# Second Order Asymptotic Functions and Applications to Quadratic Programming

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**Abstract** We introduce a new second order asymptotic function which gives information on the convexity (concavity) of the original function from its behavior at infinity. We establish several properties and calculus rules for this concept, which differs from previous notions of second order asymptotic function. Finally, we apply our new definition in order to obtain necessary and sufficient optimality conditions for quadratic programming and quadratic fractional programming.

*Keywords*: Asymptotic cone; Asymptotic function; Second order asymptotic functions; Generalized convexity; Quadratic programming; Quadratic fractional programming.

## 1 Introduction

The notion of asymptotic (or recession) directions of an unbounded set has been introduced in order to study its behavior at infinity almost 100 years ago and was rediscovered in the 1950's in connection with some applications in economics. The study of asymptotic directions was pursued during the sixties and seventies, and the concept was developed both for convex and nonconvex sets, and extended to infinite dimensional spaces.

The most important advances in the convex case were presented by Rockafellar in [23] (see [2] for an excellent account). For the nonconvex case, we mention the developments by Dedieu in [7, 8], Luc [18], Amara in [1] and Luc and Penot in [19] among others. A related notion, mainly motivated by optimization problems, is the concept of asymptotic function. A careful analysis of the behaviour of the asymptotic function associated to the objective function, along the asymptotic directions of the feasible set is crucial for determining the existence of minimizers. These notions are a valuable tool in continuous convex optimization, variational inequalities, equilibrium problems, etc, particularly in the nonsmooth case, because they do not require any differentiability assumption.

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Similarly to the first order case, the concept of second order asymptotic directions and functions was introduced in [14], where it was used for establishing necessary or sufficient optimality conditions and characterizing the efficient points in vector optimization problems, in cases where the first order asymptotic notions are not adequate. A further step in the study of second order directions and functions for the convex case was presented in [11].

On the other hand, when dealing with nonconvex functions, the usual notion of asymptotic function does not provide adequate information on the level sets of the original function The issue of finding a correct definition of asymptotic function in the quasiconvex case was dealt with in [1, 21]. Penot proposed in [21] the notion of *incident asymptotic function* based on *incident asymptotic directions* (see also [19]). In [10], three differents notions were suggested for dealing with this situation.

All these notions were compared in [12], where applications to quasiconvex optimization were given. The authors conclude that the notion of $q$-asymptotic function is the "best" suited for quasiconvex functions. Several connections of these concepts with others issues of generalized convexity were developed in [16], supporting the conclusions of [12]. Applications to fractional programming and multiobjective optimization problems can be found in [15].

Using the study of the second order asymptotic function for the convex case [11] and the first order (nonconvex) asymptotic function of [12], a related concept of second order asymptotic function for the quasiconvex case was introduced in [12].

It happpens to be the case that none of those notions provides the natural geometric interpretation of a "second order". Broadly speaking, the first order asymptotic function is the "slope" of the original function at infinity, that is to say, a sort of first derivative at infinite, while all the above mentioned definitions of second order asymptotic functions provide no information on the convexity (concavity) of the function at infinity, i.e., they are not akin to a second derivative at infinity.

In this paper, based on results on second order derivatives (see [5, 13, 20]), we propose a new definition of second order asymptotic function, which is specially adequate for dealing with quadratic functions (convex or not). This new definition has the expected natural geometric meaning, i.e., they provide information on the convexity (concavity) of the funtion at the infinity.

The structure of the paper is as follows. Preliminaries, notation and basic definitions on first and second order asymptotic cones and functions are given in the next section. In Section 3 we introduce our new definition of second order asymptotic function and develop several properties and calculus rules. Comparison with previous second order asymptotic function in the convex and quasiconvex case are also given. In Section 4 we obtain necessary and sufficient conditions for the nonemptiness and compactness of the solution sets in quadratic programming (convex and quasiconvex) and quadratic fractional programming.

## 2 Preliminaries and Asymptotic Analysis

In this paper, we denote the duality pairing between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. We follow the convention that $0 \times (+\infty) = (+\infty) \times 0 = 0$ and indef $:= (+\infty) - (+\infty)$. Given a set $K \subseteq \mathbb{R}^n$, we denote its closure by $\operatorname{cl} K$, its boundary by $\operatorname{bd} K$, its topological interior by $\operatorname{int} K$, its relative interior by $\operatorname{ri} K$, its polar (positive) cone by $K^*$ and the orthogonal complement of its affine hull by $K^{\perp}$.

Given any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its effective domain is defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We say that $f$ is a proper function if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\operatorname{dom} f$ is nonempty (clearly, $f(x) = +\infty$ for every $x \notin \operatorname{dom} f$). We denote by $\operatorname{epi} f := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$ its epigraph. Given $\lambda \in \mathbb{R}$, $S_\lambda(f) := \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ is called the *level set of $f$ at height $\lambda$*. Those sets are important in convex and nonconvex analysis, because convexity of a function $f$ is characterized by its epigraph and quasiconvexity by its level sets, that is to say,

$$\begin{aligned} f \text{ is convex} \quad &\Longleftrightarrow \quad \operatorname{epi} f \text{ is a convex set.} \\ f \text{ is quasiconvex} \quad &\Longleftrightarrow \quad S_\lambda(f) \text{ is a convex set, for all } \lambda \in \mathbb{R}. \end{aligned}$$

If $f$ is differentiable, then $f$ is said to be pseudoconvex in an open convex set $D \subseteq \mathbb{R}^n$ if

$$x_1, x_2 \in D,\ f(x_1) > f(x_2) \Longrightarrow \nabla f(x_1)^\top (x_2 - x_1) < 0.$$

We mention that every convex function is pseudoconvex and every pseudoconvex function is quasiconvex. For further results on generalized convexity we refer to [4].

An important tool for estabhishing existence of solutions of optimization problems is the notion of *coerciveness*. We recall several variants of this notion which will be useful in the sequel (see [22]).

**Definition 2.1.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. We say that $f$ is;*

*a) coercive if*
$$\lim_{\|x\| \to +\infty} f(x) = +\infty; \tag{2.1}$$

*b) supercoercive if*
$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} > 0; \tag{2.2}$$

*c) hypercoercive if*
$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty; \tag{2.3}$$

Clearly, hypercoerciveness implies supercoerciveness, which implies coerciveness. Both converse statements fail to hold. We recall that $f$ is coercive iff $S_\lambda(f)$ is a bounded set for every $\lambda \in \mathbb{R}$.

A function $f$ is said to be Lipschitz continuous if there exist $\kappa > 0$ such that

$$|f(y) - f(z)| \leq \kappa \|y - z\|, \quad \forall\, y, z \in \mathrm{dom}\, f. \tag{2.4}$$

We recall now some Linear Algebra definitions. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we say that

a) $A$ is positive definite if $x^\top Ax > 0$ for all $x \neq 0$ (i.e., all its Eigenvalues are positive);

b) $A$ is positive semidefinite if $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$ (i.e., all its Eigenvalues are nonnegative).

A function $f : \mathbb{R} \to \mathbb{R}$ is *quadratic* if it is of the form $f(x) = \frac{1}{2}x^\top Ax + b^\top x + \alpha$, with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Furthermore, $f$ is strictly convex iff $A$ is positive definite; and $f$ is convex iff $A$ is positive semidefinite. Throughout this paper, we always assume, without loss of generality, that the matrix $A$ defining a quadratic function is symmetric.

If a function $f$ is quadratic, then $f$ is convex on $\mathbb{R}^n$ iff $f$ is quasiconvex on $\mathbb{R}^n$ (see [4, Theorem 6.3.1]). Thus, a quadratic function $f$ can be quasiconvex but not convex only on a proper subset $K$ of $\mathbb{R}^n$. We say that $f$ is *merely quasiconvex* on $K$ if $f$ is a quasiconvex function without being convex on $K$ [4, page 120]. A necessary condition for a quadratic $f$ to be merely quasiconvex is the existence of one negative Eigenvalue (see [4, Remark 6.3.1]). The characterization of quasiconvex quadratic functions is deeply analyzed in [4, Chapter 6]. Finally, if $K$ is a convex set with nonempty interior, and $f$ merely quasiconvex on $K$ then $f$ is bounded from above (see [4, Exercise 4.1]).

## 2.1 First Order Asymptotic Analisys

As explained in [2], the notions of asymptotic cone and associated asymptotic function have been employed in optimization theory in order to handle unbounded situations. In particular, when standard compactness and differentiability assumptions are absent.

We now recall that, given a nonempty set $K \subseteq \mathbb{R}^n$, its asymptotic cone is defined by

$$K^\infty := \left\{ u \in \mathbb{R}^n : \ \exists\, t_k \to +\infty, \ \exists\, x_k \in K \ \text{ with } \ \frac{x_k}{t_k} \to u \right\}.$$

When $K$ is a closed convex set, it is known that the notion of asymptotic (recession) cone (see [2, 23, 24]) is equivalent to

$$K^\infty = \left\{ u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall\, \lambda \geq 0 \right\} \ \text{for any } x_0 \in K. \tag{2.5}$$

The asymptotic function of a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the function $f^\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ for which

$$\text{epi } f^\infty := (\text{epi } f)^\infty.$$

From this definition, it is easy to conclude that (see [7, 8])

$$f^{\infty}(u) = \inf \left\{ \liminf_{k \to +\infty} \frac{f(u_k t_k)}{t_k} : t_k \to +\infty, u_k \to u \right\}. \tag{2.6}$$

Moreover, when $f$ is lower semicontinuous (lsc from now on) and convex for all $x_0 \in \operatorname{dom} f$, we have, in view of [2, Proposition 2.5.2],

$$f^{\infty}(u) = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t}, \tag{2.7}$$

and

$$f^{\infty}(u) = \sup_{x \in \operatorname{dom} f} f(x+u) - f(x). \tag{2.8}$$

Note that if $f$ is convex but not necessarily lsc, then (2.7) also holds, if we take $x_0 \in \operatorname{ri} \operatorname{dom} f$. Thus, for the convex case, $f^{\infty}$ could be equivalently defined by a supremum on the effective domain of $f$, or the supremum of the ratio of variation of the function $f(x+tu) - f(x)$ to $t$ as $t$ goes to infinity.

It is easy to check that $\operatorname{dom} f^{\infty} \subseteq (\operatorname{dom} f)^{\infty}$, where the inclusion could be strict even for convex functions (for example, for $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = e^x$). If $\operatorname{dom} f$ is bounded, then $(\operatorname{dom} f)^{\infty} = \{0\}$.
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If $f$ is coercive then $f^{\infty}(u) > 0$ for all $u \neq 0$. In addition, if $f$ is convex and lsc, then by [2, Proposition 3.1.3] we have

$$f \text{ is coercive} \Longleftrightarrow f^{\infty}(u) > 0,\ \forall\, u \neq 0 \Longleftrightarrow \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact}. \tag{2.9}$$

If the function is nonconvex (for example, quasiconvex), it has been seen that the usual notion of asymptotic function (even under a coerciveness condition) is not useful for obtaining information on the original function from its behavior at infinity. In fact, consider the quasiconvex, continuous and coercive function $f(x) = \sqrt{|x|}$ for all $x \in \mathbb{R}$. Here $f^{\infty} \equiv 0$ and no information of the nonemptiness and compactness of the solution set is detected.

In order to deal with this situation several attempts to describe the behavior of the original function through a correct definition of asymptotic function have been considered in the literature (e.g., in [1, 10, 21]). The comparison between them can be found in [12], where the authors deduce that, the "most adequate" definition is the $q$-asymptotic function introduced by Flores-Bazán at al. in [10].

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The $q$-asymptotic function is defined by

$$f^{\infty}_q(u) := \sup_{\substack{x \in \operatorname{dom} f \\ t>0}} \frac{f(x+tu) - f(x)}{t}. \tag{2.10}$$

One of the most important properties of $f^{\infty}_q$ is the following: given a proper, lsc and quasiconvex function $f$, [10, Theorem 4.7] implies that

$$f^{\infty}_q(u) > 0,\ \forall\, u \neq 0 \Longleftrightarrow \operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset \text{ and compact}. \tag{2.11}$$

Interesting properties of these notions and applications to quasiconvex optimization problems can be found in [10, 12, 15] while connections with generalized convexity theory have been presented in [16].

## 2.2 Second Order Asymptotic Analisys

The usual asymptotic (recession) function is a very useful tool in convex continuous optimization, but in some especific cases (e.g., for noncoercive functions) the information from the behavior of a convex function at infinity is not good enough. Indeed, consider the convex function $f(x) = -\log x$ for $x > 0$ with dom $f = ]0, +\infty[$, then $f^\infty(u) = 0$ for $u \geq 0$. Thus no information about the unboundedness from below of the original function is detected.

In order to deal with this situation, in [14] the notion of "second order asymptotic cone" for arbitrary sets has been introduced in [14], as follows. Given $K \subseteq \mathbb{R}^n$ and $u \in \mathbb{R}^n$, a direction $v \in \mathbb{R}^n$ is said to be a *second order asymptotic direction* of $K$ at $u$ if there exist sequences $x_k \in K$, $s_k, t_k \in \mathbb{R}$, with $s_k, t_k \to +\infty$ such that

$$v := \lim_{k \to +\infty} \left( \frac{x_k}{s_k} - t_k u \right). \tag{2.12}$$

The set of all such directions $v$ is a closed cone denoted by $K^{\infty 2}[u]$ and termed the *second order asymptotic cone* of $K$ at $u$. It is nonempty precisely when $u \in K^\infty$. If $u = 0$ then $K^{\infty 2}[0] = K^\infty$.

When $K$ is a convex subset of $\mathbb{R}^n$, a characterization of $K^{\infty 2}[u]$, reminding the one for $K^\infty$ given by (2.5), has been established in [11].

**Proposition 2.1.** *([11, Proposition 3.4]) Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set with $x \in \mathrm{ri}\ K$. Then the following assertions are equivalent:*

*a) $u \in K^\infty$ and $v \in K^{\infty 2}[u]$.*

*b) For all $s > 0$ there exists $\bar{t} > 0$ such that for every $t > \bar{t}$, $x + tu + sv \in K$.*

*c) There exist sequences $s_k, t_k \to +\infty$, such that $x + s_k t_k u + s_k v \in K$.*

This characterization does not depend on the choice of $x \in \mathrm{ri}\ K$, as was noted in [11]. Furthermore, the point $x \in \mathrm{ri}\ K$ cannot be replaced by $x \in K$ in the general case (see [11, Example 3.9]). Note that the convexity of $K$ is needed for $a) \Rightarrow b)$. Finally, [11, Proposition 3.6] gives a sufficient condition for the equality between the first and second order asymptotic cones in the convex case.

A second order asymptotic function, associated to the second order cone, has been defined in [11]. Given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, take $u \in \mathbb{R}^n$ such that $f^\infty(u) \in \mathbb{R}$. Then the *second order asymptotic function* of $f$ at $u$, denoted by $f^{\infty 2}(u; \cdot)$, is defined as:

$$\mathrm{epi}\ f^{\infty 2}(u; \cdot) := (\mathrm{epi}\ f)^{\infty 2}[(u; f^\infty(u))]. \tag{2.13}$$

Since $(\mathrm{epi}\ f)^{\infty 2}[(u; f^\infty(u))]$ is a closed cone, then $f^{\infty 2}(u; \cdot)$ is lsc and positively homogeneous. By [11, Proposition 2.2], we have $f^{\infty 2}(u; 0) = 0$ or $-\infty$, while $f^{\infty 2}(u; 0) = 0$ if and only if $f^{\infty 2}(u; \cdot)$ is proper.

From (2.13) we derive the next formula (see [11] for details). Let $u \in \mathbb{R}^n$ be such that $f^\infty(u) \in \mathbb{R}$. Then for all $v \in \mathbb{R}^n$,

$$f^{\infty 2}(u;v) = \inf\Big\{ \liminf_{k\to\infty} \left( \frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) : \\ x_k \in \mathrm{dom}\, f, s_k, t_k \to +\infty,\ \frac{x_k}{s_k} - t_k u \to v \Big\}. \quad (2.14)$$

If $u = 0$ and $f^\infty$ is a proper function, then (2.14) coincides with (2.6).

By [11, Proposition 4.10] we obtain that, if $f$ is convex and $x \in \mathrm{ridom}\, f$, then for all $u \in (\mathrm{dom}\, f)^\infty$ such that $f^\infty(u) \in \mathbb{R}$ and $v \in (\mathrm{dom}\, f)^{\infty 2}[u]$, we have

$$f^{\infty 2}(u;v) = \sup_{s>0} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s} \quad (2.15)$$

$$= \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}. \quad (2.16)$$

If the function $f$ is quasiconvex, then (2.15)-(2.16) do not hold. Thus, motivated by the good properties of (2.10) in the first order case and (2.15) in the second order case, the following slight modification of (2.15) for the quasiconvex case has been introduced in [12].

Given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and $u \in \mathbb{R}^n, u \neq 0$, such that $f^\infty(u) \in \mathbb{R}$, then the *q-second order asymptotic function* of $f$ at $u$ is defined as

$$f_q^{\infty 2}(u;v) := \sup_{\substack{x\in \mathrm{dom}\, f\\ s>0}} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}. \quad (2.17)$$

For applications of second order asymptotic cones and functions to convex and quasiconvex optimization problems we refer the reader to [12, 14, 15].

# 3 A new Second Order Asymptotic Function

All previous definitions of second order asymptotic functions give no information on the convexity (concavity) of the original function at the infinity. In order to obtain such information, we introduce a new definition of second order asymptotic function. We first consider the next example.

**Example 3.1.** *([11, Example 4.12 (b)]) Take the quadratic convex function $f(x) = \frac{1}{2}x^\top Ax + b^\top x + \alpha$. It is known that $f^\infty(u) = \langle b, u\rangle$ if $u \in \ker A$, while $f^\infty(u) = +\infty$ if $u \notin \ker A$. An application of* (2.16) *yields immediately that*

$$f^{\infty 2}(u;v) = \begin{cases} \langle b, v\rangle, & \text{if } u \in \ker A \text{ and } v \in \mathbb{R}^n, \\ +\infty, & \text{if } u \notin \ker A. \end{cases}$$

*Thus the second order asymptotic function $f^{\infty 2}$ does not provide any new information on the behavior of the original function at infinity.*

We introduce now the following definition of second order asymptotic function, based on the well-known properties of the generalized second order derivatives, carefully studied in [5, 13, 20] and references therein.

**Definition 3.1.** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function and take* $x \in \mathrm{dom}\ f$. *Given* $u, v \in (\mathrm{dom}\ f)^\infty$, *we define the second order asymptotic function of* $f$ *at* $x$ *in the direction* $(u; v)$, *and we denote its value at* $(x; u; v)$ *as* $f^{\infty\infty}(x; u; v)$, *by*

$$f^{\infty\infty}(x; u; v) := \lim_{h \to +\infty} \frac{f(x + h(u + v)) - f(x + hu) - f(x + hv) + f(x)}{h^2}, \tag{3.1}$$

*if the limit in* (3.1) *exists.*

**Remark 3.1.** *a) The characterization of the families of functions for which* (3.1) *is independent of the point is an open problem. Thus, we denote by* $\Phi$ *the set of all functions for which* $f^{\infty\infty}$ *is independent of the choice of the point* $x$.

*b) If* $f$ *is any function for which* (3.1) *is independent of the choice of the point* $x \in \mathrm{dom}\ f$, *we simply write* $f^{\infty\infty}(\cdot; \cdot)$.

*c) We will refer to* $f^{\infty 2}$ *as the directional second order asymptotic function by its dependence of the choice of the direction* $u \in \mathbb{R}^n$.

As the next proposition shows, the class $\Phi$ is nonempty; it contain Lipschitz functions with convex domain.

**Proposition 3.1.** *If* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a proper lsc Lipschitz function and* $\mathrm{dom}\ f$ *is convex, then*

$$f^{\infty\infty}(x; u; v) \equiv 0, \ \forall\ x \in \mathrm{dom}\ f, \ \forall\ u, v \in (\mathrm{dom}\ f)^\infty.$$

*Thus,* $f \in \Phi$.

*Proof.* Suppose that there exists $\kappa > 0$ such that (2.4) holds for every $x, y \in \mathrm{dom}\ f$. Take any $x \in \mathrm{dom}\ f$ and $u, v \in (\mathrm{dom}\ f)^\infty$. Since $\mathrm{dom}\ f$ is a convex closed set, we have that $x + hu + \lambda v \in \mathrm{dom}\ f$ for all $h, \lambda > 0$.

Observe that,

$$\begin{aligned} \lim_{h \to +\infty} \left|\frac{f(x + hu + hv)}{h^2}\right| &= \lim_{h \to +\infty} \left|\frac{f(x + hu + hv) - f(x)}{h^2}\right| \\ &\leq \lim_{h \to +\infty} \frac{\kappa}{h^2}(h\|u\| + h\|v\|) = 0. \end{aligned}$$

The same computation holds for the remaining fractions in the definition of $f^{\infty\infty}$. Thus, $f \in \Phi$. $\square$

The basic properties of $f^{\infty\infty}$ are listed in the next proposition.

**Proposition 3.2.** *Consider $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. If $f \in \Phi$ then the following assertions hold:*

*a) $f^{\infty\infty}$ is a symmetric function of the variables $u, v$, and*

$$f^{\infty\infty}(u; 0) = f^{\infty\infty}(0; v) = 0, \ \forall \ u, v \in (\text{dom } f)^{\infty}.$$

*b) $(\beta f)^{\infty\infty} = \beta(f^{\infty\infty})$ for all $\beta \in \mathbb{R}$.*

*c) If $(\text{dom } f)^{\infty} = \{0\}$ then $f^{\infty\infty} \equiv 0$.*

*d) $f^{\infty\infty}$ is simultaneously positively homogeneous as a function of the variables $u, v$, meaning that, for all $u, v \in (\text{dom } f)^{\infty}$ we have*

$$f^{\infty\infty}(\alpha u; \alpha v) = \alpha^2 f^{\infty\infty}(u; v), \ \forall \ \alpha > 0. \tag{3.2}$$

*e) If $u, w, z \in (\text{dom } f)^{\infty}$, then*

$$f^{\infty\infty}(u; w + z) = f^{\infty\infty}(u + z; w) + f^{\infty\infty}(u; z) - f^{\infty\infty}(z; w), \tag{3.3}$$

*for all $u, w, z \in (\text{dom } f)^{\infty}$, provided that the right hand side of (3.3) is well defined, i.e., the three limits exist and if one of them is equal to $+\infty$, then the remaining two are strictly greater than $-\infty$.*

*f) Let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i = 1, 2, \ldots, k$, be such that $\bigcap_{i=1}^{k} \text{dom } g_i \neq \emptyset$. If $g_i \in \Phi$ for all $i \in \{1, 2, \ldots, k\}$, then for all $u, v \in \left(\bigcap_{i=1}^{k} \text{dom } g_i\right)^{\infty}$ it holds that*

$$(g_1 + g_2 \ldots + g_k)^{\infty\infty}(u; v) = \sum_{i=1}^{k} (g_i)^{\infty\infty}(u; v), \tag{3.4}$$

*provided that the right hand side of (3.4) is well defined, i.e., all limits in $(g_i)^{\infty\infty}$ exist and, if $(g_i)^{\infty\infty}(u; v) = +\infty$ for some $i$, then $(g_j)^{\infty\infty}(u; v) > -\infty$ for all $j \neq i$.*

*Proof.* a), b) and c) are straightforward.
d) Take $\alpha > 0$. For all $u, v \in (\text{dom } f)^{\infty}$ we have

$$\begin{aligned} f^{\infty\infty}(\alpha u; \alpha v) &= \lim_{h \to +\infty} \frac{f(x + h\alpha u + h\alpha v) - f(x + h\alpha u) - f(x + h\alpha v) + f(x)}{h^2} \\ &= \lim_{h \to +\infty} \alpha^2 \frac{f(x + h\alpha u + h\alpha v) - f(x + h\alpha u) - f(x + h\alpha v) + f(x)}{\alpha^2 h^2}. \end{aligned}$$

Take $\varepsilon = h\alpha$, so that if $h \to +\infty$ then $\varepsilon \to +\infty$. Hence

$$\begin{aligned} f^{\infty\infty}(\alpha u; \alpha v) &= \alpha^2 \lim_{\varepsilon \to +\infty} \frac{f(x + \varepsilon u + \varepsilon v) - f(x + \varepsilon u) - f(x + \varepsilon v) + f(x)}{\varepsilon^2} \\ &= \alpha^2 f^{\infty\infty}(u; v), \ \forall \ \alpha > 0. \end{aligned}$$

e) Take $u, w, z \in (\mathrm{dom}\ f)^\infty$. Then

$$f^{\infty\infty}(u; w+z) = \lim_{h\to+\infty} \frac{f(x+hu+h(w+z)) - f(x+hu) - f(x+h(w+z)) + f(x)}{h^2}. \quad (3.5)$$

Adding the following term to (3.5)

$$\frac{\pm f(x+h(u+z)) \pm f(x+hw) \pm f(x+hz) \pm f(x)}{h^2},$$

we obtain

$$f^{\infty\infty}(u; w+z) = f^{\infty\infty}(u+z; w) + f^{\infty\infty}(u; z) - f^{\infty\infty}(z; w),$$

for all $u, w, z \in (\mathrm{dom}\ f)^\infty$.
f) Set $g := g_1 + g_2 \ldots + g_k$. Then $\mathrm{dom}\ g = \bigcap_{i=1}^k \mathrm{dom}\ g_i$. Take any $x \in \mathrm{dom}\ g$. We obtain that

$$\begin{aligned}\sum_{i=1}^k (g_i)^{\infty\infty}(u; v) &= \sum_{i=1}^k \lim_{h\to+\infty} \frac{g_i(x+hu+hv) - g_i(x+hu) - g_i(x+hv) + g_i(x)}{h^2} \\ &= \lim_{h\to+\infty} \frac{g(x+hu+hv) - g(x+hu) - g(x+hv) + g(x)}{h^2} \\ &= (g_1 + g_2 \ldots + g_k)^{\infty\infty}(u; v).\end{aligned}$$

□

**Remark 3.2.** *a) If $f \in \Phi$, then $f^{\infty\infty}(u; u)$ describes the convexity (concavity) of the function $f$ at infinity in the direction $u \in (\mathrm{dom}\ f)^\infty$.*
*b) If we take $u = u + v$ in (3.3), then*

$$f^{\infty\infty}(u+v; w+z) - f^{\infty\infty}(u+v; z) = f^{\infty\infty}(u+v+z; w) - f^{\infty\infty}(z; w), \quad (3.6)$$

*for all $u, v, w, z \in (\mathrm{dom}\ f)^\infty$.*
*c) For a quadratic $f$, its growth rate is completely determined (see Proposition 4.1 in Section 4), but for higher order functions we do not obtain a real number. In fact, considering the quasiconvex and continuous function $f(x) = x^3$, for all $x \in \mathbb{R}^n$ we have*

$$f^{\infty\infty}(x; u; u) = \begin{cases} -\infty, & \text{if } u < 0, \\ 0, & \text{if } u = 0, \\ +\infty, & \text{if } u > 0. \end{cases}$$

In order to deal with such a situation, we can continue with the same reasoning line and define the third order asymptotic function $f^{\infty\infty\infty}$ as follows:

$$f^{\infty\infty\infty}(x; u; v; w) = \lim_{h\to+\infty} \frac{f(x+hu+hv+hw) - f(x+hu+hv) - f(x)}{h^3} \ldots$$
$$\frac{-f(x+hu+hw) + f(x+hu) - f(x+hv+hw) + f(x+hv) + f(x+hw)}{h^3}.$$

Taking now $f(x) = x^3$ and $u = v = w \in \mathbb{R}$, we have

$$f^{\infty\infty\infty}(x; u; u; u) = \lim_{h \to +\infty} \frac{f(x + 3hu) - 3f(x + 2hu) + 3f(x + hu) - f(x)}{h^3}$$
$$= \lim_{h \to +\infty} \frac{3^3h^3u^3 - 3 \cdot 2^3h^3u^3 - 3h^3u^3}{h^3} = 6u^3. \quad (3.7)$$

Note that $f^{\infty\infty\infty}$ does not depend on the choice of the point $x \in \mathrm{dom}\ f$.

In a similar way, we can define $m$-order asymptotic functions. We leave this task for the interested reader; this paper is devoted to second order asymptotic functions.

A natural geometric interpretation is given in the next proposition. We mention at this point that the definition $f^{\infty\infty}$ is not useful when $f$ is convex and $u, v \in \mathrm{dom}\ f^{\infty}$, because the growth rate of $f$ at infinity is linear and henceforth there is no convexity (concavity) at infinity.

**Proposition 3.3.** *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper, lsc and convex, then for all $x \in \mathrm{dom}\ f$ we have*

$$f^{\infty\infty}(x; u; v) = 0,\ \forall\ u, v \in \mathrm{dom}\ f^{\infty}. \quad (3.8)$$

*As a consequence, $f$ trivially belongs to $\Phi$.*

*Proof.* Take $x \in \mathrm{dom}\ f$. Since $u, v \in \mathrm{dom}\ f^{\infty}$, we have $u + v \in \mathrm{dom}\ f^{\infty}$ and

$$f^{\infty\infty}(x; u; v) = \lim_{h \to +\infty} \frac{f(x + h(u + v)) - f(x + hu) - f(x + hv) + f(x)}{h^2}$$
$$= \lim_{h \to +\infty} \frac{1}{h} \left( \frac{f(x + h(u + v)) - f(x) - f(x + hu) + f(x) - f(x + hv) + f(x)}{h} \right)$$
$$= 0(f^{\infty}(u + v) - f^{\infty}(u) - f^{\infty}(v)) = 0.$$

Since $u, v, u + v \in \mathrm{dom}\ f^{\infty}$, we conclude that $(f^{\infty}(u + v) - f^{\infty}(u) - f^{\infty}(v)) \in \mathbb{R}$, completing the proof. □

## 3.1 Comparison with other Second Order Asymptotic Functions

This subsection is devoted to show that our definition (3.1) is independent from the previous second order asymptotic functions defined in the literature, namely $f^{\infty 2}$ and $f_q^{\infty 2}$.

The next example shows that our new definition of second order asymptotic function is different from $f^{\infty 2}$ in the convex case.

**Example 3.2.** *Let $f : \mathbb{R} \to \mathbb{R}$ be the straight line $f(x) = ax + b$. Note that $f \in \Phi$ because it is a convex function with $\mathrm{dom}\ f^{\infty} = \mathbb{R}$. Take $u = 1$ for simplicity, so that $f^{\infty}(1) = a$, the slope of the straight line. The new second order asymptotic function is given by*

$$f^{\infty\infty}(1; 1) = \lim_{h \to +\infty} \frac{a(x + 2h) + b - 2a(x + h) - 2b + ax + b}{h^2} = \lim_{h \to +\infty} \frac{0}{h^2} = 0.$$

*Thus, $f^{\infty\infty}(1;1)=0$, which fits the geometric interpretation, because a straight line is neither strictly convex nor strictly concave.*

*On the other hand,*

$$\begin{aligned}
f^{\infty 2}(1;1) &= \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x+t+s)-tf^{\infty}(1)-f(x)}{s}\\
&= \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{a(x+t+s)+b-ta-ax-b}{s}\\
&= \lim_{s\to+\infty}\frac{sa}{s}=a.
\end{aligned}$$

*Thus $f^{\infty 2}(1;1)<f^{\infty\infty}(1;1)$ for $a<0$, and $f^{\infty\infty}(1;1)<f^{\infty 2}(1;1)$ for $a>0$. We conclude that $f^{\infty\infty}$ and $f^{\infty 2}$ differ for this $f$.*

In many situations, even in the convex case, our new second order definition gives us information while other second order asymptotic functions fail to do it.

**Example 3.3.** *Take $\alpha\in]1,3[\setminus\{2\}$. Consider $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ given by*

$$f(x)=\begin{cases}+\infty, & \text{if } x<0,\\ x^{\alpha}, & \text{if } x\geq 0.\end{cases}$$

*It is easy to prove that $f^{\infty}\equiv+\infty$ for every $\alpha\in]1,3[\setminus\{2\}$. Hence, we cannot compute $f^{\infty 2}$ because there is no $u\in\mathbb{R}$ such that $f^{\infty}(u)\in\mathbb{R}$.*

*On the other hand, taking $u=v>0$, we get, for all $x\geq 0$,*

$$\begin{aligned}
f^{\infty\infty}(x;u;u) &= \lim_{h\to+\infty}\frac{(x+2hu)^{\alpha}-2(x+hu)^{\alpha}+x^{\alpha}}{h^2}\\
&= \lim_{h\to+\infty}\frac{\alpha(x+2hu)^{\alpha-1}-\alpha(x+hu)^{\alpha-1}}{h}\\
&= \alpha\lim_{h\to+\infty}\left(2(\alpha-1)(x+2hu)^{\alpha-2}-(\alpha-1)(x+hu)^{\alpha-2}\right),
\end{aligned}$$

*applying twice L'Hospital's rule. Thus, we get that $f^{\infty\infty}(u;u)=0$ for $\alpha\in]1,2[$. Since $(x+2hu)^{\alpha-2}$ goes to $+\infty$ faster than $(x+hu)^{\alpha-2}$ for $\alpha\in]2,3[$, then $f^{\infty\infty}(u;u)=+\infty$.*

*Summarizing, for all $x\geq 0$ and $u\geq 0$ we have*

$$f^{\infty\infty}(x;u;u)=\begin{cases}0, & \text{if } 1<\alpha<2,\\ +\infty, & \text{if } \alpha>2.\end{cases}$$

*Thus, the function $f$ exhibits no convexity (concavity) at the infinity for $1<\alpha<2$.*

The next example shows that our definition of second order asymptotic function provides information in quasiconvex cases for which the remaining definitions fail to do it.

**Example 3.4.** *Consider the quasiconvex quadratic function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0, \end{cases} \qquad f^{\infty}(u) = \begin{cases} +\infty, & \text{if } u > 0, \\ -\infty, & \text{if } u \leq 0. \end{cases}$$

*Note $f_q^{\infty 2}$ cannot be computed because does not exist $u \in \mathbb{R}$ such that $f^{\infty}(u) \in \mathbb{R}$.*

*On the other hand, take any $x \in \mathbb{R}$. For $u = v > 0$ and $i = 2$, we have*

$$\begin{aligned}
f^{\infty\infty}(x; u; u) &= \lim_{h \to +\infty} \frac{f(x + hu + hu) - f(x + hu) - f(x + hu) + f(x)}{h^2} \\
&= \lim_{h \to +\infty} \frac{(x + hu + hu)^2 - (x + hu)^2 - (x + hu)^2 \pm x^2}{h^2} \\
&= \lim_{h \to +\infty} \frac{2(x + hu)hu + h^2u^2 - x^2 - 2xhu - h^2u^2 \pm x^2}{h^2} \\
&= \lim_{h \to +\infty} \frac{2h^2u^2 - x^2 \pm x^2}{h^2} = 2u^2.
\end{aligned}$$

*For $u = v < 0$, an analogous computation gives $f^{\infty\infty}(x; u; u) = -2u^2$. Thus*

$$f^{\infty\infty}(x; u; u) = \begin{cases} 2u^2, & \text{if } u \geq 0, \\ -2u^2, & \text{if } u < 0. \end{cases}$$

*Notice that $f^{\infty\infty}$ does not depend on the choice of the point $x$ and that* epi $f_i^{\infty\infty}$ *is not a cone.*

*Hence, $f$ is convex in the direction $u = 1$ and concave on direction $u = -1$.*

The previous examples show that our definition (3.1) is an efficient tool for describing the convexity (concavity) of the function at infinity, and that it can provide useful information in situations where the remaining second order asymptotic functions fail to do it.

# 4 Main Results in Quadratic Programming

This section is devoted to show the potential of our new definition of second order asymptotic function. It is well known that first and second order derivatives provide necessary and sufficient conditions in continuous optimization problems. We will show that the first and second order asymptotic functions can do the same at infinity for some optimization problems.

## 4.1 Quadratic Programming

In this section, we develop necessary and sufficient conditions for the nonemptiness and compactness of the solution set in quadratic (convex or quasiconvex) minimization problems. Henceforth, we always work with a nonempty closed convex set $K \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x^{\top}Ax + b^{\top}x + \alpha$, where $A \in \mathbb{R}^{n \times n}$ is symmetric, $b$ belongs to $\mathbb{R}^n$ and $\alpha$ belongs to $\mathbb{R}$.

We show next that quadratic functions belong to $\Phi$, showing that the family $\Phi$ extend beyond the family of Lipschitz continuous functions.

**Proposition 4.1.** *If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is a quadratic function as above, then $f \in \Phi$.*

*Proof.* We first prove that $f \in \Phi$. Take any $x, y \in K$ and $u, v \in K^\infty$. Then

$$\begin{aligned}
&\lim_{h\to+\infty} \frac{f(x+hu+hv) - f(y+hu+hv)}{h^2} \\
&= \lim_{h\to+\infty} \frac{f(x) - f(y) + h(x^\top Au - y^\top Au) + h(x^\top Av - y^\top Av)}{h^2} \\
&= \lim_{h\to+\infty} \left( \frac{f(x) - f(y)}{h^2} + \frac{(x^\top Au - y^\top Au)}{h} + \frac{(x^\top Av - y^\top Av)}{h} \right) \\
&= 0.
\end{aligned}$$

We proceed in the same way for the remaining terms, concluding finally that $f^{\infty\infty}$ is independent of the point $x \in K$. $\square$

We show next that our second order asymptotic function fully characterizes strict convexity of a quadratic function in the whole space.

**Proposition 4.2.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then the following assertions are equivalent:*

*a)* $f^{\infty\infty}(u; u) > 0$ *for all* $u \neq 0$.

*b)* $f$ *is strictly convex on* $\mathbb{R}^n$.

*c)* $\operatorname{argmin}_{\mathbb{R}^n} f$ *is nonempty and compact.*

*Proof.* Let $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ be an unbounded sequence, so that $\frac{x_k}{\|x_k\|} \to u$. Thus

$$\begin{aligned}
\lim_{k\to+\infty} \frac{f(x_k)}{\|x_k\|^2} &= \lim_{k\to+\infty} \frac{\frac{1}{2}x_k^\top Ax_k + b^\top x_k + \alpha}{\|x_k\|^2} \\
&= \lim_{k\to+\infty} \left( \frac{1}{2} \frac{x_k^\top}{\|x_k\|} A \frac{x_k}{\|x_k\|} + \frac{b^\top x_k}{\|x_k\|^2} + \frac{\alpha}{\|x_k\|^2} \right) \\
&= \frac{1}{2} u^\top Au \\
&= 2 f^{\infty\infty}(u; u).
\end{aligned}$$

Then, $f^{\infty\infty}(u; u) > 0$ for all $u \in \mathbb{R}^n \backslash \{0\}$ iff $u^\top Au > 0$ for all $u \in \mathbb{R}^n \backslash \{0\}$ iff $f$ is strictly convex iff $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$ and compact. $\square$

The previous characterization is not true for an arbitrary proper, closed and convex set $K$, as the following example shows.

**Example 4.1.** *Let $K = \mathbb{R} \times [1, 1]$. Consider the function $f(x) = x^\top Ax$ with*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

*The function is nonconvex and $K^\infty = \mathbb{R} \times \{0\}$. But $f^{\infty\infty}(u; u) = u^\top Au > 0$, for any $u \in \mathbb{R} \times \{0\}, u \neq 0$.*

**Remark 4.1.** *a) If $f^{\infty\infty}(u;u) > 0$ for all $u \in K^{\infty}\backslash\{0\}$, then $f$ is hypercoercive on $K$.*

*b) If $f^{\infty\infty}(u;u) > -\infty$ for all $u \in K^{\infty}\backslash\{0\}$, then $f$ is prox-bounded in the sense of [24, Definition 1.23], meaning that there exists $\alpha \in \mathbb{R}$ such that the funtion $g(\cdot) = f(\cdot) + \frac{1}{2}\alpha\|\cdot\|^2$ is bounded from below on $K$.*

*c) Proposition 4.2 cannot be extended to nonquadratic functions. In fact, consider the continuous function given by*

$$f(x) = \begin{cases} -|x| + 1, & \text{if } |x| \leq 1, \\ x^2 - 1, & \text{if } |x| > 1. \end{cases}$$

*It is easy to see that $f^{\infty}(u) = 0$ if $u = 0$, and $f^{\infty}(u) = +\infty$ elsewhere. Take $u \in \mathbb{R}$, and proceed with the computation as in Proposition 3.1, concluding that $f^{\infty\infty}(x;u;u) > 0$ for all $u \neq 0$ and all $x \in \mathbb{R}$, but $f$ is not convex.*

*d) If $f$ is bounded from below, then $f^{\infty\infty}(u;u) \geq 0$ for all $u \in K^{\infty}$. In fact, since $f$ is bounded from below, there exists $M \in \mathbb{R}$ such that $f(x) \geq M$ for all $x \in K$.*

*Given any $\{x_k\}_{k\in\mathbb{N}} \subseteq K$ such that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u \in K^{\infty}$ we get, with the same argument as in the proof of Proposition 4.2,*

$$f^{\infty\infty}(u;u) = \lim_{k\to+\infty} \frac{f(x_k)}{\|x_k\|^2} \geq \lim_{k\to+\infty} \frac{M}{\|x_k\|^2} = 0, \ \forall\ u \in K^{\infty}.$$

*e) Under the assumptions of Proposition 4.2 since $f$ is strictly convex, there exists $\overline{x} \in \mathbb{R}^n$ such that $\operatorname{argmin}_{\mathbb{R}^n} f = \{\overline{x}\}$.*

*f) We cannot weaken the inequality in Proposition 4.2(a), meaning that if we just assume that $f^{\infty\infty}(u;u) \geq 0$ for all $u \in \mathbb{R}^n$ we cannot conclude the nonemptiness or compactness of the set of minimizers of $f$. In fact, if we take $u \in \operatorname{Ker} A \cap b^{\perp}$, then $f^{\infty}(u) = 0$ and $f^{\infty\infty}(u;u) = 0$, and no conclusion follows.*

The literature on semidefinite quadratic programming is very rich; we refer to [9, Example 7.7] for a characterization of the nonemptiness and compactness of the set of minimizers based upon asymptotic analysis.

**Proposition 4.3.** *Let $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ be a quadratic function. If $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$, then*

*a) $f^{\infty\infty}(u;u) \geq 0$ for all $u \in K^{\infty}$.*

*b) If $u \in K^{\infty}$ and $f^{\infty\infty}(u;u) = 0$, then for each $x \in K$ it holds that $\langle \nabla f(x), u\rangle \geq 0$.*

*Proof.* a) Since $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$, then this item holds by Remark 4.1 d).
b) Take $x \in K$ and $u \in K^\infty$. Since $K$ is a closed convex set, $x + tu \in K$ for all $t > 0$. Thus

$$f(x + tu) = f(x) + t\langle \nabla f(x), u\rangle + t^2 f^{\infty\infty}(u; u), \ \forall\ t > 0. \tag{4.1}$$

Take now $u \in K^\infty$ such that $f^{\infty\infty}(u; u) = 0$. Suppose for the sake of contradiction that $\langle \nabla f(x), u\rangle < 0$ for some $x \in K$. Thus, in view of (4.1), for all $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$ we have

$$f(x + tu) = f(x) + t\langle \nabla f(x), u\rangle < f(\overline{x}),$$

for $t > 0$ large enough, a contradiction. Thus (b) also holds. □

**Remark 4.2.** *a) If $A$ is also positive semidefinite, so that $f$ is a convex quadratic function, then item a) is obvious. Thus, if* $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$ *then*

$$u \in K^\infty, f^{\infty\infty}(u; u) = 0 \implies \langle \nabla f(x), u\rangle \geq 0, \ \forall\ x \in K.$$

*b) If the closed convex set $K$ is defined by $K := \{x \in \mathbb{R}^n : Dx \geq c\}$ with $D \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$, then items a) and b) in Proposition 4.3 are also sufficient (see Eaves' Theorem [17, Theorem 2.2]).*

Next we present our second main theorem, which provides a full characterization of nonemptiness and compactness of set of minimizers of quasiconvex quadratic functions. A similar necessary condition in a particular case for boundedness from below can be found in [6, Section 3.1]; the sufficient condition seems to be new.

**Theorem 4.1.** *Let $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ be a quadratic function and $K$ a closed and convex set. If* $\operatorname{argmin}_{\mathbb{R}^n} f$ *is nonempty and compact then*

$$S := \{u \in K^\infty : \ f^{\infty\infty}(u; u) \leq 0, -u \in \{\nabla f(K)\}^*\} = \{0\}. \tag{4.2}$$

*In addition, if $f$ is quasiconvex then the reverse implication is also true.*

*Proof.* ($\Rightarrow$): If $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$ then the results of Proposition 4.3 a) and b) hold. We claim that $S = \{0\}$. Suppose, for the sake of contradiction, that there exists $u \in S$, $u \neq 0$. Then $f^{\infty\infty}(u; u) = 0$ for all $u \in K^\infty$.

Take $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$. Hence

$$\begin{aligned} f(\overline{x} + tu) &= f(\overline{x}) + t\langle \nabla f(\overline{x}), u\rangle + t^2 f^{\infty\infty}(u; u), \\ &= f(\overline{x}) + t\langle \nabla f(\overline{x}), u\rangle \end{aligned}$$

for all $t > 0$ with $\langle \nabla f(\overline{x}), u\rangle \leq 0$.

If $\langle \nabla f(\overline{x}), u\rangle < 0$ then $f(\overline{x} + tu) = f(\overline{x}) + t\langle \nabla f(\overline{x}), u\rangle < f(\overline{x})$, which is impossible. Thus $\langle \nabla f(\overline{x}), u\rangle = 0$ and $f(\overline{x} + tu) = f(\overline{x})$ for all $t > 0$, so that $u \in \operatorname{argmin}_{\mathbb{R}^n} f$, which is a contradiction. Hence $S = \{0\}$.

($\Leftarrow$): If $f$ is also quasiconvex, then we compute the $q$-asymptotic function:

$$\begin{aligned}
f_q^{\infty}(u) \leq 0 \quad &\Longleftrightarrow \quad \sup_{x\in K}\sup_{t>0}\left(x^\top Au + \frac{1}{2}tu^\top Au + b^\top u\right) \leq 0 \\
&\Longleftrightarrow \quad \sup_{x\in K}\left(x^\top Au + b^\top u + \sup_{t>0}\frac{1}{2}tu^\top Au\right) \leq 0 \\
&\Longleftrightarrow \quad u^\top Au \leq 0 \ \wedge \ \sup_{x\in K}\langle Ax+b, u\rangle \leq 0 \\
&\Longleftrightarrow \quad f^{\infty\infty}(u;u) \leq 0 \ \wedge \ -u \in \{\nabla f(K)\}^*.
\end{aligned}$$

Therefore, by [10, Theorem 4.7] we have that, if $S = \{0\}$ then $\mathrm{argmin}_K f$ is nonempty and compact. $\square$

Our result applies for a family of quadratic functions larger than the convex ones. For example, the *merely quasiconvex* functions (see [4, page 120]).

If $K = \mathbb{R}^n$ then $f$ is a convex quadratic function. Note that, in our result, no conditions of the type $u \in \mathrm{Ker}\ A$ is needed, i.e., our result extend the characterization given in [9, Example 7.7] for convex quadratic functions.

## 4.2 Quadratic Fractional Programming

Now we focus our attention in another mathematical programming problem, namely Quadratic Fractional Programming. Given a quadratic function $f$ as above and a linear function $g(x) = d^\top x + \beta$ with $d \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, we define the feasible domain $D$ as $D := \{x \in \mathbb{R}^n : \ g(x) > 0\}$, and the objective function $h$ as $h(x) := \frac{f(x)}{g(x)}$.

The problem is defined as:

$$\min_{x\in D} h(x) = \min_{x\in D} \frac{\frac{1}{2}x^\top Ax + b^\top x + \alpha}{d^\top x + \beta}, \tag{4.3}$$

and it consists of minimizing a ratio of two functions. This problem have been mainly studied in the literature by its economics applications, like, among others, the minimization of cost/time or maximization of return/risk.

Before introducing our main result of this subsection, we recall that if $f$ is convex on $D$ then $h$ is pseudoconvex on $D$ by [4, Theorem 3.2.10]; If $h$ is pseudoconvex on $D$ then $h$ is quasiconvex on $D$ by [4, Theorem 3.2.9]. Finally, if $A$ is not positive semidefinite, by [4, Theorem 7.2.1], $f$ is quasiconvex on $D$ iff $f$ is pseudoconvex on $D$. Characterization of convexity and pseudoconvexity of the function $h$ can be found in [4, Chapter 7].

The next result extends and generalizes [15, Example 4.1] from the linear fractional problem to the quadratic fractional problem.

**Theorem 4.2.** *Let $h : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a quadratic fractional function with $D$ as above. Thus $\mathrm{argmin}_D h \neq \emptyset$ and bounded iff*

$$\widehat{S} := \{u \in D^\infty \cap \{d\}^\perp : \ f^{\infty\infty}(u;u) \leq 0, -u \in \{\nabla f(D)\}^*\} = \{0\}. \tag{4.4}$$

*Proof.* We use [10, Theorem 4.7]. Making the corresponding computation, we have that

$$h_q^\infty(u) \le 0 \iff \sup_{x\in D}\sup_{t>0}\left(\frac{\frac{\frac{1}{2}(x+tu)^\top A(x+tu)+b^\top(x+tu)+\alpha}{d^\top(x+tu)+\beta}-\frac{\frac{1}{2}x^\top Ax+b^\top x+\alpha}{d^\top x+\beta}}{t}\right)\le 0$$

$$\iff \sup_{x\in D}\sup_{t>0}\left(\frac{tg(x)x^\top Au+\frac{1}{2}t^2g(x)u^\top Au+tg(x)b^\top u-td^\top uf(x)}{t(g(x)+td^\top u)g(x)}\right)\le 0$$

$$\iff \sup_{x\in D}\sup_{t>0}\left(\frac{g(x)(x^\top Au+\frac{1}{2}tu^\top Au+b^\top u)-d^\top uf(x)}{(g(x)+td^\top u)g(x)}\right)\le 0.$$

Since $u$ is an asymptotic direction, and the null vector belongs to any asymptotic set, we only consider the following two cases:

i) If $u\in\{d\}^\perp$, then

$$\begin{aligned}
&\sup_{x\in D}\sup_{t>0}\left(\frac{g(x)(x^\top Au+\frac{1}{2}tu^\top Au+b^\top u)}{(g(x))^2}\right)\le 0\\
\iff& \sup_{x\in D}\left(\frac{x^\top Au+b^\top u}{g(x)}+\sup_{t>0}\frac{\frac{1}{2}tu^\top Au}{g(x)}\right)\le 0\\
\iff& u^\top Au\le 0 \ \wedge\ \sup_{x\in D}\left(x^\top Au+b^\top u\right)\le 0\\
\iff& f^{\infty\infty}(u;u)\le 0 \ \wedge\ \langle Ax+b,u\rangle\le 0,\ \forall\, x\in D\\
\iff& f^{\infty\infty}(u;u)\le 0 \ \wedge\ -u\in\{\nabla f(D)\}^*.
\end{aligned}$$

Therefore, in this case $\mathrm{argmin}_D h$ is nonempty and bounded iff $\widehat{S}_1=\{0\}$.

ii) If $u\in \mathrm{Ker}\ A$, then

$$\sup_{x\in D}\sup_{t>0}\left(\frac{g(x)b^\top u-d^\top uf(x)}{(g(x)+td^\top u)g(x)}\right)\le 0. \tag{4.5}$$

Then, using an argument similar to the one in [15, Example 4.1], and since $u$ is an asymptotic direction (i.e., $u$ could be the null vector), we have

$$h_q^\infty(u)\le 0 \iff u\in\{d\}^\perp\cap\{-b\}^*.$$

It follows that $\widehat{S}_2=\{u\in D^\infty\cap\{d\}^\perp:\ u\in \mathrm{Ker}\ A, -u\in\{b\}^*\}$.

Its easy to check that $\widehat{S}_2\subseteq\widehat{S}_1$. In fact, if $u\in\widehat{S}_2$ then $f^{\infty\infty}(u;u)=u^\top Au=0$ and

$$\langle Ax+b,u\rangle = x^\top Au+b^\top u=b^\top u\le 0\Longrightarrow u\in\widehat{S}.$$

Finally, $\mathrm{argmin}_D h$ is nonempty and bounded iff $\widehat{S}=\widehat{S}_1=\{0\}$. ☐

**Remark 4.3.** *a) Our asymptotic approach is simpler than the one [3], where the usual asymptotic (recession) directions and functions are used for approximating a nonconvex function.*

*b) If $A = 0$, then Theorem 4.2 coincides with [15, Example 4.1] (the linear fractional problem). If $f$ is quasiconvex and $d = 0$, Theorem 4.2 coincides with Theorem 4.1 on any closed convex set $K \subseteq \mathbb{R}^n$.*

*c) Notice that we do not need to compute the function $h^{\infty\infty}$ in the proof of Theorem 4.2 but it is possible to prove that $h \in \Phi_i$ for $i \in \{1, 2\}$ whenever $u \notin D^{\infty} \cap \{d\}^{\perp}$. The proof of this fact is similar to the computation in the proof of Proposition 4.1.*

### Conclusions

Both applications show that the first and second order asymptotic functions in nonconvex programming is a relevant tool for minimization problems with unbounded data. Characterizations of the nonemptiness and compactness of the solution set for quadratic minimization problems and quadratic fractional minimization problems are a consequence of the study of the behavior of the functions at infinity, and no local descriptions are needed. Thus, the set of minimizers and the first and second order derivatives are still related at infinity, as expected.

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