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Geometric View of Conformally Invariant Equations

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This thesis is dedicated
to my family.

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Abstract

This work is about conformally invariant equations from a geometric point of view. In other words, given a solution to an elliptic conformally invariant equation in a subdomain of the sphere, following ideas of Espinar-Gálvez-Mira, we construct an elliptic hypersurface in the Hyperbolic space. We can related analytic conditions of the solution to the conformally invariant elliptic equation and the geometry of the hypersurface.

In this work we show a non-existence theorem for degenerate elliptic problems for conformal metrics on the closed hemisphere $\overline{\mathbb{S}^m_+}$ with minimal boundary. Using the work of Fabian M. Spiegel [46], we can generalize the above result to simply-connected locally conformally flat manifolds with boundary. On the compact annulus $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$, we prove a uniqueness result for degenerate problem with minimal boundary. We prove a non-existence theorem for degenerate problems on the compact annulus $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$, under the hypothesis that there is a solution on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ that satisfies certain property. We show that a solution for elliptic problems of conformal metrics on the punctured domain $\overline{\mathbb{S}^m} \setminus \{\mathbf{n}\}$ with minimal boundary is rotationally invariant. For the non-degenerate case on the punctured domain $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ we have that solutions with minimal boundary are rotationally invariant.

Keywords: Conformally invariant equations, geometric methods, Maximum principle, non-existence, rotationally invariant metrics, horospherically concave hypersurfaces, non-degenerate elliptic problems, degenerate elliptic problems.

Resumo

Este trabalho é sobre equações conformemente invariantes do ponto de vista geométrico. Em outras palavras, dada uma solução de uma equação conformemente elíptica num subdomínio da esfera, seguindo as ideias de Espinar-Gálvez-Mira, podemos construir uma superfície elíptica no espaço hiperbólico. Podemos relacionar condições analíticas da solução de uma equação elíptica conformemente invariante e a geometria da hipersuperfície.

Neste trabalho mostramos um teorema de não existência para problemas elípticos de métricas conformes no hemisfério fechado $\overline{\mathbb{S}^m_+}$ com bordo mínimo. Usando o trabalho de Fabian M. Spiegel [46], podemos generalizar o resultado acima para variedades riemannianas simplesmente conexas, conformemente planas localmente com bordo. No anel compacto, provamos um resultado de unicidade para problemas degenerados com bordo mínimo. Provamos um teorema de não existência para problemas degenerados no anel compacto $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$, sob a hipótese de que existe uma solução em $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ que satisfaz certa propriedade. Mostramos que uma solução para o problemas elípticos de métricas conformes no domain $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ com bordo mínimo é rotacionalmente invariante. Para o caso não degenerado no domain $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ temos que as soluções com bordo mínimo são rotacionalmente invariantes.

Palavras-chave: Equações conformemente invariantes, métodos geométricos, princípio do máximo, não existência, métricas rotacionalmente invariantes, hipersuperfícies horoesfericamente côncavas, problemas elípticos não degenerados, problemas elípticos degenerados.

Contents

Introduction

Let $(\mathscr{M}, g_0)$ be a compact Riemannian manifold without boundary and consider $g = e^{2\rho} g_0$, $\rho \in C^\infty(\mathscr{M})$, a conformal metric. The *Yamabe Problem* is to find a conformal metric $g = e^{2\rho} g_0$ such that the scalar curvature of $(\mathscr{M}, g)$ is constant. Yamabe [49] claimed the solution to such problem, but the proof given by himself had a mistake. This problem got the attention of the community and it was solved by a series of works by mathematicians as Trudinger [48], Aubin [1] and finally Schoen [43], who gave a complete answer to the problem.

When the manifold $(\mathscr{M}, g_0)$ is complete but not compact, the existence of a conformal metric solving the Yamabe Problem does not hold in general, as we can see in the work of Zhiren [50].

When the manifold $(\mathscr{M}, g_0)$ is compact with boundary, the Yamabe Problem with Boundary is to find a conformal metric $g = e^{2\rho} g_0$ such that the scalar curvature is constant and the mean curvature of the boundary of $\mathscr{M}$ is constant. This line was started by J. Escobar [14, 15, 16, 17] and continued by F.C. Marques [38] among others. We will also refer to this problem as the *Escobar Problem*.

Regarding to the existence of solutions, J. Escobar proved the following

Theorem [17]. *Let $\Omega \subset \mathbb{R}^m$, $m > 6$, be a bounded domain with smooth boundary. There exists a smooth metric g conformally related to the Euclidean metric such that the scalar curvature of g is zero and the mean curvature of the boundary with respect to the metric g is (positive) constant.*

Also, J. Escobar proved:

Theorem [15]. *Any bounded domain in a Euclidean space $\mathbb{R}^m$, with smooth boundary and $m \geq 3$, admits a metric conformal to the Euclidean metric having (non-zero) constant scalar curvature and minimal boundary.*

Regarding to the classification of solutions to the Escobar Problem, in the case that $(\mathscr{M}, g_0)$ is the closed Euclidean ball $\overline{\mathbb{B}^m}$, J. Escobar showed that the solution to the Yamabe Problem with Boundary must have constant sectional curvature. Even, he proved that the space of solutions in the Euclidean ball is empty when the scalar curvature is zero and the mean curvature is a non-positive constant [14].

The existence of solutions to the Yamabe problem on non-compact manifolds $(\mathscr{M}, g_0)$ with compact boundary is proved for a large class of manifolds in the work of F. Schwartz [45]. He proved that the

Riemannian manifolds that are *positive* and their ends are *large* have a conformal metric zero of scalar curvature and constant mean curvature on its boundary (see [45] for details). Even more, F. Schwartz proved the following:

> **Theorem [45].** *Any smooth function f on $\partial\mathscr{M}$ can be realized as the mean curvature of a complete scalar flat metric conformal to g_0.*

The Yamabe Problem opened the door to a rich subject in the last few years: *the study of conformally invariant equations*. More precisely, *given a smooth functional $f(x_1,\ldots,x_m)$, does there exist a conformal metric $g = e^{2\rho}g_0$ on $\mathscr{M}$ such that the eigenvalues λ_i of its Schouten tensor satisfies $f(\lambda_1,\ldots,\lambda_m) = c$ on $\mathscr{M}$?*

Given $(\mathscr{M}, g)$ a Riemannian manifold, for $m \geq 3$, the Schouten tensor of g is given by

$$\mathrm{Sch}(g) := \frac{1}{m-2}\left(\mathrm{Ric}(g) - \frac{\mathrm{Scal}(g)}{2(m-1)}g\right)$$

where $\mathrm{Ric}(g)$ and $\mathrm{Scal}(g)$ are the Ricci tensor and the scalar curvature function of g respectively.

Note that, when $f(x_1,\ldots,x_m) = x_1 + \cdots + x_m$ we have the Yamabe Problem. Of special interest is when we consider $f(\lambda) \equiv \sigma_k(\lambda)^{1/k}$, $\lambda = (\lambda_1,\ldots,\lambda_m)$, where $\sigma_k(\lambda)$ is the $k-$th elementary symmetric polynomial of its arguments $\lambda_1,\ldots,\lambda_m$ and set it to be a constant, i.e., $\sigma_k(\lambda) =$ constant, such problem is known as the σ_k-Yamabe Problem. This is an active research topic and has interactions with other fields as Mathematical General Relativity [5, 25].

Interesting problems arise in this context of conformally invariant equations. One of them is the classification of complete conformal metrics satisfying a Yamabe type equation on a subdomain of the sphere, in the line of Y.Y. Li [29, 30]. Also, it is interesting to find non-trivial solutions to conformal metrics on subdomains of the sphere prescribing the scalar curvature in the interior, or other elliptic combination of the Schouten tensor, and the mean curvature of the boundary, such problem is related to the Min-Oo conjecture when we consider the scalar curvature inside. S. Brendle, F.C. Marques and A. Neves [4] showed the existence of such non-trivial metric in the hemisphere, however such metric is not conformal to the standard one on the sphere. In other words, could one find conditions on the interior and the boundary that imply that such conformal metric is unique (see [36])? In this work we will focus in the case $\mathscr{M}$ is the $m-$dimensional sphere $\mathbb{S}^m$ or a subdomain of it.

Let us explain in more detail the meaning of a fully non-linear conformally invariant elliptic equation. Originally, these type of equations are second order elliptic partial differential equations on $\mathbb{R}^m$. The problem is to find a function $u > 0$ satisfying an identity of the type

$$\mathscr{F}(\cdot, u, \nabla u, \nabla^2 u) = c,$$

where c is a constant.

Such kind of equation is called *conformally invariant* if for all Möbius transformation ψ in $\mathbb{R}^m$ and any positive function $u \in C^2(\mathbb{R}^m)$, it holds

$$\mathscr{F}\left(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi\right) = \mathscr{F}\left(\cdot, u, \nabla u, \nabla^2 u\right) \circ \psi, \tag{1}$$

where u_ψ is defined by

$$u_\psi := |J\psi|^{\frac{m-2}{2m}} u \circ \psi,$$

and $J\psi$ is the Jacobian of ψ. For more details see [33].

One can check that if there is a smooth positive function $u : \mathbb{R}^m \to \mathbb{R}$ such that

$$\mathscr{F}\left(\cdot, u, \nabla u, \nabla^2 u\right) = c = cte,$$

then, from (1), we have that

$$\mathscr{F}\left(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi\right) = c$$

for any Möbius transformation in $\mathbb{R}^m$.

Aobing Li and YanYan Li proved a fundamental relation between solutions of this type of equations and the eigenvalues of the Schouten tensor of a conformal metric related to such solution. Specifically:

Theorem [33]. *Let $\mathscr{F}(\cdot, u, \nabla u, \nabla^2 u)$ be conformally invariant on $\mathbb{R}^m$. Then*

$$\mathscr{F}\left(\cdot, u, \nabla u, \nabla^2 u\right) = \mathscr{F}\left(0, 1, 0, -\frac{m-2}{2} A^u\right),$$

where

$$A^u := -\frac{2}{m-2} u^{-\frac{m+2}{m-2}} \nabla^2 u + \frac{2m}{(m-2)^2} u^{-\frac{2m}{m-2}} \nabla u \otimes \nabla u - \frac{2}{(m-2)^2} u^{-\frac{2m}{m-2}} |\nabla u|^2 I$$

and I is the $m \times m$ identity matrix. Moreover, $\mathscr{F}(0,1,0,\cdot)$ is invariant under orthogonal conjugation, i.e.,

$$\mathscr{F}\left(0, 1, 0, -\frac{m-2}{2} O^{-1} A O\right) = \mathscr{F}\left(0, 1, 0, -\frac{m-2}{2} A\right) \quad \forall A \in \mathscr{S}^{m\times m},\ O \in O(n),$$

where $\mathscr{S}^{m\times m}$ is the set of $m \times m$ symmetric matrices .

Thus the behavior of $\mathscr{F}\left(\cdot, u, \nabla u, \nabla^2 u\right)$ is determined by the matrix A^u, such matrix is nothing but the Schouten tensor of the conformal metric $g = u^{\frac{4}{n-2}} g_{Eucl}$. Then, in order to define a *conformally invariant equation*, we use functions $F \in C^1(U) \cap \in C^O\left(\overline{U}\right)$, where U is an open subset of $\mathscr{S}^{m\times m}$, such that the following conditions hold:

1. for all $O \in O(m)$: $O^{-1} A O \in U$ for all $A \in U$,

2. for all $t > 0$: $tA \in U$ for all $A \in U$,

3. for all $P \in \mathscr{P}$: $P \in U$, where $\mathscr{P} \subset \mathscr{S}^{m\times m}$ is the set of $m \times m$ positive definite symmetric matrices,

4. for all $P \in \mathscr{P}$: $A + P \in U$ for all $A \in U$,

5. $0 \in \partial U$.

Also, the second order differential equation will be *elliptic* if the function F satisfies

1. for all $O \in O(m)$: $F\left(O^{-1}AO\right) = F(A)$ for all $A \in U$,
2. $F > 0$ in U,
3. $F|_{\partial U} = 0$,
4. for every $M \in U$:
$$\left(\frac{\partial F}{\partial M_{ij}}\right) \in \mathscr{P}.$$

The above conditions on (F,U) allow us to simplify the function to a functional acting on the eigenvalues of the Schouten tensor, i.e., on the eigenvalues of A^u. In order to make this explicit, let us define the following subsets:

$$\begin{aligned}\Gamma_m =& \{x \in \mathbb{R}^m : x_i > 0, i = 1, \ldots, m\},\\ \Gamma_1 =& \{x \in \mathbb{R}^m : x_1 + \cdots + x_m > 0\}.\end{aligned}$$

Let $\Gamma \subset \mathbb{R}^m$ be a symmetric open convex cone and $f \in C^1(\Gamma) \cap C^0\left(\overline{\Gamma}\right)$ such that

1. $\Gamma_m \subset \Gamma \subset \Gamma_1$,
2. f is symmetric,
3. $f > 0$ in Γ,
4. $f|_{\partial\Gamma} = 0$,
5. f is homogeneous of degree 1,
6. for all $x \in \Gamma$ it holds $\nabla f(x) \in \Gamma_m$.

Now, we will see how to obtain the open set $U \subset \mathbb{R}^m$ and the function $F : \overline{U} \to \mathbb{R}$ satisfying the above properties from the data Γ and f. The pair (f,Γ) is called *elliptic data*. From $f : \overline{\Gamma} \to \mathbb{R}$ we define

$$U = \left\{A \in \mathscr{S}^{m\times m} : \lambda(A) \in \Gamma\right\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_m)$ are the eigenvalues of A. Since Γ is symmetric it is well defined. Also we define

$$F(A) = f(\lambda(A)).$$

Observe that the function $F : U \to \mathbb{R}$ is in $C^1(U)$ and it can be continuously extended to $\overline{U}$ such that $F|_{\partial U} = 0$. Then, this function $F : \overline{U} \to \mathbb{R}$ and the set U satisfy the properties listed above.

Hence, the problem with elliptic data (f,Γ) for conformal metrics in a domain $\Omega \subset \mathbb{S}^m$ is to find a conformal metric $g = e^{2\rho} g_0$ to the standard metric g_0 such that

$$f(\lambda(g)) = c \quad \text{in} \quad \Omega,$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$ and c is a constant. We can also distinguish two cases, when $c > 0$ is a positive constant, without loss of generality we can consider $c = 1$, the problem is called *non-degenerate*. When the constant satisfies $c = 0$, the problem is called *degenerate*.

By the stereographic projection, any domain of $\mathbb{R}^m$ corresponds to a domain in $\mathbb{S}^m$. Moreover, the stereographic projection is conformal, hence, any conformal equation in a domain of $\mathbb{R}^m$ can be seen as a conformal equation in the corresponding domain in $\mathbb{S}^m$, and vice-versa. Therefore, henceforth we will consider conformally invariant equations in subdomains of the sphere $(\mathbb{S}^m, g_0)$ endowed with its standard metric.

Now, take $g = e^{2\rho} g_0$ on $\Omega \subseteq \mathbb{S}^m$. The Yamabe problem for $\mathrm{Scal}(g) = 1$ and $h(g) = c$, where $h(g)$ is the boundary mean curvature with respect to the outward unit normal vector field, is equivalent to find a smooth function ρ on Ω such that

$$\begin{cases} \lambda_1 + \cdots + \lambda_m = \frac{1}{2(m-1)}, & \text{in} \quad \Omega, \\ h(g) = c, & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Posed in this form, problem (2) can be generalized to other functions of the eigenvalues of the Schouten tensor. For instance, one may consider the σ_k-Yamabe problem on $\mathbb{S}^m_+$ considering the k-symmetric function of the eigenvalues of the Schouten tensor [40, 41]. In this work we are interested in the fully nonlinear case of this problem, in the line opened by A. Li and Y.Y. Li [35]. Namely, given (f,Γ) an elliptic data and, $b \geq 0$ and $c \in \mathbb{R}$, find $\rho \in C^\infty(\overline{\mathbb{S}^m_+})$ so that $g = e^{2\rho} g_0$ is a solution of the problem

$$\begin{cases} f(\lambda(g)) = b, \quad \lambda(g) \in \Gamma \text{ in } \mathbb{S}^m_+, \\ \\ \qquad h(g) = c, \quad \text{on } \partial\mathbb{S}^m_+. \end{cases} \tag{3}$$

M.P. Cavalcante and J.M. Espinar [7] have shown by geometric methods that

Theorem [7]. *If $g = e^{2\rho} g_0$ is a conformal metric on $\overline{\mathbb{S}^m_+}$ that satisfies*

$$\begin{cases} f(\lambda(g)) & = 1, \quad \mathit{in} \quad \mathbb{S}^m_+, \\ \quad h(g) & = c, \quad \mathit{on} \quad \partial\mathbb{S}^m_+, \end{cases}$$

then, there is a conformal diffeomorphism $\Phi : \mathbb{S}^m \to \mathbb{S}^m$ such that $g = \Phi^*\left(g_0\big|_{\overline{\mathbb{S}^m_+}}\right)$.

Using analytic methods, A. Li and Y.Y. Li [35] proved the result above. Nevertheless, M.P. Cavalcante and J.M. Espinar went further and they dealt with annular domains, as J. Escobar did [14] for the scalar curvature, in the fully nonlinear elliptic case. Let us denote by $\mathbf{n} \in \mathbb{S}^m_+ \subset \mathbb{S}^m$ the north pole and let $r < \pi/2$. Denote by $B_r(\mathbf{n})$ the geodesic ball in $\mathbb{S}^n$ centered at $\mathbf{n}$ of radius r. Note that, by the choice of r, $\partial \mathbb{S}^m_+ \cap \partial B_r(\mathbf{n}) = \emptyset$.

Denote by $\mathbb{A}(r) = \mathbb{S}^m_+ \setminus \overline{B_r(\mathbf{n})}$ the annular region determined by $\mathbb{S}^m_+$ and $B_r(\mathbf{n})$. Note that the mean curvature of $\partial B_r(\mathbf{n})$ with respect to g_0 and the inward orientation along $\partial \mathbb{A}(r)$ is a constant $h(r)$ depending only on r. Let us consider the problem of finding a conformal metric on $\mathbb{A}(r)$ satisfying an elliptic condition in the interior and whose boundary components $\partial B_r(\mathbf{n})$ and $\partial \mathbb{S}^m_+$ are minimal.

In other words, given (f, Γ) an elliptic data, find $\rho \in C^\infty(\mathbb{A}(r))$ so that the metric $g = e^{2\rho} g_0$ satisfies

$$\begin{cases} f(\lambda(g)) = 1, & \text{in} \quad \mathbb{A}(r), \\ \\ \quad h(g) = 0, & \text{on} \quad \partial B_r(\mathbf{n}) \cup \partial \mathbb{S}^m_+. \end{cases} \tag{4}$$

In the above situation, M.P. Cavalcante and J.M. Espinar obtained:

Theorem [7]. *Let $\rho \in C^\infty(\overline{\mathbb{A}(r)})$ be a solution to* (4)*. Then, $g = e^{2\rho} g_0$ is rotationally symmetric metric on $\overline{\mathbb{A}(r)}$.*

This work is organized as follows. In Chapter 1, we establish the preliminary results from differential geometry necessary along this work. There, we review the different models of the Hyperbolic Space and its isometries. Also we explore the Hyperbolic Space as a conformally compact manifold and, in particular, we study its conformal infinity. Then, we recall the definition of the Schouten tensor of a conformal metric to the standard metric on the sphere, a capital object in this work. To close this chapter, we define the notion of elliptic data in the context of elliptic problems for conformal metrics on domains of the sphere and, the notion elliptic data in the context elliptic problems for hypersurfaces of the Hyperbolic space. Most of this chapter can be tracked down from the references [2, 3, 8, 19, 47].

Chapter 2 is devoted to the local relationship between horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$ and conformal metrics on $\mathbb{S}^m$. We begin by giving the definition of the Hyperbolic Gauss map for an oriented immersed hypersurface in $\mathbb{H}^{m+1}$. In such definition, we use the boundary at infinity of the Hyperbolic space, also called ideal boundary of $\mathbb{H}^{m+1}$, that is, the sphere $\mathbb{S}^m$. There are sufficient and necessary conditions for the hyperbolic Gauss map to be a local diffeomorphism. One of these conditions is related to the regularity of the light cone map of an oriented hypersurface that will be defined in Subsection 2.1.2. Others conditions are related to the principal curvatures of the given oriented hypersurface.

Then, we define one of the important objects in our study, horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$. These hypersurfaces are oriented and they have the property that its Hyperbolic Gauss map is a local diffeomorphism. The importance of this class of hypersurfaces is that, locally, we can give a conformal metric over the image of the Hyperbolic Gauss map (conformal to the standard metric g_0 on the sphere $\mathbb{S}^m$). Suppose that $g = e^{2\rho} g_0$ is this conformal metric, $\rho \in C^\infty(\Omega)$, where Ω is a small open set that is contained in the image of the Hyperbolic Gauss map, the function ρ has a geometric interpretation that is related to tangent horospheres to the original hypersurface. In the Poincaré ball model, ρ is the signed hyperbolic distance between the tangent horosphere and the origin of the Poincaré ball model.

We have to recall now the Local Representation Theorem:

Local Representation Theorem [19]. *Let $\phi : \Omega \subseteq \mathbb{S}^m \longrightarrow \mathbb{H}^{m+1}$ be a piece of horospherically concave hypersurface with Gauss map $G(x) = x$. Then, it holds*

$$\phi = \frac{e^{\rho}}{2}\left(1 + e^{-2\rho}\left(1 + |\nabla\rho|^2\right)\right)(1,x) + e^{-\rho}(0, -x + \nabla\rho).$$

Moreover, the eigenvalues λ_i of the Schouten tensor of the horospherical metric $\hat{g} = e^{2\rho} g_0$ and the principal curvatures κ_i of ϕ are related by

$$\lambda_i = \frac{1}{2} - \frac{1}{1+\kappa_i}.$$

Conversely, given a conformal metric $\hat{g} = e^{2\rho} g_0$ defined on a domain of the sphere $\Omega \subseteq \mathbb{S}^m$ such that the eigenvalues of its Schouten tensor are all less than $1/2$, the map ϕ given by (2.7) *defines an immersed, horospherically concave hypersurface in $\mathbb{H}^{m+1}$ whose Gauss map is $G(x) = x$ for $x \in \Omega$ and whose horospherical metric is the given metric $\hat{g}$.*

The Local Representation Theorem says that the function ρ is all that we need to recover the original hypersurface. Such theorem is of great importance, because we can obtain horospherically concave hypersurfaces with injective Gauss map from conformal metrics on domains Ω of the sphere $\mathbb{S}^m$ if we impose certain conditions. This conformal metric is called the horospherical metric of the horospherically concave hypersurface in $\mathbb{H}^{m+1}$.

In Section 2.4, we study how isometries in the Hyperbolic space $\mathbb{H}^{m+1}$ affect the horospherical metric, more precisely, how the horosherical metric of the hypersurface changes when we apply an isometry to this hypersurface. In particular, in Section 2.5, if the horospherically concave hypersurface is invariant under an isometry in $\mathbb{H}^{m+1}$ then the associated horospherical metric is invariant under the conformal diffeomorphism of $\mathbb{S}^m$ associated to the isometry, and vice-versa.

We finalize this chapter introducing the elliptic problems for conformal metric on domains on the sphere $\mathbb{S}^m$, elliptic problems for hypersurfaces in the hyperbolic space $\mathbb{H}^{m+1}$ and how they are related under the Local Representation Theorem.

We see how a horospherically concave hypersurface in $\mathbb{H}^{m+1}$ gives rise to a (locally) well defined conformal metric on a subdomain on $\mathbb{S}^m$. Also, such metric is global if we assume that the Hyperbolic Gauss map is injective. So, in Chapter 3 we study the opposite, that is, given a subdomain $\Omega \subset \mathbb{S}^m$ and $\rho \in C^{\infty}(\Omega)$, consider the conformal metric $g = e^{2\rho} g_0$, then the question is: what can we say about the hypersurface given by the representation formula?

It is known [3, 19] that if we impose certain conditions on the given conformal metric $g = e^{2\rho} g_0$, we have a horospherically concave hypersurface with injective map Gauss. Moreover, we can obtain that such horospherically concave hypersurface is proper. Specifically,

Theorem 3.1. *Given $\rho \in C^1(\Omega)$, the map $\phi : \Omega \to \mathbb{H}^{m+1}$ is proper if, and only if, $|\rho|_{1,\infty}(x) \to \infty$ when $x \to p$, for every $p \in \partial\Omega$.*

Using this theorem, we can give a condition on a complete conformal metric that guaranties that the associated map is proper.

Theorem 3.3. *Let $g = e^{2\rho} g_0$ be a complete metric on Ω, such that $\sigma = e^{-\rho}$ is the restriction of a continuous function that is defined on $\overline{\Omega}$. Then $\phi : \Omega \to \mathbb{H}^{m+1}$ is a proper map.*

In the following, we make use of the parallel flow of a horospherically concave hypersurface, this flow is defined using the negative of the orientation of the hypersurface. More precisely, let η be the normal vector field of ϕ, that is the orientation of ϕ, then for every $t > 0$, we define the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ as the map that we get from ϕ using the vector field $-\eta$ in a time t. More precisely,

$$\phi_t(x) = \gamma(t, \phi(x), -\eta(x)) \quad \forall x \in \Omega,$$

where $\gamma(\cdot, \phi(x), -\eta(x))$ is the geodesic in the Hyperbolic space $\mathbb{H}^{m+1}$ passing through $\phi(x)$ and has velocity $-\eta(x)$ at that point.

In fact, the map ϕ_t is a horospherically concave hypersurface in the Hyperbolic space $\mathbb{H}^{m+1}$ for every $t > 0$. The horospherically metric of $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is the conformal metric $g_t = e^{2t} g$, where g is the horospherical metric of ϕ. It is remarkable that the property of properness is invariant under the parallel flow.

Proposition 3.4. *Assume that $\phi : \Omega \to \mathbb{H}^{m+1}$ is proper, then $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is also proper for every $t \in \mathbb{R}$.*

Using the Local Representation Theorem we can say that the horospherically concave hypersurfaces that we get using the parallel flow of an horospherically concave hypersurface correspond to dilations of the horospherical metric of the original horospherically concave hypersurface.

Also, we will see that if we impose some extra conditions on the conformal metric, then we get embeddedness of horospherically concave hypersurfaces using the parallel flow.

Theorem 3.6. *Let $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$ be such that $\sigma = e^{-\rho} \in C^\infty(\Omega \cup \mathscr{V}_1)$ satisfies:*

1. *$\sigma \cdot \sigma$ can be extended to a $C^{1,1}$ function on $\overline{\Omega}$.*
2. *$\langle \nabla\sigma, \nabla\sigma \rangle$ can be extended to a Lipschtiz function on $\overline{\Omega}$.*

Then, there is $t_0 > 0$ such that for all $t > t_0$ the map $\varphi_t : \Omega \cup \mathscr{V}_1 \to \mathbb{H}^{m+1}$ associated to $\rho_t = \rho + t$ is an embedded horospherically concave hypersurface.

In Section 4 of Chapter 3, we will study how conditions on the boundary of a complete conformal metric (cf. Definition 3.5) influence the boundary of the associated horospherically concave hypersurface. We begin with geodesic balls of $\mathbb{S}^m$ and we identify their boundaries with ideal boundaries of totally geodesic hypersurfaces in $\mathbb{H}^{m+1}$, that is, given a geodesic ball of the sphere $\mathbb{S}^{m+1}$, its boundary is the ideal boundary of a totally geodesic hypersurface in the Hyperbolic space $\mathbb{H}^{m+1}$ and the converse is true. More specifically,

Proposition 3.9. *The ideal boundary of a totally geodesic hypersurface of $\mathbb{H}^{m+1}$ given by $E(a,0)$, $a=(a_0,\bar{a})$, $\ll a,a \gg = 1$, is the boundary of a geodesic ball $B_r(p)$ of $\mathbb{S}^m$, where $p=\frac{1}{|\bar{a}|}\bar{a}$ and $r\in(0,\pi)$ satisfies $\cot(r)=a_0$. Reciprocally, given the boundary of a geodesic ball $\partial B_r(p)\subset\mathbb{S}^m$, the ideal boundary of the totally geodesic hypersurface $E(a,0)\subset\mathbb{H}^{m+1}$ is $\partial B_r(p)$, where $a=(\cot(r),\csc(r)p)$.*

In the case of domains that are closed geodesic balls $\overline{B_r(p)}$ of the sphere $\mathbb{S}^m$, where $0<r\leq\pi/2$, if we impose the condition that the horospherical metric $g=e^{2\rho}g_0$ has constant mean curvature along the boundary $\partial B_r(p)$, then we get information about the location of the boundary of the horospherically concave hypersurface, we get this from the Local Representation Theorem. There, we will see that the boundary lies in an equidistant hypersurface.

Proposition 3.10. *Assuming that $\mathscr{V}_1$ contains a component which is the boundary of a geodesic ball $\partial B_r(p)$, $p\in\mathbb{S}^m$, $r\in(0,\pi)$, and $h(g)=c=cte$ along $\partial B_r(p)$, then*

$$\phi\left(\partial B_r(p)\right)\subset E(a,-c),$$

where $E(a,-c)$ is the totally geodesic hypersurface equidistant to $E(a,0)$ given by

$$E(a,-c)=\left\{y\in\mathbb{H}^{m+1} : \ll y,a \gg = -c\right\}$$

and $a=(\cot(r),\csc(r)p)$.

We will see how, using the parallel flow, we can get horospherically hypersurfaces in one of the components in the Hyperbolic space $\mathbb{H}^{m+1}$ determined by the equidistant hypersurface where its boundary is contained. For simplicity we give the statement assuming that $p=\mathbf{n}$ and $r=\pi/2$.

Theorem 3.12. *Let $g=e^{2\rho}g_0$, $\rho\in C^\infty(\Omega\cup\mathscr{V}_1)$, be a conformal metric on Ω such that $\partial\mathbb{S}^m_+\subset\mathscr{V}_1$ and*

$$h(g)=c \quad on\ \partial\mathbb{S}^m_+,$$

where $c\in\mathbb{R}$ is a constant. Assume that

$$\lim_{x\to q}\left(1+e^{2\rho(x)}+|\nabla\rho(x)|^2\right)=+\infty \quad for\ all\ q\in\mathscr{V}_2.$$

Then, there exists $t_0\geq 0$ such that for every $t>t_0$, the set $\varphi_t\left(\Omega\cup\mathscr{V}_1'\right)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $\mathbf{n}$ at its ideal boundary, where $\mathscr{V}_1'=\mathscr{V}_1\setminus\partial\mathbb{S}^m_+$.

Finally, we see how the parallel flow affects to the elliptic problem for conformal metrics.

In Chapter 4, we deal with degenerate and non-degenerate elliptic problems for conformal metrics on the closed hemisphere $\overline{\mathbb{S}^m_+}$, on compact annulus on the sphere $\mathbb{S}^m$ and on semi-annulus on the sphere $\mathbb{S}^m$.

As said above, J. Escobar proved in [14] that for the Yamabe problem with boundary for the case $\overline{\mathbb{B}^m}$, if the scalar curvature is zero then the mean curvature can not be negative. In this work, we generalize this for degenerate fully nonlinear conformally invariant equations, that is,

Theorem 4.1. *Let (f,Γ) be an elliptic data for conformal metrics and let $c \leq 0$ be a constant. Then, there is no conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+}$, where $\rho \in C^\infty\left(\overline{\mathbb{S}^m_+}\right)$, such that*

$$\begin{cases} f(\lambda(g)) &= 0 \quad on \quad \overline{\mathbb{S}^m_+}, \\ h(g) &= c \quad on \quad \partial \mathbb{S}^m_+, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$.

That theorem can be extended to m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with boundary $\partial\mathscr{M}$ that is umbilic, and $Scal(g_0) \geq 0$ on $\mathscr{M}$, using a result of F. M. Spiegel [46],

Theorem 4.2. *Set (f,Γ) an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathscr{M}, g_0)$ be a m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with umbilic boundary, and $Scal(g_0) \geq 0$ on $\mathscr{M}$. Then, there is no conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\mathscr{M})$, such that*

$$\begin{cases} f(\lambda(g)) &= 0 \quad in \quad \mathscr{M}, \\ h(g) &= c \quad on \quad \partial \mathscr{M}, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho} g_0$.

Also, using [46], we can extended the result of Cavalcante-Espinar [7] for the non-degenerate case. Specifically,

Theorem 4.3. *Set (f,Γ) an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathscr{M}, g_0)$ be a m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with umbilic boundary, and $Scal(g_0) \geq 0$ on $\mathscr{M}$. If there exists a conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\mathscr{M})$, such that*

$$\begin{cases} f(\lambda(g)) &= 1 \quad in \quad \mathscr{M}, \\ h(g) &= c \quad on \quad \partial \mathscr{M}, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho} g_0$, then $\mathscr{M}$ is isometric to a geodesic ball on the standard sphere $\mathbb{S}^m$.

Next, we deal with degenerate problems on the compact annulus $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$. Using the same techniques that M. Cavalcante and J. Espinar [7], every solution to the degenerate problem on $\overline{\mathbb{A}(r)}$ with minimal boundary is rotationally invariant. Also, if there is a solution to such problem then it is unique up to dilations.

Theorem 4.6. *Set $r \in (0,\pi/2)$. If there is a solution $g = e^{2\rho} g_0$ of the following problem*

$$\begin{cases} f(\lambda(g)) &= 0 \qquad on\ \overline{\mathbb{A}(r)}, \\ h(g) &= 0 \qquad on\ \partial\mathbb{A}(r), \end{cases}$$

then it is rotationally invariant and unique up to dilations.

Now, we assume that there is a solution $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ of

$$\begin{cases} f(\lambda(g)) &= 0 \qquad \text{in } \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}, \\ h(g) &= 0 \qquad \text{on } \partial\mathbb{S}^m_+, \end{cases} \tag{5}$$

such that $\sigma = e^{-\rho}$ can be extended to a C^2 function $\tilde{\sigma}$ on $\overline{\mathbb{S}^m_+}$ with $\tilde{\sigma}(\mathbf{n}) = 0$. Such solution is called *punctured solution* of the problem (5), and we have the following theorem for $r \in (0,\pi/2)$,

Theorem 4.10. *Set $r \in (0,\pi/2)$. If the problem (5) admits a punctured solution, then there is no solution to the following degenerate elliptic problem:*

$$\begin{cases} f(\lambda(g)) &= 0 \qquad on\ \overline{\mathbb{A}(r)}, \\ h(g) &= 0 \qquad on\ \partial\mathbb{A}(r). \end{cases}$$

In the case of the ring $\mathbb{A}\left(r, \frac{\pi}{2}\right]$, $0 < r < \dfrac{\pi}{2}$, if we impose some conditions, we obtain,

Theorem 4.11. *Let $r \in (0,\pi/2)$, $c \geq 0$ be a non-negative constant and $g = e^{2\rho} g_0$ be a conformal metric on $\mathbb{A}\left(r, \frac{\pi}{2}\right]$ that is solution of the following degenerate elliptic problem:*

$$\begin{cases} f(\lambda(g)) &= 0 \qquad in\ \mathbb{A}(r,\pi/2], \\ h(g) &= c \qquad on\ \partial\mathbb{S}^m_+, \end{cases}$$

If $e^{2\rho} + |\nabla\rho|^2 : \mathbb{A}(r,\pi/2] \to \mathbb{R}$ is proper then $\lambda(g)$ is no bounded.

In the non-degenerate case, we have,

Theorem 4.12. *Let $0 < r < \pi/2$, $c \in \mathbb{R}$ be a constant and $g = e^{2\rho} g_0$ be a conformal metric on $\mathbb{A}\left(r, \frac{\pi}{2}\right]$ that is solution of the following non-degenerate elliptic problem:*

$$\begin{cases} f(\lambda(g)) &= 1 \qquad in\ \mathbb{A}(r,\pi/2], \\ h(g) &= c \qquad on\ \partial\mathbb{S}^m_+, \\ \lim\limits_{x\to q} \rho(x) &= +\infty \qquad \forall q \in \partial B_r(\mathbf{n}). \end{cases}$$

Let $\sigma = e^{-\rho}$. If $|\nabla\sigma|^2$ is Lipschitz then $\nabla^2(\sigma^2)$ is no bounded.

In the degenerate case case on the punctured geodesic ball with minimal boundary, we have

Theorem 4.4. *Let $g = e^{2\rho} g_0$ be a conformal metric on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ that is solution of the following degenerate elliptic problem:*

$$\begin{cases} f(\lambda(g)) & = & 0 & \text{in } \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}, \\ h(g) & = & 0 & \text{on } \partial \mathbb{S}^m_+, \end{cases}$$

Then g is rotationally invariant.

Finally, in the non-degenerate case on the punctured closed geodesic ball, we have

Theorem 4.5. *Let $g = e^{2\rho} g_0$ be a conformal metric on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ that is solution of the following non-degenerate elliptic problem:*

$$\begin{cases} f(\lambda(g)) & = & 1 & \text{in } \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}, \\ h(g) & = & 0 & \text{on } \partial \mathbb{S}^m_+, \end{cases}$$

Then g is rotationally invariant.

In the last section of Chapter 4, we see that we can extend the definition of the Schouten tensor for conformal metrics to the standard one on domains of the sphere $\mathbb{S}^2$. So, we can extend the notion of eigenvalues of the Schouten tensor and we can also speak of elliptic problem for conformal metrics on domains of the sphere $\mathbb{S}^2$. There, we observe that the Yamabe Problem reduces to the classical Liouville Problem. Hence, fully nonlinear equations for conformal metrics on domains on $\mathbb{S}^2$ can be regarded as a generalization of the Liouville Problem. We see that theorems that we got in Chapter 4 have an analogous in the case of dimension 2. It is remarkable that there is a solution to the Yamabe Problem on the compact annulus with zero scalar curvature and minimal boundary, however, in dimension higher does not exist such kind of solution. In fact, Theorem 4.10 is an extension to Escobar Theorem, that is, if $m \geq 3$, then the Yamabe Problem on the compact annulus does not have solution with scalar curvature equals to zero and minimal boundary, however, this result can not be extended to $m = 2$ as we see in Chapter 4.

Chapter 1

Preliminaries

We establish the preliminary results from differential geometry necessary along this work. Here, we review the different models of the Hyperbolic Space and its isometries. Also we explore the Hyperbolic Space as a conformally compact manifold and, in particular, we study its conformal infinity. Then, we recall the definition of the Schouten tensor of a conformal metric to the standard metric on the sphere, a capital object in this work. To close this chapter, we define the notion of elliptic data in the context of elliptic problems for conformal metric on domains of the sphere and the notion elliptic data in the context elliptic problems for hypersurfaces of the Hyperbolic space. Most of this chapter can be tracked down from the references [2, 3, 8, 19, 47].

1.1 Basics on Riemannian geometry

Let $\mathscr{M}$ be a smooth oriented manifold. Now, we assume that the manifold $\mathscr{M}$ is provided with a metric, this means a symmetric and positive definite $(2,0)$-tensor, denoted by g. So, for each $p \in \mathscr{M}$, one has $g_p : T_p\mathscr{M} \times T_p\mathscr{M} \to \mathbb{R}$. For an arbitrary local chart (U, φ) at $p \in U$, g_p can be written as

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j,$$

where $g_{ij} \in C^\infty(U)$ so that $g_{ij} = g_{ji}$ and $\otimes$ is the tensorial product. Hence, the pair $(\mathscr{M}, g)$, a manifold $\mathscr{M}$ provided with a metric g, is called a **Riemannian manifold**. The functions g_{ij}, $i, j = 1, \ldots, n$, are the coefficients of the metric g in the local chart (U, φ).

For a Riemannian manifold $(\mathscr{M}, g)$, it is not necessary to distinguish between $\mathfrak{X}(\mathscr{M})$ and $\mathfrak{X}(\mathscr{M})^*$. In fact, we can identify each element $X \in \mathfrak{X}(\mathscr{M})$ with an unique 1-form $\omega \in \mathfrak{X}(\mathscr{M})^*$ using the equality

$$\omega(Y) = g(Y,X) \text{ for all } Y \in \mathfrak{X}(\mathscr{M}). \tag{1.1}$$

We will use the word **tensor** when valued in $\mathfrak{X}(\mathscr{M})$ and **form** when valued in $C^\infty(\mathscr{M})$. We will denote the **Lie bracket** of the vector fields in $\mathfrak{X}(\mathscr{M})$ by $[\ ,\ ]$, that is,

$$[X,Y] = XY - YX \text{ for all } X,Y \in \mathfrak{X}(\mathscr{M}).$$

Given a Riemannian manifold $(\mathscr{M},g)$, there exists a unique affine connection $\overline{\nabla}$ such that

(*i*) $\overline{\nabla}$ is symmetric, i.e.,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y];$$

(*ii*) $\overline{\nabla}$ is compatible with g, i.e.,

$$Xg(Y,Z) = g(\overline{\nabla}_X Y,Z) + g(Y,\overline{\nabla}_X Z),$$

for all $X,Y,Z \in \mathfrak{X}(\mathscr{M})$. This leads us to define such unique connection $\overline{\nabla}$ as the **Levi-Civita connection** on $\mathscr{M}$ associated to g.

Let $\overline{\nabla}$ be the Levi-Civita connection associated to a Riemannian metric g and $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ be the basis associated to a local chart $(U,\varphi \equiv (x_1,\ldots,x_n))$. Consider the functions $\Gamma_{ij}^k \in C^\infty(U)$, $i,j,k = 1,,\ldots,n$, given by

$$\overline{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tag{1.2}$$

then, the coefficients Γ_{ij}^k are called the **Christoffel symbols** of the connection $\overline{\nabla}$ on U associated to the metric g.

Associated to the Levi-Civita connection $\overline{\nabla}$ on a Riemannian manifold $(\mathscr{M},g)$, the **Curvature Tensor** $\overline{R}$ is defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]} Z. \tag{1.3}$$

It is well-known that $\overline{R}$ is $C^\infty(\Sigma)$-linear with respect to X,Y,Z and skew-symmetric with respect to X,Y.

Definition 1.1:
Let $(\mathscr{M},g)$ be a Riemannian manifold with Curvature Tensor $\overline{R}$. Given $X_p,Y_p \in T_p\mathscr{M}$ linearly independent, we define the ***sectional curvature****, $\overline{K}_p(X_p,Y_p)$, related to g at $p \in \mathscr{M}$ for the plane generated by $\{X_p,Y_p\}$, by*

$$\overline{K}_p(X_p,Y_p) = \frac{g(\overline{R}(X_p,Y_p)X_p,Y_p)}{\|X_p \wedge Y_p\|^2}, \tag{1.4}$$

where

$$\|X_p \wedge Y_p\| = \sqrt{\|X_p\|^2\|Y_p\|^2 - g(X_p,Y_p)^2}.$$

The definition of $\overline{K}_p(X_p,Y_p)$ does not depend on the choice of the vectors X_p, Y_p, just on the plane generated by them. Moreover, the curvature tensor $\overline{R}$ is completely determined by the sectional curvature when $\overline{K}$ is constant at every point and any plane, and we can recovered it as

$$\overline{R}(X,Y)Z = \overline{K}\left(g(X,Z)Y - g(Y,Z)X\right). \tag{1.5}$$

We will recall now other natural curvature tensors one can define on a Riemanniann manifold $(\mathscr{M},g)$.

Ricci tensor and Scalar curvature

Let $\{e_i\} \subset \mathfrak{X}(U)$, $U \subset \mathscr{M}$ open and connected, be a local orthonormal frame of the tangent bundle $TU \subset T\mathscr{M}$. Let us establish our definition for the **Ricci Curvature** and **Scalar Curvature** in $\mathscr{M}$, i.e,

$$\begin{aligned} \mathrm{Ric}(g)(X,X) &= \sum_{i=1}^{n} \overline{R}(X,e_i,Y,e_i), \\ \mathrm{Scal}(g) &= \sum_{i=1}^{n} \mathrm{Ric}(g)(e_i,e_i), \end{aligned}$$

respectively, here $X \in \mathfrak{X}(\mathscr{M})$.

Schouten Tensor

Let $m > 2$. Let $\mathscr{M}$ be a m-dimensional Riemannian manifold with metric g. We define the Schouten tensor of g as the following symmetric 2-tensor:

$$\mathrm{Sch}(g) = \frac{1}{m-2}\left(\mathrm{Ric}(g) - \frac{\mathrm{Scal}(g)}{2(m-1)}g\right), \tag{1.6}$$

where $\mathrm{Ric}(g)$ is the Ricci tensor of g and $\mathrm{Scal}(g)$ is the scalar curvature, as defined above.

1.1.1 Hypersurfaces Theory

Here, we will remind the most important concepts on hypersurface theory. Along these notes we denote by $(\mathscr{M},\langle,\rangle)$ a $(m+1)$-dimensional connected Riemannian manifold, and let $\Sigma \subset \mathscr{M}$ be an immersed, two-sided hypersurface in $\mathscr{M}$. Let us denote by $\vec{N}$ the unit normal vector field along Σ. Moreover, $\langle,\rangle$ is the metric on $\mathscr{M}$ and g is the induced metric on Σ. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection associated to $\langle,\rangle$ and g, respectively. Denote by $\mathfrak{X}(\Sigma)$ and $\mathfrak{X}(\mathscr{M})$ the linear spaces of smooth vector fields along Σ and $\mathscr{M}$ respectively. We also denote by I the induced metric on Σ, that is, $I \equiv g$.

Remark 1:
We will identify I, g and $\langle,\rangle$ when no confusion occurs.

Let $\{e_i\}_{i=1}^{m+1} \subset \mathfrak{X}(U)$, $U \subset \mathscr{M}$ open and connected, be a local orthonormal frame of the tangent bundle $TU \subset T\mathscr{M}$, then we denote

$$\overline{R}_{ijkl} = \langle \overline{R}(e_i,e_j)e_k,e_l\rangle$$

and, from Definition 1.1, the sectional curvatures in $\mathscr{M}$ are given by

$$\overline{K}_{ij} := \langle \overline{R}(e_i,e_j)e_i,e_j\rangle = \overline{R}_{ijij}.$$

The Gauss Formula (see [8]) says

$$\overline{\nabla}_X Y = \nabla_X Y + \langle S(X), Y\rangle \vec{N} \text{ for all } X, Y \in \mathfrak{X}(\Sigma),$$

where $S : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the **Weingarten (or Shape) operator** and it is given by

$$S(X) := -(\overline{\nabla}_X \vec{N})^T,$$

that is, $S(X)$ is the tangential component of $-\overline{\nabla}_X \vec{N}$. In fact, we do not need to take the tangential part in the above definition when we are dealing with orientable hypersurfaces in orientable manifolds, but we use the general definition for the sake of completeness.

Since $S : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a self-adjoint endomorphism, we denote the **mean curvature** and **extrinsic curvature** as

$$H = \frac{1}{m}\mathrm{Tr}(S) \quad \text{and} \quad K_e = \det(S),$$

where Tr and det denote the trace and determinant respectively.

Let $\mathfrak{X}(\Sigma)^\perp$ be the orthogonal complement of $\mathfrak{X}(\Sigma)$ in $\mathfrak{X}(\mathscr{M})$. Let us denote $\vec{S} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)^\perp$ the **Vector Second Fundamental Form** of Σ, that is,

$$\vec{S}(X,Y) := (\overline{\nabla}_X Y)^\perp, \; X, Y \in \mathfrak{X}(\Sigma),$$

here $(\cdot)^\perp$ means the normal part. Therefore, $\vec{S}$ induces a symmetric quadratic form on Σ given by

$$\langle \vec{S}(X,Y), \vec{N}\rangle = \langle S(X), Y\rangle, \; X, Y \in \mathfrak{X}(\Sigma),$$

which is called the **Second Fundamental Form**, and we also write it as

$$II(X,Y) = I\left(S(X), Y\right), \; X, Y \in X.$$

Hence, the mean curvature vector of Σ is given by

$$m\vec{H}_p = \mathrm{Tr}\left(\vec{S}_p\right) = \sum_{i=1}^{m} \vec{S}_p(v_i, v_i),$$

where $\{v_1, \ldots, v_m\}$ is a orthonormal basis of $T_p\Sigma$.

Also, we define the **Third Fundamental Form** by

$$III(X,Y) = I\left(SX, SY\right) \text{ for all } X, Y \in X,$$

where S is the shape operator of Σ. Since $S : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is self-adjoint, it is diagonalizable and hence let $\{e_1, \ldots, e_m\}$ be **principal directions**, i.e.,

$$S(e_i) = -\overline{\nabla}_{e_i} \vec{N} = \kappa_i e_i,$$

where κ_i are the **principal curvatures**, $i = 1, \ldots, m$, in other words, $\{e_1, \ldots, e_m\}$ are the eigendirections of S and κ_i, $i = 1, \ldots, m$, its eigenvalues.

We say that a point $p \in \Sigma$ is an **umbilic point** if $\kappa_1(p) = \ldots = \kappa_m(p)$, which is equivalent to say that II is proportional to I at p.

Let $\overline{R}$ and R denote the Riemann Curvature tensors of $\mathscr{M}$ and Σ respectively. Then, by the Gauss Formula we can relate $\overline{R}$ and R as

$$\overline{R}(X,Y)Z = R(X,Y)Z + \langle S(Y), Z\rangle S(X) - \langle S(X), Z\rangle S(Y) \text{ for all } X,Y,Z,W \in \mathfrak{X}(\Sigma).$$

There is another important equation that $\Sigma \subset \mathscr{M}$ must verify, the Codazzi Equation. Given $X, Y \in \mathfrak{X}(\Sigma)$, recall that $SX = -\overline{\nabla}_X \vec{N} \in \mathfrak{X}(\Sigma)$, the Gauss formula yields

$$\begin{aligned}
\overline{R}(X,Y)\vec{N} &= \overline{\nabla}_Y \overline{\nabla}_X \vec{N} - \overline{\nabla}_Y \overline{\nabla}_X \vec{N} + \overline{\nabla}_{[X,Y]} \vec{N} \\
&= \overline{\nabla}_X SY - \overline{\nabla}_Y SX - S[X,Y] \\
&= \nabla_X SY - \nabla_Y SX - S[X,Y] - \langle SX, SY\rangle \vec{N} + \langle SY, SX\rangle \vec{N} \\
&= \nabla_X SY - \nabla_Y SX - S[X,Y],
\end{aligned}$$

that is, $\overline{R}(X,Y)\vec{N} \in \mathfrak{X}(\Sigma)$ and the following **Codazzi Equation** holds

$$\overline{R}(X,Y)\vec{N} = \nabla_X SY - \nabla_Y SX - S[X,Y],\ X,Y \in \mathfrak{X}(\Sigma).$$

Assume that the ambient manifold is a Space Form $\mathscr{M} = \mathbb{M}^{m+1}(\kappa)$, $\kappa \in \mathbb{R}$, that is, a complete simply connected $(m+1)-$manifold of constant sectional curvature κ at every point $p \in \mathbb{M}^{m+1}(\kappa)$ and any tangent plane. So, by Cartan Theorem [8], we have

$$\mathbb{M}^{m+1}(\kappa) = \begin{cases} \mathbb{S}^{m+1}(\kappa) & \text{if} \quad \kappa > 0, \\ \quad \mathbb{R}^{m+1} & \text{if} \quad \kappa = 0, \\ \mathbb{H}^{m+1}(\kappa) & \text{if} \quad \kappa < 0. \end{cases}$$

Hence, from (1.5), the Riemann curvature tensor $\overline{R}$ can be recovered as

$$\overline{R}(X,Y)Z = \kappa(g(X,Z)Y - g(Y,Z)X),\, X,Y,Z \in \mathfrak{X}(\mathbb{M}^{m+1}(\kappa)).$$

Therefore, the Gauss and Codazzi Equations become:

$$\overline{K}_{ij} = \kappa_i \kappa_j + \kappa$$

and

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = 0,\ X,Y \in \mathfrak{X}(\Sigma),$$

respectively.

1.2 Hyperbolic Space $\mathbb{H}^{m+1}$

The Hyperbolic Space $\mathbb{H}^{m+1}(\kappa)$ of dimension $m+1$ is the simply connected $(m+1)-$dimensional manifold of constant sectional curvature $\kappa<0$. When $\kappa=-1$, we just denote $\mathbb{H}^{m+1}$. To describe $\mathbb{H}^{m+1}$, we will use different models.

1.2.1 Hyperboloid model

In this model, $\mathbb{H}^{m+1}(\kappa)$ is the hyperquadric in $\mathbb{L}^{m+2}$ given by

$$\mathbb{H}^{m+1}(\kappa)=\left\{(x_0,x_1,\dots,x_{m+1})\in\mathbb{L}^{m+2}:\,-x_0^2+\sum_{i=1}^{m+1}x_i^2=\frac{1}{\kappa},\,x_0>0\right\},$$

with the induced metric $\ll\,,\,\gg=-dx_0^2+\sum_{i=0}^{m+1}dx_i^2$. Here, $\mathbb{L}^{m+2}$ denotes the standard $(m+2)-$dimensional Lorentzian space, that is, $\mathbb{L}^{m+2}$ is nothing but $\mathbb{R}^{m+2}$ with the standard Lorentzian metric $\ll\,,\,\gg$, i.e., for every $x=(x_0,\dots,x_{m+1})$ and $y=(y_0\dots,y_{m+1})$ in $\mathbb{L}^{m+2}$, the standard Lorentzian metric is given by

$$\ll x,y\gg=-x_0y_0+x_1y_1+\ldots+x_{m+1}y_{m+1}.$$

Denoting $x=(x_0,\bar{x})$, where $\bar{x}=(x_1,\dots,x_{m+1})$, we can write

$$\ll x,y\gg=-x_0y_0+<\bar{x},\bar{y}>,$$

where $<\bar{x},\bar{y}>$ is the usual Euclidean inner product in $\mathbb{R}^{m+1}$.

The isometries of $\mathbb{H}^{m+1}$ are the restrictions to $\mathbb{H}^{m+1}$ of $\mathscr{O}^{\uparrow}(1,m+2)$, where $\mathscr{O}^{\uparrow}(1,m+2)$ is the set of all vectorial transformations of $\mathbb{L}^{m+2}$ that preserve the metric and keep invariant the light cone

$$\mathbb{N}_+^{m+1}=\left\{(x_0,x_1,\dots,x_{m+1})\in\mathbb{L}^{m+2}:\,-x_0^2+\sum_{i=1}^{m+1}x_i^2=0,\,x_0>0\right\}.$$

Geodesic curves

The geodesics in the Hyperbolic space $\mathbb{H}^{m+1}$, using the Hyperboloid model, have an easy formula if it is known one point of the geodesic and the velocity in that point. More precisely, the geodesic in $\mathbb{H}^{m+1}$ passing through $p\in\mathbb{H}^{m+1}$ with velocity $v\in U_p\mathbb{H}^{m+1}\equiv\mathbb{S}^n$ is given by

$$\gamma_{(p,v)}(t)=\cosh(t)\,p+\sinh(t)\,v\,,$$

where t is the arc length parameter of γ. Here, $U_p\mathbb{H}^{m+1}$ denotes the unitary tangent vectors at p.

Totally umbilic hypersurfaces

The totally umbilical hypersurfaces in $\mathbb{H}^{m+1}$ are non-trivial intersections of the hyperplanes of $\mathbb{L}^{m+2}$ with $\mathbb{H}^{m+1}$.

Totally geodesic hypersurfaces

One of the most important totally umbilic hypersurfaces in $\mathbb{H}^{m+1}$ are the totally geodesic hypersurfaces. Every totally geodesic hypersurface in $\mathbb{H}^{m+1}$ can be defined using a unit space vector $a = (a_0, \overline{a}) \in \mathbb{L}^{m+2}$, such that $\ll a, a \gg = 1$ and $a_0 > 0$. More precisally, the totally geodesic hypersurface is

$$E(a,0) = \left\{ y \in \mathbb{H}^{m+1} : \ll y, a \gg = 0 \right\}. \tag{1.7}$$

It is called the totally geodesic hypersurface associated to a.

Equidistant hypersurfaces

Other important totally umbilic hypersurfaces in $\mathbb{H}^{m+1}$ are the equidistant hypersurfaces to a totally geodesic hypersurface. Given $a = (a_0, \overline{a}) \in \mathbb{L}^{m+2}$, such that $\ll a, a \gg = 1$ and $a_0 > 0$, and given $c \in \mathbb{R}$, the following set

$$E(a,c) = \left\{ y \in \mathbb{H}^{m+1} : \ll y, a \gg = c \right\}, \tag{1.8}$$

is an equidistant hypersurface to the the totally geodesic hypersurface $E(a,0)$.

Horospheres

One of the most important totally umbilic hypersurfaces in $\mathbb{H}^{m+1}$ for our present work are horospheres. We say that $a = (a_0, \overline{a}) \in \mathbb{L}^{m+2}$ is a *light vector* of $\mathbb{L}^{m+2}$ if

$$\ll a, a \gg = 0.$$

One can easily see that every horosphere is given by

$$\mathscr{H}_a := \left\{ y \in \mathbb{H}^{m+1} : \ll y, a \gg = -1 \right\},$$

where $a = (a_0, \overline{a}) \in \mathbb{L}^{m+2}$ is a light vector such that $a_0 > 0$. Hence, we denote this horosphere by $\mathscr{H}_a$.

The *exterior* unit normal vector field along the horosphere $\mathscr{H}_a$, i.e., the unit normal vector field to the horosphere $\mathscr{H}_a$ such that its principal curvatures are -1, is given by

$$n(y) = y - a \text{ for every } y \in \mathscr{H}_a. \tag{1.9}$$

The Space of Horospheres

We can parametrize the set of horospheres in $\mathbb{H}^{m+1}$ as $\mathbb{R}^+ \times \mathbb{S}^m$ by

$$(a_0, u) \in \mathbb{R}^+ \times \mathbb{S}^m \mapsto a = (a_0, a_0 u).$$

Also, we can parametrize the set of horospheres in $\mathbb{H}^{m+1}$ as the cylinder $\mathbb{R} \times \mathbb{S}^m$ by

$$(t, x) \in \mathbb{R} \times \mathbb{S}^m \mapsto a = e^t (1, x). \tag{1.10}$$

1.2.2 Half-space model

Consider the subset of $\mathbb{R}^{m+1}$ given by

$$\mathbb{R}^{m+1}_+ = \left\{(x_1,\ldots,x_{m+1}) \in \mathbb{R}^{m+1} : x_{m+1} > 0\right\}$$

with the metric

$$ds^2_\kappa = \frac{1}{-\kappa x^2_{m+1}}\left(dx_1^2 + \cdots + dx^2_{m+1}\right) .$$

A straightforward computation shows that $(\mathbb{R}^{m+1}_+, ds^2_\kappa)$ has constant sectional curvature $\kappa < 0$. Moreover, we can see that the map

$$(x_0,\ldots,x_{m+1}) \longmapsto \frac{1}{x_0 + x_{m+1}}\left(x_1,\ldots,x_m,\frac{1}{\sqrt{-\kappa}}\right) ,$$

from $\mathbb{H}^{m+1}(\kappa)$ to $(\mathbb{R}^{m+1}_+, ds^2_\kappa)$ is an isometry between these spaces.

The expression of the metric says that the rotations around some vertical axis and horizontal translations are isometries of $\mathbb{R}^{m+1}_+$. Other isometries associated to each point $(y,0)$, $y \in \mathbb{R}^m \equiv \{x_{m+1} = 0\}$ are the following:

- *Vertical Hyperbolic Translations* given by

  $$(x,x_{m+1}) \longmapsto (e^t(x-y), e^t x_{m+1}),$$

  with $x \in \mathbb{R}^m$, $x_{m+1} > 0$, $t \in \mathbb{R}$. That is, Euclidean homoteties of center $(y,0)$ and scale e^t.

- *Reflections or hyperbolic isometries* given by

  $$(x,x_{m+1}) \longmapsto (y,0) + \frac{t^2}{|x-y|^2 + x^2_{m+1}}(x-y,x_{m+1}) ,$$

  where $x \in \mathbb{R}^m$, $x_{m+1} > 0$, $t > 0$ and $|\cdot|$ denotes the Euclidean metric in $\mathbb{R}^n$. That is, they are Euclidean inversions (in the half-plane) with respect to a sphere of radius t centered at $(y,0)$. Moreover, Euclidean symmetries with respect to vertical planes also are considered hyperbolic reflections centered at infinity.

It is easy to show that the group of isometries of the half-space model $\mathrm{Iso}(\mathbb{R}^{m+1}_+)$ corresponds to the subgroup of conformal transformations of $\mathbb{R}^{m+1}$, $\mathrm{Conf}(\mathbb{R}^{m+1})$, preserving the half-space, that is,

$$\mathrm{Iso}(\mathbb{R}^{m+1}_+) = \left\{\Phi \in \mathrm{Conf}(\mathbb{R}^{m+1}) \,:\, \Phi(\mathbb{R}^{m+1}_+) = \mathbb{R}^{m+1}_+\right\}.$$

Furthermore, it is clear that the fixed points of these isometries are either vertical affine subspaces or half-spheres of any radius, of dimension $k \leq m+1$, that intersect orthogonally the hyperplane $\{x_{m+1} = 0\}$. Therefore, these are the totally geodesics submanifolds of $(\mathbb{R}^{m+1}_+, ds^2_\kappa)$. In particular, the semicircles cutting orthogonally the hyperplane $\{x_{m+1} = 0\}$ and the vertical straight lines are the geodesics of this model.

1.2.3 Poincaré ball model

Consider the open ball $\mathbb{B}(1/\sqrt{-\kappa}) \subset \mathbb{R}^{m+1}$ centered at the origin of radius $1/\sqrt{-\kappa}$, with the metric of constant sectional curvature κ given by

$$g_\kappa = \frac{4}{(1+\kappa|x|^2)^2}(dx_1^2 + \cdots + dx_{m+1}^2)\,,$$

where $x = (x_1, \ldots, x_m)$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$. It is easy to see that the map from $(\mathbb{R}^{m+1}_+, ds^2_\kappa)$ to $(\mathbb{B}(1/\sqrt{-\kappa}), g_\kappa)$ given by

$$(y, y_{m+1}) \longmapsto \frac{1}{\sqrt{-\kappa}(|y|^2 + (y_{m+1}+1)^2)}(2y, |y|^2 + y_{m+1}^2 - 1)\,,$$

where $y \in \mathbb{R}^m$ and $y_{m+1} > 0$, is an isometry between these spaces. In fact, the group of isometries of the Poincaré ball model $\mathrm{Iso}(\mathbb{B}(1/\sqrt{-\kappa}))$ corresponds to the subgroup of conformal transformations of $\mathbb{R}^{m+1}$, $\mathrm{Conf}(\mathbb{R}^{m+1})$, preserving $\mathbb{B}(1/\sqrt{-\kappa})$, that is,

$$\mathrm{Iso}(\mathbb{B}(1/\sqrt{-\kappa})) = \left\{\Phi \in \mathrm{Conf}(\mathbb{R}^{m+1}) \,:\, \Phi(\mathbb{B}(1/\sqrt{-\kappa})) = \mathbb{B}(1/\sqrt{-\kappa})\right\}.$$

As a direct observation from the expression of the metric in this model, the restriction to $\mathbb{B}(1/\sqrt{-\kappa})$ of each element of $\mathscr{O}(m+1)$ is an isometry of this space. In particular, the rotations with respect to a vertical straight line passing through the origin and the Euclidean reflections with respect to hyperplanes passing through the origin are isometries. Moreover, the hyperbolic reflections in this model are given by the reflections with respect to hyperplanes passing through the origin and inversions with respect to spheres that meet orthogonally the sphere at infinity $\mathbb{S}^m_\infty(\kappa) = \partial\mathbb{B}(1/\sqrt{-\kappa})$.

Note that each exterior point to the ball, that is, each $p \in \mathbb{R}^{m+1} \setminus \overline{\mathbb{B}(1/\sqrt{-\kappa})}$ is the center of a unique sphere that meet orthogonally $\mathbb{S}^m_\infty(\kappa)$, whose radius r is given by $r = \sqrt{1/\kappa + |p|^2}$. So, each radial axis of the ball determine an one-parameter family of that reflections.

The totally geodesic submanifolds in this model are the spheres caps with dimension $k \leq m+1$ and the disks that meet orthogonally $\mathbb{S}^m_\infty(\kappa)$. In particular, the geodesics are traces of circles that meet orthogonally the boundary at infinity and traces of straight lines passing though the origin.

Horospheres in the Poincaré ball model

In the Poincaré model, the horosphere defined by the light vector $a = (a_0, \overline{a}) \in \mathbb{L}^{m+2}$, $a_0 > 0$, is given by

$$\mathscr{H}_a = \left\{ u \in \mathbb{B}^{m+1} : \left| u - \frac{1}{1+a_0}\overline{a} \right|^2 = \left(\frac{1}{1+a_0} \right)^2 \right\}.$$

Let $\mathscr{S}_a$ be the Euclidean sphere with centered at $c_0 = \frac{1}{1+a_0}\overline{a}$ and radius $r = \frac{1}{1+a_0}$, then

$$\mathscr{H}_a = \mathscr{S}_a \setminus \{a_\infty\},$$

where $a_\infty = \frac{1}{a_0}\bar{a}$. The point a_∞ is called the **point at infinity** of the horosphere $\mathscr{H}_a$. In other words, the point at infinity of the horosphere $\mathscr{H}_a$ is the unique intersection point of $\mathscr{S}_a$ and $\partial\mathbb{B}^{m+1} = \mathbb{S}^m$, that is, $a_\infty = \frac{1}{a_0}\bar{a}$.

Hence, in the Poincaré ball model, we can parametrize horospheres $(t,x) \in \mathbb{R}\times\mathbb{S}^m$ using (1.10) by

$$\mathscr{H}_x(t) \equiv \mathscr{H}_a,$$

where $a = (a_0, \bar{a}) \in \mathbb{L}^{m+2}$ given by

$$a_0 = e^t \text{ and } \bar{a} = e^t x.$$

1.2.4 Klein model

The Klein model of the Hyperbolic space $\mathbb{H}^{m+1}$ is the Euclidean ball $\mathbb{B}^{m+1} = \{x \in \mathbb{R}^{m+1} : |x|^2 < 1\}$ endowed with the distance between two points p and q in the open ball defined as follows: let a and b the two ideal points that we get when we intersect the Euclidean line that contains p and q ordered as a, p, q, b, i.e., if $|ap|$ is the Euclidean length of the segment $\overline{ap}$ then $|ap| < |aq|$ and $|qb| < |pb|$. Then the hyperbolic distance between p and q is defined

$$d(p,q) = \frac{1}{2}\log\left(\frac{|aq|}{|ap|}\frac{|pb|}{|qb|}\right). \tag{1.11}$$

The transformation $T : \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2} \to \mathbb{B}^{m+1} \subset \mathbb{R}^{m+1}$ defined as

$$T(y) = \frac{1}{y_0}\bar{y} \text{ for all } y \in \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}, \tag{1.12}$$

is a diffeomorphism. So, if we induce the Hyperbolic metric of $\mathbb{H}^{m+1}$ in $\mathbb{B}^{m+1}$, via T, we get a metric such that its distance is given by (1.11). The metric in the Klein model is defined for every $x \in \mathbb{B}^{m+1}$ by

$$g_K(u,v) = \frac{1}{(1-|x|^2)^2}\langle u,x\rangle\langle v,x\rangle + \frac{1}{1-|x|^2}\langle u,v\rangle \text{ for all } u,v \in T_x\mathbb{B}^{m+1},$$

where $|u|$ is the Euclidean norm of u and $\langle\cdot,\cdot\rangle$ is the Euclidean metric. This metric g_K has constant sectional curvature $\kappa = -1$.

In the Klein model the totally geodesic hypersurfaces are intersections of Euclidean hyperplanes with $\mathbb{B}^{m+1}$ as we can see. Every totally geodesic hypersurface Σ in the hyperboloid model has the following equation

$$\Sigma := \left\{y \in \mathbb{L}^{m+2} : \ll y, c \gg = 0\right\}$$

where $c \in \mathbb{L}^{m+2}$ satisfies $\ll c, c \gg = 1$. Using (1.12) we have that for every $y \in \Sigma$ it holds

$$\langle T(y), \bar{c}\rangle = c_0,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{R}^{m+1}$. Hence, every geodesic is only a Euclidean line inside of $\mathbb{B}^{m+1}$ and it has two ideal points in the boundary of $\mathbb{B}^{m+1}$. Even more, a geodesic γ in the

Hyperbolic space $\mathbb{H}^{m+1}$ has the same ideal point for the Klein model and the Poincaré model as we can see using the generic form of a geodesic in the hyperboloid model and (1.12). In fact, every geodesic in the hyperboloid model has the following form

$$\gamma(t) = \cosh(t)p + \sinh(t)v, \quad t \in \mathbb{R},$$

where $p \in \mathbb{H}^{m+1}$ and $v \in T_p\mathbb{H}^{m+1}$ with $\ll v, v \gg = 1$. Then, using (1.12), we have that the ideal points are:

$$\lim_{t\to-\infty} T(\gamma(t)) = \frac{\overline{p}-\overline{v}}{p_0 - v_0} \quad \text{and} \lim_{t\to+\infty} T(\gamma(t)) = \frac{\overline{p}+\overline{v}}{p_0 + v_0},$$

that are the same ideal points in the Poincaré model as one can easily verify.

Remark 2:
From now on, unless specifically stated, we will consider $\kappa = -1$, *and we will omit the dependence of* κ.

1.3 The $\mathbb{S}^m$ as the Boundary at infinity

In the previous section we have claimed that $\mathbb{S}^m_\infty$ is the *infinity* of $\mathbb{H}^{m+1}$, which looks obvious using the Poincaré ball model. Nevertheless, we shall make this clear.

Let X^{m+1} denote the interior of a smooth compact manifold $\overline{X^{m+1}}$ with boundary $\mathscr{M}^m = \partial X^{m+1}$. A smooth function $r : \overline{X^{m+1}} \to \mathbb{R}$ is said to be a *defining function* for $\mathscr{M}$ in X if

$$r > 0 \text{ in } X,\ r = 0 \text{ on } M \text{ and } dr \neq 0 \text{ on } \mathscr{M}.$$

A Riemannian metric g on X is then said to be *conformally compact* if for a defining function r, the conformal metric $\overline{g} = r^2 g$ extends to a metric on $\overline{X}$. The metric $\overline{g}$ restricted to $\mathscr{M}$ induces a metric $\hat{g}$ on $\mathscr{M}$, which rescales by conformal factor upon change in defining function and, hence, defines a conformal structure $(\mathscr{M}, [\hat{g}])$ called the *conformal infinity* of (X, g).

So, in the case of the Poincaré Ball Model, it is easy to check that the defining function $r : \overline{\mathbb{B}} \to \mathbb{R}$ given by

$$r(x) := \frac{1-|x|^2}{2},$$

gives the standard conformal structure of the $m-$sphere $(\mathbb{S}^m, [g_0])$, g_0 denotes the standard metric on $\mathbb{S}^m$, as the conformal infinity of $\mathbb{H}^{m+1} \equiv (\mathbb{B}^{m+1}, g_{-1})$. We denote such conformal infinity as $\partial_\infty \mathbb{H}^{m+1} \equiv \mathbb{S}^m$.

1.3.1 Boundary at infinity

Let v_i $(i = 1, 2)$ be two unit vectors in $T\mathbb{H}^{m+1}$ and let $\gamma_{v_i}(t)$, $i = 1, 2$, be two unit-speed geodesics on $\mathbb{H}^m$ satisfying $\gamma'_{v_i}(0) = v_i$. We say that two geodesics $\gamma_{v_1}(t)$ and $\gamma_{v_2}(t)$ are *asymptotic* if there exists a constant c such that the distance $d(\gamma_{v_1}(t), \gamma_{v_2}(t))$ is less than c for all $t \geqslant 0$. Similarly, two unit vectors v_1 and v_2

are asymptotic if the corresponding geodesics $\gamma_{v_1}(t)$, $\gamma_{v_2}(t)$ have this property. It is easy to find that being asymptotic is an equivalence relation on the set of unit-speed geodesics or on the set of unit vectors on $\mathbb{H}^{m+1}$. Every element of these equivalence classes is called a point at infinity.

Denote by $\partial_\infty\mathbb{H}^{m+1}$ the set of points at infinity, and denote by $\gamma(+\infty)$ or $v(\infty)$ the equivalence class of the corresponding geodesic $\gamma(t)$ or unit vector v.

We will see now that the definition of boundary at infinity $\partial_\infty\mathbb{H}^{m+1}$ given here agrees with the definition of conformal infinity given above.

It is well-known that for two asymptotic geodesics γ_1 and γ_2 in $\mathbb{H}^{m+1}$, the distance between the two curves $\gamma_1|_{[t_0,+\infty)}$, $\gamma_2|_{[t_0,+\infty)}$ is zero for any $t_0 \in \mathbb{R}$. Besides, for any $x, y \in \partial_\infty\mathbb{H}^n$, there exists a unique oriented unit speed geodesic $\gamma(t)$ such that $\gamma(+\infty) = x$ and $\gamma(-\infty) = y$.

For any point $p \in \mathbb{H}^{m+1}$, there exists a bijective correspondence between a set of unit vectors at p and $\partial_\infty\mathbb{H}^{m+1}$. In fact, for a point $p \in \mathbb{H}^{m+1}$ and a point $x \in \partial_\infty\mathbb{H}^{m+1}$, there exists a unique oriented unit speed geodesic γ such that $\gamma(0) = p$ and $\gamma(+\infty) = x$. Equivalently, the unit vector v at the point p is mapped to the point at infinity $v(\infty)$. Therefore, $\partial_\infty\mathbb{H}^{m+1}$ is bijective to a unit sphere, i.e., $\partial_\infty\mathbb{H}^{m+1} \equiv \mathbb{S}^m_\infty$, or just, if no confusion occurs, we denote the boundary at infinity by $\mathbb{S}^m$.

Set $\overline{\mathbb{H}^{m+1}} = \mathbb{H}^{m+1} \cup \partial_\infty\mathbb{H}^{m+1}$. For a point $p \in \mathbb{H}^{m+1}$, let $\mathscr{U}$ be an open set in the unit sphere of the tangent space $T_p\mathbb{H}^{m+1}$. For any $r > 0$, define

$$T(\mathscr{U}, r) := \left\{ \gamma_v(t) \in \overline{\mathbb{H}^{m+1}} : v \in \mathscr{U},\ r < t \leqslant +\infty \right\}.$$

Then we can construct a unique topology T on $\overline{\mathbb{H}^{m+1}}$, called the cone topology, as follows: the restriction of T to $\mathbb{H}^{m+1}$, $\mathrm{T}|_{\mathbb{H}^{m+1}}$, is the topology induced by the Riemannian distance; the sets $T(\mathscr{U}, r)$ containing a point $x \in \partial_\infty\mathbb{H}^{m+1}$ form a neighborhood basis at x. We call such topology the *ideal topology* of $\overline{\mathbb{H}^{m+1}}$. Clearly, the ideal topology T satisfies the following properties:

(A1) $\mathrm{T}|_{\mathbb{H}^{m+1}}$ coincides with the topology induced by the Riemannian distance;

(A2) for any $p \in \mathbb{H}^{m+1}$ and any homeomorphism $h : [0,1] \to [0,+\infty]$, the function φ, from the closed unit ball of $T_p\mathbb{H}^n$ to $\overline{\mathbb{H}^{m+1}}$, given by $\varphi(v) = \exp_p(h(\|v\|)v)$ is a homeomorphism. Moreover, φ identifies $\partial_\infty\mathbb{H}^n$ with the unit sphere;

(A3) for a point $p \in \mathbb{H}^{m+1}$, the mapping $v \to v(\infty)$ is a homeomorphism from the unit sphere of $T_p\mathbb{H}^{m+1}$ onto $\partial_\infty\mathbb{H}^{m+1}$.

(A4) with this topology, $\overline{\mathbb{H}^{m+1}}$ is the natural conformal compactification of $\mathbb{H}^{m+1}$ (see [24]).

Using the notion of the cone topology one can define the boundary at infinity of a subset of $\mathbb{H}^{m+1}$. In fact, given a subset $O \subseteq \mathbb{H}^{m+1}$, we denote $\overline{O}^\infty$ the closure in $\overline{\mathbb{H}^{m+1}}$ with the cone topology. Denote by $\partial_\infty O$ the boundary at infinity of O, that is, $\partial_\infty O = \overline{O}^\infty \cap \partial_\infty\mathbb{H}^{m+1}$. Also, denote by $\mathrm{int}(\cdot)$ the interior of a given set of points.

1.3.2 Mean Curvature of geodesic balls

As we did in the hyperbolic space $\mathbb{H}^{m+1}$, we shall study totally umbilic hypersurfaces in its conformal infinity $(\mathbb{S}^m, g_0)$. In this case, it is well-known that the only totally umbilic hypersurfaces are geodesic spheres, that is, the boundary of geodesic balls in the sphere.

Given $p \in \mathbb{S}^m$ and $0 < r < \pi$, we denote $B_r(p)$ the open ball of $\mathbb{S}^m$ with center p and radius r, i.e.,

$$B_r(p) = \left\{q \in \mathbb{S}^m \,:\, d_{\mathbb{S}^m}(q,p) < r\right\}.$$

Moreover,

$$\overline{B_r(p)} = \left\{q \in \mathbb{S}^m \,:\, d_{\mathbb{S}^m}(q,p) \leq r\right\}.$$

We denote for $r \in (0,\pi)$:

$$S_r(p) = \left\{q \in \mathbb{S}^m / d_{\mathbb{S}^m}(q,p) = r\right\}.$$

The definition of $S_r(p)$ also is valid for $r = 0$, in that case $S_0(p) = \{p\}$.

Note that for any $r \in (0,\pi)$:

$$S_r(p) = \partial B_r(p)$$

The inward unit normal vector field to $\partial B_r(p)$ is given by

$$\nu(x) = \frac{1}{\sin(r)}p - \cot(r)x,$$

for all $x \in \partial\Omega$. The (normalized) mean curvature h_0 of the boundary $\partial\Omega$ with respect the inward unit normal vector field ν is the constant $h_0 = \cot(r)$. Here, we have considered $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ with the induced Euclidean metric.

1.3.3 Schouten tensor for a conformal metric on domains on the sphere $\mathbb{S}^m$

Here g_0 denote the standard metric on the sphere $\mathbb{S}^m$. Let us denote the Schouten tensor of g_0 by Sch_0. It is not difficult to see from (1.6) that

$$\mathrm{Sch}_0 = \frac{1}{2}g_0.$$

Let $g = e^{2\rho}g_0$ be a conformal metric on a domain Ω of the sphere $\mathbb{S}^m$, $\rho \in C^\infty(\Omega)$. We have that

$$Sch_g = Sch_0 + d\rho \otimes d\rho - \nabla^2\rho - \frac{1}{2}|\nabla\rho|^2 \cdot g_0,$$

where ∇ ,∇^2 are the gradient and the hessian with respect the metric g_0 respectively, and $|\cdot|$ the norm with respect of g_0.

Let A_0 and A be the self-adjoint endomorphisms associated to Sch_0 and Sch with respect to g_0 and g, respectively. The existence and uniqueness of the above self-adjoint endomorphisms follows since the Schouten tensor is a symmetric $(2,0)-$tensor. Then, we have the following relation

$$A + \frac{1}{2}|\nabla\sigma|^2 Id = \sigma^2 A_0 + \sigma \cdot \nabla^2\sigma,$$

where $\sigma = e^{-\rho}$, Id is just the identity operator. That is,

$$A + \frac{1}{2}|\nabla\sigma|^2 Id = \frac{1}{2}\sigma^2 Id + \sigma \cdot Hess(\sigma).$$

Theorem 1.2:
Given $x \in \Omega$, let $s_1, \ldots, s_m$ denote the eigenvalues of $(\nabla^2\sigma)_x : T_x\Omega \to T_x\Omega$. Then, the eigenvalues of A_x are

$$\lambda_i = s_i \cdot \sigma(x) + \frac{1}{2}\left(\sigma^2(x) - |\nabla\sigma(x)|^2\right) \textit{ for } i = 1, \ldots, m. \tag{1.13}$$

1.4 Isometries and Conformal Diffeomorphism

Since $\mathbb{S}^m$ is the conformal infinity (endowed with the standard conformal structure) of the hyperbolic space $\mathbb{H}^{m+1}$, it is natural to expect that there is a one-to-one relation between the conformal diffeomorphism of $\mathbb{S}^m$, Conf$(\mathbb{S}^m)$, and the isometries of $\mathbb{H}^{m+1}$, Iso$(\mathbb{H}^{m+1})$. We will see here how they are related.

1.4.1 Conformal diffeomorphisms in $\mathbb{B}^{m+1}$

The space of conformal diffeomorphisms in $\mathbb{B}^{m+1}$, Conf$(\mathbb{B}^{m+1})$ are generated by:

- Inversions $I_p : \mathbb{B}^{m+1} \to \mathbb{B}^{m+1}$, where $p \in \mathbb{R}^{m+1}$ such that $|p| > 1$, that are defined by

$$I_p(y) = \frac{r^2}{|y-p|^2}(y-p) + p \text{ for every } y \in \mathbb{B}^{m+1}, \tag{1.14}$$

  here $r^2 = |p|^2 - 1$, and

- Euclidean orthogonal transformations preserving $\mathbb{B}^{m+1}$.

Clearly every conformal diffeomorphism on $\mathbb{B}^{m+1}$ extends, as a diffeomorphism, to the closure, i.e., $B : \overline{\mathbb{B}^{m+1}} \to \overline{\mathbb{B}^{m+1}}$.

Let us see how we can recover all Euclidean reflections by hyperplanes passing through the origin from the inversions in $\mathbb{R}^{m+1}$ that leave $\mathbb{B}^{m+1}$ invariant. Let $u \in \mathbb{S}^m$ be a unitary vector and $P = \{x \in \mathbb{R}^{m+1} : \langle x, u\rangle = 0\}$ be the hyperplane passing through the origin orthogonal to u. Let $R_u : \mathbb{B}^{m+1} \to \mathbb{B}^{m+1}$ be the Euclidean reflection with respect to P restricted to $\mathbb{B}^{m+1}$. If we take $p = 2u$ and $q = 3u$, then

$$I_p \circ I_q = R_u \circ I_{(-5u)}.$$

Therefore, we have recovered all Euclidean reflections preserving $\mathbb{B}^{m+1}$. Note also that every rotation in $\mathbb{R}^{m+1}$ is the composition of two Euclidean reflections in $\mathbb{R}^{m+1}$. Then, we can recover all the orthogonal transformations of $\mathbb{R}^{m+1}$ from the inversions that leave $\mathbb{B}^{m+1}$ invariant.

In conclusion, the space of conformal diffeomorphisms in $\mathbb{B}^{m+1}$ are generated by inversions $I_p : \mathbb{B}^{m+1} \to \mathbb{B}^{m+1}$, where $|p| > 1$, acting by composition.

1.4.2 Isometries in $\mathbb{H}^{m+1}$: Conformal diffeomorphisms on $\mathbb{B}^{m+1}$

When we work in the Poincaré model, the isometries are the conformal diffeomorphism on $\mathbb{B}^{m+1}$.

One can observe that inversions on $\mathrm{Conf}(\mathbb{B}^{m+1})$ correspond to reflections in the Hyperbolic space with respect to totally geodesic hypersurfaces. More precisely,

Proposition 1.3:
Let $p \in \mathbb{R}^{m+1}$ such that $|p| > 1$, then the isometry $T : \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2} \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ that is related to the inversion $I_p : \mathbb{R}^{m+1} \setminus \{p\} \to \mathbb{R}^{m+1} \setminus \{p\}$ is given by

$$T(y) = y - 2\frac{\ll y,(1,p) \gg}{r^2}(1,p) \text{ for all } y \in \mathbb{H}^{m+1},$$

where $r^2 = |p|^2 - 1$.

That is, an inversion is a reflection with respect to the totally geodesic hyperplane $\{y \in \mathbb{H}^{m+1} \big/ \ll y,(1,p) \gg = 0\}$.

Remark 3:
We would like to remark that the totally geodesic hypersurface given in Proposition 1.3 is the set $\{x \in \mathbb{B}^{m+1} : |x-p|^2 = |p|^2 - 1\}$ in the Poincaré model.

Hence, by Proposition 1.3, we have:

Proposition 1.4:
The space of isometries of $\mathbb{H}^{m+1}$ is generated by reflections with respect to totally geodesic hypersurfaces in $\mathbb{H}^{m+1}$.

As we said, these conformal diffeomorhpisms on $\mathbb{B}^{m+1}$ can be naturally extended to a conformal diffemorphisms on $\overline{\mathbb{B}^{m+1}}$. Hence, when we have an isometry $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$, we obtain a natural conformal diffeomorphism $B : \overline{\mathbb{B}^{m+1}} \to \overline{\mathbb{B}^{m+1}}$ that induces a diffeomorphism $B : \mathbb{S}^m \to \mathbb{S}^m$ which is also a conformal diffeomorphism.

1.4.3 Conformal Diffeomorphisms of $\mathbb{S}^m$

We know that every conformal diffeomorphism on the ball $\mathbb{B}^{m+1}$ induces a natural conformal diffeomorphism on $\mathbb{S}^m$ (see Subsections 1.4.1 and 1.4.2), also the reciprocal is true.

Proposition 1.5:
Every conformal diffeomorphism of $\mathbb{S}^m$ can be extended to a conformal diffeomorphism of $\overline{\mathbb{B}^m}$.

Proof. Let $B : \mathbb{S}^m \to \mathbb{S}^m$ be a conformal diffeomorphism. Let $Q : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the Euclidean orthogonal linear transformation such that $Q(B(\mathbf{n})) = \mathbf{n}$. Then $QB : \mathbb{S}^m \to \mathbb{S}^m$ is a conformal diffeomorphism. Hence, we only need to prove the proposition for conformal diffeomorphisms $B : \mathbb{S}^m \to \mathbb{S}^m$ such that $B(\mathbf{n}) = \mathbf{n}$.

Let $\Pi : \mathbb{S}^m \setminus \{\mathbf{n}\} \to \mathbb{R}^m$ be the stereographic projection and consider the conformal diffeomorphism $C = \Pi B \Pi^{-1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$. Recall that $P = \Pi$ is conformal and therefore, the composition is conformal.

Liouville's Theorem implies that C is the composition of a rotation, a translation and a dilation, i.e., there exist a orthogonal linear transformation $Q_1 : \mathbb{R}^m \to \mathbb{R}^m$, a vector $v \in \mathbb{R}^m$ and $\lambda > 0$ such that

$$C = \lambda Q_1 + v \text{ on } \mathbb{R}^m.$$

Let C_1 be the translation in $\mathbb{R}^m$ associated to the vector v, C_2 be the dilation in $\mathbb{R}^m$ by $\lambda > 0$, and C_3 be the orthogonal linear transformation Q_1, then

$$C = C_1 C_2 C_3 \text{ on } \mathbb{R}^m.$$

Observe that $B_1 = P^{-1} C_1 P$, $B_2 = P^{-1} C_2 P$ and $B_3 = P^{-1} C_3 P$ are the restrictions of conformal diffeomorphism on $\overline{\mathbb{B}^{m+1}}$.

1. B_1: If $v \neq 0$, one can see that the translation C_1 in $\mathbb{R}^m$ by v satisfies:

   $$B_1 = P^{-1} C_1 P = R_u \circ I_p \text{ on } \mathbb{S}^m \setminus \{\mathbf{n}\},$$

   where R_u is a linear Euclidean reflection in $\mathbb{R}^{m+1}$ and I_p is an inversion in $\mathbb{R}^{m+1}$ (see Subsection 1.4.1). More specifically, the reflection is with respect to the hyperplane passing through the origin and orthogonal to $(0, v)$, and the inversion I_p is with respect to the point $p = \left(-\frac{2}{|v|^2} v, 1\right)$.

   If $v = 0$, then B_1 is the restriction of the identity in $\mathbb{R}^{m+1}$.

2. B_2: If $\lambda \neq 1$, the dilation C_2 in $\mathbb{R}^m$ by λ satisfies:

   $$B_2 = P^{-1} C_2 P = R_u \circ I_p \quad \text{on} \quad \mathbb{S}^m \setminus \{\mathbf{n}\},$$

   where R_u is a linear Euclidean reflection in $\mathbb{R}^{m+1}$ and I_p is an inversion in $\mathbb{R}^{m+1}$ (see Subsection 1.4.1). More specifically, the reflection is with respect to the hyperplane passing through the origin and orthogonal to $\mathbf{n}$, and the inversion I_p with respect to the point $p = \frac{1+\lambda}{1-\lambda} e_{m+1}$.

   If $\lambda = 1$, then B_2 is the restriction of the identity in $\mathbb{R}^{m+1}$.

3. B_3: In this case, one can see that

   $$B_3(x) = P^{-1} C_3 P(x) = (C_3(x_1, \ldots, x_m), x_{m+1}) \text{ for all } x \in \mathbb{S}^m \setminus \{\mathbf{n}\},$$

   which is a restriction of a Euclidean orthogonal linear transformation in $\mathbb{R}^{m+1}$ that leaves the x_{m+1}-axis invariant.

Then,

$$B = B_1 \circ B_2 \circ B_3 \text{ on } \mathbb{S}^m \setminus \{\mathbf{n}\}.$$

is the restriction of a conformal diffeomorphism on $\overline{\mathbb{B}^{m+1}}$. This concludes the proof. □

Since inversions (1.14) generate $\mathrm{Conf}(\mathbb{B}^{m+1})$, then the maps

$$I_p(y) := \frac{r^2}{|y-p|^2}(y-p)+p,$$

where $p \in \mathbb{R}^{m+1}$, $|p| > 1$, and $r^2 = |p|^2 - 1$, generate $\mathrm{Conf}(\mathbb{S}^m)$ by composition. Therefore, we have:

Proposition 1.6:
Any isometry of $\mathbb{H}^{m+1}$ induces a unique conformal diffeomorphism on its boundary at infinity $\mathbb{S}^m$, and viceversa.

1.4.4 Isometries of $\mathbb{H}^{m+1}$ using the Hyperboloid model

We know that all the isometries are generated by inversions in the Poincaré ball model. In the Hyperboloid model they are restrictions of linear aplications on $\mathbb{L}^{m+2}$, they are reflections with respect to vector with Lorentzian norm positive (see Proposition 1.3).

Then, every isometry of the Hyperbolic space $\mathbb{H}^{m+1}$ is the restriction to $\mathbb{H}^{m+1}$ of certain linear map in $\mathbb{L}^{m+2}$. This property is very useful when we apply isometries to hypersurfaces in the Hyperbolic space $\mathbb{H}^{m+1}$. Since the map is linear, all vector fields along the new hypersurface are just compositions of the isometry with the original vectors fields.

If the isometry in the Poincaré model is a restriction to the Poincaré ball of an orthogonal linear transformation in $\mathbb{R}^{m+1}$ then the corresponding isometry in the Hyperboloid model is the restriction of a linear map of $\mathbb{L}^{m+2}$.

Proposition 1.7:
The isometry $I : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ associated to a Euclidean orthogonal linear transformation $B : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ has the following correspondence rule:

$$I(y) = I(y_0, \overline{y}) = (y_0, B\overline{y}) \text{ for every } y \in \mathbb{H}^{m+1}.$$

1.5 Conformally Invariant Equations and Geometric Equations

In this section, we will define the kind of conformally invariant equations in $\mathbb{S}^m$ and geometric equations for hypersurfaces in $\mathbb{H}^{m+1}$ we will work with.

1.5.1 Elliptic problems for conformal metrics on domains of $\mathbb{S}^m$

Consider $g = e^{2\rho} g_0$ a conformal metric, $\rho \in C^2(\Omega)$, $\Omega \subseteq \mathbb{S}^m$, and denote by $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ the eigenvalues of the Schouten tensor of g.

We want to study partial differential equations for ρ relating the eigenvalues of the Schouten tensor. For instance, the simplest example one may consider are the σ_k-Yamabe problem on $\Omega \subseteq \mathbb{S}^m$, that is, the k-symmetric functions of the eigenvalues of the Schouten tensor.

We are interested in the fully nonlinear case of this problem, in the line opened by A. Li and Y.Y. Li (see [33, 34] and references therein). Namely, given (f, Γ) an elliptic data, find $\rho \in C^2(\Omega)$ so that $g = e^{2\rho} g_0$ is a solution of the problem

$$f(\lambda(g)) = 1 \ \text{ in } \ \Omega.$$

We must properly define the meaning of elliptic data (f, Γ) for conformal metrics. Define

$$\Gamma_m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,:\, x_i > 0, i = 1, \ldots, m\}$$

and

$$\Gamma_1 = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,:\, \sum_{i=1}^{m} x_i > 0 \right\}.$$

Let $\Gamma \subset \mathbb{R}^m$ be a convex open set satisfying:

(C1) It is symmetric. If $(x_1, \ldots, x_m) \in \Gamma$, then $(x_{i_1}, \ldots, x_{i_m}) \in \Gamma$, for every permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$.

(C2) It is a cone. For every $t > 0$, we have that $t(x_1, \ldots, x_m) \in \Gamma$ for every $(x_1, \ldots, x_m) \in \Gamma$.

(C3) $\Gamma_m \subset \Gamma \subset \Gamma_1$.

Then, we are ready to define:

Definition 1.8:
We say that (f, Γ) is an **elliptic data** *for conformal metrics in $\mathbb{S}^m$ if $\Gamma \subset \mathbb{R}^m$ is a convex cone satisfying (C1), (C2) and (C3) and $f \in C^0\left(\overline{\Gamma}\right) \cap C^1(\Gamma)$ is a function satisfying*

1. *f is symmetric in Γ.*
2. *$f|_{\partial\Gamma} = 0$.*
3. *$f|_{\Gamma} > 0$.*
4. *f is homogeneous of degree 1.*
5. *$\nabla f(x) \in \Gamma_m$ for every $x \in \Gamma$.*

This elliptic data is necessary for the defintion of non-degenerate elliptic problems and degenerate elliptic problems.

Definition 1.9 (Elliptic problems of conformal metrics):
Given an (f,Γ) an elliptic data for conformal metrics and Ω a domain of $\mathbb{S}^m$:

1. *The non-degenerate elliptic problem in $\Omega \subseteq \mathbb{S}^m$ is find to a conformal metric $g = e^{2\rho} g_0$ to the standard g_0 such that*
$$f(\lambda(g)) = 1 \quad on \quad \Omega$$
where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of g.
2. *The degenerate elliptic problem in $\Omega \subseteq \mathbb{S}^m$ is find to a conformal metric $g = e^{2\rho} g_0$ to the standard g_0 such that*
$$f(\lambda(g)) = 0 \quad on \quad \Omega$$
where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of g.

Of special importance in our work are regular elliptic data that we define now:

Remark 4 (Regular elliptic data):
Let (f,Γ) be an elliptic data for conformal metrics on subdomains on $\mathbb{S}^m$. Since f is positive in Λ and it is homogeneous of degree 1, the elliptic data (f,Γ) is regular, i.e., there is constant $\lambda > 0$ such that
$$f(\lambda, \ldots, \lambda) = 1.$$
That means, that for every domain $\Omega \subset \mathbb{S}^m$, the dilation $g = e^{2t_0} g_0$, where $t_0 \in \mathbb{R}$ satisfies $e^{-2t_0} = 2\lambda$, is a solution to the non-degenerate problem associated to the pair (f,Γ) on Ω.

Let us see two results of YanYan Li [32] that we will use in Chapter 4.

Theorem 1.10 (YanYan Li, [32])**:**
Set $m \geq 2$ and Ω is a domain in $\mathbb{S}^m$ such that $\overline{\Omega} \neq \mathbb{S}^m$. Let $g_1 = e^{2\rho_1} g_0$ and $g = e^{2\rho_1} g_0$ be solutions of the degenerate problem associate to (f,Γ) on $\overline{\Omega}$. If
$$\rho_1 > \rho \quad on\ \partial\Omega$$
then
$$\rho_1 > \rho \quad on\ \overline{\Omega}.$$

Theorem 1.11 (YanYan Li, [32])**:**
Set $m \geq 3$. Let $g = e^{2\rho_1} g_0$ be a conformal metric on $\mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\}$ such that
$$f(\lambda(g)) = 0 \quad on\ \mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\},$$
then g is rotationally invariant.

1.5.2 Elliptic problems for hypersurfaces in the Hyperbolic space $\mathbb{H}^{m+1}$

Now, we want to focus on elliptic problems for hypersurfaces in $\mathbb{H}^{m+1}$ in terms of its principal curvatures. The baby case we shall think about are hypersurfaces whose, for example, mean curvature is constant. In other words, we will consider $\Sigma \subset \mathbb{H}^{m+1}$ that satisfies an elliptic equation $(\mathscr{W}, \Gamma^*)$ for a given curvature function depending on its principal curvatures (see [27] and references therein); that is,

$$\mathscr{W}(\kappa) = 1 \text{ on } \Sigma,$$

where $\kappa = (\kappa_1, \dots, \kappa_m)$ is composed by the principal curvatures of the hypersurface Σ.

We must properly define the meaning of elliptic data $(\mathscr{W}, \Gamma^*)$ for hypersurfaces. Let $U \subset \mathbb{R}^m$ be a convex open set and define its translation

$$\Gamma^* = \{x \in \mathbb{R}^m : x = y - (1, \dots, 1), y \in U\}$$

such that Γ^* satisfies

(G1) Γ^* is symmetric.

(G2) $\Gamma_m \subset \Gamma^* \subset \Gamma_1$.

Analogously, we can define:

Definition 1.12:
We say that $(\mathscr{W}, \Gamma^*)$ is an **elliptic data** *for hypersurfaces in $\mathbb{H}^{m+1}$ if given a convex cone $U \subset \mathbb{R}^m$ the new cone Γ^* defined above satisfies (G1) and (G2) and $\mathscr{W} \in C^0\left(\overline{U}\right) \cap C^1(U)$ is a function satisfying:*

1. *$\mathscr{W}$ is symmetric in U.*
2. *$\mathscr{W}|_U > 0$.*
3. *$\mathscr{W}|_{\partial U} = 0$*
4. *$\frac{\partial \mathscr{W}}{\partial y_i}(y) > 0$ for every $y \in U$ and $i = 1, \dots, m$.*

Definition 1.13 (Elliptic problems of hypersurfaces in $\mathbb{H}^{m+1}$):
Given an $(\mathscr{W}, U)$ an elliptic data for hypersurfaces in $\mathbb{H}^{m+1}$:

1. *The non-degenerate elliptic problem is find an oriented immersed hypersurface Σ in $\mathbb{H}^{m+1}$ such that*
$$\mathscr{W}(\kappa_1, \dots, \kappa_m) = 1 \quad on \quad \Sigma$$
where $\kappa_1, \dots, \kappa_m$ are the principal curvatures of Σ.

2. *The degenerate elliptic problem is find an oriented immersed hypersurface Σ in $\mathbb{H}^{m+1}$ such that*

$$\mathscr{W}(\kappa_1,\ldots,\kappa_m)=0 \quad on \quad \Sigma$$

where $\kappa_1,\ldots,\kappa_m$ are the principal curvatures of Σ.

Remark 5:
In the case of degenerate elliptic problems of hypersurfaces in $\mathbb{H}^{m+1}$: Since $\mathscr{W}(1,\ldots,1)=0$, then the horospheres with the natural orientation given by its mean curvature vector is a solution of the degenerate problem associate to the data $(\mathscr{W},U)$.

As above, of special importance in our work are regular elliptic data that we define now:

Definition 1.14:
Given $\mathscr{W}\in C^0\left(\overline{U}\right)\cap C^1(U)$ an elliptic data for hypersurfaces in $\mathbb{H}^{m+1}$, we say that $(\mathscr{W},U)$ is regular if there is a constant $\kappa_0>1$ such that

$$\mathscr{W}(\kappa_0,\ldots,\kappa_0)=1.$$

Remark 6:
In the case of non-degenerate elliptic problems of hypersurfaces in $\mathbb{H}^{m+1}$ with regular elliptic data, we have geodesic spheres as solutions of that kind of problems, such spheres are the totally umbilic hypersurfaces with constant umbilic equals to $\kappa>1$.

Chapter 2

Local Representation

In this Chapter we will recall the local equivalence between horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$ and conformal metrics on $\mathbb{S}^m$.

We begin by giving the definition of the Hyperbolic Gauss map for an oriented immersed hypersurface in $\mathbb{H}^{m+1}$. In such definition, we use the boundary at infinity of the Hyperbolic space, also called ideal boundary of $\mathbb{H}^{m+1}$, that is, the sphere $\mathbb{S}^m$. There are sufficient and necessary conditions for the hyperbolic Gauss map to be a local diffeomorphism. One of these conditions is related to the regularity of the light cone map of an oriented hypersurface that will be defined in Subsection 2.1.2. Others conditions are related to the principal curvatures of the given oriented hypersurface.

Then, we define one of the important objects in our study, horospherically concave hypersurfaces in $\mathbb{H}^{m+1}$. These hypersurfaces are oriented and they have the property that its Hyperbolic Gauss map is a local diffeomorphism. The importance of this class of hypersurfaces is that, locally, we can give a conformal metric over the image of the Hyperbolic Gauss map (conformal to the standard metric g_0 on the sphere $\mathbb{S}^m$). Suppose that $g = e^{2\rho} g_0$ is this conformal metric, $\rho \in C^\infty(\Omega)$, where Ω is a small open domain that is contained in the image of the Hyperbolic Gauss map, the function ρ has a geometric interpretation that is related to tangent horospheres to the original hypersurface. In the Poincaré ball model, ρ is the signed hyperbolic distance between the tangent horosphere and the origin of the Poincaré ball.

Beside that we get a conformal metric on the image of the Hyperbolic Gauss map, if this is injective, The Local Representation Theorem (cf. Theorem 2.10) says that the function ρ is all that we need to recover the original hypersurface. Such theorem is of great importance, because we can obtain horospherically concave hypersurfaces from conformal metrics on domains Ω of the sphere $\mathbb{S}^m$ if we impose certain conditions. This conformal metric is called the horospherical metric of the horospherically concave hypersurface in $\mathbb{H}^{m+1}$.

In Section 2.4, we study how isometries in the Hyperbolic space $\mathbb{H}^{m+1}$ affect the horospherical metric, more precisely, how the horosherical metric of the hypersurface changes when we apply an isometry to this hypersurface. In particular, Section 2.5, if the horospherically concave hypersurface is invariant under an isometry in $\mathbb{H}^{m+1}$ then the associated horospherical metric is invariant under a conformal diffeomorphism of $\mathbb{S}^m$.

We finalize this chapter introducing the elliptic problems for conformal metric on domains on the sphere $\mathbb{S}^m$ and elliptic problems for hypersurfaces in the hyperbolic space $\mathbb{H}^{m+1}$ and their relation under the Local Representation Theorem.

2.1 Horospherically Concave Hypersurfaces in $\mathbb{H}^{m+1}$

Here, we study the hypersurfaces in $\mathbb{H}^{m+1}$ with regular hyperbolic Gauss map in terms of their principal curvatures and their tangent horospheres.

2.1.1 Horospheres and the hyperbolic Gauss map

From the hypersurface theory viewpoint, horospheres are the only flat totally umbilical hypersurfaces in $\mathbb{H}^{m+1}$, and they are complete and embedded. All of this suggests that horospheres can be naturally regarded in many ways as *hyperplanes* in the hyperbolic space $\mathbb{H}^{m+1}$, even though they are not totally geodesic.

We shall work in the Hyperboloid model of $\mathbb{H}^{m+1}$. From now on, $\phi : \Sigma \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ will denote an oriented immersed hypersurface and $\eta : \Sigma \longrightarrow \mathbb{S}_1^{m+1}$ its unit normal, here $\mathbb{S}_1^{m+1}$ denotes the de-Sitter space

$$\mathbb{S}_1^{m+1} = \left\{ x \in \mathbb{L}^{m+2} : \ll x, x \gg = 1 \right\}.$$

Definition 2.1 ([11, 12, 6])**:**
Let $\phi : \Sigma \to \mathbb{H}^{m+1}$ be an oriented immersed hypersurface with unit normal vector field η. The Hyperbolic Gauss map of that oriented hypersurface is the map $G : \Sigma \to \mathbb{S}^m_\infty$, that associates every point $p \in \Sigma$ the point at infinity of the unique horosphere in $\mathbb{H}^{m+1}$ passing through $\phi(p)$ and whose inner unit normal at p agrees with $-\eta(p)$ [cf. Figure 2.1].

Let us point out here that horospheres are globally convex, what allows us to talk about either the *outward* or the *inward* orientation of a horosphere, meaning this simply that the unit normal points either at the concave or convex side of the horosphere. With respect to the inner orientation, the second fundamental form of a horosphere is positive definite. Moreover, it turns out that innerly oriented horospheres are the only hypersurfaces in $\mathbb{H}^{n+1}$ with constant hyperbolic Gauss map.

There is an equivalent definition: the *hyperbolic Gauss map* $G : \Sigma \to \mathbb{S}^m_\infty$ sends each $p \in \Sigma$ to the point $G(p)$ at the ideal boundary $\mathbb{S}^m_\infty$ reached by the unique geodesic γ of $\mathbb{H}^{m+1}$ that starts at $\phi(p)$ with initial speed $-\eta(p)$. In certain sense, the hyperbolic Gauss map is the analogous concept in the hyperbolic space to the classical Gauss map for hypersurfaces of $\mathbb{R}^{m+1}$.

It must however be remarked that the *a priori* chosen orientation for the hypersurface matters for the hyperbolic Gauss map. Indeed, if we change the orientation of Σ, then G turns into the *negative hyperbolic Gauss map* $G^- : \Sigma \to \mathbb{S}^m$, whose behavior is totally different to that of G.

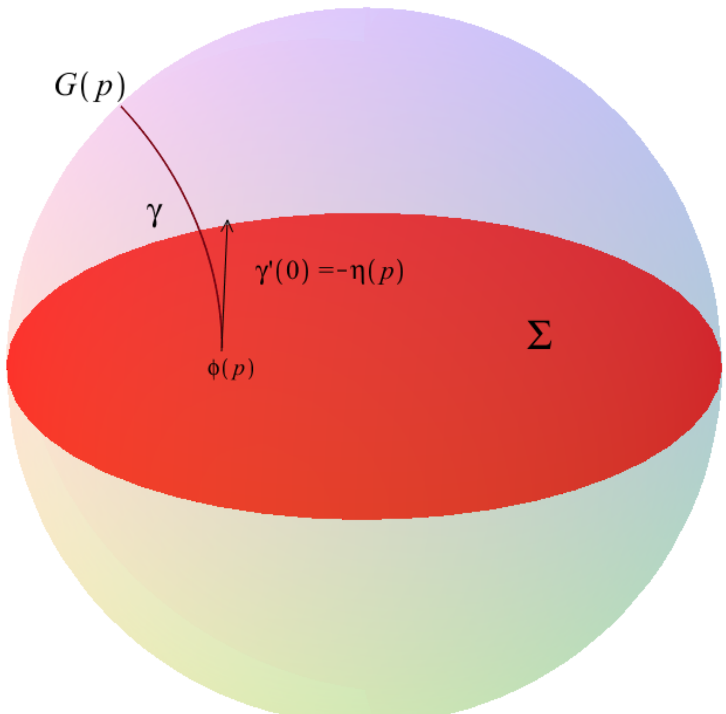


Figure 2.1: The Hyperbolic Gauss map

2.1.2 Regularity of the hyperbolic Gauss map

We shall work in the Hyperboloid model of $\mathbb{H}^{m+1}$. Let $\phi : \Sigma \to \mathbb{H}^{m+1}$ be an immersed oriented hypersurface, and let $\eta : \Sigma \to \mathbb{S}_1^{m+1}$ denote its unit normal. Then, we can define a normal map associated to ϕ taking values in the light cone as

$$\psi = \phi - \eta : \Sigma \to \mathbb{N}_+^{m+1}. \tag{2.1}$$

The map ψ is strongly related to the hyperbolic Gauss map $G : \Sigma \to \mathbb{S}_\infty^m$ of ϕ. Indeed, the ideal boundary of $\mathbb{N}_+^{m+1}$ coincides with $\mathbb{S}_\infty^m$, and can be identified with the projective quotient space $\mathbb{N}_+^{m+1}/\mathbb{R}_+$. So, with all of this, we have $G = [\psi] : \Sigma \to \mathbb{S}_\infty^m \equiv \mathbb{N}_+^{m+1}/\mathbb{R}_+$. The map ψ is called *the light cone map* of the oriented hypersurface $\phi : \Sigma \to \mathbb{H}^{m+1}$.

Moreover, if we write $\psi = (\psi_0, \dots, \psi_{m+1})$, then we may interpret G as the map $G : \Sigma \to \mathbb{S}^m$ given by

$$G = \frac{1}{\psi_0}(\psi_1, \dots, \psi_{m+1}). \tag{2.2}$$

In this way, if we label $e^\rho := \psi_0$, we get the useful relation

$$\psi = e^\rho(1, G) : \Sigma \to \mathbb{N}_+^{m+1}. \tag{2.3}$$

Observe also that, by differentiating (2.3) it follows that

$$\ll d\psi, d\psi \gg = e^{2\rho}\langle dG, dG\rangle_{\mathbb{S}^m}. \tag{2.4}$$

We introduce thus the following terminology, in analogy with the Euclidean setting.

Definition 2.2:
The smooth function $\rho : \Sigma \to \mathbb{R}$ will be called the horospherical support function*, or just the* support function*, of the hypersurface $\phi : \Sigma \to \mathbb{H}^{m+1}$.*

Besides, if $\{e_1, \dots, e_m\}$ denotes an orthonormal basis of principal directions of ϕ at p, and if $\kappa_1, \dots, \kappa_m$ are the associated principal curvatures, it is immediate that

$$\ll d\psi(e_i), d\psi(e_j) \gg = (1 + \kappa_i)^2 \delta_{ij}. \tag{2.5}$$

Thus, we have:

Lemma 2.3:
Let $\phi : \Sigma \to \mathbb{H}^{m+1}$ be an oriented hypersurface. The following conditions are equivalent at $p \in \Sigma$.

(i) The hyperbolic Gauss map G is a local diffeomorphism.

(ii) The associated light cone map ψ in (2.1) *is regular.*

(iii) All principal curvatures of Σ *are* $\neq -1$.

The regularity of the hyperbolic Gauss map gives rise to a notion of convexity specific of the hyperbolic setting, and weaker than the usual geodesic convexity notion.

Definition 2.4 ([42])**:**
Let $\Sigma \subset \mathbb{H}^{m+1}$ *be an immersed oriented hypersurface with* $\eta : \Sigma \to \mathbb{S}_1^{m+1}$ *its unit normal, and let* $\mathscr{H}_p$ *denote the horosphere in* $\mathbb{H}^{n+1}$ *that is tangent to* Σ *at* p*, and whose interior unit normal at* p *agrees with* $-\eta(p)$*. We will say that* Σ *is* horospherically concave *at* p *if there exists a neighborhood* $V \subset \Sigma$ *of* p *so that* $V \setminus \{p\}$ *does not intersect* $\mathscr{H}_p$*,* $V \setminus \{p\}$ *is contained in the concave side of* $\mathscr{H}_p$*, and in addition the distance function of the hypersurface to the horosphere does not vanish up to the second order at* p *in any direction [cf. Figure 2.2].*

As we already pointed out, in the Hyperboloid model, the (outward) unit normal vector field to the horosphere is given by $n(y) = y - a$ for every y in the horosphere [see the equation (1.9)]. Then, the horosphere that contains the point $p \in \Sigma$ and has outward unit normal $\eta(p)$ is given by $a = p - \eta(p)$.

Hence, given $\phi : \Sigma \to \mathbb{H}^{m+1}$ a horospherically concave hypersurface with unit normal vector field η, we have that the principal curvatures are greater than -1 for every $p \in \Sigma$. Let us explain this. Let $\xi = -\eta$ be the new unit normal vector field along ϕ. Take $p \in \Sigma$ and consider horosphere $\mathscr{H}_p$ that contains p and its inward unit normal agrees with $\xi(p)$. Since the hypersurface is locally outside the horoball determined by $\mathscr{H}_p$, the principal curvatures of the hypersurface ϕ with respect to ξ are less or equal to 1. Then, if we consider the original orientation of ϕ, that is, $\eta = -\xi$, then the principal curvatures of ϕ at p with respect to η are greater or equal to -1. If the Gauss map is a local diffeomorphism then all the principal curvatures are different from -1. Therefore, this definition can be immediately characterized as follows.

Corollary 2.5:
An oriented hypersurface $\Sigma \subset \mathbb{H}^{m+1}$ *is horospherically concave at* $p \in \Sigma$ *if and only if all the principal curvatures of* Σ *at* p *verify simultaneously* $\kappa_i(p) > -1$.

In particular, if Σ is horospherically concave at p any of the equivalent conditions in Lemma 2.3 holds.

2.1.3 Horospherical ovaloids

We will study first the compact case.

Definition 2.6:
A compact immersed hypersurface $\phi : \Sigma \to \mathbb{H}^{m+1}$ *will be called a* horospherical ovaloid *of* $\mathbb{H}^{m+1}$ *if it can be oriented so that it is horospherically concave at every point.*

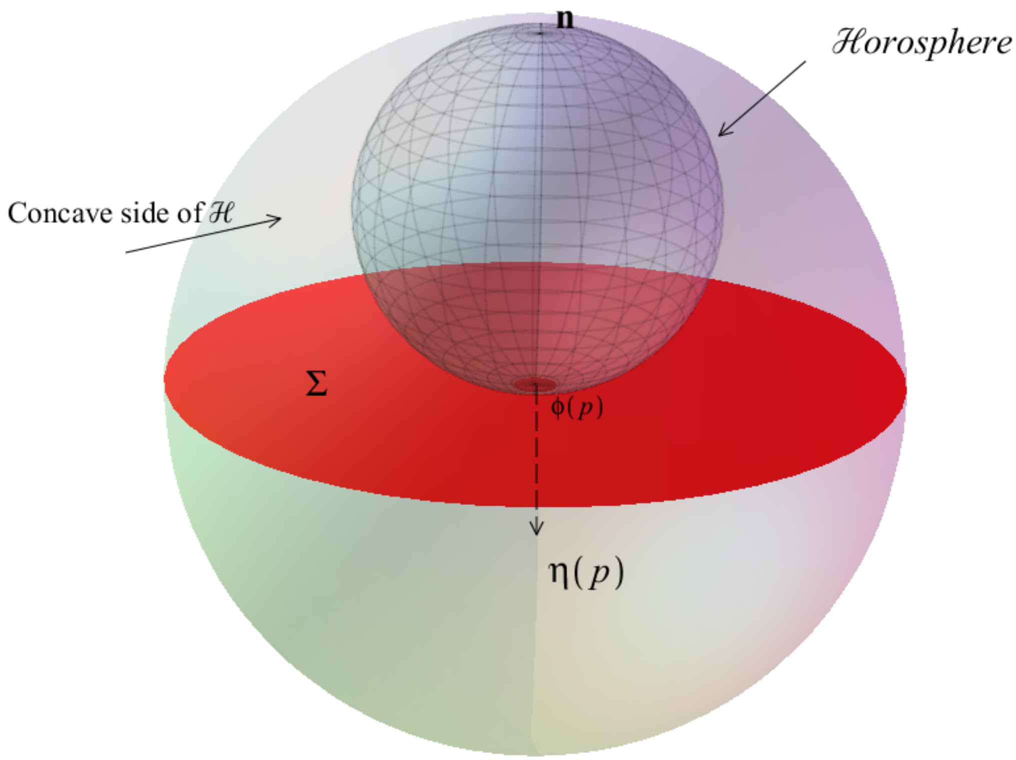


Figure 2.2: Horospherically concave hypersurface
in the Poincaré ball model

Equivalently, a compact hypersurface is a horospherical ovaloid if and only if it can be oriented so that its hyperbolic Gauss map is a global diffeomorphism. This equivalence follows directly from Lemma 2.3 and Corollary 2.5 by a simple topological argument, bearing in mind that every compact hypersurface in $\mathbb{H}^{m+1}$ has a point p at which $|\kappa_i(p)| > 1$ for every i. In particular, Σ is diffeomorphic to $\mathbb{S}^m$.

It is also immediate from the existence of this point with $|\kappa_i(p)| > 1$ that every horospherical ovaloid has a unique orientation such that $\kappa_i > -1$ everywhere for every $i = 1, \dots, n$. We call this orientation the *canonical orientation* of the horospherical ovaloid. It follows that the hyperbolic Gauss map of a canonically oriented horospherical ovaloid is always a global diffeomorphism. This is not necessarily true anymore for the other possible orientation. Let us also point out that if p is a point of a canonically oriented horospherical ovaloid $\Sigma \subset \mathbb{H}^{m+1}$, with canonical orientation η, then Σ lies around p in the concave part of the unique horosphere that passes through p and whose interior unit normal at p agrees with $-\eta(p)$.

Recall that a compact hypersurface $\Sigma \subset \mathbb{H}^{m+1}$ is a (strictly convex) *ovaloid* if all its principal curvatures are non-zero and of the same sign. Thus, any ovaloid is a horospherical ovaloid, but the converse is not true.

Nonetheless, let us point out that a horospherical ovaloid is not necessarily embedded (cf. [19]). For instance, take a regular curve $\alpha : [0,1] \to \mathbb{H}^2$ with geodesic curvature smaller than 1 at every point, and such that $\alpha(0) = \alpha(1)$ and, moreover, $\alpha'(0) = -\alpha'(1)$. Then by considering $\mathbb{H}^2$ as a totally geodesic surface of $\mathbb{H}^3$ and after rotating α across the geodesic of $\mathbb{H}^2$ that meets α orthogonally at $\alpha(0)$, we get a surface of revolution in $\mathbb{H}^3$ that is a non-embedded horospherical ovaloid.

This lack of embeddedness shows that one cannot talk in general about the outer orientation of a horospherical ovaloid, and justifies the way we introduced the canonical orientation for them.

2.1.4 Parallel Flow

Another interesting feature of canonically oriented horospherical ovaloids is its good behavior regarding the parallel flow. As usual, the *parallel flow* of an oriented hypersurface $\phi : \Sigma \to \mathbb{H}^{m+1}$ is defined for every $t \in \mathbb{R}$ as $\phi_t : \Sigma \to \mathbb{H}^{m+1}$,

$$\phi_t(p) = \exp_{\phi(p)}(-t\eta(p)) : \Sigma \to \mathbb{H}^{m+1}, \tag{2.6}$$

where exp denotes the exponential map of $\mathbb{H}^{m+1}$, and $\eta(p)$ is the canonical unit normal of ϕ at p. It is then easy to check that if ϕ is a canonically oriented horospherical ovaloid, then the *forward* flow $\{\phi_t\}_t$, $t \geq 0$, is made up by regular canonically oriented horospherical ovaloids. This is no longer true in general for the *backwards* flow (i.e. $t < 0$) due to the possible appearance of wave front singularities of the hypersurfaces (cf. [37]).

2.1.5 The horospherical metric

It will be important for our purposes to associate a natural metric to the space of horospheres in $\mathbb{H}^{m+1}$. This construction has appeared in other works previously, but we reproduce it here in order to put special emphasis on some aspects.

Let $\mathscr{M}$ denote the space of horospheres in $\mathbb{H}^{m+1}$. As we have see at the Introduction, we can parametrize $\mathscr{M}$ as the cylinder $\mathbb{R}\times\mathbb{S}^m$, where a horosphere $\mathscr{H}\in\mathscr{M}$ correspond to a point $(t,x)\in\mathbb{R}\times\mathbb{S}^n$ as x is the point at infinity of $\mathscr{H}$ and t is the (signed) hyperbolic distance of $\mathscr{H}$ to a fixed point $p\in\mathbb{H}^{m+1}$ that, without loss of generality, we can assume that p is the origin when we consider the Poincaré Ball Model of $\mathbb{H}^{m+1}$. Here, t is negative if p is contained in the convex domain bounded by $\mathscr{H}$.

Let us now construct a natural metric on this space of horospheres. Points of the form $(0,x)$ correspond to horospheres passing through the origin in the Poincaré ball model. It is then natural to endow each of these points with the canonical metric g_0 of $\mathbb{S}^m$ evaluated at x.

But now, the horosphere (t,x) is the parallel hypersurface to the horosphere corresponding to $(0,x)$, and the induced metric of $\mathbb{H}^{m+1}$ in this parallel horosphere is a dilation of the one of $\mathscr{H}\equiv(0,x)$, by the factor e^{2t}. Thus, the natural metric to define at (t,x) is the dilated metric $e^{2t}g_0$ evaluated at x, here g_0 is the standar metric on $\mathbb{S}^m$. Consequently, we may view the space of horospheres in $\mathbb{H}^{m+1}$ as the product $\mathbb{R}\times\mathbb{S}^m$ endowed with the natural degenerate metric

$$\langle,\rangle_\infty := e^{2t}g_0.$$

Observe that the vertical rulings of $\mathbb{R}\times\mathbb{S}^m$ are null lines with respect to this degenerate metric.

Definition 2.7:
Let $\phi:\Sigma\to\mathbb{H}^{n+1}$ denote an oriented hypersurface in $\mathbb{H}^{m+1}$, and let $\psi:\Sigma\to\mathbb{L}^{m+2}$ be its light map. We define the horospherical metric *g_∞ of ϕ as*

$$g_\infty := \psi^*(\ll,\gg),$$

i.e. as the pullback metric via the light map ψ of the Lorentzian metric $\ll,\gg$.

It turns out that the horospherical metric is everywhere regular if and only if the hyperbolic Gauss map of the hypersurface is a local diffeomorphism. This is a consequence of Lemma 2.3 and the following interpretation of the horospherical metric in the Hyperboloid model of $\mathbb{H}^{m+1}$. Recall that horospheres of $\mathbb{H}^{m+1}\subset\mathbb{L}^{m+2}$ are the intersections of affine degenerate hyperplanes of $\mathbb{L}^{m+2}$ with $\mathbb{H}^{m+1}$. Note that a horosphere is horospherically concave if we consider its outward orientation η, that is, the orientation pointing at the concave side. Then, a simple calculation shows that horospheres are characterized by the fact that its associated light cone map is constant, i.e., $\phi-\eta=v\in\mathbb{N}^{m+1}_+$. Moreover, if we write $v=e^\rho(1,x)$, we see that $x\in\mathbb{S}^m$ is the point at infinity of the horosphere, and that parallel horospheres correspond to collinear vectors in $\mathbb{N}^{n+1}_+$. This shows that the space of horospheres in $\mathbb{H}^{m+1}$ is naturally identified with the positive null cone $\mathbb{N}^{m+1}_+$. Thus, it is natural to endow this space with the canonical degenerate metric of the light cone, and it is quite obvious from the above construction that this light cone metric coincides with the degenerate metric $\langle,\rangle_\infty$ defined above.

Consequently,

Proposition 2.8:
The horospherical metric on a hypersurface in $\mathbb{H}^{m+1}$ is the pullback metric of its associated light cone map. Thus, it is regular if and only if the hyperbolic Gauss map is a local diffeomorphism.

Remark 7:
All this construction is clearly reminiscent of the usual identification of the space of oriented vector hyperplanes in $\mathbb{R}^{m+1}$ with the unit sphere $\mathbb{S}^m$. In this sense, just as the canonical $\mathbb{S}^m$ metric is used in order to measure geometric quantities associated to the Euclidean Gauss map of a hypersurface in $\mathbb{R}^{m+1}$, we will use the horospherical metric for measuring geometrical quantities with respect to the hyperbolic Gauss map. Let us explain in more detail this consideration, that was first done by Epstein [13].

First, observe that the ideal boundary $\mathbb{S}^m_\infty$ of $\mathbb{H}^{m+1}$ does not carry a geometrically useful metric (although it has a natural conformal structure), so we cannot endow G with a pullback metric from the ideal boundary. Nonetheless, let us observe that for defining the hyperbolic Gauss map G we need to know the exact point $p \in \mathbb{H}^{m+1}$ at which we are working (this does not happen for the Euclidean Gauss map). The additional knowledge of this point is then equivalent to the knowledge of the tangent horosphere to the hypersurface at the point. So, it is natural to use the horospherical metric for measuring lenghts associated to the hyperbolic Gauss map. An alternative justification can be found in [13] in connection with the parallel flow of hypersurfaces.

It is interesting to observe that the horospherical metric has played an important role in several different theories. For instance, it is equivalent to the Kulkarni-Pinkall metric [28] (see [42]). It also happens that the area of a Bryant surface in $\mathbb{H}^3$ with respect to the horospherical metric is exactly the total curvature of the induced metric of the surface.

2.1.6 The support function

Let $\phi : \Sigma \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ be an oriented immersion with unit normal vector field η. Remember that we can associate a horosphere $\mathscr{H}_p$ to every point $p \in \Sigma$. That horosphere is the horosphere that contains p and its interior normal agrees with $-\eta(p)$ at p. Set $(t,x) = p - \eta(p)$. Recall the geometric meaning of the hyperbolic support function:

Definition 2.9:
We define the support function as the function that associates every $p \in \Sigma$, the signed distance between the tangent horosphere $\mathscr{H}_p$ and the origin $(1,\overline{0})$.

The signed distance is positive if the origin is outside of the horoball that defines the horosphere, and negative in the other case, i.e., if the origin is inside of the horoball. That is, the support function $\rho : \Sigma \to \mathbb{R}$ is defined

$$\rho(p) = \log(t),$$

where $(t,x) = p - \eta(p) \in \mathbb{N}^{m+1}_+$, and log is the natural logarithm function.

2.1.7 Support function, Horospherical Metric and Hyperbolic Gauss map

Let $\phi : \Sigma \to \mathbb{H}^{m+1}$ be an oriented immersion with unit normal vector field η. Given $p \in \Sigma$, we associate one light vector as $(t,x) = p - \eta(p) \in \mathbb{N}^{m+1}_{+}$, so we have

$$\psi(p) = (t,x) = t\left(1, t^{-1}x\right),$$

i.e.,

$$\psi(x) = e^{\rho(p)}\left(1, G(p)\right),$$

where $G(p)$ is the horospherical Gauss map at the point p. Then, the horospherical metric satisfy

$$g_\infty(v,w) = e^{2\rho(p)} g_0\left(dG_p(v), dG_p(w)\right), \text{ for every } x \in \Sigma \text{ and } v,w \in T_x\Sigma,$$

here g_0 is the standard inner product on the sphere $\mathbb{S}^m$, at the point $G(p)$.

2.2 Injective Gauss map and Representation Formula

Let $\phi : \Sigma \to \mathbb{H}^{m+1}$ be an horospherically concave hypersurface with canonical orientation η. We assume that its horospherical Gauss map $G : \Sigma \to \mathbb{S}^m$ is regular and injective, then, $\Omega = G(\Sigma)$ is a domain in the sphere $\mathbb{S}^m$. Hence, we can parametrize the manifold Σ using the inverse of G, $G^{-1} : \Omega \to \Sigma$, then we can consider the immersion $\varphi = \phi \circ G^{-1} : \Omega \to \mathbb{H}^{m+1}$ with the unit normal vector field $\tilde{\eta} = \eta \circ G^{-1}$.

Therefore, we have that the hyperbolic Gauss map is just the inclusion of Ω in the sphere $\mathbb{S}^m$, that is, $x \in \Omega \mapsto \tilde{G}(x) = x$. If we still denote by ρ the support function of $\varphi : \Omega \subset \mathbb{S}^m \to \mathbb{H}^{m+1}$, then the horospherical metric of φ is just

$$g_\infty = e^{2\rho} \cdot g_0,$$

where g_0 is the restriction of the usual metric on $\mathbb{S}^m$ to Ω.

Thus, identifying ϕ with φ and η with $\tilde{\eta}$, one can recover the immersion ϕ in terms of the hyperbolic support function when the hyperbolic Gauss map is injective.

Theorem 2.10 (Local Representation Theorem [19])**:**
Let $\phi : \Omega \subseteq \mathbb{S}^m \longrightarrow \mathbb{H}^{m+1}$ be a piece of horospherically concave hypersurface with Gauss map $G(x) = x$. Then, it holds

$$\phi = \frac{e^\rho}{2}\left(1 + e^{-2\rho}\left(1 + |\nabla\rho|^2\right)\right)(1,x) + e^{-\rho}(0, -x + \nabla\rho). \tag{2.7}$$

Moreover, the eigenvalues λ_i of the Schouten tensor of the horospherical metric $\hat{g} = e^{2\rho} g_0$ and the principal curvatures κ_i of ϕ are related by

$$\lambda_i = \frac{1}{2} - \frac{1}{1+\kappa_i}. \tag{2.8}$$

Conversely, given a conformal metric $\hat{g} = e^{2\rho} g_0$ *defined on a domain of the sphere* $\Omega \subseteq \mathbb{S}^m$ *such that the eigenvalues of its Schouten tensor are all less than* $1/2$*, the map* ϕ *given by* (2.7) *defines an immersed, horospherically concave hypersurface in* $\mathbb{H}^{m+1}$ *whose Gauss map is* $G(x) = x$ *for* $x \in \Omega$ *and whose horospherical metric is the given metric* $\hat{g}$*.*

Remark 8:
The above Representation Formula can be seen as the hyperbolic analog to the representation formula for convex ovaloid in $\mathbb{R}^{m+1}$ *(cf. [21, 22])*

Note that Theorem 2.10 is local in nature, that is, the hyperbolic Gauss map is (locally) a global diffeomorphism (onto its local image) and, therefore, one can use Theorem 2.10 to (locally) parametrize any horospherically concave ovaloid.

In the Poincaré ball model, the Representation formula is

$$\varphi_P(x) = \frac{1 - e^{-2\rho(x)} + |\nabla e^{-\rho}(x)|^2}{\left(1 + e^{-\rho(x)}\right)^2 + |\nabla e^{-\rho}(x)|^2} x - \frac{1}{\left(1 + e^{-\rho(x)}\right)^2 + |\nabla e^{-\rho}(x)|^2} \nabla\left(e^{-2\rho}\right)(x), \quad \forall x \in \Omega. \tag{2.9}$$

In the Klein model, the Representation formula is

$$\varphi(x) = \frac{1 - e^{-2\rho(x)} + |\nabla e^{-\rho}(x)|^2}{1 + e^{-2\rho(x)} + |\nabla e^{-\rho}(x)|^2} x - \frac{1}{1 + e^{-2\rho(x)} + |\nabla e^{-\rho}(x)|^2} \nabla\left(e^{-2\rho}\right)(x), \quad \forall x \in \Omega.$$

Let $\sigma = e^{-\rho}$, then the Representation formula in the Klein model can be written

$$\varphi(x) = \frac{1 - \sigma^2(x) + |\nabla\sigma(x)|^2}{1 + \sigma^2(x) + |\nabla\sigma(x)|^2} x - \frac{1}{1 + \sigma^2(x) + |\nabla\sigma(x)|^2} \nabla\left(\sigma^2\right)(x), \quad \forall x \in \Omega. \tag{2.10}$$

This formula in the Klein model will be used in Chapters 3 and 4.

2.3 From conformal metric to hypersurfaces

A consequence of Theorem 2.10 is that, if G is injective, the horospherical metric $g = e^{2\rho} g_0$ is well-defined on Ω. Now, we want to invesitgate the converse.

In this section we work with the hyperboloid model of the Hyperbolic space $\mathbb{H}^{m+1}$, that is, $\mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ and $\ll, \gg$ will denote the Lorentzian metric of $\mathbb{L}^{m+2}$.

Let Ω be a domain in the sphere $\mathbb{S}^m$ and $\rho : \Omega \to \mathbb{R}$ a smooth function. Then, we can define the map $\phi : \Omega \to \mathbb{H}^{m+1}$ by

$$\phi(x) = \lambda(x)(1, x) + e^{-\rho(x)}(0, -x + \nabla\rho(x)),$$

where $x \in \Omega$, $\lambda(x) = \frac{e^{\rho(x)}}{2}\left(1 + e^{-2\rho(x)}\left(1 + |\nabla\rho(x)|^2\right)\right)$, and $\nabla\rho(x)$ is the gradient of ρ w.r.t. g_0 at the point x.

The map defined above is not necessary an immersion. Such map will be an immersion if the conformal metric $g = e^{2\rho} g_0$ satisfies certain conditions. More precisely we have:

Theorem 2.11:
The map $\phi : \Omega \to \mathbb{H}^{m+1}$ defined as

$$\phi(x) = \lambda(x)\,(1,x) + e^{-\rho(x)}\,(0,-x+\nabla\rho(x))\,,$$

for $x \in \Omega$, is an immersion, if only if, all the eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$ are different from $1/2$.

The above Theorem can be found in [19]. Nevertheless, we will give a different proof here. We can re-write ϕ as

$$\phi(x) = \lambda(x)\,(1,x) - \sigma(x)\,(0,x) - (0,\nabla\sigma_x)\,, \tag{2.11}$$

for every $x \in \Omega$, where $\sigma = e^{-\rho}$. Consider the eigenvalues of the Hessian of σ w.r.t. the standard metric g_0. Fix $x \in \Omega$ and let s be one eigenvalue of $Hess(\sigma)_x$ and $u \in T_x\Omega$ be a unit eigenvector associated to the eigenvalue s. We have that

$$d\phi_x(u) = \left(\frac{1}{2} - \lambda\right)[d\psi_x(u) + (0,u)]\,,$$

where

$$\lambda = s \cdot \sigma(x) + \frac{1}{2}\left(\sigma(x)^2 - |\nabla\sigma_x|^2\right)$$

and $\psi : \Omega \to \mathbb{H}^{m+1}$ is defined like $\psi(x) = e^{\rho(x)}\,(1,x)$ for every $x \in \Omega$. Let $\{u_1,\dots,u_m\}$ be an orthonormal base of $T_x\Omega$ that diagonalize the symmetric operator $Hess(\sigma)_x$, that is, there are $s_1,\dots,s_m \in \mathbb{R}$ such that

$$Hess(\sigma)_x(u_i) = s_i \cdot u_i \text{ for } i = 1,\dots,m.$$

We have,

Proposition 2.12:
Let $\{u_1,\dots,u_m\}$ be an orthonormal base of $T_x\Omega$, with respect to g_0, that diagonalize the symmetric operator $Hess_x(\sigma)$, then

$$\ll d\phi_x(u_i), d\phi_x(u_j) \gg = \left(\frac{1}{2} - \lambda_i\right)\left(\frac{1}{2} - \lambda_j\right)\delta_{ij},$$

where $i,j = 1,\dots,m$ and

$$\lambda_i = s_i \cdot \sigma(x) + \frac{1}{2}\left(\sigma(x)^2 - |\nabla\sigma_x|^2\right).$$

We note that the map $\eta = \phi - \psi$ defines a unit normal vector field along ϕ, and also ψ is its light map. Its horospherical Gauss map is just the inclusion $\Omega \hookrightarrow \mathbb{S}^m$ as we can see easily. In particular, this is an oriented immersion with injective Hyperbolic Gauss map, whose horospherical metric is $g = e^{2\rho} g_0$.

We summarize this in the following theorem:

Theorem 2.13:
If all eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$ are different of $\frac{1}{2}$, then the map $\phi : \Omega \to \mathbb{H}^{m+1}$ defines an immersion, and $\eta = \phi - \psi$ is a unit normal vector field along ϕ such that the horosperical Gauss map of ϕ is the inclusion $\Omega \hookrightarrow \mathbb{S}^m$.

Using Proposition 2.12, we obtain the reciprocal statement:

Theorem 2.14:
Let $\phi : \Omega \to \mathbb{H}^{m+1}$ be an oriented hypersurface whose horospherical Gauss is the inclusion $\Omega \hookrightarrow S^m$ and support function ρ. Then, all the eigenvalues of the Schouten tensor of its horospherical metric $g = e^{2\rho} g_0$ are different of $\frac{1}{2}$.

2.4 Changing the horospherical metric by an isometry in $\mathbb{H}^{m+1}$

Let $\phi : \Omega \to \mathbb{H}^{m+1}$ be an oriented horospherically concave hypersurface with unit normal vector field η and $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ be an isometry. We can consider the hypersurface $T\phi : \Omega \to \mathbb{H}^{m+1}$ with normal $T\eta : \Omega \to \mathbb{S}_1^{m+1}$. Let g be the horospherical metric of ϕ and g_1 be the horosppherical metric of $T\phi$.

In this section we will see that there is a natural relation between the horospherical metrics g_1 and g_0.

Proposition 2.15:
Let $\phi : \Omega \to \mathbb{H}^{m+1}$ be an oriented horospherically concave hypersurface with unit normal vector field η such that its hyperbolic Gauss map is the inclusion $\Omega \hookrightarrow \mathbb{S}^m$, g is the horospherical metric of ϕ and $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ is an isometry. If g_1 is the horospherical metric of $T\phi : \Omega \to \mathbb{H}^{m+1}$ with respect the orientation $T\eta$ and B is the conformal diffeomorphism on $\mathbb{S}^m$ associated to T, then

$$B^* (g_1) = g \text{ on } \Omega.$$

Proof. First, we compute the light map of $T\phi$,

$$\psi' = T\phi - T\eta = T(\psi) \text{ on } \Omega,$$

where ψ is the light map of ϕ. Let $\psi = e^\rho (1, G)$ where G is the inclusion $\Omega \hookrightarrow \mathbb{S}^m$ then

$$\psi' = e^\rho T(1, G) \text{ on } \Omega.$$

Therefore, $\psi'(x) = e^{\rho(x)} \left|\det(JB^{-1})_{BG(x)}\right|^{\frac{1}{m+1}} (1, BG(x))$, where $J(B^{-1})$ is the Jacobian of B^{-1}. We observe that $\hat{\Omega} = BG(\Omega)$ is the image of the hyperbolic Gauss map of $T\phi$. Then

$$g_1 = e^{2\rho G^{-1}B^{-1}} \left|\det(JB^{-1})\right|^{\frac{2}{m+1}} g_0 \text{ on } \hat{\Omega}.$$

Since $G(x) = x$ for all $x \in \Omega$, we have

$$g_1 = e^{2\rho B^{-1}} \left|\det(dB^{-1})\right|^{\frac{2}{m+1}} g_0 \text{ on } \hat{\Omega}.$$

Then, $B^*(g_1) = e^{2\rho} \left|\det(dB^{-1})_B\right|^{\frac{2}{m+1}} |\det(dB)|^{\frac{2}{m+1}} g_0$ or, in other words, $B^*(g_1) = g$ on $\hat{\Omega}$. This concludes the proof. □

2.5 Invariance of hypersurfaces and invariance of conformal metrics

Let $\rho \in C^\infty(\Omega)$ be such that its associated map $\phi : \Omega \to \mathbb{H}^{m+1}$ is a horospherically concave hypersurface with unit normal vector field $\eta : \Omega \to \mathbb{S}_1^{m+1} \subset \mathbb{L}^{m+1}$, i.e., the canonical orientation of ϕ. Let $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ be an isometry such that it leaves invariant $\Sigma = \phi(\Omega)$, i.e., $T(\Sigma) = \Sigma$.

In this subsection we prove that if we consider the horospherically concave hypersurface $T \circ \phi : \Omega \to \mathbb{H}^{m+1}$ with the orientation $\eta_T = \eta \circ (\phi^{-1}T\phi)$, then the image of its hyperbolic Gauss map agrees with the image of hyperbolic Gauss map of ϕ, in particular, the horospherically metric of $T\phi$ on Ω coincides with the one of ϕ on Ω.

Proposition 2.16:
Let $\phi : \Omega \to \mathbb{H}^{m+1}$ be horospherically concave hypersurface with orientation $\eta : \Omega \to \mathbb{S}_1^{m+1} \subset \mathbb{L}^{m+2}$ whose hyperbolic Gauss map is injective and $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ an isometry. If T leaves ϕ invariant, i.e., $T(Im(\phi)) = \phi$ then the hypersurface $T\phi : \Omega \to \mathbb{H}^{m+1}$ with orientation $\eta\phi^{-1}T\phi$ has the same horospherical metric that $\phi : \Omega \to \mathbb{H}^{m+1}$.

Proof. We assume that the hyperbolic Gauss map of ϕ is the inclusion $\Omega \hookrightarrow \mathbb{S}^m$ and its horospherical metric is $g = e^{2\rho} g_0$ on Ω. First, we compute the light map cone of $T\phi$ as

$$\psi'(x) = T\phi(x) - \eta\left(\phi^{-1}T\phi(x)\right) \text{ for all } x \in \Omega.$$

Let $y = \phi^{-1}T\phi(x)$ then

$$\psi'(x) = \phi(y) - \eta(y) = e^{\rho(y)}(1, y) \text{ for all } y \in \Omega = \phi^{-1}T\phi(\Omega).$$

Then the image of the hyperbolic Gauss map of $T\phi : \Omega \to \mathbb{H}^{m+1}$ is Ω and its horospherical metric is given by $g'_\infty = e^{2\rho} g_0$ on Ω. This concludes the proof. □

Next, we show that if $\phi : \Omega \to \mathbb{H}^{m+1}$ is invariant under an isometry $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ as above, then $B^*(g) = g$ on Ω, where $B : \overline{\mathbb{B}^{m+1}} \to \overline{\mathbb{B}^{m+1}}$ is the associated conformal diffeomorphism to T.

Proposition 2.17:
Let $g = e^{2\rho} g_0$ be a conformal metric on $\Omega \subset \mathbb{S}^{m+1}$ such the map associated $\phi : \Omega \to \mathbb{H}^{m+1}$ is a horospherically concave hypersurface in $\mathbb{H}^{m+1}$ with orientation η. Let $T : \mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ be an isometry and $B : \overline{\mathbb{B}^{m+1}} \to \overline{\mathbb{B}^{m+1}}$ be the associated conformal diffeomorphism to T. If T leaves invariant $\Sigma = Im(\phi)$ and $T\eta = \eta_T$ then

$$B^*(g) = g \text{ on } \Omega,$$

i.e., g is invariant under $B|_{\mathbb{S}^m} : \mathbb{S}^m \to \mathbb{S}^m$.

In particular, if $\Sigma = \phi : \Omega \to \mathbb{H}^{m+1}$ is invariant under a group of isometries on $\mathbb{H}^{m+1}$, then the metric g is invariant under the associated group of conformal diffeomorphisms in $\mathbb{B}^{m+1}$.

Proof. Let g_1 be the horospherical metric of $T\phi$, then by Proposition 2.16, g_1 is defined on Ω and $g_1 = g$. Thus, Proposition 2.15 implies

$$B^*(g_1) = g = B^*(g) \text{ on } \Omega,$$

which concludes the proof. □

2.6 Conformal problems on $\mathbb{S}^m$ and Weingarten problems in $\mathbb{H}^{m+1}$

In this section we see how we can get an elliptic data for problems for hypersurfaces in the Hyperbolic space $\mathbb{H}^{m+1}$ from an elliptic data of an elliptic problem for conformal metrics.

The key formula to get a good transformation from an elliptic data of conformal metrics to elliptic data of hypersurfaces in $\mathbb{H}^{m+1}$ is the formula (2.8), that can be written as

$$\kappa_i = \frac{2}{1-2\lambda_i} - 1 \qquad \forall i = 1, \ldots, m.$$

We consider a problem for conformal metrics on $\Omega \subset \mathbb{S}^m$ given by an elliptic function $f : \Gamma \to \mathbb{R}$. First, let $V \subset \mathbb{R}^m$ be the open set defined by

$$V = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,:\, x_i \neq \frac{1}{2}, \text{ for } i = 1, \ldots, m \right\} \subset \mathbb{R}^m,$$

and

$$W = \{ y = (y_1, \ldots, y_m) \in \mathbb{R}^m \,:\, y_i \neq -1, \text{ for } i = 1, \ldots, m \} \subset \mathbb{R}^m.$$

We have the following diffeomorphism $\mathscr{A} : V \subset \mathbb{R}^m \to W \subset \mathbb{R}^m$ defined by

$$\mathscr{A}(x_1, \ldots, x_m) = \left(\frac{2}{1-2x_1} - 1, \ldots, \frac{2}{1-2x_m} - 1 \right) \text{ for } (x_1, \ldots, x_m) \in U. \tag{2.12}$$

Let

$$\Lambda = \Lambda(\Gamma) = \left\{ x = (x_1, \dots, x_m) \in \Gamma : \, x_i < \frac{1}{2}, i = 1, \dots, m \right\},$$

and we consider the following open set of $\mathbb{R}^m$ given by $U = \mathscr{A}(\Lambda)$. The open set $U \subset \mathbb{R}^m$ has the following properties:

1. It is symmetric.
2. It is convex.
3. $\Gamma_m \subset \Gamma_U \subset \Gamma_1$.

where Γ_U is the translation $\Gamma_U = U - (1, \dots, 1)$.

Then, the elliptic function $\mathscr{W} : \overline{U} \to \mathbb{R}$ for problems of hypersurfaces in $\mathbb{H}^{m+1}$ associated to f is defined by

$$\mathscr{W}(y) = f\left(\mathscr{A}^{-1}(y)\right) \text{ for every } y \in \overline{U}.$$

This function satisfies the following properties:

1. $\mathscr{W}$ is symmetric in U.
2. $\mathscr{W}|_U > 0$.
3. $\mathscr{W}|_{\partial U} = 0$
4. For every $y \in U$, it holds $\dfrac{\partial \mathscr{W}}{\partial y_i}(y) > 0$ for $i = 1, \dots, m$.

Also, if f is regular data elliptic for problems of conformal metrics, i.e., there is a constant $\lambda > 0$ with $2\lambda < 1$ such that $f(\lambda, \dots, \lambda) = 1$, then, there is a constant $\kappa = (1 + 2\lambda)(1 - 2\lambda)^{-1} > 1$ such that

$$\mathscr{W}(\kappa, \dots, \kappa) = 1,$$

that is, $(\mathscr{W}, U)$ is regular data elliptic for problems of hypersurfaces in $\mathbb{H}^{m+1}$.

Moreover, one can observe that we can perform the inverse process and, from an elliptic data $(\mathscr{W}, U)$ for hypersurfaces in $\mathbb{H}^{m+1}$ we can get an elliptic data f for conformal metrics.

Chapter 3

Hypersurfaces via Conformal metrics

We saw how a horospherically concave hypersurface in $\mathbb{H}^{m+1}$ gives rise to a (locally) well defined conformal metric on a subdomain on $\mathbb{S}^m$. Also, such metric is global if we assume that the hyperbolic Gauss map is injective.

In this chapter we study the opposite case, that is, given a subdomain $\Omega \subset \mathbb{S}^m$ and $\rho \in C^\infty(\Omega)$, consider the conformal metric $g = e^{2\rho} g_0$, then the question is: what can we say about the hypersurface given by the representation formula?

We will see that if we impose certain conditions on the given conformal metric $g = e^{2\rho} g_0$, we will obtain a horospherically concave hypersurface with injective Gauss map, moreover, such horospherically concave hypersurface is proper. Also, with some extra conditions on the conformal metric, we get an embedded horospherically concave hypersurfaces using the parallel flow. We see in this chapter, in the case of domains that are closed geodesic balls $B_p(r)$ of the sphere $\mathbb{S}^m$, where $r \leq \pi/2$, that if we impose the condition of constant mean curvature on the boundary $\partial B_p(r)$ then we get information about the location of the boundary of the horospherically concave hypersurface, that we get from the Local Representation Theorem. There, we will see that the boundary is in an equidistant hypesurface.

Moreover, using the parallel flow, we can get horospherically hypersurfaces in one of the components in the Hyperbolic space $\mathbb{H}^{m+1}$ that is determined by the equidistant hypersurface where its boundary is contained. Finally, we will study how the parallel flow affects the elliptic problem for conformal metrics.

First, we introduce some notation. Let $k \geq 0$ be an integer and set

1. $C^k(\Omega) = \{f : \Omega \to \mathbb{R} : f$ is k times differentiable and the k-derivative is continuous on $\Omega\}$.

2. We define the norm $|\cdot|_{k,\infty}$ in $C^k(\Omega)$: Given $f \in C^k(\Omega)$ then

$$|f|_{k,\infty} = \sum_{i=0}^{k} \left| f^{(i)} \right|_{\infty},$$

where

$$\left| f^{(i)} \right|_{\infty} = \sup_{\Omega} \left| f^{(i)} \right| \text{ for } i = 0, \dots, k.$$

In this chapter we use the Klein model of the Hyperbolic space $\mathbb{H}^{m+1}$, unless we state the contrary. Given $\rho \in C^{\infty}(\Omega)$, the horospherically concave hypersurface associated to ρ in the Klein model is $\varphi : \Omega \to \mathbb{H}^{m+1}$ where

$$\varphi(x) = x - \frac{2}{1+e^{2\rho(x)}+|\nabla\rho(x)|^2}x + \frac{2}{1+e^{2\rho(x)}+|\nabla\rho(x)|^2}\nabla\rho(x), \quad x \in \Omega.$$

Remark 9:
Observe that φ is nothing but the map $\phi : \Omega \to \mathbb{H}^{m+1}$, given in Theorem 2.10, composed with the isometry that takes the Hyperboloid model of $\mathbb{H}^{m+1}$ onto the Klein model of $\mathbb{H}^{m+1}$. Setting $\sigma = e^{-\rho}$, this is the formula (2.10) that we saw after the Local Representation Theorem.

We say that the map $\phi = (\phi_0, \dots, \phi_{m+1}) : \Omega \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ is $C^{k,\alpha}$ if for all $i = 0, \dots, m+1$, the function $\phi_i : \Omega \to \mathbb{R}$ is in $C^{k,\alpha}(\Omega)$. First, notice that from the Local Representation Theorem we have that, if $\rho \in C^{k,\alpha}(\Omega)$, then $\phi : \Omega \to \mathbb{H}^{m+1}$ is a $C^{k-1,\alpha}-$map. And moreover, from the expression of the normal, we have that $\eta \in C^{k-1,\alpha}(\Omega)$. As a consequence, the coefficients of the first and second form of Σ are of class $C^{k-2,\alpha}$ (as functions defined on Ω).

The map φ also can be seen as a map from Ω to $\mathbb{R}^{m+1}$ since the Klein model of the Hyperbolic space $\mathbb{H}^{m+1}$ is diffeomorfic to the open ball $\{x \in \mathbb{R}^{m+1} : |x|^2 < 1\}$, then we can refer the map $\varphi : \Omega \to \mathbb{R}^{m+1}$ associated to $\rho \in C^{\infty}(\Omega)$. So, we get

$$\varphi(x) = x - \frac{2\sigma^2(x)}{1+\sigma^2(x)+|\nabla\sigma(x)|^2}x - \frac{2\sigma(x)}{1+\sigma^2(x)+|\nabla\sigma(x)|^2}\nabla\sigma(x), x \in \Omega.$$

3.1 Properness

In this section we study how the behavior of $\phi : \Omega \to \mathbb{H}^{m+1}$ depends on ρ.

Theorem 3.1:
Given $\rho \in C^1(\Omega)$, the map $\phi : \Omega \to \mathbb{H}^{m+1}$ is proper if, and only if, $|\rho|_{1,\infty}(x) \to \infty$ when $x \to p$, for every $p \in \partial\Omega$.

Proof. Let $\phi : \Omega \to \mathbb{H}^{m+1} \subset \mathbb{L}^{m+2}$ be given by the representation formula. Let $T : (\mathbb{H}^{m+}, \ll\gg) \to (\mathbb{B}^{m+1}, g_K)$ the isometry that takes the hyperboloid model into the Klein model. Then, $\varphi = T \circ \phi : \Omega \to (\mathbb{B}^{m+1}, g_K)$ is given by

$$\varphi(x) = x - \frac{2\sigma^2(x)}{1+\sigma^2(x)+|\nabla\sigma(x)|^2}x - \frac{2\sigma(x)}{1+\sigma^2(x)+|\nabla\sigma(x)|^2}\nabla\sigma(x),$$

for all $x \in \Omega$, where $\sigma(x) = e^{-\rho(x)}$. Taking the Euclidean norm of φ we obtain

$$|\varphi(x)|^2 = 1 - \left(\frac{2\sigma(x)}{1+\sigma(x)^2+|\nabla\sigma(x)|^2}\right)^2 \text{ for every } x \in \Omega. \tag{3.1}$$

Hence, φ is proper if, and only if,

$$\lim_{x\to p}\left(\frac{1}{\sigma(x)}+\sigma(x)+\frac{|\nabla\sigma(x)|^2}{\sigma(x)}\right)=+\infty \text{ for all } p\in\partial\Omega.$$

This is equivalent to

$$\lim_{x\to p}\left(2\cosh(\rho(x))+\frac{|\nabla\rho(x)|^2}{e^{\rho(x)}}\right)=+\infty \text{ for all } p\in\partial\Omega.$$

Finally, that is equivalent to

$$\lim_{x\to p}\left[\rho(x)^2+|\nabla\rho(x)|^2\right]=+\infty \text{ for all } p\in\partial\Omega.$$

□

In particular, we have

Corollary 3.2:
If $\rho:\Omega\to\mathbb{R}$ is a proper smooth function then $\phi:\Omega\to\mathbb{H}^{m+1}$ is proper.

It is interesting to note that $\phi:\Omega\to\mathbb{H}^{m+1}$ is proper if only if $\beta=\Omega\to\mathbb{R}$ diverges at the boundary, where

$$\beta(x)=\left(|\nabla\rho|^2+\rho^2\right)(x),\, x\in\Omega.$$

Also, as a consequence of the proof of the above theorem, we obtain another condition on ρ that makes ϕ proper when $g=e^{2\rho}g_0$ is complete.

Theorem 3.3:
Let $g=e^{2\rho}g_0$ be a complete metric on Ω, such that $\sigma=e^{-\rho}$ is the restriction of a continuous function defined on $\overline{\Omega}$. Then $\phi:\Omega\to\mathbb{H}^{m+1}$ is a proper map.

Proof. Since $g=e^{2\rho}g_0$ is a complete metric on $\Omega\subset\mathbb{S}^m$, we have $\limsup_{x\to p}\rho(x)=+\infty$ for all $p\in\partial\Omega$, that is equivalent to $\liminf_{x\to p}[-\rho(x)]=-\infty$ for all $p\in\partial\Omega$. Let $H:\overline{\Omega}\to\mathbb{R}$ the continuous extension of $\sigma:\Omega\to\mathbb{R}$, then

$$H(p)=\lim_{x\to p}\sigma(x)=\liminf_{x\to p}\sigma(x)=0 \text{ for all } p\in\partial\Omega.$$

Thus, $\lim_{x\to p}\rho(x)=+\infty$ for all $p\in\Omega$, which implies that

$$\lim_{x\to p}\left[\rho(x)^2+|\nabla\rho(x)|^2\right]=+\infty \text{ for all } p\in\partial\Omega,$$

that is, $\phi:\Omega\to\mathbb{H}^{m+1}$ is proper. □

3.1.1 Invariance of the properness

An interesting relation between conformal metrics and horospherically concave hypersurface is how they are related by dilation and geodesic flow. Let us explain this in more detail. We assume that $\phi : \Omega \to \mathbb{H}^{m+1}$ is an horospherically concave hypersurface in $\mathbb{H}^{m+1}$. When we move the hypersurface $\phi : \Omega \to \mathbb{H}^{m+1}$ using the unit normal vector field $-\eta$, we have a family of horospherically concave hypersurfaces $\left\{\phi_t : \Omega \to \mathbb{H}^{m+1} \,:\, t > 0\right\}$. For every $t > 0$,

$$\phi_t(x) = \cosh(t)\phi(x) - \sinh(t)\eta(x) \quad \text{for all } x \in \Omega,$$

i.e.,

$$\phi_t(x) = \frac{e^{t+\rho(x)}}{2}\left[1 + e^{-2(t+\rho(x))}|\nabla\rho(x)|^2\right](1,x) + e^{-(t+\rho(x))}(0,-x+\nabla\rho(x)) \quad \text{for all } x \in \Omega,$$

then the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is well-defined with horospherical metric $g = e^{2t}g = e^{2(t+\rho)}g_0$. That is, the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is just obtained from the conformal metric $g_t = e^{2t}g$ by the Local Representation Theorem. Since the eigenvalues of the Schouten tensor of g_t are just the dilation by a factor of e^{-2t} of the eigenvalues of the Schouten tensor of g, i.e., given $x \in \Omega$ and let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the Schouten tensor of g at the point x, then the eigenvalues of the Schouten tensor of g_t at the point $x \in \Omega$ are

$$\lambda_{i,t} = e^{-2t}\lambda_i \leq \lambda_i < \frac{1}{2} \quad \text{for all } i = 1, \ldots, m, \tag{3.2}$$

then the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is a horospherically concave hypersurface, and clearly its horospherical metric is $g_t = e^{2t}g = e^{2(t+\rho)}g_0$. In conclusion, if we take $t > 0$, the conformal metric $g_t = e^{2t}g$ give rise to a horospherically concave hypersurface $\phi_t : \Omega \to \mathbb{H}^{m+1}$ with the natural orientation η_t given by

$$\eta_t(x) = \phi_t(x) - e^{t+\rho(x)}(1,x) \quad \text{for all } x \in \Omega. \tag{3.3}$$

Then, one observation is the following:

Proposition 3.4:
Assume that $\phi : \Omega \to \mathbb{H}^{m+1}$ is proper, then $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is also proper for every $t \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$. Since ϕ is proper, we have that $\lim_{x\to p}\left[\rho(x)^2 + |\nabla\rho(x)|^2\right] = +\infty$ for all $p \in \partial\Omega$, then

$$\lim_{x\to p}\left[[\rho(x)+t]^2 + |\nabla\rho(x)|^2\right] = +\infty \text{ for all } p \in \partial\Omega.$$

Therefore, $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is proper. $\square$

3.2 From immersed to embedded

So far we have seen that the geodesic flow preserves the regularity of a horospherically concave hypersurface. In this section, we will study how, under the geodesic flow, such immersed horospherically concave hypersurface becames embedded.

We start by defining the meaning of a complete conformal metric in our situation.

Definition 3.5:
Let $\Omega \subset \mathbb{S}^m$ be an open domain such that $\partial\Omega = \mathscr{V}_1 \cup \mathscr{V}_2$ where $\mathscr{V}_1$ and $\mathscr{V}_2$ are disjoint submanifolds. We say that a conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$, is complete if given a divergent curve $\gamma : [0,1) \to \Omega$ then either

- $\lim\limits_{t\to 1} \gamma(t) \in \mathscr{V}_1$ *and* $\int_0^1 |\gamma'(t)|_g \, dt < +\infty$, *or*
- $\lim\limits_{t\to 1} \gamma(t) \in \mathscr{V}_2$ *and* $\int_0^1 |\gamma'(t)|_g \, dt = +\infty$.

In other words, g is a complete metric on the manifold with boundary $\Omega \cup \mathscr{V}_1$.

In this definition, a submanifold that composes $\mathscr{V}_2$ can be one point, that is, it is permitted submanifolds that have dimension zero, or, even $\mathscr{V}_2$ could be empty.

The next theorem shows that if we impose some extension condition on certain functions that are related to $\sigma = e^{-\rho}$ we can move along the geodesic flow and then we get an embedded hypersurface φ_t for t big. The hypersurface $\varphi_t : \Omega \cup \mathscr{V}_1 \to \mathbb{H}^{m+1}$ is obtained moving the point $\varphi(x)$ along to the geodesic $\gamma(t)$ passing through $\varphi(x)$ and tangent vector $-\eta(x)$ at that point, i.e., $\gamma(t) = \varphi_t(x)$, in other words, in the Klein model

$$\varphi_t(x) = x - 2\frac{e^{-2t}\sigma^2(x)}{1 + e^{-2t}\left[\sigma^2(x) + |\nabla\sigma(x)|^2\right]} x - \frac{e^{-2t}}{1 + e^{-2t}\left[\sigma^2(x) + |\nabla\sigma(x)|^2\right]} \nabla\sigma^2(x),$$

for all $x \in \Omega \cup \Gamma_1$.

Theorem 3.6:
Let $\rho \in C^\infty(\Omega \cup \Gamma_1)$ be such that $\sigma = e^{-\rho} \in C^\infty(\Omega \cup \mathscr{V}_1)$ satisfies:

1. *σ^2 can be extended to a $C^{1,1}$ function on $\overline{\Omega}$.*
2. *$\langle \nabla\sigma, \nabla\sigma \rangle$ can be extended to a Lipschitz function on $\overline{\Omega}$.*

Then, there is $t_0 > 0$ such that for all $t > t_0$ the map $\varphi_t : \Omega \cup \mathscr{V}_1 \to \mathbb{H}^{m+1}$ associated to $\rho_t = \rho + t$ is an embedded horospherically concave hypersurface.

Proof. Let $\zeta : \mathbb{S}^m \to \mathbb{R}$ be a $C^{1,1}$-extension of σ^2 such that $\zeta > -\dfrac{1}{3}$, and $l \in C^\infty(\mathbb{S}^m)$ be a Lipschitz extension of $|\nabla\sigma|^2$ such that also $l > -\dfrac{1}{3}$. Then for $t > 0$ we have the following Lipschitz extension of φ_t:

$$\Phi_t(x) = x - 2\frac{e^{-2t}\zeta(x)}{1+e^{-2t}\left[\zeta(x)+l(x)\right]}x - \frac{e^{-2t}}{1+e^{-2t}\left[\zeta(x)+l(x)\right]}\nabla\zeta(x), \quad x \in \mathbb{S}^m.$$

Since $\{\Phi_t\}_{t>0}$ converges to the inclusion $\mathbb{S}^m \hookrightarrow \mathbb{R}^{m+1}$ uniformly on $\mathbb{S}^m$, there is $t_0 > 0$ such that for every $t > t_0$, the map Φ_t is embedded. Then there is $t_0 > 0$ such that for every $t > t_0$ the map φ_t is embedded [23].

Also, from the equation:

$$g^{-1}\mathrm{Sch}(g) + \frac{1}{2}|\nabla\sigma|^2 Id + \langle\nabla\sigma,\cdot\rangle\nabla\sigma = \frac{1}{2}\sigma^2 Id + \nabla^2\sigma^2 \quad \text{on } \Omega$$

and the hypothesis, we have that the eigenvalues of the Schouten tensor of $g = e^{2\rho}g_0$ are bounded in Ω, so, we can choose $t_0 > 0$ large, such that for every $t > t_0$, the map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is a horospherically concave hypersurface (cf. Equation 3.2). This concludes the proof. □

3.3 Tangency at infinity using the Klein model

In the previous sections we have studied when the associated horospherically concave hypersurface $\phi : \Omega \to \mathbb{H}^{m+1}$ to $\rho \in C^\infty(\Omega)$ (we omit regularity conditions), is proper. In this section, we will study how a proper horospherically concave hypersurface approaches the boundary at infinity. To do so, we will work on the Klein Model and we will see that the hypersurface $\varphi : \Omega \cup \mathscr{V}_1 \to \mathbb{H}^{m+1}$ is tangent to the ideal boundary $\mathbb{S}^{m+1}$ of $\mathbb{H}^{m+1}$.

Proposition 3.7:
Let $\sigma = e^{-\rho}$, $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$, be such that the functions $\sigma\cdot\sigma$, $\langle\nabla\sigma,\nabla\sigma\rangle$ admit smooth extensions on $\mathbb{S}^m$. Let $\varphi : \Omega \cup \mathscr{V}_1 \to \mathbb{H}^{m+1}$ be the associated horospherical hypersurface and $\Phi : \mathbb{S}^m \to \mathbb{R}^{m+1}$ be the smooth extension on $\mathbb{S}^m$. If φ is proper then for every $x \in \mathscr{V}_2$:

$$d\Phi(x) \in T_x\mathbb{S}^m.$$

Proof. Let $\zeta : \mathbb{S}^m \to \mathbb{R}$ be a smooth extension of σ^2 such that $\zeta > -\dfrac{1}{3}$ and let $l : \mathbb{S}^m \to \mathbb{R}$ be a smooth extension of $\langle\nabla\sigma,\nabla\sigma\rangle$ such that $l > -\dfrac{1}{3}$. Then φ has the following extension:

$$\Phi(x) = x - \frac{2\zeta(x)}{1+\zeta(x)+l(x)}x - \frac{1}{1+\zeta(x)+l(x)}\nabla\zeta(x), x \in \mathbb{S}^m.$$

Since φ is proper, the equation (3.1) implies that for each $x \in \mathscr{V}_2$:

$$\frac{\sqrt{\zeta(x)}}{1+\zeta(x)+l(x)} = 0 \quad \text{and} \quad \frac{1}{1+\zeta(x)+l(x)}\nabla\zeta(x) = 0$$

then

$$\zeta(x) = 0 \quad \text{and} \quad \nabla\zeta(x) = 0 \qquad \forall x \in \Gamma_2.$$

So, for every $v \in T_x\mathbb{S}^m$, we have

$$d\Phi_x(v) = v - 2\frac{1}{1+l(x)}Hess(\zeta)_x(v), \quad \forall x \in \mathscr{V}_2,$$

where $Hess(\zeta)$ is the Hessian of ζ with respect to the standard metric on $\mathbb{S}^m$. This concludes the proof. $\square$

Using the geodesic flow, we obtain

Proposition 3.8:
Let $\sigma = e^{-\rho}$, $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$, be a function satisfying the assumptions of Proposition 3.7. Then, there exists $t_0 > 0$ such that for every $t > t_0$ it holds

$$d\Phi_t(x) \in T_x\mathbb{S}^m \textit{ and } \langle d\Phi_t(x)(v), v\rangle > \frac{1}{2}|v|^2 \textit{ for all } x \in \mathscr{V}_2 \textit{ and for all } v \in T_x\mathbb{S}^m.$$

Let us see the difference between the Klein and Poincaré models with a simple example. Let Σ be the horospherically concave hypersurface given by the function $\sigma : \Omega = \{(x,y,z) \in \mathbb{S}^2 : |z| < \cos(\pi/4)\} \to \mathbb{R}$ given by

$$\sigma(x,y,z) = \frac{2}{2+\sqrt{2}}\left(\sqrt{1-z^2} - \cos(\pi/4)\right) \text{ for all } x \in \Omega.$$

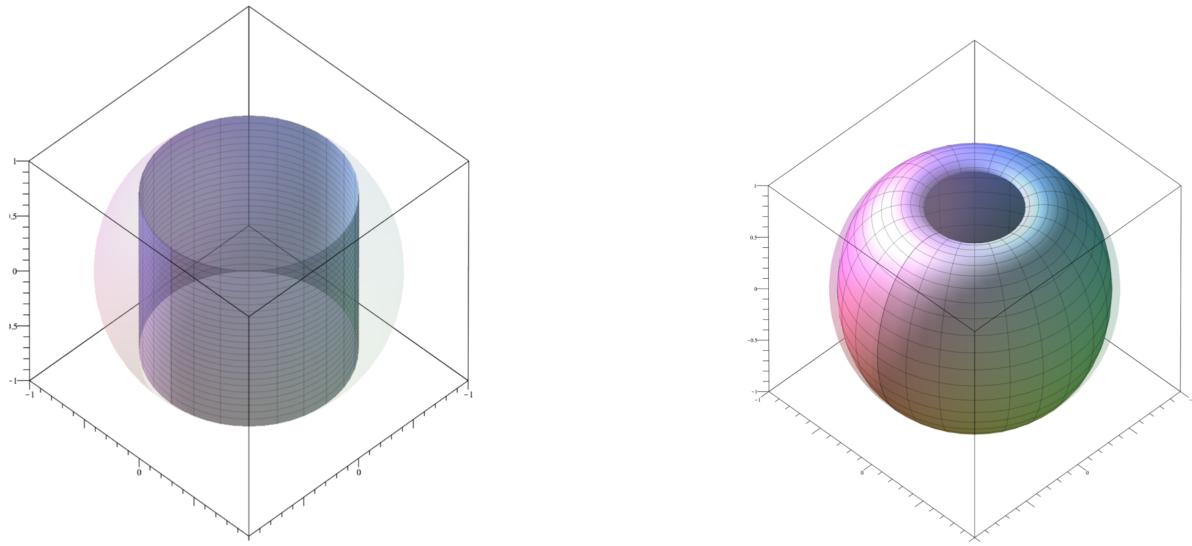
One can easily observe (cf. Figure 3.1a), that in the Poincaré Model Σ is transversal to the ideal boundary. Nevertheless, in the Klein model, Σ is tangent to the ideal boundary (cf. Figure 3.1b).

As we can see, the function σ can be smoothly extended to $\mathbb{S}^2 \setminus \{\pm e_3\}$, in fact, Figure 3.1(b) is a smooth extension of the surface in $\mathbb{R}^3$ in order to see the tangency of the surface with $\mathbb{S}^2$, the ideal boundary of $\mathbb{H}^3$.

3.4 Conditions along the boundary

Let $\Omega \subset \mathbb{S}^m$ be a domain such that $\partial\Omega = \mathscr{V}_1 \cup \mathscr{V}_2$ and $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$. Assume that ρ is complete on Ω (see Definition 3.5). We want to study how conditions on ρ along $\mathscr{V}_1$, or imposing some geometric conditions on Γ_1, influence the boundary of Σ.

In general, if $(\mathscr{M}, g_0)$ is a Riemannian manifold with boundary $\partial\mathscr{M}$, ν is a unit normal vector field along ∂M, $g = e^{2\rho}g_0$ is a conformal metric to g_0, h_0 the mean curvature of $\partial\mathscr{M}$ with respect the the



(a) Poincaré model: The surface is transversal to the ideal boundary.

(b) Klein model: The surface is tangent to the ideal boundary.

Figure 3.1: Rotational surface

metric g_0 and the unit normal vector field ν, $h(g)$ the mean curvature of the $\partial\mathscr{M}$ with respect the metric g and the normal vector field $\nu_g = \dfrac{1}{e^\rho}\nu$, then

$$e^\rho \cdot h(g) + \frac{\partial\rho}{\partial\nu} = h_0 \quad \text{on } \partial\mathscr{M}, \tag{3.4}$$

Take $\sigma = e^{-\rho}$, then we have the following relation

$$\frac{\partial\sigma}{\partial\nu} + h_0 \cdot \sigma = h(g) \quad \text{on } \partial\mathscr{M}, \tag{3.5}$$

If we consider the scaled metric $g_t = e^{2t}g$ on $\mathscr{M}$, where $t \in \mathbb{R}$, then the mean curvature of $\partial\mathscr{M}$ with respect to the unit normal vector field $\nu_t = \frac{1}{e^t}\nu_g$ is

$$h(g_t) = e^{-t}h(g) \quad \text{on } \partial\mathscr{M}. \tag{3.6}$$

Therefore, if $\partial\mathscr{M}$ is compact then $h(g_t)$ goes to 0 when t goes to infinity. In our case $\mathscr{M} = \Omega \cup \mathscr{V}_1$ and $\partial\mathscr{M} = \mathscr{V}_1$.

3.4.1 Boundary of a geodesic ball

We begin by recalling some notation. Given $p \in \mathbb{S}^m$ and $r \in \left(0, \frac{\pi}{2}\right]$, the geodesic ball of $\mathbb{S}^m$ centered at p and radius r is

$$B_r(p) = \{q \in \mathbb{S}^m \, : \, d_{\mathbb{S}^m}(q,p) < r\}$$

and its inward unit normal along $\partial B_r(p)$ is given by

$$\nu(x) = \csc(r)p - \cot(r)x \, , \quad x \in \partial B_r(p).$$

We will see that any geodesic ball $B_r(p)$ has associated a unique totally geodesic hypersurface $E(a,0) \subset \mathbb{H}^{m+1}$, here we use the Hyperboloid model,such that

$$\partial B_r(p) = \partial_\infty E(a,0),$$

where $a \in \mathbb{L}^{m+2}$ is a spacelike unit vector and $E(a,0)$ is defined by equation (1.7), i.e.,

$$E(a,0) = \left\{y \in \mathbb{H}^{m+1} \, : \ll y, a \gg = 0\right\}.$$

We can explicitly get the vector a from the center p and the radius r, and vice-versa.

Proposition 3.9:
The ideal boundary of a totally geodesic hypersurface of $\mathbb{H}^{m+1}$ given by $E(a)$, $a = (a_0, \bar{a})$, $\ll a, a \gg = 1$, is the boundary of a geodesic ball $B_r(p)$ of $\mathbb{S}^m$, where $p = \frac{1}{|\bar{a}|}\bar{a}$ and $r \in (0,\pi)$ satisfies $\cot(r) = a_0$. Reciprocally, given the boundary of a geodesic ball $\partial B_r(p) \subset \mathbb{S}^m$, the ideal boundary of the totally geodesic hypersurface $E(a,0) \subset \mathbb{H}^{m+1}$ is $\partial B_r(p)$, where $a = (\cot(r), \csc(r)p)$.

Proof. Let us see that the ideal boundary of a totally geodesic hypersurface $E(a,0)$, where $\ll a, a \gg = 1$, is the boundary of a geodesic ball of $\mathbb{S}^m$. If $a_0 \neq 0$ then, in the Poincaré model, the totally geodesic hypersurface $E(a,0)$ is given by

$$\left\{x \in \mathbb{B}^{m+1} \, : \, \left| x - \frac{1}{a_0}\bar{a} \right| = \frac{1}{a_0^2} \right\}.$$

The ideal boundary of this set is the boundary of the geodesic ball $\partial B_r(p)$ in $\mathbb{S}^m$ where

$$p = \frac{1}{|\bar{a}|}\bar{a} \text{ and } \cot(r) = a_0, \, r \in \left(0, \frac{\pi}{2}\right).$$

If $a_0 = 0$, then the ideal boundary $\partial E(a,0)$ of the totally geodesic hypersurface $E(a,0)$ is

$$\partial B_{\frac{\pi}{2}}(\bar{a}) \subset \mathbb{S}^m,$$

i.e., in this case $p = \dfrac{1}{|\overline{a}|}\overline{a}$, and $\cot\left(\dfrac{\pi}{2}\right) = 0 = a_0$.

Then, the ideal boundary of $E(a,0)$ where $\ll a,a \gg = 1$ and $a_0 \geq 0$ is the boundary $\partial B_r(p)$ of the geodesic ball $B_r(p)$ in $\mathbb{S}^m$ where

$$p = \frac{1}{|\overline{a}|}\overline{a} \quad \text{and} \quad \cot(r) = a_0 \qquad r \in \left(0, \frac{\pi}{2}\right].$$

Reciprocally, we consider the geodesic ball $B_r(p)$ in $\mathbb{S}^m$, where $r \in \left(0, \dfrac{\pi}{2}\right]$, then the boundary $\partial B_r(p)$ is the ideal boundary of the totally geodesic hypersurface $E(a,0)$, where

$$a = \left(\cot(r), \frac{1}{\sin(r)}p\right). \tag{3.7}$$

If $r \in \left(\dfrac{\pi}{2}, \pi\right)$ then $s = (\pi - r) \in \left(0, \dfrac{\pi}{2}\right)$ and $\partial B_s(-p) = B_r(p)$, i.e., $\partial B_r(p)$ is the ideal boundary of a totally geodesic hypersurface of $\mathbb{H}^{m+1}$. □

Now, we will study the boundary $\phi\left(\partial B_r(p)\right)$ of the associated horospherically hypersurface ϕ to ρ when the boundary of $B_r(p)$ has constant mean curvature with respect the metric $g = e^{2\rho} g_0$. Consider a complete conformal metric $g = e^{2\rho} g_0$ on a domain $\Omega \cup \mathscr{V}_1 \subset \overline{B_r(p)}$ such that $\partial B_r(p) \subset \mathscr{V}_1$. Let $h(g)$ be the mean curvature of $\partial B_r(p)$ with respect to g and the inward unit normal vector field $\nu_g = \frac{1}{e^\rho}\nu$ along $\partial B_r(p)$, and $h_0 = \cot(r)$.

Let $\phi : \overline{\Omega} \to \mathbb{H}^{m+1}$ be the associated horospherically concave hypersurface to the complete conformal metric g on $\Omega \cup \mathscr{V}$. Then, a straightforward computation shows

$$\ll \phi(x), h_0\,(1,x) + (0, \nu(x)) \gg = -h(g) \quad \text{along} \quad \partial B_r(p),$$

where ν is the inward unit normal vector field along $\partial B_r(p)$ with respect to the standard metric g_0.

Assume that $h(g) = c = cte$, then

$$\ll \phi(x), h_0\,(1,x) + (0, \nu(x)) \gg = -c \text{ for all } x \in \partial B_r(p), \tag{3.8}$$

where

$$\nu(x) = \csc(r)p - cot(r)x \text{ for all } x \in \partial B_r(p).$$

Since $h_0 = \cot(r)$ and $a = h_0\,(1,x) + (0, \nu(x))$, we have

$$a = \left(\cot(r), \frac{1}{\sin(r)}p\right) \text{ for all } x \in \partial B_r(p),$$

i.e., a only depends of p and r.

Remark 10:
In the particular case that $r=\pi/2$ and $p=\mathbf{n}$, the north pole, we have

$$a=(0,\ldots,0,1)=(0,e_{m+1})\,. \tag{3.9}$$

Then, from (3.8) and (3.9), we have that

$$\phi\left(\partial\mathbb{B}_r(p)\right)\subset E(a,-c)=\left\{y\in\mathbb{H}^{m+1}:\ll y,a\gg=-c\right\}$$

which is an equidistant hypersurface to $E(a,0)$, see equation (1.8). Summarizing, we have (see [7], Claim E):

Proposition 3.10:
Under the conditions above, assuming that $\mathscr{V}_1$ contains a component which is the boundary of a geodesic ball $\partial B_r(p)$, $p\in\mathbb{S}^m$, $r\in(0,\pi)$, and $h(g)=c=cte$ along $\partial B_r(p)$, then

$$\phi\left(\partial B_r(p)\right)\subset E(a,-c),$$

where $E(a,-c)$ is the totally geodesic hypersurface equidistant to $E(a,0)$ given by

$$E(a,-c)=\left\{y\in\mathbb{H}^{m+1}:\ll y,a\gg=-c\right\}$$

and $a=(\cot(r),\csc(r)p)$.

In the above conditions, we can say even more. We will see that, in fact, $\Sigma=\phi\left(\Omega\right)$ makes a constant angle with $E(a,-c)$ along $\phi\left(\partial B_r(p)\right)$. Hence, without loss of generality and for simplicity, we will assume $\mathscr{V}_1=\partial B_r(p)$. A unit normal vector field to $E(a,-c)$ is given by $N(y)=\frac{1}{\sqrt{1+c^2}}\left(a-cy\right)$, for all $y\in E(a,-c)$. Since $\partial\Sigma\subset E(a,-c)$, we have the following result (see also [7], Claim D):

Proposition 3.11:
Under the above conditions, it holds

$$\ll N,\eta\gg=\frac{-c}{\sqrt{1+c^2}}\ \text{along}\ \phi\left(\partial B_r(p)\right).$$

In other words, the angle α between Σ and $E(a,-c)$ along $\phi\left(\partial B_r(p)\right)$ is constant, where

$$\cos(\alpha)=-\frac{c}{\sqrt{1+c^2}}.$$

Proof. We will work in the Hyperboloid model for simplicity. For every $x\in\partial B_r(p)$, it holds

$$\ll N\left(\varphi(x)\right),\eta(x)\gg=\frac{1}{\sqrt{1+c^2}}\ll a-c\varphi(x),\eta(x)\gg,$$

where $a = \left(\cot(r), \dfrac{1}{\sin(r)}p\right)$, then

$$\ll N(\varphi(x)), \eta(x) \gg = \frac{1}{\sqrt{1+c^2}} \ll a, \eta(x) \gg$$
$$= \frac{1}{\sqrt{1+c^2}} \ll a, \varphi(x) - e^{\rho(x)}(1,x) \gg .$$

Since $\ll a,(1,x) \gg = 0$ for all $x \in \partial B_r(p)$, then

$$\ll N(\varphi(x)), \eta(x) \gg = -\frac{c}{\sqrt{1+c^2}} \text{ for all } x \in \partial B_r(p).$$

$\square$

In the case that $a = (0,\ldots,0,1)$, we just denote

$$E(-c) = E(a,-c).$$

3.5 Moving the hypersurface along the geodesic flow

In this section we will show the following boundary half-space property: let $\Omega \subset \mathbb{S}^m_+$ be an open set such that $\partial\Omega = \mathscr{V}_1 \cup \mathscr{V}_2$, $\mathscr{V}_1 \cap \mathscr{V}_2 = \emptyset$, where the subset $\mathscr{V}_1$ is a compact hypersurface (not necessary connected) of $\mathbb{S}^m$ that contains $\partial\mathbb{S}^m_+$, and $\mathscr{V}_2$ is a finite union of disjoint compact submanifolds of $\mathbb{S}^m$. Let $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$ be such that

$$\lim_{x\to q}\left(e^{2\rho(x)} + |\nabla\rho(x)|^2\right) = +\infty \text{ for all } q \in \mathscr{V}_2.$$

Set

$$\mathscr{V}_1' = \mathscr{V}_1 \setminus \partial\mathbb{S}^m_+,$$

we will show that, if $h(g) = c = cte$ on $\partial\mathbb{S}^m_+$, then there exists $t_1 \geq 0$ such that

$$\Sigma_t = \varphi_t\left(\Omega \cup \mathscr{V}_1'\right) \subset C_t \text{ for all } t \geq t_1,$$

where $C_t \subset \mathbb{H}^{m+1}$ is the half-space determined by the equidistant $E(-e^{-t}c)$ that contains $\mathbf{n}$ at its boundary at infinity. Specifically (see [7], Claim C):

Theorem 3.12:
Let $g = e^{2\rho}g_0$, $\rho \in C^\infty(\Omega \cup \mathscr{V}_1)$, be a conformal metric on Ω such that $\partial\mathbb{S}^m_+ \subset \mathscr{V}_1$ and

$$h(g) = c \quad \text{on } \partial\mathbb{S}^m_+,$$

where $c \in \mathbb{R}$ is a constant. Assume that

$$\lim_{x \to q} \left(1 + e^{2\rho(x)} + |\nabla\rho(x)|^2\right) = +\infty \text{ for all } q \in \mathscr{V}_2.$$

Then, there exists $t_0 \geq 0$ such that for every $t > t_0$, the set $\varphi_t\left(\Omega \cup \mathscr{V}_1'\right)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $\mathbf{n}$ at its ideal boundary.

Proof. Set $K = \mathbb{S}^m_+ \setminus \Omega$ and $int(K) = K \setminus \partial K$. For every $t > 0$ we define the continuous extensions $\Phi_t : \overline{\mathbb{S}^m_+} \setminus int(K) \to \mathbb{R}^{m+1}$ of $\varphi_t : \overline{\mathbb{S}^m_+} \setminus K \to \mathbb{H}^{m+1}$ given by

$$\Phi_t(x) = \begin{cases} \varphi_t(x) & , \quad x \in \Omega \cup \mathscr{V}_1, \\ x & , \quad x \in \mathscr{V}_2. \end{cases}$$

Observe that $\{\Phi_t\}_{t>0}$ converge to the inclusion $\overline{\mathbb{S}^m_+} \setminus int(K) \hookrightarrow \mathbb{R}^{m+1}$ when $t \to \infty$. We take an open set V such that

$$K \subset V \subset \overline{V} \subset \mathbb{S}^m_+.$$

Since $\overline{V} \setminus int(K)$ is compact, there exists $t_1 > 0$ such that, for all $t > t_1$, the set $\varphi_t\left(\overline{V} \setminus K\right)$ is in the half-space determined by the equidistant $E(|c|)$ and contains e_{m+1} at its ideal boundary.

Now we consider the map $\varphi : \overline{\mathbb{S}^m_+} \setminus V \to \mathbb{H}^{m+1}$. We will prove that there is $t_0 > t_1$ such that, for every $t > t_0$, the set $\varphi_t\left(\mathbb{S}^m_+ \setminus V\right)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $\mathbf{n}$ at its ideal boundary. That will finish the proof.

Since φ_t converges to the inclusion $\overline{\mathbb{S}^m_+} \setminus V \hookrightarrow \mathbb{R}^{m+1}$ uniformly and ∂V is compact, there is $t_1 > 0$ such that, for all $t > t_1$, $\varphi_t(\partial V)$ is in the half-space determined by $E(|c|)$ and contains e_{m+1} at its ideal boundary.

From the equation (3.6), the mean curvature of $\partial\mathbb{S}^m_+$ with respect to the scaled metric $g_t = e^{2t}g$, where $t \in \mathbb{R}$, is

$$h(g_t) = e^{-t}c \quad \text{on } \partial\mathbb{S}^m_+.$$

By the Proposition 3.10, $\phi_t\left(\partial\mathbb{S}^m_+\right) \subset E(-e^{-t}c)$. We consider the following unit normal vector field along $E(-e^{-t}c)$:

$$N(y) = \frac{1}{\sqrt{1+s_1^2}}\left[(0,\mathbf{n}) - (e^{-t}c)\cdot y\right], \quad y \in E(-e^{-t}c),$$

then the principal curvatures of the umbilic hypersurface $E(-e^{-t}c)$ with respect to N are equals to

$$\frac{ce^{-t}}{\sqrt{1+c^2e^{-2t}}}.$$

Let $\kappa_{1,t},\ldots,\kappa_{m,t}$ be the principal curvatures of φ_t. From the Local Representation Theorem, we have that for all $t > 0$:

$$\frac{1}{2} = e^{-2t}\lambda_i + \frac{1}{1+\kappa_{i,t}} \quad \text{on } \overline{\mathbb{S}^m_+} \setminus V, \quad \text{for all } i = 1,\ldots,m,$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the Schouten tensor of g.

Since $\kappa_{i,t}$ goes to 1 uniformly on $\overline{\mathbb{S}^m_+} \setminus V$ when t goes to infinity, for $i = 1, \ldots, m$, then there exists $t_0 > t_1$ such that

$$\kappa_{i,t} > \frac{1}{2} > -\frac{ce^{-t}}{\sqrt{1+c^2e^{-2t}}} \quad \text{for all } t > t_0, \quad \text{for all } i = 1, \ldots, m, \tag{3.10}$$

on $\overline{\mathbb{S}^m_+} \setminus V$.

We claim that for every $t > t_0$, the set $\varphi_t\left(\mathbb{S}^m_+ \setminus V\right)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $\mathbf{n}$ at its ideal boundary. If this were not the case, we consider the foliation by equidistant hypersurfaces $\{E(s)\}_{s\in\mathbb{R}}$ of the Hyperbolic space $\mathbb{H}^{m+1}$, given by

$$E(s) = \left\{y \in \mathbb{H}^{m+1} : \ll y, (0, e_{m+1}) \gg = s\right\}$$

We consider the first equidistant hypersurface $E(s_1)$ that intersect $\varphi_t\left(\overline{\mathbb{S}^m_+} \setminus V\right)$, i.e.,

$$E(s_1) \cap \varphi_t\left(\overline{\mathbb{S}^m_+} \setminus V\right) \neq \emptyset \text{ and } E(s) \cap \varphi_t\left(\overline{\mathbb{S}^m_+} \setminus V\right) = \emptyset \text{ for all } s < s_1.$$

In the figure 3.2a, we see an equidistant that does not intersect the hypersurface. In the figure 3.2b, we see the first equidistant that intersect the hypersurface.

Clearly $s_1 \leq -c_t = -e^{-t}c$. We note that $E(s_1) \cap \varphi_t(\partial V) = \emptyset$ since $s_1 \leq |c|$. Then

$$E(s_1) \cap \varphi_t\left(\overline{\mathbb{S}^m_+} \setminus \overline{V}\right) \neq \emptyset.$$

We claim that $E(s_1) \cap \varphi_t\left(\mathbb{S}^m_+ \setminus \overline{V}\right) = \emptyset$, otherwise there would exists an interior contact point of $\varphi_t\left(\mathbb{S}^m_+ \setminus \overline{V}\right)$, say $x \in \mathbb{S}^m_+ \setminus \overline{V}$ such that $w = \varphi_t(x) \in E(s_1)$.

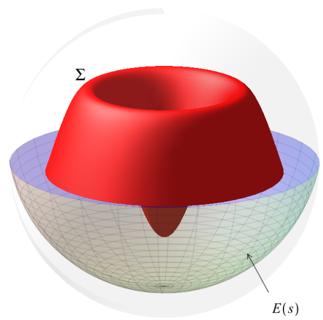
Consider the normal vector field along $E(s_1)$ that is defined in the Hyperboloid model by $N(y) = \dfrac{1}{\sqrt{1+s_1^2}}\left[(0, \mathbf{n}) + s_1 \cdot y\right]$, for all $y \in E(s)$. The principal curvatures of $E(s_1)$ with this normal are equal to

$$-\frac{s_1}{\sqrt{1+s_1^2}}.$$

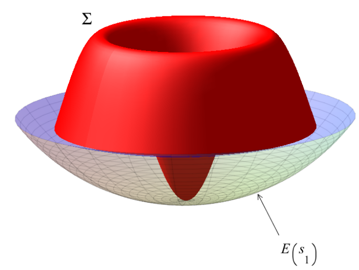
In the horospherically concave hypersurface ϕ_t, we consider the inverted orientation, i.e, the unit normal vector field $\xi_t = -\eta_t$ (see equation (3.3)). Then the principal curvatures are $\tilde{\kappa}_{i,t} = -\kappa_{i,t}$ with respect to that normal.

In the figure 3.2c, we see that the normal N to the equidistant $E(s_1)$ agrees with the inverted orientation ξ_t at the contact point.

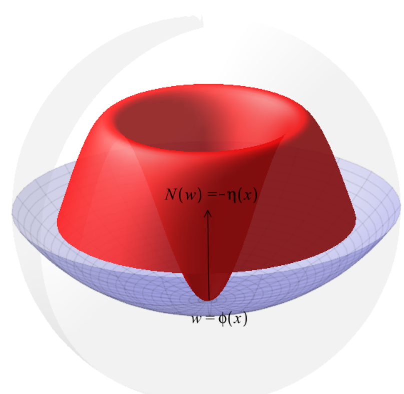
Since the Hyperbolic Gauss map is the inclusion $\overline{\mathbb{S}^m_+} \setminus \overline{V} \hookrightarrow \mathbb{R}^{m+1}$, the normal vector field ξ coincides with the normal N of the equidistant $E(s)$ at the point w.



(a) The equidistant $E(s)$ does not intersect to the hypersurface.



(b) The equidistant $E(s_1)$ intersects to the hypersurface.



(c) Inverted orientation of the $\Sigma = Im(\phi_t)$.

Figure 3.2: Process of getting the first contact equidistant in the Poincaré ball model.

The principal curvature of $E(s_1)$ with respect N at the point w satisfies

$$-\frac{s_1}{\sqrt{1+s_1^2}} \geq \frac{c_t}{\sqrt{1+c_t^2}} \text{ since } -s_1 \geq c_t.$$

Moreover, since $\varphi_t\left(\mathbb{S}^m_+ \setminus \overline{V}\right)$ is more convex than $E(s_1)$ at the point w we have that

$$\tilde{\kappa}_{i,t} \geq -\frac{s_1}{\sqrt{1+s_1^2}} \text{ at the point } w.$$

That is,

$$\kappa_{i,t} \leq \frac{s_1}{\sqrt{1+s_1^2}} \text{ at the point } w.$$

So, at the point w, we have

$$\kappa_{i,t} \leq -\frac{c_t}{\sqrt{1+c_t^2}},$$

but this is a contradiction with (3.10).

Then, for every $t > t_0$, the set $\varphi_t\left(\mathbb{S}^m_+ \setminus V\right)$ is contained in the half-space determined by $E(-e^{-t}c)$ and contains $\mathbf{n}$ at its ideal boundary.

□

3.6 Dilation and elliptic problems for conformal metric

Given a conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\Omega)$, that satisfies an elliptic problem for conformal metrics on $\Omega \subset \mathbb{S}^m$, i.e.,

$$f(\lambda(g)) = 1 \text{ on } \Omega,$$

where $f : \Gamma \to \mathbb{R}$ is an elliptic function for conformal metrics (see Subsection 1.5.1), one can naturally ask the following question: given $t_0 \in \mathbb{R}$, is the metric $g_{t_0} = e^{2t_0} g$ a solution of an elliptic problem for conformal metrics on Ω? The answer is affirmative in the non-degenerate case and in the degenerate case. Let see in the non-degenerate case.

Proposition 3.13:
Given a solution $g = e^{2\rho} g_0$, $\rho \in C^\infty(\Omega)$, of an elliptic problem with elliptic data (f,Γ) and $t_0 \in \mathbb{R}$, then $g_{t_0} = e^{2t_0} g$ is a solution of an elliptic problem that is given for the elliptic data (f_{t_0},Γ) where

$$f_{t_0}(x) = f\left(e^{2t_0}x\right) \quad \textit{for all } x \in \Gamma.$$

Proof. Since $\Gamma \subset \mathbb{R}^m$ is a cone then $e^{-2t_0}\Gamma = \Gamma$. Also $\partial\Gamma = e^{-2t_0}\partial\Gamma$, then

$$f_{t_0}(x) = f(e^{2t_0}x) = 0 \quad \text{for all } x \in \partial\Gamma,$$

and

$$\nabla f_{t_0}(x) = e^{2t_0} \nabla f(e^{2t_0} x) \in \Gamma_m \quad \text{for all } x \in \Gamma.$$

It is clear that $f_{t_0} : \overline{\Gamma} \to \mathbb{R}$ is symmetric and

$$f_{t_0}(\lambda(g_{t_0})) = f_{t_0}\left(e^{-2t_0}\lambda(g)\right) = f(e^{2t_0} e^{-2t_0} \lambda(g)) = 1 \text{ on } \Omega.$$

Then, the conformal metric $g_{t_0} = e^{2t_0} g$ is a solution of the elliptic problem for conformal metrics given by the elliptic data (f_{t_0}, Γ). □

In the case that the elliptic data is regular, i.e., there is a constant $0 < \lambda < 1/2$, such that

$$f(\lambda, \ldots, \lambda) = 1,$$

then for every $t_0 > 0$, we have that $0 < e^{-2t_0}\lambda < 1/2$ and

$$f_{t_0}\left(e^{-2t_0}(\lambda, \ldots, \lambda)\right) = 1,$$

then the ellitpic data (f_{t_0}, Γ) is regular.

We remember that if all the eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$ are less than one half then, for every $t > 0$, the eigenvalues of $g_t = e^{2(t+\rho)} g_0$ are less than one half since the eigenvalues of the Schouten tensor of g_t are obtained by multiplying e^{-2t} by the eigenvalues of the Schouten tensor of g, that is, if we move positively along the parallel flow of a horospherically concave hypersurface with injective Gauss map, we get horospherically concave hypersurfaces.

Remark 11:
Observe that if we assume that every eigenvalue of the Schouten tensor of g is less than one half, then for every $t > 0$ we have that the associated map $\phi_t : \Omega \to \mathbb{H}^{m+1}$ of g_t is a horospherically concave hypersurface. Moreover, since g_t is solution of an elliptic problem given by an elliptic data, the hypersurface $\phi_t : \Omega \to \mathbb{H}^{m+1}$ is a solution of an elliptic problem for hypersurfaces in $\mathbb{H}^{m+1}$.

Chapter 4

Escobar Type Problems

In this chapter, we will study Escobar type problems. Since the Schouten tensor is defined for manifolds with dimension greater that 2, then we will suppose that $m \geq 3$ unless there is an explicit statement to the contrary. In Section 4.1, we obtain a non-existence theorem for conformal metrics on the compact hemisphere $\overline{\mathbb{S}^m_+}$ for degenerate elliptic problems with minimal boundary condition, and, more general, with boundary that has nonpositive constant mean curvature. Our result can be extended to simply-connected locally conformally flat manifolds with umbilic boundary, we explain this in Subsection 4.1.1. In Section 4.2, we see that solutions on $\overline{\mathbb{S}^m_+}\setminus$ with minimal boundary are rotationally invariant and we, also, prove this in the non-degenerate case. In Section 4.3, we work with degenerate problems on closed annulus $\overline{\mathbb{A}(r)}$, $r \in (0, \pi/2)$. First, we prove that if there is a solution to a degenerate problem with minimal boundary then it is rotational invariant and it is unique up to dilations. Then, we will see that if there is a solution $g = e^{2\rho} g_0$ to the degenerate problem on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ with minimal boundary such that the function $\sigma = e^{-\rho}$ can be extended to a C^2 function $\tilde{\sigma}$ on $\overline{\mathbb{S}^m_+}$ with $\tilde{\sigma}(\mathbf{n}) = 0$, then there is no solution for the degenerate problem with minimal boundary on $\overline{\mathbb{A}(r)}$. In Section 4.4, we study some degenerate and non-degenerate problems on the non-compact annulus $\mathbb{A}\left(r, \frac{\pi}{2}\right]$, $0 < r < \pi/2$, we will get necessary conditions for existence of solutions, if certain conditions are satisfied.

Finally, in Section 4.5, we also see that one can establish some similar results when the dimension is 2 and we observe the difference between the dimension 2 and higher dimension case.

Along this chapter, the regularity conditions on ρ can be relaxed.

4.1 A non-existence Theorem on $\overline{\mathbb{S}^m_+}$

As we have said at the Introduction, Escobar [15, 16] proved that there is no conformal metric $g = e^{2\rho} g_{Eucl}$ on the compact Euclidean unit ball $\overline{\mathbb{B}^m}$ with zero scalar curvature and nonpositive constant mean curvature along boundary, i.e., $h(g) \leq 0$ constant. We extend here such result for degenerate elliptic equations.

We will explain the idea of our proof in the minimal case for simplicity. Assume that there exists $\rho \in C^\infty(\overline{\mathbb{S}^m_+})$ so that the conformal metric $g = e^{2\rho} g_0$ satisfies

$$\begin{cases} f(\lambda(g)) &= 0 & \text{in } \overline{\mathbb{S}^m_+}, \\ h(g) &= 0 & \text{on } \partial\mathbb{S}^m_+, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of g and (f, Γ) is an elliptic data for conformal metrics on domains of $\mathbb{S}^m$. Then, we consider the horospherically concave hypersurface Σ associated to ρ given by Theorem 2.10. We observe that the boundary of Σ is contained in a totally geodesic hypersurface E of the hyperbolic space $\mathbb{H}^{m+1}$ (cf. Proposition 3.10). Without loss of generality, we can suppose that Σ is embedded and contained in one of the half-spaces determined by the totally geodesic hypersurface E, see Theorem 3.12. Using Section 2.6, the horospherical concave hypersurface is a solution to an elliptic problem. Also, the horospheres are solutions for such problem of hypersurfaces in the Hyperbolic space $\mathbb{H}^{m+1}$.

We can parametrize the horospheres, with the south pole as the point at infinity of such family, with the signed distance of those horospheres to the origin in the Poincaré ball model. When the parameter is large negatively, the horospherically concave hypersurface Σ is inside the horosphere, i.e., it is in the convex side of the horosphere. Then we increase the parameter until we have a first horosphere that touches to the hypersurface Σ. We have a contact point between the horosphere and the hypersurface Σ. We notice that such point happens at the interior of Σ and there is no boundary contact point. Then using Theorem 1.10 we get a contradiction.

Theorem 4.1:
Let (f, Γ) be an elliptic data for conformal metrics and let $c \leq 0$ be a constant. Then, there is no conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+}$, where $\rho \in C^\infty\left(\overline{\mathbb{S}^m_+}\right)$, such that

$$\begin{cases} f(\lambda(g)) &= 0 & \text{on} & \overline{\mathbb{S}^m_+}, \\ h(g) &= c & \text{on} & \partial\mathbb{S}^m_+, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of $g = e^{2\rho} g_0$.

Proof. The proof will be done by contradiction. Assume that there exists a conformal metric on $\overline{\mathbb{S}^m_+}$, $g = e^{2\rho} g_0$, where $\rho \in C^\infty\left(\overline{\mathbb{S}^m_+}\right)$, that solves the above problem.

Since $\overline{\mathbb{S}^m_+}$ is compact, up to a dilation of g, we can assume (see Section 3.6), without loss of generality, that all the eigenvalues of $\mathrm{Sch}(g)$ are less that $1/2$. By Theorems 3.6 and 3.12, we can assume that the associated horospherically concave hypersurface $\phi : \overline{\mathbb{S}^m_+} \to \mathbb{H}^{m+1}$ is embedded and it is contained in the half-space determined by the equidistant hypersurface $E(-c)$ to the totally geodesic hypersurface $E(0)$, and such component contains $\mathbf{n}$ at its ideal boundary.

We notice that, in the Poincaré ball model, since $-c \geq 0$, the equidistant $E(-c)$ is contained in the half-space determined by the totally geodesic hyperplane $E(0)$ that contains $\mathbf{n}$ at its ideal boundary.

We consider the horospherically concave hypesurface $\Sigma = \phi\left(\overline{\mathbb{S}^m_+}\right)$ in the Hyperbolic space $\mathbb{H}^{m+1}$ that is associated to ρ.

Since, for all $t > 0$, we have

$$\frac{1}{2} = e^{-2t}\lambda_i + \frac{1}{1+\kappa_{i,t}}, \quad i = 1, \ldots, m,$$

where $k_{i,t}$ are the principal curvatures the hypersurface Σ_t that we get from the parallel flow at the time $t > 0$. Then, for t big, all the $k_{i,t}$ are positive. So, we can assume that all the principal curvatures of ϕ with respect the normal η are positives. So, we can assume the normal η points to the convex side of Σ.

Let D the bounded component that is determined by $\partial\Sigma$ in $E(-c)$. It is important to note that if we consider the topological sphere $D \cup \Sigma$, then the normal η points inward, since the hyperbolic Gauss map G is just the inclusion $\overline{\mathbb{S}^m_+} \to \mathbb{S}^m$.

Moreover, the degenerate elliptic problem for the conformal metric implies a degenerate elliptic problem for the Weingarten hypersurface in the Hyperbolic space $\mathbb{H}^{m+1}$ (see Section 2.6), where the hypersurface Σ is, now, a solution, that is,

$$\begin{cases} \mathscr{W}(\kappa_1, \ldots, \kappa_m) & = & 0 & \text{in} & \Sigma, \\ \cos(\alpha) & = & -\dfrac{c}{\sqrt{1+c^2}} & \text{on} & \partial\Sigma, \end{cases}$$

where $\kappa_1, \ldots, \kappa_m$ are the principal curvatures of Σ with its natural orietation $\eta : \overline{\mathbb{S}^m_+} \to \mathbb{S}^{m+1}_1$ given by $\eta(x) = \phi(x) - e^{\rho(x)}(1, x)$, where $x \in \overline{\mathbb{S}^m_+}$, and α is the angle between the hypersurface Σ and the equidistant hypersurface $E(-c)$ along $\partial\Sigma$, this follows from Proposition 3.11.

Recall that horospheres are solutions of the above degenerate elliptic problem for hypersurfaces in $\mathbb{H}^{m+1}$. Now, we consider a foliation of the Hyperbolic space $\mathbb{H}^{m+1}$ by horospheres, $\{H(s)\}_{s\in\mathbb{R}}$, that have the same point at the ideal boundary of the Hyperbolic Space, $p_\infty = \mathbf{s}$. We parametrize $s \in \mathbb{R}$ by the signed distance to the origin of the Poincaré ball model.

Since Σ is compact, for s large enough negatively, Σ is completely contained in the mean-convex side of the horosphere $H(s)$ (cf. Figure 4.1(a) for the case $c = 0$). We continue increasing s until we have the first contact point with Σ, let us say s_1 (cf. Figures 4.1(b), 4.1(c) 4.1(d) for the case $c = 0$). This means, for every $s < s_1$, the hypersurface Σ is in the interior of $H(s)$, $H(s) \cap \Sigma = \emptyset$, and $H(s_1) \cap \Sigma$ is not empty.

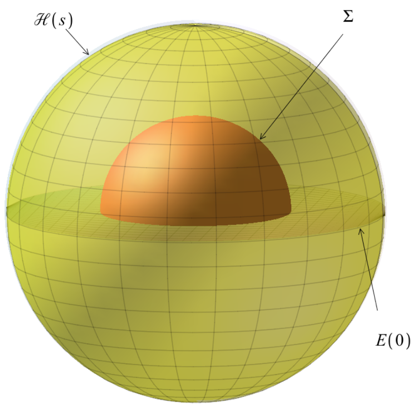
We note that

$$\cos(\alpha) = -\frac{c}{\sqrt{1+c^2}} \geq 0.$$

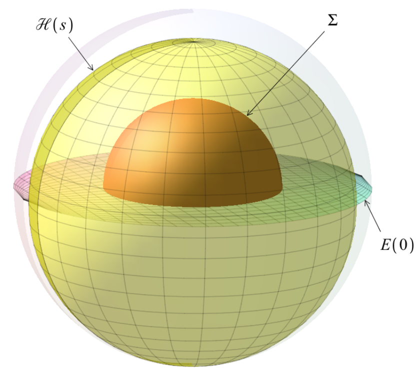
Since the angle α between the normal η and the upward normal of $E(-c)$ is acute or $\pi/2$, we get that the contact point is in the interior of Σ.

The horosphere $H(s_1)$ with its natural orientation has its own representation formula over $\mathbb{S}^m \setminus \{\mathbf{s}\}$. We consider such formula restricted to $\overline{\mathbb{S}^m_+}$ and we denote its support function by ρ_1. Since this horosphere is the first contact horosphere, we have that $\rho_1 \geq \rho$ on $\overline{\mathbb{S}^m_+}$. Also, there is no boundary contact point of Σ, so there is $x \in \mathbb{S}^m_+$ such that $\rho_1(x) = \rho(x)$ and

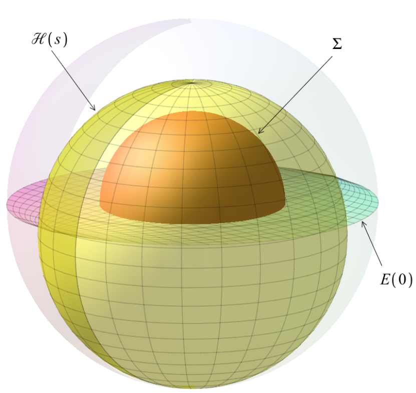
$$\rho_1 > \rho \quad \text{on } \partial\mathbb{S}^m_+,$$



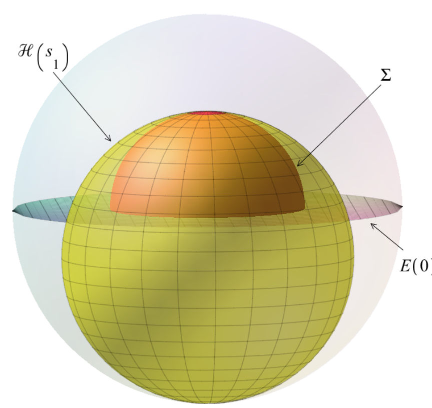
(a) The hypersurface Σ is in the convex side of the horosphere $H(s)$.



(b) The horosphere $H(s)$ does not intersect to Σ.



(c) The horosphere $H(s)$ does not intersect to the hypersurface Σ.



(d) The horosphere $H(s_1)$ touches the hypersurface Σ.

Figure 4.1: Process of getting the first contact horosphere in the Poincaré ball model. Case $c = 0$.

then, by Theorem 1.10, $\rho_1 > \rho$ on $\overline{\mathbb{S}^m_+}$, that is a contradiction, since there is $x \in \mathbb{S}^m_+$ such that $\rho_1(x) = \rho(x)$. □

In the case non-degenerate on a closed northern hemisphere, see the article of M. Cavalcante and J. Espinar. [7].

4.1.1 Simply-Connected Locally Conformally Flat Manifolds

In the case that we consider a m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with boundary $\partial\mathscr{M}$ that is umbilic, and $Scal(g_0) \geq 0$ on $\mathscr{M}$, we have,

Theorem 4.2:
Set (f,Γ) an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathscr{M}, g_0)$ be a m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with umbilic boundary, and $Scal(g_0) \geq 0$ on $\mathscr{M}$. Then, there is no conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\mathscr{M})$, such that

$$\begin{cases} f(\lambda(g)) &= 0 \quad in \quad \mathscr{M}, \\ h(g) &= c \quad on \quad \partial\mathscr{M}, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho} g_0$.

Proof. By a result of F. M. Spiegel [46], there exists a conformal diffeomorphism between the Riemannian manifold $(\mathscr{M}, g_0)$ and the standard closed hemisphere $\overline{\mathbb{S}^m_+}$. This is done by the developing map Φ of Schoen-Yau [44] and observing that umbilicity is preserved under conformal transformations. Hence, the boundary $\partial\mathscr{M}$ maps into a hypersphere in $\mathbb{S}^m$ by Φ and therefore, Φ maps $\mathscr{M}$ into the interior of a ball.

Thus, the statement follows from Theorem 4.1. □

Observe that the above argument can be used into the non-degenerate case studied by Cavalcante-Espinar [7]. Specifically,

Theorem 4.3:
Set (f,Γ) an elliptic data for conformal metrics and $c \leq 0$ a constant. Let $(\mathscr{M}, g_0)$ be a m-dimensional compact, simply-connected, locally conformally flat manifold $(\mathscr{M}, g_0)$ with umbilic boundary, and $Scal(g_0) \geq 0$ on $\mathscr{M}$. If there exists a conformal metric $g = e^{2\rho} g_0$, $\rho \in C^\infty(\mathscr{M})$, such that

$$\begin{cases} f(\lambda(g)) &= 1 \quad in \quad \mathscr{M}, \\ h(g) &= c \quad on \quad \partial\mathscr{M}, \end{cases}$$

where $\lambda(g) = (\lambda_1, \ldots, \lambda_m)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho} g_0$, then $\mathscr{M}$ is isometric to a geodesic ball on the standard sphere $\mathbb{S}^m$.

Proof. So, as above, there exists a conformal diffeomorphism from $\mathscr{M}$ to a ball into $\mathbb{S}^n$ whose boundary has constant mean curvature. Moreover, the elliptic problem on $\mathscr{M}$ pass to an elliptic problem for the pushforward metric on the geodesic ball. Thus, by [7, Theorem 1.1], such pushforward metric has constant Schouten tensor, i.e., all the eigenvalues of the Schouten tensor are equal to the same constant. Hence, $\mathscr{M}$ is isometric to the geodesic ball in the sphere by the work of F. M. Spiegel [46]. □

4.2 Punctured geodesic ball

Now, we see that any solution to degenerate problem on the punctured geodesic ball $\overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}$ with minimal boundary is rotationally invariant.

Theorem 4.4:
Let $g = e^{2\rho}g_0$ be a conformal metric on $\overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}$ that is solution of the following degenerate elliptic problem:

$$\left\{\begin{array}{rcl} f(\lambda(g)) & = & 0 \qquad in\ \overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}, \\ h(g) & = & 0 \qquad on\ \partial\mathbb{S}^m_+, \end{array}\right.$$

Then g is rotationally invariant.

Proof. Let us define $\tilde{\rho}:\mathbb{S}^m\setminus\{\mathbf{n},\mathbf{s}\}\to\mathbb{R}$ as

$$\tilde{\rho}(x)=\left\{\begin{array}{ll} \rho(x_1,\ldots,x_m) & x\in\overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}, \\ \rho(x_1,\ldots,-x_m) & x\in\overline{\mathbb{S}^m_-}\setminus\{\mathbf{s}\}. \end{array}\right.$$

First, we show that $\tilde{\rho}$ is C^1. Since

$$\frac{\partial\rho}{\partial x_{m+1}}=0 \quad \text{on } \partial\mathbb{S}^m_+$$

then $\tilde{\rho}\in C^1\left(\mathbb{S}^m\setminus\{\mathbf{n},\mathbf{s}\}\right)$. Then we have the following vector field $\nabla\tilde{\rho}:\mathbb{S}^m\setminus\{\mathbf{n},\mathbf{s}\}\to T\mathbb{S}^m$ is continuous

$$\nabla\tilde{\rho}(x)=\left\{\begin{array}{ll} \nabla\rho(x) & x\in\overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}, \\ R\nabla\rho(R(x)) & x\in\overline{\mathbb{S}^m_-}\setminus\{\mathbf{s}\}, \end{array}\right.$$

where $R:\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}$ is the Euclidean reflection $R(x_1,\ldots,x_m)=(x_1,\ldots,-x_m)$, for $(x_1,\ldots,x_m)\in\mathbb{R}^{m+1}$.

Now, let us see that $\tilde{\rho}\in C^2$, that is $\nabla\tilde{\rho}$ is C^1. Let $X_1=\nabla\rho|_{\overline{\mathbb{S}^m_+}\setminus\{\mathbf{n}\}}$ and $X_2=\nabla\tilde{\rho}|_{\overline{\mathbb{S}^m_-}\setminus\{s\}}$.

Since $X_1=X_2$ on $\partial\mathbb{S}^m_+$, given $x\in\partial\mathbb{S}^m_+$ and $v\in T_x(\partial\mathbb{S}^m_+)$ we have $\nabla_v X_1=\nabla_v X_2$.

Also, assume for a moment that $R(\nabla^2\rho(x)(e_{m+1}))=-\nabla^2\rho(x)(e_{m+1})$ for all $x\in\partial\mathbb{S}^m_+$. We have that for $x\in\partial\mathbb{S}^m_+$ and $v_m=e_{m+1}$,

$$\nabla_{-v_m}X_2=\frac{\partial X_2}{\partial(-v_m)}(x)-<X_2(x),-v_m>x=\frac{\partial X_2}{\partial(-v_m)}(x)=R\nabla_{v_m}X_1=R\left(\nabla^2\rho(x)(e_{m+1})\right)=-\nabla_{v_m}X_1$$

then $\nabla\tilde{\rho}$ is C^1, then, $\tilde{\rho}$ is C^2.

Since $\tilde{\rho} : \mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\} \to \mathbb{R}$ is C^2 and it is a solution of a degenerate problem, from Theorem 1.11, $e^{2\tilde{\rho}} g_0$ is rotationally invariant, thus g is rotationally invariant.

In order to conclude the proof, let us see that $R(\nabla^2\rho(x)(e_{m+1})) = -\nabla^2\rho(x)(e_{m+1})$ for all $x \in \partial\mathbb{S}^m_+$: for every $v \in T_x\partial\mathbb{S}^m_+$ we have that $< \nabla^2\rho(x)(v), e_{m+1} >= 0$ then $\nabla^2\rho_x(V) \subset V$ where $V = T_x\partial\mathbb{S}^m_+$. Since $\nabla^2\rho(x) : V \to V$ is symmetric, $\nabla^2\rho(x)(e_{m+1}) \| e_{m+1}$. Thus $R(\nabla^2\rho(x)(e_{m+1})) = -\nabla^2\rho(x)(e_{m+1})$.

□

In the case of a non-degenerate elliptic problem, we have the following

Theorem 4.5:
Let $g = e^{2\rho} g_0$ be a conformal metric on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ that is solution of the following non-degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) & = & 1 & \text{in } \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}, \\ h(g) & = & 0 & \text{on } \partial\mathbb{S}^m_+, \end{cases}$$

Then g is rotationally invariant.

Proof. If ρ admits smooth extension $\tilde{\rho} : \overline{\mathbb{S}^m_+} \to \mathbb{R}$ then the conformal metric $g_1 = e^{2\tilde{\rho}} g_0$, defined on $\overline{\mathbb{S}^m_+}$, is a solution to a non-degenerate problem with constant mean curvature on its boundary. Then g_1 is rotationally invariant, so, g is rotationally invariant.

If ρ does not admit smooth extension on $\overline{\mathbb{S}^m_+}$, then by [31], ρ is rotationally invariant, then g is rotationally invariant.

That concludes the proof. □

4.3 Compact Annulus

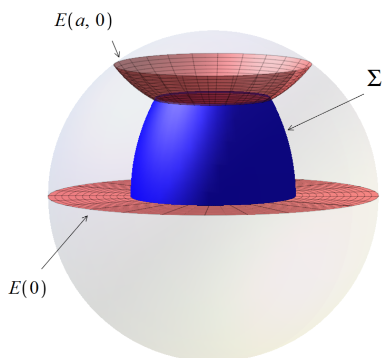
We consider the following degenerate elliptic problem on the compact annulus $\overline{\mathbb{A}(r)}$, where $r \in (0, \pi/2)$ with minimal boundary.

We prove that any solution to that problem is rotationally invariant. Moreover, it is unique up to dilations.

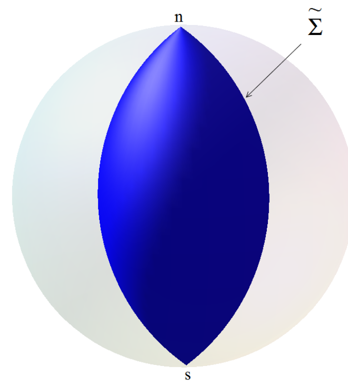
Theorem 4.6:
Set $r \in (0, \pi/2)$. If there is a solution $g = e^{2\rho} g_0$ of the following problem

$$\begin{cases} f(\lambda(g)) & = & 0 & \text{on } \overline{\mathbb{A}(r)}, \\ h(g) & = & 0 & \text{on } \partial\mathbb{A}(r), \end{cases}$$

then it is rotationally invariant and unique up to dilations.



(a) The hypersurface Σ is between two totally geodesic hypersurfaces: $E(a,0)$ and $E(0)$.



(b) Extension $\tilde{\Sigma}$ of Σ by reflection w.r.o. totally geodesic hypersurfaces.

Figure 4.2: Extension of the compact hypersurface Σ.

Proof. Let g_1 be a solution of the degenerate problem given in the theorem. First, we show that g_1 is rotationally invariant. Using the same techniques than M. P. Cavalcante and J. M. Espinar [7], we get a conformal metric on $\mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\}$ that is solution to the degenerate problem. Let us sketch this for the reader convenience. Using the Local Representation Theorem, the horospherical hypersurface associated to g_1 is contained in the slab determined by two parallel hyperplanes and, by the boundary condition, the boundaries meet orthogonally such hyperplanes (see Chapter 3, see Figure 4.2a when $g_1 = g$). Thus, we can reflect this annulus with respect to the hyperplanes and we repeat this process. Then, we obtain an embedded complete horospherically concave hypersurface $\tilde{\Sigma_1}$ whose boundary at infinity are the north and south pole and, by construction, the extension $\tilde{\Sigma_1}$ of Σ_1 is contained in the interior of an equidistant hypersurface to the geodesic joining the north and south pole, the radius of such equidistant is determined by the original annulus, then its horospherical metric $\tilde{g_1}$ it a solution to the degenerate problem on $\mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\}$ (cf. Figure 4.2b when $g_1 = g$). By analogous argument in the proof of Theorem 4.4, the metric $\tilde{g_1}$ is C^2 on $\mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\}$. Using Theorem 1.11, we have that $\tilde{g_1}$ is rotationally invariant, thus g_1 is rotationally invariant.

Now, we show that it is unique up to dilations. Again, using the parallel flow, we can assume that the associated hypersurface associated Σ_1 to g_1 is an embedded horospherically concave hypersurface.

By Theorems 3.6 and 3.12, we can suppose that the hypersurface is between the totally geodesic hypersurfaces $E(0)$ and $E(a)$, where $a = \left(\cot(r), \frac{1}{\sin(r)}\mathbf{n}\right)$. This can be visualized in the Poincaré ball model. Also, it continues there along the positive parallel flow as an embedded horospherically concave hypersurface.

Let Σ be the hypersurface associated to g. Since Σ and Σ_1 are compact, we can use the parallel flow in order that Σ_1 has no intersection with Σ. That is, if we look at the Poincaré ball model, we could say that Σ_1 is on the exterior of Σ, then again we use the parallel flow on Σ until Σ touches Σ_1. We have a contact point.

We claim that there is a boundary contact point. If not, let ρ_1 be the support function of Σ_1 and ρ be the support function of Σ. We have that there is $x \in \mathbb{A}(r)$ such that $\rho_1(x) = \rho(x)$, and, also,

$$\rho_1 > \rho \quad \text{on } \partial\mathbb{A}(r),$$

by Theorem 1.10, $\rho_1(x) > \rho(x)$, which is a contradiction.

Even more, we claim that both components of $\partial\Sigma_1$ have contact point with $\partial\Sigma$. Suppose that one component of $\partial\Sigma_1$ does not have contact point with $\partial\Sigma$, we reflect both hypersurfaces with respect to the hyperplane that contain the component of $\partial\Sigma_1$ that touches $\partial\Sigma$. We have an extension Σ_1' of Σ_1 and an extension Σ' of Σ such that they have an interior contact point and there is no boundary contact point. Then apply the same arguments of the above paragraph, we get a contraction.

Also we can say that in very level of $\mathbb{A}(r)$ there is a contact point between Σ_1 and Σ, i.e., for every $s \in (r, \pi/2)$ there is $x \in \mathbb{A}(r)$, with $d_{\mathbb{S}^m}(x, \mathbf{n}) = s$, such that the support function of Σ_1 and the function support of Σ take the same value at x. The proof is also by contradiction. If there is no such x, then we use the reflection to attain a contradiction as above.

We known that ρ_1 and ρ are rotationally invariant. Since in every level of $\mathbb{A}(r)$, there is a point in that level such that ρ_1 and ρ are equal, we have that $\rho_1 = \rho$ on that level, then $\rho_1 = \rho$ on $\overline{\mathbb{A}(r)}$. So, they are equal unless a translation through the parallel flow. That is, g_1 is a dilation of g.

This concludes the proof. □

In the case non-degenerate on the compact annulus, see the article of M. Cavalcante and J. Espinar. [7].

We will prove that the problem does not have solution if there is a solution $g = e^{2\rho} g_0$ to the following problem:

Find a conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+}$ such that

$$\begin{cases} f(\lambda(g)) = 0 & \text{in } \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}, \\ h(g) = 0 & \text{on } \partial\mathbb{S}^m_+, \end{cases} \tag{4.1}$$

such $\sigma = e^{-\rho}$ admits a C^2-extension $\tilde{\sigma} : \overline{\mathbb{S}^m_+} \to \mathbb{R}$ with $\tilde{\sigma}(\mathbf{n}) = 0$.

Definition 4.7:
We say that a conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ is a punctured solution of the problem (4.1) if it is a solution of (4.1) and $\sigma = e^{-\rho}$ admits a C^2-extension $\tilde{\sigma} : \overline{\mathbb{S}^m_+} \to \mathbb{R}$ with $\tilde{\sigma}(\mathbf{n}) = 0$.

As a first observation, we have:

Proposition 4.8:
Let $g = e^{2\rho} g_0$ be a punctured solution to the problem (4.1) then the associated map $\varphi_P : \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\} \to \mathbb{R}^{m+1}$ in the Poincaré ball model, i.e.,

$$\varphi_P(x) = \frac{1 - e^{-2\rho(x)} + \left|\nabla e^{-\rho(x)}\right|^2}{\left(1 + e^{-\rho(x)}\right)^2 + \left|\nabla e^{-\rho(x)}\right|^2} x - \frac{1}{\left(1 + e^{-\rho(x)}\right)^2 + \left|\nabla e^{-\rho(x)}\right|^2} \nabla \left(e^{-2\rho}\right)(x), \quad \forall x \in \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\},$$

admits a C^1- extension to $\overline{\mathbb{S}^m_+}$.

Also, from Theorem 4.4, we have the following proposition.

Proposition 4.9:
Punctured solutions are rotationally invariant.

Now, we are ready to prove:

Theorem 4.10:
Set $r \in (0, \pi/2)$. If the problem (4.1) admits a punctured solution, then there is no solution to the following degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) & = & 0 & \text{on } \overline{\mathbb{A}(r)}, \\ h(g) & = & 0 & \text{on } \partial \mathbb{A}(r). \end{cases}$$

Proof. We will prove this by contradiction. Suppose that there is a solution for the above problem. From the proof of Theorem 4.6, that solution can be extended to a solution $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$. Let Σ be the associated hypersurface to g. Using the parallel flow, we can assume that it is horospherically concave hypersurface through the positive parallel flow and its interior is in the component determined by $E(0)$ such that its ideal boundary contains $\mathbf{n}$.

Let g_P be a punctured solution of the problem (4.1). Since $\partial\Gamma$ is a cone, the conformal metric $e^{2s} g_P$ is also a solution of the problem (4.1), for every $s \in \mathbb{R}$.

There is $s_0 > 0$, such that for all $s \geq s_0$ the associated horospherical hypersurface ϕ_s to the punctured solution $e^{2s} g_P$ is embedded and its interior is contained in the same component for the above horospherically concave hypersurfaces. The family of hypersufaces $\{Q(s)\}_{s \geq s_0} = \{Im(\phi_s)\}_{s \geq s_0}$ converges, in the Poincaré ball model, to the inclusion $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\} \hookrightarrow \mathbb{S}^m$. Also, every $Q(s)$ admits a C^1-extension to $\mathbf{n}$ in Poincaré ball model contained in $\mathbb{R}^{m+1}$ and their tangent hyperplanes at $\mathbf{n}$ are parallel to the hyperplane $x_{m+1} = 0$.

There is a $t_1 > 0$ such the associated hypersuface Σ_1 to $g_1 = e^{2t_1} g$ intersects the family $\{Q(s)\}_{s \geq s_0}$. Without loss of generality we assume that the associated hypersurface Σ to g intersects the family $\{Q(s)\}_{s \geq s_0}$. Also, there is $s_1 \geq s_0$ such that

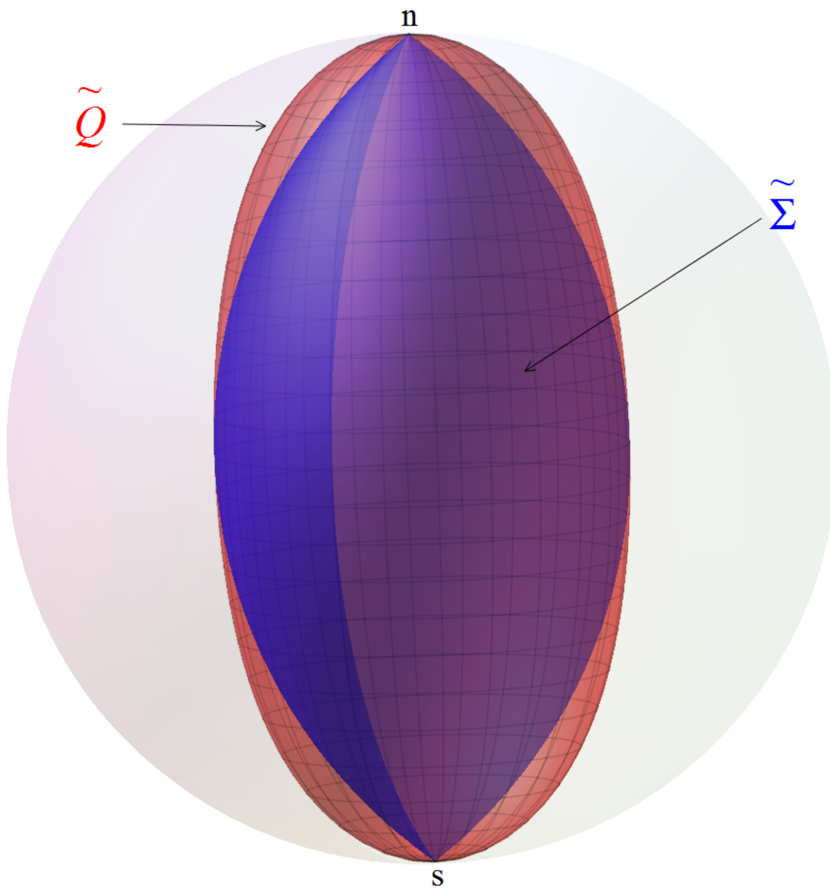


Figure 4.3: The first hypersurface $\tilde{Q}$ of a punctured solution that touches the hypesurface $\tilde{\Sigma}$.

1. $Q(s_1) \cap \Sigma \neq \emptyset$.
2. for $s > s_1$: $Q(s) \cap \Sigma = \emptyset$.

Then, we have found a first contact point. Observe that such contact point can not be at infinity since Σ is contained in the interior of the equidistant to a geodesic and the family $Q(s)$ extends to $\mathbf{n}$. Now, if the first contact point is an interior point then $Q(s_1)$ and Σ are tangent at such point. If the first contact point occurs on the boundary, $Q(s_1)$ and Σ are tangent at that point too, because all the hypersurfaces $Q(s)$ are orthogonal to the totally geodesic hypersurface $E(0)$ and Σ is orthogonal to $E(0)$ too.

We reflect with respect to the totally geodesic hypersurface $E(0)$ the hypersurfaces $Q(s_1)$ and Σ to get the horospherically concave hypersurfaces $\tilde{Q}$ and $\tilde{\Sigma}$ (cf. Figure 4.3). Their support functions are defined on $\mathbb{S}^m \setminus \{\mathbf{n}, \mathbf{s}\}$. Let ρ_1 be the support function of $\tilde{Q}$ and $\tilde{\rho}$ be the support function of $\tilde{\Sigma}$. Since there is a contact point in the interior of $\tilde{Q}$, there is $0 < \delta < \pi/2$ such that

$$\rho_1 > \tilde{\rho} \quad \text{on } \partial\Omega,$$

where $\Omega = \{x \in \mathbb{S}^m : \delta < d_{\mathbb{S}^m}(x, \mathbf{n}) < \pi - \delta\}$ and there $x_0 \in \Omega$ such that $\rho_1(x_0) = \tilde{\rho}(x_0)$, but this is a contradiction with Theorem 1.10. That concludes the proof. □

It is important to say that the σ_k-Yamabe problem on $\overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\}$ admits a punctured solution when $1 \leq k < m/2$. That punctured solution is associated to

$$\sigma(x) = \left((1 + x_{m+1})^{\beta} + (1 - x_{m+1})^{\beta} \right)^{\frac{1}{\beta}} \quad x \in \overline{\mathbb{S}^m_+} \setminus \{\mathbf{n}\},$$

where $\beta = 1 - m/(2k) < 0$, these solutions were constructed by S.-Y. A. Chang, Z. Han, and P. Yang [10].

For example, for $m = 3$ and $k = 1$, so $a = -1/2$, in Figure 4.4, we can see a slice of the associated hypersurface to $3\sigma(x_1, x_2, x_3, x_4)$ using the Poincaré ball model.

When m is even and $k = m/2$, the σ_k-Yamabe problem on the compact annulus has a solution g with $\sigma_k(\lambda(g)) = 0$ and minimal boundary.

Also, it is good to say that the assumption on the existence of the punctured solution is not a necessary condition for the non-existence of solutions to the degenerate problems on the compact annulus with minimal boundary. We know that the punctured solutions are rotationally invariant (cf. Proposition 4.9). Let us consider the σ_k-Yamabe problem when $k > m/2$, in these cases there is not solution to the degenerate problem with minimal boundary and, also, there is not a punctured solution for these problems (cf. [10]).

4.4 Semi-annulus

Now we focus on different boundary conditions when we consider an annulus on $\mathbb{S}^m$ whose boundary components are geodesic spheres, that is, the domains we will consider in this section are the sphere

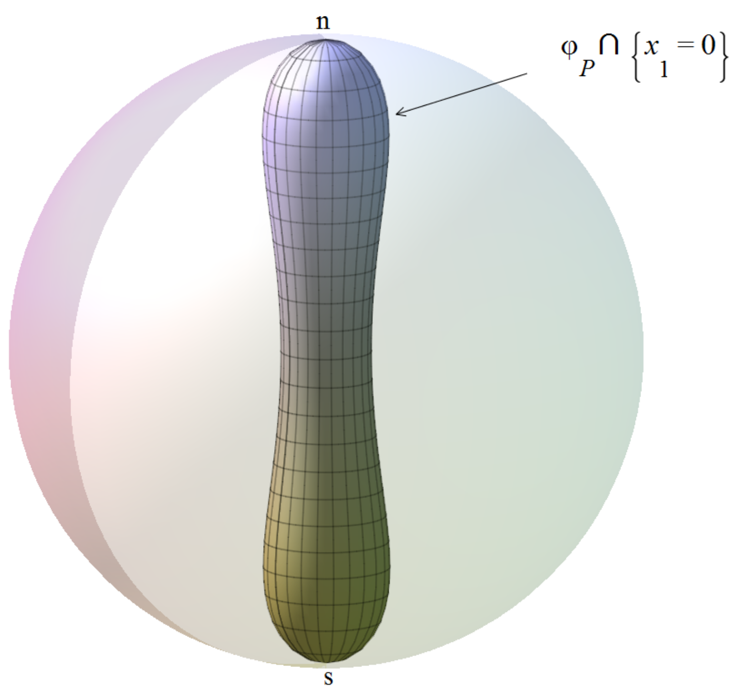


Figure 4.4: This is a slice of the hypersurface associated to 3σ using the Poincaré ball model for $m = 3$ and $k = 1$.

$\mathbb{S}^m$ minus two geodesic balls. Observe that, up to a conformal diffeomorphism, we can assume that the annulus is $\mathbb{A}(r,\pi/2] = \mathbb{A}(r) \cup \partial\mathbb{S}^m_+$, where $0 < r < \pi/2$. At one boundary component we will impose mild conditions on the metric and at the other we will impose constancy of the mean curvature of the conformal metric.

Our next result will say any conformal metric $g = e^{2\rho}g_0$ on $\mathbb{A}(r,\pi/2]$ that satisfy certain property at its end and this metric is solution to a degenerate problem with non-negative constant mean curvature on its boundary, has no bounded Schouten tensor. Specifically,

Theorem 4.11:
Let $r \in (0,\pi/2)$, $c \geq 0$ be a non-negative constant and $g = e^{2\rho}g_0$ be a conformal metric on $\mathbb{A}\left(r,\frac{\pi}{2}\right]$ that is solution of the following degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) &= 0 \quad \text{in } \mathbb{A}(r,\pi/2], \\ h(g) &= c \quad \text{on } \partial\mathbb{S}^m_+, \end{cases}$$

If $e^{2\rho} + |\nabla\rho|^2 : \mathbb{A}(r,\pi/2] \to \mathbb{R}$ is proper then $\lambda(g)$ is no bounded.

Proof. The proof is by contradiction. Suppose that $\lambda(g)$ is bounded. Using the parallel flow we can assume that $\phi_P : \mathbb{A}(r,\pi/2] \to \mathbb{R}^m$ is a proper horosphererically concave hypersurface and that property is invariant by the positive parallel flow.

We have the continuous extension $\Phi : \overline{\mathbb{A}(r)} \to \mathbb{R}^{m+1}$ of $\varphi_P : \mathbb{A}(r,\pi/2] \to \mathbb{R}^{m+1}$, defined by

$$\Phi(x) = \begin{cases} \phi_P(x) & x \in \mathbb{A}(r,\pi/2], \\ x & x \in S_r(\mathbf{n}). \end{cases}$$

We consider the foliation of $\mathbb{H}^{m+1}$ by horospheres $\{H(s)\}_{s\in\mathbb{R}}$ that have the same ideal boundary $\{\mathbf{n}\}$, and s is the signed distance between $H(s)$ and the origin of the Poincaré ball model. Since $r > 0$, we have that there is $s_1 \in \mathbb{R}$ such that $H(s_1) \cap Im(\Phi) \neq \emptyset$ and $H(s) \cap Im(\Phi) = \emptyset$ for $s > s_1$.

Also, since $c \geq 0$, the angle between $\Sigma = Im(\Phi)$ and the equidistant $E(-c)$ is obtuse and the angle between the horosphere $H = H(s_1)$ and the equidistant is acute, we have that the contact point is in the interior of Σ (see Figure 4.5 for case $c = 0$). That is, there is $x \in \mathbb{A}(r)$ where $\varphi_P(x) \in H$. Also $\phi_P^{-1}(H) \subset \mathbb{A}(r)$ is compact and there is $r < r_1 < \pi/2$ such that

$$x \in \phi_P^{-1}(H) \subset \mathbb{A}(r_1).$$

Let ρ_0 be the function support of the horosphere H restricted to $\overline{\mathbb{A}(r_1)}$ then $\rho_0(x) = \rho(x)$ and

$$\rho > \rho_0 \quad \text{on } \partial\mathbb{A}(r_1)$$

By Theorem 1.10, $\rho(x) > \rho_0(x)$, which is a contradiction. That concludes the proof. □

A similar result can be obtained if we consider that the conformal metric is a solution of a non-degenerate elliptic problem that satisfied certain mild conditions.

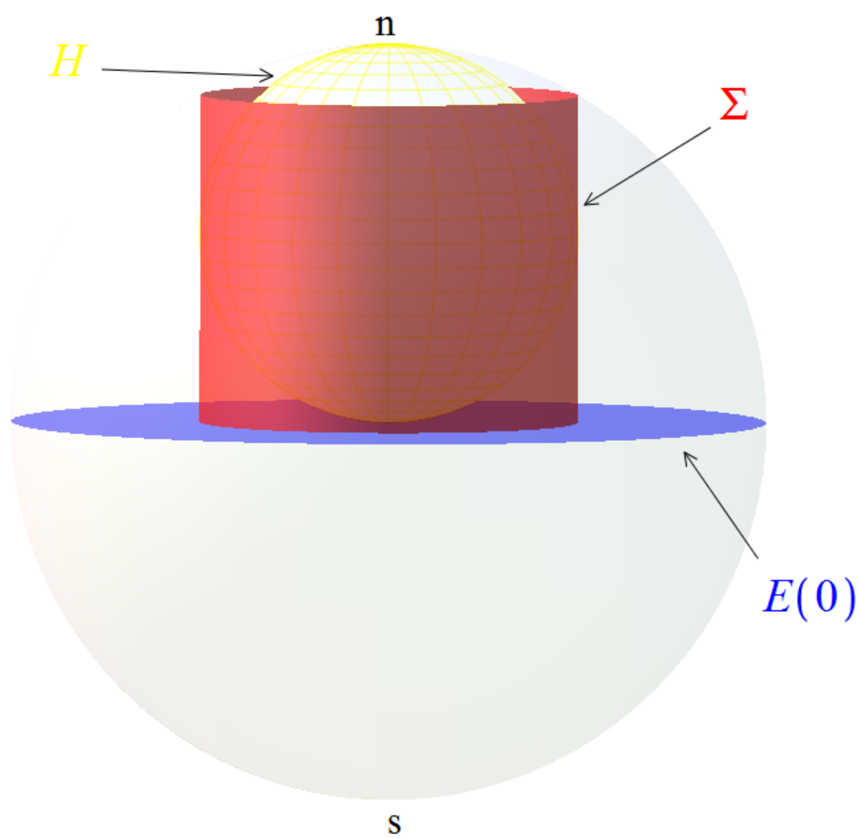


Figure 4.5: The horosphere H touching the hypersurface Σ.

Theorem 4.12:
Let $0 < r < \pi/2$, $c \in \mathbb{R}$ be a constant and $g = e^{2\rho} g_0$ be a conformal metric on $\mathbb{A}\left(r, \frac{\pi}{2}\right]$ that is solution of the following non-degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) &= & 1 & \text{in } \mathbb{A}(r,\pi/2], \\ h(g) &= & c & \text{on } \partial\mathbb{S}^m_+, \\ \lim\limits_{x\to q} \rho(x) &= & +\infty & \forall q \in \partial B_r(\mathbf{n}). \end{cases}$$

Let $\sigma = e^{-\rho}$. If $|\nabla\sigma|^2$ is Lipschitz then $\nabla^2(\sigma^2)$ is no bounded.

Proof. The proof is by contradiction. We suppose that $\nabla^2\sigma^2$ is bounded.

Using the parallel flow, we can assume that $\phi : \mathbb{A}(r,\pi/2] \to \mathbb{H}^{m+1}$ is a properly embedded horospherically concave hypersurface.

From hypothesis $h(g) = c$, we have that the boundary $\partial\Sigma$ is in $E(-c)$.

Take a closed ball Q with center the origin of the Poincaré model with big radius such that $\partial\Sigma$ is in the interior of Q (see Figure 4.6 for the case $c = 0$). Since f is homogeneous and $f(1,\dots,1) > 0$, there is a constant $\lambda_0 > 0$ such that

$$f(\lambda_0,\dots,\lambda_0) = 1,$$

and using the parallel flow, we can assume that $0 < \lambda_0 < 1/2$.

We work in the Poincaré ball model. Consider the family of totally umbilic spheres in the Hyperbolic Space with center in the $x_{m+1}-$axis, $\{Z(s)\}_{s\in(-1,1)}$, such the principal curvatures are equal to

$$k_0 = \frac{1+2\lambda_0}{1-2\lambda_0} > 1.$$

All these totally umbilic spheres, and also Σ, are solutions of the same elliptic problem for hypersurfaces in the Hyperbolic space

$$\mathscr{W}(\kappa_1,\dots,\kappa_m) = 1 \quad \text{on } \Sigma',$$

where $\kappa_1,\dots,\kappa_m$ are the principal curvatures of $\Sigma' \subset \mathbb{H}^{m+1}$.

We have the continuous extension $\Phi : \overline{\mathbb{A}(r)} \to \mathbb{R}^{m+1}$ of $\varphi_P : \mathbb{A}(r,\pi/2] \to \mathbb{R}^{m+1}$, defined by

$$\Phi(x) = \begin{cases} \phi_P(x) & x \in \mathbb{A}(r,\pi/2], \\ x & x \in S_r(\mathbf{n}). \end{cases}$$

Since $r > 0$, there is a $\delta > 0$ such that for all $s \in (1-\delta, 1)$:

1. $Z(s) \cap \Sigma = \emptyset$,

2. $Z(s) \cap Q = \emptyset$.

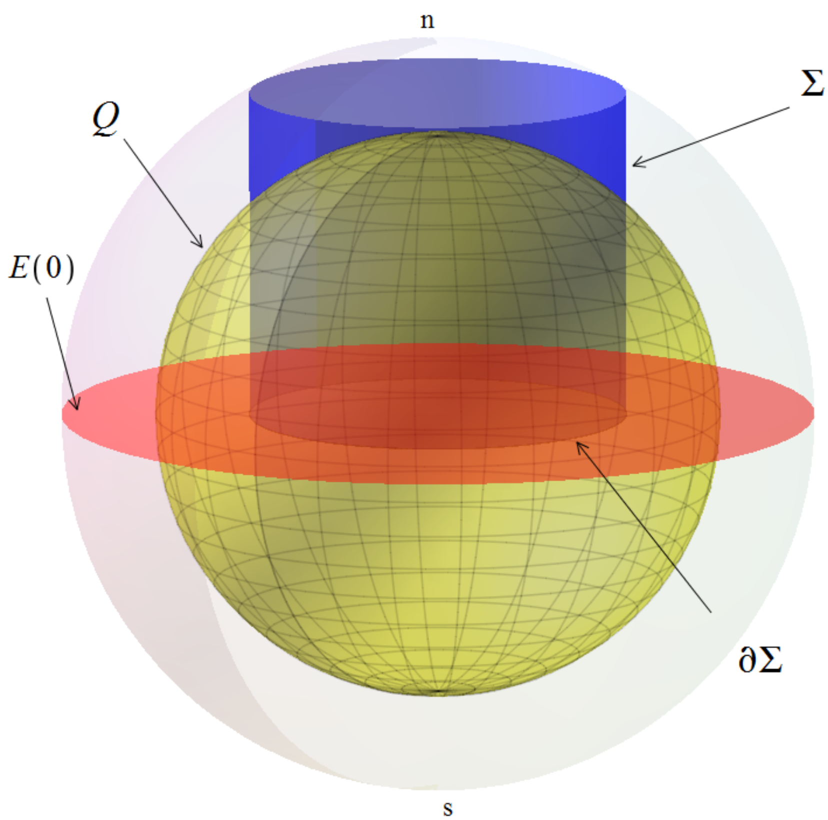


Figure 4.6: The interior of Q contains the boundary of Σ, for the case $c = 0$.

We take one of them, say $Z_0 = Z(s)$. In the Poincaré ball model, move Z_0 along of a circle with radius s and center the origin $\mathbf{0}$ until we have the first contact between the totally umbilic sphere Z_0 and the hypersurface. By item 2, the contact point is at the interior (see Figure 4.7 in the case $c = 0$). That is, there is $x \in \mathbb{A}(r)$ such that $\phi_P(x) \in Z_0$, remember that Z_0 has been rotated. We have that orientation of the hypersurface Σ is in the same direction of the natural orientation of Z_0 and Σ have the same direction.

Let ρ_0 be the function support of Z_0 restricted to $\mathbb{A}(r)$, then we have that

$$\rho \geq \rho_0 \quad \text{on } \mathbb{A}(r) \quad \text{and} \quad \rho(x) = \rho_0(x),$$

then by the strong maximum principle, $\rho = \rho_0$. So, Σ is part of a sphere, but Σ has non-empty ideal boundary. This is contradiction.

That concludes the proof. □

4.5 The 2-dimensional case

We saw that the Schouten tensor is defined for Riemannian manifolds $(\mathscr{M}^m, g_0)$ when $m \geq 3$. That is, let $(\mathscr{M}^m, g_0)$ be a Riemannian manifold where $m \geq 3$, the Schouten tensor of $(\mathscr{M}, g_0)$ is defined by the following symmetric 2-tensor

$$\mathrm{Sch}(g_0) = \frac{1}{m-2}\left(\mathrm{Ric}(g_0) - \frac{\mathrm{Scal}(g_0)}{2(m-1)} g_0\right),$$

where $\mathrm{Ric}(g_0)$ and $\mathrm{Scal}(g_0)$ are the Ricci tensor and the scalar curvature of $(\mathscr{M}, g_0)$.

The trace of the symmetric operator associated to $Sch(g_0)$ with respect to the metric g_0 gives

$$\mathrm{Tr}\left(g_0^{-1}\mathrm{Sch}(g_0)\right) = \frac{1}{2(m-1)}\mathrm{Scal}(g_0).$$

Let us consider the conformal metric $g = e^{2\rho} g_0$, where $\rho \in C^\infty(M)$, then we have the following relation:

$$\mathrm{Sch}(g) + \nabla^2\rho + \frac{1}{2}|\nabla\rho|^2 g_0 = \mathrm{Sch}(g_0) + \nabla\rho \otimes \nabla\rho,$$

where ∇ ,∇^2 are the gradient and the hessian with respect the metric g_0 respectively, and $|\cdot|$ the norm with respect of g_0.

In the case of the standard sphere $(\mathbb{S}^m, g_0)$ (see Section 1.3.3), we know that $\mathrm{Sch}(g_0) = \frac{1}{2} g_0$, then for every conformal metric $g = e^{2\rho} g_0$, we have that

$$\mathrm{Sch}(g) + \nabla^2\rho + \frac{1}{2}|\nabla\rho|^2 g_0 = \frac{1}{2} g_0 + \nabla\rho \otimes \nabla\rho. \tag{4.2}$$

So, we can take the above expression as a definition of the Schouten tensor for a conformal metric to the standard one on domains of the sphere $\mathbb{S}^2$. Hence, we can consider Yamabe type problems on $\mathbb{S}^2$.

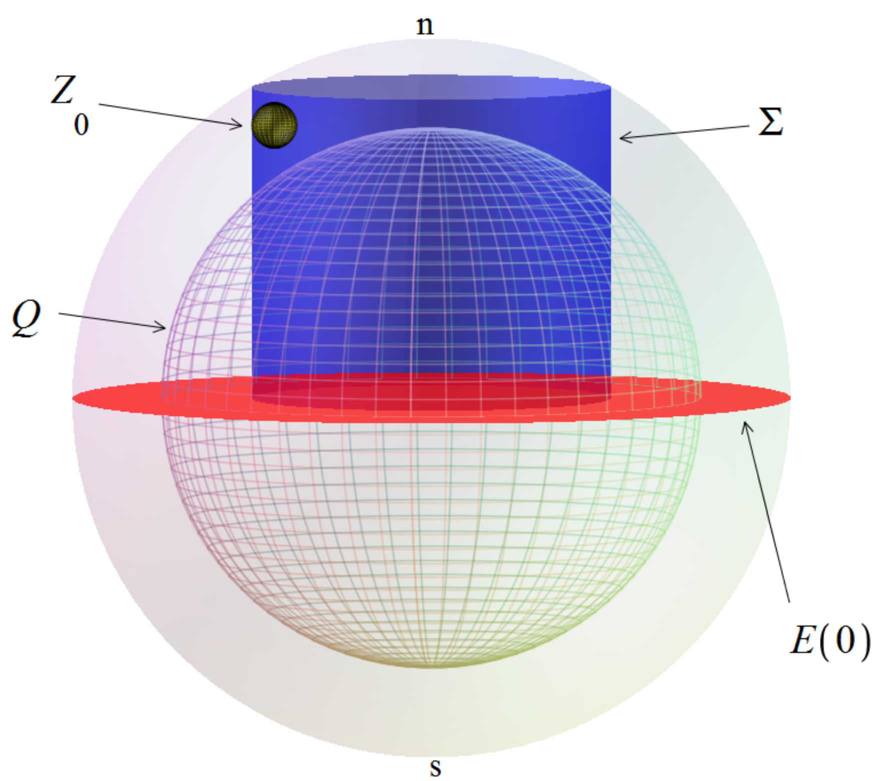


Figure 4.7: Sphere Z_0 touching the interior of Σ, for the case $c = 0$.

We begin in a general setting. The Yamabe problem for the Riemannian manifold $(\mathscr{M}, g_0)$ can be expressed as follows: find $g = e^{2\rho} g_0$, $\rho \in C^\infty(M)$, such that

$$\operatorname{Tr}\left(g^{-1}\operatorname{Sch}(g)\right) = \frac{1}{2(m-1)}c \quad \text{on} \quad \mathscr{M}$$

where c is a constant. Or

$$\lambda_1 + \cdots + \lambda_m = \frac{1}{2(m-1)}c \quad \text{on} \quad \mathscr{M},$$

where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of $g^{-1}\operatorname{Sch}(g)$ and c is a constant. If there were a solution $g = e^{2\rho} g_0$ then the Riemannian manifold $(\mathscr{M}, g)$ has constant scalar curvature equals to c.

We assume that $c = 1$ for simplicity. In the case $\mathscr{M} = \mathbb{S}^m$, $m \geq 3$, we have the problem,

$$\lambda_1 + \cdots + \lambda_m = \frac{1}{2(m-1)} \quad \text{on} \quad \mathbb{S}^m,$$

In the case $\mathbb{S}^2$, the equivalent problem is given by

$$\lambda_1 + \lambda_2 = \frac{1}{2} \quad \text{on} \quad \mathbb{S}^2,$$

and it can be interpreted as the problem of finding a conformal metric $g = e^{2\rho} g_0$ on $\mathbb{S}^2$ such that the Riemannian manifold $(\mathscr{M}, g)$ has constant scalar curvature $\operatorname{Scal}(g) = 1$ or, in other words, with constant Gaussian curvature, since from the equation (4.2)

$$\operatorname{Tr}\left(g^{-1}\operatorname{Sch}(g)\right) = e^{-2\rho}(1 - \Delta\rho) = \frac{1}{2}\operatorname{Scal}(g) = K,$$

where K is the Gaussian curvature of $g = e^{2\rho} g_0$, i.e., the Yamabe Problem reduces to the Liouville Problem.

This example says that the definition of the Schouten tensor for conformal metrics w.r.t. the standard metric on domains of the sphere $\mathbb{S}^2$, given by (4.2), makes sense. Then, we can consider more general elliptic problems for conformal metrics on $\mathbb{S}^2$, and have some analogous theorems that we saw above for domains of $\mathbb{S}^2$.

We establish the theorems in the case of domains of $\mathbb{S}^2$ without proof. First, for geodesic disk we have:

Theorem 4.13:
Let (f, Γ) be a degenerate elliptic data for conformal metrics and let $c \leq 0$ be a constant. Then, there is no conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{S}^2_+}$, where $\rho \in C^\infty\left(\overline{\mathbb{S}^2_+}\right)$, satisfying

$$\begin{cases} f(\lambda(g)) = 0 & \text{on} \quad \overline{\mathbb{S}^2_+}, \\ h(g) = c & \text{on} \quad \partial\mathbb{S}^2_+, \end{cases}$$

where $\lambda(g) = (\lambda_1, \lambda_2)$ is composed by the eigenvalues of the Schouten tensor of the metric $g = e^{2\rho} g_0$.

Second, for compact annulus, we have the following non-existence result:

Theorem 4.14:
If the problem (4.1) with $m = 2$ admits a punctured solution, then there is no solution of the following degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) &= 0 \quad \text{on } \overline{\mathbb{A}(r)}, \\ h(g) &= 0 \quad \text{on } \partial\mathbb{A}(r), \end{cases}$$

where $\lambda(g) = (\lambda_1, \lambda_2)$ is composed by the eigenvalues of the Schouten tensor of g.

In this part, it is good to say that it is possible that the punctured solution in Theorem 4.14 might not exist. For example, the Yamabe problem, or Liouville Problem for the annulus $\overline{\mathbb{A}(r)}$ in $\mathbb{S}^2$, where $0 < r < \pi/2$, has a solution with scalar curvature zero and minimal boundary, then there is no punctured solution for the Yamabe problem on $\overline{\mathbb{S}^2_+} \setminus \{\mathbf{n}\}$.

The solution of that problem is given by the conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$, where

$$e^{2\rho(x,y,z)} = \frac{1}{\sigma^2(x,y,z)} = \frac{1}{1-z^2} \quad \text{for all } (x,y,z) \in \overline{\mathbb{A}(r)}.$$

In Figure 4.8 we can see the surface associated to $\sigma(x,y,z) = \frac{1}{3}\sqrt{1-z^2}$, $(x,y,z) \in \mathbb{S}^2 \setminus \{\mathbf{n}, \mathbf{s}\}$, in the Poincaré ball model.

Also, it good to say that, for dimension $m > 2$, we can define the conformal metric $g = e^{2\rho} g_0$ on $\overline{\mathbb{A}(r)}$, $0 < r < \pi/2$, analogously, i.e.,

$$e^{2\rho(x_1,\ldots,x_{m+1})} = \frac{1}{\sigma^2(x_1,\ldots,x_{m+1})} = \frac{1}{1-x_{m+1}^2} \quad \text{for all } (x_1,\ldots,x_{m+1}) \in \overline{\mathbb{A}(r)},$$

but this conformal metric has constant scalar curvature equals to $(m-1)(m-2) > 0$. When m is even and $k = m/2$, this conformal metric is a solution for the degenerate σ_k-Yamabe problem on the compact annulus $\overline{\mathbb{A}(r)}$ with minimal boundary.

In the case of semi-annulus, we have

Theorem 4.15:
Let $r \in (0, \pi/2)$, $c \geq 0$ be a non-positive constant and $g = e^{2\rho} g_0$ be a conformal metric on $\mathbb{A}\left(r, \frac{\pi}{2}\right]$ that is solution of the following degenerate elliptic problem:

$$\begin{cases} f(\lambda(g)) &= 0 \quad \text{in } \mathbb{A}(r, \pi/2], \\ h(g) &= c \quad \text{on } \partial\mathbb{S}^m_+, \end{cases}$$

If $e^{2\rho} + |\nabla\rho|^2 : \mathbb{A}(r, \pi/2] \to \mathbb{R}$ is proper then $\lambda(g)$ is no bounded.

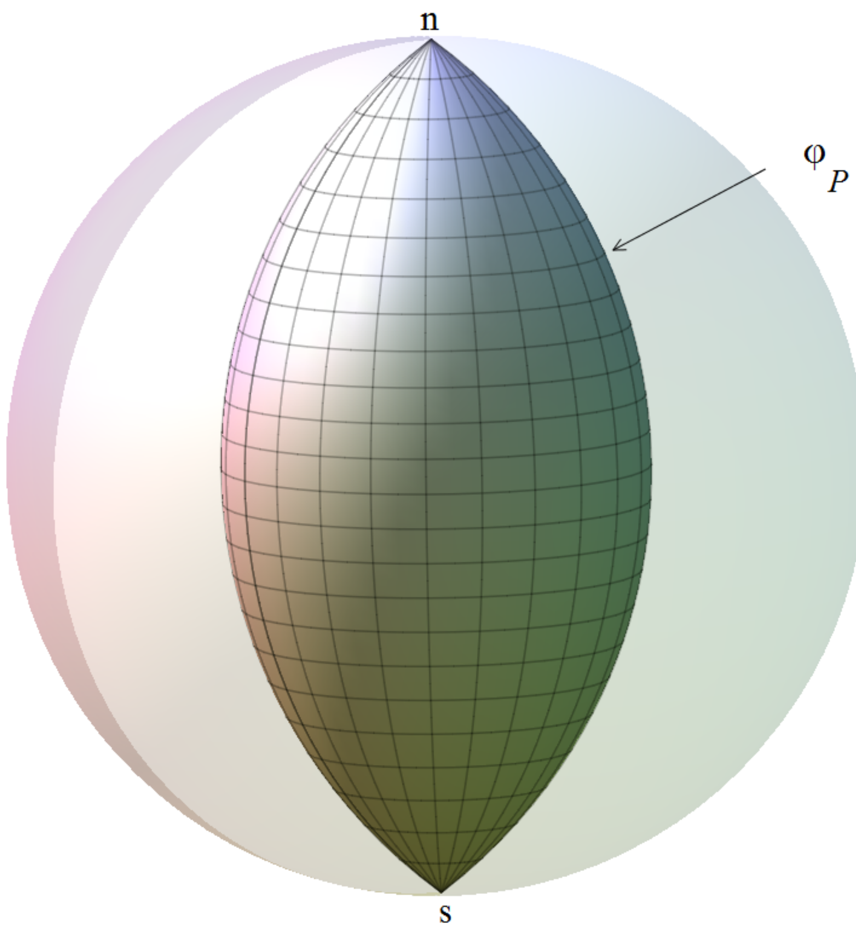


Figure 4.8: Surface associated to $\sigma = \frac{1}{3}\sqrt{1-z^2}$ in the Poincaré ball model of $\mathbb{H}^3$.

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