5.1 for linear and core sections follows from these properties for $\mathbb{E}$ and from the fact that sections of the form $\mathcal{T}e, e \in \Gamma(\mathbb{E})$ span the space of linear sections. Hence, Cor. 5.9 applies.

■

Next we show the other basic example of *VB*-Courant algebroid, which is the *standard Courant algebroid over a vector bundle.* We take it from [41], see also [29].

Proposition 5.12 ([41])**.** *Let $A\longrightarrow M$ be a vector bundle. Consider the standard Courant algebroid $\mathbb{T}A = TA \oplus T^*A \longrightarrow A$, described in Ex. 5.3, whose anchor is the projection $\mathbb{T}A \longrightarrow TA$ and the Lie bracket is given by*

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \iota_Y d\alpha).$$

Then, the double vector bundle

$$\begin{tikzcd} \mathbb{T}A \arrow[r] \arrow[d] \arrow[dr, phantom, "A \oplus T^*M"] & TM \oplus A^* \arrow[d] \\ A \arrow[r] & M, \end{tikzcd}$$

endowed with the standard Courant algebroid structure, is a VB-*Courant algebroid.*

Proof. The anchor map, which in the case of the standard Courant algebroid is just the projection over the first factor, is a double vector bundle morphism simply because of the way we define the double vector bundle structure on the Whitney sum of two double vector bundles, see Sec. C.6 of App. C. Then it remains to show properties *1,2* and *3* of Def. 5.4. To do achieve this, notice that the linear sections of $\mathbb{T}A$ are $C^\infty(M)$-spanned by sections of the form $(X, d\alpha)$, with $X \in \mathfrak{X}_{\mathrm{lin}}(A)$ (linear fields on A) and $\alpha \in \Gamma(A^*) \cong C^\infty_{\mathrm{lin}}(A)$. Likewise, the core sections are $C^\infty(M)$-spanned by (a^v, df), where a^v is the vertical lift of $a \in \Gamma(A)$, and $f \in C^\infty(M)$.

Observe that, from Cartan's rule it follows, for every $X, Y \in \mathfrak{X}_{\mathrm{lin}}(A), \alpha \in \Gamma(A^*)$,

$$\mathcal{L}_X d\alpha = dX(\alpha),$$

then

$$\begin{aligned} [\![(X, d\alpha), (Y, d\beta)]\!] &= ([X, Y], \mathcal{L}_X d\beta - \mathcal{L}_Y d\alpha + d\langle Y, \alpha\rangle) \\ &= ([X, Y], dX(\beta) - dY(\alpha) + dY(\alpha)) \\ &= ([X, Y], dX(\beta)). \end{aligned}$$

Since

$$[\![(X, d\alpha), f(Y, d\beta)]\!] = f[\![(X, d\alpha), (Y, d\beta)]\!] + \sigma(X)(f)(Y, d\beta),$$

where $\sigma : \mathbf{CDO}(A^*) \longrightarrow TM$ is the symbol map, and

$$[\![f(X, d\alpha), (Y, d\beta)]\!] = f[\![(X, d\alpha), (Y, d\beta)]\!] - \sigma(Y)(f)(X, d\alpha) + (X(\beta) + Y(\alpha)) \otimes df,$$

we conclude that the Courant bracket of linear sections is again linear.

Noting that $\langle a^v, d\alpha\rangle = a^v(\alpha) = \langle a, \alpha\rangle$ and $\langle X, df\rangle = \sigma(X)(f)$, and using the identification of linear vector fields with covariant differential operators $\mathfrak{X}_{\mathrm{lin}}(A) \cong \mathbf{CDO}(A^*)$, we have

$$\begin{aligned} [X, a^v](\beta) &= Xa^v(\beta) - a^v X(\beta) \\ &= \sigma(X)\langle a, \beta\rangle - \langle a, X(\beta)\rangle; \end{aligned}$$

since $a^v(f) = 0$ for every vertical lift $a^v \in \Gamma(TA)$, we have

$$[X, a^v](f) = 0;$$

thus, we conclude that

$$[X, a^v] = X^*(a)^v,$$

where $X^* \in \mathbf{CDO}(A)$ is the dual covariant differential operator of X.

Also note that, form Cartan's rule, we have

$$\langle \mathcal{L}_X df, Y\rangle = d\sigma(X)(f) \quad \text{and} \quad \mathcal{L}_{a^v} d\alpha = d\langle a, \alpha\rangle.$$

Then we have

$$\begin{aligned} [\![(X, d\alpha), (a^v, df)]\!] &= (X^*(a)^v, d\sigma(X)(f) - d\langle a, \alpha\rangle - d\langle a, \alpha\rangle) \\ &= (X^*(a)^v, d\sigma(X)(f)). \end{aligned}$$

In particular, noting that

$$[\![(X, d\alpha), g(a^v, df)]\!] = g[\![(X, d\alpha), (a^v, df)]\!] + \sigma(X)(g)(a^v, df)$$

and

$$[\![g(X, d\alpha), (a^v, df)]\!] = g[\![(X, d\alpha), (a^v, df)]\!],$$

we conclude that the Courant bracket of a linear section with a core section is again a core section. Moreover, since

$$[\![(a^v, df), (X, d\alpha)]\!] = -[\![(X, d\alpha), (a^v, df)]\!] + d(\sigma(X)(f) + \langle a, \alpha\rangle),$$

it follows that the Courant bracket of a core section with a linear section is also a core section.

Finally, since $a^v(f) = 0$ for every $a \in \Gamma(A)$, $f \in C^\infty(M)$, it is easy to conclude that $[\![(a^v, df), (b^v, dg)]\!] = 0$. Therefore, the standard Courant algebroid over a vector bundle is a VB-Courant algebroid.

∎

5.2 Relation with Lie 2-algebroids

In this section we aim to show the relation between Lie 2-algebroids (Def. 4.17) and VB-Courant algebroids, introduced by D. Li-Bland [41], therefore recovering the characterization of degree 2 NQ manifolds in terms of VB-Courant algebroids obtained in Prop. 3.2.1 of [41], and more explicity through splittings by M. Jotz in [29].

We begin by working out some identities involving the structure of a Lie 2-algebroid that will enable us to show that every Lie 2-algebroid yields a VB-Courant algebroid structure on the dual $D^*_{F^*}$ of its corresponding metric VB-algebroid D_{F^*}.

5.2.1 Some identities for Lie 2-algebroids

Along this subsection we are given a Lie 2-algebroid $\widehat{E}$.

Proposition 5.13. *For every ϕ_1, ϕ_2, ϕ_3, we have*

$$W([\phi_1, \phi_2], \phi_3) + W([\phi_1, \phi_3], \phi_2) = W(\phi_1, \delta(W(\phi_2, \phi_3))) = \Theta(\phi_1)(W(\phi_2, \phi_3)).$$

Proof. Let's introduce a horizontal lift, and recall the definition of the brackets on E^* given in Eq. H.4 and the curvature form K given in Eq. H.7. Using the properties of Def. 4.6

$$\begin{aligned}[\phi_1, \phi_2] &= [\eta_1 + \widehat{e_1}, \eta_2 + \widehat{e_2}] \\ &= [\eta_1, \eta_2] + [\eta_1, \widehat{e_2}] + [\widehat{e_1}, \eta_2] + [\widehat{e_1}, \widehat{e_2}] \\ &= \eta_1 \circ \partial \circ \eta_2 - \eta_2 \circ \partial \circ \eta_1 + \eta_1 \circ [e_2, \cdot] - \nabla^F_{e_2} \circ \eta_1 + \nabla^F_{\cdot} \eta_1(e_2) \\ &\quad + \partial \circ \widehat{\eta_1(e_2)} + \nabla^F_{e_1} \circ \eta_2 - \eta_2 \circ [e_1, \cdot] + \widehat{[e_1, e_2]} - K(e_1, e_2).\end{aligned} \tag{5.3}$$

Then, using formula (3.48),

$$\begin{aligned} W([\phi_1, \phi_2], \phi_3) = \eta_1 \circ \partial \circ \eta_2(e_3) - \eta_2 \circ \partial\eta_1(e_3) + \eta_1([e_2, e_3]) - \nabla^F_{e_2}\eta_1(e_3) \\ + \nabla^F_{e_3}\eta_1(e_2) + \nabla^F_{e_1}\eta_2(e_3) - \eta_2([e_1, e_3]) \\ - K(e_1, e_2)(e_3) + \eta_3 \circ \partial \circ \eta_1(e_2) + \eta_3([e_1, e_2]) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} W([\phi_1, \phi_3], \phi_2) = \eta_1 \circ \partial \circ \eta_3(e_2) - \eta_3 \circ \partial\eta_1(e_2) + \eta_1([e_3, e_2]) - \nabla^F_{e_3}\eta_1(e_2) \\ + \nabla^F_{e_2}\eta_1(e_3) + \nabla^F_{e_1}\eta_3(e_2) - \eta_3([e_1, e_2]) \\ - K(e_1, e_3)(e_2) + \eta_2 \circ \partial \circ \eta_1(e_3) + \eta_2([e_1, e_3]), \end{aligned} \tag{5.5}$$

thereby, adding (5.4)+(5.5) and cancelling terms we get

$$\begin{aligned} W([\phi_1, \phi_2], \phi_3) + W([\phi_1, \phi_3], \phi_2) &= \eta_1 \circ \partial(W(\phi_2, \phi_3)) + \nabla^F_{e_1}W(\phi_2, \phi_3) \\ &= W(\phi_1, \delta(W(\phi_2, \phi_3)), \end{aligned} \tag{5.6}$$

where we used Eq. (4.3) in the last equality.

On the other hand, using formulas (4.1), (4.2), (3.48) and (4.3),

$$\begin{aligned} \Theta(\phi_1)(W(\phi_2, \phi_3)) &= \Theta(\eta_1 + \widehat{e_1})(W(\phi_2, \phi_3)) \\ &= \eta_1 \circ \partial(W(\phi_2, \phi_3)) + \nabla^F_{e_1}W(\phi_2, \phi_3) \\ &= W(\phi_1, \delta(W(\phi_2, \phi_3))). \end{aligned} \tag{5.7}$$

■

Corollary 5.14. *For every $\phi \in \Gamma(\widehat{E})$ and $\xi \in \Gamma(F)$,*

$$\Theta(\phi)(\xi) = W(\phi, \delta(\xi))$$

holds.

Proof. Indeed, from Eq. (3.48) it is easy to see that $W : S^2(\widehat{E}) \longrightarrow F$ is surjective, therefore we can write $\xi = W(\phi_1, \phi_2)$ for some $\phi_1, \phi_2 \in \Gamma(\widehat{E})$, and the corollary follows from Eq. (5.7).

■

Proposition 5.15. *For every $\phi_1, \phi_2 \in \Gamma(\widehat{E})$ and $\varepsilon \in \Gamma(E^*)$,*

$$\widehat{\rho}(\phi_1)(\langle \phi_2, \overline{\varepsilon} \rangle) = \langle [\phi_1, \phi_2], \overline{\varepsilon} \rangle + \langle \phi_2, \overline{\Psi(\phi_1)(\varepsilon)} \rangle + \widehat{\rho}(\phi_2)(\langle \phi_1, \overline{\varepsilon} \rangle). \tag{5.8}$$

Proof. Using properties *1* and *5* of Def. 4.6 and Eq. (4.5), we have

$$\begin{aligned} \langle [\phi_1, \phi_2], \overline{\varepsilon} \rangle &= \langle \pi([\phi_1, \phi_2]), \varepsilon \rangle = \langle \Delta_\Psi(\phi_1, \pi(\phi_2)), \varepsilon \rangle \\ &= \widehat{\rho}(\phi_1)(\langle \pi(\phi_2), \varepsilon \rangle) - \widehat{\rho}(\phi_2)(\pi(\phi_1), \varepsilon \rangle) - \langle \Psi(\phi_1)(\varepsilon), \pi(\phi_2) \rangle. \end{aligned}$$

whereby, rearranging terms, we obtain (5.8).

■

Proposition 5.16. *For every $\phi_1, \phi_2 \in \Gamma(\widehat{E})$ and $\varepsilon \in \Gamma(E^*)$,*

$$\langle \partial(W(\phi_1, \phi_2)), \varepsilon \rangle = -\langle \phi_1, \overline{\Psi(\phi_2)(\varepsilon)} \rangle - \langle \phi_2, \overline{\Psi(\phi_1)(\varepsilon)} \rangle. \tag{5.9}$$

Proof. In the following computations we will use Eq. (4.5), properties *5* and *7* of Def. 4.6 and Eq. (4.3):

$$\begin{aligned} -\langle \phi_1, \overline{\Psi(\phi_2)(\varepsilon)} \rangle - \langle \phi_2, \overline{\Psi(\phi_1)(\varepsilon)} \rangle &= \langle \Delta_\Psi(\Phi_1, \pi(\phi_2)), \varepsilon \rangle - \widehat{\rho}(\phi_1)(\langle \pi(\phi_2), \varepsilon \rangle) \\ &\quad + \widehat{\rho}(\phi_2)(\langle \pi(\phi_1), \varepsilon \rangle) + \langle \Delta_\Psi(\Phi_2, \pi(\phi_1)), \varepsilon \rangle \\ &\quad - \widehat{\rho}(\phi_2)(\langle \pi(\phi_1), \varepsilon \rangle) + \widehat{\rho}(\phi_1)(\langle \pi(\phi_2), \varepsilon \rangle) \\ &= \langle [\phi_1, \phi_2] + [\phi_2, \phi_1], \overline{\varepsilon} \rangle = \langle \partial(W(\phi_1, \phi_2)), \varepsilon \rangle. \end{aligned}$$

∎

Proposition 5.17. *For every $\phi \in \Gamma(\widehat{E})$ and $f \in C^\infty(M)$,*

$$\Psi(\phi)(\rho^*(df)) = 0 \tag{5.10}$$

holds.

Proof. Since $Q^2 = 0$, we have from Eqs. (4.9) and (4.11),

$$\Psi(\phi)(\rho^*(df)) = \langle \phi, Q(Q(f)) \rangle = 0.$$

∎

5.2.2 The correspondence: Lie 2-algebroids $\leftrightsquigarrow$ *VB*-Courant algebroids

In Eqs. a), b), c), d) and e) of Rmk. E.26 we saw how to "transport" the *VB*-algebroid on D_B to $(D_B^*)_{C^*}^*$ and vice versa. We will use this insight in order to transport a Lie 2-algebroid on the linear bundle of $(D_{F^*})_E^* \cong D$ to a *VB*-Courant algebroid structure on $D_{F^*}^*$ and vice versa.

Theorem 5.18. *Lie 2-algebroid structures on a metric vector sequence $\widehat{E}$ are equivalent to* VB-*Courant algebroids on the corresponding metric double vector bundle.*

Proof. **Lie 2-algebroids $\longrightarrow$ *VB*-Courant algebroids.**

Suppose we have a Lie 2-algebroid structure on $\widehat{E}$. By Prop. 5.9, in order to obtain a *VB*-Courant algebroid structure on $D_{F^*}^*$, we only need to define the *VB*-Courant algebroid structure data on linear and core sections and verify $a \circ a^* = 0$ and properties *1,2,4* and *5* of Def. 5.1 for those types of sections. Thereby we define the data $([\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle_{D_{F^*}^*}, a)$ on $D_{F^*}^*$ as follows. Consider the isomorphism $Z : \widehat{E}_F \longrightarrow \widehat{E}$ of Prop. C.33, where $\widehat{E}_F$ is the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(F, D_{F^*}^*)$ and $\widehat{E}$ is the already introduced linear bundle corresponding to $\Gamma_{\mathrm{lin}}(E, D)$, then we define

- The linear metric $\langle \cdot, \cdot \rangle$ is the one given by Eqs. (3.55) and (3.56) of Rmk. 3.51.

- The anchor map is given by

$$a(\sigma)(f) := \widehat{\rho}(Z(\sigma))(f), \qquad a(\overline{\varepsilon})(f) := 0 \tag{5.11}$$
$$a(\sigma)(\xi) := \Theta(Z(\sigma))(\xi), \qquad a(\overline{\varepsilon})(\xi) := \langle \partial(\xi), \varepsilon\rangle,$$

for all $\sigma \in \Gamma(\widehat{E}_F), \xi \in \Gamma(F), \varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

- The bracket is given by

$$[\![\sigma_1, \sigma_2]\!] := Z^{-1}([Z(\sigma_1), Z(\sigma_2)]) \qquad [\![\sigma, \overline{\varepsilon}]\!] := \overline{\Psi(Z(\sigma))(\varepsilon)} + \overline{\rho^*(d\langle \sigma, \overline{\varepsilon}\rangle_{D^*_{F^*}})}$$
$$[\![\overline{\varepsilon_1}, \overline{\varepsilon_2}]\!] := 0 \qquad [\![\overline{\varepsilon}, \sigma]\!] := -\overline{\Psi(Z(\sigma))(\varepsilon)}, \tag{5.12}$$

for all $\sigma, \sigma_1, \sigma_2 \in \Gamma(\widehat{E}_F)$ and $\varepsilon, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$.

We first verify that for every $\tau \in F^* \otimes E \subset \widehat{E}_F$ we have $a(\tau) = a_E \circ \tau$. For $\tau = \xi \otimes \varepsilon$ this is equivalent to $a(\xi \otimes \varepsilon) = \xi \otimes a(\overline{\varepsilon})$. So, let's show this equation. For $f \in C^\infty(M)$ we have, from property *1* of Def. 4.6,

$$\begin{aligned} a(\xi \otimes \varepsilon)(f) &= \widehat{\rho}(Z(\xi \otimes \varepsilon))(f) \\ &= \widehat{\rho}(\varepsilon \otimes \xi)(f) = 0 = \xi \otimes a(\overline{\varepsilon})(f). \end{aligned}$$

For $\xi' \in \Gamma(F^*)$, we have from property *4* of Def. 4.6,

$$\begin{aligned} a(\xi \otimes \varepsilon)(\xi') &= \Theta(Z(\xi \otimes \overline{\varepsilon}))(\xi') \\ &= \langle \partial(\xi'), \varepsilon\rangle \xi = \xi \otimes a(\overline{\varepsilon})(\xi'). \end{aligned}$$

Now we need to verify properties *1,2,4* and *5* of Def. 5.1.

Property 5. We have three cases:

a) $e = \sigma, e_1 = \sigma_1, e_2 = \sigma_2$ are linear sections.
On one hand, by Eqs. (3.53) and (5.11),

$$a(\sigma)(\langle \sigma_1, \sigma_2\rangle_{D^*_{F^*}}) = \Theta(Z(\sigma))(W(Z(\sigma_1), Z(\sigma_2))).$$

On the other hand, by Eqs. (3.53) and (5.12),

$$\begin{aligned} \langle [\![\sigma, \sigma_1]\!], \sigma_2\rangle_{D^*_{F^*}} + \langle \sigma_1, [\![\sigma, \sigma_2]\!]\rangle_{D^*_{F^*}} =& W([Z(\sigma), Z(\sigma_1)], Z(\sigma_2)) \\ &+ W(Z(\sigma_1), [Z(\sigma), Z(\sigma_2)]). \end{aligned}$$

Then, from Prop. 5.13 we get property *5* in this case.

b) $e = \sigma, e_1 = \sigma_1$ are linear and $e_2 = \overline{\varepsilon}$ is core.
Using Eqs. (5.11), (5.12), (3.54), (4.9) and Prop. 5.15, we have

$$\begin{aligned} a(\sigma)(\langle \sigma_1, \overline{\varepsilon}\rangle_{D^*_{F^*}}) &= \widehat{\rho}(Z(\sigma))(\langle Z(\sigma_1), \overline{\varepsilon}\rangle) \\ &= \langle [Z(\sigma), Z(\sigma_1)], \overline{\varepsilon}\rangle + \langle Z(\sigma_1), \overline{\Psi(Z(\sigma))(\varepsilon)}\rangle \\ &\quad + \widehat{\rho}(Z(\sigma_1))(\langle Z(\sigma), \overline{\varepsilon}\rangle) \\ &= \langle [\![\sigma, \sigma_1]\!], \overline{\varepsilon}\rangle_{D^*_{F^*}} + \langle \sigma_1, [\![\sigma, \overline{\varepsilon}]\!]\rangle_{D^*_{F^*}}. \end{aligned}$$

c) $e = \overline{\varepsilon}$ core and $e_1 = \sigma_1, e_2 = \sigma_2$ linear.
Using Eqs. (3.53), (3.54), (5.11), (5.12) and Prop. 5.16, we have

$$\begin{aligned} a(\overline{\varepsilon})(\langle \sigma_1, \sigma_2 \rangle_{D^*_{F^*}} &= \langle \partial(W(Z(\sigma_1), Z(\sigma_2))), \varepsilon \rangle \\ &= -\langle Z(\sigma_1), \overline{\Psi(Z(\sigma_2))(\varepsilon)} \rangle - \langle Z(\sigma_2), \overline{\Psi(Z(\sigma_1))(\varepsilon)} \rangle \\ &= \langle [\![\overline{\varepsilon}, \sigma_1]\!], \sigma_2 \rangle_{D^*_{F^*}} + \langle \sigma_1, [\![\overline{\varepsilon}, \sigma_2]\!] \rangle_{D^*_{F^*}}. \end{aligned}$$

Property 4. First observe that property *4* is equivalent to

$$[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}(\langle e_1, e_2 \rangle) \qquad \forall e_1, e_2 \in \Gamma(E), \tag{5.13}$$

where $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E)$ is defined by

$$\langle \mathcal{D}(f), \varepsilon \rangle := a(e)(f) = \langle a^*(df), \varepsilon \rangle \qquad \forall f \in C^\infty(M), e \in \Gamma(E).$$

In our case we have, for $f \in C^\infty(M)$ and $\xi \in \Gamma(F)$,

$$\mathcal{D}(f) = \overline{\rho^*(df)} \quad \text{and} \quad \mathcal{D}(\xi) = Z^{-1}(\delta(\xi)). \tag{5.14}$$

Indeed, the first equation follows directly from Eqs. (5.11) and (4.9):

$$\langle \mathcal{D}(f), \sigma \rangle_{D^*_{F^*}} = \widehat{\rho}(Z(\sigma))(f) = \rho(\pi(Z(\sigma)))(f) = \langle \overline{\rho^*(df)}, \sigma \rangle_{D^*_{F^*}}.$$

As for the second equation, from Eqs. (5.11), (4.3), (3.53), (3.54) and Cor. 5.14, we have

$$\begin{aligned} \langle \mathcal{D}(\xi), \sigma \rangle_{D^*_{F^*}} &= \Theta(Z(\sigma))(\xi) = W(Z(\sigma), \delta(\xi)) \\ &= \langle Z^{-1}(\delta(\xi)), \sigma \rangle_{D^*_{F^*}} \end{aligned}$$

and

$$\begin{aligned} \langle \mathcal{D}(\xi), \overline{\varepsilon} \rangle_{D^*_{F^*}} &= \langle \partial(\xi), \varepsilon \rangle = \langle \delta(\xi), \overline{\varepsilon} \rangle \\ &= \langle Z^{-1}(\delta(\xi)), \overline{\varepsilon} \rangle_{D^*_{F^*}}. \end{aligned}$$

Now we verify property *4*, in its equivalent version, Eq. (5.13). We have two cases:

a) $e_1 = \sigma_1, e_2 = \sigma_2$ are linear sections.
In this case, we use Eqs. (5.12), (3.53) and (5.14), and property *7* of Def. 4.6, and get

$$\begin{aligned} [\![\sigma_1, \sigma_2]\!] &= Z^{-1}([Z(\sigma_1), Z(\sigma_2)]) \\ &= Z^{-1}(\delta(W(Z(\sigma_1), Z(\sigma_2))) \\ &= \mathcal{D}(\langle \sigma_1, \sigma_2 \rangle_{D^*_{F^*}}). \end{aligned}$$

b) $e_1 = \sigma$ linear and $e_2 = \overline{\varepsilon}$ core.
In this case we obtain directly from Eqs. (5.12) and (5.14),

$$\begin{aligned} [\![\sigma, \overline{\varepsilon}]\!] + [\![\overline{\varepsilon}, \sigma]\!] &= \overline{\Psi(Z(\sigma))(\varepsilon)} + \overline{\rho^*(d\langle \sigma, \overline{\varepsilon} \rangle)} - \overline{\Psi(Z(\sigma))(\varepsilon)} \\ &= \overline{\rho^*(d\langle \sigma, \overline{\varepsilon} \rangle_{D^*_{F^*}})} = \mathcal{D}(\langle \sigma, \overline{\varepsilon} \rangle_{D^*_{F^*}}). \end{aligned}$$

Property 2. We have three cases.

a) $e_1 = \sigma_1, e_2 = \sigma_2$ are linear sections.
We need to prove two identities:

$$a([\![\sigma_1, \sigma]\!])(f) = [a(\sigma_1), a(\sigma_2)](f) \quad \text{and} \quad a([\![\sigma_1, \sigma_2]\!])(\xi) = [a(\sigma_1), a(\sigma_2)](\xi), \tag{5.15}$$

for every $f \in C^\infty(M), \xi \in \Gamma(F)$. The first identity is a direct consequence of Prop. 4.19, and the second follows directly from Cor. H.11.

b) $e_1 = \sigma$ is linear and $e_2 = \overline{\varepsilon}$ is core.
From Eqs. (5.11), (5.12) and (4.5), we have, for every $\xi \in \Gamma(F)$,

$$\begin{aligned} a([\![\sigma, \overline{\varepsilon}]\!])(\xi) &= \langle \partial(\xi), \Psi(Z(\sigma))(\varepsilon) + \rho^*(d\langle \sigma, \overline{\varepsilon}\rangle)\rangle \\ &= \widehat{\rho}(Z(\sigma))\langle \partial(\xi), \varepsilon\rangle - \langle \Delta_\Psi(Z(\sigma), \partial(\xi)), \varepsilon\rangle. \end{aligned} \tag{5.16}$$

On the other hand, using Eqs. (5.11) and property *2* of Def. 4.17, we obtain

$$\begin{aligned} [a(\sigma), a(\overline{\varepsilon})](\xi) &= a(\sigma)(a(\overline{\varepsilon})(\xi)) - a(\overline{\varepsilon})(a(\sigma)(\xi)) \\ &= \widehat{\rho}(Z(\sigma))(\langle \partial, \overline{\varepsilon}\rangle - \langle \partial(\Theta(Z(\sigma))(\xi)), \varepsilon\rangle \\ &= \widehat{\rho}(Z(\sigma))(\langle \partial(\xi), \varepsilon\rangle - \langle \Delta_\Psi(Z(\sigma), \partial(\xi)), \varepsilon\rangle. \end{aligned} \tag{5.17}$$

From (5.16) and (5.17) it follows

$$a([\![\sigma, \overline{\varepsilon}]\!]) = [a(\sigma), a(\overline{\varepsilon})].$$

c) $e_1 = \overline{\varepsilon}$ core and $e_2 = \sigma$ linear.
On one had, from (5.11) and (5.12) we have

$$a([\![\overline{\varepsilon}, \sigma]\!])(\xi) = -\langle \partial(\xi), \Psi(Z(\sigma))(\varepsilon)\rangle. \tag{5.18}$$

On the other hand, from (5.17), (4.5) and Cor. 4.16,

$$\begin{aligned} [a(\overline{\varepsilon}), a(\sigma)](\xi) &= -[a(\sigma), a(\overline{\varepsilon})](\xi) \\ &= -\widehat{\rho}(Z(\sigma))(\langle \partial(\xi), \varepsilon\rangle) + \langle \Delta_\Psi(Z(\sigma), \partial(\xi)), \varepsilon\rangle \\ &= \rho(\partial(\xi))(\langle Z(\sigma), \overline{\varepsilon}\rangle - \langle \Psi(Z(\sigma))(\varepsilon), \partial(\xi)\rangle \\ &= -\langle \partial(\xi), \Psi(Z(\sigma))(\varepsilon)\rangle. \end{aligned} \tag{5.19}$$

From (5.18) and (5.19) we conclude that

$$a([\![\overline{\varepsilon}, \sigma]\!]) = [a(\overline{\varepsilon}), a(\sigma)].$$

Property 1. Finally we must show Jacobi identity for $[\![\cdot, \cdot]\!]$. By lemma 2.6.4 of D. Roytenberg's thesis [58], or rather by the proof of this lemma, we can conclude from properties *2*,*4* and *5*, proven above, that the *Jacobiator*

$$J(e_1, e_2, e_3) := [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_2]\!]]\!] - [\![e_1, [\![e_2, e_3]\!]]\!] \tag{5.20}$$

is completely skew-symmetric in e_1, e_2, e_3, where e_i is a linear or a core section for each $i = 1, 2, 3$. On the other hand, by property *3* of Def. 5.4, J is zero whenever two sections are core. Therefore, in order to prove property *1* of Def. 5.1, we have only two cases to work.

a) $e_1 = \sigma_1, e_2 = \sigma_2, e_3 = \sigma_3$ are linear sections.
This case follows directly from Eq. (5.12) and property *1* of Def. 4.17.

b) $e_1 = \sigma_1, e_2 = \sigma_2$ linear and $e_3 = \overline{\varepsilon}$ core.
On one hand, directly from (5.12), we have (omitting annoying long overlines on the right-hand side)

$$\begin{aligned}[\![\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]]\!] &= \Psi(Z(\sigma_1))\left(\Psi(Z(\sigma_2))(\varepsilon) + \rho^*(d\langle\sigma_2, \overline{\varepsilon}\rangle_{D^*_{F^*}})\right) + \rho^*(d\langle\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]\rangle_{D^*_{F^*}}) \\ &= \Psi(Z(\sigma_1))(\Psi(Z(\sigma_2))(\varepsilon)) + \Psi(Z(\sigma_1))(\rho^*(d\langle\sigma_2, \overline{\varepsilon}\rangle_{D^*_{F^*}})) \\ &\quad + \rho^*(d\langle Z(\sigma_1), \Psi(Z(\sigma_2))(\varepsilon)\rangle) + \rho^*\left(d(\widehat{\rho}(Z(\sigma_1))(\langle Z(\sigma_2), \overline{\varepsilon}\rangle))\right). \end{aligned} \tag{5.21}$$

Analogously,

$$\begin{aligned}[\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!] &= \Psi(Z(\sigma_2))(\Psi(Z(\sigma_1))(\varepsilon)) + \Psi(Z(\sigma_2))(\rho^*(d\langle\sigma_1, \overline{\varepsilon}\rangle_{D^*_{F^*}})) \\ &\quad + \rho^*(d\langle Z(\sigma_2), \Psi(Z(\sigma_1))(\varepsilon)\rangle) + \rho^*\left(d(\widehat{\rho}(Z(\sigma_2))(\langle Z(\sigma_1), \overline{\varepsilon}\rangle))\right). \end{aligned} \tag{5.22}$$

On the other hand, by (5.12), property *5* of Def. 4.6 and (4.5) it follows that

$$\begin{aligned}\langle[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}\rangle_{D^*_{F^*}} &= \langle\pi([Z(\sigma_1), Z(\sigma_2)], \varepsilon\rangle = \langle\Delta_\Psi(Z(\sigma_1), \pi(Z(\sigma_2))), \varepsilon\rangle \\ &= \widehat{\rho}(Z(\sigma_1))(\langle Z(\sigma_2), \overline{\varepsilon}\rangle) - \widehat{\rho}(Z(\sigma_2))(\langle Z(\sigma_1), \overline{\varepsilon}\rangle) \\ &\quad - \langle Z(\sigma_2), \overline{\Psi(Z(\sigma_1))(\varepsilon)}\rangle. \end{aligned} \tag{5.23}$$

Thereby, using (5.12), (4.58) and Eq. (5.23) above, we have (again omitting annoying overlines on the right-hand side)

$$\begin{aligned}[\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!] &= \Psi([Z(\sigma_1), Z(\sigma_2)])(\varepsilon) + \rho^*(d\langle[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}\rangle_{D^*_{F^*}}) \\ &= \Psi(Z(\sigma_1))(\Psi(Z(\sigma_2))(\varepsilon)) - \Psi(Z(\sigma_2))(\Psi(Z(\sigma_1))(\varepsilon)) \\ &\quad + \rho^*(d\langle\sigma_1, \overline{\Psi(Z(\sigma_2))(\varepsilon)}\rangle) \\ &\quad + \rho^*(d(\widehat{\rho}(Z(\sigma_1))(\langle Z(\sigma_2), \overline{\varepsilon}\rangle))) - \rho^*(d(\widehat{\rho}(Z(\sigma_2))(\langle Z(\sigma_1), \overline{\varepsilon}\rangle))) \\ &\quad - \rho^*(d\langle Z(\sigma_2), \overline{\Psi(Z(\sigma_1))(\varepsilon)}\rangle). \end{aligned} \tag{5.24}$$

From (5.21), (5.22), (5.24) and (5.10), we obtain

$$\begin{aligned}[\![\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]]\!] &= [\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!] + [\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!] \\ &\quad + \Psi(Z(\sigma_1))(\rho^*(d\langle\sigma_2, \overline{\varepsilon}\rangle_{D^*_{F^*}})) - \Psi(Z(\sigma_2))(\rho^*(d\langle\sigma_1, \overline{\varepsilon}\rangle_{D^*_{F^*}})) \\ &= [\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!] + [\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!]. \end{aligned}$$

Hence, we have verified properties *1,2,4* and *5* of Def. 5.1 for $W = \Gamma_{\mathrm{lin}}(D^*_{F^*}) \oplus \Gamma_{\mathrm{core}}(D^*_{F^*})$. It remains to check $a \circ a^* = 0$, for which it suffices to take only sections of the form df, $f \in C^\infty(M)$ and $d\xi$, $\xi \in C^\infty_{\mathrm{lin}}(F^*) \cong \Gamma(F)$. For any $f \in C^\infty(M)$ we can always find $\sigma \in \Gamma(\widehat{E}_F)$ and $\varepsilon \in \Gamma(E^*)$ such that $f = \langle\sigma, \overline{\varepsilon}\rangle_{D^*_{F^*}}$. Then, from properties *4* and *2* we have

$$\begin{aligned}a(a^*(df)) &= a(\mathcal{D}(f)) = a(\mathcal{D}(\langle\sigma, \overline{\varepsilon}\rangle_{D^*_{F^*}}) = a([\![\sigma, \overline{\varepsilon}]\!] + [\![\overline{\varepsilon}, \sigma]\!]) \\ &= [a(\sigma), a(\overline{\varepsilon})] + [a(\overline{\varepsilon}), a(\sigma)] = 0.\end{aligned}$$

Analogously, since for every $\xi \in \Gamma(F)$ we can find $\sigma_1, \sigma_2 \in \Gamma(\widehat{E}_F)$ such that $\langle \sigma_1, \sigma_2 \rangle_{D^*_{F^*}} = \xi$, it follows from the same argument above that

$$a(a^*(d\xi)) = a(\mathcal{D}\langle \sigma_1, \sigma_2 \rangle) = 0.$$

Therefore, by Prop. 5.9 we conclude that we have obtained a *VB*-Courant algebroid structure on $D^*_{F^*}$.

***VB*-Courant algebroids $\longrightarrow$ Lie 2-algebroids.**
Given a *VB*-Courant algebroid structure on $D^*_{F^*}$, we define a Lie 2-algebroid structure by

$$\begin{aligned}
&[\phi_1, \phi_2] := Z([\![Z^{-1}(\phi_1), Z^{-1}(\phi_2)]\!]), \quad \widehat{\rho}(\phi)(f) := a(Z^{-1}(\phi))(f), \qquad \langle \partial(\xi), \varepsilon \rangle := a(\overline{\varepsilon})(\xi) \\
&\overline{\Psi(\phi)(\varepsilon)} := -[\![\overline{\varepsilon}, Z^{-1}Z(\phi)]\!] = [\![Z^{-1}(\phi), \overline{\varepsilon}]\!] - \mathcal{D}(\langle \phi, \overline{\varepsilon} \rangle), \qquad \Theta(\phi)(\xi) := a(Z^{-1}(\phi))(\xi).
\end{aligned} \tag{5.25}$$

We need to verify that $([\cdot,\cdot], \widehat{\rho}, \varphi)$ is a Loday algebroid structure (Def. 4.17), where φ is defined by Eq. (4.8), properties *1-7* of Def. 4.6 and property *2* of Def. 4.6. These are simple verifications using properties *1-5* of Def. 5.1. The only properties we think are worth writing down their verifications are *5* and *6* of Def. 4.6 and *2* of Def. 4.17. In order to verify property *5* of Def. 4.6, we use property *5* of Def. 5.1,

$$\begin{aligned}
\langle [\phi_1, \phi_2], \overline{\varepsilon} \rangle &= \langle [\![Z^{-1}(\phi_1), Z^{-1}(\phi_2)]\!], \overline{\varepsilon} \rangle_{D^*_{F^*}} \\
&= a(Z^{-1}(\phi_1))(\langle Z^{-1}(\phi_2), \overline{\varepsilon} \rangle_{D^*_{F^*}}) - \langle Z^{-1}(\phi_2), [\![Z^{-1}(\phi_1), \overline{\varepsilon}]\!] \rangle \\
&= \widehat{\rho}(\phi_1)(\langle \pi(\phi_2), \varepsilon \rangle - \rho(\pi(\phi_2))(\langle \phi_1, \overline{\varepsilon} \rangle) - \langle \Psi(\phi_1)(\varepsilon), \pi(\phi_2) \rangle \\
&= \langle \Delta_\Psi(\phi_1, \pi(\phi_2)), \varepsilon \rangle.
\end{aligned} \tag{5.26}$$

In order to prove property *6* of Def. 4.6, we use property *3* of Def. 5.1, and compute for every $\phi, \widehat{E}$ and $\eta = \varepsilon \otimes \xi$,

$$\begin{aligned}
[\phi, \eta] &= Z([\![Z^{-1}(\phi), \xi \otimes \varepsilon]\!]) \\
&= Z(\xi \otimes [\![Z^{-1}(\phi), \overline{\varepsilon}]\!] + a(Z^{-1}(\phi))(\xi) \otimes \overline{\varepsilon}) \\
&= \varepsilon \otimes \Theta(\phi)(\xi) + \Psi(\Phi)(\varepsilon) \otimes \xi + \rho^*(d\langle \phi, \overline{\varepsilon} \rangle).
\end{aligned}$$

On the other hand, for every $e \in \Gamma(E)$, using property *2* of Def. 4.6 (which follows from the derivation property of $a_{D^*_{F^*}}$), we have

$$\Theta(\phi) \circ (\varepsilon \otimes \xi)(e) = \Theta(\phi)(\langle e, \varepsilon \rangle \xi) = \widehat{\rho}(\phi)(\langle e, \varepsilon \rangle)\xi + \langle e, \varepsilon \rangle \Theta(\phi)(\xi),$$

whence,

$$\varepsilon \otimes \Theta(\phi)(\xi) = \Theta(\phi) \circ (\varepsilon \otimes \xi) - \widehat{\rho}(\phi)(\langle \cdot, \varepsilon \rangle)\xi,$$

whereby, from (5.26), and Eq. (4.5), we get

$$[\phi, \eta] = \Theta(\phi) \circ \eta - \eta \circ \Delta_\Psi(\phi, \cdot).$$

Finally, we verify property *2* of Def. 4.17. By property *2* of Def. 5.1, we have

$$\begin{aligned}\langle \partial \circ \Theta(\phi)(\xi), \varepsilon\rangle &= a(\overline{\varepsilon})(a(Z^{-1}(\phi))(\xi)) \\ &= a(Z^{-1}(\phi))(a(\overline{\varepsilon})(\xi)) + a([\![\overline{\varepsilon}, Z^{-1}(\phi)]\!])(\xi) \\ &= \widehat{\rho}(\phi)(\langle \partial(\xi), \varepsilon\rangle - \langle \Psi(\phi)(\varepsilon), \partial(\xi)\rangle. \end{aligned} \tag{5.27}$$

On the other hand, since $\partial(\xi) = \pi(\mathcal{D}(\xi))$, from $a \circ a^* = 0$ we have in particular

$$\rho(\partial(\xi))(f) = a(\mathcal{D}(\xi))(f) = a \circ a^*(d\xi)(f) = 0, \qquad \forall\, \xi \in \Gamma(F), f \in C^\infty(M),$$

whereby, from Eq. (4.5),

$$\begin{aligned}\langle \Delta_\Psi(\phi, \partial(\xi)), \varepsilon\rangle &= \widehat{\rho}(\phi)(\langle \partial(\xi), \varepsilon\rangle - \rho(\partial(\xi))(\langle \phi, \overline{\varepsilon}\rangle - \langle \Psi(\phi)(\varepsilon), \partial(\xi)\rangle \\ &= \widehat{\rho}(\phi)(\langle \partial(\xi), \varepsilon\rangle - \langle \Psi(\phi)(\varepsilon), \partial(\xi)\rangle. \end{aligned} \tag{5.28}$$

From (5.27) and (5.28) we conclude that

$$\partial \circ \Theta = \Delta_\Psi \circ \partial.$$

■

Remark 5.19. In Prop. 6.36 we will wee that when the Lie 2-algebroid structure comes from a Courant algebroid $(E^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$ (cf. [59] and Sec. 6.4 below), then the corresponding *VB*-Courant algebroid corresponding to this Lie 2-algebroid is precisely the tangent prolongation of E^* on TE^* described in Prop. 5.11, now with $\mathbb{E} = E^*$.

From the above theorem, we obtain another non-trivial example of Lie 2-algebroid, the one that corresponds to the standard Courant algebroid over a vector bundle (see Prop. 5.12). Let's see explicitly how the standard Courant structure is transported to the corresponding Lie 2-algebroid following the procedure indicated in the proof of Thm. 5.18. First, using the identifications given in Sec. F.2 of the appendix, and the characterization of the dual sequence in Cor. 3.53, we can identify the pairs of the form

$$(\mathcal{T}\alpha, X^*), \qquad \alpha \in \Gamma(A^*),\ X^* \in \mathfrak{X}_{\mathrm{lin}}(A^*)$$

with sections of the dual linear bundle $\widehat{E}$ which in this case corresponds to the linear sections of the double vector bundle $(\mathbb{T}A)^*_{TM\oplus A^*}$. The identification of Cor. 3.53 is given by

$$(\mathcal{T}\alpha, X^*)(\mathcal{T}a, da) = (d\langle \alpha, a\rangle, X^*(a)) \in \Gamma(T^*M \oplus A), \qquad \forall a \in \Gamma(A);$$

and $\pi : \widehat{E} \longrightarrow E = TM \oplus A^*$ is given by

$$\langle \pi(\mathcal{T}\alpha, X^*), (a, df)\rangle = \langle \alpha, a\rangle + \sigma(X^*)(f), \qquad a \in \Gamma(A), f \in C^\infty(M).$$

Therefore, we have identified $\widehat{E} \longrightarrow E$ with $J^1A^* \oplus \mathbf{CDO}(A) \longrightarrow TM \oplus A^*$. On the other hand, the linear bundle $\widehat{E}_F$ corresponding to $\mathbb{T}A$ is $\mathfrak{X}_{\mathrm{lin}}(A) \oplus \Omega^1_{\mathrm{lin}}(A)$. Under this identifications, the map $Z : \widehat{E} \longrightarrow \widehat{E}_F$ of Prop. C.33 used in Thm. 5.18 above, is given by

$$Z(X, d\alpha) \longrightarrow (\mathcal{T}\alpha, X^*),$$

where we are identifying $\mathcal{T}\alpha$ with the first jet prolongation of α, $j^1\alpha \in J^1A^*$, and $X^* \in \mathbf{CDO}(A)$ is the covariant differential operator dual to X, when we interpret $X \in \mathfrak{X}_{\text{lin}}(A)$ as a covariant differential operator in $\mathbf{CDO}(A^*)$. Then, the procedure indicated in the proof of Thm. 5.18, gives us the Lie 2-algebroid structure on

$$(T^*M \oplus A) \otimes A^* \longrightarrow J^1A^* \oplus \mathbf{CDO}(A) \longrightarrow TM \oplus A^*$$

as follows:

- $[(\mathcal{T}\alpha_1, X_1^*)(\mathcal{T}\alpha_2, X_2^*)] = (\mathcal{T}(X_1^*(\alpha_2)), [X_1^*, X_2^*]), \qquad \alpha_i \in \Gamma(A^*),\ X_i^* \in \mathbf{CDO}(A);$
- $\widehat{\rho}(\mathcal{T}\alpha, X^*) = \sigma(X^*), \qquad \alpha \in \Gamma(A^*),\ X^* \in \mathbf{CDO}(A);$
- $\langle \partial(\alpha), (df, a)\rangle = \langle \alpha, a\rangle, \qquad \alpha \in \Gamma(A^*), a \in \Gamma(A);$
- For $\alpha \in \Gamma(A^*), a \in \Gamma(A), X^* \in \mathbf{CDO}(A), f \in C^\infty(M)$,
$$\begin{aligned}\Psi(\mathcal{T}\alpha, X^*)(df, a) &= (d\sigma(X^*)(f), X^*(a)) - (d\sigma(X^*)(f) + d\langle \alpha, a\rangle, 0)\\ &= (-d\langle \alpha, a\rangle, X^*(a));\end{aligned}$$
- $\Theta(\mathcal{T}\alpha, X^*)(\beta) = X(\beta), \qquad \alpha, \beta \in \Gamma(A^*),\ X^* \in \mathbf{CDO}(A).$

5.3 Exact V-twisted Courant algebroids

In a recent paper [24], M. Grutzmann and T. Strobl introduced a sort of vector bundle analogue for exact Courant algebroids [62], which they call *exact V-twisted Courant algebroids*. Our aim in this last section is to show that there is a canonical 1:1 correspondence between exact V-twisted Courant algebroids (actually, we slightly modify the definition of [24] by adding axiom *5* below, which is redundant in the case rank $F > 1$) and *VB*-Courant algebroids on the double dual $D^*_{F^*}$ of a self-conjugate DVB $(D; E, F; H)$, and hence, by Thm. 5.18, a canonical 1:1 correspondence with Lie 2-algebroids, or, by Thm. 4.20, a canonical 1:1 correspondence with NQ degree 2 manifolds. We remark that we obtain the correspondences between these structures *without splittings*.

Definition 5.20. An *exact V-twisted Courant algebroid* consists in the following data:

- a triplet of vector bundles, which we denote by $(F, \widehat{E}, E)$,
- a surjective vector bundle morphism $\pi : \widehat{E} \longrightarrow E$,
- an anchor map $\rho : E \longrightarrow TM$,
- a bracket $[\cdot,\cdot]$ on $\Gamma(\widehat{E})$,
- a non-degenerate surjective symmetric product $(\cdot,\cdot)$ taking values in F.
- a vector bundle morphism $\Theta : \widehat{E} \longrightarrow \mathbf{CDO}(F)$.

These structure data are subject to the following axioms:

1. $\pi \circ \pi^* = 0$, where $\pi^* : E^* \otimes F \longrightarrow \widehat{E}$ is the adjoint of π with respect to $(\cdot, \cdot)$:

$$(\pi^*(\eta), \phi) := \eta(\pi(\phi)),$$

2. i) $[\phi_1, f\phi_2] = f[\phi_1, \phi_2] + \widehat{\rho}(\phi_1)(f)\phi_2$,
   ii) $\Theta(\phi)(f\xi) = f\Theta(\phi)(\xi) + \widehat{\rho}(\phi)(f)\xi$,
   where $\widehat{\rho} : \widehat{E} \longrightarrow TM$ is given by $\widehat{\rho} := \rho \circ \pi$,
3. $[\phi_1, [\phi_2, \phi_3]] = [[\phi_1, \phi_2], \phi_3] + [\phi_2, [\phi_1, \phi_3]]$,
4. $([\phi_1, \phi_2], \phi_2) = \frac{1}{2}\Theta(\phi_1)(\phi_2, \phi_2) = (\phi_1, [\phi_2, \phi_2])$.
5. $\pi([\phi, \pi^*(\eta)]) = 0$.

Remark 5.21. The adjoint π^* is well-defined because of both, the non-degeneracy and the surjectivity of $(\cdot, \cdot)$.

Remark 5.22. The 5^{th} axiom of the definition above does not form part of the original definition of [24]. We include it in order to obtain the canonical 1:1 correspondence of Thm. 5.23 below. When $\operatorname{rank} F > 1$, axiom *5* holds automatically. Indeed, observe that axiom *1* implies that $\pi^*(E^* \otimes F)$ is isotropic with respect to $(\cdot, \cdot)$, then for any $\varepsilon \otimes \xi \in E^* \otimes F$ and any $\phi \in \widehat{E}$, take $\xi' \in \Gamma(F)$ linearly independent from ξ, then, axiom *4*, after polarization, implies that for every $\varepsilon' \in \Gamma(E^*)$,

$$\begin{aligned}\langle \pi([\phi, \pi^*(\varepsilon \otimes \xi)]), \varepsilon' \rangle \xi' &= ([\phi, \pi^*(\varepsilon \otimes \xi)], \pi^*(\varepsilon' \otimes \xi')) = \Theta(\phi)((\varepsilon \otimes \xi, \varepsilon' \otimes \xi')) - (\varepsilon \otimes \xi, [\phi, \varepsilon' \otimes \xi']) \\ &= -\langle \pi([\phi, \varepsilon' \otimes \xi']), \varepsilon \rangle \xi,\end{aligned}$$

thus the linear independence of ξ and ξ' implies

$$\langle \pi([\phi, \pi^*(\varepsilon \otimes \xi)]), \varepsilon' \rangle = \langle \pi([\phi, \varepsilon' \otimes \xi']), \varepsilon \rangle = 0.$$

Since $\varepsilon' \in \Gamma(E^*)$ was arbitrary, we conclude that $\pi([\phi, \pi^*(\varepsilon \otimes \xi)]) = 0$. By $\mathbb{R}$-bilinearity of $[\cdot, \cdot]$, item i) of axiom *2* and axiom *1*, we conclude

$$\pi([\phi, \pi^*(\eta)]) = 0$$

for any $\phi \in \Gamma(\widehat{E}), \eta \in \Gamma(E^* \otimes F)$.

In [24], an explicit example is described, showing that this equation does not hold necessarily when $\operatorname{rank} F = 1$.

Theorem 5.23. *There is a canonical 1:1 correspondence, up to isomorphisms, between exact* V*-twisted Courant algebroids and* VB*-Courant algebroids.*

Proof. Suppose we are given an exact V-twisted Courant algebroid. By axiom *1* of Def. 5.20, we have the exact *metric* vector sequence

$$0 \longrightarrow E^* \otimes F \xrightarrow{\pi^*} \widehat{E} \xrightarrow{\pi} E \longrightarrow 0, \tag{5.29}$$

which, by Thm. 3.58 is equivalent to an involutive sequence

$$0 \longrightarrow \Lambda^2 E^* \xrightarrow{\iota} \widetilde{F}^* \xrightarrow{p} F \longrightarrow 0. \tag{5.30}$$

which in turn we saw in Thm. 3.40 to be equivalent to a self-conjugate DVB

$$\begin{tikzcd} D(\widehat{E}) \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "F"] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M \end{tikzcd} \quad , \tag{5.31}$$

and this is equivalent, by Prop. 3.38, to a metric DVB structure on the dual $D_{F^*} := D(\widehat{E})^*_h$

$$\begin{tikzcd} D_{F^*} \arrow[r, "q_{E^*}"] \arrow[d, "q_{F^*}"'] \arrow[dr, phantom, "E"] & E^* \arrow[d, "q^{E^*}"] \\ F^* \arrow[r, "q^{F^*}"] & M \end{tikzcd} \quad , \tag{5.32}$$

so that the linear bundle $\widehat{F^*}$, corresponding to $\Gamma_{\text{lin}}(D_{F^*})$ fits in the exact sequence

$$0 \longrightarrow E^* \otimes E^* \xrightarrow{\iota} \widehat{F^*} \xrightarrow{p} F^* \longrightarrow 0, \tag{5.33}$$

which is the natural extended involutive sequence corresponding to (5.30).

Now we want to define a *VB*-Courant algebroid structure on $D^*_{F^*}$. As we did in Thm. 5.18, we will identify the linear bundle of $D^*_{F^*}$ with $\widehat{E}$, the linear bundle of D, via the isomorphism Z. We will use the notation σ for sections in $\Gamma_{\text{lin}}(D^*_{F^*})$ and also for sections in $\Gamma_{\text{lin}}(D)$, avoiding to write down each time the isomorphism Z and its inverse (contrary to what we did in the proof of Thm. 5.18). Also, linear sections of the form $\pi^*(\varepsilon \otimes \xi)$ we will write as $\xi \otimes \overline{\varepsilon}$, that is, as the product of the linear function $\xi \in C^\infty_{\text{lin}}(F^*)$ and the core section $\overline{\varepsilon} \in \Gamma_{\text{core}}(D^*_{F^*})$.

Before defining the structure data of the *VB*-Courant algebroid, we notice that because of the non-degeneracy and surjectivity of $(\cdot,\cdot)$ we can obtain a well-defined map $\mathcal{D} : \Gamma(F) \longrightarrow \Gamma(\widehat{E})$, given by

$$(\mathcal{D}(\xi), \sigma) := \Theta(\sigma)(\xi), \qquad \forall \xi \in \Gamma(F), \sigma \in \widehat{E}. \tag{5.34}$$

Also we can define a map, which we will denote also by $\mathcal{D}$, between the spaces $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E^*)$, given by

$$\langle \mathcal{D}(f), e \rangle := \rho(e)(f), \qquad \forall f \in C^\infty(M), e \in \Gamma(E). \tag{5.35}$$

We will use Cor. 5.9, so that we need to define the Courant algebroid structure only for linear and core sections. So we define

- $$\langle \sigma_1, \sigma_2 \rangle := (\sigma_1, \sigma_2), \quad \langle \sigma, \overline{\varepsilon} \rangle = \langle \overline{\varepsilon}, \sigma \rangle := \langle \pi(\sigma), \varepsilon \rangle, \quad \langle \overline{\varepsilon_1}, \overline{\varepsilon_2} \rangle = 0, \tag{5.36}$$

- $$a(\sigma)(\xi) := \Theta(\sigma)(\xi), \qquad a(\overline{\varepsilon})(\xi) := \langle \mathcal{D}(\xi), \overline{\varepsilon} \rangle,$$
$$a(\sigma)(f) := \widehat{\rho}(\sigma)(f), \qquad a(\overline{\varepsilon})(f) := 0, \tag{5.37}$$

- $$[\![\sigma_1, \sigma_2]\!] := [\sigma_1, \sigma_2], \qquad \langle [\![\sigma, \overline{\varepsilon_1}]\!], \overline{\varepsilon_2} \rangle := 0,$$
$$\langle [\![\sigma_1, \overline{\varepsilon}]\!], \sigma_2 \rangle := a(\sigma_1)(\langle \sigma_2, \overline{\varepsilon} \rangle) - \langle [\![\sigma_1, \sigma_2]\!], \overline{\varepsilon} \rangle, \quad [\![\overline{\varepsilon}, \sigma]\!] := -[\![\sigma, \overline{\varepsilon}]\!] + \mathcal{D}(\langle \sigma, \overline{\varepsilon} \rangle). \tag{5.38}$$

First of all, we need to verify bilinearity of $\langle \cdot, \cdot \rangle$ when we take $\sigma_2 = \xi \otimes \overline{\varepsilon} = \pi^*(\varepsilon \otimes \xi)$. This is a direct consequence of the definition of π^* (cf. axiom *1* of Def. 5.20):

$$\langle \sigma, \xi \otimes \overline{\varepsilon} \rangle := (\sigma, \pi^*(\varepsilon \otimes \xi)) = \langle \varepsilon, \pi(\sigma) \rangle \xi = \xi \langle \sigma, \overline{\varepsilon} \rangle.$$

Next we must prove consistency of the definition of $\langle [\![\sigma, \overline{\varepsilon_1}]\!], \overline{\varepsilon_2} \rangle$ with the definition of $\langle [\![\sigma_1, \overline{\varepsilon}]\!], \sigma_2 \rangle$, in the case $\sigma_2 = \xi \otimes \overline{\varepsilon}_2$. Here is where we need axiom *5*. Indeed, on one hand by the definition of $\langle [\![\sigma, \overline{\varepsilon_1}]\!], \overline{\varepsilon_2} \rangle$ and the bilinearity of $\langle \cdot, \cdot \rangle$, we have

$$\langle [\![\sigma_1, \overline{\varepsilon}]\!], \xi \otimes \overline{\varepsilon_2} \rangle = \langle [\![\sigma_1, \overline{\varepsilon}]\!], \overline{\varepsilon_2} \rangle \xi = 0,$$

and on the other hand, by the definition of $\langle [\![\sigma_1, \overline{\varepsilon}]\!], \sigma_2 \rangle$ and axiom *5* of Def. 5.20, we have

$$\begin{aligned} \langle [\![\sigma, \overline{\varepsilon}]\!], \xi \otimes \overline{\varepsilon_2} \rangle &= a(\sigma)(\langle \xi \otimes \overline{\varepsilon_2}, \overline{\varepsilon} \rangle) - \langle [\![\sigma, \xi \otimes \overline{\varepsilon_2}]\!], \overline{\varepsilon} \rangle \\ &= \langle [\sigma, \pi^*(\varepsilon_2 \otimes \xi)], \varepsilon \rangle = 0, \end{aligned}$$

thus, both ways of computing $\langle [\![\sigma, \overline{\varepsilon}]\!], \xi \otimes \overline{\varepsilon_2} \rangle$ coincide.

As for the anchor map, so far we have it defined only on linear and core sections. In order to obtain a DVB morphism $a : D^*_{F^*} \longrightarrow TF^*$ we need to check the compatibility condition (C.37) of Prop. C.24, which is equivalent to the condition

$$a(\xi \otimes \overline{\varepsilon}) = \xi \otimes a(\overline{\varepsilon}). \tag{5.39}$$

By the definitions of $a(\sigma)(f)$ and $a(\overline{\varepsilon})(f)$, and axiom *1* of Def. 5.20, it follows

$$a(\xi \otimes \overline{\varepsilon})(f) = 0 = \xi\, a(\overline{\varepsilon})(f). \tag{5.40}$$

Now, given any $\xi_2 \in \Gamma(F)$, by the surjectivity of $(\cdot, \cdot)$, we can find $\sigma_1, \sigma_2 \in \Gamma(\widehat{E})$ such that $\xi_2 = \langle \sigma_1, \sigma_2 \rangle$, whereby, using the polarized version of axiom *4* of Def. 5.20 and (5.37),

$$\begin{aligned} a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2) &= a(\xi_1 \otimes \overline{\varepsilon_1})(\langle \sigma_1, \sigma_2 \rangle) = \langle \xi_1 \otimes \overline{\varepsilon_1}, [\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!] \rangle \\ &= \xi_1\, \langle \overline{\varepsilon_1}, \mathcal{D}(\langle \sigma_1, \sigma_2 \rangle) \rangle = \xi_1\, a(\overline{\varepsilon_1})(\xi_2). \end{aligned} \tag{5.41}$$

From (5.40) and (5.41) it follows (5.39).

Now, if we want to apply Cor. 5.9, we need to verify that properties *1,2,4* and *5* of Def. 5.1 and $a \circ a^* = 0$ are satisfied by the structure defined above (only on linear and core sections).

Property 5. The case for $e = \sigma$, $e_1 = \sigma_1$ and $e_2 = \sigma_2$ linear sections, follows immediately from axiom *4* after polarization. The case with $e = \sigma_1$, $e_1 = \sigma_2$ linear and $e_2 = \overline{\varepsilon}$ core follows from Eq. (5.38). The only non-trivial case remaining is when $e = \overline{\varepsilon}$ is core, and $e_1 = \sigma_1, e_2 = \sigma_2$ are linear. From axiom *4* (after polarization), (5.38) and (5.37), we obtain the following,

$$\begin{aligned}\langle [\![\overline{\varepsilon}, \sigma_1]\!], \sigma_2\rangle + \langle \sigma_1, [\![\overline{\varepsilon}, \sigma_2]\!]\rangle &= -\langle [\![\sigma_1, \overline{\varepsilon}]\!], \sigma_2\rangle + \langle \mathcal{D}(\langle \sigma_1, \overline{\varepsilon}\rangle), \sigma_2\rangle \\ &\quad - \langle [\![\sigma_2, \overline{\varepsilon}]\!], \sigma_1\rangle + \langle \mathcal{D}(\langle \sigma_2, \overline{\varepsilon}\rangle), \sigma_1\rangle \\ &= -a(\sigma_1)(\langle \sigma_2, \overline{\varepsilon}\rangle) + \langle [\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}\rangle + a(\sigma_2)(\langle \sigma_1, \overline{\varepsilon}\rangle \\ &\quad - a(\sigma_2)(\langle \sigma_1, \overline{\varepsilon}\rangle) + \langle [\![\sigma_2, \sigma_1]\!], \overline{\varepsilon}\rangle + a(\sigma_1)(\langle \sigma_2, \overline{\varepsilon}\rangle \\ &= \langle \mathcal{D}(\langle \sigma_1, \sigma_2\rangle), \overline{\varepsilon}\rangle = a(\overline{\varepsilon})(\langle \sigma_1, \sigma_2\rangle).\end{aligned}$$

Property 4. Again the case for $e = \sigma, e_1 = \sigma_1$ and $e_2 = \sigma_2$ linear sections follows from axiom *4* after polarization. This case implies, form (5.34) that

$$[\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!] = \mathcal{D}(\langle \sigma_1, \sigma_2\rangle), \tag{5.42}$$

which in turn implies, from (5.37)

$$a(\overline{\varepsilon})(\langle \sigma_1, \sigma_2\rangle) = \langle \mathcal{D}(\langle \sigma_1, \sigma_2\rangle), \overline{\varepsilon}\rangle = \langle \overline{\varepsilon}, [\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!]\rangle.$$

Finally, the case $e = \sigma$, $e_1 = \sigma_1$ linear and $e_2 = \overline{\varepsilon}$ core follows directly from (5.38) and (5.35).

Property 2. We have three cases.

a) $\mathbf{e_1 = \sigma_1,\ e_2 = \sigma_2}$.
First we need to verify Leibniz rule with respect to the product by a function. Take $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(\widehat{E})$ and $f \in C^\infty(M)$, then, by property *5* already proven above, we have,

$$\begin{aligned}\langle [\![\sigma_1, f\sigma_2]\!], \sigma_3\rangle &= a(\sigma_1)(\langle f\sigma_2, \sigma_3\rangle) - \langle f\sigma_2, [\![\sigma_1, \sigma_3]\!]\rangle \\ &= a(\sigma_1)(f)\langle \sigma_2, \sigma_3\rangle + fa(\sigma_1)(\langle \sigma_2, \sigma_3\rangle) - f\langle \sigma_2, [\![\sigma_1, \sigma_3]\!]\rangle \\ &= \langle f[\![\sigma_1, \sigma_2]\!], \sigma_3\rangle + \langle a(\sigma_1)(f)\sigma_2, \sigma_3\rangle.\end{aligned} \tag{5.43}$$

Then, using Leibniz property together with Jacobi identity for linear sections, which we already have from axiom *3* of Def. 5.20, we conclude, in the exact manner we did in Prop. 4.19,

$$a([\![\sigma_1, \sigma_2]\!])(f) = [a(\sigma_1), a(\sigma_2)](f). \tag{5.44}$$

In order to prove the same identity for a linear function ξ instead of f, we need Leibniz rule with respect to the product of a linear function and also Jacobi identity for e_1, e_2 linear and e_3 core. More precisely, we have, for every

$\sigma_1, \sigma_2 \in \Gamma(\widehat{E})$, $\xi \in \Gamma(F)$ and $\overline{\varepsilon} \in \Gamma_{\mathrm{core}}(D^*_{F^*})$, again using property *5* already proven,

$$\begin{aligned}\langle [\![\sigma_1, \xi \otimes \overline{\varepsilon}]\!], \sigma_2 \rangle &= a(\sigma_1)(\langle \xi \otimes \overline{\varepsilon}, \sigma_2 \rangle) - \langle \xi \otimes \overline{\varepsilon}, [\![\sigma_1, \sigma_2]\!] \rangle \\ &= a(\sigma_1)(\langle \sigma_2, \overline{\varepsilon} \rangle \xi) - \langle [\![\sigma_1, \sigma_2]\!], \overline{\varepsilon} \rangle \xi \\ &= a(\sigma_1)(\langle \sigma_2, \overline{\varepsilon} \rangle) \xi + \langle \sigma_2, \overline{\varepsilon} \rangle a(\sigma_1)(\xi) - \langle [\![\sigma_1, \sigma_2]\!], \overline{\varepsilon} \rangle \xi \\ &= \langle [\![\sigma_1, \overline{\varepsilon}]\!], \sigma_2 \rangle \xi + \langle \sigma_2, \overline{\varepsilon} \rangle a(\sigma_1)(\xi) \\ &= \langle \xi \otimes [\![\sigma_1, \overline{\varepsilon}]\!] + a(\sigma_1)(\xi) \otimes \overline{\varepsilon}, \sigma_2 \rangle. \end{aligned} \tag{5.45}$$

Now, as we said above, we need Jacobi identity for e_1, e_2 linear and e_3 core. Using (5.38) repeatedly, we have, for $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(\widehat{E})$ and $\overline{\varepsilon} \in \Gamma_{\mathrm{core}}(D^*_{F^*})$,

$$\begin{aligned}\langle [\![\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]]\!], \sigma_3 \rangle &= a(\sigma_1) \langle \sigma_3, [\![\sigma_2, \overline{\varepsilon}]\!] \rangle - \langle [\![\sigma_1, \sigma_3]\!], [\![\sigma_2, \overline{\varepsilon}]\!] \rangle \\ &= a(\sigma_1)(a(\sigma_2)(\langle \sigma_3, \overline{\varepsilon} \rangle)) - a(\sigma_1)(\langle [\![\sigma_2, \sigma_3]\!], \overline{\varepsilon} \rangle) \\ &\quad - a(\sigma_2)(\langle [\![\sigma_1, \sigma_3]\!], \overline{\varepsilon} \rangle) + \langle [\![\sigma_2, [\![\sigma_1, \sigma_3]\!]]\!], \overline{\varepsilon} \rangle, \end{aligned} \tag{5.46}$$

$$\langle [\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!], \sigma_3 \rangle = a([\![\sigma_1, \sigma_2]\!])(\langle \sigma_3, \overline{\varepsilon} \rangle) - \langle [\![[\![\sigma_1, \sigma_2]\!], \sigma_3]\!], \overline{\varepsilon} \rangle, \tag{5.47}$$

$$\begin{aligned}\langle [\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!], \sigma_3 \rangle &= a(\sigma_2) \langle \sigma_3, [\![\sigma_1, \overline{\varepsilon}]\!] \rangle - \langle [\![\sigma_2, \sigma_3]\!], [\![\sigma_1, \overline{\varepsilon}]\!] \rangle \\ &= a(\sigma_2)(a(\sigma_1)(\langle \sigma_3, \overline{\varepsilon} \rangle)) - a(\sigma_2)(\langle [\![\sigma_1, \sigma_3]\!], \overline{\varepsilon} \rangle) \\ &\quad - a(\sigma_1)(\langle [\![\sigma_2, \sigma_3]\!], \overline{\varepsilon} \rangle) + \langle [\![\sigma_1, [\![\sigma_2, \sigma_3]\!]]\!], \overline{\varepsilon} \rangle. \end{aligned} \tag{5.48}$$

From (5.46), (5.47) and (5.48), after cancelling terms, we obtain

$$\langle [\![\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]]\!] - [\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!] - [\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!], \sigma_3 \rangle = 0. \tag{5.49}$$

Then, using (5.45) repeatedly, we have the following

$$\begin{aligned}[\![\sigma_1, [\![\sigma_2, \xi \otimes \overline{\varepsilon}]\!]]\!] &= [\![\sigma_1, \xi \otimes [\![\sigma_2, \overline{\varepsilon}]\!]]\!] + [\![\sigma_1, a(\sigma_2)(\xi) \otimes \overline{\varepsilon}]\!] \\ &= \xi \otimes [\![\sigma_1, [\![\sigma_2, \overline{\varepsilon}]\!]]\!] + a(\sigma_1)(\xi) \otimes [\![\sigma_2, \overline{\varepsilon}]\!] \\ &\quad + a(\sigma_2)(\xi) \otimes [\![\sigma_1, \overline{\varepsilon}]\!] + a(\sigma_1)(a(\sigma_2)(\xi)) \otimes \overline{\varepsilon}, \end{aligned} \tag{5.50}$$

$$[\![[\![\sigma_1, \sigma_2]\!], \xi \otimes \overline{\varepsilon}]\!] = \xi \otimes [\![[\![\sigma_1, \sigma_2]\!], \overline{\varepsilon}]\!] + a([\![\sigma_1, \sigma_2]\!])(\xi) \otimes \overline{\varepsilon}, \tag{5.51}$$

$$\begin{aligned}[\![\sigma_2, [\![\sigma_1, \xi \otimes \overline{\varepsilon}]\!]]\!] &= \xi \otimes [\![\sigma_2, [\![\sigma_1, \overline{\varepsilon}]\!]]\!] + a(\sigma_2)(\xi) \otimes [\![\sigma_1, \overline{\varepsilon}]\!] \\ &\quad + a(\sigma_1)(\xi) \otimes [\![\sigma_2, \overline{\varepsilon}]\!] + a(\sigma_2)(a(\sigma_1)(\xi)) \otimes \overline{\varepsilon}. \end{aligned} \tag{5.52}$$

Then, using axiom *3* of Def. 5.20 and (5.49), adding (5.50), (5.51) and (5.52), after cancelling terms, we get

$$(a([\![\sigma_1, \sigma_2]\!])(\xi) - [a(\sigma_1), a(\sigma_2)](\xi)) \otimes \overline{\varepsilon} = 0,$$

thus, since $\overline{\varepsilon}$ is arbitrary,

$$a([\![\sigma_1, \sigma_2]\!])(\xi) - [a(\sigma_1), a(\sigma_2)](\xi) = 0. \tag{5.53}$$

From Eqs. (5.44) and (5.53) we obtain

$$a([\![\sigma_1, \sigma_2]\!]) = [a(\sigma_1), a(\sigma_2)]. \tag{5.54}$$

b) $\mathbf{e_1} = \sigma,\ \mathbf{e_2} = \overline{\varepsilon}$.
Using repeatedly (5.45) and (5.39), we have, for every $\sigma \in \Gamma(\widehat{E})$, $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$ and $\xi_1, \xi_2 \in \Gamma(F)$,

$$\begin{aligned}
[\![\sigma, [\![\xi_1 \otimes \overline{\varepsilon_1}, \xi_2 \otimes \overline{\varepsilon_2}]\!]]\!] &= [\![\sigma, \xi_2 \otimes [\![\xi_1 \otimes \overline{\varepsilon_1}, \overline{\varepsilon_2}]\!]]\!] + [\![\sigma, a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2) \otimes \overline{\varepsilon_2}]\!] \\
&= \xi_2 \otimes [\![\sigma, [\![\xi_1 \otimes \overline{\varepsilon_1}, \overline{\varepsilon_2}]\!]]\!] + a(\sigma)(\xi_2) \otimes [\![\xi_1 \otimes \overline{\varepsilon_1}, \overline{\varepsilon_2}]\!] \\
&\quad + a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2) \otimes [\![\sigma, \overline{\varepsilon_2}]\!] + a(\sigma)a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2)) \otimes \overline{\varepsilon_2}
\end{aligned} \tag{5.55}$$

$$[\![[\![\sigma, \xi_1 \otimes \overline{\varepsilon_1}]\!], \xi_2 \otimes \overline{\varepsilon_2}]\!] = \xi_2 \otimes [\![[\![\sigma, \xi_1 \otimes \overline{\varepsilon_1}]\!], \overline{\varepsilon_2}]\!] + a([\![\sigma, \xi_1 \otimes \overline{\varepsilon_1}]\!])(\xi_2) \otimes \overline{\varepsilon_2} \tag{5.56}$$

$$\begin{aligned}
[\![\xi_1 \otimes \overline{\varepsilon_1}, [\![\sigma, \xi_2 \otimes \overline{\varepsilon_2}]\!]]\!] &= [\![\xi_1 \otimes \overline{\varepsilon_1}, \xi_2 \otimes [\![\sigma, \overline{\varepsilon_2}]\!]]\!] + [\![\xi_1 \otimes \overline{\varepsilon_1}, a(\sigma)(\xi_2) \otimes \overline{\varepsilon_2}]\!] \\
&= \xi_2 \otimes [\![\xi_1 \otimes \overline{\varepsilon_1}, [\![\sigma, \overline{\varepsilon_2}]\!]]\!] + a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2) \otimes [\![\sigma, \overline{\varepsilon_2}]\!] \\
&\quad + a(\sigma)(\xi_2) \otimes [\![\xi_1 \otimes \overline{\varepsilon_1}, \overline{\varepsilon_2}]\!] + a(\xi_1 \otimes \overline{\varepsilon_1})(a(\sigma)(\xi_2)) \otimes \overline{\varepsilon_2}.
\end{aligned} \tag{5.57}$$

Adding Eqs. (5.55), (5.56) and (5.57) we get, using (5.49) and cancelling terms,

$$\begin{aligned}
0 =&[a(\sigma)(a(\xi_1 \otimes \overline{\varepsilon_1})(\xi_2)) - a(\xi_1 \otimes \overline{\varepsilon_1})(a(\sigma)(\xi_2)) \\
&- a([\![\sigma, \xi_1 \otimes \overline{\varepsilon_1}]\!])(\xi_2)] \otimes \overline{\varepsilon_2}.
\end{aligned}$$

Since $\overline{\varepsilon_2}$ is arbitrary, it follows that the term in brackets [ ] is zero. Using again (5.39) and (5.45), we obtain

$$\begin{aligned}
0 =&a(\overline{\varepsilon_1}(\xi_2)a(\sigma)(\xi_1) + a(\sigma)(a(\overline{\varepsilon_1})(\xi_2))\,\xi_1 - a(\overline{\varepsilon_1})(a(\sigma)(\xi_2)\,\xi_1 \\
&- a([\![\sigma, \overline{\varepsilon_1}]\!])(\xi_2)\,\xi_1 - a(\overline{\varepsilon_1})(\xi_2)a(\sigma)(\xi_1) \\
=&[a(\sigma)(a(\overline{\varepsilon_1})(\xi_2)) - a(\overline{\varepsilon_1})(a(\sigma)(\xi_2) - a([\![\sigma, \overline{\varepsilon_1}]\!])(\xi_2)]\,\xi_1,
\end{aligned}$$

which implies, since $\xi_1 \in \Gamma(F)$ is arbitrary, that the term in brackets [ ] is zero, and since $\xi_2 \in \Gamma(F)$ is arbitrary, it follows finally

$$a([\![\sigma, \overline{\varepsilon_1}]\!]) - [a(\sigma), a(\overline{\varepsilon_1})] = 0. \tag{5.58}$$

c) $\mathbf{e_1} = \overline{\varepsilon},\ \mathbf{e_2} = \sigma$.
From the definition of a in (5.37), item b) above, and skew-symmetry of the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{X}(F)$, we have, for every $\xi \in \Gamma(F^*)$,

$$\begin{aligned}
a([\![\overline{\varepsilon}, \sigma]\!])(\xi) &= a(\mathcal{D}(\langle \sigma, \overline{\varepsilon} \rangle))(\xi) - a([\![\sigma, \overline{\varepsilon}]\!])(\xi) \\
&= [a(\overline{\varepsilon}), a(\sigma)](\xi) + a(\mathcal{D}(\langle \sigma, \overline{\varepsilon} \rangle)) \\
&= [a(\overline{\varepsilon}), a(\sigma)](\xi) + \langle \mathcal{D}(\xi), \mathcal{D}(\langle \sigma, \overline{\varepsilon} \rangle) \rangle \\
&= [a(\overline{\varepsilon}), a(\sigma)](\xi) + a(\mathcal{D}(\xi))(\langle \sigma, \overline{\varepsilon} \rangle).
\end{aligned} \tag{5.59}$$

Now, since $(\cdot,\cdot)$ is surjective, we can write $\xi = \langle \sigma_1, \sigma_2 \rangle$ for suitable $\sigma_1, \sigma_2 \in \Gamma(\widehat{E})$, which by properties *4* and *2* for the case of linear sections, which were already proven above, yields

$$\begin{aligned}
a(\mathcal{D}(\xi)) &= a(\mathcal{D}(\langle \sigma_1, \sigma_2 \rangle)) = a([\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!]) \\
&= [a(\sigma_1), a(\sigma_2)] + [a(\sigma_2), a(\sigma_2)] = 0.
\end{aligned} \tag{5.60}$$

From (5.59) and (5.60), we conclude that

$$a([\![\bar{\varepsilon},\sigma]\!]) = [a(\bar{\varepsilon}), a(\sigma)].$$

Therefore, we have obtained property *2* in the three cases.

Property 1. As we did in the proof of this same property in Thm. 5.18, relying in (the proof of) lemma 2.6.4 of [58] and the properties *2, 4* and *5* proven above, we conclude that the Jacobiator J (cf. (5.20)) is totally skew-symmetric when evaluated in linear or core sections. Again as we argued in the proof of this property in Thm. 5.18, $J(e_1, e_2, e_3)$ is zero whenever two sections are core, remaining only two non-trivial cases.

a) $e_1 = \sigma_1$ $e_2 = \sigma_2, e_3 = \sigma_3$ are linear sections.
   This case is equivalent to axiom *3* of Def. 5.20.

b) $e_1 = \sigma_1$ $e_2 = \sigma_2$ linear and $e_3 = \bar{\varepsilon}$ core.
   This case is precisely Eq. (5.49).

Thus, we have properties *1, 2, 4* and *5* checked for linear and core sections. The last thing we need to check in order to apply Cor. 5.9 is $a \circ a^* = 0$. It suffices to verify this equation only for sections of the form $df, d\xi \in T^*F$ for $f \in C^\infty(M)$ and $\xi \in \Gamma(F) \cong C^\infty_{\mathrm{lin}}(F^*)$.

For $f \in C^\infty(M)$, by (5.36), the surjectivity of π allows us to find $\sigma \in \Gamma(\widehat{E})$ and $\bar{\varepsilon} \in \Gamma_{\mathrm{core}}(D^*_{F^*})$ such that $f = \langle \sigma, \bar{\varepsilon}\rangle$. Then, using properties *4* and *2* proven above, we have

$$\begin{aligned} a \circ a^*(df) &= a(\mathcal{D}(f)) = a(\mathcal{D}(\langle \sigma, \bar{\varepsilon}\rangle)) = a([\![\sigma,\bar{\varepsilon}]\!] + [\![\bar{\varepsilon},\sigma]\!]) \\ &= [a(\sigma), a(\bar{\varepsilon}] + [a(\bar{\varepsilon}), a(\sigma)] = 0. \end{aligned}$$

For $\xi \in \Gamma(F)$, from $a^*(d\xi) = \mathcal{D}(\xi)$ and (5.60) it follows

$$a \circ a^*(d\xi) = 0.$$

Hence, by Cor. 5.9 we have obtained a *VB*-Courant algebroid structure on $D^*_{F^*}$ from the exact V-twisted Courant algebroid structure on $\widehat{E}$.

Conversely, if we start from a *VB*-Courant algebroid structure on the dual of a metric DVB $D^*_{F^*}$, then from equations (5.36), (5.37) and (5.38) we induce the corresponding exact V-twisted Courant algebroid data on the linear sequence (5.29). It is not difficult to verify that the data obtained satisfy indeed the axioms of Def. 5.20, and that these two processes of going from an exact V-twisted Courant algebroid to a *VB*-Courant algebroid and back, establish the desired canonical 1:1 correspondence.

■

Chapter 6

Poisson degree 2 manifolds

In this chapter we study the structure of a degree 2 manifold endowed with -2 graded Poisson brackets, called *Poisson degree 2 manifolds*, a particular case of Poisson k-manifolds introduced in Def. 3.4. Since Poisson structures appear naturally when performing reduction of symplectic structures, it is naturally to treat Poisson degree 2 manifolds in the context of reduction of Courant algebroids [6].

After obtaining a geometric characterization in terms of the basic exact sequence (3.14), we provide a geometric characterization of Poisson degree 2 manifolds in terms of the corresponding involutive double vector bundle, arriving to double linear Poisson brackets, invariant under the involution, giving rise to the category of what we call *involutive double linear Poisson bundles*. When we go to the dual, we obtain a *VB*-algebroid structure which is compatible with the linear metric corresponding to the transpose of the involution, a structure introduced by D. Li-Bland [41] and studied by [29] under the name of *metric VB-algebroids*.

Then we give a classification of *regular* Poisson manifolds in terms of certain Chevalley cohomology groups. Finally we recover the symplectic case, studied by D. Roytenberg [59].

6.1 The categories of degree 2 Poisson manifolds and involutive Lie algebroid sequences

Recall that we already introduced Poisson k-manifolds in Def. 3.4. A Poisson degree 2 manifold is simply a Poisson k-manifold of degree 2.

A morphism between between two degree 2 Poisson manifolds, called a *Poisson morphism*,

$$\Psi : (\mathcal{N}, \{\cdot,\cdot\}_{\mathcal{N}}) \longrightarrow (\mathcal{M}, \{\cdot,\cdot\}_{\mathcal{M}})$$

is a morphism between the degree 2 manifolds

$$\Psi = (\psi, \psi^\sharp) : \mathcal{N} = (N, \mathcal{O}_N) \longrightarrow \mathcal{M} = (M, \mathcal{O}_M),$$

such that the morphism of sheaves $\psi^\sharp : \mathcal{O}_M \longrightarrow \psi_* \mathcal{O}_N$ preserves the Poisson brackets, that is, for every $U \subset M$, and every $f, g \in \mathcal{O}_M(U)$, we have

$$\psi^\sharp(\{f,g\}) = \{\psi^\sharp(f), \psi^\sharp(g)\}.$$

Degree 2 Poisson manifolds, together with the morphisms between them, form a category.

Theorem 6.1. *Given a degree 2 manifold $\mathcal{M}$, consider its corresponding involutive sequence $(E, \widetilde{F}, p : \widetilde{F} \longrightarrow \Lambda^2 E)$, given by Prop. 3.7. Then there is a 1:1 correspondence between degree -2 Poisson brackets on $\mathcal{M}$ and the following structure on the involutive sequence:*

- *A symmetric bilinear form $\langle\cdot,\cdot\rangle$ on E^*;*
- *A Lie algebroid structure $([\cdot,\cdot],\rho)$ on $\widetilde{F}^*$;*
- *A Lie algebroid $\langle\cdot,\cdot\rangle$-preserving action, Ψ, of $\widetilde{F}^*$ on E^*;*

such that

- *The brackets between a section ζ of $\widetilde{F}^*$ and a section $\varepsilon_1 \wedge \varepsilon_2$ of $\Lambda^2 E^*$, seen inside $\widetilde{F}^*$ through the map $\Lambda^2 E^* \stackrel{\iota}{\longrightarrow} \widetilde{F}^*$, are given by*

$$[\zeta, \varepsilon_1 \wedge \varepsilon_2] = \Psi(\zeta)(\varepsilon_1) \wedge \varepsilon_2 + \varepsilon_1 \wedge \Psi(\zeta)(\varepsilon_2). \tag{6.1}$$

- *The action Ψ of $\widetilde{F}^*$ on E^* restricted to $\Lambda^2 E^*$, which we denote by $\widetilde{\Psi}$, is given by*

$$\widetilde{\Psi}(\varepsilon_1 \wedge \varepsilon_2)(\varepsilon) = \langle \varepsilon_2, \varepsilon\rangle \varepsilon_1 - \langle \varepsilon_1, \varepsilon\rangle \varepsilon_2. \tag{6.2}$$

Proof. By Prop. 3.7, we only need to prove the equivalence between degre -2 Poisson brackets on a degree 2 manifold and the data in the items of the statement. Suppose first that a degree to $\mathcal{M}$ is endowed with degree -2 Poisson brackets $\{\cdot,\cdot\}$. Then applying the brackets to degree 1 functions $\varepsilon_1, \varepsilon_2$, which are identified with (local) sections of E^*, by Leibniz rule and since the brackets have degree -2, we see that, for any $f \in C^\infty(M)$,

$$\{f\varepsilon_1, \varepsilon_2\} = f\{\varepsilon_1, \varepsilon_2\} = f\{\varepsilon_2, \varepsilon_1\} = \{\varepsilon_1, f\varepsilon_2\},$$

thus getting a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on E^*.

Again by Leibniz's rule, we see that, for a degree 2 function $\zeta \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*)$,

$$\{\zeta, \cdot\}$$

defines a derivation on $C^\infty(M)$. Hence we obtain a vector bundle map $\rho : \widetilde{F}^* \longrightarrow TM$, given by

$$\rho(\zeta)(f) := \{\zeta, f\}.$$

Setting, for $\zeta_1, \zeta_2 \in \Gamma(\widetilde{F}^*)$,

$$[\zeta_1, \zeta_2] := \{\zeta_1, \zeta_2\},$$

defines Lie brackets on $\Gamma(\widetilde{F}^*)$, by graded skew-symmetry and Jacobi identity, and actually, by Leibniz's rule, $([\cdot,\cdot],\rho)$ defines a Lie algebroid structure on $\widetilde{F}^*$.

Next, we define a vector bundle map $\Psi : \widetilde{F}^* \longrightarrow \mathbf{CDO}(E^*)$ (over the identity), where $\mathbf{CDO}(E^*)$ is the Lie algebroid whose sections are the *covariant differential operators* of sections of E (also called derivative endomorphisms), $[\cdot,\cdot]$ is given by the commutator, and the anchor is given by the symbol map, by

$$\Psi(\zeta)(\varepsilon) := \{\zeta, \varepsilon\},$$

with $\zeta \in \Gamma(\widetilde{F}^*)$ and $\varepsilon \in \Gamma(E^*)$. That Ψ actually takes values on $\mathbf{CDO}(E^*)$ and is a Lie algebroid morphism, follows from Leibniz's rule and Jacobi identity. Also from Jacobi identity it follows that Ψ preserves $\langle\cdot,\cdot\rangle$, which, by definition, means that

$$\rho(\zeta)(\langle\varepsilon_1, \varepsilon_2\rangle) = \langle\Psi(\zeta)(\varepsilon_1), \varepsilon_2\rangle + \langle\varepsilon_1, \Psi(\zeta)(\varepsilon_2)\rangle.$$

The compatibilities of $[\cdot,\cdot]$ and Ψ with the inclusion $\iota : \Lambda^2 E^* \longrightarrow \widetilde{F}^*$, given in the last two items of the statement, follow directly from Leibniz's rule.

Conversely, suppose we are given the data in the statement. We use these data to define

$$\{\mathcal{A}^1, \mathcal{A}^1\};\ \{\mathcal{A}^2, \mathcal{A}^0\};\ \{\mathcal{A}^2, \mathcal{A}^1\} \text{ and } \{\mathcal{A}^2, \mathcal{A}^2\}.$$

We need to check that the brackets defined this way are compatible with graded skew-symmetry, which follows directly from the symmetry of $\langle,\cdot,\cdot\rangle$ and the skew-symmetry of $[\cdot,\cdot]$, and Leibniz's rule. For a product $\varepsilon_1 \wedge \varepsilon_2$, it follows from the compatibility conditions (6.1) and (6.2) and for a product fg it follows from the fact that ρ takes values on the tangent bundle of M; Jacobi identity restricted to functions belonging to $\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^2$ follows from Jacobi identity of $[\cdot,\cdot]$, from the preserving brackets property of ρ and from the $\langle\cdot,\cdot\rangle$-preserving property of Ψ.

We extend the definition to the whole structure sheaf $\mathcal{O}_M$ by Leibniz's rule and graded skew-symmetry together with an induction argument. ∎

Remark 6.2. A consequence of (6.1) and (6.2) is that the brackets on $\Lambda^2 E^*$ satisfy

$$[\varepsilon_1 \wedge \varepsilon_2, \varepsilon_3 \wedge \varepsilon_4] = \langle\varepsilon_2, \varepsilon_3\rangle\varepsilon_1 \wedge \varepsilon_4 + \langle\varepsilon_2, \varepsilon_4\rangle\varepsilon_3 \wedge \varepsilon_1 - \langle\varepsilon_1, \varepsilon_3\rangle\varepsilon_2 \wedge \varepsilon_4 - \langle\varepsilon_1, \varepsilon_4\rangle\varepsilon_3 \wedge \varepsilon_2; \quad (6.3)$$

in particular, the anchor map restricted to $\Lambda^2 E^*$ is zero, so that $\Lambda^2 E^*$ actually is a Lie algebroid ideal of F, and the exact sequence

$$0 \longrightarrow \Lambda^2 E^* \longrightarrow \widetilde{F}^* \longrightarrow F^* \longrightarrow 0, \quad (6.4)$$

obtained by transposition of (3.3), turns into a Lie algebroid exact sequence.

The result above leads us to introduce the following definition.

Definition 6.3. An *involutive Lie algebroid sequence* is an involutive sequence

$$(E, \widetilde{F}, \widetilde{F} \xrightarrow{p} \Lambda^2 E),$$

endowed with the structure given by Thm. 6.1, satisfying properties (6.1) and (6.2).

An *involutive Lie algebroid morphism* $(\psi, \psi_1, \psi_2) : (\widetilde{F} \longrightarrow \Lambda^2 E) \longrightarrow (\widetilde{F'} \longrightarrow \Lambda^2 E')$ between two involutive Lie algebroid sequences, is an involutive morphism between the underlying involutive sequences, such that:

- $$\psi^*(\langle \varepsilon_1', \varepsilon_2' \rangle') = \langle \psi_1^*(\varepsilon_1'), \psi_1^*(\varepsilon_2') \rangle, \qquad \forall \varepsilon_1', \varepsilon_2' \in \Gamma((E')^*) \cong C^\infty_{\mathrm{lin}}(E'); \tag{6.5}$$

- $$\psi_2^*(\{\zeta_1', \zeta_2'\}') = \{\psi_2^*(\zeta_1'), \psi_2^*(\zeta_2')\}, \qquad \forall \zeta_1'\zeta_2' \in \Gamma(\widetilde{F'}^*) \cong C^\infty_{\mathrm{lin}}(\widetilde{F'}), \tag{6.6}$$

  where $\{\cdot,\cdot\}$ here stands for the linear Poisson structure on $\widetilde{F}$ dual to the Lie algebroid structure $([\cdot,\cdot],\rho)$ on $\widetilde{F}^*$, and similarly $\{\cdot,\cdot\}'$ stands for the linear Poisson structure on $\widetilde{F'}$ corresponding to the dual Lie algebroid on $\widetilde{F'}^*$;

- $$\psi_1^*(\Psi'(\zeta')(\varepsilon')) = \Psi(\psi_2^*(\zeta'))(\psi_1^*(\varepsilon')), \quad \forall \zeta \in \Gamma(\widetilde{F'}^*) \cong C^\infty_{\mathrm{lin}}(\widetilde{F'}), \quad \varepsilon' \in \Gamma((E')^*) \cong C^\infty_{\mathrm{lin}}(E'). \tag{6.7}$$

The compatibility of the conditions above with equations (6.1) and (6.2), follows from the fact that (ψ, ψ_1, ψ_2) is involutive.

The involutive Lie algebroid sequences, together with the morphisms between them, form the *involutive Lie algebroid sequence category.*

Corollary 6.4. *The category of degree 2 Poisson manifolds is equivalent to the category of involutive Lie algebroid sequences.*

Proof. We will construct a functor from the category of degree 2 Poisson manifolds to the involutive Lie algebroid sequence category as follows. Given a degree 2 Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\})$, by Thm. 3.12), we have an involutive sequence associated to $\mathcal{M}$, $(E, \widetilde{F} \longrightarrow \Lambda^2 E)$, and given a Poisson morphism $(\psi, \psi^\sharp)$, we have an involutive morphism associated (ψ, ψ_1, ψ_2). Simply by tracking the definitions, one can verify that the condition of preserving the Poisson brackets for $(\psi, \psi^\sharp)$ implies that (ψ, ψ_1, ψ_2) is an involutive Lie algebroid morphism between the corresponding involutive Lie algebroid sequences. Therefore, we have a functor from the category of degree 2 manifolds to the involutive Lie algebroid sequence category. Fully faithfulness and essentially surjectivity easily follow from those properties of the functor given by Thm. 3.12.

■

6.1.1 Splittings

Proposition 6.5. *There is a 1-1 correspondence between degree 2 Poisson manifolds with a splitting of (3.3) and the following data:*

- *A Lie algebroid* $(F^*, \rho, [\cdot,\cdot])$
- *A vector bundle endowed with a symmetric bilinear form* $(E^*, \langle\cdot,\cdot\rangle)$
- *An* F^**-connection* ∇ *on* $(E^*, \langle\cdot,\cdot\rangle)$ *(preserving* $\langle\cdot,\cdot\rangle$*).*
- *A* curvature form $K \in \Omega^2(F^*; \Lambda^2 E^*)$ *that satisfies*

– *The curvature condition*

$$R_\nabla = -\widetilde{\Psi} \circ K, \tag{6.8}$$

where $\widetilde{\Psi}$ is the action on $(E^*, \langle\cdot,\cdot\rangle)$ restricted to $\Lambda^2 E^*$, given by formula (6.2).

– *Bianchi identity $d_\nabla K = 0$.*

Remark 6.6. For further reference, we introduce the F^*-connection on E, dual to ∇, and denote it also by ∇.

Proof. Suppose we have a degree 2 Poisson manifold and a splitting ψ of (3.3), or equivalently of the transposed sequence (6.4), is given. By Thm. 6.1 we have equivalently the data given in the statement of that theorem, in particular, we have a vector bundle E^* endowed with a symmetric bilinear form $\langle\cdot,\cdot\rangle$. By remark 6.2, the quotient $F^* = \widetilde{F}^*/\Lambda^2 E^*$ inherits a Lie algebroid structure.

Using the action Ψ we obtain an F^*-connection by setting

$$\nabla := \Psi \circ \psi,$$

which preserves $\langle\cdot,\cdot\rangle$. Finally we define $K \in \Omega^2(F^*; \Lambda^2 E^*)$ by

$$K(X,Y) := \psi([X,Y]_{F^*}) - [\psi(X), \psi(Y)]_{\widetilde{F}^*}. \tag{6.9}$$

Of course K takes values on $\Lambda^2 E^*$ since $\pi(K(X,Y)) = 0$, where $\pi : \widetilde{F}^* \longrightarrow F^*$ is the projection. Now, by definition of ∇ and by Jacobi identity, we have

$$\begin{aligned}
\widetilde{\Psi} \circ K(X,Y)(\varepsilon) &= [K(X,Y), \varepsilon] = [\psi([X,Y]) - [\psi(X), \psi(Y)], \varepsilon] \\
&= \nabla_{[X,Y]}\varepsilon + [[\varepsilon, \psi(X)], \psi(Y)] + [\psi(X), [\varepsilon, \psi(Y)]] \\
&= \nabla_{[X,Y]}\varepsilon + \nabla_Y \nabla_X \varepsilon - \nabla_X \nabla_Y \varepsilon = -R_\nabla(X,Y)(\varepsilon).
\end{aligned}$$

Also from Jacobi identity for both brackets $[\cdot,\cdot]_{\widetilde{F}^*}$ and $[\cdot,\cdot]_{F^*}$, which will be denoted simply by $[\cdot,\cdot]$ in order not to burden the notation, we have

$$\begin{aligned}
(d_\nabla K)(X,Y,Z) &= \sum_{cyclic} (\nabla_X K(Y,Z) - K([X,Y],Z)) \\
&= \sum_{cyclic} \nabla_X(\psi([Y,Z]) - [\psi(Y), \psi(Z)]) - \sum_{cyclic} (\psi([[X,Y],Z]) - [\psi([X,Y]), \psi(Z)]) \\
&= \sum_{cyclic} [\psi(X), \psi([Y,Z])] - \sum_{cyclic} [\psi(X), [\psi(Y), \psi(Z)]] \\
&\quad - \sum_{cyclic} [\psi(X), \psi([Y,Z])] - \sum_{cyclic} \psi([[X,Y],Z]) = 0.
\end{aligned}$$

Conversely given the data in the statement, we are going to build a split degree 2 Poisson manifold. The sheaf structure is given by

$$\mathcal{O}(U) = \Gamma(\Lambda^\cdot E^*[1]|_U) \otimes \Gamma(S^\cdot F^*[1]|_U);$$

by Leibniz's rule, it is enough to define the brackets for functions with degrees 0,1 and 2. The brackets between functions with degree 0, or where one of the functions has degree 1 and the other has degree 0, we define to be 0. The other cases we define in the following way:

- $\{X, f\} := \rho(X)(f)$, for $X \in \Gamma(F^*)$ and $f \in C^\infty(M)$;
- $\{\varepsilon_1, \varepsilon_2\} := \langle \varepsilon_1, \varepsilon_2 \rangle$, for $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$;
- $\{X, \varepsilon\} := \nabla_X \varepsilon$, for $X \in \Gamma(F^*)$ and $\varepsilon \in \Gamma(E^*)$;
- $\{X_1, X_2\} := [X_1, X_2] - K(X_1, X_2)$, for $X_1, X_2 \in \Gamma(F^*)$;

all the other cases can be defined imposing graded Leibniz's rule and graded symmetry, together with an induction argument. Jacobi identity follows from Jacobi identity for $[\cdot, \cdot]$, from the bracket preserving property of ρ, from the $\langle \cdot, \cdot \rangle$-preserving property of ∇; from the curvature condition and from Bianchi identity, together with an induction argument. ∎

Proposition 6.7. *Consider the sequence*

$$\Lambda^2 E^* \xrightarrow{\iota} \widetilde{F}^* \xrightarrow{\pi} F^*, \tag{6.10}$$

the transposition of (3.3). Let $\psi, \psi' : F^* \longrightarrow \widetilde{F}^*$ be two splittings of the sequence (6.10), so that they differ by the gauge form $B := \psi - \psi' \in \Omega^1(F^*, \Lambda^2 E^*)$. Let $\sharp : E \longrightarrow E^*$ the map induced by $\langle \cdot, \cdot \rangle$. Also, for $\eta \in \Lambda^2 E^*$ and $e \in E$, let's denote

$$\eta \wedge e := \iota_e \eta = \eta(e),$$

where ι is the contraction operator. Then, for $\varepsilon \in \Gamma(E^*), e \in \Gamma(E)$ and $\alpha, \beta \in \Gamma(F^*)$,

$$\begin{aligned} \nabla'_\alpha \varepsilon &= \nabla_\alpha \varepsilon + B(\alpha) \wedge \sharp\varepsilon; \\ \nabla'_\alpha e &= \nabla_\alpha e - \sharp(B \wedge e); \end{aligned} \tag{6.11}$$

$$K'(\alpha, \beta) = K(\alpha, \beta) - d_\nabla B(\alpha, \beta) + \frac{1}{2} B \wedge B(\alpha, \beta), \tag{6.12}$$

where, in (6.11), we are using the same symbol for the F^*-connection on E^* and its corresponding adjoint on E (see Rmk. 6.6).

Proof. The proof consists of calculations:

$$\begin{aligned} \nabla'_\alpha \varepsilon &= \nabla_\alpha \varepsilon - \Psi(B(\alpha))(\varepsilon) \\ &= \nabla_\alpha \varepsilon + B(\alpha) \wedge \sharp\varepsilon; \\ \langle \nabla'_\alpha e, \varepsilon \rangle &= \rho(\alpha)\langle e, \varepsilon \rangle - \langle e, \nabla'_\alpha \varepsilon \rangle \\ &= \langle \nabla_\alpha e, \varepsilon \rangle - \langle e, B(\alpha) \wedge \sharp\varepsilon \rangle \\ &= \langle \nabla_\alpha e, \varepsilon \rangle + \langle B(\alpha) \wedge e, \sharp\varepsilon \rangle \\ &= \langle \nabla_\alpha e - \sharp(B(\alpha) \wedge e), \varepsilon \rangle \\ \therefore \qquad \nabla'_\alpha e &= \nabla_\alpha e - \sharp(B \wedge e). \\ K'(\alpha, \beta) &= K(\alpha, \beta) - \nabla_\alpha B(\beta) + \nabla_\beta B(\alpha) + B([\alpha, \beta]) + [B(\alpha), B(\beta)] \\ &= K(\alpha, \beta) - d_\nabla B(\alpha, \beta) + \frac{1}{2} B \wedge B(\alpha, \beta). \end{aligned}$$

∎

6.2 The category of involutive Poisson double vector bundles

In this section we will see the "double realization" of the involutive Lie algebroid sequence category, and show that the equivalence between the category of involutive sequences and involutive double vector bundles obtained in Ch. 3 (Sec. 3.3), induces naturally an equivalence between the involutive Lie algebroid sequence category and the category of what we call involutive Poisson double vector bundles.

Definition 6.8. Let (D, H) be an involutive double vector bundle. A Poisson structure $\{\cdot,\cdot\}$ on D is called *involutive* if it is double linear, that is, linear with respect to both vector bundle structures, and invariant under the involution H, that is

$$H^*\{f, g\} = \{H^*f, H^*g\}, \qquad f, g \in C^\infty(D), \tag{6.13}$$

where $H^* : C^\infty(D) \longrightarrow C^\infty(D)$ is the pullback on functions. The whole structure $(D, H, \{\cdot,\cdot\})$ will be called *involutive Poisson double vector bundle.*

A a morphism $\Phi : (D, H, \{\cdot,\cdot\}) \longrightarrow (D', H', \{\cdot,\cdot\}')$ is called an *involutive Poisson DVB morphism* if it commutes with the involutive structures and preserves the Poisson brackets:

1. $\Phi \circ H = H' \circ \Phi$;

2. $\Phi^*\{f, g\}' = \{\Phi^*f, \Phi^*g\}, \qquad \forall f, g \in C^\infty(D')$.

Involutive Poisson double vector bundles together with the morphisms between them form a category, called the *involutive Poisson DVB category.*

Remark 6.9. It follows from the Leibniz's rule of the Poisson brackets, and Hadamard's lemma for functions, that the Poisson brackets on a point $d \in D$ are completely determined by they values on a coordinate system around this point. In particular, since every point of D has an *adapted coordinate system* around it (Cor. A.20), it turns out that a double linear Poisson structure is completely determined by its action on double-linear functions, on pullbacks of linear functions on the side bundles and on pullbacks of linear functions on the base M. Likewise, in order to verify the brackets preserving condition *2* above, it is enough to check this for adapted coordinates, or equivalently, for functions of the types just mentioned (double-linear, pullbacks of linear functions and basic functions on the sides).

Theorem 6.10. *The category of involutive Lie algebroid sequences is equivalent to the category of involutive Poisson double vector bundles.*

Proof. Let's define a functor

$$\mathfrak{P} : \left\{ \begin{array}{c} \text{involutive Lie algd. sequences} \\ + \\ \text{involutive Lie algd. morphisms} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{involutive Poisson DVB} \\ + \\ \text{involutive Poisson DVB morphims,} \end{array} \right\} \tag{6.14}$$

and show that is essentially surjective and fully faithful, thus establishing the desired equivalence of categories.

Given an involutive Lie algebroid sequence, whose underlying involutive sequence is $(\widetilde{F}\longrightarrow\Lambda^2 E)$, consider the corresponding involutive –actually self-conjugate– double vector bundle $(D; E, F; H)$ given by the equivalence of Thm. 3.40. From the proof of this theorem, it follows that sections of $\widetilde{F}^*$ correspond with double-linear functions on D which are invariant under H, whose set we will denote by $C^\infty_{\mathrm{lin}}(D)^H$. Denoting this correspondence by

$$\begin{aligned} C^\infty_{\mathrm{lin}}(D)^H &\longrightarrow \Gamma(\widetilde{F}^*) \\ \gamma \quad &\longrightarrow \mathfrak{A}(\gamma), \end{aligned}$$

and recalling that we denoted by q_h and q_v, respectively, the horizontal and vertical projections of D over E, we define

P1. $\{\gamma_1, \gamma_2\} := \mathfrak{A}^{-1}([\mathfrak{A}(\gamma_1), \mathfrak{A}(\gamma_2)]), \qquad \forall \gamma_1, \gamma_2 \in C^\infty_{\mathrm{lin}}(D)^H;$

P2h. $\{\gamma, q_h^*(\varepsilon)\} := q_h^*(\Psi(\mathfrak{A}(\gamma))(\varepsilon)), \qquad \forall \gamma \in C^\infty_{\mathrm{lin}}(D)^H, \varepsilon \in C^\infty_{\mathrm{lin}}(E) \cong \Gamma(E^*);$

P2v. $\{\gamma, q_v^*(\varepsilon)\} := q_v^*(\Psi(\mathfrak{A}(\gamma))(\varepsilon)), \qquad \forall \gamma \in C^\infty_{\mathrm{lin}}(D)^H, \varepsilon \in C^\infty_{\mathrm{lin}}(E) \cong \Gamma(E^*);$

P3. $\{\gamma, q_h^*(q^E)^*(f)\} := q_h^*(q^E)^*(\rho(\mathfrak{A}(\gamma))(f)), \qquad \forall \gamma \in C^\infty_{\mathrm{lin}}(D)^H, f \in C^\infty(M);$

P4. $\{q_h^*(\varepsilon_1), q_v^*(\varepsilon_2)\} := \{q_v^*(\varepsilon_1), q_h^*(\varepsilon_2)\} := q_h^*(q^E)^*(\langle \varepsilon_1, \varepsilon_2 \rangle), \quad \forall \varepsilon_1, \varepsilon_2 \in C^\infty_{\mathrm{lin}}(E) \cong \Gamma(E^*)$

P5. $\{q_h^*(\varepsilon_1), q_h^*(\varepsilon_2)\} := \{q_v^*(\varepsilon_1), q_v^*(\varepsilon_2)\} := 0, \quad \forall \varepsilon_1, \varepsilon_2 \in C^\infty_{\mathrm{lin}}(E) \cong \Gamma(E^*);$

P6. $\{q_h^*(\varepsilon), q_h^*(q^E)^*(f)\} := \{q_v^*(\varepsilon), q_h^*(q^E)^*(f)\} := 0, \quad \forall \varepsilon \in C^\infty(E), f \in C^\infty(M);$

P7. $\{q_h^*(q^E)^*(f), q_h^*(q^E)^*(g)\} := 0, \qquad f, g \in C^\infty(M).$

That the definitions above are consistent with Leibniz's rule follows from the compatibility conditions *1* and *2* of the definition of an involutive Lie algebroid sequence, Leibniz rule of the Lie brackets and derivation property of the anchor.

Remark 6.11. In order to have the Poisson brackets defined on the whole space of double-linear functions, it remains to define them on double linear functions that correspond to the symmetric part of $E \otimes E$, that is, functions of the type

$$q_h^*(\varepsilon_1) q_v^*(\varepsilon_2) + q_h^*(\varepsilon_2) q_v^*(\varepsilon_1),$$

which we do just by demanding that Leibniz rule must hold. With this, we have defined the Poisson in all kinds of functions that appear in adapted coordinate systems, and thus the brackets are defined in the whole space of functions. Therefore, *it is enough to have the Poisson brackets defined on those double linear functions that are H-invariant.*

Since double linear Poisson brackets are characterized by having bidegree $(-1, -1)$ when they act on bigraded functions (see lemma E.34), we see that Eqs. P1-P7 above do define a double linear Poisson structure on D. We need to check that it is H-invariant, which follows immediately from the observation that equation $h_A = -\mathrm{Id}_E$ (recall that D is self-conjugate) implies, for every $\varepsilon \in C^\infty_{\mathrm{lin}}(E)$ and $e \in E$,

$$H^*(q_h^*(\varepsilon))(0_v(e)) = \varepsilon(q_h \circ H \circ 0_v(e)) = \varepsilon(h_A(e)) = -\varepsilon(e)$$
$$= -q_v^*(\varepsilon)(0_v(e)),$$

from which, $H^*(q_h^*(\varepsilon)) = -q_v^*(\varepsilon)$; and analogously, $h_B = \mathrm{Id}_E$ implies $H^*(q_v^*(\varepsilon)) = q_H^*(\varepsilon)$. Using these equations, and the symmetry of the pairing $\langle\cdot,\cdot\rangle$, we obtain H-invariance of $\{\cdot,\cdot\}$.

Now, given an involutive Lie algebroid morphism (ψ, ψ_1, ψ_2), by Thm. 3.40 we obtain an involutive DVB morphism $(\Phi, \varphi_E, \varphi_M)$, with $\varphi_E = \psi_1$ and $\varphi_M = \psi$. That it is a Poisson map between the corresponding involutive Poisson double vector bundles follows directly from Eqs. (6.5), (6.6) and (6.7), and the way we define the Poisson structures, given by P1-P7. Therefore we have a functor between the two categories. From these equations also follows fully faithfulness of such functor.

Finally, essentially sujectivity follows from the observation that, when we have an H-invariant Poisson structure on an involutive double vector bundle, then the pullback DVB $h_A^*(D)$ used in the proof of Prop. 3.29, inherits the Poisson structure from D, pushed by the isomorphism $\widetilde{\Phi}$. Also we saw in the proof of Prop. 3.29 that $h_A^*(D)$ inherits an involutivity structure $\widetilde{H}$, induced by H through the map $\widetilde{\Phi}$. Then, the $\widetilde{H}$-invariance of the induced Poisson brackets on $h_A^*(D)$ follows easily from the H-invariance of the Poisson brackets on D. Thus, we have found an involutive Poisson double vector bundle that is self-conjugate. The Poisson structure in this case determines, using Eqs. P1-P7 above, an involutive Lie algebroid structure on the corresponding involutive sequence attached to $h_A^*(D)$.

■

6.2.1 Metric *VB*-algebroids

Definition 6.12. A *metric VB-algebroid* is a metric DVB $(D, \langle\cdot,\cdot\rangle_A)$ with a *VB*-algebroid structure on D_B, such that $\sharp : D \longrightarrow D_A^*$ is a *VB*-algebroid isomorphism, where $(D_A^*)_{C^*}$ is endowed with the dual *VB*-algebroid structure given by Cor. E.25.

Remark 6.13. Given a double vector bundle D, the 1:1 correspondence, established in Prop. 3.38, between involutive structures on D and linear metrics on the dual $(D_A^*)_{C^*}$ induces naturally a 1:1 correspondence between involutive Poisson structures on D and metric *VB*-algebroid structures on D_A^*, thus recovering the result of M. Jotz [29] which establishes an equivalence between the category of degree 2 Poisson manifolds and the category of metric *VB*-algebroids. We offer below a direct proof of that correspondence.

Theorem 6.14. *There is a canonical 1:1 correspondence, between degree 2 Poisson manifolds and metric* VB*-algebroids.*

Proof. By theorem 3.40 we already have a canonical 1:1 correspondence between degree 2 manifolds and involutive double vector bundles, which in turn are equivalent to metric double vector bundles, as seen in Prop. 3.38. Let's show that the Poisson brackets on $\mathcal{M}$ canonically induce a *VB*-algebroid structure on $D_{F^*} := D(\widehat{F^*})^*$ such that Φ turns into a VB-algebroid isomorphism, where $\Phi : D_{F^*} \longrightarrow D^*_{F^*}$ is the symmetric isomorphism introduced in Prop 3.38.

We want to endow with a VB-algebroid structure on the double vector bundle

$$\begin{array}{ccc} D_{F^*} & \xrightarrow{q_E} & E \\ {\scriptstyle q_{F^*}}\big\downarrow & & \big\downarrow{\scriptstyle q^E} \\ F^* & \xrightarrow{q^{F^*}} & M, \end{array} \tag{6.15}$$

with core bundle E^*. By remark 2.12, we only need to define the Lie algebroid structure for core and linear sections, and for basic and linear functions. We will use the data

$$\langle \cdot, \cdot \rangle, \qquad ([\cdot, \cdot], \rho), \qquad \Psi, \tag{6.16}$$

given in theorem 6.1. So we define, for $f \in C^\infty(M), \varepsilon, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$ and $\zeta, \zeta_1, \zeta_2 \in \Gamma(\widehat{F}^*)$,

$$\begin{aligned} &\rho_D(\varepsilon)(\widetilde{f}) := 0; \ \rho_D(\varepsilon_1)(\varepsilon_2) = -\langle \partial(\varepsilon_1), \varepsilon_2 \rangle := -\langle \varepsilon_1, \varepsilon_2 \rangle; \\ &\rho_D(\zeta)(\widetilde{f}) := \rho(\zeta)(f); \quad \rho_D(\zeta)(\varepsilon) := \Psi(\zeta)(\varepsilon) =: [\zeta, \varepsilon]_D; \\ &[\varepsilon_1, \varepsilon_2]_D := 0; \quad [\zeta_1, \zeta_2]_D := [\zeta_1, \zeta_2], \end{aligned} \tag{6.17}$$

where we are extending the brackets from $\widetilde{F}^*$ to $\widehat{F^*} = \widetilde{F}^* \oplus S^2E^*$, by Leibniz's rule, and we are using the same notation for sections of E^* that correspond to linear functions on E or that correspond to core sections of D_{F^*}. Analogously as noticed in the proof of Thm. 6.1, the properties satisfied by the data (6.16), imply that we actually get a VB-algebroid structure with the definitions we made in (6.17).

Now, to prove that $\Phi : D_{F^*} \longrightarrow D^*_{F^*}$ is a Lie algebroid isomorphism, amounts to prove that the Lie algebroid structures on $F^* \oplus E \oplus E^*$ induced by Θ from D_{F^*} and by $\widetilde{\Theta}$ from $D^*_{F^*}$, coincide. By Thm. E.21 our task becomes to show that the corresponding representations up to homotopy coincide. By Thm. E.32, this consists in showing

$$\partial^* = \partial; \qquad \nabla^* = \nabla; \qquad -K^* = K. \tag{6.18}$$

Now, by our definition of ∂, given in (6.17), it follows that $\partial = \sharp$, where $\sharp : E^* \longrightarrow E$ is the morphism induced by the bilinear form $\langle \cdot, \cdot \rangle$. Since this form is symmetric, it follows $\partial^* = \partial$.

From Prop. E.7, (E.12) and (6.17), it follows immediately that $\nabla = \nabla^*$.

Finally since, by construction, $\widetilde{F}^* \subset \widehat{F^*}$ is a Lie subalgebroid, and the splitting ψ takes values on $\widetilde{F}^*$ (see Rmk. 3.42 and the proof of Thm. 3.40), it follows that K, defined in (E.15), takes values on $\Lambda^2 E^*$, which implies that

$$-K^* = K.$$

Conversely, if $(D_{F^*}, E; \Phi)$ is an involutive VB-algebroid, then introducing a horizontal lift $\psi : F^* \longrightarrow \widetilde{F}^*$ of the corresponding involutive sequence we obtain decompositions $\Theta, \widetilde{\Theta}$ for D_{F^*} and $D^*_{F^*}$, respectively. Since Φ is a statomorphism and preserves the Lie algebroid structures, it follows that the Lie algebroid structures on $F^* \oplus E \oplus E^*$ induced by Θ and

$\widetilde{\Theta}$ from D_{F^*} and $D^*_{F^*}$ respectively, coincide, which means that (6.18) holds. This implies that if we define

$$\langle \varepsilon_1, \varepsilon_2 \rangle := -\langle \partial(\varepsilon_1), \varepsilon_2 \rangle,$$

we obtain a symmetric bilinear form, where we are using the same symbol $\langle \cdot, \cdot \rangle$ to denote in the left the bilinear form on E^* and on the right the duality pairing between E and E^*.

Since $K = -K^*$, it follows that K takes values on $\Lambda^2 E^*$, which means that the Lie brackets on $\widehat{F^*}$ leave invariant $\widetilde{F}^*$, and since the anchor map ρ is zero on $S^2 E^*$, it follows that actually the Lie algebroid structure $([\cdot,\cdot];\rho)$ on $\widehat{F^*}$ restricts to a Lie subalgebroid structure on $\widetilde{F}^*$.

Since the statomorphism $\Phi : D_{F^*} \longrightarrow D^*_{F^*}$ preserves the Lie algebroid structures, it follows from Eqs. (E.33) and (E.34) (see the proof of Thm. E.32 in the appendix) that

$$\rho_D(\zeta)(\varepsilon) = [\zeta, \varepsilon],$$

where again we are denoting by the same ε a linear function on E (left-hand side) and a core section of D_{F^*} (right-hand side). So we define $\Psi : \widetilde{F}^* \longrightarrow \mathbf{CDO}(E^*)$ by

$$\Psi(\zeta)(\varepsilon) := \rho_D(\zeta)(\varepsilon) = [\zeta, \varepsilon].$$

Therefore we have obtained the data that characterizes a degree -2 Poisson structure on $\mathcal{M}_{E,\widetilde{F}}$, given in Thm. 6.1. It remains to check the compatibility conditions with the inclusion $\Lambda^2 E^* \hookrightarrow \widetilde{F}^*$, given in Eqs. (6.1) and (6.2). They follow immediately from Leibniz's rule.

Clearly, the two processes, one to get an involutive VB-algebroid from degree -2 Poisson brackets and vice versa, are inverses one of the other. So we have obtained the desired canonical 1:1 correspondence.

∎

Proposition 6.15. *Consider a* VB*-algebroid D, so that D^*_A is endowed with the dual VB-algebroid structure (see Thm. E.32 in the appendix). Then,*

$$D \underset{A}{\oplus} D^*_A$$

is a metric VB*-algebroid, with the* VB*-algebroid structure given by Prop. E.22 (App. E).*

Proof. Let's introduce a decomposition for D, which induces a decomposition for D^*_A, and therefore, by Cor. C.44 and Prop. C.7, we obtain a naturally induced decomposition on $D \underset{A}{\oplus} D^*_A$.

By Thm. E.21, Prop. E.20, Thm. E.32 and Prop. E.31 we obtain representations up to homotopy corresponding to the VB-algebroid structures on D and D^*_A, given by

$$\partial + \nabla + K \quad \text{and} \quad \partial^* + \nabla^* - K^*$$

and by Prop. E.22, it follows that the representation up to homotopy corresponding to the VB-algebroid structure on $D \underset{A}{\oplus} D^*_A$ is given by the data

$$(\partial, \partial^*) + (\nabla, \nabla^*) + (K, -K^*), \tag{6.19}$$

which evidently, under a natural identification, satisfies Eq. (6.18). By (the proof of) Thm. 6.14 and Prop. 3.44, it follows that $D \underset{A}{\oplus} D^*_A$ is a metric VB-algebroid.

■

Example. Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid. Consider the *tangent prolongation VB-algebroid*, TA, and the *cotangent VB-algebroid*, T^*A, seen in Ch. F. Then we are in the context of Prop. 6.15, and therefore

$$\begin{array}{ccc} TA \underset{A}{\oplus} T^*A & \xrightarrow{(q_{TM},\pi_{A^*})} & TM \oplus A^* \\ {\scriptstyle q_A}\Big\downarrow & & \Big\downarrow{\scriptstyle q^{TM\oplus A^*}} \\ A & \xrightarrow{q^A} & M \end{array} \tag{6.20}$$

is a metric VB-algebroid.

■

6.3 Classification of regular degree 2 Poisson manifolds

Definition 6.16. We say that a degree 2 Poisson manifold is *regular* when the metric, that is, the symmetric form $\langle\cdot,\cdot\rangle$ has constant rank.

Let's consider in this section a regular, degree 2, Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\})$, and the corresponding pair of vector bundles with their additional structure $(E^*, \langle\cdot,\cdot\rangle)$, $(F^*, \rho, [\cdot,\cdot])$. Fix a splitting of the sequence (6.10), say $\psi : F^* \longrightarrow \widetilde{F}^*$. As we saw in Prop. 6.5, by setting $\nabla_X \varepsilon = \{\psi(X), \varepsilon\}$ we get an F^*-connection on $(E^*, \langle\cdot,\cdot\rangle)$, which preserves $\langle\cdot,\cdot\rangle$, and defining $K(X,Y) := \psi([X,Y]) - \{\psi(X), \psi(Y)\}$, we get (6.8) and $d_\nabla K = 0$.

If we take another splitting ψ', then, as seen in Prop. 6.7, the two splittings are related by $\psi' = \psi - B$, with $B \in \Omega^1(F^*, \Lambda^2 E^*)$, and the corresponding operators ∇' and K' are related to ∇ and K by formulas (6.11) and (6.12). On the other hand, since the metric has constant rank, it follows that ker $\sharp$ and im$\widetilde{\Psi}$ are vector sub-bundles of E^* and $\mathbb{A}_{E^*}$, respectively, where $\mathbb{A}_{E^*}$ is the bundle of covariant differential operators, or infinitesimal automorphisms of E that preserve $\langle\cdot,\cdot\rangle$, that is, the gauge Lie algebroid of $(E^*, \langle\cdot,\cdot\rangle)$, which is well defined –and also comes with a vector bundle structure– since the metric has constant rank. Moreover, im$\widetilde{\Psi}$ is an ideal, in fact, given $\varepsilon_1 \wedge \varepsilon_2 \in \Gamma(\Lambda^2 E^*)$, $\varepsilon \in \Gamma(E^*)$ and $\zeta \in \mathbb{A}_{E^*}$,

$$\begin{aligned} [\zeta, \widetilde{\Psi}(\varepsilon_1 \wedge \varepsilon_2)](\varepsilon) &= \zeta \circ \widetilde{\Psi}(\varepsilon_1 \wedge \varepsilon_2)(\varepsilon) - \widetilde{\Psi}(\varepsilon_1 \wedge \varepsilon_2)(\varepsilon) \circ \zeta(\varepsilon) \\ &= \zeta(\langle \varepsilon_2, \varepsilon\rangle \varepsilon_1 - \langle \varepsilon_1, \varepsilon\rangle \varepsilon_2) - \langle \varepsilon_2, \zeta(\varepsilon)\rangle \varepsilon_1 + \langle \varepsilon_1, \zeta(e)\rangle \varepsilon_2 \\ &= X_\zeta(\langle \varepsilon_2, \varepsilon\rangle)\varepsilon_1 + \langle \varepsilon_2, \varepsilon\rangle \zeta(\varepsilon_1) - X_\zeta(\langle \varepsilon_1, \varepsilon\rangle)\varepsilon_2 - \langle \varepsilon_1, \varepsilon\rangle \zeta(\varepsilon_2) \\ &\quad - X_\zeta(\langle \varepsilon_2, \varepsilon\rangle)\varepsilon_1 + \langle \zeta(\varepsilon_2), \varepsilon\rangle \varepsilon_1 + X_\zeta(\langle \varepsilon_1, \varepsilon\rangle)\varepsilon_2 - \langle \zeta(\varepsilon_1), \varepsilon\rangle \varepsilon_2 \\ &= \widetilde{\Psi}(\zeta(\varepsilon_1) \wedge \varepsilon_2 - \varepsilon_1 \wedge \zeta(\varepsilon_2))(\varepsilon), \end{aligned}$$

where $X_\zeta \in TM$ is the symbol of ζ. So, coker $\widetilde{\Psi} = \mathbb{A}_{E^*}/\mathrm{im}\widetilde{\Psi}$ inherits the Lie brackets from $\mathbb{A}_{E^*}$, and since im$\widetilde{\Psi} \subset$ ker $\rho_{\mathbb{A}_{E^*}}$, it follows that coker $\widetilde{\Psi}$ inherits the whole Lie algebroid structure.

From the formulas (6.12) and (6.2), it follows that ∇ induces a well defined Lie algebroid representation of F^* on coker $\widetilde{\Psi}$, that is, ∇ induces a Lie algebroid morphism

$$\overline{\nabla} : F^* \longrightarrow \operatorname{coker} \widetilde{\Psi}$$

which doesn't depend on the splitting ψ. In turn, coker $\widetilde{\Psi}$ acts on ker $\sharp$, since $\mathbb{A}_{E^*}$ leaves ker $\sharp$ invariant and if $[\zeta] = [\zeta'] \in \operatorname{coker} \widetilde{\Psi}$, then $\zeta' = \zeta + \widetilde{\Psi}(\vartheta)$, with $\vartheta \in \Lambda^2 E^*$, hence, for any $\varepsilon \in \Gamma(\ker \sharp)$ we have

$$\zeta'(\varepsilon) = \zeta(\varepsilon) + \widetilde{\Psi}(\vartheta)(\varepsilon) = \zeta(\varepsilon).$$

Thus $\overline{\nabla}$ gives us a well defined action from F^* on ker $\sharp$. Observe that, through the natural inclusion $\operatorname{End}(E^*) \subset \operatorname{End}(\Lambda^2 E^*)$, we extend ∇ to $\Lambda^2 E^*$, and also the action $\overline{\nabla}$ to an action on $\ker \widetilde{\Psi} = \Lambda^2(\ker \sharp)$, and we have the following classification result.

Theorem 6.17. *Consider the Chevalley cohomology $H^\bullet(F^*; \ker \widetilde{\Psi})$ with differential induced by $\overline{\nabla}$. If it is possible to find $K \in \Omega^2(F^*, \Lambda^2 E^*)$ such that, for some lift ∇ of $\overline{\nabla}$, $\widetilde{\Psi} \circ K = -R_\nabla$ holds, then $d_\nabla K$ takes values on* ker $\widetilde{\Psi}$*, is closed with respect to $d_{\overline{\nabla}}$, and the class $[d_\nabla K] \in H^3(F^*; \ker \widetilde{\Psi})$ is well defined, that is, it doesn't depend on the lift ∇, nor on K. Moreover, (∇, K) gives rise to a Poisson structure if and only if $[d_\nabla K] = 0$. In this case, all the Poisson structures coming from $\overline{\nabla}$ are in 1:1 correspondence with the elements in $H^2(F^*; \ker \widetilde{\Psi})$.*

Proof.
- Choose a lift $\nabla : F^* \longrightarrow \mathbb{A}_{E^*}$ of $\overline{\nabla}$. First let's observe that, using the definition of $\widetilde{\Psi}$ and the fact that ∇ takes values on $\mathbb{A}_{E^*}$,

$$\widetilde{\Psi} \circ \nabla_X K(Y, Z) = \nabla_X \circ \widetilde{\Psi}(K(Y, Z)) - \widetilde{\Psi}(K(Y, Z)) \circ \nabla_X, \tag{6.21}$$

for $X, Y, Z \in \Gamma(F^*)$. To verify this, suppose, for sake of simplicity, that $K(Y, Z) = \varepsilon_1 \wedge \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$, then, for $\varepsilon \in \Gamma(E^*)$,

$$\begin{aligned} \nabla_X K(Y, Z) &= \nabla_X \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_1 \wedge \nabla_X \varepsilon_2 \quad \text{and} \\ \widetilde{\Psi}(K(Y, Z))(\varepsilon) &= \langle \varepsilon_2, \varepsilon \rangle \varepsilon_1 - \langle \varepsilon_1, \varepsilon \rangle \varepsilon_2, \end{aligned}$$

whence

$$\widetilde{\Psi}(\nabla_X K(Y, Z))(\varepsilon) = \langle \varepsilon_2, \varepsilon \rangle \nabla_X \varepsilon_1 - \langle \nabla_X \varepsilon_1, \varepsilon \rangle \varepsilon_2 + \langle \nabla_X \varepsilon_2, \varepsilon \rangle \varepsilon_1 - \langle \varepsilon_1, \varepsilon \rangle \nabla_X \varepsilon_2, \tag{6.22}$$

and on the other hand,

$$\begin{aligned} \nabla_X \circ \widetilde{\Psi}(K(Y, Z))(\varepsilon) =& \langle \nabla_X \varepsilon_2, \varepsilon \rangle \varepsilon_1 + \langle \varepsilon_2, \nabla_X \varepsilon \rangle \varepsilon_1 + \langle \varepsilon_2, \varepsilon \rangle \nabla_X \varepsilon_1 \\ & - \langle \nabla_X \varepsilon_1, \varepsilon \rangle \varepsilon_2 - \langle \varepsilon_1, \nabla_X \varepsilon \rangle \varepsilon_2 - \langle \varepsilon_1, \varepsilon \rangle \nabla_X \varepsilon_2; \\ -\widetilde{\Psi}(K(Y, Z)) \circ \nabla_X(\varepsilon) =& - \langle \varepsilon_2, \nabla_X \varepsilon \rangle \varepsilon_1 + \langle \varepsilon_1, \nabla_X \varepsilon \rangle \varepsilon_2. \end{aligned}$$

Adding both terms and comparing with (6.22) we get (6.21).

Then, for $X, Y, Z \in \Gamma(F^*)$, $\varepsilon \in \Gamma(E^*)$, using the condition $\widetilde{\Psi} \circ K = -R_\nabla$ given in the hypothesis,

$$\begin{aligned}
\widetilde{\Psi} \circ d_\nabla K(X,Y,Z)(\varepsilon) &= \widetilde{\Psi} \circ \left\{ \sum_{cyclic} (\nabla_X K(Y,Z) - K([X,Y],Z)) \right\} (\varepsilon) \\
&= \sum_{cyclic} (\nabla_X(\widetilde{\Psi} \circ K(Y,Z)(\varepsilon)) - \widetilde{\Psi} \circ K(Y,Z)(\nabla_X \varepsilon) - \widetilde{\Psi} \circ K([X,Y],Z))(\varepsilon) \\
&= \sum_{cyclic} (-\nabla_X [\nabla_Y, \nabla_Z] + \nabla_X \nabla_{[Y,Z]} + [\nabla_Y, \nabla_Z]\nabla_X - \nabla_{[Y,Z]}\nabla_X \\
&\qquad\qquad + [\nabla_{[X,Y]}, \nabla_Z] - \nabla_{[[X,Y],Z]})(\varepsilon) \\
&= \sum_{cyclic} (-[\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_X, \nabla_{[Y,Z]}] - [\nabla_Z, \nabla_{[X,Y]}] \\
&\qquad\qquad - \nabla_{[[X,Y],Z]})(\varepsilon) = 0.
\end{aligned}$$

Thus, $(d_\nabla K)(X,Y,Z) \in \ker \widetilde{\Psi}$.

On the other hand, the condition $-R_\nabla = \widetilde{\Psi} \circ K$ implies that the extension of ∇ to $\Lambda^2 E^*$ satisfies, for $\eta \in \Gamma(\Lambda^2 E^*)$,

$$-R_{\widetilde{\nabla}}(X,Y)(\eta) = \widetilde{\nabla}_{[X,Y]}\eta - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y](\eta) = [K(X,Y), \eta],$$

where the brackets $[\cdot,\cdot]$ are the ones given in Rmk. 6.2. Thereby, performing a calculation similar to the one done in lemma E.16, we obtain, for $\eta \in \Omega^k(F^*; \Lambda^2 E^*)$,

$$d^2_{\widetilde{\nabla}}(\eta) = -[K \overset{\wedge}{,} \eta],$$

whence

$$d_\nabla(d_\nabla K) = -[K \overset{\wedge}{,} K] = [K \overset{\wedge}{,} K] = 0,$$

so that $d_\nabla K$ is $(d_{\overline{\nabla}})$-closed. To see that the class $[d_\nabla K]$ is well defined, consider first another lift ∇' of $\overline{\nabla}$, then $\nabla' = \nabla - \widetilde{\Psi} \circ B$ for some $B \in \Omega^1(F^*, \Lambda^2 E^*)$. Take $K' := K - d_\nabla B + \frac{1}{2} B \wedge B$, then

$$\widetilde{\Psi} \circ K' = -R_{\nabla'} \text{ and } d_{\nabla'} K' = d_\nabla K,$$

so that $[d_{\nabla'} K'] = [d_\nabla K]$. Now let ∇ fixed. If K, K' satisfy

$$\widetilde{\Psi} \circ K = \Psi \circ K' = -R_\nabla,$$

then

$$\mu := K - K' \in \Omega^2(F^*, \ker \widetilde{\Psi}), \text{ and}$$
$$d_\nabla K - d_\nabla K' = d_\nabla \mu,$$

hence $[d_\nabla K] = [d_\nabla K']$. Thus, $[d_\nabla K]$ depends only on $\overline{\nabla}$.

- If $\overline{\nabla}$ comes from a Poisson structure, we already saw that a splitting ψ gives us a map ∇ that lifts $\overline{\nabla}$ and K that satisfies $\widetilde{\Psi}\circ K = -R_\nabla$ and $d_\nabla K = 0$. Conversely, suppose that we have a morphism $\overline{\nabla}$ and we can find a lift ∇ and K satisfying $\widetilde{\Psi}\circ K = -R_\nabla$ and $[d_\nabla K] = 0$. Then there exists $\mu \in \Omega^2(F^*, \ker \widetilde{\Psi})$ such that $d_\nabla \mu = d_\nabla K$, but then
$$\widetilde{\Psi}\circ (K-\mu) = \widetilde{\Psi}\circ K \text{ and } d_\nabla(K-\mu) = 0,$$
whence we can define a Poisson structure observing that with the data we already have, the only cases we need to define the Poisson brackets are, for $X, Y \in \Gamma(F^*)$, $\eta, \nu \in \Gamma(\Lambda^2 E^*)$ and $\varepsilon \in \Gamma(E^*)$,
  - $\{X+\eta, \varepsilon\} := \nabla_X(\varepsilon) + \widetilde{\Psi}(\eta)(\varepsilon)$ and
  - $\{X+\eta, Y+\nu\} := ([\eta,\nu] + \nabla_X\nu - \nabla_Y\eta + (K-\mu)(X,Y)) + [X,Y]$

  It is easy to see that the bracket obtained this way, extending it to any pair of functions by linearity, skew-symmetry and Leibniz's rule, satisfies the Jacobi identity and thus defines a Poisson bracket.

- If $\overline{\nabla}$ comes from some Poisson structure, fix a lift ∇. To find all the Poisson structures that induce ∇ we need to find all the maps $K \in \Omega^2(F^*, \Lambda^2 E^*)$ which satisfy $\widetilde{\Psi}\circ K = -R_\nabla$ and $d_\nabla K = 0$. In order to do this, notice that if K and K' are two such maps, then $\mu := K - K'$ takes values on $\ker \widetilde{\Psi}$ and $d_\nabla \mu = 0$, so $[\mu] \in H^2(F^*; \ker \widetilde{\Psi})$. Conversely, if K is a map such that (∇, K) gives rise to a Poisson structure, and we take $[\mu] \in H^2(F^*; \ker \widetilde{\Psi})$, then it follows straightforward that the pair $(\nabla, K-\mu)$ also gives rise to a Poisson structure. Now if the pairs (∇, K) and (∇, K') are equivalent, that is, they are related by a map $B \in \Omega^1(F^*, \Lambda^2 E^*)$, then this map actually takes values on $\ker \widetilde{\Psi}$, since it must leave ∇ unchanged. Then $K' = K - d_\nabla B + \frac{1}{2} B\wedge B = K - d_\nabla B$, so that in this case $[\mu] = [d_\nabla B] = 0$. So we obtain a linear map from the space of classes of equivalences of pairs (∇, K), with ∇ fixed, to $H^2(F^*; \ker \widetilde{\Psi})$ which is well defined and injective, thus it is a 1:1 correspondence. (Actually we had to choose a class $[(\nabla, K)]$ to correspond to $0 \in H^2(F^*; \ker \widetilde{\Psi})$. What we get without making choices is that the set of classes of pairs (∇, K) is an affine space modeled on $H^2(F^*; \ker \widetilde{\Psi})$.)

■

Corollary 6.18. *Given a degree 2 manifold, there is a canonical 1:1 correspondence between degree -2 Poisson brackets with non-degenerate metric and the following data:*

- *A non-degenerate metric $\langle\cdot,\cdot\rangle$ on E^*,*
- *A Lie algebroid structure $([\cdot,\cdot], \rho)$ on F^*.*

Proof. Since $\langle\cdot,\cdot\rangle$ is non-degenerate, we can use this metric to get the identification $\Lambda^2 E^* \cong \mathfrak{so}(E^*)$. Under this identification we have $\widetilde{\Psi} = \mathrm{Id}$, in particular, $\ker \widetilde{\Psi} = \operatorname{coker} \widetilde{\Psi} = 0$. Therefore, the only possible Lie algebroid morphism
$$\overline{\nabla} : F^* \longrightarrow \operatorname{coker} \widetilde{\Psi}$$

is the zero map, and any $\langle\cdot,\cdot\rangle$-preserving connection ∇ on E^* is a lift of $0 = \overline{\nabla}$. Also, $K := -R_\nabla$ satisfies the condition of the theorem. Since in this case the Chevalley cohomology $H^\bullet(F^*; \ker \widetilde{\Psi})$ is zero, the theorem tells us that there is one, and only one Poisson structure comming from $(E^*, \langle\cdot,\cdot\rangle)$ and $(F^*, \rho, [\cdot,\cdot])$. ■

6.4 Example: The symplectic case

In this section we show how the theory developed applies to the well-known symplectic case, studied by D. Roytenberg [59], shedding new light on the known results.

6.4.1 The symplectic structure

Definition 6.19. A *symplectic degree 2 manifold* is a degree 2 Poisson manifold, $(\mathcal{M}, \{\cdot,\cdot\})$, such that the Poisson brackets are non-degenerate.

The next characterization of symplectic degree 2 manifolds is sketched in [59]. We prove it here in full detail.

Proposition 6.20. *There is a canonical 1:1 correspondence between symplectic degree 2 manifolds and pseudo-euclidean vector bundles $(E^*, \langle\cdot,\cdot\rangle)$, that is, vector bundles E^* endowed with a non-degenerate metric on the fibers $\langle\cdot,\cdot\rangle$.*

Remark 6.21. Actually, in [59] the correspondence is with E, which causes no harm in this case, since the metric is non-degenerate, so that we can identify E with E^*. In the general, degenerate case, we only get a metric on the bundle that corresponds to the degree 1 functions, which is E^* according to the usual conventions. In order to keep coherence with the rest of the present work, we prefer to remain with E^*, instead of switching to E through the metric.

Proof. To say that the brackets $\{\cdot,\cdot\}$ are non-degenerate is equivalent to the condition, in local coordinates $\{x^i, \varepsilon^\mu, \alpha^\nu\}$, that the matrix

$$\begin{pmatrix} \{x^i, x^j\}(x) & \{\varepsilon^\mu, x^j\}(x) & \{\alpha^\nu, x^j\}(x) \\ \{x^i, \varepsilon^\lambda\}(x) & \{\varepsilon^\mu, \varepsilon^\lambda\}(x) & \{\alpha^\nu, \varepsilon^\lambda\}(x) \\ \{x^i, \alpha^\kappa\}(x) & \{\varepsilon^\mu, \alpha^\kappa\}(x) & \{\alpha^\nu, \alpha^\kappa\}(x) \end{pmatrix}, \tag{6.23}$$

is invertible, where, for a function $f \in \mathcal{O}_M$, $f(x)$ denotes the evaluation of the degree zero component of $f|_x \in \mathcal{O}|_x$ at x (cf. [68]). Checking the degrees of the components of this matrix, we see that it is invertible if and only if

$$\det(\{\alpha^\nu, x^j\}(x)) \neq 0 \quad \text{and} \quad \det(\{\varepsilon^\mu, \varepsilon^\lambda\}(x)) \neq 0, \tag{6.24}$$

which is equivalent to the conditions

- $\rho : F^* \longrightarrow TM$ is an isomorphism,
- $\langle\cdot,\cdot\rangle$ is non-degenerate.

We saw in Cor. 6.18 that there is a canonical 1:1 correspondence between the degree 2 Poisson manifolds with $\langle\cdot,\cdot\rangle$ non-degenerate and

- A Lie algebroid $(F^*, [\cdot,\cdot], \rho)$,
- A pseudo-euclidean vector bundle $(E^*, \langle\cdot,\cdot\rangle)$.

Since ρ preseves the Lie brackets, it follows that, when $\{\cdot,\cdot\}$ are non-degenerate, $(F^*, [\cdot,\cdot], \rho)$ is canonically isomorphic to $(TM, [\cdot,\cdot], \mathrm{Id})$. In this way we obtain a canonical 1:1 correspondence between symplectic degree 2 manifolds $(\mathcal{M}, \{\cdot,\cdot\})$ and pseudo-euclidean vector bundles $(E^*, \langle\cdot,\cdot\rangle)$. ∎

The arguments in the preceding proof can be generalized to obtain the following conclusion.

Corollary 6.22. *If a graded n-manifold $\mathcal{M}$ admits a symplectic structure ω homogeneous of degree k, then necessarily, $n = k$. Moreover, if the dimension of $\mathcal{M}$ is $(p|q_1|\dots|q_k)$, then*

$$p = q_k,\ q_1 = q_{k-1}, \dots, q_i = q_{k-i}.$$

Proof. When we calculate the corresponding matrix, in a similar way as we did in Eq. (6.23), we see that this matrix will be invertible only if there are coordinate functions of degree k, α^ν, in the same quantity as there are in degree 0, x^i, and

$$\det(\{\alpha^\nu, x^i\}) \neq 0.$$

If there were coordinate functions of degree $l > k$, β^r, then $\{\beta^r, \gamma^s\}$ will be an homogeneous function of degree ≥ 1, for any other coordinate function γ^s, since we admit only non-negative degrees. Then the evaluation of the degree zero component of $\{\beta^r, \gamma^s\}(x)$ at x will be zero, but this implies that the r-column of the matrix of ω is zero, preventing it from being invertible, a contradiction.

Now, the above argument also shows that the only non-zero entries in the matrix of ω are the ones of the form

$$\{\theta_i^r, \theta_{k-i}^s\}(x), \tag{6.25}$$

thereby, for fixed i, the entries in (6.25) must form an invertible sub-matrix, in particular it must be square, which implies that $q_i = q_{k-i}$. ∎

Proposition 6.23. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a symplectic degree 2 manifold. Then its associated sequence (6.10):*

$$0 \longrightarrow \Lambda^2 E^* \xrightarrow{\iota} \widetilde{F}^* \xrightarrow{\pi} F^* \longrightarrow 0 \tag{6.26}$$

is canonically isomorphic, as a Lie algebroid sequence, to the Atiyah sequence corresponding to the pseudo-euclidean vector bundle $(E^*, \langle\cdot,\cdot\rangle)$:

$$0 \longrightarrow \mathfrak{so}(E^*) \xrightarrow{\iota} \mathbb{A}_{E^*} \xrightarrow{\pi} TM \longrightarrow 0. \tag{6.27}$$

Proof. By Thm. 6.1, we obtain the following morphism of Lie algebroid sequences:

$$\begin{tikzcd} \Lambda^2 E^* \arrow[r, "\iota"] \arrow[d, "\widetilde{\Psi}"] & \widetilde{F}^* \arrow[r, "\pi"] \arrow[d, "\Psi"] & F^* \arrow[d, "\rho"] \\ \mathfrak{so}(E^*) \arrow[r, "\iota"] & \mathbb{A}_{E^*} \arrow[r, "\pi"] & TM \end{tikzcd}, \tag{6.28}$$

By Prop. 6.20, ρ is an isomorphism and $\langle\cdot,\cdot\rangle$ is non-degenerate, thereby $\widetilde{\widetilde{\Psi}}$ is an isomorphism too. Hence, by the exactness of the sequences and the commutativity of the diagrams in Eq. (6.28), it follows that Ψ is an isomorphism.

$\blacksquare$

Remark 6.24. The map $\widehat{\Psi} : \widehat{F^*} \longrightarrow \mathbf{CDO}(E^*)$ given by

$$\widehat{\Psi}(\zeta+\lambda) = \Psi(\zeta) + (\sharp \otimes \mathrm{Id}_{E^*})(\lambda),$$

is an isomorphism, where $\zeta \in \widetilde{F}^*, \lambda \in S^2E^*$.

Proposition 6.25. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a symplectic degree 2 manifold with associated exact sequence (6.27). Then the corresponding involutive double vector bundle $D\left(\widehat{F^*}^*\right)$, given by Thm. 3.40 is canonically isomorphic to*

$$\begin{tikzcd} \flat^*(T^*E) \arrow[r, "q_2"] \arrow[d, "q_1"'] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M, \end{tikzcd} \tag{6.29}$$

where $\flat : E \longrightarrow E^*$ is the inverse of the isomorphsim $\sharp : E^* \longrightarrow E$ given by the non-degenerate metric $\langle\cdot,\cdot\rangle$, and

$$\flat^*(T^*E) = E \underset{(\flat, E^*, q_{E^*})}{\times} T^*E := \{(\varepsilon, v) \in E \times T^*E | \flat(\varepsilon) = q_{E^*}(v)\} \subset E \times T^*E$$

is the pull-back bundle, so that we have the following diagram

$$\begin{tikzcd} \flat^*(T^*E) \arrow[r, "p_2"] \arrow[d, "p_1"] & T^*E \arrow[r, "q_E"] \arrow[d, "q_{E^*}"] & E \arrow[d, "q^E"] \\ E \arrow[r, "\flat"] & E^* \arrow[r, "q^{E^*}"] & M \end{tikzcd}, \tag{6.30}$$

and the maps q_1 and q_2 of diagram (6.29) are given p_1 and $q_E \circ p_2$, respectively.

The isomorphism

$$\begin{tikzcd} D\left(\widehat{F^*}^*\right) \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "F^*"] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M \end{tikzcd} \xrightarrow{\quad\Theta\quad} \begin{tikzcd} \flat^*(T^*E) \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "T^*M"] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M \end{tikzcd} \tag{6.31}$$

is the identity on each side bundle, and the induced map on the core θ_{T^*M} is given by $(\rho^{-1})^*$, where $\rho : F^* \longrightarrow TM$ is the anchor map.

Proof. We have

$$\widehat{F^*}^* = \widetilde{F} \oplus S^2(E) \quad \text{and} \quad \mathbf{CDO}(E^*)^* = \mathbb{A}^*_{E^*} \oplus \mathbf{sym}(E),$$

where

$$\mathbf{sym}(E) := \{\vartheta \in \mathfrak{gl}(E) | \vartheta^t = \vartheta\},$$

where $\vartheta^t : E \longrightarrow E$ is the adjoint with respect to the metric $\langle \cdot, \cdot \rangle$:

$$\vartheta^t := \sharp \circ \vartheta^* \circ \flat.$$

Define $\Phi : \widehat{F^*}^* \longrightarrow \mathbf{CDO}(E^*)^*$ by

$$\Phi(\omega + \lambda) := (\Psi^{-1})^*(\omega) + (\flat \otimes \mathrm{Id}_E)(\lambda), \tag{6.32}$$

for $\omega \in \widetilde{F}^*$ and $\lambda \in S^2(E)$. Then it is easy to see that the diagram

$$\begin{tikzcd} \widehat{F^*}^* \arrow[r, "p"] \arrow[d, "\Phi"] & E \otimes E \arrow[d, "\flat \otimes \mathrm{Id}_E"] \\ \mathbf{CDO}(E^*)^* \arrow[r, "p"] & E \otimes E \end{tikzcd} \tag{6.33}$$

commutes, so that $(\Phi; \flat, \mathrm{Id}_E; \mathrm{Id}_M)$ is a *DVS* morphism (see Def. D.2). Note that $\varphi_{F^*} := \Phi|_{F^*} : F^* \longrightarrow T^*M$ is given by

$$\varphi_{F^*} = (\rho^*)^{-1}. \tag{6.34}$$

Consider the involutive double vector bundle

$$\begin{tikzcd} D\left(\widehat{F^*}^*\right) \arrow[r, "q_2"] \arrow[d, "q_1"'] & E \arrow[d, "q^E"] \\ E \arrow[r, "q^E"] & M, \end{tikzcd}$$

given in the proof of Thm. 3.40, and the double vector bundle

$$\begin{tikzcd} T^*E \arrow[r, "q_E"] \arrow[d, "q_{E^*}"'] & E \arrow[d, "q^E"] \\ E^* \arrow[r, "q^{E^*}"] & M, \end{tikzcd}$$

introduced in Eq. (F.2), but here E is playing the role of A and we are using the notation q_E instead of π_E and q_{E^*} instead of π_{E^*}. Then, by Prop. D.5, we obtain a morphism of double vector bundles

$$(D(\Phi); \flat, \mathrm{Id}_E; \mathrm{Id}_M) : D\left(\widehat{F^*}^*\right) \longrightarrow T^*E,$$

given by

$$D(\Phi)(\omega, e_1, e_2) = ((\Psi^{-1})^*(\chi) + (\flat \otimes \mathrm{Id}_E)(\lambda), \flat(e_1), e_2), \tag{6.35}$$

where $\omega = \chi + \lambda \in \widehat{F^*}^* = \widetilde{F} \oplus S^2E$, and we are using

$$T^*(E) \cong D(\mathbf{CDO}(E^*)^*), \tag{6.36}$$

identification that follows from propositions D.6, C.32 together with Eq. (F.6) and the comments following it (or instead, we can use directly Eq. (F.5) with E playing the role of A).

Since $\flat, \mathrm{Id}_E$ and $(\rho^{-1})^*$ are isomorphisms, it follows that $D(\Phi)$ is an isomorphism.

By Prop. B.16 it follows that $\flat^*(T^*E)$ is a double vector bundle (see Eqs. (6.29) and (6.30)) and that

$$(p_2; \flat, \mathrm{Id}_E; \mathrm{Id}_M) : (\flat^*(T^*E); E, E; M) \longrightarrow (T^*E; E^*, E; M)$$

is a morphism of double vector bundles.

Now, since $q_{E^*} \circ D(\Phi) = \flat \circ q_1$, it follows, by the universal property of the pull-back bundle, that there exists a unique vector bundle map $(q_1, D(\Phi))$, such that the following diagram commutes

$$\begin{tikzcd} D\left(\widehat{F^*}^*\right) \arrow[drr, bend left, "D(\Phi)"] \arrow[dr, dotted, "{(q_1, D(\Phi))}"] \arrow[ddr, bend right, "q_1"'] & & \\ & \flat^*(T^*E) \arrow[r, "p_2"] \arrow[d, "p_1"] & T^*E \arrow[d, "q_{E^*}"] \\ & E \arrow[r, "\flat"] & E^*. \end{tikzcd} \tag{6.37}$$

We define $\Theta := (q_1, D(\phi))$ and claim that this is a DVB isomorphism. Indeed, we saw that p_2 and $D(\Phi)$ are double vector bundle isomorphisms, whence,

$$\Theta = D(\Phi)^{-1} \circ p_2 \tag{6.38}$$

is a double vector bundle isomorphism, and moreover, since $D(\Phi)_2 = \mathrm{Id}_E$ it follows that Θ is the identity on the second side bundle. By the diagram (6.37) above it follows that Θ is also the identity on the first side bundle. Finally, since $p_2|_{T^*M} = \mathrm{Id}_{T^*M}$, it follows from (6.38) and (6.34) that

$$\Theta_{F^*} = (D(\Phi)^{-1} \circ p_2)|_{T^*M} = \varphi_{F^*}^{-1} \circ \mathrm{Id}_{T^*M} = (\rho^{-1})^*.$$

■

Corollary 6.26. *The map $p_2 : \flat^*(T^*E) \longrightarrow T^*E$ is a double vector bundle isomorphism. If we denote by $(p_2)_1, (p_2)_2$ the induced maps on the side bundles, and by $(p_2)_{T^*M}$ the induced map between the core bundles, we have*

$$(p_2)_1 = \flat; \qquad (p_2)_2 = \mathrm{Id}_E; \qquad (p_2)_{T^*M} = \mathrm{Id}_{T^*M}.$$

In particular, if we introduce a decomposition

$$\Theta = (q_{E^*}, q_E, q_{T^*M}) : T^*E \longrightarrow E^* \oplus E \oplus T^*M,$$

then we obtain an induced decomposition

$$\Theta' = (p_1, p_2, q_{T^*M}) : \flat^*(T^*E) \longrightarrow E \oplus E \oplus T^*M,$$

and if $(x^i, \mathbf{e}^s, \varepsilon^r, \alpha^t)$ is an adapted coordinate system for T^*E, then $(x^i, \flat^*(\mathbf{e}^s), \varepsilon^r, \alpha^t)$ is an adapted coordinate system for $\flat^*(T^*E)$.

Proof. The first part of the corollary is already done in the proof of Prop. 6.25. The statement about the adapted coordinate system follows immediately from the first part of the corollary, and from the proof of Cor. A.20.

■

Now we want to understand the metric VB-algebroid structure on $D\left(\widehat{F^*}^*\right)^* = D_{F^*}$ –or equivalently on $\flat^*(T^*E)^*$– which corresponds to the symplectic structure on $\mathcal{M}$, according to Thm. 6.14. The VB-algebroid structure on D_{F^*} is equivalent to the corresponding Poisson structure on $D = D\left(\widehat{F^*}^*\right)$ (see Prop. E.24, where the correspondence between both structures is worked out explicitly).

Proposition 6.27. *The double-linear Poisson structure on $D\left(\widehat{F^*}^*\right)$ corresponding to the metric* VB*-algebroid structure on $D_{F^*} = D\left(\widehat{F^*}^*\right)^*$ (see Def. E.23 and Prop. E.24), is the pull-back of the canonical Poisson structure on T^*E by the isomorphism*

$$D(\Phi) : D\left(\widehat{F^*}^*\right) \xrightarrow{\cong} T^*E,$$

given in Eq. (6.35).

Proof. We know that the Poisson structure is completely determined by the action of the brackets on functions $f \in C^\infty(M)$, $\varepsilon \in \Gamma(E) \cong C^\infty_{\text{lin}}(E)$ and on functions that correspond to core and linear sections of D_{F^*}.

Now, by Eq. (6.36), Prop. C.32 and Prop. D.6, it follows that the double-linear functions on $D\left(\widehat{F^*}^*\right)$ are identified with the sections of $\mathbf{CDO}(E^*)$, and by Prop. 6.23, it follows that, for double-linear functions $\nu_1, \nu_2 \in C^\infty_{\text{lin}}\left(D\left(\widehat{F^*}^*\right)\right) \cong \Gamma(\widehat{F^*})$, we have $\widehat{\Psi}(\nu_i) \in \Gamma(\mathbf{CDO}(E^*)) \cong C^\infty_{\text{lin}}(T^*E)$ (see Rmk. 6.24) and

$$\{\nu_1, \nu_2\} = \widehat{\Psi}^{-1}[\widehat{\Psi}(\nu_1), \widehat{\Psi}(\nu_2)] = \widehat{\Psi}^{-1}\{\widehat{\Psi}(\nu_1), \widehat{\Psi}(\nu_2)\}, \tag{6.39}$$

therefore, from (6.35) and Rmk. 6.24,

$$\{\nu_1, \nu_2\} = D(\Phi)^*\{(D(\Phi)^{-1})^*(\nu_1), (D(\Phi)^{-1})^*(\nu_2)\}.$$

Next, for a double-linear function $\nu \in C^\infty_{\text{lin}}\left(D\left(\widehat{F^*}^*\right)\right)$ and a basic function $\widetilde{f} = (q^E \circ q_1)^* f$, with $f \in C^\infty(M)$, we have, using (6.28),

$$\begin{aligned}\{\nu, \widetilde{f}\} &= (q^E \circ q_1)^* \rho(\pi(\nu))(f) = (q^E \circ q_E \circ D(\Phi))^*(\pi(\widehat{\Psi}(\nu))(f)) \\ &= D(\Phi)^*(q^E \circ q_E)^*\{\widehat{\Psi}(\nu), (q^E \circ q_E)^* f\},\end{aligned}$$

whence

$$\{\nu, \widetilde{f}\} = D(\Phi)^*\{(D(\Phi)^{-1})^*\nu, (D(\Phi)^{-1})^*\widetilde{f}\}.$$

Next consider a double-linear function ν and a linear function $\varepsilon \in C^\infty_{\text{lin}}(E) \cong \Gamma(E^*)$. We have the following

$$\{\nu, (q_2)^*\varepsilon\} = (q_2)^*\widehat{\Psi}(\nu)(\varepsilon) = D(\Phi)^*(q_E)^*\{\widehat{\Psi}(\nu), (q_E)^*\varepsilon\},$$

whence

$$\{\nu, (q_2)^*\varepsilon\} = D(\Phi)^*\{(D(\Phi)^{-1})^*\nu, (D(\Phi)^{-1})^*(q_2)^*\varepsilon\}.$$

Now, for $e \in \Gamma(E) \cong \Gamma_{\text{core}}TE$ and for $\eta \in \Gamma(\mathbf{CDO}(E^*)) \cong \Gamma_{\text{lin}}TE$, observe that $[\eta, e] \in \Gamma_{\text{core}}TE^*$ and, for $\varepsilon \in \Gamma(E^*) \cong C^\infty_{\text{lin}}(E)$,

$$\begin{aligned}[\eta, e](\varepsilon) &= \eta(e(\varepsilon)) - e(\eta(\varepsilon)) \\ &= (\pi \circ \eta)(\langle e, \varepsilon\rangle) - \langle e, \eta(\varepsilon)\rangle \\ &= \langle \eta^*(e), \varepsilon\rangle = \eta^*(e)(\varepsilon),\end{aligned}$$

where $\eta^* \in \mathbf{CDO}(E^*)$ was given in Eq. (F.7). Hence,

$$[\eta, e] = \eta^*(e). \tag{6.40}$$

On the other hand,

Therefore, for $\nu \in \Gamma(\widetilde{F^*}) \subset \Gamma(\widehat{F^*}) \cong C^\infty_{\text{lin}}\left(D\left(\widehat{F^*}^*\right)\right)$, we have $\Psi(\nu) \in \Gamma(\mathbb{A}_{E^*})$, whence, for $\varepsilon \in \Gamma(E^*)$ we have $\sharp \circ \Psi(\nu)(\varepsilon) = \Psi(\nu)^*(\sharp(\varepsilon))$ (this is equivalent to the compatibility of the operator $\Psi(\nu) \in \Gamma(\mathbb{A}_{E^*})$ with the metric $\langle\cdot,\cdot\rangle$ on E^*), whence $\Psi(\nu)(\varepsilon) = \flat(\Psi(\nu)^*(\sharp\varepsilon))$ and thereby, taking into account that $q_1^* = D(\Phi)^* \circ q_E^* \circ \sharp$,

$$\begin{aligned}\{\nu, (q_1)^*\varepsilon\} &= (q_1)^*[\nu, \varepsilon] = (q_1)^*\Psi(\nu)(\varepsilon) \\ &= D(\Phi)^* \circ (q_E)^* \circ \sharp \circ \flat(\Psi(\nu)^*(\sharp(\varepsilon))) \\ &= D(\Phi)^* \circ (q_E)^*(\Psi(\nu)^*(\sharp(\varepsilon))) \\ &= D(\Phi)^* \circ (q_E)^*[\Psi(\nu), \sharp(\varepsilon)] \\ &= D(\Phi)^*\{\Psi(\nu), (q_E)^*\sharp(\varepsilon)\} \\ &= D(\Phi)^*\{(D(\Phi)^{-1})^*\nu, (D(\Phi)^{-1})^*(q_1)^*\varepsilon\}.\end{aligned}$$

This holds when $\nu \in \Gamma(\widetilde{F^*}) \subset \Gamma(\widehat{F^*})$. Now, when $\nu = \lambda \in \Gamma(S^2(E^*)) \subset \Gamma(\widehat{F^*})$, we have on one hand, from Leibniz's rule

$$\{\nu, (q_1)^*\varepsilon\} = -(q_1)^*\lambda(\sharp(\varepsilon)).$$

On the other hand, $(D(\Phi)^{-1})^*(\lambda) = \lambda \circ \sharp$ and taking into account Eq. (6.40), it follows

$$\{(D(\Phi)^{-1})^*\lambda, (D(\Phi)^{-1})^*(q_1)^*\varepsilon\} = (q_E)^*[\lambda \circ \sharp, \sharp(\varepsilon)] = (q_E)^*(-\sharp \circ \lambda)(\sharp(\varepsilon)),$$

thereby

$$\begin{aligned}D(\Phi)^*\{(D(\Phi)^{-1})^*\lambda, (D(\Phi)^{-1})^*(q_1)^*\varepsilon\} &= -D(\Phi)^* \circ (q_E)^* \circ \sharp \circ \lambda(\sharp(\varepsilon)) \\ &= -(q_1)^*\lambda(\sharp(\varepsilon)) = \{\lambda, (q_1)^*\varepsilon\}.\end{aligned}$$

Finally, for $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*) \cong C^\infty_{\text{lin}}(E)$,

$$\begin{aligned}\{(q_1)^*\varepsilon_1, (q_2)^*\varepsilon_2\} &= (q^{E^*} \circ q_1)^*\{\varepsilon_1, \varepsilon_2\} = (q^{E^*} \circ q_1)^*\langle\varepsilon_1, \varepsilon_2\rangle \\ &= D(\Phi)^* \circ (q_E)^* \circ (q^E)^*\langle\sharp\varepsilon_1, \varepsilon_2\rangle \\ &= D(\Phi)^*\{(q_E)^*\sharp^*\varepsilon_1, (q_{E^*})^*\varepsilon_2\} \\ &= D(\Phi)^*\{(D(\Phi)^{-1})^*(q_1)^*\varepsilon_1, (D(\Phi)^{-1})^*(q_2)^*_2\}.\end{aligned}$$

■

Below we obtain as a corollary the important fact that we always can find *Darboux* coordinates in a symplectic degree 2 manifold. This result is mentioned, and used, in [59], but it is not proven there.

Corollary 6.28. *Consider the canonical Darboux coordinates on T^*E, $(x^i, \varepsilon_a, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \varepsilon_a})$, which satisfy*

$$\left\{x^i, \frac{\partial}{\partial x^j}\right\} = \delta^i_j, \qquad \left\{\varepsilon_a, \frac{\partial}{\partial \varepsilon_b}\right\} = \delta^b_a,$$

and the Poisson brackets of any other pair of coordinate functions is zero. If we choose the ε_a's to be an orthonormal frame of E^* at each point, then this coordinate system induces a Darboux coordinate system on $D(\widehat{F^*})$ by pull-back under the isomorphism $D(\Phi)$, $(q^i, \varepsilon_a, p_i, \xi^a)$, which satisfies

$$\{q^i, p_j\} = \delta^i_j, \qquad \{\xi^a, \overline{\varepsilon}^b\} = \pm 1,$$

and the other brackets are zero. In particular, $(q^i, \varepsilon_a, p_i)$ is a coordinate system for $\mathcal{M}$ that satisfies

$$\{q^i, p_j\} = \delta^i_j, \qquad \{\varepsilon_a, \varepsilon_b\} = \pm 1,$$

and the other brackets are zero.

Proof. It is a direct consequence of Cor. 6.26 and Prop. 6.27.

■

6.4.2 Integrable, symplectic vector fields

We want to describe an integrable 1-vector field Q on a symplectic degree 2 manifold that is compatible with the symplectic form in the following sense.

Definition 6.29. A vector field Q on a symplectic manifold $(\mathcal{M}, \omega)$ is called *symplectic* if

$$\mathcal{L}_Q \omega = 0.$$

It can be shown, by the same calculations as in the classical case (but this time taking care of the corresponding degrees) that Q is symplectic if and only if

$$Q(\{f, g\}) = \{Q(f), g\} + (-1)^{|f|}\{f, Q(g)\}. \tag{6.41}$$

An alternative way to characterize a homogeneous vector field of degree k (see Rmk. 4.2) is the condition

$$[\epsilon, Q] = kQ,$$

where ϵ is the Euler vector field $\epsilon(f) := |f|f$ (see [59]).

Analogously, we can characterize a homogeneous k-form $\omega \in \Omega^k(\mathcal{M})$ by the condition

$$\mathcal{L}_\epsilon \omega = k\omega,$$

thus yielding an alternative way to characterize the homogeneity of a symplectic 2-form, ω, without mentioning the corresponding Poisson brackets:

$$\mathcal{L}_\epsilon \omega = 2\omega.$$

A consequence of this is that a symplectic 2-form is automatically exact, for

$$2\omega = \mathcal{L}_\epsilon \omega = d\iota_\epsilon \omega + \iota_\epsilon d\omega = d\iota_\epsilon \omega,$$

whence

$$\omega = d(\frac{1}{2}\iota_\epsilon \omega).$$

Next proposition shows that when a 1-vector field is symplectic, then it comes from a degree 3 hamiltonian, that is,

$$Q = \{\theta, \cdot\},$$

and moreover, Q is integrable, that is, $Q^2 = 0$ if and only if θ is integrable with respect to the symplectic structure, that is, $\{\theta, \theta\} = 0$. This result belongs to D. Roytenberg [59]. In Rmk. 6.33 we will show this result from a different perspective.

Proposition 6.30 ([59])**.** *There is a 1:1 canonical correspondence between integrable, symplectic 1-vector fields Q, and integrable 3-hamiltonians, that is, homogeneous degree 3 functions θ, satisfying*

$$\{\theta, \theta\} = 0.$$

Proof. If Q is a symplectic 1-vector field, we have, on one hand,

$$0 = \mathcal{L}_Q \omega = \iota_Q d\omega + d\iota_Q \omega = d\iota_Q \omega.$$

On the other hand, since $[\epsilon, Q] = Q$, we have

$$\begin{aligned}\iota_Q \omega = \iota_{[\epsilon,Q]}\omega &= [\mathcal{L}_\epsilon, \iota_Q](\omega) \\ &= \mathcal{L}_\epsilon \iota_Q \omega - \iota_Q \mathcal{L}_\epsilon \omega = \mathcal{L}_\epsilon \iota_Q \omega - 2\iota_Q \omega,\end{aligned}$$

whence

$$3\iota_Q \omega = \mathcal{L}_\epsilon \iota_Q \omega = d\iota_\epsilon \iota_Q \omega + \iota_\epsilon d\iota_Q \omega = d\iota_\epsilon \iota_Q \omega.$$

Thereby,

$$\iota_Q \omega = d(\frac{1}{3}\iota_\epsilon \iota_Q \omega),$$

that is, Q is hamiltonian: $Q = X_\theta$, with $\theta = \frac{1}{3}\omega(Q, \epsilon)$.

Observe that, in particular, it follows that θ is homogeneous of degree 3 and $Q = \{\theta, \cdot\}$.

Since, for any $f \in \mathcal{O}_M$,

$$\{\theta, \{\theta, f\}\} = \{\{\theta, \theta\}, f\} - \{\theta, \{\theta, f\}\},$$

it follows that $Q^2 = X_{\frac{1}{2}\{\theta,\theta\}}$, therefore, taking the non-degeneracy of $\{\cdot, \cdot\}$ into account, Q is integrable if and only if $\{\theta, \theta\} = 0$.

It is immediate that given a degree 3 function θ, with $\{\theta, \theta\} = 0$, we obtain an integrable, symplectic 1-vector field

$$Q := \{\theta, \cdot\},$$

and the two processes we described $Q \rightsquigarrow \theta$ and $\theta \rightsquigarrow Q$ are inverses one of the other. ■

A consequence of Prop. 6.27, is that the space $\Gamma(\widehat{E}) \cong \Gamma_{\text{lin}}\left(D\left(\widehat{F^*}^*\right)\right)$ is $C^\infty(M)$-spanned by the sections of the form $d\varepsilon$ (observe that sections of the form $\varepsilon \otimes df$ can be written as $d(f\varepsilon) - fd\varepsilon$), and the space $\Gamma(F) \cong \Gamma_{\text{core}}\left(D\left(\widehat{F^*}^*\right)\right)$ is $C^\infty(M)$-spanned by the sections of the form df. Here $\varepsilon \in \Gamma(E^*) \cong C^\infty_{\text{lin}}(E)$ and $f \in C^\infty(M)$, and d is the differential dual to the Lie algebroid structure on $D_{F^*} = D\left(\widehat{F^*}^*\right)^*$. With this in mind, given a degree 3 function on $\mathcal{M}$, θ, it is possible to express the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ that characterize θ (see Thm. 3.59) in terms of the Poisson brackets $\{\cdot,\cdot\}$. This is the content of the following result.

Proposition 6.31. *Given a degree 3 function on $\mathcal{M}$, $\theta \in \mathcal{A}^3$. Consider the corresponding pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, which characterizes θ, given by Thm. 3.59. Then, for any $f \in C^\infty(M)$, $\varepsilon \in \Gamma(E^*) \cong C^\infty_{\text{lin}}(E)$,*

$$\theta_1^\sharp(df) = -\{\theta, f\}, \qquad \theta_2^\sharp(d\varepsilon) = \{\theta, \varepsilon\}. \tag{6.42}$$

In particular, θ is completely determined by the quantities

$$\{\{\theta, f\}, \varepsilon\} \quad \text{and} \quad \{\{\theta, \varepsilon_1\}, \varepsilon_2\}, \qquad \forall f \in C^\infty(M), \varepsilon, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*). \tag{6.43}$$

Proof. If we introduce a splitting we have

$$\theta = \theta_1 + \theta_2,$$

with $\theta_1 \in \Gamma(E^* \otimes F^*)$ and $\theta_2 \in \Gamma(\Lambda^3 E^*)$. Because Eqs. (6.42) behave well with respect to linear operations, we don't loose generality if we prove them for

$$\theta = \theta_1 + \theta_2 = \varepsilon_0 \otimes \zeta + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3. \tag{6.44}$$

Then,

$$-\{\theta, f\} = -\{\varepsilon_0 \otimes \zeta, f\} = -\{\zeta, f\}\varepsilon_0 = -\langle \zeta, df\rangle \varepsilon_0 = \theta_1^\sharp(df).$$

Analogously,

$$\begin{aligned}
\{\theta, \varepsilon\} &= \{\varepsilon_0 \otimes \zeta + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3, \varepsilon\} \\
&= \langle \varepsilon_0, \varepsilon\rangle \zeta - \varepsilon_0 \wedge \{\varepsilon, \zeta\} \\
&\quad + \langle \varepsilon, \varepsilon_1\rangle \varepsilon_2 \wedge \varepsilon_3 - \langle \varepsilon, \varepsilon_2\rangle \varepsilon_1 \wedge \varepsilon_3 + \langle \varepsilon, \varepsilon_3\rangle \varepsilon_1 \wedge \varepsilon_2 \\
&= -\langle \overline{\varepsilon_0}, d\varepsilon\rangle \zeta + \varepsilon_0 \wedge \langle \zeta, d\varepsilon\rangle \\
&\quad - \langle \overline{\varepsilon_1}, d\varepsilon\rangle \varepsilon_2 \wedge \varepsilon_3 + \langle \overline{\varepsilon_2}, d\varepsilon\rangle \varepsilon_1 \wedge \varepsilon_3 - \langle \overline{\varepsilon_3}, d\varepsilon\rangle \varepsilon_1 \wedge \varepsilon_2.
\end{aligned}$$

Now if we apply the definition of $\theta_2^\sharp$ given in the proof of Thm. 3.59 to θ given in Eq. (6.44), evaluated in $de \in \Gamma(\widehat{E})$, we find that

$$\theta_2^\sharp(d\varepsilon) = -\langle \overline{\varepsilon_0}, d\varepsilon\rangle \zeta + \varepsilon_0 \wedge \langle \zeta, d\varepsilon\rangle - \langle \overline{\varepsilon_1}, d\varepsilon\rangle \varepsilon_2 \wedge \varepsilon_3 + \langle \overline{\varepsilon_2}, d\varepsilon\rangle \varepsilon_1 \wedge \varepsilon_3 - \langle \overline{\varepsilon_3}, d\varepsilon\rangle \varepsilon_1 \wedge \varepsilon_2,$$

whence,

$$\theta_2^\sharp(d\varepsilon) = \{\theta, \varepsilon\}.$$

The last assertion of the proposition follows immediately from the observation made before the proposition and from the identities

$$\begin{aligned} \langle \varepsilon_1, \varepsilon_2\rangle = \{\varepsilon_1, \varepsilon_2\}, \qquad \langle \gamma, d\varepsilon\rangle = \{\gamma, \varepsilon\}, \qquad & \varepsilon, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*), f \in C^\infty(M), \gamma \in \Gamma(\widetilde{F}^*). \\ \langle \gamma, df\rangle = \{\gamma, f\}, \qquad & \end{aligned} \tag{6.45}$$

■

Recall that we already introduced Courant algebroids in Def. 5.1. In our next theorem, we show a characterization of Courant algebroids as symplectic NQ degree 2 manifolds. This result is due to D. Roytenberg [59]. However the proof we provide is new in many aspects, specially in the fact that we use strongly our characterization of degree 3 functions as a pair of vector bundle morphisms (Thm. 3.59).

Theorem 6.32 ([59])**.** *There is a canonical 1:1 correspondence between integrable, symplectic 1-vector fields Q, and Courant algebroids. The correspondence is given by*

$$a(\varepsilon)(f) := \{Q(f), \varepsilon\} = \{Q(\varepsilon), f\}, \qquad [\![\varepsilon_1, \varepsilon_2]\!] := \{Q(\varepsilon_1), \varepsilon_2\}, \tag{6.46}$$

where $\varepsilon, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

Proof. We have already seen in Prop. 6.30 that there is a canonical 1:1 correspondence between integrable, symplectic 1-vector fields Q and degree 3 functions θ satisfying $\{\theta, \theta\} = 0$, where the correspondence is explicitly given by $\theta \rightsquigarrow Q = \{\theta, \cdot\}$. Thus, we will show actually a correspondence between integrable degree 3 functions and Courant algebroid structures.

By Thm. 3.59 we know that a degree 3 function, θ, is equivalent to the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, which, by Prop. 6.31, are determined by Eqs. (6.42), which in turn are completely determined by Eqs. (6.43), and those equations give precisely the Courant algebroid data defined in Eq. (6.46). Explicitly we have

$$\langle \theta_1^\sharp(df), \sharp(\varepsilon)\rangle = -a(\varepsilon)(f), \qquad \langle \theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle = [\![\varepsilon_1, \varepsilon_2]\!]. \tag{6.47}$$

So, suppose we begin with the Courant algebroid data $([\![\cdot, \cdot]\!], a)$. In order to see that we actually obtain a degree 3 function θ via Eqs. (6.46), we define

$$\langle \theta_2^\sharp(d\varepsilon), df\rangle := a(\varepsilon)(f), \tag{6.48}$$

and observe $\theta_2^\sharp$ is completely determined by Eq. (6.48) and the second equation of (6.47). That $\theta_2^\sharp$ actually takes values on $\widetilde{F}^*$ follows from property 5 of Def. 5.1. Also by the first equation of (6.47) and by Eq. (6.48), we already have property 1 of Thm. 3.59. Indeed, we have

$$\langle \theta_2^\sharp(d\varepsilon), df\rangle = a(\varepsilon)(f) = -\langle \theta_1^\sharp(df), \sharp(\varepsilon)\rangle = \langle \theta_1^\sharp(df), d\varepsilon\rangle, \tag{6.49}$$

where we also used $\pi(d\varepsilon) = -\sharp(\varepsilon)$, which follows from the definition of the anchor map of the metric VB-algebroid corresponding to $(\mathcal{M}, \{\cdot, \cdot\})$, given in Eq. (6.17).

Now we need to show that properties 2 and 3 of Thm. 3.59 hold for the pair $(\theta_1^\sharp, \theta_2^\sharp)$ we have defined. For the proof of property 2 we compute

$$\begin{aligned} \langle (\varepsilon_1 \otimes df)^* \circ (\theta_1^\sharp)^* - \theta_1^\sharp \circ (\varepsilon_1 \otimes df), d\varepsilon_2 \rangle &= -\theta_1^\sharp(df)\langle \varepsilon_1, \varepsilon_2\rangle + \langle (\theta_1^\sharp)^*(\sharp(\varepsilon_2)), df\rangle \varepsilon_1 \\ &= \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle - a(\varepsilon_2)(f)\varepsilon_1, \end{aligned} \tag{6.50}$$

where $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E^*)$ is the "differential" given by

$$\langle \mathcal{D}(f), \varepsilon\rangle := a(\varepsilon)(f), \qquad \forall \varepsilon \in \Gamma(E^*), \tag{6.51}$$

that is, $\mathcal{D}$ is the adjoint of a, with respect to the non-degenerate metric $\langle \cdot, \cdot \rangle$ on E^*. On the other hand, by properties 3 and 4 of Def. 5.1,

$$\begin{aligned} \langle \theta_2^\sharp(\varepsilon_1 \otimes df), d\varepsilon_2\rangle &= \langle \theta_2^\sharp(d(f\varepsilon_1) - f d\varepsilon_1), d\varepsilon_2\rangle \\ &= \langle \theta_2^\sharp(d(f\varepsilon_1)), d\varepsilon_2\rangle - f\langle \theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle \\ &= [\![f\varepsilon_1, \varepsilon_2]\!] - f[\![\varepsilon_1, \varepsilon_2]\!] \\ &= f[\![\varepsilon_1, \varepsilon_2]\!] + \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle - a(\varepsilon_2)(f)\varepsilon_1 - f[\![\varepsilon_1, \varepsilon_2]\!] \\ &= \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle - a(\varepsilon_2)(f)\varepsilon_1. \end{aligned}$$

Comparing with Eq. (6.50) we get property 2. of Thm. 3.59.

At this point it is demanded to observe that we have two ways to calculate $\langle \theta_2^\sharp(d\varepsilon_1), d(f\varepsilon_2)\rangle$. We must show they cast the same result:

- On one hand, by property 3 of Def. 5.1,

$$\langle \theta_2^\sharp(d\varepsilon_1), d(f\varepsilon_2)\rangle = [\![\varepsilon_1, f\varepsilon_2]\!] = f[\![\varepsilon_1, \varepsilon_2]\!] + a(\varepsilon_1)(f)\varepsilon_2.$$

- On the other hand, by Eq. (6.48),

$$\begin{aligned} \langle \theta_2^\sharp(d\varepsilon_1), d(f\varepsilon_2)\rangle &= f\langle \theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle + \langle \theta_2^\sharp(d\varepsilon_1), \varepsilon_2 \otimes df\rangle \\ &= f[\![\varepsilon_1, \varepsilon_2]\!] + a(\varepsilon_1)(f)\varepsilon_2. \end{aligned}$$

Finally, property 3 of Thm. 3.59 follows from property 4 of Def. 5.1:

- On one hand, by property 4. of Def. 5.1 and Eq. (6.47),

$$\langle \theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle + \langle \theta_2^\sharp(d\varepsilon_2), d\varepsilon_1\rangle = \mathcal{D}(\langle \varepsilon_1, \varepsilon_2\rangle).$$

On the other hand, by Eq. (3.48) and using $d\varepsilon = (d\varepsilon - \nabla.\varepsilon) + \nabla.\varepsilon$,

$$\begin{aligned} & W(d\varepsilon_1, d\varepsilon_2) = -\langle \nabla.\varepsilon_1, \varepsilon_2\rangle - \langle \nabla.\varepsilon_2, \varepsilon_1\rangle = -d(\langle \varepsilon_1, \varepsilon_2\rangle) \\ \Longrightarrow \quad & \theta_1^\sharp(W(d\varepsilon_1, d\varepsilon_2)) = \mathcal{D}(\langle \varepsilon_1, \varepsilon_2\rangle). \end{aligned}$$

Therefore

$$\langle \theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle + \langle \theta_2^\sharp(d\varepsilon_2), d\varepsilon_1\rangle = \theta_1^\sharp(W(d\varepsilon_1, d\varepsilon_2)).$$

- By Eqs. (6.48) and (6.50),

$$\begin{aligned}\langle \theta_2^\sharp(d\varepsilon_1), \varepsilon_2 \otimes df\rangle + \langle \theta_2^\sharp(\varepsilon_2 \otimes df), d\varepsilon_1\rangle &= a(\varepsilon_1)(f)\varepsilon_2 + \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle - a(\varepsilon_1)(f)\varepsilon_2 \\ &= \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle.\end{aligned}$$

On the other hand, by Eq. (3.48),

$$\begin{aligned} & W(d\varepsilon_1, \varepsilon_2 \otimes df) = -\langle \varepsilon_1, \varepsilon_2\rangle df \\ \Longrightarrow \qquad & \theta_1^\sharp(W(d\varepsilon_1, \varepsilon_2 \otimes df)) = \mathcal{D}(f)\langle \varepsilon_1, \varepsilon_2\rangle.\end{aligned}$$

Therefore

$$\langle \theta_2^\sharp(d\varepsilon_1), \varepsilon_2 \otimes df\rangle + \langle \theta_2^\sharp(\varepsilon_2 \otimes df), d\varepsilon_1\rangle = \theta_1^\sharp(W(d\varepsilon_1, \varepsilon_2 \otimes df)).$$

- Finally, $\langle \theta_2^\sharp(\varepsilon_i \otimes df_i), \varepsilon_j \otimes df_j\rangle = 0 = W(\varepsilon_i \otimes df_i, \varepsilon_j \otimes df_j)$, $i, j \in \{1, 2\}$, therefore

$$\langle \theta_2^\sharp(\varepsilon_1 \otimes df_1), \varepsilon_2 \otimes df_2\rangle + \langle \theta_2^\sharp(\varepsilon_2 \otimes df_2), \varepsilon_1 \otimes df_1\rangle = \theta_1^\sharp(W(\varepsilon_1 \otimes df_1, \varepsilon_2 \otimes df_2)) = 0.$$

Accordingly, we have shown that from the pair $([\![\cdot,\cdot]\!], a)$ satisfying properties 3, 4 and 5 of Def. 5.1, we obtain a degree 3 function θ, or equivalently, the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ satisfying properties 1, 2 and 3 of Thm. 3.59. And conversely, if we start from such a pair $(\theta_1^\sharp, \theta_2^\sharp)$, we can define the pair $([\![\cdot,\cdot]\!], a)$ by Eqs. (6.47), and the calculations above also show that from properties 1, 2 and 3 of Thm. 3.59 we obtain properties 3 and 4 of Def. 5.1. Property 5 of Def. 5.1, follows from the fact that $\theta_2^\sharp$ takes values on $\widetilde{F}^*$. Indeed, notice that, for $\varepsilon \in \Gamma(E^*)$, $\theta_2^\sharp(d\varepsilon) \in \widetilde{F}^* \cong \mathbb{A}_{E^*}$ (see Prop. 6.23), therefore, under the identification $\widetilde{F}^* \xrightarrow{\Psi} \mathbb{A}_{E^*}$, we have, for any $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$,

$$\begin{aligned} a(\varepsilon)(\langle \varepsilon_1, \varepsilon_2\rangle) &= (\pi \circ \theta_2^\sharp(d\varepsilon))(\langle \varepsilon_1, \varepsilon_2\rangle) \\ &= \langle \theta_2^\sharp(d\varepsilon)(\varepsilon_1), \varepsilon_2\rangle + \langle \varepsilon_1, \theta_2^\sharp(d\varepsilon)(\varepsilon_2)\rangle \\ &= \langle [\theta_2^\sharp(d\varepsilon), \varepsilon_1], \varepsilon_2\rangle + \langle \varepsilon_1, [\theta_2^\sharp(d\varepsilon), \varepsilon_2]\rangle \\ &= \langle \langle \theta_2^\sharp(d\varepsilon), d\varepsilon_1\rangle, \varepsilon_2\rangle + \langle \varepsilon_1, \langle \theta_2^\sharp(d\varepsilon), d\varepsilon_2\rangle\rangle \\ &= \langle [\![\varepsilon, \varepsilon_1]\!], \varepsilon_2\rangle + \langle \varepsilon_1, [\![\varepsilon, \varepsilon_2]\!]\rangle. \end{aligned} \tag{6.52}$$

It remains to verify that the integrability equation $\{\theta, \theta\} = 0$ is equivalent to properties 1 and 2 of Def. 5.1. First observe that property 4 is equivalent to

$$[\![\varepsilon_1, \varepsilon_2]\!] + [\![\varepsilon_2, \varepsilon_1]\!] = \mathcal{D}(\langle \varepsilon_1, \varepsilon_2\rangle).$$

Since $\langle \cdot, \cdot\rangle$ is non-degenerate, and the Lie bracket of vector fields $[a(\varepsilon_1), a(\varepsilon_2)]$ is skew-symmetric, we can conclude that

$$a \circ a^* = 0,$$

where $a^* : T^*M \longrightarrow E^*$ is the composition of the pull-back of a with the inverse of the metric isomorphism $\flat : E \longrightarrow E^*$, that is

$$\langle a^*(\alpha), \varepsilon\rangle = \langle a(\varepsilon), \alpha\rangle, \qquad \forall \varepsilon \in \Gamma(E^*), \alpha \in \Gamma(T^*M).$$

It follows that,

$$\langle a^*(\alpha), a^*(\beta)\rangle = \langle a(a^*\beta), \alpha\rangle = 0, \qquad \forall \alpha, \beta \in \Gamma(T^*M). \tag{6.53}$$

Resuming the task of characterizing integrability of θ, notice that this is a local issue, thereby we can work in a Darboux coordinate system $(q^i, \varepsilon_a, p_i)$ as the one given by Prop. 6.28. In these coordinates we have

$$\{\theta, \theta\} = A^{ij} p_i p_j + B_j^{ab} \varepsilon_a \varepsilon_b p_j + C^{abcd} \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d, \tag{6.54}$$

where $A^{ij}, B^j_{ab}, C^{abcd} \in C^\infty(U)$ are real functions in $U \subset M$ given by

$$\begin{aligned} A^{ij} &= \{\{\{\theta,\theta\}, q^i\}, q^j\}, \\ B_j^{ab} &= \pm\{\{\{\{\theta,\theta\}, \varepsilon_a\}\varepsilon_b\}, q^j\}, \\ C^{abcd} &= \pm\{\{\{\{\{\theta,\theta\}, \varepsilon_a\}, \varepsilon_b\}, \varepsilon_c\}, \varepsilon_d\}. \end{aligned}$$

Now using Jacobi identity we have the following:

$$\begin{aligned} A^{ij} &= \{\{q^j, \{q^i, \theta\}\}, \theta\} + \{\{q^i, \theta\}, \{q^j, \theta\}\} \\ &\quad + \{\{q^j, \theta\}, \{q^i, \theta\}\} + \{\theta, \{q^j, \{q^i, \theta\}\}\} \\ &= 2\{\{q^j, \{q^i, \theta\}\}, \theta\} + 2\{\{q^i, \theta\}, \{q^j, \theta\}\} \\ &= 2\{\{q^i, \theta\}, \{q^j, \theta\}\} \\ &= 2\langle a^*(dq^i), a^*(dq^j)\rangle. \end{aligned} \tag{6.55}$$

$$\begin{aligned} \{\{\{\theta,\theta\},\varepsilon_a\},\varepsilon_b\} &= -2\{\{\varepsilon_b, \{\varepsilon_a, \theta\}\}, \theta\} - 2\{\{\varepsilon_a, \theta\}, \{\varepsilon_b, \theta\}\} \\ \implies \qquad \pm B_j^{ab} &= \{\{\varepsilon_b, \{\varepsilon_a, \theta\}\}, \{q^j, \theta\}\} + 2\{\{q^j, \{\varepsilon_a, \theta\}\}, \{\varepsilon_b, \theta\}\} \\ &\quad + 2\{\{\varepsilon_a, \theta\}, \{q^j, \{\varepsilon_b, \theta\}\}\} + \{\{q^j, \theta\}, \{\varepsilon_b, \{\varepsilon_a, \theta\}\}\} \\ &= 2a([\![\varepsilon_a, \varepsilon_b]\!])(q^j) + 2a(\varepsilon_b)(a(\varepsilon_a)(q^j)) - 2a(\varepsilon_a)(a(\varepsilon_b)(q^j)) \\ &= 2\langle a([\![\varepsilon_a, \varepsilon_b]\!]) - [a(\varepsilon_a), a(\varepsilon_b)], dq^j\rangle. \end{aligned} \tag{6.56}$$

$$\begin{aligned} \{\{\{\{\theta,\theta\},\varepsilon_a\},\varepsilon_b\},\varepsilon_c\} &= -2\{\{\{\varepsilon_b, \{\varepsilon_a, \theta\}\}, \theta\}, \varepsilon_c\} - 2\{\{\{\varepsilon_a, \theta\}, \{\varepsilon_b, \theta\}\}, \varepsilon_c\} \\ &= 2([\![[\![\varepsilon_a, \varepsilon_b]\!], \varepsilon_c]\!] + [\![\varepsilon_b, [\![\varepsilon_a, \varepsilon_c]\!]]\!] - [\![\varepsilon_a, [\![\varepsilon_b, \varepsilon_c]\!]]\!]) \\ \implies \qquad \pm C^{abcd} &= 2\langle [\![[\![\varepsilon_a, \varepsilon_b]\!], \varepsilon_c]\!] + [\![\varepsilon_b, [\![\varepsilon_a, \varepsilon_c]\!]]\!] - [\![\varepsilon_a, [\![\varepsilon_b, \varepsilon_c]\!]]\!], \varepsilon_d\rangle. \end{aligned} \tag{6.57}$$

Thereby, $\{\theta, \theta\} = 0$ if and only if $A^{ij} = B_j^{ab} = C^{abcd} = 0$, and taking Eqs. (6.53), (6.55), (6.56) and (6.57) into account, this is equivalent to properties 1 and 2 of Def. 5.1, as we wanted.

■

Remark 6.33. An interesting consequence of what was observed in the proof of the corollary above is that we can obtain an alternative proof, providing a new insight, of the fact that symplectic 1-vector fields on symplectic degree 2 manifolds are automatically hamiltonian. The proof consists on obtaining from Q a pair $([\![\cdot,\cdot]\!], a)$ satisfying properties 3, 4 and 5 of Def. 5.1. Define

$$a(\varepsilon)(f) := \{Q(\varepsilon), f\}, \qquad [\![\varepsilon_1, \varepsilon_2]\!] := \{Q(\varepsilon_1), \varepsilon_2\}, \quad e, \varepsilon_1, \varepsilon_2 \in \Gamma(E^*), f \in C^\infty(M). \tag{6.58}$$

Then, from (graded) Leibniz rule of $\{\cdot,\cdot\}$, we have

$$\begin{aligned}[\![\varepsilon_1, f\varepsilon_2]\!] &= \{Q(\varepsilon_1), f\varepsilon_2\} = \{Q(\varepsilon_1), f\}\varepsilon_2 + f\{Q(\varepsilon_1), \varepsilon_2\}\\ &= a(\varepsilon_1)(f)\varepsilon_2 + f[\![\varepsilon_1, \varepsilon_2]\!],\end{aligned}$$

hence, we have property 3 of Def. 5.1.

If Q is symplectic, then Q satisfies Eq. (6.41). In particular,

$$0 = Q\{\varepsilon, f\} = \{Q(\varepsilon), f\} - \{\varepsilon, Q(f)\},$$

thereby, $\{Q(\varepsilon), f\} = \{Q(f), \varepsilon\}$, whence, if we define $\mathcal{D}(f) := Q(f)$, so that Eq. (6.51) holds, we have

$$\begin{aligned}\mathcal{D}(\langle\varepsilon_1, \varepsilon_2\rangle) &= Q(\{\varepsilon_1, \varepsilon_2\}) = \{Q(\varepsilon_1), \varepsilon_2\} - \{\varepsilon_1, Q(\varepsilon_2)\}\\ &= [\![\varepsilon_1, \varepsilon_2]\!] + [\![\varepsilon_2, \varepsilon_1]\!],\end{aligned}$$

which is equivalent to property 4 of Def. 5.1.

Finally, property 5 of Def. 5.1 is obtained easily from (graded) Jacobi identity of $\{\cdot,\cdot\}$:

$$\begin{aligned}a(\varepsilon)(\langle\varepsilon_1, \varepsilon_2\rangle) &= \{Q(\varepsilon), \{\varepsilon_1, \varepsilon_2\}\}\\ &= \{\{Q(\varepsilon), \varepsilon_1\}, \varepsilon_2\} + \{\varepsilon_1, \{Q(\varepsilon), \varepsilon_2\}\}\\ &= \langle[\![\varepsilon, \varepsilon_1]\!], \varepsilon_2\rangle + \langle\varepsilon_1, [\![\varepsilon, \varepsilon_2]\!]\rangle.\end{aligned}$$

Therefore, by what was done in the proof of Thm. 6.32, we obtain a pair $(\theta_1^\sharp, \theta_2^\sharp)$ which satisfies properties 1, 2 and 3 of Thm. 3.59, thereby obtaining a degree 3 function θ, such that

$$\langle\theta_2^\sharp(d\varepsilon), df\rangle = a(\varepsilon)(f), \qquad \langle\theta_2^\sharp(d\varepsilon_1), d\varepsilon_2\rangle = [\![\varepsilon_1, \varepsilon_2]\!]. \tag{6.59}$$

From Eqs. (6.45), it follows that

$$\{\theta_2^\sharp(d\varepsilon), f\} = a(\varepsilon)(f), \qquad \{\theta_2^\sharp(d\varepsilon_1), \varepsilon_2\} = [\![\varepsilon_1, \varepsilon_2]\!].$$

Bringing to the picture the second equation in (6.42) and Eq. (6.58), we get

$$\{Q(\varepsilon_1), \varepsilon_2\} = \{\{\theta, \varepsilon_1\}, \varepsilon_2\}, \qquad \text{and} \qquad \{Q(\varepsilon_1), f\} = \{\{\theta, \varepsilon_1\}, f\},$$

also, using Jacobi identity and the second equation above, we get

$$\{Q(f), \varepsilon\} = \{\{\theta, f\}, \varepsilon\}.$$

Therefore, since $\{\cdot,\cdot\}$ is non-degenerate, we conclude

$$Q = \{\theta, \cdot\}.$$

■

The “cotangent” Lie 2-algebroid of a Courant algebroid

Equations (6.59) show the parallel that can be drawn between the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$ corresponding to Courant algebroids, and the Poisson morphism $\pi^\sharp : T^*M \longrightarrow TM$ corresponding to Poisson manifolds. The parallel also extends to our Lie 2-algebroid structure, which was defined in Thm. 4.9, being related to the Courant algebroid structure in a very similar way the cotangent Lie algebroid is related to its corresponding Poisson structure, as Eqs. *1-5* of the following theorem show.

Proposition 6.34. *Consider a Courant algebroid structure* $(\llbracket\cdot,\cdot\rrbracket, a)$ *on the pseudo-euclidean bundle* $(E^*, \langle\cdot,\cdot\rangle)$*. By Thm. 6.32 we have equivalently the Q-structure on the corresponding symplectic degree 2 manifold* $\mathcal{M}$*. Consider the Lie 2-algebroid structure* $([\cdot,\cdot], \Theta, \Psi, \partial, \rho)$ *corresponding to this degree 2 NQ-manifold given by Thm. 4.20. Then*

1. $[d\varepsilon_1, d\varepsilon_2] = d\llbracket\varepsilon_1, \varepsilon_2\rrbracket, \qquad \forall \varepsilon_1, \varepsilon_2 \in \Gamma(E^*).$
2. $\Theta(d\varepsilon)(df) = d(a(\varepsilon)(f)), \qquad \forall \varepsilon \in \Gamma(E^*), f \in C^\infty(M).$
3. $\Psi(d\varepsilon_1)(\varepsilon_2) = -\llbracket\varepsilon_2, \varepsilon_1\rrbracket, \qquad \forall \varepsilon_1, \varepsilon_2 \in \Gamma(E^*).$
4. $\langle\partial(df), \varepsilon\rangle = a(\varepsilon)(f) \qquad \forall \varepsilon \in \Gamma(E^*), f \in C^\infty(M).$
5. $\widehat{\rho}(d\varepsilon)(f) = a(\varepsilon)(f), \qquad \forall \varepsilon \in \Gamma(E^*), f \in C^\infty(M).$

Proof. Let's prove Eq. *1.* We want to show

$$\langle[d\varepsilon_1, d\varepsilon_2], \overline{\varepsilon}\rangle = \langle d\llbracket\varepsilon_1, \varepsilon_2\rrbracket, \overline{\varepsilon}\rangle, \qquad \langle[d\varepsilon_1, d\varepsilon_2], \gamma\rangle = \langle d\llbracket\varepsilon_1, \varepsilon_2\rrbracket, \gamma\rangle \quad \forall \varepsilon_1, \varepsilon_2, \varepsilon \in \Gamma(E^*), \gamma \in \widetilde{F}^*. \tag{6.60}$$

Let's show the first equality of (6.60). By Eq. (4.15), we have

$$\langle[d\varepsilon_1, d\varepsilon_2], \overline{\varepsilon}\rangle = \widehat{\rho}(d\varepsilon_1)(\langle d\varepsilon_2, \overline{\varepsilon}\rangle) - \widehat{\rho}(d\varepsilon_2)(\langle d\varepsilon_1, \overline{\varepsilon}\rangle) + \langle T(d\varepsilon_1, d\varepsilon_2), Q(\varepsilon)\rangle. \tag{6.61}$$

Now,

1.
$$\begin{aligned}\widehat{\rho}(d\varepsilon_1)(\langle d\varepsilon_2, \overline{\varepsilon}\rangle) &= -\{Q(\{\varepsilon_2, \varepsilon\}), \varepsilon_1\} \\ &= -\{\{Q(\varepsilon_2), \varepsilon\}, \varepsilon_1\} + \{\{\varepsilon_2, Q(\varepsilon)\}, \varepsilon_1\}.\end{aligned}$$

2.
$$\begin{aligned}\widehat{\rho}(d\varepsilon_2)(\langle d\varepsilon_1, \overline{\varepsilon}\rangle) &= -\{Q(\{\varepsilon_1, \varepsilon\}), \varepsilon_2\} = \{\{\varepsilon_1, \varepsilon\}, Q(\varepsilon_2)\} \\ &= -\{\{Q(\varepsilon_2), \varepsilon_1\}, \varepsilon\} - \{\varepsilon_1, \{Q(\varepsilon_2), \varepsilon\}\}.\end{aligned}$$

3.
$$\begin{aligned}\langle T(d\varepsilon_1, d\varepsilon_2), Q(\varepsilon)\rangle &= \langle d\varepsilon_2, \overline{\langle d\varepsilon_1, Q(\varepsilon)\rangle}\rangle = -\{\{Q(\varepsilon), \varepsilon_1\}, \varepsilon_2\} \\ &= \{\{\varepsilon_2, \varepsilon_1\}, Q(\varepsilon)\} - \{\varepsilon_1, \{\varepsilon_2, Q(\varepsilon)\}\} \\ &= -\{Q(\{\varepsilon_2, \varepsilon_1\}, \varepsilon\} - \{\varepsilon_1\{\varepsilon_2, Q(\varepsilon)\}\} \\ &= -\{\{Q(\varepsilon_2), \varepsilon_1\}, \varepsilon\} + \{\{\varepsilon_2, Q(\varepsilon_1)\}, \varepsilon\} - \{\varepsilon_1\{\varepsilon_2, Q(\varepsilon)\}\}.\end{aligned}$$

Putting items 1, 2 and 3 into Eq. (6.61), we obtain

$$\langle [d\varepsilon_1, d\varepsilon_2], \overline{\varepsilon}\rangle = \{\{\varepsilon_2, Q(\varepsilon_1)\}, \varepsilon\} = \langle d[\![\varepsilon_1, \varepsilon_2]\!], \overline{\varepsilon}\rangle.$$

Now we will prove the second equality of (6.60). By Eq. (4.14) we have

$$\begin{aligned}\langle [d\varepsilon_1, d\varepsilon_2], \gamma\rangle =& \Psi(d\varepsilon_1)(\langle d\varepsilon_2, \gamma\rangle) - \Psi(d\varepsilon_2)(\langle d\varepsilon_1, \gamma\rangle) \\ &+ \rho^*(d(\langle T(d\varepsilon_2, d\varepsilon_1), \gamma\rangle)) - \langle d\varepsilon_2, Q(\gamma)^\sharp_2(d\varepsilon_1)\rangle.\end{aligned} \tag{6.62}$$

Now,

1.

$$\Psi(d\varepsilon_1)(\langle d\varepsilon_2, \gamma\rangle) = -\{Q(\{\gamma, \varepsilon_2\}), \varepsilon_1\} = -\{\{Q(\gamma), \varepsilon_2\}, \varepsilon_1\} - \{\{\gamma, Q(\varepsilon_2)\}, \varepsilon_1\}. \tag{6.63}$$

In order to compute the term $\{\{Q(\gamma), \varepsilon_2\}, \varepsilon_1\}$, observe that

$$\{\{Q(\gamma), \varepsilon_2\}, \varepsilon_1\} = -\{\{\varepsilon_1, Q(\gamma)\}, \varepsilon_2\} + \{Q(\gamma), \{\varepsilon_1, \varepsilon_2\}\}.$$

Now,

$$Q(\{\gamma, \{\varepsilon_1, \varepsilon_2\}\}) = \{Q(\gamma), \{\varepsilon_1, \varepsilon_2\}\} + \{\gamma, Q(\{\varepsilon_1, \varepsilon_2\})\},$$

Whence,

$$\{\{Q(\gamma), \varepsilon_2\}, \varepsilon_1\} = -\{\{\varepsilon_1, Q(\gamma)\}, \varepsilon_2\} + Q(\{\gamma, \{\varepsilon_1, \varepsilon_2\}\}) - \{\gamma, Q(\{\varepsilon_1, \varepsilon_2\})\}.$$

Thereby, putting this into Eq. (6.63), we have

$$\begin{aligned}\Psi(d\varepsilon_1)(\langle d\varepsilon_2, \gamma\rangle) =& \{\{\varepsilon_1, Q(\gamma)\}, \varepsilon_2\} - Q(\{\gamma, \{\varepsilon_1, \varepsilon_2\}\}) \\ &+ \{\gamma, Q(\{\varepsilon_1, \varepsilon_2\})\} - \{\{\gamma, Q(\varepsilon_2)\}, \varepsilon_1\}.\end{aligned}$$

2.

$$\begin{aligned}\Psi(d\varepsilon_2)(\langle d\varepsilon_1, \gamma\rangle &= -\{Q(\{\gamma, \varepsilon_1\}), \varepsilon_2\} = -Q(\{\{\gamma, \varepsilon_1\}, \varepsilon_2\}) - \{\{\gamma, \varepsilon_1\}, Q(\varepsilon_2)\} \\ &= Q(\{\{\gamma, \varepsilon_2\}, \varepsilon_1\}) - Q(\{\gamma, \{\varepsilon_1, \varepsilon_2\}\}) - \{\{\gamma, \varepsilon_1\}, Q(\varepsilon_2)\} \\ &= Q(\{\{\gamma, \varepsilon_2\}, \varepsilon_1\}) - Q(\{\gamma, \{\varepsilon_1, \varepsilon_2\}\}) - \{\{\gamma, Q(\varepsilon_2)\}, \varepsilon_1\} \\ &\quad - \{\{Q(\varepsilon_2), \varepsilon_1\}, \gamma\}.\end{aligned}$$

3.

$$\rho^*(d(\langle T(d\varepsilon_2, d\varepsilon_1), \gamma\rangle)) = -Q(\langle d\varepsilon_1, \overline{\langle d\varepsilon_2, \gamma\rangle}\rangle) = Q(\{\{\gamma, \varepsilon_2\}, \varepsilon_1\}).$$

4.

$$\langle d\varepsilon_2, Q(\gamma)^\sharp_2(d\varepsilon_1)\rangle = \{\{Q(\gamma), \varepsilon_1\}, \varepsilon_2\}.$$

Putting items 1, 2, 3 and 4 into Eq. (6.62), we obtain

$$\begin{aligned}\langle [d\varepsilon_1, d\varepsilon_2], \gamma\rangle &= \{\{Q(\varepsilon_2), \varepsilon_1\}, \gamma\} + \{\gamma, Q(\{\varepsilon_1, \varepsilon_2\})\}\\ &= \{\{Q(\varepsilon_2), \varepsilon_1\}, \gamma\} + \{\gamma, \{Q(\varepsilon_1), \varepsilon_2\}\} - \{\gamma, \{\varepsilon_1, Q(\varepsilon_2)\}\}\\ &= \{\gamma, \{Q(\varepsilon_1), \varepsilon_2\}\} = \langle d[\![\varepsilon_1, \varepsilon_2]\!], \gamma\rangle.\end{aligned}$$

Therefore we have Eq. *1* of the statement of the theorem. Now let's prove Eq. *2* of the theorem. We want to show

$$\langle \Theta(d\varepsilon)(df), \gamma\rangle = \langle d(a(\varepsilon)(f)), \gamma\rangle, \qquad \forall \varepsilon \in \Gamma(E^*), f \in C^\infty(M), \gamma \in \Gamma(\widetilde{F}^*). \tag{6.64}$$

By Eq. (4.12) we have

$$\langle \Theta(d\varepsilon)(df), \gamma\rangle = \widehat{\rho}(d\varepsilon)(\langle df, \gamma\rangle) + \langle \partial(df), \langle d\varepsilon, \gamma\rangle\rangle - \langle df, Q(\gamma)^\sharp_2(d\varepsilon)\rangle. \tag{6.65}$$

Let's compute each term of the right-hand side of this equation.

1.

$$\begin{aligned}\widehat{\rho}(d\varepsilon)(\langle df, \gamma\rangle) &= \{Q(\{\gamma, f\}), \varepsilon\} = \{Q(\varepsilon), \{\gamma, f\}\}\\ &= \{\{Q(\varepsilon), \gamma\}, f\} + \{\gamma, \{Q(\varepsilon), f\}\}.\end{aligned}$$

2.

$$\begin{aligned}\langle \partial(df), \langle d\varepsilon, \gamma\rangle\rangle &= \langle df, Q(\{\gamma, \varepsilon\})\rangle = \{Q(\{\gamma, \varepsilon\}), f\}\\ &= \{\{Q(\gamma), \varepsilon\}, f\} + \{\{\gamma, Q(\varepsilon)\}, f\}.\end{aligned}$$

3. $\langle df, Q(\gamma)^\sharp_2(d\varepsilon)\rangle = \{\{Q(\gamma), \varepsilon\}, f\}$.

Putting items 1, 2 and 3. into Eq. (6.65), we get

$$\langle \Theta(d\varepsilon)(df), \gamma\rangle = \{\gamma, \{Q(\varepsilon), f\}\} = \langle d(a(\varepsilon)(f)), \gamma\rangle.$$

Thereby, we have Eq. *2* of the theorem. Next let's take care of Eq. *3* of the theorem:

$$\Psi(d\varepsilon_1)(\varepsilon_2) = -\langle d\varepsilon_1, Q(\varepsilon_2)\rangle = -\{Q(\varepsilon_2), \varepsilon_1\} = -[\![\varepsilon_2, \varepsilon_1]\!].$$

Next, we prove Eq. *4* of the theorem:

$$\langle \partial(df), \varepsilon\rangle = \langle df, Q(\varepsilon)\rangle = \{Q(\varepsilon), f\} = a(\varepsilon)(f).$$

Finally, let's prove Eq. *5* of the theorem:

$$\widehat{\rho}(d\varepsilon)(f) = -\langle d\varepsilon, \overline{Q(f)}\rangle = \{\varepsilon, Q(f)\} = a(\varepsilon)(f).$$

∎

Remark 6.35. Eq. *1* of Prop. 6.34 may tempt us to conclude that

$$[\phi_1, \phi_2] = \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1 + d(\langle\theta_2^\sharp(\phi_2), \phi_1\rangle), \qquad \forall\phi_1, \phi_2 \in \Gamma(\widehat{E}), \tag{6.66}$$

in analogy with the cotangent Lie algebroid brackets in the degree 1 case. The temptation is there because, for $\phi_1 = d\varepsilon_1, \phi_2 = d\varepsilon_2$, Eq. (6.66) is true, as it is easily verified. However, Eq. (6.66) is *not true* in the general case. We can see this by contradiction. If Eq. (6.66) were valid, then by property *3* of Thm. 3.59 and property *7* of Def. 4.6, we should have

$$d(\theta_1^\sharp(W(\phi_1, \phi_2))) = \delta(W(\phi_1, \phi_2)), \qquad \forall\phi_1, \phi_2 \in \Gamma(\widehat{E}), \tag{6.67}$$

and since $W : S^2(\widehat{E}) \longrightarrow F$ is surjective, we should have

$$d(\theta_1^\sharp(\xi)) = \delta(\xi), \qquad \forall\xi \in \Gamma(F). \tag{6.68}$$

By evaluating both sides of Eq. (6.68) on $f\xi$, we should conclude that

$$\theta_1^\sharp(\xi) \otimes df = \rho^*(df) \otimes \xi.$$

Now observe the following:

$$\langle\rho^*(df), \varepsilon\rangle = \rho(\varepsilon)(f) = -\langle\varepsilon, Q(f)\rangle = \langle\varepsilon, -\{\theta, f\}\rangle = \langle\theta_1^\sharp(df), \varepsilon\rangle,$$

therefore we should have

$$\theta_1^\sharp(\xi) \otimes df = \theta_1^\sharp(df) \otimes \xi,$$

which is impossible to hold if $\xi \in \Gamma(F)$ and $f \in C^\infty(M)$ are arbitrary.

In Sec. 7.1 we find a formula relating the Lie 2-algebroid brackets of Eq. (4.14) and the brackets given by Eq. (6.66).

■

Proposition 6.36. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a symplectic degree 2 manifold. Consider its associated metric double vector bundle. We state that this DVB can be canonically identified with TE^*, with the linear metric described in Prop. 5.11.*

If $\mathcal{M}$ is endowed with an integrable, symplectic 1-vector field Q, then the corresponding VB-*Courant algebroid structure is given by the tangent prolongation of E^* described in Prop. 5.11.*

Proof. Let $(D_{F^*}, E; \Phi)$ be the associated metric DVB that corresponds to $\mathcal{M}$, so that the self-conjugate DVB is

$$D \cong D\left(\widehat{F^*}^*\right) = (D_{F^*})^*_E.$$

By Prop. 6.25 we have $(D_{F^*})^*_E \cong \flat^*(T^*E)$. Then, Cor. B.17 allows us to conclude that

$$D_{F^*} \cong (\flat^*(T^*E^*))^*_E \cong \flat^*((T^*E)^*_{E^*}) \cong \flat^*(TE^*),$$

where, for the last identification, we used the Legendre transform $\Upsilon_E : T^*E \longrightarrow T^*E^*$ (cf. Eq. (F.4)). Therefore, from Prop. B.18, we conclude that

$$\sharp^*(D_{F^*}) \cong \sharp^*(\flat^*(TE^*)) \cong TE^*.$$

We saw in the proof of Thm. 6.32 that $W(d\varepsilon_1, d\varepsilon_2) = -d\langle\varepsilon_1, \varepsilon_2\rangle$. Also, since d is the differential corresponding to the metric *VB*-algebroid, from (6.17) we conclude that $\langle d\varepsilon_1, \overline{\varepsilon_2}\rangle = -\langle\varepsilon_1, \varepsilon_2\rangle$. Now, taking into account that the Legendre transform Υ_E is $-\mathrm{Id}$ on the side bundle E, from Cor. 3.50, Eq. (3.48) and Cor. F.2 we conclude that the linear metric on $\sharp^*(D_{F^*}) \cong TE^*$ given by Thm. 3.38 is precisely the one described in Prop. 5.11.

As for the *VB*-Courant algebroid, under the identifications above, it can be seen that the corresponding *VB*-Courant algebroid structure on TE coincides with the tangent prolongation of E described in Prop. 5.11.

■

Integrability of θ in terms of the morphisms $(\theta_1^\sharp, \theta_2^\sharp)$

A very important consequence of Eq. *1* of Prop. 6.34 is the following characterization of integrability of a degree 3 function $\theta \in \mathcal{A}^3$ on a symplectic manifold.

Proposition 6.37. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a symplectic degree 2 manifold. Let $\theta \in \mathcal{A}^3$ be a degree 3 hamiltonian. Then θ is integrable, that is, $\{\theta, \theta\} = 0$ if and only if*

$$\theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]) = [\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)], \qquad \forall \varepsilon_1, \varepsilon_2 \in \Gamma(E^*) \cong C^\infty_{\mathrm{lin}}(E), \tag{6.69}$$

where d is the differential dual to the VB-algebroid structure on D_{F^*}, the brackets on the left-hand side are the preLie 2-algebroid brackets, given by Eqs. (4.14) and (4.15), and the brackets in the right-hand side are the VB-algebroid brackets on D_{F^*} (or the Lie algebroid brackets on $\widetilde{F}^*$).

Remark 6.38. The results of this section can be obtained directly from the more general results obtained in Sec. 7.2. However we chose to provide here alternative proofs exploiting the nondegeneracy of the Poisson brackets, so that it appears more transparently the role of the Courant algebroid structure, which determines already θ together with its integrability condition, as we saw in Sec. 6.4.2.

Proof. Define a pair $([\![\cdot,\cdot]\!], a)$ from θ by Eq. (6.46), where $Q = \{\theta, \cdot\}$. Then, for any $\varepsilon_3 \in \Gamma(E^*)$ we have

$$\begin{aligned}\langle\theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]), d\varepsilon_3\rangle &= \langle\theta_2^\sharp(d[\![\varepsilon_1, \varepsilon_2]\!]), d\varepsilon_3\rangle \\ &= \{\{\theta, [\![\varepsilon_1, \varepsilon_2]\!]\}, \varepsilon_3\} \\ &= [\![[\![\varepsilon_1, \varepsilon_2]\!], \varepsilon_3]\!].\end{aligned} \tag{6.70}$$

On the other hand,

$$\begin{aligned}\langle[\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)], d\varepsilon_3\rangle &= \{\{\{\theta, \varepsilon_1\}, \{\theta, \varepsilon_2\}\}, \varepsilon_3\} \\ &= -\{\{\varepsilon_3, \{\theta, \varepsilon_1\}\}, \{\theta, \varepsilon_2\}\} - \{\{\theta, \varepsilon_1\}, \{\varepsilon_3, \{\theta, \varepsilon_2\}\}\} \\ &= -[\![\varepsilon_2, [\![\varepsilon_1, \varepsilon_3]\!]]\!] + [\![\varepsilon_1, [\![\varepsilon_2, \varepsilon_3]\!]]\!].\end{aligned} \tag{6.71}$$

From Eqs. (6.70) and (6.71) it follows that

$$\langle[\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)], d\varepsilon_3\rangle - \langle\theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]), d\varepsilon_3\rangle = [\![\varepsilon_1, [\![\varepsilon_2, \varepsilon_3]\!]]\!] - [\![[\![\varepsilon_1, \varepsilon_2]\!], \varepsilon_3]\!] - [\![\varepsilon_2, [\![\varepsilon_1, \varepsilon_3]\!]]\!]. \tag{6.72}$$

Analogously, for any $f \in C^\infty(M)$,

$$\begin{aligned}\langle \theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]), df\rangle &= \langle \theta_2^\sharp(d[\![\varepsilon_1, \varepsilon_2]\!]), df\rangle \\ &= \{\{\theta, [\![\varepsilon_1, \varepsilon_2]\!]\}, f\} = a([\![\varepsilon_1, \varepsilon_2]\!])(f).\end{aligned} \tag{6.73}$$

On the other hand

$$\begin{aligned}\langle [\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)], df\rangle &= \{\{\{\theta, \varepsilon_1\}, \{\theta, \varepsilon_2\}\}, f\} \\ &= -\{\{f, \{\theta, \varepsilon_1\}\}, \{\theta, \varepsilon_2\}\} - \{\{\theta, \varepsilon_1\}, \{f, \{\theta, \varepsilon_2\}\}\} \\ &= -a(\varepsilon_2)a(\varepsilon_1)(f) + a(\varepsilon_1)a(\varepsilon_2)(f) = [a(\varepsilon_1), a(\varepsilon_2)](f).\end{aligned} \tag{6.74}$$

From Eqs. (6.73) and (6.74) it follows

$$\langle [\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)], df\rangle - \langle \theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]), df\rangle = [a(\varepsilon_1), a(\varepsilon_2)](f) - a([\![\varepsilon_1, \varepsilon_2]\!])(f). \tag{6.75}$$

Now, we saw in the proof of Thm. 6.32 that the pair $([\![\cdot, \cdot]\!])$ obtained from a degree 3 hamiltonian θ satisfies automatically properties 3 and 4 of Def. 5.1. We saw also in the proof of Thm. 6.32 that a consequence of property *4* of Def. 5.1, and the non-degeneracy of $\langle \cdot, \cdot \rangle$, is that

$$\langle a^*(\alpha), a^*(\beta)\rangle = 0 \qquad \forall \alpha, \beta \in \Gamma(T^*M),$$

where $a^* : T^*M \longrightarrow E$ is given by $\langle a^*(\alpha), \varepsilon\rangle := \langle a(\varepsilon), \alpha\rangle$, $\forall \varepsilon \in \Gamma(E^*)$.

Therefore, from Eqs. (6.72) and (6.75), together with the last part of the proof of Thm. 6.32, it follows that

$$\theta_2^\sharp([d\varepsilon_1, d\varepsilon_2]) = [\theta_2^\sharp(d\varepsilon_1), \theta_2^\sharp(d\varepsilon_2)] \Longleftrightarrow \{\theta, \theta\} = 0.$$

■

Now we would like to find out how the data of the Lie 2-algebroid corresponding to θ is related to the VB-algebroid structure on D_{F^*} through the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, in the spirit of what we have already done for $\phi_1 = d\varepsilon_1, \phi_2 = d\varepsilon_2$ in Eq. (6.69).

Proposition 6.39. *Let θ be an integrable, degree 3 hamiltonian on a symplectic degree 2 manifold. Then, for $\phi \in \Gamma(\widehat{E}), f \in C^\infty(M)$, we have*

$$\widehat{\rho}(\phi)(f) = \pi(\theta_2^\sharp(\phi))(f), \tag{6.76}$$

where $\pi : \mathbb{A}_{E^*} \longrightarrow TM$ and we are using the identification $\widetilde{F}^* \cong \mathbb{A}_{E^*}$.

Proof. Set $Q = \{\theta, \cdot\}$, then

$$\begin{aligned}\widehat{\rho}(\phi)(f) &= \rho(p_{E^*}(\phi))(f) = -\langle p_{E^*}(\phi), Q(f)\rangle \\ &= \langle p_{E^*}(\phi), \theta_1^\sharp(df)\rangle = \langle \phi, \theta_1^\sharp(df)\rangle \\ &= \langle \theta_2^\sharp(\phi), df\rangle = \langle p_F \circ \theta_2^\sharp(\phi), df\rangle \\ &= \pi(\theta_2^\sharp(\phi))(f).\end{aligned}$$

■

Proposition 6.40. *Let θ be an integrable, degree 3 hamiltonian on a symplectic degree 2 manifold. Then, for $\phi \in \Gamma(\widehat{E}), \xi \in \Gamma(F)$, we have*

$$\theta_1^\sharp(\Theta(\phi)(\xi)) = [\theta_2^\sharp(\phi), \theta_1^\sharp(\xi)], \tag{6.77}$$

where the brackets on the right-hand side concern to the involutive VB-algebroid D_{F^*}.

Proof. It is enough to prove Eq. (6.77) for $\phi = d\varepsilon$ and $\xi = df$ and to show that both sides of the equation behave in the same way with respect to the $C^\infty(M)$-module structure. Let's begin by showing this. It is immediate that both sides are tensorial with respect to ϕ. So let's see what happens with respect to ξ. Using Eq. (6.76), we have

$$\begin{aligned}\theta_1^\sharp(\Theta(\phi)(f\xi)) &= f\theta_1^\sharp(\Theta(\phi)(\xi)) + \widehat{\rho}(\phi)(f)\theta_1^\sharp(\xi)\\ &= f\theta_1^\sharp(\Theta(\phi)(\xi)) + \pi(\theta_2^\sharp(\phi))(f)\theta_1^\sharp(\xi).\end{aligned}$$

On the other hand,

$$[\theta_2^\sharp(\phi), \theta_1^\sharp(f\xi)] = f[\theta_2^\sharp(\phi), \theta_1^\sharp(\xi)] + \pi(\theta_2^\sharp(\phi))(f)\theta_1^\sharp(\xi).$$

Therefore, both sides of (6.77) behave the same way when we multiply by $f \in C^\infty(M)$ ϕ or ξ. So, to end the proof, let's verify Eq. (6.77) for $\phi = d\varepsilon$ and $\xi = df$:

On one hand, by Eq. *2* of Prop. 6.34

$$\begin{aligned}\langle\theta_1^\sharp(\Theta(d\varepsilon_1)(df)), d\varepsilon_2\rangle &= -\{\theta_1^\sharp(d(a(\varepsilon_1)(f))), \varepsilon_2\}\\ &= \{\{\theta, a(\varepsilon_1)(f)\}, \varepsilon_2\} = a(\varepsilon_2)(a(\varepsilon_1)(f)).\end{aligned} \tag{6.78}$$

On the other hand,

$$\begin{aligned}\langle[\theta_2^\sharp(d\varepsilon_1), \theta_1^\sharp(df)], d\varepsilon_2\rangle &= -\{\{\{\theta, \varepsilon_1\}, -\{\theta, f\}\}, \varepsilon_2\}\\ &= \{\{\varepsilon_2, \{\theta, \varepsilon_1\}\}, \{\theta, f\}\} + \{\{\theta, \varepsilon_1\}, \{\varepsilon_2, \{\theta, f\}\}\}\\ &= -a([\![\varepsilon_1, \varepsilon_2]\!]) + a(\varepsilon_1)(a(\varepsilon_2)(f))\\ &= -[a(\varepsilon_1), a(\varepsilon_2)] + a(\varepsilon_1)(a(\varepsilon_2)(f))\\ &= a(\varepsilon_2)(a(\varepsilon_1)(f)).\end{aligned} \tag{6.79}$$

From Eqs. (6.78) and (6.79) we obtain

$$\theta_1^\sharp(\Theta(d\varepsilon_1)(df)) = [\theta_2^\sharp(d\varepsilon_1), \theta_1^\sharp(df)],$$

which is Eq. (6.77) for $\phi = de$ and $\xi = df$, as we wanted.

∎

Chapter 7

Integrability of degree 3 functions on a degree 2 manifold

In this chapter we put together the previous results in order to obtain in Thm. 7.25 a characterization, in terms of vector bundles and maps, of degree 3 functions θ on a degree 2 Poisson manifold $(\mathcal{M}, \{\cdot,\cdot\})$ satisfying the homological equation $\{\theta,\theta\} = 0$.

7.1 An alternative formula for the bracket of an exact Lie 2-algebroid

In this section we find an alternative formula of the Lie 2-algebroid bracket defined by Eqs. (4.14) and (4.15), or equivalently by the derived bracket formula, see Prop. 4.12, in the case where the 1-vector field Q is exact, that is, it comes from an integrable 3-hamiltonian $\theta \in \mathcal{A}^3$:

$$Q = \{\theta, \cdot\}, \qquad \{\theta,\theta\} = 0. \tag{7.1}$$

Of course, we are in the context that the corresponding degree 2 manifold is endowed with degree -2 Poisson brackets $\{\cdot,\cdot\}$. The mentioned formula will prove useful in proving the characterization of integrability of a 3-hamiltonian next section.

We advice the reader to keep in mind the ideas and calculations made in Sec. G.2, for the degree 1 case. Here we follow closely those ideas, though in the degree 2 case some complications appear mainly due to the non-skew symmetry of $\theta_2^\sharp$, so that we have to keep track of the symmetric part, given by item *3* of Thm. 3.59.

Definition 7.1. Consider a degree 2 Poisson manifold endowed with a Q-structure (see Def. 4.1). The corresponding Lie 2-algebroid (Thm. 4.20) is called *exact* if Q is exact, i.e., it has the form (7.1).

Lemma 7.2. *Consider a Poisson degree 2 manifold* $(\mathcal{M}, \{\cdot,\cdot\})$ *and its corresponding metric VB-algebroid* $(D_{F^*}, [\cdot,\cdot], \rho)$*. Recall that a horizontal lift of (3.14) provides simultaneously a splitting for* $\mathcal{M}$*, a decomposition for* D_{F^*} *and a horizontal lift for (3.44). This last horizontal lift induces a projection* $p_{E^*\otimes F} : \widehat{E} \longrightarrow E^* \otimes F$*. With this considerations,* *we have*

$$\langle \phi, [\gamma, \widehat{\theta}] \rangle = \langle \phi, \widehat{\{\gamma, \theta\}} \rangle + \Big(p_{E^* \otimes F} (\mathcal{L}_\gamma \widehat{\pi(\phi)}) \Big)^* \circ (\theta_1^\sharp)^*, \qquad \forall \gamma \in \mathcal{A}^2, \theta \in \mathcal{A}^3, \phi \in \widehat{E}, \quad (7.2)$$

where, for any degree 3 function $\theta \in \mathcal{A}^3$, $\widehat{\theta} \in \Gamma(\Lambda^2(D_{F^*})_E)$ *is the embedding (3.67), described in some detail in the proof of Thm. 3.59, and we are denoting the same* γ *for a degree 2 function* $\gamma \in \mathcal{A}^2$ *and its canonically corresponding involutive section* $\gamma \in \Gamma(\widetilde{F}^*) \subset \Gamma(\widehat{F^*}) \cong \Gamma_{\text{lin}}((D_{F^*})_E)$. $\mathcal{L}$ *stands for the Lie derivative on* $\Gamma(\Lambda^\cdot (D_{F^*})^*_E)$, *associated to the Lie algebroid structure on* $D_{F^*} \longrightarrow E$.

Proof. Since both sides of (7.2) are $\mathbb{R}$-bilinear, it is suffices to show the following two fundamental cases

i) $\widehat{\theta} = e \otimes \zeta$, with $\varepsilon \in \Gamma(E^*)$ and $\zeta \in \Gamma(F)$.

ii) $\widehat{\theta} = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$, with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma(E^*)$.

Let's prove item *i*). Again because of $\mathbb{R}$-bilinearity of the brackets, we can assume $\gamma = \zeta_0 + \varepsilon_1 \wedge \varepsilon_2$, $\zeta_0 \in \Gamma(F)$ and $\varepsilon_1, \varepsilon_2 \in \Gamma(E^*)$. Then, taking into account the relation between the VB-algebroid data and the corresponding -2 Poisson brackets (see Thms. 6.1 and 6.14), and using also Eq. (6.9) or equivalently (E.15), we have

$$\begin{aligned}
[\gamma, \widehat{\theta}] &= [\gamma, \varepsilon \otimes \zeta] = [\gamma, \varepsilon] \otimes \zeta + \varepsilon \otimes [\gamma, \zeta] \\
&= \{\gamma, \varepsilon\} \otimes \zeta + \varepsilon \otimes [\zeta_0, \zeta] + \overline{\varepsilon} \wedge \overline{\varepsilon_1} \otimes \{\varepsilon_2, \widehat{\zeta}\} - \overline{\varepsilon} \wedge (\varepsilon_1 \wedge \{\varepsilon_2, \widehat{\zeta}\}) \\
&\quad + \overline{\varepsilon} \wedge (\{\varepsilon_1, \widehat{\zeta}\} \wedge \varepsilon_2) - \overline{\varepsilon} \wedge K(\zeta_0, \zeta) \\
&= \widehat{\{\gamma, \theta\}} - \varepsilon \otimes \overline{\varepsilon_1} \wedge \overline{\{\varepsilon_2, \widehat{\zeta}\}} - \varepsilon \otimes \overline{\{\varepsilon_1, \widehat{\zeta}\}} \wedge \overline{\varepsilon_2} + \varepsilon \otimes K(\zeta_0, \zeta) \\
&= \widehat{\{\gamma, \theta\}} - \overline{p_{\Lambda^2 E^*}(\{\gamma, \widehat{\zeta}\})} \otimes \varepsilon,
\end{aligned}$$

where $p_{\Lambda^2 E^*} : \widetilde{F}^* \longrightarrow \Lambda^2 E^*$ is the projection induced by the horizontal lift, and the bar means that we are considering $\Lambda^2 E^*$ as the second exterior power of the core bundle $E^* \subset D_{F^*}$. Therefore, writing, through the horizontal lift, $\phi = \widehat{e} + \eta$, we arrive to

$$\langle \phi, [\gamma, \widehat{\theta}] \rangle = \langle \phi, \widehat{\{\gamma, \theta\}} \rangle - \varepsilon \otimes (\langle \widehat{e}, \{\gamma, \widehat{\zeta}\} \rangle). \tag{7.3}$$

On the other hand, using Eq. (G.2),

$$\Big(p_{E^* \otimes F} (\mathcal{L}_\gamma \widehat{\pi(\phi)}) \Big)^* \circ (\theta_1^\sharp)^* = \varepsilon \otimes \langle \mathcal{L}_\gamma \widehat{e}, \zeta \rangle = -\varepsilon \otimes (\langle \widehat{e}, \{\gamma, \widehat{\zeta}\} \rangle). \tag{7.4}$$

From Eqs. (7.3) and (7.4), we get (7.2) in case *i*).

In the above calculations, when performing the bracket with ζ, we are considering implicitly its lift into $\widetilde{F}^*$.

Now let's work out case *ii*). In this case we have $\theta_1 = 0$. Hence (7.2) is equivalent to $[\gamma, \widehat{\theta}] = \widehat{\{\gamma, \theta\}}$. Again recalling the relation between the VB-algebroid structure and the corresponding -2 Poisson brackets, and also the interpretation of $\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ as a 2-section of $(D_{F^*})_{E^*}$ given in Eq. (3.73), we have

$$
\begin{aligned}
[\gamma, \widehat{\theta}] &= [\gamma, \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] = [\gamma, \overline{\varepsilon_1} \wedge \overline{\varepsilon_2} \otimes \varepsilon_3 - \overline{\varepsilon_1} \wedge \overline{\varepsilon_3} \otimes \varepsilon_2 + \overline{\varepsilon_2} \wedge \overline{\varepsilon_3} \otimes \varepsilon_1] \\
&= \rho(\gamma)(\varepsilon_3) \otimes \overline{\varepsilon_1} \wedge \overline{\varepsilon_2} + \varepsilon_3 \otimes [\gamma, \overline{\varepsilon_1}] \wedge \overline{\varepsilon_2} + \varepsilon_3 \otimes \overline{\varepsilon_1} \wedge [\gamma, \overline{\varepsilon_2}] \\
&\quad - \rho(\gamma)(\varepsilon_2) \otimes \overline{\varepsilon_1} \wedge \overline{\varepsilon_3} - \varepsilon_2 \otimes [\gamma, \overline{\varepsilon_1}] \wedge \overline{\varepsilon_3} - \varepsilon_2 \otimes \overline{\varepsilon_1} \wedge [\gamma, \overline{\varepsilon_3}] \\
&\quad \rho(\gamma)(\varepsilon_1) \otimes \overline{\varepsilon_2} \wedge \overline{\varepsilon_3} + \varepsilon_1 \otimes [\gamma, \overline{\varepsilon_2}] \wedge \overline{\varepsilon_3} + \varepsilon_1 \otimes \overline{\varepsilon_2} \wedge [\gamma, \overline{\varepsilon_3}] \\
&= \varepsilon_1 \wedge \varepsilon_2 \wedge \{\gamma, \varepsilon_3\} + \{\gamma, \varepsilon_1\} \wedge \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_1 \wedge \{\gamma, \varepsilon_2\} \wedge \varepsilon_3 \\
&= \widehat{\{\gamma, \theta\}}.
\end{aligned}
$$

■

Lemma 7.3. *With the same considerations made in the statement of Lem. 7.2, for every* $\varepsilon \in \Gamma(E^*)$*, we have*

$$
[\overline{\varepsilon}, \widehat{\theta}] = -\overline{p_{\Lambda^2 E^*}(\{\theta, \varepsilon\})} = -\overline{Q(\varepsilon) + (\theta_1^\sharp)^*(\sharp(\varepsilon))}, \tag{7.5}
$$

where, as already explained in the proof of lemma 7.2, $p_{\Lambda^2 E^*} : \widetilde{F} \longrightarrow \Lambda^2 E^*$ *is the projection induced by the horizontal lift, and the bar means that we are considering* $\Lambda^2 E^*$ *as the second exterior power of the core bundle* $E^* \subset D_{F^*}$*, so that* $\Lambda^2 E^* \subset \Lambda^2 (D_{F^*})_E$*.* $\sharp : E^* \longrightarrow E$ *is the core map given in Eq. (6.17), by the metric that comes from the Poisson structure.*

Proof. As we already did in the previous lemma, we can suppose without loss of generality that $\theta = \varepsilon_0 \otimes \zeta + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$. Thus, we compute

$$
\begin{aligned}
\{\varepsilon, \varepsilon_0 \otimes \zeta + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3\} =& \{\varepsilon, \varepsilon_0\} - \varepsilon_0 \wedge \{\varepsilon, \zeta\} \\
&+ \{\varepsilon, \varepsilon_1\}\varepsilon_2 \wedge \varepsilon_3 - \{\varepsilon, \varepsilon_2\}\varepsilon_1 \wedge \varepsilon_3 + \{\varepsilon, \varepsilon_3\}\varepsilon_1 \wedge \varepsilon_2.
\end{aligned}
$$

On the other hand, taking into account Eq. (3.73), we compute using Leibniz's rule for the Schouten bracket and Eqs. (6.17)

$$
\begin{aligned}
[\overline{\varepsilon} \otimes \zeta + \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] =& [\overline{\varepsilon}, \overline{\varepsilon_0} \otimes \zeta] + [\overline{\varepsilon}, \varepsilon_1 \otimes \overline{\varepsilon_2} \wedge \overline{\varepsilon_3} - \varepsilon_2 \otimes \overline{\varepsilon_1} \wedge \overline{\varepsilon_3} + \varepsilon_3 \otimes \overline{\varepsilon_1} \wedge \overline{\varepsilon_2}] \\
=& \overline{\varepsilon_0} \wedge [\overline{\varepsilon}, \zeta] - \langle \varepsilon, \varepsilon_1 \rangle \overline{\varepsilon_2} \wedge \overline{\varepsilon_3} + \langle \varepsilon, \varepsilon_2 \rangle \overline{\varepsilon_1} \wedge \overline{\varepsilon_3} - \langle \varepsilon, \varepsilon_3 \rangle \overline{\varepsilon_1} \wedge \overline{\varepsilon_2}.
\end{aligned}
$$

Comparing the two equations above we conclude that

$$
[\overline{\varepsilon}, \widehat{\theta}] = \overline{-\{\varepsilon, \theta\} + \{\varepsilon, \varepsilon_0\}\zeta} = -\overline{p_{\Lambda^2 E^*}(\{\theta, \varepsilon\})}.
$$

The second equality in Eq. (7.5) follows from the observation that $Q(\varepsilon) = \{\theta, \varepsilon\} = \{\varepsilon, \theta\}$ and $\{\varepsilon, \varepsilon_0\}\zeta = -(\theta_1^\sharp)^*(\sharp(\varepsilon))$.

■

Lemma 7.4. *On an exact Lie 2-algebroid we have*

$$
\widehat{\rho}(\phi)(f) = \rho(\theta_2^\sharp(\phi))(f) \quad \text{and} \quad \rho^*(d_{T^*M} f) = \theta_1^\sharp(df), \qquad \forall \phi \in \Gamma(\widehat{E}), f \in C^\infty(M). \tag{7.6}
$$

Here we are using the same notation ρ for two different maps. In the first equation it stands for the anchor of the metric VB-algebroid $\rho_D : D_{F^*} \longrightarrow TE$, given in Eq. (6.17). In the second equation it stands for the anchor $\rho : E \longrightarrow TM$ defined in Eq. (4.9).

In the second equation, the operator d stands for the de Rham differential corresponding to the metric VB-algebroid. The operator d_{T^*M} stands for the canonical de Rham differential of the cotangent bundle.

Proof. First observe that Eq. (6.42) is valid in the Poisson case in general, since for its proof we didn't use the non-denegeracy of the Poisson brackets. Using this equation, the definitions, and item 1 of Thm. 3.59, we have

$$\begin{aligned}\widehat{\rho}(\phi)(f) &= -\langle \phi, \overline{Q(f)}\rangle = -\langle \phi, \overline{\{\theta, f\}}\rangle = \langle \phi, \theta_1^\sharp(df)\rangle \\ &= \langle \theta_2^\sharp(\phi), df\rangle = \rho(\theta_2^\sharp(\phi))(f).\end{aligned}$$

Analogously,

$$\rho(e)(f) = -\langle e, Q(f)\rangle = -\langle e, \{\theta, f\}\rangle = \langle e, \theta_1^\sharp(df)\rangle,$$

from which it follows $\rho^*(d_{T^*M}f) = \theta_1^\sharp(df)$.

■

Lemma 7.5. *On an exact Lie 2-algebroid*

$$\Psi(\phi)(\varepsilon) = \rho(\theta_2^\sharp(\phi))(\varepsilon) - \theta_1^\sharp(W(\phi, d\varepsilon)) \tag{7.7}$$

holds for every $\phi \in \Gamma(\widehat{E}), \varepsilon \in \Gamma(E^*)$.

Proof. Ψ is defined in Eq. (4.11). Using Eq. (6.42), which is valid in the general Poisson case as already observed in the proof of lemma 7.4, and item 3 of Thm. 3.59, we have

$$\begin{aligned}\Psi(\phi)(\varepsilon) &= -\langle \phi, Q(\varepsilon)\rangle = -\langle \phi, \{\theta, \varepsilon\}\rangle \\ &= -\langle \phi, \theta_2^\sharp(d\varepsilon)\rangle \\ &= \langle \theta_2^\sharp(\phi), d\varepsilon\rangle - \theta_1^\sharp(W(\phi, d\varepsilon)) \\ &= \rho(\theta_2^\sharp(\phi))(\varepsilon) - \theta_1^\sharp(W(\phi, d\varepsilon)).\end{aligned}$$

■

Theorem 7.6. *Let $(\widehat{E}, \widehat{\rho}, \partial, \Psi, \Theta, [\cdot,\cdot])$ be an exact Lie 2-algebroid. Then the bracket $[\cdot,\cdot]$ is given by the explicit formula*

$$[\phi_1, \phi_2] = \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1 + d(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) - R_\theta(\phi_1, \phi_2), \tag{7.8}$$

where d and $\mathcal{L}$ are the de Rham differential and Lie derivative, respectively, corresponding to the metric VB-algebroid D_{F^*} that comes from the Poisson degree 2 manifold under the correspondence given in Thm. 6.14.

The (bidifferential) operator $R_\theta : \Gamma(\widehat{E}) \times \Gamma(\widehat{E}) \longrightarrow \Gamma(\widehat{E})$ is defined by

$$\begin{aligned}\langle R_\theta(\phi_1,\phi_2),\gamma\rangle := \theta_1^\sharp(&W(\phi_1, d(\langle\phi_2,\gamma\rangle)) - W(\phi_2, d(\langle\phi_1,\gamma\rangle)) - d(\langle T(\phi_2,\phi_1),\gamma\rangle)\\ &+ W(\mathcal{L}_\gamma\phi_1,\phi_2))\\ = \theta_1^\sharp(&W(\phi_2,\langle\gamma,d\phi_1\rangle) + \langle\overline{\langle\phi_2,\gamma\rangle}, d\phi_1\rangle), \qquad \forall\phi_1,\phi_2\in\Gamma(\widehat{E}),\gamma\in\Gamma(\widetilde{F}^*)\end{aligned} \tag{7.9}$$

where $W : S^2\widehat{E} \longrightarrow F$ was defined in Prop. 3.48 and $T : \widehat{E}\otimes\widehat{E} \longrightarrow \widetilde{F}$ is the bilinear map defined in Eq. (3.45).

Remark 7.7. Observe that Eq. (7.51) is *not enough* to define the operator R_θ, for the simple reason that a linear section $\phi \in \Gamma(\widehat{E})$ is not completely determined by its pairing with *involutive* sections $\Gamma(\widetilde{F}^*)$ only, because the pairing with sections of Λ^2E^* is not enough to determine the part of ϕ that projects over E (consider, for example, the case that $\operatorname{rank} E = 1$, then $\Lambda^2E^* = 0$). In the next theorem, Thm. 7.6, we will show that R_θ takes values actually on $E^*\otimes F$, so that the pairing above does determine the value of R_θ.

Another observation, which follows from Eq. (7.51), is that R_θ is tensorial on its second entry.

Remark 7.8. The same formulas (7.8) and (7.51) hold in the case of an exact *pre*Lie 2-algebroid. Integrability is not used anywhere in the proof.

Proof. The idea is to introduce a horizontal lift, which allows to correspond to the 3-hamiltonian θ a bivector $\widehat{\theta} \in \Gamma(\Lambda^2(D_{F^*})_E)$, as we did in the proof of Thm. 3.59, and exploiting the identity

$$\langle\phi,\widehat{\theta}\rangle = -\theta_2^\sharp(\phi) + (\theta_1^\sharp\circ\eta)^* = -\theta_2^\sharp(\phi) + \eta^*\circ(\theta_1^\sharp)^*, \tag{7.10}$$

for every $\phi = \widehat{e} + \eta \in \widehat{E}$ —which follows from the very way we define $\theta_2^\sharp$ in Eqs. (3.68)-(3.72)—, we can use the results for bivectors already available from sections G.2, G.3 and G.4. In particular, we are going to use Eq. (G.13), which applied to the bivector $\widehat{\theta}$ gives

$$\langle\phi_2,\langle\phi_1,[\gamma,\widehat{\theta}]\rangle\rangle = \rho(\gamma)(\langle\phi_2,\langle\phi_1,\widehat{\theta}\rangle\rangle) - \langle\phi_2,\langle\mathcal{L}_\gamma\phi_1,\widehat{\theta}\rangle\rangle - \langle\mathcal{L}_\gamma\phi_2,\langle\phi_1,\widehat{\theta}\rangle\rangle. \tag{7.11}$$

Now, applying Eq. (7.10) to the degree 3 function $-Q(\gamma) = \{\gamma,\theta\}$, we get

$$\langle\phi_1,\widehat{\{\gamma,\theta\}}\rangle = Q(\gamma)_2^\sharp(\phi_1) - \eta_1^*\circ(Q(\gamma)_1^\sharp)^*. \tag{7.12}$$

On the other hand, from the identity

$$\mathcal{L}_\gamma(\varepsilon\otimes\xi) = \varepsilon\otimes\mathcal{L}_\gamma\xi + \langle d\varepsilon,\gamma\rangle\otimes\xi,$$

which is obtained immediately from Cartan's formula (G.21), and lemma 7.10 below, it follows that $\mathcal{L}_\gamma$ preserves $E^*\otimes F \subset \widehat{E}$, hence

$$p_{E^*\otimes F}(\mathcal{L}_\gamma\phi_1) = p_{E^*\otimes F}(\mathcal{L}_\gamma\widehat{e_1}) + \mathcal{L}_\gamma\eta_1. \tag{7.13}$$

From Eqs. (7.12) and (7.13), applying lemma 7.2 and Eq. (7.10) to θ, Eq. (7.11) turns into

$$\begin{aligned}-\langle\phi_2, Q(\gamma)_2^\sharp(\phi_1) - \eta_1^*\circ(Q(\gamma)_1^\sharp)^* + (p_{E^*\otimes F}(\mathcal{L}_\gamma\widehat{e_1}))^*\circ(\theta_1^\sharp)^*\rangle = \rho(\gamma)(\langle\phi_2, \theta_2^\sharp(\phi_1) - \eta_1^*\circ(\theta_1^\sharp)^*\rangle)\\ -\langle\phi_2, \theta_2^\sharp(\mathcal{L}_\gamma\phi_1) - (p_{E^*\otimes F}(\mathcal{L}_\gamma\widehat{e_1}) + \mathcal{L}_\gamma\eta_1)^*\circ(\theta_1^\sharp)^*\rangle - \langle\mathcal{L}_\gamma\phi_2, \theta_2^\sharp(\phi_1) - \eta_1^*\circ(\theta_1^\sharp)^*\rangle,\end{aligned}$$

from which, using Eq. (G.2),

$$\begin{aligned}-\langle\phi_2, Q(\gamma)_2^\sharp(\phi_1)\rangle &= \langle\phi_2, [\gamma, \theta_2^\sharp(\phi_1)]\rangle - \langle\phi_2, \theta_2^\sharp(\mathcal{L}_\gamma\phi_1) - (\mathcal{L}_\gamma\eta_1)^*\circ(\theta_1^\sharp)^*\rangle\\ &\quad - \rho(\gamma)(\langle\phi_2, \eta_1^*\circ(\theta_1^\sharp)^*\rangle) + \langle\mathcal{L}_\gamma\phi_2, \eta_1^*\circ(\theta_1^\sharp)^*\rangle - \langle\phi_2, \eta_1^*\circ(Q(\gamma)_1^\sharp)^*\rangle\\ &= \langle\phi_2, [\gamma, \theta_2^\sharp(\phi_1)]\rangle - \langle\phi_2, \theta_2^\sharp(\mathcal{L}_\gamma\phi_1) - (\mathcal{L}_\gamma\eta_1)^*\circ(\theta_1^\sharp)^*\rangle\\ &\quad - \langle\phi_2, [\gamma, \eta_1^*\circ(\theta_1^\sharp)^*]\rangle - \langle\phi_2, \eta_1^*\circ(Q(\gamma)_1^\sharp)^*\rangle. \end{aligned} \tag{7.14}$$

Remark 7.9. In the calculations that follow, we are going to assume that $\eta_1 = \varepsilon_1\otimes\xi_1$ and $\widehat{\theta_1} = \varepsilon\otimes\zeta$, where $\varepsilon_1, \varepsilon\in\Gamma(E^*)$, $\xi_1\in\Gamma(F^*)$ and $\zeta\in\Gamma(F)$. By the same argument that we explained in the proof of lemma 7.2, there is no loose of generality in making this assumption.

With the above remark made, observe that, since

$$W(\eta_1, \phi_2) = \eta_1(e_2) = \langle e_2, \varepsilon_1\rangle\xi_1 \quad\text{and}\quad \eta_1^*\circ(\theta_1^\sharp)^* = \langle\xi_1, \zeta\rangle\overline{\varepsilon_1}\otimes\varepsilon, \tag{7.15}$$

it follows the identity

$$\langle\phi_2, \eta_1^*\circ(\theta_1^\sharp)^*\rangle = \langle e_2, \varepsilon_1\rangle\langle\xi_1, \zeta\rangle\varepsilon = \langle e_2, \varepsilon_1\rangle\theta_1^\sharp(\xi_1) = \theta_1^\sharp(W(\eta_1, \phi_2)). \tag{7.16}$$

Also observe that, since $\mathcal{L}_\gamma$ preserves $E^*\otimes F\subset\widehat{E}$, we can apply Eq. (7.16) to $\mathcal{L}_\gamma\eta_1$, so that

$$\langle\phi_2, (\mathcal{L}_\gamma\eta_1)^*\circ(\theta_1^\sharp)^*\rangle = \theta_1^\sharp(W(\mathcal{L}_\gamma\eta_1, \phi_2)). \tag{7.17}$$

On the other hand, from Eq. (G.2) and item 3 of Thm. 3.59, we obtain

$$\begin{aligned}\rho(\gamma)(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle) &= \langle\mathcal{L}_\gamma\phi_1, \theta_2^\sharp(\phi_2)\rangle + \langle\phi_1, [\gamma, \theta_2^\sharp(\phi_2]\rangle\\ &= -\langle\phi_2, \theta_2^\sharp(\mathcal{L}_\gamma\phi_1)\rangle + \theta_1^\sharp(W(\mathcal{L}_\gamma\phi_1, \phi_2)) + \langle\phi_1, [\gamma, \theta_2^\sharp(\phi_2)]\rangle. \end{aligned} \tag{7.18}$$

Putting together Eqs. (7.17) and (7.18), we get

$$\begin{aligned}\langle\phi_2, \theta_2^\sharp(\mathcal{L}_\gamma\phi_1) - (\mathcal{L}_\gamma\eta_1)^*\circ(\theta_1^\sharp)^*\rangle &= -\rho(\gamma)(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle) + \theta_1^\sharp(W(\mathcal{L}_\gamma\phi_1, \phi_2))\\ &\quad + \langle\phi_1, [\gamma, \theta_2^\sharp(\phi_2)]\rangle - \theta_1^\sharp(W(\mathcal{L}_\gamma\eta_1, \phi_2)). \end{aligned} \tag{7.19}$$

Finally, we need to compute the last two terms of Eq. (7.14). We begin with the first of them. Using (7.15) we have

$$[\gamma, \eta_1^*\circ(\theta_1^\sharp)^*] = \rho(\gamma)(\langle\xi_1, \zeta\rangle)\varepsilon\otimes\overline{\varepsilon_1} + \langle\xi_1, \zeta\rangle\rho(\gamma)(\varepsilon)\otimes\overline{\varepsilon_1} + \langle\xi_1, \zeta\rangle\varepsilon\otimes[\gamma, \overline{\varepsilon_1}],$$

we obtain

$$\langle\phi_2, [\gamma, \eta_1^* \circ (\theta_1^\sharp)^*]\rangle = \rho(\gamma)(\langle\xi_1, \zeta\rangle)\langle e_2, \varepsilon_1\rangle\varepsilon + \langle\xi_1, \zeta\rangle\langle e_2, \varepsilon_1\rangle\rho(\gamma)(\varepsilon) + \langle\xi_1, \zeta\rangle\langle e_2, [\gamma, \overline{\varepsilon_1}]\rangle\varepsilon. \tag{7.20}$$

As for the second term of the last part of Eq. (7.14), we have

$$\begin{aligned}\langle\phi_2, \eta_1^* \circ (Q(\gamma)_1^\sharp)^*\rangle &= -\langle\phi_2, \eta_1^* \circ ([\gamma, \widehat{\theta}]_1^\sharp)^*\rangle = -\langle\phi_2, \langle\xi_1, \zeta\rangle\rho(\gamma)(\varepsilon) \otimes \overline{\varepsilon_1} + \langle\xi_1, [\gamma, \zeta]\rangle\varepsilon \otimes \overline{\varepsilon_1}\rangle \\ &= -\langle\xi_1, \zeta\rangle\langle e_2, \varepsilon_1\rangle\rho(\gamma)(\varepsilon) - \langle\xi_1, [\gamma, \zeta]\rangle\langle e_2, \varepsilon_1\rangle\varepsilon. \end{aligned}\tag{7.21}$$

Putting Eqs. (7.20) and (7.21), we get

$$\begin{aligned}\langle\phi_2, [\gamma, \eta_1^* \circ (\theta_1^\sharp)^*]\rangle + \langle\phi_2, \eta_1^* \circ (Q(\gamma)_1^\sharp)^*\rangle &= \rho(\gamma)(\langle\xi_1, \zeta\rangle)\langle e_2, \varepsilon_1\rangle\varepsilon + \langle\xi_1, \zeta\rangle\langle e_2, \rho(\gamma)(\varepsilon_1)\rangle\varepsilon \\ &\quad - \langle\xi_1, [\gamma, \zeta]\rangle\langle e_2, \varepsilon_1\rangle\varepsilon, \end{aligned}\tag{7.22}$$

where we used the fact that, on a metric VB-algebroid, $[\gamma, \overline{\varepsilon_1}] = \overline{\rho(\gamma)(\varepsilon_1)}$ holds for every $\gamma \in \Gamma(\widetilde{F}^*)$ and every $\varepsilon_1 \in \Gamma(E^*)$. On the other hand, observe that

$$\begin{aligned}\langle\mathcal{L}_\gamma\eta_1, \zeta\rangle &= \langle\mathcal{L}_\gamma(\varepsilon_1 \otimes \xi_1), \zeta\rangle = \rho(\gamma)(\langle\varepsilon_1 \otimes \xi_1, \zeta\rangle) - \langle\varepsilon_1 \otimes \xi_1, [\gamma, \zeta]\rangle \\ &= \rho(\gamma)(\langle\xi_1, \zeta\rangle)\varepsilon_1 + \langle\xi_1, \zeta\rangle\rho(\gamma)(\varepsilon_1) - \langle\xi_1, [\gamma, \zeta]\rangle\varepsilon_1,\end{aligned}$$

whence

$$\begin{aligned}\langle\phi_2, (\mathcal{L}_\gamma\eta_1)^* \circ (\theta_1^\sharp)^*\rangle =& \langle e_2, \varepsilon_1\rangle\rho(\gamma)(\langle\xi_1, \zeta\rangle)\varepsilon + \langle e_2, \rho(\gamma)(\varepsilon_1)\rangle\langle\xi_1, \zeta\rangle\varepsilon \\ &- \langle e_2, \varepsilon_1\rangle\langle\xi_1, [\gamma, \zeta]\rangle\varepsilon. \end{aligned}\tag{7.23}$$

Comparing Eqs. (7.22) and (7.23), we conclude that

$$\langle\phi_2, [\gamma, \eta_1^* \circ (\theta_1^\sharp)^*]\rangle + \langle\phi_2, \eta_1^* \circ (Q(\gamma)_1^\sharp)^*\rangle = \langle\phi_2, (\mathcal{L}_\gamma\eta_1)^* \circ (\theta_1^\sharp)^*\rangle = \theta_1^\sharp(W(\mathcal{L}_\gamma\eta_1, \phi_2)). \tag{7.24}$$

Therefore, putting (7.19) and (7.24) into (7.14) we arrive to the identity

$$\begin{aligned}-\langle\phi_2, Q(\gamma)_2^\sharp(\phi_1)\rangle =& \langle\phi_2, [\gamma, \theta_2^\sharp(\phi_1)]\rangle - \langle\phi_1, [\gamma, \theta_2^\sharp(\phi_2)]\rangle \\ &+ \rho(\gamma)(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle) - \theta_1^\sharp(W(\mathcal{L}_\gamma\phi_1, \phi_2)). \end{aligned}\tag{7.25}$$

Now, from Eqs. (7.7) and (G.2), we get

$$\begin{aligned}\Psi(\phi_1)(\langle\phi_2, \gamma\rangle) &= \rho(\theta_2^\sharp(\phi_1))(\langle\phi_2, \gamma\rangle) - \theta_1^\sharp(W(\phi_1, d(\langle\phi_2, \gamma\rangle))) \\ &= \langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \gamma\rangle + \langle\phi_2, [\theta_2^\sharp(\phi_1), \gamma]\rangle - \theta_1^\sharp(W(\phi_1, d(\langle\phi_2, \gamma\rangle))). \end{aligned}\tag{7.26}$$

Finally, putting Eqs. (7.25) and (7.26) into Eq. (4.14), and also using Eq. (7.6), we have the following

$$\begin{aligned}\langle[\phi_1, \phi_2], \gamma\rangle &= \langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \gamma\rangle - \theta_1^\sharp(W(\phi_1, d(\langle\phi_2, \gamma\rangle))) \\ &\quad - \langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1, \gamma\rangle + \theta_1^\sharp(W(\phi_2, d(\langle\phi_1, \gamma\rangle))) \\ &\quad + \theta_1^\sharp(d(\langle T(\phi_2, \phi_1), \gamma\rangle)) + \rho(\gamma)(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle) - \theta_1^\sharp(W(\mathcal{L}_\gamma\phi_1, \phi_2)) \\ &= \langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1 + d(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle), \gamma\rangle \\ &\quad - \theta_1^\sharp(W(\phi_1, d(\langle\phi_2, \gamma\rangle)) - W(\phi_2, d(\langle\phi_1, \gamma\rangle)) \\ &\qquad - d(\langle T(\phi_2, \phi_1), \gamma\rangle) + W(\mathcal{L}_\gamma\phi_1, \phi_2)).\end{aligned}$$

Thus it remains to show the second equality of Eq. (7.51), which is equivalent to

$$\begin{aligned} &W(\phi_1, d(\langle \phi_2, \gamma\rangle)) - W(\phi_2, d(\langle \phi_1, \gamma\rangle)) - d(\langle \phi_1, \overline{\langle \phi_2, \gamma\rangle}\rangle) + W(\mathcal{L}_\gamma \phi_1, \phi_2) \\ &\quad = W(\phi_2, \langle \gamma, d\phi_1\rangle) + \langle \overline{\langle \phi_2, \gamma\rangle}, d\phi_1\rangle. \end{aligned} \tag{7.27}$$

On one hand we have, by Cartan's formula,

$$-W(\phi_2, d(\langle \phi_1, \gamma\rangle)) + W(\mathcal{L}_\gamma \phi_1, \phi_2) = W(\phi_2, \mathcal{L}_\gamma \phi_1 - d(\langle \phi_1, \gamma\rangle)) = W(\phi_2, \langle \gamma, d\phi_1\rangle), \tag{7.28}$$

On the other hand, by Eqs. (3.46) and (6.17), for every $\zeta \in \Gamma(F^*)$, we have

$$\begin{aligned} \langle W(\phi_1, d(\langle \phi_2, \gamma\rangle)), \zeta\rangle &= \left\langle \phi_1, \overline{\langle d(\langle \phi_2, \gamma\rangle), \widehat{\zeta}\rangle}\right\rangle + \left\langle d(\langle \phi_2, \gamma\rangle), \overline{\langle \phi_1, \widehat{\zeta}\rangle}\right\rangle \\ &= \left\langle \phi_1, \overline{\rho(\widehat{\zeta})(\langle \phi_2, \gamma\rangle)}\right\rangle + \rho_D\left(\overline{\langle \phi_1, \gamma\rangle}\right)(\langle \phi_2, \gamma\rangle) \\ &= \left\langle \phi_1, \overline{\rho(\widehat{\zeta})(\langle \phi_2, \gamma\rangle)}\right\rangle - \left\langle \langle \phi_1, \widehat{\zeta}\rangle, \langle \phi_2, \gamma\rangle\right\rangle_D. \end{aligned} \tag{7.29}$$

By Cartan's formula, Eq. (G.2) and Eq. (6.17),

$$\begin{aligned} -\langle d(\langle \phi_1, \overline{\langle \phi_2, \gamma\rangle}), \widehat{\zeta}\rangle &= \left\langle -\mathcal{L}_{\overline{\langle \phi_2, \gamma\rangle}} \phi_1 + \iota_{\overline{\langle \phi_2, \gamma\rangle}} d\phi_1, \widehat{\zeta}\right\rangle \\ &= \left\langle \phi_1, \left[\overline{\langle \phi_2, \gamma\rangle}, \widehat{\zeta}\right]\right\rangle - \rho_D\left(\overline{\langle \phi_2, \gamma\rangle}\right)\left(\langle \phi_1, \widehat{\zeta}\rangle\right) + \left\langle \langle \overline{\langle \phi_2, \gamma\rangle}, d\phi_1\rangle, \widehat{\zeta}\right\rangle \\ &= -\left\langle \phi_1, \overline{\rho(\widehat{\zeta})(\langle \phi_2, \gamma\rangle)}\right\rangle + \left\langle \langle \phi_2, \gamma\rangle, \langle \phi_1, \widehat{\zeta}\rangle\right\rangle_D + \left\langle \langle \overline{\langle \phi_2, \gamma\rangle}, d\phi_1\rangle, \widehat{\zeta}\right\rangle. \end{aligned} \tag{7.30}$$

From (7.29) and (7.30), we get

$$W(\phi_1, d(\langle \phi_2, \gamma\rangle)) - d(\langle \phi_1, \overline{\langle \phi_2, \gamma\rangle}\rangle = \langle \overline{\langle \phi_2, \gamma\rangle}, d\phi_1\rangle. \tag{7.31}$$

Hence, from Eqs. (7.28) and (7.31) we arrive to Eq. (7.27).

■

Lemma 7.10. *Consider a VB-algebroid $(D; A, B; M)_C$, with Lie algebroid structure over B, as usual. If $\mathcal{L}$ is the corresponding Lie derivative, then*

$$\mathcal{L}_{\overline{c}}\phi \in \Gamma_{\text{core}}(D_B^*) \cong \Gamma(A^*) \qquad \forall \overline{c} \in \Gamma_{\text{core}}(D_B) \cong \Gamma(C), \phi \in \Gamma_{\text{lin}}(D_B^*) \cong \widehat{C^*}_B,$$

and

$$\mathcal{L}_\gamma \overline{\kappa} \in \Gamma_{\text{core}}(D_B^*) \cong \Gamma(A^*), \qquad \forall \overline{\kappa} \in \Gamma_{\text{core}}(D_B^*) \cong \Gamma(A^*), \gamma \in \Gamma_{\text{lin}}(D_B) \cong \Gamma(\widehat{A}).$$

Proof. By lemma 2.8, in order to prove the first statement, we need to check that $\langle \mathcal{L}_{\overline{c}}\phi, \overline{c_0}\rangle = 0 \;\forall \overline{c_0} \in \Gamma_{\text{core}}(D_B)$ and $\langle \mathcal{L}_{\overline{c}}\phi, \gamma\rangle$ is fiberwise constant for every $\gamma \in \Gamma_{\text{lin}}(D_B)$. By Eq. (G.2), the definition of a VB-algebroid and lemma 2.8,

$$\langle \mathcal{L}_{\overline{c}}\phi, \overline{c_0}\rangle = \rho(\overline{c})(\langle \phi, \overline{c_0}\rangle - \langle \phi, [\overline{c}, \overline{c_0}]\rangle = 0.$$

Analogously,

$$\langle \mathcal{L}_{\overline{c}}\phi, \gamma\rangle = \rho(\overline{c})(\langle \gamma, \phi\rangle) - \langle \phi, [\overline{c}, \gamma]\rangle,$$

which is fiberwise constant due to the definition of VB-algebroid and to lemma 2.8.

Similarly it is proved that $\mathcal{L}_\gamma \overline{\kappa} \in \Gamma_{\text{core}}(D_B^*)$.

■

Theorem 7.11 (Continuation of Thm. 7.6)**.** *In the conditions and with the notations of Thm. 7.6, we have, for every* $\phi_1, \phi_2 \in \Gamma(\widehat{E})$ *and* $\varepsilon \in \Gamma(E^*)$,

$$\langle [\phi_1, \phi_2], \overline{\varepsilon} \rangle = \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)} \phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)} \phi_1 + d(\langle \phi_1, \theta_2^\sharp(\phi_2) \rangle), \overline{\varepsilon} \right\rangle. \tag{7.32}$$

In particular, it follows that the operator R_θ *takes values on* $\Gamma(E^* \otimes F)$.

Proof. From Eqs. (G.13) and (7.5), we obtain, taking into account lemma 7.10, Eq. (3.68) and Eq. (7.10),

$$\begin{aligned} -\langle \phi_2, \overline{\langle \phi_1, -Q(\varepsilon) - (\theta_1^\sharp)^*(\sharp(\varepsilon)) \rangle} \rangle =& \rho(\overline{\varepsilon})(\langle \phi_2, \theta_2^\sharp(\phi_1) - \eta_1^* \circ (\theta_1^\sharp)^* \rangle) + \langle \phi_2, \theta_1^\sharp(\mathcal{L}_{\overline{\varepsilon}} \phi_1) \rangle \\ & - \langle \mathcal{L}_{\overline{\varepsilon}} \phi_2, \theta_2^\sharp(\phi_1) \rangle, \end{aligned}$$

from which, using Eq. (G.2), lemma 7.10 again, and Prop. E.8,

$$\begin{aligned} \langle \phi_2, \overline{\langle \phi_1, Q(\varepsilon) \rangle} \rangle =& \langle \mathcal{L}_{\overline{\varepsilon}} \phi_2, \theta_2^\sharp(\phi_1) \rangle + \langle \phi_2, [\overline{\varepsilon}, \theta_2^\sharp(\phi_1)] \rangle - \langle \phi_2, [\overline{\varepsilon}, \eta_1^* \circ (\theta_1^\sharp)^*] \rangle \\ & + \langle \phi_2, \theta_1^\sharp(\mathcal{L}_{\overline{\varepsilon}} \phi_1) \rangle - \langle \mathcal{L}_{\overline{\varepsilon}} \phi_2, \theta_2^\sharp(\phi_1) \rangle - \langle \phi_2, \langle \eta_1, (\theta_1^\sharp)^*(\sharp(\varepsilon)) \rangle \rangle \\ =& \langle \phi_2, [\overline{\varepsilon}, \theta_2^\sharp(\theta_1^\sharp)] \rangle + \langle \phi_2, \eta_1^* \circ (\theta_1^\sharp)^*(\sharp(\varepsilon)) \rangle + \phi_2, \theta_1^\sharp(\mathcal{L}_{\overline{\varepsilon}} \phi_1) \rangle \\ & - \langle \phi_2, \eta_1^* \circ (\theta_1^\sharp)^*(\sharp(\varepsilon)) \rangle \\ =& \langle \phi_2, [\overline{\varepsilon}, \theta_2^\sharp(\phi_1)] \rangle + \langle \phi_2, \theta_1^\sharp(\mathcal{L}_{\overline{\varepsilon}} \phi_1) \rangle. \end{aligned}$$

With this equation, we compute from Eq. (4.15), using also (7.6), (G.2) and item *1* of Thm. 3.59,

$$\begin{aligned} \langle [\phi_1, \phi_2], \overline{\varepsilon} \rangle =& \widehat{\rho}(\phi_1)(\langle \phi_2, \overline{\varepsilon} \rangle) - \widehat{\rho}(\phi_2)(\langle \phi_1, \overline{\varepsilon} \rangle) + \langle \phi_2, \overline{\langle \phi_1, Q(\varepsilon) \rangle} \rangle \\ =& \rho(\theta_2^\sharp(\phi_1))(\langle \phi_2, \overline{\varepsilon} \rangle) - \rho(\theta_2^\sharp(\phi_2))(\langle \phi_1, \overline{\varepsilon} \rangle) + \langle \phi_2, [\overline{\varepsilon}, \theta_2^\sharp(\phi_1)] \rangle + \langle \phi_2, \theta_1^\sharp(\mathcal{L}_{\overline{\varepsilon}} \phi_1) \rangle \\ =& \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)} \phi_2, \overline{\varepsilon} \right\rangle + \langle \phi_2, [\theta_2^\sharp(\phi_1), \overline{\varepsilon}] \rangle - \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)} \phi_1, \overline{\varepsilon} \right\rangle - \langle \phi_1, [\theta_2^\sharp(\phi_2), \overline{\varepsilon}] \rangle \\ & + \langle \phi_2, [\overline{\varepsilon}, \theta_2^\sharp(\phi_1)] \rangle + \left\langle \theta_2^\sharp(\phi_2), \mathcal{L}_{\overline{\varepsilon}} \phi_1 \right\rangle \\ =& \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)} \phi_2, \overline{\varepsilon} \right\rangle - \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)} \phi_1, \overline{\varepsilon} \right\rangle + \rho(\overline{\varepsilon})(\langle \phi_1, \theta_2^\sharp(\phi_2) \rangle) \\ =& \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)} \phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)} \phi_1 + d(\langle \phi_1, \theta_2^\sharp(\phi_2) \rangle), \overline{\varepsilon} \right\rangle. \end{aligned}$$

■

Lemma 7.12. *On a Poisson degree 2 manifold, given a 3-function* $\theta \in \mathcal{A}^3$, *which defines a 1-vector field* $Q = \{\theta, \cdot\}$,

$$\partial = -\sharp \circ \theta_1^\sharp, \qquad \langle \partial(\xi), \varepsilon \rangle = \rho\left(\overline{\theta_1^\sharp(\xi)}\right)(\varepsilon),$$

holds, where ∂ *is defined by Eq. (4.10),* $\theta_1^\sharp$ *by Thm. 3.59 and* ρ *and* $\sharp$ *by Eq. (6.17).*

Proof. Regarding to the first equation, observe that the equation

$$\theta_2^\sharp(de) = \{\theta, \varepsilon\} = Q(\varepsilon)$$

Prop. 6.31 for the symplectic case, remains valid in the Poisson case with the exact same proof. Then, for any $\xi \in \Gamma(F)$ and $\varepsilon \in \Gamma(E^*)$,

$$\begin{aligned} -\langle \sharp \circ \theta_1^\sharp(\xi), \varepsilon \rangle &= \langle \theta_1^\sharp(\xi), -\sharp(\varepsilon) \rangle = \langle \xi, \theta_2^\sharp(d\varepsilon) \rangle \\ &= \langle \xi, Q(\varepsilon) \rangle = \langle \partial(\xi), \varepsilon \rangle \end{aligned}$$

As for the second equation, follows immediately from the first and Eq. (6.17).

∎

Proposition 7.13. *In the same conditions of Thm. 7.6, and with the same notations,*

$$\Theta(\phi)(\xi) = \mathcal{L}_{\theta_2^\sharp(\phi)}\xi + \mathcal{L}_{\overline{\theta_1^\sharp(\xi)}}\phi - d(\langle \xi, \theta_2^\sharp(\phi) \rangle), \qquad \forall \phi \in \Gamma(\widehat{E}), \xi \in \Gamma(F). \tag{7.33}$$

Proof. If we put in Eq. (7.14) ξ instead of ϕ instead of ϕ_1 and ξ instead of ϕ_2, we obtain, for every $\gamma \in \Gamma(\widetilde{F}^*)$,

$$-\langle \xi, Q(\gamma)_2^\sharp(\phi) \rangle = \langle \xi, [\gamma, \theta_2^\sharp(\phi)] \rangle - \langle \xi, \theta_2^\sharp(\mathcal{L}_\gamma \phi) \rangle. \tag{7.34}$$

Then, starting from the definition of Θ given in Eq. (4.12), we compute using Eqs. (7.6), (G.2), item *1* of Thm. 3.59 and lemma 7.12,

$$\begin{aligned} \langle \Theta(\phi)(\xi), \gamma \rangle =& \widehat{\rho}(\phi)(\langle \xi, \gamma \rangle) + \langle \partial(\xi), \langle \phi, \gamma \rangle \rangle - \langle \xi, Q(\gamma)_2^\sharp(\phi) \rangle \\ =& \rho(\theta_2^\sharp(\phi))(\langle \xi, \gamma \rangle) + \rho\left(\overline{\theta_1^\sharp(\xi)}\right)(\langle \phi, \gamma \rangle) + \langle \xi, [\gamma, \theta_2^\sharp(\phi)] \rangle - \langle \xi, \theta_2^\sharp(\mathcal{L}_\gamma \phi) \rangle \\ =& \left\langle \mathcal{L}_{\theta_2^\sharp(\phi)}\xi, \gamma \right\rangle + \left\langle \mathcal{L}_{\overline{\theta_1^\sharp(\xi)}}\phi, \gamma \right\rangle - \rho(\gamma)(\langle \theta_1^\sharp(\xi), \phi \rangle) \\ =& \left\langle \mathcal{L}_{\theta_2^\sharp(\phi)}\xi + \mathcal{L}_{\overline{\theta_1^\sharp(\xi)}}\phi - d(\langle \xi, \theta_2^\sharp(\phi) \rangle), \gamma \right\rangle. \end{aligned}$$

∎

7.2 Integrable degree 3 functions on a degree 2 Poisson manifold

Finally we want to obtain a characterization of the integrability of a degree 3 function $\theta \in \mathcal{A}^3$ on a Poisson degree 2 manifold $(\mathcal{M}, \{\cdot, \cdot\})$, where integrability means, as we already said before, Poisson self-commuting: $\{\theta, \theta\} = 0$.

Since $Q := \{\theta, \cdot\}$ is a 1-vector field, by Thm. 4.9 it defines a preLie 2-algebroid structure on the dual sequence (3.44). On the other hand, by Thm. 6.1, the Poisson brackets define a Lie algebroid structure on the involutive bundle $\widetilde{F}^*$. Moreover, by Thm. 3.59, θ induces a vector bundle morphism $\theta_2^\sharp : \widehat{E} \longrightarrow \widetilde{F}^*$. Therefore, in the spirit of Prop. G.19, it is natural to expect a characterization of integrability in terms of a (possibly twisted) bracket-preserving condition for $\theta_2^\sharp$, which consists in relating the Lie 2-algebroid brackets on $\widehat{E}$ with the Lie algebroid brackets on $\widetilde{F}^*$ through $\theta_2^\sharp$.

We will proceed in a similar way to what we did for the degree 1 case in Sec. G.4. In particular, the calculations will rely strongly in the vector field and derived bracket viewpoint, more precisely, in propositions 3.61 and 4.12, keeping in mind that in the present case $Q = \{\theta, \cdot\}$.

Lemma 7.14. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded 2-Poisson manifold. A degree 3 function $\theta \in \mathcal{A}^3$ is integrable, i.e. $\{\theta,\theta\} = 0$ if and only if*

a)
$$\{\theta,\theta\}(\phi_1,\phi_2) := \iota_{\phi_2}\iota_{\phi_1}\{\theta,\theta\} = 0, \qquad \forall \phi_1\phi_2 \in \Gamma(\widehat{E}) \cong \mathfrak{X}(\mathcal{M})_{-1}$$

and

b)
$$\{\theta,\theta\}(\xi_1,\xi_2) := \iota_{\xi_2}\iota_{\xi_1}\{\theta,\theta\} = 0 \qquad \forall \xi_1,\xi_2 \in \Gamma(F) \cong \mathfrak{X}(\mathcal{M})_{-2}.$$

If rank$E > 1$*, then condition b) is not necessary.*

Proof. The "only if" part of the lemma is trivial. So let's prove the "if" part. Choosing local coordinates $\{x^i, \varepsilon^\mu, \alpha^\nu\}$, the degree 4 function $\{\theta,\theta\}$ reads

$$\{\theta,\theta\} = A_{ij}\alpha^i\alpha^j + B_{abj}\varepsilon^a\varepsilon^b\alpha^j + C_{abcd}\varepsilon^a\varepsilon^b\varepsilon^c\varepsilon^d, \tag{7.35}$$

where $A_{ij}, B_{abj}, C_{abcd} \in C^\infty(U)$. Then the hypothesis of the lemma imply

$$\begin{aligned}
0 &= \frac{\partial}{\partial\alpha^j}\frac{\partial}{\partial\alpha^i}\{\theta,\theta\} = A_{ij},\\
0 &= \frac{\partial}{\partial\varepsilon^b}\frac{\partial}{\partial\varepsilon^a}\{\theta,\theta\} = B_{abj}\alpha^j,\\
0 &= \frac{\partial}{\partial\varepsilon^b}\frac{\partial}{\partial\varepsilon^d}\{\theta,\theta\} = C_{abcd}\varepsilon^c\varepsilon^d.
\end{aligned}$$

Notice that if rank$E = 1$ then $\varepsilon^c\varepsilon^d = 0$, but in this case we have $\{\theta,\theta\} = A_{ij}\alpha^i\alpha^j$, hence $C_{abcd} = 0$ trivially. Therefore, the equations above imply $A_{ij} = B_{abj} = C_{abcd} = 0$, that is, $\{\theta,\theta\} = 0$.

If rank$E > 1$, we can find local coordinates $\varepsilon^1, \varepsilon^2 \in \Gamma(E^*) \cong \mathcal{A}^1$ such that $\varepsilon^1\varepsilon^2 \neq 0$. Then setting $\phi_1 = \varepsilon^1 \otimes \frac{\partial}{\partial\alpha^i} \in \Gamma(\widehat{E}) \cong \mathfrak{X}(\mathcal{M})_{-1}$ and $\phi_2 = \varepsilon^2 \otimes \frac{\partial}{\partial\alpha^j} \in \Gamma(\widehat{E}) \cong \mathfrak{X}(\mathcal{M})_{-1}$ we get from condition a) of the lemma

$$0 = \iota_{\phi_2}\iota_{\phi_1}\{\theta,\theta\} = \varepsilon^1\varepsilon^2\frac{\partial}{\partial\alpha^j}\frac{\partial}{\partial\alpha^i}\{\theta,\theta\} = A_{ij}\varepsilon^1\varepsilon^2,$$

from which, since $\varepsilon^1\varepsilon^2 \neq 0$, it follows $A_{ij} = 0$. Therefore, when rank$E > 1$, we can conclude that $\{\theta,\theta\} = 0$ using only condition a) of the lemma.

■

Lemma 7.15. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded Poisson degree 2 manifold and $\theta \in \mathcal{A}^3$ a degree 3 function, to which there corresponds the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, given by Thm. 3.59. Consider the 1-vector field $Q := \{\theta, \cdot\}$ and its corresponding preLie 2-algebroid, given by Thm. 4.9. Also consider the dual bundle $\widehat{E}$, which is the linear bundle that fits on the corresponding dual sequence (3.44). Then, for every $\phi_1, \phi_2, \phi_3 \in \Gamma(\widehat{E})$, we have*

$$\begin{aligned}\langle \phi_3, \{\theta,\theta\}(\phi_1,\phi_2)\rangle =& \langle \phi_3, Q(\theta_2^\sharp(\phi_2))_2^\sharp(\phi_1)\rangle + \Psi(\phi_3)(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) \\ &- \langle \phi_3, Q(\theta_2^\sharp(\phi_1))_2^\sharp(\phi_2)\rangle - \langle \phi_3, \theta_2^\sharp([\phi_1,\phi_2])\rangle,\end{aligned} \tag{7.36}$$

where $\{\theta,\theta\}(\phi_1,\phi_2) \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*)$ was defined in lemma 7.14.

Proof. Using propositions 4.12 and 3.61, we have

$$\begin{aligned}\theta_2^\sharp([\phi_1,\phi_2]) =& -\iota_{[\phi_1,\phi_2]}\theta = [[\iota_{\phi_1}, Q], \iota_{\phi_2}](\theta) \\ =& \iota_{\phi_1} Q \iota_{\phi_2}(\theta) + Q\iota_{\phi_1}\iota_{\phi_2}(\theta) - \iota_{\phi_2}\iota_{\phi_1}Q(\theta) - \iota_{\phi_2}Q\iota_{\phi_1}(\theta) \\ =& Q(\theta_2^\sharp(\phi_2))_2^\sharp(\phi_1) - Q(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) \\ & - \{\theta,\theta\}(\phi_1,\phi_2) - Q(\theta_2^\sharp(\phi_1))_2^\sharp(\phi_2),\end{aligned}$$

from which, using Eq. (4.11) and rearranging terms we obtain Eq. (7.36). ∎

Lemma 7.16. *In the conditions of lemma 7.15, and with the same notations, we have, for every $\phi_1, \phi_2 \in \Gamma(\widehat{E})$, $\xi \in \Gamma(F)$,*

$$\begin{aligned}\left\langle \phi_2, \overline{\{\theta,\theta\}(\phi_1,\xi)}\right\rangle =& -\left\langle \phi_2, \overline{\langle \phi_1, Q(\theta_1^\sharp(\xi))\rangle}\right\rangle + \widehat{\rho}(\phi_2)\left(\left\langle \phi_1, \overline{\theta_1^\sharp(\xi)}\right\rangle\right) \\ &- \left\langle \phi_2, \overline{Q(\theta_2^\sharp(\phi_1))_1^\sharp(\xi)}\right\rangle - \left\langle \overline{\theta_1^\sharp(\theta(\phi_1)(\xi))}, \phi_2\right\rangle,\end{aligned} \tag{7.37}$$

where, as usual, $\{\theta,\theta\}(\phi_1,\xi) := \iota_\xi\iota_{\phi_1}\{\theta,\theta\}$, and the bar over a section of $\Gamma(E^*)$ means that it is considered as a core section $\Gamma_{\mathrm{core}}((D_{F^*})_{E^*})$.

Proof. The proof runs in the same way as for lemma 7.15. From Prop. 4.12, we have

$$-\iota_{\Theta(\phi_1)(\xi)}\theta = [[\iota_{\phi_1}, Q], \iota_\xi](\theta) = \iota_{\phi_1}Q\iota_\xi(\theta) + Q\iota_{\phi_1}\iota_\xi(\theta) - \iota_\xi\iota_{\phi_1}Q(\theta) - \iota_\xi Q\iota_{\phi_1}(\theta),$$

therefore, using Prop. 3.61, and omitting, to simplify the notation, the bar over sections of $\Gamma(E^*) \cong \Gamma_{\mathrm{core}}((D_{F^*})_{E^*})$,

$$\begin{aligned}\langle \phi_2, \theta_1^\sharp(\Theta(\phi_1)(\xi))\rangle =& -\langle \phi_2, \iota_{\Theta(\phi_1)(\xi)}\theta\rangle = -\langle \phi_2, \langle \phi_1, Q(\theta_1^\sharp(\xi))\rangle\rangle + \widehat{\rho}(\phi_2)(\langle \phi_1, \theta_1^\sharp(\xi)\rangle) \\ &- \langle \phi_2, \{\theta,\theta\}(\phi_1,\xi)\rangle - \langle \phi_2, Q(\theta_2^\sharp(\phi_1))_1^\sharp(\xi)\rangle,\end{aligned}$$

from which, rearranging terms, we get (7.37). ∎

Lemma 7.17. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded Poisson degree 2 manifold and let $\theta \in \mathcal{A}^3$ be a degree 3 function, with its associated pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$. Consider the dual bundle $\widehat{E}$, which is the linear bundle that fits on the corresponding dual sequence (3.44). Then, for every $\phi_1, \phi_2, \phi_3 \in \Gamma(\widehat{E})$, we have*

$$\sum_{cyclic} \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_1, \theta_2^\sharp(\phi_2) \right\rangle = \sum_{cyclic} \left[- \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3, \theta_2^\sharp(\phi_2) \right\rangle + \rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_3,\phi_2))) \right],$$

where the operators $\mathcal{L}$, d and ρ belong to the metric VB-algebroid structure corresponding to the Poisson degree 2 manifold (Thm. 6.14).

Proof. The proof is just a computation using Cartan's formula and item *3* of Thm. 3.59 and then a rearrangement of terms.

$$\begin{aligned}
\sum_{cyclic} \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_1, \theta_2^\sharp(\phi_2) \right\rangle =& \langle\langle \theta_2^\sharp(\phi_3), d\phi_1\rangle, \theta_2^\sharp(\phi_2)\rangle + \rho(\theta_2^\sharp(\phi_2))(\langle \phi_1, \theta_2^\sharp(\phi_3)\rangle) \\
&+ \langle\langle \theta_2^\sharp(\phi_2), d\phi_3\rangle, \theta_2^\sharp(\phi_1)\rangle + \rho(\theta_2^\sharp(\phi_1))(\langle \phi_3, \theta_2^\sharp(\phi_2)\rangle) \\
&+ \langle\langle \theta_2^\sharp(\phi_1), d\phi_2\rangle, \theta_2^\sharp(\phi_3)\rangle + \rho(\theta_2^\sharp(\phi_3))(\langle \phi_2, \theta_2^\sharp(\phi_1)\rangle) \\
= -\langle\langle \theta_2^\sharp(\phi_2), d\phi_1\rangle, \theta_2^\sharp(\phi_3)\rangle &- \rho(\theta_2^\sharp(\phi_2))(\langle \phi_3, \theta_2^\sharp(\phi_1)\rangle) + \rho(\theta_2^\sharp(\phi_2))(\theta_1^\sharp(W(\phi_1,\phi_3))) \\
-\langle\langle \theta_2^\sharp(\phi_1), d\phi_3\rangle, \theta_2^\sharp(\phi_2)\rangle &- \rho(\theta_2^\sharp(\phi_1))(\langle \phi_2, \theta_2^\sharp(\phi_3)\rangle) + \rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_3,\phi_2))) \\
-\langle\langle \theta_2^\sharp(\phi_3), d\phi_2\rangle, \theta_2^\sharp(\phi_1)\rangle &- \rho(\theta_2^\sharp(\phi_3))(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) + \rho(\theta_2^\sharp(\phi_3))(\theta_1^\sharp(W(\phi_2,\phi_1))) \\
= \sum_{cyclic} &\left[- \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3, \theta_2^\sharp(\phi_2) \right\rangle + \rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_3,\phi_2))) \right]
\end{aligned}$$

■

Lemma 7.18. *Consider the operator $R_\theta : \Gamma(\widehat{E}) \otimes \Gamma(\widehat{E}) \longrightarrow \Gamma(\widehat{E})$ defined in Eq. (7.51). We have yet another expression for it:*

$$\langle R_\theta(\phi_1,\phi_2), \gamma\rangle = \theta_1^\sharp \left(W(\phi_1, d(\langle \phi_2, \gamma\rangle)) + W(\phi_2, \langle \gamma, d\phi_1\rangle) - d(\langle \phi_1, \overline{\langle \phi_2, \gamma\rangle}\rangle) \right). \tag{7.38}$$

Proof. From Cartan's formula and the bilinearity of the symmetric tensor W it follows

$$W(\mathcal{L}_\gamma\phi_1, \phi_2) - W(\phi_2, d(\langle\phi_1,\gamma\rangle)) = W(\phi_2, \mathcal{L}_\gamma\phi_1 - d(\langle\phi_1,\gamma\rangle)) = W(\phi_2, \langle\gamma, d\phi_1\rangle),$$

by which Eq. (7.38) follows immediately from the first equation of (7.51).

■

Theorem 7.19. *Let $(\mathcal{M}, \{\cdot,\cdot\})$ be a graded Poisson degree 2 manifold, and let $\theta \in \mathcal{A}^3$ be a degree 3 function on $\mathcal{M}$, to which there corresponds the pair of morphisms $(\theta_1^\sharp, \theta_2^\sharp)$, given by Thm. 3.59. Consider the 1-vector field $Q := \{\theta, \cdot\}$ and its corresponding preLie 2-algebroid, given by Thm. 4.9. Also consider the metric VB-algebroid D_{F^*} corresponding to the Poisson degree 2 manifold, according to Thm. 6.14. Then* *i)* $[\theta_2^\sharp(\phi_1), \theta_2^\sharp(\phi_2)] - \theta_2^\sharp([\phi_1,\phi_2]) = \frac{1}{2}\Big[\{\theta,\theta\}(\phi_1,\phi_2) + \theta_2^\sharp(R_\theta(\phi_1,\phi_2))\Big], \quad \forall \phi_1,\phi_2 \in \Gamma(\widehat{E}),$

where $\{\theta,\theta\}(\phi_1,\phi_2) \in \mathcal{A}^2 \cong \Gamma(\widetilde{F}^*)$ *was defined in lemma 7.14, and* $R_\theta(\phi_1,\phi_2)$ *was defined in Eq. (7.51).*

ii) θ *is integrable, i.e.* $\{\theta,\theta\} = 0$*, if and only if*

$$[\theta_2^\sharp(\phi_1), \theta_2^\sharp(\phi_2)] - \theta_2^\sharp([\phi_1,\phi_2]) = \frac{1}{2}\theta_2^\sharp(R_\theta(\phi_1,\phi_2)) \qquad \forall \phi_1,\phi_2 \in \Gamma(\widehat{E}) \tag{7.39}$$

Proof. Let's prove the equation of item *i)* in the theorem. For every $\phi_1,\phi_2,\phi_3 \in \Gamma(\widehat{E})$, taking into account Eqs. (7.36) (4.14) (7.6), item *3* of Thm. 3.59 and Eq. (7.6), we have

$$\begin{aligned}
\langle \phi_3, \{\theta,\theta\}(\phi_1,\phi_2)\rangle =& \langle \phi_3, Q(\theta_2^\sharp(\phi_2))_2^\sharp(\phi_1)\rangle + \Psi(\phi_3)(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) \\
& - \langle \phi_3, Q(\theta_2^\sharp(\phi_1))_2^\sharp(\phi_2)\rangle - \langle \phi_3, \theta_2^\sharp([\phi_1,\phi_2])\rangle \\
=& \Psi(\phi_1)(\langle \phi_3, \theta_2^\sharp(\phi_2)\rangle) - \Psi(\phi_3)(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) + \theta_1^\sharp(d(\langle T(\phi_3,\phi_1), \theta_2^\sharp(\phi_2)\rangle)) \\
& - \langle [\phi_1,\phi_3], \theta_2^\sharp(\phi_2)\rangle + \Psi(\phi_3)(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) - \Psi(\phi_2)(\langle \phi_3, \theta_2^\sharp(\phi_1)\rangle) \\
& + \Psi(\phi_3)(\langle \phi_2, \theta_2^\sharp(\phi_1)\rangle) - \theta_1^\sharp(d(\langle T(\phi_3,\phi_2), \theta_2^\sharp(\phi_1)\rangle)) + \langle [\phi_2,\phi_3], \theta_2^\sharp(\phi_1)\rangle \\
& + \langle [\phi_1,\phi_2], \theta_2^\sharp(\phi_3)\rangle - \theta_1^\sharp(W(\phi_3, [\phi_1,\phi_2])) \\
=& \Psi(\phi_1)(\langle \phi_3, \theta_2^\sharp(\phi_2)\rangle) - \Psi(\phi_2)(\langle \phi_3, \theta_2^\sharp(\phi_1)\rangle) + \Psi(\phi_3)(\langle \phi_2, \theta_2^\sharp(\phi_1)\rangle) \\
& - \langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3, \theta_2^\sharp(\phi_2)\rangle + \langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_1, \theta_2^\sharp(\phi_2)\rangle - \rho(\theta_2^\sharp(\phi_2))(\langle \phi_1, \theta_2^\sharp(\phi_3)\rangle) \\
& + \langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_3, \theta_2^\sharp(\phi_1)\rangle - \langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_2, \theta_2^\sharp(\phi_1)\rangle + \rho(\theta_2^\sharp(\phi_1))(\langle \phi_2, \theta_2^\sharp(\phi_3)\rangle) \\
& + \langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \theta_2^\sharp(\phi_3)\rangle - \langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1, \theta_2^\sharp(\phi_3)\rangle + \rho(\theta_2^\sharp(\phi_3))(\langle \phi_1, \theta_2^\sharp(\phi_2)\rangle) \\
& + \theta_1^\sharp(d(\langle T(\phi_3,\phi_1), \theta_2^\sharp(\phi_2)\rangle)) - \theta_1^\sharp(d(\langle T(\phi_3,\phi_2), \theta_2^\sharp(\phi_1)\rangle)) - \theta_1^\sharp(W(\phi_3, [\phi_1,\phi_2])) \\
& + \langle R_\theta(\phi_1,\phi_3), \theta_2^\sharp(\phi_2)\rangle - \langle R_\theta(\phi_2,\phi_3), \theta_2^\sharp(\phi_1)\rangle - \langle R_\theta(\phi_1,\phi_2), \theta_2^\sharp(\phi_3)\rangle.
\end{aligned}$$

Now we introduce in the calculations Eqs. (7.7), (7.38) and once more item *3* of Thm. 3.59 and Eq. (7.6), and continuing the calculations above we obtain, after performing some cancellations,

$$\begin{aligned}
\langle \phi_3, \{\theta,\theta\}(\phi_1,\phi_2)\rangle =& - \langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3, \theta_2^\sharp(\phi_2)\rangle - \langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1, \theta_2^\sharp(\phi_3)\rangle - \langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_2, \theta_2^\sharp(\phi_1)\rangle \\
& + \langle \mathcal{L}_{\theta_2^\sharp(\phi_3)}\phi_1, \theta_2^\sharp(\phi_2)\rangle + \langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_3, \theta_2^\sharp(\phi_1)\rangle + \langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \theta_2^\sharp(\phi_3)\rangle \\
& + \rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_2,\phi_3))) - \theta_1^\sharp(W\phi_1, d(\langle \phi_3, \theta_2^\sharp(\phi_2)\rangle))) \\
& - \rho(\theta_2^\sharp(\phi_2))(\theta_1^\sharp(W(\phi_1,\phi_3))) - \theta_1^\sharp(W\phi_2, d(\langle \phi_3, \theta_2^\sharp(\phi_1)\rangle))) \\
& + \rho(\theta_2^\sharp(\phi_3))(\theta_1^\sharp(W(\phi_1,\phi_2))) - \theta_1^\sharp(W\phi_3, d(\langle \phi_2, \theta_2^\sharp(\phi_1)\rangle))) \\
& + \theta_1^\sharp(W(\phi_3, \langle \theta_2^\sharp(\phi_2), d\phi_1\rangle)) + \theta_1^\sharp(W(\phi_1, d(\langle \phi_3, \theta_2^\sharp(\phi_2)\rangle))) \\
& - \theta_1^\sharp(W(\phi_3, \langle \theta_2^\sharp(\phi_1), d\phi_2\rangle)) - \theta_1^\sharp(W(\phi_2, d(\langle \phi_3, \theta_2^\sharp(\phi_1)\rangle))) \\
& -\theta_1^\sharp\Big(W(\phi_3, \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2)\Big) + \theta_1^\sharp\Big(W(\phi_3, \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1)\Big) - \theta_1^\sharp(W(\phi_3, d(\theta_1^\sharp(W(\phi_1,\phi_2))))) \\
& + \theta_1^\sharp(W(\phi_3, R_\theta(\phi_1,\phi_2))) + \langle \phi_3, \theta_2^\sharp(R_\theta(\phi_1,\phi_2))\rangle - \theta_1^\sharp(W(\phi_3, R_\theta(\phi_1,\phi_2))).
\end{aligned}$$

Finally we introduce lemma 7.17 into the calculations, namely we apply it to the second line of the equation above, and after cancelling some terms, we get

$$\begin{aligned}\langle\phi_3,\{\theta,\theta\}(\phi_1,\phi_2)\rangle =& -2\sum_{cyclic}\left(\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3,\theta_2^\sharp(\phi_2)\rangle\right)+2\rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_2,\phi_3)))\\&+2\rho(\theta_2^\sharp(\phi_3))(\theta_1^\sharp(W(\phi_1,\phi_2)))-2\theta_1^\sharp\left(W(\phi_3,\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2)\right)\\&+2\theta_1^\sharp(W(\phi_3,\langle\theta_2^\sharp(\phi_2),d\phi_1\rangle))+\langle\phi_3,\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\rangle. \qquad (7.40)\end{aligned}$$

On the other hand, let's compute

$$\langle\phi_3,\theta_2^\sharp([\phi_1,\phi_2])\rangle-\langle\phi_3,[\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]\rangle.$$

Using item *3* of Thm. 3.59, Eq. (7.6) and Cartan's formula, we have

$$\begin{aligned}\langle\phi_3,\theta_2^\sharp([\phi_1,\phi_2])\rangle =& -\langle\theta_2^\sharp(\phi_3),[\phi_1,\phi_2]\rangle+\theta_1^\sharp(W(\phi_3,[\phi_1,\phi_2]))\\=& -\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_2^\sharp(\phi_3)\rangle+\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1,\theta_2^\sharp(\phi_3)\rangle-\rho(\theta_2^\sharp(\phi_3))(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle)\\&+\langle\theta_2^\sharp(\phi_3),R_\theta(\phi_1,\phi_2)\rangle+\theta_1^\sharp\left(W(\phi_3,\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2)\right)-\theta_1^\sharp\left(W(\phi_3,\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1)\right)\\&+\theta_1^\sharp\left(W(\phi_3,d(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle))\right)-\theta_1^\sharp(W(\phi_3,R_\theta(\phi_1,\phi_2)))\\=& -\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_2^\sharp(\phi_3)\rangle+\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1,\theta_2^\sharp(\phi_3)\rangle-\rho(\theta_2^\sharp(\phi_3))(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle)\\&-\langle\phi_3,\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\rangle+\theta_1^\sharp(W(\phi_3,R_\theta(\phi_1,\phi_2)))+\theta_1^\sharp\left(W(\phi_3,\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2)\right)\\&-\theta_1^\sharp(W(\phi_3,\langle\theta_2^\sharp(\phi_2),d\phi_1\rangle))+\theta_1^\sharp(W(\phi_3,d(\langle\theta_2^\sharp(\phi_2),\phi_1\rangle)))\\&+\theta_1^\sharp\left(W(\phi_3,d(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle))\right)-\theta_1^\sharp(W(\phi_3,R_\theta(\phi_1,\phi_2))) \qquad (7.41)\end{aligned}$$

The other term we compute using Eq. (G.2) and item *3* of Thm. 3.59:

$$\begin{aligned}-\langle\phi_3,[\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]\rangle =& -\rho(\theta_2^\sharp(\phi_1))(\langle\phi_3,\theta_2^\sharp(\phi_2)\rangle))+\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3,\theta_2^\sharp(\phi_2)\rangle\\=&\rho(\theta_2^\sharp(\phi_1))(\langle\phi_2,\theta_2^\sharp(\phi_3)\rangle)-\rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_2,\phi_3)))\\&+\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3,\theta_2^\sharp(\phi_2)\rangle. \qquad (7.42)\end{aligned}$$

Before adding up the terms, we need a last formula. Using Cartan's formula and item *3* of Thm. 3.59, we have

$$\begin{aligned}-\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_2^\sharp(\phi_3)\rangle =& -\langle\theta_2^\sharp(\phi_3),\langle\theta_2^\sharp(\phi_1),d\phi_2\rangle\rangle-\rho(\theta_2^\sharp(\phi_3))(\langle\phi_2,\theta_2^\sharp(\phi_1)\rangle)\\=&\langle\theta_2^\sharp(\phi_1),\langle\theta_2^\sharp(\phi_3),d\phi_2\rangle\rangle+\rho(\theta_2^\sharp(\phi_3))(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle)\\&-\rho(\theta_2^\sharp(\phi_3))(\theta_1^\sharp(W(\phi_1,\phi_2))). \qquad (7.43)\end{aligned}$$

Now putting (7.43) into (7.41) and the adding up with (7.42) we get, after cancelling terms,

$$\begin{aligned}\langle\phi_3,\theta_2^\sharp([\phi_1,\phi_2])\rangle-\langle\phi_3,[\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]\rangle=\sum_{cyclic}&\left(\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_3,\theta_2^\sharp(\phi_2)\rangle\right)\\&-\rho(\theta_2^\sharp(\phi_1))(\theta_1^\sharp(W(\phi_2,\phi_3)))-\rho(\theta_2^\sharp(\phi_3))(\theta_1^\sharp(W(\phi_1,\phi_2)))\\&+\theta_1^\sharp\left(W(\phi_3,\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2)\right)-\theta_1^\sharp(W(\phi_3,\langle\theta_2^\sharp(\phi_2),d\phi_1\rangle))\\&-\langle\phi_3,\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\rangle.\end{aligned}\tag{7.44}$$

Comparing Eqs. (7.40) and (7.44), we obtain item i) of the theorem.

In order to prove item ii), observe that in view of item i) we just proved, Eq. (7.39) is equivalent to $\{\theta,\theta\}(\phi_1,\phi_2)=0$. Therefore, keeping in mind lemma 7.14, it is enough to verify that $\{\theta,\theta\}(\xi_1,\xi_2)=0$, which is what we are going to do now. From Eqs. (4.38),(7.2) and (4.10), we have

$$\begin{aligned}0=[[\iota_{\xi_1},Q],\iota_{\xi_2}](\theta)=&\iota_{\xi_1}Q\iota_{\xi_2}(\theta)+Q\iota_{\xi_1}\iota_{\xi_2}(\theta)-\iota_{\xi_2}\iota_{\xi_1}Q(\theta)-\iota_{\xi_2}Q\iota_{\xi_1}(\theta)\\=&-\langle\partial(\xi_1),\theta_1^\sharp(\xi_2)\rangle-\{\theta,\theta\}(\phi_1,\phi_2)+\langle\partial(\xi_2),\theta_1^\sharp(\xi_1)\rangle,\end{aligned}$$

where we used that $\iota_{\xi_1}\iota_{\xi_2}(\theta)=0$ since it is a degree -1 function. Therefore,

$$\{\theta,\theta\}(\xi_1,\xi_2)=-\langle\partial(\xi_1),\theta_1^\sharp(\xi_2)\rangle+\langle\partial(\xi_2),\theta_1^\sharp(\xi_1)\rangle.$$

Now, from lemma 7.12 and the symmetry of $\sharp$ we have

$$\begin{aligned}\langle\partial(\xi_1),\theta_1^\sharp(\xi_2)\rangle&=-\langle\sharp(\theta_1^\sharp(\xi_1)),\theta_1^\sharp(\xi_2)\rangle=-\langle\sharp(\theta_1^\sharp(\xi_2)),\theta_1^\sharp(\xi_1)\rangle\\&=\langle\partial(\xi_2),\theta_1^\sharp(\xi_1)\rangle,\qquad\qquad\forall\xi_1,\xi_2\in\Gamma(F^*).\end{aligned}$$

Therefore, we conclude that $\{\theta,\theta\}(\xi_1,\xi_2)=0$, as we wanted.

■

Corollary 7.20. *In the conditions of Thm. 7.19, if θ is integrable, then*

$$p\left([\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]\right)=p\left(\theta_2^\sharp([\phi_1,\phi_2])\right)\qquad\forall\phi_1,\phi_2\in\Gamma(\widehat{E}),\tag{7.45}$$

where $p:\widetilde{F}^*\longrightarrow F^*$ is the projection of the exact sequence (3.14).

Proof. From Thm. 7.19 it follows that

$$p\left([\theta_2^\sharp(\phi_1),\theta_2^\sharp(\phi_2)]\right)-p\left(\theta_2^\sharp([\phi_1,\phi_2])\right)=p\left(\frac{1}{2}\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\right).\tag{7.46}$$

From Thm. 7.11, we know that $R_\theta(\phi_1,\phi_2)\in\Gamma(E^*\otimes F)$. Then, by item *2* of Thm. 3.59, it follows that

$$\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\in\Gamma(\Lambda^2E^*),$$

from which, $p\left(\frac{1}{2}\theta_2^\sharp(R_\theta(\phi_1,\phi_2))\right)=0$. Putting this in Eq. (7.46), we obtain (7.45).

■

Lemma 7.21. *In the same conditions of lemma 7.17, for every $\phi_1, \phi_2 \in \Gamma(\widehat{E})$, $\xi \in \Gamma(F)$,*

$$\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1, \theta_1^\sharp(\xi)\right\rangle - \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\xi, \theta_2^\sharp(\phi_2)\right\rangle + \left\langle \mathcal{L}_{\theta_1^\sharp(\xi)}\phi_2, \theta_2^\sharp(\phi_1)\right\rangle = -\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \theta_1^\sharp(\xi)\right\rangle$$
$$+\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi, \theta_2^\sharp(\phi_1)\right\rangle - \left\langle \mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1, \theta_2^\sharp(\phi_2)\right\rangle + \rho(\theta_1^\sharp(\xi))(\theta_1^\sharp(W(\phi_1,\phi_2))).$$

Proof. The proof consists just in a computation using Cartan's formula and items *1* and *3* of Thm. 3.59.

$$\begin{aligned}
&\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1, \theta_1^\sharp(\xi)\right\rangle - \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\xi, \theta_2^\sharp(\phi_2)\right\rangle + \left\langle \mathcal{L}_{\theta_1^\sharp(\xi)}\phi_2, \theta_2^\sharp(\phi_1)\right\rangle \\
=&\left\langle \iota_{\theta_2^\sharp(\phi_2)}d\phi_1, \theta_1^\sharp(\xi)\right\rangle + \rho(\theta_1^\sharp(\xi))(\langle\phi_1, \theta_2^\sharp(\phi_2)\rangle) - \left\langle \iota_{\theta_2^\sharp(\phi_1)}d\xi, \theta_2^\sharp(\phi_2)\right\rangle - \rho(\theta_2^\sharp(\phi_2))(\langle\xi, \theta_2^\sharp(\phi_1)\rangle) \\
&+\left\langle \iota_{\theta_1^\sharp(\xi)}d\phi_2, \theta_2^\sharp(\phi_1)\right\rangle + \rho(\theta_2^\sharp(\phi_1))(\langle\phi_2, \theta_1^\sharp(\xi)\rangle) \\
=&-\left\langle \iota_{\theta_2^\sharp(\phi_1)}d\phi_2, \theta_1^\sharp(\xi)\right\rangle - \rho(\theta_1^\sharp(\xi))(\langle\phi_2, \theta_2^\sharp(\phi_1)\rangle) + \rho(\theta_1^\sharp(\xi))(\theta_1^\sharp(W(\phi_1,\phi_2))) \\
&+\left\langle \iota_{\theta_2^\sharp(\phi_2)}d\xi, \theta_2^\sharp(\phi_1)\right\rangle + \rho(\theta_2^\sharp(\phi_1))(\langle\xi, \theta_2^\sharp(\phi_2)\rangle) \\
&-\left\langle \iota_{\theta_1^\sharp(\xi)}d\phi_1, \theta_2^\sharp(\phi_2)\right\rangle - \rho(\theta_2^\sharp(\phi_2))(\langle\phi_1, \theta_1^\sharp(\xi)\rangle) \\
=&-\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \theta_1^\sharp(\xi)\right\rangle + \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi, \theta_2^\sharp(\phi_1)\right\rangle - \left\langle \mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1, \theta_2^\sharp(\phi_2)\right\rangle \\
&+\rho(\theta_1^\sharp(\xi))(\theta_1^\sharp(W(\phi_1,\phi_2))).
\end{aligned}$$

∎

Proposition 7.22. *In the conditions of Thm. 7.19, and with the same notations,*

$$\langle\phi_2, \{\theta,\theta\}(\phi_1,\xi)\rangle = 2\left[-\left\langle \mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2, \theta_1^\sharp(\xi)\right\rangle + \left\langle \mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi, \theta_2^\sharp(\phi_1)\right\rangle - \left\langle \mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1, \theta_2^\sharp(\phi_2)\right\rangle\right]. \tag{7.47}$$

Proof. Using lemma 7.16, Eqs. (4.15), (4.12), lemmas 7.4, 7.12, Thm. 7.11, Prop. 7.13 and items *1* and *3* of Thm. 3.59, we will compute the left-hand side of (7.47). As we did in the proof of lemma 7.16, we omit the bar over sections of $\Gamma(E^*)$ in order to simplify the notation.

$$\begin{aligned}
\langle\phi_2,\{\theta,\theta\}(\phi_1,\xi)\rangle =&\rho(\theta_2^\sharp(\phi_1))(\langle\phi_2,\theta_1^\sharp(\xi)\rangle)-\rho(\theta_2^\sharp(\phi_2))(\langle\phi_1,\theta_1^\sharp(\xi)\rangle)-\langle[\phi_1,\phi_2],\theta_1^\sharp(\xi)\rangle\\
&+\rho(\theta_2^\sharp(\phi_2))(\langle\phi_1,\theta_1^\sharp(\xi)\rangle)-\rho(\theta_2^\sharp(\phi_2))(\langle\xi,\theta_2^\sharp(\phi_1)\rangle)-\rho(\theta_1^\sharp(\xi))(\langle\phi_2,\theta_2^\sharp(\phi_1)\rangle)\\
&+\langle\Theta(\phi_2)(\xi),\theta_2^\sharp(\phi_1)\rangle-\langle\Theta(\phi_1)(\xi),\theta_2^\sharp(\phi_2)\rangle\\
=&\rho(\theta_2^\sharp(\phi_1))(\langle\phi_2,\theta_1^\sharp(\xi)\rangle)-\rho(\theta_2^\sharp(\phi_2))(\langle\xi,\theta_2^\sharp(\phi_1)\rangle)-\rho(\theta_1^\sharp(\xi))(\langle\phi_2,\theta_2^\sharp(\phi_1)\rangle)\\
&-\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_1^\sharp(\xi)\right\rangle+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1,\theta_1^\sharp(\xi)\right\rangle-\rho(\theta_1^\sharp(\xi))(\langle\phi_1,\theta_2^\sharp(\phi_2)\rangle)\\
&+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi,\theta_2^\sharp(\phi_1)\right\rangle+\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_2,\theta_2^\sharp(\phi_1)\right\rangle-\rho(\theta_2^\sharp(\phi_1))(\langle\xi,\theta_2^\sharp(\phi_2)\rangle)\\
&-\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\xi,\theta_2^\sharp(\phi_2)\right\rangle-\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1,\theta_2^\sharp(\phi_2)\right\rangle+\rho(\theta_2^\sharp(\phi_2))(\langle\theta_2^\sharp(\phi_1),\xi\rangle)\\
=&-\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_1^\sharp(\xi)\right\rangle+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi,\theta_2^\sharp(\phi_1)\right\rangle-\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1,\theta_2^\sharp(\phi_2)\right\rangle\\
&+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\phi_1,\theta_1^\sharp(\xi)\right\rangle-\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\xi,\theta_2^\sharp(\phi_2)\right\rangle+\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_2,\theta_2^\sharp(\phi_1)\right\rangle\\
&-\rho(\theta_1^\sharp(\xi))(\theta_1^\sharp(W(\phi_1,\phi_2))).
\end{aligned}$$

Finally, using lemma 7.21, we arrive to the right-hand side of Eq. (7.47).

■

Proposition 7.23. *In the conditions of Thm. 7.19, we have*

$$[\theta_2^\sharp(\phi_1),\theta_1^\sharp(\xi)]-\theta_1^\sharp(\Theta(\phi_1)(\xi))=\frac{1}{2}\{\theta,\theta\}(\phi_1,\xi). \tag{7.48}$$

In particular, if θ is integrable, we have

$$[\theta_2^\sharp(\phi_1),\theta_1^\sharp(\xi)]=\theta_1^\sharp(\Theta(\phi_1)(\xi)). \tag{7.49}$$

Proof. Let's compute:

$$\begin{aligned}
\langle\phi_2,\theta_1^\sharp(\Theta(\phi_1)(\xi))\rangle-\langle\phi_2,[\theta_2^\sharp(\phi_1),\theta_1^\sharp(\xi)]\rangle=&\langle\Theta(\phi_1(\xi),\theta_2^\sharp(\phi_2)\rangle-\langle\phi_2,[\theta_2^\sharp(\phi_1),\theta_1^\sharp(\xi)]\rangle\\
=&\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\xi,\theta_2^\sharp(\phi_2)\right\rangle+\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1,\theta_2^\sharp(\phi_2)\right\rangle-\rho(\theta_2^\sharp(\phi_2))(\langle\xi,\theta_2^\sharp(\phi_1)\rangle)\\
&-\rho(\theta_2^\sharp(\phi_1))(\langle\phi_2,\theta_1^\sharp(\xi)\rangle)+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_1^\sharp(\xi)\right\rangle\\
=&\left\langle\iota_{\theta_2^\sharp(\phi_1)}d\xi,\theta_2^\sharp(\phi_2)\right\rangle+\rho(\theta_2^\sharp(\phi_2))(\xi,\theta_2^\sharp(\phi_1)\rangle)+\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1,\theta_2^\sharp(\phi_2)\right\rangle\\
&-\rho(\theta_2^\sharp(\phi_2))(\langle\xi,\theta_2^\sharp(\phi_1)\rangle)-\rho(\theta_2^\sharp(\phi_1))(\langle\phi_2,\theta_1^\sharp(\xi)\rangle)+\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_1^\sharp(\xi)\right\rangle\\
=&\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_1)}\phi_2,\theta_1^\sharp(\xi)\right\rangle-\left\langle\mathcal{L}_{\theta_2^\sharp(\phi_2)}\xi,\theta_2^\sharp(\phi_1)\right\rangle+\left\langle\mathcal{L}_{\theta_1^\sharp(\xi)}\phi_1,\theta_2^\sharp(\phi_2)\right\rangle.
\end{aligned}$$

Therefore, from Prop. 7.22 we obtain Eq. (7.48), from which immediately follows Eq. (7.49) if $\{\theta,\theta\}=0$.

■

7.2.1 Degenerate Courant algebroids

Let's summarize the vector bundle information we obtained to characterize integrable degree 3 functions on a degree 2 Poisson manifold.

1. **Associated to any degree 2 manifold** $\mathcal{M}$ we obtained

   a) the involutive vector sequence that characterizes, up to isomorphisms, such degree 2 manifold,

$$(E, \widetilde{F}, \widetilde{F} \xrightarrow{p} \Lambda^2 E), \text{ with inclusion } \ker p =: F \xrightarrow{\iota} \widetilde{F}$$

   to which we have naturally associated

   b) the dual sequence $\widehat{E} \xrightarrow{\pi} E$, where

$$\widehat{E} = \{\phi \in \operatorname{Hom}(\widetilde{F}^*, E^*) | \exists e \in E \ s.t.\ \phi(\lambda) = \iota_e \lambda,\ \forall \lambda \in \Lambda^2 E^*\}; \quad \pi(\phi) = e, \tag{7.50}$$

   and we have the natural identification

$$\ker \pi \cong \operatorname{Hom}(F^*, E^*) \cong \operatorname{Hom}(E, F), \quad \widetilde{\phi} \in \operatorname{Hom}(E, F) \longrightarrow (\widetilde{\phi})^* \circ \iota^* \in \operatorname{Hom}(\widetilde{F}^*, E^*).$$

   c) A map $T : \widehat{E} \otimes \widehat{E} \longrightarrow \widetilde{F}$, given by

$$\langle T(\phi_1, \phi_2), \zeta \rangle = \langle \phi_1(\zeta), \pi(\phi_2) \rangle, \qquad \forall \phi_1, \phi_2 \in \widehat{E},\ \zeta \in \widetilde{F}.$$

   d) A map $W : S^2 \widehat{E} \longrightarrow F$, given by

$$W(\phi_1, \phi_2) = T(\phi_1, \phi_2) + T(\phi_2, \phi_1).$$

2. **Associated to -2 Poisson brackets** $\{\cdot, \cdot\}$ **on** $\mathcal{M}$ we have the unique involutive Lie algebroid structure on the transposed involutive sequence $(\Lambda^2 E^* \xrightarrow{p^*} \widetilde{F}^*)$, that characterizes them, and comprises the following maps:

   a) A symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E^*;

   b) A Lie algebroid structure $([\cdot, \cdot], \rho)$ on $\widetilde{F}^*$;

   c) A Lie algebroid $\langle \cdot, \cdot \rangle$-preserving action, $\Psi : \widetilde{F}^* \longrightarrow \mathbf{CDO}(E^*)$, of $\widetilde{F}^*$ on E^*;

   such that

   – The brackets between a section γ of $\widetilde{F}^*$ and a section $\varepsilon_1 \wedge \varepsilon_2$ of $\Lambda^2 E^* \overset{p^*}{\hookrightarrow} \widetilde{F}^*$, are given by

$$[\gamma, \varepsilon_1 \wedge \varepsilon_2] = \Psi(\gamma)(\varepsilon_1) \wedge \varepsilon_2 + \varepsilon_1 \wedge \Psi(\gamma)(\varepsilon_2).$$

   – The action Ψ of $\widetilde{F}^*$ on E^* restricted to $\Lambda^2 E^*$, which we denote by $\widetilde{\Psi}$, is given by

$$\widetilde{\Psi}(\varepsilon_1 \wedge \varepsilon_2)(\varepsilon) = \langle \varepsilon_2, \varepsilon \rangle \varepsilon_1 - \langle \varepsilon_1, \varepsilon \rangle \varepsilon_2.$$

Out of these structure maps, we get two differential operators:

d) $d : \Gamma(E^*) \longrightarrow \Gamma(\widehat{E})$, given by

$$\begin{aligned} d(\varepsilon)(\gamma) &= \Psi(\gamma)(\varepsilon), \\ \text{and} \quad \pi(d(\varepsilon)) &= -\sharp(\varepsilon) \quad \forall \varepsilon \in \Gamma(E^*), \gamma \in \Gamma(\widetilde{F}^*); \end{aligned}$$

e) $\mathcal{L} : \Gamma(\widetilde{F}^*) \times \Gamma(\widehat{E}) \longrightarrow \Gamma(\widehat{E})$, given by

$$\begin{aligned} \mathcal{L}_\gamma \phi(\gamma') &= \Psi(\gamma)(\phi(\gamma')) - \langle \phi, [\gamma, \gamma'] \rangle \\ \text{and} \quad \langle \pi(\mathcal{L}_\gamma \phi), \varepsilon \rangle &= \rho(\gamma)(\langle \pi(\phi), \varepsilon \rangle) - \phi(\Psi(\gamma)(\varepsilon)) \rangle, \end{aligned}$$

$\forall \gamma, \gamma' \in \Gamma(\widetilde{F}^*), \phi \in \Gamma(\widehat{E}), \varepsilon \in \Gamma(E^*)$.

3. **Associated to a degree 3 function θ on $\mathcal{M}$**, there are two compatible morphisms which characterize it

a) $\theta_2^\sharp : \widehat{E} \longrightarrow \widetilde{F}^*$, $\quad \theta_1^\sharp : F \longrightarrow E^*$. The compatibility conditions are
   * $\langle \iota^*(\theta_2^\sharp(\phi)), \xi \rangle = \langle \pi(\phi), \theta_1^\sharp(\xi) \rangle$, $\quad \forall \phi \in \widehat{E}, \ \xi \in F$;
   * $\theta_2^\sharp(\widetilde{\phi}) = (\theta_1^\sharp \circ \widetilde{\phi})^* - \theta_1^\sharp \circ \widetilde{\phi} \in \Lambda^2 E^*$, $\quad \forall \widetilde{\phi} \in \mathrm{Hom}(E, F) \subset \widehat{E}$;
   * $\phi_1(\theta_2^\sharp(\phi_2)) + \phi_2(\theta_2^\sharp(\phi_1)) = \theta_1^\sharp(W(\phi_1, \phi_2))$, $\quad \forall \phi_1, \phi_2 \in \widehat{E}$.

Using these maps and the preceding ones, we get two differential operators:

b) $R_\theta : \Gamma(\widehat{E}) \times \Gamma(\widehat{E}) \longrightarrow \mathrm{Hom}(E, F)$, given by

$$\begin{aligned} R_\theta(\phi_1, \phi_2)(\iota^*(\gamma)) = \theta_1^\sharp(&W(\phi_1, d(\phi_2(\gamma))) - W(\phi_2, d(\phi_1(\gamma))) \\ &- d(\langle T(\phi_2, \phi_1), \gamma \rangle) + W(\mathcal{L}_\gamma \phi_1, \phi_2)), \end{aligned} \tag{7.51}$$

$\forall \phi_1, \phi_2 \in \Gamma(\widehat{E}), \gamma \in \widetilde{F}^*$.

c) $[\cdot, \cdot]_\theta : \Gamma(\widehat{E}) \times \Gamma(\widehat{E}) \longrightarrow \Gamma(\widehat{E})$, given by

$$[\phi_1, \phi_2]_\theta = \mathcal{L}_{\theta_2^\sharp(\phi_1)} \phi_2 - \mathcal{L}_{\theta_2^\sharp(\phi_2)} \phi_1 + d(\phi_1(\theta_2^\sharp(\phi_2))) - R_\theta(\phi_1, \phi_2), \qquad \forall \phi_1, \phi_2 \in \Gamma(\widehat{E}).$$

4. **Associated to the homological equation** $\{\theta, \theta\} = 0$ we have integrability equation that characterizes it:

$$[\theta_2^\sharp(\phi_1), \theta_2^\sharp(\phi_2)] - \theta_2^\sharp([\phi_1, \phi_2]_\theta) = \frac{1}{2} \theta_2^\sharp(R_\theta(\phi_1, \phi_2)) \in \Gamma(\Lambda^2 E^*), \qquad \forall \phi_1, \phi_2 \in \Gamma(\widehat{E}).$$

So now we define the structure which gathers all the vector bundle information that characterizes integrable degree 3 functions on a degree 2 Poisson manifold, and call it a degenerate Courant algebroid, since this information is equivalent to a Courant algebroid when the Poisson brackets are symplectic (see Sec. 7.2.2 below).

Definition 7.24. Given an involutive sequence $(E, \widetilde{F}, \widetilde{F} \xrightarrow{p} \Lambda^2 E)$, with $\ker p = F \overset{\iota}{\hookrightarrow} \widetilde{F}$, a *degenerate Courant algebroid* is given by an involutive Lie algebroid structure on the transposed sequence $\Lambda^2 E^* \xrightarrow{p^*} \widetilde{F}^*$, as described in item *2* above, and a pair of morphisms $\theta_1^\sharp : F \longrightarrow E^*$, $\theta_2^\sharp : \widehat{E} \longrightarrow \widehat{F}^*$, where $\widehat{E}$ is the vector bundle given by (7.50), satisfying the compatibility conditions of item *3.a*) above and the integrability equation of item *4*, where $[\cdot,\cdot]_\theta$ and R_θ are given in items *3.b*) and *3.c*).

Therefore, we have obtained the following result.

Theorem 7.25. *Given a degree 2 manifold $\mathcal{M}$, consider the involutive sequence $\widetilde{F} \longrightarrow \Lambda^2 E$ we can associate to it. Then -2 Poisson brackets $\{\cdot,\cdot\}$ together with a degree 3 function θ on $\mathcal{M}$ satisfying $\{\theta,\theta\} = 0$ are equivalent to a degenerate Courant algebroid on $\widetilde{F} \longrightarrow \Lambda^2 E$.*

Remark 7.26. D. Li-Bland [41] introduced the notion of *LA*-Courant algebroids, which are *VB*-Courant algebroids endowed with a metric *VB*-algebroid structure over the second fibration (the horizontal one) satisfying a compatibility condition. There he shows that Q structures on a degree 2 Poisson manifold that are derivations of the Poisson bracket, are in 1:1 correspondence with *LA*-Courant algebroids on the metric double vector bundle that corresponds to the degree 2 manifold. See also [29], where the correspondence is proved in an explicit fashion using splittings, representations up to homotopy and Dorfman 2-representations compatible in a suitable way. Since Q structures that come from degree 3 functions are automatically derivations of the Poisson bracket[1], it follows that degenerate Courant algebroids are examples of *LA*-Courant algebroids. The correspondences with the *VB*-Courant and metric *VB*-algebroid structures were explained, respectively, in sections 5.2.2 and 6.2.1.

7.2.2 Example: Courant algebroids and quotients

As we saw in Sec. 6.4.2, when the Poisson brackets are symplectic, i.e. non-degenerate, the whole structure of an integrable degree 3 function on a degree 2 symplectic manifold is already encoded by its corresponding Courant algebroid. Thus, what we will do in this section is to recover, in the symplectic case, the whole degenerate Courant structure through the Courant algebroid structure. To describe it, we will follow the same scheme given by items *1,2,3* and *4* in Sec. 7.2.1.

So let $(E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], a)$ be a Courant algebroid.

1. The involutive vector sequence is given in this case by transpose of the *Atiyah sequence* associated to the pseudo-euclidean vector bundle $(E^* \overset{\sharp}{\cong} E, \langle\cdot,\cdot\rangle)$:

$$\Lambda^2 E \cong \mathfrak{so}(E) \longrightarrow \mathbb{A}_E \longrightarrow TM,$$

as seen in Prop. 6.23, where the isomorphism of vector bundle sequences is given by (6.28), so that, after identifying $E^* \cong E$ through the metric, we have the vector

[1]This follows directly from Jacobi identity of the Poisson bracket.

sequence isomorphism:

$$
\begin{tikzcd}
\Lambda^2 E^* \arrow[r, "\widetilde{\iota}"] \arrow[d, "\widetilde{\Psi}"] & \widetilde{F}^* \arrow[r, "\widetilde{p}"] \arrow[d, "\Psi"] & F^* \arrow[d, "\rho"] \\
\mathfrak{so}(E) \arrow[r, "\widetilde{\iota}"] & \mathbb{A}_E \arrow[r, "\widetilde{p}"] & TM
\end{tikzcd}
.
$$

The dual sequence is given by the *first jet bundle* of E, J^1E, which is also naturally identified with the bundle of *linear 1-forms* of E^*, $\Omega^1_{\mathrm{lin}}(E^*)$, the identification map is given by

$$
\begin{aligned}
J^1E &\xrightarrow{\cong} \Omega^1_{\mathrm{lin}}(E^*) \\
j^1e &\longrightarrow de, \qquad\qquad e \in \Gamma(E) \cong C^\infty_{\mathrm{lin}}(E^*),
\end{aligned}
$$

where d is the usual (de Rham) differential applied to the function e of E^*. Now, in order to see an element of Ω^1_{lin} as an element of $\widehat{E}$ in (7.50), we use identification of covariant differential operators with *linear fields* (see [46]), and thus, given

$$X \in \widetilde{F}^* \cong \mathbb{A}_E \subset \mathbf{CDO}(E) \cong \mathfrak{X}_{\mathrm{lin}}(E^*),$$

we set, for $e \in \Gamma(E) \cong C^\infty_{\mathrm{lin}}(E^*)$,

$$de(X) = X(e), \qquad \pi(de) = -e. \tag{7.52}$$

With these identifications, the maps T and W of items *1.c)* and *2.d)* are given by

$$\langle T(de_1, de_2), X\rangle = -\langle X(e_1), e_2\rangle; \qquad W(de_1, de_2) = -d\langle e_1, e_2\rangle, \tag{7.53}$$

for every $e_1, e_2 \in \Gamma(E) \cong C^\infty_{\mathrm{lin}}(E^*)$, $X \in \mathbb{A}_E \subset \mathfrak{X}_{\mathrm{lin}}(E^*)$.

2. The involutive Lie algebroid data is already given by the Atiyah sequence, viewed as a Lie algebroi sequence:

a) The bilinear form on $E \cong E^*$ is the pseudo-euclidean metric.

b) The Lie bracket is given by the commutator, viewing $\mathbb{A}_E$ inside $\mathfrak{X}_{\mathrm{lin}}(E^*)$, and the anchor is given by the symbol of X, viewed as a covariant differential operator, or equivalently, the base map, viewing $X \in \mathfrak{X}$ as a vector bundle map $X : E^*_M \longrightarrow TE^*_{TM}$ over some base map $\widetilde{X} : M \longrightarrow TM$, then $\rho(X) = \widetilde{X}$.

c) The action of $\mathbb{A}_E$ on E can be viewed simply as the action of vector fields as derivations of functions.

d) $d : \Gamma(E) \cong C^\infty_{\mathrm{lin}}(E^*) \longrightarrow \Gamma(\widehat{E}) = \Omega^1_{\mathrm{lin}}(E^*)$ is just the differential of functions, restricted to functions linear on the fibers.

e) The operator $\mathcal{L}$ is just the Lie derivative of 1-forms under the identifications $\widetilde{F}^* \cong \mathfrak{X}_{\mathrm{lin}}(E^*)$, $\widehat{E} \cong \Omega^1_{\mathrm{lin}}(E^*)$, and the formula in this case, for 1-forms of type de is simply

$$\mathcal{L}_X de = d(X(e)) \qquad \forall X \in \mathbb{A}_E, e \in \Gamma(E).$$

3. *a*) $\theta_2^\sharp : \Omega^1_{\text{lin}}(E^*) \longrightarrow \mathfrak{X}_{\text{lin}}(E^*)$ is given by

$$\theta_2^\sharp(de) = \text{ad}_e, \qquad \text{where ad} : \Gamma(E) \longrightarrow \Gamma(\mathbb{A}_E), \ e \longrightarrow \text{ad}_e := [\![e, \cdot]\!].$$

This defines a section in $\Gamma(\mathbb{A}_E)$. In fact, from Leibniz rule for $[\![\cdot,\cdot]\!]$ it follows that ad_e is a covariant differential operator, with symbol given by $\sigma(\theta_2^\sharp(de)) = a(e)$; that ad_e preserves the metric is precisely what property 5 of Def. 5.1 says. $\theta_1^\sharp : T^*M \longrightarrow E$ is given by

$$\theta_1^\sharp(df) = -\mathcal{D}(f), \qquad \text{where } \mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E), \ \langle \mathcal{D}(f), e\rangle := a(e)(f).$$

It was shown in the proof of Thm. 6.32 that properties 3, 4 and 5 of Def. 5.1 allow to obtain the three compatibility conditions for $\theta_1^\sharp$ and $\theta_2^\sharp$.

b) The operator R_θ is zero for exact 1-forms, $R_\theta(de_1, de_2) = 0$.
For 1-forms $\phi_i = f_i de_i, \ i = 1, 2$, we have

$$R_\theta(f_1de_1, f_2de_2) = f_2R_\theta(f_1de_1, de_2) = f_2(\mathcal{D}\langle e_1, e_2\rangle \otimes df_1 - \mathcal{D}(f_1) \otimes d\langle e_1, e_2\rangle).$$

c) The bracket $[\cdot,\cdot]_\theta$ is given by (see Prop. 6.34)

$$[de_1, de_2]_\theta = d[\![e_1, e_2]\!], \qquad \forall e_1, e_2 \in \Gamma(E).$$

4. Finally, the integrability condition in this case is given by

$$[\theta_2^\sharp(de_1), \theta_2^\sharp(de_1)] = \theta_2^\sharp([de_1, de_2]_\theta),$$

which is equivalent to the Jacobi identity for $[\![\cdot,\cdot]\!]$, indeed, for every $e_1, e_2, e_3 \in \Gamma(E)$, we have

$$\langle[\theta_2^\sharp(de_1), \theta_2^\sharp(de_2)], de_3\rangle = [\![e_2, [\![e_2, e_3]\!]]\!] - [\![e_2, [\![e_1, e_3]\!]]\!];$$

on the other hand,

$$\langle\theta_2^\sharp([de_1, de_2]_\theta), de_3\rangle = \langle\theta_2^\sharp(d[\![e_1, e_2]\!]), de_3\rangle = [\![[\![e_1, e_2]\!], e_3]\!].$$

The two equalities above show that the integrability equation is equivalent to Jacobi identity of $[\![\cdot,\cdot]\!]$.

Example 7.27. Courant algebroid with a symmetry group. An example of a degenerate Courant algebroid which is not the one that corresponds to a Courant algebroid appears when we have a Lie group G acting on a Courant algebroid $\mathbb{E} \cong \mathbb{E}^*$, preserving everything: the vector bundle structure, the metric, the Courant brackets and the anchor.

We will describe once more the data of the corresponding degenerate Courant algebroid in the same scheme of items *1,2,3* and *4* of Sec. 7.2.1.

1. The base manifold is the quotient M/G, and the fiber bundles E^*, $\widetilde{F}^*$, F^* and $\widehat{E}$ are also given by the quotients

$$E^* = \mathbb{E}/G, \quad \widetilde{F}^* = \mathbb{A}_{\mathbb{E}}/G, \quad F^* = TM/G, \quad \text{and} \quad \widehat{E} = \Omega^1_{\text{lin}}(\mathbb{E})/G,$$

and we have natural identifications of the sections of each quotient, with G-invariant sections on the original bundle. Likewise, functions on M/G are naturally identified with G-invariant functions on M. So we use these G-invariant functions and sections to determine the whole structure in terms of the data we found in the previous example above, that corresponds to the Courant algebroid. We only need to verify in each case that the data obtained is again G-invariant, so that it corresponds to data on the quotient.

- For G-invariant $\phi_1, \phi_2 \in \Gamma(\Omega^1_{\text{lin}}(\mathbb{E}))^G$ and $X \in \Gamma(\mathbb{A}_{\mathbb{E}})^G \subset \mathfrak{X}_{\text{lin}}(\mathbb{E})^G$, and for any infinitesimal generator $Y \in \Gamma(\mathbb{A}_{\mathbb{E}})$ of the G-action on $(\mathbb{E}, \langle\cdot,\cdot\rangle)$, with symbol $\sigma_Y \in \Gamma(TM)$, taking into account the equivariance of the projection $\pi : \Omega^1_{\text{lin}}(\mathbb{E}) \longrightarrow \mathbb{E}^* \overset{\sharp}{\cong} \mathbb{E}$, we have

$$\begin{aligned}\sigma_Y\langle T(\phi_1,\phi_2), X\rangle =&\sigma_Y\langle X(\phi_1), \pi(\phi_2)\rangle \\ =&\langle Y(X(\phi_1)), \pi(\phi_2)\rangle + \langle X(\phi_1), Y(\pi(\phi_2))\rangle \\ =&0.\end{aligned}$$

  Therefore, $T(\phi_1, \phi_2)$ descends to the quotient, and so does

$$W(\phi_1,\phi_2) = T(\phi_1,\phi_2) + T(\phi_2,\phi_1).$$

2. We already have by hypothesis that G preserves $\langle\cdot,\cdot\rangle$, so it descends to the quotient to a bilinear symmetric pairing (now possibly degenerate). Let's see the rest of the involutive Lie algebroid structure.

- For $X_1, X_2 \in \Gamma(\mathbb{A}_{\mathbb{E}})^G$, $e \in \Gamma(\mathbb{E})^G$, $f \in C^\infty(M)^G$ and $Y \in \Gamma(\mathbb{A}_{\mathbb{E}})$ an infinitesimal generator of the action, we have

$$[Y, [X_1, X_2]] = [[Y, X_1], X_2] + [X_1, [Y, X_2]] = 0;$$

$$\sigma_Y(\rho(X_1)(f)) = \sigma_Y(\sigma_{X_1}(f)) = 0;$$
$$Y(\Psi(X_1)(e) = Y(X_1(e)) = 0;$$

so that the involutive Lie algebroid structure also descends to the quotient.

3. To check that the morphism $\theta_2^\sharp$ descends to the quotient, take an invariant 1-form $fde_1 \in \Gamma(\Omega^1_{\text{lin}})^G$, an infinitesimal generator $X \in \Gamma(\mathbb{A}_{\mathbb{E}})$ and an arbitrary $e_2 \in \Gamma(\mathbb{E})$, then we have on one hand, using $[\mathcal{L}_X, d] = 0$, that the invariance of fde_1, that is, $\mathcal{L}_X(fde_1)$ is equivalent to the equation

$$fdX(e_1) = -\sigma_X(f)de_1$$

Then, using the fact that the action preserves the Courant bracket,

$$\begin{aligned}[X,\theta_2^\sharp(fde_1)](e_2) =& X(\theta_2^\sharp(fde_1)(e_2)) - \theta_2^\sharp(fde_1)(X(e_2)) \\ =& \sigma_X(f)[\![e_1,e_2]\!] + f[\![X(e_1),e_2]\!] + f[\![e_1,X(e_2)]\!] \\ & - \theta_2^\sharp(fde_1)(X(e_2)) \\ =& \sigma_X(f)\theta_2^\sharp(de_1)(e_2) + \theta_2^\sharp(fd(X(e_1)))(e_2) + \theta_2^\sharp(fde_1)(X(e_2)) \\ & - \theta_2^\sharp(fde_1)(X(e_2)) \\ =& -\theta_2^\sharp(fd(X(e_1)))(e_2) + \theta_2^\sharp(fd(X(e_1)))(e_2) = 0.\end{aligned}$$

For the equivariance of $\theta_1^\sharp$, take $\alpha \in \Gamma(T^*M)^G$ and extending any $e \in \mathbb{E}$ to a section such that $X(e) = 0$, we have, using the equivariance of $a : \mathbb{E} \longrightarrow TM$,

$$\begin{aligned}\langle X(\theta_1^\sharp(\alpha)), e\rangle &= \sigma_X(\langle\theta_1^\sharp(\alpha), e\rangle) - \langle\theta_1^\sharp(\alpha), X(e)\rangle \\ &= \sigma_X(\langle a(e), \alpha\rangle \\ &= \langle[\sigma_X, a(e)], \alpha\rangle + \langle a(e), \mathcal{L}_{\sigma_X}\alpha\rangle = 0.\end{aligned}$$

Therefore, $\theta_2^\sharp$ and $\theta_1^\sharp$ are equivariant maps, so they descend to the quotient.

Since R_θ and $[\cdot,\cdot]_\theta$ are already determined by the structure above, it follows that these operators are also G-invariant, so they descend to the quotient and satisfy the integrability equation

$$[\theta_2^\sharp(\phi_1), \theta_2^\sharp(\phi_2)] - \theta_2^\sharp([\phi_1,\phi_2]_\theta) = \frac{1}{2}\theta_2^\sharp(R_\theta(\phi_1,\phi_2)).$$

Appendices

Appendix A

Double vector bundles

In this appendix we develop the basics of double vector bundles, including a detailed study of their local structure. Since the interest on this subject is increasing, as more applications to Poisson geometry, Lie theory and mathematical physics appear, we decided to present the main properties and facts of double vector bundles from scratch, so that besides facilitating the reading, Ch. 2 together with appendices A–F may serve as a reference into the field, including some new results and a new approach emphasizing the point of view of the linear and the core bundles, which in a precise sense (see App. D and also the reference [12]) encode the whole structure of a double vector bundle, reducing their complicated structure to a triple of plain vector bundles, which are much more simpler to deal with. It is worth remarking that in section A.4 we show that every double vector bundle admits a local decomposition, a result that is regarded as folklore but a proof of which we weren't able to find in the literature, in spite that [23] refers to [20] for this result, we didn't find it there. The main references we used for this appendix are [32],[46] and [23]. The concept of double vector bundle was originally introduced by J. Pradines [56].

A.1 More on double vector bundles and morphisms

Proposition A.1 ([32],[46]). *If Φ is a morphism of double vector bundles, setting*

$$\begin{aligned}
\varphi_A &:= q_{A'} \circ \Phi \circ 0_A; \\
\varphi_B &:= q_{B'} \circ \Phi \circ 0_B; \\
\varphi_M &:= q^{A'} \circ q_{A'} \circ \Phi \circ 0_A \circ 0^A = q^{B'} \circ q_{B'} \circ \Phi \circ 0_B \circ 0^B,
\end{aligned}$$

we obtain maps $\varphi_A : A \longrightarrow A'$, $\varphi_B : B \longrightarrow B'$, $\varphi_M : M \longrightarrow M'$ such that each of (Φ, φ_B), (Φ, φ_A), (φ_A, φ_M) and (φ_B, φ_M) is a morphism of the relevant vector bundles.

Also

$$\begin{aligned}\varphi_A \circ q_A &:= q_{A'} \circ \Phi;\\ \varphi_B \circ q_B &:= q_{B'} \circ \Phi;\\ \varphi_M \circ q^A \circ q_A &:= q^{A'} \circ q_{A'} \circ \Phi;\\ \varphi_M \circ q^B \circ q_B &:= q^{B'} \circ q_{B'} \circ \Phi.\end{aligned}$$

Proof. It is a direct verification using fiber preserving and linearity of Φ with respect to both fibrations, and the fact that the projections and zero sections are vector bundle maps. ∎

We may denote a morphism by

$$(\Phi; \varphi_A, \varphi_B, \varphi_M) : (D; A, B; M) \longrightarrow (D'; A', B'; M').$$

Proposition A.2 ([32],[46])**.** *The definition of a double vector bundle is symmetric with respect to A and B. That is, a commutative square like (2.1) is a double vector bundle if and only if the square*

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}\ , \tag{A.1}$$

is also a double vector bundle.

Proof. We have to check conditions (a), (b) and (c) from definition 2.1, which are obviously symmetric with respect to the side bundles A and B. ∎

A.2 The core bundle and core sequences

Let's call q_A and q_B, respectively, the projections from D over A and B, and let's call q^A and q^B the projections from A and B, respectively, over M. Then q_A is as vector bundle morphism over q^B and q_B is a vector bundle morphism over q^A.

We refer to the vector bundle $D \longrightarrow A$ as D_A, and to the vector bundle $D \longrightarrow B$ as D_B.

Proposition A.3 ([32])**.** *The vector bundle structures on D_A and D_B coincide on the intersection $C := \ker q_A \cap \ker q_B$. And C is itself a vector bundle over M, with projection $q^C := q^A \circ q_A|_C = q^B \circ q_B|_C$.*

Proof. First let's prove that C is a vector bundle over M with the structure inherited from D_A. Notice that $\ker q_A = D_A|_{0^A(M)}$, so, after the identification $M \cong 0^A(M)$, it follows that $\ker q_A$ is a vector bundle bundle over M with the structure inherited from D_A. Now, $q_B|_{\ker q_A} : \ker q_A \longrightarrow B$ is a vector bundle morphism over the identity $\mathrm{Id} : M \longrightarrow M$, and since $0_B(B) \subseteq \ker q_A$, it follows that $q_B|_{\ker q_A}$ has full rank. Then

$$C = \ker q_A \cap \ker q_B = \ker\left(q_B|_{\ker q_A}\right),$$

is a vector subbundle of $\ker q_A$ over M. Of course, by a completely symmetric argument, C is also a vector subbundle of $\ker q_B$, and since $q^A \circ q_A = q^B \circ q_B$, it follows that the projections over M corresponding to both structures coincide, we denote this projection by $q^C : C \longrightarrow M$. Finally we see that if $d_1, d_2 \in C$ satisfy $q^C(d_1) = q^C(d_2) = m$, then

$$\begin{aligned} d_1 \underset{A}{+} d_2 &= (d_1 \underset{B}{+} (0_B \circ 0^B(m))) \underset{A}{+} ((0_B \circ 0^B(m)) \underset{B}{+} d_2) \\ &= (d_1 \underset{A}{+} (0_A \circ 0^A(m))) \underset{B}{+} (d_2 \underset{A}{+} (0_A \circ 0^A(m))) \\ &= (d_1 \underset{B}{+} d_2). \end{aligned}$$

This shows that the two vector structures on each fiber of C coincide.

■

Proposition A.4 ([32]). *Let $(D; A, B; M)$ be a double vector bundle, and C its core bundle. Then*

1. $\ker q_B$ *with the vector bundle structure induced from D_B is canonically isomorphic to the Whitney sum $A \oplus C$.*
2. $\ker q_B$ *with the vector bundle structure induced from D_A is canonically isomorphic to the vector bundle $(q^A)^*C$.*
3. $\ker q_A$ *with the vector bundle structure induced from D_A is canonically isomorphic to the Whitney sum $B \oplus C$.*
4. $\ker q_A$ *with the vector bundle structure induced from D_B is canonically isomorphic to the vector bundle $(q^B)^*C$.*

Proof.

1. Consider the map $p_A : \ker q_B \longrightarrow \ker q_B$ given by
$$p_A(d) := 0_A(q_A(d)).$$
Since $q_B : D_A \longrightarrow B$ is a vector bundle morphism, it follows that $p_A(d) \in \ker q_B$, so p_A is a well defined, linear map. Moreover, it's a projection, since
$$p_A^2(d) = 0_A(q_A(0_A(q_A(d)))) = 0_A(q_A(d)) = p_A(d).$$
Then, defining $p_C : \ker q_B \longrightarrow \ker q_B$ by
$$p_C(d) := d \underset{B}{-} p_A(d) = d \underset{B}{-} 0_A(q_A(d))$$
we have
$$\begin{aligned} p_C^2(d) &= p_C(d) \underset{B}{-} p_A(p_C(d)) \\ &= d \underset{B}{-} p_A(d) \underset{B}{-} p_A(d \underset{B}{-} p_A(d)) \\ &= d \underset{B}{-} p_A(d) \underset{B}{-} p_A(d) \underset{B}{+} p_A^2(d) \\ &= d \underset{B}{-} p_A(d) = p_C(d). \end{aligned}$$

Notice that, for $d \in C$, we have $p_C(d) = d$, and for all $d \in \ker q_B$, we have

$$q_A(p_C(d)) = q_A(d) \underset{B}{-} q_A(0_A(q_A(d))) = q_A(d) \underset{B}{-} q_A(d) = 0,$$

hence $p_C(d) \in \ker q_B \cap \ker q_A = C$. So p_C is a projection over C. Identifying $A \cong 0_A(A)$, we obtain the map

$$P := (p_A, p_C) : \ker q_B \longrightarrow A \oplus C$$

over the identity Id_M. Let $d \in \ker q_B$. If $P(d) = 0$, then $q_A(d) = 0$, and since we already have $q_B(d) = 0$, it follows that $d \in C$. But, $P(d) = 0$ implies $p_C(d) = 0$, whence $d = 0$. So, P is an isomorphism.

2. Consider the map
$$\tau_A : (q^A)^*C \longrightarrow D_A$$
given by $\tau_A(a, c) := 0_A(a) \underset{B}{+} c$. Notice that both $0_A(a)$ and c project to $0^B(m)$ under q_B, so the addition $0_A(a) \underset{B}{+} c$ is defined and also projects to $0^B(m)$ under q_B. Therefore τ_A injects $(q^A)^*C$ into $\ker q_B$. Since, as manifolds, $(q^A)^*C$ and $\ker q_B$ are already isomorphic, it will suffice to show that τ_A is linear with respect to the structure over A.

$$\begin{aligned}\tau_A((a, c_1) + (a, c_2)) = \tau_A(a, c_1 + c_2) &= 0_A(a) \underset{B}{+} (c_1 \underset{A}{+} c_2) = (0_A(a) \underset{A}{+} 0_A(a)) \underset{B}{+} (c_1 \underset{A}{+} c_2) \\ &= (0_A(a) \underset{B}{+} c_1) \underset{A}{+} (0_A(a) \underset{B}{+} c_2) \\ &= \tau_A(a, c_1) \underset{A}{+} \tau_A(a, c_2).\end{aligned}$$

Items 3 and 4 are analogous.

■

As a corollary we obtain Mackenzie's *core sequences*, which are instrumental to understand the structure of double vector bundles and their decompositions.

Corollary A.5 ([46]). *There is an exact sequence*

$$0 \longrightarrow (q^A)^*C \longrightarrow D_A \longrightarrow (q^A)^*B \longrightarrow 0 \tag{A.2}$$

of vector bundles over A, and an exact sequence

$$0 \longrightarrow (q^B)^*C \longrightarrow D_B \longrightarrow (q^B)^*A \longrightarrow 0, \tag{A.3}$$

where the injections are the ones considered in the proof of Prop. A.4: $\tau_A(a, c) = 0_A(a) \underset{B}{+} c$; $\tau_B(b, c) = 0_B(b) \underset{A}{+} c$. And the projections are the obvious induced maps (q_A, q_B) and (q_B, q_A), respectively.

Definition A.6. The sequences (A.2) and (A.3) are called *core sequences* over A and over B, respectively.

The following corollary will be useful in order to understand the relation of a splitting of the core section (A.2) and a decomposition of D (see Def. 2.7). Further, this corollary, along with the one that follows it (Cor. A.8), will facilitate the study of duality for double vector bundles.

Corollary A.7 ([32])**.** *Let $v, w \in D$ such that*

$$q_B(v) = q_B(w) = b \in B,$$
$$q_A(v) = q_A(w) = a \in A.$$

There exists $k \in C$ such that $q^C(k) = q^B(b) = q^A(a)$ and, using the identifications of Prop. A.4,

$$v = w \underset{B}{+} (b, k),$$
$$v = w \underset{A}{+} (a, k).$$

Proof. Since $q_A(v \underset{B}{-} w) = q_A(v) - q_A(w) = 0$, it follows that $v \underset{B}{-} w \in \ker q_A$, and from Prop. A.4 there exists a unique $k \in C$ such that $v \underset{B}{-} w = (b, k)$. By the same reasons we have $v \underset{A}{-} w = (a, k')$, for a unique $k' \in C$. To conclude the proposition we need to show that $k = k'$. In order to achieve this, recall that the identification from Prop. A.4 means that $(b, k) = 0_B(b) \underset{A}{+} k$ and $(a, k') = 0_A(a) \underset{B}{+} k'$. Then

$$\begin{aligned} w \underset{A}{+} (0_A(a) \underset{B}{+} k') = v &= w \underset{B}{+} (0_B(b) \underset{A}{+} k) \\ &= (w \underset{A}{+} 0_A(a)) \underset{B}{+} (0_B(b) \underset{A}{+} k) \\ &= (w \underset{B}{+} 0_B(b)) \underset{A}{+} (0_A(a) \underset{B}{+} k) \\ &= w \underset{A}{+} (0_A(a) \underset{B}{+} k). \end{aligned}$$

By the uniqueness of k' satisfying $v = w \underset{A}{+} (0_A(a) \underset{B}{+} k')$, it follows that $k = k'$. ∎

Corollary A.8. *Let $v, v', w, w' \in D$ be such that*

$$\begin{aligned} &q_A(v) = q_A(w) = a, \\ &q_B(v') = q_B(w') = a', \\ &q_B(v) = q_B(w) = q_B(v') = q_B(w') = b, \\ &v \underset{B}{+} v' = w \underset{B}{+} w'. \end{aligned} \tag{A.4}$$

Then

$$v \underset{A}{-} w = (a, c) \in \ker q_B$$
$$v' \underset{A}{-} w' = (a', -c) \in \ker q_B.$$

Proof. We have $v \underset{A}{-} w \in \ker q_B$, then

$$v \underset{A}{-} w = (a, c),$$

for some $c \in C$. Analogously,

$$v' \underset{A}{-} w' = (a, c'),$$

for some $c' \in C'$.

From Cor. A.7 we have also

$$v \underset{A}{-} w = (b, c) \text{ and } v' \underset{A}{-} w' = (b, c').$$

By (A.4) we obtain

$$(b, c) = v \underset{A}{-} w = w' \underset{A}{-} v' = (b, -c'),$$

thus, $c' = -c$. ■

Proposition A.9 ([32]). *Let $\Phi : (D; A, B; M) \longrightarrow (D'; A', B'; M')$ be a morphism of double vector bundles. Then*

1. $\Phi(\ker q_B) \subset \ker q_{B'}$,
2. $\Phi(\ker q_A) \subset \ker q_{A'}$,
3. $\Phi(C) \subset C'$.

We denote the restriction $\varphi_C := \Phi|_C : C \longrightarrow C'$.

Proof.

1. Recall from Prop. A.1 that Φ induces vector bundle maps $\varphi_A : A \longrightarrow A'$ and $\varphi_B : B \longrightarrow B'$. Since Φ preserves fibers, it follows that $\varphi_B \circ q_B = q_{B'} \circ \Phi$, hence, if $q_B(d) = 0$, $q_{B'}(\Phi(d)) = 0$.
2. Analogous to 1.
3. From 1 and 2 it follows that

$$\Phi(C) = \Phi(\ker q_B \cap \ker q_A) \subset \ker q_{B'} \cap \ker q_{A'} = C'.$$

■

A.3 More on core and linear sections

Proposition A.10. *Let* $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B; M)'_C$ *be a double vector morphism which is the identity on the common side bundle,* B*. Then the induced map on sections, which we denote by the same* Φ*, satisfies* $\Phi(\Gamma_{\text{core}}(D_B)) \subset \Gamma_{\text{core}}(D'_B)$*.*

Proof. Let $\widetilde{\alpha} = \iota \circ \alpha \circ q^B \underset{A}{+} 0_B$ be the core section corresponding to the section $\alpha \in \Gamma(C)$. By Prop. A.9, and since Φ is the identity on M, it follows that $\Phi \circ \iota \circ \alpha$ is a section of C'. Then, by the linearity of Φ with respect to A and A', and recalling that Φ is the identity on B,

$$\Phi(\widetilde{\alpha}) = \Phi \circ \iota \circ \alpha \circ q^B \underset{A'}{+} 0_B$$

is the core section on D'_B corresponding to the section $\Phi \circ \iota \circ \alpha \in \Gamma(C')$.

■

Proposition A.11. *Let* $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B; M)'_C$ *be a double vector morphism which is the identity on the common side bundle,* B*. Then the induced map on sections (denoted by the same* Φ*), satisfies* $\Phi(\Gamma_{\text{lin}}(D_B)) \subset \Gamma_{\text{lin}}(D'_B)$*.*

Proof. Let $\gamma \in \Gamma_{\text{lin}}(D_B)$. We need to prove that $\Phi(\gamma) : D \longrightarrow D'_{A'}$ preserves fibers and the linear structure.

Let $b_1, b_2 \in B_m$. Then $q_A(\gamma(b_1)) = q_A(\gamma(b_2))$, whence

$$q_{A'}(\Phi(\gamma)(b_1)) = q_{A'}(\Phi(\gamma(b_1))) = \varphi_A(q_A(\gamma(b_1))) = q_{A'}(\Phi(\gamma(b_2))),$$

and

$$\begin{aligned}\Phi(\gamma)(b_1 + b_2) &= \Phi(\gamma(b_1 + b_2)) = \Phi(\gamma(b_1) \underset{A}{+} \gamma(b_2)) \\ &= \Phi(\gamma(b_1)) \underset{A'}{+} \Phi(\gamma(b_2)) = \Phi(\gamma)(b_1) \underset{A'}{+} \Phi(\gamma)(b_2).\end{aligned}$$

It follows that $\Phi(\gamma)$ is linear, thus $\Phi(\Gamma_{\text{lin}}(D_B)) \subset \Gamma_{\text{lin}}(D'_B)$.

■

Remark A.12. Actually, the property of preserving core and linear sections allows us to characterize double vector bundle morphisms which are the identity on B. We will be able to prove this (Prop. C.27) only after studying the local structure of double vector bundles and their morphisms.

Propositions A.10 and A.11 were taken from [23], where the proofs were left as an exercise.

A.4 Decompositions and local structure

In this section we provide a thorough treatment of decompositions and splittings of double vector bundles and core sequences, respectively, and exploit the relationship between them, which will lead in particular to a simple proof of the existence of a local decomposition for a double vector bundle. The study presented here of both, decompositions and the local structure, will be instrumental for the understanding of the linear bundle in App. C.

The relationship between decompositions and splittings is also the key to show that a decomposition of a double vector bundle automatically induces decompositions of its two duals (Sec. B.3), and horizontal lifts of each associated linear sequence (Subsec. C.2).

A.4.1 Decompositions and splittings of core sections

Recall the definition of decomposition given in Def. 2.7. The following characterization of a decomposition, although simple, is very useful and we haven't found it elsewhere.

Proposition A.13. *A decomposition of* $(D; A, B; M)_C$ *is equivalent to a map* $q_C : D \longrightarrow C$ *which is a vector bundle map with respect to both structures* D_A *and* D_B*, and satisfies* $q_C \circ \iota = \mathrm{Id}_C$*, where* $\iota : C \longrightarrow D$ *is the inclusion. Every decomposition* $\Theta : D \longrightarrow A \oplus B \oplus C$ *has the form*

$$\Theta = (q_A, q_B, q_C).$$

Proof. Suppose that we have such a map. Define

$$\begin{aligned} \Theta : &D \longrightarrow A \oplus B \oplus C \\ &d \longrightarrow (q_A, q_B, q_C). \end{aligned}$$

Clearly, $\Theta : D_A \longrightarrow (q^A)^* B \oplus_A (q^A)^* C$ and $\Theta : D_A \longrightarrow (q^B)^* A \oplus_B (q^B)^* C$ are vector bundle morphisms, and Θ is the identity on A, B and C.

Conversely, suppose that we have a decomposition $\Theta : D \longrightarrow A \oplus B \oplus C$. Define $q_C : D \longrightarrow C$ simply by $q_C = \Theta_C$, where Θ_C is the natural projection of Θ on C, $D \xrightarrow{\Theta} A \oplus B \oplus C \longrightarrow C$.

The linearity of q_C with respect to both structures readily follows form the corresponding linearities of Θ, and $q_C \circ \iota = \mathrm{Id}_C$ follows from the fact that Θ is the identity on C. It remains to prove that the projections Θ_A and Θ_B on A and B are, respectively, q_A and q_B.

Let's denote by $p_A : A \oplus B \oplus C \longrightarrow A$ and by $p_B : A \oplus B \oplus C \longrightarrow B$ the projections. Since $\Theta : D_A \longrightarrow (q^A)^* B \oplus_A (q^A)^* C$ preserves fibers, we have, for every $d \in D$,

$$\Theta(d) = \Theta(0_A(q_A(d))),$$

and since $p_A \circ \Theta \circ 0_A = \mathrm{Id}_A$, it follows that

$$\Theta_A(d) = p_A \circ \Theta(d) = p_A \Theta(0_A(q_A(d))) = q_A(d).$$

Analogously, it is shown that $\Theta_B := p_B \circ \Theta = q_B$.

■

Definition A.14. A splitting of the core sequence (A.2) is a vector bundle map

$$\theta : (q^A)^* B \longrightarrow D_A$$

over the identity, such that $(q_A, q_B) \circ \theta = \mathrm{Id}_{(q^A)^* B}$. Analogously it can be defined a splitting of the core sequence over B, (A.3).

Remark A.15. A splitting of (A.2) can always be found by choosing a complement K of $\ker(q_A, q_B) \cong (q^A)^*C$ which can be done, say, by taking the orthogonal complement with respect to a Riemannian metric on D_A. Then $(q_A, q_B)|_K : K \longrightarrow (q^A)^*B$ is an isomorphism. The splitting is obtained by setting $\theta := ((q_A, q_B)|_K)^{-1}$.

Now we are interested in determine when a splitting of (A.2) provides a decomposition of D. We begin with an important step.

Proposition A.16. *A splitting of the exact sequence (A.2) gives a map $\Theta : D \longrightarrow A \oplus B \oplus C$ inducing the identity on A, B and C, which is a vector bundle isomorphism with respect to the fibration over A.*

Proof. Take $d \in D$ and let $w = \theta((q_A, q_B)(d))$. Then $q_B(w) = q_B(d)$ and $q_A(w) = q_A(d)$. By Cor. A.7 there exists a unique $k \in C$ such that

$$d = w \underset{A}{+} (q_A(d), k) = w \underset{B}{+} (q_B(d), k).$$

Define $q_C(d) := k$. We claim that $\Theta : D \longrightarrow A \oplus B \oplus C$ given by

$$\Theta(d) := (q_A(d), q_B(d), q_C(d))$$

is the desired isomorphism. First we see that q_C preserves fibers, because if $q_A(d_1) = q_A(d_2)$, then $q^A(q_A(d_1)) = q^A(q_A(d_2)) = m$, and by definition we must have $q^C(q_C(d_1)) = q^C(q_C(d_2)) = m$. In order to check linearity, observe that

$$q_C(d) = (d \underset{A}{-} w) \underset{B}{-} 0_A(q_A(d)) = (d \underset{B}{-} w) \underset{A}{-} 0_B(q_B(d)). \tag{A.5}$$

Let $d_1, d_2 \in D$ such that $q_A(d_1) = q_A(d_2) = a$, then $q_A(d_1 \underset{A}{+} d_2) = a$. By the linearity of q_B and θ with respect to the fibration over A, it follows

$$\begin{aligned}
q_C(d_1 \underset{A}{+} d_2) &= ((d_1 \underset{A}{+} d_2) \underset{B}{-} \theta(q_A(d_1 \underset{A}{+} d_2), q_B(d_1 \underset{A}{+} d_2))) \underset{A}{-} 0_B(q_B(d_1 \underset{A}{+} d_2)) \\
&= [(d_1 \underset{A}{+} d_2) \underset{B}{-} (\theta(q_A(d_1 \underset{A}{+} d_2), q_B(d_1)) \underset{A}{+} \theta(q_A(d_1 \underset{A}{+} d_2), q_B(d_2)))] \\
&\quad \underset{A}{-} 0_B(q_B(d_1)) \underset{A}{-} 0_B(q_B(d_2)) \\
&= [(d_1 \underset{B}{-} \theta(q_A(d_1), q_B(d_1))) \underset{A}{-} (0_B(q_B(d_1))] \underset{A}{+} [(d_2 \underset{B}{-} \theta(q_A(d_2), q_B(d_2))) \underset{A}{-} 0_B(q_B(d_2)))] \\
&= q_C(d_1) \underset{A}{+} q_C(d_2).
\end{aligned}$$

Thus q_C is a vector bundle morphism, hence Θ is a vector bundle morphism. By Prop. A.4, D_A and $A \oplus B \oplus C$, as a vector bundle over A, have the same rank, equal to $\mathrm{rank}B + \mathrm{rank}C$, so it is enough to prove that Θ is injective. Suppose that $\Theta(d) = 0$, then $q_A(d) = q_B(d) = 0$, so $d = k$, by the way we defined k. But then $d = q_C(d) = 0$.

Finally, it is obvious that Θ induces the identity on A and B. If $d \in C \subset D$, then $q_A(d) = q_B(d) = 0$, so, by the definition of q_C we have $q_C(d) = d$, so Θ induces the identity also on C.

∎

Now we are able to characterize those splittings of (A.2) that are equivalent to decompositions of D. Although simple, this characterization is instrumental for what comes next, and seems to be new in the literature.

Corollary A.17. *A splitting of the exact sequence (A.2), which is simultaneously a splitting of the exact sequence (A.3), is equivalent to a decomposition of D. More specifically, in order to get a decomposition of D it is necessary and sufficient to get a splitting θ of (A.2), which preserves the fibration over B and satisfies*

$$\theta(a_1 + a_2, b) = \theta(a_1, b) \underset{B}{+} \theta(a_2, b) \tag{A.6}$$

for $a_1, a_2 \in A_m$ and $b \in B_m$, where the suffix m stands for the fiber over m.

Proof. From the proof of Prop. A.16 it follows that, if θ, seen as a map from $(q^B)^*A \longrightarrow D_B$, is also a vector bundle morphism (over the identity), that is, if θ satisfies the conditions of the corollary, then, the induced map $q_C : D \longrightarrow C$ will be linear also with respect to the fibration over B, thus Θ will be a double vector bundle morphism inducing the identity on A, B and C, that is, it will be a decomposition of D.

Conversely, suppose we have a decomposition $\Theta : D \longrightarrow A \oplus B \oplus C$. Then notice that

$$(q_A, q_C) \circ \tau_A = \mathrm{Id}_{(q^A)^*C},$$

which is equivalent to have a splitting of (A.2), since this condition implies that $\tau_A((q^A)^*C)$ is complementary to $K := \ker(q_A, q_C)$, that is,

$$D_A = K \oplus_A \ker(q_A, q_B).$$

Explicitly, the splitting $\theta : (q^A)^*B \longrightarrow D_A$ is obtained by setting

$$\theta(a, b) = \vartheta^{-1}(a, b),$$

where $\vartheta := (q_A, q_B)|_K$.

It remains to check that θ preserves the fibration over B and satisfies (A.6). By definition, we have

$$q_B(\theta(a, b)) = q_B(\vartheta(a, b)) = b, \tag{A.7}$$

which implies that θ preserves the fibration over B.

By the same argument of (A.7), we have $q_A(\vartheta(a, b)) = a$.

Now let $a_1, a_2 \in A_m$, then

$$q_A(\theta(a_1 + a_2, b)) = a_1 + a_2,$$

and taking into account (A.7), that $q_C \circ \theta = 0$ and the linearities of q_A and q_C, we get

$$\Theta(\theta(a_1 + a_2, b)) = (a_1 + a_2, b, 0) = \Theta(\theta(a_1, b) \underset{B}{+} \theta(a_2, b)),$$

which is equivalent to (A.6).

∎

A.4.2 The local structure of double vector bundles

The next result establishes the existence of local decompositions for a double vector bundle. We weren't able to find a proof of this result elsewhere in the literature, although it is assumed as known in several places[1].

Proposition A.18. *There always exists a local decomposition for a double vector bundle.*

Proof. Choose a local frame $\{a_1, \dots, a_A\}$ for A over a suitable open set $U \subset M$. Taking any splitting θ of (A.2) we obtain, for any $b \in B|_U$, elements $\theta(a_1, b), \dots, \theta(a_A, b)$. Now, any $(a, b) \in (q^A)^* B|_U$ can be written in the form $(a, b) = \sum_{i=1}^{A} \alpha_i(a_i, b)$, where $\alpha_i \in \mathbb{R}$ are unique real numbers (or, in the case $b : U \longrightarrow B$ is a smooth section –and therefore $(a, b) : U \longrightarrow (q^A)^* B_U$ is a smooth section too–, $\alpha_i : U \longrightarrow \mathbb{R}$ is a smooth function for every $i = 1, \dots, A$).

Define

$$\theta'(a, b) = \alpha_1 \underset{B}{\cdot} \theta(a_1, b) \underset{B}{+} \dots \underset{B}{+} \alpha_A \underset{B}{\cdot} \theta(a_A, b).$$

We claim that θ' is a splitting of (A.2), restricted to $U \subset M$, satisfying the conditions of Cor. A.17. First let's check that θ' is actually a splitting:

$$(q_A, q_B)(\theta'(a, b)) = (q_A, q_B)(\sum_B \alpha_i \underset{B}{\cdot} \theta(a_i, b)) = (\sum \alpha_i a_i, b) = (a, b), \tag{A.8}$$

where the suffix B means that we are using the bundle structure D_B to perform the addition. The property of preserving fibers and linearity over A is immediate form the corresponding property for θ.

It follows immediately from (A.8) that θ' preserves fibers over B, for $q_B(\theta'(a, b)) = b$.

Finally, we need to verify linearity. Observe that if $\widetilde{a}_1 = \sum \alpha_{i_1} a_i$ and $\widetilde{a}_2 = \sum \alpha_{i_2} a_i$, then

$$\widetilde{a}_1 + \widetilde{a}_2 = \sum \alpha_{i_1} a_i + \sum \alpha_{i_2} a_i = \sum (\alpha_{i_1} + \alpha_{i_2}) a_i.$$

Therefore

$$\begin{aligned} \theta'(\widetilde{a}_1 + \widetilde{a}_2, b) &= \sum_B (\alpha_{i_1} + \alpha_{i_2}) \underset{B}{\cdot} \theta(a_i, b) \\ &= \sum_B (\alpha_{i_1} \underset{B}{\cdot} \theta(a_i, b) \underset{B}{+} \alpha_{i_2} \underset{B}{\cdot} \theta(a_i, b))a = \sum_B \alpha_{i_1} \underset{B}{\cdot} \theta(a_i, b) \underset{B}{+} \sum_B \alpha_{i_2} \underset{B}{\cdot} \theta(a_i, b) \\ &= \theta'(\widetilde{a}_1, b) \underset{B}{+} \theta'(\widetilde{a}_2, b). \end{aligned}$$

Thus, by Cor. A.17 θ' induces a decomposition of $D|_U$.

∎

Remark A.19. We shall later see, in Cor. C.8, that global decompositions always exist.

As a first consequence of the local decomposition result above, we show the existence of special coordinates in a DVB, called *adapted coordinates*.

[1] See, for example, [23], where they sketch a proof for the global decomposition citing [20] for the local result, but we haven't found the proof of this result there.

Corollary A.20. *Given a double vector bundle $(D; A, B; M)_C$, for each $m \in M$ there exists an open set $U \subset M$ containing m, such that, on $D|_U$ we can find coordinates $x^i, \alpha^a, \beta^b, \kappa^c$, simultaneously adapted to both structures D_A and D_B, that is, the coordinates x^i are constant on each slice*

$$D_m := \{d \in D : q^A \circ q_A(d) = q^B \circ q_B(d) = m\},$$

the coordinates α^a are constant on each fiber over A, the coordinates β^b are constant on each fiber over B and the following equalities hold:

$$\begin{aligned}
\alpha^a(v \underset{A}{+} w) &= \alpha^a(v) = \alpha^a(w) & \qquad \alpha^a(v \underset{B}{+} w) &= \alpha^a(v) + \alpha^a(w)\\
\beta^b(v \underset{A}{+} w) &= \beta^b(v) + \beta^b(w) & \qquad \beta^b(v \underset{B}{+} w) &= \beta^b(v) = \beta^b(w)\\
\kappa^c(v \underset{A}{+} w) &= \kappa^c(v) + \kappa^c(w) & \qquad \kappa^c(v \underset{B}{+} w) &= \kappa^c(v) + \kappa^c(w).
\end{aligned}$$

Proof. Choose an open coordinate neighborhood of m, $U \subset M$, such that A, B and C trivialize over U (for example this can be done by choosing a convex set with respect to a Riemannian metric on M). Then we have adapted coordinates $x^i, \alpha^a, \beta^b, \kappa^c$ on $(A \oplus B \oplus C)|_U$. From Prop. A.18 and its proof, we get a double vector bundle isomorphism $\Theta : D|_U \longrightarrow (A \oplus B \oplus C)|_U$. The desired coordinates are obtained pulling back the coordinates $x^i, \alpha^a, \beta^b, \kappa^c$ by Θ, which we denote by the same letters.

■

Definition A.21. A coordinate system for a double vector bundle, which satisfies the properties of Cor. A.20 will be called an *adapted coordinate system*, and it's coordinates will be called *adapted coordinates*.

Remark A.22. Notice that an adapted coordinate system provides simultaneously local trivializations for both vector bundle structures of a double vector bundle.

A.4.3 Local structure of DVB morphisms

Next proposition gives a very nice characterization of double vector bundle morphisms, which will allow (see Cor. A.24) to encode the data of a DVB morphism between decomposed double vector bundles by three linear maps and one bilinear map between vector bundles. This simplifies significatively the study of double vector bundle morphisms.

Proposition A.23. *Let $\Phi : D \longrightarrow D'$ be a map, and let $(x^i, \alpha^a, \beta^b, k^c)$ and $(\tilde{x}^{\tilde{i}}, \tilde{\alpha}^{\tilde{a}}, \tilde{\beta}^{\tilde{b}}, \tilde{k}^{\tilde{c}})$ be respective coordinate systems for D and D' given by Cor. A.20. Then Φ is a double vector bundle morphism if and only if*

$$\begin{aligned}
\tilde{x}^{\tilde{i}} \circ \Phi &= \Phi^{\tilde{i}},\\
\tilde{\alpha}^{\tilde{a}} \circ \Phi &= \Phi^{\tilde{a}}_a \alpha^a,\\
\tilde{\beta}^{\tilde{b}} \circ \Phi &= \Phi^{\tilde{b}}_b \beta^b,\\
\tilde{k}^{\tilde{c}} \circ \Phi &= \Phi^{\tilde{c}}_c k^c + \Phi^{\tilde{c}}_{ab} \alpha^a \beta^b,
\end{aligned} \tag{A.9}$$

where $\Phi^{\tilde{i}}, \Phi^{\tilde{a}}_a, \Phi^{\tilde{b}}_b, \Phi^{\tilde{c}}_c, \Phi^{\tilde{c}}_{ab}$ are functions on the domain of (x^i) in M.

Proof. Suppose that Φ is a double vector bundle morphism, and let's prove (A.9). The first three equations follow from Prop. A.1. So we only need to prove the last equation. Since Φ preserves the vector bundle structures over A and A', respectively, we have

$$\tilde{k}^{\tilde{c}} \circ \Phi(x^i, \alpha^a, \beta^b, k^c) = (\Phi_A)^{\tilde{c}}_c(x^i, \alpha^a)\kappa^c(x^i) + (\Phi_A)^{\tilde{c}}_b(x^i, \alpha^a)\beta^b(x^i). \tag{A.10}$$

On the other hand, analogously we have

$$\tilde{k}^{\tilde{c}} \circ \Phi(x^i, \alpha^a, \beta^b, k^c) = (\Phi_B)^{\tilde{c}}_c(x^i, \beta^b)\kappa^c(x^i) + (\Phi_B)^{\tilde{c}}_a(x^i, \beta^b)\alpha^a(x^i). \tag{A.11}$$

Since we must have (A.10)=(A.11), it follows that

$$(\Phi_A)^{\tilde{c}}_a(x^i, \alpha^a) = (\Phi_B)^{\tilde{c}}_c(x^i, \beta^b) =: \Phi^{\tilde{c}}_c(x^i),$$

for $0_A(A) \cap 0_B(B) = 0_C(M)$.

(A.10)=(A.11) also implies

$$(\Phi_A)^{\tilde{c}}_b(x^i, \alpha^a)\beta^b(x^i) = (\Phi_B)^{\tilde{c}}_a(x^i, \beta^b)\alpha^a(x^i)$$

Then, taking $d = (x^i, 1_a, \beta^b, 0)$, where $1_a = (0, \ldots, 0, 1, 0, \ldots, 0) \in A$, the 1 being exactly in the a^{th} place, we see that

$$(\Phi_A)^{\tilde{c}}_b(x^i, 1_a)\beta^b(x^i) = (\Phi_B)^{\tilde{c}}_a(x^i, \beta^b);$$

and taking $d = (x^i, \alpha^a, 1_b, 0)$, where $1_b = (0, \ldots, 0, 1, 0, \ldots, 0) \in B$, it follows

$$(\Phi_B)^{\tilde{c}}_a(x^i, 1_b)\alpha^a(x^i) = (\Phi_A)^{\tilde{c}}_b(x^i, \alpha^a).$$

Thus, for $d = (x^i, \alpha^a, \beta^b, k^c)$,

$$(\Phi_A)^{\tilde{c}}_b(x^i, \alpha^a)\beta^b(x^i) = (\Phi_B)^{\tilde{c}}_a(x^i, 1_b)\alpha^a\beta^b = (\Phi_B)^{\tilde{c}}_a(x^i, \beta^b)\alpha^a = (\Phi_A)^{\tilde{c}}_b(x^i, 1_a)\beta^b\alpha^a.$$

In particular,

$$(\Phi_B)^{\tilde{c}}_a(x^i, 1_b) = (\Phi_A)^{\tilde{c}}_b(x^i, 1_a) =: \Phi^{\tilde{c}}_{ab}(x^i),$$

hence

$$(\Phi_A)^{\tilde{c}}_b(x^i, \alpha^a) = \Phi^{\tilde{c}}_{ab}(x^i)\alpha^a,$$

and

$$(\Phi_B)^{\tilde{c}}_a(x^i, \beta^b) = \Phi^{\tilde{c}}_{ab}(x^i)\beta^b.$$

Therefore,

$$\tilde{k}^{\tilde{c}} \circ \Phi(x^i, \alpha^a, \beta^b, k^c) = \Phi^{\tilde{c}}_c(x^i)\kappa^c + \Phi^{\tilde{c}}_{ab}(x^i)\alpha^a\beta^b.$$

For the converse, suppose that a map $\Phi : D \longrightarrow D'$ satisfies (A.9). From those equations it is immediate that Φ preserves both fibrations. Linearity with respect to both structures also follows from (A.9), taking into account the linearity of the coordinates given by Cor. A.20.

■

Corollary A.24. *Let*

$$\Phi : A \oplus B \oplus C \longrightarrow A' \oplus B' \oplus C',$$

be a double vector bundle morphism between two decomposed double vector bundles. Then

$$\Phi(a, b, c) = (\varphi_A(a), \varphi_B(b), \varphi_C(c) + \Psi(a, b)),$$

where the mapping

$$\Psi : A \oplus B \longrightarrow C'$$

is bilinear.

Proof. The proof follows easily from Props. A.1 and A.9, along with Cor. A.23. ∎

A first application of Cor. A.24 is the explicit computation of the inverse of a DVB morphism, which in particular shows that the inverse is automatically a DVB morphism too.

Proposition A.25. *Let $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ be a double vector bundle morphism with inverse Φ^{-1}. Then φ_A, φ_B and φ_C are isomorphisms; Φ^{-1} is also a double vector bundle morphism, that is Φ^{-1} is an isomorphism of double vector bundles, and*

$$\varphi_{A'} = \varphi_A^{-1};$$
$$\varphi_{B'} = \varphi_B^{-1};$$
$$\varphi_{C'} = \varphi_C^{-1}.$$

If D, D' are decomposed, then

$$\Phi^{-1}(a', b', c') = (\varphi_A^{-1}(a'), \varphi_B^{-1}(b'), \varphi_C^{-1}(c') - \varphi_C^{-1} \circ \Psi(\varphi_A^{-1}(a'), \varphi_B^{-1}(b'))). \tag{A.12}$$

Proof. That Φ^{-1} is also a double vector bundle morphism follows from the corresponding fact for vector bundles. Now recall that, by definition, $\varphi_A = q_A \circ \Phi \circ 0_A$ and $\varphi_{A'} = q_A \circ \Phi^{-1} \circ 0_{A'}$. Since $(\Phi, \varphi_A) : D_A \longrightarrow D'_{A'}$ is a vector bundle morphism, we have

$$\Phi \circ 0_A = 0_{A'} \circ \varphi_A = 0_{A'} \circ q_{A'} \circ \Phi \circ 0_A.$$

Then,

$$\varphi_{A'} \circ \varphi_A = q_A \circ \Phi^{-1} \circ 0_{A'} \circ q_{A'} \circ \Phi \circ 0_A = q_A \circ \Phi^{-1} \circ \Phi \circ 0_A = q_A \circ 0_A = \mathrm{Id}_A.$$

Interchanging Φ with Φ^{-1} we get $\varphi_A \circ \varphi_{A'} = \mathrm{Id}_{A'}$. Thus φ_A is a vector bundle isomorphism and $\varphi_A^{-1} = \varphi_{A'}$. Analogously it is shown that φ_B is a vector bundle isomorphism, with $\varphi_B^{-1} = \varphi_{B'}$. Finally, $\varphi_{C'} \circ \varphi_C = \Phi^{-1} \circ \Phi|_C = \mathrm{Id}_C$, and interchanging Φ with Φ^{-1} we get $\varphi_C \circ \varphi_{C'} = \mathrm{Id}_{C'}$. Thus φ_C is also a vector bundle isomorphism, with $\varphi_C^{-1} = \varphi_{C'}$.

Now let $D = A \oplus B \oplus C$ and $D' = A' \oplus B' \oplus C'$ be decomposed double vector bundles. For $a \in A_m$, consider the linear mapping $\Phi_a : D_a \cong B \oplus C \longrightarrow D_{\varphi(a)} \cong B' \oplus C'$, which is given by, according to Cor. A.24,

$$\Phi_a(b, c) = \begin{pmatrix} \varphi_B & 0 \\ \Psi_a & \varphi_C \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}. \tag{A.13}$$

Here $\Psi_a : B \longrightarrow C'$ is given by $\Psi_a(b) := \Psi(a,b)$, where Ψ is the bilinear mapping given in Cor. A.24. So, in order to calculate $\Phi^{-1}(a',b',c')$ we only need to calculate $\Phi_a^{-1} : D'_{a'} \longrightarrow D_{\varphi_A^{-1}(a')}$, which reduces to invert the matrix in (A.13), and this is easy:

If

$$\Phi_a^{-1} = \begin{pmatrix} X & Y \\ Z & U \end{pmatrix},$$

then, from

$$\begin{pmatrix} X & Y \\ Z & U \end{pmatrix} \begin{pmatrix} \varphi_B & 0 \\ \Psi_a & \varphi_C \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

we get the equations

$$\begin{aligned} &X \circ \varphi_B + Y \circ \Psi_a = \mathrm{Id}; \qquad && Y \circ \varphi_C = 0; \\ &Z \circ \varphi_B + U \circ \Psi_a = 0; && U \circ \varphi_C = \mathrm{Id}; \end{aligned}$$

from which we find

$$\begin{aligned} &Y = 0; \qquad && U = \varphi_C^{-1}; \\ &X = \varphi_B^{-1}; && Z = -\varphi_C^{-1} \circ \Psi_a \circ \varphi_A^{-1}, \end{aligned}$$

obtaining

$$\Phi_a^{-1} = \begin{pmatrix} \varphi_B^{-1} & 0 \\ -\varphi_C^{-1} \circ \Psi_a \circ \varphi_B^{-1} & \varphi_C^{-1} \end{pmatrix}, \tag{A.14}$$

which gives (A.12).

■

In particular, we can describe the behaviour of the change of two decompositions, and obtain an explicit computation of the projection q_C on the core bundle corresponding to a decomposition.

Proposition A.26. *Let $(D; A, B; M)_C$ be a double vector bundle. If Θ, Θ' are two decompositions, then*

$$\Theta \circ (\Theta')^{-1}(a,b,c) = (a, b, c + \Psi(a,b)), \tag{A.15}$$

where $\Psi : A \oplus B \longrightarrow C$ is bilinear (over the identity). In particular,

$$q_C(d) = q'_C(d) + \Psi(q_A(d), q_B(d)). \tag{A.16}$$

Proof. Set $\Phi := \Theta \circ (\Theta')^{-1} : A \oplus B \oplus C \longrightarrow A \oplus B \oplus C$. By Cor. A.24, and since Θ and Θ' induce the identity on A, B and C, it follows (A.15), and denoting by $p_C : A \oplus B \oplus C \longrightarrow C$ the projection, we get

$$\begin{aligned} q_C(d) = p_C \circ \Theta(d) &= p_C \circ \Theta \circ (\Theta')^{-1} \circ \Theta'(d) \\ &= p_C \circ \Theta \circ (\Theta')^{-1}(q_A(d), q_B(d), q'_C(d)) \\ &= q'_C(d) + \Psi(q_A(d), q_B(d)). \end{aligned}$$

.

■

Appendix B

More on duality of double vector bundles

We dedicate a whole appendix to study duality of double vector bundles. Because of the symmetry with respect to each structure (see Prop. A.2), it suffices to study one of the duals, and the other one will enjoy the same properties. So we study the dual with respect to the fibration over A, denoted by D^*_A, providing a thorough construction of this bundle. In particular we show that it is also a double vector bundle, and exhibit explicitly its structure. Then we move on to the study of dual morphisms, splittings and decompositions. We end appendix B describing canonical isomorphisms between suitable duals, which are important to understand the duality of VB-algebroid structures and involutivity of double vector bundles.

B.1 Construction of the dual of a double vector bundle

In this section we draw on the local structure studied in the previous chapter to describe the dual of a double vector bundle. We relayed on [32] and [46], although we supply more details.

Proposition B.1 ([32],[46]). *Let $(D; A, B; M)_C$ be a double vector bundle. The duals D^*_A and D^*_B are also double vector bundles:*

$$\begin{tikzcd} D^*_B \arrow[r, "\pi_B"] \arrow[d, "\pi_{C^*}"'] \arrow[dr, phantom, "A^*"] & B \arrow[d, "q^B"] \\ C^* \arrow[r, "q^{C^*}"] & M \end{tikzcd} ; \qquad \begin{tikzcd} D^*_A \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "B^*"] & C^* \arrow[d, "q^{C^*}"] \\ A \arrow[r, "q^A"] & M \end{tikzcd} , \tag{B.1}$$

with cores A^* and B^*, respectively.

Observe that we are denoting by the same symbol, π_{C^*}, different mappings.

Proof. By Prop. A.2, it is enough to prove that $(D^*_A; A, C^*; M)_{B^*}$ is a double vector bundle, with core B^*. We already have the dual vector bundle structure over A, D^*_A, so we need to describe the vector bundle structure over C^*, $(D^*_A)_{C^*}$, and verify the compatibility between both structures.

The projection $\pi_{C^*}: D^*_A \longrightarrow C^*$ is defined by

$$\langle \pi_{C^*}(d), c\rangle = \langle d, 0_A(\pi_A(d)) \underset{B}{+} c\rangle_A, \tag{B.2}$$

for every $c \in C_m$, where $m = q^A \circ \pi_A(d)$, and $\langle \cdot, \cdot \rangle_A$ denotes the duality pairing between D_A and D^*_A.

Recalling that, by Prop. A.4, $\ker q_A = B \oplus C$, the zero section $0_{C^*}: C^* \longrightarrow D^*_A$ is defined by observing that

$$\pi_A^{-1}(0^A(M)) = (\ker q_A)^* = B^* \oplus C^*, \tag{B.3}$$

thus we have a natural inclusion $0_{C^*}: C^* \longrightarrow \pi_A^{-1}(0^A(M)) \subset D^*_A$

$$0_{C^*}(\kappa) = (0, \kappa).$$

To check that actually $\pi_{C^*} \circ 0_{C^*} = \mathrm{Id}_{C^*}$, let $\kappa \in C^*_m$ and $c \in C_m$, then by (B.2)

$$\langle \pi_{C^*}(0_{C^*}(\kappa)), c\rangle = \langle (0,\kappa), (0,c)\rangle_A = \langle \kappa, c\rangle,$$

thence, $\pi_{C^*}(0_{C^*}(\kappa)) = \kappa$.

Let's check that π_{C^*} is a vector bundle morphism. First we see that π_{C^*} preserves fibers by definition. Now, let $d_1, d_2 \in D^*_A$ with $\pi_A(d_1) = \pi_A(d_2) = a$, then

$$\begin{aligned}
\langle \pi_{C^*}(d_1 \underset{A}{+} d_2), c\rangle &= \langle d_1 \underset{A}{+} d_2, 0_A(a) \underset{B}{+} c\rangle_A \\
&= \langle d_1, 0_A(a) \underset{B}{+} c\rangle_A + \langle d_2, 0_A(a) \underset{B}{+} c\rangle_A \\
&= \langle \pi_{C^*}(d_1) + \pi_{C^*}(d_2), c\rangle.
\end{aligned}$$

Then, $\pi_{C^*}: D^*_A \longrightarrow C^*$ is a vector bundle morphism.

Now we want to define an addition structure on the fibers of $(D^*_A)_{C^*}$. Let $d_1, d_2 \in D^*_A$ with $\pi_{C^*}(d_1) = \pi_{C^*}(d_2)$. We define $d_1 \underset{C^*}{+} d_2$ by the conditions

- $\pi_A(d_1 \underset{C^*}{+} d_2) = \pi_A(d_1) + \pi_A(d_2)$;
- $$\langle d_1 \underset{C^*}{+} d_2, v_1 \underset{B}{+} v_2\rangle_A = \langle d_1, v_1\rangle_A + \langle d_2, v_2\rangle_A, \tag{B.4}$$

  for every $v_1, v_2 \in D$ with $q_A(v_1) = \pi_A(d_1)$ and $q_A(v_2) = \pi_A(d_2)$, and of course $q_B(v_1) = q_B(v_2)$.

Let's see that $d_1 \underset{C^*}{+} d_2$ is well defined. First we need to express any $v \in D$ with $q_A(v) = \pi_A(d_1) + \pi_A(d_2)$ in the form $v = v_1 \underset{B}{+} v_2$, with $q_A(v_1) = \pi_A(d_1)$ and $q_A(v_2) = \pi_A(d_2)$. In order to get such $v_1, v_2 \in D$, we can work locally and thus suppose $D = A \oplus B \oplus C$. So we have

$$v = (\pi_A(d_1) + \pi_A(d_2), q_B(v), q_C(v));$$

Define

$$v_1 := (\pi_A(d_1), q_B(v), q_C(v)) \text{ and } v_2 := (\pi_A(d_2), q_B(v), 0).$$

Now we want to see that the formula (B.4) doesn't depend on the representation $v = v_1 \underset{B}{+} v_2$, that is, if we also have $v = v_3 \underset{B}{+} v_4$, with $q_A(v_3) = \pi_A(d_1)$ and $q_A(v_4) = \pi_A(d_2)$, then we must show that

$$\langle d_1, v_3\rangle_A + \langle d_2, v_4\rangle_A = \langle d_1, v_1\rangle_A + \langle d_2, v_2\rangle_A,$$

or equivalently

$$\langle d_1, v_1 \underset{A}{-} v_3\rangle_A = \langle d_2, v_4 \underset{A}{-} v_2\rangle_A. \tag{B.5}$$

We have the following

$$\begin{aligned} q_A(v_1) &= q_A(v_3) = \pi_A(d_1), \\ q_A(v_2) &= q_A(v_4) = \pi_A(d_2), \\ q_B(v_1) &= q_B(v_2) = q_B(v_3) = q_B(v_4) = q_B(v), \\ v_1 \underset{B}{+} v_2 &= v_3 \underset{B}{+} v_4 = v. \end{aligned}$$

Then, by Cor. A.8 it follows that

$$\begin{aligned} v_1 \underset{A}{-} v_3 &= (\pi_A(d_1), c), \\ v_2 \underset{A}{-} v_4 &= (\pi_A(d_2), -c), \end{aligned}$$

whence $v_4 \underset{A}{-} v_2 = (\pi_A(d_2), c)$, and thence

$$\begin{aligned} \langle d_1, v_1 \underset{A}{-} v_3\rangle_A &= \langle d_1, 0_A(\pi_A(d_1)) \underset{B}{+} c\rangle_A \\ &= \langle \pi_{C^*}(d_1), c\rangle = \langle \pi_{C^*}(d_2), c\rangle \\ &= \langle d_2, 0_A(\pi_A(d_2)) \underset{B}{+} c\rangle_A \\ &= \langle d_2, v_4 \underset{A}{-} v_2\rangle_A, \end{aligned}$$

which gives (B.5).

That formula (B.4) really defines a linear functional on the fibers of D_A follows from the interchange law:

Let $w_1 = v_1 \underset{B}{+} v_2$ and $w_2 = v_3 \underset{B}{+} v_4$, then

$$\begin{aligned}
\langle d_1 \underset{C^*}{+} d_2, w_1 \underset{A}{+} w_2\rangle_A &= \langle d_1 \underset{C^*}{+} d_2, (v_1 \underset{B}{+} v_2) \underset{A}{+} (v_3 \underset{B}{+} v_4)\rangle_A \\
&= \langle d_1 \underset{C^*}{+} d_2, (v_1 \underset{A}{+} v_3) \underset{B}{+} (v_2 \underset{A}{+} v_4)\rangle_A \\
&= \langle d_1, v_1 \underset{A}{+} v_3\rangle_A + \langle d_2, v_2 \underset{A}{+} v_4\rangle_A \\
&= (\langle d_1, v_1\rangle_A + \langle d_2, v_2\rangle_A) + (\langle d_1, v_3\rangle_A + \langle d_2, v_4\rangle_A) \\
&= \langle d_1 \underset{C^*}{+} d_2, w_1\rangle_A + \langle d_1 \underset{C^*}{+} d_2, w_2\rangle_A.
\end{aligned}$$

Thus, we have obtained a well defined element $d_1 \underset{C^*}{+} d_2 \in D_A^*$.

The scalar product is defined, for $d \in D_A^*$ and $t \in \mathbb{R}$, by the conditions

$$\pi_A(t \underset{C^*}{\cdot} d) = t\pi_A(d) \text{ and}$$

$$\langle t \underset{C^*}{\cdot} d, t \underset{B}{\cdot} v\rangle_A = t\langle d, v\rangle_A,$$

for every $v \in D$ such that $q_A(v) = \pi_A(d)$.

It is easy to check that these operations satisfy the properties of a vector space on each fiber. In order to see that we actually have a vector bundle structure $D_A^* \xrightarrow{\pi_{C^*}} C^*$, it remains to show that we can find local trivializations. Use the adapted coordinates given by Cor. A.20 to obtain local frames $\mathbf{b}_s$ and $\mathbf{c}_t$ dual, to β^s and κ^t. Recall that in the proof of Cor. A.20 we obtained a local decomposition $\Theta : D|_U \longrightarrow (A \oplus B \oplus C)|_U$. We can use this isomorphism in order to get local frames on $(D_A)|_U$, which we denote by $\mathbf{B}_s$ and $\mathbf{C}_t$, which are given by

$$\mathbf{B}_s(a) := \Theta^{-1}(a, \mathbf{b}_s(q^A(a)), 0),$$

and

$$\mathbf{C}_t(a) := \Theta^{-1}(a, 0, \mathbf{c}_t(q^A(a))).$$

Now we can define coordinates on $(D_A^*)_U$, $(\widetilde{x}^i, \widetilde{\alpha}^r, \widetilde{\beta}_s, \widetilde{\kappa}_t)$, given by

$$\begin{aligned}
\widetilde{x}^i(d) &:= x^i(q^A \circ \pi_A(d)); \\
\widetilde{\alpha}^r(d) &:= \alpha^r(\pi_A(d)); \\
\widetilde{\beta}_s(d) &:= \langle d, \mathbf{B}_s(\pi_A(d))\rangle_A; \\
\widetilde{\kappa}_t(d) &:= \langle d, \mathbf{C}_t(\pi_A(d))\rangle_A.
\end{aligned} \tag{B.6}$$

We claim that (B.6) is an adapted coordinate system for $(D_A^*)|_U$. Indeed, let $d_1, d_2 \in D_A^*$, with $\pi_{C^*}(d_1) = \pi_{C^*}(d_2)$, and set $a_1 := \pi_A(d_1)$; $a_2 := \pi_A(d_2)$; $m := q^A(a_1) = q^A(a_2)$, then

$$\begin{aligned}
\widetilde{\alpha}^r(d_1 \underset{C^*}{+} d_2) &= \alpha^r(\pi_A(d_1 \underset{C^*}{+} d_2)) = \alpha^r(\pi_A(d_1) + \pi_A(d_2)) \\
&= \alpha^r(\pi_A(d_1)) + \alpha^t(\pi_A(d_2)) = \widetilde{\alpha}^r(d_1) + \widetilde{\alpha}^r(d_2). \\
\widetilde{\beta}_s(d_1 \underset{C^*}{+} d_2) &= \langle d_1 \underset{C^*}{+} d_2, \mathbf{B}_s(\pi_A(d_1 \underset{C^*}{+} d_2))\rangle_A \\
&= \langle d_1 \underset{C^*}{+} d_2, \Theta^{-1}(a_1 + a_2, \mathbf{b}_s(m), 0)\rangle_A \\
&= \langle d_1 \underset{C^*}{+} d_2, \Theta^{-1}(a_1, \mathbf{b}_s(m), 0) \underset{B}{+} \Theta^{-1}(a_2, \mathbf{b}_s(m), 0)\rangle_A \\
&= \langle d_1, \mathbf{B}_s(a_1)\rangle + \langle d_2, \mathbf{B}_s(a_2)\rangle_A \\
&= \widetilde{\beta}_s(d_1) + \widetilde{\beta}_s(d_2).
\end{aligned}$$

Now, let $d_1, d_2 \in D_A^*$ with $\pi_A(d_1) = \pi_A(d_2) = a$, then

$$\begin{aligned}
\widetilde{\beta}_s(d_1 \underset{A}{+} d_2) &= \langle d_1 \underset{A}{+} d_2, \mathbf{B}_s(a)\rangle_A = \langle d_1, \mathbf{B}_s(a)\rangle_A + \langle d_2, \mathbf{B}_s(a)\rangle_A \\
&= \widetilde{\beta}_s(d_1) + \widetilde{\beta}_s(d_2). \\
\widetilde{\kappa}_t(d_1 \underset{A}{+} d_2) &= \langle d_1 \underset{A}{+} d_2, \mathbf{C}_t(a)\rangle_A = \langle d_1, \mathbf{C}_t(a)\rangle_A + \langle d_2, \mathbf{C}_t(a)\rangle_A \\
&= \widetilde{\kappa}_s(d_1) + \widetilde{\kappa}_s(d_2).
\end{aligned}$$

Thus, the coordinate system $(\widetilde{x}^i, \widetilde{\alpha}^r, \widetilde{\beta}_s, \widetilde{\kappa}_t)$ is an adapted coordinate system, as we claimed. By remark A.22 we have obtained a local trivialization for $(D_A^*)_{C^*} \longrightarrow C^*$, thus we have two vector bundle structures

$$\begin{array}{ccc}
D_A^* & \xrightarrow{\pi_{C^*}} & C^* \\
{\scriptstyle \pi_A}\big\downarrow & & \big\downarrow{\scriptstyle q^{C^*}} \\
A & \xrightarrow{q^A} & M.
\end{array} \tag{B.7}$$

Also from the coordinate system obtained, we conclude that the core bundle of D_A^* is B^*, but it can also be proved directly: on one hand we have $\ker \pi_A = B^* \oplus C^*$; on the other hand $\pi_{C^*}|_{\ker \pi_A}(\kappa) = \kappa$, for every $\kappa \in C^*$, and $\langle \pi_{C^*}|_{\ker \pi_A}(\beta), c\rangle = \langle \beta, c\rangle_A = 0$, for every $\beta \in B^*$, so it follows that

$$\ker \pi_A \cap \ker \pi_{C^*} = \ker(\pi_{C^*}|_{\ker \pi_A}) = B^*.$$

Let's see that $\pi_A : (D_A^*)_{C^*} \longrightarrow A$ is a vector bundle morphism. By the definition of π_{C^*}, we have $q^{C^*} \circ \pi_{C^*}(d) = q^A \circ \pi_A(d)$, thus π_A preserves fibers. By the definition of $\underset{C^*}{+}$ it follows that π_A preserves addition. Thus π_A is a vector bundle morphism.

Finally, we need to check the interchange law. Let $d_1, d_2, d_3, d_4 \in D_A^*$ with

$$\pi_A(d_1) = \pi_A(d_3);\ \pi_A(d_2) = \pi_A(d_4);\ \pi_{C^*}(d_1) = \pi_{C^*}(d_2);\ \pi_{C^*}(d_3) = \pi_{C^*}(d_4).$$

Let $v = v_1 \underset{B}{+} v_2 \in D$ with $q_A(v_1) = \pi_A(d_1)$ and $q_A(v_2) = \pi_A(d_2)$, then

$$\langle (d_1 \underset{A}{+} d_3) \underset{C^*}{+} (d_2 \underset{A}{+} d_4), v_1 \underset{B}{+} v_2 \rangle_A = \langle d_1 \underset{A}{+} d_3, v_1 \rangle_A + \langle d_2 \underset{A}{+} d_4, v_2 \rangle_A. \tag{B.8}$$

On the other hand

$$\begin{aligned}
\langle (d_1 \underset{C^*}{+} d_2) \underset{A}{+} (d_3 \underset{C^*}{+} d_4), v_1 \underset{B}{+} v_2 \rangle_A &= \langle d_1 \underset{C^*}{+} d_2, v_1 \underset{B}{+} v_2 \rangle_A + \langle d_3 \underset{C^*}{+} d_4, v_1 \underset{B}{+} v_2 \rangle_A \\
&= \langle d_1, v_1 \rangle_A + \langle d_2, v_2 \rangle_A + \langle d_3, v_1 \rangle_A + \langle d_4, v_2 \rangle_A \\
&= \langle d_1 \underset{A}{+} d_3, v_1 \rangle_A + \langle d_2 \underset{A}{+} d_4, v_2 \rangle_A.
\end{aligned} \tag{B.9}$$

From (B.8) and (B.9) it follows the interchange law:

$$(d_1 \underset{A}{+} d_3) \underset{C^*}{+} (d_2 \underset{A}{+} d_4) = (d_1 \underset{C^*}{+} d_2) \underset{A}{+} (d_3 \underset{C^*}{+} d_4).$$

Thus we conclude that (B.7) is a double vector bundle, with core B^*.

■

B.2 Dual morphisms

In this short section we describe the behaviour of transposed DVB morphisms. To simplify the situation, we only consider isomorphisms, which otherwise suffices for our purposes.

Proposition B.2 ([32])**.** *If $\Phi : (D; A, B; M) \longrightarrow (D'; A', B'; M')$ is a double vector bundle isomorphism, then $\Phi^*_A : D'^*_{A'} \longrightarrow D^*_A$ and $\Phi^*_B : D'^*_{B'} \longrightarrow D^*_B$ are also double vector bundle isomorphisms.*

Proof. It suffices to prove the statement of the proposition for Φ^*_A. This map is already a vector bundle morphism $D'^*_{A'} \longrightarrow D^*_A$, so it remains to prove that it is also a vector bundle morphism $(D'^*_{A'})_{C'^*} \longrightarrow (D^*_A)_{C^*}$. First we see that Φ^*_A preserves the fibration:

$$\begin{aligned}
\langle \pi_{C^*} \circ \Phi^*_A(w'), c \rangle &= \langle \Phi^*_A(w'), 0_A(\varphi_A^{-1}(a')) \underset{B}{+} c \rangle_A \\
&= \langle w', 0_{A'}(a') \underset{B'}{+} \varphi_C(c) \rangle_{A'} = \langle \pi_{C'^*}(w'), \varphi_C(c) \rangle,
\end{aligned} \tag{B.10}$$

where $w' \in D'^*_{A'}$, with $\pi_{A'}(w') = a'$.

Now let $w'_1, w'_2 \in D'^*_{A'}$, with $\pi_{C'^*}(w'_1) = \pi_{C'^*}(w'_2)$, and let $d_1, d_2 \in D$ with $q_A(d_i) = \pi_A(\Phi^*_A(w'_i))$, $i = 1, 2$, and $q_B(d_1) = q_B(d_2)$, then

$$\begin{aligned}
\langle \Phi^*_A(w'_1 \underset{C'^*}{+} w'_2), d_1 \underset{B}{+} d_2 \rangle_A &= \langle w'_1 \underset{C'^*}{+} w'_2, \Phi(d_1) \underset{B'}{+} \Phi(d_2) \rangle_{A'} \\
&= \langle w'_1, \Phi(d_1) \rangle_{A'} + \langle w'_2, \Phi(d_2) \rangle_{A'} \\
&= \langle \Phi^*_A(w'_1), d_1 \rangle_A + \langle \Phi^*_A(w'_2), d_2 \rangle_A \\
&= \langle \Phi^*_A(w'_1) \underset{C}{+} \Phi^*_A(w'_2), d_1 \underset{B}{+} d_2 \rangle_A,
\end{aligned}$$

thence

$$\Phi_A^*(w_1' \underset{C'^*}{+} w_2') = \Phi_A^*(w_1') \underset{C}{+} \Phi_A^*(w_2'),$$

so that Φ_A^* is also a vector bundle morphism $(D'^*_{A'})_{C'^*} \longrightarrow (D_A^*)_{C^*}$.

∎

Corollary B.3 ([32])**.** *The induced morphisms of Φ_A^* on the base manifold, and on the side and core bundles are given by:*

- $\varphi_{M'} = \varphi_M^{-1}$
- $\varphi_{A'} = \varphi_A^{-1}$,
- $\varphi_{C'^*} = \varphi_C^*$,
- $\varphi_{B'^*} = \varphi_B^*$.

Proof. The first two equalities, about $\varphi_{M'}$ and $\varphi_{A'}$, follows directly from the definition of Φ_A^*. The second equality follows from (B.10).

For the third equality we rely on Prop. A.4. Let $\beta' \in B'^* \subset \ker \pi_{A'} \cong B'^* \oplus C'^*$. Then $\Phi_A^*(\beta') \in \ker \pi_A \cong B^* \oplus C^*$, so let $d \in \ker q_A \cong B \oplus C$, then $d = b + c$, for $b \in B, c \in C$, and $\Phi(d) = \varphi_B(b) + \varphi_C(c) \in \ker q_{A'} \cong B' \oplus C'$. Thence

$$\langle \Phi_A^*(\beta'), d\rangle_A = \langle \beta', \Phi(d)\rangle_{A'} = \langle \beta', \varphi_B(b) + \varphi_C(c)\rangle = \langle \beta', \varphi_B(b)\rangle.$$

Thus, $\varphi_{B'^*}(\beta') = \Phi_A^*(\beta') = \varphi_B^*(\beta') \in B^* \subset D_A^*$.

∎

Corollary B.4 ([32])**.** *Let $\Phi : A \oplus B \oplus C \longrightarrow A' \oplus B' \oplus C'$ be a double vector bundle isomorphism between decomposed double vector bundles. Then,*

$$\begin{aligned} \Phi_A^* : A' \oplus C'^* \oplus B'^* &\longrightarrow A \oplus C^* \oplus B^* \\ (a', \kappa', \beta') &\longrightarrow (\varphi_A^{-1}(a'), \varphi_C^*(\kappa'), \varphi_B^*(\beta') + \Psi^*_{\varphi_A^{-1}(a')}(\kappa')), \end{aligned}$$

where

$$\begin{aligned} \Psi_a : &B \longrightarrow C' \\ &b \longrightarrow \Psi(a, b), \end{aligned}$$

where $\Psi : A \oplus B \longrightarrow C'$ is given by Cor. A.24.

Proof. By Cor. A.24 we know that

$$\Phi_A^*(a', \kappa', \beta) = (\varphi_{A'}(a'), \varphi_{C'^*}(\kappa'), \varphi_{B'^*}(\beta') + \Psi'(a', \kappa')).$$

So we need to show that $\varphi_{A'} = \varphi_A^{-1}$, $\varphi_{C'^*} = (\varphi_C)^*$, $\varphi_{B'^*} = (\varphi_B)^*$ and $\Psi'_{a'} = (\Psi_{\varphi_A^{-1}(a')})^*$.

Now, let $(\Phi_A^*)_{a'}(\kappa', \beta') := \Phi_A^*(a', \kappa', \beta')$. By definition,

$$(\Phi_A^*)_{a'} = \Phi^*_{\varphi^{-1}(a')},$$

thus, $\varphi_{A'} = \varphi_A^{-1}$ follows directly from the definition of Φ_A^*. Writing in matrix form, we see that

$$(\Phi_A^*)_{a'} = \Phi^*_{\varphi^{-1}(a')} = \begin{pmatrix} \varphi_B & 0 \\ \Psi_{\varphi_A^{-1}(a')} & \varphi_C \end{pmatrix}^t = \begin{pmatrix} (\varphi_B)^t & (\Psi_{\varphi_A^{-1}(a')})^t \\ 0 & (\varphi_C)^t \end{pmatrix},$$

from which it follows $\varphi_{C'^*} = (\varphi_C)^*$, $\varphi_{B'^*} = (\varphi_B)^*$ and $\Psi'_{a'} = (\Psi_{\varphi_A^{-1}(a')})^*$.

■

B.3 Dual splittings and decompositions

In this section we obtain the fundamental results relating the decompositions and splittings of a DVB to the induced ones on the corresponding duals. Up to our knowledge, the results presented in this section are new, the only exception being the first part of Prop. B.5, which is hinted in [46].

Proposition B.5. *The core sequence of D_A^* (over A) is the result of transposing the core sequence (A.2) of D_A:*

$$0 \longrightarrow (q^A)^*(B^*) \longrightarrow D_A^* \longrightarrow (q^A)^*(C^*) \longrightarrow 0 \tag{B.11}$$

A splitting of the core sequence (A.2) induces a splitting of the dual core sequence (B.11).

Proof. Let $c \in C$, then

$$\begin{aligned} \langle (\tau_A)^*(w), (\pi_A(w), c)\rangle &= \langle w, \tau_A(\pi_A(w), c)\rangle_A \\ &= \langle w, 0_A(\pi_A(w)) \underset{B}{+} c\rangle_A \\ &= \langle \pi_{C^*}(w), c\rangle. \end{aligned}$$

Then $(\pi_A, \pi_{C^*}) = (\tau_A)^*$.

Now let $(a, \beta) \in (q^A)^*(B^*)$. Take any $v \in D$ with $q_A(v) = \pi_A(0_A(a) \underset{C^*}{+} \beta) = a$ and $q_B(v) = b$. Then

$$\begin{aligned} \langle 0_A(a) \underset{C^*}{+} \beta, v\rangle_A &= \langle 0_A(a) \underset{C^*}{+} \beta, v \underset{B}{+} 0_B(b)\rangle_A \\ &= \langle 0_A(a), v\rangle_A + \langle \beta, 0_B(b)\rangle_A \\ &= \langle \beta, b\rangle. \end{aligned}$$

On the other hand,

$$\langle (q_A, q_B)^*(a, \beta), v\rangle = \langle (a, \beta), (a, b)\rangle = \langle \beta, b\rangle.$$

Thus,

$$\tau_{B^*}(a, \beta) = 0_A(a) \underset{C^*}{+} \beta = (q_A, q_B)^*(a, \beta).$$

Now let $\theta : (q^A)^*B \longrightarrow D_A$ be a splitting of (A.2). By transposing, we get a projection

$$D_A^* \xrightarrow{\theta^*} (q^A)^*(B^*).$$

Since $\theta^* \circ \tau_{B^*} = [(q_A, q_B) \circ \theta]^* = (\mathrm{Id}_{(q^A)^* B})^* = \mathrm{Id}_{(q^A)^*(B^*)}$, the exactness of (B.11) implies that $(\pi_A, \pi_{C^*})|_{\ker \theta^*}$ is invertible. Define $\widetilde{\theta} := [(\pi_A, \pi_{C^*})|_{\ker \theta^*}]^{-1}$. By definition it follows that $\widetilde{\theta}$ is a splitting of (B.11).

■

Proposition B.6. *If we have a splitting θ of (A.2) which is simultaneously a splitting of (A.3) then the induced splitting of (B.11), $\widetilde{\theta}$, is simultaneously a splitting of the core sequence*

$$0 \longrightarrow (q^{C^*})^*(B^*) \longrightarrow (D^*_A)_{C^*} \longrightarrow (q^{C^*})^* A \longrightarrow 0. \tag{B.12}$$

The induced decompositions by these splittings,

$$\Theta = (q_A, q_B, q_C) : D \longrightarrow A \oplus B \oplus C \qquad \text{and} \qquad \widetilde{\Theta} = (\pi_A, \pi_{C^*}, \pi_{B^*}) : D^*_A \longrightarrow A \oplus C^* \oplus B^*,$$

preserve the duality pairing:

$$\langle v, w \rangle_A = \langle \Theta(v), \widetilde{\Theta}(w) \rangle := \langle q_B(v), \pi_{B^*}(w) \rangle + \langle q_C(v), \pi_{C^*}(w) \rangle, \tag{B.13}$$

for $v \in D,\ w \in D^*_A$, with $q_A(v) = \pi_A(w)$.

Proof. We already have that $\widetilde{\theta}$ preserves the fibration over C^*, for

$$\pi_{C^*}(\widetilde{\theta}(a,k)) = p_2(\pi_A, \pi_{C^*} \circ \theta(a,k)) = p_2(a,k) = k,$$

where $p_2 : (q^A)^*(C^*) \longrightarrow C^*$ is the natural projection.

Now we need to verify

$$\widetilde{\theta}(a_1 + a_2, k) = \widetilde{\theta}(a_1, k) \underset{C^*}{+} \widetilde{\theta}(a_2, k),$$

or equivalently

$$(\pi_A, \pi_{C^*}, \pi_{B^*})(\widetilde{\theta}(a_1 + a_2, k)) = (\pi_A, \pi_{C^*}, \pi_{B^*})(\widetilde{\theta}(a_1, k) \underset{C^*}{+} \widetilde{\theta}(a_2, k)). \tag{B.14}$$

By definition of $\widetilde{\theta}$ and of the addition $\underset{C^*}{+}$, we have

$$\pi_A(\widetilde{\theta}(a_1 + a_2, k)) = \pi_A(\widetilde{\theta}(a_1, k) \underset{C^*}{+} \widetilde{\theta}(a_2, k)) = a_1 + a_2,$$

and

$$\pi_{C^*}(\widetilde{\theta}(a_1 + a_2, k)) = \pi_{C^*}(\widetilde{\theta}(a_1, k) \underset{C^*}{+} \widetilde{\theta}(a_2, k)) = k.$$

Now, recall that the projection over the core bundle induced by a splitting θ was defined in the proof of Prop. A.16, eq. (A.5). That definition implies, in our situation, that $\pi_{B^*} = p_2 \circ \theta^*$. Then, again by the definition of $\widetilde{\theta}$, $\theta^* \circ \widetilde{\theta}(a,k) = (a, 0)$ holds, so that

$$\pi_{B^*}(\widetilde{\theta}(a_1 + a_2, k)) = 0.$$

On the other hand, the linearity of θ with respect to the fibration over B and the definition of $\underset{C^*}{+}$ imply, for every $b \in B$,

$$
\begin{aligned}
\langle \pi_{B^*}(\widetilde{\theta}(a_1,k) \underset{C^*}{+} \widetilde{\theta}(a_2,k)), b\rangle &= \langle \theta^*(\widetilde{\theta}(a_1,k) \underset{C^*}{+} \widetilde{\theta}(a_2,k)), (a_1+a_2,b)\rangle_A \\
&= \langle \widetilde{\theta}(a_1,k) \underset{C^*}{+} \widetilde{\theta}(a_2,k), \theta(a_1,b) \underset{B}{+} \theta(a_2,b)\rangle_A \\
&= \langle \widetilde{\theta}(a_1,k), \theta(a_1,b)\rangle_A + \langle \widetilde{\theta}(a_2,k), \theta(a_2,b)\rangle_A \\
&= \langle \theta^*(\widetilde{\theta}(a_1,k), (a_1,b))\rangle_A + \langle \theta^*(\widetilde{\theta}(a_2,k)), (a_2,b)\rangle_A \\
&= 0.
\end{aligned}
$$

Thus,

$$\pi_{B^*}(\widetilde{\theta}(a_1,k) \underset{C^*}{+} \widetilde{\theta}(a_2,k)) = 0,$$

and so we have obtained (B.14).

Now we verify (B.13). Let's compute the first term from the right-hand side of (B.13)

$$
\begin{aligned}
\langle q_B(v), \pi_{B^*}(w)\rangle &= \langle q_B(v), p_2 \circ \theta^*(w)\rangle \\
&= \langle (q_A, q_B)(v), \theta^*(w)\rangle_A \\
&= \langle \theta(q_A, q_B)(v), w\rangle_A. \qquad \text{(B.15)}
\end{aligned}
$$

In order to compute the second term of the right-hand side of (B.13), recall the definition of q_C given in (A.5), and the definition of π_{C^*}. We get the following

$$
\begin{aligned}
\langle q_C(v), \pi_{C^*}(w)\rangle &= \langle 0_A(\pi_A(w)) \underset{B}{+} q_C(v), w\rangle_A \\
&= \langle v \underset{A}{-} \theta(q_A, q_B)(v), w\rangle_A. \qquad \text{(B.16)}
\end{aligned}
$$

Adding (B.15) and (B.16) we get (B.13).

■

Corollary B.7. *Given a double vector bundle D, a decomposition of D induces canonically decompositions for D^*_A and D^*_B.*

If θ is the splitting corresponding to the decomposition fo D, then the projections $\pi_{B^*} : D^*_A \longrightarrow B^*$ and $\pi_{A^*} : D^*_B \longrightarrow A^*$, which correspond to the decompositions, are given by

$$\pi_{B^*} = p_2 \circ \theta^* \text{ and } \pi_{A^*} = p_1 \circ \theta^*.$$

Corollary B.8. *Given a double vector bundle D and a decomposition Θ, with its corresponding decomposition $\widetilde{\Theta}$ on the dual D^*_A,*

$$\widetilde{\Theta}^{-1}(a,\kappa,\beta) = \Theta^*(a,\kappa,\beta),$$

holds for $a \in A_m$, $\kappa \in C^*_m$ and $\beta \in B^*_m$.

Proof. For any $v \in D^*_A$ and $d \in D$, with $\pi_A(v) = q_A(d) = a$, we have

$$\langle \Theta^*(\widetilde{\Theta}(v)), d\rangle_A = \langle \widetilde{\Theta}(v), \Theta(d)\rangle = \langle v, d\rangle_A,$$

where we used (B.13) in the last equality. It follows that

$$\Theta^* \circ \widetilde{\Theta} = \mathrm{Id};$$

and since $\Theta, \widetilde{\Theta}$ are isomorphisms, we conclude that

$$\widetilde{\Theta}^{-1} = \Theta^*.$$

■

Corollary B.9. *Let $(D; A, B; M)_C$ be a double vector bundle. Let $a \in A_m$, $b \in B_m$, $c \in C_m$, $\beta \in B^*_m$, $k \in C^*$, $v \in D_A$ with $q_A(v) = a$ and $q_B(v) = b$, and $w \in D^*_A$, with $\pi_A(w) = a$ and $\pi_{C^*}(w) = k$. Then*

$$\langle 0_A(a) \underset{B}{+} c, w\rangle_A = \langle c, k\rangle; \qquad \langle v, 0_A(a) \underset{C^*}{+} \beta\rangle_A = \langle b, \beta\rangle.$$

Proof. We can work locally, so that we have a decomposition of D and the induced dual decomposition on D^*_A given in Prop. B.6, and consequently we get maps $q_C : D \longrightarrow C$ and $\pi_{B^*} : D^*_A \longrightarrow B^*$. Then, by (B.13),

$$\langle 0_A(a) \underset{B}{+} c, w\rangle_A = \langle q_B(0_A(a) \underset{B}{+} c), \pi_{B^*}(w)\rangle + \langle q_C(0_A(a) \underset{B}{+} c), \pi_{C^*}(w)\rangle = \langle c, k\rangle.$$

Analogously it can be shown that $\langle v, 0_A(a) \underset{C^*}{+} \beta\rangle_A = \langle b, \beta\rangle$.

■

Corollary B.10. *Given a double vector bundle D and a decomposition Θ, the dual splitting*

$$\widetilde{\theta} : (q^A)^* C^* \longrightarrow D^*_A,$$

of (B.11) given in Prop. B.5, corresponding to the splitting θ of (A.2), is given by

$$\widetilde{\theta}(a, \kappa) = (q_A, q_C)^*(a, \kappa), \tag{B.17}$$

where $q_C : D \longrightarrow C$ is the projection induced by Θ, and $(q_A, q_C)^*$ is the transpose of $(q_A, q_C) : D_A \longrightarrow (q^A)^* C$.

Proof. In the proof of Prop. B.5 we defined, for $a \in A_m$ and $\kappa \in C^*_m$,

$$\widetilde{\theta}(a, \kappa) = [(\pi_A, \pi_{C^*})|_{\ker \theta^*}]^{-1}(a, \kappa). \tag{B.18}$$

So, (B.17) reads

$$[(\pi_A, \pi_{C^*})|_{\ker \theta^*}]^{-1} = (q_A, q_C)^*(a, \kappa). \tag{B.19}$$

To prove this take any $c \in C_m$, and let $d \in D$ with $q_A(d) = a$ and $q_C(d) = c$. Then

$$\langle (q_A, q_C)^*(a, \kappa), d\rangle_A = \langle (a, \kappa), (a, c)\rangle_A = \langle \kappa, c\rangle. \tag{B.20}$$

On the other hand, in order to compute $\langle\widetilde{\theta}(a,\kappa),d\rangle_A$, we claim that

$$[(\pi_A,\pi_{C^*})|_{\ker\theta^*}]^{-1}(a,\kappa)=\Theta^*(a,\kappa,0). \tag{B.21}$$

To verify (B.21) we need to check two things:

$$1)\ \Theta^*(a,\kappa,0)\in\ker\theta^*; \qquad 2)\ (\pi_A,\pi_{C^*})(\Theta^*(a,\kappa,0))=(a,\kappa).$$

So, let's compute:

$$\begin{aligned}\langle\theta^*\circ\Theta^*(a,\kappa,0),(a,b)\rangle_A&=\langle(a,\kappa,0),\Theta\circ\theta(a,b)\rangle_A\\&=\langle(a,\kappa,0),\Theta\circ\Theta^{-1}(a,b,0)\rangle_A\\&=\langle\kappa,0\rangle+\langle0,b\rangle=0,\end{aligned}$$

hence $\theta^*(\Theta^*(a,\kappa,0))=0$, which yields 1). Now, recalling from Cor. B.8 that $\Theta^*=\widetilde{\Theta}^{-1}$, it follows that

$$(\pi_A,\pi_{C^*})\circ\Theta^*(a,\kappa,0)=(p_A,p_{C^*})\circ\widetilde{\Theta}\circ\Theta^*(a,\kappa,0)=(p_A,p_{C^*})(a,\kappa,0)=(a,\kappa),$$

yielding 2). Thence (B.21) is true.

From (B.18) and (B.21) we get

$$\langle\widetilde{\theta}(a,\kappa),d\rangle_A=\langle\Theta^*(a,\kappa,0),d\rangle_A=\langle(a,\kappa,0),(a,b,c)\rangle_A=\langle\kappa,c\rangle. \tag{B.22}$$

From (B.20) and (B.22), follows (B.19).

∎

B.4 Canonical isomorphisms between dual DVB's

In this section we discuss the isomorphisms mentioned in the introduction of this chapter, that will enable us, in Sec. E.3 to understand the relation between a *VB*-algebroid structure, and its dual. The material in this section is mainly based on [46] (however, see also [32]).

We begin exhibiting a duality relation between the two duals of a double vector bundle, which is natural up to sign (we could choose the opposite signature).

Proposition B.11 ([46]). *There is a natural (up to sign) duality between the bundles D^*_A and D^*_B over C^*, given by*

$$(v|w)=\langle v,d\rangle_A-\langle w,d\rangle_B, \tag{B.23}$$

where $v\in D^*_A$, $w\in D^*_B$ have $\pi_{C^*}(v)=\pi_{C^*}(w)=k$ and d is any element of D with $q_A(d)=\pi_A(v)$ and $q_B(d)=\pi_B(w)$.

Proof. First we see that (B.23) is well defined, for if $d'\in D$ satisfies $q_A(d')=q_A(d)=\pi_A(v)=a$ and $q_B(d')=q_B(d)=\pi_B(w)=b$, then, by Cor. A.7 there exist $c\in C$ such that $d=d'\underset{A}{+}(a,c)$, whence

$$\langle v,d\rangle_A=\langle v,d'\rangle_A+\langle k,c\rangle,$$

by (B.2). Also by Cor. A.7 we have $d = d' \underset{B}{+} (b, c)$, whence

$$\langle w, d\rangle_B = \langle w, d'\rangle_B + \langle k, c\rangle.$$

Thus,

$$\langle v, d\rangle_A - \langle w, d\rangle_B = \langle v, d'\rangle_A - \langle w, d'\rangle_B,$$

and so the pairing (B.23) is well defined. Now we need to check that this pairing is bilinear. Let $v_1, v_2 \in D_A^*$, $w \in D_B^*$, with $\pi_{C^*}(v_1) = \pi_{C^*}(v_2) = \pi_{C^*}(w) = k$, and let $d_1, d_2 \in D$ with $q_A(d_1) = \pi_A(v_1)$, $q_A(d_2) = \pi_A(v_2)$ and $q_B(d_1) = q_B(d_2) = \pi_B(w)$. Then, by (B.4) we have

$$\begin{aligned}(v_1 \underset{C^*}{+} v_2|w) &= \langle v_1 \underset{C^*}{+} v_2, d_1 \underset{B}{+} d_2\rangle_A - \langle w, d_1 \underset{B}{+} d_2\rangle_B \\ &= \langle v_1, d_1\rangle_A + \langle v_2, d_2\rangle_A - \langle w, d_1\rangle_B - \langle w, d_2\rangle_B \\ &= (v_1|w) + (v_2|w).\end{aligned}$$

Analogously, it is shown that $(v|w_1 \underset{C^*}{+} w_2) = (v|w_1) + (v|w_2)$, for $v \in D_A^*$, $w_1, w_2 \in D_B^*$ with $\pi_{C^*}(v) = \pi_{C^*}(w_1) = \pi_{C^*}(w_2)$. So indeed $(\cdot|\cdot)$ is bilinear.

It remains to prove that it is non-degenerate. Let $v \in D_A^*$ and suppose $(v|w) = 0$ for all $w \in D_B^*$ with $\pi_{C^*}(v) = \pi_{C^*}(w) = k \in C_m^*$. Then take $w = 0_{C^*}(k) \underset{B}{+} \alpha$, where $\alpha \in A_m$ is arbitrary. We have $\pi_B(w) = 0^B(m)$ and

$$(v|w) = \langle v, d\rangle_A - \langle 0_{C^*}(k) \underset{B}{+} \alpha, d\rangle_B.$$

Since $\pi_B(w) = q_B(w)$, it follows that $d \in \ker q_B \cong A \oplus C$, so that $d = (a, c)$, for some $c \in C$, where $a = \pi(v)$, whence

$$\langle 0_{C^*}(k) \underset{B}{+} \alpha, d\rangle_B = \langle k, c\rangle + \langle \alpha, a\rangle.$$

Taking $c = 0$, that is, $d = 0_A(a)$, we get

$$0 = (v|w) = \langle \alpha, a\rangle.$$

Since α was arbitrary, we conclude that $\pi_A(v) = 0$, thus $v \in \ker \pi_A \cong B^* \oplus C^*$, so that $v = (\beta, k)$, for some $\beta \in B_m^*$.

Now take $w = 0_B(b)$, with $b \in B_m$ arbitrary. We have $d \in \ker q_A \cong B \oplus C$. Take $d = (b, 0) = 0_B(b)$. Then

$$0 = (v|w) = \langle(\beta, k), (b, 0)\rangle_A - \langle 0_B(b), 0_B(b)\rangle_B = \langle \beta, b\rangle,$$

where we are using the same symbol 0_B for the zero sections of $D_B \longrightarrow B$ and $D_B^* \longrightarrow B$. Then, since b was arbitrary,

$$v = (0, k) = 0_{C^*}(k),$$

so that $(\cdot|\cdot)$ is indeed non-degenerate.

∎

Corollary B.12. *If $D = A \oplus B \oplus C$ is a decomposed DVB, then, for $v \in D^*_A = A \oplus C^* \oplus B^*$ and $w \in D^*_B = B \oplus C^* \oplus A^*$, such that $v = (a, \kappa, \beta)$ and $w = (b, \kappa, \alpha)$, we have*

$$(v|w) = \langle \beta, b\rangle - \langle \alpha, a\rangle. \tag{B.24}$$

Proof. By Prop. B.11, choosing $d = (a, b, c)$, we have, by Eq. (B.23),

$$\begin{aligned}(v|w) &= \langle \beta, b\rangle + \langle \kappa, c\rangle - \langle \kappa, c\rangle - \langle \alpha, a\rangle \\ &= \langle \beta, b\rangle - \langle \alpha, a\rangle.\end{aligned}$$

∎

Now we get into the matter of exhibiting the isomorphisms mentioned at the beginning of this section.

Proposition B.13 ([46]). *The pairing (B.23) induces isomorphisms of double vector bundles*

$$\begin{aligned} &\Upsilon_A : (D^*_A)_{C^*} \longrightarrow (D^*_B)^*_{C^*}, \qquad && \langle \Upsilon_A(v), w\rangle_{C^*} = (v|w) \\ &\Upsilon_B : (D^*_B)_{C^*} \longrightarrow (D^*_A)^*_{C^*}, \qquad && \langle \Upsilon_B(w), v\rangle_{C^*} = (v|w)\end{aligned}$$

with $(\Upsilon_A)^* = \Upsilon_B$. Both isomorphisms induce the identity on the sides $C^* \longrightarrow C^*$.

Υ_A is the identity on the cores $B^* \longrightarrow B^*$, and induces $-\mathrm{Id}$ on the side bundles $A \longrightarrow A$.

Υ_B is the identity on the side bundles $B \longrightarrow B$, and induces $-\mathrm{Id}$ on the cores $A^* \longrightarrow A^*$.

Proof. That Υ_A and Υ_B are vector bundles (with respect to the vector structure over C^*) inducing the identity on C^* follows directly from Prop. B.11. Also, by definition, we have $(\Upsilon_A)^* = \Upsilon_B$. Let's prove that Υ_A is linear also with respect to the structures over A. First we check that Υ_A preserves the fibration over A. We need to compute $\pi_A(\Upsilon_A(v))$, so let $\alpha \in A^* \subset D^*_B$, we have, using Cor. B.9,

$$\langle \pi_A(\Upsilon_A(v)), \alpha\rangle = \langle \Upsilon_A(v), 0_{C^*}(k) \underset{B}{+} \alpha\rangle_{C^*} = (v|0_{C^*}(k) \underset{B}{+} \alpha),$$

where $k := \pi_{C^*}(\Upsilon_A(v)) = \pi_{C^*}(v)$.

Now let $d \in D$ with $q_A(d) = \pi_A(v) = a \in A_m$ and $q_B(d) = \pi_B(0_{C^*}(k) \underset{B}{+} \alpha) = 0^B(m)$. Then $d = 0_A(a) \underset{B}{+} c$, for some $c \in C_m$, so that, using again Cor. B.9,

$$\begin{aligned}(v|0_{C^*}(k) \underset{B}{+} \alpha) &= \langle v, 0_A(a) \underset{B}{+} c\rangle_A - \langle 0_{C^*}(k) \underset{B}{+} \alpha, 0_A(a) \underset{B}{+} c\rangle_B \\ &= \langle k, c\rangle - (\langle k, c\rangle + \langle \alpha, a\rangle) \\ &= -\langle \pi_A(v), \alpha\rangle.\end{aligned}$$

Thus, $\langle \pi_A(\Upsilon_A(v)), \alpha\rangle = -\langle \pi_A(v), \alpha\rangle$, that is

$$\pi_A(\Upsilon_A(v)) = -\pi_A(v), \tag{B.25}$$

in particular Υ_A preserves the fibration over A.

To check linearity, let $v_1, v_2 \in D_A^*$, with $\pi_A(v_1) = \pi_A(v_2)$. Any $w \in D_B^*$, with $\pi_{C^*}(w) = \pi_{C^*}(v_1) + \pi_{C^*}(v_2)$ can be written in the form $w = w_1 \underset{B}{+} w_2$, with $\pi_{C^*}(w_1) = \pi_{C^*}(v_1)$ and $\pi_{C^*}(w_2) = \pi_{C^*}(v_2)$. Then

$$\begin{aligned}\langle \Upsilon_A(v_1 \underset{A}{+} v_2), w\rangle_{C^*} &= \langle \Upsilon_A(v_1 \underset{A}{+} v_2), w_1 \underset{B}{+} w_2\rangle_{C^*}\\ &= (v_1 \underset{A}{+} v_2 | w_1 \underset{B}{+} w_2)\\ &= \langle v_1 \underset{A}{+} v_2, d\rangle_A - \langle w_1 \underset{B}{+} w_2, d\rangle_B\\ &= \langle v_1, d\rangle_A + \langle v_2, d\rangle_A - \langle w_1, d\rangle_B - \langle w_2, d\rangle_B\\ &= (v_1|w_1) + (v_2|w_2). \end{aligned} \tag{B.26}$$

On the other hand,

$$\begin{aligned}\langle \Upsilon_A(v_1) \underset{A}{+} \Upsilon_A(v_2), w\rangle_{C^*} &= \langle \Upsilon_A(v_1) \underset{A}{+} \Upsilon_A(v_2), w_1 \underset{B}{+}\rangle_{C^*}\\ &= \langle \Upsilon_A(v_1), w_1\rangle_{C^*} + \langle \Upsilon_A(v_2), w_2\rangle_{C^*}\\ &= (v_1|w_1) + (v_2|w_2). \end{aligned} \tag{B.27}$$

From (B.26) and (B.27) we get

$$\Upsilon_A(v_1 \underset{A}{+} v_2) = \Upsilon_A(v_1) \underset{A}{+} \Upsilon_A(v_2).$$

Therefore Υ_A is a morphism of double vector bundles. It follows from (B.25) that Υ_A induces $-\mathrm{Id}$ on the side bundles $A \longrightarrow A$. Now, for $\beta \in B_m^* \subset D_A^*$, we want to compute $\Upsilon_A(\beta)$. Since Υ_A is a double vector bundle morphism, we have $\Upsilon_A(\beta) \in B^* \subset (D_B^*)_{C^*}$, so, let's take $b \in B_m \subset \ker \pi_{C^*} \subset (D_B^*)_{C^*}$, we have

$$\langle \Upsilon_A(\beta), b\rangle_{C^*} = (\beta|b) = \langle \beta, d\rangle_A - \langle b, d\rangle_B,$$

for any $d \in D$ with $q_A(d) = \pi_A(\beta) = 0$ and $q_B(d) = \pi_B(d) = b$. In particular, $d := 0_B(b)$ satisfies the requirements, and we get

$$\langle \Upsilon_A(\beta), b\rangle_{C^*} = \langle \beta, b\rangle,$$

thence $\Upsilon_A(\beta) = \beta$, that is, Υ_A induces the identity on the cores $B^* \longrightarrow B^*$.

The corresponding claims for Υ_B follow analogously.

■

It will be useful to have formulas for the isomorphisms Υ_A and Υ_B in terms of a decomposition.

Proposition B.14. *Let $\Theta : D \longrightarrow A \oplus B \oplus C$ be a decomposition ofr a DVB. Then we have induced decompositions on the duals (Prop. B.6)*

$$\widetilde{\Theta}_A : D_A^* \longrightarrow A \oplus C^* \oplus B^* \quad \text{and} \quad \widetilde{\Theta}_B : D_B^* \longrightarrow B \oplus C^* \oplus A^*.$$

Then, terms of these decompositions, the isomorphisms Υ_A and Υ_B are given by

$$\begin{aligned}(\widetilde{\Theta}_B^{-1})^*_{C^*} \circ \Upsilon_A \circ \widetilde{\Theta}_A^{-1} : A \oplus C^* \oplus B^* &\longrightarrow C^* \oplus A \oplus B^*\\ (a, \kappa, \beta) &\longrightarrow (\kappa, -a, \beta), \end{aligned} \tag{B.28}$$

and

$$\begin{aligned}(\widetilde{\Theta}_A^{-1})^*_{C^*} \circ \Upsilon_B \circ \widetilde{\Theta}_B^{-1} : B \oplus C^* \oplus A^* &\longrightarrow C^* \oplus B \oplus A^* \\ (b, \kappa, \alpha) &\longrightarrow (\kappa, b, -\alpha).\end{aligned} \tag{B.29}$$

Proof. Let $v \in D_A^*$ and $w \in D_B^*$ with $\pi_{C^*}(v) = \pi_{C^*}(w) = \kappa$, such that $\widetilde{\Theta}_A(v) = (a, \kappa, \beta)$ and $\widetilde{\Theta}_B(w) = (b, \kappa, \alpha)$. Then, by the definitions, and using Eqs. (B.13) and (B.24),

$$\begin{aligned}\langle (\widetilde{\Theta}_B^{-1})^*_{C^*} \circ \Upsilon_A \circ \widetilde{\Theta}_A^{-1}(a, \kappa, \beta), (b, \kappa, \alpha)\rangle_{C^*} &= \langle (\widetilde{\Theta}_B^{-1})^*_{C^*} \circ \Upsilon_A(v), \widetilde{\Theta}_B(w)\rangle_{C^*} \\ &= \langle \Upsilon_A(v), w\rangle_{C^*} = (v|w) \\ &= \langle \beta, b\rangle - \langle \alpha, a\rangle \\ &= \langle (\kappa, -a, \beta), (b, \kappa, \alpha)\rangle_{C^*},\end{aligned}$$

which implies that

$$(\widetilde{\Theta}_B^{-1})^*_{C^*} \circ \Upsilon_A \circ \widetilde{\Theta}_A^{-1}(a, \kappa, \beta) = (\kappa, -a, \beta).$$

The statement about Υ_B is proved analogously.

■

We end this section by showing that the dual isomorphisms, with respect to each fibration of a double vector bundle, are also dual to each other. More precisely we have the following.

Proposition B.15. *Let $\Phi : D \longrightarrow D'$ be an isomorphism of double vector bundles. Then the dual isomorphisms given in Prop. B.2, $\Phi_A^* : D'^*_{A'} \longrightarrow D_A^*$ and $\Phi_B^* : D'^*_{B'} \longrightarrow D_B^*$ satisfy*

$$(\Phi_A^*)^*_{C'^*} = (\Phi^{-1})^*_{B'} = (\Phi_B^*)^{-1}, \tag{B.30}$$

where we are using the identifications $(D_A^*)^*_{C^*} \cong D_B^*$ and $(D'^*_{A'})^*_{C'^*} \cong D^*_{B'}$, given by Υ_B and $\Upsilon_{B'}$, respectively, in Prop. B.13.

Proof. Take $w \in D_B^*$, $\alpha' \in D'^*_{A'}$ and compute using (B.23):

$$\begin{aligned}((\Phi_A^*)^*_{C'^*}(w)|\alpha')_{C'^*} &= (w|\Phi_A^*(\alpha'))_{C^*} = \langle \Phi_A^*(\alpha'), d\rangle_A - \langle w, d\rangle_B \\ &= \langle \alpha', \Phi(d)\rangle_{A'} - \langle w, d\rangle_B,\end{aligned} \tag{B.31}$$

for any $d \in D$ with $q_A(d) = \pi_A(\Phi_A^*(\alpha')) = \varphi_A^{-1}(\pi_{A'}(\alpha'))$ and $q_B(d) = \pi_B(w)$.

On the other hand, notice that, by Cor. B.3, $\Phi(d) \in D'$ satisfies

$$q_{A'}(\Phi(d)) = \varphi_A(q_A(d)) = \varphi_A(\varphi_A^{-1}(\pi_{A'}(\alpha'))) = \pi_{A'}(\alpha'),$$

and, taking also into account Cor. A.25,

$$\pi_{B'}((\Phi^{-1})^*_B(w)) = \varphi_{B'}^{-1}(\pi_B(w)) = (\varphi_B^{-1})^{-1}(\pi_B(w)) = \varphi_B(\pi_B(w)) = \varphi_B(q_B(d)) = q_{B'}(\Phi(d));$$

then we can use $\Phi(d)$ in the definition of $(\cdot|\cdot)_{C^*}$ to compute $((\Phi^{-1})^*_B(w)|\alpha')_{C'^*}$, and we get the following

$$\begin{aligned}((\Phi^{-1})^*_B(w)|\alpha')_{C'^*} &= \langle \alpha', \Phi(d)\rangle_{A'} - \langle (\Phi^{-1})^*_B(w), \Phi(d)\rangle_{B'} = \langle \alpha', \Phi(d)\rangle_{A'} - \langle w, \Phi^{-1}(\Phi(d))\rangle_B \\ &= \langle \alpha', \Phi(d)\rangle_{A'} - \langle w, d\rangle_B.\end{aligned} \tag{B.32}$$

From (B.31) and (B.32) we get (B.30).

■

B.5 The pull-back in the DVB category

Proposition B.16. *Let*

$$\begin{tikzcd} D \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd}$$

be a double vector bundle, and let $\varphi : E \longrightarrow A$ be an isomorphism. Consider the pull-back *bundle*

$$\varphi^* D = E \underset{(\varphi,A,q_A)}{\times} D := \{(e,d) \in E \times D | \varphi(e) = q_A(d)\} \subset A \times D$$

and the projections $p_1 : \varphi^* D \longrightarrow E$, $p_2 : \varphi^* D \longrightarrow D$ over the first and second factor, respectively, so that the following diagram commutes

$$\begin{tikzcd} \varphi^* D \arrow[r, "p_2"] \arrow[d, "p_1"] & D \arrow[r, "q_B"] \arrow[d, "q_A"] & B \arrow[d, "q^B"] \\ E \arrow[r, "\varphi"] & A \arrow[r, "q^A"] & M \end{tikzcd} . \tag{B.33}$$

Let $p_E := p_1$ and $q_B^\varphi := q_B \circ p_2$, then

$$\begin{tikzcd} \varphi^* D \arrow[r, "q_B^\varphi"] \arrow[d, "p_E"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ E \arrow[r, "q^E"] & M \end{tikzcd} , \tag{B.34}$$

is endowed with a double vector bundle structure such that p_2 is a DVB isomorphism.

Proof. We need to describe a vector bundle structure $\varphi^* D \longrightarrow B$ and show its compatibility with the vector bundle structure $\varphi^* D \longrightarrow E$. Notice that $p_2 : \varphi^* D \longrightarrow D$ is an isomorphism of vector bundles over φ, since we can define $p_2^{-1} : D \longrightarrow \varphi^* D$ by

$$p_2^{-1}(d) := (\varphi^{-1}(q_A(d)), d) \qquad \forall d \in D,$$

which satisfies

$$p_2 \circ p_2^{-1}(d) = d \quad \text{and} \quad p_2^{-1} \circ p_2(e,d) = (e,d).$$

It follows that $\varphi^*(D) \xrightarrow{q_B^\varphi} B$ is a fibration. We define the zero section by

$$0_B^\varphi := p_2^{-1} \circ 0_B,$$

so that we have $q_B^\varphi \circ 0_B^\varphi = \mathrm{Id}_B$.

Let $(e_1, d_1), (e_2, d_2) \in \varphi^* D$ with $q_B^\varphi(e_1, d_1) = q_B^\varphi(e_2, d_2)$, then $q_B(d_1) = q_B(d_2)$. Define

$$(e_1, d_1) \underset{q_B^\varphi}{+} (e_2, d_2) := (e_1 + e_2, d_1 \underset{B}{+} d_2). \tag{B.35}$$

Then, for any $b \in B$ and any $(e,d) \in \varphi^* D$, with $q_B^\varphi(e,d) = b$,

$$\begin{aligned} 0_B^\varphi(b) \underset{q_B^\varphi}{+} (e,d) &= (\varphi^{-1}(q_A \circ 0_B(b)), 0_B(b)) \underset{q_B^\varphi}{+} (e,d) \\ &= (0 + e, 0_B(b) \underset{B}{+} d) = (e,d). \end{aligned}$$

Therefore, we have obtained a vector bundle structure on $\varphi^* D \xrightarrow{q_B^\varphi} B$. Now,

$$q_E((e_1,d_1) \underset{q_B^\varphi}{+} (e_2,d_2)) = q_E(e_1 + e_2, d_1 \underset{B}{+} d_2) = e_1 + e_2 = q_E(e_1,d_1) + q_E(e_2,d_2),$$

then q_E is a vector bundle morphism. q_B^φ is also a vector bundle morphism, since $q_B^\varphi = q_B \circ p_2$ is the composition of vector bundle morphisms.

Finally, let $(e_1,d_1),(e_2,d_2),(e_1,d_3),(e_2,v_4) \in \varphi^* D$, with $q_B(d_1) = q_B(d_2)$ and $q_B(d_3) = q_B(d_4)$, then

$$q_A(d_1) = \varphi(e_1) = q_A(d_3) \quad \text{and} \quad q_A(d_2) = \varphi(e_2) = q_A(d_4),$$

and we have the following

$$\begin{aligned} ((e_1,d_2) \underset{q_B^\varphi}{+} (e_2,d_2)) \underset{q_E}{+} ((e_1,d_3) \underset{q_B^\varphi}{+} (e_2,d_4)) &= (e_1 + e_2, d_1 \underset{B}{+} d_2) \underset{q_E}{+} (e_1 + e_2, d_3 \underset{B}{+} d_4) \\ &= (e_1 + e_2, (d_1 \underset{B}{+} d_2) \underset{A}{+} (d_3 \underset{B}{+} d_4)) \\ &= (e_1 + e_2, (d_1 \underset{A}{+} d_3) \underset{B}{+} (d_2 \underset{A}{+} d_4)) \\ &= (e_1, d_1 \underset{A}{+} d_3) \underset{q_B^\varphi}{+} (e_2, d_2 \underset{A}{+} d_4) \\ &= ((e_1,d_1) \underset{q_E}{+} (e_1,d_3)) \underset{q_B^\varphi}{+} ((e_2,d_2) \underset{q_E}{+} (e_2,d_4)). \end{aligned}$$

Thereby, $\varphi^* D$ is a double vector bundle. Also we see immediately that

$$(p_2; \varphi, \mathrm{Id}_B; \mathrm{Id}_M) : (\varphi^* D; E, B; M) \longrightarrow (D; A, B; M)$$

is a morphism of double vector bundles.

∎

Corollary B.17. *In the conditions of Prop. B.16, there is a canonical isomorphism*

$$(\varphi^* D)^*_E \cong \varphi^*(D^*_A)$$

Proof. From Prop. B.16 we have that $p_2 : \varphi^* D \longrightarrow D$ is a DVB isomorphism. Then by Prop. B.2 it follows that

$$\Theta := ((p_2)^*_E)^{-1} : (\varphi^* D)^*_E \longrightarrow D^*_A$$

is a DVB isomorphism. Again from Prop. B.16 we have a DVB isomorphism $p_2' : \varphi^*(D^*_A) \longrightarrow D^*_A$. Then, setting $\Theta' := (p_2')^{-1}$, we obtain the canonical isomorphism

$$\Phi := \Theta' \circ \Theta : (\varphi^* D)^*_E \longrightarrow \varphi(D^*_A).$$

∎

Proposition B.18. *Let* $(D; A, B; M)_C$ *and* $(D'; A', B'; M')_{C'}$ *be two DVB's. Let* $\varphi : E \longrightarrow A$, $\varphi' : E' \longrightarrow A'$ *be two vector bundle isomorphisms. Then any DVB morphism* $\Phi : D \longrightarrow D'$ *induces a DVB morphism*

$$\widetilde{\Phi} : \varphi^* D \longrightarrow \varphi^*(D').$$

If Φ *is an isomorphism, then* $\widetilde{\Phi}$ *is an isomorphism too.*

Proof. Define

$$\widetilde{\Phi}(e, d) := (\varphi_E(e), \Phi(d)), \qquad \varphi_E(e) := (\varphi')^{-1} \circ \varphi_A \circ \varphi(e), ,$$

which is well-defined since, for $(e, d) \in \varphi^* D$ we have $q_A(d) = \varphi(e)$, whereby

$$q_{A'} \circ \Phi(d) = \varphi_A(q_A(d)) = \varphi_A \circ \varphi(e) = \varphi' \circ \varphi_E(e).$$

Since φ_E is an isomorphism, it follows that when Φ is an isomorphism, $\widetilde{\Phi}$ is an isomorphism too, with inverse given by

$$\widetilde{\Phi}^{-1}(e', d') = (\varphi_E^{-1}(e'), \Phi^{-1}(d')).$$

∎

Appendix C

The linear bundle and global structure

In this appendix we study in detail the structure of linear sections of a double vector bundle. Since we can reduce most of what can be said about double vector bundles to statements about linear and core sections, it turns out to be a fundamental fact that linear sections feet into a vector bundle structure, as is the case (in a more trivial way) for core sections. Therefore we reduce many questions on double vector bundles to questions on linear and core bundles, which are just vector bundles. This is a powerful viewpoint in order to tackle global issues related to double vector bundles.

Other highlights in this appendix are

- The proof of Prop. C.17, exhibiting a canonical isomorphism between certain two linear bundles, in which relies the characterization of degree 2 manifolds as involutive DVB's (Thm. 3.40).
- The relation of DVB morphisms with pairs of morphisms between the corresponding linear and core bundles.
- The introduction of the *double-linear bundle* (see Prop. C.32), first studied in [12], which leads to the equivalence of the category of DVB with that of DVB-sequences (see Ch. D), established in [12].
- The construction of the Whitney sum of two DVB's, sharing a common side bundle. In particular this is again a DVB. We describe its corresponding linear sequence, and show how to induce horizontal lifts on it.

We took as a starting point for this appendix the few remarks and hints about the linear bundle, and its exact sequence, provided in [23]. We also benefited from [12], in particular the concept of double-linear bundle was taken from there.

C.1 The structure of the module of linear sections

We begin studying the structure of $\Gamma_{\mathrm{lin}}(D_B)$ (the space of linear sections over B) of a decomposed double vector bundle. Since we already proved the existence of local decompositions, the result below provides information about the local structure of $\Gamma_{\text{lin}}(D_B)$ for any double vector bundle.

The following proposition is hinted in [23], but we decided to provide here full details.

Proposition C.1. *Let $D = A \oplus B \oplus C$ be a decomposed double vector bundle. Consider the structure over B, $D_B = (q^B)^*A \oplus_B (q^B)^*C$. Then, the space of linear sections $\Gamma_{\text{lin}}(D_B)$ is isomorphic to $\Gamma(A \oplus \text{Hom}(B, C))$, as $C^\infty(M)$-modules (see remark 2.6).*

Proof. Any section $\gamma \in \Gamma(D_B)$ is a map $\gamma : B \longrightarrow (q^B)^*A \oplus_B (q^B)^*C$ whose projection on B yields the identity and therefore consists of two bundle maps over the identity on M:

$$\gamma_1 := q_A \circ \gamma : B \longrightarrow A \qquad \text{and} \qquad \gamma_2 := q_C \circ \gamma : B \longrightarrow C,$$

Now assume that γ is linear, then it is a vector bundle morphism $\gamma : B \longrightarrow (q^A)^*B \oplus_A (q^A)^*C$, which has the form

$$\gamma(b) = (\gamma_1(b), b, \gamma_2(b)).$$

Being γ a bundle morphism, it must preserve fibers. This means that

$$q^B(b_1) = q^B(b_2) = m \Rightarrow \gamma_1(b_1) = \gamma_1(b_2).$$

This implies that γ_1 induces a map $\alpha : M \longrightarrow A$, given by $\alpha(m) = \gamma_1(b)$, for any $b \in B_m$, and since γ_1 is a bundle map over the identity, it follows that α is actually a section of A.

Now let's see what we get from the condition of γ preserving the linear structure:

$$\begin{aligned}\gamma(b_1 + b_2) &= (\gamma_1(b_1 + b_2), b_1 + b_2, \gamma_2(b_1 + b_2)) \\ &= (\alpha(m), b_1 + b_2, \gamma_2(b_1 + b_2)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(b_1) \underset{A}{+} \gamma(b_2) &= (\alpha(m), b_1, \gamma_2(b_1)) \underset{A}{+} (\alpha(m), b_2, \gamma_2(b_2)) \\ &= (\alpha(m), b_1 + b_2, \gamma_2(b_1) + \gamma_2(b_2)).\end{aligned}$$

Then, the linearity of γ is equivalent to the linearity of γ_2. So we have established a map

$$\begin{aligned}\Psi : \Gamma_{\text{lin}}(D_B) &\longrightarrow \Gamma(A \oplus \text{Hom}(B, C)) \\ \gamma \quad &\longrightarrow \quad (\alpha\ ,\ \gamma_2).\end{aligned}$$

It is easy to check that Ψ preserve the $C^\infty(M)$-module structure. Also Ψ is invertible, since given (α, γ_2), we define $\gamma_1 : B \longrightarrow A$ by $\gamma_1(b) := \alpha \circ q^B$. It is immediate that $\gamma(b) := (b, \gamma_1, \gamma_2) \in (q^B)^*A \oplus_B (q^B)^*C$ is a linear section of D_B, and that the correspondence $(\alpha, \gamma_2) \longrightarrow \gamma$ preserves the module structure. Thus, Ψ is an isomorphism of modules.

■

Now we are in conditions to give a global description of the space $\Gamma_{\text{lin}}(D_B)$ as the space of sections of certain vector bundle, namely, the *linear bundle*. We also provide a explicit, useful description for the fibers of the linear bundle (see Eq. (C.2)), which is borrowed from [12].

Although the linear bundle was already introduced in the literature ([23],[12]), we haven't found an explicit construction of it, which we provide in the proposition below.

Proposition C.2. *The space $\Gamma_{\text{lin}}(D_B)$ is locally free as a $C^\infty(M)$-module, with rank equal to* $\operatorname{rank} A + \operatorname{rank} B \operatorname{rank} C$*, and we have $\Gamma_{\text{lin}}(D_B) \cong \Gamma(\widehat{A})$ for some vector bundle $\widehat{A}$. Moreover, $\widehat{A}$ fits in the exact sequence*

$$0 \longrightarrow B^* \otimes C \xrightarrow{\iota} \widehat{A} \xrightarrow{p} A \longrightarrow 0. \tag{C.1}$$

We have an explicit description for the fibers of $\widehat{A}$, given by

$$\widehat{A}_m := \{\sigma \in \operatorname{Hom}(B_m, D_a) : a \in A_m,\ q_B \circ \sigma = \operatorname{Id}_{B_m}\}. \tag{C.2}$$

Proof. For the proof of the first part of the proposition, we work locally. By Prop. A.18, we can find a double vector bundle isomorphism $D \longrightarrow \widetilde{D}$, where $\widetilde{D} = A \oplus B \oplus C$ is a decomposed double vector bundle. By Prop. A.11, it follows that $\Gamma_{\text{lin}}(D_B) \cong \Gamma_{\text{lin}}(\widetilde{D}_B)$. Now, by Prop. C.1 $\Gamma_{\text{lin}}(\widetilde{D}_B) \cong \Gamma(A \oplus \operatorname{Hom}(B, C))$, and since $\Gamma(A \oplus \operatorname{Hom}(B, C))$ is a locally free $C^\infty(M)$-module whose rank equals to $\operatorname{rank} A + \operatorname{rank} B \operatorname{rank} C$, it follows the same assertion for $\Gamma_{\text{lin}}(D_B)$.

Now let

$$\widehat{A} := \bigcup_{m \in M} \widehat{A}_m,$$

where $\widehat{A}_m$ is defined in (C.2). We claim that $\widehat{A}$ is a vector bundle over M and that $\Gamma_{\text{lin}}(D_B) \cong \Gamma(\widehat{A})$. Of course, the projection $\pi : \widehat{A} \longrightarrow M$ is defined, for $\sigma \in \widehat{A}_m$, $\pi(\sigma) = m$. Addition in the fibers is also defined in the obvious way (using the D_B structure), observing that for $\sigma_1, \sigma_2 \in \widehat{A}_m$, by the interchange law,

$$\begin{aligned}
(\sigma_1 + \sigma_2)(b_1 + b_2) &= \sigma_1(b_1 + b_2) \underset{B}{+} \sigma_2(b_1 + b_2) \\
&= (\sigma_1(b_1) \underset{A}{+} \sigma_1(b_2)) \underset{B}{+} (\sigma_2(b_1) \underset{A}{+} \sigma_2(b_2)) \\
&= (\sigma_1(b_1) \underset{B}{+} \sigma_2(b_1)) \underset{A}{+} (\sigma_1(b_2) \underset{B}{+} \sigma_2(b_2)) \\
&= (\sigma_1 + \sigma_2)(b_1) \underset{A}{+} (\sigma_1 + \sigma_2)(b_2),
\end{aligned}$$

so that we also have $\sigma_1 + \sigma_2 \in \widehat{A}_m$, thus addition is well-defined. Also scalar multiplication is defined using the D_B structure, and since $\underset{B}{\cdot}$ is a vector bundle morphism with respect to the D_A structure,

$$\begin{aligned}
(t \cdot \sigma)(b_1 + b_2) :&= t \underset{B}{\cdot} \sigma(b_1 + b_2) \\
&= t \underset{B}{\cdot} (\sigma(b_1) \underset{A}{+} \sigma(b_2)) \\
&= t \underset{B}{\cdot} \sigma(b_1) \underset{A}{+} t \underset{B}{\cdot} \sigma(b_2) \\
&= (t \cdot \sigma)(b_1) \underset{A}{+} (t \cdot \sigma)(b_2),
\end{aligned}$$

we have also that $t \underset{B}{\cdot} \sigma \in \widehat{A}_m$, for $\sigma \in \widehat{A}_m$.

Now we need to endow $\widehat{A}$ with a vector bundle atlas, which will show both, that $\widehat{A}$ is a differential manifold and that $\widehat{A} \longrightarrow M$ is locally trivial. We can work locally, so let's take a decomposition $D \xrightarrow{\cong} \widetilde{D} = A \oplus B \oplus C$. By the linearity of $q_C : D \longrightarrow C$ with respect to the structure over A, we have, for $\sigma \in \widehat{A}_m$ and $b_1, b_2 \in B_m$,

$$q_C \circ \sigma(b_1 + b_2) = q_C(\sigma(b_1) \underset{A}{+} \sigma(b_2)) = q_C \circ \sigma(b_1) + q_C \circ \sigma(b_2),$$

thence $q_C \circ \sigma \in \mathrm{Hom}(B, C)_m$. Since q_C is linear also with respect to the structure over B, we have, for $\sigma_1, \sigma_2 \in \widehat{A}_m$ and $b \in B_m$,

$$\begin{aligned} q_C \circ (\sigma_1 + \sigma_2)(b) &= q_C(\sigma_1(b) \underset{B}{+} \sigma_2(b)) \\ &= q_C \circ \sigma_1(b) + q_C \circ \sigma_2(b) \\ &= (q_C \circ \sigma_1 + q_C \circ \sigma_2)(b). \end{aligned}$$

Finally observe that, by the definition of $\widehat{A}_m$, we have a well-defined map

$$\begin{array}{ccc} \widehat{A}_m & \longrightarrow & A_m \\ \sigma & \longrightarrow & q_A \circ \sigma, \end{array} \tag{C.3}$$

given by $q_A \circ \sigma := q_A \circ \sigma(b)$, for any $b \in B_m$, and that, for $\sigma_1, \sigma_2 \in \widehat{A}_m$,

$$q_A \circ (\sigma_1 + \sigma_2) = q_A(\sigma_1(b) \underset{B}{+} \sigma_2(b)) = q_A \circ \sigma_1 + q_A \circ \sigma_2.$$

Thus we obtain an isomorphism

$$\begin{array}{ccccc} \widehat{A}_m & \cong & A_m & \oplus & \mathrm{Hom}(B, C)_m \\ \sigma & \longrightarrow & (q_A \circ \sigma & , & q_C \circ \sigma), \end{array}$$

with inverse

$$\begin{array}{rl} (a, \tau) \in A_m \oplus \mathrm{Hom}(B, C)_m \longrightarrow & \sigma \in \mathrm{Hom}(B_m, D_a) \\ & \sigma(b) := (a, b, \tau(b)). \end{array} \tag{C.4}$$

Actually, since the preceding isomorphism is valid for every $m \in U \subset M$ over which D is decomposed, we have obtained a map, for a suitable open neighborhood U of $m_0 \in M$,

$$\varphi : \widehat{A}|_U \longrightarrow (A \oplus \mathrm{Hom}(B, C))|_U,$$

where $\widehat{A}|_U := \bigcup_{m \in U} \widehat{A}_m$, and such that for each $m \in U$ fixed, the map $\varphi|_{\widehat{A}_m}$ is an isomorphism of vector spaces.

Now suppose that we have such maps corresponding to two different open sets U_1, U_2, φ_1 and φ_2, respectively. Let $\Theta_1 = (q_A, q_B, q_{1C})$ and $\Theta_2 = (q_A, q_B, q_{2C})$ be the corresponding decompositions of D and let $V = U_1 \cap U_2$, then, by Prop. A.26, we have

$$\Theta_2 \circ \Theta_1^{-1}(a, b, c) = (a, b, c + \Psi(a, b)), \tag{C.5}$$

where $\Psi : A \oplus B \longrightarrow C$ is bilinear, and

$$q_{2C} = q_{1C} + \Psi(q_A, q_B).$$

Therefore, the transition map $\varphi_2 \circ \varphi_1^{-1}$ on V, for $(a, \tau) \in (A \oplus \mathrm{Hom}(B, C))|_V$, is given by

$$\varphi_2 \circ \varphi_1^{-1}(a, \tau) = (a, \tau + \Psi_a).$$

Thus, since Ψ is bilinear, we see that the transition map $\varphi_2 \circ \varphi_1^{-1}$ depends linearly, on each fiber, of $(a, \tau) \in A_m \oplus \mathrm{Hom}(B, C)_m$, and smoothly of $m \in M$.

Hence $\widehat{A}$ is a vector bundle over M. Now, let $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$, then, by definition, γ is a vector bundle morphism $\gamma : B \longrightarrow D_A$ over a map $\alpha : M \longrightarrow A$, which is a section, since

$$q^A \circ \alpha = q^A \circ \alpha \circ q^B \circ 0^B = q^A \circ q_A \circ \gamma \circ 0^B = q^B \circ q_B \circ \gamma \circ 0^B = q^B \circ 0^B = \mathrm{Id}_M.$$

Thus, for each $m \in M$ we obtain the data

- $a := \alpha(m) \in A_m$
- $\sigma := \gamma|_{B_m} \in \mathrm{Hom}(B_m, D_a)$,

which satisfies $q_B \circ \sigma = \mathrm{Id}_{B_m}$. In this way, we obtain a map $\eta : M \longrightarrow \widehat{A}$, given by

$$\eta(m) := \gamma|_{B_m}.$$

Since $\gamma|_{B_m} \in \widehat{A}_m$, it follows actually that $\eta \in \Gamma(\widehat{A})$.

Conversely, given $\eta \in \Gamma(\widehat{A})$, we define $\gamma : B \longrightarrow D$ by

$$\gamma(b) = \eta(q^B(b))(b).$$

Since $q_B \circ \gamma(b) = q_B \circ \eta(q^B(b))(b) = b$, it follows that γ is a section of D_B. Let's show that actually $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$. First of all we have a map $\alpha : M \longrightarrow A$ given by

$$\alpha(m) := q_A \circ \eta(m).$$

We have

$$q_A \circ \gamma(b) = q_A \circ \eta(q^B(b))(b) = \alpha \circ q^B(b).$$

Finally, since $\eta(m) \in \mathrm{Hom}(B_m, D_{\alpha(m)})$, it follows that $\gamma : B \longrightarrow D_A$ is a vector bundle morphism over α, thus $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$. Hence

$$\Gamma_{\mathrm{lin}}(D_B) \cong \Gamma(\widehat{A}).$$

To finish the proof, we have to show that there is an inclusion $\iota : B^* \otimes C \longrightarrow \widehat{A}$ and a projection $p : \widehat{A} \longrightarrow A$ such that $\iota(B^* \otimes C)$ seats in $\ker p$. The exactness will follow by rank reasons. The projection $p : \widehat{A} \longrightarrow A$ is simply $p(\sigma) := q_A \circ \sigma$. The inclusion $\iota : B^* \otimes C \longrightarrow \ker p$ is given by

$$\begin{aligned} \phi \in (B^* \otimes C)_m \longrightarrow \iota(\phi) &\in \mathrm{Hom}(B_m, D_{0^A(m)}) \\ \iota(\phi)(b) &= \phi(b) \underset{A}{+} 0_B(b) = (b, \phi(b)). \end{aligned} \tag{C.6}$$

That actually $\iota(\phi) \in \mathrm{Hom}(B_m, D_{0^A(m)})$ follows from the fact that $0_B : B \longrightarrow D_A$ is a vector bundle morphism.

∎

Definition C.3. The vector bundle $\widehat{A}$ from Prop. C.2 is called the *linear bundle* associated to D_B.

Remark C.4. Of course there is a second linear bundle, the one associated to D_A, which we denote by $\widehat{B}$, whose fiber over $m \in M$ is given by

$$\widehat{B}_m = \{\omega \in \mathrm{Hom}(A_m, D_b) : b \in B_m,\ q_A \circ \omega = \mathrm{Id}_{A_m}\},$$

and fits in the exact sequence

$$0 \longrightarrow A^* \otimes C \xrightarrow{\iota} \widehat{B} \xrightarrow{p} B \longrightarrow 0. \tag{C.7}$$

C.2 Horizontal lifts and global decompositions

Next we introduce the key concept of *horizontal lift*, which is instrumental to treat global questions and explicit calculations. The importance of horizontal lifts lies in that they allow us to encode the whole data of a decomposition in a single vector bundle map. This a first benefit of the linear bundle viewpoint, which, as explained in the introduction to this chapter, reduces many aspects of the study of a double vector bundle to the study of a plain vector bundle.

Definition C.5. A *horizontal lift* of $\widehat{A}$ is a section $\psi : A \longrightarrow \widehat{A}$ of the exact sequence (C.1).

In the following proposition we prove that once we introduce a horizontal lift of the linear sequence (C.1), corresponding to $\Gamma_{\mathrm{lin}}(D_B)$, we automatically have an induced horizontal lift for the linear sequence (C.7), corresponding to $\Gamma_{\mathrm{lin}}(D_A)$.

Proposition C.6. *The correspondence*

$$\begin{aligned} b \in B \longrightarrow \overline{\psi}(b) \in \widehat{B} \\ \overline{\psi}(b)(a) := \psi(a)(b). \end{aligned} \tag{C.8}$$

associates, canonically, to each horizontal lift $\psi : A \longrightarrow \widehat{A}$ of (C.1), a horizontal lift $\overline{\psi} : B \longrightarrow \widehat{B}$ of (C.7).

Proof. Let's check that $\overline{\psi}$ is well-defined, that is, for $b \in B_m$ we need to verify that $\overline{\psi}(b) \in \widehat{B}_m$. For $a \in A_m$ we have

$$q_B \circ \overline{\psi}(b)(a) = q_B(\psi(a)(b)) = b,$$

thus $\overline{\psi}(b)(a) \in D_b$ for every $a \in A_m$. Let $a_1, a_2 \in A_m$, then, since $\psi : A \longrightarrow \widehat{A}$ is a vector bundle morphism,

$$\begin{aligned} \overline{\psi}(b)(a_1 + a_2) &= \psi(a_1 + a_2)(b) = (\psi(a_1) + \psi(a_2))(b) \\ &= \psi(a_1)(b) \underset{B}{+} \psi(a_2)(b) = \overline{\psi}(b)(a_1) + \overline{\psi}(b)(a_2). \end{aligned}$$

Finally, since ψ is a horizontal lift,

$$q_A \circ \overline{\psi}(b)(a) = q_A \circ \psi(a)(b) = a.$$

The above shows that $\overline{\psi}(b) \in \widehat{B}_m$.

It remains to check that $p \circ \overline{\psi} = \mathrm{Id}_B$. For $b \in B_m$ and any $a \in A_m$ we have

$$p \circ \overline{\psi}(b) = q_B(\overline{\psi}(b)(a)) = q_B(\psi(a)(b)) = b.$$

Therefore $\overline{\psi} : B \longrightarrow \widehat{B}$ is a horizontal lift.

■

Of course, if we expect any profit from working with linear sequences and horizontal lifts, we must have an explicit relation between horizontal lifts and splittings of core sequences and the corresponding decompositions. This is what we do below.

Proposition C.7. *Let $(D; A, B; M)_C$ be a double vector bundle. Consider the linear bundle $\widehat{A}$ corresponding to linear sections of D_B.*

There is a canonical 1:1 correspondence between horizontal lifts of (C.1) and decompositions of D; or equivalently, by Prop. C.6, between horizontal lifts of (C.7), and decompostions of D.

Proof. Let $\psi : B \longrightarrow \widehat{B}$ be horizontal lift of (C.7). We will use Cor. A.17 to obtain a decomposition of D. So we need a splitting θ of (A.2) which preserves the fibration over B and satisfies (A.6). For $(a, b) \in (q^A)^* B_m$, define

$$\theta(a, b) := \psi(b)(a).$$

Let's verify that θ is a splitting of (A.2) satisfying the required conditions. By definition we have

$$q_A(\theta(a, b)) = q_A(\psi(b)(a)) = a$$

and

$$q_B(\theta(a, b)) = q_B(\psi(b)(a)) = b.$$

Thus $(q_A, q_B)(\theta(a, b)) = (a, b)$, so θ is a splitting of (A.2). Also from this, it follows that θ preserves the fibration over B. Equation (A.6) follows from the linearity of $\psi(b) : A_m \longrightarrow D_b$.

Conversely, suppose we have a decomposition $\Theta : D \longrightarrow A \oplus B \oplus C$. By Cor. A.17 we have a map

$$\theta : (q^A)^* B \longrightarrow D_A$$

which is a splitting of (A.2), that preserves the fibration over B and satisfies (A.6).

Define, for $a \in A_m$,

$$\begin{aligned} \psi : & B_m \longrightarrow \widehat{B}_m \\ & b \longrightarrow \psi(b); \quad \psi(b)(a) := \theta(a, b). \end{aligned} \tag{C.9}$$

We need to show that ψ is well defined and that it is in fact a horizontal lift. Since

$$q_A \circ \psi(b)(a) = a \quad \text{and} \quad q_B \circ \psi(b)(a) = b,$$

it follows that $\psi(b) \in \widehat{B}_m$ and that $\psi(b)(a) \in D_b$ for every $a \in A_m$. To check linearity, let $a_1, a_2 \in A_m$, then, by (A.6)

$$\psi(b)(a_1 + a_2) = \theta(a_1 + a_2, b) = \theta(a_1, b) \underset{B}{+} \theta(a_2, b) = \psi(b)(a_1) \underset{B}{+} \psi(b)(a_2).$$

We conclude that $\psi(b) \in \widehat{B}_m$.

Finally, for $b \in B_m$, and any $a \in A_m$,

$$p \circ \psi(b) = q_B \circ \psi(b)(a) = q_B \circ \theta(a, b) = b,$$

that is, $p \circ \psi = \mathrm{Id}_B$.

■

An important by-product of the linear bundle viewpoint we obtain below, is the global decomposition for any double vector bundle, which moreover is explicit as soon as we have a fibre, non-degenerate metric. Compare for example with [23] where they sketch an argument based on local existence of decompositions and show indirectly the global existence through a Čech cohomology argument.

Corollary C.8. *Given a double vector bundle $(D; A, B; M)_C$. There always exists a decompostion $D \xrightarrow{\sim} A \oplus B \oplus C$.*

Proof. Choosing a Riemannian (fibre) metric on $\widehat{A}$, we obtain a horizontal lift of (C.1), and thence by Prop. C.7 we get the corresponding decomposition of D.

■

Corollary C.9. *Let D be a double vector bundle and Θ a decomposition. Then the horizontal lift $\widetilde{\psi} : C^* \longrightarrow \widehat{C^*}$, corresponding to the dual decomposition $\widetilde{\Theta}$ of D^*_A given in Cor. B.7, is given by*

$$\widetilde{\psi}(\kappa)(a) = (q_A, q_C)^*(a, \kappa). \tag{C.10}$$

Proof. It follows directly from (C.9) and Cor. B.10.

■

In the following proposition we find explicit formulas, in terms of a decomposition of D, for the decompositions of the linear bundles corresponding $\Gamma_{\mathrm{lin}}(D_B)$ and $\Gamma_{\mathrm{lin}}((D^*_A)_{C^*})$. Among other consequences, these identifications enables to establish a somewhat surprising isomorphism between both linear bundles (see Prop. C.17).

Proposition C.10. *Given a double vector bundle D, let $\Theta : D \longrightarrow A \oplus B \oplus C$ be a decomposition, then we have isomorphisms*

$$K : \widehat{A} \longrightarrow A \oplus \mathrm{Hom}(B, C) \quad \textit{and} \quad H : \widehat{A}_* \longrightarrow A \oplus \mathrm{Hom}(C^*, B^*),$$

where $\widehat{A}_*$ is the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(C^*, (D^*_A)_{C^*})$. The isomorphisms are given by

$$K(\sigma) = (a, \sigma_1), \quad \textit{where } \sigma_1 := q_C \circ \sigma,\ \sigma \in \mathrm{Hom}(B_m, D_a),$$

where q_C is the projection $D \longrightarrow C$ corresponding to Θ, and

$$H(\omega) = (a, \omega_1), \quad \textit{with } \omega_1 := \pi_{B^*} \circ \omega,\ \omega \in \mathrm{Hom}(C^*_m, (D^*_A)_a),$$

where π_{B^*} is the projection $D^*_A \longrightarrow B^*$ corresponding to the induced decomposition on D^*_A.

Proof. That K and H are vector bundle morphisms follows from the fact that $p : \widehat{A} \longrightarrow A$ and $p_* : \widehat{A}_* \longrightarrow A$ are vector bundle morphisms, and from the linearity of q_C and π_{B^*}. So it will suffice to show inverses for K and H, which are easily seen to be given by

$$K^{-1}(a, \sigma_1)(b) = \Theta^{-1}(a, b, \sigma_1(b)), \tag{C.11}$$

for $(a, \sigma_1) \in A_m \oplus \mathrm{Hom}(B, C)_m,\ b \in B_m$ and, by Cor. B.8,

$$H^{-1}(a, \omega_1)(\kappa) = \Theta^*_A(a, \omega_1(\kappa), \kappa), \tag{C.12}$$

for $(a, \omega_1) \in A_m \oplus \mathrm{Hom}(C^*, B^*)_m,\ \kappa \in C^*_m$.

■

Another important requirement to have any success when working with linear bundles and horizontal lifts, comprises to have explicit formulas relating the splittings and horizontal lifts induced by two decompositions, in terms of the bilinear mapping Ψ obtained in Prop. A.26. This what we address in the following proposition.

Proposition C.11. *Let $(D; A, B; M)_C$ be a double vector bundle, and let Θ, Θ' be two decompositions of D, which are related by formula (A.15) in Prop. A.26. Let θ, θ' be the corresponding splittings of (A.3) (and simultaneously of (A.2) also), and let ψ, ψ' be the corresponding horizontal lifts of (C.1). Then*

$$\begin{aligned} \theta(a, b) &= \theta'(a, b) \underset{A}{-} (a, \Psi(a, b)) \\ &= \theta'(a, b) \underset{B}{-} (b, \Psi(a, b)); \end{aligned} \tag{C.13}$$

and

$$\psi(a) = \psi'(a) - \iota_A(\Psi_a), \tag{C.14}$$

where $\Psi_a : B \longrightarrow C$ is the vector bundle morphism (over the identity) given by $\Psi_a(b) = \Psi(a, b)$, and $\iota_A : B^* \otimes C \longrightarrow \widehat{A}$ is the inclusion.

Proof. Recall that $\theta(a, b) = \Theta^{-1}(a, b, 0)$ and analogously is defined θ'. From (A.15) it follows that

$$(\Theta \circ (\Theta')^{-1})^{-1}(a, b, c) = (a, b, c - \Psi(a, b)).$$

Then

$$\begin{aligned} \theta(a, b) &= (\Theta')^{-1} \circ \Theta' \circ \Theta^{-1}(a, b, 0) \\ &= (\Theta')^{-1} \circ (\Theta \circ (\Theta')^{-1})^{-1}(a, b, 0) \\ &= (\Theta')^{-1}(a, b, -\Psi(a, b,)) \\ &= (\Theta')^{-1}(a, b, 0) \underset{A}{-} (\Theta')^{-1}(a, 0, \Psi(a, b)) \\ &= \theta'(a, b) \underset{A}{-} (a, \Psi(a, b)) \\ &= \theta'(a, b) \underset{B}{-} (b, \Psi(a, b)), \end{aligned}$$

thus obtaining (C.13). From this also follows

$$\begin{aligned}\psi(a)(b) = \theta(a,b) &= \theta'(a,b) \underset{B}{-} (b, \Psi(a,b)) = \psi'(a)(b) \underset{B}{-} (b, \Psi_a(b)) \\ &= (\psi'(a) - \iota_A(\Psi_a))(b),\end{aligned}$$

from which we get (C.14).

∎

Remark C.12. Analogously it is shown that the corresponding horizontal lifts of (C.7), $\overline{\psi}, \overline{\psi'}$ are related by

$$\overline{\psi}(b) = \overline{\psi'}(b) - \iota_B(\Psi_b),$$

where $\iota_B : A^* \otimes C \longrightarrow \widehat{B}$ is the inclusion, and $\Psi_b : A \longrightarrow C$ is given by $\Psi_b(a) := \Psi(a,b)$.

Corollary C.13. *Given a double vector bundle D, let Θ, Θ' be two decompositions and let π_{B^*}, π'_{B^*} be the projections corresponding to the induced decompositions on the dual D^*_A. Then, for $v \in D^*_A$,*

$$\pi'_{B^*}(v) = \pi_{B^*}(v) + (\Psi_{\pi_A(v)})^*(\pi_{C^*}(v)), \tag{C.15}$$

Proof. We can write the first equation in (C.13) in the following way

$$\theta'(a,b) = \theta(a,b) \underset{A}{+} (0_A(a) \underset{B}{+} \Psi_a(b)),$$

where we are considering C inside D. Then, dualizing with respect to A, it follows that

$$(\theta')^*(v) = \theta^*(v) \underset{A}{+} (\pi_A(v) \underset{C^*}{+} (\Psi_{\pi_A(v)})^*(\pi_{C^*}(v))),$$

that is

$$(\theta')^* = \theta^* \underset{A}{+} (\pi_A, (\Psi_{\pi_A})^* \circ \pi_{C^*}). \tag{C.16}$$

Recalling that $\pi_{B^*} = p_2 \circ \theta^*$, we get

$$\pi'_{B^*} = \pi_{B^*} + (\Psi_{\pi_A})^* \circ \pi_{C^*}.$$

∎

Remark C.14. Analogously it is shown that, if π_{A^*}, π'_{A^*} are the projections corresponding to the induced decompositions on the dual D^*_B, then, for $w \in D^*_B$,

$$\pi'_{A^*}(w) = \pi_{A^*}(w) + (\Psi_{\pi_B(w)})^*(\pi_{C^*}(w)), \tag{C.17}$$

where $\Psi_b(a) := \Psi(a,b)$.

Now that we know how different splittings and horizontal lifts relate (Prop. C.11), we need the same information for the induced splittings and horizontal lifts induced on the dual of a double vector bundle.

Proposition C.15. *Given a double vector bundle D and two decompositions Θ, Θ' as in Prop. C.11. Let $\widetilde{\theta}, \widetilde{\theta}'$ be the induced splittings of the core sequence (B.11) corresponding to the dual D^*_A, as given in Prop. B.5. Then, for $a \in A_m$ and $\kappa \in C^*_m$,*

$$\widetilde{\theta}'(a,\kappa) = \widetilde{\theta}(a,\kappa) \underset{A}{-} (a, \Psi^*_a(\kappa)), \tag{C.18}$$

where, as usual, $\Psi_a : B \longrightarrow C$ is given by $\Psi_a(b) = \Psi(a,b)$, with Ψ given in (A.15).

If $\widetilde{\psi}, \widetilde{\psi}' : C^* \longrightarrow \widehat{C^*}$ are the corresponding horizontal lifts, they are related by

$$\psi'(\kappa) = \psi(\kappa) - \iota_A(\Psi^*_\kappa), \tag{C.19}$$

where $\iota_A : A^* \otimes B^* \longrightarrow \widehat{C^*}$ is the inclusion, and $\Psi^*_\kappa : A \longrightarrow B^*$ is given by

$$\Psi^*_\kappa(a) := \Psi^*_a(\kappa).$$

Proof. By Cor. B.10, (C.18) is equivalent to

$$(q_A, q_C)^*(a,\kappa) = (q_A, q'_C)^*(a,\kappa) \underset{A}{+} (a, \Psi^*_a(\kappa)). \tag{C.20}$$

We will prove (C.20) computing each side of the equation separately. Let $d \in D$ with $q_A(d) = a$, $q_B(d) = b$ and $q_C(d) = c$. As for the left-hand side, we have

$$\langle (q_A, q_C)^*(a,\kappa), d \rangle_A = \langle (a,\kappa), (q_A(d), q_C(d)) \rangle_A = \langle \kappa, c \rangle. \tag{C.21}$$

Now, computing the right-hand side we have the following

$$\begin{aligned} \langle (q_A, q'_C)^*(a,\kappa) \underset{A}{+} (a, \Psi^*_a(\kappa)), d \rangle_A &= \langle (a,\kappa), (q_A(d), q'_C(d)) \rangle_A + \langle 0_A(a) \underset{C^*}{+} \Psi^*_A(\kappa), d \rangle_A \\ &= \langle \kappa, c - \Psi_a(b) \rangle + \langle \Psi^*_a(\kappa), b \rangle \\ &= \langle \kappa, c \rangle - \langle \kappa, \Psi_a(b) \rangle + \langle \kappa, \Psi_a(b) \rangle = \langle \kappa, c \rangle. \end{aligned} \tag{C.22}$$

From (C.21) and (C.22) we obtain (C.20).

Finally, the identity $\langle \Psi^*_\kappa(a), b \rangle = \langle \kappa, \Psi(a,b) \rangle$, for every $a \in A_m$, $b \in B_m$ and $\kappa \in C^*$, implies that $\Psi^*_\kappa \in A^* \otimes B^*$, and we have

$$\begin{aligned} \widetilde{\psi}'(\kappa)(a) = \theta'(a,\kappa) &= \theta(a,\kappa) \underset{A}{-} (a, \Psi^*_a(\kappa)) \\ &= \widetilde{\psi}(\kappa)(a) \underset{A}{-} (0_A(a) \underset{C^*}{+} \Psi^*_\kappa(a)) \\ &= \widetilde{\psi}(\kappa)(a) - \iota_A(\Psi^*_\kappa)(a), \end{aligned} \tag{C.23}$$

thus proving (C.19).

∎

Corollary C.16. *In the situation of Prop. C.15, let $\widehat{A}_*$ be the linear bundle of $(D^*_A)_{C^*}$, corresponding to $\Gamma_{\text{lin}}(C^*, D^*_A)$, and let ψ_*, ψ'_* be the horizontal lifts corresponding to the splittings $\widetilde{\theta}, \widetilde{\theta}'$, respectively. Then*

$$\psi'_*(a)(\kappa) = \psi_*(a)(\kappa) - \iota_{C^*}(\Psi^*_a)(\kappa). \tag{C.24}$$

Proof. Follows directly from (C.23) and the way ψ_* and ψ'_* are defined.

■

Finally, we end this section with the important –and surprising– existence of an isomorphism between the linear bundle corresponding to D_B and the linear bundle corresponding to $(D^*_A)_{C^*}$, which as already mentioned, is the key for the characterization of degree 2 manifolds in terms of involutive DVB's.

Proposition C.17. *Given a double vector bundle D, let $\widehat{A}$ be its linear bundle, corresponding to $\Gamma_{\text{lin}}(B, D)$, and let $\widehat{A}_*$ be the linear bundle of $(D^*_A)_{C^*}$, corresponding to $\Gamma_{\text{lin}}(C^*, D^*_A)$, as in Prop. C.10. If Θ is a decomposition of D, then we obtain a well-defined isomorphism*

$$T : \widehat{A} \longrightarrow \widehat{A}_*$$

given by

$$T := H^{-1} \circ \Delta \circ K, \tag{C.25}$$

where H, K are the isomorphisms of Prop. C.10, and

$$\Delta : A \oplus \text{Hom}(B, C) \longrightarrow A \oplus \text{Hom}(C^*, B^*)$$

is given by

$$\Delta(a, \sigma_1) = (a, -\sigma_1^*). \tag{C.26}$$

Proof. The proof consists in verifying that if Θ' is another decomposition of D, and H', K' are the corresponding isomorphisms given by Prop. C.10, then

$$H^{-1} \circ \Delta \circ K = (H')^{-1} \circ \Delta \circ K'. \tag{C.27}$$

Let's begin computing T. For $\sigma \in \widehat{A}_m$, with $p(\sigma) = a$ and $q_C \circ \sigma = \sigma_1 \in \text{Hom}(B, C)$ we have

$$\begin{aligned} T\sigma &= H^{-1} \circ \Delta \circ K(\sigma) \\ &= H^{-1} \circ \Delta(p(\sigma), q_C \circ \sigma) \\ &= H^{-1}(a, -\sigma_1^*), \end{aligned}$$

from which, for $\kappa \in C^*_m$ and $d \in D$ with $\Theta(d) = (a, b, c)$,

$$\begin{aligned} \langle T\sigma(\kappa), d\rangle_A &= \langle \Theta^*_A(a, -\sigma_1^*(\kappa), \kappa), d\rangle_A \\ &= \langle (a, -\sigma_1^*(\kappa), \kappa), (a, b, c\rangle_A \\ &= \langle -\sigma_1^*(\kappa), b\rangle + \langle \kappa, c\rangle \\ &= \langle \kappa, c - \sigma_1(b)\rangle. \end{aligned}$$

On the other hand, since $q'_C = q_C - \Psi(q_A, q_B)$, we get $\sigma'_1 = \sigma_1 - \Psi_a$, where $\sigma'_1 := q'_C \circ \sigma$. So we have, denoting $T' = (H')^{-1} \circ \Delta \circ K'$,

$$\begin{aligned} \langle T'\sigma(\kappa), d\rangle_A &= \langle (\Theta')^*_A(a, -(\sigma'_1)^*(\kappa), \kappa), d\rangle_A \\ &= \langle (a, -(\sigma_1 - \Psi_a)^*(\kappa), \kappa), (a, b, c - \Psi(a, b))\rangle_A \\ &= \langle \kappa, -\sigma_1(b) + \Psi(a, b)\rangle + \langle \kappa, c - \Psi(a, b)\rangle \\ &= \langle \kappa, c - \sigma_1(b)\rangle. \end{aligned}$$

Thus, it follows $T\sigma = T'\sigma$, which implies that the morphism $\widehat{A} \longrightarrow \widehat{A}_*$ is well defined.

■

C.3 Global behaviour of DVB morphisms

In this section we explain partially how morphisms of DVB's relate to morphisms of the corresponding linear sections. A full understanding will be achieved only after we introduce the double-linear bundle in next section, and ultimately after the *double realization* procedure [12] is understood. This procedure will be reviewed in App. D.

Proposition C.18. *Let $(D; A, B; M)_C$ and $(D'; A', B; M)_{C'}$ be two double vector bundles with a common side bundle, B. A morphism of double vector bundles*

$$\Phi : (D; A, B; M)_C \longrightarrow (D', A', B; M)_{C'}$$

which is the identity over B induces a vector bundle morphism $\widehat{\Phi}_B : \widehat{A} \longrightarrow \widehat{A'}$ over φ_M, where $\widehat{\Phi}_B$ is given by

$$\widehat{\Phi}_B(\sigma) = \Phi \circ \sigma,$$

for $\sigma \in \mathrm{Hom}(B_m, D_a)$, with $q_B \circ \sigma = \mathrm{Id}_{B_m}$.

Proof. We need to check that $\widehat{\Phi}_B$ is well-defined and that this mapping is a vector bundle morphism. Let $\sigma \in \widehat{A}$. Since Φ is a vector bundle morphism $D_A \longrightarrow D'_{A'}$, it follows that $\widehat{\Phi}_B(\sigma) \in \mathrm{Hom}(B_m, D'_{\varphi_A(a)})$. For the well-definition, it remains to check that $q_B \circ \Phi(\sigma) = \mathrm{Id}_{B_m}$. Since $\varphi_B = \mathrm{Id}_B$, it follows that

$$q_B \circ \widehat{\Phi}_B(\sigma) = q_B \circ \Phi \circ \sigma = \varphi_B \circ q_B \circ \sigma = \mathrm{Id}_{B_m},$$

thus $\widehat{\Phi}_B$ maps $\widehat{A}_m$ into $\widehat{A}'_m$. Linearity on the fibers readily follows from linearity of Φ with respect to the structures over B.

∎

Remark C.19. In order to get a vector bundle morphism $\widehat{\Phi}_B : \widehat{A} \longrightarrow \widehat{A'}$ for a more general double vector bundle morphism $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$, we need the extra condition requiring φ_B to be an isomorphism. In this case we have

$$\widehat{\Phi}_B(\sigma) := \Phi \circ \sigma \circ \varphi_B^{-1}. \tag{C.28}$$

Corollary C.20. *If $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B', M')_{C'}$ is an isomorphism, then Φ^*_B induces a vector bundle morphism $(\widehat{\Phi^*_B})_{B'} : \widehat{C'^*}_{B'} \longrightarrow \widehat{C^*}_B$ given by*

$$\sigma' \in (\widehat{C'^*}_{B'})_m \longrightarrow \Phi^*_B \circ \sigma' \circ \varphi_B \in (\widehat{C^*}_B)_m.$$

Here $\widehat{C^*}_B$ is the linear vector bundle corresponding to the linear sections of D^*_B.

Proof. Follows directly from Rmk. C.19 above and Cor. B.3.

∎

Now we want to understand the action of the induced morphism $(\widehat{\Phi^*_A})_{C'^*} : \widehat{A'}_{C'^*} \longrightarrow \widehat{A}_{C^*}$, where $\widehat{A'}_{C'^*}$ is the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(C'^*, D'^*_{A'})$ and $\widehat{A}_{C^*}$ is the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(C^*, D^*_A)$. In order to achieve this we first need two results of interest by their own. The first about the inverse of an induced morphism, and the second about the expression of the induced morphism when we introduce decompositions.

Proposition C.21. *If $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ is an isomorphism, then $\widehat{\Phi}_B$ is an isomorphism, with inverse*

$$(\widehat{\Phi}_B)^{-1} = \widehat{\Phi^{-1}}_{B'}.$$

Proof. For any $\sigma \in \widehat{A}$ we have from Eq. (C.28) and Prop. A.25,

$$\widehat{\Phi^{-1}}_{B'} \circ \widehat{\Phi}_B(\sigma) = \widehat{\Phi^{-1}}_{B'}(\Phi \circ \sigma \circ \circ \varphi_B^{-1}) = \Phi^{-1} \circ \Phi \circ \sigma \circ \varphi_B^{-1} \circ \varphi_B = \sigma.$$

Analogously it is shown that $\widehat{\Phi}_B \circ \widehat{\Phi^{-1}}_{B'}(\sigma') = \sigma'$.

∎

Proposition C.22. *Let $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ be a DVB morphism such that φ_B is an isomorphism. Let's introduce decompositions on D and D' so that we obtain the isomorphisms $K : \widehat{A} \longrightarrow A \oplus \mathrm{Hom}(B, C)$ and $K' : \widehat{A'} \longrightarrow A' \oplus \mathrm{Hom}(B', C')$ given in Prop. C.10. Then the induced morphism $\widehat{\Phi}_B : \widehat{A} \longrightarrow \widehat{A'}$ in terms of the decompositions has the expression*

$$K' \circ \widehat{\Phi}_B \circ K^{-1}(a, \tau) = (\varphi_A(a), (\varphi_C \circ \tau + \Psi_a) \circ \varphi_B^{-1}), \qquad \forall (a, \tau) \in A \oplus \mathrm{Hom}(B, C), \tag{C.29}$$

with $\Psi_a : B \longrightarrow C'$ given by

$$\Psi_a(b) := q_{C'} \circ \Phi \circ \Theta^{-1}(a, b, 0), \tag{C.30}$$

where $q_{C'} : D' \longrightarrow C'$ corresponds to the decomposition of D' and $\Theta : D \longrightarrow A \oplus B \oplus C$ is the decomposition of D.

Proof. If $\Theta' : D' \longrightarrow A' \oplus B' \oplus C'$ is the decomposition of D', then by Cor. A.24, we have

$$\Theta' \circ \Phi \circ \Theta^{-1}(a, b, c) = (\varphi_A(a), \varphi_B(b), \varphi_C(c) + \Psi(a, b)), \tag{C.31}$$

where $\Psi : A \oplus B \longrightarrow C'$ is bilinear. It is immediate from the equation (C.31) above that

$$\Psi(a, b) = q_{C'} \circ \Phi \circ \Theta(a, b, 0). \tag{C.32}$$

Let's take $a \in A_m, b \in B_m, \tau \in \mathrm{Hom}(B, C)_m)$, and for any $b' \in B_m$ compute, using Eqs. (C.11), (C.28) and (C.31),

$$\begin{aligned} K' \circ \widehat{\Phi}_B \circ K^{-1}(a, \tau)(b') &= K' \left(\Phi \circ \Theta^{-1}(a, b, \tau(\varphi_B^{-1}(b'))) \right) \\ &= \left(q_{A'} \circ \Phi \circ \Theta^{-1}(a, b, \tau(\varphi_B^{-1}(b')), q_{C'} \circ \Phi \circ \Theta^{-1}(a, b, \tau(\varphi_B^{-1}(b')) \right) \\ &= (\varphi_A(a), \varphi_C \circ \tau \circ \varphi_B^{-1}(b') + \Psi(a, \varphi_B^{-1}(b'))), \end{aligned}$$

hence, by Eqs. (C.32) and (C.30),

$$K' \circ \widehat{\Psi}_B \circ K^{-1}(a, \tau) = (\varphi_A(a), (\varphi_C \circ \tau + \Psi_a) \circ \varphi_B^{-1}).$$

∎

Proposition C.23. *Let $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ be a DVB isomorphism. Then $(\widehat{\Phi^*_A})_{C'^*} : \widehat{A}_{C^*} \longrightarrow \widehat{A'}_{C'^*}$ is given by*

$$(\widehat{\Phi^*_A})_{C'^*} = T \circ (\widehat{\Phi}_B)^{-1} \circ (T')^{-1}, \tag{C.33}$$

where $\widehat{A}_{C^*}, \widehat{A'}_{C'^*}$ are the linear bundles corresponding to $\Gamma_{\mathrm{lin}}(C^*, D^*_A)$ and $\Gamma_{\mathrm{lin}}(C'^*, D'^*_{A'})$, respectively, and $T : \widehat{A} \longrightarrow \widehat{A}_{C^*}$, $T' : \widehat{A'} \longrightarrow \widehat{A'}_{C'^*}$ are the isomorphisms given by Prop. C.17.

Proof. Using the isomorphisms K and H of Prop. C.10, we see that Eq. (C.33) is equivalent to

$$\begin{aligned} H \circ (\widehat{\Phi^*_A})_{C'^*} \circ (H')^{-1} &= H \circ T \circ (\widehat{\Phi}_B)^{-1} \circ (T')^{-1} \circ (H')^{-1} \\ &= H \circ T \circ (K^{-1} \circ K) \circ (\widehat{\Phi}_B)^{-1} \circ ((K')^{-1} \circ K') \circ (T')^{-1} \circ (H')^{-1} \\ &= (H \circ T \circ K^{-1}) \circ (K \circ (\widehat{\Phi}_B)^{-1} \circ (K')^{-1}) \circ (K' \circ (T')^{-1} \circ (H')^{-1}). \end{aligned} \tag{C.34}$$

Thereby, we can work only with morphisms between decomposed linear bundles and use the formula obtained in Prop. C.22 above. So let's take $(a', \tau') \in A' \oplus \mathrm{Hom}(C'^*, B'^*)$. On one hand, by Prop. B.4, we have

$$H \circ (\widehat{\Phi^*_A})_{C'^*} \circ (H')^{-1}(a', \tau') = \Big(\varphi_A^{-1}(a'), (\varphi_B^* \circ \tau' + \Psi^*_{\varphi_A^{-1}(a')}) \circ (\varphi_C^{-1})^*\Big) \tag{C.35}$$

On the other hand, by Prop. C.21 and formula (A.12) of Prop. A.25, we have

$$\begin{aligned} H \circ T \circ (\widehat{\Phi}_B)^{-1} \circ (T')^{-1} \circ (H')^{-1}(a', \tau') &= H \circ T \circ (\widehat{\Phi^{-1}}_{B'}(a', -(\tau')^*) \\ &= H \circ T \Big(\varphi_A^{-1}(a'), (-\varphi_C^{-1} \circ (\tau')^* - \varphi_C^{-1} \circ \Psi_{\varphi_A^{-1}(a')} \circ \varphi_B^{-1}) \circ \varphi_B\Big) \\ &= \Big(\varphi_A^{-1}(a'), \varphi_B^* \circ \tau' \circ (\varphi_C^{-1})^* + \Psi^*_{\varphi_A^{-1}(a')} \circ (\varphi_C^{-1})^*\Big). \end{aligned} \tag{C.36}$$

From (C.35) and (C.36), we obtain (C.34).

■

In the following proposition we obtain a converse for Prop. C.18.

Proposition C.24. *Let $(D; A, B; M)_C$ and $(D'; A', B; M)_{C'}$ be two double vector bundles with a common side bundle, B. Then any pair of vector bundle morphisms*

$$\varphi_C : C \longrightarrow C' \qquad \text{and} \qquad \varphi_{\widehat{A}} : \widehat{A} \longrightarrow \widehat{A'}$$

satisfying the compatibility condition

$$\varphi_{\widehat{A}}(\tau) = \varphi_C \circ \tau \in \mathrm{Hom}(B, C') \subset \widehat{A'}, \qquad \forall \tau \in \mathrm{Hom}(B, C) \subset \widehat{A}, \tag{C.37}$$

determine a unique DVB morphism $\Phi : D \longrightarrow D'$ over the identity on B, such that $\Phi|_C = \varphi_C$ and $\widehat{\Phi} = \varphi_{\widehat{A}}$.

Proof. First of all, observe that because of the compatibility condition (C.37) we obtain a well-defined morphism $\varphi_A : A \longrightarrow A'$ by setting

$$\varphi_A(a) := \pi' \circ \varphi_{\widehat{A}}(\widehat{a}),$$

where $\widehat{a} \in \widehat{A}$ is any horizontal lift of a, and $\pi' : \widehat{A'} \longrightarrow A'$ is the projection. In order to define the map Φ, introduce provisionally a decomposition $D \cong A \oplus B \oplus C$, so that we also obtain the induced decomposition $\widehat{A} \cong A \oplus \mathrm{Hom}(B, C)$, and let's take any $d \in D$, then we have $d = (a, b, c)$.

First suppose $b \neq 0$. Then pick any $\tau \in \mathrm{Hom}(B, C)$ with $\tau(b) = c$. Thereby we obtain

$$\sigma = (a, \tau) \in \widehat{A}$$

which satisfies $\sigma(b) = d$. Then define

$$\Phi(d) := \varphi_{\widehat{A}}(\sigma)(b). \tag{C.38}$$

We need to show that Φ is well-defined, i.e. the definition above doesn't depend on the particular σ chosen, and that Φ defined in that way actually is DVB morphism. Suppose that we have also $\sigma' \in \widehat{A}$ with $\sigma'(b) = d$. Then $q_A \circ \sigma = q_A \circ \sigma' = a$ (see Eq. (C.3)), whence

$$\tau := \sigma - \sigma' \in \mathrm{Hom}(B, C),$$

that is

$$q_A(\sigma(b) \underset{B}{-} \sigma'(b)) = q_A \circ \sigma(b) - q_A \circ \sigma'(b) = 0.$$

Then, using Eq. (C.37),

$$\begin{aligned} \varphi_{\widehat{A}}(\sigma - \sigma')(b) &= \varphi_{\widehat{A}}(\tau)(b) = \varphi_C \circ \tau(b) \\ &= \varphi_C(\sigma(b) \underset{B}{-} \sigma'(b)) = \varphi_C(d \underset{B}{-} d) = 0. \end{aligned}$$

Therefore, $\varphi_{\widehat{A}}(\sigma)(b) = \varphi_{\widehat{A}}(\sigma')(b)$, which means that Eq. (C.38) gives us a well-defined element in D'.

Now we can extend Φ by a continuity argument, or else in the case that $b = 0$, that is, $b \in \ker q_B = A \oplus C$, simply define

$$\Phi(d) = \Phi(a, c) = (\varphi_A(a), \varphi_C(c)) \in \ker q_B = A' \oplus C'.$$

The compatibility condition (C.37) guarantees that Φ is smooth. Let's prove that Φ is a DVB morphism. It is immediate to verify that Φ preserves both fibrations, and the induced maps on the side bundles are φ_A and Id_B, respectively. As for linearity, take $d_1, d_2 \in D_a$, with $q_B(d_1) = b_1$ and $q_B(d_2) = b_2$. Again by a continuity argument, we don't loose generality if we suppose that b_1 and b_2 are linearly independent (the set of such vector is an open dense in B_m). Then we can find $\sigma \in \widehat{A}$ with $\sigma(b_1) = d_1$ and $\sigma(b_2) = d_2$. Then $\sigma(b_1 + b_2) = d_1 \underset{A}{+} d_2$ and

$$\Phi(d_1 \underset{A}{+} d_2) = \varphi_{\widehat{A}}(\sigma)(b_1 + b_2) = \varphi_{\widehat{A}}(\sigma)(b_1) \underset{A'}{+} \varphi_{\widehat{A}}(\sigma)(b_2) = \Phi(d_1) \underset{A'}{+} \Phi(d_2).$$

For $d_1, d_2 \in D_b$, we can find $\sigma_1, \sigma_2 \in \widehat{A}$ such that $\sigma_1(b) = d_1$ and $\sigma_2(b) = d_2$. Then $(\sigma_1 + \sigma_2)(b) = \sigma_1(b) \underset{B}{+} \sigma_2(b) = d_1 \underset{B}{+} d_2$, so that

$$\Phi(d_1 \underset{B}{+} d_2) = \varphi_{\widehat{A}}(\sigma_1 + \sigma_2)(b) = \varphi_{\widehat{A}}(\sigma_1)(b) \underset{B}{+} \varphi_{\widehat{A}}(\sigma_2)(b) = \Phi(d_1) \underset{B}{+} \Phi(d_2).$$

■

Remark C.25. We can obtain a more general result than Prop. C.24 above, by letting D' to have a different side bundle B' and including in the hypothesis one more vector bundle morphism $\varphi_B : B \longrightarrow B'$. Then the compatibility condition (C.37) must be modified to the condition

$$\varphi_{\widehat{A}}(\tau) \circ \varphi_B = \varphi_C \circ \tau, \qquad \forall \tau \in B^* \otimes C \cong \mathrm{Hom}(B, C) \subset \widehat{A}.$$

We can prove that there is a unique DVB morphism $\Phi : D \longrightarrow D'$ such that the induced vector bundle morphisms between the side bundles B, B', the core bundles C, C' and the linear bundles $\widehat{A}, \widehat{A'}$ are φ_B, φ_C and $\varphi_{\widehat{A}}$, respectively. The proof given above to Prop. C.24 works just fine, with the only modification that we define, instead of (C.38),

$$\Phi(d) := \varphi_{\widehat{A}}(\sigma)(\varphi_B(b)).$$

An important observation, already suggested by the proposition above, is that a DVB morphism Φ that preserves a common side bundle is completely determined by its action on linear and core sections. This is the content of the following simple, though useful, lemma.

Lemma C.26. *Let*

$$\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B; M)_{C'}$$

be a DVB morphism over the identity on B. Then Φ induces maps

$$\widetilde{\Phi}_{\mathrm{lin}} : \Gamma_{\mathrm{lin}}(D_B) \longrightarrow \Gamma_{\mathrm{lin}}(D'_B) \qquad \textit{and} \qquad \widetilde{\Phi}_{\mathrm{core}} : \Gamma_{\mathrm{core}}(D_B) \longrightarrow \Gamma_{\mathrm{core}}(D'_B) \tag{C.39}$$

which completely determine Φ. More precisely, given any $d \in D$, we can find $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$ and $\beta \in \Gamma_{\mathrm{core}}(D_B)$ such that

$$\Phi(d) = \widetilde{\Phi}_{\mathrm{lin}}(\gamma)(q_B(d)) \underset{B}{+} \widetilde{\Phi}_{\mathrm{core}}(\beta)(q_B(d)).$$

Proof. We have already seen that a DVB morphism preserves core and linear sections (Props. A.10 and A.11), therefore we obtain the maps $\widetilde{\Phi}_{\mathrm{lin}}$ and $\widetilde{\Phi}_{\mathrm{core}}$ stated in (C.39).

Now let $d \in D_m$, and choose a decomposition $D \cong A \oplus B \oplus C$, so that $d = (a, b, c)$. Suppose first that $b \neq 0$. Then we can find $\gamma_2 \in \Gamma(\mathrm{Hom}(B, C))$ such that $\gamma_2(m)(b) = c$. Also we can find $\alpha \in \Gamma(A)$ such that $\alpha(m) = a$. Then, by Prop. C.1, we obtain a linear section $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$

$$\gamma := (\alpha, \gamma_2) \in \Gamma(A \oplus \mathrm{Hom}(B, C)) \cong \Gamma_{\mathrm{lin}}(D_B),$$

such that $\gamma(b) = (\alpha(m), b, \gamma_2(m)(b)) = (a, b, c) = d$, which implies that

$$\Phi(d) = \Phi(\gamma(q_B(d))) = \widetilde{\Phi}_{\mathrm{lin}}(\gamma)(q_B(d)).$$

Now, in the case $q_B(d) = 0$, choose $\beta \in \Gamma(C)$ such that $\beta(m) = c$, and define $\widetilde{\beta} \in \Gamma_{\mathrm{core}}(D_B)$ by

$$\widetilde{\beta}(b) := 0_B(b) \underset{A}{+} \overline{\beta(q^B(b))}.$$

Then, setting $\gamma := (\alpha, 0) \in \Gamma(A \oplus \mathrm{Hom}(B, C)) \cong \Gamma_{\mathrm{lin}}(D_B)$, we obtain

$$\gamma(q_B(d)) \underset{B}{+} \widetilde{\beta}(q_B(d)) = (a, 0, 0) \underset{B}{+} (0, 0, c) = (a, 0, c) = d,$$

whence

$$\begin{aligned}\Phi(d) &= \Phi(\gamma(q_B(d)) \underset{B}{+} \widetilde{\beta}(q_B(d))) = \Phi(\gamma(q_B(d))) \underset{B}{+} \Phi(\widetilde{\beta}(q_B(d))) \\ &= \widetilde{\Phi}_{\mathrm{lin}}(\gamma)(q_B(d)) \underset{B}{+} \widetilde{\Phi}_{\mathrm{core}}(\widetilde{\beta})(q_B(d)).\end{aligned}$$

■

Next proposition provides a converse for Props. A.10 and A.11, thus characterizing, in terms of core and linear sections, when a vector bundle morphism, which is the identity on a common side bundle, becomes a DVB morphism. This characterization was stated, in [23] and its proof left as an exercise.

Proposition C.27. *Let $(D; A, B; M)_C$ and $(D'; A', B; M)_{C'}$ be two double vector bundles with a common side bundle, B. A vector bundle morphism $\Phi : D_B \longrightarrow D'_B$ over the identity is a double vector bundle morphism, that is, preserves also the structures over A and A', respectively, if and only if the induced map on sections over B, also denoted by Φ, satisfies $\Phi(\Gamma_{\mathrm{lin}}(D_B)) \subset \Gamma_{\mathrm{lin}}(D'_B)$ and $\Phi(\Gamma_{\mathrm{core}}(D_B)) \subset \Gamma_{\mathrm{core}}(D'_B)$.*

Proof. We have already seen that a double vector bundle morphism preserves linear and core sections. So let's prove that this condition suffices in order to Φ be a double vector bundle morphism. We can work locally, and so suppose that D and D' are decomposed (Prop. A.18). We need to prove that $\Phi : D_A \longrightarrow D'_{A'}$ preserves fibers and the linear structure.

Let $d \in D_m$. By lemma C.26 we can find $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$ and $\widetilde{\beta} \in \Gamma_{\mathrm{core}}(D_B)$ such that

$$\Phi(d) = \widetilde{\Phi}_{\mathrm{lin}}(\gamma)(q_B(d)) \underset{B}{+} \widetilde{\Phi}_{\mathrm{core}}(\widetilde{\beta})(q_B(d)).$$

Since $\widetilde{\Phi}_{\mathrm{lin}}(\gamma)$ is a linear section on D'_B, it projects over a section $\alpha' \in \Gamma(A')$, thereby

$$q_{A'} \circ \Phi(d) = q_{A'} \circ \widetilde{\Phi}_{\mathrm{lin}}(\gamma)(q_B(d)) + q_{A'} \circ \widetilde{\Phi}_{\mathrm{core}}(\widetilde{\beta})(q_B(d)) = \alpha'(m),$$

where, by definition, $\alpha'(m) = q_{A'} \circ \gamma'(0^B(m))$. Now, by the construction of γ (see the proof of lemma C.26), we have $\gamma(0^B(m)) = (a, 0, 0)$. Hence,

$$q_{A'} \circ \Phi(d) = \alpha'(m) = q_{A'} \circ \Phi \circ \gamma(0^B(m)) = q_{A'} \circ \Phi(a, 0, 0) = q_{A'} \circ \Phi \circ 0_A(q_A(d)),$$

which means that $\Phi : D_A \longrightarrow D'_{A'}$ preserves fibers.

Now let's see linearity. Let $d_1, d_2 \in D$ such that $q_A(d_1) = q_A(d_2) = a$. Then $d_1 = (a, b_1, c_1)$ and $d_2 = (a, b_2, c_2)$. Hence

$$d_1 \underset{A}{+} d_2 = (a, b_1 + b_2, c_1 + c_2).$$

Let $\gamma \in \Gamma_{\mathrm{lin}}(D_B)$ such that

$$\gamma(b_1 + b_2) = \gamma(b_1) \underset{A}{+} \gamma(b_2) = (a, b_1, 0) \underset{A}{+} (a, b_2, 0) = (a, b_1 + b_2, 0),$$

and let $\widetilde{\beta} \in \Gamma_{\mathrm{core}}(D_B)$ such that

$$\widetilde{\beta}(b_1 + b_2) = (0, b_1 + b_2, c_1 + c_2).$$

Then $(\gamma \underset{B}{+} \widetilde{\beta})(b) = d_1 \underset{A}{+} d_2$, with $b := b_1 + b_2$; so, using that Φ preserves linear and core sections,

$$\Phi(d_1 \underset{A}{+} d_2) = \Phi(\gamma \underset{B}{+} \widetilde{\beta}) = \gamma'(b) \underset{B}{+} \widetilde{\beta}'(b),$$

where $\gamma' := \Phi(\gamma) \in \Gamma_{\mathrm{lin}}(D'_B)$ and $\widetilde{\beta}' := \Phi(\widetilde{\beta}) \in \Gamma_{\mathrm{core}}(D'_B)$. Hence

$$q_B(\Phi(d_1 \underset{A}{+} d_2)) = q_B(\gamma'(b) \underset{B}{+} \widetilde{\beta}'(b)) = b = b_1 + b_2. \tag{C.40}$$

On the other hand, since $\Phi : D_B \longrightarrow D'_B$ is a bundle map over the identity,

$$q_B(\Phi(d_1) \underset{A'}{+} \Phi(d_2)) = q_B(\Phi(d_1)) + q_B(\Phi(d_2)) = b_1 + b_2. \tag{C.41}$$

Therefore, from (C.40) and (C.41), it follows

$$q_B(\Phi(d_1 \underset{A}{+} d_2)) = q_B(\Phi(d_1) \underset{A'}{+} \Phi(d_2)). \tag{C.42}$$

It remains to show $q_{C'}(\Phi(d_1 \underset{A}{+} d_2)) = q_{C'}(\Phi(d_1) \underset{A'}{+} \Phi(d_2))$. To compute $q_{C'}(\Phi(d_1 \underset{A}{+} d_2))$, notice that $\Phi(\Gamma_{\mathrm{core}}(D_B)) \subset \Gamma_{\mathrm{core}}(D'_B)$ implies, for $c \in C$, after choosing a section $\beta : M \longrightarrow C$ with $\beta(m) = c$, and taking its corresponding core section $\widetilde{\beta}$,

$$\Phi(c) = \Phi(\widetilde{\beta}(0^B(m))) = \widetilde{\beta}'(0^B(m)),$$

whence $\Phi(C) \subset C'$. So Φ induces a vector bundle morphism $\varphi_C := \Phi|_C : C \longrightarrow C'$ over the identity. So we have

$$\begin{aligned} q_{C'}(\Phi(d_1 \underset{A}{+} d_2)) &= q_{C'}(\gamma'(b)) + \widetilde{\beta}'(b) \\ &= q_{C'}(\gamma'(b_1) \underset{A'}{+} \gamma'(b_2)) + \varphi_C(c_1 + c_2) \\ &= \gamma'_2(b_1) + \gamma'_2(b_2) + \varphi_C(c_1) + \varphi_C(c_2). \end{aligned} \tag{C.43}$$

Now we compute $\Phi(d_1)$ and $\Phi(d_2)$. As we already observed, we can find core sections $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ such that $\widetilde{\beta}_1(b_1) = (0, b_1, c_1)$, and $\widetilde{\beta}_2(b_2) = (0, b_2, c_2)$. Then

$$\begin{aligned} d_1 &= (a, b_1, c_1) = \gamma(b_1) \underset{B}{+} \widetilde{\beta}_1(b_1) \\ \text{and} \quad d_2 &= (a, b_2, c_2) = \gamma(b_2) \underset{B}{+} \widetilde{\beta}_2(b_2). \end{aligned}$$

So we have

$$\Phi(d_1) = \Phi(\gamma(b_1)) \underset{B}{+} \Phi(\widetilde{\beta}_1(b_1)),$$

hence $q_{C'}(\Phi(d_1)) = \gamma_2'(b_1) + \varphi_C(c_1)$. Analogously we can show that $q_{C'}(\Phi(d_1)) = \gamma_2'(b_2) + \varphi_C(c_2)$. Therefore, taking (C.43) into account,

$$q_{C'}(\Phi(d_1) \underset{A'}{+} \Phi(d_2)) = \gamma_2'(b_1) + \varphi_C(c_1) + \gamma_2'(b_2) + \varphi_C(c_2) = q_{C'}(\Phi(d_1 \underset{A}{+} d_2)).$$

■

C.4 Linear and core sections on the dual

Lemma C.28. *A section $\gamma \in \Gamma(D_A)$ is linear if and only if*

a) $\langle \gamma, \phi \rangle_A$ is a fiberwise linear function for every $\phi \in \Gamma_{\mathrm{lin}}(D_A^*)$, and

b) $\langle \gamma, \widetilde{\xi} \rangle_A$ is fiberwise constant for every $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D_A^*)$.

A section $\widetilde{\beta} \in \Gamma(D_A)$ is core if and only if

c) $\langle \widetilde{\beta}, \phi \rangle_A$ is fiberwise constant for every $\phi \in \Gamma_{\mathrm{lin}}(D_A^*)$ and

d) $\langle \widetilde{\beta}, \widetilde{\xi} \rangle_A = 0$ for every $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D_A^*)$.

Proof. If $\gamma \in \Gamma_{\mathrm{lin}}(D_A)$ and $\phi \in \Gamma_{\mathrm{lin}}(D_A^*)$, then, for $a_1, a_2 \in A$, we have, using Eq. (B.4),

$$\begin{aligned} \langle \gamma(a_1 + a_2), \phi(a_1 + a_2) \rangle_A &= \langle \gamma(a_1) \underset{B}{+} \gamma(a_2), \phi(a_1) \underset{C^*}{+} \phi(a_2) \rangle_A \\ &= \langle \gamma(a_1), \phi(a_1) \rangle_A + \langle \gamma(a_2), \phi(a_2) \rangle_A, \end{aligned}$$

which means that the function $a \longrightarrow \langle \gamma(a), \phi(a) \rangle_A$ is linear.

If $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D_A^*)$, with $\xi \in \Gamma(B^*)$, then by Cor. B.9, and if $\gamma \in \Gamma_{\mathrm{lin}}(D_A)$ maps over the section $\alpha \in \Gamma(B)$, we have

$$\langle \gamma(a), \widetilde{\xi}(a) \rangle_A = \langle q_B \circ \gamma(a), \xi(q^A(a)) \rangle = \langle \alpha(m), \xi(m) \rangle,$$

which means that the function $a \longrightarrow \langle \gamma(a), \widetilde{\xi}(a) \rangle_A$ is fiberwise constant.

Conversely, if $\gamma \in \Gamma(D_A)$ satisfies items *a)* and *b)* of the lemma, then for every $\widetilde{\xi} \in \Gamma_{\mathrm{core}}(D_A^*)$, with $\xi \in \Gamma(B^*)$, item *b)* implies that the function $a \longrightarrow \langle \gamma(a), \widetilde{\xi}(a) \rangle_A$ is fiberwise constant. By Cor. B.9, we obtain that the function

$$a \longrightarrow \langle q_B \circ \gamma(a), \xi(q^A(a)) \rangle$$

is fiberwise constant, which implies that $q_A \circ \gamma$ is fiberwise constant. Thereby, γ preserves fibers and we obtain a section $\alpha \in \Gamma(B)$ such that

$$q_B \circ \gamma(a) = \alpha(q^A(a)).$$

Now, item *a)* implies that, for every $\phi \in \Gamma_{\text{lin}}(D_A^*)$, the map $a \longrightarrow \langle \gamma(a), \phi(a) \rangle_A$ is linear, whereby, for $a_1, a_2 \in A$ with $a_1 + a_2 \neq 0$, and any $v \in D^*_{a_1+a_2}$ choosing $\phi \in \Gamma_{\text{lin}}(D_A^*)$ such that $\phi(a_1 + a_2) = v$ (which is always possible since $a_1 + a_2 \neq 0$), and using Eq. (B.4), we have

$$\begin{aligned}
\langle \gamma(a_1 + a_2), d \rangle_A &= \langle \gamma(a_1 + a_2), \phi(a_1 + a_2) \rangle_A = \langle \gamma(a_1), \phi(a_1) \rangle_A + \langle \gamma(a_2), \phi(a_2) \rangle_A \\
&= \langle \gamma(a_1) \underset{B}{+} \gamma(a_2), \phi(a_1) \underset{C^*}{+} \phi(a_2) \rangle_A \\
&= \langle \gamma(a_1) \underset{B}{+} \gamma(a_2), \phi(a_1 + a_2) \rangle_A = \langle \gamma(a_1) \underset{B}{+} \gamma(a_2), d \rangle_A.
\end{aligned}$$

By a continuity argument, we conclude that

$$\langle \gamma(a_1 + a_2), d \rangle_A = \langle \gamma(a_1) \underset{B}{+} \gamma(a_2), d \rangle_D$$

for every $d \in D^*_{a_1+a_2}$ with $a_1, a_2 \in A$ arbitrary (not necessarily satisfying $a_1 + a_2 \neq 0$), which means $\gamma : A \longrightarrow D_B$ is a vector bundle morphism over α, that is, $\gamma \in \Gamma_{\text{lin}}(D_A)$.

Now let's prove the second part of the lemma, the one concerning to the characterization of core sections. If $\widetilde{\beta} \in \Gamma_{\text{core}}(D_A)$, with $\beta \in \Gamma(C)$ and $\phi \in \Gamma_{\text{lin}}(D_A^*)$ which maps over $\alpha \in \Gamma(C^*)$, then by Cor. B.9

$$\langle \widetilde{\beta}(a), \phi(a) \rangle_A = \langle \beta(q^A(a)), \pi_{C^*} \circ \phi(a) \rangle = \langle \beta(m), \alpha(m) \rangle,$$

which means that $a \longrightarrow \langle \widetilde{\beta}(a), \phi(a) \rangle_A$ is fiberwise constant.

If $\widetilde{\xi} \in \Gamma_{\text{core}}(D_A^*)$, with $\xi \in \Gamma(B^*)$, then, for every $a \in A$, $\widetilde{\xi}(a) \in \ker \pi_{C^*}$, whereby Cor. B.9 implies that

$$\langle \widetilde{\beta}(a), \widetilde{\xi}(a) \rangle_A = 0.$$

Conversely, let $\widetilde{\beta} \in \Gamma(D_A)$ be any section satisfying items *c)* and *d)* of the lemma. Then, using Cor. B.9, item *d)* implies that

$$\langle \widetilde{\beta}(a), \widetilde{\xi}(a) \rangle_A = \langle q_B \circ \widetilde{\beta}(a), \xi(q^A(a)) \rangle,$$

which in turn implies that $q_B \circ \widetilde{\beta}(a) = 0 \; \forall a \in A$, that is, $\widetilde{\beta}$ takes values on $\ker q_B \cong A \oplus C$, so that we can write

$$\widetilde{\beta}(a) = (a, \widetilde{\beta}_C(a)) = 0_A(a) \underset{B}{+} \widetilde{\beta}_C(a),$$

with $\widetilde{\beta}_C := p_C \circ \widetilde{\beta} : A \longrightarrow C$, where $p_C : \ker q_B \cong A \oplus C \longrightarrow C$ is the projection. Now, item *c)* and Cor. B.9 implies that, for $\phi \in \Gamma_{\text{lin}}(D_A^*)$ over $\alpha \in \Gamma(C^*)$,

$$\langle \widetilde{\beta}(a), \phi(a) \rangle_A = \langle \widetilde{\beta}_C(a), \pi_{C^*} \circ \phi(a) \rangle = \langle \widetilde{\beta}_C(a), \alpha(q^A(a)) \rangle$$

is fiberwise constant. Since, for any $\alpha \in \Gamma(C^*)$ we can find $\phi = \widehat{\alpha} \in \Gamma_l inear(D^*_A)$ over α, we conclude that $\widetilde{\beta}_C$ is fiberwise constant, which means that

$$\widetilde{\beta}_C(a) = \beta(q^A(a))$$

for $\beta \in \Gamma(C)$ defined by $\beta(m) = \widetilde{\beta}_C(0^A(m))$. Therefore,

$$\widetilde{\beta}(a) = 0_A(a) \underset{B}{+} \iota \circ \beta \circ q^A(a),$$

that is, $\widetilde{\beta}$ is a core section.

∎

Now we use the lemma to prove the characterization of linear sections of a DVB stated in Cor. 2.9.

Corollary C.29. *There is a canonical 1:1 correspondence between sections in* $\Gamma_{\text{lin}}(D^*_B) \cong \Gamma(\widehat{C^*}_B)$ *and pairs of linear maps*

$$f_{\widehat{A}} : \widehat{A} \longrightarrow B^* \quad \textit{and} \quad f_C : C \longrightarrow \mathbb{R},$$

such that the following compatibility condition is satisfied

$$f_{\widehat{A}}(\tau) = f_C \circ \tau, \qquad \forall \tau \in \text{Hom}(B, C) \subset \widehat{A}.$$

Proof. If we take $\phi \in \Gamma_{\text{lin}}(D^*_B) \cong \Gamma(\widehat{C^*}_B)$, then by the easy part of Lem. 2.8, we obtain linear functions $f_{\widehat{A}} : \widehat{A} \longrightarrow B^*$ and $f_C : C \longrightarrow \mathbb{R}$, by setting

$$f_{\widehat{A}}(\sigma) := \langle \sigma, \phi \rangle_B \quad \text{and} \quad f_C(\mathbf{c}) := \langle \widetilde{\mathbf{c}}, \phi \rangle_B,$$

where of course we are identifying $C^\infty_{\text{lin}}(B) \cong B^*$. Since Lem. 2.8 already tells us that $f_{\widehat{A}}$ and f_C are linear, we only need to check the compatibility condition, which is just linearity of $\langle \cdot, \cdot \rangle_B$ with respect to the product of a linear function. Indeed, there is no loss of generality if we take τ of the form $\beta \otimes \mathbf{c} \in \text{Hom}(B, C)$. In this case we have

$$\begin{aligned} f_{\widehat{A}}(\tau) &= \langle \beta \otimes \widetilde{\mathbf{c}}, \phi \rangle_B = \beta \langle \widetilde{\mathbf{c}}, \phi \rangle_B \\ &= f_C(\mathbf{c})\beta = f_C \circ \tau. \end{aligned}$$

Conversely, if we have such $f_{\widehat{A}}$ and f_C, arguing similarly as we did, for example, in the proof of Prop. C.24, for every $d \in D$ with $b = q_B(d) \neq 0$ we can find $\sigma \in \widehat{A}$ such that $\sigma(b) = d$. Then we define $\phi(b) \in (D^*_B)_b$ by

$$\langle \phi(b), d \rangle_B := f_{\widehat{A}}(\sigma)(b),$$

and extend ϕ to the whole bundle B by setting, for $d = (a, c) \in \ker q_B = A \oplus C$,

$$\langle \phi(0^B(m)), (a, c) \rangle_B = f_C(c).$$

We must check that $\phi(b)$ is well-defined. If $\sigma, \sigma' \in \widehat{A}$ satisfy $\sigma(b) = \sigma'(b) = d$, then

$$q_A((\sigma - \sigma')(b)) = q_A(d \underset{B}{-} d) = 0,$$

whence the compatibility condition implies that $\sigma-\sigma' = \tau \in \mathrm{Hom}(B, C) \subset \widehat{A}$ and $\tau(b) = 0$. Then,

$$f_{\widehat{A}}(\sigma)(b) - f_{\widehat{A}}(\sigma')(b) = f_{\widehat{A}}(\sigma - \sigma')(b) = f_{\widehat{A}}(\tau)(b) = f_C(\tau(b)) = 0.$$

Therefore we have a well-defined section $\phi \in \Gamma(D^*_B)$, which is smooth as it is easily verified, for example by using adapted coordinates. Also observe that, by definition, for $c \in C \subset D$ we have $\langle \phi(b), c\rangle_B = f_C(c)$. Therefore, ϕ satisfies conditions *a)* and *b)* of Lem. C.26, from which we conclude that ϕ is a linear section.

It is clear that the two processes described above are inverses one of the other, so we have the desired canonical 1:1 correspondence.

■

C.5 The double-linear bundle

In this section we introduce the so-called *double-linear functions* on a DVB, D, which are equivalent to linear sections of any of the two duals of D, which therefore also form a vector bundle structure, which we call the *double-linear bundle*. This enables a better understanding of the isomorphism $T : \widehat{A} \longrightarrow \widehat{A}_*$, allowing to arrive to it without resorting to decompositions (see Rmk. C.37). Most important, Prop. C.33 is instrumental to show the duality between representations up to homotopy corresponding a VB-algebroid and to the *VB*-algebroid structure induced on one of its duals, cf. Thm. E.32.

Finally we show how, for any DVB morphism, the double-linear bundle enables to associate a morphism between duals of certain linear vector bundles (isomorphic to the corresponding double-linear bundle, see Prop. C.32), which is actually a morphism of exact sequences. For more details see Prop. C.38. Moreover, in the next chapter we will see that the double-linear bundle is the bridge between the category of double vector bundles and the category of double vector sequences, cf. Thm. D.8.

Definition C.30. Let $(D; A, B; M)_C$ be a double vector bundle. A function $\mu \in C^\infty(D)$ is called *double-linear* if it is linear with respect to both vector bundle structures:

$$\mu(d_1 \underset{A}{+} d_2) = \mu(d_1) + \mu(d_2),$$

for $d_1, d_2 \in D$ with $q_A(d_1) = q_A(d_2)$; and

$$\mu(d_3 \underset{B}{+} d_4) = \mu(d_3) + \mu(d_4),$$

for $d_3, d_4 \in D$ with $q_B(d_3) = q_B(d_4)$.

The set of double-linear functions is called *double-linear bundle*, and is denoted by $C^\infty_{\mathrm{lin}}(D)$.

Remark C.31. It is immediate that $C^\infty_{\mathrm{lin}}(D)$ is a vector subspace of $C^\infty(D)$.

Proposition C.32. *$C^\infty_{\mathrm{lin}}(D)$ admits a vector bundle structure over M, whose fiber over $m \in M$ is given by*

$$C^\infty_{\mathrm{lin}}(D)_m = \{\mu|_{D_m} : \mu \in C^\infty_{\mathrm{lin}}(D)\},$$

where D_m is the slice of D over m:

$$D_m := \{d \in D : q^A \circ q_A(d) = q^B \circ q_B(d) = m\}.$$

We have

$$C^\infty_{\mathrm{lin}}(D)_m = \{\nu \in C^\infty(D_m) : \nu \text{ is double} - \text{linear}\}. \tag{C.44}$$

$C^\infty_{\mathrm{lin}}(D)$ fits in the exact sequence

$$0 \longrightarrow A^* \otimes B^* \xrightarrow{\iota} C^\infty_{\mathrm{lin}}(D) \xrightarrow{p} C^* \longrightarrow 0, \tag{C.45}$$

and

$$\widehat{C^*}_A \cong \widehat{C^*}_B \cong C^\infty_{\mathrm{lin}}(D) \tag{C.46}$$

holds, where $\widehat{C^*}_A$ is the linear vector bundle corresponding to the linear sections of D^*_A, and $\widehat{C^*}_B$ the linear vector bundle corresponding to the linear sections of D^*_B.

Proof. By remark C.31, we have naturally defined vector space structures on $C^\infty_{\mathrm{lin}}(D)_m$ for each $m \in M$. We need to prove that $C^\infty_{\mathrm{lin}}(D)$ has a differential manifold structure and that it is locally trivial. Let's take adapted local coordinates on an open set $D|_U$, $U \subset M$, with $m_0 \in M \subset D$ (see Def. A.21). For $f \in C^\infty(D)$ apply Hadamard's lemma fiberwise twice, we get,

$$\begin{aligned} f(x^i, \alpha^a, \beta^b, \kappa^c) &= f(m) + \alpha^a f_a(d) + \beta^b f_b(d) + \kappa^c f_c(d) \\ &= f(m) + \alpha^a (f_a(m) + \alpha^a g_a(d) + \beta^b g_b(d) + \kappa^c g_c(d)) \\ &\qquad + \beta^b (f_b(m) + \alpha^a h_a(d) + \beta^b h_b(d) + \kappa^c g_c(d)) \\ &\qquad + \kappa^c (f_c(m) + \alpha^a l_a(d) + \beta^b l_b(d) + \kappa^c l_c(d)), \end{aligned}$$

where $d = (x^i, \alpha^a, \beta^b, \kappa^c)$ and $m = q^A \circ q_A(d)$. Now, if $f = \mu$ is double-linear, we get $\mu(m) = 0$, and applying again Hadamard's lemma, now to each function $g_\alpha, h_\alpha, g_\alpha$, for $\alpha = a, b, c$, the double-linearity leads us to conclude that

$$\begin{array}{lll} f_a(m) = f_b(m) = f_c(m) = 0; & & \\ g_a(d) = 0; & g_b(d) = g_b(m); & g_c(d) = 0; \\ h_a(d) = h_a(m); & h_b(d) = 0; & h_c(d) = 0; \\ l_a(d) = 0; & l_b(d) = 0; & h_c(d) = h_c(m). \end{array}$$

Therefore,

$$\mu(x^i, \alpha^a, \beta^b, \kappa^c) = \alpha^a \beta^b \mu_{ab}(m) + \kappa^c \mu_c(m), \tag{C.47}$$

where $\mu_{ab}(m) = g_b(m) + h_a(m)$ and $\mu_c(m) = h_c(m)$. Of course any collection of functions $\mu_{ab}, \mu_c \in C^\infty(U)$ furnishes a double-linear function given by formula (C.47). Thus, we have obtained a bijection

$$C^\infty_{\mathrm{lin}}(D)|_U \longrightarrow (A^* \otimes B^* \oplus C^*)|_U.$$

We use this bijection to endow $C^\infty_{\text{lin}}(D)|_U$ with a vector bundle structure. If we use another adapted coordinate system, then using formulas (A.9) we get

$$\mu_{ab} = \Phi^{\tilde{a}}_a \Phi^{\tilde{b}}_b \mu_{\tilde{a}\tilde{b}} + \Phi^{\tilde{c}}_{ab}\mu_{\tilde{c}}; \qquad \mu_c = \Phi^{\tilde{c}}_c \mu_{\tilde{c}},$$

so that $\{\mu_{ab}, \mu_c\}$ transform linearly. Therefore, $C^\infty_{\text{lin}}(D)$ is a vector bundle over M with

$$\operatorname{rank} C^\infty_{\text{lin}}(D) = \operatorname{rank} A \operatorname{rank} B + \operatorname{rank} C. \tag{C.48}$$

Also, from (C.47) we see that, given a double-linear function ν on D_m, we can extend it to a double-linear function μ on $D|_U$, by setting μ_{ab} and μ_c constants on U. Now take a bump-function on U, and extend it to a function ϕ on D, constant on the slices D_m. Then it is immediate to see that

$$\widetilde{\mu}(d) = \begin{cases} \phi(d)\mu(d) & \text{if } d \in D|_U \\ 0 & \text{if } d \notin D|_U \end{cases} \tag{C.49}$$

defines a double-linear function on D. This proves (C.44).

Now let $\lambda \in A^*_m \otimes B^*_m$, then we define $\iota(\lambda) \in C^\infty_{\text{lin}}(D)_m$ by

$$\iota(\lambda)(d) := \langle \lambda, q_A(d) \otimes q_B(d) \rangle, \tag{C.50}$$

which is obviously double-linear. Thus we have an injective map

$$\iota : A^* \otimes B^* \longrightarrow C^\infty_{\text{lin}}(D).$$

Let this time $\nu \in C^\infty_{\text{lin}}(D)_m$, then we define $p(\nu) \in C^*$ by

$$\langle p(\nu), c \rangle = \nu(c)$$

for every $c \in C_m$, where, in the right-hand side we are considering C_m embedded into D_m. Using a decompositon of D we see that p is surjective, with left-inverse q^*_C. Since $C = \ker q_A \cap \ker q_B$ it follows that $p \circ \iota = 0$. By rank reasons (see eq. (C.48)), we obtain the exact sequence (C.45).

To prove $\widehat{C^*}_B \cong C^\infty_{\text{lin}}(D)$ let's take $\nu \in C^\infty_{\text{lin}}(D)_m$ and associate $\sigma^\nu_B \in \widehat{C^*}_B$ by setting

$$\langle \sigma^\nu_B(b), d \rangle_B := \nu(d), \qquad b \in B_m, d \in D_b. \tag{C.51}$$

We need to check that σ^ν_B is indeed an element in $(\widehat{C^*}_B)_m$. Because of the linearity of ν with respect to the structure over B, it follows that actually $\sigma^\nu_B(b) \in (D^*_B)_b$. Now, for any $c \in C_m$,

$$\langle \pi_{C^*}(\sigma^\nu_B(b)), c \rangle = \langle \sigma^\nu_B(b), 0_B(b) \underset{A}{+} c \rangle_B = \nu(0_B(b) \underset{A}{+} c) = \nu(c),$$

which implies that $\sigma^\nu_B(b) \in (D^*_B)_\kappa$, with $\kappa \in C^*_m$ defined by $\langle \kappa, c \rangle = \nu(c)$. Now let $b_1, b_2 \in B_m$, then, for $d_1, d_2 \in D_a$ with $q_B(d_1) = b_1$ and $q_B(d_2) = b_2$,

$$\begin{aligned} \langle \sigma^\nu_B(b_1 + b_2), d_1 \underset{A}{+} d_2 \rangle_B = \nu(d_1 \underset{A}{+} d_2) &= \nu(d_1) + \nu(d_2) = \langle \sigma^\nu_B(b_1), d_1 \rangle_B + \langle \sigma^\nu_B(b_2), d_2 \rangle_B \\ &= \langle \sigma^\nu_B(b_1) \underset{C^*}{+} \sigma^\nu_B(b_2), d_1 \underset{A}{+} d_2 \rangle_B. \end{aligned}$$

Thus, $\sigma_B^\nu \in \mathrm{Hom}(B_m, (D_B^*)_\kappa)$, and since $\sigma_B^\nu(b) \in D_b$, it follows that $\sigma_B^\nu \in (\widehat{C^*}_B)_m$. Hence, we have obtained a map

$$\begin{aligned} Z_B : C^\infty_{\mathrm{lin}}(D) &\longrightarrow \widehat{C^*}_B \\ \nu \quad &\longrightarrow \sigma_B^\nu, \end{aligned} \tag{C.52}$$

which is linear in the fibers as follows directly from the definition, so that Z_B is a vector bundle morphism. It is actually an isomorphism, for, if $\sigma \in \mathrm{Hom}(B_m, (D_A^*)_\kappa)$, define

$$\nu_\sigma(d) := \langle \sigma(b), d\rangle_B, \qquad b = q_B(d).$$

Let $d_1, d_2 \in D_b$, then

$$\begin{aligned} \nu_\sigma(d_1 \underset{B}{+} d_2) &= \langle \sigma(b), d_1 \underset{B}{+} d_2\rangle_B = \langle \sigma(b), d_1\rangle_B + \langle \sigma(b), d_2\rangle_B \\ &= \nu_\sigma(d_1) + \nu_\sigma(d_2). \end{aligned}$$

Now let $d_1, d_2 \in D_a$, $a \in A_m$, then $q_B(d_1 \underset{A}{+} d_2) = b_1 + b_2$, where $b_1 = q_B(d_1)$ and $b_2 = q_B(d_2)$. So we have the following

$$\begin{aligned} \nu_\sigma(d_1 \underset{A}{+} d_2) &= \langle \sigma(b_1 + b_2), d_1 \underset{A}{+} d_2\rangle_B = \langle \sigma(b_1) \underset{C^*}{+} \sigma(b_2), d_1 \underset{A}{+} d_2\rangle_B \\ &= \langle \sigma(b_1), d_1\rangle_B + \langle \sigma(b_2), d_2\rangle_B = \nu_\sigma(d_1) + \nu_\sigma(d_2). \end{aligned}$$

Therefore $\nu_\sigma \in C^\infty_{\mathrm{lin}}(D)_m$ and, for $b \in B_m$ and $d \in D_b$,

$$\langle Z_B(\nu_\sigma)(b), d\rangle_B = \nu_\sigma(d) = \langle \sigma(b), d\rangle_B,$$

that is, $Z_B(\nu_\sigma) = \sigma$, which shows that Z_B is an isomorphism. Hence $\widehat{C^*}_B \cong C^\infty_{\mathrm{lin}}(D)$ as we wanted.

Analogously it is shown that $\widehat{C^*}_A \cong C^\infty_{\mathrm{lin}}(D)$, with the isomorphism

$$\begin{aligned} Z_A : C^\infty_{\mathrm{lin}}(D) &\longrightarrow \widehat{C^*}_A \\ \nu \quad &\longrightarrow \sigma_A^\nu, \end{aligned} \tag{C.53}$$

where

$$\langle \sigma_A^\nu(a), d\rangle_A := \nu(d), \qquad d \in D_a. \tag{C.54}$$

∎

Proposition C.33. *Let $Z := Z_B \circ Z_A^{-1}$, where Z_A and Z_B where given in (C.53) and (C.52), respectively. The following diagram commutes*

$$\begin{tikzcd} A^* \otimes B^* \arrow[r, "\iota_A"] \arrow[d, "*"] & \widehat{C^*}_A \arrow[r, "p_A"] \arrow[d, "Z"] & C^* \arrow[d, "\mathrm{Id}"] \\ B^* \otimes A^* \arrow[r, "\iota_B"] & \widehat{C^*}_B \arrow[r, "p_B"] & C^* \end{tikzcd} \quad , \tag{C.55}$$

where $* : A^* \otimes B^* \longrightarrow B^* \otimes A^*$ is the transpose: $*(\phi) = \phi^*$.

Proof. $p_B \circ Z = p_A$ follows directly from the definitions of Z_A and Z_B. So it remains to prove only that

$$Z \circ \iota_A(\phi) = \iota_B(\phi^*), \tag{C.56}$$

for every $\phi \in (A^* \otimes B^*)_m$. So let $a \in A_m$, $b \in B_m$ and $d \in D$, with $q_A(d) = a$ and $q_B(d) = b$. Let's compute:

$$\begin{aligned}\langle \iota_B \circ *(\phi)(b), d\rangle_B &= \langle 0_B(b) \underset{C^*}{+} \phi^*(b), d\rangle_B = \langle \phi^*(b), a\rangle \qquad \text{(from (B.9))}\\ &= \langle \phi(a), b\rangle. \end{aligned} \tag{C.57}$$

On the other hand, by the definitions of Z_A and Z_B, we obtain

$$\begin{aligned}\langle Z \circ \iota_A(\phi)(b), d\rangle_B &= \langle Z_B \circ Z_A^{-1} \circ \iota_A(\phi)(b), d\rangle_B\\ &= \langle Z_B \circ \iota(\phi)(b), d\rangle_B = \iota(\phi)(d), \text{ where } \iota : A^* \otimes B^* \longrightarrow C^\infty_{\text{lin}}(D) \text{ is the inclusion}\\ &= \langle \phi, a \otimes b\rangle = \langle \phi(a), b\rangle. \end{aligned} \tag{C.58}$$

From (C.57) and (C.58) it follows (C.56).

∎

Proposition C.34. *Let $(D; A, B; M)_C$ be a double vector bundle. Consider its flip $(D; B, A; M)_C$ (see Def. 2.5). Let's denote the first DVB by D_A and the second by D_B. Then we have the exact sequences*

$$A^* \otimes B^* \xrightarrow{\iota_A} C^\infty_{\text{lin}}(D_A) \xrightarrow{p} C^* \quad \text{and} \quad B^* \otimes A^* \xrightarrow{\iota_B} C^\infty_{\text{lin}}(D_B) \xrightarrow{p} C^*,$$

which are related by

$$\begin{CD} A^* \otimes B^* @>{\iota_A}>> C^\infty_{\text{lin}}(D_A) @>{p}>> C^* \\ @V{*}VV @V{\text{Flip}^*}VV @V{\text{Id}}VV \\ B^* \otimes A^* @>{\iota_B}>> C^\infty_{\text{lin}}(D_B) @>{p}>> C^* \end{CD}\ , \tag{C.59}$$

where Flip : $D_B \longrightarrow D_A$ *is the canonical identification of a double vector bundle with its flip, and* Flip* *is the restriction of the pull-back map to the set of double linear functions.*

Proof. We have already introduced ι_A, which is given, for $\lambda \in A^* \otimes B^*$, by

$$\iota_A(\lambda)(d) = \langle \lambda, q_A(d) \otimes q_B(d)\rangle.$$

Analogously, we can define, for $\eta \in B^* \otimes A^*$,

$$\iota_B(\eta)(d) = \langle \eta, q_B(d) \otimes q_A(d)\rangle.$$

For any $\lambda \in A^* \otimes B^*$ and $\tau \in B \otimes A$ we have

$$\langle \lambda^*, \tau\rangle = \langle \lambda, \tau^*\rangle.$$

This can be verified taking λ and τ of the form $\alpha \otimes \beta$ and $b \otimes a$, respectively. The general case follows by linearity. Then we have, for any $\lambda \in A^* \otimes B^*$,

$$\begin{aligned}\iota_B(\lambda^*)(d) &= \langle \lambda^*, q_B(d) \otimes q_A(d) \rangle \\ &= \langle \lambda, q_A(d) \otimes q_B(d) \rangle \\ &= \iota_A(\lambda)(d),\end{aligned}$$

where we are omitting the identification Flip and denoting indistinctly $d \in D_B$ and $d \in D_A$. From this follows (C.59).

■

Remark C.35. With the identification given in Prop. C.34, we see that it is more accurate to see the isomorphisms Z_A and Z_B defined in (C.53) and (C.52), respectively, as mappings

$$Z_A : C^\infty_{\mathrm{lin}}(D_A) \longrightarrow \widehat{C^*}_A \quad \text{and} \quad Z_B : C^\infty_{\mathrm{lin}}(D_B) \longrightarrow \widehat{C^*}_B,$$

so that the isomorphism $Z : \widehat{C^*}_A \longrightarrow \widehat{C^*}_B$ fits in the diagram

$$\begin{tikzcd}
A^* \otimes B^* \arrow[r, "\iota_A"] \arrow[d, "\mathrm{Id}"] & \widehat{C^*}_A \arrow[r, "p_A"] \arrow[d, "Z_A^{-1}"'] \arrow[ddd, bend left, "Z"] & C^* \arrow[d, "\mathrm{Id}"] \\
A^* \otimes B^* \arrow[r, "\iota_A"] \arrow[d, "*"] & C^\infty_{\mathrm{lin}}(D_A) \arrow[r, "p"] \arrow[d, "\mathrm{Flip}^*"'] & C^* \arrow[d, "\mathrm{Id}"] \\
B^* \otimes A^* \arrow[r, "\iota_B"] \arrow[d, "\mathrm{Id}"] & C^\infty_{\mathrm{lin}}(D_B) \arrow[r, "p"] \arrow[d, "Z_B"'] & C^* \arrow[d, "\mathrm{Id}"] \\
B^* \otimes A^* \arrow[r, "\iota_B"] & \widehat{C^*}_B \arrow[r, "p_B"] & C^*
\end{tikzcd} \quad . \tag{C.60}$$

Proposition C.36. *Consider the isomorphism $\Upsilon_A : D_A^* \longrightarrow (D_B^*)^*_{C^*}$ given in Prop. B.13. Let $\widehat{\Upsilon}_A : \widehat{C^*}_A \longrightarrow \widehat{C^*}_{B_*}$ be the induced isomorphism between the linear bundles, given by Remark C.19. Then,*

$$Z = T^{-1} \circ \widehat{\Upsilon}_A. \tag{C.61}$$

Here $Z : \widehat{C^*}_A \longrightarrow \widehat{C^*}_B$ is the morphism given in Prop. C.33, and $T^{-1} : \widehat{C^*}_{B_*} \longrightarrow \widehat{C^*}_B$ is the isomorphism given in Prop. C.17, with $(D_B^*)^*_{C^*}$ playing the role of D and we identify D_B^* with $((D_B^*)^*_{C^*})^*_{C^*}$.

Proof. Pick a decomposition for D, which, as we have already seen, induces decompositions for the duals and horizontal lifts for the several linear bundles involved. In particular, we have (non-canonical) isomorphisms $\widehat{C^*}_A \cong C^* \oplus \mathrm{Hom}(A, B^*)$ and $\widehat{C^*}_B \cong C^* \oplus \mathrm{Hom}(B, A^*)$. It follows from diagram (C.55) and Eqs. (C.54) and (C.51) that, in terms of these decompositions, the isomorphism Z is given by, for $(\kappa, \sigma_1) \in C^* \oplus \mathrm{Hom}(A, B^*)$,

$$Z(\kappa, \sigma_1) = (\kappa, \sigma_1^*). \tag{C.62}$$

On the other hand, since $\upsilon_A = -\mathrm{Id}$, where υ_A is the vector bundle morphism induced on the side bundle A by Υ_A, we get

$$\begin{aligned}\widehat{\Upsilon}_A(\iota(\sigma_1))(a) &= \Upsilon_A \circ \iota(\sigma_1) \circ \upsilon_A^{-1}(a) = \Upsilon \circ \iota(\sigma_1)(-a) \\ &= \Upsilon_A(0_A(-a) \underset{C^*}{+} \sigma_1(-a)) = 0_A(a) \underset{C^*}{+} (-\sigma_1)(a),\end{aligned}$$

whence, in terms of the decomposition,

$$\widehat{\Upsilon}_A(\kappa, \sigma_1) = (\kappa, -\sigma_1).$$

From (C.25) it follows that, in terms of the decomposition,

$$T^{-1} \circ \widehat{\Upsilon}_A(\kappa, \sigma_1) = \Delta(\kappa, -\sigma_1) = (\kappa, \sigma_1^*). \tag{C.63}$$

From (C.62) and (C.63) it follows (C.61).

∎

Remark C.37. In particular we obtained the formula

$$T = \widehat{\Upsilon}_A \circ Z^{-1}$$

which expresses T without the need of an auxiliary decomposition. Of course in order to obtain T corresponding to the linear bundle $\widehat{A}$ as in Prop. C.17, we must begin with the right double vector bundle, namely, D_B^*, and C^* playing the role of A. Applying Prop. C.36 to this DVB instead of D, we obtain the formula, this time for $T : \widehat{A} \longrightarrow \widehat{A}_*$, the precise map of Prop. C.17,

$$Z = T^{-1} \circ (\widehat{\Upsilon}_{C^*}),$$

where Z is the isomorphism given in Prop. C.33, but this time D_B^* is playing the role of D, so that now Z is an isomorphism between $\widehat{A}_{C^*}$, the linear bundle of $(D_B^*)_{C^*}^*$, and $\widehat{A}_B$, the linear bundle of $(D_B^*)_B^* \cong D$. Thus, the formula for T is

$$T = \widehat{\Upsilon}_{C^*} \circ Z^{-1}. \tag{C.64}$$

We saw in Prop. C.18 and Rmk. C.19 that under certain circumstances, a DVB morphism induces a vector bundle morphism on the corresponding linear bundles. The process described there requires that the base morphism is a bijection. The problem is that we are inducing the morphism on the "functorialy wrong" linear bundle. The double-linear bundle enables to establish an induced morphism on the "right" linear bundle, for *any* given DVB morphism. This is what we address in the following proposition.

Proposition C.38. *Let $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ be a double vector morphism. Then, Φ induces a vector bundle morphism $\widehat{\Phi} : (\widehat{C^*}_B)^* \longrightarrow (\widehat{C'^*}_{B'})^*$, over φ_M. Explicitly, using the identification (C.46), let $\eta \in C^\infty_{\mathrm{lin}}(D)^*_m$, then we define $\widehat{\Phi}(\eta) \in C^\infty_{\mathrm{lin}}(D')^*_{\varphi_M(m)}$ by*

$$\langle \widehat{\Phi}(\eta), \nu' \rangle := \langle \eta, \Phi^* \nu' \rangle, \tag{C.65}$$

where $\nu' \in C^\infty_{\mathrm{lin}}(D')_{\varphi_M(m)}$, *and* $\Phi^*\nu' \in C^\infty_{\mathrm{lin}}(D)_m$ *is defined as usual by* $\Phi^*\nu'(d) := \nu'(\Phi(d))$.

The following diagram commutes:

$$\begin{array}{ccccc} C & \xrightarrow{p^*} & (\widehat{C^*}_B)^* & \xrightarrow{\iota^*} & A\otimes B \\ \downarrow{\scriptstyle \varphi_C} & & \downarrow{\scriptstyle \widehat{\Phi}} & & \downarrow{\scriptstyle \varphi_A\otimes\varphi_B} \\ C' & \xrightarrow{p'^*} & (\widehat{C'^*}_{B'})^* & \xrightarrow{\iota'^*} & A'\otimes B' \end{array}, \tag{C.66}$$

where we are identifying $C \cong C^{**}$, $C' \cong C'^{**}$, $(A^*\otimes B^*)^* \cong A\otimes B$ *and* $(A'^*\otimes B'^*)^* \cong A'\otimes B'$ *in the canonical way.*

Proof. It is evident that (C.65) determines a well-defined vector bundle morphism $\widehat{\Phi} : (\widehat{C^*}_B)^* \longrightarrow (\widehat{C'^*}_{B'})^*$ over φ_M. So we only need to check that the diagram (C.66) commutes, which consists, under the identification (C.46), of straightforward computations:

- for $c \in C_m$ and $\eta' \in C^\infty_{\mathrm{lin}}(D')^*_{\varphi_M(m)}$ we have

$$\begin{aligned}\langle \widehat{\Phi}\circ p^*(c), \eta'\rangle &= \langle p^*(c), \Phi^*\eta'\rangle = \Phi^*\eta'(c) = \eta'(\Phi(c)) = \eta'(\varphi_C(c)) \\ &= \langle p'^*(\varphi_C(c)), \eta'\rangle,\end{aligned}$$

  thence, $\widehat{\Phi}\circ p^*(c) = p'^*(\varphi_C(c))$.

- for $\eta \in C^\infty_{\mathrm{lin}}(D)^*_m$ and $\lambda' \in (A'^*\otimes B'^*)_{\varphi_M(m)}$ we have

$$\langle \iota'^*\circ\widehat{\Phi}(\eta), \lambda'\rangle = \langle \iota'(\lambda'), \widehat{\Phi}(\eta)\rangle = \langle \eta, \Phi^*\iota'(\lambda')\rangle.$$

  On the other hand,

$$\langle \varphi_A\otimes\varphi_B\circ\iota^*(\eta), \lambda'\rangle = \langle \eta, \iota((\varphi_A^*\otimes\varphi_B^*)_m(\lambda'))\rangle.$$

  Thus, in order to show that $\iota'^*\circ\widehat{\Phi}(\eta) = \varphi_A\otimes\varphi_B\circ\iota^*(\eta)$ it suffices to show

$$\Phi^*\iota'(\lambda') = \iota((\varphi_A^*\otimes\varphi_B^*)_m(\lambda')). \tag{C.67}$$

  So let $d \in D_m$, we have

$$\begin{aligned}\Phi^*\iota'(\lambda')(d) &= \iota'(\lambda')(\Phi(d)) = \langle \lambda', q_A(\Phi(d))\otimes q_B(\Phi(d))\rangle \\ &= \langle \lambda', \varphi_A(q_A(d))\otimes\varphi_B(q_B(d))\rangle = \langle (\varphi_A^*\otimes\varphi_B^*)_m(\lambda'), q_A(d)\otimes q_B(d)\rangle \\ &= \iota((\varphi_A^*\otimes\varphi_B^*)_m(\lambda'))(d),\end{aligned}$$

  which gives (C.67).

∎

Remark C.39. If $\Phi_1 : D \longrightarrow D'$ and $\Phi_2 : D' \longrightarrow D''$ are DVB morphisms, then the corresponding induced morphisms given by Prop. C.38 satisfy

$$\widehat{\Phi_2 \circ \Phi_1} = \widehat{\Phi_2} \circ \widehat{\Phi_1}.$$

Indeed, let $\eta \in C^\infty_{\text{lin}}(D)^*$ and $\nu'' \in C^\infty_{\text{lin}}(D'')$, then

$$\begin{aligned}
\langle \widehat{\Phi_2 \circ \Phi_1}(\eta), \nu'' \rangle &= \langle \eta, (\Phi_2 \circ \Phi_1)^*(\nu'') \rangle \\
&= \langle \eta, \Phi_1^*(\Phi_2^*(\nu'')) \rangle \\
&= \langle \widehat{\Phi_1}(\eta), \Phi_2^*(\nu'') \rangle \\
&= \langle \widehat{\Phi_2}(\widehat{\Phi_1}(\eta)), \nu'' \rangle.
\end{aligned}$$

When $\Phi : D \longrightarrow D'$ is an isomorphism, there rises the natural question about the relation of the induced morphism of Prop. C.38 and the induced morphism given in Cor. C.20. We answer this in the next proposition.

Proposition C.40. *If $\Phi : (D; A, B; M)_C \longrightarrow (D'; A', B'; M')_{C'}$ is an isomorphism, then the following diagram commutes*

$$\begin{tikzcd}
C^\infty_{\text{lin}}(D') \arrow[r, "\Phi^*"] \arrow[d, "Z_{A'}"] & C^\infty_{\text{lin}}(D) \arrow[d, "Z_A"] \\
\widehat{C'^*}_{A'} \arrow[r, "\widehat{\Phi^*_A}"] & \widehat{C^*}_A
\end{tikzcd}, \tag{C.68}$$

where $\widehat{\Phi^*_A} : \widehat{C'^*}_{A'} \longrightarrow \widehat{C^*}_A$ is the vector bundle isomorphism induced by Φ that was given in Cor. C.20, but here we are choosing the side bundle A instead of B.

Proof. On one hand, by Cor. C.20 we have

$$\widehat{\Phi^*_A}(\sigma') = \Phi^*_A \circ \sigma' \circ \varphi_A, \qquad \forall\ \sigma' \in \widehat{C'^*}_{A'},$$

whereby, for every $d \in D$

$$\langle \widehat{\Phi^*_A} \circ Z_{A'}(\nu')(q_A(d)), d \rangle_A = \langle \sigma^{\nu'}_{A'} \circ \varphi_A(q_A(d)), \Phi(d) \rangle = \nu'(\Phi(d)).$$

On the other hand,

$$\langle Z_A \circ \Phi^*(\nu')(q_A(d)), d \rangle_A = \Phi^*(\nu'(d)) = \nu'(\Phi(d)),$$

thus we conclude that $\widehat{\Phi^*_A} \circ Z_{A'} = Z_A \circ \Phi^*(\nu')$.

∎

Proposition C.41. *If, in the situation of Prop. C.38, we choose decompositions $D \cong A \oplus B \oplus C$ and $D' \cong A' \oplus B' \oplus C'$, which in turn induce decompositions $C^\infty_{\text{lin}}(D)^* \cong A \otimes B \oplus C$ and $C^\infty_{\text{lin}}(D')^* \cong A' \otimes B' \oplus C'$, then the induced morphism $\widehat{\Phi}$ has the expression*

$$\widehat{\Phi}(\eta, c) = ((\varphi_A \otimes \varphi_B)(\eta), \varphi_C(c) + \Psi(\eta)), \qquad \eta \in A \otimes B, c \in C, \tag{C.69}$$

where $\Psi : A \otimes B \longrightarrow C'$, is given in the expression of Φ in Cor. A.24.

Proof. First notice that, from Eqs. (C.46), (C.10) and (C.54), we have that the horizontal lift of (C.45), induced by the decomposition, is given by

$$\begin{aligned} \psi : C^* &\longrightarrow C^\infty_{\mathrm{lin}}(D) \\ \kappa &\longrightarrow q_C^*(\kappa), \qquad q_C^*(\kappa)(d) = \langle \kappa, q_C(d)\rangle. \end{aligned}$$

Analogously, $\psi' : (C')^* \longrightarrow C^\infty_{\mathrm{lin}}(D')$ is given by $q_{C'}^*$.

Now, given $(\lambda', \kappa') \in (A')^* \otimes (B')^* \oplus (C')^*$, on one hand, by Cor. A.24 and Eq. (C.50), we have

$$\begin{aligned} \Phi^*(\iota'(\lambda'))(d) &= \iota'(\lambda')(\Phi(a,b,c)) = \iota'(\lambda')(\varphi_A(a), \varphi_B(b), \varphi_C(c) + \Psi(a,b)) \\ &= \langle \lambda', \varphi_A(a) \otimes \varphi_B(b)\rangle \\ &= \lambda', \varphi_A \otimes \varphi_B(a \otimes b)\rangle \\ &= (\varphi_A \otimes \varphi_B)^*(\lambda')(d). \end{aligned}$$

On the other hand

$$\begin{aligned} \Phi^*(\psi'(\kappa')) &= \psi'(\kappa')(\Phi(a,b,c)) = \psi'(\kappa')(\varphi_A(a), \varphi_B(b), \varphi_C(c) + \Psi(a,b)) \\ &= \langle \kappa', \varphi_C(c) + \Psi(a,b)\rangle \\ &= \varphi_C^*(\kappa')(q_C(d)) + \Psi(\kappa')(a \otimes b) \\ &= \psi(\varphi_C^*(\kappa'))(d) + \Psi(\kappa')(d), \end{aligned}$$

where we are identifying $\Psi \in \mathrm{Hom}(A \otimes B, C') \cong A^* \otimes B^* \otimes (C')^* \cong \mathrm{Hom}(C', A^* \otimes B^*)$. Therefore from the two equations above we obtain

$$\Phi^*(\lambda' + \kappa') = (\varphi_A \otimes \varphi_B)^*(\lambda') + \Psi(\kappa') + \varphi_C^*(\kappa'),$$

whereby, for $\eta \in A \otimes B$ and $c \in C$, and denoting the induced horizontal lift by $\widetilde{\psi} : A \otimes B \longrightarrow C^\infty_{\mathrm{lin}}(D)^*$, we get

$$\begin{aligned} \langle \widehat{\Phi}(p^*(c) + \widetilde{\psi}(\eta)), \iota'(\lambda') + \psi'(\kappa')\rangle &= \langle p^*(c) + \widetilde{\psi}(\eta), \iota'((\varphi_A \otimes \varphi_B)^*(\lambda') + \Psi(\kappa')) + \psi'(\varphi_C^*(\kappa'))\rangle \\ &= \langle (\varphi_A \otimes \varphi_B(\eta), \lambda'\rangle + \langle \Psi(\eta), \kappa'\rangle + \langle \varphi_C(c), \kappa'\rangle \\ &= \langle p^*(\varphi_C(c) + \Psi(\eta)) + \widetilde{\psi}((\varphi_A \otimes \varphi_B)(\eta)), \iota'(\lambda') + \psi'(\kappa')\rangle. \end{aligned}$$

Thus, $\widehat{\Phi}(c, \eta) = ((\varphi_A \otimes \varphi_B)(\eta), \varphi_C(c) + \Psi(\eta))$.

■

C.6 The Whitney sum in the DVB category

In this section we briefly discuss the Whitney sum of two DVB's over a common side bundle. We describe its second side bundle, the core and linear bundles, and the induced horizontal lifts. The main reason we introduce this material is to understand the main example that prompts the main aspects of the theory developed in this work: $TA \oplus_A T^*A$.

Proposition C.42. *Let $(D; A, B; M)_C$ and $(D'; A, B'; M)_{C'}$ be double vector bundles with one side bundle in common, A. Then the pull-back bundle*

$$D \underset{(q_A, A, q'_A)}{\times} D',$$

denoted simply by $D \underset{A}{\oplus} D'$ is again a double vector bundle with side bundles A and $B \underset{(q^B, M, q^{B'})}{\times} B' = B \oplus B'$, and core bundle $C \underset{(q^C, M, q^{C'})}{\times} C' = C \oplus C'$.

Proof. We need to describe the vector bundle structure over $B \oplus B'$. The projection is

$$q_{B \oplus B'} := (q_B, q_{B'}),$$

and the zero section is given by

$$0_{B \oplus B'} := (\widetilde{0}_B, \widetilde{0}_{B'}),$$

where

$$\widetilde{0}_B : B \oplus B' \xrightarrow{pr_1} B \xrightarrow{0_B} D,$$
$$\widetilde{0}_{B'} : B \oplus B' \xrightarrow{pr_2} B' \xrightarrow{0_{B'}} D'.$$

Now, for $(d_1, d'_1), (d_2, d'_2)$, with $q_{B \oplus B'}(d_1, d'_1) = q_{B \oplus B'}(d_2, d'_2)$, addition is given by

$$(d_1, d'_1) \underset{B \oplus B'}{+} (d_2, d'_2) := (d_1 \underset{B}{+} d_2, d'_1 \underset{B'}{+} d'_2).$$

It is easy to verify that adapted coordinate systems for D and D' provide an adapted coordinate system for $D \underset{A}{\times} D'$. From this observation follows the rest of the statement. ∎

Proposition C.43. *Let D, D' be double vector bundles with a common side bundle, A. Then there is a natural isomorphism*

$$\Gamma_{\text{lin}}((D \underset{A}{\times} D')_{B \oplus B'}) \cong \Gamma_{\text{lin}}(D_B) \underset{A}{\times} \Gamma_{\text{lin}}(D'_{B'}), \tag{C.70}$$

where

$$\Gamma_{\text{lin}}(D_B) \underset{A}{\times} \Gamma_{\text{lin}}(D'_{B'}) := \{(\sigma, \sigma') \in \Gamma_{\text{lin}}(D_B) \times \Gamma_{\text{lin}}(D'_{B'}) | q_A \circ \sigma = q'_A \circ \sigma'\}.$$

Consider the linear bundles corresponding to $\Gamma_{\text{lin}}(D_B)$ and $\Gamma_{\text{lin}}(D'_{B'})$, respectively, fitting in the exact sequences

$$B^* \otimes C \xrightarrow{\iota} \widehat{A} \xrightarrow{p} A, \qquad (B')^* \otimes C' \xrightarrow{\iota'} \widehat{A}' \xrightarrow{p'} A.$$

Then the pull-back bundle

$$\widehat{A} \underset{(p, A, p')}{\times} \widehat{A}',$$

which we will denote simply by $\widehat{A} \underset{A}{\times} \widehat{A}'$, *fits in the exact sequence*

$$(B^* \otimes C) \oplus ((B')^* \otimes C') \xrightarrow{(\widetilde{\iota},\widetilde{\iota'})} \widehat{A} \underset{(p,A,p')}{\times} \widehat{A}' \xrightarrow{\pi} A,$$

where

$$\widetilde{\iota} = \iota \circ pr_1 : (B^* \otimes C) \oplus ((B')^* \otimes C') \longrightarrow \widehat{A},$$
$$\widetilde{\iota'} = \iota \circ pr_2 : (B^* \otimes C) \oplus ((B')^* \otimes C') \longrightarrow \widehat{A}'$$

and

$$\pi = p \circ pr_1 = p' \circ pr_2.$$

Finally, there is a canonical 1:1 correspondence between pairs of horizontal lifts ψ, ψ' *of* $\widehat{A}$ *and* $\widehat{A}'$, *respectively, and horizontal lifts*

$$(\psi, \psi') : A \longrightarrow \widehat{A} \underset{A}{\times} \widehat{A}'.$$

Proof. The proof consists in very simple verifications that we leave as an exercise. ∎

The following corollary is an immediate consequence of Prop. C.43, and its proof again we leave as a simple verification exercise.

Corollary C.44. *The linear bundle corresponding to* $\Gamma_{\text{lin}}((D \underset{A}{\times} D')_{B \oplus B'})$ *is given by*

$$(B^* \otimes C') \oplus ((B')^* \otimes C) \oplus (\widehat{A} \underset{A}{\times} \widehat{A}'),$$

the corresponding linear sequence is

$$(B \oplus B')^* \otimes (C \oplus C') \longrightarrow (B^* \otimes C') \oplus ((B')^* \otimes C) \oplus (\widehat{A} \underset{A}{\times} \widehat{A}') \longrightarrow A$$

and a horizontal lift of this sequence is equivalent to a pair of horizontal lifts (ψ, ψ') *of* $\widehat{A}$ *and* $\widehat{A'}$, *respectively.*

∎

Appendix D

Double Realization

In Prop. C.38 we saw that the right vector bundle which "reads" the information of a double vector bundle is the dual of the double-linear bundle:

$$C^\infty_{\text{lin}}(D)^* \cong (\widehat{C^*}_A)^* \cong (\widehat{C^*}_B)^*,$$

since it is in this bundle that we can naturally induce a vector bundle morphism from a DVB morphism. This motivates the introduction of the concept of a DVB-sequence. The material on this appendix is based, up to some changes on the emphasis and the approach, on [12].

D.1 Double vector sequences and the double realization process

Definition D.1. An exact sequence of vector bundles over M,

$$0 \longrightarrow C \xrightarrow{\iota} \Omega \xrightarrow{p} A \otimes B \longrightarrow 0 \tag{D.1}$$

over the identity map Id_M is called a *double vector sequence*, or briefly a *DVB sequence*, and will be denoted by $(\Omega \longrightarrow A \otimes B; M)_C$.

We refer to A, B as side bundles, and C the core. We usually identify C with $\iota(C)$.

Definition D.2. A morphism of DVB sequences, or simply a *DVS morphism*

$$(\Phi; \varphi_A, \varphi_B; \varphi_M) : \ (\Omega \xrightarrow{p} A \otimes B; M)_C \longrightarrow (\Omega' \xrightarrow{p'} A' \otimes B'; M')_{C'},$$

consists of maps $\Phi : \Omega \longrightarrow \Omega'$, $\varphi_A : A \longrightarrow A'$, $\varphi_B : B \longrightarrow B'$ and $\varphi_M : M \longrightarrow M'$, each of (Φ, φ_M), (φ_A, φ_M), (φ_B, φ_M) is a morphism of the relevant vector bundles, such that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{p} & A \otimes B \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\varphi_A \otimes \varphi_B} \\ \Omega' & \xrightarrow{p'} & A' \otimes B' \end{array} \tag{D.2}$$

commutes.

Remark D.3. In the definition above, $\varphi_C := \Phi|_C$ is also a morphism of vector bundles over φ_M.

DVB sequences together with the above morphisms form a category.

Proposition D.4 ([12])**.** *Given a double vector sequence $(\Omega \xrightarrow{p} A \otimes B; M)_C$, there is an associated double vector bundle*

$$\begin{tikzcd} D(\Omega) \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd}$$

Here $D(\Omega)$ is given by

$$D(\Omega) = \{(\omega, a, b) \in \Omega \oplus A \oplus B | p(\omega) = a \otimes b\}.$$

The projections are given by

$$q_A(\omega, a, b) = a; \qquad q_B(\omega, a, b) = b.$$

Proof. First let's show that $D(\Omega) \longrightarrow A$ is a vector bundle. Consider the vector bundle over A: $\Omega \oplus A \oplus B \longrightarrow A$, and the map

$$\widetilde{p} : \Omega \oplus A \oplus B \longrightarrow (q^A)^*(A \otimes B), \tag{D.3}$$
$$(\omega, a, b) \longrightarrow (a, p(\omega) - a \otimes b). \tag{D.4}$$

$\widetilde{p}$ is a vector bundle map over Id_A, since

$$\begin{aligned} \widetilde{p}((\omega_1, a, b_1) \underset{A}{+} (\omega_2, a, b_2)) &= \widetilde{p}(\omega_1 + \omega_2, a, b_1 + b_2) \\ &= (a, p(\omega_1 + \omega_2) - a \otimes (b_1 + b_2)) \\ &= (a, p(\omega_1) + p(\omega_2) - (a \otimes b_1 + a \otimes b_2)) \\ &= (a, p(\omega_1) - a \otimes b_1) \underset{A}{+} (a, p(\omega_2) - a \otimes b_2) \\ &= \widetilde{p}(\omega_1, a, b_1) \underset{A}{+} \widetilde{p}(\omega_2, a, b_2). \end{aligned}$$

Also, since p is surjective, it follows that $\mathrm{im}(\widetilde{p}|_{\Omega \oplus A}) = (q^A)^*(A \otimes B)$, and

$$\ker \widetilde{p} = D(\Omega),$$

which implies that $D(\Omega) \longrightarrow A$ is a vector bundle. Analogously it is shown that $D(\Omega) \longrightarrow B$ is a vector bundle.

The zero sections are

$$0_A(a) = (0, a, 0); \qquad 0_B(b) = (0, 0, b).$$

The compatibility of the projections is easily verified:

$$\begin{aligned} q_A((\omega_1, a_1, b) \underset{B}{+} (\omega_2, a_2, b)) &= q_A(\omega_1 + \omega_2, a_1 + a_2, b) = a_1 + a_2 \\ &= q_A(\omega_1, a_1, b) + q_A(\omega_2, a_2, b). \end{aligned}$$

Finally, the interchange law again is easy:

$$\begin{aligned} ((\omega_1, a_1, b_1) \underset{A}{+} (\omega_2, a_1, b_2)) \underset{B}{+} ((\omega_3, a_2, b_1) \underset{A}{+} (\omega_4, a_2, b_2)) = \\ &= (\omega_1 + \omega_2, a_1, b_1 + b_2) \underset{B}{+} (\omega_3 + \omega_4, a_2, b_1 + b_2) \\ &= (\omega_1 + \omega_2 + \omega_3 + \omega_4, a_1 + a_2, b_1 + b_2) \\ &= (\omega_1 + \omega_3, a_1 + a_2, b_1) \underset{A}{+} (\omega_2 + \omega_4, a_1 + a_2, b_2) \\ &= ((\omega_1, a_1, b_1) \underset{B}{+} (\omega_3, a_2, b_1)) \underset{A}{+} ((\omega_2, a_1, b_2) \underset{B}{+} (\omega_4, a_2, b_2)). \end{aligned}$$

It remains to show that the core bundle is C, which follows from observing that $(\omega, a, b) \in \ker q_A \cap q_B$ if and only if $a = b = 0$ and $p(\omega) = 0$, that is, if and only if $(\omega, a, b) = (\iota(c), 0, 0)$ for some $c \in C$.

■

Proposition D.5 ([12]). *For any morphism of double vector sequences*

$$(\Phi; \varphi_A, \varphi_B; \varphi_M) : \ (\Omega \xrightarrow{p} A \otimes B; M)_C \longrightarrow (\Omega' \xrightarrow{p'} A' \otimes B'; M')_{C'},$$

define $D(\Phi) : D(\Omega) \longrightarrow D(\Omega')$ *by*

$$D(\Phi)(\omega, a, b) = (\Phi(\omega), \varphi_A(a), \varphi_B(b)),$$

for any $(\omega, a, b) \in D(\Omega)$. *Then*

$$(D(\Phi); \varphi_A, \varphi_B; \varphi_M)$$

is a morphism of double vector bundles.

Proof. First we need to check that $D(\Phi)$ is well-defined. Let $(\omega, a, b) \in D(\Omega)$, then $p(\omega) = a \otimes b$. Since the diagram (D.2) commutes, we have

$$\begin{aligned} p'(\Phi(\omega)) &= (\varphi_A \otimes \varphi_B)(p(\omega)) = (\varphi_A \otimes \varphi_B)(a \otimes b) \\ &= \varphi_A(a) \otimes \varphi_B(b). \end{aligned}$$

Thus $(\Phi(\omega), \varphi_A(a), \varphi_B(b)) \in D(\Omega')$, and so $D(\Phi)$ is well-defined indeed.

The verifications that $D(\Phi)$ is a vector bundle morphism with respect to both structures follow directly from the definitions.

■

D.2 The inverse process

Recall that we have a process to pass from a double vector bundle $(D; A, B; M)_C$ to a DVB sequence, namely considering the transpose of the sequence (C.45):

$$0 \longrightarrow C \longrightarrow C^{\infty}_{\text{lin}}(D)^* \longrightarrow A \otimes B \longrightarrow 0,$$

On the other hand, we saw in the previous section a process to pass from a DVB sequence $(\Omega \xrightarrow{p} A \otimes B; M)_C$ to a double vector bundle, given by Prop. D.4, $D(\Omega)$. In the next proposition we show that these two processes are mutually inverse.

Proposition D.6. *Let* $(\Omega \xrightarrow{p} A \otimes B; M)_C$ *be a double vector sequence. Then there is a canonical isomorphism*

$$(C^{\infty}_{\text{lin}}(D(\Omega))^* \xrightarrow{p} A \otimes B) \cong (\Omega \xrightarrow{p} A \otimes B). \tag{D.5}$$

Conversely, if $(D; A, B; M)_C$ *is a double vector bundle, then there is a canonical double vector bundle isomorphism*

$$D(C^{\infty}_{\text{lin}}(D)^*) \cong D. \tag{D.6}$$

Proof. In order to prove (D.5), we will build a map $\Phi : \Omega^* \longrightarrow C^{\infty}_{\text{lin}}(D(\Omega))$ and verify that it is a vector bundle isomorphism over the identity, which induces a DVS isomorphism

$$(\Phi^*; \text{Id}_A, \text{Id}_B; \text{Id}_M) : (C^{\infty}_{\text{lin}}(D(\Omega))^* \xrightarrow{p} A \otimes B) \xrightarrow{\cong} (\Omega \xrightarrow{p} A \otimes B). \tag{D.7}$$

Given $\xi \in \Omega^*_m,\ m \in M$, define

$$\Phi(\xi) = \nu_\xi \in C^{\infty}_{\text{lin}}(D(\Omega))_m, \qquad \nu_\xi(\omega, a, b) := \xi(\omega),\ \forall (\omega, a, b) \in D(\Omega)_m.$$

Let's check that ν_ξ is actually a double linear function in $D(\Omega)_m$.

$$\begin{aligned}\nu_\xi((\omega_1, a_1, b) \underset{B}{+} (\omega_2, a_2, b)) &= \nu_\xi(\omega_1 + \omega_2, a_1 + a_2, b) = \xi(\omega_1 + \omega_2)\\ &= \xi(\omega_1) + \xi(\omega_2) = \nu_\xi(\omega_1, a_1, b) + \nu_\xi(\omega_2, a_2, b).\end{aligned}$$

Linearity with respect to the structure over A is analogous.

Hence the map Φ is well-defined. To verify that it is a vector bundle morphism, we compute

$$\nu_{\xi_1+\xi_2}(\omega, a, b) = (\xi_1 + \xi_2)(\omega) = \xi_1(\omega) + \xi_2(\omega) = \nu_{\xi_1}(\omega, a, b) + \nu_{\xi_2}(\omega, a, b).$$

Thus Φ is a vector bundle morphism over Id_M. Since Ω^* and $C^{\infty}_{\text{lin}}(D(\Omega))$ have the same rank, in order to check that Φ is an isomorphism, it remains to check only injectivity. For this purpose, let's take bases $\{a_i\}$ and $\{b_j\}$ for A_m and B_m, for some $m \in M$. Since $p : \Omega \longrightarrow A \otimes B$ is surjective, we can find also find $\omega_{ij} \in \Omega_m$ such that $p(\omega_{ij}) = a_i \otimes b_j$. We complete to a base $\{\omega_{ij}, c_k\}$ for Ω_m, where $\{c_k\}$ is a base for $\ker p_m$. Then, if $\Phi(\xi) = \nu_\xi = 0$ for some $\xi \in \Omega^*$, we get

$$\xi(\omega_{ij}) = \nu_\xi(\omega_{ij}, a_i, b_j) = 0 \quad \text{and} \quad \xi(c_k) = \nu_\xi(c_k, 0, 0) = 0,$$

which implies $\xi = 0$.

Thus, Φ is a vector bundle isomorphism. To check that $(\Phi^*, \mathrm{Id}_A, \mathrm{Id}_B, \mathrm{Id}_M)$ in (D.7) is a DVS isomorphism, we need to check commutativity of the diagram

$$\begin{tikzcd} C^\infty_{\mathrm{lin}}(D(\Omega))^* \arrow[r, "\iota^*"] \arrow[d, "\Phi^*"'] & A \otimes B \arrow[d, "\mathrm{Id}"] \\ \Omega \arrow[r, "p"] & A \otimes B. \end{tikzcd}$$

For $\eta \in C^\infty_{\mathrm{lin}}(D(\Omega))^*$ and $\lambda \in A^* \otimes B^*$ we have

$$\langle p \circ \Phi^*(\eta), \lambda \rangle = \langle \eta, \Phi \circ p^*(\lambda) \rangle. \tag{D.8}$$

On the other hand

$$\langle \iota^*(\eta), \lambda \rangle = \langle \eta, \iota(\lambda) \rangle. \tag{D.9}$$

Then we need to show that $\Phi \circ p^*(\lambda) = \iota(\lambda)$ $\forall \lambda \in A^* \otimes B^*$. Now,

$$\begin{aligned} \Phi \circ p^*(\lambda)(\omega, a, b) &= \langle p^*(\lambda)\omega \rangle = \langle \lambda, p(\omega) \rangle \\ &= \langle \lambda, a \otimes b \rangle = \langle \lambda, q_A(\omega, a, b) \otimes q_B(\omega, a, b) \rangle \\ &= \iota(\lambda)(\omega, a, b), \end{aligned}$$

as we wanted.

Now, given a double vector bundle $(D; A, B; M)_C$, we want to show (D.6). So, now we need a map $\Phi : D \longrightarrow D(C^\infty_{\mathrm{lin}}(D)^*)$. Recall that

$$D(C^\infty_{\mathrm{lin}}(D)^*) = \{(\eta, a, b) \in C^\infty_{\mathrm{lin}}(D)^* \oplus A \oplus B | \iota^*(\eta) = a \otimes b\}.$$

For $m \in M$, let $d \in D_m$ with $q_A(d) = a$ and $q_B(d) = b$. Define $\eta \in C^\infty_{\mathrm{lin}}(D)^*_m$, depending on $d \in D_m$, by

$$\langle \eta, \mu \rangle := \mu(d), \qquad \forall \mu \in C^\infty_{\mathrm{lin}}(D)_m.$$

We claim that $(\eta, a, b) \in D(C^\infty_{\mathrm{lin}}(D)^*)_m$. We need to verify $\iota^*(\eta) = a \otimes b$. So let $\lambda \in A^* \otimes B^*$, and compute

$$\langle \iota^*(\eta), \lambda \rangle = \langle \iota(\lambda), \eta \rangle = \iota(\lambda)(d) = \langle q_A(d) \otimes q_B(d), \lambda \rangle = \langle a \otimes b, \lambda \rangle.$$

Hence, we have a well-defined map $\Phi : D \longrightarrow D(C^\infty_{\mathrm{lin}}(D)^*)$. Directly from the definition we see that Φ preserves both fibrations. It also preserves the linear structures, since

$$\Phi(d_1 \underset{A}{+} d_2) = (\eta, a, b_1 + b_2),$$

where $\eta(\mu) = \mu(d_1 \underset{A}{+} d_2) = \mu(d_1) + \mu(d_2) =: \eta_1 + \eta_2$. Then

$$\Phi(d_1 \underset{A}{+} d_2) = (\eta_1 + \eta_2, a, b_1 + b_2) = (\eta_1, a, b_1) + (\eta_2, a, b_2) = \Phi(d_1) \underset{A}{+} \Phi(d_2).$$

Analogously, we have $\Phi(d_1 \underset{B}{+} d_2) = \Phi(d_1) \underset{B}{+} \Phi(d_2)$.

Now, since, by definition of Φ, $\varphi_A = \mathrm{Id}_A$ and $\varphi_B = \mathrm{Id}_B$, in order to show that Φ is an isomorphism, it suffices to show that $\varphi_C = \mathrm{Id}_C$. Let $c \in C_m \subset D_m$, then $q_A(c) = 0^A(m)$ and $q_B(c) = 0^B(m)$. Define $\eta \in C^\infty_{\mathrm{lin}}(D)^*_m$ by $\langle \eta, \mu \rangle := \mu(c)$, for every $\mu \in C^\infty_{\mathrm{lin}}(D)_m$. Since $\iota^*(\eta) = 0$, it follows that $\eta \in \ker \iota^* = p^*(C^{**}) \cong C$. Under this identification, the definition of η shows that $\eta = c$. Thus, taking that identification into account, we have $\Phi(c) = (c, 0, 0)$, which means that $\varphi_C = \mathrm{Id}_C$. Therefore Φ is a double vector bundle isomorphism.

$\blacksquare$

Corollary D.7. *Given an exact sequence of vector bundles*

$$0 \longrightarrow B^* \otimes C \longrightarrow \widehat{A} \longrightarrow A \longrightarrow 0, \tag{D.10}$$

there is a unique, up to canonical isomorphisms, DVB

$$\begin{tikzcd} D(\widehat{A}) \arrow[r, "q_B"] \arrow[d, "q_A"'] \arrow[dr, phantom, "C"] & B \arrow[d, "q^B"] \\ A \arrow[r, "q^A"] & M \end{tikzcd}$$

such that its linear bundle corresponding to $\Gamma_{\mathrm{lin}}(D_B)$ *is (canonically isomorphic to)* $\widehat{A}$.

Proof. Consider the transpose of (D.10)

$$0 \longrightarrow A^* \longrightarrow \widehat{A}^* \longrightarrow B \otimes C^*.$$

Then, by Prop. D.4, we obtain a double vector bundle

$$\begin{tikzcd} D(\widehat{A}^*) \arrow[r, "q_{C^*}"] \arrow[d, "q_B"'] \arrow[dr, phantom, "A^*"] & C^* \arrow[d, "q^{C^*}"] \\ B \arrow[r, "q^B"] & M \end{tikzcd}$$

By Prop. D.6, we have $C^\infty_{\mathrm{lin}}(D(\widehat{A}^*))^* \cong \widehat{A}^*$, thus $C^\infty_{\mathrm{lin}}(D(\widehat{A}^*)) \cong \widehat{A}$. Then, by Prop. C.32, the linear bundle corresponding to $\Gamma_{\mathrm{lin}}(D(\widehat{A}^*)^*_B)$ is isomorphic to $\widehat{A}$. Therefore, $D(\widehat{A}) := D(\widehat{A^*})^*_B$ satisfies the requirements of the statement.

$\blacksquare$

With the above material at hand (Props. D.4 and D.6), the following proposition, which establishes the equivalence of categories between double vector bundles and DVB sequences, becomes an exercise in category theory. A proof can be found in [12]. This result will allow to obtain the equivalence between the category of degree 2 manifolds and the category of *involutive DVB's* in Sec. 3.3.

Theorem D.8 ([12])**.** *The correspondence*

$$\begin{aligned} \mathfrak{D} : (\Omega \xrightarrow{p} A \otimes B; M)_C &\rightsquigarrow (D(\Omega); A, B; M)_C \\ \mathfrak{D} : (\Phi; \varphi_A, \varphi_B; \varphi_M) &\rightsquigarrow (D(\Phi); \varphi_A, \varphi_B; \varphi_M) \end{aligned}$$

is a covariant functor from the category of double vector sequences to the category of double vector bundles.

The correspondence given in Prop. C.38,

$$\begin{aligned} \mathfrak{S} : (D; A, B; M)_C &\rightsquigarrow (C^\infty_{\mathrm{lin}}(D)^* \xrightarrow{p} A \otimes B; M)_C \\ \mathfrak{S} : (\Phi; \varphi_A, \varphi_B; \varphi_M) &\rightsquigarrow (\widehat{\Phi}; \varphi_A, \varphi_B; \varphi_M) \end{aligned}$$

is a covariant functor from the category of double vector bundles to the category of double vector sequences.

These functors give an equivalence of categories, that is, there are natural transformations

$$\delta : \mathrm{Id}_{DVB} \longrightarrow \mathfrak{D} \circ \mathfrak{S} \quad \textit{and} \quad \epsilon : \mathfrak{S} \circ \mathfrak{D} \longrightarrow \mathrm{Id}_{DVS},$$

where Id_{DVB} *is the identity functor on the category of double vector bundles, and* Id_{DVS} *is the identity functor on the category of double vector sequences.*

Definition D.9. The functor $\mathfrak{D}$ is called *double realization.*

Appendix E

More on VB-algebroids and representations up to homotopy

In this appendix we expand, providing full details, the recent theory of VB-algebroids and (2-term) representations up to homotopy (see Sec. 2.2), developed in [23]. The characterization of VB-algebroids in terms of data on a linear bundle and its core bundle (Prop. E.9), is the inspiration, and guide, to characterize NQ degree 2 manifolds in terms of geometric data in Ch. 4. Also the duality theory, which we develop thoroughly in Sec. E.3, will enable us to characterize degree -2 Poisson brackets on a degree 2 manifold, in terms of the so-called *metric VB-algebroid* introduced in D. Li-Bland's thesis [41], and treated through splittings by M. Jotz in [29], where they actually receive this name.

E.1 Structure of VB-algebroids

In view of Thm. D.8, it is natural to have a characterization of VB-algebroids in terms of the exact sequence (C.1). In the next proposition we give the first step towards such characterization, which will be given in Prop. E.9.

Proposition E.1 ([23]). *Consider a VB-algebroid, and its linear bundle $\widehat{A}$. Under the isomorphism $\Gamma_{\mathrm{lin}}(D_B) \cong \Gamma(\widehat{A})$, given in Prop. C.2, we can endow $\widehat{A}$ with a natural Lie algebroid structure, with bracket $[\cdot,\cdot]_{\widehat{A}}$ and anchor $\rho_{\widehat{A}}$ given by*

$$\begin{aligned} [X,Y]_{\widehat{A}} &:= [X,Y]_D, \\ \rho_{\widehat{A}}(X) &:= \rho_A(\alpha), \end{aligned} \tag{E.1}$$

where X projects over $\alpha \in \Gamma(A)$.

Moreover, since $[X,Y]$ is a linear section over $p([X,Y]) \in \Gamma(A)$ we can endow A with a natural Lie algebroid structure with anchor ρ_A and brackets given by

$$[\alpha_1,\alpha_2]_A := p([\widehat{\alpha_1},\widehat{\alpha_2}]_{\widehat{A}}), \tag{E.2}$$

where $\widehat{\alpha_i} \in \Gamma(D_B)$ is any horizontal lift of α_i.

In particular, the map $p : \widehat{A} \longrightarrow A$ is a Lie algebroid morphism, and the exact sequence (C.1) becomes a Lie algebroid sequence.

Proof. The proof consists of simple verifications. Here we carry out only the less obvious ones. Let $f \in C^\infty(M)$, then, since $\rho_D : D \longrightarrow TB$ is a vector bundle morphism over $\rho_A : A \longrightarrow TM$, and committing some abuse of notation because of the identification $\Gamma_{\text{lin}}(D_B) \cong \Gamma(\widehat{A})$, we have

$$\begin{aligned}[X, fY]_{\widehat{A}} = [X, ((q^B)^*f)Y]_D &= ((q^B)^*f)[X,Y]_D + \rho_D(X)((q^B)^*f)Y \\ &= f[X,Y]_{\widehat{A}} + \rho_A(\alpha)(f)Y \\ &= f[X,Y]_{\widehat{A}} + \rho_{\widehat{A}}(X)(f)Y.\end{aligned}$$

Hence $([\cdot,\cdot]_{\widehat{A}}, \rho_{\widehat{A}})$ actually provides a Lie algebroid structure on $\widehat{A}$.

We claim that $\iota_A(B^* \otimes C) \subset \widehat{A}$ is a Lie algebroid ideal. By linearity and Leibniz rule of $[\cdot,\cdot]_{\widehat{A}}$, it is enough to show the assertion for elements of the form $\iota_A(\beta \otimes \mathbf{c})$, with $\beta \in \Gamma(B^*)$ and $\mathbf{c} \in \Gamma(C)$. But in this situation, by definition it follows that $\iota_A(\beta \otimes \mathbf{c})$ is just the product $\beta \cdot \widetilde{\mathbf{c}}$, where β is seen as a linear function on B and $\widetilde{\mathbf{c}}$ is the core section corresponding to $\mathbf{c}$. Then we can apply the Leibniz rule to get, for $X \in \Gamma(\widehat{A})$ with $p(X) = \alpha \in \Gamma(A)$,

$$[X, \iota_A(\beta \otimes \mathbf{c})]_{\widehat{A}} = \rho_D(X)(\beta) \cdot \widetilde{\mathbf{c}} + \beta \cdot [X, \widetilde{\mathbf{c}}]_D. \tag{E.3}$$

Now, since ρ_D is a double vector bundle morphism, it takes linear sections of D_B to linear sections of TB, which implies that $\rho_D(X) \in \Gamma(TB)$ is a linear tangent field, which means that preserves linear functions. Then $\rho_D(X)(\beta)$ is again a linear function on B. Thus,

$$\rho_D(X)(\beta) \cdot \widetilde{\mathbf{c}} \in \Gamma(\iota_A(B^* \otimes C));$$

also $\beta \cdot [X, \widetilde{\mathbf{c}}]_D \in \Gamma(\iota_A(B^* \otimes C))$, since $[\Gamma_{\text{lin}}(D_B), \Gamma_{\text{core}}(D_B)]_D \subset \Gamma_{\text{core}}(D_B)$. Hence $\iota_A(B^* \otimes C)$ is an ideal of $\widehat{A}$, as we had asserted.

Noticing that, by (E.1), $\iota_A(B^* \otimes C) \subset \ker \rho_{\widehat{A}}$, it follows that $A \cong \widehat{A}/\ker p = \widehat{A}/\iota_A(B^* \otimes C)$ inherits a natural Lie algebroid structure characterized by the condition that $p : \widehat{A} \longrightarrow A$ is a Lie algebroid morphism, which implies in particular that the Lie brackets on A are given by (E.2) and the anchor is given by ρ_A.

■

Definition E.2. The vector bundle $\widehat{A}$ endowed with its Lie algebroid structure is called the *fat algebroid*.

Corollary E.3. *The vector bundle $B^* \otimes C$ inherits a Lie algebroid structure with zero anchor map, and brackets given by*

$$[\phi, \phi'] := [\iota_A(\phi), \iota_A(\phi')]_{\widehat{A}}.$$

Definition E.4. Recall, from remark 2.11 that $\rho_D : D \longrightarrow TB$ is a double vector bundle morphism. Then by Prop. A.9, we have a vector bundle map $\rho_D|_C : C \longrightarrow B$, where $B \subset TB$ is seen as the "vertical bundle", which is precisely the core bundle of TB. We define the *core-anchor* by $\partial := -\rho_D|_C$.

Proposition E.5 ([23])**.** *The core-anchor $\partial : C \longrightarrow B$ is given by*

$$(q^B)^* \langle \partial(\mathbf{c}), \beta \rangle = -\rho_D(\widetilde{\mathbf{c}})(\beta), \tag{E.4}$$

for $\mathbf{c} \in \Gamma(C)$ and $\beta \in \Gamma(B^*)$, where the sections of B^* are identified with the linear functions on B.

The brackets on $B^* \otimes C$ (see Cor. E.3) are given explicitly by

$$[\phi, \phi'] = \phi \partial \phi' - \phi' \partial \phi. \tag{E.5}$$

Proof. Equation (E.4) follows directly from the definition and the way vertical tangent vectors act. In order to prove (E.5), we observe, as we did in the proof of Prop. E.1, that it is enough to prove it for elements of the form $\beta \otimes \mathbf{c} \in B^* \otimes C$, because of linearity and Leibniz rule of the brackets. Now, by Eq. (E.3), and since $[\Gamma_{\text{core}}(D_B), \Gamma_{\text{core}}(D_B)] = 0$, for $\phi = \beta \otimes \mathbf{c},\ \phi' = \beta' \otimes \mathbf{c}'$, we have

$$\begin{aligned}
[\phi, \phi'] = [\beta \otimes \mathbf{c}, \beta' \otimes \mathbf{c}'] &= \rho_D(\iota_A(\beta \otimes \mathbf{c}))(\beta') \cdot \mathbf{c}' + \beta' \cdot [\iota_A(\beta \otimes \mathbf{c}), \widetilde{\mathbf{c}'}]_D \\
&= \beta \cdot \rho_D(\widetilde{\mathbf{c}})(\beta')\mathbf{c}' - \beta' \cdot \rho_D(\widetilde{\mathbf{c}'})(\beta)\mathbf{c} \\
&= -\langle \partial(\mathbf{c}), \beta' \rangle \beta \otimes \mathbf{c}' + \langle \partial(\mathbf{c}'), \beta \rangle \beta' \otimes \mathbf{c} \\
&= \phi \partial \phi' - \phi' \partial \phi,
\end{aligned}$$

as we wanted.

■

E.1.1 Lie algebroid representations; A-connections.

Definition E.6. Let A be a Lie algebroid over M. An *A-connection on a vector bundle E over M* is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E),\ (\alpha, \varepsilon) \longrightarrow \nabla_\alpha \varepsilon$ such that:

$$\nabla_{f\alpha} \varepsilon = f \nabla_\alpha \varepsilon,\ \nabla_\alpha f \varepsilon = f \nabla_\alpha \varepsilon + \rho(\alpha)(f)\varepsilon,$$

for all $f \in C^\infty(M),\ \varepsilon \in \Gamma(E)$ and $\alpha \in \Gamma(A)$. The *A-curvature of* ∇ is the tensor given by

$$R_\nabla(\alpha, \beta)(\varepsilon) := \nabla_\alpha \nabla_\beta \varepsilon - \nabla_\beta \nabla_\alpha \varepsilon - \nabla_{[\alpha,\beta]} \varepsilon,$$

for all $\alpha, \beta \in \Gamma(A),\ \varepsilon \in \Gamma(E)$. The A-connection ∇ is called *flat* if $R_\nabla = 0$. A *representation of* A is a vector bundle E together with a flat A-connection ∇ on E.

Proposition E.7 ([23])**.** *The fat algebroid has natural representations (flat connections) ζ^C and ζ^{B^*} on C and B^*, respectively, given by*

$$\widetilde{\zeta^C_X(\mathbf{c})} := [X, \widetilde{\mathbf{c}}]_D, \tag{E.6}$$
$$\zeta^{B^*}_X(\beta) := \rho_D(X)(\beta), \tag{E.7}$$

for $X \in \Gamma(\widehat{A}), \mathbf{c} \in \Gamma(C)$ and $\beta \in \Gamma(B^*)$, where, for $\mathbf{c} \in \Gamma(C)$, $\widetilde{\mathbf{c}} \in \Gamma_{\text{core}}(D_B)$ is the corresponding core section.

Dualizing the representation ζ^{B^*} to a representation ζ^B on B by

$$\langle \zeta_X^B(\mathbf{b}), \beta \rangle := \rho_{\widehat{A}}(X)\langle \mathbf{b}, \beta \rangle - \langle \mathbf{b}, \zeta_X^{B^*}(\beta) \rangle, \tag{E.8}$$

we obtain the following relations between ζ^C and ζ^B:

$$\partial \zeta_X^C = \zeta_X^B \partial; \tag{E.9}$$

$$\phi \zeta_X^B - \zeta_X^C \phi = [\phi, X]_{\widehat{A}}, \tag{E.10}$$

for all $X \in \Gamma(\widehat{A})$ and $\phi \in \mathrm{Hom}(B, C) \cong B^* \otimes C$.

Proof. For $f \in C^\infty(M)$, note that $fX \in \Gamma(\widehat{A})$ corresponds, under the isomorphism $\Gamma(\widehat{A}) \cong \Gamma_{\mathrm{lin}}(D_B)$, to $((q^B)^* f)X \in \Gamma_{\mathrm{lin}}(D_B)$. Similarly, under the isomorphism $\Gamma(C) \cong \Gamma_{\mathrm{core}}(C)$, $f\mathbf{c} \in \Gamma(C)$ corresponds to $((q^B)^* f)\widetilde{\mathbf{c}} \in \Gamma_{\mathrm{core}}(D_B)$. Then, using Leibniz rule we have

$$\widetilde{\zeta_{fX}^C(\mathbf{c})} = [((q^B)^* f)X, \widetilde{\mathbf{c}}]_D = ((q^B)^* f)[X, \widetilde{\mathbf{c}}]_D - \rho_D(\widetilde{\mathbf{c}})((q^B)^* f)X = ((q^B)^* f)[X, \widetilde{\mathbf{c}}],$$

for $\rho_D(\mathbf{c})$ is a vertical field on TB, and $(q^B)^* f$ is constant on fibers. Thus $\zeta_{fX}^C = f\zeta_X^C$.

Again by Leibniz rule,

$$\begin{aligned}
\widetilde{\zeta_X^C(f\mathbf{c})} &= [X, ((q^B)^* f)\widetilde{\mathbf{c}}]_D = \rho_D(X)((q^B)^* f)\widetilde{\mathbf{c}} + ((q^B)^* f)[X, \mathbf{c}] \\
&= \rho_{\widehat{A}}\widetilde{(X)(f)}\mathbf{c} + f\widetilde{[X, \mathbf{c}]}.
\end{aligned}$$

Hence, $\zeta_X^C(f\mathbf{c}) = \rho_{\widehat{A}}(X)(f)\mathbf{c} + f\zeta_X^C(\mathbf{c})$. Finally, from the Jacobi property for $[\cdot,\cdot]_D$ it follows that ζ^C is actually a Lie algebroid representation of $\widehat{A}$ on C, that is, a flat $\widehat{A}$-connection on C.

Now, let's see that the equation for ζ^{B^*} also defines a flat $\widehat{A}$-connection, this time on B^*. We already observed in the proof of Prop. E.1 that since $\rho_D(X)$ is a linear vector field for $X \in \Gamma_{\mathrm{lin}}(D_B)$, it preserves linear functions. Now let's compute

$$\zeta_{fX}^{B^*}(\beta) = \rho_D(((q^B)^* f)X)(\beta) = ((q^B)^*(f))\rho_D(X)(\beta) = f\zeta_X^{B^*}(\beta),$$

where the last equality needs to be well understood, since we are committing an abuse of notation, that comes from the identification of sections of B^* with linear functions on B. Next we verify Leibniz rule, where we will commit the same abuse of notation

$$\begin{aligned}
\zeta_X^{B^*}(f\beta) &= \rho_D(X)(((q^B)^* f)\beta) = \rho_D(X)((q^B)^* f)\beta + ((q^B)^* f)\rho_D(X)(\beta) \\
&= \rho_{\widehat{A}}(X)(f)\beta + f\zeta_X^{B^*}(\beta).
\end{aligned}$$

Since $\rho_D : D_B \longrightarrow TB$ preserves Lie brackets, it also follows that ζ^{B^*} is a flat $\widehat{A}$-connection on B^*.

Finally let's prove (E.9) and (E.10).

$$\begin{aligned}
\langle \partial \zeta_X^C(\mathbf{c}), \beta \rangle &= \langle \partial([X, \widetilde{\mathbf{c}}]_D, \beta \rangle = -\rho_D([X, \widetilde{\mathbf{c}}]_D)(\beta) \\
&= -[\rho_D(X), \rho_D(\widetilde{\mathbf{c}})](\beta) = -\rho_D(X)\rho_D(\widetilde{\mathbf{c}})(\beta) + \rho_D(\widetilde{\mathbf{c}})\rho_D(X)(\beta) \\
&= \rho_D(X)\langle \partial(\mathbf{c}), \beta \rangle - \langle \partial(\mathbf{c}), \zeta_X^{B^*}(\beta) \rangle \\
&= \langle \zeta_X^B(\partial(\mathbf{c})), \beta \rangle,
\end{aligned}$$

thence follows (E.9).

Next, in order to avoid confusion due to our abuse of denoting by the same symbol a section $X \in \Gamma(\widehat{A})$ and its corresponding induced linear section of D_B, we will introduce the distinction similar to the one we have been making between sections of C and the corresponding induced core section of D_B. Namely, for $X \in \Gamma(\widehat{A})$ we denote by $\widetilde{X} \in \Gamma_{\text{lin}}(D_B)$ the corresponding linear section. Analogously, we will denote by $\widetilde{f} := (q^B)^* f$ the function on B which corresponds to $f \in C^\infty(M)$. With this arrangement, we are in conditions to perform the necessary calculations which will lead to (E.10) with, hopefully, reasonable clarity.

By linearity, it is enough to prove (E.10) for $\phi = f\beta \otimes \mathbf{c}$, with, as usual, $f \in C^\infty(M)$, $\beta \in \Gamma(B^*)$ and $\mathbf{c} \in \Gamma(C)$. First, observe that (E.10) is equivalent to

$$\widetilde{\phi\zeta_X^B(\mathbf{b})} - \widetilde{\zeta_X^C\phi(\mathbf{b})} = \widetilde{[\phi, X]}_{\widehat{A}}(\mathbf{b}). \tag{E.11}$$

So, we begin computing the left hand-side of the equation above.

$$\begin{aligned}\widetilde{\phi\zeta_X^B(\mathbf{b})} - \widetilde{\zeta_X^C\phi(\mathbf{b})} =& \widetilde{f}\langle\widetilde{\zeta_X^B(\mathbf{b})}, \beta\rangle\widetilde{\mathbf{c}} - [\widetilde{X}, \widetilde{f}\langle\beta, \mathbf{b}\rangle\widetilde{\mathbf{c}}]_D = \widetilde{f}\rho_D(\widetilde{X})\widetilde{\langle\mathbf{b}, \beta\rangle}\widetilde{\mathbf{c}} - \widetilde{f}\langle\mathbf{b}, \widetilde{\rho_D(\widetilde{X})}(\beta)\rangle\widetilde{\mathbf{c}} \\ & - \rho_D(\widetilde{X})(\widetilde{f})\widetilde{\langle\beta, \mathbf{b}\rangle}\widetilde{\mathbf{c}} - \widetilde{f}\rho_D(\widetilde{X})(\widetilde{\langle\beta, \mathbf{b}\rangle})\widetilde{\mathbf{c}} - \widetilde{f\langle\beta, \mathbf{b}\rangle}[\widetilde{X}, \widetilde{\mathbf{c}}]_D \\ =& -\widetilde{f}\langle\mathbf{b}, \widetilde{\rho_D(\widetilde{X})}(\beta)\rangle\widetilde{\mathbf{c}} - \rho_D(\widetilde{X})(\widetilde{f})\widetilde{\langle\beta, \mathbf{b}\rangle}\widetilde{\mathbf{c}} - \widetilde{f\langle\beta, \mathbf{b}\rangle}[\widetilde{X}, \widetilde{\mathbf{c}}]_D.\end{aligned}$$

Now, computing the right hand-side of (E.11) we have

$$\begin{aligned}\widetilde{[\phi, X]}_{\widehat{A}} &= [\iota_A\widetilde{(f\beta \otimes \mathbf{c})}, \widetilde{X}]_D \\ &= -\rho_D(\widetilde{X})(\widetilde{f})\beta \cdot \widetilde{\mathbf{c}} - \widetilde{f}\rho_D(\widetilde{X})(\beta)\widetilde{\mathbf{c}} - \widetilde{f}\beta \cdot [\widetilde{X}, \widetilde{\mathbf{c}}]_D,\end{aligned}$$

hence

$$\widetilde{[\phi, X]}_{\widehat{A}}(\mathbf{b}) = -\rho_D(\widetilde{X})(\widetilde{f})\langle\beta, \mathbf{b}\rangle\widetilde{\mathbf{c}} - \widetilde{f}\langle\widetilde{\rho_D(\widetilde{X})(\beta)}, \mathbf{b}\rangle\widetilde{\mathbf{c}} - \widetilde{f\langle\beta, \mathbf{b}\rangle}[\widetilde{X}, \widetilde{\mathbf{c}}]_D.$$

Comparing both sides we obtain (E.11).

■

Proposition E.8 ([23])**.** *The representations of $\widehat{A}$ introduced on Prop. E.7, can be pulled back to representations of $B^* \otimes C \cong \operatorname{Hom}(B, C)$, θ_ϕ^C, θ_ϕ^B, on C and B, respectively. Explicitly, these representations are given by*

$$\begin{aligned}\theta_\phi^C(\mathbf{c}) &= \phi \circ \partial(\mathbf{c}); \\ \theta_\phi^B(\mathbf{b}) &= \partial \circ \phi(\mathbf{b}),\end{aligned}$$

for $\phi \in \operatorname{Hom}(B, C)$, $\mathbf{c} \in \Gamma(C)$, and $\mathbf{b} \in \Gamma(B)$.

Proof. By $C^\infty(M)$-linearity, it is enough to proof the proposition for $\phi = \beta_1 \otimes \mathbf{c}_1$. We have

$$\begin{aligned}\widetilde{\theta_\phi^C(\mathbf{c})} = \zeta^C_{\widetilde{\iota_A(\phi)}}(\mathbf{c}) &= [\widetilde{\iota_A(\phi)}, \widetilde{\mathbf{c}}]_D = [\beta_1 \cdot \widetilde{\mathbf{c}}_1, \widetilde{\mathbf{c}}]_D \\ &= -\rho_D(\widetilde{\mathbf{c}})(\beta_1) \cdot \widetilde{\mathbf{c}}_1 = \langle\widetilde{\partial(\mathbf{c}), \beta_1}\rangle\widetilde{\mathbf{c}}_1 \\ &= \widetilde{\phi \circ \partial(\mathbf{c})}.\end{aligned}$$

Thus follows $\theta^C_\phi(\mathbf{c}) = \phi \circ \partial(\mathbf{c})$.

To compute θ^B recall that $\iota_A(B^* \otimes C) \subset \ker \rho_{\widehat{A}}$, so we have

$$\begin{aligned}\langle \theta^B_\phi(\mathbf{b}), \beta\rangle &= \langle \zeta^B_{\iota_A(\phi)}(\mathbf{b}), \beta\rangle = -\langle \mathbf{b}, \rho_D(\beta_1 \cdot \widetilde{\mathbf{c}_1})(\beta)\rangle \\ &= -\langle \mathbf{b}, \beta_1\rangle \rho_D(\widetilde{\mathbf{c}_1})(\beta) \\ &= \langle \mathbf{b}, \beta_1\rangle \langle \partial(\mathbf{c}_1), \beta\rangle.\end{aligned}$$

Hence, $\theta^B_\phi(\mathbf{b}) = \partial \circ \phi(\mathbf{b})$.

∎

Finally we are able to give the announced characterization of a VB-algebroid structure in terms of data on the exact sequence (C.1) and the core bundle.

Proposition E.9. *There is a canonical 1:1 correspondence between VB-algebroids structures on the double vector bundle $(D; A, B; M)_C$ (over B) and the following structure on the exact sequence (C.1):*

- *an* anchor map $\rho : A \longrightarrow TM$,
- *a Lie algebroid structure* $([\cdot,\cdot], \widehat{\rho})$ *on* $\widehat{A}$,
- *a* core map $\partial : C \longrightarrow B$,
- *and two Lie algebroid representations*

$$\begin{aligned}\zeta^C : &\widehat{A} \longrightarrow \mathbf{CDO}(C) \\ &X \longrightarrow \zeta^C_X\end{aligned} \qquad \text{and} \qquad \begin{aligned}\zeta^B : &\widehat{A} \longrightarrow \mathbf{CDO}(B) \\ &X \longrightarrow \zeta^B_X.\end{aligned}$$

These structure data are related by:

1. $\widehat{\rho} = \rho \circ p_A$, *where* $p_A : \widehat{A} \longrightarrow A$ *is the projection,*
2. $\partial \zeta^C_X = \zeta^B_X \partial$,
3. $[X, \phi] = \zeta^C_X \phi - \phi \zeta^B_X$,
4. $\zeta^C_\phi = \phi \circ \partial$ *and* $\zeta^B_\phi = \partial \circ \phi$,

for all $X \in \Gamma(\widehat{A})$ *and* $\phi \in \Gamma(B^* \otimes C) \cong \Gamma(\mathrm{Hom}(B, C)) \subset \Gamma(\widehat{A})$.

Proof. If D_B is endowed with a VB-algebroid structure, we have already obtained the structure data of the statement, and showed that relations *1, 2, 3* and *4* are satisfied. So let's see that we can build the inverse process.

Given a sequence like (C.1), by the double realization process we can find the corresponding double vector bundle $(D; A, B; M)_C$ such that $\Gamma(\widehat{A}) \cong \Gamma_{\text{lin}}(D_B)$.

In order to define the anchor map $\rho_D : D_B \longrightarrow TB$, it is enough to set its action on core and linear sections (cf. lemma C.26). Define

$$\rho_D(\mathbf{c}) := -\partial \mathbf{c} \in \Gamma(B) \cong \Gamma_{\text{core}}(TB); \text{ and } \rho_D(X) := \zeta^B_X \in \Gamma(\mathbf{CDO}(B)) \cong \Gamma_{\text{lin}}(TB),$$

for all $\mathbf{c} \in \Gamma(C) \cong \Gamma_{\text{core}}(D_B)$ and $X \in \Gamma(\widehat{A}) \cong \Gamma_{\text{lin}}(D_B)$.

Also the brackets $[\cdot,\cdot]$ on $\Gamma(D_B)$ will be completely determined by their action on core and linear sections (see Rmk. 2.12). Because of the identification $\Gamma(\widehat{A}) \cong \Gamma_{\text{lin}}(D_B)$ we have already defined the brackets for linear sections. For the other cases define

$$[\mathbf{c_1}, \mathbf{c_2}] := 0 \qquad \text{and} \qquad [X, \mathbf{c}] := \zeta^C_X(\mathbf{c}),$$

for all $\mathbf{c}, \mathbf{c_1}, \mathbf{c_2} \in \Gamma(C) \cong \Gamma_{\text{core}}(D_B)$, $X \in \Gamma(\widehat{A}) \cong \Gamma_{\text{lin}}(D_B)$. It is easy to see that conditions *1, 2, 3* and *4* imply the Lie algebroid structure conditions for $([\cdot,\cdot]_D, \rho_D)$.

∎

E.2 Horizontal lifts and Representations up to homotopy

The aim of this section is to show how 2-term representations up to homotopy come into scene after introducing a decomposition on a *VB*-algebroid, and moreover, we show that once we fix a decomposition, there is a canonical 1:1 correspondence between *VB*-algebroid structures on $D \cong A \oplus B \oplus C$ and representations up to homotopy of A on $C \xrightarrow{\partial} B[1]$, cf. Thm. E.21.

In the end we construct the Whitney sum of two *VB*-algebroids and describe its corresponding representation up to homotopy after introducing decompositions on the respective *VB*-algebroids.

E.2.1 Horizontal lifts

Definition E.10. Let $(D, [\cdot,\cdot]_D, \rho_D)$ be a VB-algebroid. Consider the fat bundle $\widehat{A}$ and a horizontal lift $\psi : A \longrightarrow \widehat{A}$. We can pull-back the $\widehat{A}$-connections ζ^C and ζ^B by ψ to *induced A-connections* ∇^C and ∇^B on C and B, respectively. If we denote $\psi(X) = \widehat{X}$, for $X \in \Gamma(A)$, we have

$$\nabla^C_X := \zeta^C_{\widehat{X}}; \qquad \nabla^B_X := \zeta^B_{\widehat{X}}. \tag{E.12}$$

Remark E.11. The induced connections ∇^C and ∇^B depend on the choice of the horizontal lift.

Corollary E.12 ([23])**.** *The following equations hold*

$$\partial \circ \nabla^C_X = \nabla^B_X \circ \partial \tag{E.13}$$

$$\phi \circ \nabla^B_X - \nabla^C_X \circ \phi = [\phi, \widehat{X}], \tag{E.14}$$

for $X \in \Gamma(A)$ *and* $\phi \in \mathrm{Hom}(B, C)$.

Proof. Follows directly from Eqs. (E.9) and (E.10).

∎

Proposition E.13 ([23])**.** *Define* $K \in \Omega^2(A; \mathrm{Hom}(B, C))$ *by*

$$K(X, Y) := \widehat{[X, Y]_A} - [\widehat{X}, \widehat{Y}]_{\widehat{A}}, \tag{E.15}$$

for $X, Y \in \Gamma(A)$. Then

$$R_{\nabla^C}(X,Y) = -\theta^C_{K(X,Y)} = -K(X,Y)\circ\partial, \tag{E.16}$$
$$R_{\nabla^B}(X,Y) = -\theta^B_{K(X,Y)} = -\partial\circ K(X,Y), \tag{E.17}$$
$$\sum_{cyclic}([\widehat{X}, K(Y,Z)] - K([X,Y],Z)) = 0, \tag{E.18}$$

for $X, Y, Z \in \Gamma(A)$, where R_{∇^C} and R_{∇^B} are, respectively, the curvatures of the connections ∇^C and ∇^B, and we are omitting the suffixes for the brackets in (E.18) for sake of clearness.

Proof. First notice that K indeed takes values on $\mathrm{Hom}(B,C)$, since, by the definition of $[\cdot,\cdot]_A$ we have

$$p(K(X,Y)) = [X,Y]_A - [X,Y]_A = 0.$$

Next,

$$\begin{aligned} R_{\nabla^C}(X,Y)(\widetilde{\mathbf{c}}) &= \nabla^C_X\nabla^C_Y\widetilde{\mathbf{c}} - \nabla^C_Y\nabla^C_X\widetilde{\mathbf{c}} - \nabla_{[X,Y]}\widetilde{\mathbf{c}} \\ &= [\widehat{X},[\widehat{Y},\widetilde{\mathbf{c}}]] - [\widehat{Y},[\widehat{X},\widetilde{\mathbf{c}}]] - [\widehat{[X,Y]},\widetilde{\mathbf{c}}] \\ &= [[\widehat{X},\widehat{Y}],\widetilde{\mathbf{c}}] - [\widehat{[X,Y]},\widetilde{\mathbf{c}}] \\ &= [-\widetilde{K(X,Y)},\widetilde{\mathbf{c}}] = -\theta^C_{K(X,Y)}(\widetilde{\mathbf{c}}). \end{aligned}$$

Now, to prove (E.17), recall that $\iota_A(B^*\otimes C)\subset\ker\rho_{\widehat{A}}$. In order to perform the calculations in a reasonably clean way, the linear sections corresponding to sections of the form $\widehat{X}\in\Gamma(\widehat{A})$ will be denoted the same, being clear from the context that the section is actually a section in $\Gamma_{\mathrm{lin}}(D_B)$ instead of $\Gamma(\widehat{A})$.

$$\begin{aligned} \langle -\theta^B_{K(X,Y)}(\mathbf{b}),\beta\rangle &= -\langle \mathbf{b}, -\rho_D(\widetilde{K(X,Y)})(\beta)\rangle = -\langle \mathbf{b},\rho_D([\widehat{X},\widehat{Y}] - \widehat{[X,Y]})(\beta)\rangle \\ &= -\langle \mathbf{b},\rho_D(\widehat{X})(\rho_D(\widehat{Y})(\beta))\rangle + \langle \mathbf{b},\rho_D(\widehat{Y})(\rho_D(\widehat{X})(\beta))\rangle + \langle \mathbf{b},\rho_D(\widehat{[X,Y]})(\beta)\rangle \\ &= \langle \nabla^B_X\mathbf{b},\rho_D(\widehat{Y})(\beta)\rangle - \langle \nabla^B_Y\mathbf{b},\rho_D(\widehat{X})(\beta)\rangle - \langle \nabla^B_{[X,Y]}\mathbf{b},\beta\rangle \\ &= -\langle \nabla^B_Y\nabla^B_X\mathbf{b},\beta\rangle + \langle \nabla^B_X\nabla^B_Y\mathbf{b},\beta\rangle - \langle \nabla^B_{[X,Y]}\mathbf{b},\beta\rangle \\ &= \langle R_{\nabla^B}(X,Y)(\mathbf{b}),\beta\rangle. \end{aligned}$$

Finally, by Jacobi identity for both $[\cdot,\cdot]_{\widehat{A}}$ and $[\cdot,\cdot]_A$, and taking into account that

$$\sum_{cyclic}[\widehat{X},\widehat{[Y,Z]}] = \sum_{cyclic}[\widehat{Z},\widehat{[X,Y]}],$$

we have

$$\begin{aligned} \sum_{cyclic}([\widehat{X},K(Y,Z)] - K([X,Y],Z)) &= \sum_{cyclic}([\widehat{X},\widehat{[Y,Z]} - [\widehat{Y},\widehat{Z}]] - \widehat{[[X,Y],Z]} + [\widehat{[X,Y]},\widehat{Z}]) \\ &= -\sum_{cyclic}[\widehat{X},[\widehat{Y},\widehat{Z}]] - \sum_{cyclic}\widehat{[[X,Y],Z]} + \sum_{cyclic}[\widehat{X},\widehat{[Y,Z]}] + \sum_{cyclic}[\widehat{[X,Y} \\ &= -\sum_{cyclic}[\widehat{X},[\widehat{Y},\widehat{Z}]] - \sum_{cyclic}\widehat{[[X,Y],Z]} = 0. \end{aligned}$$

■

Definition E.14. We refer to the 2-form $K \in \Omega^2(A; \mathrm{Hom}(B, C))$ as the *curvature form* corresponding to the VB-algebroid D_B.

E.2.2 Representations up to homotopy

Now we will introduce the concept of a representation up to homotopy. We need some preliminary remarks.

Let

$$E = \bigoplus_n E^n$$

be a graded vector bundle and A a Lie algebroid. Then the space of E-valued A-differential forms, $\Omega(A; E)$, is graded by total degree:

$$\Omega(A; E) = \bigoplus_{i+j=p} \Omega^i(A; E^j). \tag{E.19}$$

The space $\Omega(A; E)$ is naturally endowed with a $\Omega(A)$-module structure, given by

$$\omega\eta(\alpha_1, \ldots, \alpha_{p+q}) = \sum \mathrm{sgn}(\sigma)\omega(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)})\eta(\alpha_{\sigma(p+1)}, \ldots, \alpha_{\sigma(p+q)}), \tag{E.20}$$

for $\omega \in \Omega^p(A),\ \eta \in \Omega^q(A; E)$, where the sum is over all (p, q)-shuffles.

Let E, F be graded vector bundles. We can form the new graded space $\mathrm{Hom}(E, F)$ of degree-preserving morphisms, whose degree k part, denoted by $\mathrm{Hom}^k(E, F)$, consists of vector bundle maps which increase the degree by k. Then $\Omega(A; \mathrm{Hom}(E, F))$ is also endowed with the graded structure given by (E.19), with $\mathrm{Hom}(E, F)$ instead of E.

There is another natural wedge product type map

$$\cdot \wedge \cdot : \Omega(A; \mathrm{End}(E)) \otimes \Omega(A; E) \longrightarrow \Omega(A; E),$$

which, for $T \in \Omega^p(A; \mathrm{End}^k(E)),\ \eta \in \Omega^q(A; E)$, is given by

$$T \wedge \eta(\alpha_1, \ldots, \alpha_{p+q}) = \sum (-1)^{qk}\mathrm{sgn}(\sigma)T_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(p)}}(\eta(\alpha_{\sigma(p+1)}, \ldots, \alpha_{\sigma(p+q)})),$$

where, again, the sum is over all (p, q)-shuffles.

Lemma E.15 ([2])**.** *There is a canonical 1:1 correspondence between degree n elements of $\Omega(A; \mathrm{End}(E))$ and operators F on $\Omega(A; E)$ which increase the degree by n and which are $\Omega(A)$-linear in the graded sense:*

$$F(\omega \wedge \eta) = (-1)^{n|\omega|}\omega \wedge F(\eta),\ \forall \omega \in \Omega(A), \eta \in \Omega(A; E).$$

Explicitly, $T \in \Omega(A, \mathrm{End}(E))$ induces the operator $\widehat{T}$ given by

$$\widehat{T}(\eta) = T \wedge \eta.$$

Proof. It is a routine to see that the operator $\widehat{T}$ actually defines a degree n, $\Omega(A)$-linear operator on $\Omega(A;E)$, for $T \in \Omega(A;\mathrm{End}(E))^n$. Let's prove the converse. Since F is $\Omega(A)$-linear, its action is completely determined by what it does on $\Omega^0(A;E) = \Gamma(E)$. Because F has degree n, it sends each $\Gamma(E^k)$ into the sum

$$\Gamma(E^{k+n}) \oplus \Omega^1(A;E^{k+n-1}) \oplus \Omega^2(A;E^{k+n-2})) \oplus \dots.$$

Denote by $F_0, F_1, F_2, \dots$ the components of $F|_{\Gamma E}$. Then we define, for each k, $T_k \in \Omega^k(A;\mathrm{End}^{n-k}(E))$, by

$$(T_k)_{\alpha_1,\dots,\alpha_k}(\varepsilon) := F_k(\varepsilon)(\alpha_1,\dots,\alpha_k) \in \Gamma(E),$$

for $\varepsilon \in \Gamma(E)$ and $\alpha_1,\dots,\alpha_k \in \Gamma(A)$. By the way we defined F_k, it follows that $F = \widehat{T}$, where

$$T = \sum_k F_k \in \Omega(A;\mathrm{End}(E))^n.$$

∎

Lemma E.16 ([2]). *Given a Lie algebroid A and a vector bundle E over M, there is a canonical 1:1 correspondence between A-connections ∇ on E and degree $+1$ operators d_∇ on $\Omega(A;E)$ which satisfy the derivation rule. Moreover, (E,∇) is a representation if and only if $d_\nabla^2 = 0$.*

Proof. The operator d_∇ is defined, for $\eta \in \Omega^k(A;E)$ by

$$\begin{aligned} d_\nabla\eta(\alpha_1,\dots,\alpha_{k+1}) = &\sum_{i<j}(-1)^{i+j}\eta([\alpha_i,\alpha_j],\dots,\hat{\alpha_i},\dots,\hat{\alpha_j},\dots,\alpha_{k+1}) \\ &+ \sum_i(-1)^{i+1}\nabla_{\alpha_i}\eta(\alpha_1,\dots,\hat{\alpha_i},\dots,\alpha_{k+1}). \end{aligned}$$

With respect to the product by functions $f \in C^\infty(M)$ the derivation rule reads

$$d_\nabla(f\eta) = (d_A f)\eta + f d_\nabla\eta,$$

which follows directly from the derivation rule of ∇. For 1-forms $\omega \in \Omega^1(A)$, the derivation rule reads

$$d_\nabla(\omega\eta) = (d_A\omega)\eta - \omega d_\nabla\eta,$$

which follows from the definitions of d_A, d_∇ and the wedge product (E.20), after a cumbersome calculation.

The derivation rule for the general case follows from an inductive argument.

Of course, given d_∇, we recover ∇ simply by restriction to 0-forms $\varepsilon \in \Omega^0(A;E) = \Gamma(E)$.

Now, if $d_\nabla^2 = 0$, then, for $\varepsilon \in \Gamma(E)$, we have

$$\begin{aligned} 0 = d_\nabla^2\varepsilon(\alpha_1,\alpha_2) &= d_\nabla(\nabla.\varepsilon)(\alpha_1,\alpha_2) \\ &= -\nabla_{[\alpha_1,\alpha_2]}\varepsilon + \nabla_{\alpha_1}\nabla_{\alpha_2}\varepsilon - \nabla_{\alpha_2}\nabla_{\alpha_1}\varepsilon = R_\nabla(\alpha_1,\alpha_2)\varepsilon. \end{aligned}$$

Thus, $R_\nabla = 0$. It follows from the same equation above, that if ∇ is flat, then $d_\nabla^2\varepsilon = 0$, for $\varepsilon \in \Gamma(E) = \Omega^0(A;E)$. By the derivation rule, it follows that for $\omega \in \Omega^k(A)$ and $\varepsilon \in \Gamma(E)$,

$$\begin{aligned} d_\nabla^2(\omega\varepsilon) &= d_\nabla((d_A\omega)\varepsilon + (-1)^k\omega d_\nabla\varepsilon)) \\ &= (d_A^2\omega)\varepsilon + (-1)^{k+1}(d_A\omega)d_\nabla\varepsilon + (-1)^k(d_A\omega)d_\nabla\varepsilon + (-1)^{2k}\omega d_\nabla^2\varepsilon = 0. \end{aligned}$$

Since $\Omega(A;E)$ is generated, as an $\Omega(A)$-module, by $\Gamma(E)$, it follows that $d_\nabla^2 = 0$.

■

Remark E.17. We will need to recall how to extend d_∇ to an operator on $\Omega(A;\mathrm{End}(E))$, where E is a graded vector bundle, in which case, as we already saw, $\mathrm{End}(E)$ acquires a natural grading. This is done demanding the following Leibniz type rule:

$$d_\nabla(T(\varepsilon)) = (d_\nabla T)(\varepsilon) + (-1)^{|T|}T d_\nabla\varepsilon,$$

where $T \in \Omega(A;\mathrm{End}(E))$ and $\varepsilon \in \Gamma(E)$, and thus $T(\varepsilon) \in \Omega(A;E)$.

Definition E.18. Let A be a Lie algebroid over M. A *representation up to homotopy of* A, also called a *degree 1 A-superconnection*, consists of a graded vector bundle E over M and an operator, called the structure operator,

$$D : \Omega(A;E) \longrightarrow \Omega(A;E)$$

which increases the total degree by one and satisfies $D^2 = 0$ and the graded derivation rule:

$$D(\omega\eta) = d_A(\omega)\eta + (-1)^k\omega D(\eta)$$

for all $\omega \in \Omega^k(A)$ and $\eta \in \Omega(A;E)$, where d_A is the associated De-Rham operator on $\Omega(A) = \Gamma(\Lambda^\cdot A^*)$. The cohomology of the resulting complex is denoted by $H^\bullet(A;E)$.

Definition E.19. A morphism $\Phi : E \longrightarrow F$ between two representations up to homotopy of A is a degree zero linear map

$$\Phi : \Omega(A;E) \longrightarrow \Omega(A;F)$$

which is $\Omega(A)$-linear and commutes with the structure differentials D_E and D_{F^*}.

Proposition E.20 ([23],[2])**.** *There is a canonical 1:1 correspondence between representations up to homotopy (E,D) of A concentrated in two consecutive degrees, say 0 and 1 (so that E is zero in all the other degrees), and the following data*

1. *Two vector bundles C and B and a vector bundle map $\partial : C \longrightarrow B$*
2. *A-connections on C and B, ∇^C and ∇^B, compatible with ∂, which means that $\nabla^B\partial = \partial\nabla^C$.*
3. *A 2-form $K \in \Omega^2(A;\mathrm{Hom}(B,C))$, called the* curvature form *such that*

$$R_{\nabla^C} = -K\circ\partial,\ R_{\nabla^B} = -\partial\circ K,$$

and

$$d_\nabla K = 0,$$

where we are considering $\nabla := \nabla^C + \nabla^B$ *as an* A*-connection on* $E := C \oplus B$*, and viewing* $\mathrm{Hom}(B, C)$ *naturally seated in* $\mathrm{End}(E)$*, so that we consider the extension* d_∇ *to* $\Omega(A; \mathrm{End}(E))$*, and view* K *as an element in* $\Omega^2(A; \mathrm{End}(E))$*.*

Proof. We have $E = E^0 \oplus E^1$. Denote $E^0 = C$ and $E^1 = B$. Due to the derivation rule and the fact that $\Omega(A; E)$ is generated as an $\Omega(A)$-module by $\Gamma(E)$, the operator D will be uniquely determined by what it does on $\Gamma(E)$, or more precisely, by what it does on C and B. Since D has total degree 1, we have

$$D(\Gamma(C)) \subset \Gamma(B) \oplus \Omega(A; C) \quad \text{and} \quad D(\Gamma(B)) \subset \Omega(A; B) \oplus \Omega^2(A; C),$$

hence

$$D(\Omega^p(A; C)) \subset \Omega^p(A; B) \oplus \Omega^{p+1}(A; C) \quad \text{and} \quad D(\Omega^p(A; B)) \subset \Omega^{p+1}(A; B) \oplus \Omega^{p+2}(A; C).$$

Thus, we obtain a decomposition $D = D_0 + D_1 + D_2$, such that $D_i(\Omega^p(A; E)) \subset \Omega^{p+i}(A; E)$. From the derivation rule for D, we deduce that D_i for $i \neq 1$ is a (graded) $\Omega(A)$-linear map, for, if $\omega\eta \in \Omega^p(A; E^k)$, then $d_A(\omega)\eta \in \Omega^{p+1}(A; E^k)$, so that this term corresponds to the image of D_1. By Lemma E.15, it follows that D_0 and D_2 are given by the wedge product with an element in $\Omega(A; \mathrm{End}(E))$. More precisely, D_0 is given by $T_0 \in \mathrm{End}^1(E)$ and D_2 is given by $T_2 \in \Omega^2(A; \mathrm{End}^{-1}(E))$.

Now, since T_0 is of degree 1, it follows that $T_0(C) \subset B$ and $T_0(B) = 0$, thence T_0 is completely determined by the vector bundle morphism $\partial := T_0|_C : C \longrightarrow B$. Analogously, we have a canonical isomorphism $\mathrm{End}^{-1}(E) \cong \mathrm{Hom}(B, C)$, so that T_2 is completely determined by an element $K \in \Omega^2(A; \mathrm{Hom}(B, C))$.

On the other hand, D_1 satisfies the derivation rule on each of the vector bundles C and B, then by Lemma E.16, it comes from A-connections on these bundles, ∇^C and ∇^B, which put together give an A-connection $\nabla := \nabla^C + \nabla^B$ on $E := C \oplus B$, which can be extended to $\Omega(A; \mathrm{End}(E))$.

Conversely, given $\partial, \nabla^C, \nabla^B$ and K, we can form the operator degree 1 operator on $\Omega(A; E)$, $D := \partial + d_\nabla + K$, where ∂ is seen as an element in $\Omega^0(A; \mathrm{End}^1(E))$, K is seen as an element in $\Omega^2(A; \mathrm{End}^{-1}(E))$ and d_∇ is the extension of $\nabla := \nabla^C + \nabla^B$ to a degree 1 operator on $\Omega(A; E)$. By what we already saw, D defined in this way is a degree 1 operator that satisfies the derivation rule.

Now let's compute $D^2\mathbf{c}$ and $D^2\mathbf{b}$ for $\mathbf{c} \in \Gamma(C)$ and $\mathbf{b} \in \Gamma(B)$.

$$\begin{aligned} D^2\mathbf{c} &= D(\partial\mathbf{c} + d_\nabla\mathbf{c}) \\ &= d_\nabla(\partial\mathbf{c}) + d_\nabla^2\mathbf{c} + K \wedge (\partial\mathbf{c}) \\ &= (d_\nabla\partial) \wedge \mathbf{c} - \partial \wedge (d_\nabla\mathbf{c}) + R_\nabla(\mathbf{c}) + (K \circ \partial)(\mathbf{c}). \end{aligned}$$

Thus, $D^2\mathbf{c} = 0 \iff \nabla^B \circ \partial = \partial \circ \nabla^C$ and $R_{\nabla^C} = -K \circ \partial$.

Analogously,

$$\begin{aligned} D^2\mathbf{b} &= D(d_\nabla \mathbf{b} + K \wedge \mathbf{b}) \\ &= d_\nabla^2 \mathbf{b} + (d_\nabla K) \wedge \mathbf{b} - K \wedge (d_\nabla \mathbf{b}) + (\partial \wedge K) \wedge \mathbf{b} + K \wedge (d_\nabla \mathbf{b}) \\ &= R_\nabla(\mathbf{b}) + (\partial \circ K)(\mathbf{b}) + (d_\nabla K)(\mathbf{b}), \end{aligned}$$

so that $D^2\mathbf{b} = 0 \iff R_{\nabla^B} = -\partial \circ K$ and $d_\nabla K = 0$.

So far we have $D^2\varepsilon = 0$ for every $\varepsilon \in \Gamma(E)$ if and only if $D = \partial + d_\nabla + K$ satisfies the equations in the statement. Now, for $\omega \in \Omega^k(A)$ and $\varepsilon \in \Gamma(E)$ we have by the derivation rule

$$\begin{aligned} D^2(\omega\varepsilon) &= D(d_A(\omega)\varepsilon + (-1)^k \omega D\varepsilon) \\ &= d_A^2(\omega)\varepsilon + (-1)^{k+1} d_A \omega D\varepsilon + (-1)^k d_A \omega D\varepsilon + (-1)^{2k} \omega D^2 \varepsilon = 0. \end{aligned}$$

Since $\Omega(A; E)$ is generated, as an $\Omega(A)$-module by $\Gamma(E)$ it follows that $D^2 = 0$. Therefore, $D^2 = 0$ if and only if ∂, ∇ and K satisfy the equations in the statement.

■

E.2.3 *VB*-algebroids $\leftrightsquigarrow$ 2-term representations up to homotopy

Theorem E.21 ([23])**.** *There is a canonical 1:1 correspondence between VB-algebroid structures on the decomposed double vector bundle $A \oplus B \oplus C$, or equivalently between VB-algebroids D with a splitting, and representations up to homotopy of A on $E = E_0 \oplus E_1 = C \oplus B[1]$.*

Proof. Given a VB-algebroid with a splitting, we saw in Prop. E.5, Cor. E.12 and Prop. E.13 that we obtain the data of the statement in Prop. E.20, so that we get a representation up to homotopy of A on $C \oplus B[1]$.

Conversely, given a representation up to homotopy of A on $C \oplus B[1]$, then we need to build a VB-algebroid structure on $D = (q^B)^* A \underset{B}{\oplus} (q^B)^* C$. We will use the data in the statement of Prop. E.20 to define this structure. Let's define the anchor map. For a core section $\widetilde{\mathbf{c}} \in \Gamma_{\text{core}}(D_B)$, which corresponds to a section $\mathbf{c} \in \Gamma(C)$, we define

$$\rho_D(\widetilde{\mathbf{c}}) = -\widetilde{\partial \mathbf{c}} \in \Gamma_{\text{vert}}(TB) = \Gamma_{\text{core}}(TB).$$

We saw in Prop. C.1 (with a different notation there) that a linear section on D has the form

$$\widetilde{X}(b) = (X(q^B(B)), b, \phi(b)), \text{ that is } \widetilde{X} = \widehat{X} \underset{B}{+} \iota(\phi),$$

where $X \in \Gamma(A)$ and $\phi \in \Gamma(\mathrm{Hom}(B, C))$. We define

$$\rho_D(\widetilde{X}) := \nabla_X^{B^*} \underset{B}{-} \partial^* \circ \phi^* \in \Gamma_{\text{lin}}(TB),$$

where ∇^{B^*} is the dual A-connection of ∇^B (with respect to ρ_A). By lemma C.26, ρ_D is completely determined by knowing its action on core and linear sections. Namely, let $d = (a, b, c) \in D$, then consider a linear section $\widetilde{X} \in \Gamma_{\text{lin}}(D_B)$ with $\widetilde{X}(b) = (a, b, 0)$, which is the horizontal lift of a section $X \in \Gamma(A)$, and a core section $\widetilde{\mathbf{c}} \in \Gamma_{\text{core}}(D_B)$ with $\widetilde{\mathbf{c}}(b) = (0, b, c)$, coming from a section $\mathbf{c} \in \Gamma(C)$, then

$$\rho_D(d) = \underset{B}{-}\partial(\mathbf{c}) \underset{B}{+} \nabla_X^{B^*}.$$

Also from Prop. C.27 it follows that ρ_D is a double vector bundle morphism.

Now we define $[\cdot,\cdot]_D$ satisfying the conditions of Def. 2.10. From remark 2.12 it suffices to define it for core and linear sections. For $\widetilde{\mathbf{c_1}}, \widetilde{\mathbf{c_2}} \in \Gamma_{\text{core}}(D_B)$ we define

$$[\widetilde{\mathbf{c_1}}, \widetilde{\mathbf{c_2}}]_D := 0;$$

for $\widetilde{X} = \widehat{X} \underset{B}{+} \iota(\phi) \in \Gamma_{\text{lin}}(D_B)$ and $\widetilde{\mathbf{c}} \in \Gamma_{\text{core}}(D_B)$, we define

$$[\widetilde{X}, \widetilde{\mathbf{c}}]_D := \widetilde{\nabla_X^C \mathbf{c}} \underset{B}{+} \widetilde{\phi \circ \partial(\mathbf{c})}$$

Next, for $\widetilde{X_1} = \widehat{X_1}, \widetilde{X_2} = \widehat{X_2} \in \Gamma_{\text{lin}}(D_B)$, we define

$$[\widetilde{X_1}, \widetilde{X_2}]_D := \widehat{[X_1, X_2]_A} \underset{B}{-} K(X, Y);$$

for $\widetilde{X_1} = \widehat{X}$ and $\widetilde{X_2} = \iota(\phi)$, we define

$$[\widetilde{X_1}, \widetilde{X_2}]_D := \iota(\nabla_X^C \circ \phi) \underset{B}{-} \iota(\phi \circ \nabla_X^B);$$

finally, for $\widetilde{X_1} = \iota(\phi_1)$ and $\widetilde{X_2} = \iota(\phi_2)$, we define

$$[\widetilde{X_1}, \widetilde{X_2}]_D := \iota(\phi_1 \circ \partial \circ \phi_2 - \phi_2 \circ \partial \circ \phi_1).$$

As we already saw, the equations above are satisfied when the structure data $\partial, \nabla^C, \nabla^B$ and K come from a VB-algebroid. It follows that the structure defined by them satisfy the axioms of a Lie algebroid, and of course, the conditions of a VB-algebroid.

■

E.2.4 The Whitney sum of two *VB*-algebroids

Proposition E.22. *Let $(D, [\cdot,\cdot]_D, \rho_D)$ and $(D', [\cdot,\cdot]_{D'}, \rho_{D'})$ be two* VB*-algebroids, such that the induced Lie algebroid on the side bundle is the same for both* VB*-algebroids, $(A, [\cdot,\cdot], \rho)$. Then there is a natural* VB*-algebroid structure on the pull-back bundle (see Prop. C.42)*

$$D \underset{A}{\oplus} D',$$

such that for any horizontal lift (ψ, ψ') of the corresponding linear sequence (see Cor. C.44), the induced representation up to homotopy is the sum of the corresponding representations induced by (D, ψ) and (D', ψ') (in the proof it is made precise what we mean with the sum of the two representations).

Proof. We will use Prop. E.9 to define a VB-algebroid structure on $D \underset{A}{\oplus} D'$. We already have, by hypothesis, the anchor map $\rho : A \longrightarrow TM$.

The core map $\partial : C \oplus C' \longrightarrow B \oplus B'$ we define by

$$\partial(\mathbf{c}, \mathbf{c}') = (\partial_D(\mathbf{c}), \partial_{D'}(\mathbf{c}')), \tag{E.21}$$

where $\partial_D, \partial_{D'}$ are the core maps of D and D', respectively.

Now we need to define brackets on the linear bundle, which, by Cor. C.44, is given by

$$(B^* \otimes C') \oplus ((B')^* \otimes C) \oplus (\widehat{A} \underset{A}{\times} \widehat{A}').$$

By definition, a section of $\widehat{A} \underset{A}{\times} \widehat{A}'$ is a pair of sections (X, Y) of $\widehat{A}$ and $\widehat{A}'$, respectively, such that

$$p \circ X = p' \circ Y.$$

We define, for sections $(X_1, Y_1), (X_2, Y_2) \in \widehat{A} \underset{A}{\times} \widehat{A}'$,

$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2]_D, [Y_1, Y_2]_{D'}). \tag{E.22}$$

For $\phi_1, \phi_2 \in (B^* \otimes C') \oplus ((B')^* \otimes C) \subset \mathrm{Hom}((B \oplus B')^*, (C \oplus C'))$ we define

$$[\phi_1, \phi_2] := \phi_1 \circ \partial \circ \phi_2 - \phi_2 \circ \partial \circ \phi_1.$$

It remains to define $[(X, Y), \phi]$ for $(X, Y) \in \widehat{A} \underset{A}{\times} \widehat{A}'$ and $\phi \in (B^* \otimes C') \oplus ((B')^* \otimes C)$. First we need to define Lie algebroid representations $\zeta^{C \oplus C'} : \widehat{A} \underset{A}{\times} \widehat{A}' \longrightarrow \mathbf{CDO}(C \oplus C')$ and $\zeta^{B \oplus B'} : \widehat{A} \underset{A}{\times} \widehat{A}' \longrightarrow \mathbf{CDO}(B \oplus B')$. Set

$$\zeta^{C \oplus C'}_{(X,Y)}(\mathbf{c}, \mathbf{c}') := (\zeta^C_X, \zeta^{C'}_Y)(\mathbf{c}') \quad \text{and} \quad \zeta^{B \oplus B'}_{(X,Y)}(\mathbf{b}, \mathbf{b}') := (\zeta^B_X, \zeta^{B'}_Y)(\mathbf{b}'). \tag{E.23}$$

Now we define

$$[(X, Y), \phi] := \zeta^{C \oplus C'}_{(X,Y)} \phi - \phi \zeta^{B \oplus B'}_{(X,Y)}. \tag{E.24}$$

The last ingredient to have the data required by Prop. E.9 is to define $\zeta^{B \oplus B'}$ and $\zeta^{C \oplus C'}$ on $(B^* \otimes C) \oplus ((B')^* \otimes C)$. We do it in the natural way:

$$\zeta^{C \oplus C'}_\phi := \phi \circ \partial \quad \text{and} \quad \zeta^{B \oplus B'}_\phi := \partial \circ \phi. \tag{E.25}$$

By construction, and because D and D' are already VB-algebroids, it follows that the data defined above satisfy conditions 1., 2., 3. and 4. of Prop. E.9, and thus we have a VB-algebroid structure on $D \underset{A}{\oplus} D'$.

Now let ψ, ψ' be horizontal lifts for $\widehat{A}$ and $\widehat{A}'$, respectively. Then, by Thm. E.21, we obtain two representations up to homotopy, corresponding to the VB-algebroid structures on D and D', respectively, which are given by, according to Prop.E.20,

$$\partial + \nabla + K \quad \text{and} \quad partial' + \nabla' + K', \tag{E.26}$$

respectively.

Let's denote by $\widetilde{A}$ the linear bundle corresponding to $(D \underset{A}{\oplus} D')_{B\oplus B'}$, so that

$$\widetilde{A} := (B^* \otimes C') \oplus ((B')^* \otimes C) \oplus (\widehat{A} \underset{A}{\times} \widehat{A}').$$

By Cor. C.44, (ψ, ψ') provides a horizontal lift for $\widetilde{A}$, and the VB-algebroid structure we just obtained on $D \underset{A}{\oplus} D'$ thereby induces a representation up to homotopy of A on

$$(C \oplus C') \oplus (B \oplus B')[1] \cong (C \oplus B[1]) \oplus (C' \oplus B'[1]),$$

denoted by $\widetilde{D}$, which can be seen therefore as an operator on

$$\Omega(A; (C \oplus B[1]) \oplus (C' \oplus B'[1]),$$

so that $\widetilde{D}$ decomposes in to components

$$\widetilde{D} = (D_1, D_2),$$

with D_1 taking values on $C \oplus B[1]$ and D_2 taking values on $C' \oplus B'[1]$.

We claim that D_1 and D_2 are the representations up to homotopy corrsponding to the VB-algebroids on D and D', respectively, that is, we have

$$\begin{aligned}\widetilde{D} &= \widetilde{\partial} + \widetilde{\nabla} + \widetilde{K}\\ &= (\partial, \partial') + (\nabla, \nabla') + (K, K').\end{aligned}$$

We already obtained $\widetilde{\partial} = (\partial, \partial')$ in Eq. (E.21). Also, $\widetilde{\nabla} = (\nabla, \nabla')$ follows immediately from the fact that the horizontal lift is given by (ψ, ψ') and from Eq. (E.23). So it remains to check that $\widetilde{K} = (K, K')$. We have

$$\begin{aligned}\widetilde{K}(X,Y) &= (\psi([X,Y]), \psi'([X,Y]) - [(\psi(X), \psi'(X)), (\psi(Y), \psi'(Y))]\\ &= (\psi([X,Y]), \psi'([X,Y])) - ([\psi(X), \psi(Y)]_D, [\psi'(X), \psi'(Y)]_{D'}\\ &= (\psi([X,Y]) - [\psi(X), \psi(Y)]_D, \psi'([X,Y]) - [\psi'(X), \psi'(Y)]_{D'})\\ &= (K(X,Y), K'(X,Y)).\end{aligned}$$

■

E.3 Duality

E.3.1 The dual *VB*-algebroid

In this subsection we show characterize a VB-algebroid structure in terms of its dual Poisson structure, which leads to the discovery of a second VB-algebroid structure on the dual with respect to the second fibration of the VB-algebroid. We describe in some detail this structure, which is dual to the first in a sense that we will make precise in terms of the corresponding representation up to homotopy in the next subsection. Finally we are able to characterize a VB-algebroid isomorphism in terms of its transpose.

Definition E.23. A Poisson structure on a double vector bundle $(D; A, B; M)$ is called *double-linear Poisson structure* if it is linear with respect to both fibrations: the one over A and the one over B.

The following characterization of a VB-algebroid structure in terms of a double-linear Poisson structure on the dual provides a great insight. We take it from [23], although we provide more details in the proof.

Proposition E.24 ([23]). *A double vector bundle $(D; A, B; M)_C$, such that D_B is equipped with a Lie algebroid structure, satisfies the VB-algebroid compatibility conditions if and only if the induced Poisson structure on D_B^* is linear over C^*. Therefore, there is a canonical 1:1 correspondence between* VB-*algebroid structures on D and double-linear Poisson structures on D_B^*.*

Proof. We can work locally. Take an adapted coordinate system, given by Cor. A.20, $\{x^i, \beta^b, \alpha^a, \kappa^c\}$ in $D_B^* \cong B \oplus A^* \oplus C^*$.

The functions α^a are double-linear, then we can identify them with linear sections of D_B, whence, if D_B is equipped with a Lie algebroid structure, $\{\alpha^a, \alpha^{\tilde{a}}\}$ is also a linear section of D_B, thus a double-linear function on D_B^*, in particular linear over C^*.

Next, we claim that the functions κ^c, restricted to a slice $(D_B^*)_m$ are identified with elements in $C \subset D_B$. Recall that, for $d \in (D_B^*)_m$, with $\pi_{C^*}(d) = \kappa$,

$$\langle c, d \rangle_B = \langle c, \kappa \rangle.$$

We define $c_{\kappa^c} \in D_m$ by

$$\langle c_{\kappa^c}, d \rangle_B := \kappa^c(d).$$

In order to verify that this equation actually defines an element in $C \subset D$, we need to check

- $\kappa^c(d_1 \underset{B}{+} d_2) = \kappa^c(d_1) + \kappa^c(d_2)$
- If $\pi_{C^*}(d_1) = \pi_{C^*}(d_2)$, then $\kappa^c(d_1) = \kappa^c(d_2)$.

Both conditions are satisfied since κ^c are adapted coordinates. Hence, the functions κ^c are canonically identified with core sections in D_B, and actually, any core section is a linear combination of the κ^c's. Then

$$\{\alpha^a, \kappa^c\} = \mu_c(x)\kappa^c.$$

The remaining cases are easy to compute:

$\{\alpha^a, \beta^b\} = \rho(\alpha^a)(\beta^b) = \beta^b \mu_b(x)$, for the anchor is a double vector bundle morphism and so $\rho(\alpha)$ is a linear tangent field on B;

$$\{\kappa^c, \kappa^{\tilde{c}}\} = 0;\ \{\alpha^a, x^i\} = \rho_i^a(x);\ \{\kappa^c, x^i\} = \{\beta^b, \beta^{\tilde{b}}\} = \{\beta^b, x^i\} = \{x^i, x^j\} = 0.$$

Hence, the induced Poisson brackets $\{\cdot, \cdot\}$ from the Lie algebroid structure on D_B are linear over C^*. We also see that, conversely, if the Poisson brackets $\{\cdot, \cdot\}$ induced from a Lie algebroid structure on D_B are linear over C^*, then they must satisfy the equations above and therefore the Lie algebroid structure is actually a VB-algebroid structure.

■

A first –and perhaps the most important– by-product of the above characterization is the occurrence of a dual (in a sense clarified later, cf. Thm. E.32) *VB*-algebroid structure associated to a given *VB*-algebroid.

Corollary E.25 ([23])**.** *A VB-algebroid structure on D_B induces a dual VB-algebroid structure on $(D_B^*)^*_{C^*}$, which in turn induces an isomorphic VB-algebroid structure on D_A^* over C^* via the isomorphism Υ_{C^*}, given in Prop. B.13, with D_B^* playing the role of D (see Rmk. C.37).*

Proof. Since the induced Poisson structure on D_B^* is linear with respect to the vector bundle structure over C^*, it induces a Lie algebroid structure on $(D_B^*)^*_{C^*}$, and since the Poisson structure is double-linear, the Lie algebroid structure is actually a *VB*-algebroid structure.

■

Remark E.26. It is useful to write down the explicit formulas for the *VB*-algebroid structure on $(D_B^*)^*_{C^*}$ and on D_A^* in terms of the *VB*-algebroid structure on D. We will use the isomorphisms Z_B and Z_A introduced in (C.52) and (C.53), respectively, but in this case D_B^* plays the role of D, in particular C^* plays the role of A and B plays the role of A.

a) Let $X \in \Gamma_{\mathrm{lin}}((D_B^*)^*_{C^*}) = \Gamma(\widehat{A}_{C^*})$ and $f \in C^\infty(M)$, then
$$\rho_{(D_B^*)^*_{C^*}}(X)(f) = \{Z_{C^*}^{-1}(X), f\} = \rho_D(Z_B \circ Z_{C^*}^{-1}(X))(f) = \rho_D(Z(X))(f).$$

b) Let $\beta \in \Gamma(B^*)$ and $\mathbf{c} \in \Gamma(C)$, then
$$\rho_{(D_B^*)^*_{C^*}}(\overline{\beta})(\mathbf{c}) = \{\beta, \mathbf{c}\} = -\rho_D(\overline{\mathbf{c}})(\beta).$$

c) $\overline{\rho_{(D_B^*)^*_{C^*}}(X)(\mathbf{c})} = \overline{\{Z_{C^*}^{-1}(X), \mathbf{c}\}} = [Z_B \circ Z_{C^*}^{-1}(X), \overline{\mathbf{c}}]_D = [Z^{-1}(X), \overline{\mathbf{c}}]_D.$

d) $[X, \overline{\beta}]_{(D_B^*)^*_{C^*}} = \overline{\{Z_{C^*}^{-1}(X), \beta\}} = \overline{\rho_D(Z_B \circ Z_{C^*}^{-1}(X))(\beta)} = \overline{\rho_D(Z^{-1}(X))(\beta)}.$

e) Finally, for $X, Y \in \Gamma(\widehat{A}_{C^*})$,
$$\begin{aligned} [X,Y]_{(D_B^*)^*_{C^*}} &= Z_{C^*}(\{Z_{C^*}^{-1}(X), Z_{C^*}^{-1}(Y)\}) = Z_{C^*} \circ Z_B^{-1}([Z_B \circ Z_{C^*}^{-1}(X), Z_B \circ Z_{C^*}^{-1}(Y)]) \\ &= Z([Z^{-1}(X), Z^{-1}(Y)]). \end{aligned}$$

From the expressions above, we readily obtain the description of the *VB*-algebroid structure on D_A^* obtained from $(D_B^*)^*_{C^*}$ by the isomorphism Υ_{C^*}. We just need to keep in mind that Υ_{C^*} is the identity on A and B^* and it is $-\mathrm{Id}$ on C^*, and also recall Eq. (C.64) from Rmk. C.37. Identifying $\Gamma(C)$ with $\Gamma_{\mathrm{core}}(D_B)$ and $\Gamma(B^*)$ with $\Gamma_{\mathrm{core}}(D_A^*)$ in order to avoid writing annoying long bars as the ones that appear in items c) and d) above, we have the following:

$$\begin{gathered} \rho_{D_A^*}(X)(f) = \rho_D(T^{-1}(X))(f); \qquad \rho_{D_A^*}(\beta)(\mathbf{c}) = \rho_D(\mathbf{c})(\beta); \qquad \rho_{D_A^*}(X)(\mathbf{c}) = [T^{-1}(X), \mathbf{c}]_D; \\ [X,\beta]_{D_A^*} = \rho_D(T^{-1}(X))(\beta); \qquad [X,Y]_{D_A^*} = T[T^{-1}(X), T^{-1}(Y)]_D. \end{gathered} \tag{E.27}$$

Definition E.27. Let D, D' be two *VB*-algebroids. A *VB-algebroid morphism* from D to D' is a DVB morphism $\Phi : D \longrightarrow D'$ that is also a Lie algebroid morphism.

Remark E.28. See [47] for the definition of Lie algebroid morphism. In the case that we have a Lie algebroid *isomorphism* $(\Phi, \varphi) : A \longrightarrow A'$:

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\pi"'] & A' \arrow[d, "\pi'"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd} ,$$

we can talk about the *push forward* $\Phi_* : \Gamma(A) \longrightarrow \Gamma(A')$ on sections and $\varphi_* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M')$ on vector fields, given by

$$\Phi_*(X)(m') = \Phi(X(\varphi^{-1}(m'))) \quad \text{and} \quad \varphi_*(V)(m') = d\varphi(V)(\varphi^{-1}(m')),$$

for every $X \in \Gamma(A)$, $V \in \mathfrak{X}(M)$ and $m' \in M'$. In this situation, Φ is a Lie algebroid morphism if and only if

$$\varphi_*(\rho_A(X))(f) = \rho_{A'}(\Phi_*(X))(f), \qquad \forall\ X \in \Gamma(A), f \in C^\infty(M') \tag{E.28}$$

and

$$\Phi_*([X,Y]_A) = [\Phi_*(X), \Phi_*(Y)]_{A'}, \qquad \forall\ X, Y \in \Gamma(A). \tag{E.29}$$

In the case of *VB*-algebroids, by Rmk. 2.12 we can spell out the condition of being a Lie algebroid isomorphism by the following equations, which involve only linear and core sections and fiberwise linear and constant functions:

a) $(\varphi_M)_*(\rho_{D_B}(X))(f) = \rho_{D'_{B'}}(\widehat{\Phi}_B(X))(f);\quad (\varphi_B)_*(\rho_{D_B}(X))(\beta') = \rho_{D'_{B'}}(\widehat{\Phi}_B(X))(\beta');$
$(\varphi_B)_*(\rho_{D_B}(\overline{\mathbf{c}}))(\beta') = \rho_{D'_{B'}}(\varphi_C(\overline{\mathbf{c}}))(\beta');$

b) $\varphi_C([X, \overline{\mathbf{c}}]_{D_B}) = [\widehat{\Phi}_B(X), \varphi_C(\overline{\mathbf{c}})]_{D'_{B'}};\quad \widehat{\Phi}_B([X,Y]_{D_B}) = [\widehat{\Phi}_B(X), \widehat{\Phi}_B(Y)]_{D'_{B'}}.$

Observe that we are identifying a fiberwise constant function on D_B with a function $f \in C^\infty(M)$. For such functions we have $\rho_{D_B}(X)(f) = \rho_A(q_A(X))(f)$, where $\rho_A : A \longrightarrow TM$ is the base morphism corresponding to $\rho_{D_B} : D_A \longrightarrow TB_{TM}$ given in Def. 2.10. Therefore, it makes sense the expression

$$(\varphi_M)_*(\rho_{D_B}(X))$$

on the first equation of item a) above, and it reads $(\varphi_M)_*(\rho_A(q_A(X)))$.

Proposition E.29. *A map $\Phi : D \longrightarrow D'$ between two* VB*-algebroids is a* VB*-algebroid isomorphism if and only if $\Phi^*_A : (D')^*_{A'} \longrightarrow D^*_A$ is a* VB*-algebroid isomorphism.*

Proof. By the the involutivity of dualization, it is enough to prove one way of the statement, that is, to prove that if $\Phi : D \longrightarrow D'$ is a *VB*-algebroid isomorphism, then

$$\Phi^*_A : (D')^*_{A'} \longrightarrow D^*_A$$

is a VB-algebroid isomorphism, that is, Φ_A^* satisfies satisfies the equations of items a) and b) of Rmk. E.28 above. We will use repeatedly those equations applied to Φ, Eqs. E.27, Eq. (C.33) of Prop. C.22, Cor. B.3 and the following identity for the push-forward of vector fields:

$$\varphi_*(X)(f) = (\varphi^{-1})^*(X(\varphi^*(f))),$$

which follows immediately from the definitions. So let's begin the computations.

-
$$\begin{aligned}(\varphi_M^{-1})_*\left(\rho_{(D')^*_{A'}}(X')\right)(f') &= (\varphi_M)^*\left(\rho_{(D')^*_{A'}}(X')((\varphi_M^{-1})^*(f'))\right)\\ &= (\varphi_M)^*\left(\rho_{D'_{B'}}((T')^{-1}(X'))((\varphi_M^{-1})^*(f'))\right)\\ &= (\varphi_M)^*(\varphi_M^{-1})^*\left(\rho_{D_B}(\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'))(\varphi_M^*((\varphi_M^{-1})^*(f')))\right)\\ &= \rho_{D_A^*}(T\circ\widehat{\Phi}_B^{-1}\circ(T')(X'))(f')\\ &= \rho_{D_A^*}(\widehat{\Phi_A^*}(X'))(f').\end{aligned}$$

-
$$\begin{aligned}(\varphi_C^*)_*\left(\rho_{(D')^*_{A'}}(X')\right)(\mathbf{c}) &= \varphi_C^{-1}\left(\rho_{(D')^*_{A'}}(X')(\varphi_C(\mathbf{c}))\right)\\ &= \varphi_C^{-1}\left([(T')^{-1}(X'),\varphi_C(\mathbf{c})]\right)_{D'_{B'}}\\ &= \varphi_C^{-1}\circ\varphi_C\left([\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'),\mathbf{c}]_{D_B}\right)\\ &= \rho_{D_A^*}(T\circ\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'))(\mathbf{c})\\ &= \rho_{D_A^*}(\widehat{\Phi_A^*}(X'))(\mathbf{c}).\end{aligned}$$

-
$$\begin{aligned}(\varphi_C^*)_*\left(\rho_{(D')^*_{A'}}(\beta'))\right)(\mathbf{c}) &= (\varphi_M)^*\left(\rho_{(D')^*_{A'}}(\beta')(\varphi_C(\mathbf{c}))\right)\\ &= (\varphi_M)^*\left(\rho_{D'_{B'}}(\varphi_C(\mathbf{c}))(\beta')\right)\\ &= (\varphi_M)^*\circ(\varphi_M^{-1})^*\left(\rho_{D_B}(\mathbf{c})((\varphi_B)^*(\beta'))\right)\\ &= \rho_{D_A^*}((\varphi_B)^*(\beta'))(\mathbf{c}).\end{aligned}$$

-
$$\begin{aligned}(\varphi_B)^*\left([X',\beta']_{(D')^*_{A'}}\right) &= (\varphi_B)^*\left(\rho_{D'_{B'}}((T')^{-1}(X'))(\beta')\right)\\ &= (\varphi_B)^*\circ(\varphi_B^{-1})^*\left(\rho_{D_B}(\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'))((\varphi_B)^*(\beta'))\right)\\ &= [T\circ\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'),(\varphi_B)^*(\beta')]_{D_A^*}\\ &= [\widehat{\Phi_A^*}(X'),(\varphi_B)^*(\beta')]_{D_A^*}.\end{aligned}$$

•

$$
\begin{aligned}
\widehat{\Phi_A^*}\left([X',Y']_{(D')^*_{A'}}\right) &= \widehat{\Phi_A^*}\circ T'\left([(T')^{-1}(X'),(T')^{-1}(Y')]_{(D'_{B'}}\right)\\
&= \widehat{\Phi_A^*}\circ T'\circ\widehat{\Phi}_B\left([\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'),\widehat{\Phi}_B^{-1}\circ(T')^{-1}(Y')]_{D_B}\right)\\
&= T\left([\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'),\widehat{\Phi}_B^{-1}(T')^{-1}(Y')]_{D_B}\right)\\
&= [T\circ\widehat{\Phi}_B^{-1}\circ(T')^{-1}(X'),T\circ\widehat{\Phi}_B^{-1}(T')^{-1}(Y')]_{D_A^*}\\
&= [\widehat{\Phi_A^*}(X'),\widehat{\Phi_A^*}(Y')]_{D_A^*}.
\end{aligned}
$$

Therefore Φ_A^* satisfies the equations of items a) and b) of Rmk. E.28, and thus we conclude that it is a VB-algebroid isomorphism.

■

E.3.2 The dual representation up to homotopy

In this subsection we introduce the notion of *duality* between to representations up to homotopy. The main result (Thm. E.32) we show is that, given a VB-algebroid structure on D_B, the representation up to homotopy corresponding to the dual VB-algebroid $(D_A^*)_{C^*}$ (see Cor. E.25) is dual to the representation corresponding to D_B. This result does not appear in [23], and is instrumental for the geometric characterization of -2 Poisson brackets we provide in terms of metric VB-algebroids (Thm. 6.14). After we obtained this result, we learned that it was independently obtained by T. Drummond, M. Jotz and C. Ortiz, [18].

Definition E.30. Let (E,D) be a representation up to homotopy of a Lie algebroid A. We define the *dual representation* (E^*,D^*), where E^* is simply the dual (graded) vector bundle, and D^* is characterized by the condition

$$d_A(\nu\wedge\eta)=D^*(\nu)\wedge\eta+(-1)^{|\nu|}\nu\wedge D(\eta), \tag{E.30}$$

for all $\eta\in\Omega(A;E)$ and $\nu\in\Omega(A;E^*)$, where $\wedge$ is the operation

$$\Omega(A;E^*)\otimes\Omega(A;E)\longrightarrow\Omega(A)$$

such that, for $\nu\in\Omega^p(A;(E^*)^j),\ \eta\in\Omega^q(A;E^k)$, $\nu\wedge\eta\in\Omega^{p+q}(A)$ is given by

$$(\alpha_1,\dots,\alpha_{p+q})\longrightarrow\sum(-1)^{qj}\mathrm{sgn}(\sigma)\langle\nu(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(p)}),\eta(\alpha_{\sigma(p+1)},\dots,\alpha_{\sigma(p+q)})\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between E^* and E, and the sum is taken, as usual, over all (p,q)-shuffles.

Proposition E.31. *Let (E,D) be a representation up to homotopy of A, concentrated in degrees 0 and 1. We saw in Prop. E.20 that $E=C\oplus B[1]$ and $D=\partial+\nabla+K$, where $\nabla=\nabla^C+\nabla^B$. Now Consider the dual representation D^* on $E^*=B^*[-1]\oplus C^*$. Then*

$$D^*=\partial^*+\nabla^*-K^*.$$

Proof. By Prop. E.20, we have $D^* = \partial' + \nabla' + K'$. We will use Eq. (E.30) to compute ∂', ∇' and K'. Let $\varepsilon \in \Gamma(E)$ and $\varepsilon^* \in \Gamma(E^*)$, then

$$\begin{aligned} d_A\langle \varepsilon^*, \varepsilon\rangle &= D^*(\varepsilon^*) \wedge \varepsilon + (-1)^{|\varepsilon^*|}\varepsilon^* \wedge D(\varepsilon) \\ \Longrightarrow \rho_A(X)\langle \varepsilon^*, \varepsilon\rangle &= \langle \nabla'_X \varepsilon^*, \varepsilon\rangle + (-1)^{|\varepsilon^*|}(-1)^{|\varepsilon^*|\cdot 1}\langle \varepsilon^*, \nabla_X \varepsilon\rangle \\ &= \langle \nabla'_X \varepsilon^*, \varepsilon\rangle + \langle \varepsilon^*, \nabla_X \varepsilon\rangle, \end{aligned} \tag{E.31}$$

thence, $\nabla' = \nabla^*$. From Eq. (E.31) we also obtain, for $\mathbf{c} \in \Gamma(C)$ and $\mathbf{b}^* \in \Gamma(B^*[-1])$

$$\begin{aligned} 0 &= \langle \partial'(\mathbf{b}^*), \mathbf{c}\rangle + (-1)^{|\mathbf{b}^*|}\langle \mathbf{b}^*, \partial(\mathbf{c})\rangle \\ &= \langle \partial'(\mathbf{b}^*), \mathbf{c}\rangle - \langle \mathbf{b}^*, \partial(\mathbf{c})\rangle, \end{aligned}$$

thence, $\partial' = \partial^*$.

Finally, to compute K' we use again Eq. (E.31) with $\mathbf{c}^* \in \Gamma(C^*)$ and $\mathbf{b} \in \Gamma(B[1])$. We have, for $X_1, X_2 \in \Gamma(A)$,

$$\begin{aligned} 0 &= K'(\mathbf{c}^*) \wedge \mathbf{b} + (-1)^{|\mathbf{c}^*|}\mathbf{c}^* \wedge K(\mathbf{b}) \\ \Longrightarrow 0 &= \langle K'(X_1, X_2)(\mathbf{c}^*), \mathbf{b}\rangle + \langle \mathbf{c}^*, K(X_1, X_2)(\mathbf{b})\rangle, \end{aligned}$$

whence $K' = -K^*$. ∎

The

Theorem E.32. *Let D be a VB-algebroid. If we choose a splitting (or equivalently a horizontal lift) θ, then by Theorem E.21 we obtain a representation up to homotopy of A on $E = C \oplus B[1]$. By Corollary E.25 we have an induced VB-algebroid structure on D^*_A. By Cor. B.7, θ induces a splitting $\widetilde{\theta}_A$ for the corresponding dual D^*_A (and also a splitting $\widetilde{\theta}_B$ for D^*_B). In this way, again by Theorem E.21, we obtain a representation up to homotopy of A on $E^* = B^*[-1] \oplus C^*$. We assert that the Lie algebroid structure on A induced by the VB-algebroid structure on D^*_A coincides with the Lie algebroid structure induced by the VB-algebroid structure on D, and the representation up to homotopy obtained on E^* coincides with the representation dual to the representation D on E.*

Remark E.33. Be careful, we are using the same symbol D to denote in one occasion a double vector bundle and in another a representation up to homotopy. However, the context makes it clear which object we are referring to in each situation.

Proof. The induced *VB*-algebroid structure is obtained from the isomorphism

$$\Upsilon_{C^*} : (D^*_B)^*_{C^*} \longrightarrow D^*_A,$$

which corresponds to the isomorphism Υ_A, given in Prop. B.13, being that in our situation D^*_B plays the role of D. This isomorphism is the identity on the sides $A \longrightarrow A$ and on the cores $B^* \longrightarrow B^*$, while it is $-\mathrm{Id}$ on the side bundles $C^* \longrightarrow C^*$.

We need to prove two things: that the Lie algebroid structures induced on A by D_B and $(D^*_B)^*_{C^*}$ coincide, and that the assertion in the statement about the duality relation between the corresponding representations up to homotopy holds.

Let's begin proving the equality of the Lie algebroid structures on A.

If we denote by $\widehat{A}_{C^*}$ the linear bundle corresponding to $(D^*_B)^*_{C^*}$, from (C.55) with D^*_B playing the role of D, so that $\widehat{A}$ plays the role of $\widehat{C^*}_A$ and $\widehat{A}_{C^*}$ plays the role of $\widehat{C^*}_B$, we obtain

$$p_{\widehat{A}_{C^*}} = * \circ p_{\widehat{A}} \circ Z^{-1},$$

thence, denoting by $\widehat{A}_A$ the linear bundle corresponding to $(D^*_A)_{C^*}$,

$$p_{\widehat{A}_A} = \Upsilon_{C^*} \circ * \circ p_{\widehat{A}} \circ Z^{-1} = - * \circ p_{\widehat{A}} \circ Z^{-1}.$$

Now, identifying, as usual, sections of the linear bundle with linear sections of the corresponding double vector bundle, and also sections of A with the corresponding horizontal lifts, we have, directly from the definitions, the identity

$$[X_1, X_2]_{D_B} = Z^{-1}\left([X_1, X_2]_{(D^*_B)^*_{C^*}}\right), \tag{E.32}$$

in particular we obtain that the Lie algebroid structures induced on A by D_B and $(D^*_B)^*_{C^*}$ coincide, since, by (C.55), $p_A \circ Z \circ \psi = \mathrm{Id}_A$.

Now Let's prove the duality relation between the corresponding representations up to homotopy. Consider the splitting θ, which induces the splitting $\widetilde{\theta}_A$ on D^*_A. By Thm. E.21, these splittings yield representations up to homotopy corresponding to the respective VB-algebroid structures on D_B and D^*_A. The representation up to homotopy D of A on $C \oplus B[1]$, corresponding to the VB-algebroid D_B is comprised by the data $D = \partial + \nabla + K$ given in Prop. E.20. Correspondingly, we denote by $D' = \partial' + \nabla' + K'$ the representation up to homotopy raised by the VB-algebroid D^*_A. By Prop. E.31, we need to show that

$$\partial' = \partial^*; \qquad \nabla' = \nabla^*; \quad \text{and} \quad K' = -K^*.$$

Consider sections $\mathbf{c} \in \Gamma(C) \cong \Gamma_{\mathrm{core}}(D_B)$ and $\beta \in \Gamma(B^*) \cong C^\infty_{\mathrm{lin}}(B)$. Then, denoting by $\rho_{D_{C^*}}$ the anchor map of the VB-algebroid $(D^*_B)^*_{C^*}$, and taking into account also the identification $\Gamma(C) \cong C^\infty_{\mathrm{lin}}(C^*)$,

$$\rho_{D_{C^*}}(\beta)(\mathbf{c}) = \{\beta, \mathbf{c}\} = -\rho_D(\mathbf{c})(\beta).$$

Now, since, Υ_{C^*} is $-\mathrm{Id}$ on C^*, denoting by $\rho_{D^*_A}$ the anchor map of the VB-algebroid D^*_A it follows that,

$$\langle \partial'(\beta), \mathbf{c}\rangle = -\rho_{D^*_A}(\beta)(\mathbf{c}) = \rho_{D_{C^*}}(\beta)(\mathbf{c}) = -\rho_D(\mathbf{c})(\beta) = \langle \partial(\mathbf{c}), \beta\rangle,$$

whence $\partial' = \partial^*$.

Next, let $X \in \Gamma(A)$, and $\mathbf{c} \in \Gamma(C)$. As above, we make several identifications without changing notation, for sake of readability. Namely, we identify $\Gamma(A)$ with $\Gamma(\psi(A))$, where $\psi(A) \subset \widehat{A}$ is the image of the horizontal lift in the linear bundle; viewing X as a section of D_B we can also identify it, through the isomorphism Z_B (C.52), with a double-linear function on D^*_B, also, through the isomorphism Z (Prop. C.33), with a linear section of $(D^*_B)^*_{C^*}$, and also, through the isomorphism T (Prop. C.17), with a linear section of

$(D_A^*)_{C^*}$. Again, we also identify $\Gamma(C) \cong \Gamma_{\text{core}}(D_B)$ and $\Gamma(C) \cong C^\infty_{\text{lin}}(C^*)$. We have the following

$$\rho_{D_A^*}(X)(\mathbf{c}) = (\Upsilon_{C^*}^{-1})^* \rho_{D_{C^*}}(X)(\Upsilon_{C^*}^*(\mathbf{c})) = -\rho_{D_{C^*}}(X)(-\mathbf{c}) = \rho_{D_{C^*}}(X)(\mathbf{c}),$$

then

$$((\nabla')^C)^*_X \mathbf{c} = \rho_{D_A^*}(X)(\mathbf{c}) = \rho_{D_{C^*}}(X)(\mathbf{c}) = \{X, \mathbf{c}\} = [X, \mathbf{c}]_D = \nabla^C_X \mathbf{c}, \tag{E.33}$$

whence $(\nabla')^C = (\nabla^C)^*$. Analogously, since Υ_{C^*} is the identity on the sides $A \longrightarrow A$ and on the cores $B^* \longrightarrow B^*$, we have

$$\begin{aligned}(\nabla^B)^*_X(\beta) = \rho_D(X)(\beta) = \{X, \beta\} = [X, \beta]_{D_{C^*}} = \Upsilon_{C^*}\left([\Upsilon_{C^*}^{-1} X, \Upsilon_{C^*}^{-1}\beta]_{D_{C^*}}\right) \\ = [X, \beta]_{D_A^*} = (\nabla')^{B^*}_X \beta,\end{aligned} \tag{E.34}$$

that is $(\nabla')^{B^*} = (\nabla^B)^*$. Thence $\nabla' = \nabla^*$.

Finally, we compute K'. Observe that, if we denote by $p_{\widehat{A}} : \widehat{A} \longrightarrow B^* \otimes C$ the projection induced by the horizontal lift ψ, then

$$K(X_1, X_2) = -p_{\widehat{A}}([X_1, X_2]),$$

where, again, we are identifying A with $\psi(A) \subset \widehat{A}$.

On the other hand, denoting by K_{C^*} the curvature form corresponding to the VB-algebroid $(D_B^*)^*_{C^*}$, we get, using Eq. (E.32),

$$\begin{aligned}K_{C^*}(X_1, X_2) = -p_{\widehat{A}_{C^*}}\left([X_1, X_2]_{(D_B^*)^*_{C^*}}\right) = -p_{\widehat{A}_{C^*}}\left(Z\left([X_1, X_2]_{D_B}\right)\right) \\ = - * \circ p_{\widehat{A}}\left([X_1, X_2]_{D_B}\right) = K(X_1, X_2)^*,\end{aligned}$$

thence,

$$\begin{aligned}K'(X_1, X_2) = \Upsilon_{C^*}^{-1}(K_{C^*}(\Upsilon_{C^*}(X_1), \Upsilon_{C^*}(X_2))) = \Upsilon_{C^*}^{-1}(K_{C^*}(X_1, X_2)) \\ = -K_{C^*}(X_1, X_2) = -K(X_1, X_2)^*,\end{aligned}$$

that is, $K' = -K^*$.

■

E.3.3 The de Rham operator as a representation up to homotopy

In this subsection we want to explore the relation between the de Rham operator d_D of a VB-algebroid and its corresponding representation up to homotopy operator D. Basically what we do is to fill-in the details of what is said to this regard in Sec. 4.6 of [23]. The results of this subsection are not needed in the rest of this work, we present them just for completeness and illustrative purposes.

Lemma E.34. *Let D be a double vector bundle. There is a subalgebra $C^{\cdot,\cdot}(D)$ of $C^\infty(D)$, the* bi-graded *functions, which allow a bi-graded structure, that is, each function $f \in$*

$C^{\cdot,\cdot}(D)$ can be endowed with a well-defined bi-degree (p,q), which is compatible with the algebra structure, in the sense that if $f \in C^{p,q}(D)$ and $g \in C^{r,s}(D)$, then $fg \in C^{p+r,q+s}(D)$.

Locally, given an adapted coordinate system $(x^i, \alpha^a, \beta^b, \kappa^c)$, the algebra $C^{\cdot,\cdot}(D)$ is $C^\infty(M)-$spanned by α^a, with degree $(1,0)$; β^b, with degree $(0,1)$; and κ^c, with degree $(1,1)$.

Proof. First locally, given adapted coordinates $(x^i, \alpha^a, \beta^b, \kappa^c)$, we assign to these coordinate functions the following bi-degrees:

- x^i are assigned bi-degree $(0,0)$;
- α^a are assigned bi-degree $(1,0)$;
- β^b are assigned bi-degree $(0,1)$;
- κ^c are assigned bi-degree $(1,1)$.

It follows from Prop. A.23 that this bi-grading is well defined globally, and can be extended to the polynomial algebra generated locally by these coordinate functions, which yields the sub-algebra $C^{\cdot,\cdot}(D)$.

∎

Remark E.35. Notice that the bi-degree $(1,1)$ functions coincide with the double-linear functions, introduced in Def. C.30.

Lemma E.36. *Consider a double vector bundle D. There is a subalgebra $\Omega^{\cdot,\cdot}(D_B)$ of $\Omega(D_B) = \Gamma(\Lambda^\cdot D_B^*)$ which allows a bi-grading.*

Proof. We may identify $\Gamma(D_B^*)$ with the space of functions on D that are linear over B. This space of linear functions contains a subspace, formed by the bi-graded functions of bi-degree $(1,q)$, $q \geq 0$. Denote this subspace by $\Gamma_{\text{pol}}(D_B^*)$. This space is single-graded, and the grading can be extended to $\Lambda^\cdot(\Gamma_{\text{pol}}(D_B^*))$, which is a subalgebra of $\Omega(D_B)$. Since the exterior algebra also has a natural grading, we obtain a bi-graded structure in the subalgebra

$$\Omega^{\cdot,\cdot}(D_B) := \Lambda^\cdot(\Gamma_{\text{pol}}(D_B^*)).$$

∎

Definition E.37. The space of *cochains linear over A* is formed by the elements of $\Omega^{\cdot,\cdot}(D_B)$ of bi-degree $(p,1)$, and is denoted by $\Omega_{\text{lin}}(D_B)$.

Lemma E.38. *By a choice of a decomposition of D, the exterior algebra of its dual D_B^*, $\Gamma(\Lambda^\cdot D_B^*)$, can be decomposed as*

$$\Gamma(\Lambda^\cdot D_B^*) \cong \Gamma(\Lambda^\cdot A^*) \otimes C^\infty(B) \otimes \Gamma(\Lambda^\cdot C^*). \tag{E.35}$$

Using this decomposition, we may describe the subspace $\Omega_{\text{lin}}(D_B)$ as

$$\Omega_{\text{lin}}(D_B) = \Lambda\Gamma(A^*) \otimes (C^\infty_{\text{lin}}(B) \oplus \Lambda^1\Gamma(C^*)) = \Omega(A) \otimes (\Gamma(B^*) \oplus \Gamma(C^*)). \tag{E.36}$$

Proof. The decomposition of Eq. (E.35) is immediate from $D_B^* \cong (q^B)^*(A^*) \underset{B}{\oplus} (q^B)^*(C^*)$. Now from this decomposition we obtain

$$\Omega^{\cdot,\cdot}(D_B) \cong \Gamma(\Lambda^{\cdot} A^*) \otimes C^{\mathrm{pol}}(B) \otimes \Gamma(\Lambda^{\cdot} C^*),$$

where $C^{\mathrm{pol}}(B)$ is the subalgebra of $C^\infty(B)$ of polynomial functions, which are described locally by the algebra of polynomials in the linear coordinates β^b. By the way the grading (over A) of $\Gamma_{\mathrm{pol}}(D_B^*)$ is extended to $\Omega^{\cdot,\cdot}(D_B)$, it follows that the product by a linear function of B or the wedge product by a section of C^* contribute to the bi-grading (p,q) by increasing q. Thus, it follows that the elements of bi-degree $(p,1)$ are necessarily the elements in

$$\Lambda^p\Gamma(A^*) \otimes (C^\infty_{\mathrm{lin}}(B) \oplus \Lambda^1\Gamma(C^*)),$$

space which equals to

$$\Omega^p(A) \otimes (\Gamma(B^*) \oplus \Gamma(C^*)).$$

■

Proposition E.39. *Let D_B be a decomposed VB-algebroid. Consider the induced de Rham differential d_D on $\Gamma(\Lambda^{\cdot} D_B^*)$. Then the space $\Omega_{\mathrm{lin}}(D_B) \subset \Gamma(\Lambda^{\cdot} D_B^*)$ is invariant under d_D, and the restriction of d_D to this subspace is a representation up to homotopy of A on $E' = B^*[-1] \oplus C^*$, which we denote by*

$$D' := d_D|_{\Omega_{\mathrm{lin}}(D_B)}.$$

Under the identification

$$E' = B^*[-1] \oplus C^* \cong \ker(\pi_A : (D_B^*)^*_{C^*} \longrightarrow A),$$

we get a duality pairing between E' and

$$E = C \oplus B[1] \cong \ker(q_A : D \longrightarrow A),$$

namely, the pairing $(\cdot|\cdot)$ over A given by Prop. B.11, between D and $(D_B^*)^*_{C^*}$, where D_B^* is playing the role of D in Prop. B.11.

With respect to this pairing, D' coincides with the representation dual to $D = \partial+\nabla+K$, given in Prop. E.21.

Proof. We need to verify that $d_D|_{\Omega_{\mathrm{lin}}(D_B)} = D^*$, which, by Prop. E.31, amounts to show that $d_D|_{\Omega_{\mathrm{lin}}(D_B)} = \partial^* + \nabla - K^*$, where the adjoints are with respect to the pairing $(\cdot|\cdot)$ over A –this is important in order to get the right signs, the difference being that the pairing between C and C^* using $(\cdot|\cdot)$ has the opposite sign. Let's compute.

For $\beta \in \Gamma(B^*)$ and $\mathbf{c} \in \Gamma(C)$ we have

$$(d_D(\beta)|\mathbf{c}) = -\langle d_D(\beta), \widetilde{\mathbf{c}}\rangle_B = -\rho_D(\widetilde{\mathbf{c}})(\beta) = \langle \partial(\mathbf{c}), \beta\rangle = (\partial(\mathbf{c})|\beta),$$

thus, if $d_D|_{\Omega_{\mathrm{lin}}(D_B)} = \partial' + \nabla' + K'$, we have so far $\partial' = \partial^*$.

For $\beta \in \Gamma(B^*), \mathbf{b} \in \Gamma(B)$ and $X \in \Gamma(A)$, we have

$$(d_D(\beta)|\mathbf{b})(X) = \langle\langle d_D(\beta), \widehat{X}\rangle_B, \mathbf{b}\rangle = \langle \rho_D(X)(\beta), \mathbf{b}\rangle = \langle \nabla^*_X(\beta), \mathbf{b}\rangle = (\nabla^*_X(\beta)|\mathbf{b});$$

for $\kappa \in \Gamma(C^*), \mathbf{c} \in \Gamma(C)$ and $X \in \Gamma(A)$,

$$\begin{aligned}(d_D(\kappa)|\mathbf{c})(X) &= -\iota_{\widetilde{\mathbf{c}}}\iota_{\widehat{X}} d_D(\widehat{\kappa}) = -\rho_D(\widehat{X})\langle\widehat{\kappa}, \widetilde{\mathbf{c}}\rangle_D + \rho_D(\widehat{\mathbf{c}})\langle\widehat{\kappa}, \widehat{X}\rangle + \langle\widehat{\kappa}, [\widehat{X}, \widetilde{\mathbf{c}}]\rangle_D \\ &= -\rho(X)\langle\kappa, \mathbf{c}\rangle + \langle\kappa, \nabla_X \mathbf{c}\rangle = -\langle\nabla_X^{*_C}\kappa, \mathbf{c}\rangle = (\nabla_X^*\kappa|\mathbf{c}),\end{aligned}$$

where ∇^{*_C} is denoting the connection dual with respect to the usual pairing between C and C^*. Thus, $\nabla' = \nabla^*$.

Finally, for $\kappa \in \Gamma(C^*), X, Y \in \Gamma(A)$ and $\mathbf{b} \in \Gamma(B)$,

$$\begin{aligned}(d_D(\kappa)(X, Y)|\mathbf{b}) &= \langle\langle d_D(\widehat{\kappa}), \widehat{X} \wedge \widehat{Y}\rangle_B, \mathbf{b}\rangle \\ &= \langle\rho_D(\widehat{X})(\langle\widehat{\kappa}, \widehat{Y}\rangle_B), \mathbf{b}\rangle - \langle\rho_D(\widehat{Y})(\langle\widehat{\kappa}, \widehat{X}\rangle_B), \mathbf{b}\rangle - \langle\langle\widehat{\kappa}, [\widehat{X}, \widehat{Y}]_D\rangle_D, \mathbf{b}\rangle \\ &= \langle K(X, Y)(\mathbf{b}), \kappa\rangle = -(K(X, Y)(\mathbf{b})|\kappa) \\ &= -(K(X, Y)^*(\kappa)|\mathbf{b}),\end{aligned}$$

hence, $K' = -K^*$.

■

Appendix F

The main examples: TA and T^*A

F.1 The double vector bundle structure on TA and its duals

Consider a vector bundle $A \xrightarrow{q^A} M$, then its tangent bundle comes with a double vector bundle structure

$$\begin{tikzcd} TA \arrow[r, "q_A"] \arrow[d, "q_{TM}"'] \arrow[dr, phantom, "A"] & A \arrow[d, "q^A"] \\ TM \arrow[r, "q^{TM}"] & M \end{tikzcd} \tag{F.1}$$

where $q_{TM} := dq^A$, where d is the differential. Addition with respect to the vertical structure ($TA \xrightarrow{dq^A} TM$) is again given by a differential:

$$d+ : T(A \times A) \cong TA \times TA \longrightarrow TA,$$

where $+ : A \times A \longrightarrow A$ is the addition in $A \xrightarrow{q^A} M$.

Finally, the zero section 0_{TM} is once more given by a differential: $0_{TM} := d(0^A) : TM \longrightarrow TA$.

The two duals corresponding to TA are

$$\begin{tikzcd} T^*A \arrow[r, "\pi_A"] \arrow[d, "\pi_{A^*}"'] \arrow[dr, phantom, "T^*M"] & A \arrow[d, "q^A"] \\ A^* \arrow[r, "q^{A^*}"] & M \end{tikzcd} ; \qquad \begin{tikzcd} TA^* \arrow[r, "\pi_{A^*}"] \arrow[d, "\pi_{TM}"'] \arrow[dr, phantom, "A^*"] & A^* \arrow[d, "q^A"] \\ TM \arrow[r, "q^{TM}"] & M \end{tikzcd} \tag{F.2}$$

with cores T^*M and A^*, respectively.

Dualizing TA^* with respect to A^*, we obtain the double vector bundle

$$\begin{tikzcd} T^*A^* \arrow[r, "q_{A^*}"] \arrow[d, "q_A"'] \arrow[dr, phantom, "T^*M"] & A^* \arrow[d, "q^{A^*}"] \\ A \arrow[r, "q^A"] & M \end{tikzcd}, \tag{F.3}$$

which is identified with T^*A through the isomorphism Υ_{A^*} given in Prop. B.13. This isomorphism

$$\Upsilon_{A^*} : T^*A^* \xrightarrow{\cong} T^*A \tag{F.4}$$

is called *Legendre transform*.

F.2 The linear bundles: jets and covariant differential operators

The linear bundle corresponding to $TA \longrightarrow A$ is the bundle of *linear vector fields*. The linear vector fields, which by definition are the linear sections of $TA \longrightarrow A$, can be characterized by the property of living invariant the linear functions of A, when acting on functions as vector fields. This fact allows us to identify linear vector fields with covariant differential operators of A^*, since $C^\infty_{\text{lin}}(A) \cong \Gamma(A^*)$, and fits in the exact sequence

$$A \otimes A^* \longrightarrow \mathbf{CDO}(A^*) \xrightarrow{p} TM, \tag{F.5}$$

which in turn is canonically identified with the bundle of covariant differential operators of A

$$A^* \otimes A \longrightarrow \mathbf{CDO}(A) \longrightarrow TM, \tag{F.6}$$

through the map

$$\begin{array}{rccc} T : & \mathbf{CDO}(A^*) & \longrightarrow & \mathbf{CDO}(A) \\ & \zeta^* & \longrightarrow & \zeta, \end{array}$$

where ζ is defined by

$$\langle \zeta(\mathbf{a}), \alpha \rangle := p(\zeta^*)(\langle \mathbf{a}, \alpha \rangle) - \langle \zeta^*(\alpha), \mathbf{a} \rangle, \tag{F.7}$$

for $\mathbf{a} \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$.

Notice that, because of what we observed above, $\mathbf{CDO}(A)$ is identified with the linear bundle of $TA^* \longrightarrow A^*$, which is the TM-dual of TA. With this considerations, it is easily seen that the map T, defined above, coincides with the map, also denoted by T, given by Prop. C.17.

The linear bundle corresponding to $TA \longrightarrow TM$ is $C^\infty(M)$-spanned by sections of the form

$$\mathcal{T}\mathbf{a} : TM \longrightarrow TA, \qquad \mathbf{a} \in \Gamma(A), \tag{F.8}$$

where $\mathcal{T}\mathbf{a}$ is the differential of $\mathbf{a}$, viewed as a map $\mathbf{a} : M \longrightarrow A$. Under the correspondence

$$\mathcal{T}\mathbf{a} \leftrightsquigarrow j^1\mathbf{a}, \quad \forall \mathbf{a} \in \Gamma(A), \tag{F.9}$$

where $j^1\mathbf{a}$ is the first jet prolongation of $\mathbf{a}$, the linear bundle corresponding to $\Gamma_{\text{lin}}(TA \longrightarrow TM)$ can be identified with the first jet bundle associated to A, and fits in the exact sequence

$$0 \longrightarrow T^*M \otimes A \longrightarrow J^1A \longrightarrow A \longrightarrow 0, \tag{F.10}$$

where the inclusion $T^*M \otimes A \xrightarrow{\iota} J^1A$ is given by

$$\iota(df \otimes \mathbf{a}) := j^1(f\mathbf{a}) - fj^1(\mathbf{a}). \tag{F.11}$$

We use the notation $\mathcal{T}$ for the differential, to distinguish it from the differential $d\mathbf{a} \in \Gamma(T^*A^*)$, where we see $\mathbf{a}$ as a linear function on A^* thanks to the identification $\Gamma(A) \cong C^\infty_{\text{lin}}(A^*)$. Actually, if we denote by $\widehat{A}_{TM}$ the linear bundle corresponding to $TA \longrightarrow TM$ (which is isomorphic to J^1A, as we just saw), and if we denote by $\widehat{A}_{A^*}$ the linear bundle corresponding to $T^*A^* \longrightarrow A^*$, then the assignment

$$\begin{aligned} \Phi : \widehat{A}_{TM} &\longrightarrow \widehat{A}_{A^*} \\ \mathcal{T}\mathbf{a} &\longrightarrow d\mathbf{a} \end{aligned} \tag{F.12}$$

determines, extending by $C^\infty(M)$-linearity, a vector bundle morphism, that coincides exactly with the isomorphism Z of Prop. C.33. To verify this we first need the following lemma.

Lemma F.1. *Let $A \longrightarrow M$ be a vector bundle. Take $v \in TA^*$ with $q_{A^*}(v) = \alpha \in A^*_m$ and $q_{TM}(v) = x \in T_mM$. Let $\mathbf{a} \in \Gamma(A)$, then*

$$\langle v, d\mathbf{a}(\alpha)\rangle_{A^*} = \langle v, \mathcal{T}\mathbf{a}(x)\rangle_{TM}. \tag{F.13}$$

Proof. After introducing a decomposition for TA^* through a connection (see Sec. F.4 below), let $q_{\overline{A^*}}(v) = \alpha'$, where $\overline{A^*}$ is the core bundle of TA^*, which is isomorphic to A^*. Then, from Eq. (B.13) we obtain

$$\langle v, d\mathbf{a}(\alpha)\rangle_{A^*} = \langle v, (d\mathbf{a}(\alpha) - \langle \nabla.\mathbf{a}, \alpha\rangle) + \langle \nabla.\mathbf{a}, \alpha\rangle\rangle_{A^*} = \langle \alpha', \mathbf{a}(m)\rangle + \langle \nabla_x\mathbf{a}, \alpha\rangle. \tag{F.14}$$

Similarly,

$$\langle v, \mathcal{T}\mathbf{a}(x)\rangle_{TM} = \langle v, (\mathcal{T}\mathbf{a}(x) - \nabla_x\mathbf{a}) + \nabla_x\mathbf{a}\rangle_{TM} = \langle \alpha', \mathbf{a}(m)\rangle + \langle \nabla_x\mathbf{a}, \alpha\rangle. \tag{F.15}$$

From (F.14) and (F.15) we obtain (F.13).

■

Corollary F.2. *The map $\Phi : \widehat{A}_{TM} \longrightarrow \widehat{A}_{A^*}$, defined by (F.12), coincides with the isomorphism Z given in Prop. C.33. Its inverse is given by $\Phi^{-1}(d\mathbf{a}) = \mathcal{T}\mathbf{a}$.*

Proof. Let's revisit Prop. C.33. If we take into account the definitions of Z_B (Eq. (C.52)) and Z_A (Eq. (C.53)), given by Eqs. (C.51) and (C.54), respectively, then we see that $Z = Z_B \circ Z_A^{-1}$ is determined by the equation, for $\sigma \in (\widehat{C^*}_A)_m$,

$$\langle Z(\sigma)(b), v\rangle_B = \langle \sigma(a), v\rangle_A, \qquad \forall a \in A_m, b \in B_m, v \in D_a \cap D_b.$$

In our case, where $D = TA^*$, $\widehat{C^*}_A = \widehat{A}_{TM}$ and $\widehat{C^*}_B = \widehat{A}_{A^*}$, the equation above, for a section $\sigma \in \Gamma(\widehat{A}_{TM}) \cong \Gamma_{\mathrm{lin}}(TA_{TM})$ reads

$$\langle Z(\sigma)(\alpha), v\rangle_{A^*} = \langle \sigma(x), v\rangle_{TM}, \tag{F.16}$$

where $x \in \Gamma(TM), \alpha \in \Gamma(A^*)$, and $v \in \Gamma(TA^*)$ is such that $q_{TM}(v) = x$ and $q_{A^*} = \alpha$.

Now, if we take σ of the form $\sigma = \mathcal{T}\mathbf{a}$, the equation (F.13) shows precisely that

$$\langle \Phi(\mathcal{T}\mathbf{a})(\alpha), v\rangle_A = \langle \mathcal{T}\mathbf{a}(x), v\rangle_{TM},$$

which, in view of Eq. (F.16), shows that $Z = \Phi$.

Since, by what we discussed above, $Z^{-1} = Z_A \circ Z_B^{-1}$ is determined by the equation

$$\langle Z^{-1}(\tau)(a), v\rangle_A = \langle \tau(b), v\rangle_B, \qquad \forall a \in A_m, b \in B_m, v \in D_a \cap D_b,$$

it follows that $\Phi^{-1}(d\mathbf{a}) = \mathcal{T}\mathbf{a}$.

■

F.3 The *VB*-algebroid structures

Given a Lie algebroid structure on A, $([\cdot,\cdot],\rho)$, we can construct canonically a *VB*-algebroid structure on $TA \longrightarrow TM$ by the following formulas

$$\begin{array}{lll} \rho(\overline{\mathbf{a}})(f) := 0; & \rho(j^1\mathbf{a})(f) := \rho(\mathbf{a})(f); & [\overline{\mathbf{a}}, \overline{\mathbf{b}}] := 0; \\ \rho(\overline{\mathbf{a}})(df) := \rho(\mathbf{a})(f); & \rho(j^1\mathbf{a})(df) := d(\rho(\mathbf{a})(f)); & [j^1\mathbf{a}, \overline{\mathbf{b}}] := \overline{[\mathbf{a}, \mathbf{b}]}; \\ & & [j^1\mathbf{a}, j^1\mathbf{b}] := j^1[\mathbf{a}, \mathbf{b}]; \end{array} \tag{F.17}$$

where, for example, $\overline{\mathbf{a}}$ is the core section corresponding to $\mathbf{a} \in \Gamma(A)$.

The linear bundle corresponding to $T^*A \longrightarrow A^*$ is denoted by J^1A_*, which fits in the exact sequence

$$A \otimes T^*M \longrightarrow J^1A_* \longrightarrow A,$$

and we have a canonical isomorphism

$$T : J^1A \xrightarrow{\cong} J^1A_*,$$

given by Prop. C.17.

By Cor. E.25, we have a *VB*-algebroid structure on T^*A, which comes from T^*A^* by the Legendre transform Υ_{A^*}, see Eq. (F.4), where the *VB*-algebroid structure on T^*A^* is induced from the *VB*-algebroid structure of $TA \longrightarrow TM$, since the TM-dual of TA, namely TA^*, has an induced Poisson structure which is linear with respect to both fibrations (this is the content of Prop. E.24 and its corollary E.25).

From Eq. (E.27) and Cor. F.2, we get the following formulas

$$\begin{array}{lll} \rho(\overline{df})(g) = 0; & \rho(d\mathbf{a})(f) = \rho(\mathbf{a})(f); & [\overline{df}, \overline{dg}] = 0; \\ \rho(\overline{df})(\mathbf{a}) = -\rho(\mathbf{a})(f); & \rho(d\mathbf{a})(\mathbf{b}) = [\mathbf{a}, \mathbf{b}]; & [d\mathbf{a}, \overline{df}] = \overline{d\rho(\mathbf{a})(f)}; \\ & & [d\mathbf{a}, d\mathbf{b}] = d[\mathbf{a}, \mathbf{b}], \end{array} \tag{F.18}$$

where, as usual, we are identifying fiberwise linear functions on a vector bundle with sections of its dual. Thus, we see that the VB-algebroid structure on T^*A^* is given by the cotangent Lie algebroid structure corresponding to the linear Poisson structure on A^*, induced by the Lie algebroid structure on A.

F.4 Natural horizontal lifts induced from a connection

We know that a decomposition of TA is equivalent to a horizontal lift of (F.5) and also equivalent to a horizontal lift of (F.10). Now, a horizontal lift of (F.5), or of (F.10), is equivalent to the choice of a connection on A. Indeed, the correspondence is as follows: given a connection ∇ of A, the horizontal lift of (F.5) is given by

$$\psi : TM \longrightarrow \mathbf{CDO}(A^*), \qquad \psi(x) := \nabla^*_x, \tag{F.19}$$

where ∇^* is the connection dual to ∇, while the horizontal lift of (F.10) is given by

$$\overline{\psi} : A \longrightarrow J^1A, \qquad \overline{\psi}(\mathbf{a}) := j^1(\mathbf{a}) - \nabla.\mathbf{a}, \tag{F.20}$$

where $j^1 : \Gamma(A) \longrightarrow \Gamma(J^1A)$ is the map that assigns to a section of A its first jet prolongation. A natural question is whether this horizontal lift is the one that corresponds to ψ, according to Prop. C.6. The answer is affirmative, and we give it in the next proposition. First we need a lemma.

Lemma F.3. *Let $v \in TA$ and $w \in TA^*$ with $q_{TM}(v) = \pi_{TM}(w) = x$, so that $v = \frac{d}{dt}\varsigma(t)|_{t=0}$ and $w = \frac{d}{dt}\tau(t)|_{t=0}$, where ς and τ are curves in A and A^*, respectively, such that $q^A(\varsigma(t)) = q^{A^*}(\tau(t)) =: m(t)$ and $\frac{d}{dt}q^A \circ \varsigma(t)|_{t=0} = \frac{d}{dt}q^{A^*} \circ \tau(t)|_{t=0} = x$. Then*

$$\langle v, w\rangle_{TM} = x(\langle \varsigma(t), \tau(t)\rangle) = \frac{d}{dt}\bigg|_{t=0} \langle \varsigma(t), \tau(t)\rangle. \tag{F.21}$$

Proof. We can work locally, and suppose that $A = M \times V$ and $A^* = M \times V^*$, where V is a vector space. Then $TA = TM \times V \times V$ and $TA^* = TM \times V^* \times V^*$, so that $v = (x, a_1, a_2) \in T_mM \times V \times V$ and $w = (x, \alpha_1, \alpha_2) \in T_mM \times V^* \times V^*$. Then

$$v = \frac{d}{dt}\bigg|_{t=0} (m(t), a_1 + ta_2), \qquad w = \frac{d}{dt}\bigg|_{t=0} (m(t), \alpha_1 + t\alpha_2). \tag{F.22}$$

Now, on one hand, by Eq. (B.13), from Prop. B.6, we have

$$\langle v, w\rangle_{TM} = \langle a_1, \alpha_2\rangle + \langle a_2, \alpha_1\rangle. \tag{F.23}$$

On the other hand, by (F.22),

$$x(\langle \varsigma(t), \tau(t)\rangle) = \frac{d}{dt}\bigg|_{t=0} \langle a_1 + ta_2, \alpha_1 + t\alpha_2\rangle = \langle a_1, \alpha_2\rangle + \langle a_2, \alpha_1\rangle. \tag{F.24}$$

Comparing (F.23) and (F.24), we get (F.21).

∎

Proposition F.4. *The horizontal lift $\overline{\psi}$ given in Eq. (F.20) is precisely the one given by Prop. C.6, that is,*

$$\overline{\psi}(a)(x) = \psi(x)(a), \tag{F.25}$$

holds for every $a \in A_m,\ x \in T_m M$, where ψ is given in Eq. (F.19).

Proof. On one hand we have, for $\alpha \in \Gamma(A^*)$,

$$\langle \psi(x)(a), d\alpha(a) \rangle_A = \langle \nabla^*_x \alpha, a \rangle = x(\alpha, \mathbf{a} \rangle) - \langle \nabla_x \mathbf{a}, \alpha \rangle, \tag{F.26}$$

where $\mathbf{a} \in \Gamma(A)$ is an extension of a, that is, $\mathbf{a}(m) = a$.

On the other hand, by lemma F.1, with A^* playing the role of A, and taking into account the correspondence given in (F.9),

$$\begin{aligned} \langle \overline{\psi}(a)(x), d\alpha(a) \rangle_A = \langle \overline{\psi}(a)(x), \mathcal{T}\alpha(x) \rangle_{TM} &= \langle \mathcal{T}\mathbf{a}(x) - \nabla_x \mathbf{a}, \mathcal{T}\alpha(x) \rangle_{TM} \\ &= x(\langle \mathbf{a}, \alpha \rangle) - \langle \nabla_x \mathbf{a}, \alpha \rangle, \end{aligned} \tag{F.27}$$

where we used in the last line

$$\langle \mathcal{T}\mathbf{a}(x), \mathcal{T}\alpha(x) \rangle = x(\langle \mathbf{a}, \alpha \rangle),$$

which follows from (F.21) of lemma F.3.

Comparing (F.26) and (F.27) we get

$$\langle \overline{\psi}(a)(x), d\alpha \rangle_A = \langle \psi(x)(a), d\alpha \rangle_A. \tag{F.28}$$

Now, using Eq. (B.13) and the fact that ψ and $\overline{\psi}$ are horizontal lifts, we have, for $\overline{\alpha} \in \Gamma_{\text{core}}(TA^*)$,

$$\langle \overline{\psi}(a)(x), \overline{\alpha}(m) \rangle_{TM} = \langle a, \alpha(m) \rangle = \langle \psi(a)(x), \overline{\alpha}(m) \rangle_{TM}. \tag{F.29}$$

From (F.28) and (F.29), we conclude (F.25).

∎

F.5 The representations up to homotopy

Proposition F.5. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid. Consider the prolonged tangent Lie VB-algebroid $TA \longrightarrow TM$, defined by Eqs. (F.17). Let's introduce a connection $\widetilde{\nabla}$ on A, which provides a horizontal lift for (F.10), given by (F.20). By Thm. E.21 there corresponds a representation up to homotopy of A on $E = A \oplus T[1]M$. This representation is encoded by the data $D = \partial + \nabla + K$, given by Prop. E.20, where $\nabla = \nabla^A + \nabla^{TM}$. We claim that*

1. $\partial = -\rho$,
2. $\nabla^A_{\mathbf{a}_1} \mathbf{a}_2 = [\mathbf{a}_1, \mathbf{a}_2] + \widetilde{\nabla}_{\rho(\mathbf{a}_2)} \mathbf{a}_1$,
3. $\nabla^{TM}_{\mathbf{a}} \mathbf{x} = [\rho(\mathbf{a}), \mathbf{x}] + \rho(\widetilde{\nabla}_{\mathbf{x}} \mathbf{a})$,

4. $K(\mathbf{a}_1,\mathbf{a}_2)(\mathbf{x}) = [\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_1,\mathbf{a}_2] + [\mathbf{a}_1,\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_2] - \widetilde{\nabla}_{\mathbf{x}}[\mathbf{a}_1,\mathbf{a}_2] + \widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_1}\mathbf{x}}\mathbf{a}_2 - \widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_2}\mathbf{x}}\mathbf{a}_1,$

for $\mathbf{a},\mathbf{a}_1,\mathbf{a}_2 \in \Gamma(A), \mathbf{x} \in \Gamma(TM)$.

Proof. From $\rho(\overline{\mathbf{a}})(df) = \rho(\mathbf{a})(f)$ and Eq. (E.4), it follows $\partial = -\rho$.

From Eq. (E.12) and $[j^1\mathbf{a}_1,\overline{\mathbf{a}_2}] = \overline{[\mathbf{a}_1,\mathbf{a}_2]}$ it follows

$$\begin{aligned}\nabla^A_{\mathbf{a}_1}\mathbf{a}_2 &= [\widehat{\mathbf{a}}_1,\overline{\mathbf{a}_2}] = [j^1\mathbf{a}_1 - \widetilde{\nabla}.\mathbf{a}_1,\overline{\mathbf{a}_2}]\\ &= [\mathbf{a}_1,\mathbf{a}_2] - [\nabla.\mathbf{a}_1,\overline{\mathbf{a}_2}]\\ &= [\mathbf{a}_1,\mathbf{a}_2] + \widetilde{\nabla}_{\rho(\mathbf{a}_2)}\mathbf{a}_1,\end{aligned}$$

where, for the last line, we used Prop. E.8.

From Eq. (E.12) and $\rho(j^1\mathbf{a})(df) = d\rho(\mathbf{a})(f)$ it follows, for any $\eta \in \Gamma(T^*M)$,

$$\begin{aligned}\nabla^{T^*M}_{\mathbf{a}}\eta &= \rho(\widehat{\mathbf{a}})(\eta) = \rho(j^1\mathbf{a} - \widetilde{\nabla}.\mathbf{a})(\eta)\\ &= d(\langle\rho(\mathbf{a}),\eta\rangle) - \langle\rho(\widetilde{\nabla}.\mathbf{a}),\eta\rangle,\end{aligned}$$

where, for the last line, we used Prop. E.8. Therefore

$$\begin{aligned}\langle\nabla^{TM}_{\mathbf{a}},\eta\rangle &= \rho(\mathbf{a})(\langle\mathbf{x},\eta\rangle) - \langle\mathbf{x},\nabla^{T^*M}_{\mathbf{a}}\eta\rangle\\ &= \rho(\mathbf{a})(\langle\mathbf{x},\eta\rangle) - \mathbf{x}(\langle\rho(\mathbf{a}),\eta\rangle) + \langle\rho(\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}),\eta\rangle.\end{aligned} \tag{F.30}$$

Now, from Cartan's calculus we have

$$\begin{aligned}\rho(\mathbf{a})(\langle\mathbf{x},\eta\rangle) &= \langle[\rho(\mathbf{a}),\mathbf{x}],\eta\rangle + \langle\mathbf{x},\mathcal{L}_{\rho(\mathbf{a})}\eta\rangle\\ &= \langle[\rho(\mathbf{a}),\mathbf{x}],\eta\rangle + \mathbf{x}(\langle\rho(\mathbf{a}),\eta\rangle) + d\eta(\rho(\mathbf{a}),\mathbf{x}).\end{aligned} \tag{F.31}$$

Observe that, in order to compute $\langle\nabla^{TM}_{\mathbf{a}}\mathbf{x},\eta\rangle$ we are interested only in the value of η at one point $m \in M$. Given any $w \in T^*_m M$ we can always find $\eta \in \Gamma(T^*M)$ with $\eta(m) = w$ and $d\eta = 0$, whence, from (F.30) and (F.31) we obtain

$$\nabla^{TM}_{\mathbf{a}}\mathbf{x} = [\rho(\mathbf{a}),\mathbf{x}] + \rho(\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}).$$

Finally, from Eq. (E.15) and $[j^1\mathbf{a}_1,j^1\mathbf{a}_2] = j^1[\mathbf{a}_1,\mathbf{a}_2]$ we have

$$\begin{aligned}K(\mathbf{a}_1,\mathbf{a}_2) &= \widehat{[\mathbf{a}_1,\mathbf{a}_2]} - [\widehat{\mathbf{a}}_1,\widehat{\mathbf{a}}_2]\\ &= j^1[\mathbf{a}_1,\mathbf{a}_2] - \widetilde{\nabla}.[\mathbf{a}_1,\mathbf{a}_2] - [j^1\mathbf{a}_1 - \widetilde{\nabla}.\mathbf{a}_1, j^1\mathbf{a}_2 - \widetilde{\nabla}.\mathbf{a}_2]\\ &= -\widetilde{\nabla}.[\mathbf{a}_1,\mathbf{a}_2] + [j^1\mathbf{a}_1,\widetilde{\nabla}.\mathbf{a}_2] + [\widetilde{\nabla}.\mathbf{a}_1,j^1\mathbf{a}_2] - [\widetilde{\nabla}.\mathbf{a}_1,\widetilde{\nabla}.\mathbf{a}_2].\end{aligned} \tag{F.32}$$

Now, using Eq. (E.14), we compute, for $\mathbf{a} \in \Gamma(A)$ and $\phi \in \Gamma(T^*M \otimes A)$,

$$\begin{aligned}[j^1\mathbf{a},\phi] &= [\widehat{\mathbf{a}} + \widetilde{\nabla}.\mathbf{a},\phi]\\ &= [\widehat{\mathbf{a}},\phi] + [\widetilde{\nabla}.\mathbf{a},\phi]\\ &= \nabla^A_{\mathbf{a}} \circ \phi - \phi \circ \nabla^{TM}_{\mathbf{a}} + [\widetilde{\nabla}.\mathbf{a},\phi].\end{aligned}$$

Then

$$[j^1\mathbf{a}_1,\widetilde{\nabla}.\mathbf{a}_2](\mathbf{x})+[\widetilde{\nabla}.\mathbf{a}_1,j^1\mathbf{a}_2](\mathbf{x})=\nabla^A_{\mathbf{a}_1}\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_2-\widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_2}\mathbf{x}}\mathbf{a}_1+[\widetilde{\nabla}.\mathbf{a}_1,\widetilde{\nabla}.\mathbf{a}_2](\mathbf{x}) \\ -\nabla^A_{\mathbf{a}_2}\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_1+\widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_2}\mathbf{x}}\mathbf{a}_1-[\widetilde{\nabla}.\mathbf{a}_2,\widetilde{\nabla}.\mathbf{a}_1](\mathbf{x}). \quad \text{(F.33)}$$

Now, from items 1. and 3. of the statement,

$$\nabla^A_{\mathbf{a}_i}\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_j=[\mathbf{a}_i,\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_j]+\widetilde{\nabla}_{\rho(\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_j)}\mathbf{a}_i. \quad \text{(F.34)}$$

From (E.5) and item 1. of the statement, we have

$$[\widetilde{\nabla}.\mathbf{a}_1,\widetilde{\nabla}.\mathbf{a}_2](\mathbf{x})=\widetilde{\nabla}_{\rho(\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_1)}\mathbf{a}_2-\widetilde{\nabla}_{\rho(\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_2)}\mathbf{a}_1. \quad \text{(F.35)}$$

Therefore, from (F.32), (F.33), (F.34) and (F.35), we obtain

$$K(\mathbf{a}_1,\mathbf{a}_2)(\mathbf{x})=[\mathbf{a}_1,\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_2]-[\mathbf{a}_2,\widetilde{\nabla}_{\mathbf{x}}\mathbf{a}_1]-\widetilde{\nabla}_{\mathbf{x}}[\mathbf{a}_1,\mathbf{a}_2]+\widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_1}\mathbf{x}}\mathbf{a}_2-\widetilde{\nabla}_{\nabla^{TM}_{\mathbf{a}_2}\mathbf{x}}\mathbf{a}_1. \quad \text{(F.36)}$$

■

Corollary F.6. *In the conditions of Prop. F.5, consider the cotangent Lie VB-algebroid $T^*A \longrightarrow A$, defined by Eqs. (F.18). The corresponding representation up to homotopy, raised by the horizontal lift induced by $\widetilde{\nabla}$, of A on $E^* = T^*[-1]M \oplus A^*$ is given by $D' = \partial' + \nabla' + K'$, with $\nabla' = \nabla^{T^*M} + \nabla^{A^*}$, such that*

1. $\partial' = -\rho^*$,
2. $\nabla^{T^*M}_{\mathbf{a}}\eta = \mathcal{L}_{\rho(\mathbf{a})}\eta - \langle \widetilde{\nabla}.\mathbf{a}, \rho^*(\eta)\rangle$,
3. $\nabla^{A^*}_{\mathbf{a}}\alpha = \mathcal{L}_{\mathbf{a}}\alpha - \rho^*(\langle \widetilde{\nabla}.\mathbf{a}, \alpha\rangle)$,
4. $K'(\mathbf{a}_1,\mathbf{a}_2)(\alpha) = -K(\mathbf{a}_1,\mathbf{a}_2)^*(\alpha)$, *with K given in Eq. (F.36).*

Proof. It is a direct consequence of Prop. F.5, Thm. E.32 and Prop. E.31.

■

Appendix G

The degree 1 case: the geometry of Lie algebroids

In this appendix we work out some of the geometric structures that will guide us when we treat degree 2 manifolds, the main subject of this work. We benefited from [48] and [37].

G.1 Degree -1 Poisson brackets

Proposition G.1. *Given a Poisson 1-manifold* $(\mathcal{M}, \{.,.\})$*, consider the corresponding vector bundle* $A \longrightarrow M$*, so that* $\mathcal{M} \cong A[1]$ *and* $\mathcal{A}^1 \cong \Gamma(A^*)$*. Then* $\{.,.\}$ *induces a Lie algebroid structure on* A^* *by the following data:*

- $\{s, f\} = \rho(s)(f)$*, for* $f, s \in \Gamma(\Lambda^\cdot A^*)$ *with* $|f| = 0,\ |s| = 1$
- $\{s_1, s_2\} = [s_1, s_2]_{A^*}$*, for* $s_1, s_2 \in \Gamma(\Lambda^\cdot A^*)$ *with* $|s_1| = |s_2| = 1$.

Conversely, starting with a Lie algebroid $(A^*, \rho, [.,.])$ *over* M*, consider the degree 1 manifold* $A[1]$*, so that* $C^\infty(A[1]) = \Gamma(\Lambda^\cdot A^*)$*, and set*

- $\{f, g\} = 0$*, for* $f, g \in C^\infty(A[1])$ *with* $|f| = |g| = 0$
- $\{s, f\} = \rho(s)(f)$*, for* $f, s \in C^\infty(A[1])$ *with* $|f| = 0,\ |s| = 1$
- $\{s_1, s_2\} = [s_1, s_2]$*, for* $s_1, s_2 \in C^\infty(A[1])$ *with* $|s_1| = |s_2| = 1$.

Extending this product to any $s_1, s_2 \in C^\infty(A[1])$ *via linearity and (graded) Leibniz's rule, we obtain a degree 1 Poisson structure on* $A[1]$.

Proof. Let $(\mathcal{M}, \{.,.\})$ be a degree 1 Poisson manifold. Note that the first equation in the theorem determines the value of $\rho(s)$ for every $s \in \Gamma(A^*)$. So, the first two equations in the theorem determine ρ and $[.,.]_{A^*}$, which we will denote simply by $[.,.]$. Now we will check that ρ is a morphism of vector bundles and that $[.,.]$ is a Lie bracket. The first is a direct consequence of the Leibniz's rule of $\{.,.\}$. (Since $\{.,.\}$ has degree -1, it follows that $\{f, g\} = 0$ when $|f| = |g| = 0$.) To see that $[.,.]$ is in fact a Lie bracket notice that

$\mathbb{R}$-bilinearity follows immediately from $\mathbb{R}$-bilinearity of the Poisson bracket. The graded anti-commutativity and graded identity of the Poisson bracket implies, respectively, that $[.,.]$ is anti-symmetric and satisfies Jacobi identity.

Now we check compatibility of the brackets with the anchor. In fact, it's a consequence of the graded Leibniz's rule:

$$[s_1, fs_2] = f\{s_1, s_2\} + \{s_1, f\}s_2 = f[s_1, s_2] + \rho(s_1)(f)s_2.$$

For the converse, first observe that we have two different ways to define $\{s_1, fs_2\}$:

- $\{s_1, fs_2\} = f\{s_1, s_2\} + \{s_1, f\}s_2$ or, since $|fs_2| = 1$, we could define directly
- $\{s_1, fs_2\} = [s_1, fs_2]$

By the compatibility of the brackets with the anchor, we see immediately that this two ways give the same result. Similarly, since $X(\cdot)$ is a derivation for any vector field,

$$\{s, fg\} = \rho(s)(fg) = g\rho(s)(f) + f\rho(s)(g) = \{s, f\}g + f\{s, g\}.$$

Thus $\{.,.\}$ is well defined, and satisfies the graded anti-commutativity and Leibniz's rule by definition. So, it only remains to prove that graded Jacobi's identity holds for $\{.,.\}$. First, we see that it is valid for functions of degree 0 or 1, the only case which is not trivial being when we take $\{f, \{s_1, s_2\}\}$. In this case it follows from the well known fact that the anchor is a morphism of Lie algebras. For functions of degree greater than 1, Jacobi identity follows combining the graded anti-commutativity and Leibniz's rule with an induction argument.

∎

We end this section mentioning some concepts and results which are related to the (super) geometry of 1-manifolds.

Definition G.2. A Lie bialgebroid is a pair (A, A^*) of Lie algebroids in duality, where the Lie brackets satisfy the compatibility condition

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y],$$

for all $X, Y \in \Gamma(A)$, where d_* is the de Rham differential on $\Gamma(\Lambda^{\cdot} A)$ defined by the Lie algebroid structure of A^*, and $[\cdot,\cdot]$ is the Schouten bracket on $\Gamma(\Lambda^{\cdot} A)$ coming from the Lie algebroid bracket on $\Gamma(A)$.

Proposition G.3. *On a degree k symplectic manifold a Q-structure is equivalent to a degree $(k+1)$ integrable hamiltonian.*

Proof. This is found in lemma 2.2 of [59]. See our Prop. 6.30, where the $k = 2$ case is worked out in detail.

∎

Proposition G.4. *There is a canonical 1:1 correspondence between Q-structures on a degree 1 Poisson manifold and bialgebroids.*

Proof. This is basically the content of Prop. 3.3 of [33].

■

Proposition G.5. *There is a canonical 1:1 correspondence between degree 2 integrable hamiltonians on a degree 1 symplectic manifold and Poisson brackets on the body of the manifold.*

Proof. This is the classical characterization of Poisson brackets in terms of self-commuting bivectors with respect to the Schouten bracket. See for example [43].

■

For what's next, we introduce some very useful notions from the so-called Schouten calculus.

G.2 Cartan-Schouten calculus

In the rest of this appendix, we work out calculations on the exterior algebra of a Lie algebroid $(A, [\cdot,\cdot], \rho)$ and its dual A^*, exploiting the de Rham differential and the Lie derivative, that will serve as guidance for the degree 2 case.

Definition G.6. Let A be a Lie algebroid. Then, for any $X \in \Gamma(A)$ we define the Lie derivative on $\Gamma(\Lambda^\cdot A \oplus \Lambda^\cdot A^*)$ as the unique degree 0 differential operator such that

i) $\mathcal{L}_X f := \rho(X)(f), \quad \mathcal{L}_X Y := [X, Y], \qquad \forall\, f \in C^\infty(M), Y \in \Gamma(A),$

ii) $\langle \mathcal{L}_X \omega, Y\rangle := \mathcal{L}_X\langle \omega, X\rangle - \langle \omega, \mathcal{L}_X Y\rangle, \qquad \forall\, \omega \in \Gamma(A^*), Y \in \Gamma(A),$

iii) $\mathcal{L}_X(V \wedge W) = (\mathcal{L}_X V) \wedge W + V \wedge (\mathcal{L}_X W), \quad \forall\, V, W \in \Gamma(\Lambda^\cdot A)$, or $V, W \in \Gamma(\Lambda^\cdot A^*)$.

Definition G.7. For any vector bundle $A \longrightarrow M$ we define a $C^\infty(M)$-bilinear, symmetric pairing

$$\langle\cdot,\cdot\rangle : \Gamma(\Lambda^\cdot A \oplus \Lambda^\cdot A^*) \times \Gamma(\Lambda^\cdot A \oplus \Lambda^\cdot A^*) \longrightarrow \Gamma(\Lambda^\cdot A \oplus \Lambda^\cdot A^*)$$

by

i) $\langle P, Q\rangle = \langle \omega, \eta\rangle := 0$ for every $P, Q \in \Gamma(\Lambda^\cdot A)$ and $\omega, \eta \in \Gamma(\Lambda^\cdot A^*)$.

ii) If $P = P_1 \wedge \cdots \wedge P_k \in \Gamma(\Lambda^k A)$ and $\omega = \omega_1 \wedge \cdots \wedge \omega_k \in \Gamma(\Lambda^k A^*)$, then

$$\langle P, \omega\rangle \in \Gamma(\Lambda^0 A) = \Gamma(\Lambda^0 A^*) = C^\infty(M), \quad \langle P, \omega\rangle := \det(\omega_i(P_j)).$$

Observe that the pairing in this case is non-degenerate, therefore induces isomorphisms

$$\Lambda^k A \cong (\Lambda^k A^*)^* \quad \text{and} \quad \Lambda^k A^* \cong (\Lambda^k A)^*. \tag{G.1}$$

iii) If $P \in \Gamma(\Lambda^k A)$ and $\omega \in \Gamma(\Lambda^l A^*)$, with $k < l$, then we define $\langle P, \omega\rangle \in \Gamma(\Lambda^{l-k} A^*)$ by

$$\langle\langle P, \omega\rangle, Q\rangle := \langle P \wedge Q, \omega\rangle, \qquad \forall\, Q \in \Lambda^{l-k} A.$$

iv) If $P \in \Gamma(\Lambda^k A)$ and $\omega \in \Gamma(\Lambda^l A^*)$, with $k > l$, then we define $\langle P, \omega\rangle \in \Gamma(\Lambda^{k-l} A)$ by

$$\langle\langle P, \omega\rangle, \eta\rangle := \langle P, \omega \wedge \eta\rangle \qquad \forall\, \eta \in \Lambda^{k-l} A^*.$$

Notice that in items *iii*) and *iv*) we are using the canonical isomorphisms (G.1).

Proposition G.8 (For example [48]). *Let A be a Lie algebroid. For any $X \in \Gamma(A)$, $P \in \Gamma(\Lambda^{\cdot} A)$ and $\omega \in \Gamma(\Lambda^{\cdot} A^*)$*

$$\mathcal{L}_X \langle P, \omega \rangle = \langle \mathcal{L}_X P, \omega \rangle + \langle P, \mathcal{L}_X \omega \rangle, \tag{G.2}$$

holds, where the Lie derivative $\mathcal{L}_X$ and the pairing $\langle \cdot, \cdot \rangle$ are understood according to Defs. G.6 and G.7, respectively.

Proof. By item *ii*) of Def. G.6 we have already (G.2) for $P \in \Gamma(A)$ and $\omega \in \Gamma(A^*)$.

Next, assuming that (G.2) is already true when $P \in \Gamma(\Lambda^k A)$ and $\omega \in \Gamma(\Lambda^k A^*)$, we will prove (G.2) for $P \in \Gamma(\Lambda^k A)$ and $\Gamma(\Lambda^l A^*)$ when $k > l$ and when $k < l$. Since the argument is completely symmetric for both cases, we will only work out here the case $k > l$. Take $\eta \in \Gamma(\Lambda^{k-l} A^*)$ arbitrary, then, on one hand, using (G.2) for $\langle P, \omega \rangle \in \Gamma(\Lambda^{k-l} A$ and η, we have

$$\mathcal{L}_X \langle P, \omega \wedge \eta \rangle = \mathcal{L}_X \langle \langle P, \omega \rangle, \eta \rangle = \langle \mathcal{L}_X \langle P, \omega \rangle, \eta \rangle + \langle \langle P, \omega \rangle, \mathcal{L}_X \eta \rangle. \tag{G.3}$$

On the other hand, using (G.2) for P and $\omega \wedge \eta \in \Gamma(\Lambda^k A^*)$, we have

$$\begin{aligned} \mathcal{L}_X \langle P, \omega \wedge \eta \rangle &= \langle \mathcal{L}_X P, \omega \wedge \eta \rangle + \langle P, \mathcal{L}_X (\omega \wedge \eta) \rangle \\ &= \langle \langle \mathcal{L}_X P, \omega \rangle, \eta \rangle + \langle \langle P, \mathcal{L}_X \omega \rangle, \eta \rangle + \langle \langle P, \omega \rangle, \mathcal{L}_X \eta \rangle. \end{aligned} \tag{G.4}$$

Comparing Eqs. (G.3) and (G.4), we get

$$\begin{aligned} \langle \mathcal{L}_X \langle P, \omega \rangle, \eta \rangle &= \langle \langle \mathcal{L}_X P, \omega \rangle, \eta \rangle + \langle \langle P, \mathcal{L}_X \omega \rangle, \eta \rangle \\ &= \langle \langle \mathcal{L}_X P, \omega \rangle + \langle P, \mathcal{L}_X \omega \rangle, \eta \rangle. \end{aligned}$$

Since η is arbitrary, Eq. (G.2) follows in this case.

So it remains to show (G.2) only in the case $P \in \Gamma(\Lambda^k A)$ and $\omega \in \Gamma(\Lambda^k A^*)$. We will prove by induction. We already observed that the case $n = 1$ is included in the definition of $\mathcal{L}_X$ (Def. G.6). Now let's prove the case $n = k$ assuming that the case $n = k - 1$ is true. Because of the linearity of the operator $\mathcal{L}_X$ and the bilinearity of the pairing $\langle \cdot, \cdot \rangle$, there is no loose of generality if we prove Eq. (G.2) only for decomposable sections

$$P = P_1 \wedge \cdots \wedge P_k \quad \text{and} \quad \Omega = \omega_1 \wedge \cdots \wedge \omega_k.$$

By the cofactors formula for the determinant, we have

$$\begin{aligned} \langle P, \Omega \rangle &= \langle P_1 \wedge \cdots \wedge P_k, \omega_1 \wedge \cdots \wedge \omega_k \rangle = \det(\langle P_i, \omega_j \rangle) \\ &= \langle P_1, \omega_1 \rangle \det A_{11} - \langle P_1, \omega_2 \rangle \det A_{12} + \cdots + (-1)^{1+k} \langle P_1, \omega_k \rangle \det A_{1k}, \end{aligned} \tag{G.5}$$

where $\widetilde{P} = P_2 \wedge \cdots \wedge P_k$, $\Omega_j = \omega_1 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_k$ and

$$\det A_{1j} = \langle \widetilde{P}, \Omega_j \rangle = \langle P_2 \wedge \cdots \wedge P_k, \omega_1 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_k \rangle,$$

and the hat $\widehat{\omega_j}$ means, as usual, that this element is missing in the wedge product.

Using the induction hypothesis, and Eq. (G.5), we have

$$\begin{aligned}
\mathcal{L}_X\langle P,\Omega\rangle &= \sum_j(-1)^{1+j}\mathcal{L}_X(\langle P_1,\omega_j\rangle\langle\widetilde{P},\Omega_j\rangle)\\
&= \sum_j(-1)^{1+j}(\langle\mathcal{L}_X P_1,\omega_j\rangle+\langle P_1,\mathcal{L}_X\omega_j\rangle)\langle\widetilde{P},\Omega_j\rangle\\
&\quad+\sum_j(-1)^{1+j}\langle P_1,\omega_j\rangle(\langle\mathcal{L}_X\widetilde{P},\Omega_j\rangle+\langle\widetilde{P},\mathcal{L}_X\Omega_j\rangle)\\
&= \sum_j(-1)^{1+j}(\langle\mathcal{L}_X P_1,\omega_j\rangle\langle\widetilde{P},\Omega_j\rangle+\langle P_1,\omega_j\rangle\langle\mathcal{L}_X\widetilde{P},\Omega_j\rangle)\\
&\quad+\sum_j(-1)^{1+j}(\langle P_1,\mathcal{L}_X\omega_j\rangle\langle\widetilde{P},\Omega_j\rangle+\langle P_1,\omega_j\rangle\langle\widetilde{P},\mathcal{L}_X\Omega_j\rangle).
\end{aligned}\tag{G.6}$$

On the other hand, let's compute the right-hand side of the equation we want to prove, i.e. Eq. (G.2).

$$\begin{aligned}
\langle\mathcal{L}_X P,\Omega\rangle+\langle P,\mathcal{L}_X\Omega\rangle &= \langle\mathcal{L}_X(P_1\wedge\widetilde{P}),\Omega\rangle+\langle P,\mathcal{L}_X\Omega\rangle\\
&= \langle(\mathcal{L}_X P_1)\wedge\widetilde{P}+P_1\wedge\mathcal{L}_X\widetilde{P},\Omega\rangle\\
&\quad+\left\langle P,\sum_j\omega_1\wedge\cdots\wedge\mathcal{L}_X\omega_j\wedge\cdots\wedge\omega_k\right\rangle\\
&= \sum_j(-1)^{1+j}(\langle\mathcal{L}_X P_1,\omega_j\rangle\langle\widetilde{P},\Omega_j\rangle+\langle P_1,\omega_j\rangle\langle\mathcal{L}_X\widetilde{P},\Omega_j\rangle)\\
&\quad+\sum_{j,l}(-1)^{1+j}\langle P_1,\overline{\omega_{l,j}}\rangle\langle\widetilde{P},\overline{\Omega_{l,j}}\rangle,
\end{aligned}\tag{G.7}$$

where $\overline{\omega_{l,j}}$ is the j^{th} factor of the wedge product

$$\overline{\Omega_l}=\omega_1\wedge\cdots\wedge\mathcal{L}_X\omega_l\wedge\cdots\wedge\omega_k,\tag{G.8}$$

in particular notice that

$$\overline{\omega_j}_{,j}=\mathcal{L}_X\omega_j\quad\text{and}\quad\overline{\omega_{l,j}}=\omega_j,\quad\text{for}j\neq l.\tag{G.9}$$

$\overline{\Omega_l},j$ is defined by

$$\overline{\Omega_{l,j}}=\overline{\omega_{l,j}}\wedge\cdots\wedge\widehat{\overline{\omega_{l,j}}}\wedge\cdots\wedge\overline{\omega_{l,k}}.\tag{G.10}$$

Now,

$$\mathcal{L}_X\Omega_j=\sum_j\omega_1\wedge\cdots\wedge\mathcal{L}_X\omega_l\wedge\cdots\wedge\widehat{\omega_j}\wedge\cdots\wedge\omega_k=\sum_{j\neq l}\overline{\Omega_{l,j}},$$

whence, using the notation given in (G.8), (G.9) and (G.10),

$$\begin{aligned}\sum_j (-1)^{1+j}(\langle P_1, \mathcal{L}_X\omega_j\rangle\langle \widetilde{P}, \Omega_j\rangle + \langle P_1, \omega_j\rangle\langle \widetilde{P}, \mathcal{L}_X\Omega_j\rangle) &= \sum_j (-1)^{1+j}\langle P_1, \overline{\omega_j}_{,j}\rangle\langle \widetilde{P}, \overline{\Omega_j}_{,j}\rangle \\ &\quad + \sum_j (-1)^{1+j}\langle P_1, \omega_j\rangle \left\langle \widetilde{P}, \sum_{l\neq j} \overline{\Omega_l}_{,j}\right\rangle \\ &= \sum_j (-1)^{1+j}\langle P_1, \overline{\omega_j}_{,j}\rangle\langle \widetilde{P}, \overline{\Omega_j}_{,j}\rangle + \sum_{l\neq j} (-1)^{1+j}\langle P_1, \overline{\omega_l}_{,j}\rangle\langle \widetilde{P}, \overline{\Omega_l}_{,j}\rangle \\ &= \sum_{j,l} (-1)^{1+j}\langle P_1, \overline{\omega_l}_{,j}\rangle\langle \widetilde{P}, \overline{\Omega_l}_{,j}\rangle. \qquad \text{(G.11)}\end{aligned}$$

From (G.6), (G.11) and (G.7) follows (G.2).

∎

Proposition G.9. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid and d its corresponding de Rham differential. Then, for any 2-section $\pi \in \Gamma(\Lambda^2 A)$ 1-section $X \in \Gamma(A)$ and dual sections $\phi_1, \phi_2 \in \Gamma(A^*)$,*

$$[\pi, f] = -\pi^\sharp(df) \qquad \text{(G.12)}$$

and

$$\rho(X)\langle \pi^\sharp(\phi_1), \phi_2\rangle = \langle\langle [X,\pi], \phi_1\rangle, \phi_2\rangle + \langle \pi^\sharp(\mathcal{L}_X\phi_1), \phi_2\rangle + \langle \pi^\sharp(\phi_1), \mathcal{L}_X\phi_2\rangle, \qquad \text{(G.13)}$$

where $\pi^\sharp : A^* \longrightarrow A$ is defined by contraction

$$\pi^\sharp(\phi) := \langle \pi, \phi\rangle, \qquad \text{(G.14)}$$

so that

$$\langle \pi^\sharp(\phi_1), \phi_2\rangle = \langle \pi, \phi_1 \wedge \phi_2\rangle = \langle\langle \pi, \phi_1\rangle, \phi_2\rangle.$$

Proof. In order to prove formula (G.12), since both sides are tensorial in π, we don't lose generality if we suppose that $\pi = X_1 \wedge X_2$, for $X_1, X_2 \in \Gamma(A)$. Then we have

$$\begin{aligned}[\pi, f] &= [X_1 \wedge X_2, f] = [f, X_1]X_2 - X_1[f, X_2] \\ &= -\rho(X_1)(f)X_2 + \rho(X_2)(f)X_1 \\ &= -\langle df, X_1\rangle X_2 + \langle df, X_2\rangle X_1 \\ &= -\pi^\sharp(df).\end{aligned}$$

Formula (G.13) follows from Eq. (G.2), using Defs. G.6 and G.7, as follows.

$$\rho(X)(\langle \pi^\sharp(\phi_1), \phi_2\rangle) = \mathcal{L}_X\langle \pi^\sharp(\phi_1), \phi_2\rangle = \langle \mathcal{L}_X\pi^\sharp(\phi_1), \phi_2\rangle + \langle \pi^\sharp(\phi_1), \mathcal{L}_X\phi_2\rangle.$$

Now,

$$\begin{aligned}\mathcal{L}_X\pi^\sharp(\phi_1) &= \mathcal{L}_X\langle \pi, \phi_1\rangle = \langle \mathcal{L}_X\pi, \phi_1\rangle + \langle \pi, \mathcal{L}_X\phi_1\rangle \\ &= \langle [X,\pi], \phi_1\rangle + \pi^\sharp(\mathcal{L}_X\phi_1).\end{aligned}$$

Hence,

$$\rho(X)(\langle\pi^\sharp(\phi_1),\phi_2\rangle) = \langle\langle[X,\pi],\phi_1\rangle,\phi_2\rangle + \langle\pi^\sharp(\mathcal{L}_X\phi_1),\phi_2\rangle + \langle\pi^\sharp(\phi_1),\mathcal{L}_X\phi_2\rangle.$$

■

Proposition G.10. *Let $(A,[\cdot,\cdot],\rho)$ be a Lie algebroid, and let $\pi \in \Lambda^2(A)$ be an integrable 2-section, that is, it satisfies $[\pi,\pi]=0$. Then*

$$d_* := [\pi,\cdot] : \Gamma(\Lambda^k A) \longrightarrow \Gamma(\Lambda^{k+1}A)$$

defines a de Rham differential on $\Gamma(\Lambda^\cdot A)$, so that we obtain a Lie algebroid structure on the dual bundle A^*, with anchor

$$\rho_*(\phi)(f) = \langle df,\phi\rangle = \rho(\pi^\sharp(\phi))(f), \qquad \forall\phi\in\Gamma(A^*), f\in C^\infty(M), \tag{G.15}$$

and Lie bracket is given by the derived bracket formula

$$\langle[\phi_1,\phi_2]_*,X\rangle = \rho_*(\phi_1)(\langle\phi_2,X\rangle) - \rho_*(\phi_2)(\langle\phi_1,X\rangle) - d_*X(\phi_1,\phi_2). \tag{G.16}$$

An explicit formula for $[\cdot,\cdot]_*$ is given by

$$[\phi_1,\phi_2]_* = \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_2 - \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1 + d(\langle\phi_1,\pi^\sharp(\phi_2)\rangle), \tag{G.17}$$

where d is the de Rham differential corresponding to the Lie algebroid structure on A.

Proof. Since π has degree 2 and the Schouten bracket has degree -1, it follows that $[\pi,\cdot]$ is a degree 1 operator on $\Gamma(\Lambda^\cdot A)$. By (graded) Jacobi identity of the Schouten bracket it follows

$$d_*^2 = [\pi,[\pi,\cdot]] = \frac{1}{2}[[\pi,\pi],\cdot] = 0.$$

Also from (graded) Leibniz property of the Schouten bracket, it follows that d_* is a graded 1-differential. Hence the operator d_* is a de Rham differential, which implies that A^* is endowed with a Lie algebroid structure coming from d_*, with anchor ρ_* defined by

$$\rho_*(\phi)(f) = \iota_\phi d_*(f) = \langle\phi,d_*(f)\rangle = \langle\phi,[\pi,f]\rangle.$$

From Eq. (G.12) and the skew-symmetry of π, we also have

$$\rho_*(\phi)(f) = \langle\phi,[\pi,f]\rangle = -\langle\phi,\pi^\sharp(df)\rangle = \langle\pi^\sharp(\phi),df\rangle,$$

and the Lie bracket is given by the derived bracket formula (G.16). It is easy to verify that the bracket defined in this way satisfies skew-symmetry, $\mathbb{R}$-bilinearity and Leibniz rule. As for Jacobi identity, Eq. (G.16) implies

$$\begin{aligned} d_*^2 f(\phi_1,\phi_2) &= \rho_*(\phi_1)(\langle d_*f,\phi_2\rangle) - \rho_*(\phi_2)(\langle d_*f,\phi_1\rangle) - \langle d_*f,[\phi_1,\phi_2]_*\rangle \\ &= \rho_*(\phi_1)(\rho_*(\phi_2)(f)) - \rho_*(\phi_2)(\rho_*(\phi_1)(f)) - \rho_*([\phi_1,\phi_2]_*)(f), \end{aligned}$$

from which, $d_*^2 = 0$ implies

$$\rho_*([\phi_1,\phi_2]_*) = [\rho_*(\phi_1),\rho_*(\phi_2)], \tag{G.18}$$

where the bracket on the right-hand side stands for the commutator of vector fields on M.

Now, using again Eq. (G.16), also Eq. (G.18) and the well-known formula for the exterior derivative

$$\langle d_* P, \phi_1 \wedge \phi_2 \wedge \phi_3 \rangle = \sum_{cyclic} (\rho(\phi_1)(\langle P, \phi_2 \wedge \phi_3 \rangle) + \langle P, \phi_1 \wedge [\phi_2, \phi_3] \rangle), \tag{G.19}$$

where $P \in \Gamma(\Lambda^2 A)$ and $\phi_1, \phi_2, \phi_3 \in \Gamma(A^*)$, we have, for every $X \in \Gamma(A), \phi_1 \phi_2, \phi_3 \in \Gamma(A^*)$,

$$\langle -d_*^2 X, \phi_1 \wedge \phi_2 \wedge \phi_3 \rangle = \langle X, [\phi_1, [\phi_2, \phi_3]_*]_* - [[\phi_1, \phi_2]_*, \phi_3]_* - [\phi_2, [\phi_1, \phi_3]_*]_* \rangle,$$

hence, $d_*^2 = 0$ implies Jacobi identity for $[\cdot, \cdot]_*$.

It remains to show formula (G.17). For this we use formulas (G.2) and (G.13). We have the following

$$\begin{aligned}
\langle [\phi_1, \phi_2]_*, X \rangle &= \rho_*(\phi_1)(\langle \phi_2, X \rangle) - \rho_*(\phi_2)(\langle \phi_2, X \rangle) - \langle \phi_2, \langle \phi_1, [\pi, X] \rangle \rangle \\
&= \rho(\pi^\sharp(\phi_1))(\langle \phi_2, X \rangle) - \rho(\pi^\sharp(\phi_2))(\langle \phi_1, X \rangle) - \langle \phi_2, \langle \phi_1, [\pi, X] \rangle \rangle \\
&= \langle \mathcal{L}_{\pi^\sharp(\phi_1)} \phi_2, X \rangle + \langle \phi_2, [\pi^\sharp(\phi_1), X] \rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_2)} \phi_1, X \rangle - \langle \phi_1, [\pi^\sharp(\phi_2), X] \rangle \\
&\quad + \rho(X)(\langle \pi^\sharp(\phi_1), \phi_2 \rangle) - \langle \pi^\sharp(\mathcal{L}_X \phi_1), \phi_2 \rangle - \langle \pi^\sharp(\phi_1), \mathcal{L}_X \phi_2 \rangle \\
&= \langle \mathcal{L}_{\pi^\sharp(\phi_1)} \phi_2, X \rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_2)} \phi_1, X \rangle + \rho(X)(\langle \pi^\sharp(\phi_2), \phi_1 \rangle) \\
&= \langle \mathcal{L}_{\pi^\sharp(\phi_1)} \phi_2 - \mathcal{L}_{\pi^\sharp(\phi_2)} \phi_1 + d(\langle \pi^\sharp(\phi_2), \phi_1 \rangle), X \rangle.
\end{aligned}$$

■

G.3 Derived brackets

Before continuing, it is necessary to make a brief digression on the derived bracket formula we used in Prop. G.10 in order to obtain the dual Lie algebroid bracket from the differential $d_* = [\pi, \cdot]$. We will follow the discussion found in [50].

Definition G.11. Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a $\mathbb{Z}$-graded algebra. Let $D : A \longrightarrow A$ be a linear endomorphism of the graded vector space A. Let $d \in \mathbb{Z}$. The linear endomorphism D is said to be a *derivation of degree* d of the graded algebra A if

i) as a linear endomorphism of a graded vector space, D is homogeneous of degree d, that is, for any homogeneous element $a \in A$ of degree p, $D(a)$ is homogeneous of degree $p + d$.

ii) for all $p \in \mathbb{Z}$, $a \in A^p$ and $b \in A$,

$$D(ab) = D(a)b + (-1)^{dp} a D(b).$$

Example G.12. The de Rham differential d corresponding to a Lie algebroid A, is a derivation of degree 1 on $\Gamma(\Lambda^\cdot A^*)$.

For any $X \in \Gamma(A)$, the Lie derivative $\mathcal{L}_X$ is a derivation of degree 0 on $\Gamma(\Lambda^\cdot A)$ and on $\Gamma(\Lambda^\cdot A^*)$, and the contraction operator ι_X defined by $\iota_X(\omega) := \langle X, \omega \rangle$ is a derivation of degree -1 on $\Gamma(\Lambda^\cdot A^*)$.

Proposition G.13. *Let A be a graded algebra. Let $D_1 : A \longrightarrow A$ and $D_2 : A \longrightarrow A$ be two derivations of A, of degree d_1 and d_2, respectively. Their bracket*

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{d_1 d_2} D_2 \circ D_1,$$

is a derivation of degree $d_1 + d_2$.

Proof. For any $a \in A^p$ and $b \in A$, we have

$$\begin{aligned}
[D_1, D_2](ab) &= D_1 \circ D_2(ab) - (-1)^{d_1 d_2} D_2 \circ D_1(ab) \\
&= D_1(D_2(a)b + (-1)^{d_2 p} a D_2(b)) - (-1)^{d_1 d_2} D_2(D_1(a)b + (-1)^{d_1 p} a D_1(b)) \\
&= D_1(D_2(a))b + (-1)^{d_1(d_2+p)} D_2(a) D_1(b) \\
&\quad + (-1)^{d_2 p} D_1(a) D_2(b) + (-1)^{(d_1+d_2)p} a D_1(D_2(b)) \\
&\quad - (-1)^{d_1 d_2} D_2(D_1(a))b - (-1)^{d_2 p} D_1(a) d_2(b) \\
&\quad - (-1)^{d_1(d_2+p)} D_2(a) D_1(b) - (-1)^{d_1 d_2 + (d_1+d_2)p} a D_2(D_1(b)) \\
&= [D_1, D_2](a)b + (-1)^{(d_1+d_2)p} a [D_1, d_2](b).
\end{aligned}$$

Definition G.14. A $\mathbb{Z}$*-graded Lie algebra* is a $\mathbb{Z}$-graded algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$, whose composition law, denoted by $(a, b) \longrightarrow [a, b]$ and called the *graded bracket*, satisfies the following properties:

i) it is homogeneous of degree d, i.e., for $a \in A^p$ and $b \in A^q$, we have $[a, b] \in A^{a+b+d}$,

ii) it is graded anticommutative, i.e., for $a \in A^p$ and $b \in A^q$,

$$[a, b] = -(-1)^{pq} [b, a],$$

iii) it satisfies the graded Jacobi identity, i.e., for $a \in A^p$, $b \in A^q$ and $c \in A^r$,

$$[a, [b, c]] = [[a, b], c] + (-1)^{(a+d)b} [b, [a, c]].$$

Example G.15. For a Lie algebroid A, the Schouten bracket $[\cdot, \cdot]$ endows $\Gamma(\Lambda^{\cdot} A)$ with a graded Lie bracket of degree -1.

Let $E = \bigoplus_{p \in \mathbb{Z}} E^p$ be a graded vector space. For each $p \in \mathbb{Z}$, let $A^p \subset \mathrm{End}(E, E)$ be the space of linear endomorphisms of E which are homogeneous of degree p. Then $A = \bigoplus_{p \in \mathbb{Z}} A^p$ equipped with the composition of applications as composition law, is a $\mathbb{Z}$-graded associative algebra, and the graded commutator, defined for homogeneous elements $a \in A^p$ and $b \in A^q$ by

$$[a, b] := ab - (-1)^{pq} ba,$$

and extended by bilinearity to the whole space A, endows A with a $\mathbb{Z}$-graded Lie algebra structure.

Proposition G.16. *Given a Lie algebroid $(A, [\cdot, \cdot], \rho)$, and a degree 1 graded differential d_* on $\Gamma(\Lambda^{\cdot} A)$. Consider the anchor map ρ_* and bracket $[\cdot, \cdot]_*$ on the dual A^*, given by Eqs. (G.15) and (G.16), respectively. Then the formula*

$$\iota_{[\phi_1, \phi_2]_*}(P) = [[\iota_{\phi_1}, d_*], \iota_{\phi_1}](P) \tag{G.20}$$

holds for every $P \in \Gamma(\Lambda^{\cdot} A)$, where the bracket on the right-hand side stands for the commutator of differentials (see Prop. G.13).

Proof. Since ι_ϕ is a derivation of degree -1 and d_* is a derivation of degree 1, it follows from Prop. G.13 that

$$D := [[\iota_{\phi_1}, d_*], \iota_{\phi_1}]$$

is a derivation of degree -1 on the algebra $\mathcal{A} := \Gamma(\Lambda^{\cdot} A)$. In particular, from item *ii*) of Def. G.11, it follows, for any $f \in C^\infty(M) = \mathcal{A}^0$ and $X \in \Gamma(A) = \mathcal{A}^1$,

$$D(fX) = D(f)X + fD(X) = fD(X),$$

which means that there is a unique section $\phi \in \Gamma(A^*)$ such that

$$D(X) = \langle \phi, X \rangle = \iota_\phi(X), \qquad \forall X \in \Gamma(A).$$

We claim that $\phi = [\phi_1, \phi_2]_*$. In order to verify this, we need to prove that the degree -1 derivations ι_ϕ and $\iota_{[\phi_1,\phi_2]_*}$, coincide on $\mathcal{A}^1$. So, let's take any $X \in \Gamma(A) = \mathcal{A}^1$, then

$$\begin{aligned}
\langle [\phi_1, \phi_2]_*, X \rangle &= [[\iota_{\phi_1}, d_*], \iota_{\phi_2}](X) \\
&= [\iota_{\phi_1} d_* + d_* \iota_{\phi_1}, \iota_{\phi_2}](X) \\
&= \iota_{\phi_1} d_* \iota_{\phi_2}(X) + d_* \iota_{\phi_1} \iota_{\phi_2}(X) - \iota_{\phi_2} \iota_{\phi_1} d_*(X) - \iota_{\phi_2} d_* \iota_{\phi_1}(X) \\
&= \rho_*(\phi_1)(\langle \phi_2, X \rangle) - \langle \phi_2, \langle \phi_1, d_*(X) \rangle \rangle - \rho_*(\phi_2)(\langle \phi_1, X \rangle) \\
&= \rho_*(\phi_1)(\langle \phi_2, X \rangle) - \rho_*(\phi_2)(\langle \phi_1, X \rangle) - \langle \phi_2, \langle \phi_1, [\pi, X] \rangle \rangle \\
&= \iota_{[\phi_1,\phi_2]_*}(X),
\end{aligned}$$

as we wanted.

∎

Lemma G.17. *In a Lie algebroid A, we have, for every $X \in \Gamma(A)$ and every $\omega \in \Gamma(\Lambda^{\cdot} A^*)$,* Cartan's formula *holds:*

$$\mathcal{L}_X = [\iota_X, d] = \iota_X d + d\iota_X, \tag{G.21}$$

where $d : \Gamma(\Lambda^{\cdot} A^*) \longrightarrow \Gamma(\Lambda^{\cdot} A^*)$ is the corresponding de Rham differential.

Proof. Since ι_X has degree -1 and d has degree 1, it follows from Prop. G.13 that $D := \iota_X d + d\iota_X$ is a degree 0 derivation. By definition we know that $\mathcal{L}_X$ is a degree 0 derivation as well. Therefore it is enough to check that D and $\mathcal{L}_X$ coincide on $\mathcal{A}^0 = C^\infty(M)$ and $\mathcal{A}^1 = \Gamma(A^*)$.

For $f \in C^\infty(M)$ we have

$$D(f) = \iota_X df + d\iota_X f = \iota_X df = \rho(X)(f) = \mathcal{L}_X f.$$

For $\alpha \in \Gamma(A^*)$ and $Y \in \Gamma(A)$, we have

$$\begin{aligned}
\langle D(\alpha), Y \rangle &= \langle \iota_X d\alpha + d\iota_X \alpha, Y \rangle \\
&= \rho(X)(\langle \alpha, Y \rangle) - \rho(Y)(\langle \alpha, X \rangle) - \langle \alpha, [X, Y] \rangle + \rho(Y)(\langle \alpha, X \rangle) \\
&= \rho(X)(\langle \alpha, Y \rangle) - \langle \alpha, [X, Y] \rangle \\
&= \langle \mathcal{L}_X \alpha, Y \rangle.
\end{aligned}$$

∎

G.4 Degree 2 integrable hamiltonians on Poisson 1-manifolds

Lemma G.18. *In a Lie algebroid A, for any 2-section $\pi \in \Gamma(\Lambda^2 A)$*

$$\langle [\pi,\pi], \phi_1 \wedge \phi_2 \wedge \phi_3\rangle = \langle \phi_3, \langle \phi_2, \langle \phi_1, [\pi,\pi]\rangle\rangle\rangle = -2 \sum_{cyclic} \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle,$$

holds for every $\phi_1, \phi_2, \phi_3 \in \Gamma(A^*)$.

Proof. Consider the degree 1 derivation $d_* := [\pi, \cdot]$, where $[\cdot,\cdot]$ is the Schouten bracket on $\Gamma(\Lambda^\cdot A)$. Then, from Eq. (G.20), we have

$$\begin{aligned}\langle [\phi_1,\phi_2]_*, \pi\rangle &= \iota_{[\phi_1,\phi_2]_*}\pi = [[\iota_{\phi_1}, d_*], \iota_{\phi_2}](\pi)\\ &= \iota_{\phi_1} d_* \iota_{\phi_2}(\pi) + d_*\iota_{\phi_1}\iota_{\phi_2}(\pi) - \iota_{\phi_2}\iota_{\phi_1} d_*(\pi) - \iota_{\phi_2} d_* \iota_{\phi_1}(\pi)\\ &= \langle \phi_1, [\pi, \langle \pi, \phi_2\rangle]\rangle + [\pi, \langle \pi, \phi_2 \wedge \phi_1\rangle]\\ &\quad - \langle [\pi,\pi], \phi_1\wedge\phi_2\rangle - \langle \phi_2, [\pi, \langle \pi, \phi_1\rangle],\end{aligned}$$

from which we get,

$$\langle [\pi,\pi], \phi_1\wedge\phi_2\rangle = \langle [\pi, \langle\pi,\phi_2\rangle], \phi_1\rangle + [\pi, \langle \pi, \phi_2\wedge\phi_1\rangle] - \langle [\pi, \langle \pi,\phi_1\rangle], \phi_2\rangle - \langle [\phi_1,\phi_2]_*, \pi\rangle. \tag{G.22}$$

Now, bringing Eqs. (G.14), (G.15) and (G.16), we have, for any $\phi_i, \phi_j, \phi_3 \in \Gamma(A^*)$,

$$\begin{aligned}\langle\langle [\pi, \langle \pi, \phi_j\rangle], \phi_i\rangle, \phi_3\rangle &= \langle d_*\pi^\sharp(\phi_j), \phi_i \wedge \phi_3\rangle = d_*\pi^\sharp(\phi_j)(\phi_i,\phi_3)\\ &= \rho(\pi^\sharp(\phi_i))(\langle \phi_3, \pi^\sharp(\phi_j)\rangle) - \rho(\pi^\sharp(\phi_3))(\langle \phi_i, \pi^\sharp(\phi_j)\rangle)\\ &\quad - \langle [\phi_i,\phi_3]_*, \pi^\sharp(\phi_j)\rangle.\end{aligned} \tag{G.23}$$

Putting Eq. (G.23) into Eq. (G.22) with $i = 1, j = 2$ in the first term of the right-hand side of (G.22) and with $i = 2, j = 1$ in the third term, we get

$$\begin{aligned}\langle [\pi,\pi], \phi_1\wedge\phi_2\wedge\phi_3\rangle &= \langle\langle [\pi,\pi], \phi_1\wedge\phi_2\rangle, \phi_3\rangle\\ &= \rho(\pi^\sharp(\phi_1))(\langle\phi_3, \pi^\sharp(\phi_2)\rangle) - \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle)\\ &\quad - \langle [\phi_1,\phi_3]_*, \pi^\sharp(\phi_2)\rangle + \rho(\pi^\sharp(\phi_3))(\langle\phi_1, \pi^\sharp(\phi_2)\rangle)\\ &\quad - \rho(\pi^\sharp(\phi_2))(\langle \phi_3, \pi^\sharp(\phi_1)\rangle) + \rho(\pi^\sharp(\phi_3))(\langle\phi_2, \pi^\sharp(\phi_1)\rangle)\\ &\quad + \langle [\phi_2,\phi_3]_*, \pi^\sharp(\phi_1)\rangle + \langle [\phi_1,\phi_2]_*, \pi^\sharp(\phi_3)\rangle\end{aligned}$$

and now using Eq. (G.17) we continue calculations

$$\begin{aligned}&= \rho(\pi^\sharp(\phi_1))(\langle \phi_3, \pi^\sharp(\phi_2)\rangle) - \rho(\pi^\sharp(\phi_2))(\langle\phi_3, \pi^\sharp(\phi_1)\rangle) + \rho(\pi^\sharp(\phi_3))(\langle \phi_2, \pi^\sharp(\phi_1)\rangle)\\ &\quad - \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_1, \pi^\sharp(\phi_2)\rangle - \rho(\pi^\sharp(\phi_2))(\langle \phi_1, \pi^\sharp(\phi_3)\rangle)\\ &\quad + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_3, \pi^\sharp(\phi_1)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_2, \pi^\sharp(\phi_1)\rangle + \rho(\pi^\sharp(\phi_1))(\langle \phi_2, \pi^\sharp(\phi_3)\rangle)\\ &\quad + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_2, \pi^\sharp(\phi_3)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle + \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle)\\ &= -\langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_1, \pi^\sharp(\phi_2)\rangle\\ &\quad + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_3, \pi^\sharp(\phi_1)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_2, \pi^\sharp(\phi_1)\rangle\\ &\quad + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_2, \pi^\sharp(\phi_3)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle\end{aligned}$$

Now, from Cartan's formula, observe for example that

$$\begin{aligned}
\langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_1, \pi^\sharp(\phi_2)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_3, \pi^\sharp(\phi_1)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_2, \pi^\sharp(\phi_3)\rangle& \\
= \langle \iota_{\pi^\sharp(\phi_3)} d\phi_1, \pi^\sharp(\phi_2)\rangle + \langle d(\iota_{\pi^\sharp(\phi_3)}\phi_1), \pi^\sharp(\phi_2)\rangle& \\
+ \langle \iota_{\pi^\sharp(\phi_2)} d\phi_3, \pi^\sharp(\phi_1)\rangle + \langle d(\iota_{\pi^\sharp(\phi_2)}\phi_3), \pi^\sharp(\phi_1)\rangle& \\
+ \langle \iota_{\pi^\sharp(\phi_1)} d\phi_2, \pi^\sharp(\phi_3)\rangle + \langle d(\iota_{\pi^\sharp(\phi_1)}\phi_2), \pi^\sharp(\phi_3)\rangle& \\
= -\langle \iota_{\pi^\sharp(\phi_2)} d\phi_1, \pi^\sharp(\phi_3)\rangle - \langle d(\iota_{\pi^\sharp(\phi_1)}\phi_3), \pi^\sharp(\phi_2)\rangle& \\
- \langle \iota_{\pi^\sharp(\phi_1)} d\phi_3, \pi^\sharp(\phi_2)\rangle - \langle d(\iota_{\pi^\sharp(\phi_3)}\phi_2), \pi^\sharp(\phi_1)\rangle& \\
- \langle \iota_{\pi^\sharp(\phi_3)} d\phi_2, \pi^\sharp(\phi_1)\rangle - \langle d(\iota_{\pi^\sharp(\phi_2)}\phi_1), \pi^\sharp(\phi_3)\rangle& \\
= -\langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle - \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_2, \pi^\sharp(\phi_1)\rangle.&
\end{aligned}$$

Putting this equality into the last equation, we get finally

$$\langle [\pi,\pi], \phi_1 \wedge \phi_2 \wedge \phi_3\rangle = -2 \sum_{cyclic} \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle,$$

as we wanted.

■

Proposition G.19. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid. Then a 2-section $\pi \in \Lambda^2 A$ is integrable if and only if the induced morphism $\pi^\sharp : A^* \longrightarrow A$ preserves brackets:*

$$\pi^\sharp([\phi_1, \phi_2]_*) = [\pi^\sharp(\phi_1), \pi^\sharp(\phi_2)], \qquad \forall \phi_1, \phi_2 \in \Gamma(A^*),$$

where $[\cdot,\cdot]_*$ is the Lie bracket given in Prop. G.10.

Proof. The statement follows immediately from the identity

$$\pi^\sharp([\phi_1, \phi_2]_*) - [\pi^\sharp(\phi_1), \pi^\sharp(\phi_2)] = -\frac{1}{2}\langle [\pi,\pi], \phi_1 \wedge \phi_2\rangle, \tag{G.24}$$

which we will prove now. Let's compute the right-hand side of (G.24). To do this, we take any $\phi_3 \in \Gamma(A^*)$, then, using Eq. (G.17), (G.2) and (G.21) and skew-symmetry, we have

$$\begin{aligned}
&\langle \phi_3, \pi^\sharp([\phi_1, \phi_2]_*) - [\pi^\sharp(\phi_1), \pi^\sharp(\phi_2)]\rangle = -\langle \pi^\sharp(\phi_3), [\phi_1, \phi_2]_*\rangle - \langle \phi_3, [\pi^\sharp(\phi_1), \pi^\sharp(\phi_2)]\rangle \\
&= -\langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_2, \pi^\sharp(\phi_3)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle - \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle) \\
&\quad - \rho(\pi^\sharp(\phi_1))(\langle \phi_3, \pi^\sharp(\phi_2)\rangle) + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle \\
&= -\langle \iota_{\pi^\sharp(\phi_1)} d\phi_2, \pi^\sharp(\phi_3)\rangle - \langle d(\iota_{\pi^\sharp(\phi_1)}\phi_2), \pi^\sharp(\phi_3)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle \\
&\quad - \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle) - \rho(\pi^\sharp(\phi_1))(\langle \phi_3, \pi^\sharp(\phi_2)\rangle) + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle \\
&= \langle \iota_{\pi^\sharp(\phi_3)} d\phi_2, \pi^\sharp(\phi_1)\rangle + \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle) + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle \\
&\quad - \rho(\pi^\sharp(\phi_3))(\langle \phi_1, \pi^\sharp(\phi_2)\rangle) + \rho(\pi^\sharp(\phi_1))(\langle \phi_2, \pi^\sharp(\phi_3)\rangle) + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle \\
&= \langle \mathcal{L}_{\pi^\sharp(\phi_3)}\phi_2, \pi^\sharp(\phi_1)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_2)}\phi_1, \pi^\sharp(\phi_3)\rangle + \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle \\
&= \sum_{cyclic} \langle \mathcal{L}_{\pi^\sharp(\phi_1)}\phi_3, \pi^\sharp(\phi_2)\rangle.
\end{aligned} \tag{G.25}$$

Taking into account lemma G.18, (G.25) is equivalent to (G.24).

■

Appendix H

Lie 2-algebroids + splitting $\leftrightsquigarrow$ *split* Lie 2-algebroids

In this appendix we establish the equivalence between Lie 2-algebroids with a splitting and *split* Lie 2-algebroids, case, proved by G. Bonavolontà and N. Poncin [4]. As an application we show that *split* Lie 2-algebroids are equivalent to Lie 2-algebroids endowed with a splitting, thus establishing the connection between Lie 2-algebroids in the way we defined them, and split Lie 2-algebroids already present in the literature, which we recall below.

H.1 Split degree 2 NQ-manifolds and split Lie 2-algebroids

Definition H.1. A split Lie n-algebroid (or an n-term L_∞-algebroid) is a graded vector bundle $\mathcal{E} = E_0 \oplus E_{-1} \oplus \cdots \oplus E_{-n+1}$ over a manifold M endowed with the following structure:

- An anchor map $\rho : E_0 \longrightarrow TM$;
- $n+1$ brackets $l_i : \Gamma(\Lambda^i \mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$ with degree $2-i$, for $i = 1, \ldots, n+1$, such that

  1. $$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma \in \mathrm{Sh}(i,j-1)} \mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0, \tag{H.1}$$
     where $x_i \in \Gamma(E_{-i})$, $\mathrm{Sh}(i, j-1)$ denotes the set of $(i, j-1)$-shuffles, and $\mathrm{ksgn}(\sigma)$ is the Koszul sign for a permutation $\sigma \in S_k$, that is,
     $$x_1 \wedge \cdots \wedge x_k = \mathrm{Ksgn}(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}.$$
  2. l_2 satisfies the Leibniz rule with respect to ρ:
     $$l_2(x_0, fx) = fl_2(x_0, x) + \rho(x_0)(f)x, \ \forall x_0 \in \Gamma(E_0), f \in C^\infty(M), x \in \Gamma(\mathcal{E}).$$
  3. For $i \neq 2$ the l_i's are $C^\infty(M)$-linear.

Example H.2. A split Lie 1-algebroid is a Lie algebroid in the usual sense. Indeed, we have $\mathcal{E} = E_0$, and l_1 vanishes as it is of degree 1. The Lie infinity algebra conditions (Eq. (H.1)) reduce to the Jacobi identity.

Example H.3. A split Lie n-algebroid over a point is exactly a Lie n-algebra, i.e., an n-term Lie infinity algebra.

The following characterization of split Lie 2-algebroids in terms of geometric data appears, in the case that the base is a point, in the work on Lie 2-algebras of Baez and Crans [3]. With minor modifications we obtain the geometric description in the general case, where the base is a manifold.

Proposition H.4 ([3]). *A split Lie 2-algebroid structure on $\mathcal{E} = A \oplus B$ consistis on the following data:*

- *a vector bundle map (over the identity) $\partial : B \longrightarrow A$;*
- *a skew-symmetric bracket $[\cdot,\cdot] : A \times A \longrightarrow A$ together with an anchor-map $\rho : A \longrightarrow TM$;*
- *a vector bundle map (over the identity) $\nabla : A \longrightarrow \mathbf{CDO}(B)$, $x \longrightarrow \nabla_x$ (from A to the bundle of covariant differential operators of B);*
- *a vector bundle map $\Upsilon : \Lambda^3 A \longrightarrow B$.*

These maps satisfy:

1. $[x, fy] = f[x, y] + \rho(x)(f)y$;
2. $\nabla_x fh = f\nabla_x h + \rho(x)(f)h$;
3. $\partial(\nabla_x h) = [x, \partial h]$;
4. $\nabla_{\partial h} k = -\nabla_{\partial k} h$
5. $\partial \circ \Upsilon(x, y, z) = -[[x, y], z] + [[x, z], y] + [x, [y, z]]$,
6. $\Upsilon(\partial h, x, y) = \nabla_x \nabla_y h - \nabla_y \nabla_x h - \nabla_{[x,y]} h$;
7.
$$\begin{aligned}\Upsilon([w, y], x, z) + \Upsilon([x, z], w, y) + \nabla_y \Upsilon(w, x, z) + \nabla_w \Upsilon(x, y, z) = \nabla_z \Upsilon(w, x, y) + \nabla_x \Upsilon(w, y, z)\\ + \Upsilon([w, z], y, x) + \Upsilon([w, z], x, y) + \Upsilon([x, y], w, z) + \Upsilon([y, z], w, x),\end{aligned} \tag{H.2}$$

for all $w, x, y, z \in \Gamma(A)$ and $h, k \in \Gamma(B)$.

The following theorem provides a characterization of *split* NQ-manifolds, establishing a canonical 1:1 correspondence with *split* Lie n-algebroids. The proof is in [4], and we do not provide it here, for reasons of space-time, though we will use this result to establish a canonical 1:1 correspondence between NQ degree 2 manifolds and Lie 2-algebroids (without splitting). However, we will provide later, through the derived brackets method, a direct proof of this correspondence.

Theorem H.5 ([4]). *There is a canonical 1:1 correspondence between split NQ-manifolds (graded manifolds with a Q-structure and a fixed splitting) and split Lie n-algebroids.*

Remark H.6 ([65]). In the degree 2 case, which is the one that interests us, the explicit correspondence is given by:

$$\begin{aligned} Q(f) &= -\rho^*(df),\ \forall f \in C^\infty(M), \\ \langle Q(\varepsilon), e_1 \wedge e_2\rangle &= \rho(e_1)(\langle e_2, \varepsilon\rangle) - \rho(e_2)(\langle e_1, \varepsilon\rangle) - \langle [e_1, e_2], \varepsilon\rangle, \\ \langle Q(\varepsilon), \xi\rangle &= \langle \partial(\xi), \varepsilon\rangle, \\ \langle Q(\zeta), e_1 \wedge e_2 \wedge e_3\rangle &= \langle \Upsilon(e_1, e_2, e_3), \zeta\rangle, \\ \langle Q(\zeta), e \wedge \xi\rangle &= \langle \nabla_e \xi, \zeta\rangle - \rho(e)(\langle \xi, \zeta\rangle), \end{aligned} \tag{H.3}$$

where $\varepsilon \in \Gamma(E^*), \zeta \in \Gamma(F^*), \xi \in \Gamma(F)$ and $e, e_i \in \Gamma(E)$.

H.2 Equivalence between both structures

Now we describe the equivalence beween a split Lie 2-algebroid structure, and a Lie 2-algebroid structure afeter we choose a splitting (horizontal lift).

Proposition H.7. *Consider a preLie 2-algebroid. When a horizontal lift is chosen, the structures we obtain through Eqs. (4.9)-(4.15) coincide with the structure data obtained through Eqs. (H.3).*

Proof. Consider the structure data obtained from Q via Eqs. (4.9)-(4.15), and fix a horizontal lift. Of course, ρ and ∂ defined in $(H.3)$ coincide with the same maps defined in Eqs. (4.9) and (4.10).

Now, define a bracket on $\Gamma(E)$ by

$$[e_1, e_2] := \Delta_\Psi(\widehat{e_1}, e_2). \tag{H.4}$$

It follows immediately from the definition of Δ_Ψ, given in Eq. (4.5), that the bracket above coincides with the one given in the second equation of (H.3), and

$$\langle [e_1, e_2], \varepsilon\rangle = \langle [\widehat{e_1}, \widehat{e_2}], \bar{\varepsilon}\rangle, \tag{H.5}$$

follows readily from Eq. (4.15). We want to warn here that we are using the same notation $[\cdot,\cdot]$ for two different brackets, one on $\Gamma(E)$ and the other on $\Gamma(\widehat{E})$. We hope the context will make it always clear which bracket we are referring to in each situation.

We also see that ∇^F, introduced in Eq. (4.2) coincides with ∇ in the last equation of (H.3).

Finally, define a vector bundle map $\Upsilon : \Lambda^2 E \otimes E \longrightarrow F$ by

$$\Upsilon(e_1, e_2, e_3) := -K(e_1, e_2)(e_3), \tag{H.6}$$

where $K : \Omega^2(E, \mathrm{Hom}(E, F))$ is the *curvature form*

$$K(e_1, e_2) := \widehat{[e_1, e_2]} - [\widehat{e_1}, \widehat{e_2}]. \tag{H.7}$$

It follows from Eq. (H.5) that $K(e_1, e_2)$ is actually an element in $\mathrm{Hom}(E, F) \cong E \otimes F$, viewed inside $\widehat{E}$. We need to show that Υ defined in Eq. H.6 actually coincides with the map $\Upsilon : \Lambda^3 E \longrightarrow F$ defined according to the fourth equation in (H.3). From Eq. (3.78) we see that this amounts to show that, for $e_1, e_2, e_3 \in \Gamma(E)$ and $\zeta \in \Gamma(F^*)$

$$\langle e_3, \langle [\widehat{e_1}, \widehat{e_2}], \widehat{\zeta} \rangle \rangle = -\langle e_3, \langle \widehat{e_2}, Q(\widehat{\zeta})^\sharp_2(\widehat{e_1}) \rangle \rangle \tag{H.8}$$

holds, where $\widehat{\zeta} \in \widetilde{F}^*$ is the horizontal lift of ζ. But (H.8) follows immediately from Eq. (4.14).

■

Theorem H.8 (**Integrability equations revisited**). *Properties* 1 *and* 2 *of Def. 4.17 are equivalent to Eqs.* 3-7 *of Prop. H.4.*

Proof. In Prop. H.7, we have showed that, when we introduce a horizontal lift, the preLie 2-algebroid structure induced by Q coincides with the split Lie-2 algebroid structure defined in (H.3). Let's prove the equivalence between Eq. *2* of Def. 4.17 and Eq. *3* of Prop. H.4. On one hand we have

$$\begin{aligned} \partial \circ \Theta(\phi)(\xi) &= \partial \circ \Theta(\widehat{e} + \eta)(\xi) \\ &= \partial \circ \Theta(\widehat{e})(\xi) + \partial \circ \Theta(\eta)(\xi) \\ &= \partial(\nabla^F_e \xi) + \partial \circ \eta \circ \partial(\xi), \end{aligned} \tag{H.9}$$

where we used property *4* of Def. 4.6. On the other hand,

$$\begin{aligned} (\Delta_\Psi \circ \partial)(\phi)(\xi) &= \Delta_\Psi(\phi, \partial(\xi)) = \Delta_\Psi(\widehat{e} + \eta, \partial(\xi)) \\ &= \Delta_\Psi(\widehat{e}, \partial(\xi)) + \Delta_\Psi(\eta, \partial(\xi)) \\ &= [e, \partial(\xi)] + \partial \circ \eta \circ \partial(\xi), \end{aligned} \tag{H.10}$$

where, again, we used property *4* of Def. 4.6.

Comparing (H.9) and (H.10), we see that Eq. *2* of Def. 4.17 and Eq. *3* of Prop. H.4 are equivalent.

Now let's prove that, using the equivalence we have already established, property *1* of Def. 4.17 is equivalent to Eqs. *4, 5, 6* and *7* of Prop. H.4.

In our situation, property *1* of Def. 4.17 is equivalent to the equation

$$J(\phi_1, \phi_2, \phi_3) = 0, \tag{H.11}$$

where the *Jacobiator*, J, for a bracket $[\cdot, \cdot]$, is defined by

$$J(\phi_1, \phi_2, \phi_3) := [\phi_1, [\phi_2, \phi_3]] - [[\phi_1, \phi_2], \phi_3] - [\phi_2, [\phi_1, \phi_3]].$$

Now,

$$\begin{aligned} J(\phi_1, \phi_2, \phi_3) &= J(\widehat{e_1} + \eta_1, \widehat{e_2} + \eta_2, \widehat{e_3} + \eta_3) \\ &= J(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}) + J(\widehat{e_1}, \eta_2, \eta_3) + J(\widehat{e_1}, \widehat{e_2}, \eta_3) + J(\widehat{e_1}, \eta_2, \widehat{e_3}) \\ &\quad + J(\eta_1, \widehat{e_2}, \widehat{e_3}) + J(\eta_1, \eta_2, \eta_3) + J(\eta_1, \widehat{e_2}, \eta_3) + J(\eta_1, \eta_2, \widehat{e_3}). \end{aligned} \tag{H.12}$$

Then, we have to calculate each one of the eight Jacobiators on the right hand side of (H.12), and verify that the nullity of these eight terms is equivalent to Eqs. *4*, *5*, *6* and *7* of Prop. H.4 –given the preceding equivalences already proven.

Computation of $J(\widehat{e_1}, \widehat{e_2}, \widehat{e_3})$

$$\begin{aligned}[\widehat{e_1}, [\widehat{e_2}, \widehat{e_3}]] &= [\widehat{e_1}, \widehat{[e_2, e_3]}] - [\widehat{e_1}, K(e_2, e_3)] \\ &= [e_1, \widehat{[e_2, e_3]}] - K(e_1, [e_2, e_3]) - \nabla^F_{e_1} \circ K(e_2, e_3) + K(e_2, e_3) \circ [e_1, \cdot].\end{aligned} \tag{H.13}$$

Analogously,

$$[\widehat{e_2}, [\widehat{e_1}, \widehat{e_3}]] = [e_2, \widehat{[e_1, e_3]}] - K(e_2, [e_1, e_3]) - \nabla^F_{e_2} \circ K(e_1, e_3) + K(e_1, e_3) \circ [e_2, \cdot]. \tag{H.14}$$

Finally

$$\begin{aligned}[[\widehat{e_1}, \widehat{e_2}], \widehat{e_3}] &= [\widehat{[e_1, e_2]}, \widehat{e_3}] - [K(e_1, e_2), \widehat{e_3}] \\ &= [\widehat{[e_1, e_2], e_3}] - K([e_1, e_2], e_3) + \nabla^F_{e_3} \circ K(e_1, e_2) \\ &\quad - K(e_1, e_2) \circ [e_3, \cdot] - \nabla^F_{\cdot} K(e_1, e_2)(e_3) - \partial(K\widehat{(e_1, e_2)}(e_3)).\end{aligned} \tag{H.15}$$

Putting together Eqs. (H.13), (H.14) and (H.15), we conclude that

$$J(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}) = 0 \tag{H.16}$$

if and only if Eqs. *5* and *7* of Prop. H.4 are true.

Computation of $J(\eta_1, \widehat{e_2}, \eta_3)$**.**

$$[\eta_1, [\widehat{e_2}, \eta_3]] = \eta_1 \circ \partial \circ \nabla^F_{e_2} \circ \eta_3 - \nabla^F_{e_2} \circ \eta_3 \circ \partial \circ \eta_1 - \eta_1 \circ \partial \circ \eta_3 \circ [e_2, \cdot] + \eta_3 \circ [e_2, \cdot] \circ \partial \circ \eta_1. \tag{H.17}$$

Next,

$$\begin{aligned}[[\eta_1, \widehat{e_2}], \eta_3] &= [\eta_1 \circ [e_2, \cdot] - \nabla^F_{e_2} \circ \eta_1 + \nabla^F_{\cdot} \eta_1(e_2) + \partial(\widehat{\eta_1(e_2)}), \eta_3] \\ &= \eta_1 \circ [e_2, \cdot] \circ \partial \circ \eta_3 - \eta_3 \circ \partial \circ \eta_1 \circ [e_2, \cdot] - \nabla^F_{e_2} \circ \eta_1 \circ \partial \circ \eta_3 + \eta_3 \circ \partial \circ \nabla^F_{e_2} \circ \eta_1 \\ &\quad + \nabla^F_{\cdot} \eta_1(e_2) \circ \partial \circ \eta_3 - \eta_3 \circ \partial \circ \nabla^F_{\cdot} \eta_1(e_2) + \nabla^F_{\partial(\eta_1(e_2))} \circ \eta_3 - \eta_3 \circ [\partial(\eta_1(e_2)), \cdot].\end{aligned} \tag{H.18}$$

Finally,

$$[\widehat{e_2}, [\eta_1, \eta_2]] = \nabla^F_{e_2} \circ \eta_1 \circ \partial \circ \eta_3 - \eta_1 \circ \partial \circ \eta_3 \circ [e_2, \cdot] - \nabla^F_{e_2} \circ \eta_3 \circ \partial \circ \eta_1 + \eta_3 \circ \partial \circ \eta_1 \circ [e_2, \cdot]. \tag{H.19}$$

Putting together Eqs. (H.17), (H.18) and (H.19), and using Eq. *3* of Prop. H.4, we conclude that

$$J(\eta_1, \widehat{e_2}, \eta_3) = 0 \tag{H.20}$$

for arbitrary η_1, e_2, η_3 if and only if Eq. *4* of Prop. H.4 holds.

Computation of $J(\widehat{e_1}, \widehat{e_2}, \eta_3)$.

$$\begin{aligned}[\widehat{e_1}, [\widehat{e_2}, \eta_3]] &= [\widehat{e_1}, \nabla^F_{e_2} \circ \eta_3 - \eta_3 \circ [e_2, \cdot]] \\ &= \nabla^F_{e_1} \nabla^F_{e_2} \circ \eta_3 - \nabla^F_{e_2} \circ \eta_3 \circ [e_1, \cdot] - \nabla^F_{e_1} \circ \eta_3 \circ [e_2, \cdot] + \eta_3 \circ [e_2, \cdot] \circ [e_1, \cdot].\end{aligned} \tag{H.21}$$

Next,

$$\begin{aligned}[[\widehat{e_1}, \widehat{e_2}], \eta_3] &= [\widehat{[e_1, e_2]} - K(e_1, e_2), \eta_3] \\ &= \nabla^F_{[e_1,e_2]} \circ \eta_3 - \eta_3 \circ [[e_1, e_2], \cdot] - K(e_1, e_2) \circ \partial \circ \eta_3 + \eta_3 \circ \partial \circ K(e_1, e_2).\end{aligned} \tag{H.22}$$

Finally,

$$[\widehat{e_2}, [\widehat{e_1}, \eta_3]] = \nabla^F_{e_2} \nabla^F_{e_1} \circ \eta_3 - \nabla^F_{e_1} \circ \eta_3 \circ [e_2, \cdot] - \nabla^F_{e_2} \circ \eta_3 \circ [e_1, \cdot] + \eta_3 \circ [e_1, \cdot] \circ [e_2, \cdot]. \tag{H.23}$$

Putting together Eqs. (H.21), (H.22) and (H.23),

$$\begin{aligned}J(\widehat{e_1}, \widehat{e_2}, \eta_3) &= R_{\nabla^F}(e_1, e_2) \circ \eta_3 - \eta_3 \circ J(e_1, e_2, \cdot) \\ &\quad + K(e_1, e_2) \circ \partial \circ \eta_3 - \eta_3 \circ \partial \circ K(e_1, e_2),\end{aligned}$$

thereby, using Eqs. *5* of Prop. H.4, we conclude that

$$J(\widehat{e_1}, \widehat{e_2}, \eta_3) = 0. \tag{H.24}$$

for arbitrary e_1, e_2, η_3 if and only if Eq. *6* of Prop. H.4 is true.

Remark H.9. Observe that at this point we already have obtained all the equations *3*-*7* of Prop.H.4 from the properties of a split Lie 2-algebroid. So it remains to show that those equations imply the nullity of the remaining Jacobiators in (H.12).

Computation of $J(\widehat{e_1}, \eta_2, \eta_3)$.

$$\begin{aligned}[\widehat{e_1}, [\eta_2, \eta_3]] &= [\widehat{e_1}, \eta_2 \circ \partial \circ \eta_3 - \eta_3 \circ \partial \circ \eta_2] \\ &= \nabla^F_{e_1} \circ \eta_2 \circ \partial \circ \eta_3 - \eta_2 \circ \partial \circ \eta_3 \circ [e_1, \cdot] - \nabla^F_{e_1} \eta_3 \circ \partial \circ \eta_2 + \eta_3 \circ \partial \circ \eta_2 \circ [e_1, \cdot].\end{aligned} \tag{H.25}$$

Next,

$$\begin{aligned}[[\widehat{e_1}, \eta_2], \eta_3] &= [\nabla^F_{e_1} \circ \eta_2 - \eta_2 \circ [e_1, \cdot], \eta_3] \\ &= \nabla^F_{e_1} \circ \eta_2 \circ \partial \circ \eta_3 - \eta_3 \circ \partial \circ \nabla^F_{e_1} \circ \eta_2 - \eta_2 \circ [e_1, \cdot] \circ \partial \circ \eta_3 + \eta_3 \circ \partial \circ \eta_2 \circ [e_1, \cdot].\end{aligned} \tag{H.26}$$

Finally,

$$\begin{aligned}[\eta_2, [\widehat{e_1}, \eta_3]] &= [\eta_2, \nabla^F_{e_1} \circ \eta_3 - \eta_3 \circ [e_1, \cdot] \\ &= \eta_2 \circ \partial \circ \nabla^F_{e_1} \circ \eta_3 - \nabla^F_{e_1} \circ \eta_3 \circ \partial \circ \eta_2 - \eta_2 \circ \partial \circ \eta_3 \circ [e_1, \cdot] + \eta_3 \circ [e_1, \cdot] \circ \partial \circ \eta_2.\end{aligned} \tag{H.27}$$

Putting together Eqs. (H.25), (H.26) and (H.27), and using Eq. *3* of Prop. H.4, we conclude that

$$J(\widehat{e_1}, \eta_2, \eta_3) = 0. \tag{H.28}$$

Computation of $J(\widehat{e_1},\eta_2,\widehat{e_3})$.

$$\begin{aligned}[]
[\widehat{e_1},[\eta_2,\widehat{e_3}]] &= [\widehat{e_1},\eta_2\circ[e_3,\cdot]-\nabla^F_{e_3}\circ\eta_2+\nabla^F_{\cdot}\eta_2(e_3)+\widehat{\partial\eta(e_3)}] \\
&= \nabla^F_{e_1}\circ\eta_2\circ[e_3,\cdot]-\eta_2\circ[e_3,\cdot]\circ[e_1,\cdot]-\nabla^F_{e_1}\nabla^F_{e_3}\circ\eta_2+\nabla^F_{e_3}\circ\eta_2\circ[e_1,\cdot] \\
&\quad+\nabla^F_{e_1}\circ\nabla^F_{\cdot}\eta_2(e_3)-\nabla^F_{\cdot}\eta_2(e_3)\circ[e_3,\cdot]+[e_1,\widehat{\partial(\eta_2(e_3))}]-K(e_1,\partial(\eta_2(e_3))).
\end{aligned} \tag{H.29}$$

Next,

$$\begin{aligned}[]
[[\widehat{e_1},\eta_2],\widehat{e_3}] &= [\nabla^F_{e_1}\circ\eta_2-\eta_2\circ[e_1,\cdot],\widehat{e_3}] \\
&= \nabla^F_{e_1}\circ\eta_2\circ[e_3,\cdot]-\nabla^F_{e_3})\nabla^F_{e_1}\circ\eta_2+\nabla^F_{\cdot}\nabla^F_{e_1}\eta_2(e_3)+\partial(\widehat{\nabla^F_{e_1}\eta_2}(e_3) \\
&\quad-\eta_2\circ[e_1,\cdot]\circ[e_3,\cdot]+\nabla^F_{e_3}\circ\eta_2\circ[e_1,\cdot]-\nabla^F_{\cdot}\eta([e_1,e_3])-\partial\circ\widehat{\eta_2([e_1},e_3]).
\end{aligned} \tag{H.30}$$

Finally

$$\begin{aligned}[]
[\eta_2,[\widehat{e_1},\widehat{e_3}]] &= [\eta_2,\widehat{[e_1,e_3]}-K(e_1,e_3)] \\
&= \eta_2[[e_1,e_3],\cdot]-\nabla^F_{[e_1,e_3]}\circ\eta_2+\nabla^F_{\cdot}\eta_2([e_1,e_3])+\partial(\eta_2\widehat{([e_1},e_3])) \\
&\quad-\eta_2\circ\partial\circ K(e_1,e_3)+K(e_1,e_3)\circ\partial\circ\eta_2.
\end{aligned} \tag{H.31}$$

Putting together (H.29), (H.30) and (H.31) we get

$$\begin{aligned}
J(\widehat{e_1},\eta_2,\widehat{e_3}) &= R_{\nabla^F}(e_3,e_1)\circ\eta_2+\eta_2\circ J(e_1,e_3,\cdot)-R_{\nabla^F}(\cdot,e_1)(\eta_2(e_3)) \\
&\quad+K(e_3,e_1)\circ\partial\circ\eta_2+\eta_2\circ\partial\circ K(e_1,e_3)-K(\cdot,e_1)(\partial(\eta_2(e_3))) \\
&\quad+[e_1,\partial(\eta_2(e_3))]-\partial\circ\nabla^F_{e_1}\eta_2(e_3),
\end{aligned}$$

thereby, using Eqs. *3, 6* and *7* of Prop. H.4, we get

$$J(\widehat{e_1},\eta_2,\widehat{e_3})=0. \tag{H.32}$$

Computation of $J(\eta_1,\widehat{e_2},\widehat{e_3})$.

$$\begin{aligned}[]
[\eta_1,[\widehat{e_2},\widehat{e_3}]] &= [\eta_1,\widehat{[e_2,e_3]}-K(e_2,e_3)] \\
&= \eta_1\circ[[e_2,e_3],\cdot]-\nabla^F_{[e_2,e_3]}\circ\eta_1+\nabla^F_{\cdot}\eta_1([e_2,e_3])+\partial(\eta_1\widehat{([e_2},e_3])) \\
&\quad-\eta_1\circ\partial\circ K(e_2,e_3)+K(e_2,e_3)\circ\partial\circ\eta_1.
\end{aligned} \tag{H.33}$$

Next,

$$\begin{aligned}[]
[[\eta_1,\widehat{e_2}],\widehat{e_3}] &= [\eta_1\circ[e_2,\cdot]-\nabla^F_{e_2}\circ\eta_1+\nabla^F_{\cdot}\eta_1(e_2)+\partial\,\widehat{\circ\,\eta_1(e_2)},\widehat{e_3}] \\
&= \eta_1\circ[e_2,\cdot]\circ[e_3,\cdot]-\nabla^F_{e_3}\circ\eta_1\circ[e_2,\cdot]+\nabla^F_{\cdot}\eta_1([e_2,e_3])+\partial\circ\widehat{\eta_1([e_2},e_3]) \\
&\quad-\nabla^F_{e_2}\circ\eta_1\circ[e_3,\cdot]+\nabla^F_{e_3}\circ\nabla^F_{e_2}\circ\eta_1-\nabla^F_{\cdot}\nabla^F_{e_2}\eta_1(e_3)-\partial(\widehat{\nabla^F_{e_2}\eta_1}(e_3)) \\
&\quad+\nabla^F_{[e_3,\cdot]}\eta_1(e_2)-\nabla^F_{e_3}\circ\nabla^F_{\cdot}\eta_1(e_2)+\nabla^F_{\cdot}\nabla^F_{e_3}\eta_1(e_2)+\partial(\widehat{\nabla^F_{e_3}\eta_1}(e_2)) \\
&\quad-[e_3,\widehat{\partial(\eta_1(e_2))}]+K(\partial(\eta_1(e_2)),e_3).
\end{aligned} \tag{H.34}$$

Finally,

$$[\widehat{e_2},[\eta_1,\widehat{e_3}]] = \nabla^F_{e_2}\circ\eta_1\circ[e_3,\cdot] - \eta_1\circ[e_3,\cdot]\circ[e_2,\cdot] - \nabla^F_{e_2}\circ\nabla^F_{e_3}\circ\eta_1 + \nabla^F_{e_3}\circ\eta_1\circ[e_2,\cdot]$$
$$+ \nabla^F_{e_2}\circ\nabla^F_{\cdot}\eta_1(e_3) - \nabla^F_{[e_2,\cdot]}\eta_1(e_3) + [e_2,\widehat{\partial(\eta_1(e_3))}] - K(e_2,\partial(\eta_1(e_3))). \tag{H.35}$$

Putting together Eqs. (H.33), (H.34) and (H.35), we get

$$J(\eta_1,\widehat{e_2},\widehat{e_3}) = \eta_1\circ J(e_2,e_3,\cdot) + R_{\nabla^F}(e_2,e_3)\circ\eta_1 + R_{\nabla^F}(e_3,\cdot)(\eta_1(e_2)) + R_{\nabla^F}(\cdot,e_2)(\eta_1(e_3))$$
$$+ \eta_1\circ\partial\circ K(e_2,e_3) + K(e_2,e_3)\circ\eta_1 + K(e_3,\cdot)(\eta_1(e_2)) + K(\cdot,e_2)(\eta_1(e_3))$$
$$+ \partial(\widehat{\nabla^F_{e_2}\eta_1}(e_3)) - [e_2,\widehat{\partial(\eta_1(e_3))}] - \partial(\widehat{\nabla^F_{e_3}\eta_1}(e_2)) + [e_3,\widehat{\partial(\eta_1(e_2))}].$$

Thereby, using Eqs. *3*, *5* and *6* of Prop. H.4, we get

$$J(\eta_1,\widehat{e_2},\widehat{e_3}) = 0. \tag{H.36}$$

Computation of $J(\eta_1,\eta_2,\eta_3)$**.**

$$[\eta_1,[\eta_2,\eta_3]] = [\eta_1,\eta_2\circ\partial\circ\eta_3 - \eta_3\circ\partial\circ\eta_2]$$
$$= \eta_1\circ\partial\circ\eta_2\circ\partial\circ\eta_3 - \eta_2\circ\partial\eta_3\circ\partial\circ\eta_1 - \eta_1\circ\partial\circ\eta_3\circ\partial\circ\eta_2 + \eta_3\circ\partial\circ\eta_2\circ\partial\circ\eta_1. \tag{H.37}$$

Next,

$$[[\eta_1,\eta_2],\eta_3]] = [\eta_1\circ\partial\circ\eta_2 - \eta_2\circ\partial\circ\eta_1,\eta_3]$$
$$= \eta_1\circ\partial\circ\eta_2\circ\partial\circ\eta_3 - \eta_3\circ\partial\eta_1\circ\partial\circ\eta_2 - \eta_2\circ\partial\circ\eta_1\circ\partial\circ\eta_3 + \eta_3\circ\partial\circ\eta_2\circ\partial\circ\eta_1. \tag{H.38}$$

Finally,

$$[\eta_2,[\eta_1,\eta_3]] = [\eta_2,\eta_1\circ\partial\circ\eta_3 - \eta_3\circ\partial\circ\eta_1]$$
$$= \eta_2\circ\partial\circ\eta_1\circ\partial\circ\eta_3 - \eta_1\circ\partial\eta_3\circ\partial\circ\eta_2 - \eta_2\circ\partial\circ\eta_3\circ\partial\circ\eta_1 + \eta_3\circ\partial\circ\eta_1\circ\partial\circ\eta_2. \tag{H.39}$$

Putting together Eqs. (H.37), (H.38) and (H.39), we conclude

$$J(\eta_1,\eta_2,\eta_3) = 0. \tag{H.40}$$

Computation of $J(\eta_1,\eta_2,\widehat{e_3})$**.**

$$[\eta_1,[\eta_2,\widehat{e_3}]] = [\eta_1,\eta_2\circ[e_3,\cdot] - \nabla^F_{e_3}\circ\eta_2 + \nabla^F_{\cdot}\eta_2(e_3) + \partial\widehat{(\eta_2(e_3))}]$$
$$= \eta_1\circ\partial\circ\eta_2\circ[e_3,\cdot] - \eta_2\circ[e_3,\cdot]\circ\partial\circ\eta_1 - \eta_1\circ\partial\circ\nabla^F_{e_3}\circ\eta_2 + \nabla^F_{e_3}\circ\eta_2\circ\partial\circ\eta_1$$
$$+ \eta_1\circ\partial\circ\nabla^F_{\cdot}\eta_2(e_3) - \nabla^F_{\cdot}\eta_2(e_3)\circ\partial\circ\eta_1 + \eta_1\circ[\partial(\eta_2(e_3)),cdot] - \nabla^F_{\partial(\eta_2(e_3))}\circ\eta_1$$
$$+ \nabla^F_{\cdot}\circ\eta_1\circ\partial\circ\eta_2(e_3) + \partial\circ\eta_1\widehat{\circ\partial\circ}\eta_2(e_3). \tag{H.41}$$

Next,

$$
\begin{aligned}
[[\eta_1,\eta_2],\widehat{e_3}] &= [\eta_1\circ\partial\circ\eta_2-\eta_2\circ\partial\eta_1,\widehat{e_3}] \\
&= \eta_1\circ\partial\circ\eta_2\circ[e_3,\cdot]-\nabla^F\circ\eta_1\circ\partial\circ\eta_2+\nabla^F_{\cdot}\circ\eta_1\circ\partial\circ\eta_2(e_3)+\partial\circ\eta_1\circ\partial\circ\eta_2(e_3) \\
&\quad-\eta_2\circ\partial\circ\eta_1\circ[e_3,\cdot]+\nabla^F_{e_3}\circ\eta_2\circ\partial\circ\eta_1-\nabla^F_{\cdot}\circ\eta_2\circ\partial\circ\eta_1(e_3)-\partial\circ\eta_2\,\widehat{\circ\,\partial\circ}\,\eta_1(e_3).
\end{aligned}
\tag{H.42}
$$

Finally,

$$
\begin{aligned}
[\eta_2,[\eta_1,\widehat{e_3}]] &= \eta_2\circ\partial\circ\eta_1\circ[e_3,\cdot]-\eta_1\circ[e_3,\cdot]\circ\partial\circ\eta_2-\eta_2\circ\partial\circ\nabla^F_{e_3}\circ\eta_1+\nabla^F_{e_3}\circ\eta_1\circ\partial\circ\eta_2 \\
&\quad+\eta_2\circ\partial\circ\nabla^F_{\cdot}\eta_1(e_3)-\nabla^F_{\cdot}\eta_1(e_3)\circ\partial\circ\eta_2+\eta_2\circ[\partial(\eta_1(e_3)),\cdot]-\nabla^F_{\partial(\eta_1(e_3))}\circ\eta_2 \\
&\quad+\nabla^F_{\cdot}\eta_2\circ\partial\circ\eta_1(e_3)+\partial\circ\widehat{\eta_2\partial\circ\eta_1}(e_3).
\end{aligned}
\tag{H.43}
$$

Putting together Eqs. (H.41), (H.42) and (H.43) and using Eqs. *3* and *4* of Prop. H.4, we arrive to

$$
J(\eta_1,\eta_2,\widehat{e_3})=0. \tag{H.44}
$$

So, we have showed Eqs. *3*-*7* of Prop. H.4 are equivalent to (H.11), and the proof of the theorem is complete.

■

Corollary H.10. *There is a canonical 1:1 correspondence between split Lie 2-algebroids and Lie 2-algebroids endowed with a splitting.*

Proof. The sequence (3.44) together with a splitting $\psi: E\longrightarrow\widehat{E}$ is equivalent to the pair of vector bundles (F,E), or equivalently, to the Whitney sum $\mathcal{E}=E\oplus F$. We saw at the beginning of the proof of Thm. 4.20 how to obtain the data $(\partial,\rho,[\cdot,\cdot]_E,\nabla,\Upsilon)$, provided we have a splitting, namely we have

$$
\begin{aligned}
[e_1,e_2]_E &= \Delta_\Psi(\widehat{e_1},e_2), \\
\nabla_e\xi &= \Theta(\widehat{e})(\xi), \\
\Upsilon(e_1,e_2,e_3) &= -K(e_1,e_2)(e_3),
\end{aligned}
$$

where K was introduced in Eq. (H.7).

Conversely, if we start with the data $(\partial,\rho,[\cdot,\cdot]_E,\nabla,\Upsilon)$, we can obtain the data $(\widehat{\rho},\partial,[\cdot,\cdot],\Psi,\Theta)$ that defines a Lie 2-algebroid in the following way.

- $\widehat{\rho}$ is defined from ρ by property *1* of Def. 4.6.
- ∂ is the same map in both structures.
- From properties *4* and *5* we can define Δ_Ψ from $[\cdot,\cdot]_E$ and ∂ (recall that thanks to the splitting we can write $\phi=\eta+\widehat{e}$). Thereby, we can obtain Ψ from Δ_Ψ and ρ using Eq. (4.5).
- Θ is obtained from ∇ and ∂, using property *4* of Def. 4.6.

- Finally, $[\cdot,\cdot]$ is obtained from $[\cdot,\cdot]_E$, Υ, ∇ and ∂, using Eq. (H.7) and properties *6,7* and *4* of Def. 4.6.

Therefore, we have obtained a preLie 2-algebroid structure. It is routine to verify that properties *1-7* of the definition are satisfied, indeed, we built the structure data so that those properties are satisfied.

That conditions *3-7* of Prop. H.4 are equivalent to conditions *1* and *2* of Def. 4.17 is precisely what we do in the proof of Thm. H.8.

■

Corollary H.11. *Consider a Lie 2-algebroid structure on the dual sequence (3.44). Then*

$$\Theta : \widehat{E} \longrightarrow \mathbf{CDO}(F)$$

is a representation of $(\widehat{E},[\cdot,\cdot])$ *by first-order differential operators (see Def. I.14), that is,*

$$\Theta([\phi_1,\phi_2]) = [\Theta(\phi_1),\Theta(\phi_2)], \qquad \forall \phi_1,\phi_2 \in \Gamma(\widehat{E}). \tag{H.45}$$

Proof. Let's compute $[\Theta(\phi_1),\Theta(\phi_2)](\xi) - \Theta([\phi_1,\phi_2])(\xi)$:

a)

$$\begin{aligned}\Theta(\phi_1)(\Theta(\phi_2)(\xi)) &= \Theta(\phi_1)(\nabla^F_{e_2}\xi + \eta_2\circ\partial(\xi))\\ &= \nabla^F_{e_1}\nabla^F_{e_2}\xi + \nabla^F_{e_1}\eta_2(\partial(\xi)) + \eta_1\circ\partial(\nabla^F_{e_2}\xi) + \eta_1\circ\partial\circ\eta_2\partial(\xi).\end{aligned}$$

Similarly

b)

$$\Theta(\phi_2)(\Theta(\phi_1)(\xi)) = \nabla^F_{e_2}\nabla^F_{e_1}\xi + \nabla^F_{e_2}\eta_1(\partial(\xi)) + \eta_2\circ\partial(\nabla^F_{e_1}\xi) + \eta_2\circ\partial\circ\eta_1\partial(\xi).$$

c) Now we want to calculate $\Theta([\phi_1,\phi_2])(\xi)$. Let's begin with $[\phi_1,\phi_2]$. Using (H.7) and Eqs. *4, 6* and *7* of Def. 4.6, we have

$$\begin{aligned}[\phi_1,\phi_2] &= [\widehat{e_1}+\eta_1, \widehat{e_2}+\eta_2]\\ &= \widehat{[e_1,e_2]} - K(e_1,e_2) + \nabla^F_{e_1}\circ\eta_2 - \eta_2\circ[e_1,\cdot]\\ &\quad + \eta_1\circ[e_2,\cdot] - \nabla^F_{e_2}\circ\eta_1 + \nabla^F_{\cdot}\eta_1(e_2) + \partial\circ\eta_1(e_2)\\ &\quad + \eta_1\circ\partial\circ\eta_2 - \eta_2\circ\partial\circ\eta_1,\end{aligned}$$

thereby

$$\begin{aligned}\Theta([\phi_1,\phi_2])(\xi) &= \nabla^F_{[e_1,e_2]}\xi - K(e_1,e_2)\circ\partial(\xi) + \nabla^F_{e_1}\circ\eta_2\circ\partial(\xi) - \eta_2\circ[e_1,\cdot]\circ\partial(\xi)\\ &\quad + \eta_1\circ[e_2,\cdot]\circ\partial(\xi) - \nabla^F_{e_2}\circ\eta_1\circ\partial(\xi) + \nabla^F_{\partial(\xi)}\eta_1(e_2) + \nabla^F_{\partial\circ\eta_1(e_2)}\xi\\ &\quad + \eta_1\circ\partial\circ\eta_2\circ\partial(\xi) - \eta_2\circ\partial\circ\eta_1\circ\partial(\xi).\end{aligned} \tag{H.46}$$

Using Eqs. *3* and *4* of Prop. H.4, and comparing items a) and b) above with Eq. (H.46) of item c), we obtain

$$[\Theta(\phi_1),\Theta(\phi_2)](\xi) - \Theta([\phi_1,\phi_2])(\xi) = R_{\nabla^F}(e_1,e_2)(\xi) + K(e_1,e_2)(\partial(\xi)).$$

Hence, taking Eq. (H.6) into account (and the skew-symmetry of Υ), and using Eq. *6* of Prop. H.4 we get

$$[\Theta(\phi_1),\Theta(\phi_2)](\xi) - \Theta([\phi_1,\phi_2])(\xi) = 0.$$

■

Appendix I

First-order differential operators and quasi pseudoalgebra brackets

In this chapter we study first-order differential operators in order to describe certain generalizations of Lie algebroids that will provide the appropriate geometric structure which describes NQ degree 2 manifolds. The whole chapter is based on [19] and [53].

Definition I.1. Let $E \longrightarrow M$ be a vector bundle. An $\mathbb{R}$-linear map $D : \Gamma(E) \longrightarrow \Gamma(E)$ is a *k-th order differential operator* if, for any $k+1$ functions $f_0, \ldots, f_k \in C^\infty(M)$, we have:

$$[f_k, [f_{k-1}, [\ldots [f_0, D], \ldots]] = 0. \tag{I.1}$$

Here the bracket $[f, D] : \Gamma(E) \to \Gamma(E)$ is defined as the commutator

$$[f, D](s) = D(f \cdot s) - f \cdot D(s).$$

The set of k-th order differential operators is denoted by $\mathcal{D}^k(E)$.

Example I.2. The 0-th order differential operators are just the vector bundle endomorphisms.

The covariant differential operators of E, sections of the Lie algebroid $\mathbf{CDO}(E)$, also called derivative endomorphisms, which are introduced after Prop. 6.1, are a particular case of first-order differential operators.

Proposition I.3. *An $\mathbb{R}$-linear operator $D : \Gamma(E) \longrightarrow \Gamma(E)$ is a first-order differential operator if and only if there exists a section $\sigma_D \in \Gamma(TM \otimes \mathrm{End}(E))$ such that, for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$,*

$$D(fs) = fD(s) + \sigma_D(df \otimes s). \tag{I.2}$$

In this case σ_D is unique.

Proof. Suppose $D \in \mathcal{D}^1(E)$. Consider the bilinear map of vector spaces $B : C^\infty(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ given by

$$B(f, s) = D(fs) - fD(s).$$

Then (I.1), in the case $k = 1$, implies that

$$B(f, gs) = D(fgs) - fD(gs) = gD(fs) - g(fD(s)) = gB(f, s)$$

and

$$\begin{aligned} B(fg, s) &= D(fgs) - fgD(s) = fD(gs) + gD(fs) - 2fgD(s) \\ &= f(D(gs) - gD(s)) + g(D(fs) - f(D(s)) \\ &= fB(g, s) + gB(f, s) \end{aligned}$$

This implies that B is represented by a section $\sigma_D \in \Gamma(TM \otimes \mathrm{End}(E))$, so that

$$D(fs) - fD(s) = \sigma_D(df \otimes s).$$

Conversely, if there is $\sigma_D \in \Gamma(TM \otimes \mathrm{End}(E))$ such that (I.2) holds, then

$$\begin{aligned} [f_1, [f_0, D]](s) &= D(f_1 f_0 s) - f_0 D(f_1 s) - f_1 D(f_0 s) + f_0 f_1 D(s) \\ &= f_1 D(f_0 s) + f_0 \sigma_D(df_1 \otimes s) - f_0 f_1 D(s) - f_0 \sigma_D(df_1 \otimes s) \\ &\quad - f_1 D(f_0 s) + f_0 f_1 D(s) \\ &= 0. \end{aligned}$$

By the way B was defined, we see that it is uniquely determined by D, therefore σ_D is unique.

■

Proposition I.4. *An $\mathbb{R}$-linear operator $D : \Gamma(E) \longrightarrow \Gamma(E)$ is in $\mathcal{D}^1(E)$ if and only if it factors through the first jet bundle $J^1(E)$, that is, if and only if there exists a vector bundle morphism*

$$\Theta_D : J^1(E) \longrightarrow E,$$

such that $D = \Theta_D \circ j^1$.

In the affirmative case, Θ_D is unique.

Proof. If there is Θ_D with $D = \Theta_D \circ j^1$, then, for $s \in \Gamma(E)$,

$$\begin{aligned} [f_1, [f_0, D]](s) &= [f_1, [f_0, \Theta_D \circ j^1]](s) = [f_1, \Theta_D(j^1(f_0 s)) - f_0 \Theta_D(j^1(s))] \\ &= [f_1, \Theta_D(f_0 j^1(s) + df_0 \otimes s) - f_0 \Theta_D(j^1(s))] \\ &= \Theta_D(df_0 \otimes f_1 s) - f_1 \Theta_D(df_0 \otimes s) = 0, \end{aligned}$$

then $D \in \mathcal{D}^1(E)$.

If $D \in \mathcal{D}^1(E)$ then by Prop. I.3 there exists $\sigma_D \in \Gamma(TM \otimes \mathrm{End}(E))$ with the property (I.2). Recall that the first jet bundle $J^1(E)$ fits in the exact sequence

$$0 \longrightarrow T^*M \otimes E \longrightarrow J^1(E) \xrightarrow{\pi} E \longrightarrow 0. \tag{I.3}$$

Then, given $\eta \in \Gamma(J^1(E))$, then we obtain a section $\pi(\eta) \in \Gamma(E)$, and $\pi(\eta - j^1(\pi(\eta))) = 0$, so that $\eta - j^1(\pi(\eta)) \in \Gamma(T^*M \otimes E)$. Thus we can define $\Theta_D \in \mathrm{Hom}(J^1(E), E)$ by

$$\Theta_D(\eta) := D(\pi(\eta)) + \sigma_D(\eta - j^1(\pi(\eta))).$$

Then $\Theta_D \circ j^1(s) = D(s) + \sigma_D(j^1 s - j^1 s) = D(s)$.

■

Corollary I.5. *The first-order differential operators are the sections of a vector bundle*

$$\mathrm{Diff}^1(E) \longrightarrow M,$$

the natural vector bundle structure induced by the bijection

$$\mathcal{D}^1(E) \ni D \longrightarrow \Theta_D \in \mathrm{Hom}(J^1(E), E),$$

so that

$$\mathrm{Diff}^1(E) \cong \mathrm{Hom}(J^1(E), E).$$

The map

$$D \in \mathcal{D}^1(E) = \Gamma(\mathrm{Diff}^1(E)) \longrightarrow \sigma_D \in \Gamma(TM \otimes \mathrm{End}(E)),$$

introduced in Prop. I.3, induces a vector bundle morphism, called the symbol map*,*

$$\sigma : \mathrm{Diff}^1(E) \longrightarrow \mathrm{Hom}(T^*M, \mathrm{End}(E)).$$

Proof. The only thing that remains to be verified is that the map of modules

$$D \in \mathcal{D}^1(E) \longrightarrow \sigma_D \in \Gamma(\mathrm{Hom}(T^*M, \mathrm{End}(E))$$

is $C^\infty(M)$-linear. By (I.2), we have, for $f_0 \in C^\infty(M)$,

$$\sigma_D(df_0) = [D, f_0],$$

then, given $f \in C^\infty(M)$, we have

$$\begin{aligned} \sigma_{(fD)}(df_0)(s) &= [fD, f_0](s) \\ &= fD(f_0 s) - f_0 f D(s) \\ &= f[D, f_0](s) = f\sigma_D(df_0)(s), \qquad \forall s \in \Gamma(E). \end{aligned}$$

■

Proposition I.6. $\mathrm{Diff}^1(E)$ *fits in the exact sequence*

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{Diff}^1(E) \xrightarrow{\sigma} \mathrm{Hom}(T^*M, \mathrm{End}(E)) \longrightarrow 0, \tag{I.4}$$

and the diagram

$$\begin{tikzcd} \mathbf{CDO}(E) \arrow[r, "p"] \arrow[d, "\cap"'] & TM \arrow[d, "\iota"] \\ \mathrm{Diff}^1(E) \arrow[r, "\sigma"] & TM \otimes \mathrm{End}(E), \end{tikzcd} \tag{I.5}$$

commutes, where $\iota : TM \longrightarrow TM \otimes \mathrm{End}(E)$ *is given by* $\iota(X) = X \otimes \mathrm{Id}_E$.

Proof. In order to prove that σ is surjective, it is enough to work locally, so we can assume that there are sections $\{e_1, \dots, e_n\}$ that span the module $\Gamma(E)$. Given $X \otimes A \in \Gamma(TM \otimes \mathrm{End}(E))$, define $D \in \mathrm{Diff}^1(E)$ by

$$D(fe_i) := fA(e_i) + df(X)A(e_i).$$

Then

$$\sigma_D(df)(e_i) = D(fe_i) - fD(e_i) = df(X)A(e_i),$$

so that $\sigma_D = X \otimes A$. Surjectivity follows by $C^\infty(M)$-linearity, since the sections of the form $X \otimes A$ span the module $\Gamma(TM \otimes \mathrm{End}(E))$.

Now, $\sigma_D = 0$ if and only if $[D, f] = 0 \; \forall f \in C^\infty(M)$, if and only if $D \in \mathrm{Hom}(E)$, thus we obtain (I.4).

That the diagram (I.5) commutes follows simply by the definitions of the maps p, ι and σ.

∎

Proposition I.7. *Let*

$$\mathcal{D}(E) := \bigcup_{i=0}^{\infty} \mathcal{D}^i(E)$$

then $\mathcal{D}(E)$ *is a Lie subalgebra of* $\mathrm{End}_{\mathbb{R}}\Gamma(E)$*, the* $\mathbb{R}$*-linear endomorphisms of* $\Gamma(E)$ *with the commutator bracket:*

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1, \qquad D_1, D_2 \in \mathrm{End}_{\mathbb{R}}\Gamma(E).$$

$\mathcal{D}^1(E)$*, however, is* not *a Lie subalgebra of* $\mathcal{D}(E)$ *(unless* $\mathrm{rank}E = 1$*). Actually we have*

$$[D_1, D_2] \in \mathcal{D}^{k+l}(E), \qquad \textit{for } D_1 \in \mathcal{D}^k, D_2 \in \mathcal{D}^l(E). \tag{I.6}$$

Proof. The whole proposition reduces to prove (I.6). We apply induction on the order of the operators. First observe that

$$[D_1, D_2] \in \Gamma(\mathrm{End}(E)), \text{ for } D_1, D_2 \in \mathcal{D}^0(E) = \Gamma(\mathrm{End}(E)).$$

The following observation is that, for $f \in C^\infty(M)$

$$D \in \mathcal{D}^k(E) \Longleftrightarrow [f, D] \in \mathcal{D}^{k-1}(E), \tag{I.7}$$

as follows directly from the definition. Next, by the Jacobi identity for the commutator of operators, we have

$$[f, [D_1, D_2]] = [[f, D_1], D_2] + [D_1, [f, D_2]]. \tag{I.8}$$

By (I.7) we can apply the induction hypothesis to each term of the right-hand side of (I.8, and thus we obtain, that, for $D_1, \in \mathcal{D}^k(E), D_2 \in \mathcal{D}^l(E)$

$$[f, [D_1, D_2]] \in \mathcal{D}^{k+l-1}(E),$$

whence, by (I.7),

$$[D_1, D_2] \in \mathcal{D}^{k+l}(E).$$

∎

Definition I.8. A first-order bidifferential operator is an $\mathbb{R}$-bilinear map

$$\Delta : \Gamma(E_1) \times \Gamma(E_2) \longrightarrow \Gamma(E),$$

where E_1, E_2, E are vector bundles, such that

$$[f_1, [f_0, \Delta]_i]_j = 0,\ 1 \leq i, j \leq 2, \tag{I.9}$$

for all $f_0, f_1 \in C^\infty(M)$, where

$$[f, \Delta]_1(s_1, s_2) := \Delta(fs_1, s_2) - f\Delta(s_1, s_2)$$

and

$$[f, \Delta]_2(s_1, s_2) := \Delta(s_1, fs_2) - f\Delta(s_1, s_2).$$

A first-order bidifferential operator $\Delta = [\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E)$ (not necessarily skew-symmetric) is called a *quasi pseudoalgebra bracket on* $\Gamma(E)$.

$[\cdot, \cdot]$ is called a *pseudolagebra bracket* if in addition of being a first-order bidifferential operator, the *adjoint map* $\mathrm{ad}_s := [s, \cdot] : \Gamma(E) \longrightarrow \Gamma(E)$ is a covariant differential operator (also called derivative endomorphism).

We call the pair $(E, [\cdot, \cdot])$ a *quasi pseudoalgebra structure* (resp. a *pseudo algebra structure*).

Proposition I.9. *A $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]$ on $\Gamma(E)$ is a quasi pseudoalgebra structure if and only if there are two vector bundle maps*

$$\varphi^l, \varphi^r : E \longrightarrow TM \otimes \mathrm{End}(E)$$

called generalized anchor maps*, left and right, such that, for all $s_1, s_2 \in E$ and all $f \in C^\infty(M)$,*

$$[s_1, fs_2] = f[s_1, s_2] + \varphi^l(s_1)(df \otimes s_2), \qquad [fs_1, s_2] = f[s_1, s_2] - \varphi^r(s_2)(df \otimes s_1). \tag{I.10}$$

The generalized anchor maps are actual anchor maps when they take values in $TM \otimes \mathrm{Id}_E$.

Proof. If $[\cdot, \cdot]$ is a first-order bidifferential operator, define $C : \Gamma(E) \times C^\infty(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ by

$$C(s_1, f, s_2) := [f, B]_2(s_1, s_2) = [s_1, fs_2] - f[s_1, s_2].$$

Then, given $g \in C^\infty(M)$,

$$C(gs_1, f, s_2) = [gs_1, fs_2] - f[gs_1, s_2].$$

On the other hand, since $[\cdot, \cdot]$ is first-order, we have

$$\begin{aligned} 0 =& [g, [f, [s_1, s_2]]_2]_1 = [g, [s_1, fs_2] - f[s_1, s_2]]_1 \\ &= [gs_1, fs_2] - f[gs_1, s_2] - g[s_1, fs_2] + gf[s_1, s_2], \end{aligned}$$

whence

$$C(gs_1, f, s_2) = g[s_1, fs_2] - gf[s_1, s_2] = gC(s_1, f, s_2),$$

which means that C is $C^\infty(M)$-linear on the first entry. Analogously it is shown that C is $C^\infty(M)$-linear in the third entry from the identity $[g,[f,[s_1,s_2]]_2]_2 = 0$. Finally, again from $[g,[f,[s_1,s_2]]_2]_2 = 0$ we have

$$[s_1, gfs_2] + gf[s_1,s_2] = f[s_1, gs_2] + g[s_1, fs_2],$$

whence

$$\begin{aligned} C(s_1, fg, s_2) &= [s_1, fgs_2] - fg[s_1,s_2] = [s_1, fgs_2] + fg[s_1,s_2] - 2fg[s_1,s_2] \\ &= f[s_1, gs_2] - fg[s_1,s_2] + g[s_1, fs_2] - gf[s_1,s_2] \\ &= fC(s_1,g,s_2) + gC(s_1,f,s_2), \end{aligned}$$

which means that C is a derivation in the second entry. Therefore C is represented by a section of $E \otimes TM \otimes E \otimes E$, or equivalently, by a vector bundle morphism

$$\varphi^l : E \longrightarrow TM \otimes \operatorname{End}(E).$$

Analogously we can find the map φ^r.

Conversely, if there are maps φ^l, φ^r satisfying (I.10) then (I.9) follows immediately from the tensoriality of the generalized anchor maps.

■

Corollary I.10. *If* $[\cdot,\cdot]$ *is a pseudoalgebra bracket on* $\Gamma(E)$, *then there is a vector bundle morphism*

$$\rho : E \longrightarrow TM$$

called the anchor map, *such that*

$$[s_1, fs_2] = f[s_1,s_2] + \rho(s_1)(f)s_2.$$

Proof. Define a map on sections $\rho : \Gamma(E) \longrightarrow \Gamma(TM)$ by

$$\rho(s) := p(\mathrm{ad}_s),$$

where $p : \mathbf{CDO}(E) \longrightarrow TM$ is the symbol map, which is a surjective vector bundle morphism. Then, for $f \in C^\infty(M)$, since $\ker p = \operatorname{End}(E)$,

$$\rho(fs) = p(\mathrm{ad}_{fs}) = p(f\mathrm{ad}_s - \varphi^r(df)) = p(f\mathrm{ad}_s) = f\rho(s).$$

Then ρ is $C^\infty(M)$-linear, and therefore induces a vector bundle morphism.

■

Remark I.11. By the preceding corollary, the right generalized anchor map of a pseudoalgebra is given by

$$\varphi^r = \rho \otimes \mathrm{Id}.$$

Therefore, we will use the notation

$$([\cdot,\cdot], \rho, \varphi)$$

to encode a pseudoalgebra structure, where $\varphi := \varphi^l$.

Corollary I.12. *Lie algebroids and Courant algebroids define quasi pseudoalgebra structures on the space of sections of the corresponding vector bundles defining the algebroids.*

■

Corollary I.13. *Let $\mathcal{A} \subset \mathcal{D}^1(E) = \Gamma(\mathrm{Diff}^1(E))$ be a subset invariant under the commutator:*

$$[D_1, D_2] \subset \mathcal{A}, \qquad \forall D_1, D_2 \in \mathcal{A}.$$

Then the commutator defines a first-order bidifferential operator on $\mathcal{A}$.

Proof. Let $f \in C^\infty(M)$ and $s \in \Gamma(E)$, then

$$\begin{aligned} [D_1, fD_2](s) &= D_1(fD_2(s)) - fD_2(D_1(s)) \\ &= fD_1D_2(s) + \sigma_{D_1}(df)(D_2(s)) - fD_2D_1(s) \\ &= f[D_1, D_2](s) + \sigma_{D_1}(df)(D_2(s)). \end{aligned} \tag{I.11}$$

Now, observe that we have define a canonical injection $\iota : \mathrm{End}(E) \longrightarrow \mathrm{End}(\mathrm{Diff}^1(E))$, given by

$$\iota(\eta)(D) = \eta \circ D, \qquad \forall \eta \in \mathrm{End}(E),\ D \in \mathrm{Diff}^1(E).$$

It is immediate to see that $\iota(\eta)(D)$ is actually a first-order differential operator, with $\sigma_{\iota(\eta)} = \eta \circ \sigma_D$.

Then (I.11) shows that the vector bundle morphism $\varphi : \mathrm{Diff}^1(E) \longrightarrow TM \otimes \mathrm{End}(\mathrm{Diff}^1(E))$ given by

$$\varphi(D) := \iota \circ \sigma_D,$$

satisfies

$$[D_1, fD_2] = f[D_1, D_2] + \varphi(D_1)(df \otimes D_2), \qquad \forall D_1, D_2 \in \mathcal{A}, \forall f \in C^\infty(M).$$

Analogously, we have

$$[fD_1, D_2] = f[D_1, D_2] - \varphi(D_2)(df \otimes D_1), \qquad \forall D_1, D_2 \in \mathcal{A}, \forall f \in C^\infty(M).$$

Thus, Prop. I.9 applies, with $\varphi^l = \varphi^r = \varphi$.

■

Definition I.14. Let $(A, [\cdot,\cdot])$ be a quasi pseudoalgebra structure. A *representation of $(A, [\cdot,\cdot])$ by first-order differential operators* is a vector bundle morphism

$$\Psi : A \longrightarrow \mathrm{Diff}^1(E)$$

such that

$$\Psi([\alpha_1, \alpha_2]) = [\Psi(\alpha_1), \Psi(\alpha_2)], \qquad \forall \alpha_1, \alpha_2 \in \Gamma(A), \tag{I.12}$$

where the bracket on the right-hand side is, of course, the commutator.

Notice that Eq. (I.12) already implies that $\Psi(\Gamma(A)) \subset \mathcal{D}^1(E)$ is invariant under the commutator. Therefore

$$(\Psi(\Gamma(A)), [\cdot,\cdot])$$

defines a quasi pseudoalgebra structure.

Example I.15. Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid, in particular $(A, [\cdot,\cdot])$ defines a pseudoalgebra structure. Then any Lie algebroid representation

$$\Psi : A \longrightarrow \mathbf{CDO}(E)$$

is a particular case of a representation by first-order differential operators.

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