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Kakeya sets and applications to analysis

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Rio de Janeiro

July of 2016

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Thesis to obtain the master's degree at IMPA

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Abstract. This dissertation presents the Kakeya sets and the relationship with analysis. The construction discussed is a variant of the one originally given by Besicovitch which is simple but it is surprising the connection with problems in other areas seemingly unrelated, the Fefferman theorem is treated as a disproof of the disk conjecture, this is an example of use the Kakeya construction to obtain an analysis result. The abstract definition of multiplier give us certain important results that we use to study the Bochner-Riesz multipliers as example of operators with kernel less singular that the ball multiplier and obtain a critical value for the boundedness of them,

In the class of the figures in which a segment of length 1 can be turned around through 360°, remaining always within the figure, which one has the smallest area?

The three-cornered hypocycloid inscribed in a circle of diameter 3/2 also belongs to the class ..., thus if we let one end of the segment describe the hypocycloid while keeping the segment touching the hypocycloid, we have the other end of the segment also moving on the hypocycloid and so the whole of the segment remains all the time within the hypocycloyd. The area of the hypocycloid is $\frac{\pi}{8}$. That is exactly half of the area of a circle of diameter 1. It was conjectured that the hypocycloid was the figure of minimum area.

My solution shows that the hypocycloid conjecture is false, and that in fact, there are figures of arbitrarily small area which permit a unit segment to change its direction by 360 while moving continuously within them.

A. S. Besicovitch in *The Kakeya Problem.*

Let ABC be any triangle of altitude h and area α. Divide its base AB into n equal parts and join the points of division to the vertex C. Then the triangle ABC is divided into n elementary triangles. Perform an arbitrary translation of each elementary triangle along the side AB (i.e. translation which leaves the base of an elementary triangle on the line AB). Now the question is, *is it possible to choose the number n and to perform the translations in such a way that the area covered by the elementary triangles in their new position is as small as we please.*

A. S. Besicovitch in *On Kakeya Problem and a similar one.*

Rougly speaking, the idea is as follows. By duality it suffices to consider the case $p > 2$. Let R be a large number, and let T be a cilindrical tube in $\mathbb{R}^n$ with length R and radius $\sqrt{R}$ and oriented in some direction ω_T. Let ψ_T be a bump function adapted to the tube T, and let $\tilde{T}$ be a shift of T by $2R$ units in the ω_T direction. Then a computation shows that

$$\left|S_1(e^{2\pi i\omega_T\cdot x}\psi_T(x))\right| \approx 1$$

for all $x \in \tilde{T}$. To exploit this computation, one uses the Besicovitch construction to find a collection ...

Fefferman's theorem is an example of how a geometric construction can be used to show the unboundedness of various oscillatory integral operators. The point is that while the action of these operators on general functions is rather complicated, their action on "wave packets" such as $e^{2\pi i\omega_T\cdot x}\psi_T(x)$ is fairly easy to analyze . . .

Terence Tao in *From Rotating Needles to Stability of Waves: Emerging Conections between Combinatorics, Analysis, and PDE.*

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Introduction

The Fourier transform is an important operator in analysis and PDE because is linear and it allows us to change differential polynomials by multiplication operators of polynomial functions (in particular it diagonalizes the laplacian). As a consequence of this if we apply the Fourier transform to certain linear PDE we obtain linear ODE which are simple. Now the problem is to return to obtain the solution of the initial PDE, this problem is difficult because the surjectivity of the Fourier transform depends of the domain of definition, however with a suitable conditions on the space of solutions is possible to return and thus to solve the PDE.

Define for every $R > 0$ the operator $\widehat{S_R f} = \chi_{B(0,R)}\widehat{f}$, if f is a well behaved function (it is suffices to take f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, see [7, Chapter 7] for a definition) we have that $S_R f(x) = \int_{\|\xi\|\leq R} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$. We are interested to know if $\lim_{R\to\infty} \|S_R f - f\|_p = 0$. By the uniform boundedness principle and a suitable dense subset of $L^p(\mathbb{R}^n)$ we note that it is equivalent to find a constant $C_p > 0$ independent of R such that $\|S_R f\|_p \leq C_p \|f\|_p$. Moreover a simple calculation shows that is enough consider the case $R = 1$. In this case the operator S_R is known as the ball multiplier. By definition:

$$\widehat{S_1 f} = \chi_B \widehat{f}$$

here B is the unit ball.

By the Plancherel theorem the answer is affirmative for $p = 2$ in any dimension $n \geq 1$. What about with the other values of $1 < p < \infty$?. In one dimension we will see that S_1 is bounded; however for $n \geq 2$ the answer is surprising as stated the following:

Fefferman's theorem: The operator S_1 is not bounded for every $n \geq 2$ and p different of 2.

To prove this theorem we begin with a revision of the main topics in Harmonic Analysis and study certain class of operators that generalize the definition of the Fourier integrals. The characterization of these operators give us de Leeuw's theorem and the duality property. Together, they show that is enough to give the proof of Fefferman's theorem for dimension 2 and $p > 2$. At this stage we are under influence of the plane geometry.

The Kakeya sets are compact sets that contains a unit line segment (needle) in every direction, in

1926 A. S. Besicovitch showed that there exist Kakeya sets of arbitrarily small area. We use the Schönberg's construction which is a variant of this to obtain a family of disjoint rectangles and a family of significant overlapping sets. Quite surprinsingly, this construction will give us a proof of Fefferman's result.

CHAPTER 1

Preliminaries in Harmonic Analysis

In this chapter we fix the notation and prove some basic facts about the Fourier transform that we use in the next chapters. The principal value $p.v(\frac{1}{x})$ is treated as a tempered distribution and is used to define the Hilbert transform. The chapter concludes with an appendix that dicusses properties of the Bessel functions; the most important of these are the assymtotic properties. The book [7, cap 7] is a good reference for the results in this chapter.

1.1 The Fourier Transform and its properties

For $1 \leq p < \infty$ we define the set $L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty \right\}$ this is a Banach space with norm $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}$ the space

$$L^\infty(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid (\exists C > 0) \, (|f(x)| \leq C, \quad a.e \quad x \in \mathbb{R}^n) \}$$

is also a Banach space with the norm $\|f\|_\infty = ess \sup(|f|) = \inf \{ C > 0 \mid |f(x)| \leq C, \quad a.e \quad x \in \mathbb{R}^n \}$. The functions of $L^1(\mathbb{R}^n)$ are called integrable, in this space is possible to define a function $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$, given by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \exp(-2\pi i x \cdot \xi) dx$, the map taking f to $\widehat{f}$ is called the Fourier transform and is denoted by $\mathscr{F}$. Note that $\left\| \widehat{f} \right\|_\infty \leq \|f\|_1$ this implies that the map $\mathscr{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is continuous.

The recognized Riemann-Lebesgue lemma says that $\lim_{\|\xi\| \to \infty} \widehat{f}(\xi) = 0$ and the dominated convergence theorem give us that $\widehat{f}$ is continuous, this implies that $Im(\mathscr{F}) \subset C_0(\mathbb{R}^n)$ the set of continuous functions that vanishes at infinity. However, for our purpose we need that the Fourier transform can be defined in a dense subset of $L^p(\mathbb{R}^n)$ such that the image of $\mathscr{F}$ is contained in this set, for this we define the

Schwartz class of functions with rapid decrease:

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid p_{\alpha,\beta}(f) < \infty, \alpha, \beta \in \mathbb{N}^n\}$$

where $p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right|$ and we use the multi-index notation $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + ... + \alpha_n$.

The space $\mathcal{S}(\mathbb{R}^n)$ is a Frechet space with topology induced by the countable family of seminorms $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$ and the set of smooth functions with compact support satisfies $C_0^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}^n)$. The following result is stronger than this.

Proposition 1.1. *The set $C_0^\infty(\mathbb{R}^n)$ is dense on $\mathcal{S}(\mathbb{R}^n)$ with the metric topology given by the family of seminorms $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$.*

Proof. We see that $\overline{C_0^\infty(\mathbb{R}^n)} = \mathcal{S}(\mathbb{R}^n)$ in the metric topology of $\mathcal{S}(\mathbb{R}^n)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$, for every $x \in \overline{B(0,1)}$, $supp(\phi) \subset \overline{B(0,2)}$, if $f \in \mathcal{S}(\mathbb{R}^n)$ we define $f_k(x) = \phi(\frac{x}{k})f(x)$, by the Leibnitz rule:

$$\partial^\beta f_k(x) = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{1}{k^{|\gamma|}} (\partial^\gamma\phi)\left(\frac{x}{k}\right)(\partial^{\beta-\gamma}f)(x)$$
$$= \sum_{\gamma\leq\beta,|\gamma|>0} \binom{\beta}{\gamma} \frac{1}{k^{|\gamma|}} (\partial^\gamma\phi)\left(\frac{x}{k}\right)(\partial^{\beta-\gamma}f)(x) + \phi\left(\frac{x}{k}\right)(\partial^\beta f)(x)$$

for every $\beta \in \mathbb{N}^n$, then:

$$\left|x^\alpha\partial^\beta f_k(x) - x^\alpha\partial^\beta f(x)\right| \leq \sum_{\gamma\leq\beta,|\gamma|>0} \binom{\beta}{\gamma} \frac{1}{k^{|\gamma|}} \left|(\partial^\gamma\phi)\left(\frac{x}{k}\right)\right| \left|x^\alpha(\partial^{\beta-\gamma}f)(x)\right|$$
$$+ \left|\phi\left(\frac{x}{k}\right) - 1\right| \left|x^\alpha(\partial^\beta f)(x)\right|$$

for every $x \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$. We note that $f_k \in C_0^\infty(\mathbb{R}^n)$, $supp(f_k) \subset \overline{B(0,2k)}$, also by the mean value inequality:

$$\left|\phi\left(\frac{x}{k}\right) - 1\right| \leq \left|\phi\left(\frac{x}{k}\right) - \phi(0)\right| \leq \sup\left\{\|\phi'(t)\| \mid t \in \mathbb{R}^n\right\} \frac{\|x\|}{k} \leq \sum_{\omega\in\mathbb{N}^n,|\omega|=1} \|\phi\|_{0,\omega} \frac{\|x\|}{k}$$
$$\leq \left(\sum_{\omega\in\mathbb{N}^n,|\omega|=1} \|\phi\|_{0,\omega}\right)\left(\sum_{i=1}^n \frac{|x_i|}{k}\right)$$

for every $x \in \mathbb{R}^n$. Then

$$\left|x^\alpha \partial^\beta f_k(x) - x^\alpha \partial^\beta f(x)\right| \leq \sum_{\gamma \leq \beta, |\gamma|>0} \binom{\beta}{\gamma} \frac{1}{k^{|\gamma|}} \left\|\phi\right\|_{0,\gamma} \left\|f\right\|_{\alpha,\gamma}$$
$$+\frac{1}{k}\left(\sum_{\omega \in \mathbb{N}^n, |\omega|=1} \left\|\phi\right\|_{0,\omega}\right)\left(\sum_{\theta \in \mathbb{N}^n, |\theta|=|\alpha|+1} \left\|f\right\|_{\theta,\beta}\right)$$

for every $x \in \mathbb{R}^n$, hence:

$$p_{\alpha,\beta}(f_k - f) \leq \sum_{\gamma \leq \beta, |\gamma|>0} \binom{\beta}{\gamma} \frac{1}{k^{|\gamma|}} \left\|\phi\right\|_{0,\gamma} \left\|f\right\|_{\alpha,\gamma}$$
$$+\frac{1}{k}\left(\sum_{\omega \in \mathbb{N}^n, |\omega|=1} \left\|\phi\right\|_{0,\omega}\right)\left(\sum_{\theta \in \mathbb{N}^n, |\theta|=|\alpha|+1} \left\|f\right\|_{\theta,\beta}\right) \rightarrow_{k\to\infty} 0$$

then $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ with the topology induced by the family of seminorms $\{p_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$.

◊

The inversion theorem tell us $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has period 4, hence $\mathscr{F}^3 = \mathscr{F}^{-1}$. (See [7], Pages 182-189). By the theorem of change of variables we have:

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

With this space we have the complete machinery, because $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for every $1 \leq p \leq \infty$, and $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an topological isomorphism, i.e is a homeomorphism and linear transformation.

As $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ we have that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, letting $f, g \in \mathcal{S}(\mathbb{R}^n)$ using the inversion formula:

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi\right) \overline{g(x)} dx \\ = \int_{\mathbb{R}^n} \widehat{f}(\xi) \left(\int_{\mathbb{R}^n} \overline{g(x)} e^{2\pi i x \cdot \xi} dx\right) d\xi &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \end{aligned} \tag{1.1}$$

the Parseval formula. If $g = f$ we obtain that

$$\left\|f\right\|_2 = \left\|\widehat{f}\right\|_2 \tag{1.2}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, then $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isometric isomorphism with respect to the norm $\left\|\cdot\right\|_2$, as $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ this map can be extended uniquely to a map $\Psi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, obviously this map is an isometric isomorphism and is called the Fourier-Plancherel transform, in this thesis we continue using the notation $\mathscr{F}, \mathscr{F}^{-1}$ and $\widehat{f}$, $\check{f}$ for $f \in L^2(\mathbb{R}^n)$ instead of Ψ, Ψ^{-1} and $\Psi(f)$,

$\Psi^{-1}(f)$ respectively..

By a simple argument of density the formula (1.2) is true for every $f \in L^2(\mathbb{R}^n)$. This is called the Plancherel theorem, using the polarization identity we obtain

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R}^n)}, \qquad \forall f, g \in L^2(\mathbb{R}^n). \tag{1.3}$$

That is precisely the Parseval formula.

In the following theorem we summarize some properties of the Fourier transform in $\mathcal{S}(\mathbb{R})$.

Proposition 1.2. *If $f \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, $\lambda > 0$, $A \in O(n)$ we define $f_\lambda(x) = f(\lambda x)$, using the notations $M_a(x) = e^{2\pi i x\cdot a}$, $\tau_a f(x) = f(x+a)$: $\widehat{M_a f} = \tau_{-a}\widehat{f}$, $\widehat{f_\lambda} = \frac{1}{\lambda^n}(\widehat{f})_{\frac{1}{\lambda}}$, $\widehat{f\circ A} = \widehat{f}\circ A$.*

Proof. Let $a \in \mathbb{R}^n$, then $\widehat{M_a f}(\xi) = \int_{\mathbb{R}^n} M_a(x) f(x) e^{-2\pi i x\cdot\xi}dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot(\xi-a)}dx = \widehat{f}(\xi - a) = (\tau_{-a}\widehat{f})(\xi)$.

Let $\lambda > 0$, then $\widehat{f_\lambda}(\xi) = \int_{\mathbb{R}^n}(f_\lambda)(x)e^{-2\pi i x\cdot\xi}dx = \int_{\mathbb{R}^n} f(\lambda x)e^{-2\pi i x\cdot\xi}dx = \int_{\mathbb{R}^n} f(y)e^{-2\pi i\frac{y}{\lambda}\cdot\xi}\frac{dy}{\lambda^n} = \frac{1}{\lambda^n}\int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot\frac{\xi}{\lambda}}dy = \frac{1}{\lambda^n}\widehat{f}(\frac{\xi}{\lambda}) = \frac{1}{\lambda^n}(\widehat{f})_{\frac{1}{\lambda}}(\xi)$.

Let $A \in O(n)$, by definition of Fourier transform $\widehat{f}(A\xi) = \int_{\mathbb{R}^n} f(x)\exp(-2\pi i x\cdot(A\xi))dx$,
but $A \in O(n)$ implies that $x \cdot A\xi = < x, A\xi > = < A^* x, \xi > = < A^{-1}x, \xi > = (A^{-1}x)\cdot\xi$, so:
$\widehat{f}\circ A(\xi) = \widehat{f}(A\xi) = \int_{\mathbb{R}^n} f(x)\exp(-2\pi i(A^{-1}x)\cdot\xi)dx = \int_{\mathbb{R}^n} f(Ay)\exp(-2\pi i y\cdot\xi)$
$|\det(A)|\, dy = \int_{\mathbb{R}^n} f(Ay)\exp(-2\pi i y\cdot\xi)dy = \widehat{f\circ A}$. For all $\xi \in \mathbb{R}^n$ this completes the proof. ◊

The elements of the topological dual space $\mathcal{S}(\mathbb{R})' = \{T : \mathcal{S}(\mathbb{R}) \to \mathbb{C} \mid T \quad \text{is linear and continuous}\}$ are called tempered distributions. It is easy to see that $T \in \mathcal{S}(\mathbb{R})'$ if and only if there exists $C > 0, N \in \mathbb{N}$ such that:

$$|T(\phi)| \leq C \sum_{|\alpha|,|\beta|\leq N} p_{\alpha,\beta}(\phi), \qquad \forall\phi \in \mathcal{S}(\mathbb{R}).$$

We define the convolution of two functions $f, g \in \mathcal{S}(\mathbb{R})$ by $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$. We also define the convolution of a tempered distribution $T \in \mathcal{S}(\mathbb{R})'$ and a function $f \in \mathcal{S}(\mathbb{R})$ as the distribution $(T * f)(\phi) = T(\overline{f} * \phi)$, here $\overline{f}(x) = f(-x)$. We conclude the section with a powerful interpolation theorem

Theorem 1.1. *Consider a linear operator T, which maps the measure space (X,μ) to the measure space (Y,ν). Suppose that $p_0, q_0, p_1, q_1 \in [1,\infty]$ and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0,1)$. If $q_0 = q_1 = \infty$, we further suppose that ν is semifinite. If T maps $L^{p_0}(\mu) + L^{p_1}(\mu)$ into $L^{q_0}(\nu) + L^{q_1}(\nu)$ and we have $\|Tf\|_{q_0} \leq M_0\|f\|_{p_0}$ for $f \in L^{p_0}$ and $\|Tf\|_{q_1} \leq M_1\|f\|_{p_1}$ for $f \in L^{p_1}$ for constants $M_0, M_1 > 0$. Then T is bounded on L^p and furthermore, $\|Tf\|_q \leq M_0^{1-t}M_1^t\|f\|_p$ for $f \in L^p$.

The proof of this theorem can be found in [9, Pags. 52-53].

1.2 Principal Values

Sometimes we have a family of integrable functions $\{Q_t\}_{t>0}$ such that $Q(x) = \lim_{t\to 0^+} Q_t(x)$ is not locally integrable, so we can not define neither the convolution with a regular function nor its Fourier transform. This problem can be solved using principal values. For our purposes we consider the special case when $Q_t(x) = \frac{x}{\pi(x^2+t^2)}$ and $Q(x) = \frac{1}{\pi x}$, for x nonzero, that is not locally integrable (take $[0,1]$ which is compact). This motivates the following definition:

Definition 1.1. We define $p.v(\frac{1}{x}) : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$, $p.v(\frac{1}{x})(\varphi) = \lim_{\epsilon\to o} \int_{|x|>\epsilon} \frac{\varphi(x)dx}{x}$

We obtain an alternative expression for $p.v(\frac{1}{x})$, take $\epsilon > 0$:

$$\int_{|x|>\epsilon} \frac{\varphi(x)dx}{x} = \int_{-\infty}^{-\epsilon} \frac{\varphi(x)dx}{x} + \int_{\epsilon}^{\infty} \frac{\varphi(x)dx}{x},$$

let $u = -x$ in the first integral, then:

$$\int_{|x|>\epsilon} \frac{\varphi(x)dx}{x} = \int_{\infty}^{\epsilon} \frac{\varphi(-u)du}{u} + \int_{\epsilon}^{\infty} \frac{\varphi(x)dx}{x} = \int_{\epsilon}^{\infty} \frac{(\varphi(x)-\varphi(-x))dx}{x}$$

as

$$\lim_{x\to 0} \frac{\varphi(x)-\varphi(-x)}{x} = \lim_{x\to 0} \frac{\varphi(x)-\varphi(0)}{x} + \lim_{x\to 0} \frac{\varphi(0)-\varphi(-x)}{x} = \varphi'(0) + \varphi'(0) = 2\varphi'(0)$$

there exist $\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \frac{(\varphi(x)-\varphi(-x))dx}{x}$, hence:

$$p.v\left(\frac{1}{x}\right)(\varphi) = \int_{0}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x} dx$$

note that $p.v(\frac{1}{x})$ is linear, also

$$\begin{aligned}\left|p.v\left(\frac{1}{x}\right)(\varphi)\right| &= \left|\int_{0}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x} dx\right| \le \int_{0}^{1} \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| dx + \int_{1}^{\infty} \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| dx \\ &= \int_{0}^{1} \frac{1}{x} \left|\int_{-x}^{x} \varphi'(t)dt\right| dx + \int_{1}^{\infty} \frac{|\varphi(x)|+|\varphi(-x)|}{x} \le \int_{0}^{1} \frac{1}{x}(2x) \|\varphi\|_{0,1} dx + \int_{1}^{\infty} \frac{2\|\varphi\|_{1.0}}{x^2} dx \\ &= 2\|\varphi\|_{0,1} \int_{0}^{1} dx + 2\|\varphi\|_{1.0} \int_{1}^{\infty} \frac{1}{x^2} = 2\|\varphi\|_{0,1} + 2\|\varphi\|_{1.0} \left[-\frac{1}{x}\right]_{1}^{\infty} = 2(\|\varphi\|_{0,1} + \|\varphi\|_{1,0}) = 2 \sum_{|\alpha|,|\beta|\le 1} \|\varphi\|_{\alpha,\beta}\end{aligned}$$

then $p.v(\frac{1}{x})$ is continuous, this implies that $p.v(\frac{1}{x}) \in \mathcal{S}(\mathbb{R})'$ is a tempered distribution, if $\psi_\epsilon(x) = \frac{1}{x}\chi_{\{|x|>\epsilon\}}$ then ψ_ϵ define tempered distributions for every $\epsilon > 0$, moreover $\lim_{\epsilon\to 0^+} \psi_\epsilon = p.v(\frac{1}{x})$ in $\mathcal{S}(\mathbb{R})'$, in fact:

$$\left|p,v\left(\frac{1}{x}\right)(\varphi)-\psi_\epsilon(\varphi)\right|=\left|\int_0^\epsilon\frac{\varphi(x)-\varphi(-x)}{x}dx\right|\leq 2\,\|\varphi\|_{0,1}\int_0^\epsilon dx=2\epsilon\,\|\varphi\|_{0,1}\to_{\epsilon\to0^+}0$$

Proposition 1.3. $\lim\limits_{t\to0^+}Q_t=\frac{1}{\pi}p.v(\frac{1}{x})$ *in* $\mathcal{S}(\mathbb{R})'$

Proof. As $\lim\limits_{\epsilon\to0^+}\psi_\epsilon=p.v(\frac{1}{x})$ we have that if $\varphi\in\mathcal{S}(\mathbb{R})$:

$$\begin{aligned}\lim_{t\to0^+}\left(Q_t-\frac{1}{\pi}p.v\left(\frac{1}{x}\right)\right)(\varphi)&=\lim_{t\to0^+}\left(Q_t(\varphi)-\frac{1}{\pi}p.v\left(\frac{1}{x}\right)(\varphi)\right)\\&=\lim_{t\to0^+}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x\varphi(x)}{x^2+t^2}dx-\frac{1}{\pi}\int_{|x|>t}\frac{\varphi(x)}{x}dx\right)\\&=\frac{1}{\pi}\lim_{t\to0^+}\left(\int_{|x|<t}\frac{x\varphi(x)}{x^2+t^2}dx+\int_{|x|>t}\frac{x\varphi(x)}{x^2+t^2}dx-\int_{|x|>t}\frac{\varphi(x)}{x}dx\right)\\&=\frac{1}{\pi}\lim_{t\to0^+}\left(\int_{|x|<t}\frac{x\varphi(x)}{x^2+t^2}dx+\int_{|x|>t}\left(\frac{x}{x^2+t^2}-\frac{1}{x}\right)\varphi(x)dx\right)\\&=\frac{1}{\pi}\lim_{t\to0^+}\left(\int_{|x|<t}\frac{x\varphi(x)}{x^2+t^2}dx+\int_{|x|>t}\frac{t^2\varphi(x)}{x(x^2+t^2)}dx\right)\end{aligned}$$

let $u=\frac{x}{t}$ then:

$$\lim_{t\to0^+}\left(Q_t-\frac{1}{\pi}p.v\left(\frac{1}{x}\right)\right)(\varphi)=\frac{1}{\pi}\lim_{t\to0^+}\left(\int_{|u|<1}\frac{u\varphi(tu)}{u^2+1}du+\int_{|u|>1}\frac{\varphi(tu)}{u(u^2+1)}du\right)$$

as $\varphi\in\mathcal{S}(\mathbb{R})$ the functions $(u\to\frac{u\varphi(tu)}{u^2+1})\in L^1(\mathbb{R})$ and $(u\to\frac{\varphi(tu)}{u(u^2+1)})\in L^1(\mathbb{R})$ for every $t>0$, by the dominated convergence theorem:

$$\lim_{t\to0^+}\left(Q_t-\frac{1}{\pi}p.v\left(\frac{1}{x}\right)\right)(\varphi)=\frac{1}{\pi}\left(\int_{|u|<1}\frac{u\varphi(0)}{u^2+1}du+\int_{|u|>1}\frac{\varphi(0)}{u(u^2+1)}du\right)=0$$

given that the functions $u\to\frac{u}{u^2+1}$ and $u\to\frac{1}{u(u^2+1)}$ are odd functions and the sets $\{u\in\mathbb{R}\mid|u|<1\}$, $\{u\in\mathbb{R}\mid|u|>1\}$ are symmetric with respect the origin. ◊

1.3 Harmonic extensions and the Hilbert transform

In this section we study the harmonic extension of a function $f\in\mathcal{S}(\mathbb{R})$ to the upper half-plane and use this to define the Hilbert transform $\mathscr{H}$ and prove the Riesz Theorem that asserts that $\mathscr{H}\in\mathscr{B}(L^p(\mathbb{R}))$ for $1<p<\infty$. By abuse of notation we identify the vector $(x,t)\in\mathbb{R}^2$ with the complex number $z=x+it$.

1.3.1 Harmonic extensions

Let $f \in \mathcal{S}(\mathbb{R})$. The harmonic extension of f to $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is given by $u(z) = P_t * f(x)$ where $z = x + it$ and $P_t(x) = \frac{t}{\pi(x^2+t^2)}$ is the Poisson kernel. (Note that this extension is not unique if $f = 0$ take the functions $u_1 \equiv 0$, $u_2(x, y) = y$, however the extension that we use is the given by the Poisson kernel). We are going to prove that in fact, u is harmonic in $\mathbb{H}$. As $P_t \in L^1(\mathbb{R})$ for every $t > 0$ we take the Fourier transform in the x variable $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$, basic properties of the exponential function proves that $\widehat{P_t} \in L^1(\mathbb{R})$ for every $t > 0$. Using inverse Fourier transform:

$$\begin{aligned} u(z) = P_t * f(x) = (\widehat{P_t}\widehat{f})^{\vee}(x) &= \int_{\mathbb{R}} \widehat{P_t}(\xi)\widehat{f}(\xi)e^{2\pi ix\xi}d\xi \\ = \int_{\mathbb{R}} e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi &= \int_{-\infty}^{0} e^{2\pi t\xi}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi + \int_{0}^{\infty} e^{-2\pi t\xi}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi \\ = \int_0^{\infty} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi &+ \int_{-\infty}^{0} \widehat{f}(\xi)e^{2\pi i\overline{z}\xi}d\xi. \end{aligned} \tag{1.4}$$

If we define $iv(z) = \int_0^{\infty} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi - \int_{-\infty}^{0} \widehat{f}(\xi)e^{-2\pi i\overline{z}\xi}d\xi$ and $F = u + iv$, $F(z) = \int_0^{\infty} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi$ is holomorphic, in fact if $\Delta \subset \mathbb{H}$ is a triangle

$$\int_{\Delta} F(z)dz = \int_{\Delta}\int_0^{\infty} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi dz = \int_0^{\infty}\int_{\Delta} \widehat{f}(\xi)e^{2\pi iz\xi}dzd\xi$$

(Fubini's theorem because $f \in \mathcal{S}(\mathbb{R})$ and so $\widehat{f} \in \mathcal{S}(\mathbb{R})$)

$$= \int_0^{\infty} \widehat{f}(\xi)\int_{\Delta} e^{2\pi iz\xi}dzd\xi = 0$$

the last step followed from the fact the function $z \in H \mapsto e^{2\pi iz\xi}$ is holomorphic for every $\xi \in (0, \infty)$. Morera's theorem implies that F is holomorphic. Clearly u is real. Note that v is also real, in fact:

$$\begin{aligned} \overline{iv(z)} &= \overline{\int_0^{\infty} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi} - \overline{\int_{-\infty}^{0} \widehat{f}(\xi)e^{-2\pi i\overline{z}\xi}d\xi} = \int_0^{\infty} \overline{\widehat{f}(\xi)}\,\overline{e^{2\pi iz\xi}}d\xi \\ &- \int_{-\infty}^{0} \overline{\widehat{f}(\xi)}\,\overline{e^{-2\pi i\overline{z}\xi}}d\xi = \int_0^{\infty} \widehat{f}(-\xi)e^{-2\pi i\overline{z}\xi}d\xi + \int_{-\infty}^{0} \widehat{f}(-\xi)e^{-2\pi iz\xi}d\xi \end{aligned}$$

on every integral let $w = -\xi$ on each integral.

$$\begin{aligned} \overline{iv(z)} &= \int_0^{-\infty} \widehat{f}(w)e^{2\pi i\bar{z}w}(-dw) - \int_{\infty}^{0} \widehat{f}(w)e^{2\pi izw}(-dw) \\ &= \int_{-\infty}^{0} \widehat{f}(w)e^{2\pi i\bar{z}w}dw - \int_0^{\infty} \widehat{f}(w)e^{2\pi izw}dw = -iv(z) \end{aligned}$$

hence $-i\overline{v(z)} = -iv(z)$, so $\overline{v(z)} = v(z)$ and v is real. Therefore u and v are real and $F = u + iv$ is holomorphic we have that u and v are harmonic.

We will need an alternative expresion for v:

$$iv(z)=\int_0^\infty \widehat{f}(\xi)e^{2\pi iz\xi}d\xi-\int_{-\infty}^0 \widehat{f}(\xi)e^{2\pi i\bar{z}\xi}d\xi=\int_0^\infty \widehat{f}(\xi)e^{2\pi ix\xi-2\pi t\xi}d\xi-\int_{-\infty}^0 \widehat{f}(\xi)e^{2\pi ix\xi+2\pi t\xi}d\xi$$
$$=\int_0^\infty e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi-\int_{-\infty}^0 e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi=\int_0^\infty sng(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi$$
$$+\int_{-\infty}^0 sng(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi=\int_{-\infty}^\infty sng(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi$$

where sgn is the sign function, hence $v(z)=\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2\pi ix\xi}d\xi$ moreover

$$v(x,t)=\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}\int_{-\infty}^\infty f(\eta)e^{-2\pi i\eta\xi}d\eta e^{2\pi ix\xi}d\xi=\int_{-\infty}^\infty\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}f(\eta)e^{-2\pi i\eta\xi}e^{2\pi ix\xi}d\eta d\xi$$
$$=\int_{-\infty}^\infty\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}f(\eta)e^{2\pi i(x-\eta)\xi}d\eta d\xi=\int_{-\infty}^\infty f(\eta)\left(\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}e^{2\pi i(x-\eta)\xi}d\xi\right)d\eta.$$

Let $Q_t(x)=\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}e^{2\pi ix\xi}d\xi$. Then

$$v(x,t)=\int_{-\infty}^\infty f(\eta)Q_t(x-\eta)d\eta=(Q_t*f)(x).$$

Note that $Q_t(x)=\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}e^{2\pi ix\xi}d\xi=(-isng(\xi)e^{-2\pi t|\xi|})^\vee(x)$, by the Fourier´s inversion Theorem we have $\widehat{Q_t}(\xi)=-isgn(\xi)e^{-2\pi t|\xi|}$ but we can to find an explicit expression:

$$Q_t(x)=\int_{-\infty}^\infty -isng(\xi)e^{-2\pi t|\xi|}e^{2\pi ix\xi}d\xi=\int_{-\infty}^0 ie^{2\pi t\xi+2\pi ix\xi}d\xi+\int_0^\infty -ie^{-2\pi t\xi+2\pi ix\xi}d\xi$$
$$=i\int_{-\infty}^0 e^{2\pi\xi(t+ix)}d\xi-i\int_0^\infty e^{2\pi\xi(ix-t)}d\xi=i\left[\frac{e^{2\pi\xi(t+ix)}}{2\pi(t+ix)}\right]_{-\infty}^0-i\left[\frac{e^{2\pi\xi(ix-t)}}{2\pi(ix-t)}\right]_0^\infty$$
$$=\frac{i}{2\pi(t+ix)}+\frac{i}{2\pi(ix-t)}=\frac{i}{2\pi}\left(\frac{1}{t+ix}+\frac{1}{ix-t}\right)=\frac{i}{2\pi}\left(\frac{ix-t+t+ix}{-x^2-t^2}\right)=\frac{x}{\pi(x^2+t^2)}$$

as $P_t(x)+iQ_t(x)=\frac{t}{\pi(x^2+t^2)}+\frac{ix}{\pi(x^2+t^2)}=\frac{t+ix}{\pi(x^2+t^2)}=\frac{i(x-it)}{\pi(x^2+t^2)}=\frac{i}{\pi(x+it)}=\frac{i}{\pi z}$ is holomorphic in $\mathbb{H}$ we have that Q_t is the conjugate Poisson kernel.

As consequence of the proposition 1.3

$$\lim_{t\to 0^+} Q_t*f(x)=\frac{1}{\pi}p.v\left(\frac{1}{x}\right)(f)(x)$$

we see that:

$$p.v\left(\frac{1}{x}\right)(f)(x)=\lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{f(y-x)}{x}dx,$$

in fact, for every $\varphi \in \mathcal{S}(\mathbb{R})$:

$$\left(p.v\left(\frac{1}{x}\right)*f\right)(\varphi) = p.v\left(\frac{1}{x}\right)(\bar{f}*\varphi) = \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{(\bar{f}*\varphi)(x)}{x}dx$$
$$= \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{1}{x}\left(\int_{\mathbb{R}}\bar{f}(x-y)\varphi(y)dy\right)dx = \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\int_{\mathbb{R}}\frac{f(y-x)\varphi(y)}{x}dydx$$
$$= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\int_{|x|>\epsilon}\frac{f(y-x)\varphi(y)}{x}dxdy = \int_{\mathbb{R}}\varphi(y)\left(\lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{f(y-x)}{x}dx\right)dy$$

then

$$\left(p.v\left(\frac{1}{x}\right)*f\right)(y) = \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{f(y-x)}{x}dx,$$

hence

$$\lim_{t\to 0^+} Q_t * f(x) = \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy.$$

By the continuity of $\mathscr{F}: \mathcal{S}(\mathbb{R})' \to \mathcal{S}(\mathbb{R})'$ we have

$$\left(\frac{1}{\pi}p.v\left(\frac{1}{x}\right)\right)^{\wedge}(\xi) = \lim_{t\to 0^+}\widehat{Q_t}(\xi) = \lim_{t\to 0^+} -isng(\xi)e^{-2\pi t|\xi|} = -isng(\xi)$$

.

1.3.2 Hilbert Transform

Definition 1.2. Hilbert Transform

We define by any one of the following equivalent expressions $\mathscr{H}: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$,

$$\mathscr{H}f = \lim_{t\to 0^+} Q_t * f,$$

$$\mathscr{H}f = \frac{1}{\pi}p.v\left(\frac{1}{x}\right)*f,$$

$$(\mathscr{H}f)^{\wedge}(\xi) = -isgn(\xi)\widehat{f}(\xi)$$

the third equation and the Plancherel theorem allows us extend the definition to $L^2(\mathbb{R})$ and see that $\|\mathscr{H}f\|_2 = \left\|\widehat{f}\right\|_2 = \|f\|_2$, for $f \in L^2(\mathbb{R})$. Moreover if $f \in \mathcal{S}(\mathbb{R})$:

$$\mathscr{H}(\mathscr{H}f)(\xi) = (-isng(\xi)(\mathscr{H}f)^{\wedge}(\xi))^{\vee} = ((-isng(\xi))^2\widehat{f}(\xi))^{\vee} = -f(\xi) \tag{1.5}$$

for every $\xi \in \mathbb{R}$, $\mathscr{H}(\mathscr{H}f) = -f$, and if f is real then u and v are real functions, in the above notation $u(x,t) = P_t * f(x)$, $v(x,t) = Q_t * f(x)$ in particular $\mathscr{H}f(x) = \lim_{t\to 0} v(x,t)$, then the Hilbert transform of a real function is real, using the Parseval identity:

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{H}f(\xi)g(\xi)d\xi &= \langle \mathscr{H}f, g\rangle_{L^2(\mathbb{R})} = \langle \widehat{\mathscr{H}f}, \widehat{g}\rangle_{L^2(\mathbb{R})} = \langle -isng(\xi)\widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R})} \\ &= -\langle \widehat{f}, -isgn(\xi)\widehat{g}\rangle_{L^2(\mathbb{R})} = -\langle f, \mathscr{H}g\rangle_{L^2(\mathbb{R})} = -\int_{\mathbb{R}} f(\xi)\mathscr{H}g(\xi)d\xi\end{aligned} \tag{1.6}$$

The following theorem will be useful in the chapter 2:

Theorem 1.2. *(Riesz)*
$\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded for every $1 < p < \infty$, i.e there exists $C_p > 0$ such that

$$\|\mathscr{H}f\|_p \le C_p \|f\|_p ,$$

for every $f \in L^p(\mathbb{R})$

Proof. If we have $F : \mathbb{H} \to \mathbb{C}$, $F = u + iv$ holomorphic and $f \in \mathcal{S}(\mathbb{R})$, $f(x) = u(x, 0)$ then $\mathscr{H}f(x) = v(x, 0)$ this implies that $F(x, 0) = f(x) + i\mathscr{H}f(x)$ then applying this to $F^2 = (u^2 - v^2) + 2iuv$ we have that $H(f^2 - (Hf)^2) = 2fHf$ applying H and using (1.5) we have $-f^2 + (\mathscr{H}f)^2 = 2\mathscr{H}(f\mathscr{H}f)$, so

$$(\mathscr{H}f)^2 = f^2 + 2\mathscr{H}(f\mathscr{H}f) \tag{1.7}$$

now we have the following result

$\circledast$ if $\|\mathscr{H}f\|_p \le C_p \|f\|_p$, $\forall f \in \mathcal{S}(\mathbb{R})$ then $\|\mathscr{H}f\|_{2p} \le (2C_p + 1) \|f\|_{2p}$, $\forall f \in \mathcal{S}(\mathbb{R})$.

In fact, $\|\mathscr{H}f\|_{2p}^2 = \left\|(\mathscr{H}f)^2\right\|_p \le \left\|f^2\right\|_{2p} + 2\left\|\mathscr{H}(f\mathscr{H}f)\right\|_p \le \left\|f^2\right\|_{2p} + 2C_p\left\|f\mathscr{H}f\right\|_p$, but $\|f\mathscr{H}f\|_p = \|f^p(\mathscr{H}f)^p\|_1^{1/p} \le \|f^p\|_2^{1/p}\|(Hf)^p\|_2^{1/p} = (\|f^p\|_2^2)^{\frac{1}{2p}}(\|(\mathscr{H}f)^p\|_2^2)^{\frac{1}{2p}} = \|f\|_{2p}\|\mathscr{H}f\|_{2p}$, then

$$\|\mathscr{H}f\|_{2p}^2 \le \|f\|_{2p}^2 + 2C_p\|f\|_{2p}\|\mathscr{H}f\|_{2p} .$$

We have two posibilities:

$\divideontimes$ If $\|\mathscr{H}f\|_{2p} \le \|f\|_{2p}$ then $\|\mathscr{H}f\|_{2p}^2 \le \|f\|_{2p}^2 + 2C_p\|f\|_{2p}^2 = (2C_p + 1)\|f\|_{2p}^2$
$\le (2C_p + 1)^2\|f\|_{2p}^2$, so $\|\mathscr{H}f\|_{2p} \le (2C_p + 1)\|f\|_{2p}$.

$\divideontimes$ If $\|f\|_{2p} \le \|\mathscr{H}f\|_{2p}$, then $\|\mathscr{H}f\|_{2p}^2 \le \|f\|_{2p}^2 + 2C_p\|f\|_{2p}\|\mathscr{H}f\|_{2p} \le (2C_p + 1)\|f\|_{2p}\|\mathscr{H}f\|_{2p}$, so $\|\mathscr{H}f\|_{2p} \le (2C_p + 1)\|f\|_{2p}$.

this complete the proof of $\circledast$. By induction we prove that for every $k \ge 1$, $\|\mathscr{H}f\|_{2^k} \le (2^k - 1)\|f\|_{2^k}$,for $k = 1$ is obvious because $\mathscr{H}$ is and isometric isomorphism, suppose that this is true for k, for $k + 1$ using $\circledast$: $\|\mathscr{H}f\|_{2^{k+1}} \le (2(2^k - 1) + 1)\|f\|_{2^{k+1}} = (2^{k+1} - 1)\|f\|_{2^{k+1}}$. The Riesz-Thorin theorem (See [8]) applied to $\|\mathscr{H}f\|_2 = \|f\|_2$ and $\|\mathscr{H}f\|_{2^k} \le (2^k - 1)\|f\|_{2^k}$ implies that $\mathscr{H}$ is bounded in $L^p(\mathbb{R}^n)$ for all $p \in [2, 2^k]$ for every $k \ge 1$ therefore $\mathscr{H}$ is bounded in $L^p(\mathbb{R}^n)$ for every $p \in [2, \infty)$.

For $1 < p < 2$ we use the following argument of duality given that the conjugate exponent satisfies $q \geq 2$, by (1.6):

$$\begin{aligned} \|\mathscr{H}f\|_p &= \sup\left\{\left|\int_{\mathbb{R}} \mathscr{H}f(x)g(x)dx\right| \mid \|g\|_q \leq 1\right\} = \sup\left\{\left|\int_{\mathbb{R}} f(x)\mathscr{H}g(x)dx\right| \mid \|g\|_q \leq 1\right\} \\ &\leq \|f\|_p \sup\left\{\left|\|\mathscr{H}g\|_q\right| \mid \|g\|_q \leq 1\right\} \leq C_q \|f\|_p \end{aligned} \tag{1.8}$$

then $\|\mathscr{H}f\|_p \leq C_p \|f\|_p$, for every $f \in \mathcal{S}(\mathbb{R})$, $1 < p < \infty$, but $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ so $\|\mathscr{H}f\|_p \leq C_p \|f\|_p$ for every $f \in L^p(\mathbb{R})$, this complete the proof. $\Diamond$

1.4 Other important estimates

This section compiles some technical results which we will use later. It may be skipped on a first reading.

1.4.1 Approximation of the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$

Proposition 1.4. *For every $f \in L^p(\mathbb{R}^n)$, $\|f\|_p = \sup\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\}$, $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. In fact, by the Holder inequality:

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \leq \int_{\mathbb{R}^n} |f(x)|\,|g(x)|\,dx \leq \|f\|_p \|g\|_q$$

for every $g \in C_0^\infty(\mathbb{R}^n)$, so

$$C := \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \leq \|f\|_p$$

note that if $f = 0$ we have the inequality immediately, if f is nonzero let $E = \{x \in \mathbb{R}^n \mid f(x) = 0\}$, as f is mensurable E is mensurable, define $g = \frac{|f|^p}{f(\|f\|_p + \chi_E)^{p-1}}$, as $\frac{1}{p} + \frac{1}{q} = 1$ we have $\frac{p}{q} = p - 1$ so $p = (p-1)q$, also as f is mensurable g is mensurable, moreover

$$\int_{\mathbb{R}^n} |g(x)|^q\,dx = \int_{\mathbb{R}^n} \frac{|f(x)|^p\,dx}{(\|f\|_p + \chi_E)^p} = \int_{\mathbb{R}^n - E} \frac{|f(x)|^p\,dx}{\|f\|_p^p} = \int_{\mathbb{R}^n} \frac{|f(x)|^p\,dx}{\|f\|_p^p} = 1$$

so $g \in L^q(\mathbb{R}^n)$, $\|g\|_q = 1$, moreover:

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| = \left|\int_{\mathbb{R}^n} \frac{|f(x)|^p\,dx}{(\|f\|_p + \chi_E(x))^{p-1}}\right| = \int_{\mathbb{R}^n - E} \frac{|f(x)|^p\,dx}{\|f\|_p^{p-1}} = \frac{1}{\|f\|_p^{p-1}} \int_{\mathbb{R}^n} |f(x)|^p\,dx = \|f\|_p$$

as $g \in L^q(\mathbb{R}^n)$ there exists $\{g_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$, such that $\lim\limits_{k\to\infty} \|g_k - g\|_q = 0$ in particular there exists $\{g_{k_l}\}_{l\in\mathbb{N}}$ subsequence of $\{g_k\}_{k\in\mathbb{N}}$ such that $g(x) = \lim\limits_{l\to 0} g_{k_l}(x)$ a.e $x \in \mathbb{R}^n$, as $\lim\limits_{l\to\infty} \|g_{k_l}\|_p = \|g\|_q = 1$ we can put $h_l = \frac{g_{k_l}}{\|g_{k_l}\|_q} \in C_0^\infty(\mathbb{R}^n)$, $\|h_l\|_q = 1$, $\lim\limits_{l\to\infty} h_l(x) = g(x)$ a.e $x \in \mathbb{R}^n$, so $\{fh_l\}_{l\in\mathbb{N}} \subset L^1(\mathbb{R}^n)$ satisfies that $\lim\limits_{l\to\infty} (fh_l)(x) = (fg)(x)$ a.e $x \in \mathbb{R}^n$, moreover $fg \in L^1(\mathbb{R}^n)$, the dominated convergence theorem implies that $\lim\limits_{l\to\infty} \|fh_l - fg\|_1 = 0$, in special:

$$C \geq \lim_{l\to\infty} \left| \int_{\mathbb{R}^n} f(x)h_l(x)dx \right| = \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| = \|f\|_p$$

this completes the proof. ◊

1.4.2 Fourier transform of radial functions and Bessel functions

The Fourier transform of a radial function is obtained in terms of the Bessel function for this is necessary study some properties of that functions in the origin and in the infinity.

Definition 1.3. Let $k \in \mathbb{R}$, $k > -\frac{1}{2}$ the Bessel function of order k is defined by:

$$J_k(t) = \frac{\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{its}(1-s^2)^{k-\frac{1}{2}} ds$$

the following properties will be used in this thesis.

Proposition 1.5. *(1) If $f(x) = f_0(\|x\|)$, $x \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$, then*

$$\widehat{f}(\xi) = 2\pi \|\xi\|^{1-\frac{n}{2}} \int_0^\infty f_0(s) J_{\frac{n}{2}-1}(2\pi \|\xi\| s) s^{\frac{n}{2}} ds$$

(2) If $j > -\frac{1}{2}$, $k > -1$, and $t > 0$, then

$$J_{j+k+1}(t) = \frac{t^{k+1}}{2^k \Gamma(k+1)} \int_0^1 J_j(ts) s^{j+1} (1-s^2)^k ds$$

(3) $J_\mu(t) = O(t^\mu)$ if $t \to 0$ and $J_\mu(t) \sim t^{-\frac{1}{2}}$ if $t \to \infty$.

Proof. (1) Let $s = \|x\|$, $x = su$, $r = \|\xi\|$, $\xi = rv$ then by the polar coordinates formula:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx = \int_{\mathbb{R}^n} f_0(\|x\|) e^{-2\pi i x\cdot\xi} dx = \int_0^\infty \int_{S^{n-1}} f_0(s) e^{-2\pi i r s u\cdot v} s^{n-1} d\sigma(v) ds$$
$$= \int_0^\infty f_0(s) s^{n-1} \int_{S^{n-1}} e^{-2\pi i r s u\cdot v} d\sigma(v) ds$$

we calculate the inner integral, for this let:

$$L_\theta(u) = \left\{v \in S^{n-1} \mid u \cdot v = \cos\theta\right\}$$

by influence of the geometry this set is called a paralell, note that $\sigma(L_\theta(Au)) = \sigma(L_\theta(u))$ for every $A \in O(n)$, in fact, $v \in L_\theta(Au) \Leftrightarrow (Au)\cdot v = u \cdot (A^{-1}v) = \cos\theta \Leftrightarrow A^{-1}v \in L_\theta(u) \Leftrightarrow v \in A(L_\theta(u))$ so $L_\theta(Au) = A(L_\theta(u))$, as σ is invariant by rotations $\sigma(L_\theta(Au)) = \sigma A(L_\theta(u)) = \sigma(L_\theta(u))$ this allows us to choose $u = e_n$, then $L_\theta(e_n) = \left\{v \in S^{n-1} \mid v_n = \cos\theta\right\}$, but $v \in L_\theta(e_n) \Leftrightarrow \|v\| = 1, v_n = \cos(\theta), \left\|v^{(n-1)}\right\|^2 = 1 - v_n^2 = 1 - \cos^2\theta = \sin^2\theta \Leftrightarrow \left\|v^{(n-1)}\right\| = \sin\theta$, then $L_\theta(e_n) = \left\{(v^{(n-1)}, \cos\theta) \in \mathbb{R}^n \mid \left\|v^{(n-1)}\right\| = \sin\theta\right\}$ but σ is invariant by traslations then $\sigma(L_\theta(e_n)) = \sigma(S^{n-2}_{\sin\theta} \times \{0\}) = \sigma_{n-2}(S^{n-2}_{\sin\theta}) = \sigma_{n-2}(S^{n-2})\sin^{n-2}\theta$ so

$$\int_{S^{n-1}} e^{-2\pi i r s u\cdot v} d\sigma(v) = \int_0^\pi \int_{L_\theta(u)} e^{-2\pi i r s u\cdot v} d\sigma(v) d\theta = \int_0^\pi \int_{L_\theta(u)} e^{-2\pi i r s \cos\theta} d\sigma(v) d\theta$$
$$= \int_0^\pi e^{-2\pi i r s\cos\theta} \int_{L_\theta(u)} d\sigma(v) d\theta = \int_0^\pi e^{-2\pi i r s \cos\theta} \sigma(L_\theta(u)) d\theta = \int_0^\pi e^{-2\pi i r s\cos\theta} \sigma_{n-2}(S^{n-2}) \sin^{n-2}\theta d\theta$$

but $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ then $\sigma(S^{n-1}) = n\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, take $t = -\cos(\theta)$, if $\theta = 0 \Rightarrow t = -1$, if $\theta = \pi \Rightarrow t = 1$, $dt = \sin(\theta)d\theta$, then

$$\int_{S^{n-1}} e^{-2\pi i r s u\cdot v} d\sigma(v) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^1 e^{2\pi i r s t}(1-t^2)^{\frac{n-3}{2}} dt = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{(\pi r s)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi r s)$$
$$= 2\pi (rs)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(2\pi r s)$$

so

$$\widehat{f}(\xi) = \int_0^\infty f_0(s) s^{n-1}(2\pi)(rs)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(2\pi r s) ds = 2\pi \|\xi\|^{1-\frac{n}{2}} \int_0^\infty f_0(s) J_{\frac{n}{2}-1}(2\pi\|\xi\| s) s^{n/2} ds$$

(2) As $J_k(t) = \frac{\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{its}(1-s^2)^{k-\frac{1}{2}} ds = \frac{\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} (\int_{-1}^0 e^{its}(1-s^2)^{k-\frac{1}{2}} ds$ $+\int_0^1 e^{its}(1-s^2)^{k-\frac{1}{2}} ds)$ for the first integral let $\overline{s} = -s$, if $s = -1 \Rightarrow \overline{s} = 1$, if $s = 0 \Rightarrow \overline{s} = 0$, $d\overline{s} = -ds$, hence $J_k(t) = \frac{\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \left(\int_1^0 e^{-it\overline{s}}(1-\overline{s}^2)^{k-\frac{1}{2}}(-d\overline{s}) + \int_0^1 e^{its}(1-s^2)^{k-\frac{1}{2}} ds\right) = \frac{2\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 \cos(ts)(1-s^2)^{k-\frac{1}{2}} ds$ using the Taylor's expansion for cos we have

$$\cos(its) = \sum_{j=0}^\infty \frac{(-1)^j (ts)^{2j}}{(2j)!} = \sum_{j=0}^\infty \frac{(-1)^j t^{2j} s^{2j}}{(2j)!}$$

hence

$$J_k(t) = \frac{2\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \sum_{j=0}^\infty \frac{(-1)^j t^{2j}}{(2j)!} \int_0^1 s^{2j}(1-s^2)^{k-\frac{1}{2}} ds$$

take $u = s^2$, $du = 2sds$, $ds = \frac{du}{2\sqrt{u}}$ then:

$$\int_0^1 s^{2j}(1-s^2)^{k-\frac{1}{2}}ds = \int_0^1 u^j(1-u)^{k-\frac{1}{2}}\frac{du}{2\sqrt{u}} = \frac{1}{2}\int_0^1 u^{j-\frac{1}{2}}(1-u)^{k-\frac{1}{2}}du$$
$$= \frac{1}{2}\int_0^1 u^{(j+\frac{1}{2})-1}(1-u)^{(k+\frac{1}{2})-1}du = \frac{1}{2}B\left(j+\frac{1}{2}, k+\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(j+k+1)}$$

given that $j > -\frac{1}{2}$, so

$$J_k(t) = \frac{2\left(\frac{t}{2}\right)^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j t^{2j}}{(2j)!}\cdot\frac{1}{2}\cdot\frac{\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(j+k+1)} = \frac{2\left(\frac{t}{2}\right)^k}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j t^{2j}}{(2j)!}\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+k+1)}$$

but $\Gamma(j+\frac{1}{2}) = (j-\frac{1}{2})\Gamma(j-\frac{1}{2}) = (j-\frac{1}{2})(j-\frac{3}{2})\Gamma(j-\frac{3}{2}) = ... = (j-\frac{1}{2})(j-\frac{3}{2})...(j-\frac{2j-1}{2})\Gamma(\frac{1}{2}) = \frac{(2j-1)(2j-3)...(3)(1)}{2^j}\Gamma(\frac{1}{2}) = \frac{(2j)!}{2^{2j}j!}\Gamma(\frac{1}{2})$ implies that

$$J_k(t) = \frac{2\left(\frac{t}{2}\right)^k}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j t^{2j}}{(2j)!}\frac{(2j)!\Gamma(\frac{1}{2})}{2^{2j}j!\Gamma(j+k+1)} = \sum_{j=0}^{\infty}\frac{(-1)^j(\frac{t}{2})^{k+2j}}{j!\Gamma(j+k+1)}$$

hence

$$\int_0^1 J_j(ts)s^{j+1}(1-s^2)^k ds = \int_0^1\left(\sum_{l=0}^{\infty}\frac{(-1)^l(ts/2)^{j+2l}}{l!\Gamma(l+j+1)}\right)s^{j+1}(1-s^2)^k ds$$
$$= \sum_{l=0}^{\infty}\frac{(-1)^l(\frac{t}{2})^{j+2l}}{l!\Gamma(l+k+1)}\int_0^1 s^{2j+2l+1}(1-s^2)^k ds$$

let $r = s^2$, then $dr = 2sds$,

$$\int_0^1 s^{2j+2l+1}(1-s^2)^k ds = \frac{1}{2}\int_0^1(s^2)^{j+l}(1-s^2)^k(2sds) = \frac{1}{2}\int_0^1 r^{j+l}(1-r)^k dr$$
$$= \frac{1}{2}B(j+k+1, k+1) = \frac{1}{2}\frac{\Gamma(j+k+1)\Gamma(k+1)}{\Gamma(l+j+k+2)}$$

given that $k > -1$. Hence

$$\int_0^1 J_j(ts)s^{j+1}(1-s^2)^k ds = \sum_{l=0}^{\infty}\frac{(-1)^l(\frac{t}{2})^{j+2l}}{l!\Gamma(l+j+1)}\cdot\frac{1}{2}\cdot\frac{\Gamma(j+l+1)\Gamma(k+1)}{\Gamma(l+j+k+2)}$$
$$= \sum_{l=0}^{\infty}\frac{(-1)^l(\frac{t}{2})^{j+2l}}{2l!\Gamma(l+j+k+2)}\cdot\Gamma(k+1) = \sum_{l=0}^{\infty}\frac{(-1)^l(\frac{t}{2})^{j+k+1+2l}}{2l!\Gamma(l+j+k+2)}\cdot\frac{\Gamma(k+1)}{(t/2)^{k+1}}$$
$$= \frac{2^k\Gamma(k+1)}{t^{k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l(\frac{t}{2})^{j+k+1+2l}}{l!\Gamma(l+j+k+2)} = \frac{2^k\Gamma(k+1)}{t^{k+1}}J_{j+k+1}(t)$$

so $J_{j+k+1}(t) = \frac{t^{k+1}}{2^k\Gamma(k+1)}\int_0^1 J_j(ts)s^{j+1}(1-s^2)^k ds$.

(3) These properties can be found in [8, Pags. 158-159] $\Diamond$

Corollary 1.1. *Let* $\delta > 0$, $\Phi_\delta : \mathbb{R}^n \to \mathbb{R}$,

$$\Phi_\delta(t) = \max\left\{(1 - \|t\|^2)^\delta, 0\right\} = \begin{cases} (1 - \|t\|^2)^\delta & if \quad \|t\| \leq 1 \\ \\ 0 & if \quad \|t\| > 1 \end{cases}$$

then $\widehat{\Phi_\delta}(\xi) = \pi^{-\delta}\Gamma(\delta+1)\|\xi\|^{-\frac{n}{2}-\delta} J_{\frac{n}{2}+\delta}(2\pi\|\xi\|)$.

Proof. As Φ_δ is radial the part (1) the previous proposition implies that:

$$\widehat{\Phi_\delta}(\xi) = 2\pi\|\xi\|^{1-\frac{n}{2}} \int_0^1 (1-s^2)^\delta J_{\frac{n}{2}-1}(2\pi\|\xi\| s) s^{n/2} ds = 2\pi\|\xi\|^{1-\frac{n}{2}} \int_0^1 J_{\frac{n}{2}-1}(2\pi\|\xi\| s) s^{n/2}(1-s^2)^\delta ds$$

$$= 2\pi\|\xi\|^{1-\frac{n}{2}} J_{\frac{n}{2}+\delta}(2\pi\|\xi\|) \cdot \frac{2^\delta\Gamma(\delta+1)}{(2\pi\|\xi\|)^{\delta+1}} \qquad \text{Part (2) of the previous proposition}$$

$$= \pi^{-\delta}\Gamma(\delta+1)\|\xi\|^{-\frac{n}{2}-\delta} J_{\frac{n}{2}+\delta}(2\pi\|\xi\|)$$

$\Diamond$

CHAPTER 2

Introduction to Fourier Multipliers

In this chapter we define the Fourier multipliers and study their principal properties, the Hilbert transform is used to prove that a characteristic function of an interval is a multiplier, this fact with the extension Theorem for multipliers give the generalization for a convex polyhedron in $\mathbb{R}^n$, we conclude the chapter with the complete proof of the restriction Theorem of de Leeuw.

2.1 Basic definitions and results

Definition 2.1. If $m \in L^\infty(\mathbb{R}^n)$ and $1 < p < \infty$, $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, if $\widehat{T_m f} = m\widehat{f}$ is a bounded operator we say that m is a multiplier on $L^p(\mathbb{R}^n)$ and the norm of this is $|m|_p = \|T_m\|$.

For example by the Riesz Theorem (Theorem 1.1) $m(\xi) = -isng(\xi)$ is a multiplier on $L^p(\mathbb{R})$.

Proposition 2.1. *If m is a multiplier on $L^p(\mathbb{R}^n)$, $m^a(\xi) = m(\xi + a), a \in \mathbb{R}^n$, $m_\lambda(\xi) = m(\lambda\xi), \lambda > 0$, $m \circ A$, $A \in O(n)$ are multipliers on $L^p(\mathbb{R}^n)$, moreover $|m|_p = |m^a|_p = |m_\lambda|_p = |m \circ A|_p$.*

Proof. Take $f \in \mathcal{S}(\mathbb{R}^n)$, let $a \in \mathbb{R}^n$, then using the Proposition 1.2, $T_{m^a} f(x) = \int_{\mathbb{R}^n} m^a(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$ $= \int_{\mathbb{R}^n} m(\xi + a)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi - a)e^{2\pi i x\cdot(\xi - a)}d\xi = \int_{\mathbb{R}^n} m(\xi)(\tau_{-a}\widehat{f})(\xi)e^{2\pi i x\cdot(\xi-a)}d\xi$ $= \int_{\mathbb{R}^n} m(\xi)\widehat{M_a f}(\xi)e^{2\pi i x\cdot(\xi-a)}d\xi = M_{-a}(x)\int_{\mathbb{R}^n} m(\xi)\widehat{M_a f}(\xi)e^{2\pi i x\cdot\xi}d\xi = M_{-a}(x)T_m(M_a f)(x)$.

Then $\|T_{m^a} f\|_p = \|M_{-a}T_m(M_a f)\|_p = \|T_m(M_a f)\|_p \leq |m|_p \|M_a f\|_p = |m|_p \|f\|_p$, so m^a is a multiplier and $|m^a|_p \leq |m|_p = |(m^a)^{-a}|_p \leq |m^a|_p$, i.e $|m|_p = |m^a|_p$.

Let $\lambda > 0$, if $g(x) = f(\lambda x)$, using the Proposition 1.2, $T_{m_\lambda} f(x) = \int_{\mathbb{R}^n} m_\lambda(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$ $= \int_{\mathbb{R}^n} m(\lambda\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{\mathbb{R}^n} m(\eta)\widehat{f}(\frac{\eta}{\lambda})e^{2\pi i x\cdot\frac{\eta}{\lambda}}\frac{d\eta}{\lambda^n} = \frac{1}{\lambda^n}\int_{\mathbb{R}^n} m(\xi)\widehat{f}(\frac{\xi}{\lambda})e^{2\pi i\frac{x}{\lambda}\cdot\xi}d\xi = \int_{\mathbb{R}^n} m(\xi)\widehat{g}(\xi)e^{2\pi i\frac{x}{\lambda}\cdot\xi}d\xi$ $= (T_m g)(\frac{x}{\lambda})$.

Then $\|T_{m_\lambda} f\|_p^p = \int_{\mathbb{R}^n} |(T_{m_\lambda} f)(x)|^p\, dx = \int_{\mathbb{R}^n} \left|(T_m g)(\frac{x}{\lambda})\right|^p dx = \int_{\mathbb{R}^n} |(T_m g)(z)|^p \lambda^n dz = \lambda^n \|T_m g\|_p^p \leq |m|_p^p \lambda^n \|g\|_p^p = |m|_p^p \lambda^n \int_{\mathbb{R}^n} |g(x)|^p dx = |m|_p^p \int_{\mathbb{R}^n} |f(y)|^p dy = |m|_p^p \|f\|_p^p$, so m_λ is a multiplier and $|m_\lambda|_p \leq |m|_p = |(m_\lambda)_{\lambda^{-1}}|_p \leq |m_\lambda|_p$, i.e $|m|_p = |m_\lambda|_p$.

Let $A \in O(n)$, using the Proposition 1.2, $\widehat{f \circ A} = \widehat{f} \circ A$ then:

$$\begin{aligned}(T_{m\circ A} f)(x) &= \int_{\mathbb{R}^n} (m \circ A)(\xi)\widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = \int_{\mathbb{R}^n} m(\eta)\widehat{f}(A^{-1}\eta)e^{2\pi i x\cdot A^{-1}\eta} d\eta \\ &= \int_{\mathbb{R}^n} m(\eta)\widehat{f}(A^{-1}\eta)e^{2\pi i Ax\cdot\eta} d\eta = \int_{\mathbb{R}^n} m(\eta)\widehat{f\circ A^{-1}}(\eta)e^{2\pi i Ax\cdot\eta} d\eta = (T_m(f\circ A^{-1}))(Ax).\end{aligned}$$

Then $\|T_{m\circ A} f\|_p^p = \int_{\mathbb{R}^n} |(T_{m\circ A} f)(x)|^p dx = \int_{\mathbb{R}^n} \left|(T_m(f\circ A^{-1}))(Ax)\right|^p dx = \int_{\mathbb{R}^n} \left|(T_m(f\circ A^{-1}))(y)\right|^p dy = \left\|T_m(f\circ A^{-1})\right\|_p^p \leq |m|^p \left\|f\circ A^{-1}\right\|_p^p = |m|_p^p \int_{\mathbb{R}^n} \left|(f\circ A^{-1})(x)\right|^p dx = |m|_p^p \int_{\mathbb{R}^n} |f(y)|^p dy = |m|_p^p \|f\|_p^p$, so $m \circ A$ is a multiplier and $|m\circ A|_p \leq |m|_p = \left|(m\circ A)\circ A^{-1}\right|_p \leq |m\circ A|_p$, i.e $|m|_p = |m\circ A|_p$. $\Diamond$

Let $\mathcal{M}(L^p(\mathbb{R}^n))$ be the set of multipliers of $L^p(\mathbb{R}^n)$. The following Theorem is a powerfull characterization for Fourier multipliers.

Theorem 2.1. *$m \in \mathcal{M}(L^p(\mathbb{R}^n))$ if and only if there exists $C > 0$ such that for every $f, g \in C_0^\infty(\mathbb{R}^n)$*

$$\left|\int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx\right| \leq C \|f\|_p \|g\|_q, \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, moreover

$$|m|_p = \inf\left\{C > 0 \mid \left|\int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx\right| \leq C \|f\|_p \|g\|_q, \forall f, g \in C_0^\infty(\mathbb{R}^n)\right\}.$$

Proof. Suppose that $m \in \mathcal{M}(L^p(\mathbb{R}^n))$, then $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $(T_m f)^\wedge = m\widehat{f}$ is bounded, let $f, g \in C_0^\infty(\mathbb{R}^n)$, $\|f\|_p = \|g\|_q = 1$, then $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$, $(T_m f)^\wedge = m\widehat{f} \in L^2(\mathbb{R}^n)$ we can use the Parseval identity:

$$\begin{aligned}\left|\int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx\right| &= \left|\int_{\mathbb{R}^n} (m\widehat{f})(x)\check{g}(x)dx\right| = \left|\int_{\mathbb{R}^n} (T_m f)(x)g(x)dx\right| \\ \leq \sup\left\{\sup\left\{\left|\int_{\mathbb{R}^n} (T_m f)(x)g(x)dx\right| \mid \|g\|_q = 1\right\} \mid \|f\|_p = 1\right\} &= \sup\left\{\|T_m\|_p \mid \|f\|_p = 1\right\} = \|T_m\|\end{aligned}$$

we note that (2.1) is valid if $f = 0$ or $g = 0$, if f and g are nonzero then:

$$\begin{aligned}\left|\int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx\right| &= \left|\int_{\mathbb{R}^n} m(x)\|f\|_p \left(\frac{f}{\|f\|_p}\right)^\wedge(x)\|g\|_q \left(\frac{g}{\|g\|_q}\right)^\wedge(-x)dx\right| \\ &= \left|\int_{\mathbb{R}^n} m(x)\left(\frac{f}{\|f\|_p}\right)^\wedge(x)\left(\frac{g}{\|g\|_q}\right)^\wedge(-x)dx\right| \|f\|_p \|g\|_q \leq \|T_m\| \|f\|_p \|g\|_q\end{aligned}$$

Conversely suppose that (2.1) is valid and $1 \leq p \leq 2$, using the previous Proposition with $f \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned} \|Tf\|_p &= \sup\left\{\left|\int_{\mathbb{R}^n}(Tf)(x)g(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n}(m\widehat{f})(x)\check{g}(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \leq C\|f\|_p \end{aligned}$$

since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ we have that T_m is bounded and $|m|_p = \|T_m\| \leq C$.

For $p > 2$ we use the following argument of duality:

$$\begin{aligned} \|Tf\|_p &= \sup\left\{\left|\int_{\mathbb{R}^n}(T_m f)(x)g(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n}(m\widehat{f})(x)\widehat{\overline{g}}(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n}(m\widehat{\overline{g}})(x)\check{\overline{f}}(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n}(T_m\overline{g})(x)\overline{f}(x)dx\right| \mid \|g\|_q = 1, g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &\leq \|T_m\overline{g}\|_{L^q(\mathbb{R}^n)}\left\|\overline{f}\right\|_p \leq \|T_m\|_{B(L^q(\mathbb{R}^n))}\|f\|_p \leq C\|f\|_p \end{aligned}$$

for every $f \in C_0^\infty(\mathbb{R}^n)$, given that $1 \leq q \leq 2$ and we saw that $m \in \mathcal{M}(L^p(\mathbb{R}^n))$, if $1 \leq p \leq 2$, here $\overline{f}(x) = f(-x)$, and we use that $\left\|\overline{f}\right\|_p = \|f\|_p$ for every $f \in L^p(\mathbb{R}^n)$.

Again as $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ we have that T_m is bounded. This completes the proof. $\Diamond$

Corollary 2.1. *Let* $m \in \mathcal{M}(L^p(\mathbb{R}^n))$ *then:*
(a) $|m|_2 = \|m\|_\infty$
(b) $|m|_p = |m|_q$, $\frac{1}{p} + \frac{1}{q} = 1$
(c) $\|m\|_\infty \leq |m|_p$ *for* $1 < p < \infty$.

Proof. Let $m \in \mathcal{M}(L^p(\mathbb{R}^n))$, the Plancherel Theorem implies that:

$$\|T_m f\|_2 = \|(T_m f)^\wedge\|_2 = \left\|m\widehat{f}\right\|_2 \leq \|m\|_\infty \left\|\widehat{f}\right\|_2 = \|m\|_\infty \|f\|_2$$

then $|m|_2 = \|T_m\| \leq \|m\|_\infty$ take $\epsilon > 0$ arbitrary and $A \subset \{x \in \mathbb{R}^n \mid |m(x)| > \|m\|_\infty - \epsilon\}$, A mensurable, since $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric isomorphism there exists $f \in L^2(\mathbb{R}^n)$ such that $\widehat{f} = \mathscr{F}f = \chi_A$, hence:

$$\begin{aligned} &\|T_m f\|_2^2 = \|(T_m f)^\wedge\|_2^2 = \left\|m\widehat{f}\right\|_2^2 = \|m\chi_A\|_2^2 = \int_{\mathbb{R}^n}|(m\chi_A)(x)|^2\,dx \\ &= \int_A |m(x)|^2\,dx \geq (\|m\|_\infty - \epsilon)^2 vol(A) = (\|m\|_\infty - \epsilon)^2\|\chi_A\|_2^2 = (\|m\|_\infty - \epsilon)^2\|f\|_2^2 \end{aligned}$$

then $|m|_2 = \|T_m\| \geq \|m\|_\infty - \epsilon$, for every $\epsilon > 0$, so $|m|_2 \geq \|m\|_\infty$, this proves (a). By the previous argument $|m|_2 = \|m\|_\infty$. The previous Theorem implies that if $f, g \in C_0^\infty(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} m(x)\widehat{f}(x)\widehat{g}(-x)dx \right| \leq C \|f\|_p \|g\|_q$$

but $\overline{f}, \overline{g} \in C_0^\infty(\mathbb{R}^n)$, we have:

$$\left| \int_{\mathbb{R}^n} m(x)\widehat{g}(x)\widehat{f}(-x)dx \right| = \left| \int_{\mathbb{R}^n} m(x)\widehat{\overline{f}}(x)\widehat{\overline{g}}(-x)dx \right| \leq |m|_p \left\|\overline{f}\right\|_p \|\overline{g}\|_q = |m|_p \|f\|_p \|g\|_q$$

hence $|m|_q \leq |m|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$, interchanging p and q we obtain that $|m|_p \leq |m|_q$, so $|m|_p = |m|_q$, this proves (b). As for every $f \in C_0^\infty(\mathbb{R}^n)$, $\|f\|_2 = 1$, $\|T_m f\|_2 = \left\|(T_m f)^2\right\|_1^{1/2} \leq \|T_m f\|_p^{1/2} \|T_m f\|_q^{1/2} \leq |m|_p^{1/2} |m|_q^{1/2} = |m|_p$ we have that $\|m\|_\infty = |m|_2 = \|T_m\|_{\mathscr{B}(L^2(\mathbb{R}^n))} \leq |m|_p$, this proves (c).

◊

The set $\mathcal{M}(L^p(\mathbb{R}^n))$ can be submerged in $\mathscr{B}(L^p(\mathbb{R}^n))$ as prove the following:

Theorem 2.2. *$\mathcal{M}(L^p(\mathbb{R}^n))$ is isometrically isomorphic to a Banach subalgebra of $\mathscr{B}(L^p(\mathbb{R}^n))$.*

Proof. Let $m_1, m_2 \in \mathcal{M}(L^p(\mathbb{R}^n))$ and $\alpha \in \mathbb{C}$ then for every $f \in \mathcal{S}(\mathbb{R}^n)$:

$$T_{m_1+m_2} f = ((m_1 + m_2)\widehat{f})^\vee = (m_1\widehat{f} + m_2\widehat{f})^\vee = (m_1\widehat{f})^\vee + (m_2\widehat{f})^\vee = T_{m_1} f + T_{m_2} f = (T_{m_1} + T_{m_2})f$$

$$T_{\alpha m_1} f = (\alpha m_1 \widehat{f})^\vee = \alpha(m_1\widehat{f})^\vee = \alpha T_{m_1} f$$

$$T_{m_1} T_{m_2} f = T_{m_1}(T_{m_2} f) = T_{m_1}(m_2\widehat{f})^\vee = (m_1 m_2 \widehat{f})^\vee = T_{m_1 m_2} f$$

as $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ we have that $m_1 + m_2 \in \mathcal{M}(L^p(\mathbb{R}^n))$, $\alpha m_1 \in \mathcal{M}(L^p(\mathbb{R}^n))$, $m_1 m_2 \in \mathcal{M}(L^p(\mathbb{R}^n))$. Let $\{m_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(L^p(\mathbb{R}^n))$ a Cauchy sequence, in the proof of the previous lemma we saw that $\|m\|_\infty \leq |m|_p$ for every $m \in \mathcal{M}(L^p(\mathbb{R}^n))$ so $\{m_j\}_{j\in\mathbb{N}} \subset L^\infty(\mathbb{R}^n)$ is a Cauchy sequence but this is a Banach space, there exists $m \in L^\infty(\mathbb{R}^n)$ such that $\|m_j - m\|_\infty \to_{j\to\infty} 0$.

Let $f \in \mathcal{S}(\mathbb{R}^n)$, by definition $T_{m_j} f = (m_j\widehat{f})^\vee$, $T_m f = (m\widehat{f})^\vee$, as $\left\{m_j\widehat{f}\right\}_{j\in\mathbb{N}} \cup \left\{m\widehat{f}\right\} \subset L^1(\mathbb{R}^n)$ the dominated convergence Theorem implies that:

$$(T_{m_j} f)(x) = \int_{\mathbb{R}^n} m_j(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi \to_{j\to\infty} \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = (T_m f)(x)$$

a.e $x \in \mathbb{R}^n$, as $\{m_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(L^p(\mathbb{R}^n))$ is a Cauchy sequence is bounded, let $M = \sup_{j\in\mathbb{N}} |m_j|_p < \infty$, by the Fatou lemma:

$$\int_{\mathbb{R}^n} |(T_m f)(x)|^p \, dx \leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} \left|(T_{m_j} f)(x)\right|^p dx = \liminf_{j\to\infty} \left\|T_{m_j} f\right\|_p^p \leq \liminf_{j\to\infty} |m_j|^p \|f\|_p^p \leq M^p \|f\|_p^p$$

hence $\|T_m f\|_p \leq M \|f\|_p$ for every $f \in \mathcal{S}(\mathbb{R}^n)$, but $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $T_m \in \mathscr{B}(L^p(\mathbb{R}^n))$, so $m \in \mathcal{M}(L^p(\mathbb{R}^n))$.

We note that the previous argument implies that if $\{k_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(L^p(\mathbb{R}^n))$ and $k_j(x) \to_{j\to\infty} k(x)$ a.e $x \in \mathbb{R}^n$ then $k \in \mathcal{M}(L^p(\mathbb{R}^n))$, moreover:

$$|k|_p \leq \liminf_{j\to\infty} |k_j|_p \tag{2.2}$$

for $l \in \mathbb{N}$ fixed we define $k_j := m_j - m_l$ then $k = m - m_l$, using the equation (2.2)

$$|m - m_l|_p \leq \liminf_{j\to\infty} |m_j - m_l|_p$$

as $\{m_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(L^p(\mathbb{R}^n))$ is a Cauchy sequence we have that:

$$\limsup_{l\to\infty} |m - m_l|_p \leq \lim_{l\to\infty} \liminf_{j\to\infty} |m_j - m_l|_p = \lim_{j,l\to\infty} |m_j - m_l|_p = 0$$

so $|m_l - m|_p \to_{l\to\infty} 0$, by definition $\mathcal{M}(L^p(\mathbb{R}^n))$ is a Banach algebra, on the other hand $\Phi : \mathcal{M}(L^p(\mathbb{R}^n)) \to \mathscr{B}(L^p(\mathbb{R}^n))$, $\Phi(m) = T_m$ is an isometric monomorphism, then $\mathcal{M}(L^p(\mathbb{R}^n))$ is isometrically isomorphic to $Im(\Phi)$ that is a Banach subalgebra of $\mathscr{B}(L^p(\mathbb{R}^n))$. $\Diamond$

2.2 Intervals and one dimensional multipliers

Let $I = [a, b]$, we define $\widehat{T_I f} = \chi_I \widehat{f}$, as $\widehat{M_c f} = \tau_{-c}\widehat{f}$ for every $c \in \mathbb{R}$ we have:

$$\begin{aligned}(M_c\mathscr{H}M_{-c}f)^\wedge(\xi) &= \tau_{-c}(\mathscr{H}M_{-c}f)^\wedge(\xi) = \tau_{-c}(\mathscr{H}(M_{-c}f))^\wedge(\xi) = \tau_{-c}(-isng(\xi)(M_{-c}f)^\wedge(\xi)) \\ &= \tau_{-c}(-isng(\xi)\tau_c\widehat{f}(\xi)) = -i\tau_{-c}sng(\xi)\tau_{-c}\tau_c\widehat{f}(\xi) = -i\tau_{-c}(sng(\xi))\widehat{f}(\xi) = -isng(\xi - c)\widehat{f}(\xi)\end{aligned}$$

then $m(\xi) = -isng(\xi - c)$ is a multiplier on $L^p(\mathbb{R})$ given that M_a is an isometric isomorphism of $L^p(\mathbb{R}^n)$ for every $a \in \mathbb{R}^n$ and $\mathscr{H}$ is bounded by the Riesz Theorem. Note that $\frac{i}{2}((M_a\mathscr{H}M_{-a} - M_b\mathscr{H}M_{-b})f)^\wedge(\xi) = \frac{i}{2}(-isng(\xi - a) + isng(\xi - b))\widehat{f}(\xi) = \frac{1}{2}(sng(\xi - a) - sng(\xi - b))\widehat{f}(\xi)$, but $a \leq b$ implies that $\xi \in I$, $sng(\xi - b) = -1$, $sng(\xi - a) = 1$, if $\xi > b$ then $sng(\xi - b) = 1$, $sng(\xi - a) = 1$, if $\xi < a$ then $sng(\xi-b) = -1$, $sng(\xi-a) = -1$ so $sng(\xi-b)-sng(\xi-a) = 2\chi_I(\xi)$ hence $\frac{i}{2}((M_a\mathscr{H}M_{-a} - M_b\mathscr{H}M_{-b})f)^\wedge(\xi) = \chi_I(\xi)\widehat{f}(\xi) = (T_I f)^\wedge(\xi)$, then $(T_I f)^\wedge = \frac{i}{2}((M_a\mathscr{H}M_{-a} - M_b\mathscr{H}M_{-b})f)^\wedge$ so $T_I f = \frac{i}{2}(M_a\mathscr{H}M_{-a} - M_b\mathscr{H}M_{-b})f$ for every $f \in \mathcal{S}(\mathbb{R})$, then

$$T_I = \frac{i}{2}(M_a\mathscr{H}M_{-a} - M_b\mathscr{H}M_{-b}).$$

We have the proof of the following Theorem:

Theorem 2.3. *There exists $C_p > 0$, $1 < p < \infty$ independent of I such that $\|T_I f\|_p \leq C_p \|f\|_p$, for every $f \in L^p(\mathbb{R})$.*

We note that I can be every interval in $\mathbb{R}$, even I unbounded because M_a is an isometry for every $a \in \mathbb{R}$ and the previous Theorem gives a constant independent of I, if we take $I = [-R, R]$ then the operator of partial sum $T_{[-R,R]}f(x) = \int_{-R}^{R} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi$ is bounded for every $R > 0$; the previous Theorem says that the constant $C_p > 0$ is independent of $R > 0$, this prove the following:

Corollary 2.2. *If $f \in L^p(\mathbb{R})$, $1 < p < \infty$ then* $\lim_{R\to\infty} \left\|T_{[-R,R]}f - f\right\|_p = 0$.

the proof of the previous Corollary is a particular case of the Corollary 2.3 bellow.

2.3 Higher dimensional extensions

The following Proposition extend a multiplier $m \in \mathcal{M}(L^p(\mathbb{R}^{n_1}))$ to $\overline{m} \in \mathcal{M}(L^p(\mathbb{R}^n))$, with $n > n_1$.

Theorem 2.4. *extension Theorem*
If $m \in \mathcal{M}(L^p(\mathbb{R}^{n_1}))$, let $n = n_1 + n_2$, $\overline{m} : \mathbb{R}^n \to \mathbb{R}$, $\overline{m}(\xi, \eta) = m(\xi)$, then $\overline{m} \in \mathcal{M}(L^p(\mathbb{R}^n))$, moreover $|\overline{m}|_p \leq |m|_p$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\omega = (\xi, \eta)$, $\widehat{f}(\omega) = \int_{\mathbb{R}^n} f(t)e^{-2\pi it\cdot\omega}dt = \int_{\mathbb{R}^n} f(x,y)e^{-2\pi ix\cdot\xi-2\pi iy\cdot\eta}dt$ $= \int_{\mathbb{R}^{n_1}} e^{-2\pi ix\cdot\xi}\left(\int_{\mathbb{R}^{n_2}} f(x,y)e^{-2\pi iy\cdot\eta}dy\right)dx = \int_{\mathbb{R}^{n_1}} e^{-2\pi ix\cdot\xi}(\mathscr{F}_{n_2}f)(x,\eta)dx = \mathscr{F}_{n_1}(\mathscr{F}_{n_2}f(\cdot,\eta))(\xi)$, here $\mathscr{F}_k$ is the Fourier transform acting in dimension k, if $z = (x, y)$ this implies that:

$$
\begin{aligned}
(T_{\overline{m}}f)(z) = (\overline{m}\widehat{f})^{\vee}(z) &= \int_{\mathbb{R}^n} (\overline{m}\widehat{f})(\omega)e^{2\pi iz\cdot\omega}d\omega = \int_{\mathbb{R}^n} \overline{m}(\omega)\widehat{f}(\omega)e^{2\pi iz\cdot\omega}d\omega = \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\omega)e^{2\pi iz\cdot\omega}d\omega \\
&= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} m(\xi)\widehat{f}(\omega)e^{2\pi iz\cdot\omega}d\eta d\xi = \int_{\mathbb{R}^{n_1}} m(\xi)e^{2\pi ix\cdot\xi}\int_{\mathbb{R}^{n_2}} \widehat{f}(\eta,\omega)e^{2\pi iy\cdot\eta}d\eta d\xi \\
&= \int_{\mathbb{R}^{n_1}} m(\xi)e^{2\pi ix\cdot\xi}\int_{\mathbb{R}^{n_2}} \mathscr{F}_{n_1}(\mathscr{F}_{n_2}f(\cdot,\eta))(\xi)e^{2\pi iy\cdot\eta}d\eta d\xi \\
&= \int_{\mathbb{R}^{n_1}} m(\xi)e^{2\pi ix\cdot\xi}\mathscr{F}_{n_1}\left(\int_{\mathbb{R}^{n_2}} \mathscr{F}_{n_2}f(\cdot,\eta)e^{2\pi iy\cdot\eta}d\eta\right)(\xi)d\xi \\
&= \int_{\mathbb{R}^{n_1}} m(\xi)e^{2\pi ix\cdot\xi}\mathscr{F}_{n_1}(f(\cdot,y))(\xi)d\xi = (T_m f(\cdot,y))(x)
\end{aligned}
$$

where the penultimate equality is given by the Fourier inversion Theorem, then

$$
\begin{aligned}
\|T_{\overline{m}}f\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |T_{\overline{m}}f(x)|^p\, dx = \int_{\mathbb{R}^{n_2}}\left(\int_{\mathbb{R}^{n_1}} |T_m f(\cdot,y)(x)|^p\, dx\right)dy \\
&= \int_{\mathbb{R}^{n_2}} \|T_m f(\cdot,y)\|^p_{L^p(\mathbb{R}^{n_1})}\, dy \leq \int_{\mathbb{R}^{n_2}} |m|^p_p\, \|f(\cdot,y)(x)\|^p_p\, dy \\
&= |m|^p_p \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}} |f(x,y)|^p\, dxdy = |m|^p_p \int_{\mathbb{R}^n} |f(x)|^p\, dx = |m|^p_p\, \|f\|^p_p
\end{aligned}
$$

so $\|T_{\overline{m}}f\|_{L^p(\mathbb{R}^n)} \leq |m|_p\, \|f\|_p$, $T_{\overline{m}}$ is bounded and $\overline{m}$ is a multiplier, moreover $|\overline{m}|_p \leq |m|_p$. ◊

Corollary 2.3. *If $m \in \mathcal{M}(L^p(\mathbb{R}))$, let $\overline{m} : \mathbb{R}^n \to \mathbb{R}$, $\overline{m}(\xi, \eta) = m(\xi)$, then $\overline{m} \in \mathcal{M}(L^p(\mathbb{R}^n))$, moreover $|\overline{m}|_p \leq |m|_p$.*

Proof. Is a immediate consequence of the Theorem with $n_1 = 1$. $\Diamond$

The previous Proposition using the multiplier $m = \chi_{(0,\infty)}$ implies that $\overline{m} = \chi_{(0,\infty)\times\mathbb{R}^{n-1}}$ is a multiplier of a half-space, using the Proposition 2.1 we have that χ_L is a multiplier where L is an arbitrary half-space, if we take P convex polyhedron of N faces this is the intersection of N half-spaces $L_1, ..., L_N$, but $\chi_P = \chi_{L_1}...\chi_{L_N}$ using the Theorem 2.1 χ_P is a multiplier, this prove the following Corollary:

Corollary 2.4. *There exists $C_p > 0$, $1 < p < \infty$, independent of $\lambda > 0$ such that $\|T_{\lambda P} f\|_p \leq C_p \|f\|_p$, for every $f \in L^p(\mathbb{R}^n)$, where $T_{\lambda P} = \Phi(\chi_{\lambda P})$ and $\lambda P = \{\lambda x \in \mathbb{R}^n \mid x \in P\}$.*

Proof. As λP is a polyhedron for every $\lambda > 0$ we have that $\chi_{\lambda P}$ is a multiplier, as $|\chi_{\lambda P}|_p = |\chi_{\lambda P_0}|_p$ with $P_0 = [-R_1, R_1] \times ... \times [-R_n, R_n]$ a cube with edges parallel to the axis then $\chi_{\lambda P_0}(x) = \chi_{[-\lambda R_1,\lambda R_1]}(x_1)...\chi_{[-\lambda R_n,\lambda R_n]}(x_n) = \overline{\chi_{[-\lambda R_1,\lambda R_1]}}(x)...\overline{\chi_{[-\lambda R_n,\lambda R_n]}}(x)$, then

$$\chi_{\lambda P_0} = \overline{\chi_{[-\lambda R_1,\lambda R_1]}}...\overline{\chi_{[-\lambda R_n,\lambda R_n]}}$$

by the Theorem 2.3 we have that there exists $B_p > 0$ such that $\left|\chi_{[-\lambda R_j,\lambda R_j]}\right|_p \leq B_p$ for every $\lambda > 0$, $j \in \{1, ..., n\}$ then $|\chi_{\lambda P}|_p \leq \left|\overline{\chi_{[-\lambda R_1,\lambda R_1]}}...\overline{\chi_{[-\lambda R_n,\lambda R_n]}}\right|_p \leq \left|\overline{\chi_{[-\lambda R_1,\lambda R_1]}}\right|_p ... \left|\overline{\chi_{[-\lambda R_n,\lambda R_n]}}\right|_p \leq B_p^n$ for every $\lambda > 0$, if we take $C_p = B_p^n$ then $|\chi_{\lambda P}|_p \leq C_p$, by definition $\|T_{\lambda P} f\|_p \leq C_p \|f\|_p$, for every $f \in L^p(\mathbb{R}^n)$, $\lambda > 0$. $\Diamond$

Corollary 2.5. *Let P be a convex polyhedron of $\mathbb{R}^n$ that contains the origin. If $1 < p < \infty$ then*

$$\lim_{\lambda\to\infty} \|T_{\lambda P} f - f\|_p = 0,$$

where $T_{\lambda P} = \Phi(\chi_{\lambda P})$.

Proof. We consider $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, as $\mathscr{F}$ is a topological isomorphism (i.e homeomorphism that is linear) the Proposition 1.1 implies that $\mathscr{F}^{-1}(C_0^\infty(\mathbb{R}^n))$ is dense in $\mathcal{S}(\mathbb{R}^n)$ in the metric topology. Let $f \in \mathscr{F}^{-1}(C_0^\infty(\mathbb{R}^n))$ then $\mathscr{F} f = \widehat{f} \in C_0^\infty(\mathbb{R}^n)$. There exists $\lambda_0 > 0$ such that *supp*$(\widehat{f}) \subset \lambda_0 P$ then $(\chi_{\lambda P}\widehat{f})(\xi)e^{2\pi i x\cdot\xi} = \widehat{f}(\xi)e^{2\pi i x\cdot\xi}$ for every $\lambda > \lambda_0$, so that

$$(T_{\lambda P} f)(x) = \int_{\mathbb{R}^n} (\chi_{\lambda P}\widehat{f})(\xi)e^{2\pi i x\cdot\xi} d\xi = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi = f(x)$$

Let $f \in L^p(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ there exists $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|f_k - f\|_p = 0$. Let $\epsilon > 0$ arbitrary and $k_0 \in \mathbb{N}$ such that $\|f_{k_0} - f\|_p < \frac{\epsilon}{4(1+C_p)}$, since $f_{k_0} \in \mathcal{S}(\mathbb{R}^n)$ there exists $\{f_{k_0,l}\}_{l\in\mathbb{N}} \subset \mathscr{F}^{-1}(C_0^\infty(\mathbb{R}^n))$ such that $\lim_{l\to\infty} \|f_{k_0,l} - f_{k_0}\|_{\alpha,\beta} = 0$ for every $\alpha, \beta \in \mathbb{N}^n$, then $\lim_{l\to\infty} \|f_{k_0,l} - f_{k_0}\|_p = 0$, there exists $l_0 \in \mathbb{N}$ such that $\|f_{k_0,l_0} - f_{k_0}\|_p < \frac{\epsilon}{4(1+C_p)}$, since $\lim_{\lambda\to\infty} \|T_{\lambda P} f_{k_0,l_0} - f_{k_0,l_0}\|_p = 0$ there exists $\lambda_1 > 0$ such that if $\lambda > \lambda_1$ then $\|T_{\lambda P} f_{k_0,l_0} - f_{k_0,l_0}\|_p < \frac{\epsilon}{2}$, using the Corollary 2.4 we have:

$$\begin{aligned} \|T_{\lambda P}f - f\|_p \le \|T_{\lambda P}f - T_{\lambda P}f_{k_0}\|_p + \|T_{\lambda P}f_{k_0} - f_{k_0}\|_p + \|f_{k_0} - f\|_p \le \|T_{\lambda P}f - T_{\lambda P}f_{k_0}\|_p \\ + \|T_{\lambda P}f_{k_0} - T_{\lambda P}f_{k_0,l_0}\|_p + \|T_{\lambda P}f_{k_0,l_0} - f_{k_0,l_0}\|_p + \|f_{k_0,l_0} - f_{k_0}\|_p + \|f_{k_0} - f\|_p \\ = \|T_{\lambda P}(f - f_{k_0})\|_p + \|T_{\lambda P}(f_{k_0} - f_{k_0,l_0})\|_p + \|T_{\lambda P}f_{k_0,l_0} - f_{k_0,l_0}\|_p + \|f_{k_0,l_0} - f_{k_0}\|_p + \|f_{k_0} - f\|_p \\ \le C_p\|f_{k_0} - f\|_p + C_p\|f_{k_0,l_0} - f_{k_0}\|_p + \|T_{\lambda P}f_{k_0,l_0} - f_{k_0,l_0}\|_p + \|f_{k_0,l_0} - f_{k_0}\|_p + \|f_{k_0} - f\|_p \\ = (1 + C_p)(\|f_{k_0,l_0} - f_{k_0}\|_p + \|f_{k_0} - f\|_p) + \|T_{\lambda P}f_{k_0,l_0} - f_{k_0,l_0}\|_p < \epsilon \end{aligned}$$

if $\lambda > \lambda_1$, i.e $\lim_{\lambda\to\infty} \|T_{\lambda P}f - f\|_p = 0$, this completes the proof. ◊

Corollary 2.6. *The operator $T_{cube} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for every $1 < p < \infty$.*

2.4 de Leeuw's Theorem

In this section we prove de Leeuw's Theorem. The proof given here is due to [1].

Theorem 2.5. *(de Leeuw)*
Let $m \in \mathcal{M}(L^p(\mathbb{R}^n))$, $n = n_1 + n_2$, then for almost every $x \in \mathbb{R}^n$, $m_x(\cdot) = m(x,\cdot) \in \mathcal{M}(L^p(\mathbb{R}^{n_2}))$ and $|m_x|_p \le |m|_p$. The restriction is possible in

$$\Omega = \{x \in R^{n_1} \mid (x,y) \quad \textit{is a Lebesgue point of} \quad m, \quad a.e \quad y \in \mathbb{R}^{n_2}\}.$$

Proof. Let $m \in \mathcal{M}(L^p(\mathbb{R}^n))$ by Theorem 2.1 we have that there exists $C > 0$ such that for every $f^*, g^* \in C_0^\infty(\mathbb{R}^n)$,

$$\left|\int_{\mathbb{R}^n} m(x)\widehat{f^*}(x)\widehat{g^*}(-x)dx\right| \le C\|f^*\|_p \|g^*\|_q$$

for every $x \in \Omega$, we define $m_x : \mathbb{R}^{n_2} \to \mathbb{R}$, $m_x(y) = m(x,y)$ and take $f, \varphi \in C_0^\infty(\mathbb{R}^n)$, $g, \psi \in C_0^\infty(\mathbb{R}^n)$, then $f^*(x,y) = f(x)g(y)$, $g^*(x,y) = \varphi(x)\psi(y)$ satisfies $f^*, g^* \in C_0^\infty(\mathbb{R}^n)$, let's assume first that m is continuous and define:

$$I : \mathbb{R}^{n_1} \to \mathbb{R}, \quad I(x) = \int_{\mathbb{R}^{n_2}} m(x,y)\widehat{g}(y)\widehat{\psi}(-y)dy$$

then the Fubini's Theorem implies that:

$$\begin{aligned} \left|\int_{\mathbb{R}^{n_1}} I(x)\widehat{f}(x)\widehat{\varphi}(-x)dx\right| = \left|\int_{\mathbb{R}^{n_1}}\left(\int_{\mathbb{R}^{n_2}} m(x,y)\widehat{g}(y)\widehat{\psi}(-y)dy\right)\widehat{f}(x)\widehat{\varphi}(-x)dx\right| \\ = \left|\int_{\mathbb{R}^n} m(\xi)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)dx\right| \le |m|_p\|f^*\|_p\|g^*\|_q = |m|_p\|f\|_{L^p(\mathbb{R}^{n_1})}\|g\|_{L^q(\mathbb{R}^{n_2})}\|\varphi\|_{L^p(\mathbb{R}^{n_1})}\|\psi\|_{L^q(\mathbb{R}^{n_2})} \\ = \left(|m|_p\|g\|_{L^q(\mathbb{R}^{n_2})}\|\psi\|_{L^q(\mathbb{R}^{n_2})}\right)\|f\|_{L^p(\mathbb{R}^{n_1})}\|\varphi\|_{L^p(\mathbb{R}^{n_1})} \end{aligned}$$

by the Theorem 2.1, $I \in \mathcal{M}(L^p(\mathbb{R}^{n_1}))$ moreover $|I|_p \leq |m|_p \|g\|_{L^q(\mathbb{R}^{n_2})} \|\psi\|_{L^q(\mathbb{R}^{n_2})}$.

By the Corolary 2.1:

$$\left|\int_{\mathbb{R}^{n_2}} m_x(y)\widehat{g}(y)\widehat{\psi}(-y)dy\right| = \left|\int_{\mathbb{R}^{n_2}} m(x,y)\widehat{g}(y)\widehat{\psi}(-y)dy\right| = |I(x)| \leq \|I\|_\infty \leq |m|_p \|g\|_{L^p(\mathbb{R}^{n_2})} \|\psi\|_{L^q(\mathbb{R}^{n_2})} \tag{2.3}$$

again by the Theorem 2.1 $m_x \in L^p(\mathbb{R}^{n_2})$, moreover $|m_x|_p \leq |m|_p$.

In the general case we eliminate the continuity restriction, if $m \in L^\infty(\mathbb{R}^n)$ we have that $m \in L^1_{loc}(\mathbb{R}^n)$, in fact if $K \subset \mathbb{R}^n$ is compact:

$$\int_K |m(x)|\, dx \leq \|m\|_\infty vol(K) < \infty$$

this implies that almost every $x \in \mathbb{R}^n$ is a Lebesgue point of m, (see [10] Theorem 19.21), this let us define:

$$\Omega = \{x \in R^{n_1} \mid (x,y) \quad \text{is a Lebesgue point of} \quad m, \quad a.e \quad y \in \mathbb{R}^{n_2}\}$$

also $vol(R^n - \Omega) = 0$. Let $m_\epsilon : \mathbb{R}^n \to \mathbb{R}, m_\epsilon = m \star \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}$, then for every $f^*, g^* \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} m_\epsilon(\xi)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\xi\right| &= \left|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} m(\xi-\eta)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)d\eta\right)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\xi\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi-\eta)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\eta d\xi\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi-\eta)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\xi d\eta\right|
\end{aligned}$$

let $\mu = \xi - \eta$, then $\xi = \mu + \eta$ the Theorem of change of variables applies to the internal integral:

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} m_\epsilon(\xi)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\xi\right| &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi-\eta)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\widehat{f^*}(\xi)\widehat{g^*}(-\xi)d\xi d\eta\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\mu)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\widehat{f^*}(\mu+\eta)\widehat{g^*}(-\mu-\eta)d\mu d\eta\right| \\
&= \left|\int_{\mathbb{R}^n} \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\int_{\mathbb{R}^n} m(\mu)\widehat{f^*}(\mu+\eta)\widehat{g^*}(-\mu-\eta)d\mu d\eta\right| \\
&= \left|\int_{\mathbb{R}^n} \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\int_{\mathbb{R}^n} m(\mu)(\widehat{M_\eta f^*})(\mu)(\widehat{M_{-\eta}g^*})(-\mu)d\mu d\eta\right| \qquad (2.4)\\
&\leq \int_{\mathbb{R}^n} \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\left|\int_{\mathbb{R}^n} m(\mu)(\widehat{M_\eta f^*})(\mu)(\widehat{M_{-\eta}g^*})(-\mu)d\mu\right| d\eta \\
&\leq \int_{\mathbb{R}^n} \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)\, |m|_p \|M_\eta f^*\|_p \|M_{-\eta}g^*\|_q \, d\eta \\
&= |m|_p \|f^*\|_p \|g^*\|_q \int_{\mathbb{R}^n} \epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\eta)d\eta = |m|_p \|f^*\|_p \|g^*\|_q
\end{aligned}$$

by the Theorem 2.1 $m_\epsilon \in \mathcal{M}(L^p(\mathbb{R}^n))$, moreover $|m_\epsilon|_p \leq |m|_p$, for every $\epsilon > 0$ ⋇.

The expression $m_\epsilon(\xi) = \int_{\mathbb{R}^n} m(\eta)\epsilon^{-n}\chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n}(\xi-\eta)d\eta = \frac{1}{\epsilon^n}\int_{\xi+[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n} m(\eta)d\eta$ and the dominated convergence Theorem implies that m_ϵ is continuous, if $\xi = (x,y)$ with $x \in \Omega$ then ξ is a Lebesgue point for almost every $y \in \mathbb{R}^{n_2}$, then

$$m(\xi) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^n} \int_{\xi+[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n} m(\eta)d\eta = \lim_{\epsilon\to 0} m_\epsilon(\xi)$$

in special:

$$m_x(y) = m(x,y) = \lim_{\epsilon\to 0} m_\epsilon(x,y) = \lim_{\epsilon\to 0} (m_\epsilon)_x(y)$$

a.e $y \in \mathbb{R}^{n_2}$, since m_ϵ is continuous we have that $m_\epsilon \in L^1_{loc}(\mathbb{R}^n)$, the dominated convergence Theorem, $|(m_\epsilon)_x|_p \le |m_\epsilon|_p$ for every $\epsilon > 0$ and $\divideontimes$ implies that

$$\left|\int_{\mathbb{R}^{n_2}} m_x(y)\widehat{g}(y)\widehat{\psi}(-y)dy\right| = \lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^{n_2}} (m_\epsilon)_x(y)\widehat{g}(y)\widehat{\psi}(-y)dy\right| \le \lim_{\epsilon\to 0} |(m_\epsilon)_x|_p \, \|g\|_{L^p(\mathbb{R}^{n_2})} \, \|\psi\|_{L^q(\mathbb{R}^{n_2})}$$
$$\le \lim_{\epsilon\to 0} |m_\epsilon|_p \, \|g\|_{L^p(\mathbb{R}^{n_2})} \, \|\psi\|_{L^q(\mathbb{R}^{n_2})} \le |m|_p \, \|g\|_{L^p(\mathbb{R}^{n_2})} \, \|\psi\|_{L^q(\mathbb{R}^{n_2})}$$

for every $g, \psi \in C_0^\infty(\mathbb{R}^n)$, by the Theorem 2.1 $m_x \in \mathcal{M}(L^p(\mathbb{R}^{n_2}))$, $|m_x|_p \le |m|_p$, this completes the proof.

$\Diamond$

CHAPTER 3

Bochner-Riesz multipliers vs. Ball multipliers

In this Chapter we study the Bochner-Riesz multipliers as a class of continuous operator. We know that the characteristic function of a cube is a multiplier for every value $1 < p < \infty$ but this function is discontinuous in every point of the boundary of the cube, by reasons of correspondence we expect that the Bochner-Riesz multipliers are multipliers for $1 < p < \infty$ but in this chapter we are going to see that this is false.

3.1 Introduction

Consider the operators $(T_R f)(x) = \int_{\|\xi\|<R} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$, $f \in \mathcal{S}(\mathbb{R}^n)$ so that $\widehat{(T_R f)} = \chi_{B(0,R)}\widehat{f}$ and $T_R : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is the operator with Fourier multiplier $\chi_{B(0,R)}$. We would like to know if

$$\lim_{R\to\infty} \|T_R f - f\|_p = 0, f \in L^p(\mathbb{R}^n).$$

By the Corollary 2.2 of the Chapter 2 we know that it is true for $n = 1$, $1 < p < \infty$, for $n > 1$. The higher dimensional problem is more difficult for this we consider multipliers more regular that $\chi_{B(0,R)}$, for example taking the average between 0 and R of the operators T_R, i.e

$$\frac{1}{R}\int_0^R T_t f(x)dx = \frac{1}{R}\int_0^R \int_{\|\xi\|\le t} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi dt = \frac{1}{R}\int_0^R \int_{\mathbb{R}^n} \chi_{B(0,t)}(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi dt$$
$$= \frac{1}{R}\int_{\mathbb{R}^n}\int_0^R \chi_{B(0,t)}(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}dtd\xi = \int_{\mathbb{R}^n}\left(\frac{1}{R}\int_0^R \chi_{B(0,t)}(\xi)dt\right)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, but by definition

$$\chi_{B(0,t)}(\xi) = \begin{cases} 1 & if \quad t > \|\xi\| \\ \\ 0 & if \quad t \leq \|\xi\| \end{cases}$$

then for $\|\xi\| \leq R$:

$$\frac{1}{R}\int_0^R \chi_{B(0,t)}(\xi)dt = \frac{1}{R}\left(\int_0^{\|\xi\|} \chi_{B(0,t)}(\xi)dt + \int_{\|\xi\|}^R \chi_{B(0,t)}(\xi)dt\right) = \frac{1}{R}\int_{\|\xi\|}^R dt = 1 - \frac{\|\xi\|}{R}$$

for $\|\xi\| > R \geq t$ we have $\chi_{B(0,t)}(\xi) = 0$, hence

$$\frac{1}{R}\int_0^R \chi_{B(0,t)}(\xi)dt = \begin{cases} 1 - \frac{\|\xi\|}{R}, & if \quad \|\xi\| \leq R \\ \\ 0, & if \quad \|\xi\| > R \end{cases}$$

and so

$$\frac{1}{R}\int_0^R T_t f(x)dx = \int_{\|\xi\|\leq R}\left(1 - \frac{\|\xi\|}{R}\right)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

with this we obtain the operators

$$\widehat{(T_R^\delta f)}(\xi) = \left(1 - \frac{\|\xi\|}{R}\right)_+^\delta \widehat{f}(\xi)$$

where $A_+ = \max\{A, 0\}$, let $m(\xi) = (1 - \|\xi\|)_+^\delta$ then $m_{1/R}(\xi) = \left(1 - \frac{\|\xi\|}{R}\right)_+^\delta$ by the Proposition 2.1 of the Chapter 2 we know that if $m \in \mathcal{M}(L^p(\mathbb{R}^n))$ then $m_{1/R} \in \mathcal{M}(L^p(\mathbb{R}^n))$, moreover $\left|m_{1/R}\right|_p = |m|_p$, this fact allows us consider the case $R = 1$, that is we want to find the values of $1 \leq p \leq \infty$ for which $m \in \mathcal{M}(L^p(\mathbb{R}^n))$.

Instead of considering the operator T_1^δ we are going to consider the operator

$$\widehat{(T_\delta f)}(\xi) = \left(1 - \|\xi\|^2\right)_+^\delta \widehat{f}(\xi)$$

for $\delta > 0$ these are the Bochner-Riesz multipliers. Note that

$$\begin{aligned} (1 - \|\xi\|^2)_+^\delta &= (1 - \|\xi\|)_+^\delta \left[(1 + \|\xi\|)^\delta \psi_1(\|\xi\|)\right], \\ (1 - \|\xi\|)_+^\delta &= (1 - \|\xi\|^2)_+^\delta \left[(1 + \|\xi\|)^{-\delta} \psi_2(\|\xi\|)\right] \end{aligned} \tag{3.1}$$

where $\psi_1, \psi_2 \in C_0^\infty$ such that $\psi_1 \mid_{\overline{B(0,1)}} = \psi_2 \mid_{\overline{B(0,1)}} = 1$, now we need the following Theorem due to Hormander

Theorem 3.1. *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ a radial function with $supp(\psi) \subset \left\{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq \|\xi\| \leq 2\right\}$ and $\sum_{j=-\infty}^{\infty}\left|\psi(2^{-j}\xi)\right|^2 = 1$, if $m \in L^\infty(\mathbb{R}^n)$, $\sup_{j\in\mathbb{N}}\left\|m(2^j\cdot)\psi\right\|_{H^\alpha(\mathbb{R}^n)} < \infty$ for some $\alpha > \frac{n}{2}$. Then,*

$m \in \mathcal{M}(L^p(\mathbb{R}^n))$, $1 < p < \infty$.

and the Corollary

Corollary 3.1. *Let $m \in C^k(\mathbb{R}^n - \{0\})$ with $k = \left[\frac{n}{2}\right] + 1$ and*

$$\sup_{R>0} \left(\frac{1}{R^n} \int_{R<\|\xi\|<2R} \left|\partial^\beta m(\xi)\right|^2 d\xi \right)^{1/2} \leq CR^{-|\beta|}, |\beta| \leq k$$

then $m \in \mathcal{M}(L^p(\mathbb{R}^n))$, $1 < p < \infty$. In special this condition is satisfied if

$$\left|\partial^\beta m(\xi)\right| \leq C \left\|\xi\right\|^{-\beta}, |\beta| \leq k$$

the proofs of these facts can be found in [2, Pags. 163-164].

Applying the last part of the Corollary to the functions in the inside of the brackets in (3.1) we have that these functions are Fourier multipliers (it can be seen using the Leibnitz rule and that $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^n)$) hence $T_1^\delta : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded if and only if $T_\delta : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.

Since the multipliers $\xi \to (1 - \|\xi\|^2)_+^\delta$ have singularities in S^{n-1} we descompose these as the sum of a convergent series of terms with support in the dyadic anullus $A_k = A(1 - 2^{-k+1}, 1 - 2^{-k-1}) = \left\{\xi \in \mathbb{R}^n \mid 1 - 2^{-k+1} < \|\xi\| < 1 - 2^{-k-1}\right\}$, note that $\overline{B(0,1)} = \bigcup_{k\in\mathbb{N}} A_k$.

In fact, let $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that $supp(\varphi_k) \subset A_k, 0 \leq \varphi_k \leq 1, \left\|\partial^\beta \varphi_k\right\|_\infty \leq C_\beta 2^{(k+1)\beta}$ for some $C_\beta > 0$ independent of k, $\sum_{k=1}^\infty \varphi_k(t) = 1$ if $t \in \left[\frac{1}{2}, 1\right]$, with this take $\varphi_0 : [0,1] \to \mathbb{R}$,

$$\varphi_0(t) = \begin{cases} 1 - \sum_{k=1}^\infty \phi_k(t), & if \quad 0 \leq t \leq \frac{1}{2} \\ \\ 0, & if \qquad t \geq \frac{1}{2} \end{cases}$$

so $\sum_{k=0}^\infty \varphi_k(t) = 1$, for every $t \in [0,1]$, hence:

$$(1 - \|\xi\|^2)_+^\delta = \sum_{k=0}^\infty (1 - \|\xi\|^2)^\delta \varphi_k(\|\xi\|)$$

as $\varphi_k(\|\xi\|) = 0$ if $\xi \notin A_k$, also if $\xi \in A_k$, $1 - 2^{-k+1} < \|\xi\| < 1 - 2^{-k-1}$, then $(1 - 2^{-k+1})^2 < \|\xi\|^2$, $1 - 2^{2-k} + 2^{-2k+2} < \|\xi\|^2$, $1 - \|\xi\|^2 < 2^{2-k} + 2^{-2k+2} = 2^{-k}(4 - 2^{2-k})$.
We define $\tilde{\varphi}_k(\|\xi\|) = 2^{k\delta}(1 - \|\xi\|)^\delta \varphi_k(\|\xi\|)$, so

$$(1 - \|\xi\|^2)_+^\delta = \sum_{k=0}^\delta 2^{-k\delta} \tilde{\varphi}_k(\|\xi\|)$$

then $T_\delta f = \sum_{k=0}^\infty 2^{-k\delta} T_k f$, where $(T_k f)^\wedge(\xi) = \tilde{\varphi}_k(\|\xi\|) \widehat{f}(\xi)$.

3.2 Norm-boundedness in $L^p(\mathbb{R}^n)$ for Bochner-Riesz operators

Lema 3.1. *Let* $0 < \delta < 1$, $\varphi \in C_0^\infty(\mathbb{R})$ *such that* $supp(\varphi) \subset (1-4\delta, 1-\delta)$, $0 \le \varphi \le 1$, $\left\|\partial^\beta(\varphi \circ \|\cdot\|)\right\|_\infty \le C\delta^{-|\beta|}$, *if* $T^\delta = \Phi(\varphi \circ \|\cdot\|)$, *then for every* $\epsilon > 0$,

$$\left\|T^\delta f\right\|_p \le C(\epsilon)\delta^{-(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|}\left\|f\right\|_p.$$

Proof. Since $\varphi \circ \|\cdot\| \in C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric isomorphism there exists $k \in L^2(\mathbb{R}^n)$ such that $\varphi \circ \|\cdot\| = \mathscr{F}k = \widehat{k}$, take a even positive integer.

By the Plancherel Theorem

$$\begin{aligned}\|(1+\|\cdot\|^a)k\|_2 &= \left\|((1+(\|\cdot\|^2)^{a/2})k)^\wedge\right\|_2 = \left\|((I_{L^2(\mathbb{R}^n)} + M^{a/2}_{\|\cdot\|^2})k)^\wedge\right\|_2 \\ &= \left\|(I+(-\Delta)^{a/2})\varphi\circ\|\cdot\|\right\|_2 \le \|\varphi\circ\|\cdot\|\|_2 + \left\|(-\Delta)^{a/2}\varphi\circ\|\cdot\|\right\|_2\end{aligned}$$

but $\|\varphi\circ\|\cdot\|\|_2^2 = \int_{\mathbb{R}^n}\varphi(\|\xi\|)^2 d\xi \le C^2\int_{1-4\delta<\|\xi\|<1-\delta} d\xi \le C^2(vol_n(B(0,1-\delta)) - vol_n(B(0,1-4\delta)) = C^2\omega_n((1-\delta)^n - (1-4\delta)^n) = 3C^2\omega_n\delta\sum_{j=0}^{n-1}(1-\delta)^j(1-4\delta)^{n-1-j} = \overline{C}^2\delta$, hence $\|\varphi\circ\|\cdot\|\|_2 \le \overline{C}\delta^{1/2}$, also $(-\Delta)^{a/2}\varphi\circ\|\cdot\| = (-\sum_{|\alpha|=2}\partial^\alpha)^{a/2}\varphi\circ\|\cdot\| = \sum_{j=0}^{\frac{a}{2}}\binom{\frac{a}{2}}{j}(-\sum_{|\alpha|=2}\partial^\alpha)^j\varphi\circ\|\cdot\|$, so

$$\begin{aligned}\left\|(-\Delta)^{a/2}\varphi\circ\|\cdot\|\right\|_2 &= \left\|\sum_{j=0}^{\frac{a}{2}}\binom{\frac{a}{2}}{j}\left(-\sum_{|\alpha|=2}\partial^\alpha\right)^j\varphi\circ\|\cdot\|\right\|_2 \le \sum_{j=0}^{\frac{a}{2}}\binom{\frac{a}{2}}{j}\left\|\left(-\sum_{|\alpha|=2}\partial^\alpha\right)^j\varphi\circ\|\cdot\|\right\|_2 \\ &\le \sum_{j=0}^{\frac{a}{2}}\binom{\frac{a}{2}}{j}C\delta^{-2j}\|\varphi\circ\|\cdot\|\|_2 \le 2^{\frac{a}{2}}C\delta^{-a}\overline{C}\delta^{1/2} = C_1\delta^{1/2-a}\end{aligned}$$

given that $0 \le j \le \frac{a}{2}$, $0 < \delta < 1$, implies $\delta^a < \delta^{2j}$, so $\delta^{-2j} < \delta^{-a}$, and the known formula $\sum_{l=0}^m\binom{m}{l} = 2^m$, hence $\|(1+\|\cdot\|^a)k\|_2 \le C_2\delta^{1/2}(1+\delta^{-a}) \le \overline{C}\delta^{1/2-a}$.

We claim that this fact is true for every $a > 0$, for $s > 1$ define $\theta(t) = \frac{(1+t)^s}{1+t^s}$, $t > 0$ then θ is continuous, but $\lim\limits_{t\to\infty}\theta(t) = \lim\limits_{t\to\infty}\frac{\left(\frac{1}{t}+1\right)^s}{\frac{1}{t^s}+1} = 1$, so θ is bounded, then there exists $M > 0$ such that $(1+t)^s \le M(1+t^s)$, take $s > 1$ such that as is an even positive integer we have $(1+\|\cdot\|^a)^s \le M(1+\|\cdot\|^{as})$,
so

$$1+\|\cdot\|^a \le M^{1/s}(1+\|\cdot\|^{as})^{1/s}, \qquad \text{and by the Hölder inequality}$$

$$\begin{aligned}\|(1+\|\cdot\|^a)k\|_2 &\le M^{1/s}\left\|(1+\|\cdot\|^{as})^{1/s}k\right\|_2 = M^{1/s}\left\|((1+\|\cdot\|^{as})k)^{1/s}k^{1/s'}\right\|_2 = M^{1/s}\left\|((1+\|\cdot\|^{as})k)^{2/s}k^{2/s'}\right\|_1^{1/2} \\ &\le M^{1/s}\left\|((1+\|\cdot\|^{as})k)^2\right\|_1^{\frac{1}{2s}}\left\|k^2\right\|_1^{\frac{1}{2s'}} = M^{1/s}\|(1+\|\cdot\|^{as})k\|_2^{1/s}\|k\|_2^{1/s'}\end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$, but $\|(1+\|\cdot\|^{as})k\|_2 \le \overline{C}\delta^{\frac{1}{2}-as}$, $\|k\|_2 = \|\varphi\circ\|\cdot\|\|_2 \le \overline{C}\delta^{1/2}$, hence

$$\|(1+\|\cdot\|^{a})k\|_2 \le M^{1/s}C^{\frac{1}{2s}-a}\overline{C}\delta^{\frac{1}{2s'}} = C'\delta^{\frac{1}{2}-a}.$$

If we take $a=\frac{n}{2}+\epsilon$ then

$$\|k\|_1 = \left\|(1+\|\cdot\|^{a})k(1+\|\cdot\|^{a})^{-1}\right\|_1 \le \|(1+\|\cdot\|^{a})k\|_2 \left\|(1+\|\cdot\|^{a})^{-1}\right\|_2 \le C'\delta^{\frac{1}{2}-a}\left\|(1+\|\cdot\|^{a})^{-1}\right\|_2$$

the exists $M>0$ such that $(1+\|x\|)^a \le M(1+\|x\|^a)$ for every $x\in\mathbb{R}^n$, hence

$$\begin{aligned}\left\|(1+\|\cdot\|^{a})^{-1}\right\|_2^2 &= \int_{\mathbb{R}^n}\frac{dx}{(1+\|x\|^a)^2} \le \int_{\mathbb{R}^n}\frac{Mdx}{(1+\|x\|)^{2a}} = Mn\omega_n\int_0^\infty \frac{r^{n-1}dr}{(1+r)^{n+2\epsilon}}\\ &\le Mn\omega_n\int_0^\infty\frac{dr}{(1+r)^{1+2\epsilon}} = Mn\omega_n\left[\frac{(1+r)^{-2\epsilon}}{-2\epsilon}\right]_0^\infty = \frac{Mn\omega_n}{2\epsilon} = C(\epsilon)\end{aligned}$$

so $\|k\|_1 \le C(\epsilon)\delta^{-\frac{n-1}{2}-\epsilon}$. As $T^\delta f = K\star f$ the Young inequality implies that

$$\begin{aligned}\left\|T^\delta f\right\|_1 &= \|K\star f\|_1 \le \|K\|_1\|f\|_1 \le C(\epsilon)\delta^{-\frac{n-1}{2}-\epsilon}\|f\|_1 \quad \text{and}\\ \left\|T^\delta f\right\|_\infty &= \|K\star f\|_\infty \le \|K\|_1\|f\|_\infty \le C(\epsilon)\delta^{-\frac{n-1}{2}-\epsilon}\|f\|_\infty.\end{aligned}$$

For $p=2$ we have $\left\|T^\delta f\right\|_2 = \left\|(T^\delta f)^\wedge\right\|_2 = \left\|(\phi\circ\|\cdot\|)\widehat{f}\right\|_2 \le \|\phi\circ\|\cdot\|\|_\infty\left\|\widehat{f}\right\|_2 = C\|f\|_2 \le C(\epsilon)\|f\|_2$, by the Riesz-Thorin interpolation Theorem,

$$\frac{1}{p} = \frac{1-\theta}{1}+\frac{\theta}{2} = 1-\frac{\theta}{2} = \frac{2-\theta}{2}, \qquad \theta\in[0,1],$$

hence $\theta = 2-\frac{2}{p}$, $1-\theta = \frac{2}{p}-1$, so

$$\left\|T^\delta f\right\|_p \le (C(\epsilon)\delta^{-\frac{n-1}{2}-\epsilon})^{1-\theta}C(\epsilon)^\theta\|f\|_p = C(\epsilon)\delta^{-(\frac{n-1}{2}+\epsilon)\left(\frac{2}{p}-1\right)}\|f\|_p$$

for every $p\in[1,2]$, $f\in L^p(\mathbb{R}^n)$, hence $\varphi\circ\|\cdot\|\in\mathcal{M}(L^p(\mathbb{R}^n))$, by the Corollary 2.1 we have that $\varphi\circ\|\cdot\|\in\mathcal{M}(L^q(\mathbb{R}^n))$ for $\frac{1}{p}+\frac{1}{q}=1$, moreover $|\varphi\circ\|\cdot\||_p = |\varphi\circ\|\cdot\||_q$ then

$$\left\|T^\delta f\right\|_q \le C(\epsilon)\delta^{-(\frac{n-1}{2}+\epsilon)\left(\frac{2}{p}-1\right)}\|f\|_q = C(\epsilon)\delta^{-(\frac{n-1}{2}+\epsilon)\left(1-\frac{2}{q}\right)}\|f\|_q, \quad 2\le q\le\infty.$$

hence

$$\left\|T^\delta f\right\|_p \le C(\epsilon)\delta^{-(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|}\|f\|_p, \quad 1\le p\le\infty, \quad f\in L^p(\mathbb{R}^n).$$

this completes the proof. ◊

Lema 3.2. *If $m\in\mathcal{M}(L^p(\mathbb{R}^n))$, supp$(m)$ is compact, then $\widehat{m}\in L^p(\mathbb{R}^n)$.*

Proof. As $\mathscr{F}:\delta(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is an topological isomorphism if $g\in C_0^\infty(\mathbb{R}^n)\subset\mathcal{S}(\mathbb{R}^n)$ such that

$g\mid_{supp(m)}\equiv 1$, there exists $f\in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{f}=\mathscr{F}f=g$, so $f\in L^p(\mathbb{R}^n)$ given that $m\in \mathcal{M}(L^p(\mathbb{R}^n))$, we have $T_m\in \mathscr{B}(L^p(\mathbb{R}^n))$, so $T_mf\in L^p(\mathbb{R}^n)$, but $(T_mf)^\wedge = m\widehat{f}=mg=m$, hence $\widehat{m}=\overline{T_mf}\in L^p(\mathbb{R}^n)$. ◊

Theorem 3.2. *(1) If $\delta>\frac{n-1}{2}$, $T_\delta\in L^p(\mathbb{R}^n)$,$1\le p\le\infty$.*

(2) If $0<\delta\le\frac{n-1}{2}$, $T_\delta\in L^p(\mathbb{R}^n)$ if $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{\delta}{n-1}$, $T_\delta\notin L^p(\mathbb{R}^n)$ if $\left|\frac{1}{p}-\frac{1}{2}\right|>\frac{2\delta+1}{2n}$.

The value $\frac{n-1}{2}$ of δ is called critical index.

Proof. (1) As $T_\delta f=\sum_{k=0}^\infty 2^{-k\delta}T_kf$, $(T_kf)^\wedge(\xi)=\tilde{\varphi}_k(\|\xi\|)\widehat{f}(\xi)$, $supp(\varphi_k)\subset(1-2^{-k+1},1-2^{-k-1})$, the Lemma 3.1 implies that

$$\|T_kf\|_p\le C(\epsilon)2^{(k+1)(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|}\|f\|_p$$

hence $\|T_\delta f\|_p\le\sum_{k=0}^\infty 2^{-k\delta}\|T_kf\|_p\le C(\epsilon)\sum_{k=0}^\infty 2^{(k+1)(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|-k\delta}\|f\|_p$
$=\overline{C(\epsilon)}\sum_{k=0}^\infty(2^{(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|-\delta})^k\|f\|_p$.

If $\delta>\frac{n-1}{2}$ then using $0\le\left|\frac{2}{p}-1\right|=2\left|\frac{1}{p}-\frac{1}{2}\right|\le 2\cdot\frac{1}{2}=1$ and take $\epsilon=\frac{\delta-\frac{n-1}{2}}{2}$ we have $\delta>\frac{n-1}{2}+\epsilon\ge(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|$ hence $(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|-\delta<0$ so $\sum_{k=0}^\infty(2^{(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|-\delta})^k<\infty$ and $T_\delta\in\mathscr{B}(L^p(\mathbb{R}^n))$.

(2) If $0<\delta\le\frac{n-1}{2}$ and $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{\delta}{n-1}$, as $\|T_\delta f\|_p\le\overline{C(\epsilon)}\sum_{k=0}^\infty 2^{k(\frac{n-1}{2}+\epsilon)\left|\frac{2}{p}-1\right|-k\delta}\|f\|_p$
$=\overline{C(\epsilon)}\sum_{k=0}^\infty 2^{2k(\frac{n-1}{2}+\epsilon)\left|\frac{1}{p}-\frac{1}{2}\right|-k\delta}\|f\|_p=\overline{C(\epsilon)}\sum_{k=0}^\infty(2^{(n-1+2\epsilon)\left|\frac{1}{p}-\frac{1}{2}\right|-\delta})^k\|f\|_p$

let $\epsilon_0=\frac{\delta}{\left|\frac{1}{p}-\frac{1}{2}\right|}-(n-1)>0$ and $0<\epsilon<\frac{\epsilon_0}{2}$ hence $(n-1+2\epsilon)\left|\frac{1}{p}-\frac{1}{2}\right|-\delta<0$ this implies that $\sum_{k=0}^\infty(2^{(n-1+2\epsilon)\left|\frac{1}{p}-\frac{1}{2}\right|-\delta})^k<\infty$ and $T_\delta\in\mathscr{B}(L^p(\mathbb{R}^n))$.

By the part (3) of the Proposition 1.5 we have that $J_\mu(t)=O(t^\mu)$ if $t\to 0$ and $J_\mu(t)=O(t^{\frac{1}{2}})$ if $t\to\infty$ and the Corollary 1., $\Phi_\delta:\mathbb{R}^n\to\mathbb{R}$, $\Phi_\delta(t)=(1-\|\xi\|^2)_+^\delta$ implies $\widehat{\Phi_\delta}(\xi)=\pi^{-\delta}\Gamma(\delta+1)\|\xi\|^{-\frac{n}{2}+\delta}J_{\frac{n}{2}+\delta}(2\pi\|\xi\|)$ then there exists C_1,$C_2>0$ such that

$$\left|J_{\frac{n}{2}+\delta}(2\pi\|\xi\|)\right|\le C_1\|\xi\|^{\frac{n}{2}+\delta},\qquad\text{if}\quad\|\xi\|\to 0\qquad\text{and}$$
$$J_{\frac{n}{2}+\delta}(2\pi\|\xi\|)\sim C_2\|\xi\|^{-1/2},\qquad\text{if}\quad\|\xi\|\to\infty$$

hence $\left|\widehat{\Phi_\delta}(\xi)\right|\le\pi^{-\delta}\Gamma(\delta+1)C_1$ if $\|\xi\|\to 0$ and $\widehat{\Phi_\delta}(\xi)\sim\pi^{-\delta}\Gamma(\delta+1)C_2\|\xi\|^{-(\frac{n+1}{2}-\delta)}$ if $\|\xi\|\to\infty$. So $\widehat{\Phi_\delta}\in L^p(\mathbb{R}^n)$ if and only if there exists $\alpha>0$ such that $\|\cdot\|^{-(\frac{n+1}{2}+\delta)}\in L^p(\mathbb{R}^n-B(0,\alpha))$, but $\int_{\|\xi\|>\alpha}\|\xi\|^{-(\frac{n+1}{2}+\delta)p}d\xi=n\omega_n\int_\alpha^\infty r^{-(\frac{n+1}{2}+\delta)p}r^{n-1}dr=n\omega_n\int_\alpha^\infty r^{-(\frac{n+1}{2}+\delta)p+n-1}dr=n\omega_n\left[\frac{r^{n-(\frac{n+1}{2}+\delta)p}}{n-(\frac{n+1}{2}+\delta)p}\right]_\alpha^\infty$ $<\infty$ for every $\alpha>0$ if and only if $n-(\frac{n+1}{2}+\delta)p<0$, $p>\frac{n}{\frac{n+1}{2}+\delta}=\frac{2n}{n+1+2\delta}$, hence $\widehat{\Phi_\delta}\notin L^p(\mathbb{R}^n)$ if

$p \leq \frac{2n}{n+1+2\delta}$, by the previous Lemma $\Phi_\delta \notin \mathcal{M}(L^p(\mathbb{R}^n))$, by duality $\Phi_\delta \notin \mathcal{M}(L^q(\mathbb{R}^n))$ with $\frac{1}{p}+\frac{1}{q}=1$, but $\frac{1}{p} \geq \frac{n+1+2\delta}{2n}$, $\frac{1}{q} \leq 1-\frac{1}{p} \leq 1-\frac{n+1+2\delta}{2n} = \frac{n-1-2\delta}{2n}$, $q \geq \frac{2n}{n-1-2\delta}$, hence $\Phi_\delta \notin \mathcal{M}(L^p(\mathbb{R}^n))$ if $\frac{1}{p} \leq \frac{1}{2}-\frac{1+2\delta}{2n}$ or $\frac{1}{p} \geq \frac{1}{2}+\frac{1+2\delta}{2n}$ so $\frac{1}{p}-\frac{1}{2} \leq -\frac{1+2\delta}{2n}$ or $\frac{1}{p}-\frac{1}{2} \geq \frac{1+2\delta}{2n}$, i.e $\left|\frac{1}{p}-\frac{1}{2}\right| \geq \frac{1+2\delta}{2n}$. ◊

3.3 A first negative result for ball multipliers

In this section we study the multiplier for the ball i.e. $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $\widehat{Tf} = \chi_B \widehat{f}$ in dimension greater than one, here B is the unit ball, the case $n=1$ is given by the Corollary 2.5 of the chapter 2 that says that $T_{cube} \in \mathscr{B}(L^p(\mathbb{R}^n))$, however, T is essentially different from T_{cube} as prove the following:

Lema 3.3. *$T \notin \mathscr{B}(L^p(\mathbb{R}^n))$ for $p \notin \left(\frac{2n}{n+1}, \frac{2n}{n-1}\right)$.*

Proof. If $f = \chi_B$ then $f(x) = f_0(\|x\|)$ with $f_0 = \chi_{(0,1)}$ using the Proposition 1.5 of the Chapter 1 we obtain $\widehat{f}(\xi) = 2\pi \|\xi\|^{1-\frac{n}{2}} \int_0^1 J_{\frac{n}{2}-1}(2\pi \|\xi\| s) s^{\frac{n}{2}} ds = \|\xi\|^{-\frac{n}{2}} J_{\frac{n}{2}}(2\pi \|\xi\|)$, the part (3) of this propostion implies that there exists $C > 0$ such that:

$$\left|J_{\frac{n}{2}}(2\pi \|\xi\|)\right| \leq C \|\xi\|^{\frac{n}{2}} \qquad \text{if } \|\xi\| \to 0, \qquad J_{\frac{n}{2}}(2\pi \|\xi\|) \sim \|\xi\|^{-\frac{1}{2}} \qquad \text{if } \|\xi\| \to \infty$$

then $\left|\widehat{f}(\xi)\right| \leq 2\pi C$ if $\|\xi\| \to 0$ and $\widehat{f}(\xi) \sim \|\xi\|^{-\left(\frac{n+1}{2}\right)}$ if $\|\xi\| \to \infty$, so $\widehat{f} \in L^p(\mathbb{R}^n)$ $\Leftrightarrow (\exists \alpha > 0) \left(\|\cdot\|^{-\left(\frac{n+1}{2}\right)} \in L^p(\mathbb{R}^n - B(0,\alpha))\right)$, but $\int_{\|\xi\|>\alpha} \|\xi\|^{-\left(\frac{n+1}{2}\right)p} d\xi = n\omega_n \int_\alpha^\infty r^{-\left(\frac{n+1}{2}\right)p} . r^{n-1} dr = n\omega_n \int_\alpha^\infty r^{-\left(\frac{n+1}{2}\right)p+n-1} dr = n\omega_n \left[\frac{r^{-\left(\frac{n+1}{2}\right)p+n}}{-\left(\frac{n+1}{2}\right)p+n}\right]_\alpha^\infty < \infty$, for every $\alpha > 0 \Leftrightarrow -\left(\frac{n+1}{2}\right)p + n \leq 0 \Leftrightarrow p \leq \frac{2n}{n+1}$, hence $\widehat{f} \notin L^p(\mathbb{R}^n)$ if $p < \frac{2n}{n+1}$, by the Lemma 3, $f \notin \mathcal{M}(L^p(\mathbb{R}^n))$, by the argument of duality proved in the de Leeuw's Theorem $f \notin \mathcal{M}(L^q(\mathbb{R}^n))$ with $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p} > \frac{n+1}{2n}$, $\frac{1}{q} = 1-\frac{1}{p} < 1-\frac{n+1}{2n} = \frac{n-1}{2n}$, $q > \frac{2n}{n-1}$, so $f \notin \mathcal{M}(L^p(\mathbb{R}^n))$ if $p \notin \left(\frac{2n}{n+1}, \frac{2n}{n-1}\right)$. ◊

CHAPTER 4

Kakeya sets

In this chapter we define Kakeya sets and present a construction due to I. J. Schoemberg. This construction is the perfect ingredient to complete the proof of Fefferman's Theorem.

4.1 Definition and Geometrical motivation

Definition 4.1. A compact set $K \subset \mathbb{R}^n$ is called a Kakeya or Besicovitch set if

$$(\forall x \in S^{n-1})(\exists y \in K)([y, x + y] \subset K),$$

where $[a, b] = \{(1 - t)a + tb \mid t \in [0, 1]\}$, i.e a Kakeya set contains a unit segment in every direction.

The following are simple examples:

- In $\mathbb{R}^2$, $D\left(0, \frac{1}{2}\right) = \left\{z \in \mathbb{R}^2 \mid \|z\| \leq \frac{1}{2}\right\}$
- In general in $\mathbb{R}^n$, $B\left(0, \frac{1}{2}\right) = \left\{x \in \mathbb{R}^n \mid \|x\| \leq \frac{1}{2}\right\}$.
- Equilateral triangle of height 1 and area $\frac{1}{\sqrt{3}}$.
- The deltoid or Hypocycloid of Steiner with area $\frac{\pi}{8}$.

In 1917 S. Kakeya posed the problem to find a Kakeya set of minimum area, this problem was solved by A. Besicovitch in 1927. Surprisingly there exists Kakeya sets with arbitrarily small area. We have a strong type of Kakeya set that includes a condition of continuity in the rotation:

Definition 4.2. A Kakeya needle set is a Kakeya set K such that

$$(\exists N \subset K)(\forall \theta \in [0, 2\pi])(A(\theta)N \subset K), \qquad A(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO_2(\mathbb{R})$$

here $N = \left(x_1 - \frac{1}{2}, x_2 + \frac{1}{2}\right) \times \{x_2\}$, with $x = (x_1, x_2) \in K$ i.e there exists a unit line segment (needle) contained in K that can be rotated continuosly 360°.

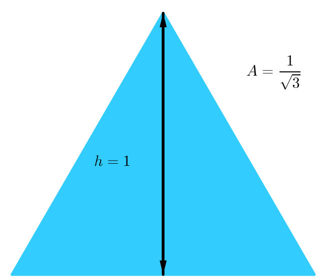

Figure 4.1: the equilateral triangle of height 1 and area $\frac{1}{\sqrt{3}}$ is a Kakeya set

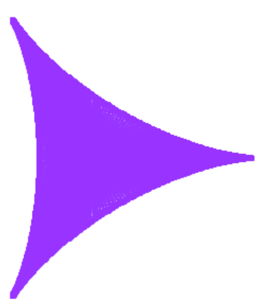

Figure 4.2: the deltoid is a Kakeya set

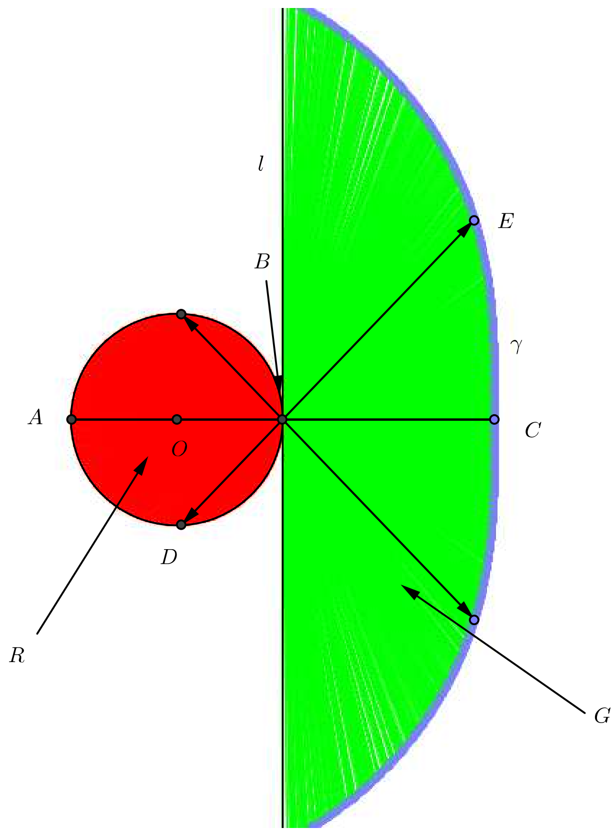

Figure 4.3: example of Kakeya set that is not Kakeya needle set

Examples:

- In $\mathbb{R}^2$, $D\left(0,\frac{1}{2}\right)=\left\{z\in\mathbb{R}^2 \mid \|z\|\leq\frac{1}{2}\right\}$.
- Equilateral triangle of height 1 and area $\frac{1}{\sqrt{3}}$.

Remark

We see that in fact the Definition 4.2 is stronger than the Definition 4.1. Figure 4.3 show an example of this, take a AC a unit line segment, let B the midpoint of AC, R a circle with center O and radius $\frac{1}{4}$ such that AB is a diameter of R, the blue curve γ is constructed as follows, if we take a point $D\in\partial R$ there exists a unique point E such that $B\in DE$ and $|DE|=1$, if we vary the point $D\in\partial R$ the set of points E obtained in this way defines this curve, let l be the perpendicular line that contains the point B, we call G the green region bounded by the line l and γ (included the boundary), note that by construction $K=R\cup G$ contains a needle through B in every direction, but the regions R and G do not contain needle that can be rotate $360°$, as $R\cap G=\{B\}$ the unique posibility is the needle AC, but this needle can not be rotated $360°$, then K is a Kakeya set that is not a Kakeya needle set.

The problem for Kakeya needle set is analogous, find a Kakeya needle sets of minimum area. For a long time was believed that the solution was the deltoid (three-cuspec hypocycloid) with area $\frac{\pi}{8}$ (see figure 4.2), however Besicovitch surprinsingly showed that there exist Kakeya needle sets of arbitrarily small area. In 1921 J. Pal showed that the solution of this problem in the convex case is given by the equilateral triangle of area $\frac{1}{\sqrt{3}}$ (see figure 4.1).

In the following section we study the construction of Kakeya sets given by I. J. Schoenberg using sprouts of triangles.

4.2 Sprouting Triangles

Let $a,b.c\in\mathbb{R}^2$ noncollinear points, suppose that the side $\overline{ab}$ is the base of the triangle Δabc and that height is h. We extend the sides $\overline{ac}$ and $\overline{bc}$ to segments $\overline{aa'}$ and $\overline{bb'}$ such that the triangles $\Delta aba'$, $\Delta abb'$ have the same height $h'>h$. Let $d=\frac{a+b}{2}$. The triangles $\Delta'=\Delta ada'$ and $\Delta''=\Delta bdb'$ are called sprouts from de height h to h'. (See Figure 4.4).

4.3 Sequence of Rectangles and Besicovitch sets

In this section we use the sprouting of triangles for construct an increasing sequence $\{E(k)\}_{k\in\mathbb{N}}$ of sets that we call Besicovitch approximation and the union E of this sequence is a Besicovitch or Kakeya

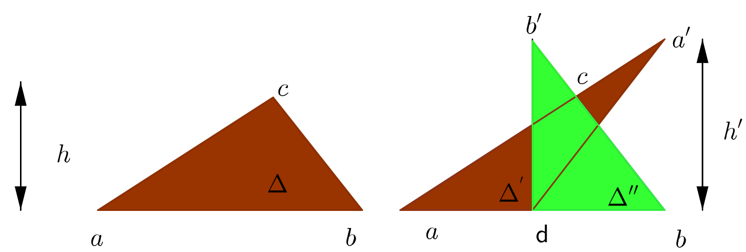

Figure 4.4: the triangles $\Delta' = \Delta ada'$ and $\Delta'' = \Delta bdb'$ arise as Δ sprouts from height h to h'

set. The following lemma is enough for our final purpose, i.e the proof of the Fefferman's Theorem.

Lema 4.1. *Let $\eta > 0$, there exists $E \subset \mathbb{R}^2$, a sequence of rectangles $\{R_j\}_{j=1}^m$, and a sequence of sets $\left\{\widetilde{R_j}\right\}_{j=1}^m$ such that:*

$(1) R_j \cap R_l = \emptyset$ *if j, l are different,*

$(2) \left|\widetilde{Rj}\right| = 2\,|R_j|$

$(3) \left|\widetilde{R_j} \cap E\right| \geq \frac{1}{120}\left|\widetilde{R_j}\right|$

$(4) |E| \leq \eta \sum_{j=1}^m |R_j|$

Proof. As the proof is extremely long we divide it into steps.

- Construction and estimates of Besicovitch sets. Let

$$\Delta_0 = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{\sqrt{3}}{2}, \frac{y}{\sqrt{3}} \leq x \leq 1 - \frac{y}{\sqrt{3}}\right\}$$

and $\{h_l\}_{l\in\mathbb{N}}$ a increasing sequence of real numbers such that $h_0 = \frac{\sqrt{3}}{2}$=height of Δ_0. Sprouting Δ_0 from h_0 to h_1 we obtain two triangles Δ_{11} and Δ_{12}, again sprouting these triangles from h_1 to h_2 we obtain triangles $\Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{24}$ continue sprouting in the step k we obtain 2^k triangles Δ_{kj}, $1 \leq j \leq 2^k$, everyone with base 2^{-k} and height h_k, we define $E(k) = \bigcup_{j=1}^{2^k} \Delta_{kj}$, we claim that $|E(k)| \leq \frac{3}{2}$, for every $k \in \mathbb{N}$.

For prove this claim we use the Figure 4.5, the triangle ΔABC with base of length $AB = b$ gives rise to the triangles ΔAMF and ΔBME, we take the difference $\Delta AMF \cup \Delta BME - \Delta ABC = \Delta GCE \cup \Delta HCF$ and called the triangles ΔGCE and ΔHCF arms and note that $|\Delta GCE| = |\Delta HCF|$, now we compute the area of each arm. We claim that $|\Delta GCE| = \frac{b}{2}\frac{(h_1-h_0)^2}{2h_1-h_0}$.

We note that the triangles ΔBME and ΔCNE are similar, thus

$$\frac{heigth(\Delta CNE)}{height(\Delta BME)} = \frac{base(\Delta CNE)}{base(\Delta BME)}, \tag{4.1}$$

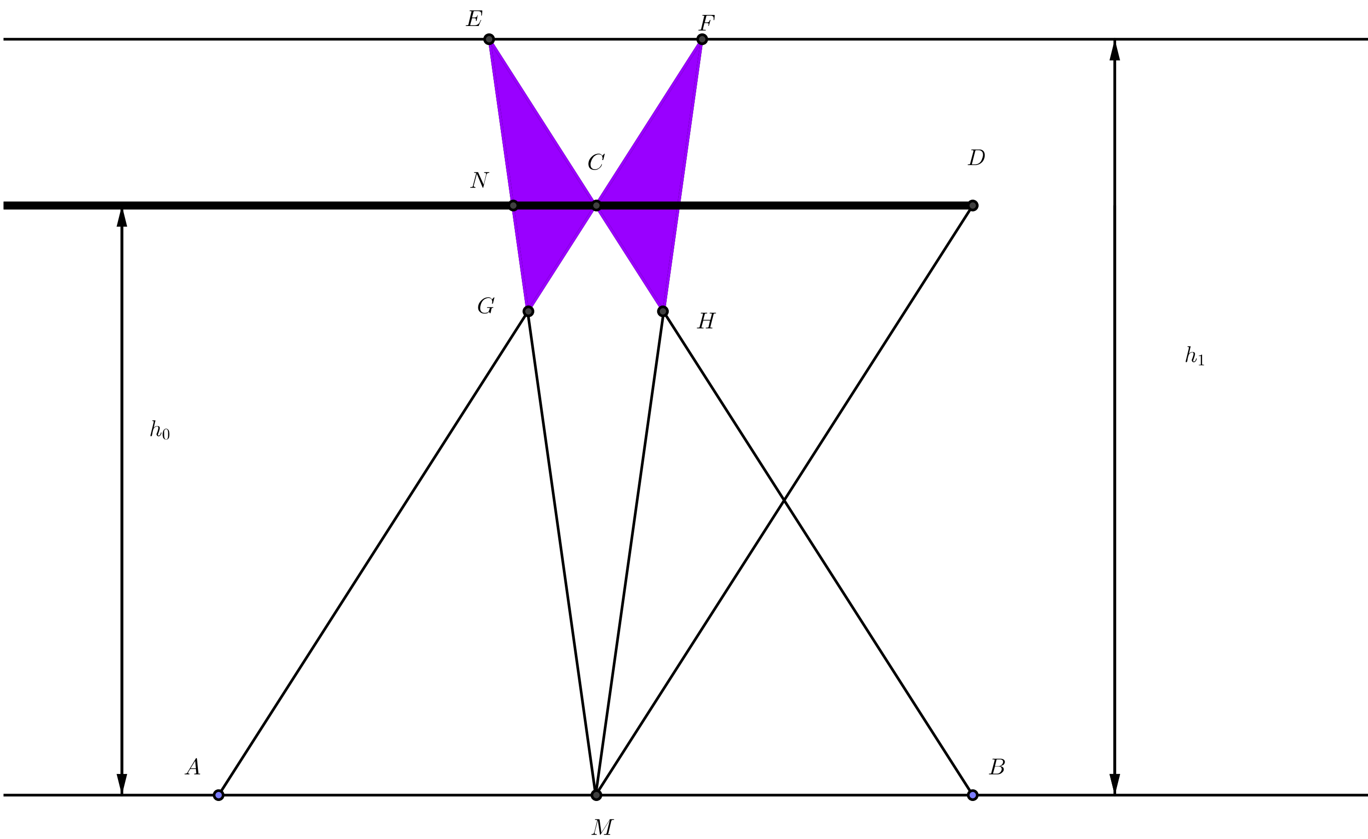

Figure 4.5: the triangle ΔABC with base of length $AB = b$ gives rise to the triangles ΔAMF and ΔBME.

i.e

$$\frac{h_1 - h_0}{h_1} = \frac{NC}{b/2}.$$

We trace the half line parallel to GC that passes by M. It intercepts to the line of equation $y = h_0$ in an unique point D, then $CD = AM = b/2$. Since the triangles ΔNGC and ΔMDN are similar we have that

$$\frac{heigth(\Delta NGC)}{height(\Delta MDN)} = \frac{base(\Delta NGC)}{base(\Delta MDN)} \tag{4.2}$$

i.e

$$\frac{height(\Delta NGC)}{h_0} = \frac{NC}{ND} = \frac{NC}{NC + CD} = \frac{NC}{NC + b/2}$$

hence

$$NC = \frac{b}{2}\left(\frac{h_1 - h_0}{h_1}\right), \qquad height(\Delta NGC) = \frac{h_0 NC}{NC + b/2}$$

so $NC + b/2 = \frac{b}{2}\left(\frac{h_1-h_0}{h_1}\right) + b/2 = \frac{b}{2}\left(\frac{2h_1-h_0}{h_1}\right)$ and $height(\Delta(NGC)) = \frac{\frac{b}{2}\left(\frac{h_1-h_0}{h_1}\right)h_0}{\frac{b}{2}\left(\frac{2h_1-h_0}{h_1}\right)} = \frac{(h_1-h_0)h_0}{2h_1-h_0}$, hence

$$\begin{aligned}
|\Delta GCE| = |\Delta NGC| + |\Delta NEC| &= \frac{NC \cdot height(\Delta NGC)}{2} + \frac{NC \cdot (h_1 - h_0)}{2} \\
= \frac{b}{4}\left(\frac{h_1 - h_0}{h_1}\right)\frac{(h_1 - h_0)h_0}{2h_1 - h_0} + \frac{b}{4}\left(\frac{h_1 - h_0}{h_1}\right)(h_1 - h_0) &= \frac{b}{4}\frac{(h_1 - h_0)^2}{2h_1 - h_0}\frac{h_0}{h_1} + \frac{b}{4}\frac{(h_1 - h_0)^2}{h_1} \\
= \frac{b}{4}\frac{(h_1 - h_0)^2}{h_1}\left(\frac{h_0}{2h_1 - h_0} + 1\right) = \frac{b}{4}\frac{(h_1 - h_0)^2}{h_1}\left(\frac{h_0 + 2h_1 - h_0}{2h_1 - h_0}\right) &= \frac{b}{4}\frac{(h_1 - h_0)^2}{h_1}\left(\frac{2h_1}{2h_1 - h_0}\right) \\
&= \frac{b}{2}\left(\frac{(h_1 - h_0)^2}{2h_1 - h_0}\right)
\end{aligned} \tag{4.3}$$

To fix ideas we take the sequence

$$h_1 = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right)$$
$$h_2 = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}+\frac{1}{3}\right)$$
$$\vdots$$
$$h_j = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{j+1}\right)$$

we note that $E(k-1) = \bigcup_{j=1}^{2^{k-1}} \Delta_{k-1,j} \subset E(k) = \bigcup_{j=1}^{2^k} \Delta_{k,j}$ for every $k \in \mathbb{N}$, we denote the arms of $\Delta_{k,j}$ by $\Delta^1_{k-1,j}$ and $\Delta^2_{k-1,j}$ called $k - arms$. This allows us to write $\bigcup_{j=1}^{2^k} \Delta_{k,j} - \bigcup_{j=1}^{2^{k-1}} \Delta_{k-1,j} = \bigcup_{l=1}^{2} \bigcup_{j=1}^{2^{k-1}} \Delta^l_{k-1,j}$, by an application of the equation (4.3) we have that $\left|\Delta^l_{k-1,j}\right| = \frac{1}{2^k}\left(\frac{h_j-h_{j-1}}{2h_j-h_{j-1}}\right)$ for every $1 \le j \le 2^{k-1}$, $l \in \{1,2\}$.

We estimate the area of $E(k)$ as the sum of the area of Δ_0 and the area of the $s - arms$ for $1 \le s \le k$, in fact $\left\{\Delta^l_{s-1,j} \mid 1 \le j \le 2^{s-1}, l \in \{1,2\}\right\} = \{s - arms\}$ and $E(k) = \Delta_0 \cup \bigcup_{s=1}^{k} \bigcup_{j=1}^{2^{s-1}} \bigcup_{l=1}^{2} \Delta^l_{s-1,j}$, this implies that:

$$|E(k)| = |\Delta_0| + \sum_{s=1}^{k}\sum_{j=1}^{2^{s-1}}\sum_{l=1}^{2}\left|\Delta^l_{sj}\right| = \frac{\sqrt{3}}{4} + \sum_{s=1}^{k}\frac{(h_s-h_{s-1})^2}{2h_s-h_{s-1}}$$

but $2h_s - h_{s-1} = h_s + (h_s - h_{s-1}) \ge \frac{\sqrt{3}}{2}$ and $h_s - h_{s-1} = \frac{\sqrt{3}}{2}\frac{1}{s+1}$, so

$$|E(k)| \le \frac{\sqrt{3}}{2} + \sum_{s=1}^{k}\frac{\frac{1}{(s+1)^2}\frac{3}{4}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\sum_{s=1}^{k}\frac{1}{(s+1)^2} = \frac{\sqrt{3}}{2}\sum_{s=1}^{k+1}\frac{1}{s^2} \le \frac{\sqrt{3}}{2}\sum_{s=1}^{\infty}\frac{1}{s^2} = \frac{\sqrt{3}}{2}\frac{\pi^2}{6} < \frac{3}{2}.$$

for every $k \in \mathbb{N}$.

- Construction of the family of rectangles $\{R_j\}_{j\in\mathbb{N}}$ that satisfies (1), (2), (3) and (4).

Note that the base of each triangle Δ_{kj}, $1 \le j \le 2^k$, $k \in \mathbb{N}$ is an dyadic interval in $[0,1]$, let

$$D([0,1]) = \left\{ I \mid I = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right], 0 \le j \le 2^k - 1, k \in \mathbb{N} \right\} = \{\text{dyadic intervals on } [0,1]\}$$

$$S(\Delta_0) = \left\{ \Delta_{kj} \mid 1 \le j \le 2^k, k \in \mathbb{N} \right\} = \{\text{sprouts of } \Delta_0\}$$

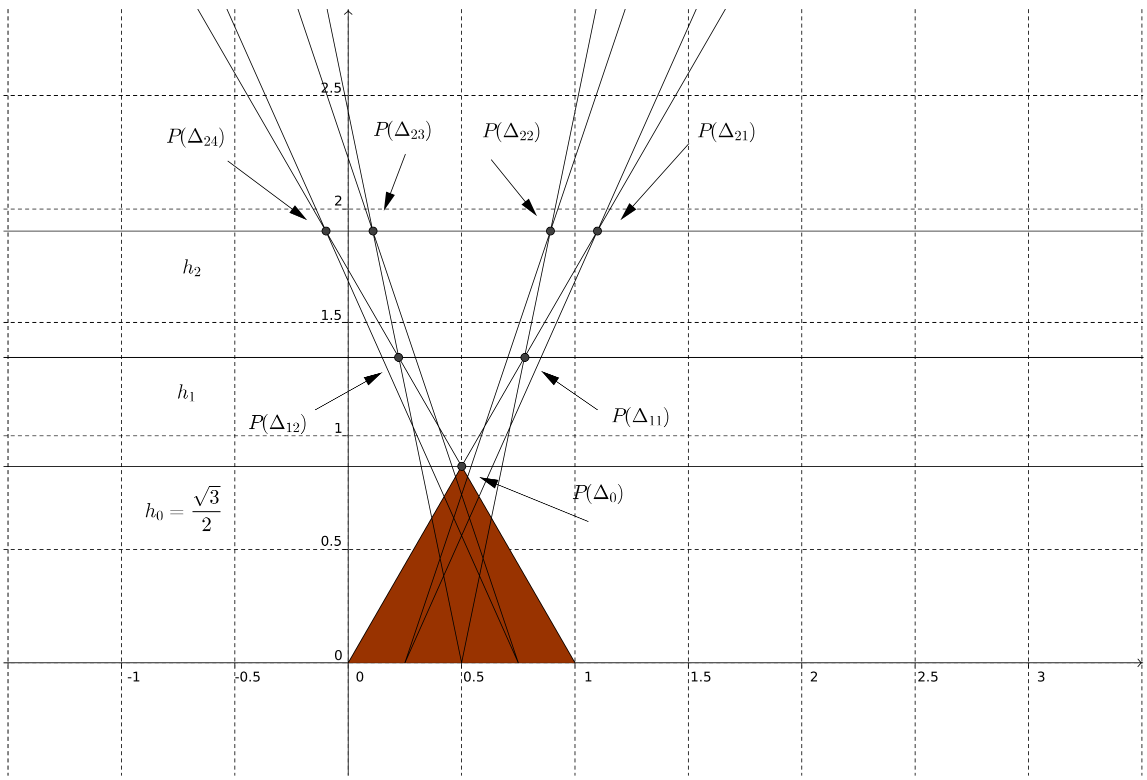

Figure 4.6: sprouting the triangle Δ_0 we obtain a bijection $\pi : S(\Delta_0) \to D([0,1])$

then the map $\pi : S(\Delta_0) \to D([0,1])$, $\pi(\Delta_{kj}) = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ is surjective, moreover by construction for every $I \in D([0,1])$, $I = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ corresponds an only $\Delta(I) \in S(\Delta_0)$ whose base is I and the upper vertex belongs to the line $y = h_k$ (See figure 4.6), so π is bijective, let $P(I)$ this vertex, using $\Delta(I)$ we are going to construct a rectangle $R(I)$, as in the figure 4.9.

We write $A_{kj} = \left(\frac{j}{2^k}, 0\right)$, $B_{kj} = \left(\frac{j+1}{2^k}, 0\right)$, then $\Delta_{kj} = \Delta A_{kj} B_{kj} C_{kj}$ where $C_{kj} = P(\Delta_{kj})$ the upper vertex, we note that $\max\left(\left\{|A_{kj}B_{kj}| \mid 1 \leq j \leq 2^k\right\} \cup \left\{|B_{kj}C_{kj}| \mid 1 \leq j \leq 2^k\right\}\right) = |A_{k1}B_{k1}|$, because $A_{k1}B_{k1}$ is a diagonal of a yellow rectangle of the figure 4.7, and the segments $A_{kj}B_{kj}$ and $B_{kj}C_{kj}$ are contained in this rectangle for $1 \leq j \leq 2^{k-1}$ but the set $E(k)$ is symmetric with respect to the line with equation $x = \frac{1}{2}$, as $|AC| = 1$, $\sin(\theta) = \frac{\sqrt{3}}{2} = \frac{h_k}{|A_{k1}C_{k1}|}$, so $|A_{k1}C_{k1}| = \frac{2\sqrt{3}}{3}h_k$.

As $\log(k+1) = \int_1^{k+1} \frac{dx}{x} = \sum_{j=1}^{k} \int_j^{j+1} \frac{dx}{x} \geq \sum_{j=1}^{k} \int_j^{j+1} \frac{dx}{j+1} = \sum_{j=1}^{k} \frac{1}{j+1} = \frac{1}{2} + ... + \frac{1}{k+1}$ we have $\frac{2\sqrt{3}}{3}h_k < \frac{3}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{k+1}\right) < \frac{3}{2}\left(1 + \log(k+1)\right) < 3\log(k+2)$ the later inequality is true because $e < 3$ and $k \geq 1$, implies $e(k+1) < 4(k+1) < k^2 + 4k + 4 = (k+2)^2$ and so $1 + \log(k+1) < 2\log(k+2)$.

Now we use the figure 4.9 and note that the previous paragraph implies that $\Delta_{kj} \subset \widetilde{R_{kj}}$ because $\max\left(|A_{kj}C_{kj}|, |A_{kj}B_{kj}|\right) < 3\log(k+2)$ and B_{kj} or C_{kj} belongs to the diagonal of the upper part of $\widetilde{R_{kj}}$. Also $\log(k+2) = \int_1^{k+2} \frac{dx}{x} = \sum_{j=1}^{k+1} \int_j^{j+1} \frac{dx}{x} \leq \sum_{j=1}^{k+1} \int_j^{j+1} \frac{dx}{j} \leq \sum_{j=1}^{k+1} \frac{1}{j} = 1 + \frac{1}{2} + ... + \frac{1}{k+1} = h_k$, as $\Delta_{kj} \subset E$, we have that $\left|\widetilde{R_{kj}} \cap E\right| \geq |\Delta_{kj}| = \frac{1}{2}2^{-k}h_k \geq 2^{-k-1}\log(k+2)$.

We apply the law of the sines to the triangle $\Delta A_{kj}B_{kj}D_{kj}$, to obtain

$$\frac{|A_{kj}D_{kj}|}{\sin(\prec A_{kj}B_{kj}D_{kj})} = \frac{|A_{kj}B_{kj}|}{\sin(\prec A_{kj}D_{kj}B_{kj})}$$
$$|A_{kj}D_{kj}| = 2^{-k}\frac{\sin(\prec A_{kj}B_{kj}D_{kj})}{\sin(\prec A_{kj}D_{kj}B_{kj})}$$

as $\sin(\prec A_{kj}D_{kj}B_{kj}) = \frac{A_{kj}C_{kj}}{C_{kj}D_{kj}} \geq \frac{A_{kj}D_{kj}}{C_{kj}D_{kj}} = \cos(\prec A_{kj}D_{kj}B_{kj})$ we have that

$$|A_{kj}D_{kj}| \leq \frac{2^{-k}}{\cos(\prec A_{kj}D_{kj}B_{kj})}$$

by the law of the cosines applied to the triangle Δ_{kj} and the estimates $h_k \leq |A_{kj}C_{kj}|, |B_{kj}C_{kj}| \leq$

$\frac{2\sqrt{3}}{3}h_k$,

$$\begin{aligned}\cos(\sphericalangle A_{kj}C_{kj}B_{kj}) &= \frac{|A_{kj}C_{kj}|^2 + |B_{kj}C_{kj}|^2 - |A_{kj}B_{kj}|^2}{2\,|A_{kj}C_{kj}\,|B_{kj}C_{kj}||} \\ &\geq \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2\cdot\frac{4}{3}h_k^2} = \frac{2h_k^2 - 2^{-2k}}{\frac{8}{3}h_k^2} \geq \frac{3}{4} - 3\cdot 2^{-2k-2} \\ &\geq \frac{3}{4} - \frac{3}{16} = \frac{9}{16} > \frac{1}{10}\end{aligned}$$

hence $|A_{kj}D_{kj}| \leq 10\cdot 2^{-k} = 5\cdot 2^{1-k}$ and

$$\begin{aligned}\left|\widetilde{R_{kj}}\cap E\right| &\geq 2^{-k-1}\log(k+2) = \frac{5}{20}\cdot 2^{1-k}\log(k+2) \\ = \frac{5}{60}\cdot 2^{1-k}\cdot 3\log(k+2) &= \frac{1}{60}\,|A_{kj}D_{kj}|\,|A_{kj}Q_{kj}| = \frac{1}{60}\,|R_{kj}| = \frac{1}{120}\left|\widetilde{R_{kj}}\right|\end{aligned}$$

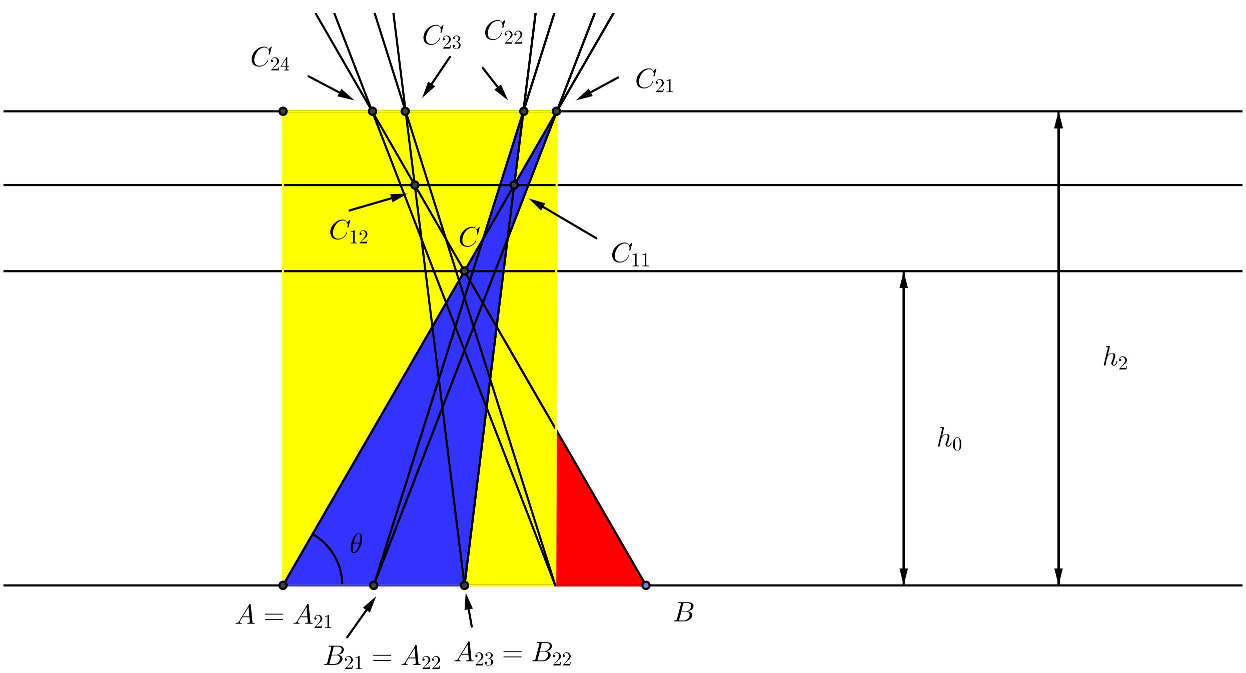

Figure 4.7: the triangles $\Delta A_{kj}B_{kj}C_{kj}$ for $1 \leq j \leq 2^{k-1}$ are contained in the yellow rectangle with diagonal $A_{k1}C_{k1}$, this figure represents this for $k = 2$.

To see (4) we use that $|E| \leq \frac{3}{2}$, if $D_k = \left\{I \in D([0,1]) \mid |I| = 2^{-k}\right\}$, $R_k = \{R(I) \mid I \in D_k\}$ we

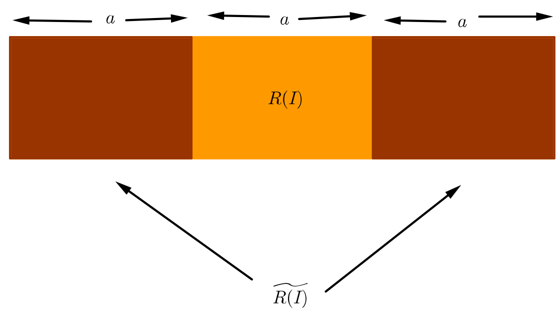

Figure 4.8: the set $\widetilde{R(I)}$ has area the double of the area of the rectangle $R(I)$.

compute a lower bound for $|A_{kj}D_{kj}|$, by the law of the sines applied to the triangle $\Delta A_{kj}B_{kj}D_{kj}$:

$$\frac{|A_{kj}D_{kj}|}{\sin(\prec A_{kj}B_{kj}D_{kj})} = \frac{|A_{kj}B_{kj}|}{\sin(\prec A_{kj}D_{kj}B_{kj})}$$
$$|A_{kj}D_{kj}| = 2^{-k}\frac{\sin(\prec A_{kj}B_{kj}D_{kj})}{\sin(\prec A_{kj}D_{kj}B_{kj})}$$

so

$$|A_{kj}D_{kj}| \geq \frac{2^{-k}\sin(\prec A_{kj}B_{kj}D_{kj})}{\sin(\prec A_{kj}D_{kj}B_{kj})} \geq 2^{-k}\sin\left(\frac{\pi}{3}\right) \geq 2^{-k}\frac{1}{2} = 2^{-k-1}$$

hence $|R(I)| = |A(I)D(I)|\,|A(I)Q(I)| \geq 2^{-k-1} \times 3\log(k+2)$ for every $R(I) \in R_k$. But $Card(R_k) = 2^k$, hence

$$\sum_{I\in D_k} |R(I)| \geq 2^k \cdot 2^{-k-1} \cdot 3\log(k+2) = \frac{3}{2}\log(k+2).$$

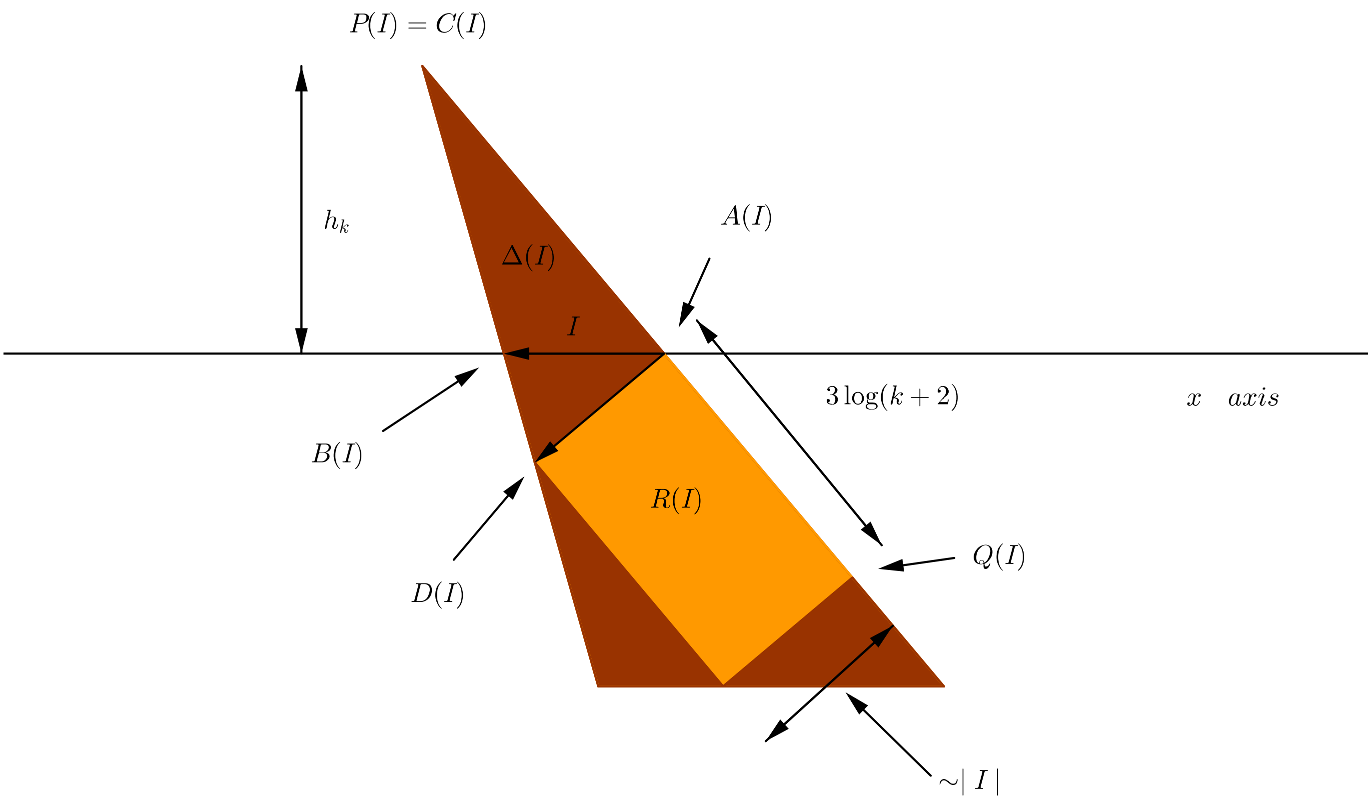

Figure 4.9: the rectangle $R(I)$ has area at least $2^{-k-1} \cdot 3\log(k+2)$

If $k+2 > e^{\frac{1}{\eta}}$ we have that $|E| \leq \frac{3}{2} < \eta \frac{3}{2} \log(k+2) \leq \eta \sum_{I \in D_k} |R(I)|$, to end we see that if I and J are different then $R(I) \cap R(J) = \emptyset$. This is a consequence of the following lemma.

Lema 4.2. *Let $I_1, I_2 \in D_k$, if I_1 lies to the left of I_2 then $P(I_2)$ lies to the left of $P(I_1)$.*

Proof. We use the notation $I_1 < I_2$ if and only if I_1 lies to the left of I_2, $P(I_2) < P(I_1)$ if and only if $P(I_2)$ lies to the left of $P(I_1)$, the statement of the lemma is $I_1 < I_2$ implies $P(I_2) < P(I_1)$.

If $\Delta \subset \mathbb{R}^2$ is a triangle with base in the x-axis and vertices in the points $(0,a)$, $(b,0)$, (c,d), $a < b$ then the lines that contains the sides have equations:

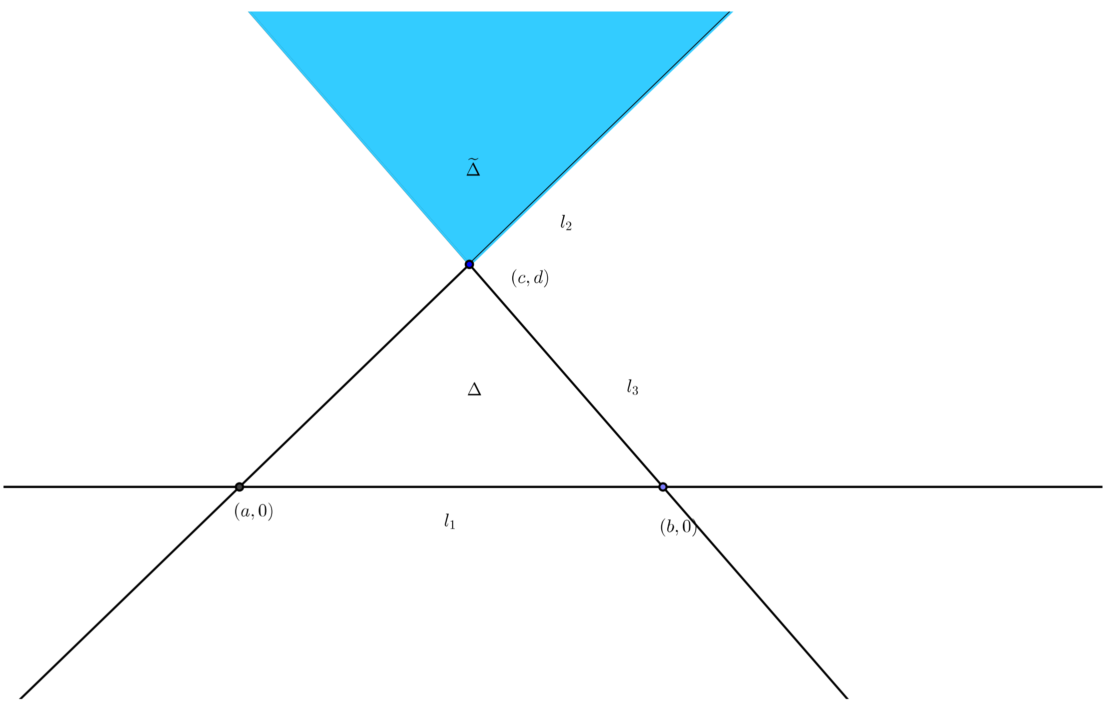

Figure 4.10: the region $\widetilde{\Delta}$ is called the shadow of Δ

$$l_1 : y = 0$$

$$l_2 : \frac{x-a}{y} = \frac{c-a}{d}, \quad x - a = \frac{c-a}{d}y, \quad x = a + \frac{c-a}{d}y$$

$$l_3 : \frac{x-b}{y} = \frac{c-b}{d}, \quad x - b = \frac{c-b}{d}y, \quad x = b + \frac{c-b}{d}y$$

we note that as Δ is a nondegenerate triangle d is nonzero, define

$$\widetilde{\Delta} = \left\{(x,y) \in \mathbb{R}^2 \mid b + \frac{c-b}{d}y \leq x \leq a + \frac{c-a}{d}y, \quad y \geq d\right\} \tag{4.4}$$

and called it the shadow of Δ (See Figure 4.10). Now we have the following claims $(\alpha), (\beta), (\gamma)$

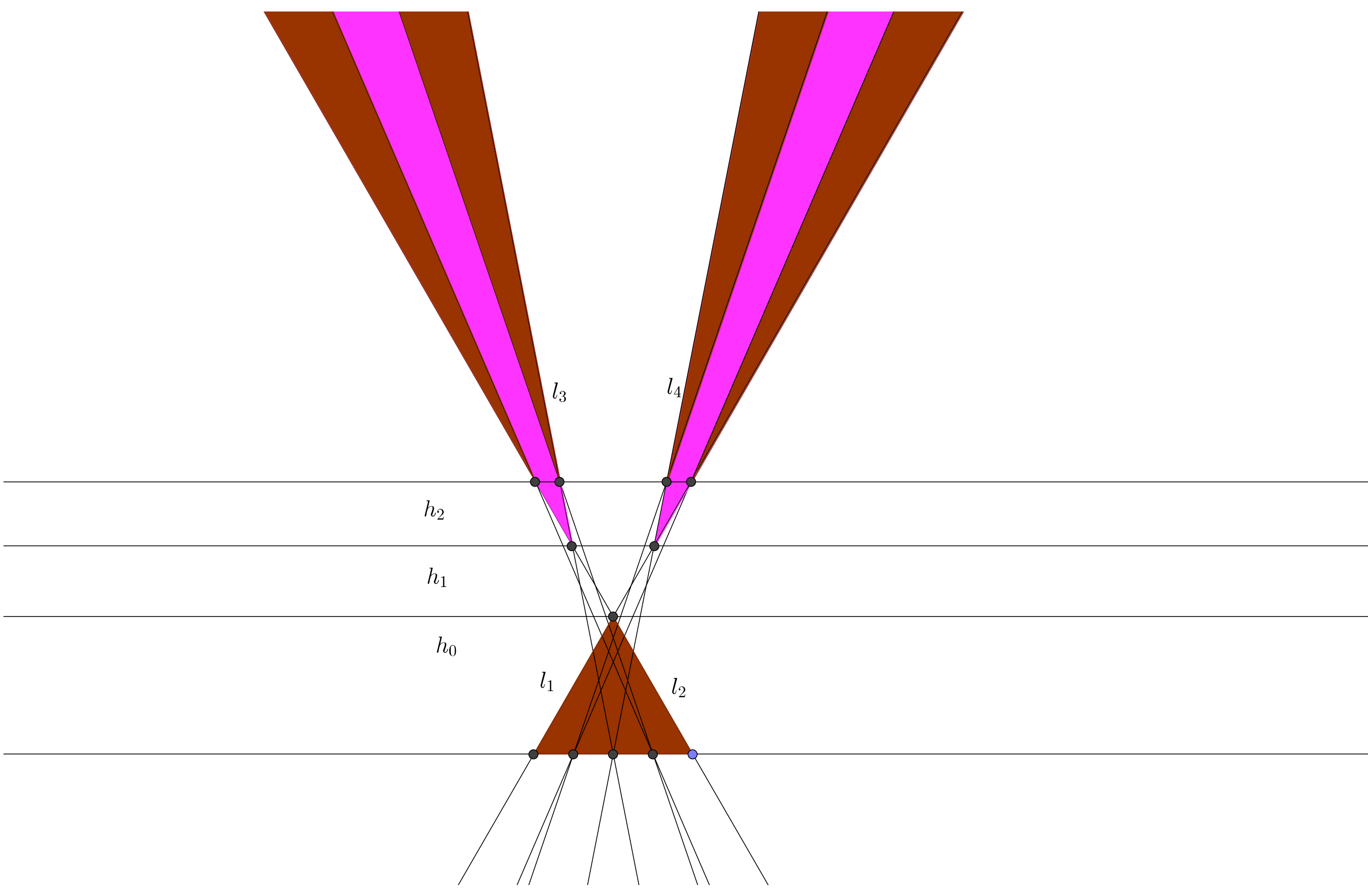

Figure 4.11: representation of the claim (α), If $I_1 \subset I_2$ then $\widetilde{\Delta(I_1)} \subset \widetilde{\Delta(I_2)}$.

and the lemma is a direct consequence of this, in fact:

(α) If $I_1 \subset I_2$ then $\widetilde{\Delta(I_1)} \subset \widetilde{\Delta(I_2)}$.

(β) If I_1, I_2 are the halves of I, $I_1 < I_2$ then $\widetilde{\Delta(I_1)} \cap \widetilde{\Delta(I_2)} = \emptyset$ and $\widetilde{\Delta(I_2)} < \widetilde{\Delta(I_1)}$.

(γ) If $I_1, I_2 \in D_k$, $I_1 \cap I_2 = \emptyset$, $I_1 < I_2$ then $\widetilde{\Delta(I_1)} \cap \widetilde{\Delta(I_2)} = \emptyset$ and $\widetilde{\Delta(I_2)} < \widetilde{\Delta(I_1)}$.

- Proof of (α). For this we assume without loss of generality that $|I_1| = 2^{-k-1}$, $|I_2| = 2^{-k}$. If $I_2 = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$ for some $0 \leq j \leq 2^k - 1$, we write $P(I_2) = (a, h_k)$ and call l_{k1} the line that passes by $P(I_2)$ and $\left(\frac{j}{2^k}, 0\right)$ and l_{k2} the line that passes by $P(I_2)$ and $\left(\frac{j+1}{2^k}, 0\right)$. We have:

$$l_{k1} : \frac{x - \frac{j}{2^k}}{y} = \frac{a - \frac{j}{2^k}}{h_k}, \quad x - \frac{j}{2^k} = \left(a - \frac{j}{2^k}\right)\frac{y}{h_k}, \quad x = \frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k}$$

$$l_{k2} : \frac{x - \frac{j+1}{2^k}}{y} = \frac{a - \frac{j+1}{2^k}}{h_k}, \quad x - \frac{j+1}{2^k} = \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k}, \quad x = \frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k}$$

By definition $\widetilde{\Delta(I_2)} = \left\{(x, y) \in \mathbb{R}^2 \mid y \geq h_k, \frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k}\right\}$, in this case $I_1 = \left[\frac{j}{2^k}, \frac{2j+1}{2^{k+1}}\right]$ or $I_1 = \left[\frac{2j+1}{2^{k+1}}, \frac{j+1}{2^k}\right]$ for the first case we find $P(I_1)$ the point of intersection of the lines l_{k1} and the line with equation $y = h_{k+1}$, then $P(I_1) = \left(\frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}, h_{k+1}\right)$, with this we find the equation of the line l_{k4}

$$l_{k4} : \frac{x - \frac{2j+1}{2^{k+1}}}{y} = \frac{\frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{2j+1}{2^{k+1}}}{h_{k+1}}$$

$$x = \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}$$

By definition

$\widetilde{\Delta(I_1)} = \left\{(x, y) \in \mathbb{R}^2 \mid y \geq h_{k+1}, \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}} \leq x \leq \frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k}\right\}$.

Let $(x, y) \in \widetilde{\Delta(I_1)}$ then $y \geq h_{k+1} \geq h_k$ and

$$\frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}} \leq x \leq \frac{j}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k}$$

to see that $(x,y) \in \widetilde{\Delta(I_2)}$ is enough to prove that

$$\begin{aligned}
&\frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}} \\
&= \frac{2j+1}{2^{k+1}} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k} - \frac{y}{2^{k+1}h_{k+1}} \Leftrightarrow \frac{j+1}{2^k} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k} - \frac{y}{2^k h_k} \\
&\leq \frac{2j+1}{2^{k+1}} + \left(a - \frac{j}{2^k}\right)\frac{y}{h_k} - \frac{y}{2^{k+1}h_{k+1}} \Leftrightarrow \frac{1}{2^{k+1}} \leq \frac{y}{2^k h_k} - \frac{y}{2^{k+1}h_{k+1}} \\
&= \frac{y}{2^k}\left(\frac{1}{h_k} - \frac{1}{2h_{k+1}}\right) = \frac{y}{2^k}\left(\frac{2h_{k+1}-h_k}{2h_k h_{k+1}}\right) \Leftrightarrow h_k h_{k+1} \leq (2h_{k+1}-h_k)y \\
&= 2h_{k+1}y - h_k y \Leftrightarrow h_k h_{k+1} + h_k y \leq 2h_{k+1}y \Leftrightarrow h_k(h_{k+1}+y) \leq 2h_{k+1}y,
\end{aligned}$$

that is clear because $y \geq h_{k+1} \geq h_k$, then $\widetilde{\Delta(I_1)} \subset \widetilde{\Delta(I_2)}$.

For the second case we find $P(I_1)$ the point of intersection of the lines l_{2k} and the line with equation $y = h_{k+1}$, then $P(I_1) = \left(\left(a - \frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k}, 0\right)$, with this we find the equation of the line l_{k3}

$$\begin{aligned}
l_{k3} : \frac{x - \frac{2j+1}{2^{k+1}}}{y} &= \frac{\frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k} - \frac{2j+1}{2^{k+1}}}{h_{k+1}} \\
x &= \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k} + \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}
\end{aligned}$$

By definition

$\widetilde{\Delta(I_1)} = \left\{(x,y) \in \mathbb{R}^2 \mid y \geq h_{k+1}, \frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k} + \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}\right\}$

Let $(x,y) \in \widetilde{\Delta(I_1)}$ then $y \geq h_{k+1} \geq h_k$, and

$$\frac{j+1}{2^k} + \left(a - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{2j+1}{2^{k+1}} + \left(\left(a - \frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k} + \frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}$$

to see that $(x,y)\in\widetilde{\Delta(I_2)}$ is enough to prove that

$$\frac{2j+1}{2^{k+1}}+\left(\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}+\frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}\leq\frac{j}{2^k}+\left(a-\frac{j}{2^k}\right)\frac{y}{h_k}$$
$$\Leftrightarrow\frac{2j+1}{2^{k+1}}+\frac{y}{2^{k+1}h_{k+1}}+\left(a-\frac{j}{2^k}\right)\frac{y}{h_k}-\frac{y}{2^k h_k}\leq\frac{j}{2^k}+\left(a-\frac{j}{2^k}\right)\frac{y}{h_k}$$
$$\Leftrightarrow\frac{2j+1}{2^{k+1}}-\frac{j}{2^k}\leq\frac{y}{2^k h_k}-\frac{y}{2^{k+1}h_{k+1}}\Leftrightarrow\frac{1}{2^{k+1}}\leq\frac{y}{2^k}\left(\frac{1}{h_k}-\frac{1}{2h_{k+1}}\right)=\frac{y}{2^k}\left(\frac{2h_{k+1}-h_k}{2h_kh_{k+1}}\right)$$
$$\Leftrightarrow h_kh_{k+1}\leq(2h_{k+1}-h_k)y=2h_{k+1}y-h_ky\Leftrightarrow h_kh_{k+1}+h_ky\leq 2h_{k+1}y\Leftrightarrow h_k(h_{k+1}+y)\leq 2h_{k+1}y$$

that is clear because $y\geq h_{k+1}\geq h_k$, then $\widetilde{\Delta(I_1)}\subset\widetilde{\Delta(I_2)}$. This completes the proof of (α).

– Proof of (β). If $|I|=2^{-k}$, $I=\left[\frac{j}{2^k},\frac{j+1}{2^k}\right]$, put $I_1=\left[\frac{j}{2^k},\frac{2j+1}{2^{k+1}}\right]$, $I_2=\left[\frac{2j+1}{2^{k+1}},\frac{j+1}{2^k}\right]$. By what we have:

$$\widetilde{\Delta(I_1)}=\left\{(x,y)\in\mathbb{R}^2\mid y\geq h_{k+1},\frac{2j+1}{2^{k+1}}+\left(\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}-\frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}\leq x\leq\frac{j}{2^k}+\left(a-\frac{j}{2^k}\right)\frac{y}{h_k}\right\}$$
$$\widetilde{\Delta(I_2)}=\left\{(x,y)\in\mathbb{R}^2\mid y\geq h_{k+1},\frac{j+1}{2^k}+\left(a-\frac{j+1}{2^k}\right)\frac{y}{h_k}\leq x\leq\frac{2j+1}{2^{k+1}}+\left(\left(a-\frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k}+\frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}\right\}.$$

So prove (β) is enough to prove that:

$$\frac{2j+1}{2^{k+1}}+\left(\left(a-\frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k}+\frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}<\frac{2j+1}{2^{k+1}}+\left(\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}-\frac{1}{2^{k+1}}\right)\frac{y}{h_{k+1}}.$$

But this is equivalent to

$$\frac{1}{2^{k+1}}+\left(a-\frac{j+1}{2^k}\right)\frac{h_{k+1}}{h_k}<\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}-\frac{1}{2^{k+1}}$$
$$\Leftrightarrow\frac{1}{2^{k+1}}+\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}-\frac{h_{k+1}}{2^kh_k}<\left(a-\frac{j}{2^k}\right)\frac{h_{k+1}}{h_k}-\frac{1}{2^{k+1}}$$
$$\Leftrightarrow\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}<\frac{h_{k+1}}{2^kh_k}\Leftrightarrow\frac{1}{2^k}<\frac{h_{k+1}}{2^kh_k}\Leftrightarrow h_k<h_{k+1},$$

that is true by hypothesis.

– Proof of (γ). Let $I_1=\left[\frac{j}{2^k},\frac{j+1}{2^k}\right]$, $I_2=\left[\frac{l}{2^k},\frac{l+1}{2^k}\right]$, $0\leq j,l\leq 2^k-1$, as $I_1\cap I_2=\emptyset$ and

$I_1 < I_2$ we have $j+1 < l$, for $n \in \mathbb{N}$ define $S_2(n) = \{r \in \mathbb{N} \mid 2^r \mid n\}$ as $1 = 2^0 \mid n$ we have that $0 \in S_2(n)$ so $S_2(n)$ is nonempty, let $\nu_2(n) = \max S_2(n)$ we call $\nu_2(n)$ the 2-valuation of n, we can write $n = 2^{\nu_2(n)}t$, by the maximality of $\nu_2(n)$, t is odd.

Let $s = \max\{\nu_2(n) \mid j+1 \leq n \leq l\}$. We claim that there exists an unique n such that $j+1 \leq n \leq l$ and $s = \nu_2(n)$, else $n_1 = 2^s t_1$, $n_2 = 2^s t_2$, n_1, n_2 different we can assume without loss of generality that $n_1 < n_2$ so $t_1 < t_2$, but as t_1 and t_2 are odd there exists $\overline{t}$ even such that $t_1 < \overline{t} < t_2$, if $\overline{t} = 2u$, $m = 2^s\overline{t} = 2^{s+1}u$ then $n_1 = 2^s t_1 < m = 2^s\overline{t} = 2^{s+1}u < 2^s t_2 = n_2$, so $j+1 \leq m = 2^{s+1}u \leq l$, that contradicts the maximality of s because $\nu_2(m) \geq s+1 > s$, put $n = 2^s t$ with t odd we define $I_1^* = \left[\frac{t-1}{2^{k-s}}, \frac{t}{2^{k-s}}\right]$, $I_2^* = \left[\frac{t}{2^{k-s}}, \frac{t+1}{2^{k-s}}\right]$, we claim that $I_1 \subset I_1^*$, $I_2 \subset I_2^*$, as $n \in \mathbb{N}$ is the unique such that $j+1 \leq n \leq l$, $s = \nu_2(n)$ we have $j < n < l+1$, also $2^s(t-1) \leq j$ and $2^s(t+1) \geq l+1$, hence $\frac{t-1}{2^{k-2}} \leq \frac{j}{2^k} \leq \frac{j+1}{2^k} \leq \frac{t}{2^{k-s}}$ also $\frac{t}{2^{k-s}} \leq \frac{l}{2^k} \leq \frac{l+1}{2^k} \leq \frac{t+1}{2^{k-s}}$, this implies that $I_1 \subset I_1^*$, $I_2 \subset I_2^*$, also $I_1^* < I_2^*$, as t is odd $t-1$ is even so $t-1 = 2w$, for some $w \in \mathbb{N}$, hence $t+1 = 2(w+1)$, and this implies that: $I^* = I_1^* \cup I_2^* = \left[\frac{t-1}{2^{k-s}}, \frac{t+1}{2^{k-s}}\right] = \left[\frac{w}{2^{k-s-1}}, \frac{w+1}{2^{k-s-1}}\right]$ so I^* is an dyadic interval with $|I^*| = \frac{1}{2^{k-s-1}}$, we note that as $l \leq 2^k - 1$ and $s \leq k-1$, so $k-s-1 \geq 0$.

As I_1^*, I_2^* are halves of a dyadic interval I^*, $I_1^* < I_2^*$ we apply (α) to obtain $\widetilde{\Delta(I_1)} \subset \widetilde{\Delta(I_1^*)}$, $\widetilde{\Delta(I_2)} \subset \widetilde{\Delta(I_2^*)}$ and (β) to obtain $\widetilde{\Delta(I_1)} \cap \widetilde{\Delta(I_2)} \subset \widetilde{\Delta(I_1^*)} \cap \widetilde{\Delta(I_2^*)} = \emptyset$, also $\widetilde{\Delta(I_2^*)} < \widetilde{\Delta(I_1^*)}$, hence $\widetilde{\Delta(I_2)} < \widetilde{\Delta(I_1)}$ and $\widetilde{\Delta(I_1)} \cap \widetilde{\Delta(I_2)} = \emptyset$.

As $P(I) \in \widetilde{\Delta(I)}$, if we suppose that $I_1 < I_2$ then $\widetilde{\Delta(I_2)} < \widetilde{\Delta(I_1)}$ in special $P(I_2) < P(I_1)$, this completes the proof. $\diamondsuit$

If $I_1, I_2 \in D_k$, I_1 and I_2 different suppose without loss of generality that $I_1 < I_2$, using the previous lemma $P(I_2) < P(I_1)$, (See Figure 4.12). If $P(I_1) = (a_1, h_k)$, $P(I_2) = (a_2, h_k)$ then $a_2 < a_1$, we write $I_1 = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$, $I_2 = \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right]$ and use that $j+1 \leq l$, moreover:

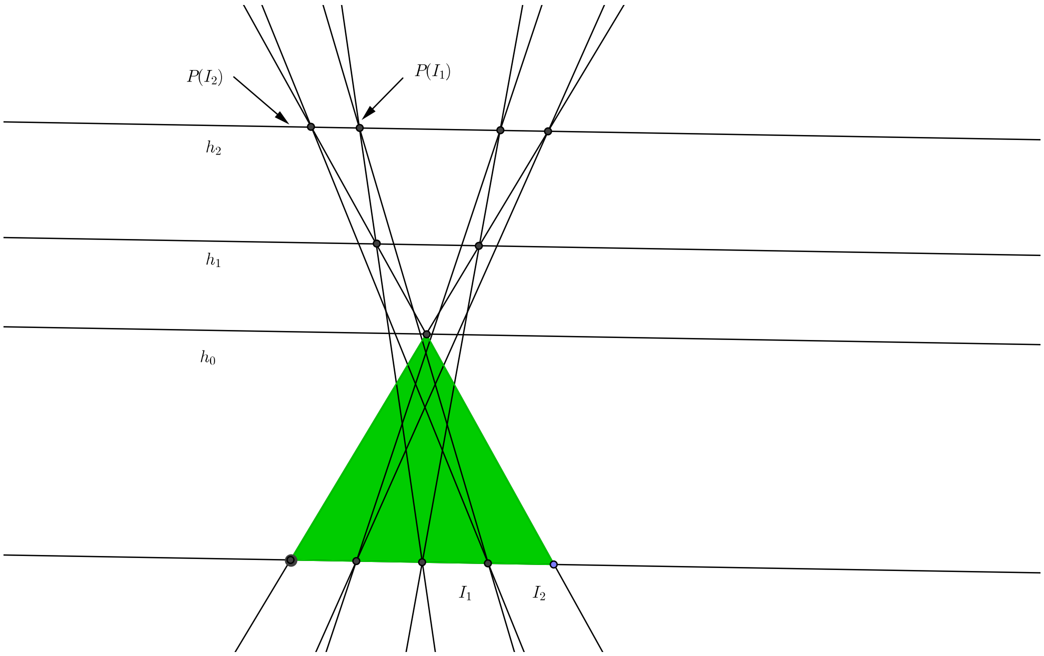

Figure 4.12: If $I_1 < I_2$ then $P(I_2) < P(I_1)$.

$\widetilde{\Delta(I_1)} = \left\{(x,y) \in \mathbb{R}^2 \mid y \geq h_k, \frac{j+1}{2^k} + \left(a_1 - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{j}{2^k} + \left(a_1 - \frac{j}{2^k}\right)\frac{y}{h_k}\right\}$ and

$\widetilde{\Delta(I_2)} = \left\{(x,y) \in \mathbb{R}^2 \mid y \geq h_k, \frac{l+1}{2^k} + \left(a_2 - \frac{l+1}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{l}{2^k} + \left(a_2 - \frac{l}{2^k}\right)\frac{y}{h_k}\right\}$.

Let $[I_1] = \left\{(x,y) \in \mathbb{R}^2 \mid y \leq 0, \frac{j}{2^k} + \left(a_1 - \frac{j}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{j+1}{2^k} + \left(a_1 - \frac{j+1}{2^k}\right)\frac{y}{h_k}\right\}$ and $[I_2] = \left\{(x,y) \in \mathbb{R}^2 \mid y \leq 0, \frac{l}{2^k} + \left(a_2 - \frac{l}{2^k}\right)\frac{y}{h_k} \leq x \leq \frac{l+1}{2^k} + \left(a_2 - \frac{l+1}{2^k}\right)\frac{y}{h_k}\right\}$ we note that $R(I_1) \subset [I_1]$, $R(I_2) \subset [I_2]$, $[I_1] \cap (\mathbb{R} \times \{0\}) = I_1 \times \{0\}$, $[I_2] \cap (\mathbb{R} \times \{0\}) = I_2 \times \{0\}$, we claim $card([I_1] \cap [I_2]) \leq 1$, for see this is enough to prove that $\frac{j+1}{2^k} + \left(a_1 - \frac{j+1}{2^k}\right)\frac{y}{h_k} \leq \frac{l}{2^k} + \left(a_2 - \frac{l}{2^k}\right)\frac{y}{h_k}$, for $y \leq 0$ but this is equivalent to $\left(a_1 - \frac{j+1}{2^k}\right)\frac{y}{h_k} - \left(a_2 - \frac{l}{2^k}\right)\frac{y}{h_k} \leq \frac{l-j-1}{2^k}$, for $y \leq 0 \Leftrightarrow (a_1 - a_2)\frac{y}{h_k} + \left(\frac{l-j-1}{2^k}\right)\frac{y}{h_k} \leq \frac{l-j-1}{2^k}$, $y \leq 0$ that is clear because $a_2 < a_1$. If $[I_1] \cap [I_2] = \emptyset$ we are done, if $[I_1] \cap [I_2]$ is not empty, let $(x,y) \in [I_1] \cap [I_2]$ then $\frac{j+1}{2^k} + \left(a_1 - \frac{j+1}{2^k}\right)\frac{y}{h_k} = \frac{l}{2^k} + \left(a_2 - \frac{l}{2^k}\right)\frac{y}{h_k}$, $y \leq 0$, as $a_1 > a_2$ and $j+1 \leq l$, $l-j-1 \leq 0$, $l \leq j+1$, $l = j+1$, hence $(a_1 - a_2)\frac{y}{h_k} = 0$, so $y = 0$ this implies that $x = \frac{l}{2^k} = \frac{j+1}{2^k}$, then $[I_1] \cap [I_2] = \left\{\left(\frac{l}{2^k}, 0\right)\right\} = \left\{\left(\frac{j+1}{2^k}, 0\right)\right\}$, then $card([I_1] \cap [I_2]) \leq 1$, as $R(I_1) \cap R(I_2) \subset [I_1] \cap [I_2] \subset \mathbb{R} \times \{0\}$ we have $R(I_1) \cap R(I_2) \subset [I_1] \cap [I_2] \subset (I_1 \cap I_2) \times [0]$, if we take for every $I \in D_k$, $R(I)$ with one of the sides contained in the line l_j with equation

$x = \frac{j}{2^k} + \left(a_I - \frac{j}{2^k}\right)\frac{y}{h_k}$, $P(I) = (a_I, h_k)$, $I = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$, as the slope of l_j is different of the slope of l_{j+1}, $0 \leq j \leq 2^k - 1$, implies that $R(I_1) \cap R(I_2) = \emptyset$ if $I_1 \cap I_2$ is not empty, $I_1 < I_2$, if $I_1 \cap I_2 = \emptyset$ then $R(I_1) \cap R(I_2) = \emptyset$ this implies the existence of the families $\{R(I)\}_{I \in D_k}$ and $\left\{\widetilde{R(I)}\right\}_{I \in D_k}$ that satisfies (1), (2), (3) and (4).

$\Diamond$

CHAPTER 5

Fefferman's Theorem as an application of Kakeya sets

5.1 Introduction

Let $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $\widehat{Tf} = \chi_B \widehat{f}$. The "disc conjecture" asserts that $T \in \mathscr{B}(L^p(\mathbb{R}^n))$ for $p \in \left[\frac{2n}{n+1}, \frac{2n}{n-1}\right]$. We saw in Theorem 3.2 that $T_\delta \in \mathscr{B}(L^p(\mathbb{R}^n))$ for every $\delta > \frac{n-1}{2}$ and $1 \leq p \leq \infty$, but the Bochner-Riesz operators and the ball multiplier are different because the disc conjecture is false. This is the statement of the Fefferman's Theorem that we are going to prove.

Theorem 5.1. *Fefferman's Theorem*

If $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $\widehat{Tf} = \chi_B \widehat{f}$, *then* $T \in \mathscr{B}(L^p(\mathbb{R}^n))$ *if and only if* $p = 2$.

5.2 Proof overview

By the Plancherel Theorem $T \in \mathscr{B}(L^2(\mathbb{R}^2))$, for every $n \geq 1$, this prove sufficiency. We see the necessity by contradiction. Note that it is enough to show that $T \notin \mathscr{B}(L^p(\mathbb{R}^2))$ for $p > 2$ because de Leeuw's Theorem says that $T \in \mathscr{B}(L^p(\mathbb{R}^n))$ implies $T \in \mathscr{B}(L^p(\mathbb{R}^{n-1}))$ and the argument of duality

$T \in \mathscr{B}(L^p(\mathbb{R}^n))$ if and only if $T \in \mathscr{B}(L^q(\mathbb{R}^n))$ for $\frac{1}{p}+\frac{1}{q}=1$, given that $p>2$ if and only if $1<q<2$. If $T \in \mathscr{B}(L^p(\mathbb{R}^2))$, $p>2$ then the following holds.

Lema 5.1. *Y. Meyer*

Let $\{v_j\}_{j\in\mathbb{N}} \subset S^1$, *and* $H_j = \left\{x \in \mathbb{R}^2 \mid x\cdot v_j \geq 0\right\}$. *Define* $\{T_j\}_{j\in\mathbb{N}} \subset \mathscr{L}(L^2(\mathbb{R}^2))$ *by* $\widehat{T_j f} = \chi_{H_j}\widehat{f}$. *Then for every sequence* $\{f_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^2)$ *the following inequality follows:*

$$\left\|\left(\sum_{j=1}^{\infty}|T_j f_j|^2\right)^{\frac{1}{2}}\right\|_p \leq C\left\|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{\frac{1}{2}}\right\|_p. \tag{5.1}$$

So to prove the Fefferman's Theorem it is enough to exhibit a counterexample for this Lemma for every $p>2$, we use the variant of the Kakeya construction given in the previous chapter to achieve this goal.

5.3 Proof details

Proof of the Meyer's Lemma

The idea is to replace every operator T_j by an operator associated to the disc, let $r>0$, $D_j^r = B(rv_j, r)$ define $\left\{T_j^r\right\}_{j\in\mathbb{N}} \subset \mathscr{L}(L^p(\mathbb{R}^2))$, $\widehat{T_j^r f} = \chi_{D_j^r}\widehat{f}$ this allows us say that $D_j^{r_1} \subset D_j^{r_2}$ if $r_1 < r_2$ and $\bigcup_{r>0} D_j^r = H_j$, in fact $x \in D_j^r \Leftrightarrow \|x - rv_j\| < r \Leftrightarrow r^2 > \|x - rv_j\|^2 = \|x\|^2 - 2rx\cdot v_j + r^2 \Leftrightarrow \|x\|^2 < 2rx\cdot v_j$, this implies that if $x \in D_j^{r_1} \Rightarrow \|x\|^2 < 2r_1 x\cdot v_j < 2r_2 x\cdot v_j \Rightarrow x \in D_j^{r_2}$.

Also $x \in \bigcup_{r>0} D_j^r \Rightarrow (\exists r>0)(x \in D_j^r) \Rightarrow \|x\|^2 < 2rx\cdot v_j \Rightarrow x\cdot v_j \geq 0 \Rightarrow x \in H_j$ and $x \in H_j \Rightarrow x\cdot v_j \geq 0 \Rightarrow (\exists r>0)(\|x\|^2 < 2rx\cdot v_j) \Rightarrow x \in D_j^r \subset \bigcup_{r>0} D_j^r$.

With these facts we obtain $\lim\limits_{r\to\infty}\chi_{D_j^r}(x) = \chi_{H_j}(x)$ for every $x \in \mathbb{R}^2$, let $f \in C_0^\infty(\mathbb{R}^2) \subset \delta(\mathbb{R}^2)$

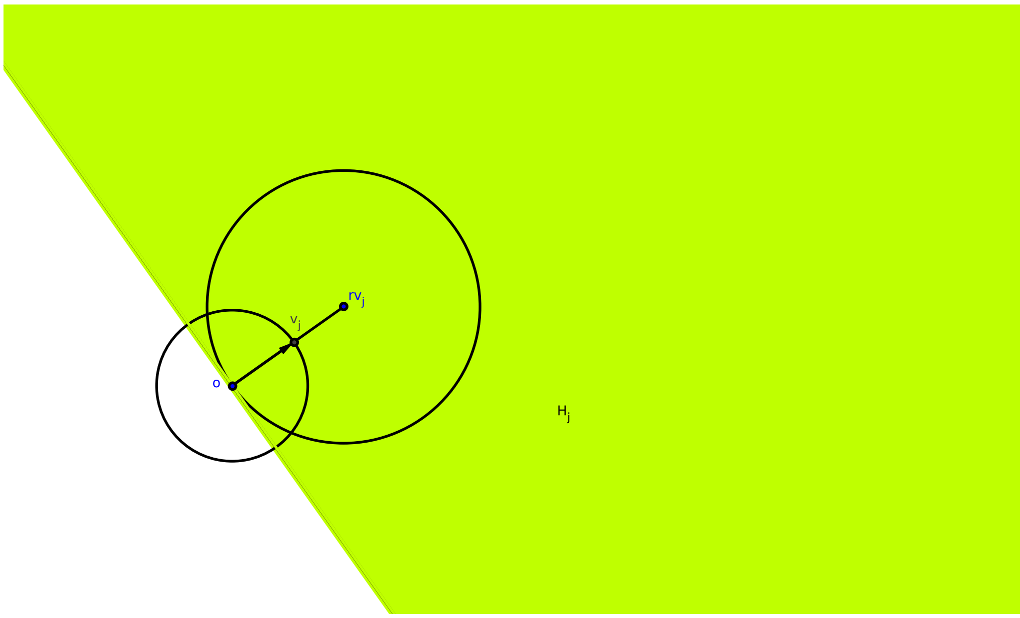

Figure 5.1: D_j^r looks much like the half plane H_j for enormous r

then $\widehat{f} \in \delta(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$, so $\widehat{T_j^r f} = \chi_{D_j^r}\widehat{f} \in L^2(\mathbb{R}^2)$, $\widehat{T_j f} = \chi_{H_j}\widehat{f} \in L^2(\mathbb{R}^2)$, moreover $\widehat{T_j f}(x) = \chi_{H_j}(x)\widehat{f}(x) = \lim_{r\to\infty} \chi_{D_j^r}(x)\widehat{f}(x) = \lim_{r\to\infty} \widehat{T_j^r f}(x)$, $(x \in \mathbb{R}^2)$, by the dominated convergence Theorem $\left\|T_j^r f - T_j f\right\|_2 = \left\|\widehat{T_j^r f} - \widehat{T_j f}\right\|_2 \to_{r\to\infty} 0$ hence $\lim_{r\to\infty} T_j^r f(x) = T_j f(x)$ a.e $x \in \mathbb{R}^2$ as $f \in \delta(\mathbb{R}^2)$, $\widehat{T_j^r f} = \chi_{D_j^r}\widehat{f} \in \delta(\mathbb{R}^2)$, $\widehat{T_j f} = \chi_{H_j}\widehat{f} \in \delta(\mathbb{R}^2)$ we have that $T_j^r f, T_j f \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$ by the dominated convergence Theorem $\lim_{r\to\infty} \left\|T_j^r f - T_j f\right\|_p = 0$.

As H_j is a half space of $\mathbb{R}^2$ Theorem 2.1 implies that $\chi_{H_j} \in \mathcal{M}(L^p(\mathbb{R}^2))$, as we are assuming that $\chi_{D(0,1)} \in \mathcal{M}(L^p(\mathbb{R}^2))$ this implies that $\chi_{D_j^r} \in \mathcal{M}(L^p(\mathbb{R}^2))$, moreover $\left|\chi_{D_j^r}\right|_p = \left|\chi_{D(0,1)}\right|_p$, for every $r > 0$, hence $\|T_j f\|_p \leq \left|\chi_{H_j}\right|_p \|f\|_p$, $\left\|T_j^r f\right\|_p \leq \left|\chi_{D(0,1)}\right|_p \|f\|_p$, for every $f \in L^p(\mathbb{R}^2)$.

If $f \in L^p(\mathbb{R}^2)$ there exists $\{f_l\}_{l\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^2)$ such that $\|f_l - f\|_p \to_{l\to\infty} 0$, let $\epsilon > 0$ arbitrary, there exists $l_0 \in \mathbb{N}$ such that $\|f_{l_0} - f\|_p < \frac{\epsilon}{2(\left|\chi_{H_j}\right|_p + \left|\chi_{D(0,1)}\right|_p)}$, there exists $r_0 > 0$ such that if $r > r_0$ then $\left\|T_j^r f_{l_0} - T_j f_{l_0}\right\|_p < \frac{\epsilon}{2}$, hence

$$\left\|T_j^r f - T_j f\right\|_p \leq \left\|T_j^r f - T_j^r f_{l_0}\right\|_p + \left\|T_j^r f_{l_0} - T_j f_{l_0}\right\|_p + \|T_j f_{l_0} - T_j f\|_p \leq \left|\chi_{D(0,1)}\right|_p \|f - f_{l_0}\|_p + \left\|T_j^r f_{l_0} - T_j f_{l_0}\right\|_p$$

$+\left|\chi_{H_j}\right|_p \|f_{l_0} - f\|_p < \left|\chi_{D(0,1)}\right|_p \frac{\epsilon}{2(\left|\chi_{H_j}\right|_p + \left|\chi_{D(0,1)}\right|_p)} + \frac{\epsilon}{2} + \left|\chi_{H_j}\right|_p \frac{\epsilon}{2(\left|\chi_{H_j}\right|_p + \left|\chi_{D(0,1)}\right|_p)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$

if $r > r_0$, so $\lim_{r\to\infty} \left\|T_j^r f - T_j f\right\|_p = 0$, then $\lim_{r\to\infty} T_j^r f(x) = T_j f(x)$ a.e $x \in \mathbb{R}^2$.

Let $\{f_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^2)$ then $\{T_j f_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^2)$, so $\lim_{r\to\infty} \left(\sum_{j=1}^m \left|T_j^r f_j(x)\right|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^m |T_j f_j(x)|^2\right)^{\frac{1}{2}}$ a.e $x \in \mathbb{R}^2$, for every $m \in \mathbb{N}$, by Fatou's Lemma:

$$\left\|\left(\sum_{j=1}^m |T_j f_j|^2\right)^{\frac{1}{2}}\right\|_p \leq \liminf_{r\to\infty} \left\|\left(\sum_{j=1}^m \left|T_j^r f_j(x)\right|^2\right)^{\frac{1}{2}}\right\|_p$$

this implies to prove (5.1) is enough to prove:

$$\left\|\left(\sum_{j=1}^m \left|T_j^r f_j(x)\right|^2\right)^{\frac{1}{2}}\right\|_p \leq C \left\|\left(\sum_{j=1}^m |f_j|^2\right)^{\frac{1}{2}}\right\|_p$$

with $C > 0$ independent of $r > 0$. As T_j^r is the operator with multiplier $\chi_{D_j^r}$ the Theorem 2.1 implies that $\left|\chi_{D_j^r}\right|_p = \left|\chi_{D_j^1}\right|_p$, $r > 0$, this tells us that it is enough to prove the case $r = 1$. However:

$$\begin{aligned}
(T_j^1 f)(x) &= \int_{D_j^1} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = \int_{\|\xi - v_j\|<1} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = \int_{\|\xi\|<1} \widehat{f}(\xi + v_j) e^{2\pi i x\cdot(\xi+v_j)} d\xi \\
&= e^{2\pi i x\cdot v_j} \int_{\|\xi\|<1} \widehat{f}(\xi + v_j) e^{2\pi i x\cdot\xi} d\xi = e^{2\pi i x\cdot v_j} \int_{\|\xi\|<1} (\widehat{M_{-v_j} f})(\xi) e^{2\pi i x\cdot\xi} d\xi = e^{2\pi i x\cdot v_j} T(M_{-v_j} f)(x) \\
&= (M_{-v_j} T(M_{-v_j} f))(x)
\end{aligned}$$

as we assume that $T \in \mathscr{B}(L^p(\mathbb{R}^2))$ there exists $C > 0$ such that $\|Tf\|_p \leq C \|f\|_p$, we claim that

$$\left\|\left(\sum_{j=1}^m |T f_j|^2\right)^{\frac{1}{2}}\right\|_p \leq C \left\|\left(\sum_{j=1}^m |f_j|^2\right)^{\frac{1}{2}}\right\|_p \tag{5.2}$$

as we assume $p > 2$, $\frac{p}{2} > 1$, there exists $g \in L^{(\frac{p}{2})'}(\mathbb{R}^2)$ such that $\|g\|_{(\frac{p}{2})'} = 1$ (here q' is de dual exponent of q) and:

$$\left\|\left(\sum_{j=1}^{m}|Tf_j|^2\right)^{\frac{1}{2}}\right\|_p^2=\left(\int_{\mathbb{R}^2}\left(\sum_{j=1}^{m}|Tf_j|^2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}}=\left\|\sum_{j=1}^{m}|Tf_j|^2\right\|_{\frac{p}{2}}=\int_{\mathbb{R}^2}\sum_{j=1}^{m}|Tf_j|^2 g$$

$$\leq C^2\int_{\mathbb{R}^2}\sum_{j=1}^{m}|f_j|^2 g\leq C^2\left(\int_{\mathbb{R}^2}\left(\sum_{j=1}^{m}|f_j|^2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^2}|g|^{(\frac{p}{2})'}\right)^{\frac{1}{(\frac{p}{2})'}}\leq C^2\left\|\left(\sum_{j=1}^{m}|f_j|^2\right)^{\frac{1}{2}}\right\|_p^2$$

take square roots we obtain (5.2), hence:

$$\left\|\left(\sum_{j=1}^{m}\left|T^1 f_j\right|^2\right)^{\frac{1}{2}}\right\|_p=\left\|\left(\sum_{j=1}^{m}\left|M_{v_j}T(M_{-v_j}f_j)\right|^2\right)^{\frac{1}{2}}\right\|_p=\left\|\left(\sum_{j=1}^{m}\left|T(M_{-v_j}f_j)\right|^2\right)^{\frac{1}{2}}\right\|_p$$

$$\leq C\left\|\left(\sum_{j=1}^{m}\left|M_{-v_j}f_j\right|^2\right)^{\frac{1}{2}}\right\|_p=C\left\|\left(\sum_{j=1}^{m}|f_j|^2\right)^{\frac{1}{2}}\right\|_p$$

then

$$\left\|\left(\sum_{j=1}^{m}|T_j f_j|^2\right)^{\frac{1}{2}}\right\|_p\leq C\left\|\left(\sum_{j=1}^{m}|f_j|^2\right)^{\frac{1}{2}}\right\|_p \tag{5.3}$$

for every $m\in\mathbb{N}$, letting $m\to\infty$ we obtain the proof of the Lemma.

Counterexample to Meyer's Lemma

Let $\eta>0$, by the Lemma 4.1 there exist a family of rectangles $\{R_j\}_{j=1}^{2^k}$, a family of switches $\left\{\tilde{R}_j\right\}_{j=1}^{2^k}$, $R_j\cap R_l=\emptyset$ if j,l are different, $E\subset\mathbb{R}^2$ Besicovitch set such that (1) $\left|\tilde{R}_j\right|=2\left|R_j\right|$, (2) $\left|\tilde{R}_j\cap E\right|\geq\frac{1}{10}\left|\tilde{R}_j\right|$, (3) $|E|\leq\eta\sum_{j=1}^{2^k}|R_j|$, take $f_j=\chi_{R_j}, 1\leq j\leq 2^k, v_j\in S^1$ parallel to the longer sides of R_j as in the figure 5.2.

We remember that $T_j: L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)$, $\widehat{T_jf}=\chi_{H_j}\widehat{f}$, we calculate some integrals that we need later, for this we use the Laplace transform $\mathcal{L}\{G\}(s)=g(s)=\int_0^\infty e^{-st}G(t)dt$, and the recognized

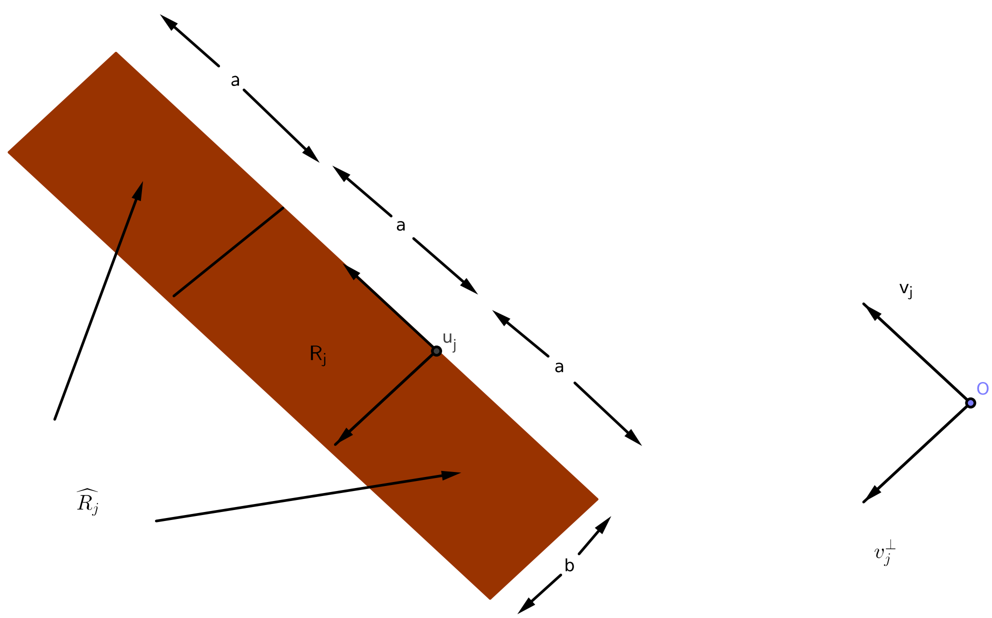

Figure 5.2: Besicovitch construction

property $\int_0^\infty \frac{G(t)}{t}dt = \int_0^\infty g(s)ds$, by a simple computation if $G(t) = \sin(\alpha t)$ then $g(s) = \frac{\alpha}{s^2+\alpha^2}$, if $G(t) = \cos(\alpha t)$ then $g(s) = \frac{s}{s^2+\alpha^2}$, with this $\int_0^\infty \frac{\sin(\alpha t)}{t}dt = \int_0^\alpha \frac{\alpha}{s^2+a^2}ds = \frac{\pi}{2}sgn(\alpha)$, also

$$\int_0^\infty \left(\frac{\cos(\alpha t) - \cos(\beta t)}{t}\right) dt = \int_0^\infty \left(\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2}\right) = \frac{1}{2}\left[\log(s^2+\alpha^2) - \log(s^2+\alpha^2)\right]_0^\infty$$
$$= \frac{1}{2}\left[\log\frac{s^2+\alpha^2}{s^2+\beta^2}\right]_0^\infty = \frac{1}{2}\left(0 - \log\left(\frac{\alpha^2}{\beta^2}\right)\right) = -\frac{1}{2}\log\frac{\alpha^2}{\beta^2}$$

this implies that $\int_{\mathbb{R}} \frac{e^{-2\pi ix\cdot\xi}}{x}dx = \int_{-\infty}^0 \frac{e^{-2\pi ix\cdot\xi}}{x}dx + \int_0^\infty \frac{e^{-2\pi ix\cdot\xi}}{x}dx$ in the first integral take $u = -x$, $x = 0 \Rightarrow u = 0$, $x \to -\infty \Rightarrow u \to \infty$, so $\int_{\mathbb{R}} \frac{e^{-2\pi ix\cdot\xi}}{x}dx = \int_\infty^0 \frac{e^{2\pi iu\cdot\xi}}{(-u)}(-du) + \int_0^\infty \frac{e^{-2\pi ix\cdot\xi}}{x}dx = \int_\infty^0 \frac{e^{2\pi ix\cdot\xi}}{x}dx + \int_0^\infty \frac{e^{-2\pi ix\cdot\xi}}{x}dx = -2i\int_0^\infty \frac{\sin(x\xi)}{x}dx = -2i\left(\frac{\pi}{2}sgn(\xi)\right)$
$= -\pi i sgn(\xi)$.

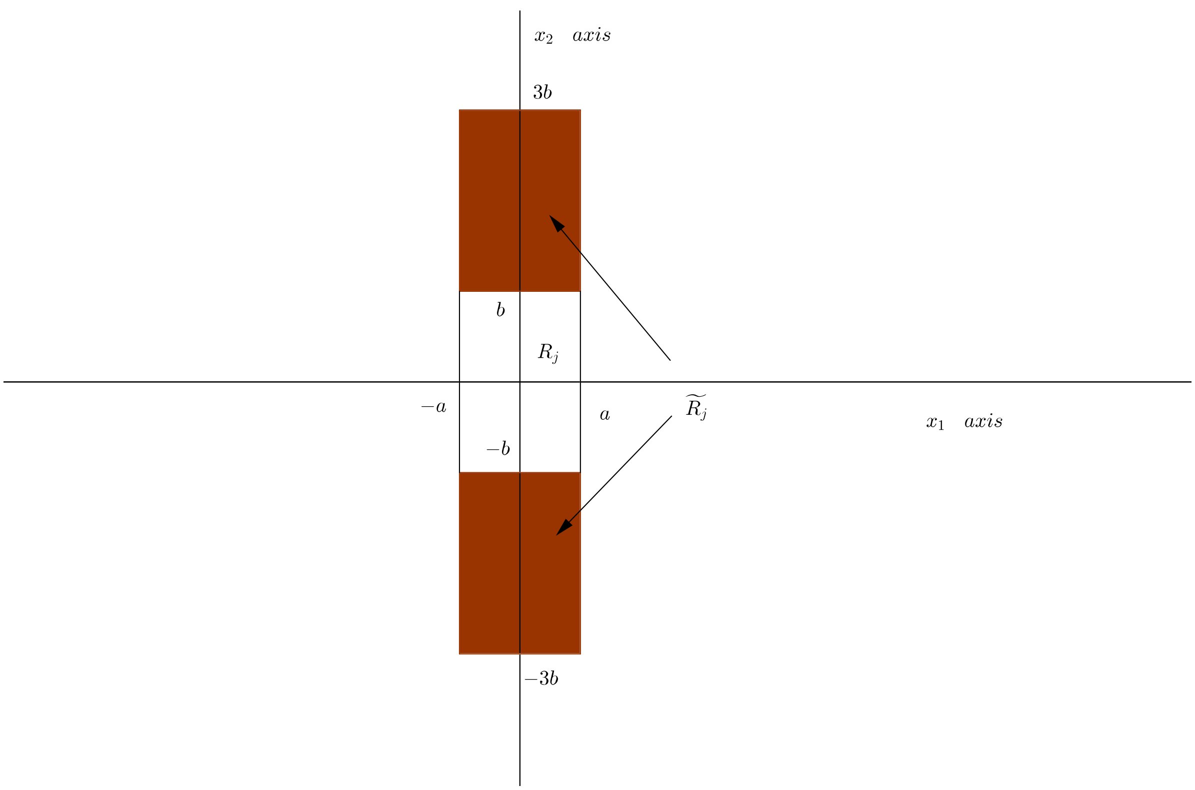

Figure 5.3: rectangle with sides parallel to the axis, $0 < a \leq b < \infty$

We assume first $R_j = [-a_j, a_j] \times [-b_j, b_j]$, $0 < a \le b < \infty$, in this case $v_j = (0,1)$. By definition

$$(T_j \chi_{R_j})(x) = (\chi_{H_j} \widehat{\chi_{R_j}})^{\vee}(x) = \int_{\mathbb{R}^2} \chi_{H_j}(\xi)\widehat{\chi_{R_j}}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{\mathbb{R}^2} \chi_{H_j}(\xi)\left(\int_{\mathbb{R}^2}\chi_{R_j}(\eta)e^{-2\pi i\xi\cdot\eta}d\eta\right)e^{2\pi i x\cdot\xi}d\xi$$
$$= \int_{\mathbb{R}^2}\chi_H(\xi)\left(\int_{-a_j}^{a_j}\int_{-b_j}^{b_j}e^{-2\pi i(\xi_1\eta_1+\xi_2\eta_2)}d\eta_1 d\eta_2\right)e^{2\pi i x\cdot\xi}d\xi$$
$$= \int_0^{\infty}\int_{\mathbb{R}}\left(\int_{-a_j}^{a_j}e^{-2\pi i\xi_1\eta_1}d\eta_1\right)\left(\int_{-b_j}^{b_j}e^{-2\pi i\xi_2\eta_2}d\eta_2\right)e^{2\pi i(x_1\xi_1+x_2\xi_2)}d\xi_1 d\xi_2$$
$$= \left(\int_{\mathbb{R}}\left(\int_{-a_j}^{a_j}e^{-2\pi i\xi_1\eta_1}d\eta_1\right)e^{2\pi i x_1\xi_1}d\xi_1\right)\left(\int_0^{\infty}\left(\int_{-b_j}^{b_j}e^{-2\pi i\xi_2\eta_2}d\eta_2\right)e^{2\pi i x_2\xi_2}d\xi_2\right)$$
$$= \left(\int_{\mathbb{R}}\widehat{\chi_{[-a_j,a_j]}}(\xi_1)e^{2\pi i x_1\xi_1}d\xi_1\right)\left(\int_0^{\infty}\left[\frac{e^{-2\pi i\xi_2\eta_2}}{-2\pi i\xi_2}\right]_{-b_j}^{b_j}e^{2\pi i x_2\xi_2}d\xi_2\right)$$
$$= \chi_{[-a_j,a_j]}(x_1)\left(\int_0^{\infty}\left(\frac{e^{2\pi i b\xi_2}-e^{-2\pi i b\xi_2}}{2\pi i\xi_2}\right)e^{2\pi i x_2\xi_2}d\xi_2\right)$$
$$= \chi_{[-a_j,a_j]}(x_1)\left(\int_0^{\infty}\left(\frac{e^{2\pi i(x_2+b_j)\xi_2}-e^{2\pi i(x_2-b_j)\xi_2}}{2\pi i\xi_2}\right)d\xi_2\right)$$
$$= \frac{\chi_{[-a_j,a_j](x_1)}}{2\pi i}\int_0^{\infty}\left(\frac{\cos(2\pi(x_2+b_j)\xi_2)-\cos(2\pi i(x_2-b_j)\xi_2)}{\xi_2}\right)d\xi_2$$
$$+\frac{\chi_{[-a_j,a_j](x_1)}}{2\pi}\int_0^{\infty}\left(\frac{\sin(2\pi i(x_2+b_j)\xi_2)-\sin(2\pi i(x_2-b_j)\xi_2)}{\xi_2}\right)d\xi_2$$
$$= \frac{\chi_{[-a_j,a_j]}(x_1)}{2\pi i}\left(-\frac{1}{2}\log\left(\frac{x_2+b_j}{x_2-b_j}\right)^2 + i\left(\frac{\pi}{2}(sgn(x_2+b_j)-sgn(x_2-b_j))\right)\right)$$
$$= \frac{\chi_{[-a_j,a_j]}(x_1)}{2\pi i}\left(-\log\left|\frac{x_2+b_j}{x_2-b_j}\right| + \pi i\chi_{[-b_j,b_j]}(x_2)\right)$$

then $\left|(T_j\chi_{R_j})(x_1,x_2)\right| \ge \frac{1}{2\pi}\chi_{[-a_j,a_j](x_1)}\left|\log\left|\frac{x_2+b_j}{x_2-b_j}\right|\right|$, but by definition $\widetilde{R_j} = [-a_j.a_j]\times([-b_j,-3b_j]\cup[b_j,3b_j])$ then $x\in\widetilde{R_j}$ if and only if $|x_1|\le a_j, \quad b_j\le|x_2|\le 3b_j$, we claim that $\max\left\{\left|\frac{x_2+b_j}{x_2-b_j}\right|,\left|\frac{x_2-b_j}{x_2+b_j}\right|\right\}\ge 2$ for every $x\in\widetilde{R_j}$, in fact $x\in\widetilde{R_j}$ implies $b_j\le|x_2|\le 3b_j$ if and only if $b_j\le x_2\le 3b_j$ or $-3b_j\le x_2\le -b_j$, in the first case $\left|\frac{x_2+b_j}{x_2-b_j}\right| = \frac{x_2+b_j}{x_2-b_j}\ge 2 \Leftrightarrow x_2+b_j\ge 2x_2-2b_j\Leftrightarrow x_2\le 3b_j$ that is clear, in the second case $\left|\frac{x_2-b_j}{x_2+b_j}\right| = \frac{b_j-x_2}{-b_j-x_2}\ge 2\Leftrightarrow b_j-x_2\ge -2b_j-2x_2\Leftrightarrow x_2\ge -3b_j$ that also is clear, then $x\in\widetilde{R_j}$ implies $\left|(T_j\chi_{R_j})(x_1,x_2)\right|\ge\frac{\log 2}{2\pi}\ge\frac{1}{10}$, hence $\left|T_j\chi_{R_j}\right|\ge\frac{1}{10}\chi_{\widetilde{R_j}}$.

If R_j is a rectangle centered at the origin then $R_j = A_j([-a,a]\times[-b,b])$ with $A_j:\mathbb{R}^2\to\mathbb{R}^2$, $A_j(x_1,x_2) = x_1v_j^{\perp}+x_2v_j$, $v_j^{\perp} = (-v_{j2},v_{j1})$, if $v_j = (v_{j1},v_{j2})$, as $A_j\in O(2)$ using the proposition 1.2

and that $\chi_{R_j} = \chi_{[-a_j,a_j]\times[-b_j,j]} \circ A_j^{-1}$:

$$T_j(\chi_{R_j}) = \left(\chi_{H_j}\widehat{\chi}_{[-a_j,a_j\times[-b_j,b_j]]} \circ A_j^{-1}\right)^{\vee} = \left((\chi_{H_j}\circ A_j)\circ A_j^{-1}\widehat{\chi}_{[-a_j,a_j]\times[-b_j,b_j]}\circ A_j^{-1}\right)^{\vee}$$
$$= \left(\left((\chi_{H_j}\circ A_j)\widehat{\chi}_{[-a_j,a_j]\times[-b_j,b_j]}\right)\circ A_j^{-1}\right)^{\vee} = \left(\left((\chi_{H_j}\circ A_j)\widehat{\chi}_{[-a_j,a_j]\times[-b_j,b_j]}\right)\right)^{\vee}\circ A_j^{-1}$$
$$= \left(\chi_{H_0}\widehat{\chi}_{[-a_j,a_j]\times[-b_j,b_j]}\right)^{\vee}\circ A_j^{-1} = T_0\chi_{[-a_j,a_j]\times[-b_j,b_j]}\circ A_j^{-1}$$

where $H_0 = \left\{x\in\mathbb{R}^2 \mid x_2\geq 0\right\}$, $\widehat{T_0 f} = \chi_{H_0}\widehat{f}$. As $x\in R_j \Leftrightarrow A_j^{-1}x\in[-a_j,a_j]\times[-b_j,b_j] \Rightarrow \left|T\chi_{R_j}(x)\right| = \left|(T_0\chi_{[-a_j,a_j]\times[-b_j,b_j]})(A_j^{-1}x)\right| \geq \frac{1}{10}\chi_{[-a_j,a_j]\times[-b_j,b_j]}(A_j^{-1}x) = \frac{1}{10}\chi_{\widetilde{R_j}}(x)$, where $\widetilde{R_j} = A_j([-a_j,a_j]\times([-3b_j,-b_j]\cup[b_j,3b_j]))$.

The general case is when R_j is a rectangle centered at the point $y\in\mathbb{R}^2$, for this we suppose that R_j is a rectangle centered at the origin. If $R_j^y = y+R_j$ then $\chi_{R_j^y} = \tau_{-y}\chi_{R_j}$ and $\widetilde{R_j^y} = y+\widetilde{R_j}$ implies $\chi_{\widetilde{R_j^y}} = \tau_{-y}\chi_{\widetilde{R_j}}$ then

$$(T_j\chi_{R_j^y})(x) = \int_{\mathbb{R}^2}\chi_{H_j}(\xi)\widehat{\chi_{R_j^y}}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{H_j}\widehat{\chi_{R_j^y}}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{H_j}\widehat{\tau_{-y}\chi_{\widetilde{R_j}}}(\xi)e^{2\pi i x\cdot\xi}d\xi$$
$$= \int_{H_j}e^{-2\pi i y\cdot\xi}\widehat{\chi_{R_j}}(\xi)e^{2\pi i x\cdot\xi}d\xi = \int_{H_j}\widehat{\chi_{R_j}}(\xi)e^{2\pi i(x-y)\cdot\xi}d\xi = (T_j\chi_{R_j})(x-y) = \tau_{-y}(T_j\chi_{R_j})(x)$$

as $x\in R_j^y \Leftrightarrow x-y\in R_j \Rightarrow \left|T_j\chi_{R_j^y}\right| = \left|(T_j\chi_{R_j})(x-y)\right| \geq \frac{1}{10}\chi_{\widetilde{R_j}}(x-y) = \frac{1}{10}\chi_{\widetilde{R_j^y}}(x)$. If we write $f_j = \chi_{R_j}$ where R_j an artitrary rectangle with v_j parallel to the longest side of R_j we have that $|(T_jf_j)(x)| \geq \frac{1}{10}$ for every $x\in\tilde{R}_j$, so

$$\int_E\left(\sum_{j=1}^{2^k}|(T_jf_j)(x)|^2\right)dx = \sum_{j=1}^{2^k}\int_E|(T_jf_j)(x)|^2\,dx \geq \sum_{j=1}^{2^k}\int_{E\cap\tilde{R}_j}|(T_jf_j)(x)|^2\,dx \geq \frac{1}{100}\sum_{j=1}^{2^k}\left|E\cap\tilde{R}_j\right|$$
$$\geq \frac{1}{12000}\sum_{j=1}^{2^k}\left|\tilde{R}_j\right| = \frac{1}{6000}\sum_{j=1}^{2^k}|R_j|$$

if the Lemma 5.1 was true, using that $p>2$, $\frac{p}{2}>1$, $\frac{1}{\frac{p}{2}}+\frac{1}{q}=1 \Leftrightarrow \frac{2}{p}+\frac{1}{q}=1 \Leftrightarrow \frac{1}{q}=1-\frac{2}{p}=\frac{p-2}{p} \Leftrightarrow$

$q = \frac{p}{p-2}$, we apply the Hölder inequality:

$$\int_E \left(\sum_{j=1}^{2^k} |(T_j f_j)(x)|^2 \right) dx = \int_{\mathbb{R}^2} \chi_E(x) \left(\sum_{j=1}^{2^k} |(T_j f_j)(x)|^2 \right) dx$$

$$\leq \left(\int_{\mathbb{R}^2} |\chi_E|^q \, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \left(\sum_{j=1}^{2^k} |(T_j f_j)(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} = |E|^{\frac{p-2}{p}} \left\| \left(\sum_{j=1}^{2^k} |(T_j f_j)(x)|^2 \right)^{\frac{1}{2}} \right\|_p^2$$

$$\leq C^2 |E|^{\frac{p-2}{p}} \left\| \left(\sum_{j=1}^{2^k} |f_j(x)|^2 \right)^{\frac{1}{2}} \right\|_p^2 = C^2 |E|^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^2} \left(\sum_{j=1}^{2^k} |f_j(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}}$$

$$\text{as } R_j \cap R_l = \emptyset \;\; if \;\; j,l \text{ are different}, = C^2 |E|^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^2} \left(\chi_{\cup_{j=1}^{2^k} R_j} \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}}$$

$$= C^2 |E|^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^2} \chi_{\cup_{j=1}^{2^k} R_j} dx \right)^{\frac{2}{p}} = C^2 |E|^{\frac{p-2}{p}} \left(\sum_{j=1}^{2^k} \int_{R_j} dx \right)^{\frac{2}{p}}$$

$$= C^2 |E|^{\frac{p-2}{p}} \left(\sum_{j=1}^{2^k} |R_j| \right)^{\frac{2}{p}} \leq C^2 \eta^{\frac{p-2}{p}} \left(\sum_{j=1}^{2^k} |R_j| \right)^{\frac{p-2}{p}} \left(\sum_{j=1}^{2^k} |R_j| \right)^{\frac{2}{p}} = C^2 \eta^{\frac{p-2}{p}} \left(\sum_{j=1}^{2^k} |R_j| \right)$$

for $0 < \eta < \left(\frac{1}{6000C^2}\right)^{\frac{p}{p-2}}$ we have that $C^2 \eta^{\frac{p-2}{p}} < \frac{1}{6000}$ hence $\int_E \left(\sum_{j=1}^{2^k} |(T_j f_j)(x)|^2 \right) dx < \frac{1}{6000} \left(\sum_{j=1}^{2^k} |R_j| \right)$,

which is a contradiction. This complete the proof.

CHAPTER 6

Conclusions and Additional Results

In this chapter we will see two elementary consequences of Fefferman's theorem.

Theorem 6.1. *If* $A \in GL(\mathbb{R}^n)$ *is a self-adjoint operator then* $\chi_{A(B)} \in \mathcal{M}(L^p(\mathbb{R}^n))$ *if and only if* $p = 2$.

Proof. By definition and the theorem of the change of variables we have that if $f \in \mathcal{S}(\mathbb{R}^n)$:

$$(T_A f)(x) = \int_{A(B)} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = \int_B (\widehat{f} \circ A)(\xi) e^{2\pi i x\cdot A\xi} d\xi = |\det(A)| \int_B \widehat{f}(A(\xi)) e^{2\pi i x\cdot A\xi} d\xi$$

but A is self-adjoint $x \cdot A\xi = Ax \cdot \xi$, using change of variables $\widehat{f} \circ A = |\det(A)|^{-1} \widehat{f \circ A^{-1}}$ then

$$(T_A f)(x) = \int_{\mathbb{R}^n} \widehat{f \circ A^{-1}}(\xi) e^{2\pi i x\cdot A\xi} d\xi = T(f \circ A^{-1})(Ax)$$

for every $x \in \mathbb{R}^n$, where T is the operator associate to χ_B, then $T_A f = T(f \circ A^{-1}) \circ A$, this implies that $\|T_A f\|_p^p = |\det(A)|^{-1} \left\|T(f \circ A^{-1})\right\|_p^p$, so $\|T_A f\|_p = |\det(A)|^{-\frac{1}{p}} \left\|T(f \circ A^{-1})\right\|_p$. Let p different of 2, as T is not bounded there exists $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that $\|f_j\|_p = 1$, $\|Tf_j\|_p \geq j$, for every $j \in \mathbb{N}$, let $g_j = |\det(A)|^{\frac{1}{p}} f_j \circ A$, then:

$$\|g_j\|_p^p = |\det(A)| \int_{\mathbb{R}^n} |f_j(Ax)|^p \, dx = |\det(A)| \int_{\mathbb{R}^n} |f_j(x)|^p \left|\det(A^{-1})'(x)\right| dx = \int_{\mathbb{R}^n} |f_j(x)|^p \, dx = \|f_j\|_p^p = 1$$

moreover $\|T_A f\|_p = |\det(A)|^{-\frac{1}{p}} \left\|T(g_j \circ A^{-1})\right\|_p = \|Tf_j\|_p \geq j$, so $\{g_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$, $\|g_j\|_p = 1$, $\|T_A f_j\|_p \geq j$, for every $j \in \mathbb{N}$, so $T_A \notin \mathscr{B}(L^p(\mathbb{R}^n))$ if p is different of 2, the Plancherel theorem implies that $T_A \in \mathscr{B}(L^2(\mathbb{R}^n))$, then $\chi_{A(B)} \in \mathcal{M}(L^p(\mathbb{R}^n))$ if and only if $p = 2$. $\Diamond$

Corollary 6.1. *If $a_1, ..., a_n > 0$ then $\chi_E \notin \mathcal{M}(L^p(\mathbb{R}^n))$ for p different of 2, where*

$$E = \left\{ \xi \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} \frac{\xi_j^2}{a_j^2} < 1 \right\}$$

is an ellipsoid.

Proof. Take the operator A associate to the matrix $\mathbf{A} = diag(a_1, ..., a_n)$ then $A \in GL(\mathbb{R}^n)$ is self-adjoint, note that $A(B) = E$, in fact $y = Ax$, $x \in B \Leftrightarrow y_j = a_j x_j$, $\sum_{j=1}^{n} x_j^2 < 1$, $1 \leq j \leq n \Leftrightarrow \sum_{j=1}^{n} \frac{y_j^2}{a_j^2} < 1 \Leftrightarrow y \in E$, applying the theorem we complete the proof. $\Diamond$

Proposition 6.1. *Let $C = \left\{ \xi \in \mathbb{R}^{n+1} \mid \xi^{(n)} \in B \right\}$ the cylinder, $n \geq 2$ then $\chi_C \in \mathcal{M}(L^p(\mathbb{R}^{n+1}))$ if and only if $p = 2$.*

Proof. Let T_C be the operator associated to χ_C, then for every $f \in C_0^0(\mathbb{R}^{n+1})$:

$$(T_C f)(x) = \int_C \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = \int_{\mathbb{R}} \int_B \widehat{f}(\xi^{(n)}, \xi_{n+1}) e^{2\pi i x^{(n)} \cdot \xi^{(n)}} e^{2\pi i x_{n+1} \xi_{n+1}} d\xi^{(n)} d\xi_{n+1}$$

as $\chi_B \notin \mathcal{M}(L^p(\mathbb{R}^n))$ for p different of 2 there exists $\{f_j\}_{j \in \mathbb{N}} \subset C_0^0(\mathbb{R}^n)$ such that $\|f_j\|_p = 1$, $\|T f_j\|_p \geq j$, for every $j \in \mathbb{N}$, we define $g_j : \mathbb{R}^{n+1} \to \mathbb{R}$, $g_j(x^{(n)}, x_{n+1}) = f_j(x^{(n)}) \chi_{(0,1)}(x_{n+1})$ then $\|g_j\|_{L^p(\mathbb{R}^{n+1})}^p = \int_{\mathbb{R}^{n+1}} |g_j(x)|^p \, dx = \int_{\mathbb{R}^{n+1}} |f_j(x)|^p \left|\chi_{(0,1)}(x_{n+1})\right|^p dx = \int_0^1 dx_{n+1} \int_{\mathbb{R}^n} |f_j(x)|^p \, dx^{(n)} = \|f_j\|_{L^p(\mathbb{R}^n)} = 1$, moreover by the recognized property of the Fourier transform of functions of independent variables $\widehat{g_j}(\xi^{(n)}, \xi_{n+1}) = \widehat{f_j}(\xi^{(n)}) \widehat{\chi_{(0,1)}}(\xi_{n+1})$ then

$$\begin{aligned}
(T_C g_j)(x) &= \int_{\mathbb{R}} \int_B \widehat{g_j}(\xi^{(n)}, \xi_{n+1}) e^{2\pi i x^{(n)} \cdot \xi^{(n)}} e^{2\pi i x_{n+1} \xi_{n+1}} d\xi^{(n)} d\xi_{n+1} \\
&= \left(\int_{\mathbb{R}} \widehat{\chi_{(0,1)}}(\xi_{n+1}) e^{2\pi i x^{n+1} \cdot \xi^{n+1}} d\xi_{n+1} \right) \left(\int_B \widehat{f_j}(\xi^{(n)}) e^{2\pi i x^{(n)} \cdot \xi^{(n)}} d\xi^{(n)} \right) = \chi_{(0,1)}(x_{n+1}) (T f_j)(x^{(n)})
\end{aligned}$$

hence

$$\begin{aligned}
\|T_C g_j\|_{L^p(\mathbb{R}^{n+1})}^p &= \int_{\mathbb{R}^{n+1}} |T_C g_j(x)|^p \, dx = \int_{\mathbb{R}^{n+1}} \left|\chi_{(0,1)}(x_{n+1})\right|^p |(T f_j)(x^n)|^p \, dx \\
&= \left(\int_0^1 dx_{n+1} \right) \left(\int_{\mathbb{R}^n} \left|(T f_j)(x^{(n)})\right|^p dx^{(n)} \right) = \|T f_j\|_{L^p(\mathbb{R}^n)}^p
\end{aligned}$$

then $\{g_j\}_{j \in \mathbb{N}} \subset L^p(\mathbb{R}^n)$, $\|g_j\|_p = 1$, $\|T_C g_j\| \geq j$, for every $j \in \mathbb{N}$, that is $T_C \notin \mathscr{B}(L^p(\mathbb{R}^{n+1}))$ by definition $\chi_C \notin \mathcal{M}(L^p(\mathbb{R}^{n+1}))$ for every p different of 2, $1 < p < \infty$.

On the other hand we have that $\chi_B \in \mathcal{M}(L^2(\mathbb{R}^n))$, note that $\chi_C(x^{(n)}, x_{n+1}) = \chi_B(x^{(n)})$, the theorem 2.4 (extension theorem) implies that $\chi_C \in \mathcal{M}(L^2(\mathbb{R}^{n+1}))$, this completes the proof. $\Diamond$

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