VB-groupoids cocycles and their applications to multiplicative structures

Leandro Ginés Egea
Advisor: Prof. Henrique Bursztyn
Co-Advisor: Prof. Thiago Drummond (UFRJ)
IMPA

June, 2016

Dedicado a
mi familia

Acknowledgements

I wish to express my deep gratitude for my wife, Anita, for her encouragement and constant support during all these years. And a special gratefulness to my daughter Felicitas, who gave to me calmness and motivation in difficult moments with her happiness.

I want to thank my parents, sister and brother who have trusted and supported me all the time. Also, I would like to thank my friends from Argentina who did the distance be not so long.

A special thank to Henrique and Thiago, for the talks, the advices, the suggestions, for their time and constant motivation throughout these years.

Thank to all my friends and colleagues from IMPA, who made me spend a very nice time in Rio.

Abstract

This thesis concerns the study of VB-groupoids and their cocycles (i.e., groupoid cocycles with additional linearity properties). We provide a description of VB-groupoid cocycles at the infinitesimal level, i.e., in terms of the underlying algebroid data. As an application, we use our results in the general study of *multiplicative geometric structures* on Lie groupoids. We pursue the viewpoint that such structures can be regarded as cocycles on suitable VB-groupoids. This approach gives a unifying framework to several multiplicative structures of interest, such as multiplicative forms, multivectors, and subbundles. In fact, with this point of view we treat general multiplicative tensors taking values in VB-groupoids. In particular, we establish a Lie theory for multiplicative tensors with coefficients in a (2-term) representation up to homotopy.

Resumo

A presente tese versa sobre o estudo de VB-grupoides e seus cociclos (ou seja, cociclos em grupoides com propriedades lineares adicionais). Obtemos uma descrição de cociclos em VB-grupoides num nível infinitesimal, ou seja, em termos de dados do algebroide associado. Como uma aplicação, usamos nossos resultados no estudo geral de estruturas geométricas multiplicativas em grupoides de Lie. Seguimos o ponto de vista que tais estruturas podem ser vistas como cociclos num VB-grupoide apropiado. Esta abordagem permite colocar no mesmo contexto varias das estruturas multiplicativas de interesse, tais como formas diferenciais, multivectores, subfibrados multiplicativos, e tratar-las conjuntamente de maneira uniforme. De fato, com este ponto de vista, tratamos tensores multiplicativos gerais tomando valores em VB-grupoides. Em particular, estabelecemos uma teoria de Lie para tensores multiplicativos com coeficientes numa representação a menos de homotopia (2-termos)

Contents

Introduction

In recent years Lie groupoids have played an increasing role in several areas of mathematics, such as non-commutative geometry [16], foliations [21, 35], singular spaces [23, 34], Poisson geometry [12, 13, 41], etc. Just as Lie groups have Lie algebras as their infinitesimal counterparts, the infinitesimal version of a Lie groupoid is a Lie algebroid, and the Lie theory relating them is very rich (see e.g. [11]); in particular, in contrast with Lie algebras, there are nontrivial obstructions to the integration of Lie algebroids [13].

An area where Lie groupoids have become a central tool is Poisson geometry. Any Poisson manifold is naturally associated with a Lie algebroid structure on its cotangent bundle, and the Lie groupoids arising in this context come equipped with compatible symplectic forms, known as *multiplicative* [41]. These *symplectic groupoids* naturally arise in the study of symmetries as well as in quantization problems, see e.g. [4]. On the other hand, multiplicative Poisson structures on Lie groups define the so-called *Poisson-Lie groups*, which are semi-classical limits of quantum groups.

The study of symplectic groupoids, Poisson-Lie groups, or more general Poisson groupoids [42] is the starting point for considering *multiplicative geometric structures* on a Lie groupoid $\mathcal{G}$, that is, geometric structures on $\mathcal{G}$ that are *compatible* with the multiplication on $\mathcal{G}$, and their description in terms of infinitesimal data obtained from the Lie algebroid $A = Lie(\mathcal{G})$ of $\mathcal{G}$. The simplest multiplicative structures are multiplicative functions, which are just *cocycles* on the Lie groupoid. Other types of multiplicative structures have drawn much attention in recent years, including multiplicative differential forms [7, 14], multivector fields [24, 33], distributions and foliations [22, 27], complex structures [28], etc. In all these contexts, a central issue is always finding the infinitesimal description of, and proving an integration theorem for, the multiplicative object. This infinitesimal-global correspondence recovers some classical results, such as the correspondence between Lie bialgebras and Poisson-Lie groups, Poisson structures and symplectic groupoids, and many others.

This thesis fits into this general program, bringing in an additional feature: the study of multiplicative tensors, such as forms and multivectors, with *values in vector bundles, or representations*. Here "vector bundles" and "representations" should be understood in the realm of Lie groupoids. While the representation of a Lie group realizes it as the automorphisms of a vector space, Lie groupoids are naturally represented on vector bundles: each arrow of the groupoid is sent to an isomorphism between fibers of the vector bundle. A difficulty in the theory is that, while Lie groups have natural adjoint and coadjoint representations, Lie groupoids do not. And in order to make sense of such natural objects, one is forced to go a step further and consider *representations up to homotopy* [1, 2].

Representations up to homotopy of Lie groupoids and Lie algebroids generalize their representations on vector bundles to representations on graded vector bundles, or complexes. In fact, one can make sense of the (co-)adjoint representation as a 2-term representation up to homotopy. Recently, it has been shown in [19, 20] that 2-term representations up to homotopy can be geometrically described by certain double structures, known as VB-groupoids (VB-algebroids), which are (categorified) *vector-bundle objects* in the contexts of Lie groupoids (algebroids). Prototypical examples are the tangent and cotangent bundles of a Lie groupoid (algebroid). More precisely, a *VB-groupoid* is a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r] & M \end{tikzcd}$$

where the vertical sides are Lie groupoids, the horizontal sides are vector bundles and the two structures on $\mathcal{E}$ are compatible in a suitable way. A VB-algebroid is defined similarly. It is proven in [19, 20] that there is a one-to-one correspondence between isomorphism classes of VB-groupoids $(\mathcal{E}, E; \mathcal{G}, M)$ (resp. VB-algebroids $(\mathcal{A}, E; A, M)$) and isomorphism classes of 2-term representations up to homotopy of $\mathcal{G}$ (resp. A) on $C_{[0]} \oplus E_{[1]}$, where C is the so-called *core bundle*.

The Lie theory underlying VB-algebroids and VB-groupoids has been recently explained in [5, 8]. Building on these results and using the infinitesimal-global correspondence for cocycles on general VB-groupoids developed in the first part of this thesis, we prove a general infinitesimal-global correspondence for multiplicative structures unifying the cases previously known, and including new ones.

One of the initial motivations for this thesis was the work [14], where the authors consider the notion of *multiplicative forms on a Lie groupoid* $\mathcal{G} \rightrightarrows M$ *with coefficients in an (ordinary) representation* $E \longrightarrow M$, and described their infinitesimal counterpart, the so-called *E-valued Spencer operators*. The kernels of multiplicative 1-form with coefficients give rise to *multiplicative distribution* on $\mathcal{G}$, that is, subbundles of $T\mathcal{G}$ which are Lie subgroupoids of the tangent groupoid $T\mathcal{G} \rightrightarrows TM$, with the particularity that the base manifold is the whole space TM. In trying to extend this viewpoint to general multiplicative distributions (i.e., those covering a subbundle of TM, as considered in [17, 27]), one realizes that one should consider (2-term) representations up to homotopy as coefficients.

In order to make sense of multiplicative forms with these more general coefficients, it is convenient to have the following interpretation of the usual multiplicativity condition, which extends the approach in [7] for the case of trivial coefficients. A representation $E \longrightarrow M$ of a Lie groupoid $\mathcal{G}$ has an associated VB-groupoid $\mathbf{s}^*E \rightrightarrows E$ over $\mathcal{G}$ with *trivial core bundle* (the *action VB-groupoid*), where $\mathbf{t}$ is the target map of $\mathcal{G}$ (see [19] or Subsection 1.3.1 below). Then a k-form $\omega \in \Omega^k(\mathcal{G}, \mathbf{s}^*E)$ with coefficients in E is multiplicative (as in [14]) if and only if the natural map

$$\omega : \bigoplus_k T\mathcal{G} \longrightarrow \mathbf{s}^*E,$$

defined by evaluation, is a morphism of Lie groupoids, where $\bigoplus_k T\mathcal{G}$ is the Whitney sum of the tangent groupoid over $\mathcal{G}$ (see Subsection 3.3). This viewpoint can now be extended by replacing $\mathbf{s}^*E$ by a general VB-groupoid, in particular those arising from representations up to homotopy.

Actually, it is a general fact that multiplicative geometric structures on a Lie groupoid can be viewed as a morphism of appropriate Lie groupoids. This observation goes back to [33], where it is shown that a Poisson structure $\pi \in \Gamma(\wedge^2 T\mathcal{G})$ on a Lie groupoid is multiplicative if and only if the induced map $\pi^\sharp : T^*\mathcal{G} \longrightarrow T\mathcal{G}$ is a morphism of Lie groupoids. This viewpoint is further explored in [7, 9, 10]. Moreover, since these geometric structures have some *linear conditions*, the morphisms which determine them are actually morphism between VB-groupoids satisfying additional linear properties. And since we can *dualize* VB-groupoids, those morphisms can be seen as *cocycles* in suitable VB-groupoids. With this in mind, we can extend the discussion to more general geometric structures and allow them to have general coefficients.

Given a VB-groupoid $\mathcal{E}$, the central idea we pursue is then to view an $\mathcal{E}$-valued multiplicative (p, q)-tensor on a Lie groupoid $\mathcal{G}$,

$$\tau \in \Gamma(\bigotimes^p T^*\mathcal{G} \otimes \bigotimes^q T\mathcal{G} \otimes \mathcal{E}),$$

as a cocycle

$$c_\tau : (\oplus_p T\mathcal{G}) \oplus (\oplus_q T^*\mathcal{G}) \oplus \mathcal{E}^* \longrightarrow \mathbb{R},$$

defined on the Whitney sums of its tangent and cotangent bundles and the dual of $\mathcal{E}$, making use of tangent and cotangent groupoid structures (Definition 3.2).

This viewpoint has some nice features:

- It allows us to work with functions (cocycles) instead of more complicated structures. Moreover, the characterization of groupoid cocycles in terms of infinitesimal data is simple.

- It allows us to put many geometric structures of interest in a common framework.

- It provides simpler proofs even of known results, besides giving new ones.

An important particular case of multiplicative tensors with coefficients are differential forms with values in a representation up to homotopy $\omega \in \Omega^q(\mathcal{G}, \mathcal{E})$, where $\mathcal{E}$ is the VB-groupoid associated with the representation. In this thesis, we obtain an infinitesimal-global correspondence for these objects extending the result in [14], which we recover when $\mathcal{E}$ is an ordinary representation (with a different proof). Moreover, when $q = 1$ the kernel $\ker(\omega)$ is a (general) multiplicative subbundle, and we recover the description of multiplicative tangent distributions and foliations in [14, 17, 27]; we also use our methods to treat multiplicative Dirac structures [36], giving an alternative viewpoint to results in [25,26]. Our methods are also well suited for other classes of multiplicative structures, such as Nijenhuis structures on Courant algebroids, that we plan to explore in the future.

Outline of the Thesis. This thesis is organized as follows.

In **Chapter 1** we recall the definitions of Lie groupoids, and Lie algebroids, VB-groupoids and VB-algebroids, as well as representations up to homotopy. We recall some basic results for these objects which are relevant for subsequent chapters.

Chapter 2 is dedicated to the study of cocycles on VB-groupoids and on VB-algebroids. We provide a description of cocycles on a VB-groupoid, satisfying additional linearity conditions, in terms of VB-algebroid data and prove an infinitesimal-global correspondence:

$$\begin{tikzcd} \text{Cocycles on VB-groupoids} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] \arrow[dr] & \text{Cocycles on VB-algebroids} \arrow[d, rightsquigarrow] \\ \text{Characterization} \arrow[r, rightsquigarrow] & \text{Infinitesimal description} \end{tikzcd}$$

In order to simplify the discussion, we first treat *linear* cocycles, and then *bilinear* cocycles in detail, and finally we extend the results to *multilinear* cocycles. The main results in this chapter are:

- **Theorem 2.15**, which says that *linear cocycles* on a VB-groupoid $\mathcal{E} \rightrightarrows E$ with core bundle C over a source-simply-connected Lie groupoid $\mathcal{G}$ are in natural one-to-one correspondence with pairs $(\mathbf{D}, \sigma)$, where $\mathbf{D} : \Gamma_{\text{lin}}(A_{\mathcal{E}}, E) \longrightarrow \Gamma(E^*)$ is a $C^\infty(M)$-linear operator defined on the space of linear section of the Lie algebroid $A_{\mathcal{E}}$ of $\mathcal{E}$, σ is an element in $\Gamma(C^*)$, satisfying suitable compatibility conditions.

- **Theorem 2.30**, which states that *bilinear cocycles* on a VB-groupoid $\mathcal{E}_1 \oplus \mathcal{E}_2$ over a source-simply-connected Lie groupoid $\mathcal{G}$ are in natural one-to-one correspondence with triples $(\mathbf{D}, \sigma_1, \sigma_2)$, where
  - $\mathbf{D} : \Gamma_{\text{lin}}(A_{\mathcal{E}_1}, E_1) \times_{\Gamma(A)} \Gamma_{\text{lin}}(A_{\mathcal{E}_2}, E_2) \longrightarrow \Gamma(E_1^* \otimes E_2^*)$ is a $C^\infty(M)$-linear operator,

– $\sigma_1 : C_1 \longrightarrow E_2^*$ and $\sigma_2 : C_2 \longrightarrow E_1^*$ are vector bundle maps,

satisfying suitable compatibility conditions.

In **Chapter 3** we apply the theory of VB-groupoid cocycles to study multiplicative (p,q)-tensor fields on a Lie groupoid $\mathcal{G}$ with coefficients on a VB-groupoid $\mathcal{E}$. The componentwise linear function $c_\tau \in C^\infty((\oplus_p T\mathcal{G}) \oplus (\oplus_q T^*\mathcal{G}) \oplus \mathcal{E}^*)$ associated to τ will be a multilinear cocycle. As an example we describe multiplicative forms with coefficients in a representation up to homotopy.

Our main theorem is:

- **Theorem 3.17**, which says that multiplicative $\mathcal{E}$-valued (p,q)-tensors on a source-simply-connected Lie groupoid $\mathcal{G}$ are in one-to-one correspondence with quadruples $(\mathbf{D}, l, r, \sigma)$ which define $A_\mathcal{E}$-valued (p,q)-tensors on A, see Def. 3.14 for an explicit description of the infinitesimal objects.

For the trivial representation, this recovers the infinitesimal description of multiplicative differential forms, multivector fields etc, as in [10]. This result can be specialized to the case of differential forms with coefficients on representations up to homotopy, leading to:

- **Theorem 3.25**, which establishes a one-to-one correspondence between multiplicative k-forms on a source-simply-connected Lie groupoid with values in a representation up to homotopy and triples $(\mathbb{D}, l, \theta)$, where
  – $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C)$ satisfies a derivation rule,
  – $l : A \longrightarrow \wedge^{k-1} T^*M \otimes C$ is a linear map,
  – $\theta \in \Omega^k(M, E)$,

  satisfying suitable compatibility equations (see (3.34)–(3.39)).

For ordinary representations, this recovers the result in [14].

In **Chapter 4**, we use the techniques developed to treat (p,q)-tensor fields with coefficients to consider *VB-subgroupoids*, or multiplicative subbundles. We consider subbundles arising as kernels of linear and bilinear cocycles, and then we apply the theory of Chapter 2. We describe the particular cases of tangent distributions and subbundles of $T\mathcal{G} \oplus T^*\mathcal{G}$. We use this approach to study multiplicative Dirac structures, recovering results about their infinitesimal description from a new viewpoint. We work out explicitly the examples of multiplicative presymplectic form and multiplicative Poisson structures. The main result here is

- **Theorem 4.26**, which says that, for a VB-groupoid $\mathcal{E} \rightrightarrows E$ with core bundle C, over a source-simply-connected Lie groupoid $\mathcal{G} \rightrightarrows M$, there is a natural one-to-one correspondence between VB-subgroupoids $\mathcal{H} \rightrightarrows H$ of $\mathcal{E}$, with core bundle K, and operators $\mathcal{D} : \Gamma_{\text{lin}}(A_\mathcal{E}, E) \longrightarrow \Gamma(H^* \otimes C/K)$ satisfying compatibility conditions described by Equations (4.11)-(4.14).

Chapter 1

Background

In this chapter we recall some definitions and results which will be useful throughout this thesis, and we fix some notation. For Lie groupoids and Lie algebroids we follow mostly [13], for VB-groupoids, VB-algebroids [19, 20] and for representation up to homotopy we rely on [1, 2, 19, 20]. These references contain the proofs of results omitted here.

1.1 Lie groupoids and Lie algebroids

A *Lie groupoid* consists of two manifolds, $\mathcal{G}$ and M, called the space of arrows and the space of objects, respectively, together with the following structure maps

- surjective submersions $\mathbf{s}, \mathbf{t} : \mathcal{G} \longrightarrow M$, called *source* and *target* maps,
- a smooth map $\mathbf{m} : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$, called the *multiplication*, where

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \ : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

  is the space of *composable* arrows,
- smooth maps, $1 : M \longrightarrow \mathcal{G}$ and $\iota : \mathcal{G} \longrightarrow \mathcal{G}$, the *unit* map and the *inversion* map,

satisfying suitable compatibility conditions: $\mathcal{G}$ is a category over M where all morphisms are invertible. We denote a Lie groupoid by $\mathcal{G} \rightrightarrows M$.

Example 1.1. Trivial examples. These first examples show how Lie groupoids are generalizations of different spaces. A Lie group G is a Lie groupoid over a point. Given a manifold M, we can see it as the Lie groupoid $M \rightrightarrows M$ where the source and target maps are the identity. Finally, every vector bundle $\pi : E \longrightarrow M$ is a Lie groupoid, where $\mathbf{s} = \mathbf{t} = \pi$, and the multiplication is the addition on the fibers.

Example 1.2. The pair groupoid. Let M be a manifold. The direct product $M \times M$ is a Lie groupoid over M, where the source map is the projection in the first component, and the target map is the projection on the second component. The multiplication is $(y, z)(x, y) = (x, z)$. This Lie groupoid is called *the pair groupoid*.

Example 1.3. General linear groupoid. Let $E \longrightarrow M$ be a vector bundle. The *general linear groupoid* of E is the Lie groupoid whose arrows are all linear isomorphisms $E_x \longrightarrow E_y$, for $x, y \in M$, and the space of objects is M. Given an isomorphism $g : E_x \longrightarrow E_y$, the source is $\mathbf{s}(g) = x$ and the target is $\mathbf{t}(g) = y$. The multiplication is given by composition of linear maps. We denote this Lie groupoid by $\mathcal{G}(E)$. This is a generalization of the general linear group $GL(V)$ for a vector space V. For details of the smooth structure of $\mathcal{G}(E)$ see e.g. [32], Chapter 1, Example 1.1.12.

Example 1.4. The fundamental groupoid. Assume that M is a connected manifold. The *fundamental groupoid* of M, denoted by $\Pi_1(M) \rightrightarrows M$, consists of homotopy classes of paths with fixed end points. The multiplication is the concatenation of paths.

Example 1.5. The action groupoid. Let $G \times M \longrightarrow M$ be an action of a Lie group on a manifold M. The *action groupoid* over M has as space of arrows the direct product $G \times M$. For $(g, x), (h, y) \in G \times M$, the source, target and multiplication are

$$\mathbf{s}(g, x) = x, \qquad \mathbf{t}(g, x) = g \cdot x, \qquad (g, x)(h, y) = (gh, y).$$

Definition 1.6. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be two Lie groupoids. A *morphism* of Lie groupoids between $\mathcal{G}$ and $\mathcal{H}$ consists of maps $F : \mathcal{G} \longrightarrow \mathcal{H}$ and $f : M \longrightarrow N$ which are compatible with all the structure maps, i.e., a morphism of Lie groupoids is a **functor**.

A *Lie algebroid* is a vector bundle $A \longrightarrow M$ together with a vector bundle map $\rho : A \longrightarrow TM$, called the *anchor map*, and a Lie bracket on the space of sections $\Gamma(A)$ such that the following Leibniz rule holds:

$$[a, fb] = f[a, b] + \mathcal{L}_{\rho(a)} fb$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$.

Example 1.7. A Lie algebra $\mathfrak{g}$ is a Lie algebroid where $M = \{*\}$ is a point. The tangent bundle $TM \longrightarrow M$ of a manifold M is a Lie algebroid, where the Lie bracket is the Lie bracket of vector fields and the anchor map is the identity $\mathrm{Id} : TM \longrightarrow TM$.

Example 1.8. Poisson Manifolds. Let (M, π) be a Poisson manifold. Its cotangent bundle $T^*M \longrightarrow M$ inherits a natural Lie algebroid structure, where the anchor map $\pi^\sharp : T^*M \longrightarrow TM$ is $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$, and the Lie bracket is determined by the Poisson bracket: $[\mathrm{d}f, \mathrm{d}g] := \mathrm{d}\{f, g\}$ together with a Leibniz rule.

Example 1.9. Let $\mathcal{F} \subseteq TM$ be an involutive distribution on M, i.e. a (smooth) linear subbundle of TM whose space of sections is closed under the usual bracket of vector fields. Then $\mathcal{F} \longrightarrow M$ is a Lie algebroid with bracket the restriction of Lie brackets of vector fields, and where the anchor map is the inclusion $\iota : \mathcal{F} \longrightarrow TM$.

Example 1.10. The Lie algebroid of a Lie groupoid. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with **s, t** the source and target maps. Let $A := \mathrm{Ker}(T\mathbf{s})\big|_M$. Since the source map is a surjective submersion, A is a vector bundle over M. The anchor map $\rho : A \longrightarrow TM$ is defined by $\rho(a_p) = (T\mathbf{t})_p a \in T_pM$. Similarly to the Lie algebra case, one can identify the space of sections of A with vector fields on $\mathcal{G}$ which are invariant by right multiplication (see e.g. [13]):

$$\overrightarrow{a}_g = TR_g(a_{\mathbf{t}(g)}) \qquad \text{for } a \in \Gamma(A),$$

where TR_g is the differential map of the right multiplication by $g \in \mathcal{G}$. Under this identification, the Lie bracket on $\Gamma(A)$ is determined by

$$\overrightarrow{[a,b]} := [\overrightarrow{a}, \overrightarrow{b}]_{\mathfrak{X}(\mathcal{G})}.$$

1.2 Multiplicative functions

Our approach to study multiplicative structures on a Lie groupoid $\mathcal{G}$ is to regard them as multiplicative functions in some Lie groupoid $\mathbb{G}$. We recall here a characterization of this kind of functions in terms of an infinitesimal data, and their relation with cohomology of Lie groupoids and Lie algebroids.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, a function $F \in C^\infty(\mathcal{G})$ is called *multiplicative* if

$$F(g_1 \cdot g_2) = F(g_1) + F(g_2) \tag{1.1}$$

for all composable pair $(g_1, g_2) \in \mathcal{G}^{(2)}$.

An immediate consequence of the definition is that

$$F|_M = 0. \tag{1.2}$$

For a function $F : \mathcal{G} \longrightarrow \mathbb{R}$ its *infinitesimal counterpart* is the differential of F restricted to the Lie algebroid A:

$$AF : A \longrightarrow \mathbb{R}, \qquad \langle AF, a\rangle = \mathrm{d}F(a) \quad \text{for all } a \in A.$$

The following result characterizes multiplicative functions and its proof can be found in [10]. We use this proposition in Chapter 2 to describe infinitesimally cocycles in VB-groupoids.

Proposition 1.11. *Let F be a smooth function in a Lie groupoid $\mathcal{G} \rightrightarrows M$. Let A be the Lie algebroid of $\mathcal{G}$. If F is multiplicative then*

$$\mathcal{L}_{\overrightarrow{a}} F = \boldsymbol{t}^* \langle AF, a\rangle \qquad \forall\ a \in \Gamma(A). \tag{1.3}$$

Moreover if a function $F \in C^\infty(\mathcal{G})$ satisfies (1.3) and (1.2), and $\mathcal{G}$ is source connected then F is a multiplicative function.

Example 1.12. Every function $f \in C^\infty(M)$ defines a multiplicative function on $\mathcal{G}$ as follows

$$F := \mathbf{t}^* f - \mathbf{s}^* f.$$

Since F is multiplicative, the infinitesimal counterpart satisfies

$$\mathbf{t}^* \langle AF, a\rangle = \mathcal{L}_{\overrightarrow{a}}(\mathbf{t}^* f - \mathbf{s}^* f) = \mathbf{t}^* \mathcal{L}_{\rho(a)} f,$$

for all $a \in \Gamma(A)$, where $\rho : A \longrightarrow TM$ is the anchor map. Note that we used

$$\mathcal{L}_{\overrightarrow{a}}(\mathbf{s}^* f)(p) = \frac{d}{dr}|_{r=0} f(\mathbf{s}(\varphi^a_r(p))) = 0$$

because $\mathbf{s}(\varphi^a_r(p)) = p$, where φ^a_r is the flow of the vector field of $\overrightarrow{a}$. Therefore

$$\langle AF, a\rangle = \mathcal{L}_{\rho(a)} f.$$

When $\mathcal{G} = M \times M \rightrightarrows M$ is the pair groupoid, functions of the form

$$F(x, y) = f(x) - f(y) \quad \text{for } f \in C^\infty(M)$$

are always multiplicative.

1.2.1 Cohomology of Lie groupoids and Lie algebroids

Given a Lie groupoid $\mathcal{G}$, denote by $\mathcal{G}^{(k)}$, $k > 0$, the space of k-composable arrows:

$$\mathcal{G}^{(k)} = \{(g_1, \dots, g_k) \in \mathcal{G}^k \ : \ \mathbf{s}(g_i) = \mathbf{t}(g_{i+1}),\ i = 1, \dots, k-1\},$$

and $\mathcal{G}^{(0)} = M$. The space of smooth groupoid k-cochains is $C^k(\mathcal{G}) = C^\infty(\mathcal{G}^{(k)})$. There is a coboundary operator $\delta : C^\bullet(\mathcal{G}) \longrightarrow C^{\bullet+1}(\mathcal{G})$ defined as follows: for $k = 0$

$$\delta(f)(g) = f(\mathbf{s}(g)) - f(\mathbf{t}(g)),$$

and for $k > 0$

$$\begin{aligned}(\delta f)(g_0, \dots, g_k) \quad = \quad & f(g_1, \dots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \dots, g_{i-1}g_i, \dots, g_k) \\ & + (-1)^{k+1} f(g_0, \dots, g_{k-1})\end{aligned} \tag{1.4}$$

where $f \in C^k(\mathcal{G})$. The coboundary operator satisfies $\delta^2 = 0$, and the cohomology of the complex $(C^\bullet(\mathcal{G}), \delta)$ is referred to as *cohomology* of the Lie groupoid.

Remark 1.13. Note that for $F \in C^\infty(\mathcal{G})$ the coboundary operator is

$$(\delta F)(g_1, g_2) = F(g_2) - F(g_1 g_2) + F(g_1).$$

Then F is multiplicative if and only if $\delta F = 0$, i.e., F is a *cocycle* on $\mathcal{G}$.

Given a Lie algebroid A, let

$$d_A : \Omega^k(A) = \Gamma(\wedge^k A^*) \longrightarrow \Omega(A)^{k+1}$$

be the Lie algebroid differential associated to the complex $\Omega(A) = \Gamma(\wedge A^*)$ given by the Koszul formula

$$\begin{aligned} d_A(\omega)(a_1, \dots, a_{k+1}) &= \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_{k+1}) \\ &\quad + \sum_i (-1)^{i+1} \mathcal{L}_{\rho(a_i)} \omega(a_1, \dots, \widehat{a_i}, \dots, a_{k+1}). \end{aligned} \tag{1.5}$$

This operator satisfies $d_A^2 = 0$ and the following derivation rule

$$d_A(\omega\eta) = d_A(\omega)\eta + (-1)^k \omega d_A(\eta)$$

for $\omega \in \Omega^k(A)$ and $\eta \in \Omega^{k+1}(A)$. The cohomology of the complex $(\Omega(A), d_A)$ is called the *cohomology* of the Lie algebroid A.

Remark 1.14. A function $f \in C^\infty(A)$ is a *morphism* of Lie algebroid or a *cocycle* if

$$d_A(f)(X, Y) := f([X, Y]) - \mathcal{L}_{\rho(X)} f(Y) + \mathcal{L}_{\rho(Y)} f(X) = 0$$

for all sections $X, Y \in \Gamma(A)$.

1.3 VB-groupoids and representations up to homotopy

Representations up to homotopy of Lie groupoids generalize their representations on vector bundles to representations on graded vector bundles, or complexes. It has been shown in [19] that 2-term representations up to homotopy can be geometrically described by certain double structures, known as VB-groupoids, which are the natural *vector-bundle objects* in the context of Lie groupoids. Prototypical examples are the tangent and cotangent groupoids. We recall here briefly VB-groupoids, representations up to homotopy and their relation. We follow mostly [19].

1.3.1 VB-groupoids

A *VB-groupoid* ([37]) is a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "Q"] \arrow[d, shift right, "\overline{s}"'] \arrow[d, shift left, "\overline{t}"] & \mathcal{G} \arrow[d, shift right, "\mathbf{s}"'] \arrow[d, shift left, "\mathbf{t}"] \\ E \arrow[r, "q_E"] & M \end{tikzcd} \tag{1.6}$$

such that the vertical sides are Lie groupoids, the horizontal sides are vector bundles and the two structures on $\mathcal{E}$ are compatible:

- $\overline{s}, \overline{t} : \mathcal{E} \longrightarrow E$ are vector bundle morphisms over $s, t : \mathcal{G} \longrightarrow M$, respectively.
- $Q : \mathcal{E} \longrightarrow \mathcal{G}$ is a Lie groupoid morphism over $q_E : E \longrightarrow M$.
- For all $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathcal{E}$ such that $Q(\eta_1) = Q(\eta_3)$, $Q(\eta_2) = Q(\eta_4)$ and (η_1, η_2) and (η_3, η_4) composable pair, then

  $$(\eta_1 + \eta_3) \cdot (\eta_2 + \eta_4) = \eta_1 \cdot \eta_2 + \eta_3 \cdot \eta_4.$$

  This is known as *the interchange law.*

Remark 1.15. There is an equivalent definition using actions. An $(\mathbb{R}, \cdot)$-action h on the space of arrows of a Lie groupoid $\mathcal{E} \rightrightarrows E$ is called *multiplicative* if $h_\lambda : \mathcal{E} \longrightarrow \mathcal{E}$ is a morphism of Lie groupoids for every $\lambda \in \mathbb{R}$ (see [8]). Bursztyn, Cabrera, and del Hoyo proved in [8] that there is a one-to-one correspondence between VB-groupoids and regular multiplicative actions.

Given a VB-groupoid there is a canonical vector bundle over M, called *(right-) core bundle* which is defined as follows:

$$C = 1^*(Ker(\overline{s}))$$

where $1 : M \longrightarrow \mathcal{G}$ is the unit map. The core is important in the structure of a VB-groupoid: there is a short exact sequence of vector bundles over $\mathcal{G}$, called the *(right-) core exact sequence*,

$$0 \longrightarrow \mathbf{t}^*C \longrightarrow \mathcal{E} \longrightarrow \mathbf{s}^*E \longrightarrow 0 \tag{1.7}$$

where the maps are $\mathbf{t}^*C \longrightarrow \mathcal{E}$, $(g, c) \longrightarrow c \cdot 0_g$, and $\mathcal{E} \longrightarrow \mathbf{s}^*E$, $\eta \longrightarrow (Q(\eta), \overline{s}(\eta))$, and any splitting of this sequence induces a decomposition of the VB-groupoid as $\mathcal{E} = \mathbf{t}^*C \oplus \mathbf{s}^*E$ (see [19]).

We give now some important examples of VB-groupoids that we will use later. We do this in detail because most of the applications we present in this thesis are related with these VB-groupoids.

Example 1.16. Tangent groupoid. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ with structure maps $(\mathbf{s}, \mathbf{t}, \mathbf{m}, 1, \iota)$, it induces a Lie groupoid structure on $T\mathcal{G}$ over TM where the structure maps are $\mathrm{T}\mathbf{s}$, $\mathrm{T}\mathbf{t}$, $\mathrm{T}m$, $\mathrm{T}\epsilon$ and $\mathrm{T}\iota$, the differentials of the structure maps of $\mathcal{G}$. Explicitly, the multiplication $\overline{m} = \mathrm{T}\mathbf{m} : T(\mathcal{G}^{(2)}) \simeq T\mathcal{G}^{(2)} \longrightarrow T\mathcal{G}$ is given as follows: for $X \in T_g(\mathcal{G}_{\mathbf{s}(g)})$, $Y \in T_h(\mathcal{G}_{\mathbf{t}(h)})$ we have (see [32] § 1.1)

$$\overline{m}_{(g,h)}(X, Y) = (\mathrm{T}R_h)_g(X) + (\mathrm{T}L_g)_h(Y) \tag{1.8}$$

where $(g, h) \in \mathcal{G}^{(2)}$, R_h and L_g are the right and left multiplication, respectively, and where

$$\mathcal{G}_{\mathbf{s}(g)} = \{\widetilde{g} \in \mathcal{G} \ / \ \mathbf{s}(\widetilde{g}) = \mathbf{s}(g)\} \quad \text{and} \quad \mathcal{G}_{\mathbf{t}(h)} = \{\widetilde{h} \in \mathcal{G} \ / \ \mathbf{t}(\widetilde{h}) = \mathbf{t}(h)\}.$$

The Lie groupoid $T\mathcal{G} \rightrightarrows TM$ is called *tangent groupoid* of $\mathcal{G}$. Moreover, considering the linear structure of the tangent bundle,

$$\begin{tikzcd} T\mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ TM \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid with core bundle A, and we refer to it as the *tangent VB-groupoid.*

Remark 1.17. Using Formula (1.8), a right invariant vector field $\overrightarrow{a} \in \mathfrak{X}(\mathcal{G})$, with $a \in \Gamma(A)$ can be write as

$$\overrightarrow{a}(g) = a(\mathbf{t}(g)) \cdot 0_g \tag{1.9}$$

where the multiplication is in the tangent groupoid.

Example 1.18. Cotangent groupoid. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let A its Lie algebroid. Denote by A^* the dual vector bundle of A with respect to M. We will define a Lie groupoid structure on $T^*\mathcal{G} \rightrightarrows A^*$. The source and target maps $\widetilde{s}, \widetilde{t} : T^*\mathcal{G} \longrightarrow A^*$ are characterized by:

$$\langle \widetilde{s}(\eta), a \rangle = -\langle \eta, 0_g \cdot T\iota(a) \rangle \quad \text{and} \quad \langle \widetilde{t}(\eta), b \rangle = \langle \eta, b \cdot 0_g \rangle,$$

for $g \in \mathcal{G}$, $\eta \in T^*_g\mathcal{G}$, $a \in A_{\mathbf{s}(g)}$ and $b \in A_{\mathbf{t}(g)}$. In particular, for a section $a \in \Gamma(A)$ we have

$$\langle \widetilde{s}(\eta), a_{(\mathbf{s}(g))} \rangle = -\langle \eta, \overleftarrow{a}_g \rangle \quad \text{and} \quad \langle \widetilde{t}(\eta), a_{(\mathbf{t}(g))} \rangle = \langle \eta, \overrightarrow{a}_g \rangle.$$

The multiplication is given by:

$$\langle \eta \cdot \mu, X \cdot Y \rangle = \langle \eta, X \rangle + \langle Y, \mu \rangle, \tag{1.10}$$

for $\eta \in T^*_g\mathcal{G}$, $\mu \in T^*_h\mathcal{G}$ with $\widetilde{s}(\eta) = \widetilde{t}(\mu)$, and where $X \in T_g\mathcal{G}$, $Y \in T_h\mathcal{G}$ are composable vectors. With this structure maps $T^*\mathcal{G} \rightrightarrows A^*$ is Lie groupoid, called the *cotangent groupoid.* Moreover

$$\begin{tikzcd} T^*\mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ A^* \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid, where the vector bundle structures are the usual ones, and where its core bundle is T^*M. We call it the *cotangent VB-groupoid*.

Another important type of example comes from representations of Lie groupoids. Recall that a *representation* of a Lie groupoid $\mathcal{G}$ is a vector bundle $E \longrightarrow M$ such that for every $g : x \longrightarrow y \in \mathcal{G}$ there is a linear isomorphism $g\cdot : E_x \longrightarrow E_y$ satisfying $g \cdot (h \cdot e) = (gh) \cdot e$ for composable arrows, and $x\cdot = Id\big|_{E_x}$, for every unit $x \in M$. Note that when $\mathcal{G} = G$ is a Lie group, E is a vector space, and a representation of this Lie groupoid is indeed a usual representation of a Lie group on the vector space E. Moreover a representation of Lie groups on a vector space V can be seen as a morphism of Lie groups $G \longrightarrow GL(V)$. Is straightforward to check that in the case of Lie groupoids, a representation on $E \longrightarrow M$ is equivalent to the existence of a Lie groupoid morphism $\Delta : \mathcal{G} \longrightarrow \mathcal{G}(E)$. We denote $(E \longrightarrow M, \Delta)$ a representation of $\mathcal{G}$ on the vector bundle E.

Example 1.19. Action groupoid associated to a representation. Let $E \longrightarrow M, \Delta)$ be a representation of $\mathcal{G}$. Take now the pull back of E by the source map

$$\mathbf{s}^*(E) = \{(e, g) \in E \times \mathcal{G} : \; e \in E_{\mathbf{s}(g)}\}.$$

There is a Lie groupoid structure on $\mathbf{s}^*(E)$ over E, where the structure maps are

$$\begin{aligned} \overline{s}(e, g) &= e \\ \overline{t}(e, g) &= g \cdot e \\ (g_1, g_2 \cdot e_2)(g_2, e_2) &= (g_1 g_2, e_2). \end{aligned}$$

This Lie groupoid is denoted by $E * \mathcal{G} \rightrightarrows E$ and is called the *action groupoid for the representation.* Moreover

$$\begin{tikzcd} E * \mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid with zero core bundle, called the *action VB-groupoid.*

As an example of this VB-groupoid, consider an orbit $\mathcal{O} \subseteq M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$, i.e.,

$$\mathcal{O} = M/\mathcal{G} = \{\mathcal{O}_x \; : \; x \in M\}$$

where $\mathcal{O}_x = \{\mathbf{t}(g) \; : \; g \in \mathbf{s}^{-1}(x)\}$ is the orbit through x. Restringing $\mathcal{G}$ to $\mathcal{O}$ we get a Lie groupoid $\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O}$. The *normal bundle* is the quotient

$$N_{\mathcal{G}_{\mathcal{O}}} = \frac{T\mathcal{G}}{T\mathcal{G}_{\mathcal{O}}}$$

and it is a Lie groupoid over $N_{\mathcal{O}} = TM/T\mathcal{O}$. This defines a VB-groupoid

$$\begin{tikzcd} N_{\mathcal{G}_{\mathcal{O}}} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}_{\mathcal{O}} \arrow[d, shift left] \arrow[d, shift right] \\ N_{\mathcal{O}} \arrow[r] & \mathcal{O} \end{tikzcd}$$

with core bundle $C = 0$. That means that this VB-groupoid comes from a representation de $\mathcal{G}_{\mathcal{O}}$ on $N_{\mathcal{O}}$. This is the *normal representation*, which appears in the theorem of linearization [15, 43, 45].

Example 1.20. Semi-direct product. Let $(C \longrightarrow M, \Delta)$ be a representation of $\mathcal{G}$. Consider the vector bundle over $\mathcal{G}$ given by the pull back of C by the target map

$$\mathbf{t}^*(C) = \{(c, g) \in C \times \mathcal{G} \ : \ c \in C_{\mathbf{t}(g)}\}.$$

There is a Lie groupoid structure on $\mathbf{t}^*(C)$ over M: the source, target and multiplication are

$$\begin{aligned} \widetilde{s}(c, g) &= \mathbf{s}(g) \\ \widetilde{t}(c, g) &= \mathbf{t}(g) \\ (c_1, g_1) \cdot (c_2, g_2) &= (c_1 + \Delta_{g_1} c_2, g_1 g_2). \end{aligned}$$

This Lie groupoid is called *semi-direct product of C and $\mathcal{G}$* and it is denoted by $C \rtimes \mathcal{G} \rightrightarrows M$. Moreover, the semi-direct product is a VB-groupoid

$$\begin{tikzcd} C \rtimes \mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r] & M \end{tikzcd}$$

with core bundle C, and we call it the *semi-direct product VB-groupoid.*

Example 1.21. Dual of a VB-groupoid. (See [19], §4 for more details.) Consider a VB-groupoid as (1.6). Let $\mathcal{E}^*$ be the dual vector bundle of $\mathcal{E}$ over $\mathcal{G}$. There is a Lie groupoid structure on $\mathcal{E}^* \rightrightarrows C^*$. The source and target map $\widehat{s}, \widehat{t}$ are determined by:

$$\langle \widehat{s}(\epsilon), c \rangle = -\langle \epsilon, 0_g \cdot c^{-1} \rangle \quad \text{and} \quad \langle \widehat{t}(\epsilon), d \rangle = \langle \epsilon, d \cdot 0_g \rangle,$$

where $\epsilon \in \mathcal{E}^*_g$, $c \in C_{\mathbf{s}(g)}$ and $d \in C_{\mathbf{t}(g)}$. The multiplication is: for $(\epsilon_1, \epsilon_2) \in (\mathcal{E}^*)^{(2)}$ where $\epsilon_i \in \mathcal{E}^*_{g_i}$,

$$\langle \epsilon_1 \cdot \epsilon_2, \eta_1 \cdot \eta_2 \rangle = \langle \epsilon_1, \eta_1 \rangle + \langle \epsilon_2, \eta_2 \rangle,$$

with $\eta_i \in \mathcal{E}_{g_i}$. With these structure maps and the usual vector bundle structures $\mathcal{E}^* \longrightarrow \mathcal{G}$ and $C^* \longrightarrow M$,

$$\begin{tikzcd} \mathcal{E}^* \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ C^* \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid with core bundle E^*.

Remark 1.22. The cotangent VB-groupoid is the dual of the tangent VB-groupoid. Also, semi-direct products and action groupoids are in duality. That means that the dual VB-groupoid of the semi-direct product VB-groupoid $C \rtimes \mathcal{G}$ is the action VB-groupoid $C^* * \mathcal{G}$. Conversely, the dual of the action VB-groupoid $E * \mathcal{G}$ is the semi-direct product VB-groupoid $E^* \rtimes \mathcal{G}$ (see [19]).

The next examples are related with operations in the category of VB-groupoid over a fix Lie groupoids $\mathcal{G} \rightrightarrows M$. They are important because we will use them in Chapters 3 and 4. Recall that the structure maps of the Lie groupoid $\mathcal{E} \rightrightarrows E$ are denoted by $(\overline{s}, \overline{t}, \overline{m}, \overline{\iota}, \overline{\epsilon})$.

Example 1.23. Sum. Let $\mathcal{E}_i \longrightarrow E_i$ be two VB-groupoids over $\mathcal{G} \longrightarrow M$ with core bundles C_i. The direct product $\mathcal{E}_1 \times \mathcal{E}_2$ is a Lie groupoid over $E_1 \times E_2$ where the structure maps are component to component. Also, since $\mathcal{E}_1 \times \mathcal{E}_2$ is a vector bundle over $\mathcal{G}$,

$$\begin{tikzcd} \mathcal{E}_1 \oplus \mathcal{E}_2 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ E_1 \oplus E_2 \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid over $\mathcal{G}$, where the core bundle is $C_1 \oplus C_2 \longrightarrow M$.

Example 1.24. VB-subgroupoid. Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G}$ with core bundle C. Recall that a *Lie subgroupoid* of $\mathcal{E}$ is a pair $(\mathcal{H}, i)$ where $\mathcal{H}$ is a Lie groupoid and $i : \mathcal{H} \longrightarrow \mathcal{E}$ is an injective Lie groupoid morphism ([32]). A *VB-subgroupoid* is a VB-groupoid

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ H \arrow[r] & M \end{tikzcd}$$

such that $\mathcal{H} \rightrightarrows H$ is a Lie subgroupoid of $\mathcal{E} \rightrightarrows E$ and $\mathcal{H}$ is a vector subbundle of $\mathcal{E}$ over $\mathcal{G}$. Note that, in particular, the side bundle H is a vector subbundle of E, and the core bundle K is a vector subbundle of C.

Example 1.25. Quotient. Let $\mathcal{H} \rightrightarrows H$ be a VB-subgroupoid of $\mathcal{E} \rightrightarrows E$, and consider the vector bundle $\mathcal{E}/\mathcal{H}$ over $\mathcal{G}$. There exists a Lie groupoid structure on $\mathcal{E}/\mathcal{H}$ over E/H. The source map $\widehat{s} : \mathcal{E}/\mathcal{H} \longrightarrow E/H$ is: for $[\eta] \in \mathcal{E}/\mathcal{H}$,

$$\widehat{s}([\eta]) := [\overline{s}(\eta)].$$

This map is well defined. Indeed, if $\eta' \in [\eta]$ there exists an element $\tau \in \mathcal{H}$ such that $\eta' = \eta +_{\mathcal{G}} \tau$. Then

$$\overline{s}(\eta') = \overline{s}(\eta +_{\mathcal{G}} \tau) = \overline{s}(\eta) + \overline{s}(\tau),$$

because $\overline{s}$ is linear, and since $\mathcal{H}$ is a subgroupoid, $\overline{s}(\tau) \in H$, which means that $[\overline{s}(\eta')] = [\overline{s}(\eta)]$. In the same way, we define the target map $\widehat{t} : \mathcal{E}/\mathcal{H} \longrightarrow E/H$. Let now $[\eta], [\mu] \in \mathcal{E}/\mathcal{H}$ such that $\widehat{s}([\eta]) = \widehat{t}([\mu])$.

Lemma 1.26. *There exist elements $\eta' \in [\eta]$ and $\mu' \in [\mu]$ such that $(\eta', \mu') \in \mathcal{E}^{(2)}$. Moreover, if $\widetilde{\eta} \in [\eta]$ and $\widetilde{\mu} \in [\mu]$ are also composable, then $[\eta' \cdot \mu'] = [\widetilde{\eta} \cdot \widetilde{\mu}]$.*

Proof. Since $\widehat{s}([\eta]) = \widehat{t}([\mu])$ follows that there is $e \in H$ such that $\overline{s}(\eta) = \overline{t}(\mu) +_E e$. Since $\overline{t}|_{\mathcal{H}}$ is onto over H, there exist $\tau \in \mathcal{H}$ such that $\overline{t}(\tau) = e$. By linearity of $\overline{t}$, we have $\overline{s}(\eta) = \overline{t}(\mu + \tau)$. Hence $\mu' = \mu + \tau \in [\mu]$ and $(\eta, \mu') \in \mathcal{E}^{(2)}$. Suppose now that $\widetilde{\eta} \in [\eta]$ and $\widetilde{\mu} \in [\mu]$ are also composable. There exist $\tau_1, \tau_2 \in \mathcal{H}$ such that

$$\widetilde{\eta} = \eta + \tau_1, \qquad \widetilde{\mu} = \mu' + \tau_2.$$

In particular we have that (τ_1, τ_2) is a composable pair:

$$\overline{s}(\widetilde{\eta}) = \overline{t}(\widetilde{\mu}) \Longrightarrow \overline{s}(\eta) + \overline{s}(\tau_1) = \overline{t}(\mu') + \overline{t}(\tau_2) \Longrightarrow \overline{s}(\tau_1) = \overline{t}(\tau_2)$$

because $\overline{s}(\eta) = \overline{t}(\mu')$. Then by the interchange law

$$\widetilde{\mu} \cdot \widetilde{\mu} = (\eta + \tau_1) \cdot (\mu' + \tau_2) = \eta \cdot \mu' + \tau_1 \cdot \tau_2,$$

with $\tau_1 \cdot \tau_2 \in \mathcal{H}$. Therefore $[\eta \cdot \mu'] = [\widetilde{\eta} \cdot \widetilde{\mu}]$. $\square$

Hence the multiplication $[\eta] \cdot [\mu] := [\eta \cdot \mu]$ is well defined. The unit map $\widehat{1} : E/H \longrightarrow \mathcal{E}/\mathcal{H}$ and the inversion map $\widehat{\iota} : \mathcal{E}/\mathcal{H} \longrightarrow \mathcal{E}/\mathcal{H}$ are

$$\widehat{1}([e]) = [\overline{1}(e)], \quad \text{and} \quad \widehat{\iota}([\eta]) = [\overline{\iota}(\eta)].$$

Therefore $\mathcal{E}/\mathcal{H} \rightrightarrows E/H$ is a Lie groupoid. And moreover

$$\begin{tikzcd} \mathcal{E}/\mathcal{H} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ E/H \arrow[r] & M \end{tikzcd}$$

is a VB-groupoid with core bundle C/K.

1.3.2 Representations up to homotopy

Even when representations of Lie groupoids are natural extensions of those of Lie groups, they are too restrictive, in the sense that not always exists a such representation in a given vector bundle, and because there is not a good definition of adjoint representation. The adjoint representation is an important object in the study of the cohomology of classifying spaces, and in the Lie algebroid setting, for deformation cohomology. To solve this situation, Arias-Abad and Crainic [1], introduced the concept of representation up to homotopy, where Lie groupoids are represented in a complex of vector bundles, rather than in vector bundles. These allows more flexibility, for example, there is not a requirement for the associativity condition.

In this thesis we only use representation up to homotopy on a 2-term graded vector bundle $C_{[0]} \oplus E_{[1]}$ over M, where C sits in degree zero, and E in degree one. To define representations up to homotopy, we recall an equivalent definition for representations of Lie groupoids on a vector bundle, which allows a natural extension to the graded case. We also give an equivalent definition of representations up to homotopy which is more conceptual. We follow [1] and [19] for this part.

Let $E \longrightarrow M$ be a vector bundle over M. Define the map $\pi_0^k : \mathcal{G}^{(k)} \longrightarrow M$ by $\pi_0^k(g_1, \ldots, g_k) = \mathbf{t}(g_1)$, for $k > 0$, and the identity for $k = 0$. Consider now the following vector bundle over $\mathcal{G}^{(k)}$:

$$\begin{tikzcd} (\pi_0^k)^*(E) \arrow[d] & E \arrow[d] \\ \mathcal{G}^{(k)} \arrow[r, "\pi_0^k"'] & M, \end{tikzcd}$$

and denote by

$$C^k(\mathcal{G}, E) = \Gamma((\pi_0^k)^* E).$$

its space of sections. The space $C^\bullet(\mathcal{G}, E) = \oplus_k C^k(\mathcal{G}, E)$ has a right graded module structure over $C^\bullet(\mathcal{G})$: for $\eta \in C^k(\mathcal{G}, E)$ and $f \in C^l(\mathcal{G}) = C^\infty(\mathcal{G}^{(l)})$

$$(\eta \star f)(g_1, \ldots, g_{k+l}) = (-1)^{kl} \eta(g_1, \ldots, g_k) f(g_{k+1}, \ldots, g_{k+l}).$$

Theorem 1.27. [19] *There is a one-to-one correspondence between representations of a Lie groupoid $\mathcal{G}$ on a vector bundle $E \longrightarrow M$ and degree one operators*

$$D : C^\bullet(\mathcal{G}, E) \longrightarrow C^{\bullet+1}(\mathcal{G}, E)$$

such that

$$D(\eta \star f) = D(\omega) \star f + (-1)^{|\eta|} \eta \star \delta(f),$$

preserve normalized functions, and $D^2 = 0$, where δ is the groupoid coboundary operator, see (1.4).

Proof. (Sketch) Given a representation $(E \longrightarrow M, \Delta)$ define an operator

$$D : C^0(\mathcal{G}, E) = \Gamma(E) \longrightarrow C^1(\mathcal{G}, E) = \Gamma(\mathbf{t}^* E)$$

as follows

$$D(e)(g) := g \cdot e_{\mathbf{s}(g)} - e_{\mathbf{s}(g)},$$

for $e \in \Gamma(E)$, and then extend by Leibniz rule to higher degrees. Conversely, given an operator $D : C^\bullet(\mathcal{G}, E) \longrightarrow C^{\bullet+1}(\mathcal{G}, E)$, define the representation by:

$$g : E_{\mathbf{s}(g)} \longrightarrow E_{\mathbf{t}(g)} \quad g \cdot e_{\mathbf{s}(g)} = D(e)(g) + e_{\mathbf{s}(g)} \qquad \text{for } e \in \Gamma(E)$$

The property $D^2 = 0$ is equivalent to the flatness condition of the representation. □

This point of view to representations can be naturally extended to the graded case. Let $V = C_{[0]} \oplus E_{[1]}$ be a graded vector bundle over M, with C sits in degree zero and E in degree one. Consider $C(\mathcal{G}, V)$ to be a graded (right) $C^\bullet(\mathcal{G})$-module with respect to the total grading

$$C(\mathcal{G}, V)^k = \Gamma((\pi_0^k)^* C) \oplus \Gamma((\pi_0^{k-1})^* E)$$

Definition 1.28. A *representation up to homotopy* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a 2-term-graded vector bundle $V = C_{[0]} \oplus E_{[1]}$ over M together with a degree one operator

$$D : C(\mathcal{G}, V)^\bullet \longrightarrow C(\mathcal{G}, V)^{\bullet+1}$$

such that

$$D(\omega \star f) = D(\omega) \star f + (-1)^{|\omega|} \omega \star \delta(f),$$

where δ is the coboundary operator, $D^2 = 0$ and preserves normalized functions.

For a more conceptual interpretation, remember that we want to represent a Lie groupoid in a complex $\partial : C \longrightarrow E$. For an element $g \in \mathcal{G}$ we expect that it acts as a map of complexes

$$g : V^\bullet_{\mathbf{s}(g)} \longrightarrow V^\bullet_{\mathbf{t}(g)}.$$

That means we would have linear maps $g^0 : C_{\mathbf{s}(g)} \longrightarrow C_{\mathbf{t}(g)}$ and $g^1 : E_{\mathbf{s}(g)} \longrightarrow E_{\mathbf{t}(g)}$ which commute with ∂:

$$g^1 \circ \partial = \partial \circ g^0.$$

Also, since there is not a requirement about associativity, if (g, h) is a composable pair, $g^0 \circ h^0$ and $(gh)^0$ do not need to be equal; and the same with $g^1 \circ h^1$ and $(gh)^1$. We summarize this in the following theorem, which is proven in [1] for a general representation up to homotopy, and in [19] for the particular case of representation up to homotopy on a 2-term graded vector bundle.

Theorem 1.29. *A representation up to homotopy of $\mathcal{G}$ on a 2-term graded vector bundle $V = C_{[0]} \oplus E_{[1]}$ over M is equivalent to a quadruple $(\Delta^0, \Delta^1, \partial, \Omega)$, where*

- *Δ^0 and Δ^1 are unital quasi-actions of $\mathcal{G} \rightrightarrows M$ on C and E, respectively. That means for each $g \in \mathcal{G}$ we have linear maps (not necessary invertible) $\Delta^0_g : C_{\boldsymbol{s}(g)} \longrightarrow C_{\boldsymbol{t}(g)}$ and $\Delta^1_g : E_{\boldsymbol{s}(g)} \longrightarrow E_{\boldsymbol{t}(g)}$ (quasi-action), and for all $p \in M$, the linear map $\Delta^0_p : C_p \longrightarrow C_p$ and $\Delta^1_p : E_p \longrightarrow E_p$ are the identity (unital).*
- *$\partial : C \longrightarrow E$ is a vector bundle map over the identity of M*
- *$\Omega \in C^2(\mathcal{G}; E \longrightarrow C) = \Gamma(\mathcal{G}^{(2)}, Hom(s^*(E), t^*(C)))$ is a normalized operator*

satisfying the following equations:

$$\Delta^1_g \partial - \partial \Delta^0_g = 0, \tag{1.11}$$

$$\Delta^0_{g_1} \Delta^0_{g_2} - \Delta^0_{g_1 g_2} + \Omega_{g_1, g_2} \partial = 0, \tag{1.12}$$

$$\Delta^1_{g_1} \Delta^1_{g_2} - \Delta^1_{g_1 g_2} + \partial \Omega_{g_1, g_2} = 0, \tag{1.13}$$

$$\Delta^0_{g_1} \Omega_{g_2, g_3} - \Omega_{g_1 g_2, g_3} + \Omega_{g_1, g_2 g_3} - \Omega_{g_1, g_2} \Delta^1_{g_3} = 0 \tag{1.14}$$

for $(g_1, g_2, g_3) \in \mathcal{G}^{(3)}$.

Remark 1.30. When we say *representation up to homotopy*, we mean the quadruple $(\Delta^0, \Delta^1, \partial, \Omega)$.

1.3.3 Representations up to homotopy vs VB-groupoids

There is an equivalence between classes of representations up to homotopy and classes of VB-groupoids. We show here, briefly this correspondence.

Let $(\Delta^0, \Delta^1, \partial, \Omega)$ be a representation up to homotopy of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on $C_{[0]} \oplus E_{[1]}$. We will associate to this data a VB-groupoid. Consider the vector bundle over $\mathcal{G}$, given by $\mathbf{t}^*C \oplus \mathbf{s}^*E$. This space has a Lie groupoid structure over E: for $(g, c, e),\ (g_i, c_i, e_i) \in \mathbf{t}^*C \oplus \mathbf{s}^*E,\ i = 1, 2$ we define

$$\begin{aligned} \widetilde{s}(g, c, e) &= e \\ \widetilde{t}(g, c, e) &= \partial c + \Delta^1_g e \\ (g_1, c_1, e_1) \cdot (g_2, c_2, e_2) &= (g_1 g_2, c_1 + \Delta^0_{g_1} c_2 - \Omega_{g_1,g_2} e_2, e_2), \end{aligned}$$

when $e_1 = \partial c_2 + \Delta^1_{g_2} e_2$. Conversely, if $\mathcal{E} \rightrightarrows E$ is a VB-groupoid over $\mathcal{G}$ with core bundle C, we can associate a representation up to homotopy. Let $h : \mathbf{s}^*E \longrightarrow \mathcal{E}$ be a horizontal lift, i.e., a section of the short exact sequence (1.7) covering the identity of $\mathcal{G}$ such that

$$h(e, 1_x) = \widetilde{1}_e.$$

This horizontal lift always exists [19]. Then we define the quadruple $(\Delta^0, \Delta^1, \partial, \Omega)$ by

$$\begin{aligned} \partial c &= \widetilde{t}(c) \\ \Delta^0_g c &= h_g(\widetilde{t}(c)) \cdot c \cdot \widetilde{0}_{g^{-1}} \\ \Delta^1_g e &= \widetilde{t}(h_g(e)) \\ \Omega_{g_1,g_2} e &= (h_{g_1 g_2}(e) - h_{g_1}(\widetilde{t}(h_{g_2}(e))) \cdot h_{g_2}(e)) \cdot \widetilde{0}_{(g_1 g_2)^{-1}}. \end{aligned}$$

Remark 1.31. The isomorphism class of the VB-groupoid associated to a representation up to homotopy is independent of the choice of the horizontal lift (see [19]).

The previous correspondence between 2-terms representations up to homotopy of $\mathcal{G} \rightrightarrows M$ and VB-groupoids over $\mathcal{G}$ together with a horizontal lift is one-to-one, and it is proved in [19]. Moreover they also proved that there is a one-to-one correspondence between **isomorphism classes of VB-groupoids** over $\mathcal{G}$ and **isomorphism classes of 2-terms representation up to homotopy** of $\mathcal{G}$.

1.4 VB-algebroids and representation up to homotopy

As in the Lie groupoid case, representations up to homotopy of Lie algebroids generalize their representations on vector bundles to representations on graded vector bundles or complexes. The particular case of 2-terms representations up to homotopy can be geometrically described by VB-algebroids, which are the natural *vector-bundle* *objects* in the context of Lie algebroids. In this section we recall the definitions, some properties and the relation between VB-algebroids and representations up to homotopy. The material of this part can be found in [20].

1.4.1 VB-algebroids

Consider a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Psi"] \arrow[d, "Q"'] & A \arrow[d, "q"] \\ E \arrow[r, "\psi"] & M \end{tikzcd} \tag{1.15}$$

where all sides are vector bundles.

Definition 1.32. A commutative square (1.15) is called a *double vector bundle* if the two linear structures in $\mathcal{A}$ are compatible. Explicitly, if the following conditions hold:

1. $Q(\eta_1 +_A \eta_2) = Q(\eta_1) + Q(\eta_2)$ for $\eta_1, \eta_2 \in \mathcal{A}_a$ for some $a \in A$.

2. $\Psi(\eta +_E \mu) = \Psi(\eta) + \Psi(\mu)$ for $\eta, \mu \in \mathcal{A}_e$ for some $e \in E$.

3. For $\eta_1, \eta_2, \mu_1, \mu_2 \in \mathcal{A}$ such that $Q(\eta_i) = Q(\mu_i),\ i = 1, 2$ and $\Psi(\eta_1) = \Psi(\eta_2)$ and $\Psi(\mu_1) = \Psi(\mu_2)$ we have

$$(\eta_1 +_E \mu_1) +_A (\eta_2 +_E \mu_2) = (\eta_1 +_A \eta_2) +_E (\mu_1 +_A \mu_2). \tag{1.16}$$

   This last equation is called *the interchange law*.

Remark 1.33. A double vector bundle is just a VB-groupoid, where the groupoid structures $\mathcal{A} \rightrightarrows E$ and $A \rightrightarrows M$ are as in Example 1.1 (vector bundles as Lie groupoids).

Remark 1.34. We also write a double vector bundle (1.15) as $(\mathcal{A}, E; A, M)$, and we say that $\mathcal{A} \longrightarrow E$ is a double vector bundle over $A \longrightarrow M$. An element $\eta \in \mathcal{A}$ will be written as

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & a \arrow[d] \\ e \arrow[r] & p \end{tikzcd}$$

or $(\eta, e; a, p)$, where $e = Q(\eta) \in E$ and $a = \Psi(\eta) \in A$.

Remark 1.35. We follow the definition of double vector bundle of [20]. Nevertheless, there is an equivalent definition in [18] in terms of scalar multiplication.

There is a canonical vector bundle over M associated to a double vector bundle: the *core bundle* $C \longrightarrow M$ is the intersection of the kernels of the projections $Q : \mathcal{A} \longrightarrow E$ and $\Psi : \mathcal{A} \longrightarrow A$:

$$c \in C \quad \Longleftrightarrow \quad \begin{tikzcd} c \arrow[r] \arrow[d] & 0_p \arrow[d] \\ 0_p \arrow[r] & p \end{tikzcd} .$$

To every section $c \in \Gamma(C)$ of the core bundle we associate a section $S_c : E \longrightarrow \mathcal{A}$ of E over $\mathcal{A}$ defined by:

$$S_c(e) := 0_e +_A \bar{c} = \begin{tikzcd} 0_e \arrow[r] \arrow[d] & 0 \arrow[d] \\ e \arrow[r] & p \end{tikzcd} + \begin{tikzcd} c \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & p \end{tikzcd} \tag{1.17}$$

where $\bar{c}$ is the element c viewed as an element of $\mathcal{A}$. This kind of section is called *core section*. We denote by $\Gamma_{\mathrm{cor}}(\mathcal{A}, E)$ the space of core sections.

Now we give some initial examples.

Example 1.36. The prolonged tangent bundle. Let $q : A \longrightarrow M$ be a vector bundle then its tangent groupoid (seeing the vector bundle as a Lie groupoid)

$$\begin{tikzcd} TA \arrow[r, "\pi"] \arrow[d, "Tq"'] & A \arrow[d, "q"] \\ TM \arrow[r, "\pi"] & M. \end{tikzcd} \tag{1.18}$$

is a double vector bundle with core bundle A, called *the prolonged tangent bundle*. Explicitly, the vertical linear structure $TA \longrightarrow TM$ is defined as follows: for a curve $\gamma(t) \subseteq A$, consider the vector $\frac{d}{dt}\big|_{t=0}\gamma(t) \in T_{\gamma(0)}A$, so

$$Tq(\frac{d}{dt}\big|_{t=0}\gamma(t)) = \frac{d}{dt}\big|_{t=0}q(\gamma(t)) \quad \in T_{q(\gamma(0))}M,$$

then

$$\frac{d}{dt}\big|_{t=0}\gamma(t) +_{TM} \frac{d}{dt}\big|_{t=0}\sigma(t) = \frac{d}{dt}\big|_{t=0}(\gamma(t) + \sigma(t))$$

for $\gamma(t),\ \sigma(t) \subseteq A$ with $\gamma(0) = \sigma(0)$.

Example 1.37. Trivial double vector bundle. Let $q_A : A \longrightarrow M$, $q_C : C \longrightarrow M$ and $q_E : E \longrightarrow M$ be vector bundles over M and consider $\mathcal{A} = A \oplus C \oplus E$. We define a double vector bundle structure on $\mathcal{A}$ over A and over E as follows:

$$\begin{aligned} (a, c_1, e_1) +_A (a, c_2, e_2) &= (a, c_1 + c_2, e_1 + e_2) \quad \text{over } A \\ (a_1, c_1, e) +_E (a_2, c_2, e) &= (a_1 + a_2, c_1 + c_2, e) \quad \text{over } E, \end{aligned}$$

i.e., the linear structure over A is $q_A^{*}*(E\oplus C) \longrightarrow A$, and over E is $q_E^*(A\oplus C) \longrightarrow E$. We call $\mathcal{A}$ the *trivial double vector bundle* with side bundles E and A, and core bundle C.

Example 1.38. Dual of a double vector bundle. Let $(\mathcal{A}, E; M, A)$ be a double vector bundle. Take the dual of $\mathcal{A}$ over A, which we denote by $\mathcal{A}^*$. We will show that

$$\begin{tikzcd} \mathcal{A}^* \arrow[r] \arrow[d] & A \arrow[d] \\ C^* \arrow[r] & M \end{tikzcd}$$

has a double vector bundle structure with core bundle E^*. For a complete description and more details see [32]. The horizontal structure $\mathcal{A}^* \longrightarrow A$ is the usual one. With respect to the vertical side, the projection $q^* : \mathcal{A}^* \longrightarrow C^*$ is defined by

$$\langle q^*(\eta), c\rangle = \langle \eta, 0_a +_E \overline{c}\rangle$$

where $c \in C_p$, $\eta \in \mathcal{A}^*_a$, and $a \in A_p$. The addition in $\mathcal{A}^* \longrightarrow C^*$ is defined by:

$$\langle \eta_1 +_{C^*} \eta_2, d\rangle = \langle \eta_1, d_1\rangle + \langle \eta_2, d_2\rangle$$

for $(\eta_1, \xi, a_1, p), (\eta_2, \xi, a_2, p) \in \mathcal{A}^*_\xi$, and where $d = d_1 +_E d_2$.

Example 1.39. Cotangent double vector bundle. Let $A \longrightarrow M$ be a vector bundle. The *cotangent double vector bundle* associated to A is obtained by dualization of its prolonged tangent bundle $(TA, TM; A, M)$:

$$\begin{tikzcd} T^*A \arrow[r] \arrow[d] & A \arrow[d] \\ A^* \arrow[r] & M \end{tikzcd} \tag{1.19}$$

with core bundle $T^*M \longrightarrow M$.

There is another type of sections on a double vector bundles, called *linear*.

Definition 1.40. A *linear section* of E over $\mathcal{A}$ is a section $\chi : E \longrightarrow A$ which is a vector bundle morphism over some section $a : M \longrightarrow A$. The space of linear sections is denoted by $\Gamma_{\mathrm{lin}}(\mathcal{A}, E)$.

The following proposition can be found in [31].

Proposition 1.41. *The space of sections of $\mathcal{A}$ over E is a $C^\infty(E)$-module generated by $\Gamma_{lin}(\mathcal{A}, E)$ and $\Gamma_{cor}(\mathcal{A}, E)$. Moreover if $(\mathcal{A}_i, E_i; A, M)$ are k double vector bundles over A with core bundles C_i, then the space of sections of $\mathcal{A} = \oplus_{i=1}^k \mathcal{A}_i$ over $E = \oplus_{i=1}^k E_i$ is generated by*

- $(X_1, \dots, X_k)$, *where* $X_i \in \Gamma_{lin}(\mathcal{A}_i, E_i)$ *are linear sections, all of them covering the same section* $a \in \Gamma(A)$, *and*
- $S^i(\xi_i) = (0, \dots, \underbrace{S_{\xi_i}}_{i}, \dots, 0)$ *with* $\xi_i \in \Gamma(C_i)$, *for* $i = 1, \dots, k$.

The core bundle has an important role in the structure of a double vector bundle: there is a canonical short exact sequence of sections of vector bundles over M,

$$0 \longrightarrow \Gamma(Hom(E, C)) \longrightarrow \Gamma_{\mathrm{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(A) \longrightarrow 0 \tag{1.20}$$

where for $T \in \Gamma(Hom(E, C))$ we associate the linear section covering the zero section of $\Gamma(A)$ given by

$$S_T(e) = 0_e +_A \overline{T(e)}.$$

The space of linear sections of a double vector bundle is isomorphic to the space of sections of some vector bundle $\widehat{A}$ over M (see [20]). Hence this short exact sequence is $C^\infty(M)$-linear, which means that (1.20) is an exact sequence is of vector bundles over M. There exists always a splitting of this exact sequence to level of vector bundles, and any splitting of this sequence gives a *decomposition* of the double vector bundle as $\mathcal{A} = A \oplus C \oplus E$ (see [17, 20]).

Remark 1.42. In some examples there exists a canonical splitting which is not $C^\infty(M)$-linear, like the case when $\mathcal{A} = TA$, the prolonged tangent bundle: the core exact sequence of vector bundles is

$$0 \longrightarrow T^*M \otimes A \longrightarrow J^1A \longrightarrow A \longrightarrow 0,$$

and at level of section it is

$$0 \longrightarrow \Omega^1(M) \otimes \Gamma(A) \longrightarrow \Gamma(J^1A) \longrightarrow \Gamma(A) \longrightarrow 0,$$

which has a canonical splitting $j^1 : \Gamma(A) \longrightarrow \Gamma(J^1A)$ which is not $C^\infty(M)$-linear. Here J^1A is the first jet bundle associated to A.

Definition 1.43. A *morphism* of double vector bundles between $(\mathcal{A}_i, E_i; A_i, M_i)$, $i = 1, 2$ is a quadruple $(F, F_{ver}; F_{hor}, f)$ of maps such that in the following diagram are all vector bundle maps

$$\begin{tikzcd}
& A_1 \arrow[rr, "F_{hor}"] \arrow[dd] & & A_2 \arrow[dd] \\
\mathcal{A}_1 \arrow[ur] \arrow[rr, "F"] \arrow[dd] & & \mathcal{A}_2 \arrow[ur] \arrow[dd] & \\
& M_1 \arrow[rr, "f"] & & M_2 \\
E_1 \arrow[ur] \arrow[rr, "F_{ver}"] & & E_2 \arrow[ur] &
\end{tikzcd}$$

Definition 1.44. A VB-algebroid is a double vector bundle

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Psi"] \arrow[d, "Q"'] & A \arrow[d, "q"] \\ E \arrow[r, "\psi"] & M \end{tikzcd}$$

where the $\mathcal{A} \longrightarrow E$ is a Lie algebroid, and the following compatibility conditions hold: the anchor map $\rho_{\mathcal{A}} : \mathcal{A} \longrightarrow TE$ is a vector bundle morphism over a map $\rho : A \longrightarrow TM$ and the Lie bracket satisfies

- $[\Gamma_{\mathrm{lin}}(\mathcal{A}, E), \Gamma_{\mathrm{lin}}(\mathcal{A}, E)] \subseteq \Gamma_{\mathrm{lin}}(\mathcal{A}, E)$
- $[\Gamma_{\mathrm{lin}}(\mathcal{A}, E), \Gamma_{\mathrm{cor}}(\mathcal{A}, E)] \subseteq \Gamma_{\mathrm{cor}}(\mathcal{A}, E)$
- $[\Gamma_{\mathrm{cor}}(\mathcal{A}, E), \Gamma_{\mathrm{cor}}(\mathcal{A}, E)] = 0.$

Remark 1.45. In [8] it is proved an equivalent definition of VB-algebroids in terms of scalar multiplication.

As a consequence of the definition, follows that the vector bundle $A \longrightarrow M$ inherits a Lie algebroid structure with anchor map ρ and Lie bracket given by

$$[a, b] := \Psi([\chi_a, \chi_b])$$

where $\chi_a, \chi_b \in \Gamma_{lin}(\mathcal{A}, E)$ are any linear sections covering a and b, respectively.

The space of linear sections of a VB-algebroid which is isomorphic to the space of sections of some vector bundle $\widehat{A}$ over M, has a Lie algebroid structure over M: its Lie bracket and anchor map are the restriction of the structure of $\mathcal{A}$ to $\Gamma_{lin}(\mathcal{A}, E)$. This Lie algebroid $\widehat{A} \longrightarrow M$ has canonical representations on E^* and C:

- $(\nabla^1)^* : \Gamma_{\mathrm{lin}}(\mathcal{A}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$ such that $(\nabla^1)^*_X$ is the derivation in E^* corresponding to the vector field $\rho_{\mathcal{A}}(X)$, that is

  $$\ell_{(\nabla^1)^*_X(\varphi)} = \rho_{\mathcal{A}}(X)(\ell_\varphi), \quad \varphi \in \Gamma(E^*) \tag{1.21}$$

  where $\ell_\varphi \in C^\infty(E)$ is the linear function defined by $\ell_\varphi(e_p) = \langle \varphi_p, e_p \rangle$.
- $\nabla^0 : \Gamma_{\mathrm{lin}}(\mathcal{A}, E) \times \Gamma(C) \longrightarrow \Gamma(C)$ characterized by

  $$S_{\nabla_X \zeta} = [X, S_\zeta]. \tag{1.22}$$

There is a canonical vector bundle map $\partial : C \longrightarrow E$ defined as follows: since the anchor map $\rho_{\mathcal{A}} : \mathcal{A} \longrightarrow TE$ is a morphism of double vector bundle from $(\mathcal{A}, E; A, M)$ to $(TE, TM; E, M)$, it restricts to the core bundles, hence

$$\partial := \rho_{\mathcal{A}}\big|_C : C \longrightarrow E. \tag{1.23}$$

We call ∂ by *core anchor map*.

Example 1.46. Tangent algebroid. Let $(A \longrightarrow M, \rho, [\cdot\ ,\cdot])$ be a Lie algebroid and consider the prolonged tangent bundle $(TA, TM; A, M)$ (Example 1.36). We will define a Lie algebroid structure over $TA \longrightarrow TM$. For linear sections $Ta, Tb \in \Gamma_{\text{lin}}(TA, TM)$ and core sections $S_c, S_d \in \Gamma_{\text{cor}}(TA, TM)$ with $a, b, c, d \in \Gamma(A)$ we define:

- $[Ta, Tb] = T[a, b]$
- $[Ta, S_c] = S_{[a,b]}$
- $[S_c, S_d] = 0.$

The anchor map $\rho_{TA} : TA \longrightarrow T(TM)$ is

- $\rho_{TA}(Ta)$ is the linear vector in $\mathfrak{X}(TM)$ corresponding to the usual derivation $\mathcal{L}_{\rho(a)} \in \mathrm{Der}(\Omega(M))$ of 1-forms.
- $\rho_{TA}(S_c) = \rho(c)^{\uparrow}$ is the vertical vector field corresponding to the section $\rho(c)$, i.e., the vector field given by

$$\rho(c)^{\uparrow}(X_p) := \frac{d}{dt}\Big|_{t=0}(X_p + t\rho(c)(p)).$$

Example 1.47. Cotangent algebroid. The *cotangent algebroid* is the vector bundle $T^*A \longrightarrow A^*$, where the anchor $\rho^* : T^*A \longrightarrow TA^*$ is determined by

- $\rho^*(R_a) = H_a$ is the *Hamiltonian lift* of $a \in \Gamma(A)$, i.e.,

$$H_a = \pi_{\text{lin}}^{\sharp}(\ell_a)$$

  where $\pi_{\text{lin}} \in \Gamma(\wedge^2 TA^*)$ is the linear Poisson structure of A^*; with $R_a : A^* \longrightarrow T^*A$ is given by

$$R_a(\xi) = (\mathrm{d}\ell_\xi)_a + q^*\mathrm{d}\langle \xi, a\rangle, \tag{1.24}$$

  where $q : A \longrightarrow M$,
- $\rho^*(S_\theta) = \rho_A^*(\theta)^{\uparrow}$, for $\theta \in \Omega^1(M)$.

The Lie bracket is determined by

$$[R_a, R_b] = R_{[a,b]} \qquad [R_a, S_\theta] = S_{\mathcal{L}_{\rho(a)}(\theta)} \qquad [S_{\theta_1}, S_{\theta_2}] = 0.$$

Example 1.48. VB-algebroid associated to a VB-groupoid. Given a VB-groupoid $\mathcal{E} \rightrightarrows E$ over $\mathcal{G} \rightrightarrows M$, then the Lie algebroid $A_{\mathcal{E}} \longrightarrow E$ of $\mathcal{E} \rightrightarrows E$ is a VB-algebroid over $A \longrightarrow M$. The compatibility conditions follow by the compatibility structures on $\mathcal{E}$. We refer the reader to [8] for more details.

Example 1.49. Sum. Given two VB-algebroids $\mathcal{A}_i \longrightarrow E_i$ over A with core bundles C_i. Let $\mathcal{A} = \mathcal{A}_1 \times_A \mathcal{A}_2$ and $E = E_1 \times_M E_2$. Then $\mathcal{A} \longrightarrow E$ is a VB-algebroid over A, with core bundle $C_1 \times_M C_2$. The Lie bracket is induced by the Lie bracket in each component:

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]),$$

and the anchor map is $\rho = (\rho_1, \rho_2)$.

Example 1.50. Dual of a VB-algebroid. Fix a VB-algebroid

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd},$$

with core bundle $C \longrightarrow M$, and dualize it with respect to A. We get the double vector bundle

$$\begin{tikzcd} \mathcal{A}^* \arrow[r] \arrow[d] & A \arrow[d] \\ C^* \arrow[r] & M \end{tikzcd}$$

with core bundle E^* (see Example 1.38). The VB-algebroid structure is determined by (see [17], APPENDIX A, for more details):

- Using the one-to-one correspondence between

  $$\begin{array}{ccc} \Gamma_{\text{lin}}(\mathcal{A}^*, C^*) & \longleftrightarrow & \Gamma_{\text{lin}}(\mathcal{A}, E) \\ X^{\perp} & \longleftrightarrow & X, \end{array}$$

  we define

  $$[X^{\perp}, Y^{\perp}] := [X, Y]^{\perp}.$$

- If $S_\eta \in \Gamma_{\text{cor}}(\mathcal{A}^*, C^*)$ for some $\eta \in \Gamma(E^*)$, then $[X^{\perp}, S_\eta]$ is the core section such that

  $$\ell_{[X^{\perp}, S_\eta]} = \rho_{\mathcal{A}}(X)(\ell_\eta).$$

1.4.2 Representations up to homotopy

One of the problems of representations of Lie algebroids is that there is not a good definition of **adjoint representation**, in the sense that we expect that it controls the deformation of the structure, as in the Lie algebra case. For this reason, and others, Arias-Abad and Crainic introduced in [2] representations up to homotopy. In this subsection, we first recall the definition of representation of Lie algebroids and we give an equivalent definition which allows a natural extension to the graded case. We follow [2, 20].

Let $A \longrightarrow M$ be a Lie algebroid. Recall that a *representation* of A is a vector bundle $E \longrightarrow M$ together with a flat A-connection $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$.

Given an A-connection $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$, let $\Omega(A, E) = \Gamma(\wedge A^* \otimes E)$ be the space of E-valued A-differential forms. We have a degree one operator d_∇ acting on this space induced by the Koszul formula: for $\omega \in \Omega^k(A, E) = \Gamma(\wedge^p A^* \otimes E)$

$$\begin{aligned} d_\nabla\omega(a_1, \ldots, a_{k+1}) &= \sum_{i<j}(-1)^{i+j}\omega([a_i, a_j], \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{k+1} \\ &\quad + \sum_i(-1)^{i+1}\nabla_{a_i}\omega(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}). \end{aligned}$$

satisfying the following derivation rule:

$$d_\nabla(\alpha \wedge \omega) = d_A(\alpha) \wedge \omega + (-1)^k\alpha \wedge d_\nabla(\omega), \tag{1.25}$$

for $\alpha \in \Omega^k(A)$, where $d_A : \Omega^\bullet(A) \longrightarrow \Omega^{\bullet+1}(A)$ is the Lie algebroid differential, see (1.5). The next result can be found in [2].

Proposition 1.51. *Let $A \longrightarrow E$ be a Lie algebroid and let $E \longrightarrow M$ be a vector bundle. There is a natural one-to-one correspondence between flat A-connections on E and degree one operators d on $\Omega(A, E)$ satisfying the derivation rule (1.25) and $d^2 = 0$.*

Consider now a 2-term graded vector bundle $V = C_{[0]} \oplus E_{[1]}$ over M. The space of V-valued A-differential forms is $\Omega(A; V) = \Gamma(\wedge A^* \otimes V)$. This space has a grading given by

$$\Omega(A; V)_k = \Gamma(\wedge^k A^* \otimes C) \oplus \Gamma(\wedge^{k-1} A^* \otimes E)$$

and it is a (naturally graded) module over the algebra $\Gamma(A) = \Gamma(\wedge^\bullet A^*)$.

Definition 1.52. A *2-term representation up to homotopy* of a Lie algebroid A is a 2-term-graded vector bundle $V = C_{[0]} \oplus E_{[1]}$ over M together with a degree one operator $d : \Omega(A; V)_\bullet \longrightarrow \Omega(A; V)_{\bullet+1}$ such that $d^2 = 0$ and satisfies the derivation rule

$$d(\alpha \wedge \omega) = d_A(\alpha) \wedge \omega + (-1)^k\alpha \wedge d(\omega),$$

for $\alpha \in \Omega^k(A)$, where $d_A : \Omega^\bullet(A) \longrightarrow \Omega^{\bullet+1}(A)$ is the Lie algebroid differential.

Remark 1.53. A representation up to homotopy can be defined on any graded vector bundle (see e.g. [2, 17]). For this thesis, we only are interested in representations up to homotopy in 2-term-graded vector bundles. When we write *representation up to homotopy* we are talking about of this particular case.

As in the Lie groupoid case, there is an equivalence of this definition in terms of structure operators.

Theorem 1.54 ([17, 20]). *A representation up to homotopy of a Lie algebroid on $C_{[0]} \oplus E_{[1]}$ is equivalent to the following data: A-connections ∇^0 and ∇^1 on C and E, respectively, a vector bundle map $\partial : C \longrightarrow E$ and a section $\Omega \in \Gamma(\wedge^2 A^* \otimes Hom(E, C))$ satisfying the following*

(i) $\partial \circ \nabla^0 = \nabla^1 \circ \partial$

(ii) $d_{\nabla^{End}}\Omega = 0$

(iii) The following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\partial"] \arrow[d, "R_{\nabla^0}"'] & E \arrow[d, "R_{\nabla^1}"] \arrow[dl, "-\Omega"'] \\ C \arrow[r, "\partial"] & E \end{tikzcd}$$

where R_{∇^0} and R_{∇^1} are the curvatures corresponding to the connections on C and E, respectively.

1.4.3 Representations up to homotopy vs. VB-algebroids

Here we explain briefly the correspondence between representations up to homotopy and VB-algebroids. This correspondence will allow us to pass from **multiplicative tensors with coefficients in a representation up to homotopy** to multiplicative functions.

Let $(\nabla^0, \nabla^1, \partial, \Omega)$ be a representation up to homotopy of A on the 2-term graded vector bundle $C \oplus E$ over M. We will construct the corresponding VB-algebroid associated to this representation. The underlying double vector bundle is the trivial double vector bundle

$$\begin{tikzcd} \mathcal{A} := A \oplus C \oplus E \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

with core bundle C. Let $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}, E)$ be the canonical horizontal lift given by

$$h(a)(e) := (a, 0, e) \qquad \text{for } e \in E.$$

The anchor map $\rho : \mathcal{A} \longrightarrow TE$ is:

- for a linear section $h(a)$, the linear vector field $\rho(h(a))$ is the one that corresponds to the derivation $(\nabla^1_a)^* : \Gamma(E^*) \longrightarrow \Gamma(E^*)$, that means

  $$\rho(h(a))(\ell_\eta) = \ell_{((\nabla^1_a)^*\eta)} \quad \text{for } \eta \in \Gamma(E^*),$$

  where $(\nabla^1_a)^*$ is the dual connection of ∇^1.

- For a core section S_c,
$$\rho(S_c) = \partial(c)^{\uparrow}$$

The Lie bracket is characterized by:

- $[S_{c_1}, S_{c_2}] = 0$
- $[h(a), S_c] = S_{\nabla^0_a c}$
- $[h(a), h(b)] = h([a,b]) + S_{\Omega(a,b)}$.

Conversely, let $(\mathcal{A}, E; A, M)$ be a VB-algebroid with core bundle C and with a horizontal lift $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}, E)$. Let

- $\partial : C \longrightarrow E$
- $\nabla^0 : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(C) \longrightarrow \Gamma(C)$
- $(\nabla^1)^* : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$

the associated canonical operators. We define:

- $\widetilde{\nabla}^0 : \Gamma(A) \times \Gamma(C) \longrightarrow \Gamma(C)$ by $\widetilde{\nabla}^0_a c := \nabla^0_{h(a)} c$,
- $\widetilde{\nabla}^1 : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ by $\widetilde{\nabla}^1_a e := (\nabla^1_{h(a)})^* e$,
- $\Omega \in \Gamma(\wedge^2 A^* \otimes \text{Hom}(E, C))$ by $\Omega(a,b) = h([a,b]) - [h(a), h(b)]$.

Then the quadruple $(\widetilde{\nabla}^0, \widetilde{\nabla}^1, \partial, \Omega)$ defines a representation up to homotopy of A on the 2-term graded vector bundle $C_{[0]} \oplus E_{[1]}$.

This correspondence between VB-algebroids over A together with a horizontal lift, and representations up to homotopy of A is one-to-one, and it is proved in [20].

Chapter 2

VB-groupoid cocycles and their infinitesimal data

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let τ be a *multiplicative structure* on $\mathcal{G}$, that is, a geometric structure which is compatible with the multiplication on $\mathcal{G}$. Our approach to study this structure is look at it as a cocycle F_τ on some Lie groupoid $\mathcal{E} \rightrightarrows E$. For the multiplicative structures that we are interested, their associated cocycles F_τ are actually defined in a VB-groupoid $\mathcal{E} \rightrightarrows E$ over $\mathcal{G}$.

In this chapter we fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ and we consider VB-groupoids $\mathcal{E} \rightrightarrows E$ over $\mathcal{G}$ and cocycles $F \in C^\infty(\mathcal{E})$ satisfying different linear conditions with respect to the linear structure $\mathcal{E} \longrightarrow \mathcal{G}$. We characterize (multi-)linear cocycles and infinitesimal (multi-)linear cocycles. Moreover we establish a correspondence of these global objects with an infinitesimal data, see Theorems 2.15, 2.30 and 2.35.

2.1 Linear cocycles

Let

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{Q} & \mathcal{G} \\ {\scriptstyle \bar{s}}\downarrow\downarrow{\scriptstyle \bar{t}} & & {\scriptstyle \mathbf{s}}\downarrow\downarrow{\scriptstyle \mathbf{t}} \\ E & \xrightarrow{q_E} & M \end{array}$$

be a VB-groupoid over $\mathcal{G} \rightrightarrows M$ with core bundle C.

Definition 2.1. A function $F : \mathcal{E} \longrightarrow \mathbb{R}$ is called a *linear $\mathcal{E}$-cocycle* if it is multiplicative with respect to the groupoid structure and linear with respect to the vector bundle structure over $\mathcal{G}$. That means

- $F(\eta_1 \cdot \eta_2) = F(\eta_1) + F(\eta_2)$ for all $(\eta_1, \eta_2) \in \mathcal{E}^{(2)}$, and
- $F(\lambda\eta + \mu) = \lambda F(\eta) + F(\mu)$ for all $\eta, \mu \in \mathcal{E}_g = Q^{-1}(g)$ and $\lambda \in \mathbb{R}$.

We say too that F is a *linear cocycle on $\mathcal{E}$*. And when there is not risk of confusion about the VB-groupoid considered, we just write *linear cocycle*.

Remark 2.2. Let $\eta_i \in \mathcal{E}$ for $i = 1, 2, 3, 4$ such that $(\eta_1, \eta_3), (\eta_2, \eta_4) \in \mathcal{E}^{(2)}$, and $Q(\eta_1) = Q(\eta_2)$ and $Q(\eta_3) = Q(\eta_4)$. Then

$$\begin{aligned} F\big((\eta_1+\eta_3)\cdot(\eta_2+\eta_4)\big) &= F(\eta_1+\eta_3)+F(\eta_2+\eta_4) \qquad (F \text{ multiplicative}) \\ &= F(\eta_1)+F(\eta_3)+F(\eta_2)+F(\eta_4) \qquad (F \text{ linear}) \\ &= F(\eta_1\cdot\eta_2)+F(\eta_3\cdot\eta_4) \qquad (F \text{ multiplicative}) \\ &= F(\eta_1\cdot\eta_2+\eta_3\cdot\eta_4) \qquad (F \text{ linear}), \end{aligned}$$

which means that a *linear $\mathcal{E}$-cocycle* is compatible with the interchange law.

Example 2.3. Let $f \in C^\infty_{\mathrm{lin}}(E)$ be a linear function. The function $F \in C^\infty(\mathcal{E})$ defined by

$$F = \bar{t}^* f - \bar{s}^* f$$

is a linear cocycle on $\mathcal{E}$, where $\bar{s}, \bar{t} : \mathcal{E} \longrightarrow E$ are the source and target maps.

Multiplicative functions $F \in C^\infty(\mathcal{E})$ satisfy $\mathcal{L}_{\overrightarrow{Y}} F = \bar{t}^*\langle AF, Y\rangle$ (Eq. (1.3)) for all sections $Y \in \Gamma(A_{\mathcal{E}})$, where $A_{\mathcal{E}} = \mathrm{Lie}(\mathcal{E})$ and $AF : A_{\mathcal{E}} \longrightarrow \mathbb{R}$ is the infinitesimal part of F. Since $\mathcal{E}$ is a VB-groupoid over $\mathcal{G}$ its Lie algebroid $A_{\mathcal{E}}$ is a VB-algebroid over A. Then the space of sections $\Gamma(A_{\mathcal{E}}, E)$ of $A_{\mathcal{E}}$ over E can be generated by its linear and core sections. Therefore we only have to check Equation (1.3) for these two kinds of sections. Hence we obtain the main result of this subsection, which characterizes linear cocycles on VB-groupoids.

Proposition 2.4. *Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over a source-connected Lie groupoid $\mathcal{G} \rightrightarrows M$. A function $F \in C^\infty(\mathcal{E})$ is a linear $\mathcal{E}$-cocycle if and only if*

$$F\big|_E = 0$$

and there exist a $C^\infty(M)$-linear map $\mathbf{D}_F : \Gamma_{lin}(A_{\mathcal{E}}, E) \longrightarrow \Gamma(E^*)$ and a section $\sigma \in \Gamma(C^*)$ such that

$$\left\{ \begin{array}{l} \mathcal{L}_{\overrightarrow{X}} F = \ell_{\mathcal{T}(\mathbf{D}_F(X))} \\ \mathcal{L}_{\overrightarrow{S_c}} F = \langle \sigma, c\rangle \circ \mathbf{t} \circ Q. \end{array} \right. \tag{2.1}$$

For the proof of this proposition we need some results about right invariant vector fields coming from core and linear sections. We start with core sections.

Define the maps:

$$\mathcal{T}, \mathcal{S} : \Gamma(C) \longrightarrow \Gamma(\mathcal{E})$$

by

$$\mathcal{T}(c)(g) := c(\mathbf{t}(g)) \cdot 0_g \tag{2.2}$$

$$\mathcal{S}(c)(g) := -0_g \cdot c(\mathbf{s}(g)) \tag{2.3}$$

where $\cdot$ is the multiplication on $\mathcal{E}$.

Remark 2.5. The source and target of the zero section $0_g \in \mathcal{E}_g$ are, respectively, $0_{\mathbf{s}(g)} \in E_{\mathbf{s}(g)}$ and $0_{\mathbf{t}(g)} \in E_{\mathbf{t}(g)}$, and since $c(\mathbf{t}(g))$ is an element of the core, then $\overline{s}(c(\mathbf{t}(g))) = 0_{t(g)} \in E_{\mathbf{t}(g)}$. Therefore we can multiply them. Moreover, $\mathcal{T}(c)(g) \in \mathcal{E}_g$ because

$$Q(c(\mathbf{t}(g)) \cdot 0_g) = Q(c(\mathbf{t}(g)) \cdot Q(0_g) = 1_{\mathbf{t}(g)} \cdot g = g$$

where we used that Q is a morphism of Lie groupoids.

Remark 2.6. We will always use the notation $\mathcal{T}, \mathcal{S}$ for these kind of maps from the core bundle to the Lie groupoid, independent of the VB-groupoid that we are considering.

Let $\mathcal{E}^*$ be the dual VB-groupoid of $\mathcal{E}$ over $\mathcal{G}$ and let $\mathcal{T}, \mathcal{S} : \Gamma(E^*) \longrightarrow \Gamma(\mathcal{E}^*)$ defined as before.

Proposition 2.7. *The pull-back maps* $\overline{t}^*, \overline{s}^* : C^\infty(E) \longrightarrow C^\infty(\mathcal{E})$ *satisfy*

$$\overline{t}^*(\ell_\varphi) = \ell_{\mathcal{T}(\varphi)} \tag{2.4}$$

$$\overline{s}^*(\ell_\varphi) = \ell_{\mathcal{S}(\varphi)} \tag{2.5}$$

for all $\varphi \in \Gamma(E^*)$.

Proof. First we show that $\overline{t}^*(\ell_\varphi)$ is a linear function on $\mathcal{E}$. If $\eta, \mu \in \mathcal{E}_g$, then

$$\overline{t}^*(\ell_\varphi)(\eta + \mu) = \ell_\varphi(\overline{t}(\eta + \mu)) = \ell_\varphi(\overline{t}(\eta) + \overline{t}(\mu)) = \ell_\varphi(\overline{t}(\eta)) + \ell_\varphi(\overline{t}(\mu)),$$

where we used that $\overline{t}$ is a linear map. For the second part, on one hand we have

$$\ell_\varphi(\overline{t}(\eta)) = \langle \varphi(\mathbf{t}(g)), \overline{t}(\eta) \rangle.$$

On the other hand

$$\ell_{\mathcal{T}(\varphi)}(\eta) = \langle \mathcal{T}(\varphi), \eta \rangle(g) = \langle \varphi(\mathbf{t}(g)) \cdot 0_g, \eta_g \rangle = \langle \varphi(\mathbf{t}(g)), \overline{t}(\eta) \rangle$$

where the last equation follows by definition of the target map in the dual VB-groupoid. In the same way, we get the condition about the source. □

Also, a section $c \in \Gamma(C)$ defines a core section of the algebroid $A_\mathcal{E}$ given by

$$S_c(e) = 0_e +_A \overline{c(p)} = \frac{d}{dr}\Big|_{r=0}(e + rc(q_E(e))).$$

Proposition 2.8. *For any section* $c : M \longrightarrow C$ *of the core bundle we have that*

$$\overrightarrow{S_c} = \mathcal{T}(c)^\uparrow \tag{2.6}$$

where the LHS is the right invariant vector field associated to the core section S_c *and the RHS is the vertical lift of the section* $\mathcal{T}(c)$.

Proof. Let $\eta \in \mathcal{E}$ and let $g = Q(\eta) \in \mathcal{G}$ and $e = \bar{t}(\eta) \in E_{t(g)}$. Then

$$\begin{aligned}
\overrightarrow{S_c}(\eta) = dR_\eta(S_c(e)) &= \frac{d}{dr}\Big|_{r=0} R_\eta(e + rc(q_E(e))) \\
&= \frac{d}{dr}\Big|_{r=0} \big((e + rc(q_E(e))) \cdot \eta\big) \\
&= \frac{d}{dr}\Big|_{r=0} \big((e + rc(q_E(e))) \cdot (\eta + 0_g)\big) \\
\text{interchange law} &= \frac{d}{dr}\Big|_{r=0} \big(e \cdot \eta + rc(q_E(e)) \cdot 0_g\big) \\
&= \frac{d}{dr}\Big|_{r=0} (\eta + r\mathcal{T}(c)(g)) \\
&= (\mathcal{T}(c))^{\uparrow}(\eta).
\end{aligned}$$

□

Now we describe the action of a linear cocycle on right invariant vector field coming from core sections.

Proposition 2.9. *For any linear cocycle $F \in C^\infty(\mathcal{E})$ we have that*

$$\mathcal{L}_{\overrightarrow{S_c}} F = F \circ c \circ \mathbf{t} \circ Q \tag{2.7}$$

for all $c \in \Gamma(C)$. Moreover, there exists a section $\sigma \in \Gamma(C^*)$ such that

$$\langle AF, S_c \rangle = q_E^* \langle \sigma, c \rangle, \tag{2.8}$$

for all $c \in \Gamma(C)$.

Proof. Let $c \in \Gamma(C)$ and let S_c be its corresponding core section of $A_{\mathcal{E}}$. Considering the right invariant vector field $\overrightarrow{S_c}$, then for $\eta \in \mathcal{E}_g$ we have

$$\begin{aligned}
\mathcal{L}_{\overrightarrow{S_c}} F(\eta) &= \frac{d}{dt}\Big|_0 F(\eta + t\mathcal{T}(c)(g)) \\
&= F(\mathcal{T}(c)(g)) \quad \text{by the linearity of } F \\
&= F(c(\mathbf{t}(g)) \cdot 0_g) \\
&= F(c(\mathbf{t}(g))) \quad \text{by multiplicativity and linearity of } F.
\end{aligned}$$

Then Equation (2.7) follows. Now taking units $e \in E$ we have that

$$\mathcal{L}_{\overrightarrow{S_c}} F(e) = F(c(q_E(e))).$$

We define $\sigma : M \longrightarrow \Gamma(C^*)$ by $\langle \sigma, c \rangle(p) = F(c(q(e)))$ for any $e \in E_p$, and together with the property $\mathcal{L}_{\overrightarrow{S_c}} F = \bar{t}^* \langle AF, S_c \rangle$ since F is multiplicative, we get (2.8). □

Now let $X \in \Gamma_{\text{lin}}(A_{\mathcal{E}}, E)$ be a linear section. Since AF is linear with respect to A, it follows that $\langle AF, X\rangle$ is a linear function on E. Then there exists a section $\mathbf{D}_F(X) \in \Gamma(E^*)$ such that

$$\langle AF, X\rangle = \ell_{\mathbf{D}_F(X)}$$

Proposition 2.10. *For any linear $\mathcal{E}$-cocycle $F \in C^\infty(\mathcal{E})$ there exists a $C^\infty(M)$-linear map $\mathbf{D}_F : \Gamma_{lin}(A_{\mathcal{E}}, E) \longrightarrow \Gamma(E^*)$ such that*

$$\mathcal{L}_{\overrightarrow{X}} F = \ell_{\mathcal{T}(\mathbf{D}_F(X))} \tag{2.9}$$

Proof. Since the function $\langle AF, X\rangle$ is linear in E, there exists a section $\mathbf{D}_F(X) \in \Gamma(E^*)$ such that

$$\langle AF, X\rangle = \ell_{\mathbf{D}_F(X)}$$

Then the multiplicativity condition together with Proposition 2.7 imply

$$\mathcal{L}_{\overrightarrow{X}} F = \overline{t}^* \langle AF, X\rangle = \overline{t}^*(\ell_{\mathbf{D}_F(X)}) = \ell_{\mathcal{T}(\mathbf{D}_F(X))},$$

Take now $h \in C^\infty(M)$. Then

$$\begin{aligned}
\ell_{\mathcal{T}(\mathbf{D}_F((h\circ q_E)X))} &= \mathcal{L}_{\overrightarrow{(h\circ q_E)X}} F \\
&= \overline{t}^*(h\circ q_E)\mathcal{L}_{\overrightarrow{X}} F \\
&= \overline{t}^*(h\circ q_E)\ell_{\mathcal{T}(\mathbf{D}_F(X))} \\
&= \ell_{\mathcal{T}(h\circ q_E)\mathcal{T}(\mathbf{D}_F(X))} \\
&= \ell_{\mathcal{T}((h\circ q_E)\mathbf{D}_F(X))}
\end{aligned}$$

Since $\mathcal{T}$ is injective we have the identity. □

We need one more result, which one can find in [10].

Proposition 2.11. *Let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ the Whitney sum (as vector bundles over $\mathcal{G}$) of the VB-groupoids $\mathcal{E}_i \rightrightarrows E_i$, $i = 1, \ldots k$ over $\mathcal{G}$, and consider its Lie algebroid $A_{\mathcal{E}}$, which naturally splits as a Whitney sum $A_{\mathcal{E}_1} \oplus \cdots \oplus A_{\mathcal{E}_k}$ over A. If $F : \mathcal{E} \longrightarrow \mathbb{R}$ is a componentwise linear function, that is, for any $g \in \mathcal{G}$, $F : \mathcal{E}_g \longrightarrow \mathbb{R}$ is a multilinear map, then its infinitesimal counterpart $AF : A_{\mathcal{E}} \longrightarrow \mathbb{R}$ is also componentwise linear function. Reciprocally, if $\mathcal{G}$ is source simply connected, then any Lie algebroid cocycle $\Lambda : A_{\mathcal{E}} \longrightarrow \mathbb{R}$ which is componentwise linear integrates to a unique componentwise linear function $F : \mathcal{E} \longrightarrow \mathbb{R}$ such that $AF = \Lambda$. Moreover in the case $\mathcal{E}_1 = \cdots = \mathcal{E}_j, j \leq k$, Λ is symmetric (resp. skew-symmetric) is the first j components if and only if F is also.*

Proof. Proposition 2.4. If F is a linear $\mathcal{E}$-cocycle, by Propositions 2.9 and 2.10 there exists a pair $(\mathbf{D}_F, \sigma)$ satisfying the condition (2.1). Conversely, we want to prove that given a pair $(\mathbf{D}, \sigma)$ satisfying condition (2.1), implies that F is a linear cocycle.

We define a function $\mu : A_{\mathcal{E}} \longrightarrow \mathbb{R}$ which will be the infinitesimal counterpart of F. Define

$$\begin{aligned}\langle \mu, X\rangle &= \ell_{\mathbf{D}(X)}\\ \langle \mu, S_c\rangle &= \langle \sigma, c\rangle \circ q_E\end{aligned}$$

for $X \in \Gamma_{\mathrm{lin}}(A_{\mathcal{E}})$ and $c \in \Gamma(C)$. By hypothesis about the pair $(\mathbf{D}, \sigma)$, and by Proposition 2.7, we have that

$$\mathcal{L}_{\overrightarrow{X}}F = \ell_{\mathcal{T}(\mathbf{D}(X))} = \overline{t}^*(\ell_{\mathbf{D}(X)}) = \overline{t}^*\langle \mu, X\rangle \tag{2.10}$$

$$\mathcal{L}_{\overrightarrow{S_c}}F = \sigma(c)\circ \mathbf{t}\circ Q = \overline{t}^*(\sigma(c)\circ q_E) = \overline{t}^*\langle \mu, S_c\rangle. \tag{2.11}$$

Remember that the space of sections $\Gamma(A_{\mathcal{E}}, E)$ is generated as $C^\infty(E)$-module by its linear and core sections, so by the equations (2.10) and (2.11), we get a well defined map at the level of sections $\mu : \Gamma(A_{\mathcal{E}}, E) \longrightarrow C^\infty(E)$. Moreover, the equations (2.10) and (2.11) imply too that μ is $C^\infty(E)$-linear. Therefore $\mu : A_{\mathcal{E}} \longrightarrow \mathbb{R}$ is well defined map and linear over E. It follows too that μ satisfies

$$\mathcal{L}_{\overrightarrow{Y}}F = \overline{t}^*\langle \mu, Y\rangle$$

for all section $Y \in \Gamma(A_{\mathcal{E}})$. Since $\mathcal{E}$ is source connected there exists a multiplicative function $F_\mu \in C^\infty(\mathcal{E})$ such that $AF_\mu = \mu$. Then we have that $\mathcal{L}_{\overrightarrow{Y}}F = \mathcal{L}_{\overrightarrow{Y}}F_\mu$ for all $Y \in \Gamma(A_{\mathcal{E}})$. Since $\mathcal{E}$ has connected $\overline{s}$-fibers, it follows that $F - F_\mu$ is constant along the $\overline{s}$-fibers. Finally $(F - F_\mu)\big|_E = 0$, which means that $(F - F_\mu) = 0$ everywhere. Then $F = F_\mu$. Now we have to check the linearity over A. Let $\alpha_i \in A_{\mathcal{E}},\ i = 1, 2$, projectable over $a \in A$ and over $e_i \in E$. Let $X \in \Gamma_{\mathrm{lin}}(A_{\mathcal{E}}, E)$ such that $X(e_1) = \alpha_1$. Then $\alpha_2 = X(e_2) +_E S_c(e_2)$ for some section $c \in \Gamma(C)$. Then

$$\begin{aligned}\mu(\alpha_1 +_A \alpha_2) &= \mu(X(e_1) +_A (X(e_2) +_E S_c(e_2)))\\ &= \mu\big((X(e_1) +_A X(e_2)) +_E S_c(e_2)\big) \quad \text{by interchange law}\\ &= \mu\big(X(e_1 + e_2) +_E S_c(e_2)\big) \quad \text{by linearity of } X\\ &= \mu\big(X(e_1 + e_2)\big) + \mu\big(S_c(e_2)\big) \quad \text{by linearity of } \mu \text{ w.r.t } E\\ &= \langle \mathbf{D}(X), e_1 + e_2\rangle + \mu\big(S_c(e_2)\big)\\ &= \langle \mathbf{D}(X), e_1\rangle + \langle \mathbf{D}(X), e_2\rangle + \mu\big(S_c(e_2)\big)\\ &= \mu(X(e_1)) + \mu(X(e_2)) + \mu\big(S_c(e_2)\big)\\ &= \mu(\alpha_1) + \mu(X(e_2) +_E S_c(e_2))\\ &= \mu(\alpha_1) + \mu(\alpha_2).\end{aligned}$$

Hence $\mu = dF$ is linear with respect to A. Then by Proposition 2.11 it follows that F is linear over $\mathcal{G}$. $\square$

2.2 Infinitesimal linear cocycles

In this section we characterize and describe functions defined on a VB-algebroid $\mathcal{A}$ over A, which are cocycles with respect to the Lie algebroid structure, and linear with respect to the vector bundle structure $\mathcal{A} \longrightarrow A$. We will prove that this description is the **infinitesimal counterpart** of linear cocycles on VB-groupoids.

Let

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & A \arrow[d] \\ E \arrow[r] & M \end{tikzcd} \tag{2.12}$$

be a VB-algebroid over A with core bundle $C \longrightarrow M$. Recall the canonical operator associated to a VB-algebroid: the flat connections $\nabla^0 : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(C) \longrightarrow \Gamma(C)$ (1.22) and $(\nabla^1)^* : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$ (1.21), and the core anchor map $\partial : C \longrightarrow E$ (1.23).

Definition 2.12. A function $f \in C^\infty(\mathcal{A})$ is a *linear $\mathcal{A}$-cocycle* if it is a cocycle on $\mathcal{A}$ which is linear with respect to the vector bundle structure $\mathcal{A} \longrightarrow A$.

We describe now these functions in terms of their actions on linear and core sections.

Proposition 2.13. *Let $f \in C^\infty(\mathcal{A})$ be a linear $\mathcal{A}$-cocycle. Then there exist a $C^\infty(M)$-linear map $\mathbf{D} : \Gamma_{lin}(\mathcal{A}) \longrightarrow \Gamma(E^*)$ and a section $\sigma \in \Gamma(C^*)$ such that*

$$\mathbf{D}(S_T) = \langle \sigma, T \rangle \quad \textit{for all } T \in \Gamma(Hom(E, C)) \tag{2.13}$$

$$\mathbf{D}([X, Y]) = (\nabla^1)^*_X \mathbf{D}(Y) - (\nabla^1)^*_Y \mathbf{D}(X) \tag{2.14}$$

$$\langle \mathbf{D}(X_a), \partial(c) \rangle = \mathcal{L}_{\rho_A(a)} \sigma(c) - \sigma(\nabla^0_X c). \tag{2.15}$$

Conversely, a pair $(\mathbf{D}, \sigma)$ satisfying (2.13), (2.14) and (2.15) defines an linear cocycle on $\mathcal{A}$.

Proof. Let $f : \mathcal{A} \longrightarrow \mathbb{R}$ be a linear cocycle. Consider f as a map at the level of sections $f : \Gamma(\mathcal{A}, E) \longrightarrow C^\infty(E)$, i.e., for a section $X \in \Gamma(\mathcal{A}, E)$ we associated the function $f \circ X : E \longrightarrow \mathbb{R}$. Take a core section S_c then

$$\langle f, S_c \rangle(e) = f(0_e +_A \overline{c(p)}) = f(\overline{c(p)})$$

for $e \in E_p$, where we used the linearity of f with respect to the vector bundle structure over A. This means that the function $\langle f, S_c \rangle \in C^\infty(E)$ is basic. So, define $\sigma \in \Gamma(C^*)$ such that $q_E^*(\langle \sigma, c \rangle) = \langle f, S_c \rangle$. On the other hand, define $\mathbf{D} := f\big|_{\Gamma_{\text{lin}}(\mathcal{A}, E)}$. Since f is a cocycle then

$$\langle f, [Y, Z] \rangle = \mathcal{L}_{\rho_{\mathcal{A}}(Y)} \langle f, Z \rangle - \mathcal{L}_{\rho_{\mathcal{A}}(Z)} \langle f, Y \rangle$$

for all $Y, Z \in \Gamma(\mathcal{A})$. Remember that if X be a linear section then

$$\ell_{(\nabla^1)^*_X \eta} = \rho_{\mathcal{A}}(X)(\ell_\eta).$$

Hence $\ell_{(\nabla^1)^*_X\mathbf{D}(Y)} = \rho_{\mathcal{A}}(X)(\ell_{\mathbf{D}(Y)}) = \mathcal{L}_{\rho_{\mathcal{A}}(X)}\langle f, Y\rangle$. This implies that the condition (2.14) for $\mathbf{D}$ holds. Now suppose that X covers the section $a \in \Gamma(A)$. Recall that the Lie bracket $[X, S_c] = S_{\nabla^0_X c}$. Then

$$\langle f, [X, S_c]\rangle = \sigma(\nabla^0_X c) \circ q_E.$$

In the other side

$$\mathcal{L}_{\rho_{\mathcal{A}}(X)}(\sigma(c) \circ q_E) = (\mathcal{L}_{\rho_A(a)}\sigma(c)) \circ q_E$$

and

$$\mathcal{L}_{\rho_{\mathcal{A}}(S_c)}\ell_{\mathbf{D}(X)} = \mathcal{L}_{(\partial(c))^\uparrow}\ell_{\mathbf{D}(X)} = \langle \mathbf{D}(X), \partial(c)\rangle \circ q_E.$$

So the third equation holds. By definition of $\mathbf{D}$ we have that $\mathbf{D}(hX) = h\mathbf{D}(X)$ for all $h \in C^\infty(M)$. Finally, the first equation, the **compatibility** between $\mathbf{D}$ and σ holds:

$$\mathbf{D}(S_T) = f \circ S_T = \langle \sigma, T\rangle.$$

Therefore the pair $(\mathbf{D}, \sigma)$ satisfies Equations (2.13), (2.14) and (2.15). Conversely, given a pair $(\mathbf{D}, \sigma)$ satisfying these conditions, define $\mu : \Gamma(\mathcal{A}) \longrightarrow C^\infty(E)$ by:

$$\langle \mu, X\rangle = \ell_{\mathbf{D}(X)} \tag{2.16}$$

$$\langle \mu, S_c\rangle = \sigma(c) \circ q_E. \tag{2.17}$$

The compatibility of $\mathbf{D}$ and σ implies that μ is well define on core linear sections. Moreover we have that

$$\mu(q_E^*(h)X) = q_E^*(h)\mu(X) \quad \text{and} \quad \mu(q_E^*(h)S_c) = q_E^*(h)\mu(S_c)$$

for all $h \in C^\infty(M)$. Then we extend μ to all sections $\Gamma(\mathcal{A}, E)$ by $C^\infty(E)$-linearity. Therefore $\mu : \mathcal{A} \longrightarrow \mathbb{R}$ is a well defined linear map with respect to the linear structure $\mathcal{A} \longrightarrow E$. The equations (2.14) and (2.15) satisfied by $\mathbf{D}$ and σ imply that μ is a morphism of Lie algebroid. Now, to check the linearity over A, let $\alpha_i \in \mathcal{A}$, $i = 1, 2$, projectable over $a \in A$ and over $e_i \in E$. Let $X \in \Gamma_{\mathrm{lin}}(\mathcal{A}, E)$ such that $X(e_1) = \alpha_1$. Then $\alpha_2 = X(e_2) +_E S_c(e_2)$ for some section $c \in \Gamma(C)$. Then

$$\begin{aligned}
\mu(\alpha_1 +_A \alpha_2) &= \mu(X(e_1) +_A (X(e_2) +_E S_c(e_2))) \\
&= \mu\big((X(e_1) +_A X(e_2)) +_E S_c(e_2)\big) \quad \text{by interchange law} \\
&= \mu\big(X(e_1 + e_2) +_E S_c(e_2)\big) \quad \text{by linearity of } X \\
&= \mu\big(X(e_1 + e_2)\big) + \mu\big(S_c(e_2)\big) \quad \text{by linearity of } \mu \text{ w.r.t } E \\
&= \langle \mathbf{D}(X), e_1 + e_2\rangle + \mu\big(S_c(e_2)\big) \\
&= \langle \mathbf{D}(X), e_1\rangle + \langle \mathbf{D}(X), e_2\rangle + \mu\big(S_c(e_2)\big) \\
&= \mu(X(e_1)) + \mu(X(e_2)) + \mu\big(S_c(e_2)\big) \\
&= \mu(\alpha_1) + \mu(X(e_2) +_E S_c(e_2)) \\
&= \mu(\alpha_1) + \mu(\alpha_2).
\end{aligned}$$

Hence μ is linear with respect to A. $\square$

Definition 2.14. Let $\mathcal{A} \longrightarrow E$ be a VB-algebroid over A with core bundle C. A pair $(\mathbf{D}, \sigma)$ satisfying (2.13), (2.14) and (2.15) is called the *infinitesimal components of a linear cocycle.*

Now we are in condition to state the global-infinitesimal correspondence between linear cocycles and infinitesimal linear cocycles.

Theorem 2.15. *Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G}$. Then every linear cocycle $F \in C^\infty(\mathcal{E})$ induces a pair $(\mathbf{D}, \sigma)$ satisfying (2.13), (2.14) and (2.15) on $A_\mathcal{E}$. Moreover, if $\mathcal{G}$ is source simply connected, there is a one-to-one correspondence between linear cocycles on $\mathcal{E}$ and such pairs given by*

$$\begin{aligned} \mathcal{L}_{\overrightarrow{X}} F &= \ell_{\mathcal{T}(\mathbf{D}(X))} \\ \mathcal{L}_{\overrightarrow{S_c}} F &= \langle \sigma, c\rangle \circ \mathbf{t} \circ Q. \end{aligned}$$

Proof. Let $F \in C^\infty(\mathcal{E})$ be linear cocycle. By Proposition 2.4 there exist $C^\infty(M)$-linear map $\mathbf{D}_F : \Gamma_{\mathrm{lin}}(A_\mathcal{E}, E) \longrightarrow \Gamma(E^*)$ and a section $\sigma \in \Gamma(C^*)$ such that

$$\begin{aligned} \mathcal{L}_{\overrightarrow{X}} F &= \ell_{\mathcal{T}(\mathbf{D}(X))} \\ \mathcal{L}_{\overrightarrow{S_c}} F &= \sigma(c) \circ \boldsymbol{t} \circ Q. \end{aligned}$$

Note that for $T \in \Gamma(Hom(E, C))$, $\overrightarrow{S_T}$ is a core linear vector field, which implies that $\mathbf{D}_F(S_T) = \sigma \circ T$. Moreover the operator $\mathbf{D}_F$ satisfies

$$\ell_{\mathbf{D}_F(X)} = \langle AF, X\rangle \ \forall\ X \in \Gamma_{\mathrm{lin}}(A_\mathcal{E}, E) \quad \text{i.e.} \quad \mathbf{D}_F = AF\big|_{\Gamma_{\mathrm{lin}}(A_\mathcal{E}, E)}.$$

Since AF is an infinitesimal linear cocycle follows that $\mathbf{D}_F$ satisfies Equation (2.14). On the other hand we have that $\langle AF, S_c\rangle = \langle \sigma, c\rangle \circ q_E$, then by Proposition 2.13 follows that Equation (2.15) holds. Conversely, let $\mu \in C^\infty(A_\mathcal{E})$ be the linear cocycle obtained from the pair $(\mathbf{D}, \sigma)$. Since $\mathcal{G}$ is source simply connected, there exists a (unique) function $F \in C^\infty(\mathcal{E})$ integrating μ, which is multiplicative and linear. □

Remark 2.16. There is an alternative proof of this theorem. It is using the properties of the Lie derivate $\mathcal{L}_{\overrightarrow{X}} F$ with respect to the Lie bracket of vector fields, taking only vector fields coming from linear and core sections.

Proof. Let $F \in C^\infty(\mathcal{E})$ be linear cocycle. By Proposition (2.4) there exist $C^\infty(M)$-linear map $\mathbf{D} : \Gamma_{lin}(A_\mathcal{E}) \longrightarrow \Gamma(E^*)$ and a section $\sigma \in \Gamma(C^*)$ such that

$$\begin{aligned} \mathcal{L}_{\overrightarrow{X}} F &= \ell_{\mathcal{T}(\mathbf{D}(X))} \\ \mathcal{L}_{\overrightarrow{S_c}} F &= \sigma(c) \circ \boldsymbol{t} \circ Q. \end{aligned}$$

Note that for $T \in \Gamma(Hom(E, C))$, $\overrightarrow{S_T}$ is a core linear vector field, which implies that $\mathbf{D}(S_T) = \sigma \circ T$. Now, if $X, Y \in \Gamma_{lin}(A_\mathcal{E})$ their Lie bracket $[X, Y] \in \Gamma_{lin}(A_\mathcal{E})$, then

$$\mathcal{L}_{[\overrightarrow{X}, \overrightarrow{Y}]} F = \ell_{\mathcal{T}(\mathbf{D}([X,Y]))} = \overline{t}^*(\ell_{\mathbf{D}([X,Y])}).$$

On the other side, we have that

$$\begin{aligned}
\mathcal{L}_{[\overrightarrow{X},\overrightarrow{Y}]}F &= \mathcal{L}_{\overrightarrow{X}}\mathcal{L}_{\overrightarrow{Y}}F - \mathcal{L}_{\overrightarrow{Y}}\mathcal{L}_{\overrightarrow{X}}F \\
&= \mathcal{L}_{\overrightarrow{X}}(\ell_{\mathcal{T}(\mathbf{D}(Y))}) - \mathcal{L}_{\overrightarrow{Y}}(\ell_{\mathcal{T}(\mathbf{D}(X))}) \\
&= \mathcal{L}_{\overrightarrow{X}}(\overline{t}^*(\ell_{\mathbf{D}(Y)})) - \mathcal{L}_{\overrightarrow{Y}}(\overline{t}^*(\ell_{\mathbf{D}(X)})) \\
&= \overline{t}^*(\mathcal{L}_{\rho(X)}\ell_{\mathbf{D}(Y)}) - \overline{t}^*(\mathcal{L}_{\rho(Y)}\ell_{\mathbf{D}(X)}).
\end{aligned}$$

Since the pullback map $\overline{t}^* : C^\infty(E) \longrightarrow C^\infty(\mathcal{E})$ is injective, follows

$$\mathbf{D}([X,Y]) = (\nabla^1)^*_X\mathbf{D}(Y) - (\nabla^1)^*_Y\mathbf{D}(X).$$

Let $c \in \Gamma(C)$, then $[X, S_c] = S_{\nabla^0_X c}$, so

$$\mathcal{L}_{[\overrightarrow{X},\overrightarrow{S_c}]}F = \mathcal{L}_{\overrightarrow{S_{\nabla^0_X c}}}F = \overline{t}^*(\sigma(\nabla^0_X c) \circ q_E).$$

In the other side, we have

$$\mathcal{L}_{\overrightarrow{X}}\mathcal{L}_{\overrightarrow{S_c}}F = \mathcal{L}_{\overrightarrow{X}}(\overline{t}^*(\sigma(c) \circ q_E)) = \overline{t}^*(\mathcal{L}_{\rho(X)}(\sigma(c) \circ q_E)) = \overline{t}^*((\mathcal{L}_{\rho_A(a)}\sigma(c)) \circ q_E),$$

and

$$\mathcal{L}_{\overrightarrow{S_c}}\mathcal{L}_{\overrightarrow{X}}F = \mathcal{L}_{\overrightarrow{S_c}}(\overline{t}^*(\ell_{\mathbf{D}(X)})) = \overline{t}^*(\mathcal{L}_{\rho(S_c)}\ell_{\mathbf{D}(X)}) = \overline{t}^*(\langle \mathbf{D}(X), \partial(c)\rangle \circ q_E).$$

Hence

$$\langle \mathbf{D}(X_a), \partial(c)\rangle = \mathcal{L}_{\rho_A(a)}\sigma(c) - \sigma(\nabla^0_X c).$$

Therefore the pair $(\mathbf{D}, \sigma)$ is the infinitesimal components of a linear cocycle on $A_{\mathcal{E}}$. Conversely, let $\mu \in C^\infty(A_{\mathcal{E}})$ be the linear cocycle obtained from the pair $(\mathbf{D}, \sigma)$ Since $\mathcal{G}$ is source simply connected, there exists a (unique) function $F \in C^\infty(\mathcal{E})$ integrating μ, which is multiplicative and linear. □

Corollary 2.17. *Let $\mathcal{G} \rightrightarrows M$ be source simply connected Lie groupoid, and let $E \longrightarrow M$ be a representation. Then linear cocycles on the action VB-groupoid $E * \mathcal{G}$ (1.19) are in one-to-one correspondence with $C^\infty(M)$-linear operators $\mathbf{D} : \Gamma_{lin}(A_{\mathcal{E}}) \longrightarrow \Gamma(E^*)$ such that*

$$\mathbf{D}([X,Y]) = (\nabla^1)^*_X\mathbf{D}(Y) - (\nabla^1)^*_Y\mathbf{D}(X).$$

Corollary 2.18. *Let $\mathcal{G} \rightrightarrows M$ be source simply connected Lie groupoid, and let $C \longrightarrow M$ be a representation. Then linear cocycles on the semidirect product $C \rtimes \mathcal{G}$ (1.20) are in one-to-one correspondence with sections $\sigma \in \Gamma(C^*)$ satisfying*

$$\mathcal{L}_{\rho(a)}\langle \sigma, c\rangle = \langle \sigma, \nabla_a c\rangle.$$

Example 2.19. Let $f \in C^\infty_{\text{lin}}(E)$ be a linear function and let $F = \overline{t}^* f - \overline{s}^* f \in C^\infty(\mathcal{E})$ as in Example 2.3. Since F is a linear cocycle, consider its infinitesimal components $(\mathbf{D}, \sigma)$. Let $X \in \Gamma_{\text{lin}}(A_{\mathcal{E}}, E)$ be a linear section. Then

$$\mathcal{L}_{\overrightarrow{X}} F = \mathcal{L}_{\overrightarrow{X}} \overline{t}^* f = \overline{t}^* (\mathcal{L}_{\rho_{A_{\mathcal{E}}}(X)} f),$$

which implies that $\mathbf{D}(X) = (\nabla^1)^*_X f$, where here we are seeing $f \in C^\infty_{\text{lin}}(E) \simeq \Gamma(E^*)$. Taking now a core section S_c, then

$$\mathcal{L}_{\overrightarrow{S_c}} F = f \circ \overline{t} \circ c \circ t \circ Q$$

which means

$$\sigma(c) = f \circ \overline{t} \circ c = f \circ \partial \circ c,$$

hence $\sigma = \partial^*(f)$. Since the connection $(\nabla^1)^* : \Gamma_{\text{lin}}(A_{\mathcal{E}}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$ is flat, it follows that $\mathbf{D}$ satisfies Equation (2.14). On the other hand, if X is a linear section covering $a \in \Gamma(A)$ and $c \in \Gamma(C)$, we have

$$\begin{aligned} \langle \mathbf{D}(X), \partial(c) \rangle &= \langle (\nabla^1)^*_X f, \partial(c) \rangle \\ &= \mathcal{L}_{\rho(a)} \langle f, \partial(c) \rangle - \langle f, \nabla^1_X \partial(c) \rangle \\ &= \mathcal{L}_{\rho(a)} \langle \partial^*(f), c \rangle - \langle f, \partial \circ \nabla^0_X c \rangle \\ &= \mathcal{L}_{\rho(a)} \langle \partial^*(f), c \rangle - \langle \partial^* f, \nabla^0_X c \rangle, \end{aligned}$$

hence Equation (2.15) holds. Conversely, given a function $f \in C^\infty_{\text{lin}}(E)$ define $\mathbf{D}(X) = (\nabla^1)^*_X f$ and $\sigma = \partial^*(f)$. The pair $(\mathbf{D}, \sigma)$ is an infinitesimal component of a linear cocycle on A, and by uniqueness, the function $F \in C^\infty(\mathcal{E})$ which integrates this infinitesimal data is $F = \overline{t}^* f - \overline{s}^* f$.

2.3 Multilinear cocycles

Many of the multiplicative structures on a Lie groupoid $\mathcal{G}$ which matters for us, when viewed as functions, are defined in a Whitney sum (as vector bundles over $\mathcal{G}$) of VB-groupoids $\mathcal{E}_i \rightrightarrows E_i$ over $\mathcal{G}$. In this section we study functions F defined on a Whitney sum of VB-groupoids which are multiplicative and multilinear, and we describe them infinitesimally. The case of bilinear cocycles is done in details and then we extend to the general case.

2.3.1 Bilinear cocycles

Let $\mathcal{E}_i \rightrightarrows E_i$ be two VB-groupoids over $\mathcal{G} \rightrightarrows M$ with core bundles C_i, $i = 1, 2$. We take the VB-groupoid sum:

$$\begin{array}{ccc} \mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2 & \longrightarrow & \mathcal{G} \\ \downarrow\downarrow & & \downarrow \\ E := E_1 \oplus E_2 & \longrightarrow & M \end{array}$$

Definition 2.20. A *bilinear cocycle* on $\mathcal{E}$ is a function $F \in C^\infty(\mathcal{E})$ such that it is multiplicative with respect to the groupoid structure and it is bilinear with respect to the vector bundle structure.

This means that the maps

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{R} \arrow[d] \\ E \arrow[r] & *, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathcal{G} \arrow[r, "id"'] & \mathcal{G} \end{tikzcd} .$$

are multiplicative and bilinear, respectively.

We know that the space of sections $\Gamma(A_\mathcal{E}, E)$ is generated, as $C^\infty(E)$-module, by the following sections (see Proposition 1.41):

- (X_1, X_2), where $X_i \in \Gamma_{\text{lin}}(A_{\mathcal{E}_i}, E_i)$ are linear sections covering the same section $a \in \Gamma(A)$,
- $S^1_{c_1} = (S_{c_1}, 0)$ and $S^2_{c_2} = (0, S_{c_2})$, where $c_i \in \Gamma(C_i)$.

Hence, bilinear cocycles can be characterized by their action on right invariant vector fields, coming from these three types of sections.

Theorem 2.21. *Let $\mathcal{E}_i \rightrightarrows E_i$, $i = 1, 2$ be two VB-groupoids over a source connected Lie groupoid $\mathcal{G} \rightrightarrows M$ and consider the VB-groupoid sum $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Then a function $F \in C^\infty(\mathcal{E})$ is a bilinear cocycle if and only if*

$$F\Big|_{E_1 \oplus E_2} = 0$$

and there exist a $C^\infty(M)$-linear map $\mathbf{D} : \Gamma_{\text{lin}}(A_{\mathcal{E}_1}, E_1) \times_{\Gamma(A)} \Gamma_{lin}(A_{\mathcal{E}_2}, E_2) \longrightarrow \Gamma(E_1^* \otimes E_2^*)$, and vector bundle morphisms $\sigma_1 : C_1 \longrightarrow E_2^*$ and $\sigma_2 : C_2 \longrightarrow E_1^*$ covering the identity of M such that

$$\begin{cases} \mathcal{L}_{(\overrightarrow{X_1}, \overrightarrow{X_2})} F = \ell_{\mathcal{T}(D(X_1, X_2))} \\ \mathcal{L}_{\overrightarrow{S_{c_1,0}}} F = \ell_{\mathcal{T}(\sigma_1(c_1))} \circ \pi_2 \\ \mathcal{L}_{\overrightarrow{S_{0,c_2}}} F = \ell_{\mathcal{T}(\sigma_2(c_2))} \circ \pi_1 \end{cases} \tag{2.18}$$

where $\pi_1 : \mathcal{E}_1 \oplus \mathcal{E}_2 \longrightarrow \mathcal{E}_1$ and $\pi_2 : \mathcal{E}_1 \oplus \mathcal{E}_2 \longrightarrow \mathcal{E}_2$ are the projections over the first and second component, respectively.

Again we first start with core sections. Let $c_i \in \Gamma(C_i)$ be sections of the core bundles and let $S_{c_1,c_2} = (S_{c_1}, S_{c_2})$ be a core section of $A_\mathcal{E}$. Then for $(\eta_1, \eta_2) \in \mathcal{E}_g$

$$\begin{aligned} \mathcal{L}_{\overrightarrow{S_{c_1,c_2}}} F(\eta_1, \eta_2) &= \frac{d}{dt}\Big|_0 F(\eta_1 + t\mathcal{T}(c_1)(g), \eta_2 + t\mathcal{T}(c_2)(g)) \\ &= F(\eta_1, \mathcal{T}(c_2)(g)) + F(\mathcal{T}(c_1)(g), \eta_2) \quad \text{by bilinearity of } F. \end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}_{\overrightarrow{S_{c_1,0}}} F(\eta_1,\eta_2) &= F(\mathcal{T}(c_1)(g),\eta_2)\\
\mathcal{L}_{\overrightarrow{S_{0,c_2}}} F(\eta_1,\eta_2) &= F(\eta_1,\mathcal{T}(c_2)(g)).
\end{aligned}$$

By the multiplicativity of F, taking units we have

$$\begin{aligned}
\langle AF, S_{c_1,0}\rangle(e_1,e_2) &= F(c_1(p),e_2)\\
\langle AF, S_{0,c_2}\rangle(e_1,e_2) &= F(e_1,c_2(p)).
\end{aligned}$$

Proposition 2.22. *Let $F \in C^\infty(\mathcal{E})$ be a bilinear cocycle. Then there exist vector bundle maps $\sigma_i : C_i \longrightarrow E_j^*$, with $i,j = 1,2,\ i \neq j$, over the identity of M such that*

$$\ell_{\sigma_1(c_1)} \circ \gamma^1 = \langle AF, S^1_{c_1}\rangle \tag{2.19}$$
$$\ell_{\sigma_2(c_2)} \circ \gamma^2 = \langle AF, S^2_{c_2}\rangle, \tag{2.20}$$

where $\gamma^1 : E_1 \oplus E_2 \longrightarrow E_2$ and $\gamma^2 : E_1 \oplus E_2 \longrightarrow E_1$ are the forgetful projections:

$$\gamma^1(e_1,e_2) = e_2 \quad \textit{and} \quad \gamma^2(e_1,e_2) = e_1.$$

Proof. Take $c_1 \in \Gamma(C_1)$. Since F is bilinear the map $\langle AF, S_{c_1}\rangle(e_1,e_2) = F(c_1(p),e_2)$ is a linear function on E_2. Then there exists a map $\sigma_1 : \Gamma(C_1) \longrightarrow \Gamma(E_2^*)$ such that $\langle AF, S^1_{c_1}\rangle = \ell_{\sigma_1(c_1)} \circ \gamma^2$. In the same way, there exists a map $\sigma_2 : \Gamma(C_2) \longrightarrow \Gamma(E_1^*)$ such that $\langle AF, S^2_{c_2}\rangle = \ell_{\sigma_2(c_2)} \circ \gamma^1$. □

Now we work with linear sections. Let $X = (X_1,X_2)$ be a linear section of $A_\mathcal{E}$ with $X_i \in \Gamma_{\mathrm{lin}}(A_{\mathcal{E}_i}, E_i)$ covering the same section $a \in \Gamma(A)$. Let $\overrightarrow{X} = (\overrightarrow{X_1}, \overrightarrow{X_2})$ be the corresponding right invariant vector field, and denote by ϕ^i_t their flows. Note that the flows $\phi^i_t : \mathcal{E}_i \longrightarrow \mathcal{E}_i$ are linear over $\mathcal{G}$ because the vector fields $\overrightarrow{X_i}$ are linear. Then

$$\begin{aligned}
\mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\eta_1+\mu_1,\eta_2) &= \frac{d}{dt}\Big|_{t=0} F(\phi^1_t(\eta_1+\mu_1),\phi^2_t(\eta_2))\\
&= \frac{d}{dt}\Big|_{t=0} F(\phi^1_t(\eta_1)+\phi^1_t(\mu_1),\phi^2_t(\eta_2))\\
&= \frac{d}{dt}\Big|_{t=0} \big(F(\phi^1_t(\eta_1),\phi^2_t(\eta_2)) + F(\phi^1_t(\mu_1),\phi^2_t(\eta_2))\big)\\
&= \mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\eta_1,\eta_2) + \mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\mu_1,\eta_2).
\end{aligned}$$

Also we have

$$\mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\eta_1,\eta_2+\mu_2) = \mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\eta_1,\eta_2) + \mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F(\eta_1,\mu_2).$$

Hence $\mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})} F \in C^\infty_{bil}(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \Gamma(\mathcal{E}_1^* \otimes \mathcal{E}_2^*)$. Since F is multiplicative, taking units we obtain $\langle AF, (X_1,X_2)\rangle \in C^\infty_{bil}(E_1 \oplus E_2)$.

Proof. *Theorem 2.21.* Since F is bilinear and multiplicative $\langle AF,(X_1,X_2)\rangle \in C^\infty_{bil}(E_1\oplus E_2)$. Define

$$\mathbf{D}:\Gamma_{\mathrm{lin}}(A_{\mathcal{E}_1})\times_{\Gamma(A)}\Gamma_{\mathrm{lin}}(A_{\mathcal{E}_2})\longrightarrow\Gamma(E_1^*\otimes E_2^*)$$

by

$$\ell_{\mathbf{D}(X_1,X_2)}=\langle AF,(X_1,X_2)\rangle.$$

Then

$$\mathcal{L}_{(\overrightarrow{X_1},\overrightarrow{X_2})}F=\widehat{t}^*\langle AF,(X_1,X_2)\rangle=\widehat{t}^*(\ell_{\mathbf{D}(X_1,X_2)})=\ell_{\mathcal{T}(\mathbf{D}(X_1,X_2))},$$

where $\widehat{t}:\mathcal{E}\longrightarrow E$ is the target map. Let $h\in C^\infty(M)$ and let $X_i\in\Gamma_{\mathrm{lin}}(A_{\mathcal{E}_i})$. Then $q_i^*(h)X_i\in\Gamma_{\mathrm{lin}}(A_{\mathcal{E}_i})$. Then

$$\begin{aligned}\langle\mathbf{D}(q_1^*(h)X_1,q_2^*(h)X_2),(e_1,e_2)\rangle &= AF(h(p)X_1(e_1),h(p)X_2(e_2))\\ &= h(p)AF(X_1(e_1),X_2(e_2))\end{aligned}$$

which implies that $\mathbf{D}$ is a $C^\infty(M)$-linear. The other two equations follow by Proposition (2.22). Conversely, define an operator μ as follow:

$$\langle\mu,(X_1,X_2)\rangle = \ell_{\mathbf{D}(X_1,X_2)} \tag{2.21}$$
$$\langle\mu,S_{c_i}\rangle = \ell_{\sigma(c_i)}\circ\gamma^j \tag{2.22}$$

for $X_i\in\Gamma_{lin}(A_{\mathcal{E}_i},E_i)$ and $c_i\in\Gamma(C_i)$, $i,j=1,2$, $i\neq j$. If $T_i\in\Gamma(Hom(E_i,C_i))$ and we consider its associated core linear right invariant vector field (S_{T_1},S_{T_2}), then by conditions (2.18) μ is well defined on all linear section. The equations (2.18) together with the $C^\infty(M)$-linearity of $\mathbf{D}$ allow to extend μ by $C^\infty(E)$-linearity to all sections in $\Gamma(A_{\mathcal{E}},E)$. Then the map $\mu:A_{\mathcal{E}}\longrightarrow\mathbb{R}$ is well defined and linear. Since the space of sections $\Gamma(A_{\mathcal{E}},E)$ is generated as $C^\infty(E)$-module by the linear and core sections, the equations (2.18) also imply that $\mathcal{L}_{\overrightarrow{Y}}F=\overline{t}^*\langle\mu,Y\rangle$ for every section Y. Then μ is a Lie algebroid function which we can integrate to a multiplicative function $F_\mu\in C^\infty(\mathcal{E})$. By a similar argument on Theorem 2.4 we get $F=F_\mu$. Finally we prove that μ is bilinear with respect to A. Let $\alpha_1,\beta_1\in A_{\mathcal{E}_1}$ be projectable over e_1, $d_1\in E_1$, respectively, and over $a\in A$, and let $\alpha_2\in A_{\mathcal{E}_2}$ be projectable over $e_2\in E_2$ and over $a\in A$. Take $(X_1,X_2)\in\Gamma_{\mathrm{lin}}(A_{\mathcal{E}},E)$ such that

$$\begin{aligned}\alpha_1 &= X_1(e_1)\\ \beta_1 &= X_1(d_1)+_{E_1}\overline{c}_1\\ \alpha_2 &= X_2(e_2).\end{aligned}$$

Then

$$\begin{aligned}
\mu(\alpha_1 +_A \beta_1, \alpha_2) &= \mu(X_1(e_1) +_A (X_1(d_1) +_{E_1} S_\xi(d_1)), X_2(e_2)) \\
&= \mu\big((X_1(e_1) +_A X_1(d_1)) +_E S_\xi(d_1), X_2(e_2)\big) \quad \text{by interchange law} \\
&= \mu\big(X_1(e_1+d_1) +_{E_1} S_\xi(d_1), X_2(e_2)\big) \quad \text{by linearity of } X_1 \\
&= \mu\big((X_1(e_1+d_1), X_2(e_2)) +_E (S_\xi(d_1), 0)\big) \\
&= \mu\big(X_1(e_1+d_1), X_2(e_2)\big) + \mu\big(S_\xi(d_1), 0\big) \quad \text{by linearity of } \mu \text{ r.t } E \\
&= \langle \mathbf{D}(X_1, X_2), (e_1 + d_1, e_2)\rangle + \mu\big(S_\xi(d_1), 0\big) \\
&= \langle \mathbf{D}(X_1, X_2), (e_1, e_2)\rangle + \langle \mathbf{D}(X_1, X_2), (d_1, e_2)\rangle + \mu\big(S_\xi(d_1), 0\big) \\
&= \mu(X_1(e_1), X_2(e_2)) + \mu(X_1(d_1), X_2(e_2)) + \mu\big(S_\xi(d_1), 0\big) \\
&= \mu(\alpha_1) + \mu(X_1(d_1) +_{E_1} S_\xi(d_1), X_2(e_2)) +_{E_2} 0) \\
&= \mu(\alpha_1, \alpha_2) + \mu(\beta_1, \alpha_2).
\end{aligned}$$

Hence μ is bilinear over A, and by Proposition 2.11 it follows that F is bilinear over $\mathcal{G}$. □

If we consider, for example a symplectic form $\omega \in \Omega^2(\mathcal{G})$ or a Poisson structure $\pi \in \Gamma(\wedge^2 T^*\mathcal{G})$ on a Lie groupoid $\mathcal{G}$, when we look at them as functions $F_\omega : T\mathcal{G} \oplus T\mathcal{G} \longrightarrow \mathbb{R}$ and $F_\pi : T^*\mathcal{G} \oplus T^*\mathcal{G} \longrightarrow \mathbb{R}$, respectively, they are skew-symmetric. The next proposition consider bilinear cocycles which are symmetric or skew-symmetric.

Proposition 2.23. *Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over a source 1-connected Lie groupoid $\mathcal{G} \rightrightarrows M$. Let $\mathcal{E}^2$ be the VB-groupoid sum of two copies of $\mathcal{E}$. Then a symmetric (resp. skew-symmetric) function $F \in C^\infty(\mathcal{E}^2)$ is a bilinear groupoid function if and only if*

$$F\big|_{E\oplus E} = 0$$

and there exists a $C^\infty(M)$-linear map $\mathbf{D} : \Gamma_{lin}(A_\mathcal{E}, E) \longrightarrow \Gamma(E^* \otimes E^*)$ and a vector bundle morphism $\sigma : C \longrightarrow E^*$ such that

- $\ell_{\mathbf{D}(X)} \in C^\infty(E \oplus E)$ *is symmetric (resp. skew-symmetric)*

and such that

$$\left\{ \begin{array}{l} \mathcal{L}_{(\overrightarrow{X}, \overrightarrow{X})} F = \ell_{\mathcal{T}(\mathbf{D}(X))} \\ \iota_{\mathcal{T}(c)} F = \mathcal{T}(\sigma(c)) \qquad (\textit{resp.} \quad \iota_{\mathcal{T}(c)} F = -\mathcal{T}(\sigma(c))) \end{array} \right. \tag{2.23}$$

Proof. Since F is a bilinear multiplicative function follows by Theorem 2.21 that there exist a triple $(\widetilde{\mathbf{D}}, \sigma_1, \sigma_2)$ where

- $\widetilde{\mathbf{D}} : \Gamma_{\mathrm{lin}}(A_\mathcal{E}, E) \times_{\Gamma(A)} \Gamma_{\mathrm{lin}}(A_\mathcal{E}, E) \longrightarrow \Gamma(E^* \otimes E^*)$
- $\sigma_1, \sigma_2 : C \longrightarrow E^*$.

Since it suffices to take linear sections of the form (X, X) with $X \in \Gamma_{\mathrm{lin}}(A_{\mathcal{E}}, E)$, the operator $\mathbf{D}$ is defined by

$$\mathbf{D}(X) = \widetilde{\mathbf{D}}(X, X).$$

The linear maps σ_i are determined by

$$\mathcal{L}_{\overrightarrow{S_{c,0}}} F = \ell_{\mathcal{T}(\sigma_1(c))} \circ \pi_2 \quad \text{and} \quad \mathcal{L}_{\overrightarrow{S_{0,c}}} F = \ell_{\mathcal{T}(\sigma_2(c))} \circ \pi_1.$$

If F is symmetric then

$$\begin{aligned} \ell_{\mathcal{T}(\sigma_1(c))} \circ \pi_2(\eta_1, \eta_2) &= \mathcal{L}_{\overrightarrow{S_{c,0}}} F(\eta_1, \eta_2) = F(\mathcal{T}(c)(g), \eta_2) \\ &= F(\eta_2, \mathcal{T}(c)(g)) = \mathcal{L}_{\overrightarrow{S_{0,c}}} F(\eta_2, \eta_1) \\ &= \ell_{\mathcal{T}(\sigma_2(c))} \circ \pi_1(\eta_2, \eta_1) \end{aligned}$$

and since $\mathcal{T}$ is injective follow that $\sigma_1 = \sigma_2 =: \sigma$. Moreover the equation

$$\iota_{\mathcal{T}(c)} F = \mathcal{T}(\sigma(c))$$

holds. Conversely, defining a map $\mu : \Gamma(A_{\mathcal{E}^2}) \longrightarrow C^\infty(E^2)$ by

$$\begin{aligned} \langle \mu, (X, X) \rangle &= \ell_{\mathbf{D}(X)} \\ \langle \mu, S_{c,0} \rangle &= \ell_{\sigma(c)} \circ \gamma^2 \\ \langle \mu, S_{0,c} \rangle &= \ell_{\sigma(c)} \circ \gamma^1 \end{aligned}$$

it follows by similar arguments as in Theorem 2.21 that μ is a well defined infinitesimal cocycle which integrates to F. By the properties of $\mathbf{D}$ and σ it follows that μ is symmetric. Hence by Proposition 2.11 we get that F symmetric. The skew-symmetric case is analogous. $\square$

2.3.2 Multilinear cocycles

We now extend the previous discussion to the case of a Whitney sum of k VB-groupoids over $\mathcal{G}$.

Let

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{Q_i} & \mathcal{G} \\ \bar{s}_i \downarrow\downarrow \bar{t}_i & & \mathbf{s} \downarrow\downarrow \mathbf{t} \\ E_i & \xrightarrow{q_i} & M \end{array}$$

be k VB-groupoids over $\mathcal{G} \rightrightarrows M$ with core bundles C_i. Let $\mathcal{E} := \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ be the Whitney sum of vector bundles over $\mathcal{G}$, which is a Lie groupoid over the Whitney sum $E := E_1 \oplus \cdots \oplus E_k$.

Definition 2.24. A function $F \in C^\infty(\mathcal{E})$ is called a *k-linear cocycle* if it is a multiplicative function with respect to the groupoid structure, and if it is a k-linear map with respect to the vector bundle structure $\mathcal{E} \longrightarrow \mathcal{G}$.

This means that

- $F((\eta_1, \dots, \eta_k)\cdot(\mu_1, \dots, \mu_k)) = F(\eta_1, \dots, \eta_k) + F(\mu_1, \dots, \mu_k)$ for all composable elements, and
- $F(\eta_1, \dots, \eta_i + \mu_i, \dots, \eta_k) = F(\eta_1, \dots, \eta_i, \dots, \eta_k) + F(\eta_1, \dots, \mu_i, \dots, \eta_k)$, for $\eta_j \in (\mathcal{E}_j)_g$, $\mu_i \in (\mathcal{E}_i)_g$, for all i.

Remark 2.25. Notation. To simplify we adopt the following notation: if $c_i \in \Gamma(C_i)$ then $\mathcal{T}(c_i) \in \Gamma(\mathcal{E}_i)$, hence

$$\iota_{\mathcal{T}(c_i)} F := F(\cdot, \dots, \overbrace{\mathcal{T}(c_i)}^{i}, \dots, \cdot) \quad \in \; \Gamma(\mathcal{E}_1^* \otimes \cdots \otimes \widehat{\mathcal{E}_i^*} \otimes \cdots \otimes \mathcal{E}_k^*)$$

Proposition 2.26. *If $\mathcal{G}$ is a source connected Lie groupoid, then a function $F \in C^\infty(\mathcal{E})$ is a k-linear cocycle if and only if*

$$F\big|_E = 0$$

and there exist a $C^\infty(M)$-linear map

$$\mathbf{D} : \Gamma_{lin}(A_{\mathcal{E}_1}, E_1) \times_{\Gamma(A)} \cdots \times_{\Gamma(A)} \Gamma_{lin}(A_{\mathcal{E}_k}, E_k) \longrightarrow \Gamma(E_1^* \otimes \cdots \otimes E_k^*)$$

and vector bundle maps $\sigma_i : C_i \longrightarrow E_1^* \otimes \cdots \otimes \widehat{E_i^*} \otimes \cdots \otimes E_k^*$ over the identity of M, for $i = 1, \dots, k$, such that

$$\begin{cases} \mathcal{L}_{\overrightarrow{X}} F = \ell_{\mathcal{T}(D(X))} \\ \iota_{\mathcal{T}(c_i)} F = \mathcal{T}(\sigma_i(c_i)) \quad i = 1, \dots, k \end{cases} \tag{2.24}$$

Proof. Since F is k-linear by an analogous argument of the bilinear case it follows that

$$\mathcal{L}_{(\overrightarrow{X_1}, \dots, \overrightarrow{X_k})} F \;\in\; C^\infty_{k-\text{linear}}(\mathcal{E}) \simeq \Gamma(\mathcal{E}_1^* \otimes \cdots \otimes \mathcal{E}_k^*),$$

with $X_i \in \Gamma_{\text{lin}}(A_{\mathcal{E}_i}, E_i)$ a linear section, and since F is multiplicative, taking units we have then that $\langle AF, (X_1, \dots X_k)\rangle \in C^\infty_{k-\text{linear}}(E) \simeq \Gamma(E_1^* \otimes \cdots \otimes E_k^*)$. Hence we define the operator $\mathbf{D}(X_1, \dots X_1)$ as the unique section in $\Gamma(E_1^* \otimes \cdots \otimes E_k^*)$ such that

$$\ell_{\mathbf{D}(X_1, \dots X_1)} = \langle AF, (X_1, \dots X_k)\rangle. \tag{2.25}$$

For core sections, if $c_i \in \Gamma(C_i)$ then

$$\mathcal{L}_{(0, \dots, \overrightarrow{S_{c_i}}, \dots, 0)} F(\eta_1, \dots, \eta_k) = F(\eta_1, \dots, \overline{c_i(\mathbf{t}(g))}, \dots, \eta_k) = \iota_{\mathcal{T}(c_i)} F \circ \pi_i(\eta_1, \dots, \eta_k)$$

by multilineartiy of F, and by multiplicative condition, taking units we have

$$\langle AF, (0, \ldots, S_{c_i}, \ldots, 0)\rangle =: \ell_{\sigma_i(c_i)} \circ \gamma^i$$

where $\gamma^i : \oplus E_j \longrightarrow \oplus_{j\neq i} E_j$ are the forgetful projections

$$\gamma^i(e_1, \ldots, e_k) = (e_1, \ldots, \widehat{e_i}, \ldots, e_k), \tag{2.26}$$

i.e., γ^i forgets its i-entry, and $\sigma_i : C_i \longrightarrow E_1^* \otimes \cdots \otimes \widehat{E_i^*} \otimes \cdots \otimes E_k^*$ are vector bundle maps. Conversely define a map

$$\langle \mu, (X_1, \ldots, X_k)\rangle = \ell_{\mathbf{D}(X_1,\ldots,X_k)} \tag{2.27}$$
$$\langle \mu, S_{c_i}\rangle = \ell_{\sigma_i(c_i)} \circ \gamma^j \tag{2.28}$$

for $X_i \in \Gamma_{lin}(A_{\mathcal{E}_i}, E_i)$ and $c_i \in \Gamma(C_i)$, $i, j = 1, \ldots, k$, $i \neq j$. By arguments similar to the previous subsection, this is a well-defined map which is a morphism of Lie algebroids and k-linear with respect to the linear structure over A. And since $\mathcal{G}$ is source connected, and using similar arguments of the previous sections, the function who integrates μ is F. □

2.4 Infinitesimal bilinear cocycles

Let

$$\begin{tikzcd} \mathcal{A}_i \arrow[r] \arrow[d] & A \arrow[d] \\ E_i \arrow[r] & M \end{tikzcd}$$

be two VB-algebroids over A with core bundles C_i. Consider the VB-algebroid sum $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. In this section we study functions defined on $\mathcal{A}$ which are bilinear with respect to the linear structure over A, and morphism with respect to the Lie algebroid structure. We give a description of this kind of functions in terms of some infinitesimal data, and then we establish a correspondence with bilinear cocycles.

Definition 2.27. A function $F \in C^\infty(\mathcal{A})$ is a *bilinear $\mathcal{A}$-cocycle* if it is a cocycle on $\mathcal{A}$ and if it is bilinear with respect to the vector bundle structure $\mathcal{A}_1 \oplus \mathcal{A}_2 \longrightarrow A$.

Proposition 2.28. *Let $f \in C^\infty(\mathcal{A})$ be a bilinear $\mathcal{A}$-cocycle. Then there exists a triple $(\mathbf{D}, \sigma_1, \sigma_2)$ where*

- $\mathbf{D} : \Gamma_{lin}(A_{\mathcal{E}_1}, E_1) \times_{\Gamma(A)} \Gamma_{lin}(A_{\mathcal{E}_2}, E_2) \longrightarrow \Gamma(E_1^* \otimes E_2^*)$ *is a $C^\infty(M)$-linear,*
- $\sigma_1 : C_1 \longrightarrow E_2^*$ *and* $\sigma_2 : C_2 \longrightarrow E_1^*$ *are vector bundle maps covering the identity of M,*

such that

$$\mathbf{D}(S_{T_1}, S_{T_2}) = \sigma_1(T_1) \circ \gamma^1 + \sigma_2(T_2) \circ \gamma^2 \quad \textit{for all } T_i \in \Gamma(Hom(E_i, C_i)) \tag{2.29}$$

and satisfy

$$\mathbf{D}([X,Y]) = (\nabla^1)^*_X \mathbf{D}(Y) - (\nabla^1)^*_Y \mathbf{D}(X) \tag{2.30}$$
$$\iota_{\partial_1(c_1)} \mathbf{D}(X) = (\nabla^1)^*_{X_2} \sigma_1(c_1) - \sigma_1(\nabla^0_{X_1} c_1) \tag{2.31}$$
$$\iota_{\partial_2(c_2)} \mathbf{D}(X) = (\nabla^1)^*_{X_1} \sigma_2(c_2) - \sigma_2(\nabla^0_{X_2} c_2) \tag{2.32}$$
$$\partial_2^* \circ \sigma_1 = \sigma_2^* \circ \partial_1, \tag{2.33}$$

where $\partial_i : C_i \longrightarrow E_i$ are the core anchor maps defined in (1.23) Reciprocally, any triple $(\mathbf{D}, \sigma_1, \sigma_2)$ satisfying the previous conditions induces a bilinear cocycle on $\mathcal{A}$.

Proof. Let $f \in C^\infty(\mathcal{A})$ be a bilinear cocycle. Consider it as a map over sections $f : \Gamma(\mathcal{A}) \longrightarrow C^\infty(E)$. Take a linear section defined by $X_a = (X_a^1, X_a^2)$. Then

$$\begin{aligned} \langle f, X_a \rangle(\lambda e_1 + d_1, e_2) &= f(X_a^1(\lambda e_1 + d_1), X_a^2(e_2)) \\ &= f(\lambda X_a^1(e_1) +_A X_a^1(d_1), X_a^2(e_2)) \\ &= \lambda f(X_a^1(e_1), X_a^2(e_2)) + f(X_a^1(d_1), X_a^2(e_2)) \end{aligned}$$

because f is bilinear over A. So $\langle f, X_a \rangle \in C^\infty_{\mathrm{bil}}(E_1 \oplus E_2) \simeq \Gamma(E_1^* \otimes E_2^*)$. We define then

$$\mathbf{D} : \Gamma_{\mathrm{lin}}(\mathcal{A}_1, E_1) \times_{\Gamma(A)} \Gamma_{\mathrm{lin}}(\mathcal{A}_2, E_2) \longrightarrow \Gamma(E_1^* \otimes E_2^*).$$

by

$$\mathbf{D} = f\big|_{\Gamma_{\mathrm{lin}}(\mathcal{A}_1, E_1) \times_{\Gamma(A)} \Gamma_{\mathrm{lin}}(\mathcal{A}_2), E_2}.$$

Now, let $c_1 \in \Gamma(C_1)$. Then

$$\begin{aligned} \langle f, (S_{c_1}, 0) \rangle(e_1, e_2) &= f(0_{e_1} +_A \overline{c_1(p)}, 0_{e_2}) \\ &= f(0_{e_1}, 0_{e_2}) + f(\overline{c_1(p)}, 0_{e_2}) \quad \text{by bilinearity} \\ &= f(\overline{c_1(p)}, 0_{e_2}) \end{aligned}$$

Note that the element 0_{e_2} projects to $e_2 \in E_2$ and to $0 \in A$ but 0_{e_2} is not the zero element of the fiber $(\mathcal{A}_2)_0$ over $0 \in A$. So this induces a map $\sigma_1 : \Gamma(C_1) \longrightarrow \Gamma(E_2^*)$ by:

$$\langle \sigma_1(c_1), e_2 \rangle = f(\overline{c_1(p)}, 0_{e_2}).$$

In the same way, we define $\sigma_2 : \Gamma(C_2) \longrightarrow \Gamma(E_1^*)$ by

$$\langle \sigma_2(c_2), e_1 \rangle = f(0_{e_1}, \overline{c_2(p)}).$$

Note that the maps σ_1 and σ_2 are both $C^\infty(M)$-linear, so we have two vector bundle morphisms $\sigma_1 : C_1 \longrightarrow E_2^*$ and $\sigma_2 : C_2 \longrightarrow E_1^*$. To see the compatibility condition, let $T_i \in \Gamma(\mathrm{Hom}(E_i, C_i))$ and consider the associated linear core sections S_{T_i}. Then

$$\begin{aligned} \langle \mathbf{D}(S_{T_1}, S_{T_2}) \rangle(e_1, e_2) &= \langle f, (S_{T_1}, S_{T_2}) \rangle(e_1, e_2) \\ &= f(S_{T_1}(e_1), e_2) + f(e_1, S_{T_2}(e_2)) \\ &= \langle \sigma_1(T_1)(e_1), e_2 \rangle + \langle e_1, \sigma_2(T_2)(e_2) \rangle. \end{aligned}$$

Now we will check the equations. Since f is a cocycle, it satisfies

$$\langle f, [Y.Z]\rangle = \mathcal{L}_{\rho_{\mathcal{A}}(Y)}\langle f, Z\rangle - \mathcal{L}_{\rho_{\mathcal{A}}(Z)}\langle f, Y\rangle$$

for all sections $Y, Z \in \Gamma(\mathcal{A})$. First, take two linear sections $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Their Lie bracket is $[X, Y] = ([X_1, Y_1], [X_2, Y_2])$. By definition of the connection $(\nabla^1)^* : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$ (see Equation (1.21)), we have that

$$X \cdot \mathbf{D}(Y) = (\nabla^1)^*_X \mathbf{D}(Y) := \mathcal{L}_{\rho_{\mathcal{A}}(X)}\mathbf{D}(Y).$$

Then the first equation holds. Let $(S_{c_1}, 0)$ be a core section. Then

$$[(X_1, X_2), (S_{c_1}, 0)] = (S_{\nabla^0_{X_1} c_1}, 0).$$

So $\langle f, [(X_1, X_2), (S_{c_1}, 0)]\rangle = \sigma_1(\nabla^0_{X_1} c_1) \circ \gamma_1$. On the other hand we have

$$\mathcal{L}_{\rho_{\mathcal{A}}(X)}(\sigma_1(c_1) \circ \gamma_1) \quad = \quad \nabla^0_{X_2}(\sigma_1(c_1)) \circ \gamma_1,$$

and

$$\mathcal{L}_{\rho_{\mathcal{A}}(S_{c_1},0)}\ell_{\mathbf{D}(X)} \quad = \quad \iota_{\partial_1(c_1)}\mathbf{D}(X) \circ \gamma_1.$$

Therefore the second equation follows. In the same way we obtain the third equation. For the last one we have that $[(S_{c_1}, 0), (0, S_{c_2})] = 0$. Also

$$\mathcal{L}_{\rho_{\mathcal{A}}(S_{c_1},0)}\sigma_2(c_2) = \langle \sigma_2(c_2), \partial_1(c_1)\rangle$$

and

$$\mathcal{L}_{\rho_{\mathcal{A}}(0,S_{c_2})}\sigma_1(c_1) = \langle \sigma_1(c_1), \partial_2(c_2)\rangle$$

which imply the last equation. Conversely, given the triple $(\mathbf{D}, \sigma_1, \sigma_2)$ satisfying the equations, we have to construct a bilinear cocycle $f : \mathcal{A} \longrightarrow \mathbb{R}$. We will define it at the level of sections and then we will prove that it is a $C^\infty(E)$-linear. Take a linear section covering $a \in \Gamma(A)$ of the form (X_1, X_2), then define

$$\langle f, (X_1, X_2)\rangle := \ell_{\mathbf{D}(X_1, X_2)}.$$

For a core section $(S_{c_1}, 0)$ with $c_1 \in \Gamma(C_1)$, define

$$\langle f, (S_{c_1}, 0)\rangle(e_1, e_2) := \langle \sigma_1(c_1), e_2\rangle.$$

Analogously, for a core section $(0, S_{c_2})$ with $c_2 \in \Gamma(C_2)$, define

$$\langle f, (0, S_{c_2})\rangle(e_1, e_2) := \langle e_1, \sigma_2(c_2)\rangle.$$

The compatibility condition implies that f is well defined in all linear sections. So extending f to all sections by linearity we have a well defined linear map $f : \Gamma(\mathcal{A}, E) \longrightarrow C^\infty(E)$. Moreover, by the $C^\infty(M)$-linearity of $\mathbf{D}$, σ_1, σ_2 we can extend

f by $C^\infty(E)$-linearity. Hence we have a linear function $f : \mathcal{A} \longrightarrow \mathbb{R}$ over E. The equations satisfied by $\mathbf{D}$, σ_1, σ_2 imply that f is a morphism of Lie algebroids. To see the bilinearity over A, let $\alpha_1, \beta_1 \in \mathcal{A}_1$ and $\alpha_2 \in \mathcal{A}_2$, projectable over $a \in A$ and over e_1, $d_1 \in E_1$ and $e_2 \in E_2$, respectively. Let $X \in \Gamma_{lin}(\mathcal{A}_1, E_1)$ such that $X(e_1) = \alpha_1$. Then $\beta_1 = X(d_1) +_{E_1} S_c(d_1)$ for some section $c \in \Gamma(C_1)$, and let $Y \in \Gamma_{lin}(\mathcal{A}_1, E_1)$ such that $Y(e_2) = \alpha_2$. Then

$$
\begin{aligned}
f(\alpha_1 +_A \beta_1, \alpha_2) &= f(X(e_1) +_A (X(d_1) +_{E_1} S_c(d_1)), Y(e_2)) \\
&= f\big((X(e_1) +_A X(d_1)) +_{E_1} S_c(d_1), Y(e_2)\big) \quad \text{by interchange law} \\
&= f\big(X(e_1 + d_1) +_{E_1} S_c(d_1), Y(e_2)\big) \quad \text{by linearity of } X \\
&= f\big(X(e_1 + d_1), Y(e_2)\big) + f\big(S_c(d_1), Y(e_2)\big) \quad \text{by linearity w.r.t } E \\
&= \langle \mathbf{D}(X, Y), (e_1 + d_1, e_2)\rangle + f\big(S_c(d_1), Y(e_2)\big) \\
&= \langle \mathbf{D}(X, Y), (e_1, e_2)\rangle + \langle \mathbf{D}(X, Y), (d_1, e_2)\rangle + f\big(S_c(d_1), Y(e_2)\big) \\
&= f(X(e_1), Y(e_2)) + f(X(d_1), Y(e_2)) + f\big(S_c(d_1), Y(e_2)\big) \\
&= f(\alpha_1, \alpha_2) + f(X(d_1) +_{E_1} S_c(d_1), Y(e_2)) \\
&= f(\alpha_1, \alpha_2) + f(\beta_1, \alpha_2).
\end{aligned}
$$

□

Definition 2.29. A triple $(\mathbf{D}, \sigma_1, \sigma_2)$ satisfying Equations (2.30), (2.31), (2.32), (2.33), and the compatibility condition (2.29) is called the *component of a bilinear $\mathcal{A}$-cocycle* over A.

Now we enunciate a global-infinitesimal correspondence between bilinear groupoid cocycles and bilinear algebroid cocycles. Let $\mathcal{E}_i \rightrightarrows E_i$, $i = 1, 2$ be two VB-groupoids over $\mathcal{G}$, and let $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ be the VB-groupoid sum. Denote by $A_{\mathcal{E}} = A_{\mathcal{E}_1} \oplus A_{\mathcal{E}_2}$ the VB-algebroid of $\mathcal{E}$.

Theorem 2.30. *Every bilinear $\mathcal{E}$-cocycle induces a triple $(\mathbf{D}, \sigma_1, \sigma_2)$ satisfying the Equations (2.29)-(2.33). Moreover if $\mathcal{G}$ is source connected, there is a one to one correspondence between bilinear $\mathcal{E}$-cocycle and such triples $(\mathbf{D}, \sigma_1, \sigma_2)$ given by*

$$
\begin{cases}
\mathcal{L}_{\overrightarrow{X}} F = \ell_{\mathcal{T}(D(X))} \\
\mathcal{L}_{\overrightarrow{S_{c_1,0}}} F = \ell_{\mathcal{T}(\sigma_1(c_1))} \circ \pi_2 \\
\mathcal{L}_{\overrightarrow{S_{0,c_2}}} F = \ell_{\mathcal{T}(\sigma_2(c_2))} \circ \pi_1
\end{cases}
$$

Proof. If $F \in C^\infty(\mathcal{E})$ is a bilinear cocycle then by Theorem 2.21 there exists a triple $(\mathbf{D}, \sigma_1, \sigma_2)$, which defines the components of a bilinear $A_{\mathcal{E}}$-cocycle over A because they are associated to the bilinear cocycle AF. □

In the case where the VB-algebroids $\mathcal{A}_i$ are the same, we can consider bilinear cocycles f which are symmetric (resp. skew-symmetric). Consider the VB-algebroid

$$\begin{tikzcd}
\mathcal{A}^2 = \mathcal{A} \oplus \mathcal{A} \arrow[r] \arrow[d] & A \arrow[d] \\
E^2 = E \oplus E \arrow[r] & M.
\end{tikzcd}$$

In this situation we have the following description

Proposition 2.31. *There is one-to-one correspondence between symmetric (resp. skew-symmetric) bilinear cocycle on $\mathcal{A}^2$ and pairs $(\mathbf{D}, \sigma)$ where the operator $\mathbf{D} : \Gamma_{lin}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes E^*)$ is a $C^\infty(M)$-linear, $\sigma : C \longrightarrow E^*$ is a vector bundle map over the identity of M such that*

- $\ell_{\mathbf{D}(X)} \in C^\infty(E^2)$ *is a symmetric (resp. skew-symmetric) function*
- $\mathbf{D}(S_T) = \langle \sigma \circ T + (\sigma \circ T)^*, id_E \rangle$ *for every $T \in \Gamma(Hom(E, C))$*

and satisfy

$$\begin{aligned}
\mathbf{D}([X,Y]) &= (\nabla^1)^*_X \mathbf{D}(Y) - (\nabla^1)^*_Y \mathbf{D}(X) \\
\iota_{\partial(c)} \mathbf{D}(X) &= (\nabla^1)^*_X \sigma(c) - \sigma(\nabla^0_X c) \\
\partial^* \circ \sigma &= \sigma^* \circ \partial \quad (\textit{resp.} \quad \partial^* \circ \sigma = -\sigma^* \circ \partial)
\end{aligned}$$

Proof. By Proposition 2.28 there is a one-to-one correspondence between bilinear cocycles and component of a infinitesimal bilinear cocycles $(\widetilde{\mathbf{D}}, \sigma_1, \sigma_2)$ over A. Since we have two copies of the same VB-algebroid, we can define

$$\mathbf{D} : \Gamma_{\text{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes E^*) \qquad \mathbf{D}(X) = \widetilde{\mathbf{D}}(X, X),$$

and the condition of symmetric (resp. skew-symmetric) of f implies that $\ell_{\mathbf{D}(X)}$ is symmetric (resp. skew-symmetric). The vector bunble maps $\sigma_1, \sigma_2 : C \longrightarrow E^*$ are determined by

$$\langle f, (S_c, 0) \rangle (e_1, e_2) := \langle \sigma_1(c), e_2 \rangle \quad \text{and} \quad \langle f, (0, S_c) \rangle (e_1, e_2) := \langle e_1, \sigma_2(c) \rangle.$$

If f is symmetric then

$$\begin{aligned}
\langle \sigma_1(c), e_2 \rangle &= \langle f, (S_c, 0) \rangle (e_1, e_2) = \langle f, (0, S_c) \rangle (e_2, e_1) \\
&= \langle \sigma_2(c), e_2 \rangle,
\end{aligned}$$

which implies that $\sigma_1 = \sigma_2$. This last condition implies that

$$\text{Eq. } (2.31) \;=\; \text{Eq. } (2.32) \;=\; \text{Eq. } (\iota_{\partial(c)} \mathbf{D}(X) = (\nabla^1)^*_X \sigma(c) - \sigma(\nabla^0_X c))$$

and

$$\text{Eq. } (2.33) \quad \Longrightarrow \quad \partial^* \circ \sigma = \sigma^* \circ \partial.$$

In the case when f is skew-symmetric, we have $\sigma_1 = -\sigma_2$, which implies that equations (2.31), (2.32) and $\iota_{\partial(c)}\mathbf{D}(X) = (\nabla^1)^*_X\sigma(c) - \sigma(\nabla^0_X c)$ are all equivalent. Also, the condition $\sigma_1 = -\sigma_2$ implies $\partial^* \circ \sigma = -\sigma^* \circ \partial$. Conversely, defining a map $\mu : \Gamma(A_{\mathcal{E}^2}) \longrightarrow C^\infty(E^2)$ by

$$\begin{aligned} \langle \mu, (X,X)\rangle &= \ell_{\mathbf{D}(X)} \\ \langle \mu, S_{c,0}\rangle &= \ell_{\sigma(c)} \circ \gamma^2 \\ \langle \mu, S_{0,c}\rangle &= \ell_{\sigma(c)} \circ \gamma^1 \end{aligned}$$

follows by similar arguments on the Proposition 2.28 that μ is well defined. By the properties of $\mathbf{D}$ and σ follow that μ is symmetric. Hence by Proposition 2.11 we get that F symmetric. The skew-symmetric case is analogous. $\square$

2.4.1 Infinitesimal multilinear cocycles

In this subsection we extend bilinear cocycles on VB-algebroids to the general case, i.e., multilinear cocycles. Take k VB-algebroids over A

$$\begin{tikzcd} \mathcal{A}_i \arrow[r] \arrow[d] & A \arrow[d] \\ E_i \arrow[r] & M \end{tikzcd}$$

with core bundles C_i and consider the VB-algebroid sum

$$\begin{tikzcd} \mathcal{A} = \oplus_{i=1}^k \mathcal{A}_i \arrow[r] \arrow[d] & A \arrow[d] \\ E = \oplus_{i=1}^k E_i \arrow[r] & M. \end{tikzcd}$$

Definition 2.32. A function $f \in C^\infty(\mathcal{A})$ is a *k-linear cocycle* if it is a Lie algebroid morphism and if it is k-linear with respect to the vector bundle structure $\oplus_{i=1}^k \mathcal{A}_i \longrightarrow A$.

Definition 2.33. An *infinitesimal k-linear structure* on $\mathcal{A}$ over A is a $(\mathbf{D}, \sigma_i)$, with $i = 1, \ldots, k$, where

$$\mathbf{D} : \Gamma_{\text{lin}}(\mathcal{A}_1, E_1) \times_{\Gamma(A)} \cdots \times_{\Gamma(A)} \Gamma_{\text{lin}}(\mathcal{A}_k, E_k) \longrightarrow \Gamma(E_1^* \otimes \cdots \otimes E_k^*)$$

is a $C^\infty(M)$-linear operator, and $\sigma_i : C_i \longrightarrow \otimes_{j\neq i}^k E_j^*$, for $i = 1, \ldots, k$, are vector bundle maps over the identity of M, such that

$$\mathbf{D}(S_{T_1}, \ldots, S_{T_k}) = \sum_{i=1}^k \langle \sigma_i(T_i), \mathrm{Id}_{E^j}\rangle \quad \text{for all } T_i \in \Gamma(\mathrm{Hom}(E_i, C_i))$$

where $\mathrm{Id}_{E^i} : \oplus_{j\neq i} E_j \longrightarrow \oplus_{j\neq i} E_j$ is the identity in each component and such that

$$\begin{array}{rcl} \mathbf{D}([X,Y]) & = & X \cdot \mathbf{D}(Y) - Y \cdot \mathbf{D}(X) \\ \iota_{\partial_i(c_i)} \mathbf{D}(X_1,\ldots,X_k) & = & (\nabla^1)^*_{(X_1,\ldots,\widehat{X_i},\ldots,X_k)} \sigma_i(c_i) - \sigma_i(\nabla^0_{X_i} c_i) \quad \text{for } i = 1,\ldots k \\ \langle \sigma_i(c_i), \partial_j(c_j) \rangle & = & \langle \sigma_j(c_j), \partial_i(c_i) \rangle \quad \text{for all } i,j = 1,\ldots k \text{ with } i \neq j \end{array}$$

Proposition 2.34. *Let $\mathcal{A} \longrightarrow E$ be a sum of k VB-algebroids over A. Then there is one-to-one correspondence between k-linear algebroid function $f \in C^\infty(\mathcal{A})$ and infinitesimal k-linear structure on $\mathcal{A}$ over A.*

Proof. The proof of this proposition is analogous to the proof of Proposition 2.28. □

We enunciate now the global-infinitesimal correspondence between these multilinear objects.

Theorem 2.35. *Let $\mathcal{E} \rightrightarrows E$ be sum of k VB-groupoids $\mathcal{E}_i \rightrightarrows E_i$ over a source simply connected Lie groupoid $\mathcal{G}$, and denote by $A_{\mathcal{E}}$ its Lie algebroid, which splits naturally as the sum of the k VB-algebroid $A_{\mathcal{E}_i}$. Then there is one-to-one correspondence between k-linear cocycles on $C^\infty(\mathcal{E})$ and infinitesimal k-linear structure on $\mathcal{A}$ over A.*

Chapter 3

Applications to multiplicative structures on Lie groupoids

In this chapter we describe **multiplicative** (p,q)-tensors on a Lie groupoid $\mathcal{G}$ with coefficient in a VB-groupoid $\mathcal{E} \rightrightarrows E$ over $\mathcal{G}$. We will see such tensors as cocycles on some VB-groupoid $\mathbb{G}$ over $\mathcal{G}$, and then we will apply all what we did in the Chapter 2. As a consequence, we generalize the description of multiplicative k-forms (with trivial coefficients [7] and with values in some representation [14]), the description of multiplicative multivector fields [24, 33], with new proofs, and we will consider *multiplicative k-forms with coefficients in a representation up to homotopy.*

Definition 3.1. Let $\pi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{G}$ be a vector bundle over a Lie groupoid $\mathcal{G} \rightrightarrows M$. A *$(p,q)$-tensor field on $\mathcal{G}$ with coefficient in $\mathcal{E}$* is a section $\tau \in \Gamma(\bigotimes^p T^*\mathcal{G} \otimes \bigotimes^q T\mathcal{G} \otimes \mathcal{E})$.

Given a (p,q)-tensor field $\tau \in \Gamma(\bigotimes^p T^*\mathcal{G} \otimes \bigotimes^q T\mathcal{G} \otimes \mathcal{E})$ we associate a componentwise linear function

$$c_\tau : (\oplus_p T\mathcal{G}) \oplus (\oplus_q T^*\mathcal{G}) \oplus \mathcal{E}^* \longrightarrow \mathbb{R}$$

defined by

$$c_\tau(X_1, \ldots, X_p, \mu_1, \ldots, \nu_q, \eta) = \langle \tau(X_1, \ldots, X_p, \mu_1, \ldots, \mu_q), \eta \rangle.$$

where $\mathcal{E}^*$ is the dual of $\mathcal{E}$ over $\mathcal{G}$.

Definition 3.2. Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G}$ with core bundle C. A (p,q)-tensor τ on $\mathcal{G}$ with coefficients in $\mathcal{E}$ is *multiplicative* if the associated componentwise linear function $c_\tau \in C^\infty((\oplus_p T\mathcal{G}) \oplus (\oplus_q T^*\mathcal{G}) \oplus \mathcal{E}^*)$ is multiplicative. We call this τ an *$\mathcal{E}$-valued multiplicative (p,q)-tensor.*

Remark 3.3. Notation. When we have vector bundles E_i over some manifold M, given a section $\varphi \in \Gamma(E_1^* \otimes \cdots \otimes E_k^*)$ we will denote by $c_\varphi : E_1 \oplus \cdots \oplus E_k \longrightarrow \mathbb{R}$ its associated componentwise linear function defined by

$$c_\varphi(e_1, \ldots, e_k) := \langle \varphi, (e_1, \ldots, e_k) \rangle.$$

Our goal now is to describe the infinitesimal counterpart of τ by studying c_τ.

3.1 Core and linear sections

We fix an $\mathcal{E}$-valued multiplicative (p,q)-tensor τ on $\mathcal{G}$, and let c_τ be its associated componentwise linear function. From now to the end of the chapter we will assume that the (p,q)-tensor is skew-symmetric in the first p-components and skew-symmetric in the q-components of $T^*\mathcal{G}$. This is done just to simplify their description but the method applies in general. The case we are interesting have this property.

Before consider the action of c_τ on linear and core sections, for an $\mathcal{E}$-valued multiplicative (p,q)-tensor, we need a proposition, which is the **same** as Proposition 2.7, but we do it explicitly in this context. We denote

- $\mathbb{G}_{\mathcal{E}}^{(p,q)} = (\oplus_p T\mathcal{G}) \oplus (\oplus_q T^*\mathcal{G}) \oplus \mathcal{E}^*$,
- $\mathbb{M}_{\mathcal{E}}^{(p,q)} = (\oplus_p TM) \oplus (\oplus_q A^*) \oplus C^*$.

And when we consider on the Whitney sum of the tangent and cotangent groupoid we will omit the subindice $\mathcal{E}$. Recall the notation for the target maps: $T\mathbf{t} : T\mathcal{G} \longrightarrow TM$, $\widetilde{t} : T^*\mathcal{G} \longrightarrow A^*$, $\bar{t} : \mathcal{E} \longrightarrow E$ and $\bar{t}^* : \mathcal{E}^* \longrightarrow C^*$.

Proposition 3.4. *The pullback map*

$$(\mathbb{T}_{\mathcal{E}}^{(p,q)})^* := \left((T\mathbf{t})^p, (\widetilde{t})^q, \bar{t}^*\right)^* : C^\infty(\mathbb{M}_{\mathcal{E}}^{(p,q)}) \longrightarrow C^\infty(\mathbb{G}_{\mathcal{E}}^{(p,q)})$$

preserves componentwise linear functions. Moreover the pullback function satisfies

$$(\mathbb{T}_{\mathcal{E}}^{(p,q)})^*(c_\varphi) = c_{\mathcal{T}(\varphi)}$$

for every $\varphi \in \Gamma(\wedge^p T^*M \otimes \wedge^q A^* \otimes C)$, where

$$\begin{array}{rcl} \mathcal{T} : \Gamma(\wedge^p T^*M \otimes \wedge^q A^* \otimes C) & \longrightarrow & \Gamma(\wedge^p T^*\mathcal{G} \otimes \wedge^q T\mathcal{G} \otimes \mathcal{E}) \\ \alpha \otimes a \otimes c & \longrightarrow & \mathcal{T}(\alpha \otimes a \otimes c) = \mathbf{t}^*\alpha \otimes \overrightarrow{a} \otimes c^{\mathcal{E}}(g) \end{array}$$

and where $c^{\mathcal{E}}(g) := c(\boldsymbol{t}(g)) \cdot 0_g$ is the core section of $\mathcal{E} \longrightarrow \mathcal{G}$ associated to $c \in \Gamma(C)$ (see equation (2.2)).

Proof. The first part of the Lemma follows from the fact that each one of the maps $T\mathbf{t}, \widetilde{t}$ and $\bar{t}^*$ is a morphism of vector bundles. The second part follows by Proposition 2.7. Nevertheless, we give here another proof using the explicit Lie groupoid structure of the tangent groupoid $T\mathcal{G}$ and of the cotangent groupoid $T^*\mathcal{G}$. Let $\Phi = \alpha \otimes a \otimes c \in \Gamma(\wedge^p T^*M \otimes \wedge^q A^* \otimes C)$ and $V_1, \ldots, V_p \in T_g\mathcal{G}$, $\mu_1, \ldots, \mu_q \in T_g^*\mathcal{G}$ and $\eta \in \mathcal{E}_g^*$. On the one hand we have

$$c_{\mathcal{T}(\Phi)}(V^p, \mu^q, \eta) = \alpha(T\mathbf{t}(V_1), \ldots, T\mathbf{t}(V_p))\overrightarrow{a}(\mu_1, \ldots, \mu_q)c^{\mathcal{E}}(\eta).$$

On the other hand

$$(\mathbb{T}_{\mathcal{E}}^{(p,q)})^*(c_\Phi)(V^p,\mu^q,\eta) = \alpha(T\mathbf{t}(V_1),\dots,T\mathbf{t}(V_p))a(\widetilde{t}(\mu_1),\dots,\widetilde{t}(\mu_q))c(\overline{t}^*(\eta)).$$

We have to prove that $\overrightarrow{a}(\mu_1,\dots,\mu_q) = a(\widetilde{t}(\mu_1),\dots,\widetilde{t}(\mu_q))$ and that $c^{\mathcal{E}}(\eta) = c(\overline{t}^*(\eta))$. For the first equation we proceed by induction on q. The case $q = 1$ follows by Remark 1.17

$$\langle \widetilde{t}(\mu), a(\mathbf{t}(g))\rangle := \langle \mu, a(\mathbf{t}(g))\cdot 0_g\rangle = \langle \mu, \overrightarrow{a}(\mu)\rangle.$$

Now assume that the result is true for $q-1$. Without loss of generality, we can assume that $a = a_1 \wedge a_2$ for $a_1 \in \Gamma(A)$ and $a_2 \in \Gamma(\wedge^{q-1}A)$. Then

$$\begin{aligned}
\overrightarrow{a_1\wedge a_2}(\mu_1,\dots,\mu_q) &= \sum_{j=1}^{q}(-1)^j\langle \mu_j, \overrightarrow{a_1}(g)\rangle \overrightarrow{a_2}(\mu_1,\dots,\widehat{\mu_j},\dots,\mu_q)\\
&= \sum_{j=1}^{q}(-1)^j\langle \widetilde{t}(\mu), a_1(\mathbf{t}(g))\rangle a_2(\widetilde{t}(\mu_1),\dots,\widehat{\widetilde{t}(\mu_j)},\dots,\widetilde{t}(\mu_q))\\
&= a_1\wedge a_2(\widetilde{t}(\mu_1),\dots,\widetilde{t}(\mu_q)).
\end{aligned}$$

This is what we wanted to prove. For the second equation we have that

$$\ell_c(\overline{t}^*(\eta)) = \langle c, \overline{t}^*(\eta)\rangle := \langle c(\mathbf{t}(g))\cdot 0_g, \eta\rangle = \langle c^{\mathcal{E}}(g), \eta\rangle.$$

□

From the definition it follows that

$$\mathcal{T}(f\varphi) = (\mathrm{t}^*f)\mathcal{T}(\varphi), \quad \forall \varphi \in \Gamma(\wedge^p T^*M \otimes \wedge^q A^* \otimes C),\ f \in C^\infty(M). \tag{3.1}$$

Remark 3.5. We write $\mathbb{T}^{(p,q)}$ for the target map $((T\mathbf{t})^p, (\widetilde{t})^q) : C^\infty(\mathbb{M}^{(p,q)}) \longrightarrow C^\infty(\mathbb{G}^{(p,q)})$. When is not risk to confusion we drop the superindices in both $\mathbb{T}_{\mathcal{E}}^{(p,q)}$ and $\mathbb{T}^{(p,q)}$.

The main result of this section is the following proposition, whose proof will be done in the last part of Subsection 3.1.2:

Proposition 3.6. *Let $\mathcal{G} \rightrightarrows M$ be a source connected Lie groupoid and let $\mathcal{E} \rightrightarrows E_1$ be a VB-groupoid over $\mathcal{G}$ with core bundle C. Then an $\mathcal{E}$-valued multiplicative (p,q)-tensor field τ on $\mathcal{G}$ is multiplicative if and only if*

$$c_\tau|_{\mathbb{M}_{\mathcal{E}}^{(p,q)}} = 0$$

and there exist

- $\mathbf{D} : \Gamma_{lin}(A_{\mathcal{E}^*}, C^*) \longrightarrow \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$
- $l : A \longrightarrow \wedge^{p-1}T^*M \otimes \wedge^q A \otimes C$

- $r : T^*M \longrightarrow \wedge^p T^*M \otimes \wedge^{q-1} A^* \otimes C$
- $F : \wedge^p TM \otimes \wedge^q A^* \longrightarrow E$

such that $\mathbf{D}$ *satisfies the Leibniz rule (3.17) and*

$$\begin{cases} \mathcal{L}_{\overrightarrow{X}} \tau = \mathcal{T}(\mathbf{D}(X)) \\ i_{\overrightarrow{a}} \tau = \mathcal{T}(l(a)) \\ i_{t^*\alpha} \tau = \mathcal{T}(r(\alpha)) \\ i_{S_\zeta} \tau = \mathcal{T}(F^*(\zeta)) \end{cases} \tag{3.2}$$

3.1.1 Core sections

The componentwise linear function c_τ associated to an $\mathcal{E}$-valued multiplicative (p,q)-tensor is defined on the VB-groupoid

$$(\bigoplus_p T\mathcal{G}) \oplus (\bigoplus_q T^*\mathcal{G}) \oplus \mathcal{E}^* = \mathbb{G}_\mathcal{E}^{(p,q)}.$$

Its VB-algebroid, which we denote by $\mathbb{A}_\mathcal{E}^{(p,q)}$, splits naturally as the sum $(\bigoplus_p TA) \oplus (\bigoplus_q T^*A) \oplus A_{\mathcal{E}^*}$. Hence there are three different kinds of core sections: one coming from the core A, one coming from the core T^*M and one coming from the core E^*. We study now the action of the function c_τ on these three kinds of sections. When there is no risk of confusion we will not write the superindices (p,q).

Let $a \in \Gamma(A)$. We denote by S_a the core section of the tangent algebroid TA generated by a and let $\overrightarrow{S_a}$ be its corresponding right invariant vector field on $T\mathcal{G}$. The local flow of this vector field is given by $\varphi^t_{S_a}(V) = V + t\overrightarrow{a}(g)$ for $V \in T_g\mathcal{G}$. Consider the core section S^i_a of $\mathbb{A}_\mathcal{E}$ defined by:

$$S^i_a = (\underbrace{0, \dots, \overbrace{S_a}^{i}, \dots, 0}_{p}, \underbrace{0, \dots, 0}_{q}, 0) : \mathbb{M}_\mathcal{E} \longrightarrow \mathbb{A}_\mathcal{E}$$

and let $\overrightarrow{S^i_a} \in \mathfrak{X}(\mathbb{G}_\mathcal{E}^{(p,q)})$ be the associated right invariant vector field. Since τ is an $\mathcal{E}$-valued multiplicative (p,q)-tensor then

$$\begin{aligned} (\mathbb{T}_\mathcal{E}^{(p,q)})^* \langle Ac_\tau, S^i_a \rangle \Big(V^p, \mu^q, \eta\Big) &= (\mathcal{L}_{\overrightarrow{Sa^i}} c_\tau)\Big(V^p, \mu^q, \eta\Big) \\ &= \frac{d}{dt}\Big|_{t=0} \langle \tau\big(V_1, \dots, V_i + t\overrightarrow{a}, \dots, V_k, \mu^q\big), \eta\rangle \\ &= \langle \tau\Big(V_1, \dots, V_{i-1}, \overrightarrow{a}, V_{i+1}, \dots, V_p, \mu^q\Big), \eta\rangle \\ &= (-1)^{i-1} c_{i_{\overrightarrow{a}}\tau} \circ \pi_{(i)}\Big(V^p, \mu^q, \eta\Big) \end{aligned}$$

for $V_i \in T\mathcal{G}$, $\mu_j \in T^*\mathcal{G}$ and $\eta \in \mathcal{E}^*$, where $V^p = (V_1, \ldots, V_p)$, $\mu^q = (\mu_1, \ldots, \mu_q)$, and $\pi_{(i)} : \mathbb{G}_{\mathcal{E}}^{(p,q)} \longrightarrow \mathbb{G}_{\mathcal{E}}^{(p-1,q)}$ is the forgetful map with respect to the i-entry of $\bigoplus_p T\mathcal{G}$ (see (2.26)). Taking now units $(v^p, \nu^q, \xi) \in \mathbb{M}_{\mathcal{E}}$ we have

$$\begin{aligned}\langle Ac_\tau, S_a^i\rangle(v^p, \nu^q, \xi) &= (\mathcal{L}_{\overrightarrow{S_a^i}} c_\tau)\Big(v^p, \nu^q, \xi\Big)\\ &= \langle \tau(v_1, \ldots, v_{i-1}, \overrightarrow{a}, v_{i+1}, \ldots, v_p, \nu^q), \xi\rangle\\ &= (-1)^{i-1}\langle \tau(\overrightarrow{a}, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p, \nu^q), \xi\rangle\end{aligned}$$

This implies that $\langle Ac_\tau, S_a^i\rangle$ is a componentwise linear function of $\gamma_{(i)}(v^p, \nu^q, \xi) \in \mathbb{M}_{\mathcal{E}}^{(p-1,q)}$, where

$$\gamma_{(i)}^{(p,q)} : \mathbb{M}_{\mathcal{E}}^{(p,q)} \longrightarrow \mathbb{M}_{\mathcal{E}}^{(p-1,q)}$$

if the forgetful map with respect to the i-entry of $\bigoplus_q TM$. Thus there is a $l(a) \in \Gamma(\wedge^{p-1}T^*M \otimes \wedge^q A \otimes C)$ such that

$$\langle Ac_\tau, S_a^i\rangle = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)}. \tag{3.3}$$

Now note that

$$\begin{aligned}\mathcal{L}_{\overrightarrow{S_a^i}} c_\tau &= (\mathbb{T}_{\mathcal{E}}^{(p,q)})^* \langle Ac_\tau, S_a^i\rangle\\ &= (\mathbb{T}_{\mathcal{E}}^{(p,q)})^* (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)}\\ &= (-1)^{i-1} c_{\mathcal{T}(l(a))} \circ \pi_{(i)},\end{aligned}$$

We summarize these facts in the following proposition:

Proposition 3.7. *For any multiplicative $\mathcal{E}$-valued (p,q)-tensor τ on $\mathcal{G}$ we have that*

$$(\mathbb{T}_{\mathcal{E}})^* \langle Ac_\tau, S_a^i\rangle = (-1)^{i-1} c_{i_{\overrightarrow{a}}\tau} \circ \pi_{(i)} \tag{3.4}$$

In particular there exists vector bundle morphism $l : A \longrightarrow \wedge^{p-1}T^*M \otimes \wedge^q A \otimes C$ such that

$$\langle Ac_\tau, S_a^i\rangle = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)}. \tag{3.5}$$

Proof. The previous discussion implies Equation (3.5). Observe that for each $a \in \Gamma(A)$

$$c_{i_{\overrightarrow{a}}\tau} = \langle \tau(\overrightarrow{a}, \cdot, \ldots, \cdot), \cdot\rangle = (\mathcal{L}_{\overrightarrow{S_a^1}} c_\tau) = c_{\mathcal{T}(l(a))} \ \in C^\infty(\mathbb{G}_{\mathcal{E}}^{(p-1,q)}),$$

then

$$(\mathbb{T}_{\mathcal{E}})^* \langle Ac_\tau, S_a^i\rangle = \mathcal{L}_{\overrightarrow{S_a^i}} c_\tau = (-1)^{i-1} c_{\mathcal{T}(l(a))} \circ \pi_{(i)} = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)}.$$

Hence Equation (3.4) holds. Finally we prove that the map l is $C^\infty(M)$-linear. For $h \in C^\infty(M)$ we have

$$\mathcal{T}(l(ha)) = i_{\overrightarrow{ha}}\tau = (\mathrm{t}^*h) i_{\overrightarrow{a}}\tau = (\mathrm{t}^*h)\mathcal{T}(l(a)) = \mathcal{T}(hl(a))$$

where the last equality follows by (3.1). Since $\mathcal{T}$ is injective follows that $l(ha) = hl(a)$. □

Consider now a 1-form $\alpha \in \Omega^1(M)$ and let S_α be the corresponding core section of the cotangent algebroid T^*A. The local flow of the right invariant vector field $\overrightarrow{S_\alpha} \in \mathfrak{X}(T^*\mathcal{G})$ associated to S_α is given by $\varphi^t_{S_\alpha}(\mu) = \mu + t(\mathbf{t}^*\alpha)$ for $\mu \in T^*_g\mathcal{G}$. Define a core section of $\mathbb{A}_\mathcal{E}$ by

$$S^j_\alpha = (\underbrace{0,\ldots,0}_{p}, \underbrace{0,\ldots,\overbrace{S_\alpha}^{j},\ldots 0}_{q}, 0) : \mathbb{M}_\mathcal{E} \longrightarrow \mathbb{A}_\mathcal{E}$$

Then for a $\mathcal{E}$-valued multiplicative (p,q)-tensor τ,

$$\begin{aligned}
(\mathbb{T}_\mathcal{E})^*\langle Ac_\tau, S^j_\alpha\rangle\Big(V^p,\mu^q,\eta\Big) &= (\mathcal{L}_{\overrightarrow{S^j_\alpha}}c_\tau)\Big(V^p,\mu^q,\eta\Big)\\
&= \frac{d}{dt}\Big|_{t=0}\langle\tau\big(V^p,\mu_1,\ldots,\mu_j+t(\mathbf{t}^*\alpha),\mu_{j+1},\ldots,\mu_q\big),\eta\rangle\\
&= \langle\tau\Big(V^p,\mu_1,\ldots,\mu_{j-1},(\mathbf{t}^*\alpha),\mu_{j+1},\ldots,\mu_q\Big),\eta\rangle\\
&= (-1)^{j-1}c_{i_{\mathbf{t}^*\alpha}}\tau\circ\pi^*_{(j)}\Big(V^p,\mu^q,\eta\Big),
\end{aligned}$$

where $i_{t^*\alpha}\tau := \tau(\underbrace{\cdot,\ldots,\cdot}_{p},\mathbf{t}^*\alpha,\cdot,\ldots,\cdot)$ and where $\pi^{(p,q)}_{(j)*} : \mathbb{G}^{(p,q)}_\mathcal{E} \longrightarrow \mathbb{G}^{(p,q-1)}_\mathcal{E}$ is the forgetful map with respect to the j-entry of $\bigoplus_q T^*\mathcal{G}$. Taking now units $(v^p,\nu^q,\xi) \in \mathbb{M}_\mathcal{E}$ then

$$\begin{aligned}
\langle Ac_\tau, S^j_\alpha\rangle(v^p,\nu^q,\xi) &= (\mathcal{L}_{\overrightarrow{S^j_\alpha}}c_\tau)\Big(v^p,\nu^q,\xi\Big)\\
&= \langle\tau(v^p,\nu_1,\ldots,\nu_{j-1},(\mathbf{t}^*\alpha),\nu_{j+1},\ldots,\nu_q),\xi\rangle\\
&= (-1)^{j-1}\langle\tau(v^p,(\mathbf{t}^*\alpha),\nu_1,\ldots,\nu_{j-1},\nu_{j+1},\ldots,\nu_q),\xi\rangle
\end{aligned}$$

This last equation implies that $\langle Ac_\tau, S^j_\alpha\rangle$ is a componentwise linear function of $\gamma^*_{(j)}(v^p,\nu,\xi) \in \mathbb{M}^{(p,q-1)}_\mathcal{E}$, where

$$\gamma^{(p,q)}_{(j)*} : \mathbb{M}^{(p,q)}_\mathcal{E} \longrightarrow \mathbb{M}^{(p,q-1)}_\mathcal{E}$$

is the forgetful map with respect to the j-entry of $\bigoplus_q T^*A$. Thus there is a $r(\alpha) \in \Gamma(\wedge^pT^*M\otimes\wedge^{q-1}A\otimes C)$ such that

$$\langle Ac_\tau, S^j_\alpha\rangle = (-1)^{j-1}c_{r(\alpha)}\circ\gamma^*_{(j)}. \tag{3.6}$$

Proposition 3.8. *For any multiplicative $\mathcal{E}$-valued (p,q)-tensor τ on $\mathcal{G}$ we have that*

$$(\mathbb{T}_\mathcal{E})^*\langle Ac_\tau, S^j_\alpha\rangle = (-1)^{j-1}c_{i_{\mathbf{t}^*\alpha}\tau}\circ\pi^*_{(j)}, \tag{3.7}$$

In particular there exists vector bundle morphism $r : T^*M \longrightarrow \wedge^pT^*M\otimes\wedge^{q-1}A\otimes C$ such that

$$\langle Ac_\tau, S^j_\alpha\rangle = (-1)^{j-1}c_{r(\alpha)}\circ\gamma^*_{(j)}. \tag{3.8}$$

Proof. The proof is analogous to the Proposition 3.7. Note that the following equation holds

$$i_{t^*\alpha}\tau = \mathcal{T}(r(\alpha)). \tag{3.9}$$

□

Finally, let $\zeta : M \longrightarrow E^*$ be a section of the core bundle E^* and let $S_\zeta : C^* \longrightarrow A_{\mathcal{E}^*}$ be the core section of $A_{\mathcal{E}^*}$ over C^* associated to ζ. Since τ is an $\mathcal{E}$-valued multiplicative (p,q)-tensor, we have

$$\begin{array}{rcl} (\mathbb{T}_\mathcal{E})^*\langle Ac_\tau, S_\zeta\rangle(V^p,\mu^q,\eta) & = & \mathcal{L}_{\overrightarrow{S_\zeta}} c_\tau(V^p,\mu^q,\eta) \\ & = & \left.\dfrac{d}{dt}\right|_{t=0}\langle\tau(V^p,\mu^q),\eta+t\zeta^\mathcal{E}(g)\rangle \\ & = & \langle\tau(V^p,\mu^q),\zeta^\mathcal{E}\rangle \\ \text{by property of units} & = & \langle\bar{t}\big(\tau(V^p,\mu^q)\big)\cdot\tau(V^p,\mu^q),\zeta(\mathbf{t}(g))\cdot 0_g\rangle \\ \text{by multiplication in the dual} & = & \langle\bar{t}\big(\tau(V^p,\mu^q)\big),\zeta(\mathbf{t}(g))\rangle+\langle\tau(V^p,\mu^q)\big),0_g\rangle \\ & = & \langle\bar{t}\big(\tau(V^p,\mu^q)\big),\zeta(\mathbf{t}(g))\rangle \\ & = & c_F(\mathbb{T}(V^p,\mu^q),\zeta(\mathbf{t}(g))) \end{array}$$

where $c_F : \mathbb{M}_\mathcal{E}^{(p,q)} \longrightarrow \mathbb{R}$ is the componentwise linear function defined on the units of $\mathbb{G}_\mathcal{E}^{(p,q)}$ covered by c_τ. Hence c_F has associated a multilinear map $F : \bigotimes_p TM \otimes \bigotimes_q A^* \longrightarrow E$. Then taking units $(v^p,\nu^q,\xi) \in \mathbb{M}_\mathcal{E}$ we have

$$\langle Ac_\tau, S_\zeta\rangle(v^p,\nu^q,\xi) = \langle F(v^p,\nu^q),\zeta\rangle = \langle(v^p,\nu^q),F^*(\zeta)\rangle.$$

The previous discussion is the proof of the following proposition

Proposition 3.9. *For any multiplicative $\mathcal{E}$-valued (p,q)-tensor τ on $\mathcal{G}$ we have that*

$$(\mathbb{T}_\mathcal{E})^*\langle A\ell_\tau, S_\zeta\rangle = \langle F\circ\mathbb{T},\zeta\rangle\circ\gamma = \langle\mathbb{T},F^*(\zeta)\rangle\circ\gamma, \tag{3.10}$$

where $\gamma : \mathbb{G}_\mathcal{E}^{(p,q)} \longrightarrow \mathbb{G}^{(p,q)}$ forgets the last component. Moreover there exist a section $\sigma(\zeta) = F^*(\zeta) \in \Gamma(\wedge^p T^*M \otimes \wedge^q A)$ such that

$$c_{\mathcal{T}(\sigma(\zeta))} = \langle F\circ\mathbb{T},\zeta\rangle = \langle\mathbb{T},F^*(\zeta)\rangle \tag{3.11}$$

Equivalently to Equation (3.11) we have

$$i_{\zeta^\mathcal{E}}\tau = \mathcal{T}(F^*(\zeta)),$$

where $i_{\zeta^\mathcal{E}}\tau \in \Gamma(\wedge^p T\mathcal{G} \otimes \wedge^q T^*\mathcal{G})$ means

$$c_{i_{\zeta^\mathcal{E}}\tau}(V^p,\mu^q) = \langle\tau(V^p,\mu^q),\zeta^\mathcal{E}(g)\rangle.$$

3.1.2 Linear sections

We keep with a fix $\mathcal{E}$-valued multiplicative (p, q)-tensor τ. Let X_a be a linear section of $A_{\mathcal{E}^*}$ over C^* covering a section $a \in \Gamma(A)$. We define a linear section of $\mathbb{A}_{\mathcal{E}}^{(p,q)}$ over $\mathbb{M}_{\mathcal{E}}^{(p,q)}$, which covers a, by

$$\chi_a = (\underbrace{Ta, \ldots, Ta}_{p-times}, \underbrace{R_a, \ldots, R_a}_{q-times}, X_a) = ((Ta)^p, (R_a)^q, X_a).$$

The right invariant vector field associated to it is

$$\overrightarrow{\chi_a} = ((\overrightarrow{Ta})^p, ((\overrightarrow{a})^{T^*})^q, \overrightarrow{X_a}),$$

and since this vector field is linear there exists a derivation $D_{\chi_a} \in \mathrm{Der}(\mathbb{G}_{\mathcal{E}}^{(p,q)^*})$ such that

$$\mathcal{L}_{\overrightarrow{\chi_a}} c_\tau = \overrightarrow{\chi_a}(c_\tau) = c_{D_{\chi_a}(\tau)}. \tag{3.12}$$

Since τ is an $\mathcal{E}$-valued multiplicative (p, q)-tensor, we have

$$\mathcal{L}_{\overrightarrow{\chi_a}} c_\tau = \overrightarrow{\chi_a}(c_\tau) = c_{D_{X_a}(\tau)} = (\mathbb{T}_{\mathcal{E}}^{(p,q)})^* \langle Ac_\tau, \chi_a \rangle. \tag{3.13}$$

Remark 3.10. When we do not need to remark the section of A which is covered by a linear section of $\mathbb{A}_{\mathcal{E}}$, we just write X for the linear section of $\mathbb{A}_{\mathcal{E}}$, and χ for the linear section of $\mathbb{A}_{\mathcal{E}}^{(p,q)}$ built from X.

Proposition 3.11. *Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G}$. For any multiplicative $\mathcal{E}$-valued (p, q)-tensor τ on $\mathcal{G}$ one has that*

$$(\mathbb{T}_{\mathcal{E}})^* \langle Ac_\tau, \chi \rangle = c_{D_\chi(\tau)} \tag{3.14}$$

In particular there exists an operator $\mathbf{D} : \Gamma_{lin}(A_{\mathcal{E}^*}) \longrightarrow \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$ such that

$$\langle Ac_\tau, \chi \rangle = c_{\mathbf{D}(X)}, \tag{3.15}$$

or equivalently

$$\mathcal{L}_{\overrightarrow{\chi}} c_\tau = c_{\mathcal{T}(\mathbf{D}(X))}, \tag{3.16}$$

and satisfying the following Leibniz rule for $f \in C^\infty(M)$

$$\mathbf{D}(fX_a) = f\mathbf{D}(X_a) + df \wedge l(a) - a \wedge r(df) \tag{3.17}$$

Remark 3.12. An explanation about the notation. Let $f \in C^\infty(M)$. The expressions $\mathrm{d}f \wedge l(a)$, $a \wedge r(\mathrm{d}f) \in \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$ mean:

$$\begin{aligned}
\langle \mathrm{d}f \wedge l(a), (v^p, \nu^q, \xi)\rangle &= \sum_{i=1}^{p} (-1)^{i-1} \mathrm{d}f(v_i) \langle l(a), (v_1, \ldots, \widehat{v_i}, \ldots v_p, \nu^q, \xi)\rangle \\
\langle a \wedge r(\mathrm{d}f), (v^p, \nu^q, \xi)\rangle &= \sum_{i=1}^{q} (-1)^{i-1} \langle a, \nu_i\rangle \langle r(\mathrm{d}f), (v^p, \nu_1, \ldots, \widehat{\nu_i}, \ldots, \xi)\rangle.
\end{aligned}$$

Proof. Since τ is multiplicative the Equation (3.13) holds for any linear section $\chi \in \Gamma(\mathbb{A}_{\mathcal{E}}, \mathbb{M}_{\mathcal{E}}^{(p,q)})$. And combining with (3.12), then the Equation (3.14) follows. Also Equation (3.13) implies, taking units, that the function $\langle Ac_\tau, \chi\rangle$ is multilinear in $C^\infty((\oplus^p TM) \oplus (\oplus^q A^*) \oplus C^*)$. Then there exists a section $\mathbf{D}(\chi) \in \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$ such that the Equation (3.15) holds. Using now the Proposition 3.4, we get the Equation (3.16). Finally we prove the Leibniz rule. For $f \in C^\infty(M)$ and $X_a \in \Gamma_{\mathrm{lin}}(A_{\mathcal{E}^*}, C^*)$ we have

$$\begin{aligned}
c_{\mathbf{D}(fX_a)}(v, \nu, c) &= \langle Ac_\tau, ((T(fa))^p, (R_{fa})^q, fX_a)\rangle(v^p, \nu^q, \xi) \\
&= \langle Ac_\tau, ((fTa + \ell_{\mathrm{d}f} S_a)^p, (fR_a + \ell_{-a} S_{\mathrm{d}f})^q, fX_a)\rangle(v^p, \nu^q, \xi) \\
&= \langle Ac_\tau, ((fTa)^p, (fR_a)^q, fX_a) + ((\ell_{\mathrm{d}f} S_a)^p, (\ell_{-a} S_{\mathrm{d}f})^q, 0)\rangle(v^p, \nu^q, \xi) \\
&= \langle Ac_\tau, ((fTa)^p, (fR_a)^q, fX_a)\rangle(v^p, \nu^q, \xi) \\
&\quad + \langle Ac_\tau, ((\ell_{\mathrm{d}f} S_a)^p, (\ell_{-a} S_{\mathrm{d}f})^q, 0)\rangle(v^p, \nu^q, \xi) \\
&= f\langle Ac_\tau, ((Ta)^p, (R_a)^q, X_a)\rangle(v^p, \nu^q, \xi) + \langle Ac_\tau, (\ell_{\mathrm{d}f} S_a)^p\rangle(v^p, \nu^q, \xi) \\
&\quad + \langle Ac_\tau, (\ell_{-a} S_{\mathrm{d}f})^q\rangle(v^p, \nu^q, \xi) \\
&= f c_{\mathbf{D}(X_a)}(v^p, \nu^q, \xi) + \sum_{i=1}^{p} \langle Ac_\tau, \ell_{\mathrm{d}f} S_a^i\rangle(v^p, \nu^q, \xi) \\
&\quad + \sum_{i=1}^{q} \langle Ac_\tau, \ell_{-a} S_{\mathrm{d}f}^i\rangle(v^p, \nu^q, \xi) \\
&= f c_{\mathbf{D}(X_a)}(v^p, \nu^q, \xi) \\
&\quad + \sum_{i=1}^{p} (-1)^{i-1} \mathrm{d}f(v_i) \langle l(a), (v_1, \ldots, \widehat{v_i}, \ldots v_p, \nu^q, \xi)\rangle \\
&\quad - \sum_{i=1}^{q} (-1)^{i-1} \langle a, \nu_i\rangle \langle r(\mathrm{d}f), (v^p, \nu_1, \ldots, \widehat{\nu_i}, \ldots, \xi)\rangle \\
&= \big(f c_{\mathbf{D}(X_a)} + c_{\mathrm{d}f \wedge l(a)} - c_{a \wedge r(\mathrm{d}f)}\big)(v^p, \nu^q, \xi)
\end{aligned}$$

where we used that

$$\begin{aligned}
\langle Ac_\tau, \ell_{\mathrm{d}f} S_a^i\rangle(v^p, \nu^q, \xi) &= (-1)^{i-1} \mathrm{d}f(v_i) c_{l(a)} \circ \gamma_{(i)}(v^p, \nu^q, \xi) \\
\langle Ac_\tau, \ell_{-a} S_{\mathrm{d}f}^i\rangle(v^p, \nu^q, \xi) &= (-1)^{i-i} \langle -a, \nu_i\rangle c_{r(\mathrm{d}f)} \circ \gamma_{(j)}^*(v^p, \nu^q, \xi)
\end{aligned}$$

Therefore the Leibniz rule holds. $\square$

Now we proof the main theorem

Proof. Proposition 3.6. If τ is an $\mathcal{E}$-valued multiplicative (p,q)-tensor on $\mathcal{G}$, then Propositions 3.11, 3.7, 3.8 and 3.9 show the existence of $(\mathbf{D}, l, r, F)$ satisfying the Equations (3.2). Conversely, let $(\mathbf{D}, l, r, F)$ satisfying those conditions. We will define a function $\lambda \in C^\infty(\mathbb{A}_{\mathcal{E}}^{(p,q)})$ which is a cocycle. Recall that we have an injective map

$$\begin{array}{rcl} \Gamma_{\text{lin}}(A_{\mathcal{E}^*}, C^*) & \longrightarrow & \Gamma_{\text{lin}}(\mathbb{A}_{\mathcal{E}}^{(p,q)}, \mathbb{M}_{\mathcal{E}}^{(p,q)}) \\ X_a & \longrightarrow & \chi_a := ((Ta)^p, (R_a)^q, X_a). \end{array}$$

The function λ is determined by

$$\begin{array}{rcl} \langle \lambda, \chi \rangle & = & c_{\mathbf{D}(X)} \\ \langle \lambda, S_a^i \rangle & = & (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)} \\ \langle \lambda, S_\alpha^j \rangle & = & (-1)^{j-1} c_{r(\alpha)} \circ \gamma_{(j)}^* \\ \langle \lambda, S_\zeta \rangle & = & c_{F^*(\zeta)} \circ \gamma \end{array}$$

where $\alpha \in \Omega^1(M)$ and $\zeta \in \Gamma(E^*)$. In the case of linear sections, it is enough to define λ only in sections χ_a coming from $X_a \in \Gamma_{\text{lin}}(A_{\mathcal{E}^*}, C^*)$. Since $(f \circ q_C^*)X_a$ is also a linear section of $A_{\mathcal{E}^*}$ over C^*, the action of λ on the section χ associated to $(f \circ q_C^*)X_a$ is given by the Leibniz rule satisfied by $\mathbf{D}$. Hence λ is well defined and it can be extended to all section of $\mathbb{A}_{\mathcal{E}}^{(p,q)}$ by $C^\infty(\mathbb{M}_{\mathcal{E}}^{(p,q)})$-linearity. Moreover

$$\mathcal{L}_{\overrightarrow{\chi_a}} c_\tau = c_{D(X_a)(\tau)} = c_{\mathcal{T}(D(X_a))} = (\mathbb{T}_{\mathcal{E}})^* \big(c_{D(X_a)} \big) = (\mathbb{T}_{\mathcal{E}})^* \langle A c_\tau, \chi_a \rangle.$$

A similar argument shows that $\mathcal{L}_{\overrightarrow{S_b^i}} c_\tau = (\mathbb{T}_{\mathcal{E}})^* \langle A c_\tau, S_b^i \rangle$, $\mathcal{L}_{\overrightarrow{S_\alpha^j}} c_\tau = (\mathbb{T}_{\mathcal{E}})^* \langle A c_\tau, S_\alpha^j \rangle$ and $\mathcal{L}_{\overrightarrow{S_\zeta}} c\ell_\tau = (\mathbb{T}_{\mathcal{E}})^* \langle A c_\tau, S_\zeta \rangle$. Then by linearity, we have that

$$\mathcal{L}_{\overrightarrow{\chi}} c_\tau = (\mathbb{T}_{\mathcal{E}})^* \langle A c_\tau, \chi \rangle$$

for all section $\chi \in \Gamma(\mathbb{A}_{\mathcal{E}}^{(p,q)})$. Since $\mathbb{G}_{\mathcal{E}}^{(p,q)}$ is source connected, the result follows by Proposition (1.11).

□

3.2 IM equations

Let $(A, \rho, [\cdot,\cdot])$ be a Lie algebroid over M, and let $\mathcal{A} \longrightarrow E$ be a VB-algebroid over $A \longrightarrow M$ with core bundle $C \longrightarrow M$. In this section we study tensor fields on A with coefficient in $\mathcal{A}$ which are *compatible* with the Lie algebroid structure of A. We give a description of these tensors in terms of an infinitesimal data, and then we establish a correspondence with multiplicative tensor fields.

A *(p,q)-tensor on A with coefficients in a vector bundle $\mathcal{A}$* is a section $\phi \in \Gamma(\wedge^p T^*A \otimes \wedge^q TA \otimes \mathcal{A})$. As in the previous section, we associate to ϕ a componentwise linear function $c_\phi : \mathbb{A}_{\mathcal{A}}^{(p,q)} := (\bigoplus_p TA) \oplus (\bigoplus_q T^*A) \oplus \mathcal{A}^* \longrightarrow \mathbb{R}$ defined by

$$c_\phi(Y^p, \mu^q, \zeta) = \langle \phi(Y^p, \mu^q), \zeta \rangle.$$

In the case when $\mathcal{A}$ is a VB-algebroid, the space $\mathbb{A}_{\mathcal{A}}^{(p,q)}$ is also a VB-algebroid over A.

Definition 3.13. A (p,q)-tensor ϕ on A with coefficients in a VB-algebroid $\mathcal{A}$ is called *Lie algebroid tensor* if its associated componentwise linear function c_ϕ is a Lie algebroid cocycle on $\mathbb{A}_{\mathcal{A}}^{(p,q)}$.

The function $c_\phi \in C^\infty(\mathbb{A}_{\mathcal{A}}^{(p,q)})$ is a Lie algebroid cocycle if and only if $d_{\mathbb{A}} c_\phi = 0$, where $d_{\mathbb{A}}$ is the Lie algebroid differential of $\mathbb{A}_{\mathcal{A}}^{(p,q)}$. This is equivalent to

$$\langle c_\phi, [U,V] \rangle = \mathcal{L}_{\rho_{\mathbb{A}}(U)} \langle c_\phi, V \rangle - \mathcal{L}_{\rho_{\mathbb{A}}(V)} \langle c_\phi, U \rangle. \tag{3.18}$$

where $\rho_{\mathbb{A}}$ is the anchor map of the Lie algebroid $\mathbb{A}_{\mathcal{A}}^{(p,q)}$. Since $\mathbb{A}_{\mathcal{A}}^{(p,q)}$ is also a VB-algebroid over A it is enough to check Equation (3.18) for core and linear sections. Because of that we define first an *infinitesimal tensor* in terms of these kinds of sections, and then we give the correspondence with Lie algebroid tensors.

Recall that given the VB-algebroid $\mathcal{A}^* \longrightarrow C^*$, dual of the VB-algebroid $\mathcal{A} \longrightarrow E$, it has canonical operators (see (1.23) and (1.22)) which we denote by

- $\partial^* : E^* \longrightarrow C^*$
- $(\widetilde{\nabla}^0) : \Gamma_{\text{lin}}(\mathcal{A}^*, C^*) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$.

Also we have an injective map $\Gamma_{\text{lin}}(\mathcal{A}^*, C^*) \longrightarrow \Gamma_{\text{lin}}(\mathbb{A}_{\mathcal{A}}^{(p,q)}, \mathbb{M}_{\mathcal{E}}^{(p,q)})$ given by

$$X_a \longrightarrow \chi_a := ((Ta)^p, (R_a)^p, X_a).$$

Definition 3.14. An $\mathcal{A}$-valued (p,q)-tensor on A is a quadruple $(\mathbf{D}, l, r, \sigma)$ where

- $\mathbf{D} : \Gamma_{\text{lin}}(\mathcal{A}^*, C^*) \longrightarrow \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$
- $l : A \longrightarrow \wedge^{p-1} T^*M \otimes \wedge^q A \otimes C$
- $r : T^*M \longrightarrow \wedge^p T^*M \otimes \wedge^{q-1} A \otimes C$
- $\sigma : E^* \longrightarrow \wedge^p T^*M \otimes \wedge^q A$

satisfying the Leibniz rule

$$\mathbf{D}(fX_a) = f\mathbf{D}(X_a) + \mathrm{d}f \wedge l(a) - a \wedge r(\mathrm{d}f), \tag{3.19}$$

the compatibility condition

$$\mathbf{D}(S_T) = \sigma \circ T \qquad \text{for } T \in \Gamma(\mathrm{Hom}(C^*, E^*)), \tag{3.20}$$

and the following equations

$$\begin{aligned}
&(3.21) & \mathbf{D}([X_a, X_b]) &= X_a \cdot \mathbf{D}(X_b) - X_b \cdot \mathbf{D}(X_a)\\
&(3.22) & i_{\rho(b)}\mathbf{D}(X_a) &= X_a \cdot l(b) - l([a,b])\\
&(3.23) & i_{\rho^*(\beta)}\mathbf{D}(X_a) &= X_a \cdot r(\beta) - r(\mathcal{L}_{\rho(a)}\beta)\\
&(3.24) & i_{\partial^*(\gamma)}\mathbf{D}(X_a) &= X_a \cdot \sigma(\gamma) - \sigma((\nabla^1)^*_{X_a}\gamma)\\
&(3.25) & i_{\rho(a)}l(b) &= -i_{\rho(b)}l(a)\\
&(3.26) & i_{\rho^*(\alpha)}r(\beta) &= -i_{\rho^*(\beta)}r(\alpha)\\
&(3.27) & i_{\rho(a)}r(\beta) &= i_{\rho^*(\beta)}l(a)\\
&(3.28) & i_{\partial^*(\gamma)}l(a) &= i_{\rho(a)}\sigma(\gamma)\\
&(3.29) & i_{\rho^*(\alpha)}\sigma(\gamma) &= i_{\partial^*(\gamma)}r(\alpha).
\end{aligned}$$

for $X_a,\ X_b \in \Gamma_{\mathrm{lin}}(\mathcal{A}^*, C^*)$; $a,\ b \in \Gamma(A)$; $\alpha,\ \beta \in \Omega^1(M)$ and $\zeta \in \Gamma(E^*)$.

Remark 3.15. The notation $X_a \cdot \mathbf{D}(X_b)$ if for a module structure on $\Gamma((\otimes^p T^*M) \otimes (\otimes^q A) \otimes C)$ over $\Gamma_{\mathrm{lin}}(\mathcal{A}^*, C^*)$. Its definition together with its properties, which we will use in the next proposition, are in the Appendix.

Proposition 3.16. *There is a one-to-one correspondence between Lie algebroid (p,q)-tensor on A with coefficients in $\mathcal{A}$ and $\mathcal{A}$-valued (p,q)-tensor on A.*

Proof. Let ϕ be a Lie algebroid (p,q)-tensor on A with coefficients in $\mathcal{A}$ and consider its associated componentwise linear function c_ϕ. We define $\mathbf{D} : \Gamma_{\mathrm{lin}}(\mathcal{A}^*, C^*) \longrightarrow \Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$ by

$$c_{\mathbf{D}(X_a)} := c_\phi(\chi_a),$$

where $a \in \Gamma(A)$. Consider the core section S^1_a. Then the function $\langle \phi, S^1_a \rangle$ is multilinear on $\mathbb{M}^{(p-1,q)}_{\mathcal{A}}$, so there exists a section $l(a) \in \Gamma(\wedge^{p-1}T^*M \otimes \wedge^q A \otimes C)$ such that

$$\langle \phi, S^1_a \rangle = c_{l(a)} \circ \gamma_{(1)}.$$

Hence we have that $\langle \phi, S^i_a \rangle = (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)}$. Note that

$$\langle \phi, S^1_{ha} \rangle = \langle \phi, (h \circ \widetilde{q}) S^1_a \rangle = (h \circ \widetilde{q}) \langle \phi, S^1_a \rangle,$$

where $\widetilde{q} : \mathbb{M}^{(p,q)}_{\mathcal{E}} \longrightarrow M$ is the projection, which implies that $l(fa) = fl(a)$. Taking now core sections of the form S^i_α and S_ζ, with $\alpha \in \Omega^1(M)$ and $\zeta \in \Gamma(E^*)$, we get the maps r and σ, respectively. Now we check the equations. First we start with linear sections. For χ_a and χ_b we have

$$[\chi_a, \chi_b] = [((Ta)^p, (R_a)^q, X_a), ((Tb)^p, (R_b)^q, X_b)] = ((T[a,b])^p, (R_{[a,b]})^q, [X_a, X_b]),$$

then

$$\begin{aligned} c_{\mathbf{D}([X_a,X_b])} &= \mathcal{L}_{\rho_{\mathbb{A}}(\chi_a)} c_{\mathbf{D}(X_b)} - \mathcal{L}_{\rho_{\mathbb{A}}(\chi_b)} c\ell_{\mathbf{D}(X_a)} \\ &= c_{X_a \cdot \mathbf{D}(X_b)} - c_{X_b \cdot \mathbf{D}(X_a)}. \end{aligned}$$

Then the Equation (IM1) holds. For Equation (IM2) we consider a linear section χ_a and a core section S^1_b. The two terms of right side of (3.18) are

$$\begin{aligned} \mathcal{L}_{\rho_{\mathbb{A}}(\chi_a)} \langle c_\phi, S^1_b \rangle &= \mathcal{L}_{((\rho(a)^T)^p, (H_a)^q, \rho_{\mathcal{A}^*}(X_a))} \big(c_{l(b)} \circ \gamma_{(1)} \big) \\ &= \big(\mathcal{L}_{(0, (\rho(a)^T)^{p-1}, (H_a)^q, \rho_{\mathcal{A}^*}(X_a))} c_{l(b)} \big) \circ \gamma_{(1)} \\ &= c_{X_a \cdot l(b)} \circ \gamma_{(1)}. \\ \mathcal{L}_{(\rho(b)^\uparrow, 0, 0)} \langle c_\phi, \chi_a \rangle &= \mathcal{L}_{(\rho(b)^\uparrow, 0, 0)} c_{\mathbf{D}(X_a)} \\ &= \langle \mathbf{D}(X_a), \rho_A(b) \rangle \circ \gamma_{(1)} \\ &= c_{i_{\rho(b)} \mathbf{D}(X_a)} \circ \gamma_{(1)}. \end{aligned}$$

while in the left-han side is

$$\langle c_\phi, [\chi_a, S^1_b] \rangle = \langle c_\phi, S^1_{([a,b])} \rangle = c_{l([a,b])} \circ \gamma_{(1)},$$

which proves the Equation (IM2). Now take a linear section χ_a and a core section S^1_β. Then the right side of (3.18) has terms

$$\begin{aligned} \mathcal{L}_{\rho_{\mathbb{A}}(\chi_a)} \langle c_\phi, S^1_\beta \rangle &= \mathcal{L}_{((\rho(a)^T)^p, (H_a)^q, \rho_{\mathcal{A}^*}(X_a))} \big(c_{r(\beta)} \circ \gamma^*_{(1)} \big) \\ &= \big(\mathcal{L}_{((\rho(a)^T)^p, 0, (H_a)^{q-1}, \rho_{\mathcal{A}^*}(X_a))} c_{r(\beta)} \big) \circ \gamma^*_{(1)} \\ &= c_{X_a \cdot r(\beta)} \circ \gamma^*_{(1)}. \\ \mathcal{L}_{(0, \rho^*(\beta)^\uparrow, 0)} \langle c_\phi, \chi_a \rangle &= \mathcal{L}_{(0, \rho^*(\beta)^\uparrow, 0)} \big(c_{\mathbf{D}(X_a)} \circ \gamma^*_{(1)} \big) \\ &= \langle \mathbf{D}(X_a), \rho^*(\beta) \rangle \circ \gamma^*_{(1)} \\ &= c_{i_{\rho^*(\beta)} \mathbf{D}(X_a)} \circ \gamma^*_{(1)}. \end{aligned}$$

The left-hand side is

$$\langle c_\phi, [\chi_a, S^1_\beta] \rangle = \langle c_\phi, [R_a, \beta] \rangle = \langle c_\phi, S_{(\mathcal{L}_{\rho(a)} \beta)} \rangle = c_{r(\mathcal{L}_{\rho(a)} \beta)} \circ \gamma^*_{(1)}.$$

Then Equation (IM3) holds. Taking now a linear section χ_a and a core section S_ζ we have

$$\begin{aligned} \mathcal{L}_{\rho_{\mathbb{A}}(\chi_a)} \langle c_\phi, S_\zeta \rangle &= \mathcal{L}_{((\rho(a)^T)^p, (H_a)^q, \rho_{\mathcal{A}^*}(X_a))} \big(c_{\sigma(\zeta)} \circ \gamma \big) \\ &= \big(\mathcal{L}_{((\rho(a)^T)^p, (H_a)^q, 0)} c_{\sigma(\zeta)} \big) \circ \gamma \\ &= c_{X_a \cdot \sigma(\zeta)} \circ \gamma \\ \mathcal{L}_{(0,0,\partial^*(\zeta)^\uparrow)} \langle c_\phi, \chi_a \rangle &= \langle \mathbf{D}(X_a), \partial^*(\zeta) \rangle \circ \gamma \\ &= c_{i_{\partial^*(\zeta)} \mathbf{D}(X_a)} \circ \gamma. \end{aligned}$$

and on the other side

$$\langle c_\phi, [\chi_a, S_\zeta]\rangle = \langle c_\phi, [X_a, S_\zeta]\rangle = \langle c_\phi, S_{(\widetilde{\nabla}^0)_{X_a}\zeta}\rangle = c_{\sigma((\widetilde{\nabla}^0)_{X_a}\zeta)} \circ \gamma.$$

Therefore IM4 holds.
Now we will take only core sections. Since their brackets are zero, the left side of (3.18) will be always equal to zero. For sections S^1_a and S^1_b we have

$$\begin{aligned}
\mathcal{L}_{(\rho(a)^\uparrow,0,0)}\langle c_\phi, S^2_b\rangle(v^p,\nu^q,\xi) &= \frac{d}{dt}\Big|_{t=0} c_{l(b)} \circ \gamma_{(2)}(v_1 + t\rho(a), v_2, \ldots, v_p, \nu^q, \xi) \\
&= \langle l(b)(\rho(a), v_2, \ldots, v_p, \nu^q), \xi\rangle.
\end{aligned}$$

Doing the same, interchanging b with a, Equation (IM5) follows. In a similar way, taking now core sections of the form S^1_α and S^1_β, Equation (IM6) follows. If we take the core sections S^1_a and S^1_α then

$$\begin{aligned}
\mathcal{L}_{(\rho(a)^\uparrow,0,0)}\langle c_\phi, S^1_\alpha\rangle(v^p,\nu^q,\xi) &= \frac{d}{dt}\Big|_{t=0} c_{r(\alpha)} \circ \gamma^*_{(1)}(v_1 + t\rho(a), v_2, \ldots, v_p, \nu^q, \xi) \\
&= \langle r(\alpha)(\rho(a), v_2, \ldots, v_p, \nu_2, \ldots, \nu_q), \xi\rangle. \\
\mathcal{L}_{(0,\rho^*(\alpha),0)}\langle c_\phi, S^1_a\rangle(v^p,\nu^q,\xi) &= \frac{d}{dt}\Big|_{t=0} c_{l(a)} \circ \gamma_{(1)}(v^p, \nu_1 + t\rho^*(\alpha), \nu_2, \ldots, \nu_q, \xi) \\
&= \langle l(a)(v_2, \ldots, v_p, \rho^*(\alpha), \nu_2, \ldots, \nu_q), \xi\rangle.
\end{aligned}$$

which proves Equation (IM7). For S^1_a and S_ζ

$$\begin{aligned}
\mathcal{L}_{\rho_{\mathbb{A}}(S^1_a)}\langle c_\phi, S_\zeta\rangle(v^p,\nu^q,\xi) &= \mathcal{L}_{((\rho(a))^\uparrow,0,0)}\langle c_\phi, S_\zeta\rangle(v^p,\nu^q,\xi) \\
&= \frac{d}{dt}\Big|_{t=0} \langle (v^1 + t\rho(a), v^2, \ldots, v^p, \nu^q), \sigma(\zeta)\rangle \\
&= \langle (\rho(a), v^2, \ldots, v^p, \nu^q), \sigma(\zeta)\rangle \\
\mathcal{L}_{\rho_{\mathbb{A}}(S_\zeta)}\langle c_\phi, S^1_a\rangle(v^p,\nu^q,\xi) &= \mathcal{L}_{(0,0,\partial^*(\zeta)^\uparrow)}(c_{l(a)} \circ \gamma_{(1)})(v^p,\nu^q,\xi) \\
&= \frac{d}{dt}\Big|_{t=0} (c_l(a) \circ \gamma_{(1)})(v^p, \nu^q, \xi + t\partial^*(\zeta)) \\
&= (c_{l(a)} \circ \gamma_{(1)})(v^p, \nu^q, \partial^*(\zeta)) \\
&= \langle l(a)(v^2, \ldots, v^p, \nu^q), \partial^*(\zeta)\rangle
\end{aligned}$$

Therefore Equation (IM8) follows. Finally, consider S^1_α and S_ζ. Then

$$\begin{aligned}
\mathcal{L}_{(0,0,\partial^*(\zeta)^\uparrow)}\langle c_\phi, S^1_\alpha\rangle(v^p,\nu^q,\xi) &= \frac{d}{dt}\Big|_{t=0} c_{r(\alpha)} \circ \gamma^*_{(1)}(v^p, \nu^q, \xi + t\partial^*(\zeta)) \\
&= \langle r(\alpha)(v^p, \nu_2, \ldots, \nu_q), \partial^*(\zeta)\rangle \\
\mathcal{L}_{(0,\rho^*(\alpha)^\uparrow,0)}\langle c_\phi, S_\zeta\rangle(v^p,\nu^q,\xi) &= \frac{d}{dt}\Big|_{t=0} \langle (v^p, \nu_1 + t\rho^*(\alpha), \nu_2, \ldots, \nu_q), \sigma(\zeta)\rangle \\
&= \langle \sigma^*(v^p, \rho^*_A(\alpha), \nu_2, \ldots, \nu_q), \zeta\rangle.
\end{aligned}$$

So Equation (IM9) holds. Finally, the Leibniz rule follows by the same arguments in the proof of Proposition 3.11. Conversely given a $(\mathbf{D}, l, r, \sigma)$ define the map $\mu : \Gamma(\mathbb{A}_{\mathcal{A}}^{(p,q)}, \mathbb{M}_{\mathcal{E}}^{(p,q)}) \longrightarrow C^{\infty}(\mathbb{M}_{\mathcal{E}}^{(p,q)})$ by

$$\begin{array}{rcl} \langle \mu, \chi_a \rangle & = & c_{\mathbf{D}(X_a)} \\ \langle \mu, S_a^i \rangle & = & (-1)^{i-1} c_{l(a)} \circ \gamma_{(i)} \\ \langle \mu, S_\alpha^j \rangle & = & (-1)^{j-1} c_{r(\alpha)} \circ \gamma_{(j)}^* \\ \langle \mu, S_\zeta \rangle & = & c_{\sigma(\zeta)} \circ \gamma. \end{array}$$

By the proof of Theorem 3.6 it follows that μ is a well defined linear map with respect to the structure $\mathbb{A}_{\mathcal{A}}^{(p,q)} \longrightarrow \mathbb{M}_{\mathcal{E}}^{(p,q)}$ and componentwise linear function with respect to $\mathbb{A}_{\mathcal{A}}^{(p,q)} \longrightarrow A$. Hence we can see $\mu = c_\phi$ where ϕ is a (p, q) tensor on A with coefficients in $\mathcal{A}$. Also the IM equations satisfied by $(\mathbf{D}, l, r, \sigma)$ imply that c_ϕ is a cocycle, which means that ϕ is a Lie algebroid tensor. $\square$

We state now the infinitesimal-global correspondence

Theorem 3.17. *Given a multiplicative $\mathcal{E}$-valued (p, q)-tensor on $\mathcal{G}$, the associated quadruple $(\mathbf{D}, l, r, \sigma)$ is an $A_{\mathcal{E}}$-valued (p, q)-tensor on A. Moreover, if $\mathcal{G}$ is source simply connected the correspondence is one-to-one, given by*

$$\begin{cases} \mathcal{L}_{\overrightarrow{X}} \tau = \mathcal{T}(\mathbf{D}(X)) \\ i_{\overrightarrow{a}} \tau = \mathcal{T}(l(a)) \\ i_{t^*\alpha} \tau = \mathcal{T}(r(\alpha)) \\ i_{S_\zeta} \tau = \mathcal{T}(\sigma(\zeta)) \end{cases} \tag{3.30}$$

Proof. If τ is a multiplicative $\mathcal{E}$-valued (p, q)-tensor on $\mathcal{G}$ then the infinitesimal counterpart Ac_τ of its associated componentwise linear function c_τ is a Lie algebroid cocycle. And by definition of $(\mathbf{D}, l, r, \sigma)$ associated to τ follows that this quadruple is the corresponding one with the cocycle Ac_τ. Hence $(\mathbf{D}, l, r, \sigma)$ is an $A_{\mathcal{E}}$-valued (p, q)-tensor on A. $\square$

Remark 3.18. If $\mathcal{E}$ is trivial VB-groupoid $\mathbb{R}$, we are in the case of usual tensor fields on a Lie groupoid $\mathcal{G}$. In this situation our result recovers the description of multiplicative (p, q)-tensors given in [7].

We will now see how to recover various results in the literature from this theorem, and get new ones.

3.3 Multiplicative k-forms with coefficients in a representation up to homotopy

In this section we apply the theory of multiplicative tensors with coefficients to the case of k-forms with coefficients in a representation up to homotopy. We will characterize such forms and we will describe them infinitesimally.

Also we show that the known cases of multiplicative forms and multiplicative forms with coefficients in a representation are particular cases of this more general approach.

Let $(\Delta^0, \Delta^1, \partial, \Omega)$ be a representation up to homotopy of $\mathcal{G}$ on $C_{[0]} \oplus E_{[1]}$, and let $\mathcal{E} := \mathbf{s}^*(E) \oplus \mathbf{t}^*(C)$ be the associated VB-groupoid (see Subsection 1.3.3). A *multiplicative k-form on $\mathcal{G}$ with coefficient in a representation up to homotopy* is a multiplicative $(k,0)$-tensor on $\mathcal{G}$ with coefficients in $\mathcal{E}$. We may think of a k-form ω on $\mathcal{G}$ with coefficient in $\mathcal{E}$ as a pair $\omega = (\omega^0, \omega^1)$ where $\omega^0 \in \Omega^k(\mathcal{G}, \mathbf{t}^*(C))$ and $\omega^1 \in \Omega^k(\mathcal{G}, \mathbf{s}^*(E))$. With this expression for ω, we can write the multiplicativity condition in terms of ω^0 and ω^1.

Proposition 3.19. *A k-form $\omega = (\omega^0, \omega^1) \in \Omega^k(\mathcal{G}; \mathcal{E})$ is multiplicative if and only if the following equations hold*

$$\begin{aligned}
&(3.31) \qquad & \omega^1 &= \boldsymbol{s}^*(\theta) \\
&(3.32) \qquad & \partial \circ \omega^0 + g \cdot \omega^1 &= \boldsymbol{t}^*(\theta) \quad \textit{for some } \theta \in \Omega^k(M, E) \\
&(3.33) \qquad & (m^*\omega^0)_{(g,h)} &= (P_1^*\omega^0)_{(g,h)} + g \cdot (P_2^*\omega^0)_{(g,h)} - \Omega_{g,h}(P_2^*\omega^1)_{(g,h)}
\end{aligned}$$

where $P_1, P_2 : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ are the projections on the first and second component, respectively.

Proof. Let $\omega = (\omega^0, \omega^1) \in \Omega^k(T\mathcal{G} \otimes \mathcal{E})$ be a k-form with coefficients in $\mathcal{E}$. By definition ω is multiplicative if its associated componentwise linear function $c_\omega : (\bigoplus_k T\mathcal{G}) \oplus \mathcal{E}^* \longrightarrow \mathbb{R}$ is multiplicative. Note that this is equivalent to the map $\omega : \bigoplus_k T\mathcal{G} \longrightarrow \mathcal{E}$ being a morphism of Lie groupoids. In particular there exists a k-form $\theta \in \Omega^k(M, E)$ such that $\overline{s} \circ \omega = \theta \circ (T\mathbf{s})^k$ and $\overline{t} \circ \omega = \theta \circ (T\mathbf{t})^k$, where $(T\mathbf{s})^k$ and $T\mathbf{t})^k$ are the source and target maps of $\bigoplus_k T\mathcal{G} \longrightarrow \bigoplus_k TM$, respectively. This implies that

$$\begin{aligned}
\theta \circ (T\mathbf{s})^k &= \omega^1 \\
\theta \circ (T\mathbf{t})^k &= \partial \circ \omega^0 + g \cdot \omega^1
\end{aligned}$$

Hence the compatibility of ω with the source and the target maps is equivalent to (3.31) and (3.32). Let now $X_g^k = (g, X_1, \ldots, X_k)$ and $Y_h^k = (h, Y_1, \ldots, Y_k)$ in $\oplus^k T\mathcal{G}$ be two composable elements. Since ω is a morphism of Lie groupoids, it follows that $\omega(X_g^k)$ and $\omega(Y_h^k)$ are composable in $\mathcal{E}$. Since

$$\begin{aligned}
\overline{s}(\omega(X_g^k)) &= \omega^1(X_g^k) \\
\overline{t}(\omega(Y_h^k)) &= \partial \circ \omega^0(Y_h^k) + h \cdot \omega^1(Y_h^k)
\end{aligned}$$

then $\omega(X_g^k)$ and $\omega(Y_h^k)$ are composable if and only if

$$(P_1^*\omega^1)_{(g,h)} = \partial \circ (P_2^*\omega^0)_{(g,h)} + h \cdot (P_2^*\omega^1)_{(g,h)}.$$

Now with respect to the multiplication we have

$$\omega(X^k_g \cdot Y^k_h) = \big(\omega^0(X^k_g \cdot Y^k_h), \omega^1(X^k_g \cdot Y^k_h)\big).$$

On the other hand

$$\begin{aligned}\omega(X^k_g)\cdot\omega(Y^k_h) &= \big(\omega^0(X^k_g)\cdot\omega^0(Y^k_h), \omega^1(X^k_g)\cdot\omega^1(Y^k_h)\big)\\ &= (gh,\ \omega^0(X^k_g) + g\cdot\omega^0(Y^k_h) - \Omega_{g,h}\omega^1(Y^k_h),\ \omega^1(Y^k_h)\end{aligned}$$

Therefore ω respects the multiplication if and only if

$$\begin{aligned}(m^*\omega^0)_{(g,h)} &= (P_1^*\omega^0)_{(g,h)} + g\cdot(P_2^*\omega^0)_{(g,h)} - \Omega_{g,h}(P_2^*\omega^1)_{(g,h)}\\ (m^*\omega^1)_{(g,h)} &= (P_2^*\omega^1)_{(g,h)}\end{aligned}$$

Note that by Equation (3.31) and by properties of the multiplication, we have

$$\begin{aligned}(m^*\omega^1)(X^k_g, Y^k_h) &= m^*(\mathbf{s}^*\theta)(X^k_g, Y^k_h) = \theta((\mathrm{d}\mathbf{s})^k(X^k_g\cdot Y^k_h) = \theta((\mathrm{d}\mathbf{s})^k(Y^k_h))\\ (P_2^*\omega^1)(X^k_g, Y^k_h) &= (\mathbf{s}\circ P_2)^*\theta(X^k_g, Y^k_h) = \theta((\mathrm{d}\mathbf{s})^k Y^k_h).\end{aligned}$$

□

Example 3.20. In the case with trivial coefficient, $\mathcal{E} = \mathbb{R} \simeq C$ with the action given by the identity map, $E = \{*\}$ is a point, and the maps $\partial = 0$ and $\Omega = 0$. Therefore we have a usual k-form $\omega \in \Omega^k(\mathcal{G})$, and it is multiplicative if and only if

$$(m^*\omega)_{(g,h)} = (P_1^*\omega)_{(g,h)} + (P_2^*\omega)_{(g,h)},$$

which is precisely the case treated in [7].

Example 3.21. If $C \longrightarrow M$ is a representation of $\mathcal{G}$, then the VB-groupoid associated is $\mathcal{E} = C \rtimes \mathcal{G}$, with trivial side bundle E. Then $\partial = 0$ and $\Omega = 0$. Therefore, a k-form ω on $\mathcal{G}$ with coefficient in $\mathcal{E}$ is multiplicative if and only if

$$(m^*\omega)_{(g,h)} = (P_1^*\omega)_{(g,h)} + g\cdot(P_2^*\omega)_{(g,h)},$$

which is the definition given in [14].

Let $\omega \in \Omega^k(\mathcal{G}, \mathcal{E})$ be a multiplicative k-form, where $\mathcal{E} = \mathbf{s}^*E \oplus \mathbf{t}^*C$ is the VB-groupoid associated to a representation up to homotopy $(\Delta^0, \Delta^1, \partial, \Omega)$ of $\mathcal{G}$. We will describe in detail the $A_{\mathcal{E}}$-valued $(k,0)$-tensor on A associated to the multiplicative $(k,0)$-tensor ω. To do this, first we need a little work.

A representation up to homotopy $(\Delta^0, \Delta^1, \partial, \Omega)$ induces a representation up to homotopy of $\mathcal{G}$ on the graded vector bundle $E^*_{[0]} \oplus C^*_{[1]}$ as follows:

- The quasi action $(\Delta^T)^0_g : E^*_{\mathbf{s}(g)} \longrightarrow E^*_{\mathbf{t}(g)}$ given by:

$$(\Delta^T)^0_g(\eta) := (\Delta^1_{g^{-1}})^*\eta, \quad \text{for } \eta \in E^*_{\mathbf{s}(g)}$$

- The quasi actions $(\Delta^T)^1_g : C^*_{\mathbf{s}(g)} \longrightarrow C^*_{\mathbf{t}(g)}$ given by:

$$(\Delta^T)^1_g(\xi) := (\Delta^0_{g^{-1}})^*\xi, \quad \text{for } \xi \in C^*_{\mathbf{s}(g)}$$

- The vector bundle map $\partial^T : E^* \longrightarrow C^*$ is the dual map ∂^*.

- The operator Ω^T is given by: $\xi \in C^*_{\mathbf{s}(g)}$ we have

$$\Omega^T_{(g,h)} : C^*_{\mathbf{s}(h)} \longrightarrow E^*_{\mathbf{t}(g)} \qquad \Omega^T_{(g,h)}(\xi) := (\Omega_{(h^{-1},g^{-1})})^*\xi,$$

  for $(g,h) \in \mathcal{G}^{(2)}$.

We check now the equations that the quadruple $(\Delta^0)^T, (\Delta^1)^T, \partial^T, \Omega^T)$ has to satisfy (see Subsection 1.3.2). The compatibility of the quasi actions with the vector bundle map ∂^T:

$$(\Delta^1)^T \circ \partial^T - \partial^T \circ (\Delta^0)^T = (\partial \circ \Delta^0)^* - (\Delta^1 \circ \partial)^* = -(\Delta^1 \circ \partial - \partial \circ \Delta^0)^* = 0.$$

For the second equation we have

$$\begin{aligned}
(\Delta^0)^T_{g_1}(\Delta^0)^T_{g_2} - (\Delta^0)^T_{g_1g_2} + \Omega^T_{(g_1,g_2)} \circ \partial^T &= (\Delta^1_{g_1^{-1}})^*(\Delta^1_{g_2^{-1}})^* - (\Delta^1_{g_2^{-1}g_1^{-1}})^* \\
&\quad + (\partial \circ \Omega^T_{(g_2^{-1},g_1^{-1})})^* \\
&= \left(\Delta^1_{g_2^{-1}}\Delta^1_{g_1^{-1}} - \Delta^1_{g_2^{-1}g_1^{-1}} + \partial \circ \Omega^T_{(g_2^{-1},g_1^{-1})}\right)^* \\
&= 0.
\end{aligned}$$

In a similar way we get the third and fourth equation. Hence $((\Delta^0)^T, (\Delta^1)^T, \partial^T, \Omega^T)$ is a representation up to homotopy of $\mathcal{G}$ on $E^*_{[0]} \oplus C^*_{[1]}$, called *dual representation.*

On the other hand the dual VB-groupoid of $\mathcal{E} = \mathbf{s}^*E \oplus \mathbf{t}^*C$ is the VB-groupoid with structure maps given by:

- The source and target maps $\widehat{s}, \widehat{t} : \mathcal{E}^* = \mathbf{s}^*(E^*) \oplus \mathbf{t}^*(C^*) \longrightarrow C^*$ are

$$\begin{aligned}
\widehat{s}(g,\eta,\xi) &= \partial^*(\eta) + (\Delta^0_g)^*\xi \\
\widehat{t}(g,\eta,\xi) &= \xi
\end{aligned}$$

  where $\eta \in E^*_{\mathbf{s}(g)}$ and $\xi \in C^*_{\mathbf{t}(g)}$. The multiplication is

$$(g_1,\eta_1,\xi_1) \cdot (g_2,\eta_2,\xi_2) = (g_1g_2, \eta_2 + (\Delta^1_{g_2})^*\eta_1 - \Omega^*_{g_1,g_2}\xi_1, \xi_1),$$

  under the compatibility condition $\xi_2 = \partial^*(\eta_1) + (\Delta^0_g)^*\xi_1$.

Lemma 3.22. *There is an isomorphism of Lie groupoids between the Lie groupoid* $\boldsymbol{s}^*(C^*) \oplus \boldsymbol{t}^*(E^*)$ *obtained by the dual representation* $((\Delta^0)^T, (\Delta^1)^T, \partial^T, \Omega^T)$*, and the Lie groupoid* $\boldsymbol{s}^*(E^*) \oplus \boldsymbol{t}^*(C^*)$ *which is obtained by dualization of the VB-groupoid* $\mathcal{E} = \boldsymbol{s}^*E \oplus \boldsymbol{t}^*C$*. Moreover this is an isomorphism of VB-groupoids.*

Proof. Let $(g,\xi,\eta)\in \mathbf{s}^*(C^*)\oplus \mathbf{t}^*(E^*)$. The inverse of this element is (see [19]):

$$(g,\xi,\eta)^{-1} = (g^{-1}, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi)$$

Define the map $\varphi : \mathbf{s}^*(C^*)\oplus \mathbf{t}^*(E^*) \longrightarrow \mathbf{s}^*(E^*)\oplus \mathbf{t}^*(C^*)$ by

$$\varphi(g,\xi,\eta) = (g, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi).$$

We show now the compatibility of φ with the source, target and multiplication. Recall that the source and target maps of $\mathbf{s}^*(C^*)\oplus \mathbf{t}^*(E^*)$, denoted by $\widetilde{s},\widetilde{t}$, are defined by (see Subsection 1.3.3):

- $\widetilde{s}(g,\xi,\eta) = \xi$
- $\widetilde{t}(g,\xi,\eta) = \partial^T\eta + (\Delta^1)^T_g\xi$

for $\xi\in C^*_{\mathbf{s}(g)}$ and $\eta\in E^*_{\mathbf{t}(g)}$. Then

$$\begin{aligned}
\widehat{s}(\varphi(g,\xi,\eta)) &= \widehat{s}(g, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi)\\
&= \partial^*(-(\Delta^1_g)^*\eta + \Omega^*_{g^{-1},g}\xi) + (\Delta^0_g)^*(\partial^*(\eta) + (\Delta^0_{g^{-1}})^*\xi)\\
&= -(\Delta^1_g\circ\partial)^*\eta + (\Omega_{g^{-1},g}\circ\partial)^*\xi + (\partial\circ\Delta^0_g)^*(\eta) + (\Delta^0_{g^{-1}}\Delta^0_g)^*\xi\\
&= (\Delta^0_{g^{-1}g})^*\xi = \xi\\
&= \widetilde{s}(g,\xi,\eta)
\end{aligned}$$

With respect to the target map

$$\begin{aligned}
\widehat{t}(\varphi(g,\xi,\eta)) &= \widehat{t}(g, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi)\\
&= \partial^T\eta + (\Delta^1)^T_g\xi\\
&= \widetilde{t}(g,\xi,\eta)
\end{aligned}$$

The compatibility with respect to the multiplication is a long computation and it is presented in the Appendix. □

The representation up to homotopy $(\Delta^0,\Delta^1,\partial,\Omega)$ of $\mathcal{G}$ on $C_{[0]}\oplus E_{[1]}$ also induces a representation up to homotopy $(\nabla^0,\nabla^1,\partial,\widetilde{\Omega})$ of A on $C_{[0]}\oplus E_{[1]}$ in the following way (see [3] for details) :

- the connection $\nabla^0 : \Gamma(A)\times\Gamma(C)\longrightarrow\Gamma(C)$ is given by

$$(\nabla^0_a c)(p) = \frac{d}{dt}\Big|_{t=0}(\Delta^0_{g(t)}c(p))$$

  where $g(t)$ is a curve in $\mathbf{s}^{-1}(p)$ with $g'(0)=a(p)$. In a similar way define the connection $\nabla^1 : \Gamma(A)\times\Gamma(E)\longrightarrow\Gamma(E)$.

- The operator Ω can also be differentiated to an operator $\widetilde{\Omega} \in \Gamma(\wedge^2 A^* \otimes \mathrm{Hom}(E, C))$ (see Definition 3.9 in [3]).

- The vector bundle map from C to E is the map ∂

Consider the dual representation (see e.g. [17]) $(\nabla^{T_0}, \nabla^{T_1}, \partial^T, \widetilde{\Omega}^T)$ of A on $E^*_{[0]} \oplus C^*_{[1]}$, where

$$\nabla^{T_0} = (\nabla^1)^* \quad \nabla^{T_1} = (\nabla^0)^* \quad \partial^T = \partial^* \quad \widetilde{\Omega^T}_{a,b} = -\widetilde{\Omega}^*_{a,b}.$$

Note that by definition of the quadruple $(\nabla^{T_0}, \nabla^{T_1}, \partial^T, \widetilde{\Omega}^T)$, it follows that this representation is induced by the dual representation $((\Delta^0)^T, (\Delta^1)^T, \partial^T, \Omega^T)$. Also this dual representation induces a VB-algebroid structure on $\mathbb{A} := A \oplus E^* \oplus C^*$

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d] & A \arrow[d] \\ C^* \arrow[r] & M \end{tikzcd}$$

with core bundle $E^* \longrightarrow M$.

Corollary 3.23. *There is a Lie algebroid isomorphism between the Lie algebroid $\mathbb{A} := A \oplus E^* \oplus C^*$ induced by the dual representation and the Lie algebroid $A_{\mathcal{E}^*}$. Moreover this is an isomorphism of VB-algebroids.*

In what follows we will use the VB-algebroid $(\mathbb{A}, C^*; A, M)$ instead of $(A_{\mathcal{E}^*}, C^*; A, M)$ because the calculus are easier and more explicit. Denote by $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(\mathbb{A}, C^*)$ the canonical horizontal lift: $h(a)(\xi) = (a, 0, \xi)$. This horizontal lift is $C^\infty(M)$-linear and satisfies

$$[h(a), h(b)] = h([a, b]) + S_{\widetilde{\Omega}^T(a,b)}$$

Now with respect to the sections of the tangent algebroid, there is also a natural inclusion $\Gamma(A) \hookrightarrow \Gamma_{\mathrm{lin}}(TA, TM)$, $\;a \mapsto Ta$. However, unlike the previous horizontal lift, this inclusion is not $C^\infty(M)$-linear:

$$T(fa) = fTa + \ell_{\mathrm{d}f} S_a.$$

Remember that the $A_{\mathcal{E}}$-valued $(k, 0)$-tensor on A associated to the $\mathcal{E}$-valued multiplicative $(k, 0)$-tensor ω on $\mathcal{G}$ is a triple $(\mathbf{D}, l, \sigma)$ where

- $\mathbf{D} : \Gamma_{\mathrm{lin}}(\mathcal{A}^*, C^*) \longrightarrow \Gamma(\wedge^p T^*M \otimes C) = \Omega^k(M, C)$

- $l : A \longrightarrow \wedge^{p-1} T^*M \otimes \wedge^q A \otimes C$

- $\sigma : E^* \longrightarrow \wedge^p T^*M$ which we view as an element $\theta \in \Omega^k(M, E)$

satisfying some equations. Note that there is not a map $r : T^*M \longrightarrow \wedge^p T^*M \otimes \wedge^{q-1}A^* \otimes C$ because there is not a cotangent algebroid. Since we have a natural inclusion of $\Gamma(A)$ in $\Gamma_{\text{lin}}(\mathcal{A}^*, C^*)$, we can define a new operator

$$\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C) \quad \text{by} \quad \mathbb{D}(a) = \mathbf{D}(h(a))$$

Note that by Proposition 3.11, and using that $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}^*, C^*)$ is $C^\infty(M)$-linear we have

$$\begin{aligned} \mathbb{D}(fa) &= \mathbf{D}(h(fa)) = \mathbf{D}(fh(a)) \\ &= f\mathbf{D}(h(a)) + \mathrm{d}f \wedge l(a) \\ &= f\mathbb{D}(a) + \mathrm{d}f \wedge l(a) \end{aligned}$$

This motivates the following definition

Definition 3.24. An $A_{\mathcal{E}}$-*valued* $(k, 0)$-tensor on A is

- $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C)$
- $l : A \longrightarrow \wedge^{k-1}T^*M \otimes C$
- $\theta \in \Omega^k(M, E)$

such that the following equations hold

$$\mathbb{D}(fa) = f\mathbb{D}(a) + \mathrm{d}f \wedge l(a) \tag{3.34}$$
$$\mathbb{D}([a, b]) = a \cdot \mathbb{D}(b) - b \cdot \mathbb{D}(a) + \widetilde{\Omega}_{a,b} \circ \theta \tag{3.35}$$
$$\iota_{\rho(b)}\mathbb{D}(a) = a \cdot l(b) - l([a, b]) \tag{3.36}$$
$$\iota_{\partial^*(\eta)}\mathbb{D}(a) = a \cdot \theta^*(\eta) - \theta^*((\nabla^1)^*_a \eta) \tag{3.37}$$
$$\iota_{\rho(a)}l(b) = -\iota_{\rho(b)}l(a) \tag{3.38}$$
$$\iota_{\partial^*(\eta)}l(a) = \iota_{\rho(a)}\theta^*(\eta) \tag{3.39}$$

We only need to check the Equation (3.35). Let $a, b \in \Gamma(A)$, then

$$\begin{aligned} \mathbb{D}([a, b]) &= \mathbf{D}(h([a, b])) \\ &= \mathbf{D}([h(a), h(b)] - S_{\widetilde{\Omega}^T_{a,b}}) \\ &= \mathbf{D}([h(a), h(b)]) - \mathbf{D}(0, S_{\widetilde{\Omega}^T_{a,b}}) \\ &= h(a) \cdot \mathbf{D}(h(b)) - h(b) \cdot \mathbf{D}(h(a)) - \theta^*(\widetilde{\Omega}^T_{a,b}) \\ &= a \cdot \mathbb{D}(b) - b \cdot \mathbb{D}(a) + \widetilde{\Omega}_{a,b} \circ \theta. \end{aligned}$$

Therefore applying Theorem 3.17 to this case we obtain a global-infinitesimal correspondence between multiplicative k-form $\omega \in \Omega^k(\mathcal{G}, \mathcal{E})$ with coefficients in a representation up to homotopy and $A_{\mathcal{E}}$-valued $(k, 0)$-tensor $(\mathbb{D}, l, \theta)$ on A.

Theorem 3.25. *Let $\mathcal{G}$ be a source simply connected Lie groupoid. There is one-to-one correspondence between multiplicative k-forms $\omega \in \Omega^k(\mathcal{G}, \mathcal{E})$ with coefficients in a representation up to homotopy and $A_{\mathcal{E}}$-valued $(k,0)$-tensor $(\mathbb{D}, l, \theta)$ on A.*

Now we will give an explicit expression for the operator $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C)$. The anchor map of the Lie algebroid $\mathbb{A} \longrightarrow C^*$ on linear sections is

$$\rho_{\mathbb{A}}(h(a))(\xi_p) = \frac{d}{dt}\Big|_{t=0} \big((\varphi_t^a(p))\big)^* \cdot \xi_p \quad \in TC^*$$

where φ_t^a is the flow of the right invariant vector field $\overrightarrow{a} \in \mathfrak{X}(\mathcal{G})$. It follows then that the flow of the right invariant vector field $\overrightarrow{h(a)}$, restricted to units, is

$$\Phi_t^{h(a)}(p, \xi_p, 0) \;\; = \;\; (\varphi_t^a(p), \big((\varphi_t^a(p))\big)^* \cdot \xi_p, 0_p).$$

This flow $\Phi_t^{h(a)}$ is a linear map over φ_t^a:

$$\big((\Phi_t^{h(a)})^*\big)^{-1} : \begin{tikzcd} \mathcal{E}^* \arrow[r] \arrow[d] & \mathcal{E}^* \arrow[d] \\ \mathcal{G} \arrow[r, "\varphi_t^a"] & \mathcal{G} \end{tikzcd},$$

which can be restricts to

$$\big((\Phi_t^{h(a)})^*\big) : \begin{tikzcd} C^* \arrow[r] \arrow[d] & (\Phi_t^{h(a)})^*(C^*) \arrow[d] \\ M \arrow[r, "\varphi_t^a"] & M \end{tikzcd}$$

Taking dual we get

$$(\Phi_t^{h(a)})^{-1} : \begin{tikzcd} (\Phi_t^{h(a)})^* C \arrow[r] \arrow[d] & C \arrow[d] \\ (\varphi_t^a)^{-1}(M) \arrow[r, "\varphi_t^a"] & M \end{tikzcd}$$

given by

$$(\Phi_t^{h(a)})_p^{-1}(c) = \varphi_t^a(p)^{-1} \cdot c$$

for $c \in C_{\mathbf{t}(\varphi_t^a(p))}$.

Proposition 3.26. *Let $\omega = (\omega^0, \omega^1) \in \Omega^k(\mathcal{G}, \mathcal{E})$ be a multiplicative k-form with coefficients in a representation up to homotopy. Then the associated operator $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C)$ is given by*

$$\mathbb{D}(a)(v_1, \ldots, v_k) = \frac{d}{dt}\Big|_{t=0} \varphi_t^a(p)^{-1} \cdot \omega^0(d\varphi_t^a(v_1), \ldots, d\varphi_t^a(v_k)) \tag{3.40}$$

Proof. For $v_i \in TM$ and $\xi \in C^*$ we have

$$\begin{aligned}
\langle \mathbb{D}(a), (v_1, \dots, v_k, \xi) \rangle &= \langle Ac_\omega, (v_1, \dots, v_k, \xi) \rangle \\
&= \frac{d}{dt}\Big|_{t=0} \langle \omega(\mathrm{d}\varphi_t^a(v_1), \dots, \mathrm{d}\varphi_t^a(v_k)), (\Phi_t^{h(a)})(\xi) \rangle \\
&= \frac{d}{dt}\Big|_{t=0} \langle \varphi_t^a(p)^{-1} \cdot \omega(\mathrm{d}\varphi_t^a(v_1), \dots, \mathrm{d}\varphi_t^a(v_k)), \xi \rangle \\
&= \frac{d}{dt}\Big|_{t=0} \langle \varphi_t^a(p)^{-1} \cdot \omega^0(\mathrm{d}\varphi_t^a(v_1), \dots, \mathrm{d}\varphi_t^a(v_k)), \xi \rangle.
\end{aligned}$$

□

Example 3.27. Multiplicative k-forms. Consider a multiplicative linear k-form ω on $\mathcal{G}$. In our context this means that ω is a multiplicative $(k,0)$-tensor on $\mathcal{G}$ with trivial coefficients, which means that $C = \mathbb{R}$ and $E = 0$. Then its associated $IM-(k,0)$ tensor on A is the pair $(\mathbb{D}, l)$ where

- $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M)$ given by

$$\mathbb{D}(a)(v_1, \dots, v_k) = \frac{d}{dt}\Big|_{t=0} \omega(d\phi_\alpha^t(v_1), \dots, d\phi_\alpha^t(v_k))$$

- $l : A \longrightarrow \wedge^{k-1} T^* M$ determined by

$$l(a) = \epsilon^*(i_{\overrightarrow{a}} \omega)$$

  where $\epsilon : M \longrightarrow \mathcal{G}$ is the unit map, and such that the following Leibniz rule holds

$$\mathbb{D}(fa) = f\mathbb{D}(a) + df \wedge l(a)$$

(See Theorem 1 in [14], in the case of the trivial representation). The pair $(\mathbb{D}, l)$ satisfies then the following conditions

$$\mathbb{D}(fa) = f\mathbb{D}(a) + \mathrm{d}f \wedge l(a) \tag{3.41}$$

$$\mathbb{D}([a,b]) = a \cdot \mathbb{D}(b) - b \cdot \mathbb{D}(a) \tag{3.42}$$

$$\iota_{\rho(b)}\mathbb{D}(a) = a \cdot l(b) - l([a,b]) \tag{3.43}$$

$$\iota_{\rho(a)} l(b) = -\iota_{\rho(b)} l(a) \tag{3.44}$$

which is what in [14] is called *k-Spencer Operator with trivial coefficients.*

On the other hand, take $a \in \Gamma(A)$, then $\sigma(a) \in \Omega^{k-1}(M)$. Define $\nu(a) = \mathbb{D}(a) - d(\sigma(a)) \in \Omega^k(M)$. Let $h \in C^\infty(M)$. Then

$$\begin{aligned}
\nu(ha) &= \mathbb{D}(ha) - d(\sigma(ha)) \\
&= h\mathbb{D}(a) - dh \wedge \sigma(a) - d(h\sigma(a)) \\
&= h\mathbb{D}(a) - dh \wedge \sigma(a) - (hd(\sigma(a)) - dh \wedge \sigma(a)) \\
&= h(\mathbb{D}(a) - d(\sigma(a))) \\
&= h\nu(a).
\end{aligned}$$

Therefore $\nu : A \longrightarrow \wedge^k T^*M$ is a vector bundle morphism over M. The next corollary recovers the results in [7].

Corollary 3.28. *Given a multiplicative k-form $\omega \in \Omega^2(\mathcal{G})$ there exists vector bundle maps $\nu : A \longrightarrow \wedge^2 T^*M$, $\mu : A \longrightarrow T^*M$ such that $\mathbb{D}(a) = \nu(a) + d(\mu(a))$. Moreover the pair (l, ν) is $IM - k$ form on A.*

Example 3.29. Multiplicative k-form with coefficient in a representation. Consider a representation of $\mathcal{G}$ on a vector bundle C, and let $\omega \in \Omega^k(\mathcal{G}, \mathbf{t}^*C)$ be a k-form on $\mathcal{G}$ with coefficient in the representation. Its associated $(k, 0)$ tensor on A with coefficients in C is the pair $(\mathbb{D}, l)$ where

- $\mathbb{D} : \Gamma(A) \longrightarrow \Omega^k(M, C)$ given by

$$\mathbb{D}(a)(v_1, \ldots, v_k) = \frac{d}{dt}\Big|_{t=0} \varphi_t^a(p)^{-1} \cdot \omega(d\phi_\alpha^t(v_1), \ldots, d\phi_\alpha^t(v_k))$$

- $l : A \longrightarrow \wedge^{k-1} T^*M \otimes C$ determined by

$$l(a) = \epsilon^*(i_{\overrightarrow{a}} \omega)$$

  and such that the following Leibniz rule holds

$$\mathbb{D}(fa) = f\mathbb{D}(a) + df \wedge l(a)$$

(See Theorem 1 in [14]). Moreover, in this case, the operators $\Omega = 0$, $\theta = 0$ and the equations for $(\mathbb{D}, l)$ are

$$\begin{aligned}
&(3.45) & \mathbb{D}(fa) &= f\mathbb{D}(a) + \mathrm{d}f \wedge l(a) \\
&(3.46) & \mathbb{D}([a, b]) &= a \cdot \mathbb{D}(b) - b \cdot \mathbb{D}(a) \\
&(3.47) & \iota_{\rho(b)}\mathbb{D}(a) &= a \cdot l(b) - l([a, b]) \\
&(3.48) & \iota_{\rho(a)} l(b) &= -\iota_{\rho(b)} l(a) \\
&(3.49) & &
\end{aligned}$$

which is precisely the definition of a C-valued k-Spencer operator over A given in [14].

Remark 3.30. The module structure on $\Omega^k(M, C)$ over $\Gamma(A)$ in this context coincides with Lie derivative operator $\mathcal{L}_a$ acting on $\Omega^k(M, C)$ defined on [14] (see Appendix).

Chapter 4

Applications to VB-subalgebroids

Let $\mathcal{A} \longrightarrow E$ be a VB-algebroid over a Lie algebroid $A \longrightarrow M$. In this chapter we study *IM-subbundles*, that is, double vector subbundles $\Delta \longrightarrow \Delta_M$ which are Lie subalgebroids of $\mathcal{A} \longrightarrow E$. The linear quotient $\mathcal{A}/\Delta$ is, indeed, a Lie algebroid. Then the idea is to consider an IM-subbundle as the kernel of a morphism of Lie algebroids $\Phi : \mathcal{A} \longrightarrow \mathcal{A}/\Delta$. Moreover, since $\mathcal{A}/\Delta$ is also a VB-algebroid over A we can dualize and then we get a function $F_\Phi : \mathcal{A} \oplus \Delta^\circ \longrightarrow \mathbb{R}$, where Δ° denote the annihilator of Δ in $\mathcal{A}^*$, which we identify with the dual over A of $\mathcal{A}/\Delta$. This function F_Φ is an *infinitesimal bilinear cocycle*, hence we can use what we did in the previous chapters.

As particular cases, we consider *IM-distributions*, *IM-subbundles* of $TA \oplus T^*A$, and finally *IM-Dirac structures*, offering a new viewpoint and giving new proofs to results in [17, 26, 27]. We remark, however, that our approach is more general and allows to consider IM-subbundles not only of TA or $TA \oplus T^*A$.

The objective of this chapter is the description in terms of some infinitesimal data of the IM-subbundles. We will mention, however, the global objects, i.e., the multiplicative geometric structures defined on Lie groupoids, whose infinitesimal counterparts are examples of IM-subbundles. We will also say something about the *integration* of IM-subbundles.

4.1 Double vector subbundles

First we study double vector subbundles, without any Lie algebroid conditions. We look at them as the kernel of double vector bundles, which are also surjective submersions.

Let

$$\begin{array}{ccc} \mathcal{A}_i & \longrightarrow & A \\ \downarrow & & \downarrow \\ E_i & \longrightarrow & M \end{array}$$

be two double vector bundles over A with core bundles C_i. Let $F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ be a surjective submersion morphism of double vector bundles covering $F_1 : E_1 \longrightarrow E_2$, $F_0 : C_1 \longrightarrow C_2$ and the identity on A. In particular, the maps F_0 and F_1 are surjective submersions. Consider the double vector bundle dual of $\mathcal{A}_2$ with respect to A

$$\begin{array}{ccc} \mathcal{A}_2^* & \longrightarrow & A \\ \downarrow & & \downarrow \\ C_2^* & \longrightarrow & M \end{array}$$

with core bundle E_2^* and define the map

$$\overline{F} : \mathcal{A}_1 \times_A \mathcal{A}_2^* \longrightarrow \mathbb{R} \qquad \text{by} \quad \overline{F}(X, \alpha) = \langle F(X), \alpha \rangle$$

where the pairing is canonical one over A.

Lemma 4.1. *$F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a double vector bundle morphism if and only if $\overline{F} : \mathcal{A}_1 \times_A \mathcal{A}_2^* \longrightarrow \mathbb{R}$ is linear with respect to $E_1 \times_M C_2^*$ and bilinear with respect to A.*

Proof. From the definition of the pairing it follows that $\overline{F}$ is linear with respect to the second entry. The linearity of $\overline{F}$ with respect to the first entry is equivalent to the linearity of F with respect to the linear structure over A. Let $(X, \alpha), (Y, \beta) \in (\mathcal{A}_1 \times_A \mathcal{A}_2^*)_{(e,\xi)}$. On the one hand we have

$$\overline{F}((X, \alpha) + (Y, \beta)) = \overline{F}(X +_{E_1} Y, \alpha + \beta) = \langle F(X +_{E_1} Y), \alpha + \beta \rangle. \tag{4.1}$$

On the other hand

$$\overline{F}(X, \alpha) + \overline{F}(Y, \beta) = \langle F(X), \alpha \rangle + \langle F(Y), \beta \rangle = \langle F(X) +_{E_2} F(Y), \alpha + \beta \rangle. \tag{4.2}$$

Then (4.1) = (4.2) if and only if F is linear with respect to the vertical linear structure. □

Define the map

$$\mathbf{D} : \Gamma_{lin}(\mathcal{A}_1, E_1) \times_A \Gamma_{lin}(\mathcal{A}_2^*, C_2^*) \longrightarrow \Gamma(E_1^* \otimes C_2)$$

by

$$\mathbf{D}(\chi_a, \phi_a)(e, \xi) = \langle F(\chi_a(e)), \phi_a(\xi) \rangle_A = \overline{F}(\chi_a(e), \phi_a(\xi)).$$

for $e \in E_1$ and $\xi \in C_2^*$.

Remark 4.2. The function $\overline{F}$ is actually a *bilinear cocycle* on $\mathcal{A}_1 \oplus \mathcal{A}_2$, thinking the double vector bundles of as VB-groupoids, where the Lie groupoids are vector bundles. Hence the operator $\mathbf{D}$ is the same as in Proposition 2.21.

Lemma 4.3. *There is a well defined operator* $\mathcal{D} : \Gamma_{lin}(\mathcal{A}_1, E_1) \longrightarrow \Gamma((Ker\ F_1)^* \otimes C_2)$ *given by*

$$\mathcal{D}(\chi_a)(e,\xi) = \mathbf{D}(\chi_a, \phi_a)(e,\xi)$$

where $\phi_a \in \Gamma_{lin}(\mathcal{A}_2^*, C_2^*)$ *is any linear section covering* $a \in \Gamma(A)$.

Proof. Remember the short exact sequence (1.20) for the double vector bundle $(\mathcal{A}_2^*, C_2^*; A, M)$:

$$0 \longrightarrow \Gamma(Hom(C_2^*, E_2^*)) \longrightarrow \Gamma_{lin}(\mathcal{A}_2^*, C_2^*) \longrightarrow \Gamma(A) \longrightarrow 0.$$

Let $\varphi \in \Gamma(Hom(C_2^*, E_2^*))$ and let $\phi_\varphi(\xi) = 0_\xi +_A \overline{\varphi(\xi)}$ the linear section associated to φ. Since F is a morphism of double vector bundles we have that $F(0_e) = 0_{F_1(e)}$ for every $e \in E_1$. Then

$$\begin{aligned}
\mathbf{D}(0, \phi_\varphi)(e,\xi) &= \langle F(0_e), \phi_\varphi(\xi)\rangle \\
&= \langle 0_{F_1(e)}, 0_\xi +_A \overline{\varphi(\xi)}\rangle \\
&= \langle F_1(e), \varphi(\xi)\rangle
\end{aligned}$$

Hence, if $e \in Ker(F_1)$ we have $\mathbf{D}(0, \phi_\varphi)(e,\xi) = 0$ for every $\varphi \in \Gamma(Hom(C_2^*, E_2^*))$. So if $\phi_a,\ \psi_a \in \Gamma_{lin}(\mathcal{A}_2^*, C_2^*)$ and $\chi_a \in \Gamma_{lin}(\mathcal{A}_1, E_1)$, then

$$(\mathbf{D}(\chi_a, \phi_a) - \mathbf{D}(\chi_a, \psi_a))(e,\xi) = \mathbf{D}(0, \phi_a - \psi_a)(e,\xi) = \mathbf{D}(0, \phi_\varphi)(e,\xi) = 0$$

for every $e \in Ker(F_1)$. $\square$

Let $\alpha \in \mathcal{A}_1$ be any element which projects to $e_p \in E_1$ and to $a_p \in A$, with $p \in M$. Denote by $a \in \Gamma(A)$ any section such that $a(p) = a_p$, and let $\chi_a : E_1 \longrightarrow \mathcal{A}_1$ be any linear section covering a. Then there exists a unique element $\delta \in \mathcal{A}_1$ such that $\alpha = \chi_a(e) +_{E_1} \delta$:

$$\begin{tikzcd} \alpha \arrow[r] \arrow[d] & a \arrow[d] \\ e \arrow[r] & p \end{tikzcd} \quad = \quad \begin{tikzcd} \chi_a(e) \arrow[r] \arrow[d] & a \arrow[d] \\ e \arrow[r] & p \end{tikzcd} \quad +_{E_1} \quad \begin{tikzcd} \delta \arrow[r] \arrow[d] & 0 \arrow[d] \\ e \arrow[r] & p \end{tikzcd} .$$

Moreover, δ can be written $\delta = 0_e +_A \overline{c_p} = S_c(e)$ where $c \in \Gamma(C_1)$ such that $c(p) = c_p$.

Proposition 4.4. *Let* $\alpha \in \mathcal{A}_1$*, and write it as* $\alpha = \chi_a(e) +_{E_1} (0_e +_A \overline{c})$ *for some linear sections* χ_a *covering* a*. Then*

$$\alpha \in Ker(F) \Leftrightarrow \begin{cases} e \in Ker(F_1) \\ \mathcal{D}(\chi_a)(e,\cdot) = -F_0(c) \end{cases}$$

Proof. We have to show that $\langle F(\alpha), \nu\rangle = 0$ for all element $\nu \in \mathcal{A}_2^*$, where the dual and the pairing is over A. The fiber $\mathcal{A}_2^* \longrightarrow A$ over a_p is generated by $0_a +_{C_2^*} \eta$ and $\phi(\xi)$ for a fixed $\phi \in \Gamma_{\mathrm{lin}}(\mathcal{A}_2^*, C_2^*)$ covering a, and $\xi \in (C_2^*)_p$, $\eta \in (E_2^*)_p$ varying . Hence

$$F(\alpha) = 0 \Leftrightarrow \begin{cases} \langle F(\alpha), 0_a +_{C_2^*} \eta\rangle = 0 \\ \langle F(\alpha), \phi(\xi)\rangle = 0 \end{cases}$$

Since $F(\alpha) = F(\alpha) +_{E_2} 0_{F_1(e)}$, the first equations becomes

$$\begin{aligned} 0 = \langle F(\alpha) +_{E_2} 0_{F_1(e)}, 0_a +_{C_2^*} \eta\rangle &= \langle F(\alpha), 0_a\rangle + \langle 0_{F_1(e)}, \overline{\eta}\rangle \\ &= 0 + \langle F_1(e), \eta\rangle. \end{aligned}$$

As η is arbitrary, the first equation holds if and only if $e \in \mathrm{Ker}(F_1)$. Since F is a double vector bundle morphism we have that $F(\alpha) = F(\chi_a(e)) +_{E_2} (0_{F_1(e)} +_{C_2} \overline{F_0(c)})$. Then the second equation is

$$\begin{aligned} 0 &= \langle F(\chi_a(e)) +_{E_2} (0_{F_1(e)} +_{C_2} \overline{F_0(c)}), \phi(\xi) +_{C_2^*} 0_\xi\rangle \\ &= \langle F(\chi_a(e)), \phi(\xi)\rangle + \langle (0_{F_1(e)} +_{C_2} \overline{F_0(c)}, 0_\xi\rangle \\ &= \mathcal{D}(\chi_a, \phi)(e, \xi) + \langle \xi, F_0(c)\rangle. \end{aligned}$$

Since ξ is arbitrary, the second equation holds if and only if $\mathcal{D}(\chi_a)(e, \cdot) = -F_0(c)$. $\square$

The next theorem characterizes double vector subbundles in terms of the operator $\mathcal{D}$

Theorem 4.5. *Let $(\mathcal{A}, E, A, M)$ be a DVB with core bundle C. Let $\Delta_M \subseteq E$ and $K \subseteq C$ be vector subbundles. Then there is a one to one correspondence between double vector subbundles $(\Delta, \Delta_M; A, M)$ with core bundle K and linear operators $\mathcal{D} : \Gamma_{lin}(\mathcal{A}, E) \longrightarrow \Gamma(\Delta_M^* \otimes C/K)$.*

Proof. The first part of the proposition is Lemma 4.3, taking as double vector bundles $(\mathcal{A}, E; A, M)$ and $(\mathcal{A}/\Delta, E/\Delta_M; A, M)$, and as maps the respective projections. For the second part, write an element $\alpha \in \mathcal{A}$ as $\alpha = \chi_a(e) +_E (0_e +_A \overline{c})$. Define

$$\Delta = \left\{ \begin{array}{l} \alpha \in \mathcal{A} \; : \; \alpha = \chi_a(e) +_E (0_e +_A \overline{c}) \text{ for } \chi_a \in \Gamma_{lin}(\mathcal{A}, E), \\ e \in \Delta_M, \; c \in C \text{ s.t. } \mathcal{D}(\chi_a)(e) = -\pi(c) \end{array} \right\}$$

where $\pi : C \longrightarrow C/K$ is the projection. Note that Δ is well defined. We will prove now that Δ is linear with respect to the two linear structures on $\mathcal{A}$. Let $\alpha_1, \alpha_2 \in \mathcal{A}_e$ with $e \in \Delta_M$, and write them as $\alpha_i = \chi_{a_i}(e) +_E (0_e +_A \overline{c}_i)$. Then

$$\begin{aligned} \alpha_1 +_E \alpha_2 &= \big(\chi_{a_1}(e) +_E (0_e +_A \overline{c}_1)\big) +_E \big(\chi_{a_2}(e) +_E (0_e +_A \overline{c}_2)\big) \\ &= (\chi_{a_1}(e) +_E \chi_{a_2}(e)) +_E (0_e +_A \overline{c_1 + c_2}) \end{aligned}$$

Since $\alpha_1, \alpha_2 \in \Delta$ we have that $\mathcal{D}(\alpha_i)(e) = -\pi(c_i)$. The section $\chi_{a_1} + \chi_{a_2}$ is a linear section covering $a_1 + a_2$, and by the linearity of $\mathcal{D}$ and of $\pi : C \longrightarrow C/K$ follow that

$\mathcal{D}(\chi_{a_1} + \chi_{a_2})(e, \cdot) = -\pi(c_1 + c_2)$. Hence $\alpha_1 + \alpha_2 \in \Delta$ over $e \in \Delta_M$. Suppose now that $\alpha_1, \alpha_2 \in \mathcal{A}_a$ and write them as $\alpha_1 = \chi_a(e_1)$ and $\alpha_2 = \chi_a(e_2) +_E (0_{e_2} +_A \overline{c})$. Then

$$\begin{aligned} \alpha_1 +_A \alpha_2 &= \Big(\chi_a(e_1)\Big) +_A \Big(\chi_a(e_2) +_E (0_{e_2} +_A \overline{c})\Big) \\ &= \Big(\chi_a(e_1) +_A \chi_a(e_2)\Big) +_E \overline{c} \\ &= \chi_a(e_1 + e_2) +_E \overline{c} \end{aligned}$$

where we have used the interchange law and the linearity of the section. Then $\alpha_1 +_A \alpha_2 \in \Delta$ over $a \in A$. Therefore Δ is linear with respect to both structures of $\mathcal{A}$. $\square$

As consequence we can characterize the linear sections of $\mathcal{A} \longrightarrow E$ which restrict to $\Delta \longrightarrow \Delta_M$. For $X \in \Gamma_{\text{lin}}(\mathcal{A}, E)$ we have

$$X|_{\Delta_M} \in \Gamma_{\text{lin}}(\Delta, \Delta_M) \Leftrightarrow \mathcal{D}(X)(e) = 0 \tag{4.3}$$

for all $e \in \Delta_M$.

Although we can say when a linear section of $\mathcal{A} \longrightarrow E$ restricts to $\Delta \longrightarrow \Delta_M$, the operator $\mathcal{D}$ is defined at the level of elements and not of sections. But we will need to use the sections of the vector bundle $\Delta \longrightarrow \Delta_M$, for example when we consider it as a Lie subalgebroid of $\mathcal{A} \longrightarrow E$.

Definition 4.6. [17] Given a double vector bundle $(\mathcal{A} \longrightarrow E; A \longrightarrow M)$ with a double vector subbundle $\Delta \subseteq \mathcal{A}$, a horizontal lift $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}, E)$ is called *adapted* to Δ if for every section $a \in \Gamma(A)$, the section $h(a)|_{\Delta_M} \in \Gamma_{\text{lin}}(\Delta, \Delta_M)$.

Proposition 4.7. *Let Δ be a double vector subbundle of $\mathcal{A}$ with side bundle Δ_M and core bundle K. Then there exists an operator*

$$\nabla : \Gamma_{lin}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes C)$$

such that

$$\mathcal{D}(X)(u) = \pi(\nabla_u(X)) \qquad u \in \Gamma(\Delta_M) \tag{4.4}$$

where $\pi : C \longrightarrow C/K$ is the projection map. Moreover if we have two operators $\nabla_1, \nabla_2 : \Gamma_{lin}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes C)$ satisfying Equation (4.4) then

$$(\nabla_1 - \nabla_2)X \in \Gamma(\Delta_M^* \otimes K).$$

Proof. Let $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}, E)$ be an adapted horizontal lift (for existence, see [17]). Let $X_a \in \Gamma_{lin}(\mathcal{A}, E)$. Then the section $h(a) -_E X_a$ is a core linear section covering $0 \in \Gamma(A)$. This means that there exists $T(X_a) \in \Gamma(Hom(E, C))$ such that

$$h(a) -_E X_a = -S_{T(X_a)}.$$

We define the operator $\nabla : \Gamma_{\text{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes C)$ characterized by

$$-S_{\nabla(X_a)} = h(a) -_E X_a.$$

Since h is adapted to the subbundle, we have that

$$h(a) = X_a -_E S_{\nabla(X_a)} \in \Gamma(\Delta, \Delta_M).$$

Then $\mathcal{D}(X_a)(u) = \pi(\nabla_u(X_a))$ for all $u \in \Gamma(\Delta_M)$. Now, suppose we have two adapted horizontal lifts h_1 and h_2 and let ∇_1 and ∇_2 be the correspondent associated operator. Then

$$h_i(a)(u) = X_a(u) -_E S_{(\nabla_i)_u(X_a)}(u) \quad \text{for } u \in \Gamma(\Delta_M).$$

Then

$$\Delta_u \ni \ (h_1(a) -_A h_2(a))(u) = -S_{(\nabla_1 - \nabla_2)_u(X_a)}(u)$$

which means that $(\nabla_1 - \nabla_2)_u(X_a) \in \Gamma(K)$. □

Remark 4.8. Let $\nabla : \Gamma_{\text{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(E^* \otimes C)$ be a connection associated to Δ, and let $f \in C^\infty(M)$. Then

$$\nabla_{fe}(X) = f\nabla_e(X).$$

Hence, we will write the connection as $\nabla : \Gamma(E) \times \Gamma_{\text{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(C)$ when we want to emphasize the $C^\infty(M)$-linearity with respect to $\Gamma(E)$.

4.1.1 Linear distributions

Let $A \longrightarrow M$ be a vector bundle and consider the prolonged tangent bundle

$$\begin{array}{ccc} TA & \longrightarrow & A \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

with core bundle $A \longrightarrow M$.

Definition 4.9. A *linear distribution* Δ of A is a double vector subbundle of the prolonged tangent bundle TA. In particular, there exist vector subbundles $\Delta_M \subseteq TM$ and $C \subseteq A$ such that

$$\begin{array}{ccc} \Delta & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Delta_M & \longrightarrow & M \end{array}$$

is a double vector bundle with core bundle $C \longrightarrow M$.

We will apply what we did for double vector subbundles of this particular case.

Proposition 4.10. *Let $A \longrightarrow M$ be a vector bundle and let $\Delta_M \subseteq TM$ and $C \subseteq A$ be vector subbundles. Then there is a one to one correspondence between*

$$\left\{ \begin{array}{l} \Delta \subseteq TA \text{ double vector subbundle with} \\ \text{side bundle } \Delta_M \text{ and core bundle } C \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{D} : \Gamma(A) \longrightarrow \Gamma((\Delta_M)^* \otimes A/C) \text{ s.t.} \\ \mathbb{D}_u(fa) = f\mathbb{D}_u(a) + (\mathcal{L}_u f)\pi(a) \end{array} \right\}$$

where $u \in \Gamma(\Delta_M)$.

Proof. Let $\mathcal{D} : \Gamma_{\text{lin}}(TA, TM) \longrightarrow \Gamma(\Delta_M^* \otimes A/C)$ be the operator associated to a linear distribution Δ. Recall that the natural inclusion $\Gamma(A) \ni a \hookrightarrow Ta \in \Gamma_{\text{lin}}(TA, TM)$, satisfies the property (see [32])

$$T(fa) = (f \circ q)Ta + \ell_{\mathrm{d}f} S_a \tag{4.5}$$

Then define the map

$$\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/C) \quad \mathbb{D}(a) = \mathcal{D}(Ta).$$

It follows that

$$\begin{aligned} \mathbb{D}(fa) &= \mathcal{D}(T(fa)) \\ &= \mathcal{D}((f \circ q)Ta + \ell_{\mathrm{d}f} S_a) \\ &= f\mathcal{D}(Ta) + \ell_{\mathrm{d}f} \pi_{TA}(S_a) \\ &= f\mathbb{D}(a) + \ell_{\mathrm{d}f} \pi(a). \end{aligned}$$

Conversely, giving an operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/C)$, let $\mathcal{D}(Ta) := \mathbb{D}(a)$. The Leibniz rule implies that we can extend the map $\mathcal{D}$ to all linear sections of TA over TM. □

Remark 4.11. Proposition 4.10 can be obtained from [17], combining Proposition 5.5 and Lemma 5.6 therein. However, here the proof is different.

In [17] there is an operator

$$\mathbb{D}^\Delta : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/C)$$

associated to a double vector subbundle Δ of TA, defined as follows:

$$\mathbb{D}^\Delta(a)(x, \cdot) = \pi(L_\chi(a))$$

where $\chi : A \longrightarrow \Delta$ is any linear vector field taking values on the distribution Δ, covering $x \in \Gamma(\Delta_M)$ and $L_\chi : \Gamma(A) \longrightarrow \Gamma(A)$ is the associated derivation to χ. We have another operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/C)$ associated to Δ which is obtained from the Proposition 4.10.

Proposition 4.12. *We have that $\mathbb{D} = \mathbb{D}^\Delta$, where $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/C)$ is the associated operator to Δ, obtained from the Proposition 4.10.*

Proof. For any choice of a linear section $\chi : A \longrightarrow \Delta$ we have

$$\pi_{TA}(\chi(a)) = 0$$

where $\pi_{TA} : TA \longrightarrow TA/\Delta$ is the natural projection. Combining Propositions 4.4 and 4.10 we have $\mathbb{D}_u(a) = -\pi(c_p)$, where $c_p \in A_p$ is the core element such that

$$\chi(a) = Ta(u) +_{TM} (u, c).$$

Now we will prove that $c = L_\chi(a)$. For this we shall see that $\chi(a_p)$ and $Ta(u) +_{TM} \xi$, where $\xi = (u, -L_\chi(a)_p)$, are equal by checking that they act equally on functions locally defined at a_p. As usual, it suffices to consider pull-back functions $f \circ q$ for $f \in C^\infty_{\mathrm{loc}}(M)$ and fiberwise linear functions ℓ_φ, for $\varphi \in \Gamma_{\mathrm{loc}}(A^*)$. For the first type, as

$$Tq(\chi(a_p)) = Tq(Ta(u) +_{TM} \xi) = u$$

it follows that they act equally on pull-back functions. For the second type, on one hand, we have by definition,

$$\langle \mathrm{d}\ell_\varphi(a_p), \chi(a_p)\rangle = \mathcal{L}_u\langle\varphi, a\rangle(p) - \langle\varphi(p), L_\chi(a)_p\rangle.$$

On the other hand, using that $\mathrm{d}\ell_\varphi : A \longrightarrow T^*A$ is a vector bundle morphism from $A \longrightarrow M$ to the tangent prolongation $T^*A \longrightarrow A^*$ covering $\varphi : M \longrightarrow A^*$, it follows that

$$\begin{aligned}
\langle \mathrm{d}\ell_\varphi(a_p), Ta(u) +_{TM} \xi\rangle &= \langle \mathrm{d}\ell_\varphi(a_p) +_{A^*} \mathrm{d}\ell_\varphi(0_p), Ta(u) +_{TM} \xi\rangle \\
&= \langle \mathrm{d}\ell_\varphi(a_p), Ta(u)\rangle + \langle \mathrm{d}\ell_\varphi(0_p), \xi\rangle \\
&= \mathcal{L}_u\langle\varphi, a\rangle(p) + \langle \mathrm{d}\ell_\varphi(0_p), \xi\rangle.
\end{aligned}$$

The result follows now from the identity

$$\langle \mathrm{d}\ell_\varphi(0_p), (u, c)\rangle = \langle\varphi(p), c_p\rangle,$$

as the identification $T_{0_p}A = T_pM \oplus A_p$ sees T_pM as the tangent space to the zero section and A_p as the tangent space to the fiber $q^{-1}(p)$. $\square$

Let $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(TA, TM)$ be an adapted horizontal lift to Δ, and let

$$\nabla^h : \Gamma_{lin}(TA, TM) \longrightarrow \Gamma(T^*M \otimes A)$$

be the associated connection operator to Δ satisfying $\pi \circ \nabla^h\Big|_{\Delta_M} = \mathcal{D}$. Since we have a natural inclusion $\Gamma(A) \hookrightarrow \Gamma_{lin}(TA, TM)$, we define a new operator by

$$\widetilde{\nabla} : \Gamma(TM) \times \Gamma(A) \longrightarrow \Gamma(A) \qquad \widetilde{\nabla}_u(a) := \nabla^h_u(Ta)$$

Note that since the inclusion $\Gamma(A) \hookrightarrow \Gamma_{lin}(TA, TM)$ is not $C^\infty(M)$-linear we have

$$\widetilde{\nabla}_u(fa) = f\widetilde{\nabla}_u a + (\mathcal{L}_u f)a,$$

which means that $\widetilde{\nabla}$ is actually a usual connection. By Proposition 4.12 we have

$$\pi \circ \widetilde{\nabla}\Big|_{\Delta_M} = \mathbb{D} = \mathbb{D}^{\Delta}.$$

Hence by Theorem 5.7 in [17], it follows that $\widetilde{\nabla}$ is *adapted* to the distribution, that is, that for every $u \in \Gamma(\Delta_M)$ the linear vector field $X_{\widetilde{\nabla}_u} : A \longrightarrow TA$ corresponding to the derivation $\widetilde{\nabla}_u : \Gamma(A) \longrightarrow \Gamma(A)$ is a section of the distribution Δ (see [17]).

Example 4.13. Involutive Distributions. Consider now a linear distribution $\Delta \subseteq TA$ which is also involutive. In particular this means that $\Delta \longrightarrow A$ is a VB-algebroid over $\Delta_M \longrightarrow M$, which implies that $\Delta_M \longrightarrow M$ has a Lie algebroid structure, which in this case means that Δ_M is an involutive distribution over M. Let $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma((\Delta_M)^* \otimes A/K)$ and let $a \in \Gamma(K)$. Since K is the core bundle of $(\Delta, A; \Delta_M, M)$ it follows that $a^{\uparrow} \in \Gamma_A(\Delta)$. By the involutive condition we have $[a^{\uparrow}, \Gamma_A(\Delta)] \subseteq \Gamma_A(\Delta)$. Using Lemma 4.2 in [27], we get that

$$Ta\Big|_{\Delta_M} \in \Gamma_{\mathrm{lin}}(\Delta, \Delta_M).$$

Then $h(a) = Ta - S_{\widetilde{\nabla}(a)} \in \Gamma_{\mathrm{lin}}(\Delta, \Delta_M)$, so $\widetilde{\nabla}_u a \in \Gamma(K)$ for all $u \in \Gamma(\Delta)$. Hence $\pi(\widetilde{\nabla}_u a) = 0$, and therefore $\mathbb{D}\big|_{\Gamma(K)} = 0$. This allows us to define a map $\mathbb{D} : \Gamma(\Delta_M) \times \Gamma(A/K) \longrightarrow \Gamma(A/K)$. Let $s : \Gamma(TM) \longrightarrow \mathfrak{X}_{\mathrm{lin}}(A)$ be the horizontal lift given by

$$s(u)(a_p) = h(a)(u_p) \in T_pA.$$

Consider the following linear vector fields on A: $s(u) + (\widetilde{\nabla}_u(a))^{\uparrow}$, $s(v) + (\widetilde{\nabla}_v(a))^{\uparrow}$ and $s([u,v]) + (\widetilde{\nabla}_{[u,v]}(a))^{\uparrow}$, for $u, v \in \Gamma(\Delta_M)$. Then

$$[s(u) + (\widetilde{\nabla}_u(a))^{\uparrow}, s(v) + (\widetilde{\nabla}_v(a))^{\uparrow}] = [s(u), s(v)] + \left(\mathbb{D}_{s(u)}(\widetilde{\nabla}_v a) - \mathbb{D}_{s(v)}(\widetilde{\nabla}_u a)\right)^{\uparrow} \tag{4.6}$$

The linear vector fields $s([u,v]) + (\widetilde{\nabla}_{[u,v]}(a))^{\uparrow}$ and (4.6) cover the same section $[u,v]$, so their difference is a linear core section $k^{\uparrow}$. But since the horizontal lift s is adapted and $[u,v] \in \Gamma(\Delta_M)$, then $k \in \Gamma(K)$. So

$$k + \widetilde{\nabla}_{[u,v]} a = \mathbb{D}_{s(u)}(\widetilde{\nabla}_v a) - \mathbb{D}_{s(v)}(\widetilde{\nabla}_u a).$$

Then applying π to both sides, we get that the induced map $\mathbb{D} : \Gamma(\Delta_M) \times \Gamma(A/K) \longrightarrow \Gamma(A/K)$ is a flat Δ_M-connection. Then we recover the next result.

Proposition 4.14. [17,27] *Let $(\Delta, A; \Delta_M, M)$ be a linear distribution on A with core bundle K. Then Δ is involutive if and only if Δ_M is involutive and the associated map $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(\Delta_M^* \otimes A/K)$ satisfies*

1. $\mathbb{D}\big|_{\Gamma(K)} = 0$
2. *The induced connection $\mathbb{D} : \Gamma(\Delta_M) \times \Gamma(A/K) \longrightarrow \Gamma(A/K)$ is a flat Δ_M-connection.*

4.1.2 Double vector subbundles of $TA \oplus T^*A$

Let $(\mathfrak{L}, U; A, M)$ be a double vector subbundle of $(TA \oplus T^*A, TM \oplus A^*; A, M)$ with core bundle $K \subseteq A \oplus T^*M$, and let

$$\mathcal{D} : \Gamma_{lin}(TA \oplus T^*A, TM \oplus A^*) \longrightarrow \Gamma(U^* \otimes \frac{A \oplus T^*M}{K})$$

be the associated canonical operator. We characterize this double vector subbundles.

Proposition 4.15. *Let $A \longrightarrow M$ be a vector bundle and let $U \subseteq TM \oplus A^*$ and $K \subseteq A \oplus T^*M$ be vector subbundles. Then there is a one to one correspondence between double vector subbundles $\mathfrak{L} \subseteq TA \oplus T^*A$ with side bundle U and core bundle K and operators*

$$\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(U^* \otimes \frac{A \oplus T^*M}{K})$$

satisfying the following Leibniz rule

$$\mathbb{D}_u(fa) = f\mathbb{D}_u(a) + (\ell_{df}a, \ell_{-a}df).$$

Proof. Define the operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(U^* \otimes \frac{A \oplus T^*M}{K})$ by

$$\mathbb{D}(a) := \mathcal{D}(Ta, R_a),$$

where $Ta : TM \longrightarrow TA$ is the linear section associated to $a : M \longrightarrow A$, and $R_a : A^* \longrightarrow T^*A$ is given by Equation (1.24). We only need to check the Leibniz rule

$$\begin{aligned}
\mathbb{D}(fa) &= \mathcal{D}(T(fa), R_{(fa)}) \\
&= \mathcal{D}(fTa + \ell_{\mathrm{d}f}S_a, fR_a + \ell_{-a}S_{\mathrm{d}f}) \\
&= \mathcal{D}(fTa, fR_a) + \mathcal{D}(\ell_{\mathrm{d}f}S_a, \ell_{-a}S_{\mathrm{d}f}) \\
&= f\mathcal{D}(Ta, R_a) + \pi_{TA \oplus T^*A}(\ell_{\mathrm{d}f}S_a, \ell_{-a}S_{\mathrm{d}f}) \\
&= f\mathbb{D}(a) + (\ell_{\mathrm{d}f}a, \ell_{-a}\mathrm{d}f).
\end{aligned}$$

□

Let $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(TA \oplus T^*A, TM \oplus A^*)$ be an adapted horizontal lift and let

$$\nabla^h : \Gamma_{\mathrm{lin}}(TA \oplus T^*A, TM \oplus A^*) \longrightarrow \Gamma((TM \oplus A^*)^* \otimes (A \oplus T^*M)$$

be the associated connection operator with the property $\pi \circ \nabla^h\Big|_U = \mathcal{D}$. Define now the operator

$$\widetilde{\nabla} : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M) \qquad \widetilde{\nabla}_u a := \nabla^h_u(Ta, R_a).$$

Note that $\pi \circ \widetilde{\nabla}\big|_U = \mathbb{D}$.

Proposition 4.16. *The map* $\nabla^h : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$ *is* $C^\infty(M)$-*linear in the first coordinate and satisfies the following Leibniz rule with respect to the second coordinate*

$$\nabla^h(fa) = f\nabla^h(a) + (\ell_{df}a, \ell_{-a}df)$$

Proof. Assume that the horizontal lift h is $C^\infty(M)$-linear. Then

$$\begin{array}{rcl} h(fa) = fh(a) & = & f(Ta, R_a) - fS_{\nabla^h(a)} \\ h(fa) & = & (T(fa), R_{fa}) - S_{\nabla^h(fa)} \end{array}$$

We know that

$$\begin{array}{rcl} T(fa) & = & fTa +_{TM} \ell_{\mathrm{d}f} S_a \\ R_{fa} & = & fR_a +_{A^*} \ell_{-a} S_{\mathrm{d}f}. \end{array}$$

Then $(T(fa), R_{fa}) = f(Ta, R_a) + (\ell_{\mathrm{d}f}S_a, \ell_{-a}S_{\mathrm{d}f})$, which implies that

$$S_{\nabla^h(fa)} = fS_{\nabla^h(a)} + (\ell_{\mathrm{d}f}S_a, \ell_{-a}S_{\mathrm{d}f}).$$

Hence

$$\nabla^h(fa) = f\nabla^h(a) + (\ell_{\mathrm{d}f}a, \ell_{-a}\mathrm{d}f).$$

With respect to the first coordinate, for $\tau \in \Gamma(TM \oplus A^*)$,

$$-S_{\nabla^h_{f\tau}a}(\tau) = h(a)(f\tau) - (Ta, R_a)(f\tau) = fh(a)(\tau) - f(Ta, R_a)(\tau) = fS_{\nabla^h_\tau a}.$$

Therefore ∇^h is $C^\infty(M)$-linear in the first argument. □

The following proposition connects this work with [26]. See Appendix for definition of Dorfman connections and for the proof of this proposition.

Proposition 4.17. *There is one-to-one correspondence between horizontal lifts* $h : \Gamma(A) \longrightarrow \Gamma(TA \oplus T^*A)$ *and* $(TM \oplus A^*)$*-Dorfman connection on* $A \oplus T^*M$

$$\Lambda : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M)$$

via the relation

$$\Lambda_{(u,\xi)}(a, \theta) = \nabla^h_{(u.\xi)}a + (0, d\langle \xi, a\rangle + \mathcal{L}_u\theta). \quad (4.7)$$

Moreover, if $(\mathfrak{L}, U; A, M)$ *be a double vector subbundle of* $(TA \oplus T^*A, TM \oplus A^*; A, M)$ *with core bundle* $K \subseteq A \oplus T^*M$*, then* h *is adapted to* $\mathfrak{L}$ *if and only if* Λ *is adapted to* $\mathfrak{L}$.

4.2 Double vector subalgebroids

In this section we endow on a double vector subbundle a Lie algebroid structure, and then we describe it in terms on an infinitesimal data. As before, we consider the particular cases of subbundles of TA and of $TA \oplus T^*A$.

Definition 4.18. [17]. Let $(\mathcal{A}, E; A, M)$ be a VB-algebroid. A double vector subbundle $(\Delta, \Delta_M; A, M)$ is a *VB-subalgebroid* of $\mathcal{A}$ if $\Delta \longrightarrow \Delta_M$ is a Lie subalgebroid of $\mathcal{A} \longrightarrow E$.

Let $(\mathcal{A}, E; A, M)$ be a VB-algebroid with core bundle C and let $(\Delta, \Delta_M; A, M)$ be a VB-subalgebroid of $\mathcal{A}$, with core bundle K. Since we are working with quotients, we will show that the (linear) quotient $\mathcal{A}/\Delta$ over A has a Lie algebroid structure over E/Δ_M such that $(\mathcal{A}/\Delta, E/\Delta_M; A, M)$ is a VB-algebroid with core bundle C/K. The compatibility of the double linear structure on

$$\begin{tikzcd} \mathcal{A}/\Delta \arrow[r] \arrow[d] & A \arrow[d] \\ E/\Delta_M \arrow[r] & M \end{tikzcd}$$

follows by the definition of double vector subbundle. Let $h : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}, E)$ be an adapted horizontal lift. Define the horizontal lift

$$\overline{h} : \Gamma(A) \longrightarrow \Gamma_{\text{lin}}(\mathcal{A}/\Delta, E/\Delta_M) \quad \text{by} \quad \overline{h}(a) := \overline{h(a)}.$$

Since h is adapted, $\overline{h}$ is well defined. With respect to core sections, for $\overline{c} \in \Gamma(C/K)$, the section $S_{\overline{c}}$ is defined by

$$S_{\overline{c}} : E/\Delta_M \longrightarrow \mathcal{A}/\Delta \qquad S_{\overline{c}}(\overline{e}) = \overline{S_c(e)}$$

Let us see that this is well defined. Let $c_1 \in \overline{c}$ and $e_1 \in \overline{e}$. That means there exist $k \in K$ and $\delta \in \Delta_M$ such that $c_1 = c + k$ and $e_1 = e + \delta$. Then

$$\begin{aligned} S_{c_1}(e_1) &= S_{c+k}(e+\delta) \\ &= 0_{e+\delta} + (\overline{c+\delta}) \\ &= (0_e + \overline{C}) + (0_\delta + \overline{k}) \\ &= S_c(e) + S_k(\delta). \end{aligned}$$

Since $S_k(\delta) \in \mathcal{A}/\Delta$ follows that $\overline{S_{c_1}(e_1)} = \overline{S_c(e)}$, which implies that the core section is well defined. With respect to the bracket, we define

- $[\overline{h}(a), \overline{h}(b)] = \overline{h[a,b]}$
- $[\overline{h}(a), S_{\overline{c}}] = S_{\overline{\nabla^0_{h(a)} c}}$

- $[S_{\overline{c_1}}, S_{\overline{c_2}}] = 0$.

The anchor map $\overline{\rho} : \mathcal{E}/\Delta \longrightarrow TE/T\Delta_M$ is determined by

- $\overline{\rho}(\overline{h}(a)) = \overline{\rho(h(a))}$,
- $\overline{\rho}(S_{\overline{c}}) = \overline{\partial(c)}^{\uparrow}$

which is well defined because Δ is a subalgebriod, h is adapted to Δ, and the map $\partial : E \longrightarrow C$ restrics to $\Delta_M \longrightarrow K$. Therefore $\mathcal{E}/\Delta \longrightarrow E/\Delta$ has a Lie algebroid structure, and hence it is a VB-algebroid over A. The projection map $F : \mathcal{A} \longrightarrow \mathcal{A}/\Delta$ is now a morphism of Lie algebroids. Indeed, the compatibility with the anchor follows by definition of $\overline{\rho}$. The core and linear sections of $\Gamma(\mathcal{A}/\Delta, E/\Delta_M)$ are induced by core and linear sections of $\Gamma(\mathcal{A}, E)$, so the compatibility of the bracket follows by

- $\overline{[h(a), h(b)]} = \overline{h([a, b]) + \Omega_{a,b}} = \overline{h([a, b])} + \overline{S_{\Omega_{a,b}}}$, and since h is adapted the section $\Omega_{a,b} \in \Gamma(\mathrm{Hom}(\Delta_M, K))$
- $\overline{[h(a), S_c]} = \overline{[S_{\nabla^0_{h(a)}c}]} = S_{\overline{\nabla^0_{h(a)}c}}$.

Hence the map $F : \mathcal{A} \longrightarrow \mathcal{A}/\Delta$ is a morphism of Lie algebroids.

Now dualizing $\mathcal{A}/\Delta$ over A we obtain the VB-algebroid

$$\begin{tikzcd} \Delta^\circ \arrow[r] \arrow[d] & A \arrow[d] \\ K^\circ \arrow[r] & M \end{tikzcd}$$

with core bundle Δ_M°, where we have identified $(\mathcal{A}/\Delta)^* \simeq \Delta^\circ$, the annihilator of Δ in $\mathcal{A}^*$; $(E/\Delta_M)^* \backsimeq \Delta_M^\circ$, the annihilator of Δ_M in E^*; and $(C/K)^* \backsimeq K^\circ$, the annihilator of K in C^*. Denote by $F : \mathcal{A} \longrightarrow \mathcal{A}/\Delta$ be the projection map and by

$$\overline{F} : \mathcal{A} \times_A \Delta^\circ \longrightarrow \mathbb{R}$$

the natural pairing between $\mathcal{A}$ and $\mathcal{A}^*$:

$$\overline{F}(\alpha, \eta) = \langle F(\alpha), \eta \rangle_A.$$

It is straightforward to check that if Δ is a Lie subalgebroid of $\mathcal{A}$ then $\overline{F}$ is a morphism of Lie algebroid, where we are considering $\mathbb{R}$ equipped with the trivial Lie algebroid structure.

Remember that we have canonically associated to a VB algebroid $\mathcal{A}$ the following operators:

- A vector bundle map $\partial : C \longrightarrow E$ (see Equation (1.23))
- A flat connection $\nabla^1 : \Gamma_{\mathrm{lin}}(\mathcal{A}, E) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$ (see Equation (1.21))

- A flat connection $\nabla^0 : \Gamma_{\text{lin}}(\mathcal{A}, E) \times \Gamma(C) \longrightarrow \Gamma(C)$ (see Equation (1.22)).

Remark 4.19. Throughout this part, we use the notation ∇^1 instead $(\nabla^1)^*$ for the action of linear sections on $\Gamma(E^*)$ to simplify the equations.

Since the map $\overline{F} : \mathcal{A} \oplus \Delta^\circ \longrightarrow \mathbb{R}$ is bilinear with respect to A, and also is a Lie algebroid morphism we have the following result as a consequence of Theorem 2.28.

Proposition 4.20. *Let $(\Delta, \Delta_M; A, M)$ be a Lie subalgebroid of $(\mathcal{A}, E; A, M)$ with core bundle K. Then $\partial(K) \subseteq \Delta_M$ and there exists an operator*

$$\mathbf{D} : \Gamma_{lin}(\mathcal{A}, E) \times_{\Gamma(A)} \Gamma_{lin}(\Delta^\circ, K^\circ) \longrightarrow (E^* \otimes C/K)$$

such that the following equations hold

$$\mathbf{D}([X,Y]) = X \cdot \mathbf{D}(Y) - Y \cdot \mathbf{D}(X) \tag{4.8}$$

$$\iota_{\partial(c)}\mathbf{D}(X) = \nabla^1_{X_2}\pi_C(c) - \pi_C(\nabla^0_{X_1}c) \tag{4.9}$$

$$\overline{\partial}(\mathbf{D}(X)(e)) = (\nabla^0_{X_2})^*\pi_E(e) - \pi_E((\nabla^1_X)^*e) \tag{4.10}$$

where the action is

$$\ell_{X\cdot\mathbf{D}(Y)} = \mathcal{L}_{\rho(X_1,X_2)}\overline{F}(Y_1, Y_2)$$

for $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \Gamma_{lin}(\mathcal{A}, E) \times_{\Gamma(A)} \Gamma_{lin}(\Delta^\circ, K^\circ)$, and the map $\overline{\partial} : C/K \longrightarrow E/\Delta_M$ is the quotient map.

From the previous section, there is a correspondence between double vector subbundles and certain operators. As our goal is to describe VB-subalgebroids, we will rewrite Proposition 4.20 in terms of the operator $\mathcal{D}$ associated to Δ.

Let $\nabla : \Gamma_{\text{lin}}(\mathcal{A}, E) \longrightarrow \Gamma(\text{Hom}(E, C))$ be the associated connection to the double vector subbundle Δ, after the choice of an horizontal lift $h : \Gamma(A) \longrightarrow \Gamma_{lin}(\mathcal{A}, E)$. Since Δ is a VB-algebroid, the linear sections are closed by the Lie bracket, so there exists a section $\Omega \in \Gamma(\wedge^2 A) \otimes \Gamma(\text{Hom}(E, C))$ such that

$$\Omega_{a,b} = h([a,b]) - [h(a), h(b)]$$

and since the horizontal lift is adapted follows that $\Omega \in \Gamma(\wedge^2 A) \otimes \Gamma(\text{Hom}(\Delta_M, K))$. Let $X_a, X_b \in \Gamma_{\text{lin}}(\mathcal{A}, E)$. By definition we have

$$\begin{aligned} S_{\nabla_{X_a}} &= h(a) - X_a \\ S_{\nabla_{X_b}} &= h(b) - X_b \\ S_{\nabla_{[X_a,X_b]}} &= h([a,b]) - [X_a, X_b]. \end{aligned}$$

On the other hand

$$\begin{aligned} [h(a), h(b)] &= [X_a + S_{\nabla_{X_a}}, X_b + S_{\nabla_{X_b}}] \\ &= [X_a, X_b] + [X_a, S_{\nabla_{X_b}}] + [S_{\nabla_{X_a}}, X_b] + [S_{\nabla_{X_a}}, S_{\nabla_{X_b}}] \\ &= [X_a, X_b] + S_{\left(\nabla^0_{X_a}\circ\nabla_{X_b} - \nabla_{X_b}\circ\nabla^1_{X_a}\right)} + S_{\left(\nabla_{X_a}\circ\nabla^1_{X_b} - \nabla^0_{X_b}\circ\nabla_{X_a}\right)} \\ &+ S_{\left(\nabla_{X_a}\circ\,\partial\,\circ\nabla_{X_b} - \nabla_{X_b}\circ\,\partial\,\circ\nabla_{X_a}\right)}. \end{aligned}$$

Then

$$\begin{aligned}
\Omega_{a,b} &= \nabla_{[X_a,X_b]} - \left(\nabla^0_{X_a}\circ\nabla_{X_b} - \nabla_{X_b}\circ\nabla^1_{X_a}\right) - \left(\nabla_{X_a}\circ\nabla^1_{X_b} - \nabla^0_{X_b}\circ\nabla_{X_a}\right)\\
&- \left(\nabla_{X_a}\circ\ \partial\ \circ\nabla_{X_b} - \nabla_{X_b}\circ\ \partial\ \circ\nabla_{X_a}\right)\\
&= \nabla_{[X_a,X_b]} - \nabla_{X_a}(\nabla^1_{X_b} + \partial\ \circ\nabla_{X_b}) + \nabla_{X_b}(\nabla^1_{X_a} + \partial\ \circ\nabla_{X_a})\\
&\quad -\nabla^0_{X_a}\circ\nabla_{X_b} + \nabla^0_{X_b}\circ\nabla_{X_a}
\end{aligned}$$

Therefore, taking $u\in\Gamma(\Delta_M)$

$$\begin{aligned}
\pi(\nabla_{[X_a,X_b]}u) &= \pi(\nabla_{X_a}(\nabla^1_{X_b} + \partial\ \circ\nabla_{X_b})u) - \pi(\nabla_{X_b}(\nabla^1_{X_a} + \partial\ \circ\nabla_{X_a})u)\\
&\quad +\pi(\nabla^0_{X_a}\circ\nabla_{X_b}u) - \pi(\nabla^0_{X_b}\circ\nabla_{X_a}u)
\end{aligned}$$

Then, the equation for $\mathcal{D}$ is

$$\begin{aligned}
\mathcal{D}_e([X,Y]) &= \pi(\nabla_X(\nabla^1_{X_b} + \partial\ \circ\nabla_Y)u) - \pi(\nabla_Y(\nabla^1_X + \partial\ \circ\nabla_X)u)\\
&\quad +\pi(\nabla^0_X\circ\nabla_Y u) - \pi(\nabla^0_Y\circ\nabla_X u).
\end{aligned}$$

We state now the main theorem of this chapter.

Theorem 4.21. *Let $(\mathcal{A},E;A,M)$ be a VB-algebroid with core bundle C, and let $(\Delta,\Delta_M;A,M)$ be a double vector subbundle with core bundle K and let*

$$\nabla:\Gamma_{lin}(\mathcal{A},E)\longrightarrow\Gamma(Hom(E,C))$$

be a connection associated to Δ. Then $\Delta\longrightarrow\Delta_M$ is a Lie subalgebroid of $\mathcal{A}\longrightarrow E$ if and only if the following equations hold

$$\begin{aligned}
&(4.11) & \partial(K) &\subseteq \Delta_M\\
&(4.12) & \iota_{\partial(c)}\mathcal{D}(X) &= -\pi(\nabla^0_X c) \qquad c\in K\\
&(4.13) & \overline{\partial}(\mathcal{D}(X)(u)) &= -\pi_E(\nabla^1_X u)\\
&(4.14) & \mathcal{D}_u([X,Y]) &= \pi(\nabla_{(\nabla^1_Y+\partial\ \circ\nabla Y)(u)}X) - \pi(\nabla_{(\nabla^1_X+\partial\ \circ\nabla X)(u)}Y)\\
&&&\quad +\pi(\nabla^0_X\circ\nabla_u Y) - \pi(\nabla^0_Y\circ\nabla_u X)
\end{aligned}$$

where $\overline{\partial}:C/K\longrightarrow E/\Delta_M$ is the quotient map.

Proof. We only need to check the converse. Taking $\mathcal{D}=\pi\circ\nabla\big|_{\Delta_M}$, it defines a double vector subbundle $\Delta\subseteq\mathcal{A}$. We will prove $\Delta\longrightarrow\Delta_M$ is a Lie subalgebroid of $\mathcal{A}\longrightarrow E$. The anchor $\rho_\Delta:\Delta\longrightarrow T\Delta_M$ has to be the restriction of the anchor $\rho_{\mathcal{A}}$ to Δ. Let $k\in\Gamma(K)$. Then $\rho_{\mathcal{A}}(S_k)=\partial(k)^\uparrow\in\mathfrak{X}(E)$. So

$$\partial(k)^\uparrow\in\mathfrak{X}(\Delta_M)\Longleftrightarrow\partial(k)\in\Gamma(\Delta_M)\Longleftrightarrow\partial(K)\subseteq\Delta_M.$$

Consider now a linear section given by $X+S_{\nabla(X)}\in\Gamma(\Delta,\Delta_M)$. Then

$$\begin{aligned}
\rho_{\mathcal{A}}(X+S_{\nabla(X)})\in\mathfrak{X}(\Delta_M) &\Longleftrightarrow \pi_E(\rho_{\mathcal{A}}(X)(u)) + \pi_E(S_{\nabla(X)})(u)) = 0\\
&\Longleftrightarrow \pi_E(\nabla^1_X u) + \pi_E(\partial(\nabla_u(X))) = 0\\
&\Longleftrightarrow -\pi_E(\nabla^1_X u) = \overline{\partial}(\pi(\nabla_u(X)))\\
&\Longleftrightarrow -\pi_E(\nabla^1_X u) = \overline{\partial}(\mathcal{D}_u(X))
\end{aligned}$$

Now we check the Lie bracket conditions. Since $K \subseteq C$ follows that $[S_{k_1}, S_{k_2}] = 0$ for all $k_1, k_2 \in \Gamma(K)$. For linear and core sections we have

$$[X + S_{\nabla(X)}, S_k] = [X, S_k] + [S_{\nabla(X)}, S_k] = S_{\nabla^0_X k} + S_{\nabla(X)\circ\partial\circ k}$$

Then $[X + S_{\nabla(X)}, S_k] \in \Gamma_{\text{cor}}(\Delta, \Delta_M)$ if and only if

$$\begin{aligned} \nabla^0_X k + \nabla(X) \circ \partial \circ k \in \Gamma(K) &\iff -\pi(\nabla^0_X k) = \pi(\nabla(X) \circ \partial \circ k) \\ &\iff -\pi(\nabla^0_X k) = \iota_{\partial(k)}\mathcal{D}(X). \end{aligned}$$

For two linear sections

$$\begin{aligned} [X + S_{\nabla_X}, Y + S_{\nabla_Y}] &= [X, Y] + [X, S_{\nabla_Y}] + [S_{\nabla_X}, Y] + [S_{\nabla_X}, S_{\nabla_Y}] \\ &= [X, Y] + S_{\left(\nabla^0_X\circ\nabla(Y)-\nabla(Y)\circ\nabla^1_X\right)} + S_{\left(\nabla(X)\circ\nabla^1_Y-\nabla^0_Y\circ\nabla(X)\right)} \\ &+ S_{\left(\nabla(X)\circ\ \partial\ \circ\nabla(Y)-\nabla(Y)\circ\ \partial\ \circ\nabla(X)\right)}. \end{aligned}$$

Then $[X + S_{\nabla_X}, Y + S_{\nabla_Y}] \in \Gamma_{\text{lin}}(\Delta, \Delta_M)$ if and only if

$$\nabla([X, Y]) - \nabla_{(\nabla^1_Y+\partial\ \circ\nabla(Y))}(X) + \nabla_{(\nabla^1_X+\partial\ \circ\nabla(X))}(Y) - \nabla^0_X \circ \nabla(Y) + \nabla^0_Y \circ \nabla(X)$$

is in $\Gamma(\text{Hom}(\Delta_M, K))$, if and only if, equation (4.14) holds. □

Example 4.22. Linear subalgebroids of TA. Consider the tangent VB-algebroid $(TA, TM; A, M)$. Recall that we have a natural inclusion $\Gamma(A)$ in $\Gamma_{\text{lin}}(TA, TM)$: $a \longrightarrow Ta$. Hence we can rewrite the canonical connections in terms of sections of $\Gamma(A)$:

- $\nabla^0 : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ is given by

$$\nabla^0_a b = [a, b]$$

- $\nabla^1 : \Gamma(A) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ is given by

$$\nabla^1_a u = [\rho(a), u].$$

Note now that these connections are not flat. And in this context the core anchor map is $\partial = \rho : A \longrightarrow TM$. Recall the operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma((\Delta_M)^* \otimes A/C)$ and the connection $\widetilde{\nabla} : \Gamma(TM) \times \Gamma(A) \longrightarrow \Gamma(A)$ defined in Subsection 4.1.1. Recall the basic connection associated to a TM-connection on A (see Example 2.5 in [17]):

$$\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A) \qquad \nabla^{\text{bas}}_a b = [a, b] + \widetilde{\nabla}_{\rho(b)} a$$

Then the equation for $\mathbb{D}$ reads

$$\begin{aligned}
\mathbb{D}_u([a,b]) &= \pi(\widetilde{\nabla}_{[\rho_A(b),u]+\rho_A(\widetilde{\nabla}_u b)}a) - \pi(\widetilde{\nabla}_{[\rho_A(a),u]+\rho_A(\widetilde{\nabla}_u a)}b) \\
&\quad +\pi([a,\widetilde{\nabla}_u b]) - \pi([b,\widetilde{\nabla}_u a]) \\
&= \pi(\widetilde{\nabla}_{\rho_A(\widetilde{\nabla}_u b)}a + [a,\widetilde{\nabla}_u b]) - \pi(\widetilde{\nabla}_{\rho_A(\widetilde{\nabla}_u a)}b + [b,\widetilde{\nabla}_u a]) \\
&\quad +\pi(\widetilde{\nabla}_{[\rho_A(b),u]}a - \widetilde{\nabla}_{[\rho_A(b),u]}b) \\
&= \pi(\nabla^{\mathrm{bas}}_a(\widetilde{\nabla}_u b)) - \pi(\nabla^{\mathrm{bas}}_b(\widetilde{\nabla}_u a)) + \pi(\widetilde{\nabla}_{[\rho_A(b),u]}a - \widetilde{\nabla}_{[\rho_A(b),u]}b) \\
&= \hat{\nabla}^{\mathrm{bas}}_a\mathbb{D}_u(b) - \hat{\nabla}^{\mathrm{bas}}_b\mathbb{D}_u(a) + \pi(\widetilde{\nabla}_{[\rho_A(b),u]}a - \widetilde{\nabla}_{[\rho_A(b),u]}b)
\end{aligned}$$

where $\hat{\nabla}^{\mathrm{bas}}$ is the A-connection on the quotient A/C given by

$$\hat{\nabla}^{\mathrm{bas}}_a\pi(b) = \pi([a,b] + \widetilde{\nabla}_{\rho_A(b)}a) = \pi(\nabla^{\mathrm{bas}}_a b).$$

Therefore, the equations of Theorem 4.21 in this case are

- $\rho_A(K) \subseteq \Delta_M$
- $\iota_{\rho_A(b)}\mathbb{D}(a) = -\pi([a,b])$
- $\overline{\rho}(\mathbb{D}_u(a)) = -\pi_{TM}([\rho_A(a),u])$
- $\mathbb{D}_u([a,b]) = \hat{\nabla}^{bas}_a\mathbb{D}_u(b) - \hat{\nabla}^{bas}_b\mathbb{D}_x(a) + \pi(\nabla_{[\rho(b),u]}a - \nabla_{[\rho(a),u]}b)$

which recover the Theorem 5.17 in [17]. And in the case when $\Delta_M = TM$, we have that $\pi\circ\widetilde{\nabla} = \mathcal{D} = \mathbb{D}$ for all $u \in \Gamma(TM)$, and observing that we can rewrite the basic connection by $\hat{\nabla}^{\mathrm{bas}}_a b = \pi([a,b]) + \mathbb{D}_{\rho_A(b)}(a)$, then the equation for $\mathbb{D}$ is

$$\mathbb{D}_u([a,b]) = \hat{\nabla}^{\mathrm{bas}}_a\mathbb{D}_u(b) - \hat{\nabla}^{\mathrm{bas}}_b\mathbb{D}_u(a) + \mathbb{D}_{[\rho(b),u]}(a) - \mathbb{D}_{[\rho(b),u]}(b).$$

Then, together with the equation

$$\iota_{\rho_A(b)}\mathbb{D}(a) = -\pi([a,b])$$

we get a *Spencer operator relative to* π, which is the infinitesimal description of wide multiplicative distribution given in [14].

Example 4.23. Linear subalgebroids of $TA \oplus T^*A$. Let $\mathfrak{L} \longrightarrow U$ be a double vector subbundle of

$$\begin{array}{ccc}
TA\oplus T^*A & \longrightarrow & A \\
\downarrow & & \downarrow \\
TM\oplus A^* & \longrightarrow & M
\end{array}$$

with core bundle $K \subseteq A \oplus T^*M$. Let $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(U^*\otimes(A\oplus T^*M)/K)$ and $\widetilde{\nabla} : \Gamma(TM\oplus A^*)\times\Gamma(A) \longrightarrow \Gamma(A\oplus T^*M)$ such that $\pi\circ\widetilde{\nabla}\big|_U = \mathbb{D}$ (see Subsection 4.1.2). The structure maps of the VB-algebroid $(TA\oplus T^*A, TM\oplus A^*; A, M)$ are:

- $\partial = (\rho_A, \rho_A^*) : A \oplus T^*M \longrightarrow TM \oplus A^*$.
- $\nabla^0 : \Gamma(A) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M)$ is given by

$$\nabla_a(b,\theta) = ([a,b], \mathcal{L}_{\rho_A(a)}\theta)$$

- $\nabla^1 : \Gamma(A) \times \Gamma(TM \oplus A^*) \longrightarrow \Gamma(TM \oplus A^*)$ is given by

$$\nabla_a(X,\alpha) = ([\rho(a), X], \mathcal{L}_a\alpha)$$

where we used the inclusion $\Gamma(A) \ni a \longrightarrow (Ta, R_a) \in \Gamma_{\text{lin}}(TA \oplus T^*A, TM \oplus A^*)$. Then Theorem 4.21 applied to this case is:

Theorem 4.24. *Let $(\mathfrak{L}, U; A, M)$ be a double vector subbundle of $(TA \oplus T^*A, TM \oplus A^*; A, M)$ with core bundle $K \subseteq A \oplus T^*M$, and let $\widetilde{\nabla} : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$ be the associated connection operator (after a choice of an adapted horizontal lift). Then $\mathfrak{L} \longrightarrow U$ is a subalgebroid of $TA \oplus T^*A \longrightarrow TM \oplus A^*$ if and only if the following equations hold:*

$$\begin{array}{llcl}
(4.15) & (\rho_A, \rho_A^*)(K) & \subseteq & U \\
(4.16) & \iota_{(\rho_A(b), \rho_A^*(\theta))}\mathbb{D}(a) & = & -\pi(\nabla^0_a(b,\theta)) \qquad \textit{for } (b,\theta) \in K \\
(4.17) & \overline{(\rho_A, \rho_A^*)}(\mathbb{D}_{(X,\alpha)}(a)) & = & -\pi_U(\nabla^1_a(X,\alpha)) \\
(4.18) & \mathbb{D}_{(X,\alpha)}([a,b]) & = & \pi(\nabla_{(\nabla^1_b(X,\alpha) + (\rho_A,\rho_A^*)(\nabla_{(X,\alpha)}b))}a) \\
& & & -\pi(\nabla_{(\nabla^1_a(X,\alpha) + (\rho_A,\rho_A^*)(\nabla_{(X,\alpha)}a))}b) \\
& & & +\pi(\nabla^0_a \circ \nabla_{(X,\alpha)}b) - \pi(\nabla^0_b \circ \nabla_{(X,\alpha)}a).
\end{array}$$

Now we connect this result with Theorem 5.9 in [26]. Given an operator $\widetilde{\nabla} : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$ define two **basic** connections (see Proposition 5.1 in [26])

- $\nabla_1^{\text{bas}} : \Gamma(A) \times \Gamma(TM \oplus A^*) \longrightarrow \Gamma(TM \oplus A^*)$, given by

$$(\nabla_1^{\text{bas}})_a(X,\alpha) = (\rho, \rho^*)(\widetilde{\nabla}_{(X,\alpha)}a) + \nabla^1_a(X,\alpha)$$

- $\nabla_0^{\text{bas}} : \Gamma(A) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M)$, given by

$$(\nabla_0^{\text{bas}})_a(b,\theta) = \widetilde{\nabla}_{(\rho,\rho^*)(b,\theta)}a + \nabla^0_a(b,\theta).$$

For $(b,\theta) \in \Gamma(K)$ we have

$$\begin{array}{lclcl}
(\nabla_0^{\text{bas}})_a(b,\theta) & \in \Gamma(K) & \Leftrightarrow & \widetilde{\nabla}_{(\rho,\rho^*)(b,\theta)}a + \nabla^0_a(b,\theta) & \in \Gamma(K) \\
& & \Leftrightarrow & \pi(\widetilde{\nabla}_{(\rho,\rho^*)(b,\theta)}a) = -\pi(\nabla^0_a(b,\theta)) & \\
& & \Leftrightarrow & \iota_{(\rho,\rho^*)(b,\theta)}\mathbb{D}(a) = -\pi(\nabla^0_a(b,\theta)) &
\end{array}$$

For $(X,\alpha)\in\Gamma(U)$ we have

$$\begin{aligned}(\nabla_1^{\text{bas}})_a(X,\alpha) \quad \in\ \Gamma(U) \quad &\Leftrightarrow\quad (\rho,\rho^*)(\widetilde{\nabla}_{(X,\alpha)}a)+\nabla^1_a(X,\alpha) \quad \in\ \Gamma(U)\\ &\Leftrightarrow\quad \pi_U((\rho,\rho^*)(\widetilde{\nabla}_{(X,\alpha)}a)) = -\pi_U(\nabla^1_a(X,\alpha))\\ &\Leftrightarrow\quad \overline{(\rho_A,\rho_A^*)}(\mathbb{D}_{(X,\alpha)}(a)) = -\pi_U(\nabla^1_a(X,\alpha))\end{aligned}$$

where we used $\pi_U\circ(\rho,\rho^*) = \overline{(\rho,\rho^*)}\circ\pi$.

The *basic curvature* (see Proposition 5.4 in [26]) is an operator

$$R^{\text{bas}}\in\Omega^2(A,\operatorname{Hom}(TM\oplus A^*, A\oplus T^*M))$$

given by

$$R^{\text{bas}}(a,b)(X,\alpha) = -\widetilde{\nabla}_{(X,\alpha)}[a,b]+\nabla^0_a(\widetilde{\nabla}_{(X,\alpha)}b)-\nabla^0_b(\widetilde{\nabla}_{(X,\alpha)}a)+\widetilde{\nabla}_{\nabla_b^{\text{bas}}(X,\alpha)}a-\widetilde{\nabla}_{\nabla_a^{\text{bas}}(X,\alpha)}b.$$

Then for $(X,\alpha)\in\Gamma(U)$ we have that $R^{\text{bas}}(a,b)(X,\alpha)\in\Gamma(K)$ if and only if

$$\begin{aligned}\pi(\widetilde{\nabla}_{(X,\alpha)}[a,b]) \;&=\; +\pi(\nabla^0_a(\widetilde{\nabla}_{(X,\alpha)}b))-\pi(\nabla^0_b(\widetilde{\nabla}_{(X,\alpha)}a))+\pi(\widetilde{\nabla}_{(\nabla_1^{\text{bas}})_b(X,\alpha)}a)\\ &\quad -\pi(\widetilde{\nabla}_{(\nabla_1^{\text{bas}})_a(X,\alpha)}b)\\ &=\; +\pi(\nabla^0_a(\widetilde{\nabla}_{(X,\alpha)}b))-\pi(\nabla^0_b(\widetilde{\nabla}_{(X,\alpha)}a))-\pi(\widetilde{\nabla}_{(\rho,\rho^*)(\widetilde{\nabla}_{(X,\alpha)}a)+\nabla^1_a(X,\alpha)}b)\\ &\quad +\pi(\widetilde{\nabla}_{(\rho,\rho^*)(\widetilde{\nabla}_{(X,\alpha)}b)+\nabla^1_b(X,\alpha)}a)\end{aligned}$$

Therefore our Theorem 4.24 is equivalent to Theorem 5.9 in [26].

4.2.1 Infinitesimal-global correspondence

Let $\mathcal{E}\rightrightarrows E$ be a VB-groupoid over $\mathcal{G}\rightrightarrows M$ with core bundle C, and let $\mathcal{H}\rightrightarrows H$ be a VB-subgroupoid of $\mathcal{E}$ with core bundle K (see Example 1.24).

Lemma 4.25. *The Lie algebroid $A_{\mathcal{H}}\longrightarrow H$ of $\mathcal{H}$ is a VB-subalgebroid of $A_{\mathcal{E}}$.*

Proof. Since $\mathcal{H}$ is a Lie subgroupoid of $\mathcal{E}$, it follows that $A_{\mathcal{H}}$ is a Lie subalgebroid of $A_{\mathcal{E}}$. Moreover, since $\mathcal{H}\rightrightarrows H$ is a VB-groupoid over $\mathcal{G}$, then by Corollary 4.1.2 in [8] $A_{\mathcal{H}}\longrightarrow H$ is a VB-algebroid over A. Hence $A_{\mathcal{H}}$ is a VB-subalgebroid of $A_{\mathcal{E}}$. □

To *integrate* IM-subbundles we need first two results from [8]:

Theorem 4.3.4. *Let $\mathcal{A}\longrightarrow E$ be a VB-algebroid over $A\longrightarrow M$, so that $\mathcal{A}\longrightarrow E$ is integrable. Then its source-simply-connected integration $\mathcal{E}\rightrightarrows E$ carries a VB-groupoid over the source-simply-connected Lie groupoid $\mathcal{G}\rightrightarrows M$ integrating $A\longrightarrow M$,*

$$\begin{array}{ccc}\mathcal{E} & \longrightarrow & \mathcal{G}\\ \downarrow\downarrow & & \downarrow\downarrow\\ E & \longrightarrow & M\end{array}$$

uniquely determined by the property that its differentiation is the given VB-algebroid.

Corollary 4.3.7. *Let $\mathcal{A}_1$ be a VB-subalgebroid of $\mathcal{A}_2$ defining, at level of basic algebroids, a Lie subalgebroid $(A_1 \longrightarrow M_1) \hookrightarrow (A_2 \longrightarrow M_2)$. For $i = 1, 2$, let $\mathcal{E}_i$ and G_i be source-simply-connected integrations of $\mathcal{A}_i$ and A_i, respectively. Then $\mathcal{E}_1$ is a VB-subgroupoid of $\mathcal{E}_2$ provided $\mathcal{G}_1$ is a Lie subgroupoid of $\mathcal{G}_2$.*

Now applying Theorem 4.21 combined with the previous two results we get the following infinitesimal-global correspondence.

Theorem 4.26. *Let $\mathcal{E} \rightrightarrows E$ be a VB-groupoid over a source 1-connected Lie groupoid $\mathcal{G} \rightrightarrows M$. Then there is one-to-one correspondence between VB-subgroupoids $\mathcal{H} \rightrightarrows H$ of $\mathcal{E}$ and operators $\mathcal{D} : \Gamma_{lin}(A_{\mathcal{E}}, E) \longrightarrow \Gamma(H^* \otimes C/K)$ satisfying Equations (4.11)-(4.14).*

4.3 IM-Dirac structures

In this section, we apply what we did before to the particular case of *IM-Dirac structures* on a Lie algebroid A, that means, we describe double vector bundles

$$\begin{array}{ccc} \mathfrak{L} & \longrightarrow & A \\ \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

whit core bundle K such that $\mathfrak{L} \longrightarrow U$ is a Lie subalgebroid of $TA \oplus T^*A \longrightarrow TM \oplus A^*$, $\mathfrak{L} \longrightarrow A$ is a Dirac structure. First we study only the case when $\mathfrak{L} \longrightarrow A$ is a Dirac structure, providing a description in terms of U and K. Then we combine with the previous section to include the Lie subalgebroid condition. As examples, we consider the cases of closed linear 2-forms, Poisson structures and involutive distributions.

4.3.1 Dirac structures

Let $(q : A \longrightarrow M, [\cdot,\cdot], \rho)$ be a Lie algebroid, and consider the VB-algebroid

$$\begin{array}{ccc} TA \oplus T^*A & \longrightarrow & A \\ {\scriptstyle \mathbb{T}q}\downarrow & & \downarrow \\ TM \oplus A^* & \longrightarrow & M \end{array}$$

with core bundle $A \oplus T^*M$, and where the map $\mathbb{T}q = (Tq, P_2)$, where $P_2 : T^*A \longrightarrow A^*$ is the cotangent prolongation.

The Courant-Dorfman bracket of two sections $(X,\alpha),\ (Y,\beta) \in \Gamma_A(TA \oplus T^*A)$ (see [6]) is

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha)$$

If $X, Y \in \mathfrak{X}(A)$ are linear vector fields, and $\alpha, \beta \in \Omega^1(A)$ are linear forms, then (X,α) and (Y,β) are linear sections, as well as their Courant-Dorfman bracket. Let $(u,\eta),(v,\xi) \in \Gamma(TM \oplus A^*)$ and let $(X,\alpha),\ (Y,\beta) \in \Gamma_A(TA \oplus T^*A)$ be any linear sections projectable to $(u,\eta),(v,\xi)$, respectively. This means

$$\begin{array}{cccc}
\begin{tikzcd} X \arrow[r] \arrow[d] & a \arrow[d] \\ u \arrow[r] & p \end{tikzcd}, &
\begin{tikzcd} \alpha \arrow[r] \arrow[d] & a \arrow[d] \\ \eta \arrow[r] & p \end{tikzcd}, &
\begin{tikzcd} Y \arrow[r] \arrow[d] & a \arrow[d] \\ v \arrow[r] & p \end{tikzcd}, &
\begin{tikzcd} \beta \arrow[r] \arrow[d] & a \arrow[d] \\ \xi \arrow[r] & p \end{tikzcd}
\end{array}$$

Define a bracket in $\Gamma(TM \oplus A^*)$ by:

$$[(u,\eta),(v,\xi)] = \mathbb{T}q([\![(X,\alpha),(Y,\beta)]\!]).$$

This bracket is well defined. Indeed if $(\widetilde{Y},\widetilde{\beta})$ is another linear section covering (v,ξ), there exists a core linear section $\nu^\uparrow$ such that $(\widetilde{Y},\widetilde{\beta}) = (Y,\beta) + \nu^\uparrow$. Then

$$[\![(X,\alpha),(\widetilde{Y},\widetilde{\beta})]\!] = [\![(X,\alpha),(Y,\beta)]\!] + [\![(X,\alpha),\nu^\uparrow]\!],$$

and the bracket $[\![(X,\alpha),\nu^\uparrow]\!]$ is a core linear section in $TA \oplus T^*A$ over A, then it projects over $0 \in TM \oplus A^*$, and then

$$\mathbb{T}q([\![(X,\alpha),(\widetilde{Y},\widetilde{\beta})]\!]) = \mathbb{T}q([\![(X,\alpha),(Y,\beta)]\!]) + \underbrace{\mathbb{T}q([\![(X,\alpha),\nu^\uparrow]\!])}_{=0} = \mathbb{T}q([\![(X,\alpha),(Y,\beta)]\!]),$$

and hence, the bracket in $\Gamma(TM \oplus A^*)$ is well defined.

Since $u \sim_q X$ and $v \sim_q Y$ then $[u,v] \sim_q [X,Y]$. So the first component of the bracket $[(u,\eta),(v,\xi)]$ is $[u,v] \in \Gamma(TM)$. In order to describe the second component $P_2(\mathcal{L}_X\beta - \iota_Y d\alpha) \in A^*$ we need a little more work.

Let $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(TA \oplus T^*A, TM \oplus A^*)$ be a horizontal lift and let ∇^h : $\Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$ the connection operator such that

$$h(a)(u,\eta) = (Ta(u), R_a(\eta)) - S_{\nabla^h_{(u,\eta)}a}(u,\eta).$$

We define $\Delta_{(u,\eta)} : \Gamma(A) \longrightarrow \Gamma(A)$ by

$$\Delta_{(u,\eta)}a = \mathrm{P}_A(\nabla^h_{(u,\eta)}a),$$

where $P_A : A \oplus T^*M \longrightarrow A$ is the projection over A. The map $\Delta_{(u,\eta)}$ is a derivation over u:

$$\begin{aligned}
\Delta_{(u,\eta)}(fa) &= \mathrm{P}_A(\nabla^h_{(u,\eta)}(fa)) \\
&= \mathrm{P}_A(f\nabla^h_{(u,\eta)}a + (\ell_{\mathrm{d}f}S_b, \ell_{-a}\mathrm{d}f)) \\
&= f\mathrm{Pr}_A(\nabla^h_{(u,\eta)}a) + \mathcal{L}_u fa \\
&= f\Delta_{(u,\eta)}a + \mathcal{L}_u fa
\end{aligned}$$

Consider $\sigma_h : \Gamma(TM\oplus A^*) \longrightarrow \Gamma_{\rm lin}(TA\oplus T^*A, A)$ the horizontal lift of the horizontal linear structure of $(TA\oplus T^*A, TM\oplus A^*; M, A)$ associated to h:

$$\sigma_h(u,\eta)(a_p) = h(a)(u_p,\eta_p) = (Ta(u), R_a(\eta)) - S_{\nabla^h_{(u,\eta)}a}(u,\eta)$$

and write $\sigma_h(u,\eta) = (X,\alpha)$ and $\sigma_h(v,\xi) = (Y,\beta)$. Then

$$\begin{aligned} X(a_p) = \mathrm{Pr}_{TA}(\sigma_h(u,\eta)(a_p)) &= \mathrm{Pr}_{TA}((Ta(u), R_a(\eta)) - S_{\nabla^h_{(u,\eta)}a}(u,\eta)) \\ &= Ta(u_p) + \mathrm{Pr}_{TA}(S_{-\nabla^h_{(u,\eta)}a}(u,\eta)) \\ &= Ta(u_p) + (0 + \mathrm{Pr}_A(-\nabla^h_{(u,\eta)}a)) \\ &= Ta(u_p) + \frac{d}{dr}\Big|_{r=0}(a_p - r\mathrm{Pr}_A(\nabla^h_{(u,\eta)}a)) \\ &= Ta(u_p) + \frac{d}{dr}\Big|_{r=0}(a_p - r\Delta_{(u,\eta)}a) \\ &=: \widehat{\Delta_{(u,\eta)}a}(a_p). \end{aligned}$$

Now we calculate $P_2(\mathcal{L}_X\beta - \iota_Y d\alpha)$. For $b\in\Gamma(A)$ we have

$$\langle P_2(\mathcal{L}_X\beta - \iota_Y d\alpha), b\rangle = \langle \mathcal{L}_X\beta - \iota_Y d\alpha, b^\uparrow\rangle = \langle \mathcal{L}_X\beta, b^\uparrow\rangle - \langle \iota_Y \mathrm{d}\alpha, b^\uparrow\rangle.$$

The first term is equal to

$$\langle \mathcal{L}_X\beta - \iota_Y d\alpha, b^\uparrow\rangle = \mathcal{L}_X\langle\beta, b^\uparrow\rangle - \langle\beta, [X, b^\uparrow]\rangle.$$

The function $\langle\beta, b^\uparrow\rangle$ is basic:

$$\langle\beta, b^\uparrow\rangle(a_p) = \langle\beta(a_p), b^\uparrow(a_p)\rangle = \langle \mathrm{P}_2(\beta(a_p)), b(p)\rangle = \langle\xi(p), b(p)\rangle.$$

Hence $\mathcal{L}_X\langle\beta, b^\uparrow\rangle = q^*\mathcal{L}_u\langle\xi, b\rangle$ because X is a linear vector field. Following Lemma B.2 in [26], we have $\langle\beta, [X, b^\uparrow]\rangle = \langle\xi, (\Delta_{(u,\eta)}b)^\uparrow\rangle$. Therefore

$$\langle\mathcal{L}_X\beta, b^\uparrow\rangle = q^*(\mathcal{L}_u\langle\xi, b\rangle) - \langle\xi, (\Delta_{(u,\eta)}b)^\uparrow\rangle.$$

The term $\langle\iota_Y\mathrm{d}\alpha, b^\uparrow\rangle$ is equal to

$$\langle\iota_Y\mathrm{d}\alpha, b^\uparrow\rangle = \mathcal{L}_Y\langle\alpha, b^\uparrow\rangle. - \mathcal{L}_{b^\uparrow}\langle\alpha, Y\rangle - \langle\alpha, [Y, b^\uparrow]\rangle.$$

As before, we have $\mathcal{L}_Y\langle\alpha, b^\uparrow\rangle = q^*(\mathcal{L}_v\langle\eta, b\rangle)$. Write $\mathcal{L}_{b^\uparrow}\langle\alpha, Y\rangle = \langle\mathcal{L}_{b^\uparrow}\alpha, Y\rangle + \langle\alpha, [b^\uparrow, Y]\rangle$. Using again Lemma B.2 in [26], we get

$$\mathcal{L}_{b^\uparrow}\alpha = -q^*(\mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b)),$$

where $\mathrm{P}_{T^*M} : A\oplus T^*M \longrightarrow T^*M$ is the projection on the second component. Hence we have

$$\langle\iota_Y\mathrm{d}\alpha, b^\uparrow\rangle = q^*(\mathcal{L}_v\langle\eta, b\rangle) + \langle q^*(\mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b)), Y\rangle = q^*(\mathcal{L}_v\langle\eta, b\rangle + \langle\mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b), v\rangle),$$

and then

$$\begin{aligned}\langle P_2(\mathcal{L}_X\beta - \iota_Y d\alpha), b\rangle &= \mathcal{L}_u\langle \xi, b\rangle - \langle \xi, \mathrm{Pr}_A(\nabla^h_{(u,\eta)}b)\rangle \\ &\quad -\mathcal{L}_v\langle \eta, b\rangle - \langle \mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b), v\rangle \\ &= \mathcal{L}_u\langle \xi, b\rangle - \mathcal{L}_v\langle \eta, b\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)}b\rangle.\end{aligned}$$

Therefore, the bracket in $\Gamma(TM \oplus A^*)$ is equal to

$$\langle [(u,\eta),(v,\xi)]_h, (b,\theta)\rangle = \langle [u,v],\theta\rangle + \mathcal{L}_u\langle \xi, b\rangle - \mathcal{L}_v\langle \eta, b\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)}b\rangle. \tag{4.19}$$

Theorem 4.27. *The triple $(TM \oplus A^*, P_{TM}, [\cdot,\cdot]_h)$ is a Dull algebroid (see [26], Definition 2.2). Moreover, if we denote by Λ^h the $(TM \oplus A^*)$-Dorfman connection on $A \oplus T^*M$ associated to h, then $[\cdot,\cdot]_h = [\![\cdot,\cdot]\!]_{\Lambda^h}$*

Proof. Remember that the Dorfman connection associated to h is

$$\Lambda^h_{(u,\eta)}(b,\theta) = \nabla^h_{(u,\eta)}b + (0, \mathrm{d}\langle \eta, b\rangle + \mathcal{L}_u\theta).$$

Then

$$\begin{aligned}\langle [\![(u,\eta),(v,\xi)]\!]_{\Lambda^h}, (b,\theta)\rangle &= \mathcal{L}_u(\langle (v,\xi),(b,\theta)\rangle) - \langle (v,\xi), \Lambda^h_{(u,\xi)}(b,\theta)\rangle \\ &= \mathcal{L}_u\langle v,\theta\rangle + \mathcal{L}_u\langle \xi,\beta\rangle \\ &\quad -\langle (v,\xi), \nabla^h_{(u,\xi)}b + (0, \mathrm{d}\langle \xi, b\rangle + \mathcal{L}_u\theta) \\ &= \mathcal{L}_u\langle v,\theta\rangle + \mathcal{L}_u\langle \xi,\beta\rangle - \langle v, \mathrm{d}\langle \eta, b\rangle \\ &\quad -\langle v, \mathcal{L}_u\theta\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)}b\rangle \\ &= \langle [(u,\eta),(v,\xi)]_h, (b,\theta)\rangle.\end{aligned}$$

□

Proposition 4.28. *Let $(X,\alpha) = \sigma_h(u,\eta) \in \Gamma_{lin}(\mathfrak{L}, A)$ for $(u,\eta) \in \Gamma(U)$. Then for a $(b,\theta) \in \Gamma(A \oplus T^*M)$*

$$[\![(X,\alpha),(b^\uparrow, q^*\theta)]\!] = (\nabla^h_{(u,\eta)}b)^\uparrow + (0, q^*(\mathcal{L}_u\theta + d\langle \eta, b\rangle)). \tag{4.20}$$

In particular, if $\Gamma(\mathfrak{L}, A)$ is closed by the Courant-Dorfman bracket, then

$$\nabla^h_{(u,\eta)}b + (0, \mathcal{L}_u\theta + d\langle \eta, b\rangle) \ \in \Gamma(K) \quad \textit{for all } (b,\theta) \in \Gamma(K).$$

Proof. Recall that

$$[X, b^\uparrow] = (\Delta_{(u,\eta)}b)^\uparrow = (\mathrm{P}_A(\nabla^h_{(u,\eta)}b))^\uparrow.$$

Also we have that

$$\mathcal{L}_X q^*\theta - \iota_{b^\uparrow}\mathrm{d}\alpha = q^*(\mathcal{L}_u\theta + \mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b) + \mathrm{d}\langle \eta, b\rangle).$$

Therefore

$$\begin{aligned}[\![(X,\alpha),(b^\uparrow, q^*\theta)]\!] &= (\mathrm{P}_A(\nabla^h_{(u,\eta)}b))^\uparrow + q^*(\mathcal{L}_u\theta + \mathrm{P}_{T^*M}(\nabla^h_{(u,\eta)}b) + \mathrm{d}\langle \eta, b\rangle) \\ &= (\nabla^h_{(u,\eta)}b))^\uparrow + (0, q^*(\mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle)).\end{aligned}$$

Hence, if $(b,\theta) \in \Gamma(K)$ and $\mathfrak{L}$ is closed by the bracket, then $[\![(X,\alpha),(b^\uparrow, q^*\theta)]\!]$ is a core linear section of $\mathfrak{L}$ over A, which implies that $\nabla^h_{(u,\eta)}b + (0, \mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle) \ \in \Gamma(K)$. □

With respect to the pairing, if $(b,\theta)\in\Gamma(A\oplus T^*M)$, then

$$\begin{aligned}\langle (X,\alpha),(b^\uparrow,q^*\theta)\rangle &= \langle q^*\theta,X\rangle+\langle b^\uparrow,\alpha\rangle\\ &= \langle q^*\theta+_A 0_\alpha,X\rangle+\langle b^\uparrow+_{A^*}0,\alpha\rangle\\ &= \langle\theta,u\rangle+\langle b,\eta\rangle.\end{aligned}$$

Proposition 4.29. *$\mathfrak{L}$ is Langrangian if and only if $K^\circ=U$ and the bracket defined in $\Gamma(TM\oplus A^*)$ is skew-symmetric for every $(u,\eta),(v,\xi)\in\Gamma(U)$.*

Proof. Let $(b,\theta)\in\Gamma(K)$ and take $(u,\eta)\in\Gamma(TM\oplus A^*)$. Consider a linear section $(X,\alpha)\in\Gamma_A(TA\oplus T^*A)$ covering (u,η). Then

$$\langle (b,\theta),(u,\eta)\rangle=0\Leftrightarrow\langle (X,\eta),(b^\uparrow,q^*\theta)\rangle=0.$$

Since $(b,\theta)\in\Gamma(K)$, the section $(b^\uparrow,q^*\theta)\in\Gamma_{\mathrm{cor}}(TA\oplus T^*A,A)$. If $(X,\alpha)\in\Gamma(\mathfrak{L})$ and $\mathfrak{L}$ is Lagrangian then $\langle (X,\eta),(b^\uparrow,q^*\theta)\rangle=0$. But then $\langle (b,\theta),(u,\eta)\rangle=0$ with $(u,\eta)\in\Gamma(U)$. Hence $U\subseteq K^\circ$. Moreover, since $\mathfrak{L}$ is Langrangian we have $(TA\oplus T^*A/\mathfrak{L})^*\simeq\mathfrak{L}^\circ=\mathfrak{L}$ and the following double vector bundle

$$\begin{tikzcd}\mathfrak{L} \arrow[r]\arrow[d] & A \arrow[d]\\ K^\circ \arrow[r] & M\end{tikzcd}$$

with core bundle U°. So, doing what we did before, interchanging U by K° and K by U°, it follows that $K^\circ\subseteq U$. Therefore $K^\circ=U$. Now take $(u,\eta),(v,\xi)\in\Gamma(U)$ and let (X,α), $(Y,\beta)\in\Gamma_A(\mathfrak{L})$ any linear sections projectable to $(u,\eta),(v,\xi)$. Then

$$[(v,\xi),(u,\eta)]=([v,u],P_2(\mathcal{L}_Y\alpha-\iota_X\mathrm{d}\beta)).$$

We have $[v,u]=-[u,v]$. For the second component we have

$$\begin{aligned}\langle P_2(\mathcal{L}_Y\alpha-\iota_X\mathrm{d}\beta),b\rangle &= \langle\mathcal{L}_Y\alpha-\iota_X\mathrm{d}\beta,b^\uparrow\rangle\\ &= \langle\mathcal{L}_Y\alpha-\mathcal{L}_X\beta-\mathrm{d}\langle\beta,X\rangle,b^\uparrow\rangle\\ &= \langle\mathcal{L}_Y\alpha-\mathcal{L}_X\beta,b^\uparrow\rangle+\langle\mathrm{d}\langle\beta,X\rangle,b^\uparrow\rangle\end{aligned}$$

and in the other hand

$$\begin{aligned}\langle P_2(\mathcal{L}_X\beta-\iota_Y\mathrm{d}\alpha),b\rangle &= \langle\mathcal{L}_X\beta-\iota_Y\mathrm{d}\alpha,b^\uparrow\rangle\\ &= \langle\mathcal{L}_X\beta-\mathcal{L}_Y\alpha,b^\uparrow+\langle\mathrm{d}\langle\alpha,Y\rangle,b^\uparrow\rangle\end{aligned}$$

Then the second component is skew-symmetric if and only if

$$-\big(\langle\mathcal{L}_Y\alpha-\mathcal{L}_X\beta,b^\uparrow\rangle+\langle\mathrm{d}\langle\beta,X\rangle,b^\uparrow\rangle\big)=\langle\mathcal{L}_X\beta-\mathcal{L}_Y\alpha,b^\uparrow\rangle+\langle\mathrm{d}\langle\alpha,Y\rangle,b^\uparrow\rangle,$$

if and only if

$$\mathcal{L}_{b^\uparrow}\langle\beta,X\rangle=-\mathcal{L}_{b^\uparrow}\langle\alpha,Y\rangle\quad\Longleftrightarrow\quad\mathcal{L}_{b^\uparrow}(\langle\beta,X\rangle+\langle\alpha,Y\rangle)=0.$$

Then if $\mathfrak{L}$ is Lagrangian, $\langle\beta, X\rangle + \langle\alpha, Y\rangle = 0$. Hence the bracket in two elements of $\Gamma(U)$ is skew-symmetric. Conversaly, if the bracket of two elements of $\Gamma(U)$ is skew-symmetric then $\langle\beta, X\rangle + \langle\alpha, Y\rangle = 0$ for all linear sections in $\Gamma(\mathfrak{L})$. For linear and core sections, we have

$$\langle(X,\alpha),(b^{\uparrow}, q^*\theta)\rangle = \langle\theta, u\rangle + \langle\eta, b\rangle = 0.$$

And for two core section

$$\langle(a^{\uparrow}, q^*\omega),(b^{\uparrow}, q^*\theta\rangle = \langle(\rho,\rho^*)(a,\omega),(b,\theta)\rangle = 0$$

because $(\rho,\rho^*)(K) \subseteq U$. Therefore, $\mathfrak{L} \subseteq \mathfrak{L}^{\circ}$. Now if we dualize the double vector bundle $(TA \oplus T^*A)/\mathfrak{L}$ over A we get a double vector bundle $\mathfrak{L}^{\circ}$ with side bundle U and core bundle U°. Hence we have two double vector bundles over A with the same side and core bundle. This means that the dimension (over A) is the same. Therefore $\mathfrak{L} = \mathfrak{L}^{\circ}$. □

Remark 4.30. This proposition can be found in [26], Theorem 4.15, (2), with a different proof.

Lemma 4.31. *If the sections of $\mathfrak{L}$ are closed with respect to the Courant bracket then $\mathfrak{L} \longrightarrow A$ is a VB-algebroid over $U \longrightarrow M$. In particular, $U \longrightarrow M$ inherits a Lie algebroid structure.*

Proof. If $[\![\Gamma(\mathfrak{L}),\Gamma(\mathfrak{L})]\!] \subseteq \Gamma(\mathfrak{L})$ then with this bracket $\mathfrak{L} \longrightarrow A$ is a Lie algebroid. If $(X,\alpha),\ (Y,\beta) \in \Gamma_A(\mathfrak{L})$ are linear sections then $[\![(X,\alpha),(Y,\beta)]\!]$ is linear. If $(b,\theta) \in \Gamma(K)$ then

$$\begin{array}{rcl}
[\![(X,\alpha),(b,\theta)^{\uparrow}]\!] & = & [\![(X,\alpha),(b^{\uparrow}, q^*\theta)]\!] \\
& = & ([X,b^{\uparrow}], \mathcal{L}_X q^*\theta - \iota_{b^{\uparrow}}\mathrm{d}\alpha) \\
& = & (\nabla^h_{(u,\eta)}b))^{\uparrow} + (0, q^*(\mathcal{L}_u\theta + \mathrm{d}\langle\eta, b\rangle)),
\end{array}$$

where $(u,\eta) = \mathbb{T}(q)(X,\alpha)$, which means that $[\![\Gamma_{\text{lin}}(\mathfrak{L}),\Gamma_{\text{cor}}(\mathfrak{L})]\!] \subseteq \Gamma_{\text{cor}}(\mathfrak{L})$. For two core sections $(a,\theta),(b,\omega) \in \Gamma(K)$ we have

$$[\![(a^{\uparrow}, q^*\theta),(b^{\uparrow}, q^*\omega)]\!] = ([a^{\uparrow}, b^{\uparrow}], \mathcal{L}_{a^{\uparrow}} q^*\omega - \iota_{b^{\uparrow}}\mathrm{d}(q^*\theta)) = 0.$$

Then $\mathfrak{L} \longrightarrow A$ is a VB-algebroid over $U \longrightarrow M$. □

Now we can characterize Dirac structures on a Lie algebroid A.

Theorem 4.32. *Let $\mathfrak{L} \subseteq TA \oplus T^*A$ be a double vector subbundle. Let $h : \Gamma(A) \longrightarrow \Gamma_{lin}(TA \oplus T^*A, TM \oplus A^*)$ be an adapted horizontal lift. Then $\mathfrak{L}$ is a Dirac structure if and only if $K^{\circ} = U$ and $(U \longrightarrow M, P_{TM}\big|_U, [\cdot,\cdot]_h)$ is a Lie algebroid.*

Proof. The Lagrangian condition follows by Theorem 4.29, and by the previous lemma, $\Gamma(\mathfrak{L})$ being closed by the Courant-Dorfman bracket implies that $(U \longrightarrow M, \mathrm{P}_{TM}\big|_U, [\cdot,\cdot]_h)$ is a Lie algebroid. Hence, it remains to prove that $(U \longrightarrow M, \mathrm{P}_{TM}\big|_U, [\cdot,\cdot]_h)$ Lie algebroid together $K = U^\circ$ imply $\mathfrak{L}$ is closed by the bracket:

$$[\![\widetilde{X}, \widetilde{Y}]\!] \in \Gamma(\mathfrak{L}, A) \tag{4.21}$$

Note that we only need to check equation 4.21 for linear an core sections. If $\widetilde{X}, \widetilde{Y}$ are core sections, then its Courant-Dorfman bracket is 0. If $\widetilde{X} = (X, \alpha) = \sigma_h(u, \eta)$, and $\widetilde{Y} = (b^\uparrow, q^*\theta)$, for $(u, \eta) \in \Gamma(U)$ and $(b, \theta) \in \Gamma(K)$, then

$$[\![\sigma_h(u,\eta), (b^\uparrow, q^*\theta)]\!] = (\nabla^h_{(u,\eta)} b)^\uparrow + (0, q^*(\mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle)).$$

We claim that $\nabla^h_{(u,\eta)} b + (0, \mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle) \in \Gamma(K)$. Indeed, since U is a Lie algebroid and since $K = U^\circ$ then

$$\begin{aligned} 0 &= \langle [(u,\eta),(v,\xi)]_h, (b,\theta)\rangle \\ &= \langle [u,v], \theta\rangle + \mathcal{L}_u\langle b, \xi\rangle - \mathcal{L}_v\langle b, \eta\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)} b\rangle \\ &= \mathcal{L}_u\langle v, \theta\rangle - \langle v, \mathcal{L}_u\theta\rangle + \mathcal{L}_u\langle b, \xi\rangle - \langle v, \mathrm{d}\langle b, \eta\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)} b\rangle \\ &= \mathcal{L}_u \underbrace{\langle (v,\xi),(b,\theta)\rangle}_{=0} - \langle v, \mathcal{L}_u\theta + \mathrm{d}\langle b, \eta\rangle - \langle (v,\xi), \nabla^h_{(u,\eta)} b\rangle \\ &= -\langle (v,\xi), \nabla^h_{(u,\eta)} b + (0, \mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle)\rangle. \end{aligned}$$

Hence $\nabla^h_{(u,\eta)} b + (0, \mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle) \in \Gamma(K)$. So

$$[\![\sigma_h(u,\eta), (b^\uparrow, q^*\theta)]\!] = (\nabla^h_{(u,\eta)} b)^\uparrow + (0, q^*(\mathcal{L}_u\theta + \mathrm{d}\langle \eta, b\rangle)) \in \Gamma(\mathfrak{L}, A).$$

Finally, we need to check the Equation 4.21 for linear sections. Let now $\widetilde{Y} = (Y, \beta) = \sigma_h(v, \xi)$. By Theorem 4.9 in [26] we have

$$[\![\sigma_h(u,\eta), \sigma_h(v,\xi)]\!] = \sigma_h([(u,\eta),(v,\xi)]) - S_{R_\Lambda((u,\eta),(v,\xi))(\cdot,0)}$$

where R_Λ is the curvature of the Dorfman connection Λ associated to the horizontal lift h (see Definition 3.3 in [26]). Since $\sigma_h([(u,\eta),(v,\xi)]) \in \Gamma(\mathfrak{L}, A)$ it is enough to prove that $S_{R_\Lambda((u,\eta),(v,\xi))(\cdot,0)}$ is a section of $\mathfrak{L}$ over A, which is the same that proving $R_\Lambda((u,\eta),(v,\xi))(\cdot,0) \in \Gamma(\mathrm{Hom}(A,K))$. Let $\widetilde{w} = (w,\delta) \in \Gamma(U)$, $a \in \Gamma(A)$, and set $\widetilde{u} = (u,\eta)$ and $\widetilde{v} = (v,\xi)$. Then using Proposition 3.4 in [26] we have

$$\langle R_\Lambda(\widetilde{u},\widetilde{v})(a,0), \widetilde{w}\rangle = \langle [[\widetilde{u},\widetilde{v}]_h, \widetilde{w}]_h + [[\widetilde{v},\widetilde{w}]_h, \widetilde{u}]_h + [[\widetilde{w},\widetilde{u}]_h, \widetilde{v}]_h, (a,0)\rangle = 0$$

because U is a Lie algebroid. Hence (4.21) is satisfied by linear sections. Therefore Equation 4.21 holds for all sections in $\Gamma(\mathfrak{L}, A)$, and together with the Lagrangian condition, it follows that $\mathfrak{L}$ is a Dirac structure. □

Remark 4.33. One can see Theorem 4.15 together with Corollary 4.16 in [26] for a different proof of this result.

4.3.2 IM-Dirac structures

An *IM-Dirac structure* on a Lie algebroid A is a Dirac structure $\mathfrak{L}$ over A, such that it is a Lie subalgebroid of $TA \oplus T^*A \longrightarrow TM \oplus A^*$. Combining the description of subalgebroids with the description of Dirac structures, we get an infinitesimal description of multiplicative Dirac structures.

Theorem 4.34. *Let $\mathfrak{L} \subseteq TA \oplus T^*A$ be a double vector subbundle. Let $h : \Gamma(A) \longrightarrow \Gamma_{lin}(TA \oplus T^*A, TM \oplus A^*)$ be an adapted horizontal lift. Then $\mathfrak{L}$ is an IM-Dirac structure if and only if $K = U^\circ$, $(U \longrightarrow M, P_{TM}\big|_U, [\cdot,\cdot]_h)$ is a Lie algebroid, and*

$$(\rho_A, \rho_A^*)(K) \subseteq U \tag{4.22}$$

$$\iota_{(\rho_A(b),\rho_A^*(\theta))}\mathbb{D}(a) = -\pi(\nabla^0_a(b,\theta)) \qquad \textit{for } (b,\theta) \in K \tag{4.23}$$

$$\begin{aligned}\mathbb{D}_{(u,\eta)}([a,b]) = {} & \pi(\nabla_{(\nabla^1_b(u,\eta)+(\rho_A,\rho_A^*)(\nabla_{(u,\eta)}b))}a) \\ & -\pi(\nabla_{(\nabla^1_a(u,\eta)+(\rho_A,\rho_A^*)(\nabla_{(u,\eta)}a))}b) \\ & +\pi(\nabla^0_a \circ \nabla_{(u,\eta)}b) - \pi(\nabla^0_b \circ \nabla_{(u,\eta)}a).\end{aligned} \tag{4.24}$$

Remark 4.35. This theorem is equivalent to Theorem 5.10 in [26].

A *multiplicative Dirac structure* on a Lie groupoid $\mathcal{G}$ is a Dirac structure $\mathfrak{D}$ such that it is a Lie subgroupoid of $T\mathcal{G} \oplus T^*\mathcal{G}$. It was proven in [36] (Theorem 5.1) that every multiplicative Dirac structure on $\mathcal{G}$ induces an IM-Dirac structure on A. Moreover when $\mathcal{G}$ is source simply connected this is a one-to-one correspondence. Hence using this result and combining with Theorem 4.34, Theorem 4.3.4 and Corollary 4.3.7 in [7], we have proven

Proposition 4.36. *If $\mathcal{G}$ is source simply connected, there is a one-to-one correspondence between multiplicative Dirac structure $\mathfrak{D}$ on $\mathcal{G}$ and subbundles $U \subseteq TM \oplus A^*$ whit an operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(U^* \otimes U^*)$ such that $(U \longrightarrow M, P_{TM}, [\cdot,\cdot]_h)$ is Lie algebroid, and such that Equations (4.22), (4.23) and (4.24) hold.*

Example 4.37. Closed 2-forms. Let $\omega \in \Omega^2(A)$ be a closed 2-form on A, and let $\omega^\sharp : TA \longrightarrow T^*A$ be the map given by

$$\omega^\sharp(X) = i_X\omega \quad \text{for } X \in TA.$$

We know that the graph of ω, $\mathrm{graph}(\omega^\sharp) = \{(X, i_X\omega) \ : \ X \in TA\} \subseteq TA \oplus T^*A$ is a Dirac structure on A. Assume too that ω is **linear**, i.e., it is a vector bundle map

$$\begin{tikzcd} TA \arrow[r, "\omega^\sharp"] \arrow[d] & T^*A \arrow[d] \\ TM \arrow[r, "r"] & A^* \end{tikzcd}$$

over some linear map $r : TM \longrightarrow A^*$ covering the identity of M. We will proof in a different way, using Theorem 4.32, that graph$(\omega^\sharp)$ is a Dirac structure.

Since $\omega^\sharp$ is also linear over A we get that $\omega^\sharp$ is a morphism of double vector bundles. Define

$$\mathfrak{L}_\omega = \text{graph}(\omega^\sharp) = \{(X, i_X\omega) \ : \ X \in TA\}.$$

Since $\omega^\sharp$ is a morphism of double vector bundles, it follows that $\mathfrak{L}_\omega$ is a double vector subbundle of $TA \oplus T^*A$, and therefore it defines a double vector bundle

$$\begin{tikzcd} \mathfrak{L}_\omega \arrow[r] \arrow[d] & A \arrow[d] \\ U \arrow[r] & M \end{tikzcd}$$

with core bundle $K \longrightarrow M$. Now we will describe who are U and K. Denote by $Pr = (Tq, P_2) : TA \oplus T^*A \longrightarrow TM \oplus A^*$ the left vertical vector bundle map of the double vector bundle $(TA \oplus T^*A, TM \oplus A^*; A, M)$, where $q : A \longrightarrow M$ and $P_2 : T^*A \longrightarrow A^*$. We know that $U = Pr(\mathfrak{L}_\omega)$. Let

$$TA \ni \begin{tikzcd} X \arrow[r] \arrow[d] & a \arrow[d] \\ Tq(X) = u \arrow[r] & p \end{tikzcd}, \qquad T^*A \ni \begin{tikzcd} i_X\omega \arrow[r] \arrow[d] & a \arrow[d] \\ P_2(i_X\omega) \arrow[r] & p \end{tikzcd}$$

and let $b \in \Gamma(A)$. Then

$$\begin{aligned} \langle P_2(i_X\omega), b\rangle &:= \langle i_X\omega, 0_a +_{TM} \overline{b}\rangle = \omega(X, 0_a +_{TM} \overline{b}) \\ &= \omega(X +_{TM} 0_u, 0_a +_{TM} \overline{b}) = \omega(X, 0_a) + \omega(0_u, \overline{b}) \\ &= \langle r(u), b\rangle. \end{aligned}$$

Therefore

$$U = \text{graph}(r) = \{(u, r(u)) \ : \ u \in TM\}.$$

The core bundle is $K = Ker(Pr_{|\mathfrak{L}_\omega}) \cap Ker(\pi) \subseteq A \oplus T^*M$. If $(a, \theta) \in K$, then

$$\begin{tikzcd} \overline{a} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & p \end{tikzcd}, \qquad \begin{tikzcd} \theta \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & p \end{tikzcd}$$

with $i_{\overline{a}}\omega = \overline{\theta}$ (because $(\overline{a}, \overline{\theta}) \in \mathfrak{L}_\omega$). Let $X \in TA$ with projections $u \in TM$ and $0 \in A$. Then

$$\begin{aligned} \langle i_{\overline{a}}\omega, X\rangle &= \omega(\overline{a}, X) = -\omega(X, \overline{a}) \\ &= -\omega(X, 0 +_{TM} \overline{a}) = -\langle r(u), a\rangle \\ &= \langle u, -r^T(a)\rangle \end{aligned}$$

which means that $\overline{-r^T(a)} = i_{\overline{a}}\omega$. Defining $\sigma : A \longrightarrow T^*M$ as $\sigma = -r^T$ we get that

$$K = \mathrm{graph}(\sigma) = \{(a, \sigma(a)) \ : \ a \in A\}.$$

And follows that $K^\circ = U$:

$$\langle (a, \sigma(a)), (u, r(u)) \rangle = \langle a, r(u) \rangle + \langle \sigma(a), u \rangle = \langle a, r(u) \rangle - \langle a, r(u) \rangle = 0$$

Proposition 4.38. *There is a canonical inclusion* $\Gamma(A) \longrightarrow \Gamma_{lin}(\mathfrak{L}, U)$.

Proof. We look for a map $h(a) : U = \mathrm{graph}(r) \longrightarrow \mathfrak{L}_\omega = \mathrm{graph}(\omega^\sharp)$. Let $u \in \Gamma(TM)$. We want $h(a)(u, r(u)) = (Ta(u), \mu(r(u)))$ such that $\iota_{Ta(u)}\omega = \mu(r(u))$. The elements $Ta(u), R_a(r(u)), \iota_{Ta(u)}\omega$ have projections

$$\begin{array}{ccc} Ta(u) & \longrightarrow & a \\ \downarrow & & \downarrow \\ u & \longrightarrow & p \end{array}, \qquad \begin{array}{ccc} R_a(r(u)) & \longrightarrow & a \\ \downarrow & & \downarrow \\ r(u) & \longrightarrow & p \end{array}, \qquad \begin{array}{ccc} \iota_{Ta(u)}\omega & \longrightarrow & a \\ \downarrow & & \downarrow \\ r(u) & \longrightarrow & p \end{array}.$$

Then

$$\begin{array}{ccc} \Phi := \iota_{Ta(u)}\omega -_{A^*} R_a(r(u)) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ r(u) & \longrightarrow & p \end{array}$$

which means that $\Phi = 0_{r(u)} +_A \overline{\varphi}$ where $\varphi \in \Omega^1(M)$. On one hand we have

$$\langle \Phi, T0(y) \rangle = \langle 0_{r(u)} +_A \overline{\varphi}, T0(y) \rangle = \langle \alpha, y \rangle \quad \forall y \in TM.$$

On the other hand

$$\begin{aligned} \langle \Phi, T0(y) \rangle &= \langle \iota_{Ta(u)}\omega -_{A^*} R_a(r(u)), Ta(y) - Ta(y) \rangle \\ &= \langle \iota_{Ta(u)}\omega, Ta(y) \rangle - \langle R_a(r(u)), Ta(y) \rangle \\ &= \omega(Ta(u), Ta(y)) \\ &= a^*\omega(u, y). \end{aligned}$$

Since ω is closed then $\omega = (r^t)^*\omega_{\mathrm{can}}$ (see Example 2.6 in [7]), where $\omega_{\mathrm{can}} \in \Omega^2(T^*M)$ is the canonical symplectic form. Using that $\omega_{\mathrm{can}} = -\mathrm{d}\theta_{\mathrm{can}}$, where θ_{can} is the tautological 1-form on T^*M and the property $\lambda^*\theta = \lambda$ for every $\lambda \in \Omega^1(M)$, it follows that

$$a^*\omega = a^*((r^t)^*(\omega_{\mathrm{can}})) = a^*(\mathrm{d}\sigma^*\theta_{\mathrm{can}}) = \mathrm{d}((\sigma \circ a)^*\theta_{\mathrm{can}}) = \mathrm{d}(\sigma(a)).$$

Hence $\varphi = \mathrm{d}\sigma(a)(u)$. Therefore

$$h(a)(u, \eta) = (Ta(u), R_a(\eta) +_{A^*} S_{\mathrm{d}\sigma(a)(u)}(u, \eta)) = (Ta(u), R_a(\eta)) - (0, S_{-\mathrm{d}\sigma(a)(u)}(u, \eta))$$

is an adapted horizontal lift. $\square$

The bracket on $\Gamma(U)$ is: let $(u, r(u)), (v.r(v)) \in \Gamma(U)$, and $(b, \theta) \in \Gamma(A \oplus T^*M)$ we have

$$\begin{aligned}
\langle [(u, r(u)), (v, r(v))], (b, \theta) \rangle &= \langle [u, v], \theta \rangle + \mathcal{L}_u \langle r(v), b \rangle - \mathcal{L}_v \langle r(u), b \rangle \\
&\quad - \langle (v, r(v)), \nabla^h_{(u, r(u))} b \rangle \\
&= \langle [u, v], \theta \rangle + \mathcal{L}_u \langle r(v), b \rangle - \mathcal{L}_v \langle r(u), b \rangle \\
&\quad + \langle (v, r(v)), \mathrm{d}(\sigma(b))(u) \rangle \\
&= \langle [u, v], \theta \rangle + \mathcal{L}_u \langle r(v), b \rangle - \mathcal{L}_v \langle r(u), b \rangle \\
&\quad + \mathcal{L}_u \langle v, \sigma(b) \rangle - \mathcal{L}_v \langle u, \sigma(b) \rangle - \langle \sigma(b), [u, v] \rangle \\
&= \langle [u, v], \theta \rangle + \mathcal{L}_u \langle r(v), b \rangle - \mathcal{L}_v \langle r(u), b \rangle \\
&\quad - \mathcal{L}_u \langle r(v), b \rangle + \mathcal{L}_v \langle r(u), b \rangle + \langle r([u, v]), b \rangle \\
&= \langle ([u, v], r([u, v])), (b, \theta) \rangle,
\end{aligned}$$

which implies that $[(u, r(u)), (v, r(v))]_h = ([u, v], r([u, v]))$. Hence, this bracket is skew symmetric for sections of U, and together with the condition $K^\circ = U$, follow that $\mathfrak{L}_\omega$ is a Lagrangian subbundle. Moreover, since the bracket is closed for sections of U, we have that $U \longrightarrow M$ is a Lie algebroid, and therefore, Theorem 4.32 holds, and we get in another way the known result that $\mathfrak{L}_\omega$ is a Dirac structure.

Remark 4.39. We can identify $U \simeq TM$ and $K \simeq A$, and then we get, naturally, that $K^\circ = U$ and that U is a Lie algebroid.

Suppose now that conditions (4.22), (4.23) and (4.24) are also satisfied. We will work little more on these equations. The first one says $(\rho, \rho^*)(K) \subseteq U$. So, for $(a, \sigma(a)), (b, \sigma(b)) \in \Gamma(K)$ we have

$$\begin{aligned}
\langle (\rho, \rho^*)(a, \sigma(a)), (b, \sigma(b)) \rangle &= \langle \rho(a), \sigma(b) \rangle + \langle \rho^*(\sigma(a)), b \rangle \\
&= \langle \rho(a), \sigma(b) \rangle + \langle \rho(b), \sigma(a) \rangle,
\end{aligned}$$

which means that $(\rho, \rho^*)(K) \subseteq U$ if and only if

$$\iota_{\rho(a)} \sigma(b) = -\iota_{\rho(b)} \sigma(a). \tag{4.25}$$

Before to analyze the other two equations, we need to know who is $\pi : A \oplus T^*M \longrightarrow \frac{A \oplus T^*M}{K}$ and $\mathbb{D}$. Using $K^\circ = U$ we get

$$\frac{A \oplus T^*M}{K} \simeq (K^\circ)^* \simeq U^* \simeq T^*M.$$

So for $(a, \theta) \in A \oplus T^*M$ and $(u, r(u)) \in U$

$$\begin{aligned}
\langle \pi(a, \theta), (u, r(u)) \rangle &= \langle (a, \theta), (u, r(u)) \rangle = \langle \theta, u \rangle + \langle a, r(u) \rangle \\
&= \langle \theta, u \rangle - \langle \sigma(a), u \rangle = \langle \theta - \sigma(a), u \rangle.
\end{aligned}$$

Therefore we can write $\pi(a,\theta) = \theta - \sigma(a)$. For $\mathbb{D}$ remember that it satisfies $\mathbb{D} = \pi \circ \nabla^h\big|_U$. Then

$$\mathbb{D}_{(u,r(u))}(a) = \pi(0, -\mathrm{d}\sigma(a)(u)) = -\mathrm{d}\sigma(a)(u).$$

Hence the left side of Equation (4.23) is

$$\langle \mathbb{D}(a), (\rho,\rho^*)(b,\sigma(b))\rangle = -\langle \mathrm{d}\sigma(a), (\rho(b), \rho^*(\sigma(b)))\rangle = -\iota_{\rho(b)}\mathrm{d}\sigma(a).$$

The right side is

$$-\pi([a,b], \mathcal{L}_{\rho(a)}\sigma(b)) = -(\mathcal{L}_{\rho(a)}\sigma(b) - \sigma([a,b])).$$

Therefore, Equation (4.23) is satisfied if and only if

$$\sigma([a,b]) = \mathcal{L}_{\rho(a)}\sigma(b) - \iota_{\rho(b)}\mathrm{d}\sigma(a). \tag{4.26}$$

Finally, for the last equation, recall the following operators

$$\nabla^0 : \Gamma(A) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M), \quad \nabla^0_a(b,\theta) = ([a,b], \mathcal{L}_{\rho(a)}\theta),$$

and

$$\nabla^1 : \Gamma(A) \times \Gamma(TM \oplus A^*) \longrightarrow \Gamma(TM \oplus A^*), \quad \nabla^1_a(u,\eta) = ([\rho(a),u], \mathcal{L}_a\eta).$$

The left hand side of Equation (4.24) is

$$\mathbb{D}_{(u,r(u))}([a,b]) = -\mathrm{d}(\sigma([a,b]))(u).$$

If the we assume that Equation (4.23) holds, then

$$\mathbb{D}_{(u,r(u))}([a,b]) = (\mathcal{L}_{\rho(b)}\mathrm{d}\sigma(a) - \mathcal{L}_{\rho(a)}\mathrm{d}\sigma(b))(u) \tag{4.27}$$

For the right hand side we have that

$$\begin{aligned}
\pi(\nabla^h_{(\nabla^1_b(u,\eta)+(\rho_A,\rho^*_A)(\nabla_{(u,\eta)}b))}a) &= \pi(\nabla^h_{([\rho(b),u],\mathcal{L}_b r(u)-\rho^*(\mathrm{d}\sigma(b)(u)))}a \\
&= -\pi(0, \mathrm{d}\sigma(a)[\rho(b),u]) \\
&= -\mathrm{d}\sigma(a)([\rho(b),u]).
\end{aligned}$$

Analogously,

$$\pi(\nabla_{(\nabla^1_a(u,\eta)+(\rho_A,\rho^*_A)(\nabla_{(u,\eta)}a))}b) = -\mathrm{d}\sigma(b)([\rho(a),u]).$$

The third term of the right hand side of Equation (4.24) is

$$\pi(\nabla^0_a \circ \nabla_{(u,\eta)}b) = \pi(\nabla^0_a(0, -\mathrm{d}\sigma(b)(u)) = \pi(0, \mathcal{L}_{\rho(a)}(-\mathrm{d}\sigma(b)(u))) = -\mathcal{L}_{\rho(a)}(\mathrm{d}\sigma(b)(u)).$$

In the same way we have

$$\pi(\nabla^0_b \circ \nabla_{(u,\eta)}a) = -\mathcal{L}_{\rho(b)}(\mathrm{d}\sigma(a)(u)).$$

Therefore the right hand side of the equation is

$$-\mathrm{d}\sigma(a)([\rho(b),u]) + \mathrm{d}\sigma(b)([\rho(a),u]) - \mathcal{L}_{\rho(a)}(\mathrm{d}\sigma(b)(u)) + \mathcal{L}_{\rho(a)}(\mathrm{d}\sigma(b)(u)),$$

and it is equal to (4.27). Hence, Equation (4.23) implies Equation (4.24).

Therefore, a multiplicative Dirac structure coming from a multiplicative closed linear 2-form, is described infinitesimally by a vector bundle map $\sigma : A \longrightarrow T^*M$ over the identity of M satisfying

- $\iota_{\rho(a)}\sigma(b) = -\iota_{\rho(b)}\sigma(a)$,
- $\sigma([a,b]) = \mathcal{L}_{\rho(a)}\sigma(b) - \iota_{\rho(b)}\mathrm{d}\sigma(a)$.

Hence with our approach we also obtain the IM 2-forms describing multiplicative closed linear 2-forms (see [7]).

Example 4.40. Poisson Manifolds. Let $\pi \in \Gamma(\wedge^2 TA)$ be a bivector on A which is linear in the following sense: there exists a vector bundle map $\sigma : A^* \longrightarrow TM$ such that

$$\begin{tikzcd} T^*A \arrow[r, "\pi^\sharp"] \arrow[d] & TA \arrow[d] \\ A^* \arrow[r, "\sigma"'] & TM \end{tikzcd}$$

is a vector bundle, where $\pi^\sharp(\alpha) = \iota_\alpha \pi$. Note that $\pi^\sharp$ is a morphism of double vector bundles between the prolonged tangent bundle and the prolonged cotangent bundle. Let $\mathfrak{L}_\pi$ the graph of $\pi^\sharp$:

$$\mathfrak{L}_\pi = \{(\pi^\sharp(\alpha), \alpha) \ : \ \alpha \in T^*A\}.$$

Since $\pi^\sharp$ is a morphism of double vector bundles we have that $\mathfrak{L}_\pi$ is a double vector subbundle of $TA \oplus T^*A$ with side bundle $U \subseteq TM \oplus A^*$ and core bundle $K \subseteq A \oplus T^*M$. The side bundle $U = \mathbb{T}q(\mathfrak{L}_\pi)$, where $\mathbb{T}q = (Tq, P_2)$. Then

$$u = Tq(\pi^\sharp(\alpha)) = (Tq \circ \pi^\sharp)(\alpha) = (\sigma \circ P_2)(\alpha) = \sigma(P_2(\alpha))$$

which implies that $U = \mathrm{graph}(\sigma) = \{(\sigma(\xi), \xi) \ : \ \xi \in A^*\}$. The core bundle K is the intersection of the kernels of the projections. So for $(a,\theta) \in K$ we have

$$\begin{tikzcd} TA \ni \overline{a} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & p \end{tikzcd} \qquad \begin{tikzcd} T^*A \ni \overline{\theta} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & p \end{tikzcd}$$

whit the condition $\overline{a} = \pi^\sharp(\overline{\theta})$. If $\alpha \in T^*A$ then

$$\langle a, P_2(\alpha)\rangle = \langle \pi^\sharp(\overline{\theta}), \alpha\rangle = \pi(\overline{\theta}, \alpha) = -\pi(\alpha, \overline{\theta}) = -\langle \sigma(P_2(\alpha)), \theta\rangle = \langle P_2(\alpha), -\sigma^T(\theta)\rangle,$$

which means that $a = -\sigma^T(\theta)$. Hence we get $K = \text{graph}(-\sigma^T)$ and note that $K^\circ = U$. Therefore $\pi \in \Gamma(\wedge^2 TA)$ is a Poisson structure if and only if $\mathfrak{L}_\pi$ is a Dirac structure, if and only if $(U \longrightarrow M, P_{TM}, [,]_h)$ is a Lie algebroid. Let see more about the Lie algebroid structure of U.

Proposition 4.41. *There is a canonical inclusion* $\Gamma(A) \longrightarrow \Gamma_{lin}(\mathfrak{L}_\pi, U)$.

Proof. We look for a map $\Phi_a : \text{graph}(\sigma) \longrightarrow \text{graph}(\pi^\sharp)$. Let $(\sigma(\xi), \xi) \in \Gamma(U)$. Note that

$$\begin{tikzcd} Ta(\sigma(\xi)) \arrow[r] \arrow[d] & a \arrow[d] \\ \sigma(\xi) \arrow[r] & p \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi^\sharp(R_a(\xi)) \arrow[r] \arrow[d] & a \arrow[d] \\ \sigma(\xi) \arrow[r] & p. \end{tikzcd}$$

Let $\Phi_a = Ta(\sigma(\xi)) -_{TM} \pi^\sharp(R_a(\xi)) = 0_{\sigma(\xi)} +_A \bar{b}$ for some $b \in \Gamma(A)$. On one hand, for $\eta \in A^*$, we have

$$\langle \Phi_a, R_0(\eta)\rangle = \langle 0_{\sigma(\xi)} +_A \bar{b}, R_0(\eta)\rangle = \langle b, \eta\rangle.$$

On the other hand we have

$$\begin{aligned} \langle \Phi_a, R_0(\eta)\rangle &= \langle Ta(\sigma(\xi)) -_{TM} \pi^\sharp(R_a(\xi)), R_a(\eta) -_{A^*} R_a(\eta)\rangle \\ &= \langle \pi^\sharp(R_a(\xi)), R_a(\eta)\rangle = \pi(R_a(\xi), R_a(\eta)) \\ &= (R_a^*\pi)(\xi, \eta) = (R_a^*\pi)(\xi)(\eta). \end{aligned}$$

Hence $\Phi_a = -S_{(R_a^*\pi)(\xi)}$. Define now the map $l : A \longrightarrow \wedge^2 A$ by $l(a) = R_a^*(\pi)$. Therefore the map

$$a \longrightarrow (Ta + S_{-l(a)}, R_a)$$

is an adapted horizontal lift. Moreover the operator $\nabla : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$ is given by

$$\nabla_{(X,\xi)} a = (l(a)(\xi), 0).$$

□

The bracket for elements in $\Gamma(U)$ associated to the horizontal lift h is

$$\begin{aligned} \langle [(\sigma(\xi), \xi), (\sigma(\eta), \eta)], (b, \theta)\rangle &= \langle [\sigma(\xi), \sigma(\eta)], \theta\rangle + \mathcal{L}_{\sigma(\xi)}\langle \eta, b\rangle - \mathcal{L}_{\sigma(\eta)}\langle \xi, b\rangle \\ &\quad -\langle (\sigma(\eta), \eta), l(b)(\xi)\rangle \\ &= \langle [\sigma(\xi), \sigma(\eta)], \theta\rangle + \mathcal{L}_{\sigma(\xi)}\langle \eta, b\rangle - \mathcal{L}_{\sigma(\eta)}\langle \xi, b\rangle \\ &\quad -l(b)(\xi, \eta) \end{aligned}$$

If we take an element $(-\sigma^T(\theta), \theta) \in \Gamma(K)$ then

$$\begin{aligned} (4.28)\, 0 &= \langle [(\sigma(\xi), \xi), (\sigma(\eta), \eta)]_h, (-\sigma^T(\theta), \theta)\rangle \\ &= \langle [\sigma(\xi), \sigma(\eta)], \theta\rangle - \mathcal{L}_{\sigma(\xi)}\langle \sigma(\eta), \theta\rangle + \mathcal{L}_{\sigma(\eta)}\langle \sigma(\xi), \theta\rangle - l(-\sigma^T(\theta))(\xi, \eta) \end{aligned}$$

Proposition 4.42. *There is a Lie algebroid structure on $A^* \longrightarrow M$ with anchor map $\sigma : A^* \longrightarrow TM$.*

Proof. Let $\xi,\ \eta \in \Gamma(A^*)$ and $b \in \Gamma(A)$. Their brackets is determined by:

$$\langle [\xi,\eta]_{A^*}, b\rangle = \mathcal{L}_{\sigma(\xi)}\langle \eta, b\rangle - \mathcal{L}_{\sigma(\eta)}\langle \xi, b\rangle - l(b)(\xi,\eta) \tag{4.29}$$

The bilinearity and skew-symmetry of the bracket follow immediately. Now we check Jacobi. Note first that by Equation (4.28) and the definition of the bracket we have that

$$\sigma([\xi,\eta]_{A^*}) = [\sigma(\xi),\sigma(\eta)].$$

Then

$$\begin{aligned}
\langle [[\xi,\eta],\delta], b\rangle &= \mathcal{L}_{\sigma([\xi,\eta])}\langle \sigma(\delta), b\rangle - \mathcal{L}_{\sigma(\delta)}\langle [\xi,\eta], b\rangle - l(b)([\xi,\eta],\delta) \\
&= \mathcal{L}_{[\sigma(\xi),\sigma(\eta)]}\langle \delta, b\rangle - \mathcal{L}_{\sigma(\delta)}\mathcal{L}_{\sigma(\xi)}\langle \eta, b\rangle + \mathcal{L}_{\sigma(\delta)}\mathcal{L}_{\sigma(\eta)}\langle \xi, b\rangle \\
&\quad -\langle \{\{\ell_\xi,\ell_\eta\},\ell_\delta\}, a\rangle \\
&= \mathcal{L}_{\sigma(\xi)}\mathcal{L}_{\sigma(\eta)}\langle \delta, b\rangle - \mathcal{L}_{\sigma(\eta)}\mathcal{L}_{\sigma(\xi)}\langle \delta, b\rangle - \mathcal{L}_{\sigma(\delta)}\mathcal{L}_{\sigma(\xi)}\langle \eta, b\rangle \\
&\quad +\mathcal{L}_{\sigma(\delta)}\mathcal{L}_{\sigma(\eta)}\langle \xi, b\rangle - \langle \{\{\ell_\xi,\ell_\eta\},\ell_\delta\}, a\rangle
\end{aligned}$$

Now it is straightforward to check that

$$\langle [[\xi,\eta],\delta], b\rangle + \langle [[\eta,\delta],\xi], b\rangle + \langle [[\delta,\xi],\eta], b\rangle = 0.$$

Therefore $(A^* \longrightarrow M, \sigma, [,]_{A^*})$ is a Lie algebroid. □

Hence we get that the bracket in $\Gamma(U)$ is

$$[(\sigma(\xi),\xi),(\sigma(\eta),\eta)]_h = (\sigma([\xi,\eta]_{A^*}), [\xi,\eta]_{A^*})$$

Now we assume that π is a multiplicative Poisson structure. Note that the condition $(\rho,\rho^*)(K) \subseteq U$ follows by the fact that $\pi^\sharp : T^*A \longrightarrow TA$ is a morphism of double vector bundles, and means

$$\begin{aligned}
0 &= \langle (\rho,\rho^*)(-\sigma^T(\theta),\theta), (-\sigma^T(w), w)\rangle \\
&= -\langle \rho(\sigma^T(\theta)), w\rangle, \sigma^T(w)\rangle - \langle \rho^*(\theta)
\end{aligned}$$

for every $\theta, w \in T^*M$, which holds if and only if $\rho \circ \sigma^T = -\sigma \circ \rho^*$. For the other two equations of Theorem 4.34, we need to know first who are $\mathbb{D}$ and the projection $\pi : A \oplus T^*M \longrightarrow (A \oplus T^*M)/K$. The operator $\mathbb{D} : \Gamma(A) \longrightarrow \Gamma(U^* \otimes U^*)$ is in this case given by

$$\mathbb{D}((\sigma(\xi),\xi),(\sigma(\eta),\eta)) = l(a)(\xi,\eta).$$

For the projection π we have

$$\begin{aligned}
\langle \pi(a,\theta), (\sigma(\xi),\xi)\rangle &= \langle (a,\theta), (\sigma(\xi),\xi)\rangle \\
&= \langle a, \xi\rangle + \langle \theta, \sigma(\xi)\rangle \\
&= \langle a + \sigma^T(\theta), \xi\rangle
\end{aligned}$$

which means that we can write $\pi(a,\theta) = \sigma^T(\theta) + a$. Now the left side of Equation (4.23) is

$$\iota_{(\rho(-\sigma^T(\theta)),\rho^*(\theta))}\mathbb{D}(a) = \iota_{\rho^*(\theta)}l(a)$$

while the left side is

$$-\pi([a, -\sigma^T(\theta)], \mathcal{L}_{\rho(a)}\theta) = [a, \sigma^T(\theta)] - \sigma^T(\mathcal{L}_{\rho(a)}\theta).$$

Therefore Equation (4.23) translates to the condition

$$\iota_{\rho^*(\theta)}l(a) = [a, \sigma^T(\theta)] - \sigma^T(\mathcal{L}_{\rho(a)}\theta).$$

Now the left side of Equation (4.24) is $l([a,b])(\xi)$. Lets see the right side. The term third term is

$$\begin{aligned} \pi(\nabla^0_a(\nabla_{(\sigma(\xi),\xi)}b)) &= \pi(\nabla^0_a(l(b)(\xi))) = \pi([a, l(b)(\xi)], 0) \\ &= [a, l(b)(\xi)]. \end{aligned}$$

Analogously we have that $\pi(\nabla^0_b(\nabla_{(\sigma(\xi),\xi)}a)) = [b, l(a)(\xi)]$. The first term of the right side of Equation (4.24) is

$$\begin{aligned} \pi(\nabla_{(\nabla^1_b(\sigma(\xi),\xi)+(\rho,\rho^*)(\nabla_{(\sigma(\xi),\xi)}b))}a) &= \pi(\nabla_{(([\rho(b),\sigma(\xi)],\mathcal{L}_b\xi)+(\rho,\rho^*)(l(b)(\xi)))}a) \\ &= \pi(\nabla_{(([\rho(b),\sigma(\xi)]+\rho(l(b)(\xi))),\mathcal{L}_b\xi)}a) \\ &= \pi(l(a)(\mathcal{L}_b\xi)) \\ &= l(a)(\mathcal{L}_b\xi). \end{aligned}$$

In a similar way we get that the second term is $l(b)(\mathcal{L}_a\xi)$. Therefore Equation (4.24) becames

$$l([a,b])(\xi) = l(a)(\mathcal{L}_b\xi) - l(b)(\mathcal{L}_a\xi) + [a, l(b)(\xi)] - [b, l(a)(\xi)]. \tag{4.30}$$

Hence we get the following description

Proposition 4.43. *Let $A \longrightarrow M$ be a Lie algebroid. A linear multiplicative Poisson structure $\pi \in \Gamma(\wedge^2 TA)$ on A is in one to one correspondence with the following data: a vector bundle morphism $\sigma : A^* \longrightarrow TM$, a linear map $l : \Gamma(A) \longrightarrow \Gamma(\wedge^2 A)$ satisfying*

$$\begin{aligned} \rho\circ\sigma^T &= -\sigma\circ\rho^* \\ \iota_{\rho^*(\theta)}l(a) &= [a, \sigma^T(\theta)] - \sigma^T(\mathcal{L}_{\rho(a)}\theta) \\ l([a,b])(\xi) &= l(a)(\mathcal{L}_b\xi) - l(b)(\mathcal{L}_a\xi) + [a, l(b)(\xi)] - [b, l(a)(\xi)] \end{aligned}$$

Corollary 4.44. *Let $A \longrightarrow M$ be a Lie algebroid. Suppose that there exist vector bundle morphism $\sigma : A^* \longrightarrow TM$, a linear map $l : \Gamma(A) \longrightarrow \Gamma(\wedge^2 A)$ satisfying*

$$\iota_{\rho^*(\theta)}l(a) = [a, \sigma^T(\theta)] - \sigma^T(\mathcal{L}_{\rho(a)}\theta) \tag{4.31}$$

$$l([a,b])(\xi) = l(a)(\mathcal{L}_b\xi) - l(b)(\mathcal{L}_a\xi) + [a, l(b)(\xi)] - [b, l(a)(\xi)]. \tag{4.32}$$

Then there is a Lie algebroid structure on $A^* \longrightarrow M$ such that the pair (A, A^*) is a Lie bialgebroid.

Proof. We have proved the $A^* \longrightarrow TM$. Define the map $\delta : \Gamma(\wedge^\bullet A) \longrightarrow \Gamma(\wedge^{\bullet+1} A)$ by: in degree 0

$$\langle \delta(f), \xi \rangle = \mathcal{L}_{\sigma(\xi)} f \qquad \text{for} \quad f \in C^\infty(M),$$

in degree 1, we define $\delta = l$. The Equation (4.31) implies that δ is a derivation a that we can extend to all degrees. It follows that δ is the coboundary operator associated to the Lie algebroid A^*. Moreover Equation (4.32) means that

$$\delta([a,b]) = [\delta(a), b] + [a, \delta(b)].$$

Hence the pair (A, A^*) is a Lie bialgebroid (see [29], Definition 3.12 and Remark 3.14). $\square$

Appendix

A Linear vector fields

Lemma .45. *Let $X \in \Gamma_{lin}(A_{\mathcal{E}}, E)$ be a linear section covering some section $a \in \Gamma(A)$. Then the right invariant vector field $\overrightarrow{X} \in \mathfrak{X}(\mathcal{E})$ is linear and covers $\overrightarrow{a} \in \mathfrak{X}(\mathcal{G})$.*

Proof. First we show that $\overrightarrow{X}$ covers $\overrightarrow{a}$. Let $\eta \in \mathcal{E}_g$ and denote by $R^{\mathcal{E}}$ and by $R^{\mathcal{G}}$ the right multiplication in $\mathcal{E}$ and in $\mathcal{G}$, respectively. Since $Q : \mathcal{E} \longrightarrow \mathcal{G}$ is a morphism of Lie groupoids we have

$$Q(R^{\mathcal{E}}_\eta(\mu)) = Q(\mu \cdot \eta) = Q(\mu) \cdot Q(\eta) = R^{\mathcal{G}}_{Q(\eta)}(Q(\mu)) \quad \Rightarrow \quad Q \circ R^{\mathcal{E}}_\eta = R^{\mathcal{G}}_{Q(\eta)} \circ Q.$$

Then

$$\begin{aligned} (TQ)_\eta(\overrightarrow{X}_\eta) &= (TQ)_\eta((TR^{\mathcal{E}}_\eta)(X_{\bar{t}(\eta)})) = T(Q \circ R^{\mathcal{E}}_\eta)_\eta(X_{\bar{t}(\eta)}) \\ &= T(R^{\mathcal{G}}_{Q(\eta)} \circ Q)_\eta(X_{\bar{t}(\eta)}) = (TR^{\mathcal{G}}_g)(TQ(X_{\bar{t}(\eta)})) \\ &= (TR^{\mathcal{G}}_g)(a_{\mathbf{t}(g))} = \overrightarrow{a}_g. \end{aligned}$$

Now we prove the linearity: $\overrightarrow{X}_{\eta +_{\mathcal{G}} \mu} = \overrightarrow{X}_\eta +_{T\mathcal{G}} \overrightarrow{X}_\mu$. Let $\gamma(r), \xi(r) \subseteq \mathcal{E}$ curves such that

$$\begin{aligned} &\gamma(0) = \bar{1}(\bar{t}(\eta)), \quad \gamma'(0) = X_{\bar{t}(\eta)}, \quad \bar{s}(\gamma(r)) = \bar{t}(\eta) \\ &\xi(0) = \bar{1}(\bar{t}(\mu)), \quad \xi'(0) = X_{\bar{t}(\mu)}, \quad \bar{s}(\xi(r)) = \bar{t}(\mu) \end{aligned}$$

with $Q(\gamma(r)) = Q(\xi(r))$. Then

$$\begin{aligned} \overrightarrow{X}_{\eta +_{\mathcal{G}} \mu} &= (TR_{\eta+\mu})_{\bar{1}(\bar{t}(\eta+\mu))}\Big(X_{\bar{t}(\eta+\mu)}\Big) \\ &= (TR_{\eta+\mu})_{\bar{1}(\bar{t}(\eta))+\bar{1}(\bar{t}(\mu))}(X_{\bar{t}(\eta)} +_A X_{\bar{t}(\mu)}) \\ &= \frac{d}{dr}\Big|_{r=0}\Big(R_{\eta+\mu}(\gamma(r) +_{\mathcal{G}} \xi(r))\Big) \\ &= \frac{d}{dr}\Big|_{r=0}\Big((\gamma(r) +_{\mathcal{G}} \xi(r)) \cdot (\eta + \mu)\Big) \\ &= \frac{d}{dr}\Big|_{r=0}\Big(\gamma(r) \cdot \eta + \xi(r) \cdot \mu\Big) \\ &= \frac{d}{dr}\Big|_{r=0}(\gamma(r) \cdot \eta) + \frac{d}{dr}\Big|_{r=0}(\xi(r) \cdot \mu) \\ &= \overrightarrow{X}_\eta + \overrightarrow{X}_\mu. \end{aligned}$$

□

Let X be a linear section of $A_{\mathcal{E}}$ and consider its corresponding linear right invariant vector field $\overrightarrow{X} \in \mathfrak{X}(\mathcal{E})$. We denote by $\overline{D}_X \in \mathrm{Der}(\mathcal{E}^*)$ the associated derivation given by:

$$\ell_{\overline{D}_X(\varphi)} = \overrightarrow{X}(\ell_\varphi) \quad \text{for } \varphi \in \Gamma(\mathcal{E}^*).$$

This derivation is $C^\infty(M)$-linear in the following sense: if $h \in C^\infty(M)$, then the section $(h \circ q_E)X$ is a linear section of $A_{\mathcal{E}}$ over E, and its associated right invariant vector field is $\overrightarrow{(h \circ q_E)X} = \overline{t}^*(h \circ q_E)\overrightarrow{X}$. Then

$$\ell_{\overline{D}_{(h\circ q_E)X}(\varphi)} = \overrightarrow{(h \circ q_E)X}(\ell_\varphi) = \overline{t}^*(h \circ q_E)\overrightarrow{X}(\ell_\varphi) = \overline{t}^*(h \circ q_E)\ell_{\overline{D}_X(\varphi)}, \tag{.33}$$

which means $\overline{D}_{(h\circ q_E)X}(\varphi) = (h \circ q_E \circ \overline{t})\overline{D}_X$.

B Module structure on the space of linear sections

Let $(\mathcal{A}, E; A, M)$ be a VB-algebroid with core bundle C. Recall the canonical flat connection $\nabla^0 : \Gamma_{\mathrm{lin}}(\mathcal{A}, E) \times \Gamma(C) \longrightarrow \Gamma(C)$ (see (1.22)) associated to a VB-algebroid characterized by

$$S_{\nabla_X c} = [X, S_c].$$

There is a module structure on $\Gamma(\wedge^p T^*M \otimes \wedge^q A \otimes C)$ over the space $\Gamma_{\mathrm{lin}}(\mathcal{A}^*, C^*)$ given by:

$$X_a \cdot (\theta \otimes \phi \otimes c) = \mathcal{L}_{\rho(a)}\theta \otimes \phi \otimes c + \theta \otimes [a, \phi] \otimes c + \theta \otimes \phi \otimes \nabla^0_{X_a} c \tag{.34}$$

Lemma .46. *We have the following equality*

$$c_{X_a\cdot(\theta\otimes\phi\otimes c)} = \mathcal{L}_{\rho_{\mathbb{A}}(\chi_a)} c_{\theta\otimes\phi\otimes c}, \tag{.35}$$

where $\chi_a = ((Ta)^p, (R_a)^q, X_a)$ and $\rho_{\mathbb{A}}$ is the anchor map of the Lie algebroid $\mathbb{A}^{(p,q)}_{\mathcal{A}}$ (see Section 3.2).

Proof. Recall that the anchor map $\rho_{\mathbb{A}} : \mathbb{A}^{(p,q)}_{\mathcal{A}} \longrightarrow \mathbb{M}^{(p,q)}_{\mathcal{E}}$ has by components the corresponding anchor of each VB-algebroid:

$$\rho_{\mathbb{A}}((Ta)^p, (R_a)^q, X_a) = ((\rho(a)^T)^p, (H_a)^q, \rho_{\mathcal{A}^*}(X_a)).$$

Note that $\mathcal{L}_{(\rho(a)^T)^p}\theta$ where $\theta \in \Omega^p(M)$ is the usual Lie derivative of forms along vector fields, i.e., $\mathcal{L}_{\rho(a)}\theta$. Also we have

$$\mathcal{L}_{(H_a)^q} c_\phi = c_{[a,\phi]},$$

where the bracket is the Schouten bracket (see for example [10] for the previous equality and Schouten bracket). We have too that $\mathcal{L}_{\rho_{\mathcal{A}^*}(X_a)} c = \nabla^0_{X_a} c$. Then using these equalities and the Appendix in [10], the lemma follows. □

Suppose now that the VB-algebroid comes from a representation $C \longrightarrow M$ of A, i.e., we have a flat A-connections on C: $\nabla : \Gamma(A) \times \Gamma(C) \longrightarrow \Gamma(C)$. Hence the module structure on $\Gamma(\wedge^p T^*M \otimes C) = \Omega^k(M, C)$ over $\Gamma(A)$ is

$$a \cdot (\theta \otimes c) = \mathcal{L}_{\rho(a)}\theta \otimes c + \theta \otimes \nabla_a c \tag{.36}$$

Let $\omega = \theta \otimes c \in \Omega^k(M, C)$. Recall the Lie derivative operator $\mathcal{L}_a$ acting on $\Omega^k(M, C)$ defined on [14]

$$\mathcal{L}_a\omega(V) = \nabla_a\omega(V) - \sum_i \omega(v_1, \ldots, [\rho(a), v_i], \ldots, v_k)$$

for a $V = (v_1, \ldots, v_k)$ with $v_i \in TM$. Using the local expression $\omega(V) = \theta(V)c \in \Gamma(C)$, we have

$$\begin{aligned}
\mathcal{L}_a\omega(V) &= \nabla_a\omega(V) - \sum_i \omega(v_1, \ldots, [\rho(a), v_i], \ldots, v_k) \\
&= \nabla_a(\theta(V)c) - \sum_i \theta(v_1, \ldots, [\rho(a), v_i], \ldots, v_k)c \\
&= \theta(V)\nabla_a c + \mathcal{L}_{\rho(a)}\theta(V)c - \sum_i \theta(v_1, \ldots, [\rho(a), v_i], \ldots, v_k)c \\
&= \theta(V)\nabla_a c + (\mathcal{L}_{\rho(a)}\theta(V) - \sum_i \theta(v_1, \ldots, [\rho(a), v_i], \ldots, v_k))c \\
&= \theta(V)\nabla_a c + (\mathcal{L}_{\rho(a)}\theta)(V)c.
\end{aligned}$$

Hence the Lie derivative operator $\mathcal{L}_a$ acting on $\Omega^k(M, C)$ defined on [14] is the same that the modulo structure on $\Omega^k(M, C)$ over $\Gamma(A)$ defined here.

C Compatibility of multiplication

Here we will prove Lemma 3.22:

Lemma .47. 3.22 *There is an isomorphism of Lie groupoids between the Lie groupoid $\boldsymbol{s}^*(C^*) \oplus \boldsymbol{t}^*(E^*)$ obtained by the dual representation $((\Delta^0)^T, (\Delta^1)^T, \partial^T, \Omega^T)$, and the Lie groupoid $\boldsymbol{s}^*(E^*) \oplus \boldsymbol{t}^*(C^*)$ which is obtained by dualization of the VB-groupoid $\mathcal{E} = \boldsymbol{s}^*E \oplus \boldsymbol{t}^*C$. Moreover this is an isomorphism of VB-groupoids.*

First we recall the structure maps of the VB-groupoids $\mathbf{s}^*(C^*) \oplus \mathbf{t}^*(E^*)$ and $\mathbf{s}^*(E^*) \oplus \mathbf{t}^*(C^*)$. Given a representation up to homotopy $(\Delta^0, \Delta^1, \partial, \Omega)$ of $\mathcal{G}$, it induces a representation up to homotopy of $\mathcal{G}$ on the graded vector bundle $E^*_{[0]} \oplus C^*_{[1]}$ as follows:

- The quasi action $(\Delta^T)^0_g : E^*_{\mathbf{s}(g)} \longrightarrow E^*_{\mathbf{t}(g)}$ given by:

$$(\Delta^T)^0_g(\eta) := (\Delta^1_{g^{-1}})^*\eta, \quad \text{for } \eta \in E^*_{\mathbf{s}(g)}$$

- The quasi actions $(\Delta^T)^1_g : C^*_{\mathbf{s}(g)} \longrightarrow C^*_{\mathbf{t}(g)}$ given by:

$$(\Delta^T)^1_g(\xi) := (\Delta^0_{g^{-1}})^*\xi, \quad \text{for } \xi \in C^*_{\mathbf{s}(g)}$$

- The vector bundle map $\partial^T : E^* \longrightarrow C^*$ is the dual map ∂^*.
- The operator Ω^T is given by: $\xi \in C^*_{\mathbf{s}(g)}$ we have

$$\Omega^T_{(g,h)} : C^*_{\mathbf{s}(h)} \longrightarrow E^*_{\mathbf{t}(g)} \qquad \Omega^T_{(g,h)}(\xi) := (\Omega_{(h^{-1},g^{-1})})^*\xi,$$

for $(g, h) \in \mathcal{G}^{(2)}$.

Hence we get the VB-groupoid $\mathbf{s}^*(C^*) \oplus \mathbf{t}^*(E^*)$ with source and target map given by:

- $\widetilde{s}(g, \xi, \eta) = \xi$
- $\widetilde{t}(g, \xi, \eta) = \partial^*(\eta) + (\Delta^0_{g^{-1}})^*\xi$

for $\xi \in C^*_{\mathbf{s}(g)}$ and $\eta \in E^*_{\mathbf{t}(g)}$. The multiplication is defined as

$$(g_1, \xi_1, \eta_1) \cdot^1 (g_2, \xi_2, \eta_2) = (g_1g_2, \xi_2, \eta_1 + (\Delta^1_{g_1^{-1}})^*\eta_2 - \Omega_{g_2^{-1},g_1^{-1}}\xi_2),$$

under the compatibility condition

$$\widetilde{s}(g_1, \xi_1, \eta_1) = \xi_1 = \partial^*(\eta_2) + (\Delta^0_{g_2^{-1}})^*\xi_2 = \widetilde{t}(g_2, \xi_2, \eta_2) \tag{.37}$$

On the other hand the dual VB-groupoid of $\mathcal{E} = \mathbf{s}^*E \oplus \mathbf{t}^*C$ is the VB-groupoid with structure maps given by:

- The source and target maps $\widehat{s}, \widehat{t} : \mathcal{E}^* = \mathbf{s}^*(E^*) \oplus \mathbf{t}^*(C^*) \longrightarrow C^*$ are

$$\begin{aligned} \widehat{s}(g, \mu, \delta) &= \partial^*(\mu) + (\Delta^0_g)^*\delta \\ \widehat{t}(g, \mu, \delta) &= \delta \end{aligned}$$

where $\mu \in E^*_{\mathbf{s}(g)}$ and $\delta \in C^*_{\mathbf{t}(g)}$. The multiplication is

$$(g_1, \mu_1, \delta_1) \cdot^2 (g_2, \mu_2, \delta_2) = (g_1g_2, \mu_2 + (\Delta^1_{g_2})^*\mu_1 - \Omega^*_{g_1,g_2}\delta_1, \delta_1),$$

under the compatibility condition $\delta_2 = \partial^*(\mu_1) + (\Delta^0_g)^*\delta_1$.

Proof. of Lemma 3.22 Let $(g, \xi, \eta) \in \mathbf{s}^*(C^*) \oplus \mathbf{t}^*(E^*)$. The inverse of this element is (see [19]):

$$(g, \xi, \eta)^{-1} = (g^{-1}, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi)$$

Define the map $\varphi : \mathbf{s}^*(C^*) \oplus \mathbf{t}^*(E^*) \longrightarrow \mathbf{s}^*(E^*) \oplus \mathbf{t}^*(C^*)$ by

$$\varphi(g, \xi, \eta) = (g, -(\Delta^0)^T_{g^{-1}}\eta + \Omega^T_{g^{-1},g}\xi,\ \partial^T\eta + (\Delta^1)^T_g\xi).$$

We show now the compatibility of φ with the source, target and multiplication. For the source we have

$$\begin{aligned}
\widehat{s}(\varphi(g,\xi,\eta)) &= \widehat{s}(g,-(\Delta^0)^T_{g^{-1}}\eta+\Omega^T_{g^{-1},g}\xi,\ \partial^T\eta+(\Delta^1)^T_g\xi) \\
&= \partial^*(-(\Delta^1_g)^*\eta+\Omega^*_{g^{-1},g}\xi)+(\Delta^0_g)^*(\partial^*(\eta)+(\Delta^0_{g^{-1}})^*\xi) \\
&= -(\Delta^1_g\circ\partial)^*\eta+(\Omega_{g^{-1},g}\circ\partial)^*\xi+(\partial\circ\Delta^0_g)^*(\eta)+(\Delta^0_{g^{-1}}\Delta^0_g)^*\xi \\
&= (\Delta^0_{g^{-1}g})^*\xi=\xi \\
&= \widetilde{s}(g,\xi,\eta)
\end{aligned}$$

With respect to the target map

$$\begin{aligned}
\widehat{t}(\varphi(g,\xi,\eta)) &= \widehat{t}(g,-(\Delta^0)^T_{g^{-1}}\eta+\Omega^T_{g^{-1},g}\xi,\ \partial^T\eta+(\Delta^1)^T_g\xi) \\
&= \partial^T\eta+(\Delta^1)^T_g\xi \\
&= \widetilde{t}(g,\xi,\eta)
\end{aligned}$$

Now we check the compatibility of the multiplication. Let $(g_1,\xi_1,\eta_1),(g_2,\xi_2,\eta_2)\in\mathbf{s}^*(C^*)\oplus\mathbf{t}^*(E^*)$ be two composable elements. Then

$$\begin{aligned}
\varphi(g_1,\xi_1,\eta_1)\cdot^2\varphi(g_2,\xi_2,\eta_2) = \Big(&g_1g_2, \\
&-(\Delta^1_{g_2})^*\eta_2+\Omega^*_{g_2^{-1},g_2}\xi_2+(\Delta^1_{g_2})^*\big(-(\Delta^1_{g_1})^*\eta_1+\Omega^*_{g_1^{-1},g_1}\xi_1\big) \\
&-\Omega^*_{g_1,g_2}\big(\partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\xi_1\big), \\
&\partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\xi_1\Big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\varphi\big((g_1,\xi_1,\eta_1)\cdot^1(g_2,\xi_2,\eta_2)\big) &= \varphi(g_1g_2,\xi_2,\eta_1+(\Delta^1_{g_1^{-1}})^*\eta_2-\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2) \\
&= \Big(g_1g_2, \\
&\quad -(\Delta^1_{g_1g_2})^*\big(\eta_1+(\Delta^1_{g_1^{-1}})^*\eta_2-\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2\big) \\
&\quad +\Omega^*_{g_2^{-1}g_1^{-1},g_1g_2}\xi_2, \\
&\quad \partial^*\big(\eta_1+(\Delta^1_{g_1^{-1}})^*\eta_2-\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2\big)+(\Delta^0_{g_2^{-1}g_1^{-1}})^*\xi_2\Big).
\end{aligned}$$

Now we have to show that

$$\begin{aligned}
&-(\Delta^1_{g_2})^*\eta_2+\Omega^*_{g_2^{-1},g_2}\xi_2+(\Delta^1_{g_2})^*\big(-(\Delta^1_{g_1})^*\eta_1+\Omega^*_{g_1^{-1},g_1}\xi_1\big)-\Omega^*_{g_1,g_2}\big(\partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\xi_1\big) \\
&=-(\Delta^1_{g_1g_2})^*\big(\eta_1+(\Delta^1_{g_1^{-1}})^*\eta_2-\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2\big)+\Omega^*_{g_2^{-1}g_1^{-1},g_1g_2}\xi_2
\end{aligned}$$

and

$$\partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\xi_1=\partial^*\big(\eta_1+(\Delta^1_{g_1^{-1}})^*\eta_2-\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2\big)+(\Delta^0_{g_2^{-1}g_1^{-1}})^*\xi_2.$$

We start for the second one:

$$\begin{aligned}
\partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\xi_1 &= \partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*(\partial^*(\eta_2)+(\Delta^0_{g_2^{-1}})^*\xi_2) \;\;\text{by Eq (.37)}\\
&= \partial^*(\eta_1)+(\Delta^0_{g_1^{-1}})^*\partial^*(\eta_2)+(\Delta^0_{g_1^{-1}})^*(\Delta^0_{g_2^{-1}})^*\xi_2\\
\text{by Eq. (1.11)} &= \partial^*(\eta_1)+\partial^*(\Delta^1_{g_1^{-1}})^*(\eta_2)+(\Delta^0_{g_1^{-1}})^*(\Delta^0_{g_2^{-1}})^*\xi_2\\
\text{by Eq. (1.12)} &= \partial^*(\eta_1)+\partial^*(\Delta^1_{g_1^{-1}})^*(\eta_2)-\partial^*\Omega^*_{g_2^{-1},g_1^{-1}}\xi_2+(\Delta^0_{g_2^{-1}g_1^{-1}})^*\xi_2.
\end{aligned}$$

Now we show the first equality. We expand the two sides and we get

$$\begin{aligned}
(\mathrm{I}) &= -(\Delta^1_{g_2})^*\eta_2+\Omega^*_{g_2^{-1},g_2}\xi_2-(\Delta^1_{g_1}\Delta^1_{g_2})^*\eta_1+(\Omega_{g_1^{-1},g_1}\Delta^1_{g_2})^*\xi_1\\
&\quad -(\partial\Omega_{g_1,g_2})^*\eta_1-(\Delta^0_{g_1^{-1}}\Omega_{g_1,g_2})^*\xi_1\\
&= -(\Delta^1_{g_2})^*\eta_2+\Omega^*_{g_2^{-1},g_2}\xi_2-(\Delta^1_{g_1}\Delta^1_{g_2}+\partial\Omega_{g_1,g_2})^*\eta_1\\
&\quad +(\Omega_{g_1^{-1},g_1}\Delta^1_{g_2}-\Delta^0_{g_1^{-1}}\Omega_{g_1,g_2})^*\xi_1\\
(\mathrm{II}) &= -(\Delta^1_{g_1g_2})^*\eta_1-(\Delta^1_{g_1^{-1}}\Delta^1_{g_1g_2})^*\eta_2+(\Omega^*_{g_2^{-1},g_1^{-1}}\Delta^1_{g_1g_2}+\Omega_{g_2^{-1}g_1^{-1},g_1g_2})^*\xi_2
\end{aligned}$$

By Equation (1.13) we have $-(\Delta^1_{g_1}\Delta^1_{g_2}+\partial\Omega_{g_1,g_2})^*\eta_1=-(\Delta^1_{g_1g_2})^*\eta_1$ and $-(\Delta^1_{g_2}-\Delta^1_{g_1^{-1}}\Delta^1_{g_1g_2})^*\eta_2=-\Omega^*_{g_1^{-1},g_1g_2}\partial^*(\eta_2)$. By Equation (1.14) we have

$$(\Omega^*_{g_2^{-1},g_1^{-1}}\Delta^1_{g_1g_2}+\Omega_{g_2^{-1}g_1^{-1},g_1g_2})^*\xi_2=(\Delta^0_{g_2^{-1}}\Omega_{g_1^{-1},g_1g_2}+\Omega_{g_2^{-1},g_2})^*\xi_2.$$

Then, putting all together we have to show that

$$-\Omega^*_{g_1^{-1},g_1g_2}(\partial^*(\eta_2)+(\Delta^0_{g_2^{-1}})^*\xi_2)+(\Omega_{g_1^{-1},g_1}\Delta^1_{g_2}-\Delta^0_{g_1^{-1}}\Omega_{g_1,g_2})^*\xi_1=0 \tag{.38}$$

Remember that $\xi_1=\partial^*(\eta_2)+(\Delta^0_{g_2^{-1}})^*\xi_2$ (see Equation (.37)) and together with the Equation (1.14), the left hand side of (.38) is $\Omega^*_{g_1^{-1}g_1,g_2}\xi_1$, and it is zero because the operator Ω is normalized (see [19], Theorem 2.13). Then (I) = (II), and therefore, the map φ is compatible with the multiplication, and hence, it is and isomorphism of Lie groupoids. Moreover, since φ is also linear with respect to the vector bundle structure over $\mathcal{G}$, it is actually an isomorphism of VB-groupoids. □

D Dull algebroids and Dorfman connections

In this section we recall the definitions of Dull algebroids and Dorfman connections. The reference for this part is [26].

Definition .48. A *dull algebroid* is a vector bundle $Q \longrightarrow M$ endowed with an anchor, i.e. a vector bundle morphism $\rho_Q : Q \longrightarrow TM$ over the identity on M and a bracket $[,]_Q$ on $\Gamma(Q)$ with

$$\rho_Q[q_1,q_2]_Q=[\rho_Q(q_1),\rho_Q(q_2)]$$

for all $q_1, q_2 \in \Gamma(Q)$, and satisfying the Leibniz identity in both terms

$$[f_1q_1, f_2q_2]_Q = f_1f_2[q_1, q_2]_Q + f_1\rho_Q(q_1)(f_2)q_2 - f_2\rho_Q(q_2)(f_1)q_1$$

for all $f_1, f_2 \in C^\infty(M), q_1, q_2 \in \Gamma(Q)$.

Definition .49. Let $(Q \longrightarrow M, \rho_Q, [,]_Q)$ be a dull algebroid. Let $B \longrightarrow M$ be a vector bundle with a fiberwise pairing $\langle\cdot,\cdot\rangle : Q \times_M B \longrightarrow \mathbb{R}$ and a map $\mathbf{d}_B : C^\infty(M) \longrightarrow \Gamma(B)$ such that

$$\langle q, \mathbf{d}_B f\rangle = \rho_Q(q)(f)$$

for all $q \in \Gamma(Q)$ and $f \in C^\infty(M)$. Then $(B, \mathbf{d}_B, \langle\cdot,\cdot\rangle)$ is called a *pre-dual* of Q and Q and B are said to be paired by $\langle\cdot,\cdot\rangle$.

Definition .50. Let $(Q \longrightarrow M, \rho_Q, [,]_Q)$ be a dull algebroid and $(B \longrightarrow M, \mathbf{d}_B, \langle\cdot,\cdot\rangle)$ be a pre-dual of Q.

1. A *Dorfman* $(Q-)$*connection* on B is an $\mathbb{R}$-bilinear map

   $$\nabla : \Gamma(Q) \times \Gamma(B) \longrightarrow \Gamma(B)$$

   such that

   (a) $\nabla_{fq}b = f\nabla_q b + \langle q, b\rangle \mathbf{d}_B f$

   (b) $\nabla_q(fb) = f\nabla_q b + \rho_Q(q)fb$

   (c) $\nabla_q(\mathbf{d}_B f) = \mathbf{d}_B(\mathcal{L}_{\rho(q)}f)$ for all $f \in C^\infty(M), q \in \Gamma(Q), b \in \Gamma(B)$.

2. The *curvature* of ∇ is the map

   $$R_\nabla : \Gamma(Q) \times \Gamma(Q) \longrightarrow \Gamma(B^* \otimes B)$$

   defined on $q_1, q_2 \in \Gamma(Q)$ by: $R_\nabla(q_1, q_2) = \nabla_{q_1}\nabla_{q_2} - \nabla_{q_2}\nabla_{q_1} - \nabla_{[q_1,q_2]_Q}$

Recall Proposition 4.17:

Proposition .51. 4.17 *There is one-to-one correspondence between horizontal lifts* $h : \Gamma(A) \longrightarrow \Gamma(TA \oplus T^*A)$ *and* $(TM \oplus A^*)$*-Dorfman connection on* $A \oplus T^*M$

$$\Lambda : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M)$$

via the relation

$$\Lambda_{(u,\xi)}(a, \theta) = \nabla^h_{(u.\xi)}a + (0, d\langle\xi, a\rangle + \mathcal{L}_u\theta). \qquad (.39)$$

Moreover, if $(\mathfrak{L}, U; A, M)$ *be a double vector subbundle of* $(TA\oplus T^*A, TM\oplus A^*; A, M)$ *with core bundle* $K \subseteq A \oplus T^*M$*, then* h *is adapted to* $\mathfrak{L}$ *if and only if* Λ *is adapted to* $\mathfrak{L}$*.*

Proof. Let $h : \Gamma(A) \longrightarrow \Gamma_{\mathrm{lin}}(TA \oplus T^*A, TM \oplus A^*)$ be a horizontal lift. Recall that the linear section $R_a : A^* \longrightarrow T^*A$ is given by

$$R_a(\xi) = (d\ell_\xi)_a - q^*(\mathrm{d}\langle \xi, a\rangle).$$

Then

$$\begin{aligned}
-S_{\nabla^h_{(u,\xi)}(a)}(u_p, \xi_p) &= h(a)(u_p, \xi_p) - (Ta(u_p), Ra(\xi_p)) \\
&= h(a)(u_p, \xi_p) - (Ta(u_p), (\mathrm{d}\ell_\xi)_{a_p} - q^*(\mathrm{d}\langle \xi, a\rangle)) \\
&= h(a)(u_p, \xi_p) - (Ta(u_p), (\mathrm{d}\ell_\xi)_{a_p}) + (0, q^*(d\langle \xi, a\rangle)) \\
&= -(\delta_{(u,\xi)}(a))^\uparrow(a_p) + q^*(\mathrm{d}\langle \xi, a\rangle)
\end{aligned}$$

which implies that

$$\nabla^h_{(u,\xi)}(a) = \delta_{(u,\xi)}(a) - (0, \mathrm{d}\langle \xi, a\rangle) \tag{.40}$$

where

$$\delta : \Gamma(TM \oplus A^*) \times \Gamma(A) \longrightarrow \Gamma(A \oplus T^*M)$$

is defined by

$$(\delta_{(u,\xi)}(a))^\uparrow(a_p) = (Ta(u_p), (\mathrm{d}\ell_\xi)_{a_p}) - h(a)(u_p, \xi_p) \tag{.41}$$

Define the map $\Lambda : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \longrightarrow \Gamma(A \oplus T^*M)$ by

$$\Lambda_{(u,\xi)}(a, \theta) = \delta_{(u,\xi)}(a) + (0, \mathcal{L}_u\theta) = \nabla^h_{(u.\xi)}a + (0, \mathrm{d}\langle \xi, a\rangle + \mathcal{L}_u\theta). \tag{.42}$$

Then using Theorem 4.1 in [26] and the relation (.42) follows the first part of the proposition. The second part of the proposition follows by Proposition 4.12 in [26]. $\square$

Bibliography

[1] ARIAS ABAD, C., CRAINIC, M., - *Representations up to homotopy and Bott's spectral sequence for Lie groupoids*, Advances in Mathematics, Vol. 248, pp. 416 - 452, 2013.

[2] ARIAS ABAD, C., CRAINIC, M., - *Representation up to homotopy of Lie algebroids*, J. Reine Angew Math., 663:91-126, 2012.

[3] ARIAS ABAD, C., Schätz, F., - *Deformations of Lie brackets and representations up to homotopy*, Indagationes Mathematicae, Vol. 22, 1-2, 2011, pp. 27-54.

[4] BATES, S., WEINSTEIN, A. - *Lectures on the geometry of quantization*, Berkeley Mathematics, Lecture Notes, Vol. 10, 1997.

[5] BRAHIC, O., CABRERA, A., ORTIZ, C., - *Obstruction to the integrability of VB-algebroids*, arXiv:1403.1990v2, 2014.

[6] BURSZTYN, H., *A brief introduction to Dirac manifolds*, Geometric and Topological Methods for Quantum Field Theory. Proceedings of the 2009 Villa de Leyva Summer School, Cambridge University Press, Cambridge (2013): 4-38.

[7] BURSZTYN, H., CABRERA, A. - *Multiplicative forms at the infinitesimal level*, Math. Ann. 353(3):663-705, 2012.

[8] BURSZTYN, H., CABRERA, A., del HOYO, M., - *Vector bundles over Lie groupoids and algebroids*, Advances in Mathematics, Vol. 290, February 2016, Pages 163-207.

[9] BURSZTYN, H., CABRERA, A., ORTIZ C., - *Linear and multiplicative 2-forms*, Lett. in Math. Phys., Vol. 90, 2009, Pages 59-83

[10] BURSZTYN, H., DRUMMOND, T., - *Lie theory of multiplicative tensors*. Work in progress.

[11] CANNAS DA SILVA, A., WEINSTEIN, A. - *Geometric models for noncommutative algebras*, Berkeley Mathematics, Lecture Notes, Vol. 10, 1999.

[12] COSTE, A., DAZORD, P., WEINSTEIN, A. - *Groupoïdes symplectiques*, Dept. Math. Univ. Claude-Bernard Lyon I, 1987.

[13] CRAINIC, M., FERNANDES, R. - *Integrability of Lie brackets*, Annals of Mathematics, 157 (2003), 575-620.

[14] CRAINIC, M., SALAZAR, M., STRUCHINER, I. - *Multiplicative forms and Spencer operators*, Mathematische Zeitschrift 279 (3-4), 939-979.

[15] CRAINIC, M., STRUCHINER, I. - *On the linearization theorem for proper Lie groupoids*, Annales scientifiques de l'École Normale Supérieure, 2013.

[16] CONNES, A. - *Noncommutative geometry*, Academic Press, San Diego, 1994.

[17] DRUMMOND, T., JOTZ, M., ORTIZ, C., - *VB-algebroid morphisms and representations up to homotopy*, Differential Geometry and its Applications, 40. pp. 332-357.

[18] GRABOWSKI, J., ROTKIEWICZ,M., - *Higher vector bundles and multi-graded symplectic manifolds*, Journal of Geometry and Physics 59.9 (2009): 1285-1305.

[19] GRACIA-SAZ, A., MEHTA, R., - *VB-groupoids and representation theory of Lie groupoids*, arXiv:1007.3658v5, 2015.

[20] GRACIA-SAZ, A., MEHTA, R., - *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math., 223(4):1236-1275, 2010.

[21] HAEFLIGER, A., - *Groupoides d'holonomie et espaces classifiants*, Astérisque 116 (1984), 70-97

[22] HAWKINGS, E., - *A groupoid approach to quantization*, J. Symplectic Geometry. 6:61-125, 2008.

[23] del HOYO, M. - *Lie groupoids and their orbispaces*, Portugaliae Mathematica, Vol. 70, Issue 2, 2013, pp. 161-209.

[24] IGLESIAS, D., GENGOUX, L., XU, P., - *Universal lifting theorem and quasi-Poisson groupoids*, J. Eur. Math. Soc., Volume 14, Issue 3, 2012, pp. 681-731.

[25] JOTZ LEAN, M., - *Dirac groupoids and Dirac bialgebroids*, arXiv:1403.2934v2, 2015.

[26] JOTZ LEAN, M., - *Dorfman connections and Courant algebroids*, arXiv:1209.6077v2, 2015.

[27] JOTZ, M., ORTIZ, C.- *Foliated groupoids and infinitesimal ideal system*, Indagationes Mathematicae (2014), Vol. 25, Number 5, 1019-1053.

[28] LAURENT-GENGOUX, C., STIENON, M., XU, P., *Integration of holomorphic Lie algebroids*, Math. Ann. (2009) 345: 895.

[29] LAURENT-GENGOUX, C., STIENON, M., XU, P., *Lectures on Poisson groupoids*, Geometry and Topology Monographs 17, (2011), 473-502.

[30] LAURENT-GENGOUX, C., TU, J., L., XU, P., - *Chern-Weil map for principal bundles over groupoids*, 2008.

[31] MACKENZIE, K., C,. H,. - *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids*, arXiv:0611799v2, 2006.

[32] MACKENZIE, K., C,. H,. - *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Notes Series, 213. Cambridge Univerity Press, Cambridge, 2005.

[33] MACKENZIE, K., XU, P., - *Lie bialgebroids and Poisson groupoids*, Duke Math. J. Volume 73, **2**, 415-452, 1994.

[34] MOERDIJK, I., - *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001) Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 205-222.

[35] MOERDIJK, I., - *Classifying topos and foliations*, Ann. Inst. Fourier. Grenoble 41 (1991), 189-209.

[36] ORTIZ, C., - *Multiplicative Dirac structures*, Pacific J. Math. 266 (2013) 329-365.

[37] PRADINES, J., - *Remarque sur le groupoïde cotangent de Weinstein-Dazord*, C. R. Acad. Sci. Paris Sér, Math., 306(13):557-560, 1988.

[38] RENAULT, J., - *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Mathematics 793. Springer-Verlag, 1980.

[39] VAN EST, W., T., - *Group cohomology and Lie algebra cohomologyin Lie groups, I, II*, Proc. Kon. Ned. Akad., **56**, (1953), 484-504.

[40] VAN EST, W., T., - *On the algebraic cohomology concepts in Lie groups, I, II* Proc. Kon. Ned. Akad., **58**, (1955), 225-233, 286-294.

[41] WEINSTEIN, A., - *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. (1987)

[42] WEINSTEIN, A., - *Coisotropic calculus and Poisson groupoids*, J. of Math. Soc. of Japan. **40** (1988), 705-727.

[43] WEINSTEIN, A., - *Linearization of regular proper groupoids*, J. Inst. Math. Jussieu 1 (2002), N 3, 493-511.

[44] WEINSTEIN, A., XU, P., - *Extensions of symplectic groupoids and quantization*, J. Reine Angew. Math. **417** (1991), 159-189.

[45] ZUNG, N., *Proper groupoids and momentum maps: linearization, affinity and convexity*, Ann. Sci. cole Norm. Sup. 39, (2006), no. 5, 841-869.