Instituto de Matemática Pura e Aplicada

Doctoral Thesis

TOPOLOGY OF LEAVES OF GENERIC LOGARITHMIC FOLIATIONS

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A minha familia
e minha galera,
pelo imenso apoio e carinho.

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Resumo

Esta tese tem como objetivo o estudo da topologia das folhas genéricas de uma folheação logarítmica genérica em espaços projetivos complexos.

Partindo da teoria de Lefschetz para seções hiperplanas de hipersuperfícies, provamos que os grupos de homotopia de uma seção hiperplana de uma folha genérica de dimensão menor que a dimensão da seção são isomorfos aos grupos de homotopia da folha genérica da mesma dimensão. Para condições genéricas sobre a 1-forma logarítmica fechada que define a folheação, explicitamos o grupo fundamental de uma folha genérica.

No caso de folheações no plano projetivo, isto é, em dimensão 2, provamos que a folha genérica de uma folheação logarítmica genérica é homeomorfa ao monstro do lago Ness, isto é, um plano ao qual colamos uma infinidade de alças.

Palavras-chave: Folheações, Logarítmica, Topologia das folhas.

Contents

Introduction

Motivation

This thesis studies the topology of generic leaves of codimension one singular holomorphic foliations on complex projective manifolds. The focus is on the topology of generic leaves of generic logarithmic foliations on the complex projective space $\mathbb{P}^{n+1}$. A non singular leaf of a codimension one holomorphic foliation on $\mathbb{P}^{n+1}$ is a immersed complex manifold of codimension one in $\mathbb{P}^{n+1}$. If this leaf is algebraic, i.e. the closure of such leaf in $\mathbb{P}^{n+1}$ is a projective variety, we know topological properties of its closure. In particular, we have the following result about hyperplane sections of a projective variety.

Theorem 0.0.1 (Lefschetz theorem of hyperplane sections)**.** *[M2, Theorem 7.4] Let X be a smooth projective variety of complex dimension n which lies in the projective space $\mathbb{P}^m$. Let $H \subset \mathbb{P}^m$ be a hyperplane whose intersection $H \cap X$ is a smooth hyperplane section of X. Then the inclusion map*

$$H \cap X \hookrightarrow X$$

induces isomorphisms of homotopy groups of dimension less than $n-1$. Furthermore, the induced homomorphism

$$\pi_{n-1}(H \cap X) \to \pi_{n-1}(X)$$

is onto.

From the above theorem, it follows that the claims below hold true for smooth hyperplane sections $H \cap X$.

(L1) If the dimension of X is greater than one, then the hiperplane section $H \cap X$ is connected.

(L2) If the dimension of X is greater than two, then the fundamental groups of $H \cap X$ and X are isomorphic.

The next claims are another known facts about the topology of a smooth hypersurface of $\mathbb{P}^{n+1}$.

(L3) Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. If n is greater equal than two, then X is simply connected.

(L4) Let $X \subset \mathbb{P}^2$ be a smooth complex curve. If the degree of X is d, then the genus of X is
$$\frac{(d-1)(d-2)}{2}.$$

But in general a generic leaf of a codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^{n+1}$ is not algebraic. Based on the above claims, Dominique Cerveau in [C, Section 2.10] proposes to study topological properties of a generic leaf $\mathcal{L}$ of $\mathcal{F}$ through the following questions.

Let $H \subset \mathbb{P}^{n+1}$ be a general hyperplane with respect to a generic leaf $\mathcal{L}$ of $\mathcal{F}$.

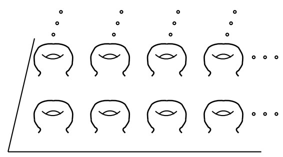
(C1) If $n > 1$, is the hyperplane section $H \cap \mathcal{L}$ connected?

(C2) If $n > 2$, are the fundamental group of $H \cap \mathcal{L}$ and $\mathcal{L}$ isomorphic?

(C3) If $n \geq 2$, is the generic leaf $\mathcal{L}$ simply connected?

Also, in [C, Section 2.10], he remarks that the generic leaf of a foliation on $\mathbb{P}^3$, defined in homogeneous coordinates by the 1-form
$$\lambda_0 \frac{dx_0}{x_0} + \lambda_1 \frac{dx_1}{x_1} + \lambda_2 \frac{dx_2}{x_2} + \lambda_3 \frac{dx_3}{x_3},$$
with $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ general complex numbers satisfying $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 0$, is biholomorphic to $\mathbb{C}^2$.

For the general hyperplane H passing through $[0:0:0:1]$ the topology of the generic leaf of the foliation restricted to H was studied by Ferrán Valdez in [V]. There he proves that for $\lambda_0, \lambda_1, \lambda_2$ and λ_3 sufficiently general the generic leaf is homeomorphic to the Loch-Ness Monster, i.e. the real plane with infinitely many handles attached.



Loch Ness Monster

In general, the non singular leaf of a holomorphic foliation on $\mathbb{P}^2$ is a non compact Riemann surface. Kerékjártó theorem (see Theorem 2.2.9) gives us a topological classification of orientable non compact real surfaces, based in the description of the following topological invariants.

(a) The space of ends of a surface S, which is compact and totally disconnected.

(b) The genus, which is finite or infinite.

In this sense, we know that for holomorphic foliations on $\mathbb{C}^2$ defined by a generic analytic vector field all leaves, except for at most countably many, are contractible and the rest are topological cylinders. This result is due to Tanya Firsova [F]. For foliations defined by generic polynomial vector fields an analogous result is not known.

Let $\mathcal{B}_n$ be the space of foliations on $\mathbb{P}^2$ defined by the 1-form

$$Q(x,y)dx - P(x,y)dy,$$

where the polynomials P, Q have degree at most n in each affine chart. Natalya Goncharuk and Yury Kudryasov in [G-K] prove that there is a dense subset $\mathcal{B}'_n$ of $\mathcal{B}_n$ such that any foliation contained in $\mathcal{B}'_n$ has a leaf with at least

$$\frac{(n+1)(n+2)}{2} - 4$$

handles. Moreover, if the polynomials satisfy

$$P(x,-y) = -P(x,y), \quad Q(x,-y) = Q(x,y)$$

in a chart, then all leaves of $\mathcal{F}$ have infinite genus.

Main Results

The object of this thesis is, more precisely, to provide a topological description of generic leaves of generic logarithmic foliations on $\mathbb{P}^{n+1}$ using homotopy theory. Here are some of the main results, which will be proved in Chapters 3 and 4.

Theorem 1 (Theorem 4.1.2 of Chapter 4)**.** *Let $\mathcal{L}$ be a generic leaf of a generic logarithmic foliation $\mathcal{F}$ on $\mathbb{P}^{n+1}$. Let $H \subset \mathbb{P}^{n+1}$ be a sufficiently general hyperplane. Then the morphisms of homotopy groups*

$$(i)_* : \pi_l(\mathcal{L} \cap H) \to \pi_l(\mathcal{L}),$$

induced by the inclusion $i : \mathcal{L} \cap H \hookrightarrow \mathcal{L}$ are

(1) isomorphims if $l < n-1$;

(2) epimorphisms if $l = n-1$.

The above theorem is an analogue of Theorem 0.0.1 for a generic leaf of a logarithmic foliation. Moreover the Theorem 1 implies that (C1) is true for generic leaves of generic logarithmic foliations, with $n > 1$, and (C2) holds true when $n > 2$.

Recall that the complement of a simple normal crossing divisor $D = D_0 + \cdots + D_k$ in $\mathbb{P}^{n+1}$ has fundamental group isomorphic to

$$\pi_1(\mathbb{P}^{n+1} - D) \cong \mathbb{Z}^{k+1}/(d_0, \ldots, d_k)\mathbb{Z},$$

where d_j is the degree of the irreducible component D_j of D. Thus we have the statement below.

Theorem 2 (Theorem 4.1.1 of Chapter 4)**.** *Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{P}^{n+1}$ defined by a closed logarithmic 1-form ω with simple normal crossing polar divisor $D = D_0 + \cdots + D_k$. If $n \geq 2$ then the fundamental group of a generic leaf $\mathcal{L}$ of $\mathcal{F}$ is isomorphic to the subgroup G of $\pi_1(\mathbb{P}^{n+1} - D)$ defined by*

$$G := \left\{ (m_0, \ldots, m_k) \in \mathbb{Z}^{k+1}/(d_0, \ldots, d_k)\mathbb{Z} | \sum_{j=0} \lambda_j m_j = 0 \right\},$$

where d_j is the degree of the irreducible component D_j of D and λ_j is the residue of ω around D_j.

Thus (C3) is true when the above group G is trivial, which happens for sufficiently generic logarithmic foliations (see Corollary 4.4.3 for more details).

In order to prove these results we exhibit a relation between the homotopy groups of $\mathbb{P}^{n+1} - D$ and of a generic leaf $\mathcal{L}$ of $\mathcal{F}$, where D is the polar locus of the 1-form defining $\mathcal{F}$. The existence of such a relation relies on the following result.

Theorem 3 (Theorem 4.3.1 of Chapter 4)**.** *Let ω be a closed logarithmic 1-form on a projective manifold X of dimension $n + 1$. Assume that D is a normal crossing polar divisor of ω and the singularities of ω outside D are isolated. Consider a normal covering space*

$$\rho : Y \to X - D,$$

over which the function

$$g(y) = \int_{y_0}^{y} \rho^* \omega \tag{1}$$

is well defined for $y \in Y$. If the singularities of ω outside D are isolated then the relative homotopy group $\pi_l(Y, g^{-1}(c))$ is zero for $l \leq n$, with $c \in \mathbb{C}$.

The above theorem is an adaptation of [S, Corollary 21] of Carlos Simpson, which concerns the topology of integral varieties of a closed holomorphic 1-form on a projective variety X.

The phenomenon observed by Cerveau is more general, as Corollary 4.4.3 proves that for sufficiently generic logarithmic foliation on $\mathbb{P}^3$ the generic leaf is simply connected. Furthermore the restriction of the foliation to a sufficiently general hyperplane gives a logarithmic foliation satisfying the hypothesis of the result below.

Theorem 4 (Theorem 3.4.7 of Chapter 3)**.** *Let $\mathcal{F}$ be a logarithmic foliation defined by a closed logarithmic 1-form ω on $\mathbb{P}^2$. Assume that the polar divisor $D = \cup_{j=0}^{k} D_j$ of ω is a supported on $k + 1 > 3$ curves and has only normal crossing singularities. If the residues $\lambda_j/\lambda_l \in \mathbb{C} - \mathbb{R}$, then a generic leaf $\mathcal{L}$ of $\mathcal{F}$ is homeomorphic to the Loch-Ness monster.*

The proof of this result relies on the description of the topological invariants (a,b) for a generic leaf of the generic logarithmic foliation on $\mathbb{P}^2$.

If $\mathcal{F}$ is a Riccati foliation on a projective surface, then we can give a precise description of the topology of a generic leaf of $\mathcal{F}$. In particular we show the result below.

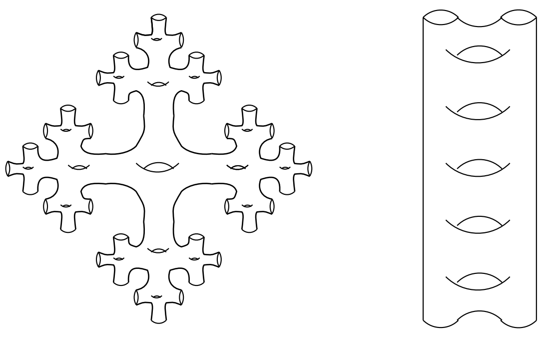
Theorem 5 (Theorem3.1.1 of Chapter 3)**.** *Let $\mathcal{F}$ be a Riccati foliation (singular or not) on a compact complex surface X. Assume that global holonomy*

$$\rho : \pi_1(\Sigma_{g,k}) \to \mathrm{PSL}(2, \mathbb{C})$$

of $\mathcal{F}$ is infinite, where $\Sigma_{g,k}$ is a open subset of the base C of the rational fibration $\pi : X \to C$ associated to the Riccati foliation $\mathcal{F}$. Then any leaf of $\mathcal{F}$ outside a countable set of leaves $\mathcal{C}_{\mathcal{L}}$ is homeomorphic to one of the following real surfaces:

1) *the plane,*
2) *the Loch Ness monster, i.e. the real plane with infinitely many handles attached,*
3) *the cylinder,*
4) *the Jacob's ladder, i.e. the cylinder with infinitely many handles attached to both directions,*
5) *the Cantor tree, i.e. the sphere without a Cantor set,*
6) *the blooming Cantor tree, i.e. the Cantor tree with infinitely many handles attached to each end,*
7) *the plane without an infinite discrete set*
8) *the Loch Ness monster without an infinite discrete set,*
9) *the Jacob's ladder without an infinite discrete set*
10) *the Cantor tree without an infinite discrete set,*
11) *the blooming Cantor tree without an infinite discrete set.*

Furthermore, any two leaves outside $\mathcal{C}_{\mathcal{L}}$ are biholomorphic.



Blooming Cantor tree — Jacob's ladder

Figure 0

It is to be observed that Ghys in [Gh] showed that a generic leaf of Riemann surfaces laminations on compact spaces is homeormophic to one of the real surfaces (1,2,3,4,5,6).

Plan

Chapter 1 starts with a presentation of basic definitions and results about singular holomorphic foliations $\mathcal{F}$ in compact complex manifolds. This chapter presents the concepts of holonomy, which describes the behaviour of the leaves in a neighborhood of a fixed leaf.

Chapter 2 begins with a survey of the classification of surfaces. In particular, we present the classification of open surfaces due to Kerékjártó. Then we restrict our attention to the normal covers and Cayley graphs associated to a group, which acts properly discontinuously on a normal cover. Next, we make a description of normal covers of bordered Riemann surfaces. Particularly, we give a total description of abelian covers, which gives us a tool for the topological description of generic leaves of a foliation with invariant algebraic curve.

In Sections 1 and 2 of Chapter 3 we give a topological description of generic leaves of Riccati foliations on projective surfaces and homogeneous foliations in $\mathbb{C}^2$. In Section 3 we study the genus of leaves of dimension one singular holomorphic foliations on a complex manifold with invariant compact complex curve. In Section 4 we give the proof of Theorem 4.

Chapter 4 provides a detailed exposition of Theorems 1, 2 and 3.

Chapter 1

Preliminaries

We start this Chapter by recalling definitions of holomorphic foliations and presenting some basic facts about them. Next we will also collect properties of logarithmic and Riccati foliations. To do this, we follow [C-M], [Bn], [I-Y], [L-S].

1.1 Foliations

There are several ways to define foliations. Here we introduce them following [I-Y].

Definition 1.1.1. [I-Y] The *standard holomorphic foliation* of dimension n (respectively of codimension m) of a polydisk

$$\mathbb{D}^{n+m} = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m \big| \quad |x| < 1, \quad |y| < 1\}$$

is the representation of $\mathbb{D}^{n+m}$ as the disjoint union of n-disks, called *plaques*,

$$\mathbb{D}^{n+m} = \bigsqcup_{|y|<1} \mathcal{L}_y, \quad \mathcal{L}_y = \big\{\{|x| < 1\} \times \{y\} \subseteq \mathbb{D}^{n+m}\big\}.$$

Definition 1.1.2. A *holomorphic foliation* $\mathcal{F}$ of dimension n of a complex analytic manifold M of dimension $n+m$ is a partition $M = \sqcup_\alpha \mathcal{L}_\alpha$ of the latter into a disjoint union of connected subsets $\mathcal{L}_\alpha$, called *leaves*, which locally is biholomorphic to the standard foliation of dimension n, i.e. each point $p \in M$ admits an open neighborhood U' and a biholomorphism $\phi : U' \to \mathbb{D}^{n+m}$ of U' onto the polydisk $\mathbb{D}^{n+m}$, which sends the connected components of $U' \cap \mathcal{L}_\alpha$, to the plaques of the standard holomorphic foliation, i.e. for each α there is a subset $Y(\alpha)$ of $\{|y| < 1\}$ such that

$$\phi(\mathcal{L}_\alpha \cap U') = \bigsqcup_{y \in Y(\alpha)} \mathcal{L}_y.$$

The pairs $\{U', \phi\}$ are called *trivial neighborhoods* of $\mathcal{F}$.

Example 1.1.3. Let ω be a closed 1-form on a complex manifold M^n without zeros. For each point $p \in M$ we are able to choose an open neighborhood U_p where the Poincaré lemma holds for ω. Explicitly, there exists a holomorphic function $f : U_p \to \mathbb{C}$ such that $df = \omega$. By the implicit function Theorem, we have a neighborhood $U'_p \subset U_p$ biholomorphic to polydisk $\mathbb{D}^{n-1} \times \mathbb{D}$ and $f \circ \phi^{-1}$ has level hypersurfaces of the form $\{y = a\}$, where ϕ is the biholomorphism. Therefore ω defines a holomorphic foliation $\mathcal{F}$ of dimension $n-1$ on M.

Definition 1.1.4. A *singular foliation* of dimension n (or codimension m) in a complex analytic manifold M^{n+m} is a holomorphic foliation $\mathcal{F}$ with complex n-dimensional leaves in the complement $M - \mathrm{Sing}(\mathcal{F})$ of an analytic subset $\mathrm{Sing}(\mathcal{F})$ of codimension ≥ 2, called the *singular locus* of $\mathcal{F}$.

Example 1.1.5. Let M be a complex manifold and $\{U_j\}_{j\in\Lambda}$ an open covering of it. Take a collection of holomorphic 1-forms $\omega_j \in \Omega^1_M(U_j)$ with singular locus $\mathrm{Sing}(\omega_j)$ of codimension ≥ 2 and such that

$$d\omega_j \wedge \omega_j = 0 \quad \text{and} \quad \omega_j = f_{ij}\omega_i \quad \text{on} \quad U_i \cap U_j = U_{ij}, \quad f_{ij} \in \mathcal{O}^*_M(U_{ij}).$$

By the Frobenius theorem, we have a foliation $\mathcal{F}$ of codimension 1 on the complement of $\cup_j \mathrm{Sing}(\omega_j)$. By definition $\cup_j \mathrm{Sing}(\omega_j)$ is an analytic subset, hence $\mathcal{F}$ is a singular foliation on M.

Definition 1.1.6. Let D be an analytic hypersurface of M and $\mathcal{F}$ be a foliation defined by holomorphic 1-forms $\omega_j \in \Omega^1_M(U_j)$ as above, where $\{U_j\}$ is an open cover of M. The hypersurface D is called *$\mathcal{F}$-invariant* if and only if in each open U_j the local equation $\{f_j = 0\} = D \cap U_j$ satisfies

$$\omega_j \wedge df_j = f_j \eta,$$

where η is a holomorphic 2-form in $\Omega^2_M(U_j)$.

1.2 Holonomy

The holonomy defined below plays a central role in the topological description of leaves of foliations. The following definitions agree with the ones given in [I-Y] and [P-S].

Definition 1.2.1. A *cross section* to a leaf $\mathcal{L}$ of a foliation $\mathcal{F}$ of codimension m on M at a point $o \in M$ is a holomorphic map $\tau : (\mathbb{C}^m, 0) \to (M, o)$ transverse to $\mathcal{L}$. Very often we identify the cross section with the image of the map τ.

Definition 1.2.2. Let $\mathcal{L}$ be a leaf of a holomorphic foliation $\mathcal{F}$ and let τ, τ' be two cross sections at the points $o, o' \in \mathcal{L}$. Let also $\gamma : [0,1] \to \mathcal{L}$ be a path connecting $o = \gamma(0)$ to $o' = \gamma(1)$. We take an open finite cover $\{U_j\}$ of $\gamma([0,1])$ such that the flow-box theorem holds in each open U_j. One can take a partition $\{0 = t_0, t_1, \ldots, t_k = 1\}$ such that the image $\gamma([t_j, t_{j+1}])$ belongs to U_j. Consider the cross sections τ_j, $j = 0, \ldots, k$, $\tau_0 = \tau$, $\tau_k = \tau'$ to $\mathcal{L}$ at the points $\gamma(t_j)$. In this way we define the correspondence map on $\gamma|_{[t_j,t_{j+1}])}$ as

$$\begin{array}{rcl} h_j : (\tau_j, \gamma(t_j)) & \to & (\tau_{j+1}, \gamma(t_{j+1})) \\ \alpha(\tau_j) & \mapsto & \alpha(\tau_{j+1}), \end{array}$$

where $\alpha(\tau_j) = \mathcal{L}_\alpha \cap \tau_j$, $\alpha(\tau_{j+1}) = \mathcal{L}_\alpha \cap \tau_{j+1}$. The set $\mathcal{L}_\alpha$ is a connected leaf of the foliation $\mathcal{F}$ restricted at U_j, such that if it is sufficiently close to $\mathcal{L} \cap U_j$ then intersects each cross section once. The composition

$$h_\gamma = h_k \circ \cdots \circ h_0 : (\tau, o) \to (\tau', o')$$

is a holomorphic map, also called the *holonomy map* along the path γ.

Definition 1.2.3 (Holonomy representation)**.** We take a point o on a leaf $\mathcal{L}$ of a foliation $\mathcal{F}$ and a cross sections τ at o. Closed paths γ starting at o contained in the corresponding leaf of the foliation induce germs of biholomorphisms $h_\gamma : (\tau, o) \to (\tau, o)$ which not depend on the homotopy class of the path. The *holonomy representation* of $\pi_1(\mathcal{L}, o)$ is the morphism defined by

$$\begin{array}{rcl} \mathrm{Hol}(\mathcal{L}, \mathcal{F}) : \pi_1(\mathcal{L}, o) & \to & \mathrm{Diff}(\tau, o) \\ {[\gamma]} & \mapsto & h_\gamma, \end{array} \tag{1.1}$$

and the holonomy group of the foliation along $\mathcal{L}$ is the image of this map (which will be confounded with the representation itself). Different points in the leaf and different sections give rise to representations conjugated by germs of holomorphic diffeomorphisms.

Example 1.2.4. [L-S] Let $\mathcal{F}$ be a singular foliation defined by a closed holomorphic 1-form ω on a complex manifold M^n. Take a leaf $\mathcal{L} \subset M - \mathrm{Sing}(\mathcal{F})$ and a closed path $\gamma : [0,1] \to \mathcal{L}$, $\gamma(0) = \gamma(1) = o$. We will prove that the holonomy map on γ is trivial. The 1-form ω is closed and regular in each point of γ, thus we can choose trivializing charts $\{(x_i^j), U_j\}_{j=1,\dots,r}$ which is also an open finite cover of $\gamma([0,1])$ satisfying:

i) the sets $U_j \cap U_{j+1}$ are simply connected,

ii) $\omega|_{U_j} = dx_n^j$,

iii) $\{x_n^j = 0\} \supset \gamma \cap U_j$.

By ii) the 1-form $dx_n^j - dx_n^{j+1}$ vanishes on $U_j \cap U_{j+1}$, thus the function $x_n^j - x_n^{j+1}$ is constant. Hence x_n^j and x_n^{j+1} coincide in $U_j \cap U_{j+1}$. Consider a partition $\{0 = t_1 < t_2 < \cdots < t_r = 1\}$ of $[0,1]$ such that $\gamma(t_{j+1}) \in U_j \cap U_{j+1}$. Then we choose some transversals τ_j at $\gamma(t_j)$ with the property that $(x_i^j)(\gamma([t_j, t_{j+1}])) \times (x_i^j)(\tau_{j+1}) \subset U_j$, and such that $(x_i^j)(\tau_{j+1}) = (x_1^j(\gamma(t_{j+1})), \dots, x_{n-1}^j(\gamma(t_{j+1})), x_n^j)$. Thus the holonomy in each $\gamma([t_j, t_{j+1}])$ is trivial. Therefore the composition of them is trivial, which is the holonomy on γ.

Example 1.2.5. Let $\mathcal{F}$ be a singular foliation on a complex surface M. By the Camacho-Sad Theorem for each singular point p of a foliation $\mathcal{F}$ there exist at least one *separatrix*, this means that for a neighborhood U of p there exist at least one local leaf $C \subset U$ such that $p \in \bar{C}$ and $\bar{C} \cap U$ is an analytic set in U, which is $\mathcal{F}$-invariant in U. Let τ a cross section at a regular point $o \in C$, thus the holonomy map on a closed curve $\gamma : I \to C \cap U$ around p and based on o has the form

$$h_\gamma(z) = \exp(2\pi i CS(C, p))z + h.o.t.,$$

where $CS(C, p)$ is the Camacho-Sad index of p in C.

1.3 Closed logarithmic forms and logarithmic foliations

As in the Example 1.1.5 we shall define logarithmic foliations by closed logarithmic 1-forms. This subsection follows [Bn], [CkSoV], [I-Y], [L-S] and [Pa2].

Definition 1.3.1. Let M be a connected complex manifold, and let $D \subset M$ be a union of complex hypersurfaces D_j. A *closed logarithmic 1-form* ω on M with poles on D is a meromorphic 1-form with the following property: for any $p \in M$ there exists a neighborhood U of p in M such that $\omega|_U$ can be written as

$$\omega_0 + \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}, \tag{1.2}$$

where ω_0 is a closed holomorphic 1-form on U, $\lambda_j \in \mathbb{C}^*$ and $f_j \in \mathcal{O}(U)$, and $\{f_j = 0\}$, $j = 1, \dots, r$, are the reduced equations of the irreducible components of $D \cap U$. The set D is known as the polar divisor of ω. The holomorphic foliation $\mathcal{F}$ of M defined by ω is called *logarithmic foliation*.

The following result is an adaptation of the [I-Y, Theorem11.26] and [L-S] for logarithmic foliations on complex manifolds M.

Theorem 1.3.2. *The holonomy group associated with any leaf of a logarithmic foliation $\mathcal{F}$ with poles in $D = \cup D_j$ is abelian and linearizable (it is isomorphic to a subgroup of $\mathbb{C}^*$). Moreover, if M is simply connected then $\mathrm{Hol}(D_j - \mathrm{Sing}(\mathcal{F}), \mathcal{F})$ is a subgroup of the group generated by $\{\exp(2\pi i \frac{\lambda_k}{\lambda_j})\}$, where λ_j is the residue of each irreducible component D_j of D.*

Proof. Let us first notice that a closed logarithmic 1-form ω on M^n is a closed holomorphic 1-form in $M - D$. In particular the holonomy group of any leaf $\mathcal{L}$ of $\mathcal{F}$ in $M - D$ is trivial, as we proved in the Example 1.2.4. The hypersurfaces D_j of Definition 1.3.1 without the singular locus of ω are leaves of $\mathcal{F}$. Take a trivializing chart $(U, (x_j))$ around a regular point o in $D_1 - \mathrm{Sing}(\mathcal{F}) = D_1'$ such that passing to the chart (x_j) we have $D_1' \cap U = \{x_n = 0\}$. In this chart the leaves are $\{x_n = const.\}$, therefore we can write ω as the 1-form

$$\lambda_1 \frac{dx_n}{x_n} + g(x_1, \dots, x_n) dx_n,$$

where $g(x_1, \dots, x_n)dx_n$ is a holomorphic 1-form. The 1-form ω is closed, thus g depend only of x_n. Consider the primitive h of $\lambda_1^{-1} g(x_n) dx_n$, we have

$$\lambda_1 \frac{dy}{y} = \lambda_1 \frac{dx_n}{x_n} + g(x_n) dx_n, \quad y = x_n \exp(h).$$

Take a closed path $\gamma : [0, 1] \to D_1'$ with $\gamma(0) = o$, thus by Example 1.2.4 and the equation above we can take an open cover $\{U_j, (x_i^j)\}$ and a partition $\{0 = t_1, \dots, t_k = 1\}$ of γ satisfying:

i) the sets $U_j \cap U_{j+1}$ are simply connected,

ii) $\gamma([t_j, t_{j+1}]) \subset U_j$

iii) $\omega|_{U_j} = \lambda_1^j \frac{dx_n^j}{x_n^j}$

iv) $\{x_n^j = 0\} \supset \gamma \cap U_j$,

also we take the transversals τ_j as in Example 1.2.4. At the intersection $U_j \cap U_{j+1}$ we have that

$$\lambda_1^j \frac{dx_n^j}{x_n^j} = \lambda_1^{j+1} \frac{dx_n^{j+1}}{x_n^{j+1}}.$$

The residues are equal around zero, thus $\lambda_1^j = \lambda_1^{j+1}$. Writing the holonomy $h_j : \tau_j \to \tau_{j+1}$ and the equations

$$\frac{dx_n^{j+1}}{x_n^{j+1}} = \frac{h_j'(x_n^j)dx_n^j}{h_j(x_n^j)} = \frac{dx_n^j}{x_n^j},$$

we obtain that $x_n^j h_j'(x_n^j) = h_j(x_n^j)$, thus h_j' is constant. Therefore h_γ is linear. Since the charts only depend of ω, we can choose the same chart $(U, (x_j))$ around o for any path $\gamma \in \pi_1(D_1', o)$. Consequently, $\mathrm{Hol}(D_1, \mathcal{F})$ is abelian and linear.

Let $\tilde{\gamma}$ be the lifting of γ to a leaf $\mathcal{L} \cap \{\cup U_j\}$ such that $\tilde{\gamma}(0) = z \in \tau$ and $\tilde{\gamma}(1) = h_\gamma(z)$. Define a path $\alpha : I \to \tau$ joining z and $h_\gamma(z)$ without passing through o. Consider the integral

$$\int_{\tilde{\gamma}*\alpha} \omega = \int_{\tilde{\gamma}} \omega + \int_\alpha \omega = \int_\alpha \lambda_1 \frac{dz}{z} = 2\pi i \mu,$$

where μ is a finite integer combination of the residues λ_j of ω. Hence $h_\gamma(z) = \exp(2\pi i\mu/\lambda_1)z$.

We now assume that M is simply connected. Thus the residues of ω only can be the residues λ_j of each irreducible component D_j of D, which is the desired conclusion. □

Consider the complex projective space $\mathbb{P}^n$, with homogeneous coordinates $x_0, \ldots, x_n$. Codimension one foliations on $\mathbb{P}^n$ are defined in homogeneous coordinates by

$$\omega = \sum_0^n F_j dx_j,$$

where $\{F_j\}$ are homogeneous polynomials of the same degree satisfying

$$\sum_{j=0}^n F_j x_j = 0, \quad d\omega \wedge \omega = 0 \quad \text{and} \quad \mathit{codim}(\mathrm{Sing}(\omega)) \geq 2.$$

Definition 1.3.3. Without the condition $\sum F_j x_j = 0$, the 1-form defines a foliation on $\mathbb{C}^{n+1}$ known as *homogeneous foliation*, which can be extended to a foliation on $\mathbb{P}^{n+1}$.

We will study these foliations in Chapter 3.

Definition 1.3.4. The *degree* of a codimension one foliation $\mathcal{F}$ on $\mathbb{P}^n$, $\deg\mathcal{F}$, is the number of tangences of the leaves of $\mathcal{F}$ with a generic one-dimensional linear subspace of $\mathbb{P}^n$.

It is easy to check that $\deg(\mathcal{F}) = d-2$ if the 1-form defining $\mathcal{F}$ has components F_j of degree $d-1$. In this way a logarithmic foliation on $\mathbb{P}^n$ can be seen as

$$\omega = \prod_{j=0}^{r} F_j \sum_{j=0}^{r} \lambda_j \frac{dF_j}{F_j} = \sum_{j=0}^{r} \lambda_j \hat{F}_j dF_j$$

for some homogeneous polynomials F_j of degree d_j and $\lambda_j \in \mathbb{C}$ such that $\sum \lambda_j d_j = 0$. The next statement from [CkSoV, Theorem 3] shows us the kind of singular locus of a logarithmic foliation in $\mathbb{P}^n$ $(n \geq 3)$ when the hypersurfaces defined by $F_j = 0$ are smooth and in general position, and $\lambda_j \neq 0$ for all $j = 0, \ldots, r$.

Theorem 1.3.5. *Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{P}^n$, with $n \geq 3$, given by*

$$\omega = \sum_{j=0}^{r} \lambda_j \frac{dF_j}{F_j},$$

and satisfying that the irreducible components of the polar divisor are smooth and intersect transversely, $\lambda_j \neq 0$, $j = 0, \ldots, r$. Then the singular locus $\mathrm{Sing}(\mathcal{F})$ *can be written as a disjoint union*

$$\mathrm{Sing}(\mathcal{F}) = Z \cup R$$

where

$$Z = \cup_{i \neq j} D_i \cap D_j$$

and R is a finite set.

Definition 1.3.6. A hypersurface D of a complex manifold M^n is *simple normal crossing* divisor if each of its irreducible components D_j, where $\cup_{j=1,\ldots,l} D_j = D$, is smooth and locally near of each point D can be represented in a chart $(z_1, \ldots, z_n) : U \to M$ as the locus $\{z_1 \cdots z_k = 0\}$ with $1 < k \leq n$.

Lemma 1.3.7. *Let ω a closed logarithmic 1-form in a smooth projective variety X. If the polar locus $D \subset X$ of ω is simple normal crossing and a singular point p is not in* $\mathrm{Sing}(D)$. *Then the connected component S_p of $\{x \in X | \omega(x) = 0\}$, which contains p, has empty intersection with D, i.e. $S_p \cap D = \emptyset$.*

Proof. [CkSoV] Suppose that $S_p \cap D$ is non empty. Let q be a point in $S_p \cap D$. Since ω is a closed logarithmic 1-form with simple normal crossing polar divisor, there is a coordinate chart $(U, (x_j))$ of q such that ω can be written as

$$\sum_{j=0}^{k} \lambda_j \frac{dx_j}{x_j} + dh,$$

where $k \leq \dim(X)$ and h is a holomorphic function. Consider the change of coordinates given by

$$y_0 = \exp(h/\lambda_0) x_0 \quad \text{and} \quad y_j = x_j \quad \text{if} \quad j \in \{1, \ldots, k\}.$$

In this new coordinates ω looks like

$$(y^{-1})^*\omega = \sum_{j=0}^{k} \lambda_j \frac{dy_j}{y_j}. \tag{1.3}$$

If k is equal zero, formula (1.3) implies that q is a regular point contradicting our assumption.

Since k is greater than 1, formula (1.3) shows that the singularities of ω in the chart U are contained in

$$\bigcup_{i\neq j} D_i \cap D_j,$$

with $i, j = 0, \ldots, k$. By the compactness of S_p, we have a finite number of this coordinates charts $(U_i, (x_j^i))$ covering S_p. Therefore, the set S_p is contained in the union of intersections of irreducible components of D. This contradicts our assumption. □

1.4 Riccati Foliations

The aim of this subsection is to introduce a family of foliations known as Riccati foliations, following [L-M], [GM] and [Bn].

Consider a 2-dimensional vector bundle V and a meromorphic connection ∇ over an analytic smooth curve Σ_g of genus g. Over a trivializing chart $(z, (y_1, y_2)) : U \subset \mathrm{V} \to \mathbb{D} \times \mathbb{C}^2$ of V, we have the meromorphic system

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

defined by ∇. Take the projectivization $\mathbb{P}\mathrm{V}$ of V, the flat sections of the meromorphic connection ∇ can be projected on $\mathbb{P}\mathrm{V} = M$, which in affine coordinate $(y : 1) = (y_1, y_2)$ results in solutions of the Riccati equation

$$\frac{dy}{dz} = -c(z)y^2 + (a(z) - d(z))y + b(z).$$

As ∇ is integrable this projection defines a foliation $\mathcal{F}$ known as *Riccati foliation* with respect the projection $\pi : M \to \Sigma_g$ with fiber $\mathbb{P}^1$ (see [Bn, p.50] for more details). Analogously to Example 1.1.5, we define in a trivialization $\{U_\alpha\}$ of $\mathcal{F}$ a collection of holomorphic vector fields $\{v_\alpha\}$

$$v_\alpha = f_{\alpha\beta} v_\beta \quad \text{on} \quad U_\alpha \cap U_\beta,$$

where v_α is the vector field defining $\mathcal{F}$ in U_α. In our case in the trivial chart the vector field looks like

$$\partial_z + (-c(z)y^2 + (a(z) - d(z))y + b(z))\partial_y.$$

We introduce the tangent bundle $T_{\mathcal{F}}$ of $\mathcal{F}$ as the line bundle on the total space M defined by the collection $\{f_{\alpha\beta}^{-1}\}$ of nonvanishing holomorphic function.

From the homological point of view, we can associate an element of $H^2(M, \mathbb{Z})$ to each line bundle over M. Since M is a ruled surface, $H^2(M, \mathbb{Q})$ is generated by the homology class of σ_h and F; where σ_h is any holomorphic section and F is any fibre. Let us choose $\sigma_h \in H^2(M, \mathbb{Z})$ with self-intersection:

$$\sigma_h \cdot \sigma_h = 0, \quad F \cdot F = 0, \quad \text{and} \quad F \cdot \sigma_h = 1.$$

By the local representation of the Riccati foliation $\mathcal{F}$ we know that the generic fibre F is transversal to the foliation. We conclude from the formula for not $\mathcal{F}$-invariant compact curve C, [Bn, Proposition 2,p.23],

$$T_{\mathcal{F}} \cdot C = C \cdot C - tang(\mathcal{F}, C) \tag{1.4}$$

that $T_{\mathcal{F}} \cdot F = 0$.

Considering the exact sequence

$$0 \to TF \to TM \to \pi^*(T\Sigma_g) \to 0, \tag{1.5}$$

where $TF \hookrightarrow TM$ is the subline bundle defined as the kernel of the Jacobian of π. Also we have the following commutative diagram

$$\begin{array}{ccccccccc} & & & & T_{\mathcal{F}} & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & TF & \longrightarrow & TM & \xrightarrow{D\pi} & N & \longrightarrow & 0; \end{array}$$

where $N = \pi^*(T\Sigma_g)$.

Example 1.4.1. Let $\rho : \tilde{C} \to C$ be the universal cover of a compact complex curve C and $\pi_1(C)$ be the fundamental group of C. Consider a representation

$$\varrho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C}),$$

where the image $\varrho(\gamma)$ of an element $\gamma \in \pi_1(C)$ is an automorphism of $\mathbb{P}^1$. We define the action of $\pi_1(C)$ on $\tilde{C} \times \mathbb{P}^1$ to be the left action

$$\begin{array}{lcl} \pi_1(C) \times (\tilde{C} \times \mathbb{P}^1) & \to & \tilde{C} \times \mathbb{P}^1 \\ (\gamma, (z, w)) & \mapsto & (\psi_\gamma(z), \varrho(\gamma)(w)), \end{array}$$

where $\psi_\gamma \in Aut(\tilde{C}, \rho)$ is given by the isomorphism between $\pi_1(C)$ and the group of deck transformations $Aut(\tilde{C}, \rho)$. Since the action of $\pi_1(C)$ on $\tilde{C} \times \mathbb{P}^1$ is free and properly discontinuous, we have that the quotient space

$$M = \tilde{C} \times \mathbb{P}^1 / \pi_1(C)$$

is a complex surface. This construction is known as the *suspension* of the homomorphism ϱ. The foliation on $\tilde{C} \times \mathbb{P}^1$ with leaves $\tilde{C} \times \{w\}$, for all $w \in \mathbb{P}^1$, is invariant by the action of $\pi_1(C)$. Therefore, it passes to a holomorphic foliation $\mathcal{F}$ of M. By [Cnd-Cln, Theorem 3.1.4,v.I] the foliation $\mathcal{F}$ is a Riccati foliation with adapted fibration $\pi : M \to C$, where π is given by

$$\begin{array}{ccc} \tilde{C} \times \mathbb{P}^1 & \longrightarrow & M \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & C \end{array} .$$

Proposition 1.4.2. *[GM]Let $\mathcal{F}$ be a Riccati foliation with $T_{\mathcal{F}} = aF$, then*

1 *Counting multiplicities there are $2-2g-a \geq 0$ fibres of the ruling that are leaves of the foliation (after removing the singular points on them), and all the rest of the leaves of the foliation intersect transversely the remaining fibres of the ruling.*

2 *After removing the fibres that are leaves of $\mathcal{F}$, the foliation is obtained by suspending a representation $\varrho : \pi_1(\Sigma_{g,r}) \to \mathrm{PSL}(2,\mathbb{C})$.*

Proof. (1) We know that $\mathcal{F}$ can be thought as a section of $T^*_{\mathcal{F}} \otimes TM$, but by the above commutative diagram also it induces a section of $T^*_{\mathcal{F}} \otimes N$, which zeros are the $\mathcal{F}$-invariant fibres. Thus the intersection number $T^*_{\mathcal{F}} \otimes N \cdot \sigma_h = 2-2g-a$ is the number (counted with multiplicities) of $\mathcal{F}$-invariant fibres.

(2) Let $\{F_1, \ldots, F_r\}$ be the $\mathcal{F}$-invariant fibres and $p_j = \pi(F_j)$ their projections. We choose a leaf $\mathcal{L}$ of the foliation and a point o therein. As $\mathcal{L}$ is a regular covering of $\Sigma_g - \{p_1, \ldots, p_r\} = \Sigma_{g,r}$ we can lift a curve $\gamma : [0,1] \to \Sigma_{g,r}$ with $\gamma(0) = \pi(o)$. The holonomy on the lifting $\widetilde{\gamma}$ of a representant γ of $\pi_1(\Sigma_{g,r}, \pi(o))$ and the transversality of the fibres give us that $h_{\widetilde{\gamma}}$ is a biholomorphis of $\mathbb{P}^1$. Hence we can define the morphism

$$\varrho : \pi_1(\Sigma_{g,r}, \pi(o)) \to \mathrm{PSL}(2,\mathbb{C}).$$

Then the foliation $\mathcal{F}$ in $M - \{F_1, \ldots, F_r\} = M^*$ is biholomorphic to the suspension of ϱ. □

Brunella [Bn, p.52-56] gives a study of invariant fibres, which is resumed in the following proposition.

Proposition 1.4.3. *[Bn, p.56] Let $\mathcal{F}$ be a Riccati foliation on a compact connected surface M, with adapted fibration $\pi : M \to \Sigma_g$. Then there exists a birational map $f : M \dashrightarrow M'$ such that:*

i) f is biregular on M^*; in particular, the transform $\mathcal{F}'$ of $\mathcal{F}$ by f is still Riccati, with adapted fibration $\pi' = \pi \circ f^{-1} : M' \to \Sigma_g$;

ii) π' has no singular fibre;

iii) each $\mathcal{F}$-invariant fibre of π' belongs to one of the following classes:

iii.1) nondegenerate fibre: around the fibre, the foliation has equation

$$\lambda w dz - z dw = 0 \qquad (z,w) \in \mathbb{D} \times \mathbb{P}^1, \quad \lambda \notin \mathbb{Z}$$

or

$$dz - z dw = 0 \qquad (z,w) \in \mathbb{D} \times \mathbb{P}^1;$$

iii.2) semidegenerate fibre: the fibre contains two saddle nodes, of the same multiplicity, whose strong separatrices are contained in the fibre;

iii.3) nilpotent fibre: the fibre contains only one singularity, generated by a vector field with nilpotent and nontrivial linear part.

Chapter 2

Topology of Riemann Surfaces

This chapter presents a description of the normal covers of orientable bordered surfaces, particularly abelian covers, via Cayley graphs. We show the following result

Theorem 2.0.1. *If $\tilde{X}_G(\Sigma_{g,n})$ is an infinite normal cover of a compact surface of genus g minus n points, $\Sigma_{g,n}$. Then $\tilde{X}_G(\Sigma_{g,n})$ is homeomorphic to one of the following surfaces*

1) *the plane,*
2) *the Loch Ness monster,*
3) *the cylinder,*
4) *the Jacob's ladder,*
5) *the Cantor tree,*
6) *the blooming Cantor tree,*
7) *the plane without an infinite discrete set*
8) *the Loch Ness monster without an infinite discrete set,*
9) *the Jacob's ladder without an infinite discrete set*
10) *the Cantor tree without an infinite discrete set,*
11) *the blooming Cantor tree without an infinite discrete set.*

Although this result seems to be well-known we could only find in the literature a proof for normal covers of compact Riemann surfaces [G1]. We use Kerékjártó's classification of non-compact Riemann surfaces together with standard in geometric group theory to deduce it. It is worthwhile mentioning that the classification of infinite normal cover of compact Riemann surfaces coincides with Ghys' classification of generic leaves of laminations by Riemann surfaces of compact spaces.

2.1 Compact Riemann surfaces

We recall that every compact surface S can be constructed from a polygon P_{2n} with $2n$-sides by identifying pairs of edges. The $2n$ edges of the polygon become a union of n circles in the surface, all intersecting in a single point. The interior of the polygon can be thought of as open disk attached to the union of the n circles. Choose a orientation of the boundary of P_{2n}. A pair of edges identified will be labeled by the letter a if the direction for attaching correspond to the orientation ∂P_{2n} or a^{-1} if it is counter the orientation.

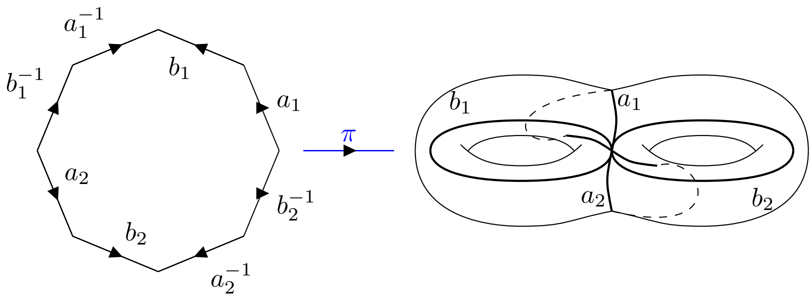


Figure 1

Example 2.1.1. We will name the boundary with a "word" formed by the labels of the edge, as shown in the examples below.

a) The sphere corresponds to a polygon with two edges and boundary cc^{-1}.

b) The real projective plane corresponds to a polygon with two edges and orientation cc.

c) The torus corresponds to a polygon with four edges and boundary $aba^{-1}b^{-1}$.

d) The orientable compact surface Σ_g of genus g corresponds to a polygon with $4g$ edges and boundary $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$. The figure above shows the case $g = 2$, where π is the attaching map.

Note that this representation is not unique. The sphere also corresponds to a polygon of four edges with boundary $abb^{-1}a^{-1}$.

Two surfaces are homeomorphic if and only if they have the same Euler characteristic, and are either both orientable or else both nonorientable. The *classification theorem of closed surfaces* states that any connected closed surface is homeomorphic to some member of one of these three families:

a) the sphere;

b) the connected sum of g tori, for $g \geq 1$;

c) the connected sum of k real projective planes, for $k \geq 1$.

2.2 Non-compact Riemann surfaces

Definition 2.2.1. A compact connected surface S is called a *bordered surface* if it is homeomorphic to a closed subset U of a compact surface Σ_g and $\Sigma_g - U$ is the union of k connected sets, where each component is homeomorphic to the disk and with $k < \infty$.

We are able to associate to a bordered surface S a polygon P_n with n-sides by identifying pairs of edges and leaving free some edges. In this case there are edges without association to any letter or direction, and this edges correspond to arcs of *boundary curves.* This identification in P_n gives a compact surface with boundary. The bordered surfaces are classified by the next theorem due to Brahana.

Theorem 2.2.2. *[B] Two triangulated bordered surfaces are homeomorphic if and only if they both have the same number of boundary curves, the same Euler characteristic, and are either both orientable or else both nonorientable.*

According to Brahana Theorem any compact surface S, with or without border, is homeomorphic to a connected sum of the sphere with m tori and n real projective planes and a finite number of open disks removed. We define the *genus* $g(S)$ of S as follows:

$$g(S) = \begin{cases} \frac{1}{2}(2 - \chi(S) - r) & \text{if } S \text{ is orientable} \\ 2 - \chi(S) - r & \text{if } S \text{ is nonorientable,} \end{cases}$$

where $\chi(S)$ is the Euler characteristic of S and r the number of boundary components of S. From the above theorems we have three topological invariants of a compact surface with or without boundary: the orientability, the number of boundary curves and the genus. In particular, the interior of a bordered surface is classified by this invariants. But not every separable non-compact surface is a subset of a compact surface. The canonical counterexample is the Loch Ness monster, which is a non compact surface with infinite genus obtained from $\mathbb{C}$ by attaching a infinite number of handles. It is necessary to consider the invariants defined below in order to state the classsifiation of non compact surfaces. We will present the classification following the works [R, G1, G2]. In what follows, we will call by a bordered surface a compact surface with boundary or its interior.

Definition 2.2.3. A *boundary component* of a surface S is a nested sequence $P_1 \supset P_2 \supset \cdots$ of connected unbounded regions in S such that:

i) the boundary of P_n in S is compact for all n;

ii) for any bounded subset A of S, this is that the closure of A in S is compact in S, $P_n \cap A = \emptyset$ for n sufficiently large.

We say that two boundary components $P_1 \supset P_2 \cdots$ and $P'_1 \supset P'_2 \cdots$ are *equivalent* if for any n there is a corresponding integer N such that $P_n \subset P'_N$ and vice versa. The equivalence class e^* is called an *end* of S.

Definition 2.2.4. The *ideal boundary* Ends(S) of a surface S is the topological space having the ends of S as elements, and endowed with the following topology: for any open subset $U \subset S$ whose boundary in S is compact, we define U^* to be the set of all ends e^*, represented by some $e = P_1 \supset P_2 \cdots$, such that $P_n \subset U$ for n sufficiently large; we take the set of all such U^* as a basis for the topology of Ends(S).

A surface is called *planar* if every compact subsurface in it is of genus zero.

Definition 2.2.5. Let e^*, represented by $e = P_1 \supset P_2 \cdots$, be an end of S. We say that e^* is *planar* and/or *orientable* if the sets P_n are planar and/or orientable for all sufficiently large n.

Following Definition 2.2.5, we shall consider the ideal boundary to be a nested triple of sets $\mathrm{Ends}(S) \supset \mathrm{Ends}'(S) \supset \mathrm{Ends}''(S)$, where Ends($S$) is the whole ideal boundary and:

a) If $e^* \in \mathrm{Ends}'(S)$ then for any representative $e = P_1 \supset P_2 \cdots$ there is $N \in \mathbb{N}$ such that P_n is not planar for $n > N$, i.e., there exist a bounded subset A in P_n, which closure is compact bordered surface in S and considering that A is a bordered surface we have that it has genus zero, $g(A) = 0$.

b) If $e^* \in \mathrm{Ends}''(S)$ then for any representative $e = P_1 \supset P_2 \cdots$ there is $N \in \mathbb{N}$ such that P_n is not orientable for $n > N$, this implies that there exist a bounded subset A of P_n, which closure is compact in S and it is nonorientable.

Definition 2.2.6. A noncompact surface S is of *infinite genus* and/or *infinitely nonorientable* if there is no bounded subset $A \subset S$ such that $S - A$ is of genus zero and/or orientable.

Remark 2.2.7. Definitions above do not depend on the representative e chosen for the equivalence class e^*.

Definition 2.2.8. We define four *orientability classes* of surfaces. The surface S may either *orientable* or *nonorientable.* If for any compact subset $K \subset S$, $S - K$ is nonorientable we say that S is *infinitely nonorientable.* If for some compact subset K, $S - K$ is orientable, then S is called of *even* or *odd nonorientability* according as K contains a subset homeomorphic to a connected sum of an even or an odd number of real projective planes without a disk.

Now we can state Kerékjártó's Theorem, see [R] for more details.

Theorem 2.2.9. *Let S and S' be two separable surfaces of the same genus and orientability class. Then S and S' are homeomorphic if and only if their ideals boundaries (considered as the triples of spaces* $(\mathrm{Ends}(S), \mathrm{Ends}'(S), \mathrm{Ends}''(S))$*) are topologically equivalent.*

Remark 2.2.10. Using the Stone Representation Theorem William S. Massey proved that the boolean ring of continuous functions from Ends(S) to $\mathbb{Z}_2$ is isomorphic to $H_e^0(S, \mathbb{Z}_2)$. Where H_e^q stands for the Alexander-Spainer cohomology. Recall that from the very definition of Alexander-Spainer cohomology, $H_e^0(S, \mathbb{Z}_2)$ is the ring of functions from S to $\mathbb{Z}_2$ continuous outside a compact set. This study of Massey gives the following statement.

Lemma 2.2.11. *Let $\{U_\alpha\}_{\alpha\in\Delta}$ be the collection of subsets of S having compact complement. For every α there is a map $\phi : H^q(U_\alpha) \to H^q_e(S)$ and the direct limit of $H^q(U_\alpha)$ (with respect to the maps $i^* : H^q(U_{\alpha'}) \to H^q(U_\alpha)$ induced by inclusions $U_\alpha \subset U_{\alpha'}$) is isomorphic to $H^q_e(S)$.*

From this lemma we obtain the following result, which will be use throughout this text.

Lemma 2.2.12. *Let S be a separable non-compact surface. Suppose that for every compact $K \subset S$ there exist a compact $K \subset K' \subset S$ such that $S - K'$ has n connected components. Then* $\mathrm{Ends}(S)$ *is a set of n points.*

Proof. By the above, we can construct a exhaustion $\{K_j\}$ such that the number of components of the complement of each K_j in S is n. We will denote by U_j the subset $S - K_j$. Applying the Lemma 2.2.11 to the collection $\{U_j\}$ it follows that $\mathrm{Ends}(S)$ is a set of n points. □

2.3 Cayley graphs and covering spaces of a bouquet of circles

This subsection presents some relations between groups and topology of graphs, like in Geometric Group Theory and Combinatory Topology, which support the further development in this text.

We start by recalling some basic definitions about normal covering spaces of topological space X. These kind of spaces include surfaces and the Cayley graphs defined below.

Definition 2.3.1. Let X be a topological space. A *normal covering space* of X is a topological space $\tilde{X}$ with a surjective map ρ satisfying the following conditions:

- $*$ For each point $p \in X$ there exist a neighborhood U such that $\rho^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by ρ homeomorphically onto U.
- $*$ For each $x \in X$ and each pair of points $\tilde{x}$, $\tilde{x}'$ in $\rho^{-1}(x)$ there is a automorphism $g : \tilde{X} \to \tilde{X}$ taking $\tilde{x}$ to $\tilde{x}'$, such that $\rho \circ g(x) = \rho(x)$ for all $x \in \tilde{X}$. These automorphisms form a group $Aut(\tilde{X}, \rho)$.

Remark 2.3.2. These are also called regular coverings, and omitting the last condition we obtain the usual definition of covering space.

Proposition 2.3.3. *[H, Prop.1.36] Let X a path-connected and locally path-connected topological space. Then for every normal subgroup $H < \pi_1(X, x_0)$ there is a normal covering space $\rho : X_H \to X$ such that $\rho_*(\pi_1(X_H, \tilde{x}_0)) = H$ for a suitably chosen base point $\tilde{x}_0$ in X_H and $Aut(X_H, \rho) \simeq \pi_1(X, x_0)/H$.*

Proposition 2.3.4. *[H, Prop.1.37] If X is a path-connected and locally path-connected topological space, then two normal covering spaces $\rho_1 : \tilde{X}_1 \to X$ and $\rho_2 : \tilde{X}_2 \to X$ are homeomorphic via a homemorphism $f : \tilde{X}_1 \to \tilde{X}_2$*

taking a base point $\tilde{x}_1 \in \rho_1^{-1}(x_0)$ *to a base point* $\tilde{x}_2 \in \rho_2^{-1}(x_0)$ *if and only if* $\rho_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = \rho_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

Definition 2.3.5. Let $\rho_u : \tilde{X} \to X$ the universal covering map, i.e. $\rho_{u*}(\pi_1(\tilde{X}, \tilde{x}))$ is the trivial subgroup of $\pi_1(X, x)$. A *fundamental domain* of this cover is a subset $\mathrm{D} \subset \tilde{X}$ such that

1. the union of $\gamma\mathrm{D}$ over all $\gamma \in \pi_1(X, x)$ covers $\tilde{X}$,

2. the collection $\gamma\mathrm{D}^\circ$ is mutually disjoint,

3. $\rho_u(\mathrm{D}) = X$ and the restriction $\rho_u|_{\mathrm{D}^\circ} : \mathrm{D}^\circ \to X$ is homeomorphic onto its image.

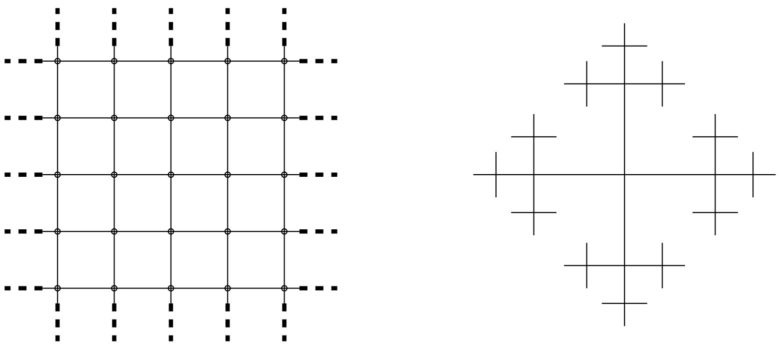
The image $\rho_u(D^\circ)$ will be confunded with the fundamental domain itself.

Definition 2.3.6. Given a group G and a generating set $\mathfrak{S} = \{a_1, \ldots, a_r\}$, one defines the *Cayley graph* of G with respect to $\mathfrak{S}$. This is a graph $\mathrm{Cayley}(G; \mathfrak{S})$ such that

a) its set of vertices is G;

b) its set of edges is (g, ga_j), with $a_j \in \mathfrak{S}$.

Example 2.3.7. The next figure represents the Cayley graphs for $\mathbb{Z}^2$ with respect to the canonical basis $\mathfrak{S}_{c2} = \{(1,0), (0,1)\}$ and for the free group F_2 with respect to the set of generators $\mathfrak{S}_2 = \{a_1, a_2\}$



$\mathrm{Cayley}(\mathbb{Z}^2, \mathfrak{S}_c 2)$ $\mathrm{Cayley}(F_2, \mathfrak{S}_2)$

Figure 2

Definition 2.3.8. Let H be a normal subgroup of G. The *quotient graph* of $\mathrm{Cayley}(G, \mathfrak{S})$ by H is the graph $\mathrm{Cayley}(G, \mathfrak{S})/H$ such that

a) its set of vertices is G/H;

b) its set of edges is (Hg, Hga_j), with $a_j \in \mathfrak{S}$. It may happen that g, ga_j defines the same class Hg'. In this case the edge (Hg, Hga_j) is a loop.

Example 2.3.9. Recall that the wedge sum $\vee_{k=1}^{n} S_{a_k}^1$ of n circles is called a *bouquet of circles*. Both graphs in Figure 2 are normal covering spaces of $\vee_{k=1}^{2} S_{a_k}^1$. Since Cayley$(F_2, \mathfrak{S}_2)$ is simply connected, it is a universal cover of $\vee_{k=1}^{2} S_{a_k}^1$. By Proposition 2.3.3 there exists a group H acting on Cayley$(F_2, \mathfrak{S}_2)$, such that Cayley$(F_2, \mathfrak{S}_2)/H \simeq$ Cayley$(\mathbb{Z}^2, \mathfrak{S}_{c2})$.

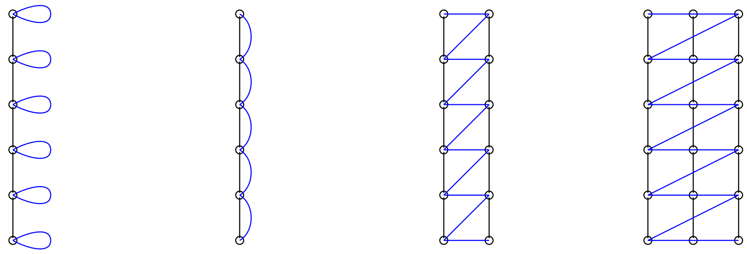
If Cayley$(G, \mathfrak{S})$ is a normal cover of some space $\vee_{k=1}^{n} S_{a_k}^1$ then the quotient graph by the normal subgroup $H < G$ still is a normal covering space of $\vee_{k=1}^{n} S_{a_k}^1$. The group of automorphism of this cover is G/H.

Example 2.3.10. The next figure shows different quotients of the form

$$\text{Cayley}(\mathbb{Z}^2, \mathfrak{S}_{c2})/(a,b)\mathbb{Z},$$

with $\gcd(a,b) = 1$ and $a \neq 0 \neq b$, and also the case $a = 0$, $b = 1$.

$a = 0, \quad b = 1$ $\qquad$ $a = 2, \quad b = 1$



$a = 1, \quad b = 1$ $\qquad$ $a = 3, \quad b = 1$

Figure 3

All these graphs are normal covering space of $\vee_{k=1}^{2} S_{a_k}^1$ with automorphism group

$$Aut\left(\text{Cayley}(\mathbb{Z}^2, \mathfrak{S}_{c2})/(a,b)\mathbb{Z}, \rho\right)$$

isomorphic to $\mathbb{Z}$.

Definition 2.3.11. A *path* $\alpha_{v,w}$ in a graph Γ connecting the vertex v to the vertex w is a finite sequence of edges $\{e_j\}_{j=1,\dots,n}$, such that $\cup_j^n \bar{e_j}$ is connected and contains v and w. The *length* $|\alpha_{v,w}|$ of $\alpha_{v,w}$ is the number of differents edges in $\{e_j\}$.

Definition 2.3.12. (Word metric) Let Cayley$(G, \mathfrak{S})$ be a Cayley graph with vertices V and edges E. Then the map

$$\begin{array}{lll} d : V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & min\{r \in \mathbb{N} | \quad r = |\alpha_{v,w}| \quad \text{for some path} \quad \alpha_{v,w}\}. \end{array}$$

The map d is a metric on V. A ball $B(N,v)$ of radius $N \in \mathbb{N}$ at a vertex v in this metric will be the union $\cup_w \alpha_{v,w}$, where $|\alpha_{v,w}| \leq N$.

Let $\rho : \tilde{S} \to S$ be a normal covering of a orientable closed surface S and $G = Aut(\tilde{S}, \rho)$ be the group of automorphisms of the cover. We know that G is generated by the automorphisms

$$\psi_j : (\tilde{S}, \tilde{o}) \to (\tilde{S}, \tilde{o}_j),$$

where $\tilde{o}_j$ is the end point of the lift in a fixed point $\tilde{o} \in \rho^{-1}(o)$ of a generator γ_j of $\pi_1(S, o)$. Therefore, we can associate the Cayley graph of G with generating set $\{\psi_j\}$ to the cover $\rho : \tilde{S} \to S$ as follows:

* its set of vertices is $\rho^{-1}(o)$.
* its set of edges is the lift of γ_j at $\tilde{o}$ with end point $\psi_j(\tilde{o})$, where $\tilde{o} \in \rho^{-1}(o)$ and ψ_j defined as above.

When S is compact the group G is called cocompact.

Definition 2.3.13. Let G be a group acting properly discontinuous on a surface $\tilde{S}$. If G is cocompact we define ends of G as follows

$$\mathrm{Ends}(G) = \mathrm{Ends}(\tilde{S})$$

.

The definition above is equivalent to [Lo, Definition 8.2.1] but phrased in a slightly different manner, more adapted for our purposes.

We can associate a surface $\tilde{S}$ to a Cayley graph of a finitely generated group G with generating set $\{a_j\}_{j=1,\dots,k}$ as follows. Take an embedding of $\bigvee_j S^1_{a_j}$ in $\mathbb{R}^3$ and a tubular neighborhood N of it. The boundary of N is a compact surface S. We take the normal cover $\tilde{S}$ of S with automorphism group isomorphic to G. This implies that any finitely generated group G is the group of automorphisms of a normal covering $\tilde{S} \to S$ of a compact Riemann surface.

Hopf [Hp] proved that any normal cover $\tilde{S}$ of a compact space S must have either zero or one or two or a Cantor set of ends. From the definition of $\mathrm{Ends}(G)$, it follows the next result.

Theorem 2.3.14 (Possible number of ends of groups.)**.** *[Lo, Theorem 8.2.8] Let G be a finitely generated group. Then G has 0, 1, 2 or infinitely many ends.*

This facts and the classification of open surfaces lead to the following statement.

Theorem 2.3.15. *[G1, Theorem 15.2] If S is an infinite normal covering surface of an orientable closed surface Σ_g, with $g > 0$, then S is homeomorphic to one of the following six surfaces:*

1) *the plane,*
2) *the Loch Ness monster, i.e. a plane having infinite handles,* $\mathrm{Ends}(S) = \mathrm{Ends}'(S)$,
3) *the cylinder,*
4) *the Jacob's ladder, i.e. a cylinder having handles converging to both ends,* $\mathrm{Ends}(S) = \mathrm{Ends}'(S)$,
5) *the Cantor tree, i.e. a sphere without a Cantor set or a branching tree,*
6) *the blooming Cantor tree, i.e. a branching tree with handles converging to each end,* $\mathrm{Ends}(S) = \mathrm{Ends}'(S)$.

We will also make use of the following nontrivial result.

Theorem 2.3.16. *[Lo, Theorem 8.2.9]*

1. *A finitely generated group has no ends if and only if it is finite.*

2. *A finitely generated group has exactly two ends if and only if it is virtually* $\mathbb{Z}$.

3. Stallings's decomposition theorem *A finitely generated group has infinitely many ends if and only if it is a nontrivial free amalgamated product over a finite group or if it is a nontrivial HNN-extension over a finite group.*

Remark 2.3.17. From item *3.* it follows that abelian groups G have no ends, one end or two ends.

2.4 Normal and abelian covers

We will now restrict our attention to normal covers of orientable compact surfaces of genus g minus n points, $\Sigma_{g,n}$.

A normal cover $\rho_G : \tilde{X}_G(\Sigma_{g,n}) \to \Sigma_{g,n}$ is quotient of the universal cover $\tilde{\Sigma}_{g,n}$ of $\Sigma_{g,n}$ by normal subgroups $G < \pi_1(\Sigma_{g,n}, o)$. They are given by covering maps $\pi_G : \tilde{\Sigma}_{g,n} \to \tilde{X}_G(\Sigma_{g,n})$, with automorphism group $Aut(\tilde{\Sigma}_{g,n}, \pi_G)$ isomorphic to G. We associate the quotient Cayley graph

$$\text{Cayley}(\pi_1(\Sigma_{g,n}, o))/G$$

to each $\tilde{X}_G(\Sigma_{g,n})$. Notice that G is the kernel of the morphism

$$\begin{array}{llll} \varrho_G : \pi_1(\Sigma_{g,n}, o) & \to & Aut(\tilde{X}_G(\Sigma_{g,n}), \rho_G) \\ \gamma & \mapsto & \quad \psi_\gamma, \end{array} \tag{2.1}$$

where ψ_γ is the automorphism taking $\tilde{o}_0 \in \rho_G^{-1}(o)$ to the end point $\tilde{o}_1 \in \rho_G^{-1}(o)$ of the lift $\tilde{\gamma}$ of γ starting at $\tilde{o}_0$.

Definition 2.4.1. An *abelian cover* of a manifold S will be a normal covering space

$$\rho_{\tilde{G}} : \tilde{X}_{\tilde{G}}(S) \to S,$$

such that the group $Aut(\tilde{X}_{\tilde{G}}(S), \rho_{\tilde{G}})$ is an abelian group. Let G be an abelian subgroup of $\pi_1(S)/[\pi_1(S), \pi_1(S)]$ corresponding to the subgroup $\tilde{G} < \pi_1(S)$, we will denote by

$$\rho_G : A_G(S) \to S$$

the above abelian cover. We will write it simply $A(S)$ when G is trivial, and in this case $A(S)$ is called the maximal abelian cover.

Example 2.4.2. The universal cover of $\bigvee^n S^1$ is the Cayley graph of the free group F_n generated by a set of n elements, $\mathfrak{S}_n = \{a_1, \ldots, a_n\}$. Abelian covers of $\bigvee^n S^1$ come from the quotient of $\text{Cayley}(F_n, \mathfrak{S}_n)$ by normal subgroups $\tilde{G} \lhd$

$\pi_1(\bigvee^n S^1) = F_n$ containing the commutator subgroup $[F_n, F_n]$. The group of automorphism of the cover $\text{Cayley}(F_n, \mathfrak{S}_n)/\tilde{G}$ with projection

$$\rho_{\tilde{G}} : \text{Cayley}(F_n, \mathfrak{S}_n)/\tilde{G} \to \bigvee_{k=1}^{n} S^1_{a_k}$$

is isomorphic to group $F_n/\tilde{G}$. As the quotient group $F_n/[F_n, F_n]$ is isomorphic to $\mathbb{Z}^n$ we have that the maximal abelian cover $A(\bigvee^n S^1)$ is $\text{Cayley}(\mathbb{Z}^n, \mathfrak{S}_{cn})$. According to Proposition 2.3.3 the abelian covers $A_G(\bigvee^n S^1)$ are associated to the quotient of $\text{Cayley}(\mathbb{Z}^n, \mathfrak{S}_{cn})$ by subgroups G of $\mathbb{Z}^n$. We can visualize $\text{Cayley}(\mathbb{Z}^n, \mathfrak{S}_{cn})$ as $\mathbb{Z}^n$ inside of $\mathbb{R}^n$ and $\mathfrak{S}_{cn}$ as translations by vectors of the canonical basis of $\mathbb{R}^n$, i.e. $\mathfrak{S}_{cn} = \{v_j = (\delta_{1j}, \dots, \delta_{nj})\}_{j=1}^n$. The case $n = 2$ is illustrated by Figures 2 and 3. In what follows, $\text{Cayley}(\mathbb{Z}^n)$ denotes the Cayley graph $\text{Cayley}(\mathbb{Z}^n, \mathfrak{S}_{cn})$.

2.4.1 Proof of Theorem 2.0.1

Note that for finite normal covers of $\Sigma_{g,n}$, we can complete these to finite ramified covers. Thus the topological classification of finite normal covers is given by the Riemann-Hurwitz formula. To prove Theorem 2.0.1 about infinite normal covers, we will describe the sets $\text{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ and $\text{Ends}'(\tilde{X}_G(\Sigma_{g,n}))$, which are topological invariants of the classification of open orientable surfaces.

We begin by recalling some definitions and facts.

Definition 2.4.3. Let K be a compact bordered subsurface in S, it is called *canonical subsurface* if it has the following properties:

i) the closure of each connected component U of $S - K$ is non-compact and meets K in exactly one simple closed curve.

ii) each connected component of $S - K$ is either planar or of infinite genus.

Moreover, a collection $K_0, K_1, \dots$ of canonical subsurfaces of S such that $S = \cup_{j=0}^{\infty} K_j$ and $K_j \subset \text{Int}(K_{j+1})$, it is called *canonical exhaustion*.

When $n > 0$ there is a continuous retraction R from $\Sigma_{g,n}$ to the wedge sum of $2g + (n - 1)$ circles $S^1_{a_k}$ touching at a single point $o \in \Sigma^{\circ}_{g,n}$, denoted by $\bigvee_{k=1}^{2g+n-1} S^1_{a_k}$. The map

$$R : \Sigma_{g,n} \to \bigvee_{k=1}^{2g+n-1} S^1_{a_k} \tag{2.2}$$

induces an isomorphism between the fundamental groups $\pi_1(\Sigma_{g,n}, o)$ and $\pi_1(\bigvee_{k=1}^{2g+n-1} S^1_{a_k}, o)$. The group is generated by the homotopy class of each circle $S^1_{a_k}$ and it is the free group F_{2g+n-1},

$$F_{2g+n-1} := \left\{ b_1^{x_1} \cdots b_k^{x_k} \cdots b_l^{x_l} \middle| x_k \in \mathbb{Z}, \quad b_k \in \{a_1, \dots, a_{2g+n-1}\} \quad \text{and} \quad l \in \mathbb{N} \right\}.$$

Definition 2.4.4. Let γ be a simple closed curve in a non-compact surface $\Sigma_{g,n} = \Sigma_{g,0} - \{p_1, \dots, p_n\}$ around a point $p_j \in \{p_1, \dots, p_n\}$, we call it a *border cycle*. Its homotopy class $[\gamma]$ is called a *boundary class* (which will be confounded with the curve itself).

Now, we can state our first result.

Lemma 2.4.5. *Let G be the kernel of ϱ_G, defined as in 2.1, and let γ a border cycle of $\Sigma_{g,n}$, $n > 0$. If there exists $a \in \mathbb{Z}^*$ such that $\gamma^a \in G$ then* $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ *contains a subset of planar ends with discrete topology. Otherwise, if there is no border cycle with this property then* $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ *is a single point.*

Proof. Let $\tilde{\gamma}$ the lift of γ in $\tilde{X}_G(\Sigma_{g,n})$ through a point $\tilde{o} \in \rho_G^{-1}(o)$. Assume that $\gamma^a \in G$ for some $a \in \mathbb{Z}^*$ and $|a|$ is minimal with this property . Therefore $\tilde{\gamma}^a$ is a closed curve and a finite cover of γ, which is boundary of a pointed disk $\mathbb{D}^* = \mathbb{D} - p$ in $\Sigma_{g,n}$. The inverse image of $\mathbb{D}^*$ under ρ_G has a connected component $\tilde{D}$, whose boundary in $\tilde{X}_G(\Sigma_{g,n})$ contains $\tilde{\gamma}^a$. Hence $\tilde{D}$ is a finite normal cover of $\mathbb{D}^*$. Therefore $\tilde{D}$ give us an element open and closed in $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$, which is planar. In the same manner we can see that each connected component of $\rho_G^{-1}(\mathbb{D}^*)$ give us an planar end, which is an element open and closed in $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$, hence $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ contains a subset of planar ends with discrete topology.

In the case that no boundary class is in the kernel G, we take a compact subset K in $\tilde{X}_G(\Sigma_{g,n})$ and let Γ be the Cayley graph corresponding to $\tilde{X}_G(\Sigma_{g,n})$. By [H, Proposition 1.33] there exists a lifting of the retraction R (2.2)

$$\tilde{R} : \tilde{X}_G(\Sigma_{g,n}) \to \Gamma,$$

Since Γ is the Cayley graph of $\pi_1(\Sigma_{g,n}, o)/G$ there exist a compact K_Γ such that the image $\tilde{R}(K) \subset K_\Gamma$ and either $\Gamma - K_\Gamma$ is connected or it has finite connected components. Therefore there is a compact connected neighbourhood K' of K_Γ in $\tilde{X}_G(\Sigma_{g,n})$ such that $K \subset K'$ and $\tilde{R}|_{K'}$ is a retraction from K' to K_Γ. Since $\rho(K')$ is compact in $\Sigma_{g,n}$, we are able to choose a representative γ of a border cycle which is simple (no self-intersections) does not meet $\rho(K')$. Since no boundary class is in G, we have that $\rho^{-1}(\gamma)$ is the union of unbounded curves, which does not intersect K'. Thus we can follow an unbounded connected component of $\rho^{-1}(\gamma)$ to connect any two points in $\tilde{X}_G(\Sigma_{g,n}) - K'$. We can apply Lemma 2.2.12 to conclude that $\mathrm{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ is a single point. □

Remark 2.4.6. Think of the surface $\Sigma_{g,n}$, with $g, n > 0$, as a punctured polygon $P_{4g}^n \subset P_{4g}$ of $4g$ edges without n points $\{p_1, \dots, p_n\}$ of its interior and with boundary $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. We can choose the generators of $\Sigma_{g,n}$ to be $\{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_{n-1}\}$, where a_j, b_k correspond to the edges of P_{4g}^n and the curves c_j ,with $j = 1, \dots, n$ are the cycles around each point p_j. These generators of $\pi_1(\Sigma_{g,n})$ will be called *canonical generators* and they have the following intersection indices:

$$a_j \cap b_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases},$$

the autointersection index is zero for any curve a_j, b_j and c_j and

$$c_j \cap a_k = 0 = c_j \cap b_k \quad \text{for any} \quad \begin{cases} j \in \{1, \ldots, n\} \\ k \in \{1, \ldots, g\} \end{cases} .$$

To prove the Theorem 2.0.1, we need a better understanding of the set End$'$ of normal covers.

Lemma 2.4.7. *For any infinite normal cover* $\tilde{X}_G(\Sigma_{g,n})$ *the set* Ends$'(\tilde{X}_G(\Sigma_{g,n}))$ *is empty, a single point, two points or infinitely many ends.*

Proof. We can attach the border cycles $\{c_j^g\}_{j=1,\ldots,n}$ of $\Sigma_{g,n}$ to the border cycles $\{c_j^0\}_{j=1,\ldots,n}$ of $\Sigma_{0,n}$ to obtain the closed surface $\Sigma_{g+n-1} = \Sigma_{g+n-1,0}$. It is possible to choose the canonical generators of $\pi_1(\Sigma_{g,n})$ such that under the monomorphism $i_* : \pi_1(\Sigma_{g,n}) \to \pi_1(\Sigma_{g+n-1})$ they coincide with the complement of the subset $\{b'_{g+1}, \ldots, b'_{g+n-1}\}$ of the canonical generators $\{a'_j, b'_j\}_{j=1,\ldots,g+n-1}$ of $\pi_1(\Sigma_{g+n-1})$, i.e. $i_*(a_j^g) = a'_j$, $i_*(b_j^g) = b'_j$ and $i_*(c_j^g) = a'_{g+j}$. Therefore we can embed any normal cover $\tilde{X}_G(\Sigma_{g,n})$ in an normal cover $\tilde{X}_H(\Sigma_{g+n-1})$ where H is the subgroup of $\pi_1(\Sigma_{g+n-1})$ which is the normal closure of the set $\{i_*(G)\} \cup \{b'_{g+1}, \ldots, b'_{g+n-1}\}$. The last condition combined with Proposition 2.3.3 ensure that the automorphism group of both covers are isomorphic. Since the surface $\tilde{X}_H(\Sigma_{g+n-1})$ is a normal cover, the Theorem 2.3.15 shows that $\tilde{X}_H(\Sigma_{g+n-1})$ is homeomorphic to one of following six surfaces: the plane, the Loch Ness monster, the cylinder, the Jacob's ladder, the Cantor tree, or the blooming Cantor tree.

If $\tilde{X}_H(\Sigma_{g+n-1})$ is homeomorphic to the plane, to the cylinder or the Cantor tree then $\tilde{X}_G(\Sigma_{g,n})$ is planar and Ends$'(\tilde{X}_G(\Sigma_{g,n}))$ is empty. Assume $\tilde{X}_H(\Sigma_{g+n-1})$ is homeomorphic to the Jacob's ladder and $\tilde{X}_G(\Sigma_{g,n})$ is nonplanar. There exists a canonical exhaustion $\{K_j\}$ of $\tilde{X}_H(\Sigma_{g+n-1})$ such that $\tilde{X}_H(\Sigma_{g+n-1}) - K_j$ has two connected components P'_j and P''_j and the sequences $\{P'_j\}$ and $\{P''_j\}$ are nested.

If for all border cycles c_l of $\Sigma_{g,n}$ there are some integers $m_l \in \mathbb{Z}^*$ such that $m_l c_l$ is contained in G, then is possible to construct a canonical exhaustion $\{K'_j\}$ of $\tilde{X}_G(\Sigma_{g,n})$ as follows

$$K'_j = K_j \cap \tilde{X}_G(\Sigma_{g,n}) - \cup_{l=1}^n \rho_G^{-1}(Q_l^j),$$

where the set $\{Q_j^l\}_{j\in\mathbb{N}}$ is the nested sequence of the planar end of $\Sigma_{g,n}$ surrounded by c_l. The sequences $\{Q'_j\} = \{P'_j \cap \tilde{X}_G(\Sigma_{g,n}) - \cup_l \rho_G^{-1}(\bar{Q}_j^l)\}$ and $\{Q''_j\} = \{P''_j \cap \tilde{X}_G(\Sigma_{g,n}) - \cup_l \rho_G^{-1}(\bar{Q}_j^l)\}$ are representatives of elements in the set of ends of $\tilde{X}_G(\Sigma_{g,n})$.

As $\tilde{X}_G(\Sigma_{g,n})$ is nonplanar, we can find a inclusion ι of the torus without a disk $\mathbb{T}^*$ in $A_G(\Sigma_{g,n})$, which closer in $\tilde{X}_H(\Sigma_{g+n-1})$ is compact. Hopf showed [Hp, Holsfsatz 3',p.90] that if e^* is an end of a normal cover $\tilde{X}_H(\Sigma_{g+n-1})$, for any open P'_l of a representation sequence $\{P'_l\}$ and a compact $\iota(\mathbb{T}^*)$ subset then there is an automorphism $\psi \in Aut(\tilde{X}_H(\Sigma_{g+n-1}), \rho_H)$ such that $\psi(\iota(\mathbb{T}^*)) \subset P'_l$. By the choice of H, the group $Aut(\tilde{X}_G(\Sigma_{g,n}), \rho_G)$ is isomorphic to $Aut(\tilde{X}_H(\Sigma_{g+n-1}), \rho_H)$. Then the sequences $\{Q'_j\}$ and $\{Q''_j\}$ represent elements in $\text{Ends}'(\tilde{X}_G(\Sigma_{g,n}))$. As any connected component of $\rho_G^{-1}(Q^l_j)$ belongs to one of the sets $K_j \cap i_*(\tilde{X}_G(\Sigma_{g,n}))$, which have finite genus, then the set $\text{Ends}'(\tilde{X}_G(\Sigma_{g,n}))$ has only two points represented by the sequences $\{Q'_j\}$ and $\{Q''_j\}$. Now assume that $m_j c_j \notin G$ for any integer $m_j \in \mathbb{Z}^*$ and $j \neq 1$. According to Lemma 2.4.5, $\text{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ has one point,. Consequently then $\text{Ends}'(\tilde{X}_G(\Sigma_{g,n}))$ is one point. If some $m_j c_j$ are in G and others are not, we can plug the holes surrounded by the connected components of $\rho^{-1}(m_j c_j)$ to reduce to the latter case.

The proof for $\tilde{X}_H(\Sigma_{g+n-1})$ homeomorphic to the Loch Ness monster or to the blooming Cantor tree is similar. □

Proof of Theorem 2.0.1. Theorem 2.3.15 gives us the possible infinite normal covers of $\Sigma_{g,0}$. For infinite normal covers $\tilde{X}_G(\Sigma_{g,n})$ of $\Sigma_{g,n}$, when $n \neq 0$, Lemma 2.4.5 implies that $\text{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ is either a single point or an infinite set of points. If $\text{Ends}(\tilde{X}_G(\Sigma_{g,n}))$ is a single point, then $\tilde{X}_G(\Sigma_{g,n})$ is homeomorphic to Loch Ness Monster or the plane. Otherwise Lemma 2.4.7 guaranties that $\tilde{X}_G(\Sigma_{g,n})$ is homeomorphic to one of the surfaces

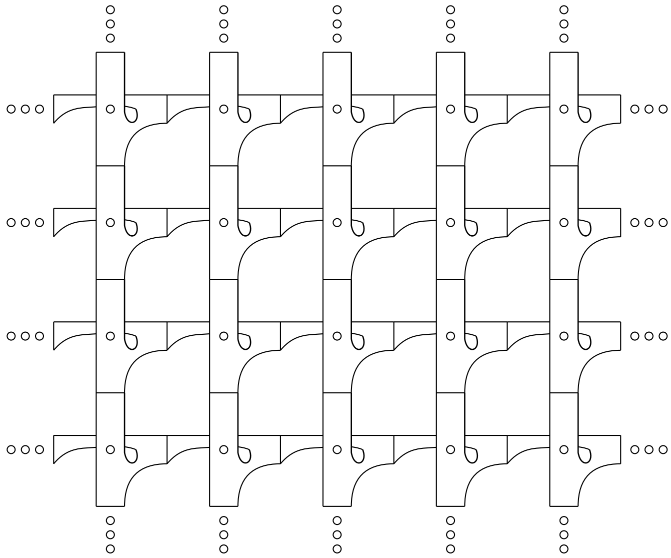
* the plane without an infinite discrete set,
* the Loch Ness monster without an infinite discrete set,
* the Jacob's ladder without an infinite discrete set
* the Cantor tree without an infinite discrete set,
* the blooming Cantor tree without an infinite discrete set.

This completes the proof. □

2.4.2 Abelian covers

In particular, we are interested in a good understanding of the topology of infinite abelian covers of $\Sigma_{g,n}$. The results below give a description of the infinite abelian covers depending of the genus g of $\Sigma_{g,n}$.

Theorem 2.4.8. *If $A_G(\Sigma_{0,n})$ is an infinite abelian cover of $\Sigma_{0,n} \simeq \mathbb{D} - \{p_1, \ldots, p_{n-1}\}$ and if $n \geq 3$ then $A_G(\Sigma_{0,n})$ is homeomorphic to one of the surfaces 2,7 or 8 of the list of Theorem 2.0.1.*



$A(\Sigma_{0,3})$
Figure 4

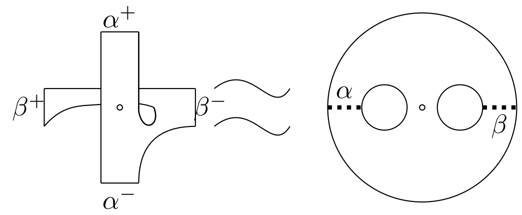
Corollary 2.4.9. *If $A_G(\Sigma_{1,n})$ is an infinite abelian cover of $\Sigma_{1,n}$ and if $n \neq 0$ then it is homeomorphic to one of the surfaces 2,3,7 or 8 of the list of Theorem 2.0.1.*

Corollary 2.4.10. *If $A_G(\Sigma_{g,n})$ is an infinite abelian cover of $\Sigma_{g,n}$ and if $g \geq 2$ then it is homeomorphic to one of the following surfaces 2,4,8 or 9 of the list of Theorem 2.0.1.*

Note that the abelian covers of $\Sigma_{0,1} \simeq \mathbb{D}$ and $\Sigma_{0,2} \simeq \mathbb{D}^*$ are clearly $\mathbb{D}$ or $\mathbb{D}^*$. For $n \geq 3$ the situation is considerably more involved (see Figure 4).

Proof of Theorem 2.4.8. The surface $\Sigma_{0,n}$ is homeomorphic to a disk without $n-1$ points, it has n border cycles γ_j. There is no loss of generality in assuming that the retraction $R : \Sigma_{0,n} \to \bigvee^{n-1} S^1_{a_k}$ identifies $\gamma_1, \dots, \gamma_{n-1}$ with the circles $S^1_{a_1}, \dots, S^1_{a_{n-1}}$. Let us lift γ_j to Cayley$(\mathbb{Z}^{n-1})$. It follows that $\tilde{\gamma}_j = v_j$ for $j = 1, \dots, n-1$ and $\tilde{\gamma}_n$ is homotopic to $\sum_{j=1}^{n-1} v_j$ in the notation of Example 2.4.2.

We first prove that the maximal abelian cover for $\Sigma_{0,3}$ is the Loch Ness monster, based on the proof given in [V, Theorem 1]. Define a fundamental domain U of $\Sigma_{0,3}$ such that it is simply connected and $\bar{U} = \Sigma_{0,3}$, as in the figure below.



Fundamental domain for $\Sigma_{0,3}$
Figure 5

If we attach the fundamental domain U at every vertex of Cayley($\mathbb{Z}^2$), then we obtain Figure 4.

The lifting of the curves $[\gamma_2, \gamma_1] = \gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2$ and $[\gamma_2^{-1}, \gamma_1]$ through the origin $(0,0)$ are closed curves in $A(\Sigma_{0,3})$. The union $V = U_{(0,0)} \cup U_{(1,0)} \cup U_{(0,1)} \cup U_{(1,1)}$ of the fundamental domains lifted to vertices $(0,0), (1,0), (0,1)$ and $(1,1)$ is homeomorphic to a ring. The boundary of V in $A(\Sigma_{0,3})$ contains two edges of kind α^- (as in the Figure 5), each one of them belongs to different components of the border of the ring(see Figure 6). Since the lifting b_2 of $[a_2^{-1}, a_1]$ intersects these edges, it follows that the intersection index of $b_1 \cap b_2$ is 1 (see Figure 6). Hence $A(\Sigma_{0,3})$ has infinite genus.

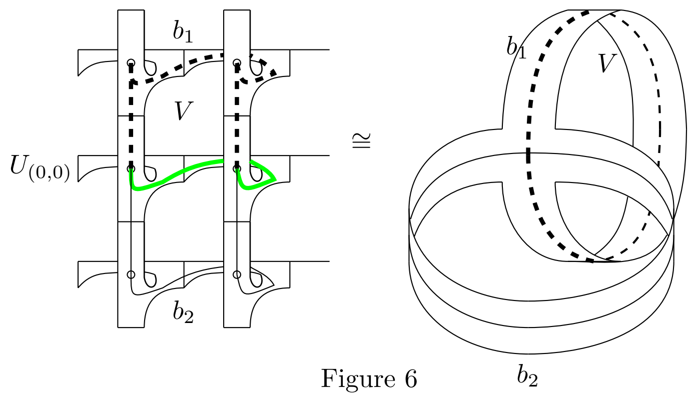
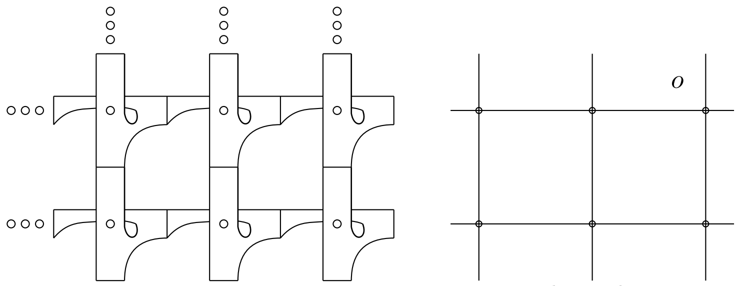


Figure 6

Since we can embed the fundamental domain for $\Sigma_{0,3}$ in the fundamental domain for $\Sigma_{0,n}$, for $n > 3$; the argument above works for the maximal abelian cover of $\Sigma_{0,n}$ for any n. By Lemma 2.4.5 the set End($A(\Sigma_{0,n})$) is a single point, thus $A(\Sigma_{0,n})$ is homoemorphic to the Loch Ness monster.

Let us now deal with the case of abelian covers which are not maximal. The Riemann-Hurwitz formula implies that the only finite covers of $\Sigma_{0,3}$ of genus zero are $\Sigma_{0,4}, \Sigma_{0,6}$. Any other finite cover has genus and contains the subgraph T (see Figure 7). The infinite abelian covers have locally structure of an open subset of a finite cover, thus the existence of genus in $A_G(\Sigma_{0,n})$ depends of the existence of two cycles with intersection 1 and the minimal subgraph which ensures that this is the case, is the subgraph T.

By Lemma 2.4.5 we have two alternatives for $\text{Ends}(A_G(\Sigma_{0,n}))$, it is a single point or it contains an infinite countable set of planar ends with discrete topology. When $\text{Ends}(A_G(\Sigma_{0,n}))$ is a single point the kernel of ϱ_G does not contain any lifting of boundary classes. Thus the kernel of ϱ_G from (2.1) is generated by elements with homotopic type $\sum b_k v_k$, which have at least two coefficients different from zero and if $0 \neq b_k$ for any k then $b_k \neq b_j$ for at least one coefficient. Therefore for each v_k there is at least one $v_{k'}$ such that $\bar{v_k} \cap \bar{v_{k'}} = \{(0, \ldots, 0)\}$ and $\bar{v_{k'}}$ is not a loop. Since v_k^a is not in G for any $a \in \mathbb{Z}^*$, we have that the lifts of γ_k^{-1} and $\gamma_{k'}^{-1} \cdot \gamma_k \cdot \gamma_{k'} \cdot \gamma_k^{-2} \cdot \gamma_{k'}^{-1}$ through $(0, \ldots, 0) = o$ is a subgraph T. Consequently, the quotient graph associated to $A_G(\Sigma_{0,n})$ contains infinite copies of the subgraph T. Whence $A_G(\Sigma_{0,n})$ has infinite genus, and so it is homeomorphic to Loch Ness monster.



subgraph T

Figure 7

When $\text{Ends}(A_G(\Sigma_{0,n}))$ contains an infinite discrete set of planar ends, the genus will depends of the existence of an inclusion of the subgraph T in the quotient graph $\text{Cayley}(\mathbb{Z}^{n-1})/G$. If $A_G(\Sigma_{0.n})$ has an inclusion of T then there is an automorphism ψ of $A_G(\Sigma_{0.n})$ such that it is infinite, hence T has infinite copies. Therefore $A_G(\Sigma_{0.n})$ has infinite genus, otherwise it is planar, i.e. the set $\text{Ends}'(A_G(\Sigma_{0,n}))$ is empty.

Assume $A_G(\Sigma_{0,n})$ has infinite genus. As $A_G(\Sigma_{0,n})$ is an infinite cover we have some boundary class γ_k which is not contained in G. Each inclusion of the subgraph T in $A_G(\Sigma_{0,n})$ intersects a connected component of $\rho_G^{-1}(\gamma_k)$, this means that the genus of $A_G(\Sigma_{0,n})$ accumulates in the corresponding end to $\rho_G^{-1}(\gamma_k)$. Therefore the set $\text{Ends}'(A_G(\Sigma_{0,n}))$ is a single point. □

Example 2.4.11. Take the abelian covers with Cayley graph equal to the graphs of the Figure 3 of the Example 2.3.10. Note that the first two covers do not contain the subgraph T, then they are planar. The last two contain infinite copies of the subgraph T. Therefore, they have infinite genus.

Lemma 2.4.12. *Let $A(\Sigma_{g,n})$ be the maximal abelian cover of $\Sigma_{g,n}$, with $g \geq 1$. Then $A(\Sigma_{g,n})$ is homeomorphic to*

∗ *the Loch Ness monster if $n > 1$ or $n = 0$ and $g > 1$,*

- *the Loch Ness monster without an infinite subset with discrete topology if* $n = 1$ *and* $g > 1$,
- *the plane if* $n = 0$ *and* $g = 1$,
- *the plane without an infinite subset with discrete topology if* $n = g = 1$.

Proof. We first prove that the maximal abelian cover for $\Sigma_{g,0} = \Sigma_g$ is the Loch Ness monster if $g > 1$, based in the proof given in [Ne].

In the notation of Remark 2.4.6, the surface Σ_g can be thought of as $4g$ polygon attached to the connected sum of $2g$ circles $S^1_{a_j}, S^1_{b_j} \subset \Sigma_g$ at point $o \in \Sigma$, $\bigvee_o^g S^1_{a_j} \vee \bigvee_o^g S^1_{b_j}$. The group $\mathbb{Z}^{2g}$ is the automorphism group of the cover $A(\Sigma_g)$. The abelian cover of $\bigvee_o^g S^1_{a_j} \vee \bigvee_o^g S^1_{b_j}$ inside $A(\Sigma_g)$ is Cayley($\mathbb{Z}^{2g}$).

Next, starting at every vertex of the graph Cayley($\mathbb{Z}^{2g}$) we attach a two-cell via the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The resulting two-complex is homeomorphic to $A(\Sigma_g)$. Every edge of $A(\Sigma_g)$ meets a pair of two-cells while every vertex meets $2g$ two-cells. For every compact $K \subset \Sigma_g$ we can construct a compact $K \subset K' \subset \Sigma_g$ such that K' is the union of closed two-cells with nonempty intersection with a closed ball in $\mathbb{R}^{2g}$ and containing K. Therefore $\Sigma_g - K'$ is connected and Ends($A(\Sigma_g)$) is just one point according to Lemma 2.2.12.

The loops $a_1 a_2 a_1^{-1} a_1^{-1} a_2^{-1} a_1$ and $b_1 b_2 b_1^{-1} b_1^{-1} b_2^{-1} b_1$, based at the origin, meet in exactly one point. From this we conclude that there exist a subsurface in $A(\Sigma_g)$ homeomorphic to a torus without a disk. As $A(\Sigma_g)$ has infinite automorphism group then it has infinite genus. Hence $End'(A(\Sigma_g))$ is non empty and we can deduce from Theorem 2.3.15 that $A(\Sigma_g)$ is homomeomorphic to the Loch Ness monster.

We now turn to the case $n > 1$. In this case we have that the boundary classes do not belong to the kernel of

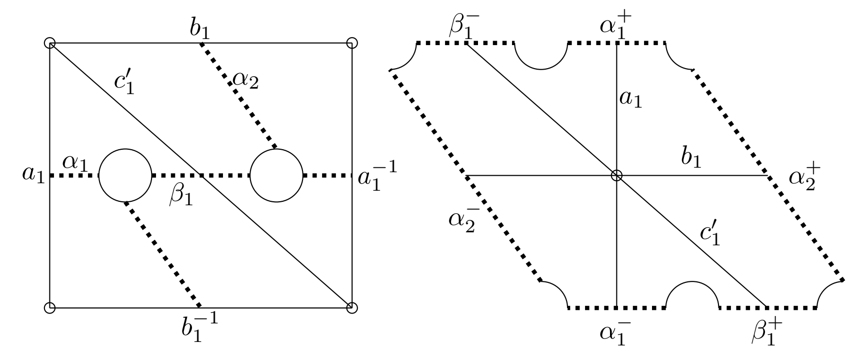
$$\varrho_{[F_{2g+n-1}, F_{2g+n-1}]} : \pi_1(\Sigma_{g,n}, o) \to Aut(A(\Sigma_{g,n}), \rho).$$

Hence $A(\Sigma_{g,n})$ has a single end. As in the proof of the Theorem 2.4.8 we will define a fundamental domain for $\Sigma_{g,n}$ using the punctured polygon P^n_{4g} defined as in Remark 2.4.6. We label the vertices of P^n_{4g} with the numbers $\{1, 2, \ldots, 4g-1, 4g\}$ in anti-clockwise order beginning with the vertex between the edges with label b_g^{-1} and a_1. Without loss of generality we can assume that the points $\{p_1, \ldots, p_n\}$ in the interior of P_{4g} are equidistributed along the diagonal d_1 which joins the vertex 1 to the vertex $2g+1$ and are ordered on d_1 as follows $1, p_1, p_2, \ldots, p_{n-1}, p_n, 2g+1$. The distance between two successive points is $|d_1|/(n+1)$. We define the curves c'_j as the union of the segments $[2, \frac{p_j + p_{j+1}}{2}]$ and $[\frac{p_j+p_{j+1}}{2}, 2g+2]$, where $\frac{p_j+p_{j+1}}{2}$ is the middle point of the segment $[p_j, p_{j+1}]$ and with $j = 1, \ldots, n-1$. Note that the set of cycles $\{a_1, b_1, \ldots, a_g, b_g, c'_1, \ldots, c'_{n-1}\}$ is a base of generators for the group $\pi_1(\Sigma_{g,n})$, such set is called the set of *Cayley generators*. We cut the polygon P^n_{4g} by

- the open segments of line $(p_j, p_{j+1}) = \beta_j$,
- the semiopen segments $[\frac{1+2}{2}, p_1) = \alpha_1$, $[\frac{j+(j+1)}{2}, p_1) = \alpha_l$ for $j \geq 2g+2$ and $4g+1 = 1$,

* the semiopen segments $[\frac{j+(j+1)}{2}, p_n) = \alpha_l$ for $2 \leq j < 2g+2$ and $4g+1 = 1$.

To construct the fundamental domain $U_{g,n}$ of $\Sigma_{g,n}$ we attach the pieces using the same attaching rules of the boundary of P_{4g}, as the Figure 8 shows.



P_4^2 Fundamental domain of $\Sigma_{1,2}$

Figure 8

Take the Cayley graph Cayley($\mathbb{Z}^{2g+n-1}$) with the generators being the Cayley generators of $\pi_1(\Sigma_{g,n})$. Now, take the cycles $\gamma_1 = a_j c_1 a_j^{-2} c_1^{-1} a_j$ and $\gamma_2 = b_j c_1^{-1} b_j^{-2} c_1 b_j$ beginning at origin. Observe that the curves $a_j \cap U_{g,n}$, $b_j \cap U_{g,n}$ have intersection one on $U_{g,n}$, hence $\gamma_1 \cap \gamma_2 = 1$. Therefore the maximal abelian cover $A(\Sigma_{g,n})$ has infinite genus.

It remains to analyze the case when $n = 1$. The surface $\Sigma_{g,1}$ corresponds to a punctured polygon P_{4g}^* of $4g$ edges with a hole inside and with exterior boundary $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The hole corresponds to a border cycle γ_1 homotopic to $a_1 \cdots a_{2g} \cdot a_1^{-1} \cdots a_{2g}^{-1}$. Their maximal abelian covers have Cayley graphs Cayeley($\mathbb{Z}^{2g}$). Attaching infinitely the polygon P_{4g}^* with the graph Cayeley($\mathbb{Z}^{2g}$) as in the proof of the case $n = 0$ we obtain that

* $A(\Sigma_{1,1})$ is homeomorphic to the plane without an infinite discrete set.
* $A(\Sigma_{g,1})$ is homeomorphic to the Loch Ness monster without an infinite discrete set.

□

The case above is slightly different from the case $g = 0$. Note that in the case $g = 0$ the generators of $\pi_1(\Sigma_{g,n})$ are of the same topological kind and in the case $g > 0$ they are not.

Lemma 2.4.13. *If $\Sigma_{g,n}$ with $g > 1$, then any infinite abelian cover $A_G(\Sigma_{g,n})$ has infinite genus.*

Proof. Suppose the assertion of the lemma is false. Let $\{a_j, b_j, c_l\}$ canonical generators. As $a_j \cap U_{g,n}$, $b_j \cap U_{g,n}$ have intersection one on the fundamental domain $U_{g,n}$, defined as above, we have that there are no integers l and k such that a_j^l and b_j^k are in G, since otherwise $A_G(\Sigma_{g,n})$ would have infinite genus. But $g > 1$ since the canonical generators contain the cycles a_1, b_1, a_2 and b_2. Therefore, we take the cycles γ_1 homotopic to $a_1 b_2 a_1^{-2} b_2^{-1} a_1$ and γ_2 homotopic to $b_1 b_2^{-1} b_1^{-2} b_2 b_1$. By the properties of the canonical generators we have that the lifts $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ of γ_1, γ_2 at the same point have intersection index $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = 1$, a contradiction. □

Proof of Corollary 2.4.10. Theorem 2.0.1 and the fact that $A_G(\Sigma_{g,n})$ is an abelian cover show that $A_G(\Sigma_{g,n})$ is homeomorphic to one of the following real surfaces: the plane, the cylinder, the Loch Ness monster, the Jacob's ladder or one of these without an infinite discrete set. Lemma 2.4.13 reduces the last list in the Loch Ness monster, the Jacob's ladder or one of these without an infinite discrete set. □

Lemma 2.4.14. *Let $\{a_1, b_1, c_1, \ldots, c_{n-1}\}$ be a canonical generators of $\pi_1(\Sigma_{1,n}, o)$ and $A_G(\Sigma_{1,n})$ an abelian cover of $\Sigma_{1,n}$. If one of the following conditions holds*

* *$a_1^j, b_1^k \in G$ for some integers $j, k \in \mathbb{Z}$,*
* *$n > 2$ and some c_l holds $c_l^{-1}, c_l \notin G$.*

Then $A_G(\Sigma_{1,n})$ has genus different from zero.

Proof. We will prove that $A_G(\Sigma_{1,n})$ contains couples of cycles with intersection index 1.

Assume the condition $a_1^j, b_1^k \in G$ for some integers $j, k \in \mathbb{Z}$, which are minimal with this property. Let γ_1 and γ_2 be the lifts of a_1^j, b_1^k at $\tilde{o} \in \rho_G^{-1}(o)$ respectively. Since $a_1^j, b_1^k \in G$ and j, k are minimimal with this property, the curves γ_1, γ_2 are closed and only intersect at $\tilde{o}$. By Remark 2.4.6 the intersection at $\tilde{o}$ is transversal. Then the abelian cover $A_G(\Sigma_{1,n})$ has couples of cycles with intersection index equal to one. Hence $g(A_G(\Sigma_{1,n})) \neq 0$.

We now turn to the case $n > 2$ and for some c_l holds $c_l^{-1}, c_l \notin G$. The construction of the couple of cycles with intersection 1 depends on whether a_1^j, b_1^k are in G or not. We will give the cycles for each case:

a) $a_1^j, b_1^k \notin G$ for every integers $j, k \in \mathbb{Z}$. We take the lifts γ_1, γ_2 at a point $\tilde{o} \in \rho_G^{-1}(o)$ of $a_1 c_l a_1^{-2} c_l^{-1} a_1$ and $b_1 c_l^{-1} b_1^{-2} c_l b_1$, respectively.

b) $a_1^j \in G$ and $b_1^k \notin G$ for every integers $k \in \mathbb{Z}$. We take the lifts γ_1, γ_2 at a point $\tilde{o} \in \rho_G^{-1}(o)$ of $a_1 c_l a_1^{-2} c_l^{-1} a_1$ and $b_1 c_l^{-1} a_1^2 b_1^{-2} c_l a_1^2 b_1$, respectively.

c) $b_1^k \in G$ and $a_1^j \notin G$ for every integers $j \in \mathbb{Z}$. This case is analogues to *b*).

The curves γ_1, γ_2 are lifts of elements in G, thus they are closed. Considering that the automorphisms $\psi_{a_1}, \psi_{b_1}, \psi_{c_l}$, defined as in (2.1), are not trivial, we have that γ_1, γ_2 intersect only at $\tilde{o}$. By Remark 2.4.6 the intersection is transversal, which completes the proof. □

The hypothesis $g > 1$ in Lemma 2.4.13 is necessary because for $\Sigma_{1,n}$ the abelian covers with group G generated by $\{b_1, c'_1, \ldots, c'_{n-1}\}$ are planar with $\mathrm{Ends}(A_G(\Sigma_{1,n}))$ infinite and with discrete topology.

Proof of Corollary 2.4.9. The existence of genus in $A_G(\Sigma_{1,n})$ for arbitrary G will depend on the construction of cycles with intersection 1 from the lifts of a_1 and b_1 adding lifts of c_k, when at last one of this cycle is not trivially contained in G, i.e. $c_k \notin G$. By Lemma 2.4.7 the set $\mathrm{Ends}'(A_G(\Sigma_{1,n}))$ has cardinality less or equal to 1, and the corollary follows. □

Chapter 3

Leaves of logarithmic foliations on surfaces

This chapter presents a topological description of generic leaves of dimension one holomorphic foliations on projective surfaces, which are orientable real surfaces. In particular, we prove that the generic leaf of a Riccati foliation is homemomorphic to one of the real surfaces listed in Theorem 2.0.1. In a similar way we get that the generic leaves of a homogeneous foliations on the projective plane are homeomorphic to the real surfaces given by Theorem 2.4.8. Also, we show in Theorem 3.4.7 that a generic leaf of a sufficiently generic logarithmic foliations on the projective plane is homeomorphic to the Loch Ness monster.

3.1 Riccati foliations

Let $\mathcal{F}$ be a Riccati foliation on a compact complex surface M, with adapted fibration $\pi : M \to \Sigma_g$. Except for a finite number of invariant fibres, say $\pi^{-1}(p_1), \ldots, \pi^{-1}(p_k)$, all the other leaves of $\mathcal{F}$ are covering spaces of $\Sigma_g - \{p_1, \ldots, p_k\}$. The latter set will be denoted by $\Sigma_{g,k}$. We can apply Theorem 2.0.1 to describe the topology of the non-algebraic leaves of $\mathcal{F}$.

Theorem 3.1.1. *Let $\mathcal{F}$ be a Riccati foliation (singular or not) on a compact complex surface X. Assume that global holonomy*

$$\rho : \pi_1(\Sigma_{g,k}) \to \mathrm{PSL}(2, \mathbb{C})$$

of $\mathcal{F}$ is infinite. Then any leaf of $\mathcal{F}$ outside a countable set of leaves is homeomorphic to one of the following real surfaces:

1) the plane,

2) the Loch Ness monster,

3) the cylinder,

4) the Jacob's ladder,

5) the Cantor tree,

6) the blooming Cantor tree,

7) the plane without an infinite discrete set

8) the Loch Ness monster without an infinite discrete set,

9) the Jacob's ladder without an infinite discrete set

10) the Cantor tree without an infinite discrete set,

11) the blooming Cantor tree without an infinite discrete set.

Furthermore, any two leaves outside of that countable set are biholomorphic.

Proof. Throughout the proof, $\mathbb{P}^1$ denotes a regular fiber $\pi^{-1}(o)$, with $o \in \Sigma_{g,k}$. Let $G = \rho(\pi_1(\Sigma_{g,k}))$ be the global holonomy group of $\mathcal{F}$. Take $p \in \mathbb{P}^1$ with an infinite G orbit and consider the leaf $\mathcal{L}_p$ of $\mathcal{F}$ through p. It is a covering space of $\Sigma_{g,k}$, whose covering map is the restriction of π to $\mathcal{L}_p$. Note that the isotropy group $\mathrm{Iso}_G(p) = \{g \in G | g(p) = p\}$ is a subgroup of $\pi_*(\pi_1(\mathcal{L}_p))$.

Since G is countable and any nontrivial element in G has at most two fixed points, it follows that there exists a countable set $\mathcal{C} \subset \mathbb{P}^1$ such that for every $p \in \mathbb{P}^1 - \mathcal{C}$ the group $\mathrm{Iso}_G(p)$ reduces to the identity. Thus any two leaves $\mathcal{L}_p$ and $\mathcal{L}_q$ with $p, q \in \mathbb{P}^1 - \mathcal{C}$ are normal covers of $\Sigma_{g,k}$ with isomorphic groups of covering transformations. It follows that $\mathcal{L}_p$ and $\mathcal{L}_q$ are biholomorphic. Theorem 2.0.1 now shows that these leaves are homeomorphic to one of the real surfaces of the list in this theorem. □

Corollary 3.1.2. *Under the hypotheses of Theorem 3.1.1. If the global holonomy is abelian then, except for a finite number of leaves corresponding to $\mathcal{F}$-invariant fibres and leaves through finite orbits of ρ, any other leaf is homeomorphic to one of the real surfaces 1,2,3,4,7,8 or 9 of the list of Theorem 3.1.1.*

Proof. As in the proof of Theorem 3.1.1, $\mathbb{P}^1$ denotes a regular fiber $\pi^{-1}(o)$, with $o \in \Sigma_{g,k}$. Since the global holonomy group G is infinite and abelian, there are not many options for the finite orbits of G. If there is a finite set on $\mathbb{P}^1$ of cardinality $n \geq 3$ invariant by G then we can map G in the symmetric group S_n, and since an automorphism of $\mathbb{P}^1$ with three fixed points must be the identity we obtain that this map is injective contradiction with the hypothesis on G. Therefore the finite orbits must correspond to one or two common invariant points for all elements of G, which will denote by $I \subset \mathbb{P}^1$. If I has length one then G is a subgroup of the affine group $\mathrm{Aff}(\mathbb{C})$. When G has two invariant points it is a subgroup of the multiplicative group $\mathbb{C}^*$ or of the dihedral group $\mathbb{C}^* \ltimes \mathbb{Z}_2 = \{z \mapsto \lambda z^{\pm 1}; \lambda \in \mathbb{C}^*\}$.

Note that the isotropy group $\mathrm{Iso}_G(p)$ of any point $p \in \mathbb{P}^1 - I$ is trivial, thus the leaf $\mathcal{L}_p$ through p is an abelian cover of $\Sigma_{g,k}$. It follows from Theorem 2.4.8 and Corollaries 2.4.9 and 2.4.10 that $\mathcal{L}_p$ must be homeomorphic to one of the real surfaces 1,2,3,4,7,8 or 9 of the list of Theorem 3.1.1. □

Example 3.1.3. Let Σ_2 the compact Riemann surface of genus 2 and $\{a_1, b_1, a_2, b_2\}$ be the canonical generators of $\pi_1(\Sigma_2, o)$ for a point o in Σ_2. Consider the homomorphism

$$\rho : \pi_1(\Sigma_2, o) \to \mathrm{PSL}(2, \mathbb{C})$$

defined by

$$\begin{array}{lcl} \rho(a_1) : \mathbb{P}^1 & \to & \mathbb{P}^1 \\ z & \mapsto & z + 1, \end{array}$$

and the biholmorphisms $\rho(b_1)$, $\rho(a_2)$ and $\rho(b_2)$ are the identity. From the suspension of this homomorphism we get a non singular Riccati foliation. Since $\rho(\pi_1(\Sigma_2, o))$ is isomorphic to $\mathbb{Z}$, Theorem 2.3.16 shows that any generic leaf $\mathcal{L}$ of $\mathcal{F}$ has two ends. We conclude from Lemma 2.4.13 that $\mathcal{L}$ has infinite genus. Therefore $\mathcal{L}$ is homeomorphic to Jacob's ladder.

Example 3.1.4. Let $M, \mathcal{F}$ as in the above example. Jacob's ladder without an infinite discrete set is obtained from a bimeromorphism $\varphi : M \to M'$, which is the composition of one blow-up b in a regular point p in the intersection of a fibre F with the infinite section and one blow-down on the closure of $b^{-1}(F) - E$, where $E = b^{-1}(p)$ (see [Bn, p.54]). The bimeromorphism sends a trivial neighborhood $U \subset M$ of a fibre F to a trivial neighborhood $U' \subset M'$ with an induced foliation, which has two singularities on the fibre F', one logarithmic and one dicritical. The holonomy around F' is trivial. The image of the infinite section is invariant by the new foliation and passes through the logarithmic singularity. Therefore the generic leaf $\mathcal{L}$ is a normal cover of $\Sigma_{2,1}$. Considering that the holonomy of the border cycle is trivial, we have that $\mathcal{L}$ is homeomorphic to Jacob's ladder deprived from a discrete set.

Example 3.1.5. For each $x \in \mathbb{C}$, let Γ_x be the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by

$$e_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

If $|x| \geq 2$, the ping-pong lemma (see [Lo, Theorem 4.4.1]) implies that the subgroup Γ_x is free of rank two.

Consider the suspension of the homomorphism

$$\begin{array}{lcl} \rho : \pi_1(\Sigma_2, o) & \to & \mathrm{PSL}(2, \mathbb{C}) \\ a_1 & \mapsto & e_1 \\ a_2 & \mapsto & e_2 \\ b_j & \mapsto & Id, \end{array}$$

where a_j and b_j, $j = 1, 2$, are the canonical generators of $\pi_1(\Sigma_2, o)$. This suspension gives a nonsingular Riccati foliation $\mathcal{F}$ on a compact complex surface M with adapted fibration $\pi : M \to \Sigma_2$.. Since Γ_x is the group of deck transformation of the cover $\pi|_{\mathcal{L}} : \mathcal{L} \to \Sigma_2$, where $\mathcal{L}$ is a generic leaf of $\mathcal{F}$, Theorem 2.3.16 shows that $Ends(\mathcal{L})$ is infinite. Considering that $\mathcal{L}$ is a normal cover of a compact surface, Theorem 2.3.15 yields $Ends(\mathcal{L})$ is a Cantor set. Figure 8 shows that the union of the lifts of a fundamental piece D_2 at each vertex of a ball $B(N, \tilde{o})$ of the associated Cayley graph Cayley(F_2) is homeomorphic to sphere without $3^{N-1}(3 + 1)$ disks, therefore $\mathcal{L}$ is planar.

Notice that $\{\overline{B(N,\tilde{o})}\}$ is a canonical exhaustion. Consequently $\mathcal{L}$ is homeomorphic to the Cantor tree.

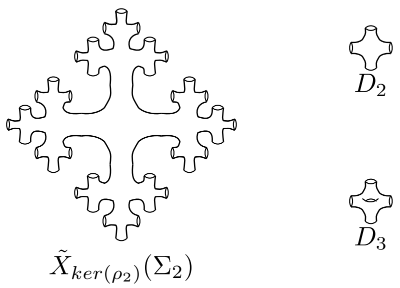


Figure 8

In a similar way the blooming Cantor tree is obtained as generic leaf given by the suspension of the foliation on $\tilde{X}_u(\Sigma_3) \times \mathbb{P}^1$ with leaves $\tilde{X}_u(\Sigma_3) \times \{w\}$ by the homomorphism

$$\begin{array}{lcl} \varrho : \pi_1(\Sigma_3, o) & \to & \mathrm{PSL}(2,\mathbb{C}) \\ a_1 & \mapsto & e_1 \\ a_2 & \mapsto & e_2 \\ a_3 & \mapsto & Id \\ b_j & \mapsto & Id, \end{array}$$

where a_j and b_j, $j = 1,2,3$, are the canonical generators of $\pi_1(\Sigma_3, o)$. The fundamental piece D_3 in this case has genus 1. Thus the lifts of a fundamental piece D_3 at the vertices of a ball $B(N,\tilde{o})$ of the associated Cayley graph Cayley(F_2) is homeomorphic to the sphere with $1 + \sum_{k=2}^{N-1} 3^{k-1}(3+1)$ handles and $3^{N-1}(3+1)$ open disks removed (see Figure 0).

The cases of the Cantor tree and blooming Cantor without an infinite discrete set are obtained in a similar way to Example 3.1.4.

3.2 Homogeneous foliations

We will now consider foliations $\mathcal{F}$ in $\mathbb{P}^2$ which are defined in an affine chart by a homogeneous 1-form

$$\omega = h_1(x,y)dx + h_2(x,y)dy,$$

where h_1, h_2 are homogeneous polynomials of the same degree ν and without common factors. Let $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be the radial vector field. If $\omega(R) = 0$ then ω must be a complex multiple of $xdy - ydx$ and the foliation defined by it is the pencil of lines through zero. From now assume that $\omega(R) \neq 0$.

Theorem 3.2.1. *Let $\omega = h_1(x,y)dx + h_2(x,y)dy$ be a homogeneous 1-form of degree at least one on $\mathbb{C}^2$. Assume that $\gcd(h_1,h_2) = 1$ and $\omega(R) \neq 0$. Consider the foliation $\mathcal{F}$ on $\mathbb{C}^2$ defined by it. Except for a finite number of leaves homeomorphic to $\mathbb{C}^*$ contained in lines through the origin all the other leaves of $\mathcal{F}$ are biholomorphic. Moreover they are homeomorphic to one of the following alternatives:*

1) the plane,

2) the Loch Ness monster,

3) the plane without an infinite discrete set,

4) the Loch Ness monster without an infinite discrete set.

5) a compact Riemann surface less a finite set.

Proof. The foliation defined by $\mathcal{F}$ extends to a foliation of $\mathbb{P}^2$ of degree $\deg(h_1) = \deg(h_2)$ leaving the line at infinity invariant. After making a linear change of coordinates we can assume that x does not divide $\omega(R) = xh_1 + yh_2$. The roots of the polynomial

$$Q_{\nu+1}(x, y) = xh_1(x, y) + yh_2(x, y) = \prod_1^k (y - t_j x)^{\nu_j}$$

correspond to $\mathcal{F}$-invariant lines through zero with finite slope. Blowing-up $(0, 0) \in \mathbb{C}^2$ we obtain a Riccati foliation $\mathcal{F}'$ which, in the coordinates x and $t = y/x$, is defined by the 1-form

$$\frac{p^*\omega}{t^{\nu+1}} = (Q_{\nu+1}(1, t)dx + xh_2(1, t)dt)$$

Moreover, the roots $\{t_j\}$ of $Q_{\nu+1}(1, t)$ are the projection of the invariant fibres on the exceptional divisor E, which contains exactly two singularities, one at the line at infinity and one at the exceptional divisor. If the root is simple then the singularities are logarithmic. In case the root has multiplicity two or higher, then the corresponding singular fibre contains two saddle nodes with the same multiplicity, whose strong separatrices are contained in the fibre.

The holonomy around each singular fiber $\pi^{-1}(t_j)$ fixes two points, which correspond to the line at infinity and the exceptional divisor. Then it belongs to the group $\mathbb{C}^*$. Thus the global holonomy $\mathrm{Hol}(E)$ of $\mathcal{F}'$ is abelian and contained in $\mathbb{C}^*$. Except for the leaves corresponding to invariant fibres, the exceptional divisor and the line at infinity, all the other leaves are homeomorphic to an abelian cover of $\Sigma_{0,k}$ with the same covering group. If the global holonomy is finite then the coverings are homeomorphic to compact Riemann surface without a finite number of points. If instead the global holonomy is infinite then the possibilities are covered by Theorem 3.1.1. Notice that the cases of Jacob's ladder and Jacob's ladder without an infinite discrete set do not appear in our list because $g(E) = 0$, see Theorem 2.4.8. □

Example 3.2.2. Let $\mathcal{F}$ be a homogeneous foliation defined by a homogeneous 1-form

$$\omega = ydx + \lambda xdy.$$

If $\omega(R) \neq 0$ and $\lambda \notin \mathbb{Q}$, the above theorem shows that the leaves $\mathcal{L}$ of $\mathcal{F}$ are biholomorphic to an infinite normal cover of $\mathbb{C}^*$. Moreover, $\mathcal{L}$ is the universal cover of $\mathbb{C}^*$. Then it is biholomorphic to $\mathbb{C}$.

In the case λ is a rational number $\mathcal{L}$ is biholomorphic to $\mathbb{C}^*$.

Example 3.2.3. Consider a homogeneous foliation $\mathcal{F}$ on $\mathbb{P}^2$ defined by the closed logaritmic 1-form

$$\omega = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} + \lambda_3 \frac{d(y-x)}{(y-x)},$$

where $\sum \lambda_j = 1$. Note that the generic leaf is an abelian cover of the sphere without three points, $\Sigma_{0,3}$. If the set $\{\lambda_1, \lambda_2, \lambda_3\}$ is $\mathbb{Z}$ linearly independent, we deduce that the Cayley graph associated to the generic leaf $\mathcal{L}$ as abelian cover of $\Sigma_{0,3}$ is Cayley($\mathbb{Z}^2$). Hence $\mathcal{L}$ is homeomorphic to the maximal abelian cover $A(\Sigma_{0,3})$, see proof of Theorem 2.4.8. Therefore $\mathcal{L}$ is homeomorphic to the Loch Ness Monster.

Requiring for the set $\{\lambda_1, \lambda_2, \lambda_3\}$ to satisfy:

$$\lambda_1 \in \mathbb{R} - \mathbb{Q}, \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = -\lambda_1,$$

we obtain that the generic leaf $\mathcal{L}$ as abelian cover has Cayley graph Cayley($\mathbb{Z}^2$)$/(0,1)$. This graph does not contain the subgraph T (see Figure 7), so that it is a planar surface. In the notation of Example 2.4.2, $(0,1)$ is a border cycle. Lemma 2.4.5 now shows that Ends($\mathcal{L}$) contains an infinite discrete set of planar ends. Hence $\mathcal{L}$ is homeomorphic to the plane without an infinite discrete set.

Assuming the data is

$$\lambda_1 \in \mathbb{R} - \mathbb{Q}, \quad \lambda_2 = \frac{1}{n} \quad \text{with} \quad n \geq 2 \quad \text{and} \quad \lambda_3 = 1 - \lambda_1 - \lambda_2.$$

It follows that $\mathcal{L}$ has Cayley graph Cayley($\mathbb{Z}^2$)$/(0,n)$. Since $n \geq 2$, we see that Cayley($\mathbb{Z}^2$)$/(0,n)$ contains the subgraph T. By Lemma 2.4.5, Ends($\mathcal{L}$) contains an infinite discrete set of planar ends. Consequently, $\mathcal{L}$ is homeomorphic to the Loch Ness monster without an infinite discrete set.

3.3 Abelian holonomy in arbitrary dimension

Let $\mathcal{F}$ be a singular foliation by curves on an arbitrary complex manifold M. Assume that $\mathcal{F}$ leaves invariant a compact curve $C \subset M$ and that the holonomy of $\mathcal{F}$ along C is abelian.

To detect topology on leaves of $\mathcal{F}$ near C, we will use the topological description of abelian covers of bordered surfaces obtained in Chapter 2.

Definition 3.3.1. Let τ be a cross section to $\mathcal{F}$-invariant curve C at a regular point $o \in C$ and let $\gamma = \gamma_{\sigma_1}^{\beta_1} \cdots \gamma_{\sigma_k}^{\beta_k}$ be a closed path, where $\{\gamma_j\}_{j \in \Lambda}$ are a canonical generators of $\pi_1(C - \mathrm{Sing}(\mathcal{F}), o)$, and $\beta_l \in \mathbb{Z}, \sigma_l \in \Lambda$. Let $p \in \tau$, $l \in \mathbb{N}$ and the leaf $\mathcal{L}_p$ of $\mathcal{F}$ through p. We define $B(l, p, \tau)$ to be

$$\cup_{|\gamma| \leq l} \tilde{\gamma}_p(I),$$

where $\tilde{\gamma}_p : [0,1] \to \mathcal{L}_p$ is the lift of γ to $\mathcal{L}_p$ at p and $|\gamma| = \sum_{j=1}^{k} |\beta_j|$. If $\tilde{\gamma}_p$ is well defined for each γ, such that $|\gamma| < l$, we call $B(l, p, \tau)$ the *graph ball* in $\mathcal{L}_p$ of radius l and center p.

Lemma 3.3.2. *Let $\mathcal{F}$ be a holomorphic foliation of dimension 1 of a complex manifold M. If a compact Riemann surface $C \subset M$ is $\mathcal{F}$-invariant and not contained in* $\mathrm{Sing}(\mathcal{F})$ *and* $\mathrm{Hol}(C,\mathcal{F})$ *is infinite. Then for each $N \in \mathbb{N}$ there is an embedding*

$$\varepsilon : (B(N, v_0), v_0) \to (\mathcal{L}_p, p),$$

where $B(N, v_0)$ is a ball in the word metric of a related Cayley graph of a normal cover of $C - \mathrm{Sing}(\mathcal{F})$ and $\mathcal{L}_p$ is a leaf through a regular point $p \in M - C$ sufficiently close to C.

Proof. Identify $C - \mathrm{Sing}(\mathcal{F})$ with $\Sigma_{g,n}$. Let τ be a cross section to C at the regular point $o \in \Sigma_{g,k}$ and $\{\gamma_j\}_{j=1}^{2g+n-1}$ be Cayley generators of $\pi_1(\Sigma_{g,n}, o)$. Since $\mathrm{Hol}(C,\mathcal{F})$ is infinite, for each $N \in \mathbb{N}$ there exists a point $p \in \tau$ sufficiently close to o such that a holonomy map h_γ on $\gamma \in \pi_1(\Sigma_{g,n}, o)$, satisfying $|\gamma| < N$, is well defined. Therefore $B(N, p, \tau)$ is a graph ball in $\mathcal{L}_p$. The principal ingredient to prove the existence of an embedding ε is that any neighbourhood of o in τ has points p, whose group $\mathrm{Iso}_{\mathrm{Hol}(C,\mathcal{F})}(p) = \{g \in \mathrm{Hol}(C,\mathcal{F}) | g(p) = p\}$ is trivial [Go, Proposition 2.7]. We will denote by G the kernel of the holonomy representation $\mathrm{Hol}(C,\mathcal{F})$ (1.1). Choose a vertex v_0 in the Cayley graph $\mathrm{Cayley}(\pi_1(\Sigma_{g,n}, o))/G$ and define a function

$$\begin{array}{lcl} \tilde{\epsilon} : \mathrm{Vertices}(B(N, p, \tau)) & \to & \mathrm{Vertices}(B(N, v_0)) \\ p & \mapsto & v_0 \\ \tilde{\gamma}_p(1) & \mapsto & \tilde{\gamma}_{v_0}(1), \end{array}$$

where $\tilde{\gamma}_{v_0}(1)$ is the lift of γ in $\mathrm{Cayley}(\pi_1(\Sigma_{g,n}, o))/G$ at the vertex v_0. Suppose $\tilde{\epsilon}(\tilde{\gamma}_p(1)) = \tilde{\epsilon}(\tilde{\gamma}'_p(1)) = v_q$, then $\tilde{\gamma}_{v_0} \cdot \tilde{\gamma}'^{-1}_{v_q}(1) = v_0$, i.e. $\gamma \cdot \gamma'^{-1} \in G$. Since $B(N, p, \tau)$ is well defined and $\mathrm{Iso}_{\mathrm{Hol}(C,\mathcal{F})}(p)$ is trivial, we have $h_{\gamma \cdot \gamma'}$ is a trivial map and $h_{\gamma'^{-1}} \circ h_{\gamma\cdot\gamma'} = h_\gamma$. Therefore $\tilde{\epsilon}$ is bijective. Since the lifts of the paths γ_j at the points $\tilde{\gamma}_p(1)$, $|\gamma| < N$, are edges of $B(N, p, \tau)$, we can extend $\tilde{\epsilon}$ to a graph isomorphism

$$\begin{array}{lcl} \epsilon : B(N, p, \tau) & \to & B(N, v_0) \\ p & \mapsto & v_0 \\ edge(\tilde{\gamma}_p(1), \widetilde{\gamma \cdot \gamma_j}^{\pm 1}_p(1)) & \mapsto & edge(\tilde{\gamma}_{v_0}(1), \widetilde{\gamma \cdot \gamma_j}^{\pm 1}_{v_0}(1)). \end{array}$$

Hence ϵ^{-1} is a homeomorphism of graphs. Since $B(N, p, \tau)$ is compact in M, the homeomorphism $\varepsilon = \epsilon^{-1}$ is an embedding. □

Let $\mathcal{F}$ and C be as above. Identify $\mathbb{Z}^{2g+n-1}$ with the abelianization of $\pi_1(C - \mathrm{Sing}(\mathcal{F}))$ and denote by ϱ the morphism from $\mathbb{Z}^{2g+n-1}$ to $\mathrm{Hol}(C,\mathcal{F}) \subset \mathrm{Diff}(\mathbb{C}^{\dim M - 1}, 0)$ induced by the holonomy representation of $\mathcal{F}$ along C:

$$\begin{array}{rcl} \varrho : \mathbb{Z}^{2g+n-1} & \to & \mathrm{Hol}(C,\mathcal{F}) \\ v_j & \mapsto & h_{\gamma_j}, \end{array} \tag{3.1}$$

where γ_j are the Cayley generators of $\pi_1(C - \mathrm{Sing}(\mathcal{F}))$.

Theorem 3.3.3. *Let $\mathcal{F}$ be a holomorphic foliation of dimension one of a complex manifold M with $\mathcal{F}$-invariant compact Riemann surface C. Assume that the set $C \cap \mathrm{Sing}(\mathcal{F})$ has cardinality n and C has genus g. If $\mathrm{Hol}(C,\mathcal{F})$ is abelian and the abelian cover $A_G(C - \mathrm{Sing}(\mathcal{F}))$ of infinite genus, where $G = \ker(\varrho)$ defined as (3.1), then $\mathcal{F}$ has leaves of arbitrary genus. Moreover, if $\mathrm{Hol}(C,\mathcal{F})$ is linearizable then there are leaves of infinite genus.*

Proof. Let τ be a cross-section to C at a regular point o of $\mathcal{F}$. By the above lemma, for each $N \in \mathbb{N}$ there exists a point $p \in \tau$ such that $B(N, p, \tau)$ is a graph ball in $\mathcal{L}_p$ homeomorphic to the graph ball $B(N, v_0)$ in the quotient graph Cayley$(\mathbb{Z}^{2g+n-1})/G$ associated with $A_G(C - \text{Sing}(\mathcal{F}))$. Let $U_{g,n}$ be a fundamental domain of $C - \text{Sing}(\mathcal{F}) = \Sigma_{g,n}$ such that it is open, simply connected and $\bar{U}_{g,n} = \Sigma_{g,n}$. Choose an open subset $U \subset \Sigma_{g,n}$ homeomorphic to $\Sigma_{g,n}$, such that the lift V_q through q of the intersection $U_{g,n} \cap U = V$ is well defined at each point $q \in \text{Vertices}(B(N, p, \tau))$.

Since $U_{g,n}$ is homeomorphic to each V_q and the attachment rules on the boundaries $\partial U_{g,n}(v)$, $\partial V_q \cap U$ match. Therefore the embedding ε of the above lemma extends to the interior of surfaces

$$\begin{aligned} S(N, p, \tau) &= \overline{\cup V_{\varepsilon(v)}} \qquad && \varepsilon(v) \in V(B(N, p, \tau)) \\ S(N, v_0) &= \overline{\cup U_{g,n}(v)} \qquad && v \in V(B(N, v_0)). \end{aligned}$$

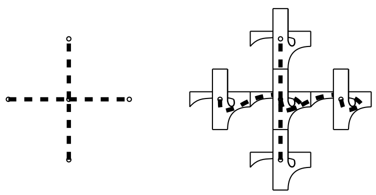
Considering that $A_G(\Sigma_{g,n})$ has infinite genus, there exists a minimal $N_0 \in \mathbb{N}$ such that $S(N_0, v_0)^\circ$ has genus different of zero, $g(S(N_0, v_0)^\circ) \neq 0$. Since $A_G(\Sigma_{g,n})$ is normal, it follows that

$$g(S((a+1)N_0, v_0)) > g(S(aN_0, v_0)) + g(S(N_0, v_0))$$

for any $a \in \mathbb{N}_{>0}$. Therefore, for $p \in \tau$ sufficiently close to C with $\text{Iso}_{\text{Hol}(C,\mathcal{F})}(p)$ trivial there exists an embedding

$$\varepsilon_N : S(N, v_0) \to S(N, p, \tau) \subset \mathcal{L}_p$$

with genus $g(S(N, p))$ arbitrary large.



$B(1, v_0)$ $S(1, v_0)$

Figure 9

Assume that $\text{Hol}(C, \mathcal{F})$ is linearizable. If $\text{Hol}(C, \mathcal{F})$ has a contracting map h then for a leaf $\mathcal{L}_p$ intersecting τ in p, the intersection $\mathcal{L}_p \cap \tau$ has points arbitrarily close to C. Since $\text{Hol}(C, \mathcal{F})$ is abelian and Iso(p) is trivial, we have that $\text{Iso}(\text{h}_\gamma(\text{p}))$ is trivial. Thus we can embed in $\mathcal{L}_p$ a surface $S(N, h^m(p), \tau)$ for any $N \in \mathbb{N}$. It follows that $\mathcal{L}_p$ has unbounded genus.

Otherwise, when $\text{Hol}(C, \mathcal{F})$ has not a contracting map there is a point $p \in \tau$ such that $h(p) \in \tau$ is well defined for all $h \in \text{Hol}(C, \mathcal{F})$ and Iso(p) is trivial. We can now proceed analogously to the above case to show that the leaf $\mathcal{L}_p$ has unbounded genus. $\square$

3.4 Generic logarithmic foliations

We will describe the topological invariants of generic leaves of generic logarithmic foliations in $\mathbb{P}^2$.

3.4.1 Ends

We begin with a description of the ends of generic leaves of logarithmic foliations. In what follows we will say end of a leave $\mathcal{L}$ for an element of a boundary component of $\mathcal{L}$ (see Definition 2.2.3).

Lemma 3.4.1. *Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{P}^2$ with polar divisor D. Let $\mathcal{L}$ be a non-algebraic leaf of $\mathcal{F}$. If E is an end of $\mathcal{L}$ then either locally the leaf $\mathcal{L}$ is a separatrix of a singularity of $\mathcal{F}$ on the complement of D, or $\overline{E} \cap D \neq \emptyset$.*

Proof. The divisor D has at least two irreducible components. Divide the set of irreducible components in two sets, say D_0 and D_∞. Let F_0 and F_∞ be homogeneous polynomials of the same degree on $\mathbb{C}^3$ defining, respectively, D_0 and D_∞. The quotient $\frac{F_0}{F_\infty}$ defines a non-constant holomorphic map $F : U \to \mathbb{C}^*$, where $U = \mathbb{P}^2 - D$ is the complement of D in $\mathbb{P}^2$.

Let K be a compact subset of $\mathcal{L}$. Let E be an end of $\mathcal{L}$ contained in a connected component of $\mathcal{L} - K$ such that the boundary $\partial_{\mathcal{L}} E$ in $\mathcal{L}$ is compact. The restriction of F to E is a holomorphic function $f : E \to \mathbb{C}$. If f is constant then $\mathcal{L}$ is an irreducible component of a fibre of the rational function and hence is algebraic contradicting our assumptions. So $f : E \to \mathbb{C}$ is a non-constant holomorphic function.

Let $V = f(E) \subset \mathbb{C}$ be the image of f. Since f is holomorphic and non-constant V is an open subset of $\mathbb{C}$. If it contains ∞ or 0 in its closure then the lemma follows by continuity, the closure of the end E intersects D_∞ or D_0 respectively.

Assume from now on that $\infty, 0 \notin \overline{f(E)}$.

Let $\mathcal{G}$ be the restriction of $\mathcal{F}$ to $U - K = U'$. The boundary $\partial E = \overline{E} - E$ in U' is mapped by f to ∂V, the boundary of V. We point out that ∂E is invariant by $\mathcal{G}$, see [Cnd-Cln, Proposition 4.1.11]. If ∂E reduces to a point the end in question accumulates at one of the finitely many singularity of $\mathcal{F}$ in U. If instead the boundary contains infinitely many points then it follows that $F(\partial E)$ contains infinitely many points. Therefore $\mathcal{G}$ contains infinitely many leaves contained in fibres of F. Thus $\mathcal{F}$ has infinite algebraic leaves. Jouanolou's Theorem implies that every leaf of $\mathcal{F}$ is algebraic, contradicting again our assumptions. □

Definition 3.4.2. Let ω be a closed meromorphic 1-form on a complex manifold M with polar divisor D . If $o \in M - D$ then we can define a *multivalued primitive* for ω on $M^* = M - D$ by the formula

$$F(z) = \int_{\gamma_{o,z}} \omega, \tag{3.2}$$

where $\gamma_{o,z} : I \to M^*$ is a path joining o to z. Let $\mathcal{U}$ a fundamental domain of M^*, such that it is open and simply connected. If $o \in \mathcal{U}$ then $F|_{\mathcal{U}}$ defines a primitive for ω in $\mathcal{U}$. Passing to the universal covering $u : \tilde{M}^* \to M^*$ and choosing a point $\tilde{o}$ in $u^{-1}(o)$, we define a *primitive* for $u^*\omega$ as the function

$$\begin{array}{rcl} \tilde{F} : \tilde{M}^* & \to & \mathbb{C} \\ z & \mapsto & \int_{\gamma_{\tilde{o},z}} u^*\omega, \end{array}$$

where $\gamma_{\tilde{o},z} : I \to \tilde{M}^*$ is any path joining $\tilde{o}$ to z. Observe that the following equation holds

$$\tilde{F}|_{\mathcal{U}_{\tilde{o}}} = F|_{\mathcal{U}} \circ u.$$

Definition 3.4.3. Let $\mathcal{F}$ be a foliation on M defined by closed meromorphic 1-form with polar divisor D. We will say that a leaf $\mathcal{L}$ of $\mathcal{F}$ is a *generic leaf* if for some connected component C of $u^{-1}(\mathcal{L})$ the value of $\tilde{F}$ on C is a regular value of $\tilde{F}$.

Theorem 3.4.4. *[Pa2, Theorem B] Let ω be a closed logarithmic 1-form with poles on a simple normal crossing divisor D in M^n. Suppose that the residues λ_j of ω are non vanishing, and that for any pair of irreducible components D_j, D_l of D with non empty intersection, the ratios λ_j/λ_l are not negative real numbers. Let F be a multivaluated primitive of ω. Then there exists a fundamental system of neighborhoods of D in M in which the fibres of $F : M - D \to (\mathbb{C},+)/H$ are connected, where H is the subgroup of $(\mathbb{C},+)$ generated by $\{2\pi i\lambda_j\}$*

Proposition 3.4.5. *Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{P}^2$ defined by a closed logarithmic form ω with normal crossing polar divisor D and the ratios λ_j/λ_l are not negative real numbers. Let $\mathcal{L}$ be a non-algebraic leaf of $\mathcal{F}$. If $\mathcal{L}$ is a generic leaf of $\mathcal{F}$ then* Ends($\mathcal{L}$) *is a single point.*

Proof. Suppose the proposition is false. Then we could find a regular leaf $\mathcal{L}$ and a compact subset K of $\mathcal{L}$ such that the complement of K in $\mathcal{L}$ has two connected components E_1 and E_2. By Theorem 3.4.4 there is a neighbourhood U of D whose intersections with the fibres of F (3.2) are connected. We can choose U such that the intersection $U \cap K$ is empty. Since $\mathcal{L}$ is a generic leaf, Lemma 3.4.1 shows that the intersections $E_j \cap U$, $j = 1, 2$, are not empty. Thus E_1 and E_2 intersect in the connected set $U \cap \mathcal{L}$, a contradiction. □

3.4.2 Topology of leaves

Proposition 3.4.6. *Let $\mathcal{F}$ be a logarithmic foliation defined by a closed logarithmic 1-form ω on a projective surface M. Assume that the polar divisor $D = \cup_{j=1}^n D_j$ of ω is non empty and is a simple normal crossing divisor. Let λ_j the residue of ω on D_j. If the residues $\lambda_j/\lambda_l \in \mathbb{C} - \mathbb{R}$ and the irreducible components D_j satisfy*

- *(*) if D_j has genus different from zero, $g(D_j) \neq 0$, the intersection* Sing($\mathcal{F}$)$\cap D_j$ *is not empty.*
- *(**) if D_j has genus zero, the intersection* Sing($\mathcal{F}$)$\cap D_j$ *contains at least three different points.*

Then any generic leaf $\mathcal{L}$ has infinite genus.

Proof. From Lemma 3.4.1, it follows that a generic leaf $\mathcal{L}$ intersects a component of any neighborhood of the polar divisor D. Thus it is possible to find a cross section τ to one of the irreducible components D_j of D at a regular point $o \in D_j$ such that the intersection $\mathcal{L} \cap \tau$ is not empty. Theorem 1.3.2 now shows that the holonomy group $\mathrm{Hol}(D_j, \mathcal{F})$ is abelian, in particular, it is a subgroup of $\mathbb{C}^*$. The latter property implies

$$\mathrm{Iso}_{\mathrm{Hol}(D_j)}(p) = Id$$

for each point $p \in \tau - o$. Since λ_k/λ_j does not belong to $\mathbb{R}$, there is a map $h_\gamma \in \mathrm{Hol}(D_j, \mathcal{F})$ such that $\{h_\gamma^n(p)\}_{n\in\mathbb{N}}$ accumulate at o, for any point $p \in \tau \cap \mathcal{L}$.

Now, we consider three cases: the case when genus of D_j is greater than 1, equal to 1, or 0.

When $g(D_j) \geq 2$ any infinite abelian cover of $D_j - \mathrm{Sing}(\mathcal{F}) = \Sigma_{g,n}$ has infinite genus by Lemma 2.4.13. Since $\lambda_k/\lambda_j \notin \mathbb{R}$ the holonomy h_{c_l} in the border cycle c_l is hyperbolic, Theorem 3.3.3 shows that $\mathcal{L}$ has infinite genus.

The case $g(D_j) = 1$ is slightly different. Let

$$\varrho : \mathbb{Z}^{2n+g-1} \to \mathrm{Hol}(C, \mathcal{F})$$

defined as (3.1). The only way for $A_{ker\varrho}(D_j - \mathrm{Sing}(\mathcal{F}))$ being a planar surface is that for every border cycle c_l of $D_j - \mathrm{Sing}(\mathcal{F})$ the holonomy h_{c_l} is the identity map, see Lemma 2.4.14. This contradicts our assumption $\lambda_k/\lambda_j \notin \mathbb{R}$. Therefore $A_{ker\varrho}(D_j - \mathrm{Sing}(\mathcal{F}))$ has infinite genus.

We now turn to the case $g(D_j) = 0$. Since the ratios $\lambda_k/\lambda_j \notin \mathbb{R}$ and $\mathrm{Sing}(\mathcal{F}) \cap D_j$ has at least three points, we have that the decomposition

$$\mathrm{Hol}(D_j, \mathcal{F}) \simeq \mathbb{Z}^r \oplus \mathbb{Z}_q,$$

has rank at least two, i.e. $r \geq 2$. Therefore the Cayley graph associated to $A_{ker\varrho}(D_j - \mathrm{Sing}(\mathcal{F}))$ contains infinite copies of the subgraph T. Thus it has infinite genus.

Theorem 3.3.3 implies infinite genus in any generic leaf $\mathcal{L}$. □

Theorem 3.4.7. *Let $\mathcal{F}$ be a logarithmic foliation defined by a closed logarithmic 1-form ω on $\mathbb{P}^2$. Assume that the polar divisor $D = \cup_{j=0}^k D_j$ of ω is supported on $k + 1 > 3$ curves and has only normal crossing singularities. If the residues $\lambda_j/\lambda_l \in \mathbb{C} - \mathbb{R}$, then a generic leaf $\mathcal{L}$ of $\mathcal{F}$ is homeomorphic to the Loch-Ness monster.*

Proof. Let D_1 be an irreducible component of D. By Bézout's Theorem D_1 satisfies the hypotheses of Proposition 3.4.6, for this reason a regular leaf has infinite genus. Proposition 3.4.5 shows that a regular leaf $\mathcal{L}$ has one end. Hence $\mathcal{L}$ is homeomorphic to the Loch Ness Monster. □

Remark 3.4.8. Consider the case when the polar divisor D of a closed logarithmic 1-form ω is supported in 3 lines. Note that ω satisfies the hypotheses of Theorem 3.2.1. If the residues $\lambda_j/\lambda_l \in \mathbb{C} - \mathbb{R}$ of ω, then a generic leaf of the foliatin defined by ω is an infinite normal cover of $\mathbb{C}^*$. Consequently, a generic leaf is homeomorphic to $\mathbb{C}$.

Chapter 4

Leaves of logarithmic foliations on projective spaces

In this chapter we study the topology of a general leaf $\mathcal{L}$ of a logarithmic foliation $\mathcal{F}$ on the projective space $\mathbb{P}^{n+1}$ $(n > 1)$, defined in homogenous coordinates by the closed logarithmic 1-form

$$\omega = (\prod_{j=0}^{k} F_j) \sum_{j=0}^{k} \lambda_j \frac{dF_j}{F_j},$$

where $F_j \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ are homogeneous polynomials of degree d_j, the hypersurfaces $D_j := \{F_j = 0\}$ are smooth and the residues satisfy $\sum_{j=0}^{k} \lambda_j d_j = 0$. We will provide answers to the following questions concerning the topology of the generic leaf $\mathcal{L}$.

(1) Is the generic leaf $\mathcal{L}$ simply connected ?

(2) If $n > 1$ and $H \subset \mathbb{P}^{n+1}$ is a sufficiently general hyperplane, are the fundamental groups of $\mathcal{L}$ and of $\mathcal{L} \cap H$ isomorphic?

Question (2) was raised by Dominique Cerveau in [C, Section 2.10].

4.1 Main results

The following results answer the questions (1,2) when $\mathcal{F}$ is sufficiently generic.

Theorem 4.1.1. *Let $\mathcal{F}$ be a logarithmic foliation defined by a logarithmic 1-form ω on $\mathbb{P}^{n+1}$, $n \geq 2$. If the polar divisor $D = \sum_{j=0}^{k} D_j$ of ω is a simple normal crossing; then the fundamental group of a generic leaf $\mathcal{L}$ of $\mathcal{F}$ is* *isomorphic to the subgroup G of $\pi_1(\mathbb{P}^{n+1} - D)$ defined by*

$$G := \left\{ (m_0, \ldots, m_k) \in \mathbb{Z}^{k+1}/(d_0, \ldots, d_k)\mathbb{Z} | \sum_{j=0} \lambda_j m_j = 0 \right\}, \qquad (4.1)$$

where d_j is the degree of the irreducible component D_j of D and λ_j is the residue of ω around D_j.

A positive answer to Question (1) when $\mathcal{F}$ is generic and the subgroup G (4.1) is trivial. This leaves us a glimpse of a relation between the fundamental group of the complement of D in $\mathbb{P}^{n+1}$ ($\pi_1(\mathbb{P}^{n+1} - D)$) and the fundamental group of a generic leaf of $\mathcal{F}$.

A Zariski-Lefschetz type Theorem [HmLe, Theorem 0.2.1] provides an isomorphism between the homotopy groups

$$\pi_l((\mathbb{P}^{n+1} - D) \cap H) \cong \pi_l(\mathbb{P}^{n+1} - D),$$

for H hyperplane sufficiently general, $l \leq (n-1)$ and $n \geq 1$. This theorem will allow us to prove the following result.

Theorem 4.1.2. *Let $\mathcal{F}$ be a logarithmic foliation defined by a logarithmic 1-form ω on $\mathbb{P}^{n+1}$, $n \geq 1$, with a simple normal crossing polar divisor $D = \sum_{j=0}^k D_j$. Let $H \subset \mathbb{P}^{n+1}$ be an hyperplane such that $H \cap D$ is a reduced divisor with simple normal crossings in H. Suppose the leaves $\mathcal{L}, \mathcal{L} \cap H$ are generic leaves of $\mathcal{F}, \mathcal{F}|_H$ respectively. Then the morphism between homotopy groups*

$$(i)_* : \pi_l(\mathcal{L} \cap H) \to \pi_l(\mathcal{L}),$$

induced by the inclusion $i : \mathcal{L} \cap H \hookrightarrow \mathcal{L}$ are

(*) isomorphisms if $l < n-1$,

(*) epimorphisms if $l = n-1$.

This provides a positive answer to Question (2) when $n > 2$.

4.2 Homotopy Theory

In this section we recall some definitions and results from Homotopy Theory that will be used throughout this chapter. We will not provide proofs, for details see [H, Chapter 4].

Definition 4.2.1. Let X a topological space with basepoint x_0, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where the homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all $t \in [0,1]$ and $I^n = [0,1]^n$. The set $\pi_n(X, x_0)$ ($n \geq 1$) has group structure with operation

$$(f * g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n), & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \ldots, s_n), & s_1 \in [\frac{1}{2}, 1]. \end{cases}$$

The definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-connected components of X.

Definition 4.2.2. The **relative homotopy group** $\pi_n(X, A, x_0)$ for a pair (X, A) with a base point $x_0 \in A$ is defined by the set of homotopy classes of maps

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0),$$

where $J^{n-1} = \partial I^n - I^{n-1}$ with $I^{n-1} = \{(s_1, \dots, s_n) \in I^n | s_n = 0\}$.

Lemma 4.2.3. *The inclusions*

$$i : (A, x_0) \hookrightarrow (X, x_0), \quad j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$$

and the restriction of $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ *to* $(\partial I^n, J^{n-1})$ *define the following long exact sequence of homotopy groups*

$$\cdots \to \pi_{k-1}(X, x_0) \to \pi_{k-1}(X, A, x_0) \to \pi_{k-2}(A, x_0) \to \cdots \pi_0(X, x_0).$$

Definition 4.2.4. A space X with basepoint x_0 is said to be n**-connected** if $\pi_k(X, x_0) = 0$ for $k \leq n$. Analogously, the pair (X, A) is called n-connected if $\pi_k(X, A, x_0) = 0$ for all $x_0 \in A$ with $0 < k \leq n$ and for $i = 0$ ($\pi_0(X, A, x_0) = 0$) means that each path-component of X contains points in A.

Lemma 4.2.5. *The following four conditions are equivalent, for* $k > 0$*:*

1) *Every map* $(D^k, \partial D^k) \to (X, A)$ *is homotopic relative to* ∂D^k *to a map* $(D^k, \partial D^k) \to (A, A)$.

2) *Every map* $(D^k, \partial D^k) \to (X, A)$ *is homotopic through such maps to* ∂D^k *to a map* $(D^k, \partial D^k) \to (A, A)$.

3) *Every map* $(D^k, \partial D^k) \to (X, A)$ *is homotopic relative to* ∂D^k *to a constant map* $(D^k, \partial D^k) \to (A, A)$.

4) $\pi_k(X, A, x_0) = 0$ *for all* $x_0 \in A$.

From now on we will assume that the spaces X and A are path connected. The base point x_0 will be omitted from the notation.

Lemma 4.2.6. *If* $\rho : (\tilde{X}, \tilde{A}) \to (X, A)$ *is a covering space with* $\tilde{A} = \rho^{-1}(A)$*, then the map* $\rho_* : \pi_n(\tilde{X}, \tilde{A}) \to \pi_n(X, A)$ *is an isomorphism for all* $n > 1$.

Lemma 4.2.7 (Transitivity)**.** *Suppose* $W \subset V \subset U$*, and* (V, W) *is* l*-connected. Then* (U, V) *is* l*-connected if and only if* (U, W) *is* l*-connected.*

Lemma 4.2.8 (Deformation)**.** *Suppose* (U, V) *is a pair, and* $U' \subset U$*, and* $V' \subset U' \cap V$*. Suppose* $f : U \times [0, 1] \to U$ *is a continuous map such that* $f(u, 0) = u$*,* $f(V \times [0, 1]) \subset V$*,* $f(U' \times [0, 1]) \subset U'$*,* $f(V, 1) \subset V'$*, and* $f(U, 1) \subset U'$*. Then* (U, V) *is* l*-connected if and only if* (U', V') *is* l*-connected.*

Lemma 4.2.9 (Excision I)**.** *Suppose* $W \subset V \subset U$*. Suppose that the* U*-closures* $\overline{W}$ *and* $\overline{U - V}$ *are disjoint. Then* (U, V) *is* l*-connected if and only if* $(U - W, V - W)$ *is* l*-connected.*

Lemma 4.2.10 (Excision II)**.** *Suppose* $W \subset V \subset U$*. Suppose that there exist* $W' \subset W$ *such that* $\overline{W'} \cap \overline{U - V} = \emptyset$ *and there is a deformation from the pair* $(U - W', V - W')$ *to the pair* $(U - W, V - W)$*. Then* (U, V) *is* l*-connected if and only if* $(U - W, V - W)$ *is* l*-connected.*

Lemma 4.2.11 (Exhaustion). *Suppose U (respectively V) is an infinite union of open subsets U_i (respectively V_i), with $V_i \subset U_i$. If the pairs (U_i, V_i) are l-connected then (U, V) is l-connected.*

The following Lefschetz-Zariski type Theorem is deduced from the homotopy exact sequence for fibre bundles and [HmLe, Theorem 0.2.1] of Hamm and Lê Dũn Tráng.

Theorem 4.2.12. *Let D be a reduced divisor with only normal crossing singularities in $\mathbb{P}^{n+1}$, $n > 0$. Let H be a hyperplane in $\mathbb{P}^{n+1}$, whose intersection $H \cap D$ is a reduced divisor with only normal crossing singularities in H. Then the inclusion map*

$$(\mathbb{P}^{n+1} - D) \cap H \hookrightarrow \mathbb{P}^{n+1} - D$$

induces isomorphisms of homotopy groups in dimension less than n. Furthermore, the induced homomorphism

$$\pi_n((\mathbb{P}^{n+1} - D) \cap H) \to \pi_n(\mathbb{P}^{n+1} - D)$$

is onto.

Finally, we also recall a theorem about the fundamental group of the complement of hypersurfaces in $\mathbb{P}^{n+1}$ due to Deligne and Fulton.

Theorem 4.2.13. *[Di, Proposition 4.1.3 and Theorem 4.1.13] Let $D \subset \mathbb{P}^{n+1}$ be a hypersurface with only normal crossing singularities and irreducible components D_i, $i = 0, \dots, k$ and $n \geq 1$. Then the fundamental group $\pi_1(\mathbb{P}^{n+1} - D)$ is abelian and is isomorphic to*

$$\frac{\mathbb{Z}^{k+1}}{\mathbb{Z}(d_0, \dots, d_k)} \equiv \mathbb{Z}^k \oplus \frac{\mathbb{Z}}{\mathbb{Z}\mathrm{gcd}(d_0, \dots, d_k)}.$$

This last result shows that when $\gcd(d_0, \dots, d_k) \neq 1$, the fundamental group $\pi_1(\mathbb{P}^{n+1} - D)$ has torsion.

4.3 Simpson-Lefschetz Theorem

The proofs of Theorems 4.1.1 and 4.1.2 rely on the following result.

Theorem 4.3.1. *Let ω be a closed logarithmic 1-form on a projective manifold X of dimension $n+1$, $n \geq 1$. Assume that polar divisor D of ω is simple normal crossing divisor. Consider a normal covering space*

$$\rho : Y \to X - D,$$

over which the function

$$g(y) = \int_{y_0}^{y} \rho^* \omega \tag{4.2}$$

is well defined for $y \in Y$. If the singularities of ω outside D are isolated then the pair $(Y, g^{-1}(c))$ is n-connected with $c \in \mathbb{C}$.

Theorem 4.3.1 is an adaptation of [S, Corollary 21] which concerns the integral varieties of a closed holomorphic 1-form on a projective manifold X. One of the key steps in its proof consists in establishing an Ehresmann type result for the function g outside an open neighborhood of the singularities of $\rho^*\omega$. Before stating this result we will need to introduce some notation and establish some preliminary results.

Singular theory

Let $\{p_i\}$ be the finite set of isolated singularities of ω in $X - D$. Fix a metric μ on X. Since X is compact, μ is complete. We can choose $\varepsilon_1 > 0$ sufficiently small such that the closed balls

$$B_\mu(p_i, \varepsilon_1) = M_i \tag{4.3}$$

are pairwise disjoint and the restriction of ω to an open neighbourhood of M_i is exact. We define primitives $g_i(x) = \int_{p_i}^x \omega$ for $x \in M_i$. Since the points p_i are isolated singularities, it follows from [M3, Theorems 4.8, 5.10] the existence of $\varepsilon_2 > 0$ sufficiently small such that

(i) $0 \in B(0, \varepsilon_2) \subset \mathbb{C}$ is the unique critical value for the primitive g_i;

(ii) the intersections $g_i^{-1}(0) \cap \partial M_i$ and $g_i^{-1}(B(0, \varepsilon_2)) \cap \partial M_i = T_i$ are smooth, and the restriction of ω to T_i is a 1-form on T_i which never vanishes.

Lemma 4.3.2. *Let $F_i = g_i^{-1}(0)$ and $E_i = g_i^{-1}(c)$ with $c \in B(0, \varepsilon_2) - \{0\}$ be fibres of g_i restricted to*

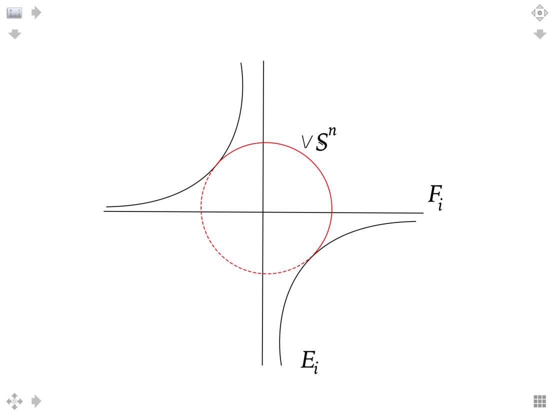
$$N_i = M_i \cap g_i^{-1}(B(0, \varepsilon_2)).$$

For small ε_2 the pair (N_i, F_i) is l-connected for every $l \in \mathbb{N}$ and the pair (N_i, E_i) is n-connected.

Proof. For ε_2 sufficiently small [M3, Theorem 5.2] implies that F_i is a deformation retract of N_i. Therefore the pair (N_i, F_i) is l-connected for $l \in \mathbb{N}$.

We learn from [M3, Theorems 5.11, 6.5] that E_i has the homotopy type of a bouquet of spheres $\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n$ for ε_2 sufficiently small. Thus the fibre E_i is $(n-1)$-connected. Since the neighbourhood N_i can be contracted to p_i the long exact sequence from Lemma 4.2.3 implies that the pair (N_i, E_i) is n-connected. □

Remark 4.3.3. We point out that [M3, Corollary 7.3] guarantees that the bouquet of spheres $\mathbb{S}^n \vee \cdots \vee \mathbb{S}^n$ is non-trivial in E_i. Thus we can picture what happened in N_i as follows:



Ehresmann type result

Let $\rho : Y \to X - D$ be a normal covering space and the function g as in (4.2). We will use $j \in J_i$ as an index set for the points $\tilde{p}_j$ of the discrete set $\rho^{-1}(p_i)$ and we will denoted the union $\cup J_i$ by J. Fix $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that

(*) in each connected component $\tilde{M}_j$ of $\rho^{-1}(M_i)$ containing the point $\tilde{p}_j$, the restriction of ρ in $\tilde{M}_j$ is a biholomorphism; and

(*) the subsets N_i, T_i, F_i, E_i satisfy the properties mentioned above for every i.

We define a primitive $\tilde{g}_j = g_i \circ \rho$ for the restriction of $\rho^*\omega$ to $\tilde{M}_j$ such that $g|_{\tilde{M}_j} = \tilde{g}_j + a_j$ for some $a_j \in \mathbb{C}$, with $j \in J_i$. The subsets $\tilde{N}_j, \tilde{T}_j, \tilde{F}_j, \tilde{E}_j$ of $\tilde{M}_j$ are the analogues of the subsets N_i, T_i, F_i, E_i of M_i.

We choose δ such that $0 < 5\delta < \varepsilon_2$. For each $b \in \mathbb{C}$, we define the subset $J(b)$ of J formed by the indexes j such that $|b - a_j| < 3\delta$. Let $U_b = B(b, \delta) \subset \mathbb{C}$ and define the open subset of the covering space Y

$$W(b) = g^{-1}(U_b) \cap \Big(\bigcup_{j \in J(b)} \tilde{N}_j^\circ \Big),$$

where $\tilde{N}_j^\circ$ denotes the interior of $\tilde{N}_j$, which satisfies

$$(g^{-1}(U_b) - W(b)) \cap \overline{W(b)} \subset \bigcup_{j \in J(b)} \tilde{T}_j.$$

We can now formulate our Ehresmann type result.

Proposition 4.3.4. *There exists a trivialization of $g^{-1}(U_b) - W(b)$ with trivializing diffeomorphism*

$$\Phi : U_b \times (g^{-1}(b) - W(b)) \to g^{-1}(U_b) - W(b),$$

such that the restriction to the boundary satisfies

$$\Phi(U_b \times (g^{-1}(b) - W(b)) \cap \overline{W(b)}) = (g^{-1}(U_b) - W(b)) \cap \overline{W(b)}.$$

Proof. For each point q in the polar divisor $D \subset X$ of the logarithmic 1-form ω, we have a coordinate chart $(V(q), \phi)$ such that

$$\omega = \phi^*(\sum_{j=1}^{r(q)} \lambda_j \frac{dz_j}{z_j} + \eta),$$

where η is a closed holomorphic 1-form and $\{\prod_{j=1}^{r(q)} z_j \circ \phi = 0\} = D \cap V(q)$. Choose $V(q)$ such that η is exact in $\phi(V(q))$ with primitive h. Consider the coordinate change $z \to y$ defined by

$$y_1 = z_1 \exp(\frac{h}{\lambda_1}), \quad \text{and} \quad y_l = z_l \quad \text{with} \quad l = 2, \ldots, n+1.$$

This gives a new coordinate chart $(V(q), \psi)$ where

$$\omega = \psi^*(\sum_{j=1}^{r(q)} \lambda_j \frac{dy_j}{y_j}). \tag{4.4}$$

We can take a finite number of points $q_\beta \in D$ with coordinate charts (V_β, ψ_β) satisfying (4.4) and such that the union $\cup_\beta V_\beta$ covers D.

Let $U_i \subset N_i$ be open balls containing the singular points of ω in $X - D$ such that the diameter of $g_i(U_i)$ is smaller than $\delta/10$. Let $\{A_\alpha\}$ be a finite cover of

$$X - (\bigcup_\beta V_\beta \cup \bigcup_i U_i)$$

such that ω is exact in A_α and the p_i's are not contained in any A_α.

We define the following C^∞ real vector fields:

(1) the vector fields u_β, v_β in V_β such that

$$D\psi_\beta(u_\beta) = \sum_{j=1}^{r(q_\beta)} \frac{y_j}{\lambda_j} \frac{\partial}{\partial y_j}, \quad D\psi_\beta(v_\beta) = \sqrt{-1} \sum_{j=1}^{r(q_\beta)} \frac{y_j}{\lambda_j} \frac{\partial}{\partial y_j},$$

(2) the vector fields u_α, v_α in A_α such that $\omega(u_\alpha) = 1$, $\omega(v_\alpha) = \sqrt{-1}$ and if the intersection $A_\alpha \cap T_i$ is non empty the vector fields u_α, v_α are tangent to T_i;

(3) the smooth vector fields u_i, v_i in $\overline{U_i}$ which vanish in p_i.

We take a partition of the unity subordinated to the open cover $\{V_\beta\} \cup \{A_\alpha\} \cup \{U_i\}$ of X and we define the vector fields

$$u = \sum \phi_\gamma u_\gamma \quad \text{and} \quad v = \sum \phi_\gamma v_\gamma,$$

which are complete since they are defined over all X. By definition, the vector fields u_β, v_β leave the divisor D invariant. Therefore the vector fields u, v also leave the divisor D invariant. It follows that the restriction of u, v to $X - D$

are still complete vector fields and they satisfy $\omega(u) = 1, \omega(v) = \sqrt{-1}$ outside of $D \cup (\cup_i U_i)$.

Let $\tilde{u}, \tilde{v}$ be the liftings of u, v with respect to Y. Notice that the vector fields $\tilde{u}, \tilde{v}$ are complete vector fields on Y, which restricted to $g^{-1}(U_b) - W(b)$ satisfy $\rho^*\omega(\tilde{u}) = 1, \rho^*\omega(\tilde{v}) = \sqrt{-1}$. It implies the existence of the diffeomorphism

$$\begin{array}{rcl} \Phi : U_b \times (g^{-1}(b) - W(b)) & \to & g^{-1}(U_b) - W(b) \\ (t_1 + b, t_2 + b) \times \{q\} & \mapsto & \Phi_1(t_1, \Phi_2(t_2, q)), \end{array}$$

where Φ_1, Φ_2 are flows of $\tilde{u}, \tilde{v}$, respectively. The vector fields $\tilde{u}, \tilde{v}$ are tangent to $\tilde{T}_j$ for every $j \in J$. In particular, they are tangent to $\cup_{j \in J(b)} \tilde{T}_j$. It follows that

$$\Phi(U_b \times (g^{-1}(b) - W(b)) \cap \overline{W(b)}) = (g^{-1}(U_b) - W(b)) - (g^{-1}(\partial U_b) \cap \overline{W(b)})$$

as wanted.

□

The proof of Theorem 4.3.1

Define the following sets

$$P(b, V) = g^{-1}(V) \cup W(b),$$

where V is contained in U_b;

$$R(b) = g^{-1}(U_b) - W(b), \quad F(b) = g^{-1}(b);$$

and the intersections

$$P^R(b, V) = P(b, V) - W(b), \quad F^R(b) = F(b) - W(b).$$

Lemma 4.3.5. *Let $V \subset U_b$ be a contractible subset. If there exists a continuous map $\xi : U_b \times [0,1] \to U_b$ such that $\xi(y, 0) = y$,$\xi(V \times [0,1]) \subset V$ and $\xi(U_b \times \{1\}) \subset V$ then the pair $(g^{-1}(U_b), P(b, V))$ is l-connected for every l.*

Proof. For each $\tilde{T}_j$ with $j \in J(b)$ we can choose a vector field ν_j tangent to the level sets of g and pointing to the interior of $W(b)$. The vector fields ν_j allow us to construct a deformation $h : W_b \times [0,1] \to W_b$ such that $h(y, 0) = y$ and the image of $h(W_b \times \{1\}) = W'(b)$ has empty intersection with $R(b)$.

The map $h(y, 1 - t)$ gives us a deformation of the pair $(g^{-1}(U_b) - W'(b), P(b, V) - W'(b))$ to the pair $(R(b), P^R(b, V))$. The excision Lemma 4.2.10 implies that the pairs $(g^{-1}(U_b), P(b, V))$ y $(R(b), P^R(b, V))$ have the same l-connectivity.

The map ξ can be lifted through the diffeomorphism Φ from Proposition 4.3.4. Hence the pair $(R(b), P^R(b, V))$ is l-connected for every l, and the lemma follows. □

Lemma 4.3.6. *Let $V \subset U_b$ be as in Lemma 4.3.5. The pair $(g^{-1}(U_b), g^{-1}(V))$ is n-connected.*

Proof. Consider the pair $(\tilde{N}_j^\circ \cap g^{-1}(U_b), \tilde{N}_j^\circ \cap g^{-1}(V)) = (U_{b,j}, V_j)$ with $j \in J(b)$. Since V is contractible and the restriction of g to $\tilde{N}_j - \tilde{F}_j$ is a trivial fibration, Lemma 4.2.8 implies that the pair $(U_{b,j}, V_j)$ is l-connected for every l if $\tilde{F}_j \subset V_j$, and n-connected if $\tilde{F}_j$ is not contained in V_j. Therefore the pair $(W(b), \cup_{j\in J(b)} V_j)$ is at least n-connected.

Let ν be the vector field defined in the proof of the previous lemma. The vector field $-\nu$ points toward the interior of $R(b)$. Analogously, we define a deformation $h : R(b) \times [0,1] \to R(b)$ such that the closure of the image $h(R(b) \times \{1\}) = R'(b)$ has empty intersection with $\overline{W(b)}$. Lemma 4.2.10 implies that the pair $(P(b,V), g^{-1}(V))$ is n-connected. From Lemmas 4.2.7(transitivity) and 4.3.5 we conclude that the pair $(g^{-1}(U_b), g^{-1}(V))$ is n-connected. □

Proof of Theorem 4.3.1. Take a triangulation Δ of $\mathbb{C}$ with equilateral triangles with sides of length δ, such that one of the vertices in V_Δ is $c \in \mathbb{C}$. Let H_l be the family of concentric hexagons with center c and vertices in V_Δ. Label by c_i the vertices V_Δ such that

(*) between c_i and c_{i+1} it always exists an edge $e_i \in \mathrm{E}_\Delta$ of the triangulation Δ, and $c_0 = c$.

(**) the vertices c_i with $6(l-1)l/2 < i \leq 6l(l+1)/2$ are in the hexagon H_l.

Consider the open sets $U_i = B_{\mathbb{C}}(c_i, \delta)$ and $W_i = \cup_{j\leq i} U_j$. Since the intersection $U_i \cap W_{i-1} = V_i$ is contractible in U_i, Lemma 4.3.6 gives that the pair $(g^{-1}(U_i), g^{-1}(V_i))$ is n-connected.

The W_i-closures of the sets $(W_i - W_{i-1})$ and $(W_{i-1} - U_i)$ are disjoint, thus the previous paragraph combined with Lemma 4.2.9 imply that the pair $(g^{-1}(W_i), g^{-1}(W_{i-1}))$ is n-connected for every i. From Lemma 4.2.7 (transitivity) we deduce that the pair $(g^{-1}(W_i), g^{-1}(W_0))$ is n-connected for every i. Taking $V = c$ in Lemma 4.3.6, we see that $(g^{-1}(W_0), g^{-1}(c))$ is n-connected. Hence $(g^{-1}(W_i), g^{-1}(c))$ is n-connected for all i. Finally, applying Lemma 4.2.11 (exhaustion) for the pairs (W_i, c) we conclude the proof of Theorem 4.3.1. □

Example 4.3.7. Let ω be closed logarithmic 1-form on $\mathbb{P}^{n+1}$, with a simple normal crossing polar divisor $D = H_0 + \cdots + H_k$ and $1 \leq k \leq n+1$. Let H_j be hyperplans of $\mathbb{P}^{n+1}$. Modulo an automorphism of $\mathbb{P}^{n+1}$ we can take $H_j = \{z_j = 0\}$ where $[z_0 : \cdots : z_{n+1}]$ are homogeneous coordinates for $\mathbb{P}^{n+1}$. Take the universal covering

$$\begin{array}{cccc} \rho : \mathbb{C}^{n+1} & \to & \mathbb{P}^{n+1} - D \\ {[1 : x_1 : \cdots : x_{n+1}]} & \mapsto & [1 : e^{2\pi\sqrt{-1}x_1} : \cdots : e^{2\pi\sqrt{-1}x_k} : x_{k+1} : \cdots : x_{n+1}]. \end{array}$$

If we denote the residues by $\mathrm{Res}(\omega, H_j) = \lambda_j$, then the pull-back $\rho^*\omega$ admits the following expression

$$2\pi\sqrt{-1}\sum_{j=0}^{k} \lambda_j dx_j,$$

which is a linear 1-form on $\mathbb{C}^{n+1}$. In this case there are no singularities outside the divisor and the primitive g is a linear map with $g^{-1}(c) \cong \mathbb{C}^n$. In particular, the pair $(\mathbb{C}^{n+1}, g^{-1}(c))$ is l-connected for every l.

Example 4.3.8. Consider the closed rational 1-form

$$\omega = d\left(\frac{x^2+y^2+z^2}{xy}\right)$$

in homogeneous coordinates $[x:y:z]$ of $\mathbb{P}^2$. The polar divisor D of ω has only two irreducible components $D_0 = \{x=0\}, D_1 = \{y=0\}$, with $D = 2D_0 + 2D_1$. The singularities of ω outside of D are the points $p_1 = [1:1:0], p_2 = [-1:1:0]$.

The 1-form ω is exact in $\mathbb{P}^2 - D$. The leaves of the foliation $\mathcal{F}$ defined by ω in $\mathbb{P}^2 - D$ coincide with

$$\{x^2+y^2+z^2-\alpha xy = 0\} - D \quad \text{with} \quad \alpha \in \mathbb{C}.$$

If we assume that Proposition 4.3.4 is true in this situtation, we would have for $\delta > 0$ sufficiently small a diffeomorphism

$$\Phi : g^{-1}(B(2,\delta)) - W(2)) \cong B(2,\delta) \times (g^{-1}(2) - W(2)).$$

But this is impossible since the set $g^{-1}(2)$ consists of two lines and the set $g^{-1}(2) - W(2)$ is not connected and the set $g^{-1}(B(2,\delta)) - W(2))$ is connected.

The construction of the vector field used to prove Proposition 4.3.4 fails in this case, since at the singular points $q_1 = [1:0:1], q_2 = [-1:0:1], q_3 = [0:1:1], q_4 = [0:-1:1]$ the vector field

$$u = \frac{x^2 y}{x^2-y^2-z^2}\frac{\partial}{\partial x} + \frac{y^2 x}{y^2-x^2-z^2}\frac{\partial}{\partial y} + \frac{2z}{xy}\frac{\partial}{\partial z}$$

cannot be extended.

4.4 Proofs of main results

Let $\mathcal{F}$ be a logarithmic foliation on the projective space $\mathbb{P}^{n+1}$ with $n \geq 1$, defined in homogenous coordinates by the logarithmic 1-form

$$\omega = (\prod_{j=0}^{k} F_j)\sum_{j=0}^{k} \lambda_j \frac{dF_j}{F_j},$$

where $k \geq 1$ and the residues satisfy $\sum_{j=0}^{k} \lambda_j d_j = 0$. The polar divisor D of ω is the reduced divisor given by $\sum_{j=0}^{k} D_j$, where $D_j := \{F_j = 0\}$ and d_j is the degree of the homogeneous polynomial F_j.

Proposition 4.4.1. *Let $\mathcal{F}$ be a logarithmic foliation as above. Suppose that the polar divisor D of ω is a simple normal crossing divisor. Then there exists a normal cover $\rho_Y : Y \to \mathbb{P}^{n+1} - D$ such that $\rho_Y^*\omega$ is exact, and for a primitive g defined by $g(y) = \int_{y_0}^{y} \rho_Y^*\omega$ the inverse image $g^{-1}(c)$ of a regular value $c \in \mathbb{C}$ is biholomorphic to a generic leaf $\mathcal{L}$ of $\mathcal{F}$.*

Proof. Consider the universal cover $\rho : Z \to \mathbb{P}^{n+1} - D$ and the subgroup of $\pi_1(\mathbb{P}^{n+1} - D)$

$$G = \{(m_0,\ldots,m_k) \in \mathbb{Z}^{k+1}/(d_0,\ldots,d_k)\mathbb{Z} | \sum_{j=0}^{k} m_j\lambda_j = 0\}.$$

Note that the torsion subgroup of $\pi_1(\mathbb{P}^{n+1} - D)$ is contained in G.

Define the covering space

$$Y = \frac{Z}{G}$$

of $\mathbb{P}^{n+1} - D$ with projection ρ_Y.

Claim. The closed 1-form $\rho_Y^* \omega$ is exact.

Aiming at a contradiction assume that the claim is false. Then there exists a closed curve $\gamma : I \to Y$ homotopically non trivial such that $\int_\gamma \rho_Y^* \omega \neq 0$. Let $\{\gamma_j\}_{j=0}^k$ be representatives of generators of $\pi_1(\mathbb{P}^{n+1} - D)$, i.e.

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_j} \frac{dF_l}{F_l} = \delta_{jl},$$

where δ_{jl} is the Kroenecker delta. Since $\pi_1(\mathbb{P}^{n+1} - D)$ is abelian we can write

$$\rho_Y \circ \gamma = \gamma_0^{m_0} * \cdots * \gamma_k^{m_k},$$

and obtain the equality

$$\int_\gamma \rho_Y^* \omega = \int_{\gamma_0^{m_0} * \cdots * \gamma_k^{m_k}} \omega.$$

The right hand side can be written as $2\pi\sqrt{-1} \sum_{j=0}^k m_j \lambda_j$. But since the homotopty class of γ is in $\pi_1(Y)$ we deduce that $\rho_Y \circ \gamma$ belongs to G. This is a sought contradiction.

Claim. Let $c \in \mathbb{C}$ be a regular value of g. Then $\rho_Y|_{g^{-1}(c)} : g^{-1}(c) \to \mathcal{L}$ is a biholomorphism.

Suppose not. Then there exist points $y_0, y_1 \in \rho_Y^{-1}(x_0)$ distinct from $x_0 \in \mathcal{L}$ such that $y_0, y_1 \in g^{-1}(c)$.

Take $\tilde{\gamma} : I \to Y$ with $\tilde{\gamma}(0) = y_0$ y $\tilde{\gamma}(1) = y_1$. Recall that Theorem 1.3.5 shows that the singularities outside D are isolated. Since $n \geq 1$, Theorem 4.3.1 implies that the pair $(Y, g^{-1}(c))$ is 1-connected. By Lemma 4.2.5 there exists γ' contained in $g^{-1}(c)$ homotopic to $\tilde{\gamma}$ with fixed extremes. Therefore $\gamma = \rho \circ \gamma'$ is a curve in $\mathcal{L}$ which is not homotopically trivial in $\mathbb{P}^{n+1} - D$. But since it is contained in a leaf of the foliation we have that

$$\int_\gamma \omega = 0.$$

Writing $\gamma = \gamma_0^{m_0} \cdots \gamma_k^{m_k}$, we deduce that $\sum_{j=0}^k m_j \lambda_j = 0$. Hence γ is homotopic to an element of G a contradiction. Thus $\rho|_{g^{-1}(c)}$ is a biholomorphism. □

Example 4.4.2. Let us consider the case where the polar divisor of the logarithmic 1-form ω has only two irreducible components, say D_0 and D_1. If the degrees d_0, d_1 are equal then the leaves $\mathcal{L}$ of the foliation $\mathcal{F}$ are contained in elements of the pencil

$$\{aF_0 + bF_1 | (a : b) \in \mathbb{P}^1\}.$$

In particular the generic leaf $\mathcal{L}$ is of the form $\{aF_0 + bF_1 = 0\} - D$ for $(a : b)$ generic. Theorem 4.2.13 implies that

$$\pi_1(\mathbb{P}^3 - D) \cong \mathbb{Z} \oplus \frac{\mathbb{Z}}{d\mathbb{Z}},$$

where $d = d_0 = d_1$. Note that the subgroup G in this case is the torsion subgroup of $\pi_1(\mathbb{P}^3 - D)$. Hence Proposition 4.4.1 and Theorem 4.3.1 imply that

$$\pi_1(\mathcal{L}) = \frac{\mathbb{Z}}{d\mathbb{Z}}.$$

We are now ready to prove Theorems 4.1.1 and 4.1.2.

Proof of Theorem 4.1.1. By Theorem 1.3.5 the singularities outside D are isolated, and we have that the covering Z/G of $\mathbb{P}^{n+1} - D$ given by Proposition 4.4.1 satisfies the hypotheses of Theorem 4.3.1, where Z is the universal cover of $\mathbb{P}^{n+1} - D$. Since $n \geq 2$, Theorem 4.3.1 and Proposition 4.4.1 show that the fundamental group $\pi_1(Z/G)$ and the fundamental group $\pi_1(\mathcal{L})$ of a generic leaf are isomorphic, which is the desire conclusion. □

Corollary 4.4.3. *Let $\mathcal{F}, \omega, D, k, n$ and G be as in Theorem 4.1.1. If the residues λ_j of ω are non resonant and* $\gcd(d_0, \ldots, d_k) = 1$ *then the generic leaf is simply connected.*

Proof. From the hypothesis on the residues λ_j and $\gcd(d_0, \ldots, d_k)$, the subgroup G of $\pi_1(\mathbb{P}^{n+1} - D)$ is trivial. Thus, we have that the universal cover Z of $\mathbb{P}^{n+1} - D$ coincides with covering space given by Proposition 4.4.1. By Theorem 4.1.1 the generic leaf is simply connected. □

Proof of Theorem 4.1.2. The inclusion $i : H - D(H) \hookrightarrow \mathbb{P}^{n+1} - D$ induces the morphisms

$$i_* : \pi_l(H - D(H)) \to \pi_l(\mathbb{P}^{n+1} - D) \tag{4.5}$$

in homotopy, where $D(H) = H \cap D$. From Theorem 4.2.12 we have that i_* is an isomorphism for $l < n$ and an epimorphism for $l = n$.

Consider the normal cover $\rho : Y \to \mathbb{P}^{n+1} - D$ given by Proposition 4.4.1. Let g be a primitive of $\rho^*\omega$. Let $Y(H) = \rho^{-1}(H - D(H))$. Notice that $Y(H)$ is a connected normal covering space of $H - D(H)$. Let g_H be the restriction of g to $Y(H)$. Let $c \in \mathbb{C}$ be a common regular value for g and g_H. Since $l \leq n - 1$, Theorem 4.3.1 implies that the morphisms

$$i_* : \pi_l(g^{-1}(c)) \to \pi_l(Y)$$

and

$$i_* : \pi_l(g_H^{-1}(c)) \to \pi_l(Y(H))$$

induced by the inclusion $Y(H) \hookrightarrow Y$, are isomorphisms if $l < n - 1$ and epimorphisms if $l = n - 1$. Considering the exact sequence from Lemma 4.2.3 we obtain the following commutative diagram for $l > 0$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_{l+1}(Y(H), g_H^{-1}(c)) & \longrightarrow & \pi_l(g_H^{-1}(c)) & \longrightarrow & \pi_l(Y(H)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \pi_{l+1}(Y, g^{-1}(c)) & \longrightarrow & \pi_l(g^{-1}(c)) & \longrightarrow & \pi_l(Y) & \longrightarrow & \cdots
\end{array}$$

From the morphism (4.5) and Lemma 4.2.6 we have that the morphisms

$$\pi_l(Y(H)) \to \pi_l(Y)$$

are isomorphisms for $l < n$ and epimorphisms for $l = n$. Analogously, Theorem 4.3.1 implies that the morphisms

$$\pi_l(Y(H), g_H^{-1}(c)) \to \pi_l(Y, g^{-1}(c))$$

are isomorphisms for $l < n$ and epimorphisms for $l = n$. Applying the five Lemma, we have that the morphisms

$$\pi_l(g_H^{-1}(c)) \to \pi_l(g^{-1}(c))$$

are isomorphisms for $l < n-1$ and epimorphisms for $l = n-1$. Hence the theorem follows from the biholomorphism given by Proposition 4.4.1. □

Corollary 4.4.4. *Let $\omega, D, G, \mathcal{F}$ satisfying the hypothesis of Corollary 4.4.3. If $d_j = 1$ and $k > n+1$, then the generic leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is $(n-1)$-connected.*

Proof. Let $l_j \in \mathbb{C}[x_0, \dots, x_{n+1}]$ be homogeneous polynomials of degree 1 such that $D_j = \{l_j = 0\}$. Define the linear inclusion

$$\begin{array}{lcl} \varphi : \mathbb{P}^{n+1} & \to & \mathbb{P}^k \\ [x_0 : \cdots : x_{n+1}] & \mapsto & [l_0 : \cdots : l_k]. \end{array}$$

Since the polar divisor D is normal crossing, the morphism φ is an embedding and $\omega = \varphi^*\tilde{\omega}$, where

$$\tilde{\omega} = \sum_{j=0}^{k} \lambda_j \frac{dz_j}{z_j}$$

with $[z_0 : \cdots : z_k]$ homogeneous coordinates of $\mathbb{P}^k$.

The image $\varphi(\mathbb{P}^{n+1}) = H_s$ is a linear subspace of $\mathbb{P}^k$ of codimension $s = k-(n+1)$. Denote by $\tilde{D}$ the polar divisor of $\tilde{\omega}$. Since the intersection $H_s \cap \tilde{D}$ is normal crossing in H_s, we can construct a descending sequence

$$\mathbb{P}^k = H_0 \supset H_1 \supset \cdots \supset H_{s-1} \supset H_s$$

of linear subspaces satisfying

(*) the codimension of H_j in H_{j-1} is 1, when $j > 0$.

(*) the intersection $H_j \cap \tilde{D}$ is normal crossing in H_j.

Let $\mathcal{F}_j$ be the foliation defined by the restriction of the 1-form $\tilde{\omega}$ to H_j. Without loss of generality we can assume that a generic leaf $\mathcal{L}_0$ of $\mathcal{F}_0$ satisfies that $\mathcal{L}_j = \mathcal{L}_0 \cap H_j$ is also a generic leaf of $\mathcal{F}_j$.

Iterating Theorem 4.1.2 we have that the pair $(\mathcal{L}_{j-1}, \mathcal{L}_j)$ is $(k-j-1)$-connected. The hypothesis that the group G is trivial, Proposition 4.4.1 and Example 4.3.7 imply that $\mathcal{L}_0$ is biholomorphic to $\mathbb{C}^{k-1}$. Hence the leaf $\mathcal{L}_j$ is at least $(k-j-2)$-connected. Since the generic leaf $\mathcal{L}$ of $\mathcal{F}$ is homeomorphic to $\mathcal{L}_s$, we conclude that it is $(n-1)$-connected. □

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