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Generic Symplectic Cocycles are Hyperbolic

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Contents

Abstract

The goal of this thesis is to give a positive and complete answer to the question about genericity of nonuniformly hyperbolic dynamics for two classes of symplectic cocycles: the locally constant ones and the Hölder continuous and dominated ones, both over shifts.

Indeed, we prove that the Avila-Bonatti-Viana criterion for simplicity of Lyapunov spectra holds generically in these two settings. Moreover, to symplectic cocycles, simplicity implies (nonuniform) hyperbolicity, since the multiplicity of zero as a Lyapunov exponent must be even.

Resumo

O objetivo da presente tese é dar uma resposta completa e positiva para a questão da genericidade de dinâmicas não uniformemente hiperbólicas para duas classes de cociclos simpléticos: os localmente constantes e os Hölder contínuos e dominados, ambos sobre shifts.

De fato, mostraremos que o critério explícito de Avila-Bonatti-Viana para simplicidade de espectros de Lyapunov vale genericamente nesses dois contextos. Além disso, para cociclos simpléticos, simplicidade implica hiperbolicidade (não-uniforme), uma vez que a multiplicidade de zero como expoente de Lyapunov é necessariamente par.

Chapter 1

Introduction

The adjective *hyperbolic* is, probably, one of the most used in many different fields of knowledge as mathematics, physics, economy, etc. In each context it gets a proper and convenient definition. In Dynamical Systems it can be found qualifying things like periodic points, sets, diffeomorphisms, homeomorphisms, measures, cocycles and many others. In most cases it implies good geometric properties of the system, mainly invariant submanifolds with well-behaved assymptotic trajectories. The theory is almost all developed over these invariant submanifolds and its properties, so its important to know how often a system (and which kind of system) has such properties.

Let M be a compact manifold and $f : M \to M$ a diffeomorphism. A compact and invariant $\Lambda \subset M$ is *hyperbolic* (or *uniformly hyperbolic*) if for every $x \in \Lambda$,

(1) each tangent space admits a splitting $T_xM = E^s(x) \oplus E^u(x)$ such that

$$Df(E^s(x)) = E^s(f(x)) \;\; ; \;\; Df(E^u(x)) = E^u(f(x))$$

(2) there are constants $C > 0$ and $0 < \lambda < 1$ such that, for all $x \in \Lambda$ and $n \geq 1$

$$\|Df^n v\| \leqslant C\lambda^n \|v\| \;\; \forall v \in E^s(x)$$

$$\|Df^{-n} v\| \leqslant C\lambda^n \|v\| \;\; \forall v \in E^u(x)$$

In this case, it is well known that these subbundle are integrable to a continuous foliation with smooth leaves. So, we get two transversal foliations into *stable* leaves $W^s(x)$ (tangent to $E^s(x)$) and *stable* leaves $W^u(x)$ (tangent to $E^u(x)$). Moreover, these sets can be described as

$$W^s(x) = \{y \in \Lambda \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}$$

$$W^u(x) = \{y \in \Lambda \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$$

After some years of research it was discovered that there were some interesting systems which did not satisfy these hyperbolicity conditions. Thus, a weaker version of the definition was "created", the *nonuniform hyperbolicity.* It is kind of the same definition as above, where the constants C and λ are allowed to vary with the point, in a certain way.

An f-invariant set Y (not necessarily compact) is *nonuniformly hyperbolic* if for every $x \in Y$ there is, as before, a Df-splitting of the tangent space of x into $E^s(x) \oplus E^u(x)$, which varies measurably with the point and satisfies

(1) There exist positive measurable functions $C, K, \varepsilon, \lambda : Y \to (0, \infty)$ such that $\lambda(x)e^{\varepsilon(x)} < 1$ for all $x \in Y$ and such that for every $n \geq 1$ we have

$$\|Df^n v\| \leqslant C(x)\lambda(x)^n e^{\varepsilon(x)n}\|v\| \quad \forall v \in E^s(x)$$

$$\|Df^{-n} v\| \leqslant C(x)\lambda(x)^n e^{\varepsilon(x)n}\|v\| \quad \forall v \in E^u(x)$$

(2) λ and ε are f-invariants. The functions C and K vary slowly along trajectories, i.e.

$$C(f^n(x)) \leqslant C(x)e^{\varepsilon(x)|n|} \text{ and } \ K(f^n(x)) \geq K(x)e^{-\varepsilon(x)|n|}$$

(3) the angle between the spaces satisfies

$$\measuredangle(E^s(x), E^u(x)) \geq K(x)$$

In this case we can see the stable and unstable sets defined above, $W^s(x)$ and $W^u(x)$, as a "measurable" foliation of smooth leaves which have similar properties as in the uniform case, but now we have to restrict x to sets known as *Pesin blocks.* This notion of nonuniformly hyperbolic diffeomorphisms was introduced by Pesin (see, for instance [19]). Some authors refer to this nonuniform hyperbolicity theory as Pesin theory.

Associated to this kind of system there is the concept of *Lyapunov exponents*, that we will explain later. The point here is that nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents: a dynamical system is nonuniformly hyperbolic if it admits an invariant measure with nonzero Lyapunov exponents almost everywhere. Doubtless, this is an efficient way to verify whether a given system is nonuniformly hyperbolic or not.

There are some recent results concerning about how frequent the nonuniform hyperbolic diffeomorphisms appear. Recently, A. Avila and J. Bochi proved, in [1], that for C^1-generic volume-preserving diffeomorphisms either there is at least one zero Lyapunov exponent at almost every point or the set of points with only nonzero exponents forms an ergodic component. Also, J. Rodriguez-Hertz (see [20]), using the results of Avila and Bochi, showed that for a generic conservative diffeomorphism of a closed connected 3-manifold M, either all Lyapunov exponents vanish almost everywhere, or else the system is nonuniformly hyperbolic and ergodic. For the symplectic case we have a result by J. Bochi saying that C^1-generic symplectomorphisms are either ergodic and Anosov or have at least two zero Lyapunov exponents at almost every point, [5].

All these definitions we have seen until now can be adapted to *linear cocycles*, which are, actually, the protagonists of this thesis. We will make all definitions in the upcoming sections. The derivative $Df : TM \to TM$ of a diffeomorphism f is one example of linear cocycle. But we will focus on another example: the random products of matrices, which can be viewed as a linear cocycle over a shift map.

In the setting of random dynamical systems, there are some classical results due to Furstenberg-Kesten [10], Furstenberg [9], Ledrappier [16], Guivarc'h-Raugi [14] and Goldsheid-Margulis [13], all of them dealing with random products of matrices and its Lyapunov spectrum (the set of Lyapunov exponents): existence of exponents, positivity of the largest one, simplicity of the spectrum (that is, Oseledets subspaces are one-dimensional).

All these results inspired many recent works. A few years ago Bonatti, Gomez-Mont and Viana [6] proved a version of Furstenberg's positivity criterion that applies to any cocycle admitting invariant stable and unstable holonomies; later Bonatti and Viana [8] extended the Guivarc'h-Raugi simplicity criterion; Avila, Viana [4] improved a little bit more the simplicity criterion and applied it to solve the Zorich-Kontsevich conjecture.

In general, it is not true that simple Lyapunov spectrum implies nonuniform hiperbolicity. The exponents can be all different, but one of them can be zero. However, if we are in the symplectic world, this sole zero can not exist. The Lyapunov spectrum of a symplectic cocycle shows a symmetry:

both λ and $-\lambda$ always appear as a Lyapunov exponent of a symplectic cocycle. In particular, if 0 belongs to the spectrum then the central bundle (the subbundle associated to the zero exponent) must have even dimension. Thus, in this setting, simplicity of the spectrum implies non-zero exponents, and then nonuniform hyperbolicity.

1.1 Statement of Results

Our results in this thesis give a positive and complete answer to the question of genericity of nonuniformly hyperbolic dynamics in the context of cocycles involving matrices that preserve a symplectic form. Moreover, we show that the explicit criterion of Avila-Bonatti-Viana for simplicity of the Lyapunov spectrum holds in an open and dense set.

Our goal is to fulfil the following general statement

Theorem. *Generic locally constant or Hölder dominated symplectic cocycles have simple Lyapunov spectrum. More than that, the set of cocycles with simple Lyapunov spectrum contains a subset that is open and dense in the whole space. In particular, these cocycles are nonuniformly hyperbolic.*

Before talk about the settings and all its precise definitions, let us try to explain the general strategy for the proof.

As we explained in the last paragraphs, simplicity of the Lyapunov spectrum is sufficient to get nonuniform hyperbolicity in the symplectic world, because of the symmetry of the spectrum. Thus, the last part of the theorem above is solved. According to the Avila-Viana criterion, in order to get simplicity, we just may verify two properties they called *twisting* and *pinching*. Then, the strategy can be summarized by the following three steps:

1. Pinching and twisting properties hold in a open and dense set of cocycles.

2. Avila-Viana criterion concludes that these cocycles have simple Lyapunov spectra.

3. Simplicity plus symplecticity implies nonuniform hyperbolicity.

At this point, anyone can realize that, the only step we have to worry about is the first one. So, we can be a little bit more precise on what we will indeed prove:

Theorem A. *Let $\mathcal{H}$ be be the set of locally constant symplectic linear cocycles. The subset $\mathcal{N} \subset \mathcal{H}$ of the cocycles satisfying the twisting and pinching properties is open and dense in the natural topology. Furthermore, every $A \in \mathcal{N}$ has simple Lyapunov spectrum and, in particular, is nonuniformly hyperbolic.*

Theorem B. *For $0 < \nu < 1$, let $\mathcal{H}^\nu$ be the set of C^ν-Hölder and dominated symplectic linear cocycles. The subset $\mathcal{N} \subset \mathcal{H}^\nu$ of the cocycles satisfying the twisting and pinching properties is open and dense in the C^ν topology. Furthermore, every $A \in \mathcal{N}$ has simple Lyapunov spectrum and, in particular, is nonuniformly hyperbolic.*

Remark. The natural topology in the set of locally constant linear cocycles is the following: "two cocycles in $\mathcal{H}$ are close if they are constant at the same cylinders with uniformly close matrices". And the C^ν topology in $\mathcal{H}^\nu$ is the topology defined by the norm

$$\|A\|_\nu = \sup_{\mathtt{x}\in\Sigma_T} \|A(\mathtt{x})\| + \sup_{\mathtt{x}\neq\mathtt{y}\in\Sigma_T} \frac{\|A(\mathtt{x}) - A(\mathtt{y})\|}{d(\mathtt{x},\mathtt{y})^\nu}$$

As a simple application of Theorem A, we can start discussing random matrices, which although might be the simplest multidimensional dynamical system possible, it has a big historical importance for the ergodic theory of products of matrices. We can find random matrices as linearization of smooth systems or as transition matrices in population dynamics, for instance. For us, we are just interested in it as a particular case of locally constant linear cocycles.

Corollary C. *Symplectic random matrices with simple Lyapunov spectrum form an open and dense set.*

Let $b > 1$ and $A_1, A_2, ..., A_b$ symplectic matrices of $Sp(2d, \mathbb{R})$ (they correspond to operators of $\mathbb{R}^{2d}$ that preserve a symplectic form ω). Let $(p_1, ..., p_b)$ be a probability vector, that is, its coordinates are strictly positive numbers such that $\sum_i p_i = 1$. Consider $L = \{L_j\}_{j=0}^\infty$ a family of independent random variables and P a probability distribution such that for all $j \geq 0$

$$P(L_j = A_i) = p_i$$

Define the power of L as $L^n = L_{n-1} \cdot ... \cdot L_1 \cdot L_0$.

The history of ergodic theory for products of random matrices really starts with a theorem of H. Furstenberg and H. Kesten, [10], on the dominated growth rate which, in our setting, says that with full probability one can find a real number λ_1 such that

$$\lim_n \frac{1}{n} \log \|L^n\| = \lambda_1.$$

This number represents the maximal asymptotic growth rate for almost all products of the random matrices L. We can realize other growth rates if we restrict ourselves to lower dimensional subspaces of $\mathbb{R}^{2d}$. The celebrated Oseledets' Multiplicative Ergodic Theorem ([18]) ensures the existence of subspaces where we can see smaller growth rates. The set of all such numbers is called the *Lyapunov spectrum* of L and we say that it is *simple* if there are $2d$ distinct numbers in it.

To prove the Corollary C we have only to show that random matrices are, indeed, an example of locally constant linear cocycles. Let us seize the opportunity to review some definitions about linear cocycles.

1.2 Linear Cocycles

Consider the trivial bundle $M \times \mathbb{R}^{2d}$, where $(M, \mathcal{B}, \mu)$ is a probability space. Given $f : M \to M$ a measure preserving transformation and $A : M \to Sp(2d, \mathbb{R})$ a measurable map, we define the *linear cocycle* associated to A over f as

$$F : M \times \mathbb{R}^{2d} \to M \times \mathbb{R}^{2d}, \quad (x, v) \mapsto (f(x), A(x).v).$$

Remark. We could make the definition above in a more general way. For instance, we could consider any vector bundle instead of the trivial one, also we could define the map A from M to any other group of matrices, say $Gl(k, \mathbb{R})$. At first, there is no loss of generality if we assume the bundle to be trivial, because every vector bundle is locally isomorphic to the trivial one. About the matrices group, we will keep focused on the symplectic group in all the text. For this reason, we made the previous definitions on that way.

We write, for any $n \in \mathbb{N}$, the iterates $F^{\pm n}(x, v) = (f^{\pm n}(x), A^{\pm n}(x).v)$

where

$$\begin{aligned} A^n(x) &= A\left(f^{n-1}(x)\right)\cdots A\left(f(x)\right)A(x) \\ A^{-n}(x) &= A\left(f^{-n}(x)\right)^{-1}\cdots A\left(f^{-1}(x)\right)^{-1} = \left[A^n\left(f^{-n}(x)\right)\right]^{-1} \end{aligned}$$

Of course, the negative iterations only make sense when f is invertible; in such case F will be invertible too.

Theorem 1.1 (Oseledets). *Assume that $f : M \to M$ is measure preserving and invertible. Suppose also that $\log^+ \|A^{\pm 1}(x)\| \in L^1(\mu)$. Then, for μ-almost every $x \in M$ there are $k = k(x)$ numbers, $\lambda_1(x), ..., \lambda_k(x)$ and a decomposition $\mathbb{R}^{2d} = E_x^1 \oplus \cdots \oplus E_x^k$ such that, for all $i = 1, ..., k$*

1. $k(f(x)) = k(x), \;\; \lambda_i(f(x)) = \lambda_i(x) \quad$ *and* $\quad A(x).E_x^i = E_{f(x)}^i$
2. $\lim\limits_{n\to+\infty} \frac{1}{n} \log \|A^n(x).v\| = \lambda_i(x)$ *for every* $v \in E_x^i$

Moreover, the functions $\lambda_i(x)$ and E_x^i depend measurably on x.

If the probability μ is ergodic then the function $k(x)$, the numbers $\lambda_i(x)$ and the dimensions of the *Oseledets subspaces* E_x^i are constant almost everywhere. The number $\dim E_x^i$ is the *multiplicity* of the *Lyapunov exponent* $\lambda_i(x)$. The set of all Lyapunov exponents counted with its multiplicity, we call *Lyapunov spectrum* of the cocycle. It is *simple* if all of the exponents have multiplicity 1.

1.3 Random Matrices

Let us see now how to consider *random matrices* as a locally constant linear cocycle, in fact we will "construct" the latter from the random matrices.

Consider $X = Sp(2d, \mathbb{R})$ and $M = X^{\mathbb{Z}}$. Set $f : M \to M$ as the two-sided shift $f\left((\alpha_j)_{j\in\mathbb{Z}}\right) = (\alpha_{j+1})_{j\in\mathbb{Z}}$, and

$$A : M \to Sp(2d, \mathbb{R}), \quad A\left((\alpha_j)_{j\in\mathbb{Z}}\right) = \alpha_0$$

The map A takes a bi-infinite sequence of symplectic matrices and gives the matrix in the zeroth position. Thus, the linear cocycle corresponding to these choices is

$$F\left((\alpha_j)_{j\in\mathbb{Z}}, v\right) = \left((\alpha_{j+1})_{j\in\mathbb{Z}}, \alpha_0 \cdot v\right)$$

Iterating it we get

$$F^n\left((\alpha_j)_{j\in\mathbb{Z}}, v\right) = \left((\alpha_{j+n})_{j\in\mathbb{Z}}, \alpha_{n-1}\cdots\alpha_1\alpha_0\cdot v\right)$$

Given a probability ν on X, we consider $\mu = \nu^{\mathbb{Z}}$, the product measure on M. It is easy to show that μ is invariant and ergodic under the shift map.

Remark. For general subshifts of finite type the Markov measure is ergodic if and only if the stochastic matrix which defines the space is irreducible.

For now, we will just generalize a little the last construction. Consider a probability space $(Y, \mathcal{S}, m)$, and define $N = Y^{\mathbb{Z}}$, $\eta = m^{\mathbb{Z}}$. Let $g : N \to N$ be the shift map and take $B : N \to Sp(2d, \mathbb{R})$ a measurable function that depends only on the zeroth coordinate: $B(x) = B_0(x_0)$ where $B_0 : Y \to Sp(2d, \mathbb{R})$ is also a measurable function. The linear cocycle $G : N \times \mathbb{R}^{2d} \to N \times \mathbb{R}^{2d}$ associated to g and B chosen as above is called *locally constant.* Note that, in this case the cocycle G is semi-cojugated to the one we defined for random matrices. We have the following diagram:

$$\begin{tikzcd} M \times \mathbb{R}^{2d} \arrow[r, "F"] & M \times \mathbb{R}^{2d} \\ N \times \mathbb{R}^{2d} \arrow[r, "G"] \arrow[u, "\Phi"] & N \times \mathbb{R}^{2d} \arrow[u, "\Phi"'] \end{tikzcd}$$

where $\Phi : N \times \mathbb{R}^{2d} \to M \times \mathbb{R}^{2d}$ is defined by $\Phi((x_k)_k, v) = ((B_0(x_k))_k, v)$ and the probability ν on $X = Sp(2d, \mathbb{R})$ is chosen such that $m = (B_0)_*\nu$.

Consider another map $\hat{B} : N \to Sp(2d, \mathbb{R})$ and the cocycle $\hat{G}$ associated to g and $\hat{B}$. We say that $\hat{G}$ is close to G, as above, when $\hat{B}$ also depends only on the zeroth coordinate and $\|\hat{B}(x) - B(y)\|$ is small whenever $x_0 = y_0$. In other words, B and $\hat{B}$ are constant in the same cylinders and the corresponding matrices are close.

Chapter 2

The Main Lemma

The goal of this chapter is to prove a lemma that will be very helpful. It is a kind of algebraic version of the dynamical theorems of this thesis. It concerns of "moving" eigenspaces of an operator of a symplectic vector space in a convenient way using a special kind of transformation called *transvections*, which will be defined in the section 2.2.

Lemma 2.1 (Main Lemma). *Let (E, ω) be a $2d$-dimensional symplectic vector space and $V \oplus W$ any decomposition of E. Suppose that $A \in SL(E)$ is an operator such that $A(V) \cap W$ has dimension $k > 0$.*

Then, there exist k symplectic transvections $\sigma_1, ..., \sigma_k$ of the space E such that all of them are close to the identity and for $\sigma = \sigma_1...\sigma_k$ we have

$$(\sigma A)(V) \cap W = \{0\}$$

Before proving it, let us recall some notions of symplectic linear algebra. If you are conversant with this subject, you can skip next section and move straight to the section about transvections.

2.1 Symplectic Basics

In this section we will see some of the basics of symplectic linear algebra. Since most of the proofs are either easy or classical, we will just outline the results without care about demonstrating them.

First of all, consider V a vector space of even dimension, say $2d$. In this theory a special role is played by the following $2d \times 2d$ matrix:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where 0 and I represent the null and identity matrices of order n, respectively. Observe that J is orthogonal and antisymmetric, i.e.

$$J^{-1} = J^t = -J$$

A real matrix T of order $2d$ is called *symplectic* if $T^tJT = J$. The set of all $2d \times 2d$ real symplectic matrices is denoted by $Sp(2d, \mathbb{R})$. A symplectic matrix always have determinant 1 and its inverse can be calculated by the formula $T^{-1} = -JT^tJ$. As a group, $Sp(2d, \mathbb{R})$ is a subgroup of $SL(2d, \mathbb{R})$.

Fix some inner product $\langle \cdot, \cdot \rangle$ in V, some orthogonal and antisymmetric matrix J and define the following bilinear form $\omega : V \times V \to \mathbb{R}$, $\omega(u, v) = \langle u, Jv \rangle$. We can easily check that ω is an alternating and nondegenerate bilinear form. Such forms are called *symplectic forms*. It is known that all of them are equivalent, hence there is no loss when we fix one form once for all. A simple calculation shows that for a symplectic matrix T we have

$$\omega(Tu, Tv) = \langle Tu, Jtv \rangle = \left\langle u, T^tJTv \right\rangle = \langle u, Jv \rangle = \omega(u, v)$$

From this equalities we can conclude that, T is symplectic if, and only if, it preserves the symplectic form ω: $T_*\omega = \omega$.

A basis $\{v_1, ..., v_{2d}\}$ of V is called a *symplectic basis* if the matrix $[\omega(v_i, v_j)]_{i,j}$ is exactly the matrix J. Every symplectic space has a symplectic basis, and all of such are isomorphic. Thus we can always consider $V = \mathbb{R}^{2d}$, for which the standard basis is a symplectic basis, considering the form defined with the matrix J we defined in the beginning of this section. In general, the matrix J is just the coefficient matrix of the symplectic form in a symplectic basis.

Consider a subspace $F \subset V$. Define $F^\omega = \{u \in V \ ; \ \omega(u, v) = 0 \ \forall v \in F\}$ the symplectic orthogonal of F. We can prove that $\dim F^\omega + \dim F = \dim V$. But, in general, we do not have the direct sum $V = F \oplus F^\omega$.

A subspace is called *isotropic* when $F \subseteq F^\omega$, and *coisotropic* when $F^\omega \subseteq F$. SinceMan, there are so many changes happening in our lifes, in different areas, and we do believe that our ministry is about to change too... we still don't know exactly what is about to happen, although we feel free to follow the voice of the Holy Spirit... as I'm saying to you, we feel that connection with you... we do believe in that vision, and we are very identified with it... but, on the other hand, we have to be cautious with all we have today... "wise heart will know the proper time and procedure." Eclesiastes 8:5 We know that God has great things to us... and we do not have the right of mess everything up. the symplectic form is alternating, every line in V is isotropic and by duality every hyperplane is coisotropic. If the subspace is both isotropic and coisotropic, then it is called *Lagrangian*, $F = F^\omega$. In this case, we have necessarily $\dim F = \frac{\dim V}{2}$.

If the restriction of ω to F is still a nondegenerate form, then F is called *symplectic*. By necessity F must be of even dimension and so (F, ω) is a symplectic space. In this case, the symplectic orthogonal F^ω will be symplectic too, and $V = F \oplus F^\omega$. Conversely, if $V = F \oplus G$ and $\omega(F, G) = 0$, then F and G are symplectic.

Let F be a Lagrangian subspace of V. Then $J(F)$ is also Lagrangian and we can write $V = F \oplus J(F)$. Moreover, if $V = F \oplus G$ with F and G Lagrangian, then any basis of F can be completed with vectors of G to a symplectic basis of V.

2.2 Transvections: an useful tool

A *transvection* is a special kind of linear transformation with a particular geometric behaviour. Finding the right parameters and composing with tranformations, we can change them a little without affect their properties in a big set, namely, in a hyperplane. The easiest way to view a transvection is as a generalization of a planar shear. Such transformation fixes a line on the plane and the other vectors are shifted parallel to this line.

A linear map $\tau : \mathbb{R}^m \to \mathbb{R}^m$ is called a *transvection* if there are a hyperplane $H \subset \mathbb{R}^m$ and a vector $v \in H$ such that the restriction $\tau|_H$ is the identity on H and for any vector $u \in \mathbb{R}^m$, $\tau(u) - u$ is a multiple of v.

We can write $(\tau - I)(u) = \lambda(u)v$, where λ is a linear functional of $\mathbb{R}^m$ satisfying $H = \ker \lambda$. So,

$$\tau(u) = u + \lambda(u).v, \quad \lambda \in (\mathbb{R}^m)^*, \quad H = \ker \lambda$$

Consider $(\mathbb{R}^{2d}, \omega)$ a symplectic space. Which transvections are symplec-

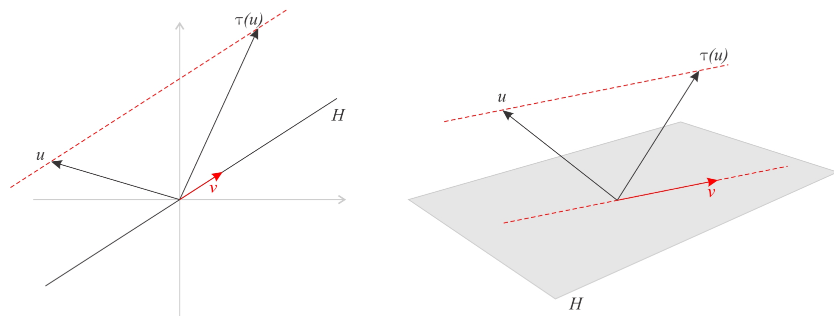

Figure 2.1: transvection on $\mathbb{R}^2$ and $\mathbb{R}^3$

tic? If $\tau \in Sp(2d, \mathbb{R})$, then

$$\begin{aligned} \omega(u, u') &= \omega(\tau(u), \tau(u')) \\ &= \omega(u + \lambda(u).v, u' + \lambda(u').v) \\ &= \omega(u, u') - \lambda(u)\omega(u', v) + \lambda(u')\omega(u, v) \end{aligned}$$

Hence, $\lambda(u)\omega(u', v) = \lambda(u')\omega(u, v)$ for every $u, u' \in \mathbb{R}^{2d}$. Choosing u' such that $\omega(u', v) = 1$ and calling $a = \lambda(u')$, we get the following general formula for a symplectic transvection:

$$\tau_{v,a}(u) = u + a.\omega(u, v).v$$

Although it will not be used here, it is interesting to point out that the set of symplectic transvections generate the whole symplectic group (for the proof, see Appendix A). This fact motivated us to use transvections to make perturbations in order to prove the main lemma in the next section.

2.3 Proof of the Main Lemma

Main Lemma Bis. *Let (E, ω) be a $2d$-dimensional symplectic vector space and $V \oplus W$ any decomposition of E. Suppose that $A \in SL(E)$ is an operator such that $A(V) \cap W$ has dimension $k > 0$.*

Then, there exist k symplectic transvections $\sigma_1, ..., \sigma_k$ of the space E such that all of them are close to the identity and for $\sigma = \sigma_1 ... \sigma_k$ we have

$$(\sigma A)(V) \cap W = \{0\}$$

proof.

Let us proceed by induction on $k = \dim A(V) \cap W$.

Define $m = \dim V$. Obviously, $k \leqslant \min\{m, 2d - m\}$.

If $k = 1$, then we can find basis of the subspaces such that

$$\begin{aligned} A(V) &= \langle\{v_1, v_2, ..., v_m\}\rangle \\ W &= \langle\{v_1, w_2, ..., w_{2d-m}\}\rangle \end{aligned}$$

Note that both $V_0 = A(V) + W$ and $V_1 = \{v_1\}^\omega$ are a hyperplanes on E.

Take $u_0 \notin V_0 \cup V_1$ and $\varepsilon > 0$. Denote by $\sigma : E \to E$ the symplectic transvection associated to u_0 and ε:

$$\sigma(u) = u + \varepsilon.\omega(u, u_0)u_0$$

Recall that if we restrict σ to the hyperplane orthogonal to the line spanned by u_0, we get the identity there; and more, for any $u \in E$ if we compute the difference vector $\sigma(u) - u$ we always get a multiple of u_0. Hence, taking a convenient ε we obtain σ as close to the identity as we want.

Now let us finish the first step of the induction. Take any $v \in (\sigma A)(V) \cap W$. So, there are numbers $\alpha_1, ..., \alpha_m$ and $\beta_1, ..., \beta_{2d-m}$ such that

$$\sum_{i=1}^{m} \alpha_i \sigma(v_i) = v = \beta_1 v_1 + \sum_{i=2}^{2d-m} \beta_i w_i \tag{2.1}$$

If we write $\sigma(v_i) = v_i + K_i u_0$, with $K_i = \varepsilon\ \omega(v_i, u_0)$, the equality above gives us the following

$$(\alpha_1 - \beta_1)v_1 + \sum_{i=2}^{m} \alpha_i v_i - \sum_{i=2}^{2d-m} \beta_i w_i + \left(\sum_{i=1}^{m} \alpha_i K_i\right) u_0 = 0 \tag{2.2}$$

Since $u_0 \notin V_0$, the set $\{v_1, ..., v_m, w_2, ..., w_{2d-m}, u_0\}$ is a basis of E, then all the coefficients in (2.2) are zero. Then

$$\begin{cases} \alpha_1 = \beta_1 \\ \alpha_i = 0, \quad i = 2, ..., m \\ \beta_i = 0, \quad i = 2, ..., 2d - m \\ \sum\limits_{i=1}^{m} \alpha_i K_i = 0 \end{cases}$$

Putting together the second and the last equations of this system, the sum is reduced to $\alpha_1 K_1 = 0$, that is, $\varepsilon.\alpha_1.\omega(v_1, u_0) = 0$. Since $u_0 \notin V_1$, we have $\omega(v_1, u_0) \neq 0$, and so we get $\alpha_1 = 0$. Thus, we conclude that the generic vector v which was in the intersection must be the zero vector, as we would like. We have got the step one of the induction. Let us proceed to the inductive step.

Suppose now that the lemma is true for $k-1$, and let $A(V) \cap W = \langle\{v_1, ..., v_k\}\rangle$. In this case we can find basis for $A(V)$ and W such that

$$\begin{aligned} A(V) &= \langle\{v_1, v_2, ..., v_m\}\rangle \\ W &= \langle\{v_1, ..., v_k, w_{k+1}, ..., w_{2d-m}\}\rangle \end{aligned}$$

As before, we define $V_0 = A(V) + W$ and $V_1 = \{v_1, ..., v_k\}^{\omega}$, which are, in this case, subspaces of E of codimension k. The definition of σ_k is exactly the same as σ in the previous case:

$$\sigma_k(u) = u + \varepsilon.\omega(u, u_0)u_0, \quad \varepsilon > 0, \ u_0 \notin V_0 \cup V_1$$

For a generic $v \in (\sigma_k A)(V) \cap W$ we will obtain the same expression (2.1), but the equality (2.2) becomes slightly different

$$\sum_{i=1}^{k} (\alpha_i - \beta_i) v_i + \sum_{i=k+1}^{m} \alpha_i v_i - \sum_{i=k+1}^{2d-m} \beta_i w_i + \left(\sum_{i=1}^{m} \alpha_i K_i \right) u_0 = 0 \qquad (2.3)$$

Clearly, in this case the set $\{v_1, ..., v_m, w_{k+1}, ..., w_{2d-m}, u_0\}$ is not a basis for the whole space E, but the choose $u_0 \notin V_0$ implies that this set is linearly independent, what is sufficient to give us the same conclusion about the coefficients of the linear combination above.

$$\begin{cases} \alpha_i = \beta_i, \quad i = 1, ..., k \\ \alpha_i = 0, \quad i = k+1, ..., m \\ \beta_i = 0, \quad i = k+1, ..., 2d-m \\ \sum\limits_{i=1}^{m} \alpha_i K_i = 0 \end{cases}$$

Here, $\alpha_i = 0$ for $i > k$, and so the sum loses its last terms. Thus, we get $\sum\limits_{i=1}^{k} \alpha_i K_i = 0$. Note that it is equivalent to

$$\varepsilon \sum_{i=1}^{k} \alpha_i.\omega(v_i, u_0) = 0 \Longleftrightarrow \omega\left(\sum_{i=1}^{k} \alpha_i v_i, u_0 \right) = 0$$

Therefore we have,

$$(\sigma_k A)(V) \cap W = \left\{ v = \sum_{i=1}^{k} \alpha_i v_i = (\alpha_1, ..., \alpha_k) \;\middle|\; \omega\left(\sum_{i=1}^{k} \alpha_i v_i, u_0\right) = 0 \right\}$$

Since $u_0 \notin V_1$ this subspace has dimension $k-1$.

By the induction hypothesis, there are $\sigma_1, ..., \sigma_{k-1}$ symplectic automorphisms, all close to the identity, and such that

$$\left[(\sigma_1 ... \sigma_{k-1})(\sigma_k A)\right](V) \cap W = \{0\}$$

By taking $\sigma = \sigma_1 ... \sigma_k$ we finish the proof of the lemma □

Observe that the conclusion of the previous lemma is an open condition, meaning that every B in a neighbourhood of σA has the same property:

$$B(V) \cap W = \{0\}$$

Suppose now that $(\sigma A)(V') \cap W' \neq \{0\}$ for another decomposition $E = V' \oplus W'$. Applying the lemma, we can find a symplectic automorphism τ, close to identity, in order to obtain $\tau\sigma A(V') \cap W' = \{0\}$. Regarding the observation above, if τ is close enough to identity we get also $\tau\sigma A(V) \cap W = \{0\}$. With it, is easy to conclude the same for any finite number of decompositions of E:

Corollary 2.2. *Let (E, ω) be a $2d$-dimensional symplectic vector space, $A \in GL(E)$ and $m \in \mathbb{N}$. If $E = V_1 \oplus W_1 = ... = V_m \oplus W_m$, then there exist a symplectic automorphism σ close to the identity and such that*

$$(\sigma A)(V_i) \cap W_i = \{0\} \quad \forall i = 1, .., m$$

If we focus on the group $Sp(E)$ of symplectic automorphisms of E, we get the following conclusion

Corollary 2.3. *Given any finite number of decompositions of a symplectic space $(E, \omega) = V_1 \oplus W_1 = ... = V_m \oplus W_m$, the set*

$$\mathcal{R} = \{A \in Sp(E) \mid A(V_i) \cap W_i = \{0\} \; \forall i = 1, ..., m\}$$

is open and dense in $Sp(E)$.

2.4 Spectrum of Symplectic Operators

Before finishing the chapter let us say a few words about the spectrum of a symplectic operator. Let T be a real symplectic matrix and $p(\lambda)$ be its characteristic polynomial. Is easy to see that $p(\lambda)$ is a reciprocal polynomial. Indeed,

$$\begin{aligned} p(\lambda) &= \det(T-\lambda I) = \det(T^t - \lambda I) = \det(-JT^{-1}J - \lambda I) \\ &= \det(-JT^{-1}J + \lambda JJ) = \det(-T^{-1} + \lambda I) = \det(T^{-1})\det(-I + \lambda T) \\ &= \lambda^{2d}\det(T - \lambda^{-1} I) = \lambda^{2d} p(\lambda^{-1}) \end{aligned}$$

Thus, if μ is an eigenvalue of a real symplectic matrix, then so are μ^{-1}, $\overline{\mu}$ and $\overline{\mu}^{-1}$. Also from this formula, we can conclude that if 1 or -1 are eigenvalues of T, then they must have even algebraic multiplicity.

Consider now two eigenvalues λ, μ of T such that $\lambda\mu \neq 1$. A standard calculation shows that the corresponding eigenvectors are ω-orthogonal to each other. Thus, if one matrix T has all its eigenvalues real and pairwise distincts we can find a basis of $\mathbb{R}^{2d}$ where the matrix of T has the form $\mathrm{diag}(\lambda_1, ..., \lambda_d, \lambda_1^{-1}, ..., \lambda_d^{-1})$. If some eigenvalue is a complex number then we have to consider 2×2 blocks like

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad \alpha^2 + \beta^2 = 1$$

Let us see how it works. First of all, as T has $2d$ distinct eigenvalues none of them is equal to 1. We can separate them into three groups: the real eigenvalues – $\{\mu_1^{\pm 1}, ..., \mu_r^{\pm 1}\}$; the complex unitary eigenvalues – $\{z_1, \overline{z}_1, ..., z_s, \overline{z}_s\}$; and the complex of modulus different from one – $\{w_1^{\pm 1}, \overline{w}_1^{\pm 1}, ..., w_t^{\pm 1}, \overline{w}_t^{\pm 1}\}$. Of course, $2r + 2s + 4t = 2d$. Denote by $E_\lambda = \ker(T - \lambda I)$ the eigenspace associated to the eigenvalue λ. These groups define a T-invariant splitting of the whole space

$$\mathbb{R}^{2d} = \left(\bigoplus_{j=1}^{r} U_j\right) \oplus \left(\bigoplus_{j=1}^{s} V_j\right) \oplus \left(\bigoplus_{j=1}^{t} W_j\right)$$

where

$$\begin{aligned} U_j &= E_{\mu_j} \oplus E_{\mu_j^{-1}} \\ V_j &= E_{z_j} \oplus E_{\overline{z}_j} \\ W_j &= E_{w_j} \oplus E_{w_j^{-1}} \oplus E_{\overline{w}_j} \oplus E_{\overline{w}_j^{-1}} \end{aligned}$$

Since each subspace above is ω-orthogonal to every other, we can safely say that they are all symplectic subspaces. If we choose a symplectic basis to each one separately and after we put them together we will get a symplectic basis for the whole space. The point here is that this splitting allows us to write the matrix of T in blocks of size at most 2 along the diagonal.

- There is a symplectic basis of each U_j such that the matrix of the restriction of T to it has the form

$$\begin{bmatrix} \mu_j & 0 \\ 0 & \mu_j^{-1} \end{bmatrix}$$

- There is a symplectic basis of each V_j such that the matrix of the restriction of T to it has the form

$$\begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix},$$

$$\text{where } z_j = \alpha_j + i\beta_j, \text{ with } \alpha_j^2 + \beta_j^2 = 1$$

- There is a symplectic basis of each W_j such that the matrix of the restriction of T to it has the form

$$\begin{bmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{bmatrix}_{4\times 4},$$

$$\text{where } B = \begin{bmatrix} \gamma_j & -\delta_j \\ \delta_j & \gamma_j \end{bmatrix}, \quad w_j = \gamma_j + i\delta_j, \text{ with } \gamma_j^2 + \delta_j^2 \neq 1$$

Chapter 3

The Locally Constant Case

The goal of this chapter is to prove Theorem A which, essentially says that among locally constant symplectic cocycles, the ones exhibiting the twisting and pinching properties belong to a dense (and open) subset. At least, this is the part of the theorem that we have to care about, since the other statements were justified.

Consider a two-sided subshift of finite type $f : \Sigma_T \to \Sigma_T$ associated to an irreducible matrix, T, with coefficients in $\{0, 1\}$, and a locally constant function $A : \Sigma_T \to Sp(2d, \mathbb{R})$. Let $F_A : \Sigma_T \times \mathbb{R}^{2d} \to \Sigma_T \times \mathbb{R}^{2d}$, $F_A(\mathtt{x}, u) = (f(\mathtt{x}), A(\mathtt{x})u)$ be the cocycle associated to A. As usual, in most of time we will refer to the map A as the cocycle. Since A is locally constant and it takes values on the symplectic group, we say that the cocycle is locally constant and symplectic.

Suppose, in addition, that the map A depends only on the 0-th symbol of the element in Σ_T, that is, A is constant in each one of the cylinders $[0; j]$ with $1 \leqslant j \leqslant d$. For $\mathtt{x} \in [0; j]$ we will write $A(\mathtt{x}) = A_j$.

The local stable and unstable set of a point $\mathtt{x}$ are defined as $W^s_{loc}(\mathtt{x}) = \{\mathtt{y} = (y_j)_{j\in\mathbb{Z}} \,;\, y_j = x_j \ \forall j \geq 0\}$ and $W^u_{loc}(\mathtt{x}) = \{\mathtt{y} = (y_j)_{j\in\mathbb{Z}} \,;\, y_j = x_j \ \forall j < 0\}$.

Given a periodic point $\mathtt{p}$ and a point $\mathtt{q} \in W^u_{loc}(\mathtt{p})$, we say that $\mathtt{q}$ is a *homoclinic point* associated to $\mathtt{p}$ if there is some $\ell \geq 1$ such that $f^\ell(\mathtt{q}) \in W^s_{loc}(\mathtt{p})$.

3.1 Pinching and Twisting

We say that a locally constant cocycle $A : \Sigma_T \to Sp(2d, \mathbb{R})$ is *simple* if there exist a periodic point $\mathtt{p}$ for the subshift $f : \Sigma_T \to \Sigma_T$ and a related homoclinic point $\mathtt{q}$ such that

[pinching] All eigenvalues of $A^{per(\mathtt{p})}(\mathtt{p})$ have distinct absolute value

[twisting] For any invariant subspaces (sums of eigenspaces) E and F of $A^{per(\mathtt{p})}(\mathtt{p})$ with complementary dimensions holds that $A^{\ell}(\mathtt{q})(E) \cap F = \{0\}$, where ℓ is some multiple of $per(\mathtt{p})$ such that $f^{\ell}(\mathtt{q}) \in W^s_{loc}(\mathtt{p})$

In [3], Avila and Viana proved that these two conditions are sufficient for the cocycle to have simplicity on the Lyapunov spectrum. We are going to verify them in our setting.

Remark. Since we are considering the symplectic group we could relax a little the twisting definition restricting to pairs of subspaces where E is isotropic and F is coisotropic. On the other hand, once we have a weaker twisting property, if we are aiming for the cocycle to be simple, then we have to consider a strong version of pinching, which Avila and Viana called *strongly pinching*. They prove that if a symplectic cocycle is strongly pinching and twists isotropic subspaces then it is simple. See [4] (sections 4.2 and 4.3).

Of course, if B is another symplectic and locally constant cocycle close enough to a simple cocycle A then it will be simple too. Remember that in this kind of perturbation we are changing just the matrices of the cocycle, that is, the base dynamics is fixed.

Proposition 3.1. *Inside every neighborhood of a symplectic locally constant cocycle A, there exist a cocycle B and a periodic orbit $\hat{\mathtt{p}} \in \Sigma_T$ such that, B is constant in the same cylinders as A, and all eigenvalues of $B^{per(\hat{\mathtt{p}})}(\hat{\mathtt{p}})$ are real and have distinct norms.*

proof. Let $\mathtt{p}$ be a periodic point for f. Up to an initial perturbation, we can assume that $A^{per(\mathtt{p})}(\mathtt{p})$ has all of its eigenvalues with multiplicity 1, with distinct norms except for a number $c \geq 0$ of pairs of complex conjugated eigenvalues. If $c = 0$ then there is nothing to prove and the theorem is done. So, from now on $c > 0$. Let us assume, for simplicity, that the point $\mathtt{p}$ is fixed.

Suppose $\mathtt{p} = (..., \underline{i}, i, ..., i, ...) \in [0; i]$ and let $\mathtt{q} \in [0; i]$ be an homoclinic point associated to $\mathtt{p}$. (The underline indicates the 0-th position).

$$\mathtt{q} = (..., i, ..., \underline{i}, i_1, ..., i_{\ell-1}, i, ..., i, ...)$$

In this case we get $A(\mathtt{p}) = A_i$ and $A^\ell(\mathtt{q}) = A_{i_{\ell-1}} \cdot ... \cdot A_{i_1} \cdot A_i$

The homoclinic point above can be chosen such that ℓ is minimal. In this case, the symbols $i, i_1, ..., i_{\ell-1}$ are different.

The following result comes from the main lemma.

Lemma 3.2. *Let $T \in Sp(2d, \mathbb{R})$ be any symplectic linear transformation of the space $\mathbb{R}^{2d}$ and $m \in \mathbb{N}$. If there are decompositions $\mathbb{R}^{2d} = V_1 \oplus W_1 = ... = V_m \oplus W_m$, then in every neighborhood of T there is a symplectic automorphism $\tilde{T}$ such that*

$$\tilde{T}(V_i) \cap W_i = \{0\} \quad \forall i = 1, .., m$$

This lemma allows us to make a little perturbation of $A_{i_{\ell-1}}$, if necessary, in order to get some transversality between any two (sums of) eigenspaces E^I, E^J of $A_i = A(\mathtt{p})$ such that $\dim E^I + \dim E^J = 2d$. That is,

$$A^\ell(\mathtt{q})(E^I) \cap E^J = \{0\}$$

It is important because with this we can talk about *dominated decomposition*.

For each n, let $\mathtt{x}_n$ be the periodic point of period $\ell + n$ defined by the itinerary $(i, i_1, ..., i_{\ell-1}, i, ..., i)$. They are kind of truncations of the orbit of the homoclinic point $\mathtt{q}$, where the symbol i appears $n+1$ times. Consider the compact sets K_n obtained as the closure of the union of the orbits of $\mathtt{x}_m$ over all $m \leqslant n$.

Lemma 3.3. *For every large enough n, the cocycle A admits a dominated decomposition $E^1 \oplus ... \oplus E^k$ over the set K_n coinciding with the eigenspaces decomposition at the point $\mathtt{p}$.*

proof. This lemma can be viewed as a particular case of the lemma 4.3 of the next chapter: the case when the holonomies $\phi^s_{p,z}$, $\phi^u_{p,z}$ are trivial, that is, all equal to identity •

The conclusion of this last lemma is robust. It holds for every cocycle B in C^0 neighborhood $\mathcal{U}$ of A. The cocycle B admits a dominated decompositon over K_n, the map

$$(B, \mathtt{x}) \longmapsto E^1_{B,\mathtt{x}} \oplus ... \oplus E^k_{B,\mathtt{x}}$$

is continuous with respect to the first variable and the spaces $E^j_{B,\mathtt{x}}$ and $E^j_{A,\mathtt{p}}$ have the same dimension and are uniformly close.

Let $j_0 = \min\{1 \leqslant j \leqslant k \; ; \; \dim E^j = 2\}$. Let μ_n be the eigenvalue of the operator $A^{\ell+n}(\mathtt{x}_n)$ restricted to the two-dimensional subspace $E^{j_0}_{A,\mathtt{x}_n}$.

Perturbing the cocycle, if necessary, we can assume that the eigenvalue μ_n is real, that is, the operator restricted to the subspace is a multiple of the identity. Let us see how to make such perturbation:

Consider the symplectic invariant space $\mathcal{E}^{j_0}_{A,\mathtt{x}_n} = E(\mu_n) \oplus E(\mu_n^{-1})$. Note that if μ_n is unitary, then $\dim \mathcal{E}^{j_0}_{A,\mathtt{x}_n} = 2$. Otherwise, $\dim \mathcal{E}^{j_0}_{A,\mathtt{x}_n} = 4$

The restriction of $A^{\ell+n}(\mathtt{x}_n)$ to $\mathcal{E}^{j_0}_{A,\mathtt{x}_n}$ has one of the two forms below

$$\begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix}_{2\times 2} \quad \text{for } \mu_n = e^{i\theta_n}$$

$$\begin{bmatrix} r_n \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix} & 0 \\ 0 & r_n^{-1} \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix} \end{bmatrix}_{4\times 4} \quad \text{for } \mu_n = r_n e^{i\theta_n}$$

For any cocycle B, define $\rho(n, B)$ as the rotation number of $B^{\ell+n}(\mathtt{x}_n)$ restricted to $\mathcal{E}^{j_0}_{B,\mathtt{x}_n}$. With respect to the notation above, $\rho(n, A) = \theta_n$.

Consider also, for any continuous arc of cocycles $\mathcal{B} = \{B_t \; ; \; t \in I\}$, the number $\delta(n, \mathcal{B})$ defined as the oscillation of $\rho(n, B_t)$ over all $t \in I$.

Lemma 3.4. *There exist a continuous arc $\mathcal{A} = \{A_t \; ; \; t \in [0,1]\}$ of locally constant cocycles in $\mathcal{U}$, depending on the 0-th coordinate and such that $A_0 = A$ and $\forall t > 0$ there is a $n_t > 1$ such that for $n \geq n_t$*

$$\delta\left(n, \{A_s\}_{s\in[0,t]}\right) > 1$$

proof. see lemma 4.4 in the next chapter. •

Hence we have for every $t \in [0,1]$ and n large enough, $A_t^{\ell+n}(\mathtt{x}_n)$ has at most c pairs of complex conjugated eigenvalues; by the previous lemma we can claim that there is a t close to 0 and a natural $n \geq 1$ such that $\rho(n, A_t) \in \mathbb{Z}$. Then $A_t^{\ell+n}(\mathtt{x}_n)$ has an real eigenvalue on $\mathcal{E}^{j_0}_{A_t,\mathtt{x}_n}$. In fact, we get for the restriction, in a convenient basis

$$A_t^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}^{j_0}_{A_t,\mathtt{x}_n}} = \begin{cases} diag(a, a, a^{-1}, a^{-1}) & \text{if } |\mu| \neq 1 \\ diag(\pm 1, \pm 1) & \text{if } |\mu| = 1 \end{cases}$$

Define, for a small ε, the symplectic matrix I_ε such that

$$I_\varepsilon|_{\mathcal{E}^{j_0}_{A_t,\mathtt{x}_n}} = \begin{cases} diag(1+\varepsilon, 1+\frac{\varepsilon}{2}, (1+\varepsilon)^{-1}, (1+\frac{\varepsilon}{2})^{-1}) & \text{if } |\mu| \neq 1 \\ diag(1+\varepsilon, (1+\varepsilon)^{-1}) & \text{if } |\mu| = 1 \end{cases}$$

and is the identity restricted to the other subspaces of the decomposition. Clearly, $\|I - I_\varepsilon\| < \varepsilon$

Now we will define the perturbation of the cocycle A_t. We need a map

$$\tilde{A} : \Sigma_T \to Sp(2d, \mathbb{R})$$

constant on each cylinder $[0; j]$ for $1 \leqslant j \leqslant d$ and such that $\tilde{A}^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}^{j_0}_{\tilde{A},\mathtt{x}_n}}$ have distinct real eigenvalues.

Put, for $\mathtt{x} \in [0; j]$

$$\tilde{A}(\mathtt{x}) = \begin{cases} A_t(\mathtt{x}) & , \text{if } j \neq i_{\ell-1} \\ I_\varepsilon \cdot A_t(\mathtt{x}) & , \text{if } j = i_{\ell-1} \end{cases} = \begin{cases} A_{t,j} & , \text{if } j \neq i_{\ell-1} \\ I_\varepsilon \cdot A_{t,i_{\ell-1}} & , \text{if } j = i_{\ell-1} \end{cases}$$

We have modified A_t far from the fixed point $\mathtt{p}$. The periodic point $\mathtt{x}_n$ starts its orbit at the same cylinder of $\mathtt{p}$, then it goes away, far from $\mathtt{p}$ for a while and finally goes back to the cylinder, going each time closer to $\mathtt{p}$. Since the symbols were chosen to be all different, there is no danger in changing the cocycle in any one of the symbols that correspond to cylinders far from $\mathtt{p}$.

It is easy to see that this perturbed cocycle is close to A_t. Indeed, denoting by $C = [0; i_{\ell-1}]$ the cylinder with 0-th symbol $i_{\ell-1}$ and by L the supremum of $\|A_t(\mathtt{x})\|$ restricted to C, we have got

$$\begin{aligned} \sup_{\mathtt{x}\in\Sigma_T} \left\| \left(A_t - \tilde{A} \right)(\mathtt{x}) \right\| &= \sup_{\mathtt{x}\in C} \left\| \left(A_t - \tilde{A} \right)(\mathtt{x}) \right\| \\ &= \sup_{\mathtt{x}\in C} \|A_t(\mathtt{x}) - I_\varepsilon A_t(\mathtt{x})\| \\ &\leqslant \|I - I_\varepsilon\| \cdot \sup_{\mathtt{x}\in C} \|A_t(\mathtt{x})\| \\ &< \varepsilon \cdot L \end{aligned}$$

Now, what could we say about the eigenvalues of the cocycle $\tilde{A}$?

Lemma 3.5. *The eigenvalues of $\tilde{A}^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}^{j_0}_{\tilde{A},\mathtt{x}_n}}$ are real and have distinct norms.*

proof. Let $\mathtt{z}_n = f^\ell(\mathtt{x}_n)$. It is obvious that $f^n(\mathtt{z}_n) = \mathtt{x}_n$. Since they belong to the same periodic orbit it is natural that the matrices $A_t^{\ell+n}(\mathtt{x}_n)$ and $A_t^{\ell+n}(\mathtt{z}_n)$ have the same eigenvalues. In fact, they are conjugate matrices

$$\begin{aligned} A_t^{\ell+n}(\mathtt{z}_n) &= A_t^\ell(\mathtt{x}_n) \cdot A_t^n(\mathtt{z}_n) \\ &= [A_t^n(\mathtt{z}_n)]^{-1} \cdot A_t^n(\mathtt{z}_n) \cdot A_t^\ell(\mathtt{x}_n) \cdot A_t^n(\mathtt{z}_n) \\ &= [A_t^n(\mathtt{z}_n)]^{-1} \, A_t^{\ell+n}(\mathtt{x}_n) \, [A_t^n(\mathtt{z}_n)] \end{aligned}$$

We also have, $\tilde{A}^{\ell+n}(\mathtt{x}_n)$ is cojugated to $I_\varepsilon A_t^{\ell+n}(\mathtt{z}_n)$

$$\begin{aligned} \tilde{A}^{\ell+n}(\mathtt{x}_n) &= \tilde{A}^n(\mathtt{z}_n) \cdot \tilde{A}^\ell(\mathtt{x}_n) \\ &= A_t^n(\mathtt{z}_n) \cdot I_\varepsilon \cdot A_t^\ell(\mathtt{x}_n) \\ &= A_t^n(\mathtt{z}_n) \cdot I_\varepsilon \cdot A_t^\ell(\mathtt{x}_n) \cdot A_t^n(\mathtt{z}_n) \cdot [A_t^n(\mathtt{z}_n)]^{-1} \\ &= A_t^n(\mathtt{z}_n) \cdot I_\varepsilon A_t^{\ell+n}(\mathtt{z}_n) \cdot [A_t^n(\mathtt{z}_n)]^{-1} \end{aligned}$$

So, the conclusion is, $\tilde{A}^{\ell+n}(\mathtt{x}_n)$ and $I_\varepsilon A_t^{\ell+n}(\mathtt{x}_n)$ have the same eigenvalues. Since the last expression has obviously distinct and real eigenvalues when restricted to its correspondent $\mathcal{E}^{j_0}$, the lemma is proved •

Thus we get a cocycle $\tilde{A}$, close to A, and a periodic point $\mathtt{x}_n$ such that $\tilde{A}^{per(\mathtt{x}_n)}(\mathtt{x}_n)$ has at most $c-1$ complex conjugated eigenvalues. Repeating this process finitely many times we will get a cocycle as required in the proposition. So, we finished the proof of the Proposition 3.1 □

For the twisting part, we may just repeat the argument we did during the proof of the proposition 3.1. That is, let $\hat{\mathtt{p}}, \hat{\mathtt{q}} \in [0; i]$ be a fixed point and a related homoclinic point, respectively.

$$\hat{\mathtt{p}} = (..., \underline{i}, i, ..., i, ...), \ \hat{\mathtt{q}} = (..., i, ..., \underline{i}, i_1, ..., i_{\ell-1}, i, ..., i, ...)$$

Remember, ℓ was chosen such that the symbols $\{i, i_1, ..., i_{\ell-1}\}$ are all different. So, we can modify the cocycle in the orbit of $\hat{\mathtt{q}}$ without affect the value of the cocycle on $\mathtt{p}$, just perturbing the matrices of the cylinders $[0, i_\alpha]$ for $\alpha = 1, ..., \ell - 1$.

Thus, by the lemma 3.2 (or applying directly the main lemma), we can perturb $\tilde{A}_{i_{\ell-1}}$ such that, given any sums of eigenspaces of $\tilde{A}(\hat{\mathtt{p}})$, E, F with complementary dimensions we have

$$\tilde{A}^{\ell}(\hat{\mathtt{q}})(E) \cap F = \sigma \tilde{A}_{i_{\ell-1}} \cdot \ldots \cdot \tilde{A}_{i_1} \cdot \tilde{A}_i(E) \cap F = \{0\}$$

This gives us the density of the twisting property and, together with the Proposition 3.1, proves the Theorem A.

Remark. If we are in the case when Σ_T does not have fixed points, there will be a periodic point whose orbit does not contain all symbols (since we are implicitly considering that Σ_T is not a single periodic orbit). Hence we can pick up a homoclinic point associated to it such that the nonperiodic part of its orbit does not contain the symbols which are in the itinerary of the periodic point. So, we can apply the ideas above to cover this case.

Chapter 4

The Dominated Case

In this chapter we will consider another class of symplectic cocycles and prove the same result about simplicity. The base dynamics is the same as before, a two-sided subshift of finite type $f : \Sigma_T \to \Sigma_T$ associated to an irreducible matrix, T, with coefficients in $\{0, 1\}$. The cocycle here will be generated by a Hölder continuous map $A : \Sigma_T \to Sp(2d, \mathbb{R})$.

$$F_A : \Sigma_T \times R^{2d} \to \Sigma_T \times \mathbb{R}^{2d}$$

We are considering here the C^ν-norm in the space of C^ν maps A, $\nu \in (0, 1)$. Let η be an f-invariant ergodic probability on Σ_T with $\operatorname{supp} \eta = \Sigma_T$ and continuous local product structure.

4.1 Holonomies

The existence of stable and unstable holonomies are necessary to give sense to the calculation in the nonconstant case. For locally constant cocycles we could "move" from one fiber to another by identification, since two points in the same local stable or unstable set had the same image by the cocycle A. Now, when we are considering general Hölder continuous cocycles, it doesn't happen, so we need to impose an extra condition on A to fix it. This extra condition is the one we call domination.

We say that a cocycle A is *dominated* if there are a distance d in Σ_T and constants $\theta < 1$ and $0 < \nu < 1$ such that, up to replacing A by some power A^N,

1. $d(f(\mathtt{x}), f(\mathtt{y})) \leqslant \theta\ d(\mathtt{x}, \mathtt{y})$ and $d(f^{-1}(\mathtt{x}), f^{-1}(\mathtt{z})) \leqslant \theta\ d(\mathtt{x}, \mathtt{z})$ for every $\mathtt{y} \in W^s_{loc}(\mathtt{x})$, $\mathtt{z} \in W^u_{loc}(\mathtt{x})$ and $\mathtt{x} \in \Sigma_T$;

2. $\mathtt{x} \mapsto A(\mathtt{x})$ is ν-Hölder continuous and $\|A(\mathtt{x})\|\,\|A(\mathtt{x})^{-1}\|\,\theta^\nu < 1$ for every $\mathtt{x} \in \Sigma_T$.

Proposition 4.1. *If A is dominated there exists a bounded continuous family $\phi^u_{\mathtt{x},\mathtt{y}}$ of linear transformations of $\mathbb{R}^{2d}$ defined for every pair $\mathtt{x}, \mathtt{y} \in \Sigma_T$ in the same local unstable manifold of f, and there exists a constant $C > 0$ such that*

1. $\phi^u_{\mathtt{x},\mathtt{x}} = \mathrm{id}$ *and* $\phi^u_{\mathtt{x},\mathtt{y}} \cdot \phi^u_{\mathtt{y},\mathtt{z}} = \phi^u_{\mathtt{x},\mathtt{z}}$*;*

2. $A(\mathtt{y})^{-1} \cdot \phi^u_{f(\mathtt{x}),f(\mathtt{y})} \cdot A(\mathtt{x}) = \phi^u_{\mathtt{x},\mathtt{y}}$ *for* $\mathtt{x}, \mathtt{y} \in [0; i, j]$*;*

3. $\left\|\phi^u_{\mathtt{x},\mathtt{y}} - \mathrm{id}\right\| \leqslant C d(\mathtt{x}, \mathtt{y})^\nu$

proof. This is a standard result and its proof can be found in many places in the literature, see for instance, [6]. □

Of course, one can define a dual family of transformations $\{\phi^s_{\mathtt{x},\mathtt{y}}\}$ for $\mathtt{x}, \mathtt{y} \in \Sigma_T$ in the same local stable manifold of f. These families are called the *unstable* and *stable holonomies* of the cocycle A.

Given a fixed point $\mathtt{p} \in [0; i]$ and a related homoclinic point

$$\mathtt{q} = (..., i, ..., \underline{i}, i_1, ..., i_{\ell-1}, i, ..., i, ...) \in [0; i]$$

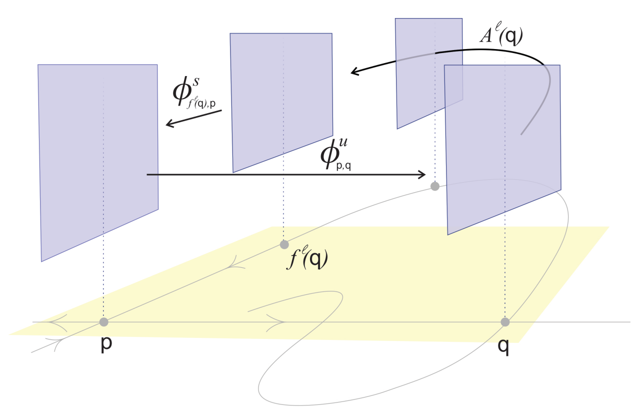

Figure 4.1: $\mathtt{q} \in W^u_{loc}(\mathtt{p}) \cap [0; i]$ e $f^\ell(\mathtt{q}) \in W^s_{loc}(\mathtt{p}) \cap [0; i]$

Suppose that ℓ is some multiple of $\mathtt{p}$ such that $f^\ell(\mathtt{q}) \in W^s_{loc}(\mathtt{p})$ and define the following operator of the fiber over $\mathtt{p}$,

$$\psi_{\mathtt{p},\mathtt{q}} = \phi^s_{f^\ell(\mathtt{q}),\mathtt{p}} \circ A^\ell(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}$$

This map is the natural substitute for $A^\ell(\mathtt{q})$ in the definition of twisting property for locally constant cocycles. It can be proved that both unstable and stable holonomies exist in the locally constant case, but its constructions will give us identity maps. Thus $\psi_{\mathtt{p},\mathtt{q}} = \mathrm{id} \circ A^\ell(\mathtt{q}) \circ \mathrm{id} = A^\ell(\mathtt{q})$.

4.2 Simplicity

We say that a dominated cocycle $A : \Sigma_T \to Sp(2d, \mathbb{R})$ is *simple* if there exist a periodic point $\mathtt{p}$ for the subshift $f : \Sigma_T \to \Sigma_T$ and a related homoclinic point $\mathtt{q}$ such that

[pinching] All eigenvalues of $A^{per(\mathtt{p})}(\mathtt{p})$ have distinct absolute value

[twisting] For any invariant subspaces (sums of eigenspaces) E and F of $A^{per(\mathtt{p})}(\mathtt{p})$ with complementary dimensions holds that $\psi_{\mathtt{p},\mathtt{q}}(E) \cap F = \{0\}$.

Proposition 4.2. *Inside every C^ν-neighborhood of a symplectic dominated cocycle A, there exist a cocycle B and a periodic orbit $\hat{\mathtt{p}} \in \Sigma_T$ such that all eigenvalues of $B^{per(\hat{\mathtt{p}})}(\hat{\mathtt{p}})$ are real and have distinct norms.*

This theorem is the analogous of Proposition 3.1 in the previous chapter. Its proof will follow the same lines, with a little more care, eventually, because of the holonomies, which didn't exist there.

proof. As before we will suppose $\mathtt{p}$ fixed and $A(\mathtt{p})$ with all its eigenvalues of multiplicity one and distinct norms except for $c > 0$ pairs of complex conjugated numbers. The protagonists of this act are (those in the figure 4.1)

$$\begin{aligned}
\mathtt{p} &= (..., \underline{i}, i, ..., i, ...) \in [0; i] \\
\mathtt{q} &= (..., i, ..., \underline{i}, i_1, ..., i_{\ell-1}, i, ..., i, ...) \in [0; i] \\
\psi_{\mathtt{p},\mathtt{q}} &= \phi^s_{f^\ell(\mathtt{q}),\mathtt{p}} \circ A^\ell(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}
\end{aligned}$$

The homoclinic point above can be chosen such that ℓ is minimal. In this case, the symbols $i, i_1, ..., i_{\ell-1}$ are different.

Given any splitting $V \oplus W = \mathbb{R}^{2d}$. Define the following subspaces:

$$V_{(u)} = \left[A^{\ell-1}(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}\right](V) \quad , \; W_{(s)} = \left[\phi^s_{f^\ell(\mathtt{q}),\mathtt{p}}\right]^{-1}(W)$$

We are interested in looking at the intersection $A(f^{\ell-1}(\mathtt{q}))(V_{(u)}) \cap W_{(s)}$ where V, W are eigenspaces of $A(\mathtt{p})$. (We will make the substitutions to clear all things up in a while)

Applying the lemma 3.2 we can compose $A(f^{\ell-1}(\mathtt{q}))$ with a symplectic transvection σ (or perhaps a finite composition of transvections), close to identity, to get, for any pair E^I, E^J of (sums of) eigenspaces of $A(\mathtt{p})$ with $\dim E^I + \dim E^J = 2d$:

$$\sigma \circ A(f^{\ell-1}(\mathtt{q}))\left(E^I_{(u)}\right) \cap E^J_{(s)} = \{0\}$$

Since $f^{\ell-1}(\mathtt{q})$ is "far" from $\mathtt{p}$, we can choose a neighbourhood of it that does not contain any other iterate of $\mathtt{q}$ and make, there, a little perturbation on the cocycle such that

$$\tilde{A}(f^{\ell-1}(\mathtt{q})) = \sigma \circ A(f^{\ell-1}(\mathtt{q}))$$

Thus, we may suppose, for simplicity, that the cocycle A itself has the property we wish

$$\begin{aligned}
A(f^{\ell-1}(\mathtt{q}))\left(E^I_{(u)}\right) \cap E^J_{(s)} \quad &= \{0\} \\
A(f^{\ell-1}(\mathtt{q})) \circ A^{\ell-1}(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}\left(E^I\right) \cap E^J_{(s)} \quad &= \{0\} \\
A^{\ell}(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}\left(E^I\right) \cap E^J_{(s)} \quad &= \{0\} \\
A^{\ell}(\mathtt{q}) \circ \phi^u_{\mathtt{p},\mathtt{q}}\left(E^I\right) \cap \left[\phi^s_{f^\ell(\mathtt{q}),\mathtt{p}}\right]^{-1}\left(E^J\right) \quad &= \{0\}
\end{aligned}$$

And this last line, is equivalent to

$$\psi_{\mathtt{p},\mathtt{q}}\left(E^I\right) \cap E^J = \{0\}$$

Consider, for each n, $\mathtt{x}_n$ as the periodic point of period $\ell + n$ defined by the itinerary $(i, i_1, ..., i_{\ell-1}, i, ..., i)$, where the symbol i appears $n+1$ times. Consider, also, the compact sets

$$K_n = \overline{\bigcup_{m \leqslant n} \mathcal{O}(\mathtt{x}_m)} \quad ; \quad K_\infty = \overline{\mathcal{O}(\mathtt{q})} = \mathcal{O}(\mathtt{q}) \cup \{\mathtt{p}\}$$

Let $\mathbb{R}^{2d} = E^1_{\mathtt{p}} \oplus E^2_{\mathtt{p}} \oplus \cdots \oplus E^k_{\mathtt{p}}$ be the splitting into eigenspaces of $A(\mathtt{p})$, ordered according to increasing norm of eigenvalues. They have dimension 1 or 2 depending on the eigenvalue to be real or complex.

Lemma 4.3. *For every large enough n, the cocycle A admits a dominated decomposition $E^1 \oplus ... \oplus E^k$ over the set K_n coinciding with the eigenspaces decomposition at the point $\mathtt{p}$.*

This lemma is due to Bonatti and Viana [8] (lemma 9.2). We will reproduce the main lines of their proof here.

proof. Using the holonomies we can transport the splitting into eigenspaces from the fiber over $\mathtt{p}$ to the fiber over $\mathtt{q}$, and then, by iteration, we can spread it over the homoclinic orbit, that is, we can induce the splitting over the compact set K_∞.

For each $1 \leqslant i \leqslant k$ define $F^i_{\mathtt{p}} = E^1_{\mathtt{p}} \oplus \cdots \oplus E^i_{\mathtt{p}}$ and $G^i_{\mathtt{p}} = E^i_{\mathtt{p}} \oplus \cdots \oplus E^k_{\mathtt{p}}$. Of course, $E^i_{\mathtt{p}} = F^i_{\mathtt{p}} \cap G^i_{\mathtt{p}}$. Following this, define

$$E^i_{\mathtt{q}} = F^i_{\mathtt{q}} \cap G^i_{\mathtt{q}} \quad \text{where}$$

$$F^i_{\mathtt{q}} = A^{-\ell}(f^\ell(\mathtt{q})) \circ \phi^s_{\mathtt{p},f^\ell(\mathtt{q})} F^i_{\mathtt{p}} \quad \text{and} \quad G^i_{\mathtt{q}} = \phi^u_{\mathtt{p},\mathtt{q}} G^i_{\mathtt{p}}$$

According to the transversality assumption we did before we can claim that $F^i_{\mathtt{q}} \oplus G^{i+1}_q = \mathbb{R}^{2d} = F^{i-1}_{\mathtt{q}} \oplus G^i_q$. Thus, $\dim E^i_{\mathtt{q}} = \dim E^i_{\mathtt{p}}$. Now, iterate the cocycle A to define this decomposition over the orbit of $\mathtt{q}$.

If we look locally, via the stable holonomy, the cocycle A is constant along local stable manifolds. In these coordinates $\phi^s = \mathrm{id}$ and so $F^j_{f^n(\mathtt{q})} = F^j_{\mathtt{p}}$ for all $n \geq \ell$ and every j. Therefore, $F^i_{f^n(\mathtt{q})} \to F^i_{\mathtt{p}}$ when $n \to \pm\infty$. Because of the transversality, we can conclude the same for G^i. Then, the splitting that we have defined over K_∞ is continuous.

This continuity together with the fact that the points of K_∞ spend all but a finite number of iterates close to $\mathtt{p}$ and the fact that the eigenvalues of $A(\mathtt{p})$ are all distinct imply that this decomposition is dominated over K_∞.

Since for large n, the set K_n is in a small neighbourhood of K_∞, using cone fields we can deduce that there exists a dominated decomposition also over such K_n. $\bullet$

Clearly the same conclusion holds for every cocycle B in C^0 neighborhood $\mathcal{U}$ of A: the cocycle B admits a dominated decompositon over K_n, the map

$$(B, \mathtt{x}) \longmapsto E^1_{B,\mathtt{x}} \oplus ... \oplus E^k_{B,\mathtt{x}}$$

is continuous, the spaces $E^j_{B,\mathtt{x}}$ and $E^j_{A,\mathtt{p}}$ have the same dimension and are uniformly close.

Let μ_n be the eigenvalue of the operator $A^{\ell+n}(\mathtt{x}_n)$ restricted to the two-dimensional subspace $E^{j_0}_{A,\mathtt{x}_n}$.

We claim that, changing a little bit the the cocycle, if necessary, we can assume that $\mu_n \in \mathbb{R}$, that is, the operator restricted to the subspace is a multiple of the identity. Let us see how to make such perturbation:

Consider the symplectic invariant space

$$\mathcal{E}^{j_0}_{A,\mathtt{x}_n} = E(\mu_n) \oplus E(\mu_n^{-1})$$

It is symplectic and has dimension 2 or 4 depending on the complex number μ_n be unitary or not, respectively.

Also here we have, for the restriction of $A^{\ell+n}(\mathtt{x}_n)$ to $\mathcal{E}^{j_0}_{A,\mathtt{x}_n}$ two possibilities

$$\begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix}_{2\times 2} \quad \text{for } \mu_n = e^{i\theta_n}$$

$$\begin{bmatrix} r_n \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix} & 0 \\ 0 & r_n^{-1} \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix} \end{bmatrix}_{4\times 4} \quad \text{for } \mu_n = r_n e^{i\theta_n}$$

For any cocycle B, define $\rho(n, B)$ as the rotation number of $B^{\ell+n}(\mathtt{x}_n)$ restricted to $\mathcal{E}^{j_0}_{B,\mathtt{x}_n}$.

Consider also, for any continuous arc of cocycles $\mathcal{B} = \{B_t\ ;\ t \in I\}$, the number $\delta(n, \mathcal{B})$ defined as the oscillation of $\rho(n, B_t)$ over all $t \in I$.

Lemma 4.4. *There exist a continuous arc $\mathcal{A} = \{A_t\ ;\ t \in [0,1]\}$ of C^ν cocycles in $\mathcal{U}$ such that $A_0 = A$ and $\forall t > 0$ there is a $n_t > 1$ such that for $n \geq n_t$*

$$\delta\left(n, \{A_s\}_{s\in[0,t]}\right) > 1$$

proof. Fix a basis coherent with the dominated decomposition $\mathbb{R}^{2d} = E^1_{\mathtt{p}} \oplus E^2_{\mathtt{p}} \oplus \cdots \oplus E^k_{\mathtt{p}}$. Given any θ, define R_θ as the linear map which is the "symplectic rotation" of angle θ restricted to $\mathcal{E}^{j_0}$, that is, it has one of the two forms described above (according to the modulus of μ_n) and it is identity along all other eigenspaces. Of course, R_θ is a symplectic transformation. Define, for small $\varepsilon > 0$

$$A_t(\mathtt{x}) = R_{t\varepsilon} \cdot A(\mathtt{x}), \quad \text{for } t \in [0,1]$$

where $\varepsilon > 0$ is chosen small enough so that the whole path is inside $\mathcal{U}$. Each periodic point $\mathtt{x}_n$ starts in the same cylinder as $\mathtt{p}$, then goes out for ℓ iterations and finally goes back spending at least n iterates close to the fixed point p. To compute the rotation number $A_t^{\ell+n}(\mathtt{x}_n)$ we have only to consider the matrices corresponding to the latter n points of the orbit. If we consider $\mathtt{x}_n$ close enough to $\mathtt{p}$, each matrix contributes adding up an angle close to $t\varepsilon + \rho_0$ (where ρ_0 is the rotation number of the restriction of $A(\mathtt{p})$ to $\mathcal{E}^{j_0}$). So the variation of the rotation number goes to infinity when $n \to \infty$.

For a more detailed explanation on this see the lemma 9.3 of [8] $\square$

We have for every $t \in [0,1]$ and n large enough, $A_t^{\ell+n}(\mathtt{x}_n)$ has at most c pairs of complex conjugated eigenvalues; by the previous lemma we can claim that there is a t close to 0 and a natural $n \geq 1$ such that $\rho(n, A_t) \in \mathbb{Z}$. Then $A_t^{\ell+n}(\mathtt{x}_n)$ has an real eigenvalue on $\mathcal{E}_t^{j_0} = \mathcal{E}_{A_t}^{j_0}$. In fact, we get for the restriction, in some basis

$$A_t^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}_{A_t}^{j_0}} = \begin{cases} diag(a, a, a^{-1}, a^{-1}) & , \text{if } |\mu| \neq 1 \\ diag(\pm 1, \pm 1) & , \text{if } |\mu| = 1 \end{cases}$$

Define, as in the last chapter, for a small ε, the same symplectic matrix I_ε such that $\|I - I_\varepsilon\| < \varepsilon$

Now we will define the perturbation of the cocycle A_t, in the same way we did. But now the requirements for the perturbed cocycle $\tilde{A} : \Sigma_T \to Sp(2d, \mathbb{R})$ are: it must be C^ν-Hölder continuous and dominated, C^ν-close to A_t and $\tilde{A}^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}_{\tilde{A},\mathtt{x}_n}^{j_0}}$ must have distinct real eigenvalues.

Put, for $\mathtt{x} \in [0; j]$

$$\tilde{A}(\mathtt{x}) = \begin{cases} A_t(\mathtt{x}) & , \text{if } j \neq i_{\ell-1} \\ I_\varepsilon \cdot A_t(\mathtt{x}) & , \text{if } j = i_{\ell-1} \end{cases}$$

The definition of the perturbation is the same as in the locally constant case. So, we can use the Lemma 3.5 to assure that $\tilde{A}^{\ell+n}(\mathtt{x}_n)|_{\mathcal{E}_{\tilde{A},\mathtt{x}_n}^{j_0}}$ has distinct real eigenvalues.

Thus we get a cocycle $\tilde{A}$ and a periodic point $\mathtt{x}_n$ such that $\tilde{A}^{per(\mathtt{x}_n)}(\mathtt{x}_n)$ has at most $c - 1$ complex conjugated eigenvalues. Repeating this process finitely many times we will get a cocycle as required in the Proposition 4.2.

We are not done, yet. We still have to prove the following

Lemma 4.5. *The cocycle $\tilde{A}$ is C^ν-Hölder continuous, dominated and it is C^ν-close to A_t.*

proof. Let us begin by proving that the perturbed cocycle is close to A_t. Consider $C = [0; i_{\ell-1}]$ the cylinder with 0-th symbol $i_{\ell-1}$ and L the supremum of $\|A_t(\mathtt{x})\|$ restricted to C.

$$\|A_t - \tilde{A}\|_\nu = \sup_{\mathtt{x}\in\Sigma_T} \left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) \right\| + \sup_{\mathtt{x}\neq\mathtt{y}} \frac{\left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) - \left(A_t - \tilde{A}\right)(\mathtt{y}) \right\|}{d(\mathtt{x},\mathtt{y})^\nu}$$

We will make the estimatives in three parts:

(I)

$$\begin{aligned} \sup_{\mathtt{x}\in\Sigma_T} \left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) \right\| &= \sup_{\mathtt{x}\in C} \left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) \right\| \\ &= \sup_{\mathtt{x}\in C} \|A_t(\mathtt{x}) - I_\varepsilon A_t(\mathtt{x})\| \\ &\leqslant \|I - I_\varepsilon\| \cdot \sup_{\mathtt{x}\in C} \|A_t(\mathtt{x})\| \\ &< \varepsilon \cdot L \end{aligned}$$

Now consider two cases for the second term of the sum, when $\mathtt{x}$ and $\mathtt{y}$ are both in the cylinder C, and when $\mathtt{x} \in C$ but $\mathtt{y} \notin C$. The case when they are both out of C is trivial (the numerador vanishes)

(II) In this case, the supremum is over all $\mathtt{x}, \mathtt{y} \in \Sigma_T$ such that $\mathtt{x} \in C$ but $\mathtt{y} \notin C$. In such case, $d(\mathtt{x},\mathtt{y}) = 1$

$$\begin{aligned} \sup \frac{\left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) - \left(A_t - \tilde{A}\right)(\mathtt{y}) \right\|}{d(\mathtt{x},\mathtt{y})^\nu} &= \sup_{\mathtt{x}\in C} \left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) - 0 \right\| \\ &< \varepsilon \cdot L \end{aligned}$$

(III) This last case, the supremum is over all $\mathtt{x} \neq \mathtt{y}$ both belonging to C,

$$\begin{aligned} \sup \frac{\left\| \left(A_t - \tilde{A}\right)(\mathtt{x}) - \left(A_t - \tilde{A}\right)(\mathtt{y}) \right\|}{d(\mathtt{x},\mathtt{y})^\nu} &\leqslant \|I - I_\varepsilon\| \sup \frac{\|A_t(\mathtt{x}) - A_t(\mathtt{y})\|}{d(\mathtt{x},\mathtt{y})^\nu} \\ &\leqslant \|I - I_\varepsilon\| \left(\|A_t\|_\nu - L\right) \\ &< \varepsilon \left(\|A_t\|_\nu - L\right) \end{aligned}$$

Hence, we can now estimate

$$\begin{aligned} \|A_t - \tilde{A}\|_\nu &= (\mathsf{I}) + \max\{(\mathsf{II}), (\mathsf{III})\} \\ &< \varepsilon L + \max\{\varepsilon L, \varepsilon\,(\|A_t\|_\nu - L)\} \\ &= \varepsilon \max\{2L, \|A_t\|_\nu\} \end{aligned}$$

Therefore, they are C^ν-close. To see that $\tilde{A}$ is dominated, we have to show that there are a distance $\tilde{d}$ in Σ_T and constant $\tilde{\theta} < 1$ such that

1. $\tilde{d}(f(\mathtt{x}), f(\mathtt{y})) \leqslant \tilde{\theta}\ \tilde{d}(\mathtt{x}, \mathtt{y})$ and $\tilde{d}(f^{-1}(\mathtt{x}), f^{-1}(\mathtt{z})) \leqslant \tilde{\theta}\ \tilde{d}(\mathtt{x}, \mathtt{z})$ for every $\mathtt{y} \in W^s_{loc}(\mathtt{x})$, $\mathtt{z} \in W^u_{loc}(\mathtt{x})$ and $\mathtt{x} \in \Sigma_T$;
2. $\mathtt{x} \mapsto A(\mathtt{x})$ is ν-Hölder continuous and $\|\tilde{A}(\mathtt{x})\|\|\tilde{A}(\mathtt{x})^{-1}\|\tilde{\theta}^\nu < 1$ for every $\mathtt{x} \in \Sigma_T$.

Let d and θ be the distance and the constant of A_t. We can assume that $d(\mathtt{x}, \mathtt{y}) = \theta^{N(\mathtt{x},\mathtt{y})}$, where $N(\mathtt{x}, \mathtt{y}) = \max\{N\ ;\ x_i = y_i\ \forall |i| \leqslant N\}$.

Let ε be the small positive number chosen to make the perturbation $\tilde{A}$. Define

$$\tilde{\theta} = (1+\varepsilon)^{-2/\nu}\theta$$
$$\tilde{d}(\mathtt{x}, \mathtt{y}) = \tilde{\theta}^{N(\mathtt{x},\mathtt{y})}$$

Of course, $\tilde{\theta} < \theta$ and $\tilde{d}$ verifies the property 1 above. Let us see the property 2: if $\mathtt{x}$ does not belong to the cylinder C, where the modification was done, we have

$$\begin{aligned} \|\tilde{A}(\mathtt{x})\|\|\tilde{A}(\mathtt{x})^{-1}\|\tilde{\theta}^\nu &= \|A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1}\|\tilde{\theta}^\nu \\ &\leqslant \|A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1}\|\theta^\nu < 1 \end{aligned}$$

And, for $\mathtt{x} \in C$

$$\begin{aligned} \|\tilde{A}(\mathtt{x})\|\|\tilde{A}(\mathtt{x})^{-1}\|\tilde{\theta}^\nu &= \|I_\varepsilon \cdot A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1} \cdot I_\varepsilon^{-1}\|\tilde{\theta}^\nu \\ &\leqslant \|I_\varepsilon\|\|I_\varepsilon^{-1}\|\|A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1}\|\tilde{\theta}^\nu \\ &= \|A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1}\|(1+\varepsilon)^2\tilde{\theta}^\nu \\ &= \|A_t(\mathtt{x})\|\|A_t(\mathtt{x})^{-1}\|\theta^\nu < 1 \end{aligned}$$

Then, the cocycle $\tilde{A}$ is dominated as required $\bullet$

This proof ends the Propositon 4.2 $\square$

For the twisting part, as in the previous chapter, we may just repeat the argument in the very beginning of the proof of the Proposition 4.2 with $\tilde{A}$ instead of A. Note that, we do not change the property of distinct norms when we compose with a transvection, because the modification is done far from the periodic point.

That said, we have proved the Theorem B.

Appendix A

Symplectic Generators

Let V be a even dimensional vector space and ω a symplectic form. Denote by $Sp(V)$ the group of symplectic operators of V and T the group **generated** by all **symplectic transvections**.

We have seen that every $\tau \in T$ can be written as

$$\tau_{u,a}(v) = v + a \cdot \omega(v,u)u$$

for some $a \in \mathbb{R}$ and some u in the fixed hyperplane of τ.

Proposition A.1. *T acts transitively on $V\backslash\{0\}$, that is, given $v \neq w \in V\backslash\{0\}$, there is $\tau \in T$ such that $\tau(v) = w$.*

proof.

Take $v \neq w \in V\backslash\{0\}$.

If $\omega(v,w) \neq 0$, consider $a = \omega(v,w)^{-1}$ and $u = v - w$. Then,

$$\begin{aligned} \tau_{u,a}(v) &= v + a \cdot \omega(v,u)u \\ &= v + \omega(v,w)^{-1} \cdot \omega(v, v-w)(v-w) \\ &= v - (v-w) \\ &= w \end{aligned}$$

For the case $\omega(v,w) = 0$, we choose a vector z out of the union of the hyperplanes $v^\omega \cup w^\omega$. Then, we have $\omega(v,z) \neq 0 \neq \omega(w,z)$. Now, we apply the previous case twice and get $\tau_1(v) = z$ and $\tau_2(z) = w$. So, $\tau_2\tau_1(v) = w$. $\square$

Remark. If the vector γ can be written as $\gamma = \alpha - \beta$ with $\omega(\alpha, \beta) \neq 0$ then we will write

$$\tau_\gamma = \tau_{\gamma, \omega(\alpha,\beta)^{-1}}$$

By the calculations in the previous proposition we can see that $\tau_\gamma(\alpha) = \beta$ and $\tau|_{\gamma^\omega} \equiv \mathrm{id}$

Definition. A ordered pair (u, v) is called *hyperbolic* if $\omega(u, v) = 1$.

Proposition A.2. *T acts transitively on hyperbolic pairs, that is, given two hyperbolic pairs (u_1, v_1) and (u_2, v_2), there is a finite product of transvections that maps u_1 to u_2 and v_1 to v_2.*

proof. By the Proposition A.1, there is τ such that $\tau(u_1) = u_2$. Denote $v_3 = \tau(v_1)$.

So, $\tau : (u_1, v_1) \mapsto (u_2, v_3)$. We would like to find a transvection σ such that $\sigma : (u_2, v_3) \mapsto (u_2, v_2)$. Composing we will get $\sigma\tau : (u_1, v_1) \mapsto (u_2, v_2)$. Let us construct σ.

If $\omega(v_2, v_3) \neq 0$, take $\gamma = v_3 - v_2$. First,

$$\begin{aligned}
\omega(u_2, \gamma) &= \omega(u_2, v_3 - v_2) \\
&= \omega(u_2, v_3) - \omega(u_2, v_2) \\
&= \omega(\tau(u_1), \tau(v_1)) - 1 \\
&= \omega(u_1, v_1) - 1 \\
&= 1 - 1 = 0
\end{aligned}$$

Hence, by the remark, $\tau_\gamma(u_2) = u_2$ and $\tau_\gamma(v_3) = v_2$. In this case, take $\sigma = \tau_\gamma$.

Now, if $\omega(v_2, v_3) = 0$ we will use $z = u_2 + v_3$ as the intermediate step.

We have $\omega(z, v_3) = \omega(u_2, v_3) = 1 \neq 0$, then take $\gamma_1 = v_3 - z = -u_2$. We still have $\omega(u_2, \gamma_1) = 0$. So, by the remark, $\tau_{\gamma_1} : (u_2, v_3) \mapsto (u_2, z)$.

Also, $\omega(z, v_2) = \omega(u_2 + v_3, v_2) = \omega(u_2, v_2) + \omega(v_3, v_2) = 1 \neq 0$. Take $\gamma_2 = z - v_2 = u_2 + v_3 - v_2$. Let us see that u_2 is in the orthogonal of γ_2:

$$\omega(u_2, \gamma_2) = \omega(u_2, u_2 + v_3 - v_2) = \omega(u_2, v_3) - \omega(u_2, v_2) = 1 - 1 = 0$$

So, by the remark, $\tau_{\gamma_2} : (u_2, z) \mapsto (u_2, v_2)$

Composing we have,

$$(u_1, v_1) \xrightarrow{\tau} (u_2, v_3) \xrightarrow{\tau_{\gamma_1}} (u_2, z) \xrightarrow{\tau_{\gamma_2}} (u_2, v_2)$$

$\square$

Proposition A.3. *$T = Sp(V)$, that is, the symplectic group is generated by transvections.*

proof. Let us proceed by induction on $n = \dfrac{\dim V}{2}$.

If $n = 1$, then $\dim V = 2$. In this case a symplectic base is a hyperbolic pair. By the proposition A.2, we have $T = Sp(V)$. (We could have used the fact that for two-dimensional vector spaces $Sp(V) = SL(V)$ and in that case it is easy to show that transvections generate $SL(V)$).

Now suppose $\dim V = 2d$, $n > 1$. Choose a hyperbolic pair $(u, v) \in V$ and denote by W the plane they span. Obviously, W is symplectic and $V = W \oplus W^\omega$.

Take any $\sigma \in T$. By the proposition A.2 there is a transvection $\tau \in T$ such that $\tau : (\sigma(u), \sigma(v)) \mapsto (u, v)$. Thus, $\tau\sigma : (u, v) \mapsto (u, v)$. That is, $\tau\sigma|_W = id_W$ and since W is symplectic, $\tau\sigma|_{W^\omega} \in Sp(W^\omega)$. By induction hypothesis, $\tau\sigma|_{W^\omega}$ is a product of transvections on W^ω. Now, any transvection on W^ω can be extended to a transvection on the whole space just including W in the fixed hyperplane. This way, we get $\tau\sigma \in T$, and hence $\sigma \in T$. $\square$

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