Influence and sharp thresholds for the random cluster model

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Introduction

The purpose of this dissertation is to provide a description of some applications of influence and sharp-thresholds theorems to the random-cluster model. Originally, these results appeared in the context of Analysis of Boolean functions and have been used in many areas.

The influence of an agent or event on another one is a topic of great interest in physics, statistics, computer science, economics, philosophy and many other areas. In the mathematical setting, the influence of a variable in a Boolean function has been a fundamental notion for many developments in Fourier analysis (of such functions), probability, combinatorics, statistical physics and percolation.

A Boolean function, $f(x) = f(x_1, ..., x_N)$, takes values on $\{0, 1\}$, and its entries x_k are Boolean variables on $\{0, 1\}$. A variable is called pivotal if by flipping the bit x_k, the value of f changes. Each variable x_k assumes 1 independently with probability $p \in [0, 1]$; thus the hypercube $\{0, 1\}^N$ is endowed with the product measure. The influence of the kth variable of a Boolean function f is the probability that the kth variable is pivotal. This definition and more general ideas of influence were introduced in [3] in the context of "collective coin flipping".

In response to a conjecture of Ben-Or and Linial [3], Kahn, Kalai and Linial [13] proved that, in the above context, there always exists a variable k so that its influence on f is at least $c\text{Var}[f] \log N/N$, where c is a positive constant independent of f and N. The proof uses harmonic analysis on $\mathbb{Z}_2^N$, which as a set is just the N-dimensional discrete cube, and the group structure allows one to make use of the tools of that theory. In [3], the authors gave an example of a function whose influence is just $\log N/N$, proving that the bound is sharp. One of the main applications of KKL theorem is to sharp thresholds of graph properties [7].

Intimately related to the combinatorial notions of influence and pivotality, threshold phenomena occur when the probability of an event changes swiftly as some underlying parameter varies. They play an important role in probability theory and statistics, physics and computer science, and are related to issues studied in economics and political science [14]. Sharp threshold theorems were originally introduced for product measures and are a powerful tool for the study of phase transitions [4].

In many cases, however, the variables x_k are not independent. An important example concerns the random-cluster model. This may be viewed as a parametric family of probability measures $\phi_{p,q}$ on a finite graph G, having two parameters, an edge-weight $p \in [0,1]$ and a cluster-weight $q \in (0,\infty)$. The probability of a configuration is proportional to

$$p^{|open\ edges|}(1-p)^{|closed\ edges|}q^{|clusters|}.$$

For $q \geq 1$, this model can be extended to infinite-volume lattices where it exhibits a phase transition at some critical parameter $p_c(q)$, which depends on the lattice.

Graham and Grimmett [11], extended the KKL theorem, and a sharp threshold result, to monotonic measures. As a consequence, they derived a lower bound for the probability of an open crossing of a rectangle on the square lattice for the random-cluster model. Afterwards the same authors proved a sharp threshold theorem [12], now with no assumption of symmetry, for such probabilities for the random- cluster model near the self-dual point.

In the case of planar graphs, the dual of the random-cluster model is random-cluster model also, with the same cluster-weight q and p, p_d related by $p_d/(1-p_d) = q(1-p)/p$. The unique fixed point of the mapping $p \mapsto p_d$ is the self-dual point $p_{sd}(q)$, given by $\frac{\sqrt{q}}{(1+\sqrt{q})}$. Thus the self-duality of the square lattice gives rise to the conjecture that $p_c(q) = p_{sd}(q),\ q \in [1,\infty)$. The inequality $p_c(q) \geq p_{sd}(q)$ was proved in [9] using Zhang's argument (the same used to prove that $p_c \geq \frac{1}{2}$ for bond percolation in two dimensions).

On the other hand, the reverse inequality, $p_c(q) \leq p_{sd}(q)$, was more intricate. There were two steps enough to imply it: firstly, that the probability of crossing a box $[-m,m]^2$ approaches 1 as $m \to \infty$, when $p > p_{sd}(q)$; and secondly, that this implies the existence of an infinite cluster. The first of these two claims was proved in [12] and is in this text. Beffara and Duminil-Copin [2] proved the conjecture, by generalizing the Russo-Seymour-Welsh theorem for percolation to the random-cluster model; and also by showing that the probability of crossings goes to 1 when $p > p_{sd}(q)$.

The dissertation presents a collection of influence and sharp thresholds theorems and a resultant theorem about box crossings for the random-cluster model. The text is organized as follows. In Chapter 1, we present some basic definitions and results of analysis of Boolean functions. Chapter 2 introduces monotonic measures and extends the notions and results about influence and sharp thresholds to this context. Finally, Chapter 3 is devoted to the random-cluster model and to the theorems about box crossings.

Chapter 1

Boolean functions

This chapter is devoted to the basics of Analysis of Boolean functions and the KKL theorem. We follow [8, Chapters 1, 3, 4, 5]; for more details on this topic, see [16]

1.1 Introduction

Let $\Omega_N := \{-1,1\}^N$ be the hypercube. An element of Ω_N will be denoted by $\omega = \omega_N = (x_1, \cdots, x_N)$, where $x_1, \cdots, x_N$ are its N bits. A function from Ω_N into $\{-1,1\}$ or $\{0,1\}$ is called *Boolean function*, which is canonically identified with a subset $A_f \subseteq \Omega$ by $A_f := f^{-1}(\{1\}) = \{\omega : f(\omega) = 1\}$.

Ω_N will be endowed with the product measure $\mathbb{P}_p = \mathbb{P}_p^N = ((1-p)\delta_{-1} + p\delta_1)^{\otimes N}$, $p \in [0,1]$, and $\mathbb{E}_p$ will denote the corresponding expectation. When $p = \frac{1}{2}$, we will write $\mathbb{P} = \mathbb{P}^N_{\frac{1}{2}} = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes N}$ and $\mathbb{E}$ will denote the corresponding expectation.

Given an element $\omega \in \Omega_N$, we will often consider, for each $i \in [N] := \{1, \cdots, N\}$, $\omega^i \in \Omega_N$, obtained from ω by flipping the i-th coordinate and keeping unchanged the others. We also define, for $\epsilon \in [0,1]$, the random element $\omega^\epsilon \in \Omega_N$ drawn according to the rule: for each $i \in [N]$ independently, $x_i^\epsilon = x_i$, with probability $1-\epsilon$, and x_i^ϵ is uniformly random, with probability ϵ. Notice that ω and ω^ϵ have the same distribution and, for each $i \in [N]$ (since $x_i \in \{-1,1\}$), they satisfy $\mathbb{E}[x_i] = \mathbb{E}[x_i^\epsilon] = 0$ and $\mathbb{E}[x_i x_i^\epsilon] = (1-\epsilon)$.

Let us consider some examples.

1 *Dictator.* $DICT_N(x_1, \cdots, x_N) := x_1$; the first bit determines the result.

2 *Parity.* $PAR_N(x_1, \cdots, x_N) := \prod_{i=1}^{N} x_i$; this function tells us whether the number of -1's is even or odd.

3 *Majority.* For N odd, $MAJ_N(x_1, \cdots, x_N) := sgn(\sum_{i=1}^{N} x_i)$.

4 *Iterated 3-majority.* For any positive integer k, define inductively $M_3 = MAJ_3^{\otimes 1} = MAJ_3$ and $M_3^{k+1} = MAJ_3^{k+1} = (x^{(1)}, \cdots, x^{(3)}) = M_3(M_3^{\otimes k}(x^{(1)}), M_3^{\otimes k}(x^{(2)}), M_3^{\otimes k}(x^{(3)}))$, $x^{(i)} \in \Omega_{N^k}$, $i = 1, 2, 3$. Thus the bits are indexed by the leaves of a rooted 3-ary tree of depth k and one iteratively applies M_3 to obtain values at the vertices at level $k-1$ and so on until the root is assigned a value.

5 *Tribes* Let k, b be positive integers and consider k subsequent blocks of size b. Define $TR_{k,b}$ to be 1 if there exists at least one block which contains all 1's, and 0 otherwise.

6 *Clique containment.* If $r = \binom{N}{2}$ for any positive integer N, then Ω_r can be identified with the set of labelled graphs on N vertices ($x_i = 1$ if, and only if, the i-th edge is present). Recall that a clique of size k of a graph $G = (V, E)$ is a complete graph on k vertices embedded in G. Now for any $1 \leq k \leq \binom{N}{2}$, let $CLIQ_N^k$ be the indicator function of the event that the random graph G_ω defined by $\omega \in \Omega_r$ contains a clique of size k.

We say that a function $f : \Omega_N \to \{-1, 1\}$ is *monotone (increasing)* if $f(\psi) \leq f(\omega)$, whenever $\psi \leq \omega$ coordinate-wise. A subset $A \subseteq \Omega_N$ is said to be monotone when its indicator function is monotone. We call a Boolean function *symmetric*, if $f(\omega^\pi) = f(\omega)$ for all permutations $\pi \in S_N$ (where S_N is the permutation group of N elements and $\omega^\pi = (\omega_{\pi(1)}, \cdots, \omega_{\pi(N)})$); and *transitive-symmetric*, if for all $i, i' \in [N]$ there exists a permutation $\pi \in S_N$ taking i to i' such that $f(\omega^\pi) = f(\omega)$, for all $\omega \in \Omega_N$. Finally, when $f : \Omega_N \to \{-1, 1\}$ satisfies $\mathbb{E}[f] = 0$, it is called *balanced.*

1.2 The Fourier decomposition of a function on Ω_N

We consider the space $L^2(\Omega_N)$, of real functions on $\Omega_N = \{-1,1\}$ endowed with the inner product:

$$\langle f,g\rangle := \sum_{x_1,\cdots x_N} 2^{-N} f(x_1,\cdots,x_N) g(x_1,\cdots,x_N) \tag{1.1}$$

$$= \mathbb{E}[fg] \qquad (\ \forall f,g \in L^2(\Omega_N)\ ). \tag{1.2}$$

For each subset $S \subseteq [N]$, let χ_S be the function on Ω_N defined for any $\omega = (x_1,\cdots,x_N)$ by ($\chi_\emptyset \equiv 1$)

$$\chi_S(\omega) := \prod_{i\in S} x_i. \tag{1.3}$$

Lemma 1.1. *The family of 2^N functions $\{\chi_S\}_{S\subseteq[N]}$ forms an orthonormal basis of $L^2(\Omega_N)$.*

Proof. Let $S_1, S_2 \subseteq [N]$ be arbitrary subsets. Then

$$\langle \chi_{S_1}, \chi_{S_2}\rangle = \sum_{x_1,\cdots,x_N} 2^{-N} \prod_{i\in S_1} x_i \prod_{j\in S_2} x_j = \sum_{x_1,\cdots,x_N} 2^{-N} \prod_{i\in S_1\cup S_2\setminus S_1\cap S_2} = 0, \tag{1.4}$$

for χ_S is a balanced function, $\forall S \subseteq [N]$. Also,

$$\langle \chi_S, \chi_S\rangle = \sum_{x_1,\cdots x_N} 2^{-N} \prod_{i\in S} x_i^2 = 1; \tag{1.5}$$

hence the family is orthonormal.

By identifying canonically each $f \in L^2(\Omega_N)$ with a vector in R^N, we see that $\dim L^2(\Omega_N) = 2^N$. Since there are 2^N functions χ_S, we conclude that they form an (orthonormal) basis of $L^2(\Omega_N)$. □

Thus, any function $f \in \Omega_N$ can be decomposed as

$$f = \sum_{S\subseteq[N]} \hat{f}(S)\chi_S, \tag{1.6}$$

where $\{\hat{f}(S)\}_{S\subseteq[N]}$ are the so-called *Fourier coefficients* of f, that satisfy

$$\hat{f}(S) = \langle f, \chi_S\rangle = \mathbb{E}[f\chi_S]. \tag{1.7}$$

Notice that $\hat{f}(\emptyset) = \mathbb{E}[f]$ and, since $\{\chi_S\}_{S\subseteq[N]}$ is an orthonormal basis, the Parseval's formula is valid:

$$\begin{aligned}\mathbb{E}[f^2] = \langle f, f\rangle &= \langle \sum_{S\subseteq[N]} \hat{f}(S)\chi_S, \sum_{S'\subseteq[N]} \hat{f}(S')\chi_{S'}\rangle \\ &= \sum_{S\subseteq[N]} \sum_{S'\subseteq[N]} \hat{f}(S)\hat{f}(S')\langle\chi_S, \chi_{S'}\rangle \\ &= \sum_{S\subseteq[N]} \hat{f}(S)^2\langle\chi_S, \chi_S\rangle \\ &= \sum_{S\subseteq[N]} \hat{f}(S)^2 \qquad (1.8)\end{aligned}$$

Definition 1.1. *For any $f \in L^2(\Omega_N)$, the energy spectrum E_f is defined by*

$$E_f(m) := \sum_{|S|=m} \hat{f}^2(S) \qquad (1.9)$$

1.3 Pivotality and Influence

Given a Boolean function f and a variable $i \in [N]$, we say that i is *pivotal* for (f, w) if $f(\omega) \neq f(\omega^i)$. The event $[i\ is\ pivotal\ for\ f]$ is the set of configurations for which i is pivotal for (f, w). Notice that this event is measurable with respect $\{x_j\}_{j\neq i}$; it is independent of the bit x_i. The pivotal set for f, $\mathcal{P}_f$, is the random set of $[N]$ given by

$$\mathcal{P}_f(\omega) := \{i \in [N] : i\ is\ pivotal\ for\ (f, \omega)\}. \qquad (1.10)$$

The *influence* of the i-th bit, $I_i(f)$, is defined by $I_i(f) := \mathbb{P}[i\ is\ pivotal\ for\ f] = \mathbb{P}\{i \in \mathcal{P}\}$. The influence vector, $Inf(f)$, is the collection of all the influences, $\{I_i\}_{i\in[N]}$. The *total influence*, $I(f)$, is defined by

$$I(f) := \sum_{i=1}^{N} I_i(f) = ||Inf(f)||_1 = \mathbb{E}[|\mathcal{P}|]. \qquad (1.11)$$

Lemma 1.2. *For any Boolean function f, $I(f) = \frac{|\partial_E(A_f)|}{2^{N-1}}$, where $\partial_E(A_f)$ denotes the edge boundary of $A_f \subseteq \Omega_N$ (e.g., it is the set of edges where exactly one of the endpoints is in A).*

Proof. By the definitions above and the uniformity of $\mathbb{P}$, we have

$$I(f) = \sum_{i=1}^{N} I_i(f) = \sum_{i=1}^{N} \mathbb{P}\{\omega : f(\omega) \neq f(\omega^i)\} = \sum_{i=1}^{N} \frac{1}{2^N} |\{\omega : f(\omega) \neq f(\omega^i)\}|. \tag{1.12}$$

Now, $\forall i \in [N]$, it is clear that ω and ω^i are neighbours and, if $f(\omega) \neq f(\omega^i)$, then either $\omega \in A_f$ and $\omega^i \in A_f$ or $\omega \neq A_f$ and $\omega^i \in A_f$. On the other hand, if the edge $e = \langle \omega, \omega^i \rangle \in \partial_E(A_f)$, then $f(\omega) \neq f(\omega^i)$ and $\omega, \omega^i \in \{\omega : f(\omega) \neq f(\omega^i)\}$. Thus

$$\sum_{i=1}^{N} |\{\omega : f(\omega) \neq f(\omega^i)\}| = 2|\partial_E(A_f)| \tag{1.13}$$

□

Now, we evaluate the influences for some examples introduced before.

- For $f(\omega) = DICT_N(\omega) = x_1$, $\mathcal{P} = \{1\}$ and $I_1(f) = I(f) = 1$.
- For $f(\omega) = PAR_N(\omega) = \prod_{i=1}^{N} x_i$, $\mathcal{P} = [N]$, $I_i(f) = 1$ ($\forall i \in [N]$) and $I(f) = N$.
- For $f(\omega) = MAJ_N(x_1, \cdots, x_N) = sgn(\sum_{i=1}^{N} x_i)$ $(Nodd)$, $i \in [N]$ is pivotal if, and only if, $\sum_{j \neq i} x_j = 0$; hence

$$\begin{cases} \mathcal{P}(\omega) = \emptyset & if\ \forall i \in [N],\ \sum_{j\neq i} x_j \neq 0 \\ \\ |\mathcal{P}(\omega)| = \frac{N+1}{2} & if\ \exists i \in [N];\ \sum_{j \neq i} x_j = 0, \end{cases} \tag{1.14}$$

  where in the second case, the pivotal variables are those whose bit is equal to majority's value. Thus

$$I_i(f) = \mathbb{P}[\sum_{j \neq i} x_j = 0] = \frac{\binom{N-1}{\frac{N-1}{2}}}{2^{N-1}}, \tag{1.15}$$

  for all $i \in [N]$, for f is symmetric. By Stirling's approximation, this quantity is $\sim \sqrt{\frac{2}{\pi N}}$.

- For iterated 3-majority, $i \in [3^k]$ is pivotal if, and only if, the other two bits in the subtree are different. Since x_i is independent of the other bits and MAJ_3 is applied k times, we have that

$$I_i(f) = \mathbb{P}[i \text{ } is \text{ } pivotal \text{ } for \text{ } f] = 2^{-k}, \tag{1.16}$$

for the probability of the event $\{given\ a\ variable,\ the\ other\ two\ bits\ in\ the\ same\ subtree\ are\ different\}$ is 2^{-1}.

Theorem 1.1. *Let A be an increasing event in Ω_N. Then*

$$\frac{d(\mathbb{P}_p(A))}{dp} = \sum_i I_i^p(A) \tag{1.17}$$

Proof. Let us consider that each variable x_i has its own parameter p_i and let $\mathbb{P}_{p_1,\cdots,p_N}$ and $\mathbb{E}_{p_1,\cdots,p_N}$ be the corresponding probability measure and expectation. It suffices to show that

$$\frac{\partial(\mathbb{P}_{p_1,\cdots,p_N}(A))}{\partial p_i} = I_i^{(p_1,\cdots,p_N)}(A). \tag{1.18}$$

Without loss of generality, take $i = 1$. Now,

$$\mathbb{P}_{p_1,\cdots,p_N}(A) = \mathbb{P}_{p_1,\cdots,p_N}(A \backslash \{1 \in \mathcal{P}_A\}) + \mathbb{P}_{p_1,\cdots,p_N}(A \cap \{1 \in \mathcal{P}_A\}). \tag{1.19}$$

The event in the first term is measurable with respect to the other variables and hence this term does not depend on p_1, while the second term is

$$p_1 \mathbb{P}_{p_1,\cdots,p_N}(\{1 \in \mathcal{P}_A\}), \tag{1.20}$$

for A is increasing implies that $A \cap \{1 \in \mathcal{P}_A\}$ is the event $\{x_1 = 1\} \cap \{1 \in \mathcal{P}_A\}$. □

Proposition 1.1. *For any monotone Boolean function on Ω_N, f,*

$$I^{\frac{1}{2}}(f) \leq I^{\frac{1}{2}}(MAJ_N) \tag{1.21}$$

Proof. Since $[f = 1]$ is a monotone event, we apply Russo formula so that

(in the following, $|\omega|$ denotes the number of positive bits in ω)

$$I^p(f) = \frac{d}{dp}\mathbb{P}_p[f=1] = \frac{d}{dp}\sum_{\omega\in\Omega_N} 1_{\{f=1\}}(\omega)p^{|\omega|}(1-p)^{N-|\omega|} \tag{1.22}$$

$$= \sum_{\omega\in\Omega} 1_{\{f=1\}}(\omega)[|\omega|p^{|\omega|-1}(1-p)^{N-|\omega|} - p^{|\omega|}(N-|\omega|)(1-p)^{N-|\omega|-1}] \tag{1.23}$$

$$= \sum_{\omega\in\Omega_N} 1_{\{f=1\}}(\omega)p^{|\omega|-1}(1-p)^{N-|\omega|-1}(|\omega|-Np) \tag{1.24}$$

$$\leq \sum_{|\omega|>Np} 1_{\{f=1\}}(\omega)p^{|\omega|-1}(1-p)^{N-|\omega|-1}(|\omega|-Np) \tag{1.25}$$

$$\leq \sum_{|\omega|>Np} p^{|\omega|-1}(1-p)^{N-|\omega|-1}(|\omega|-Np). \tag{1.26}$$

By taking $f(\omega) = 1$, if $|\omega| > Np$, and $f(\omega) = -1$ *or* 0, otherwise, we get equality in the last estimate; when $p = \frac{1}{2}$, such f is MAJ_N. □

Corollary 1.1. *The total influence at $p = \frac{1}{2}$ of any monotone Boolean is at most $O(\sqrt{N})$.*

Proof. Since $I(MAJ_N) \sim \sqrt{\frac{2}{\pi}}\sqrt{N}$, the conclusion follows from Proposition 1. □

Proposition 1.2. *If $f : \Omega_N \to \{0,1\}$, then for all k,*

$$I_k(f) = 4\sum_{S:k\in S} \hat{f}(S)^2 \qquad \textit{and} \tag{1.27}$$

$$I(f) = 4\sum_{S\subseteq[N]} |S|\hat{f}(S)^2 \tag{1.28}$$

Proof. Let us consider $f : \Omega_N \to \mathbb{R}$ and introduce the functions, for each $k \in [N]$,

$$\begin{aligned}\nabla_k f :&\Omega_N \to \mathbb{R}\\ &\omega \mapsto f(\omega) - f(\sigma_k(\omega)),\end{aligned} \tag{1.29}$$

where σ_k acts on Ω_N by flipping the k-th bit.

Notice that

$$\nabla_k f(\omega) = \sum_{S\subseteq[N]} \hat{f}(S)[\chi_S(\omega) - \chi_S(\sigma_k(\omega))]$$
$$= \sum_{S\subseteq[N];k\in S} 2\hat{f}(S)\chi_S(\omega), \tag{1.30}$$

from which it follows that for all $S \subseteq [N]$,

$$\nabla_k \hat{f}(S) = \begin{cases} 2\hat{f}(S), & if\ k \in S \\ 0, & otherwise. \end{cases} \tag{1.31}$$

Since f takes values in $\{0,1\}$, $\nabla_k f \in \{-1,0,1\}$ and, therefore, $I_k(f) = \mathbb{P}[|\nabla_k(f)| = 1] = \mathbb{E}[|\nabla_k f|] = ||\nabla_k f||_1 = ||\nabla_k f||_2^2 = \mathbb{E}[(\nabla_2 f)^2]$. By using the Parseval formula for $\nabla_k f$ and (1.31), we obtain $I_k(f) = 4\sum_{S:k\in S} \hat{f}(S)^2$.

By summing over k and exchanging the order of summation, we obtain

$$I(f) = \sum_k 4 \sum_{S:k\in S} \hat{f}(S)^2 = \sum_{S\subseteq[N]} 4|S|\hat{f}(S)^2. \tag{1.32}$$

□

If f takes values in $\{-1,1\}$, then $\nabla_k f \in \{-2,0,2\}$ and $I_k f = \sum_{S:k\in S} \hat{f}(S)^2$, $I(f) = \sum_S |S|\hat{f}(S)^2$.

Proposition 1.3. *If $f : \Omega_N \to \{0,1\}$ is monotone, then for all k,*

$$I_k(f) = 2\hat{f}(\{k\}). \tag{1.33}$$

If f maps into $\{-1,1\}$, then $I_k(f) = \hat{f}(\{k\})$.

Proof.

$$\hat{f}(\{k\}) := \mathbb{E}[f\chi_{\{k\}}] = \mathbb{E}[f\chi_{\{k\}}1_{\{k\notin\mathcal{P}\}}] + \mathbb{E}[f\chi_{\{k\}}1_{\{k\in\mathcal{P}\}}] \tag{1.34}$$

The first term is zero, for if k is not pivotal for f, then χ_k and $f1_{k\notin\mathcal{P}}$ are independent; thus, $\mathbb{E}[f\chi_{\{k\}}1_{\{k\notin\mathcal{P}\}}] = \mathbb{E}[\chi_{\{k\}}]\mathbb{E}[f1_{\{k\notin\mathcal{P}\}}] = 0$ ($\chi_{\{k\}}$ is balanced).

Notice that, since f is monotone, $[f = 1] \cap \{k \in \mathcal{P}\} = [x_k] = 1 \cap \{k \in \mathcal{P}\}$. Thus, as the fact of k being pivotal is independent of the bit x_k,

$$\mathbb{E}[f\chi_{\{k\}}1_{\{k\in\mathcal{P}\}}] = \mathbb{P}[f = 1, x_k = 1, k \in \mathcal{P}] \tag{1.35}$$
$$= \mathbb{P}[x_k = 1]\mathbb{P}[k \in \mathcal{P}] = \frac{I_k(f)}{2} \tag{1.36}$$

□

Proposition 1.4. *If $f : \Omega_N \to \{-1, 1\}$ is monotone, then $I(f) \leq \sqrt{N}$.*

Proof. By Proposition 3, we have that

$$I(f) = \sum_{k=1}^{N} I_k(f) = \sum_{k=1}^{N} \hat{f}(\{k\}) \leq \big(\sum_{k=1}^{N} \hat{f}^2(\{k\})\big)^{\frac{1}{2}} \sqrt{N}, \tag{1.37}$$

by the Cauchy-Schwarz inequality. The Parseval formula tells us that

$$\sum_{k=1}^{N} \hat{f}^2(\{k\}) \leq \mathbb{E}[f^2] \leq 1. \tag{1.38}$$

□

Theorem 1.2 (Poincar inequality)**.** *Let $f : \Omega_N \to \{-1, 1\}$ be a Boolean funtion. Then*

$$Var[f] \leq \sum_{i=1}^{N} I_i(f). \tag{1.39}$$

Hence there exists $i \in [N]$ such that $I_i(f) \geq \frac{Var[f]}{N}$.

Proof. Notice that $2\mathbb{P}[f(\omega) \neq f(\tilde{\omega})] = Var[f]$, where $\omega, \tilde{\omega}$ are i.i.d. uniforms on Ω_N. Indeed,

$$\mathbb{E}[(f(\omega) - f(\tilde{\omega}))^2] = \mathbb{E}[f^2(\omega) - 2f(\omega)f(\tilde{\omega}) + f^2(\omega)] \tag{1.40}$$

$$= 2\mathbb{E}[f^2(\omega)] - 2\mathbb{E}[f(\omega)]\mathbb{E}[f(\tilde{\omega})] \tag{1.41}$$

$$2Var[f], \tag{1.42}$$

by independence and identical distributions. On the other hand, since f takes values in $\{-1, 1\}$, $\mathbb{E}[(f(\omega) - f(\tilde{\omega}))^2] = 4\mathbb{P}[f(\omega) \neq f(\tilde{\omega})]$.

Now define $\omega_i \in \Omega_N$ by

$$\omega_i(j) = \begin{cases} \omega(j), & if\ j \leq i \\ \tilde{\omega}(j), & if\ j > i \end{cases} \quad (i, j \in [N]),\ \omega_0 = \tilde{\omega}\ and\ \omega_N = \omega, \tag{1.43}$$

so that $\omega_i \sim Unif(\Omega_N)$, because $\omega, \tilde{\omega}$ are i.i.d. $Unif(\Omega_N)$.

If $f(\omega) \neq f(\tilde{\omega})$, then $f(\omega_i) \neq f(\omega_{i+1})$ for some i. Thus,

$$\mathbb{P}[f(\omega) \neq f(\tilde{\omega})] \leq \sum_{i=0}^{N} \mathbb{P}[f(\omega_i \neq \omega_{i+1})] \tag{1.44}$$

$$= \sum_{i=0}^{N} \mathbb{P}[\omega(i+1) \neq \tilde{\omega}(i+1), i+1 \in \mathcal{P}] \tag{1.45}$$

$$= \frac{1}{2} \sum_{i=0}^{N} I_{k+1}(f), \tag{1.46}$$

by the independence of the events and the fact that $\mathbb{P}[\omega(i+1) \neq \tilde{\omega}(i+1)] = \frac{1}{2}$. Therefore, $Var[f] = 2\mathbb{P}[f(\omega) \neq f(\tilde{\omega})] \leq I[f]$.

An alternative proof is possible by using the Fourier decomposition of f. Notice that

$$Var[f] = \mathbb{E}[f^2] - \mathbb{E}[f]^2 = 1 - \hat{f}(\emptyset)^2; \text{ and by Proposition 2,} \tag{1.47}$$

$$I(f) := \sum_{i=1}^{N} I_i(f) = \sum_{S \subseteq [N]} |S| \hat{f}(S)^2 \tag{1.48}$$

$$\geq \sum_{S \subseteq} [N] \hat{f}(S)^2 - \hat{f}(\emptyset)^2 = ||f||_2^2 - \hat{f}(\emptyset)^2 \tag{1.49}$$

$$= 1 - \hat{f}(\emptyset)^2 = Var[f]. \tag{1.50}$$

□

1.4 The Kahn, Kalai and Linial Theorem

Theorem 1.3 (Hypercontractivity). *Consider $\mathbb{R}^N$ with standard Gaussian measure. Let K_t be the heat kernel on $\mathbb{R}^N$. If $1 < q < 2$, then there exists $t = t(q) > 0$ (independent of the dimension N) such that for any $f \in L^q(\mathbb{R}^N)$,*

$$||K_t * f||_2 \leq ||f||_q \tag{1.51}$$

For any $\rho \in [0, 1]$ and any $f : \Omega_N \to \mathbb{R}$, we define the *noise operator* by

$$T_\rho f : \omega \to \mathbb{E}[f(\omega^{1-\rho})|\omega]. \tag{1.52}$$

This noise operator acts in a very simple way on the Fourier coefficients:

$$(T_\rho f)(\omega) = \mathbb{E}[\sum_{S\subseteq[N]} \hat{f}(S)\chi_S(\omega^{1-\rho})|\omega] \tag{1.53}$$

$$= \sum_{S\subseteq[N]} \hat{f}(S)\mathbb{E}[\chi_S(\omega^{1-\rho})|\omega] \tag{1.54}$$

$$= \sum_{S\subseteq[N]} \hat{f}(S)\mathbb{E}[\prod_{i\in S}\omega^{1-\rho}(i)|\omega] \tag{1.55}$$

$$= \sum_{S\subseteq[N]} \hat{f}(S)\prod_{i\in S}\mathbb{E}[\omega^{1-\rho}|\omega] \tag{1.56}$$

$$= \sum_{S\subseteq[N]} \hat{f}(S)\rho^{|S|}\prod_{i\in S}\omega(i) \tag{1.57}$$

$$= \sum_{S\subseteq[N]} \hat{f}(S)\rho^{|S|}\chi_S(\omega). \tag{1.58}$$

Theorem 1.4 (Bonomi-Gross-Beckner)**.** *For any $f : \Omega_N \to \mathbb{R}$ and any $\rho \in [0,1]$*

$$||T_\rho f||_2 \leq ||f||_{1-\rho^2} \tag{1.59}$$

Theorem 1.5 (Kahn-Kalai-Linial)**.** *If $f : \Omega_N \to \{0,1\}$ is a Boolean function, then there exist a universal $c > 0$ and $i \in [N]$ such that*

$$I_i(f) \geq cVar[f]\frac{(\log N)}{N} \tag{1.60}$$

Proof. We divide the analysis into the following two cases.

Case 1. Suppose that there is some $k \in [N]$ such that $I_k(f) \geq N^{-3/4}Var[f]$. Then the bound (1.60) is satisfied for a small $c > 0$ (for all $n \in \mathbb{N}$, $\log(n) \geq \log\log(n) \Rightarrow \frac{1}{4}\log(n) \geq c\log\log(n)$, $c < \frac{1}{4}$, $n^{-3/4} \Rightarrow c'\frac{\log(n)}{n}$).

Case 2. Suppose that for all $k \in [N]$, $I_k(f) = ||\nabla_k(f)||_2^2 \leq Var[f]N^{-3/4}$. We will show that in this case, most of the Fourier spectrum of f is concentrated on high frequencies. Let $M \geq 1$, whose value will be chosen later. We want to bound from above the bottom part (up to M) of the Fourier spectrum of f.

$$\sum_{1\leq|S|\leq M}\hat{f}(S)^2 \leq \sum_{1\leq|S|\leq M}|S|\hat{f}(S)^2 \leq 2^{2M}\sum_{|S|\geq 1}(\frac{1}{2})^{2|S|}|S|\hat{f}(S)^2 \tag{1.61}$$

$$= \frac{1}{4}2^{2M}\sum_k ||T_{\frac{1}{2}}(\nabla_k f)||_2^2 \quad \text{(see Proposition 2.)} \tag{1.62}$$

By applying the Theorem 4 with $\rho = \frac{1}{2}$ to the above sum, we obtain

$$\sum_{1\le|S|\le M} \hat{f}(S)^2 \le \frac{1}{4} 2^{2M} \sum_k ||\nabla_k f||_{5/4}^2 \tag{1.63}$$

$$\le 2^{2M} \sum_k I_k(f)^{8/5} \qquad ((\nabla_k f)^{5/4} \in \{0,1\}) \tag{1.64}$$

$$\le 2^{2M} N Var[f]^{8/5} N^{-\frac{3}{4}\cdot\frac{8}{5}} \qquad \text{(hypothesis Case 2)} \tag{1.65}$$

$$\le 2^{2M} N^{-\frac{1}{5}} Var[f] \qquad (\text{f is Boolean, } Var[f] \le 1) \tag{1.66}$$

Now with $M := \lfloor \frac{1}{20} \log_2 N \rfloor$,

$$\sum_{1\le|S|\le\frac{1}{20}\log_2 N} \le N^{1/10-1/5} Var[f] = N^{-1/10} Var[f]. \tag{1.67}$$

This shows that under the assumption in Case 2, most of the Fourier spectrum is concentrated above $\Omega(\log N)$. Thus

$$\sup_k I_k(f) \ge \frac{\sum_k I_k(f)}{N} = \frac{4\sum_{|S|\ge1} |S|\hat{f}(S)^2}{N} \ge \frac{1}{N}\Big[\sum_{|S|>M} |S|\hat{f}(S)^2\Big] \tag{1.68}$$

$$\ge \frac{M}{N}\Big[\sum_{|S|>M} \hat{f}(S)^2\Big] = \frac{M}{N}\Big[Var(f) - \sum_{1\le|S|\le M} \hat{f}(S)^2\Big] \tag{1.69}$$

$$\ge \frac{M}{N} Var[f][1 - N^{-1/10}] \tag{1.70}$$

$$\ge Var[f] \frac{\log N}{N}, \tag{1.71}$$

with $c_1 = \frac{1}{20\log 2}(1 - 2^{-1/10})$. By combining with the constant given in Case 1, this completes the proof. □

The above theorem is sharp. Indeed, it turns out that the *tribes* function has all influences smaller than $c(\log N)/N$, for some $c < \infty$. Let $b > 0$ be a parameter, which will be determined later. Consider a partition $[N] = B_1 \cup \cdots \cup B_{n/b}$ into N/b disjoint parts, called *tribes*, of size b each. Now we choose the parameter b that makes this function $f = TR_{\frac{N}{b},b}$ balanced. The probability that at least one bit in a given tribe is -1 is $1 - 2^{-b}$, so

$$\mathbb{P}[f = 0] = (1 - 2^{-b})^{\frac{N}{b}}. \tag{1.72}$$

We define b as the number that makes this probability equal to $1/2$ (since the exact b may not be an integer or not a divisor of N, we ignore a probable small error). Using the estimate $(1-2^{-b})^{\frac{N}{b}} \sim e^{-2^{-b}N/b}$ and solving for b, we see that $b+\log_2 b = \log_2 N - \log_2(\ln 2 + o(1))$, whence $b = \log_2 N - \log_2\log_2 N + O(1)$.

In order to determine the influences $I_k(f)$, consider the tribe that x_k belongs to, say $k \in B_1$. Then flipping the variable x_k affects the value of $f(\omega)$ if, and only if, for each tribe B_j with $j \neq 1$, $x_i = -1$ for at least one $i \in B_j$, and $x_i \in B_1 \backslash \{k\}$. Thus,

$$I_k(f) = (1-2^{-b})^{N/b-1} \cdot 2^{-b+1} = \frac{\frac{1}{2}\cdot 2 \cdot 2^{-b}}{1-2^{-b}} = c\big(\frac{\log N}{N}\big). \tag{1.73}$$

Theorem 1.6. *There exits a universal $c > 0$ such that for any $f : \Omega_N \to \{0,1\}$,*

$$||I(f)|| = ||Inf(f)||_1 \geq cVar[f]\log\big(\frac{1}{||Inf(f)||_\infty}\big) \tag{1.74}$$

Proof. Let $f : \Omega_N \to \{0,1\}$ and $\delta := ||Inf(f)||_\infty = \sup_k I_k(f)$. Assume that $\delta \leq 1/1000$. Exactly like the previous theorem,

$$\sum_{1\leq|S|\leq M} \hat{f}(S)^2 \leq 2^{2M}\sum_k I_k(f)^{8/5} \tag{1.75}$$

$$\leq 2^{2M}\delta^{3/5}\sum_k I_k(f) \tag{1.76}$$

$$= 2^{2M}\delta^{3/5} I(f). \tag{1.77}$$

Now,

$$Var[f] = \sum_{|S|\geq 1}\hat{f}(S)^2 \leq \sum_{1\leq|S|\leq M}\hat{f}(S)^2 + \frac{1}{M}\sum_{|S|>M}|S|\hat{f}(S)^2 \tag{1.78}$$

$$\leq \big[2^{2M}\delta^{3/5} + \frac{1}{M}\big] I(f). \tag{1.79}$$

Choose $M = \frac{3}{10}\log_2(\delta^{-1}) - \frac{1}{2}\log_2\log_2\delta^{-1}$. Since $\delta \leq 1/1000$, $M \geq \frac{1}{10}\log_2(\delta^{-1})$, which leads us to

$$Var[f] \leq \Big[\frac{1}{\log_2(1/\delta)} + \frac{10}{\log_2(1/\delta)}\Big] I(f) \tag{1.80}$$

which gives $I(f) = ||Inf(f)||_1 \geq \frac{1}{11\log 2}Var[f]\log\big(\frac{1}{||Inf(f)||_\infty}\big)$. This gives us the result for $\delta \leq 1/1000$.

If $\delta \geq 1/1000$, by the Poincar inequality ($I(f) \geq Var[f]$), the claim is true if we take c to be $1/\log 1000$. Since $1/\log 1000 > 1/11 \log 2$, we have the result with $c = 1/11 \log 2$. ☐

In [5, Theorem, 1], this result is extended to the case $\mathbb{P}_p[f = 1] = p \leq \frac{1}{2}$ and for any product space X^N, for a probability space X. In this context, the influence of the k-th variable on f is defined as follows: for $x = (x_1, \cdots x_N - 1) \in X^{N-1}$, consider the set $s_k(x) = \{(x_1, \cdots, x_{k-1}, t, x_k, \cdots, x_{N-1}) : t \in X\}$.

$$I_k(f) = \mathbb{P}_p\{x \in X^{N-1} : f \text{ is not constant on } s_k(x)\}. \tag{1.81}$$

Theorem 1.7 (BKKKL)**.** *Consider $[0,1]^N$ as a measure space with the uniform measure. Let $f : [0,1]^N \to \{0,1\}$, with $\mathbb{P}_p[f = 1] = p \leq \frac{1}{2}$. Then there exist a constant $c > 0$ and $k \in [N]$ such that*

$$I_k(f) \geq cp\frac{\log N}{N} \tag{1.82}$$

The proof of this theorem is modified in [7, Theorem 3.4], by using a convexity argument, to give the following:

Theorem 1.8. *For every function $f : X \to \{0,1\}$ with $\mathbb{P}_p[f = 1] = p \leq \frac{1}{2}$, if $I_k(f) \leq \delta$ for every k, then there exists a constant $c > 0$ such that*

$$\sum_{k=1}^{N} I_k(f) \geq cp \log(1/\delta) \tag{1.83}$$

Here we follow the approach of [6], according to which Theorem 1.7 is derived from the discrete case, or more specifically, from Theorem 1.8 with $X = \{0,1\}^N$. It is worth noting that the Influence theorem in the discrete case away from the uniform measure is also proved in [18].

Proof.

Claim 1.1. *Given a function $g : [0,1]^N \to \{0,1\}$, there is a monotone function $f : [0,1]^N \to \{0,1\}$ such that $I_k(g) \geq I_k(f)$ for every k.*

The proof of this claim is a combinatorial argument and is presented in [5, Lemma 1]. Restrict g to the segment $s_k(x)$. Define $T_k(g)$ as the function which is monotone on $s_k(x)$ and satisfies

$$\mathbb{P}_{N-1}(T_k(g)^{-1}(0) \cap s_k(x)) = \mathbb{P}_{N-1}(g^{-1}(0) \cap s_k(x)) \qquad \forall x \in [0,1]^{N-1}. \tag{1.84}$$

Notice that $I_k(g) = I_k(T_k(g))$ and $I_j(g) \geq I_j(T_k(g))$, $j \neq k$. Repeated applications of these operations yields a function which is fixed under all T_k, hence monotone.

Thus we may consider that $f : [0,1]^N \to \{0,1\}$ is a monotone function. Let $k = 3\log N$ and subdivide $[0,1]^N$ into 2^{kn} equal subcubes, by subdividing each one of the base intervals into 2^k equal parts. Since f is monotone and assumes 0/1-values, f is constant on each of the small subcubes except on the 'mixed' subcubes (where f changes from 0 to 1).

The number of 'mixed' subcubes is no more than the number of subcubes that touch the boundary of $[0,1]^N$. Indeed, for $N = 2$, each segment $s_k(x)$, $k = 1, 2$, $x \in [0,1]$ has at most one point separating, within the segment, the points that take 0 (before the point) and those that assume 1 (after the point). Hence, there are at most $2 \cdot 2^k$ mixed cubes. Suppose this is valid for all $1 < m < N$, for a given N. Since each fiber $s_k(x)$, $k = 1, \cdots, N$, $x \in [0,1]^{N-1}$ has at most $(N-1)2^k$ 'mixed' $N-1$-dimensional subcubes, and there are 2^k subcubes along the axis excluded in the fiber, we conclude that the cube $[0,1]^N$ has at most $N2^k$ 'mixed' subcubes. Thus, this property holds for all $N \in \mathbb{N}$.

Now f corresponds in a natural way to a function on the discrete cube $g : \{0,1\}^{kN} \to \{0,1\}$, by replacing the interval $[r2^{-k}, (r+1)2^{-k}]$ with the binary expansion of r:

$$g(\omega_1(1), \cdots, \omega_1(k), \cdots, \omega_N(1), \cdots, \omega_N(k)) \tag{1.85}$$

$$= f(\sum_{j=1}^{k} \omega_1(j)2^{j-k}, \cdots, \sum_{j=1}^{k} \omega_N(j)2^{j-k}). \tag{1.86}$$

Every variable $i \in [N]$ is replaced by k variables $i_j \;:\; 1 \leq j \leq k$, the k bits of the binary expansion of r. We write a vector in $\{0,1\}^{kN}$ as $(\omega_{-i}, \omega) = (\omega_1, \cdots, \omega_{i-1}, \omega, \omega_{i+1}, \cdots, \omega_N)$, $\omega_{-i} \in \{0,1\}^{k(N-1)}$ and $\omega \in \{0,1\}^k$ to emphasize the entries corresponding to the ith variable.

For each $i \in [N]$ and a fixed $\omega_{-i} \in \{0,1\}^{k(N-1)}$, define $g_i^{\omega_{-i}} : \{0,1\}^k \to \{0,1\}$ by $\omega \mapsto g(\omega_{-i}, \omega)$. Notice that

$$\sum_{j=1}^{k} I_j(g_i^{\omega_{-i}}) = \mathbb{E}[\sum_{j=1}^{k} \mathbb{P}[j \textit{ is pivotal for } g_i^{\omega_{-i}} | \omega_{-i}]], \tag{1.87}$$

and that $I_j(g_i^{\omega_{-i}}) = \mathbb{P}[j \textit{ is pivotal for } g_i^{\omega_{-i}} | \omega_{-i}]$. By [5, Lemma 3], if $h : \{0,1\}^m \to \{0,1\}$ is a monotone function, then $\sum_{k=1}^{m} I_k(h) \leq 2$. Thus, the expression in (1.87) is less than 2.

By noting $I_i(f) = \mathbb{E}[\mathbb{P}[f \textit{ is not constant in } \omega_{-i}|\omega_{-i}]]$ and that this conditional probability is 1 in $\{x \in [0,1]^{N-1};\ f \textit{ is not constant in } s_i(x)\}$, we obtain for each $i \in [N]$

$$\sum_{j=1}^{k} I_{i_j}(g) \leq 2\mathbb{P}\{x \in [0,1]^{N-1};\ f \textit{ is not constant in } s_i(x)\} \tag{1.88}$$

$$= 2\mathbb{E}[\mathbb{P}[f \textit{ is not constant in } \omega_{-i}|\omega_{-i}]] \tag{1.89}$$

$$= 2I_i(f) \tag{1.90}$$

This implies, by Theorem 1.8, that

$$\sum_{i,j} I_{i_j}(g) \geq cp \log N \tag{1.91}$$

and, using (1.88)-(1.90),

$$\sum_{i} I_i(f) \geq \frac{c}{2} p \log N. \tag{1.92}$$

In particular, there exists a variable i such that

$$I_i(f) \geq \frac{c}{2} p \frac{\log N}{N} \tag{1.93}$$

□

Chapter 2

Monotonic measures

We introduce the monotonic measures and show that they feature positive association and the FKG inequality. By defining the conditional influence, we present an influence theorem for such measures, an analogue to the KKL Theorem, which leads to a sharp threshold theorem in this context. Corresponding results are valid for probability measures on the cube $[0,1]^N$ that are absolutely continuous with respect to Lebesgue measure. This chapter is based on [10, Chapter 2] and [11].

2.1 Stochastic ordering of measures

Let E be a finite, $|E| = N$, or countably infinite set. Consider $\Omega = \Omega_E := \{0,1\}^E$, whose members are 0/1-vectors $\omega = (\omega(e) : \ e \in E)$, and $\mathcal{F}$, the set of all subsets of Ω. Henceforth, thinking of our applications, E will be the edge-set of a graph, and thus we regard the variables $i \in [N]$ as edges $e \in E$.

Given a configuration ω and an edge $e \in E$, we will often consider the configurations ω^e, ω_e, obtained from ω by setting 1, in the first case, and 0, in the second, to the edge e and maintaining the other edges unchanged. An edge e is said to be *open* in $\omega \in \Omega$ if $w(e) = 1$, and *closed* otherwise.

A probability measure is said to be positive if $\mu(\omega) > 0$ for all $\omega \in \Omega$. Given two probability measures μ_1, μ_2 in $(\Omega, \mathcal{F})$, we write $\mu_1 \leq_{st} \mu_2$ and say that μ_1 is *stochastically dominated* by μ_2 if

$$\mathbb{E}_{\mu_1}[X] \leq \mathbb{E}_{\mu_2}[X] \qquad \text{(for all increasing r. v. } X \text{ on } \Omega). \tag{2.1}$$

For two probability measures ϕ_1, ϕ_2 on $(\Omega, \mathcal{F})$, a *coupling* of ϕ_1 and ϕ_2 is a probability measure κ on $(\Omega, \mathcal{F}) \times (\Omega, \mathcal{F})$ with ϕ_1 as the first marginal and ϕ_2

as the second one. The next theorem concerning couplings is just stated, we will not prove it, for future use (for a proof, see [15, Section IV.1.2, Theorem 2.4]).

Theorem 2.1. *Let μ_1, μ_2 be probability measures on $(\Omega, \mathcal{F})$. Then, $\mu_1 \leq_{st} \mu_2$ if, and only if, there exists a coupling κ satisfying $\kappa(S) = 1$, where $S = \{(\omega_1, \omega_2) \in \Omega^2 : \omega_1 \leq \omega_2\}$ is the sub-diagonal of the product space Ω^2.*

For $\omega_1, \omega_2 \in \Omega$, we denote by $\omega_1 \wedge \omega_2$ and $\omega_1 \vee \omega_2$ the *minimum* and *maximum* configurations, respectively, given by

$$\omega_1 \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\} \qquad (e \in E) \tag{2.2}$$
$$\omega_1 \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\} \qquad (e \in E). \tag{2.3}$$

Theorem 2.2 (Holley inequality)**.** *Let μ_1, μ_2 be positive probability measures on $(\Omega, \mathcal{F})$ such that*

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \qquad (\omega_1, \omega_2 \in \Omega). \tag{2.4}$$

Then

$$\mathbb{E}_{\mu_1}[X] \leq \mathbb{E}_{\mu_2}[X] \qquad \textit{(for increasing functions } X : \Omega \to \mathbb{R}\textit{)}, \tag{2.5}$$

that is $\mu_1 \leq_{st} \mu_2$.

Proof. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$. We may construct a reversible Markov chain with state space Ω and unique invariant measure μ by choosing a suitable generator satisfying the detailed balance equations. Let $G : \Omega^2 \to \mathbb{R}$ be given by

$$G(\omega_e, \omega^e) = 1, \quad G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)} \qquad (\omega \in \Omega, e \in E). \tag{2.6}$$

We let $G(\omega, \omega') = 0$ for all other pairs ω, ω' with $\omega \neq \omega'$. The diagonal elements $G(\omega, \omega)$ are chosen so that

$$\sum_{\omega' \in \Omega} G(\omega, \omega') = 0 \qquad (\omega \in \Omega). \tag{2.7}$$

It is straightforward that

$$\mu(\omega)G(\omega, \omega') = \mu(\omega')G(\omega', \omega) \qquad (\omega, \omega' \in \Omega), \tag{2.8}$$

and therefore G generates a Markov chain on Ω which is reversible with respect to μ. Now we check that the chain is irreducible. Given $\omega, \omega' \in \Omega$, one may flip the zero (*closed*) edges one by one thus arriving at the unit vector 1 (*open* configuration), and then one may flip again the states of each edge one by one thus arriving at ω'. Since each such transition probability is positive, the chain is irreducible. It follows that the chain has unique invariant measure μ.

Let μ_1, μ_2 satisfy the hypothesis of the theorem, and let S be the set of all ordered pairs (π, ω) of configurations in Ω satisfying $\pi \leq \omega$. We define $H : S \times S \to \mathbb{R}$ by

$$H(\pi_e, \omega; \pi^e, \omega^e) = 1, \tag{2.9}$$

$$H(\pi, \omega^e; \pi_e, \omega_e) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.10}$$

$$H(\pi^e, \omega^e; \pi_e, \omega^e) = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.11}$$

for all $(\pi, \omega) \in S$ and $e \in E$; all other off-diagonal values of H are set to be 0. The diagonal terms $H(\pi, \omega; \pi, \omega)$ are chosen in such a way that

$$\sum_{(\pi', \omega') \in S} H(\pi, \omega; \pi', \omega') = 0 \qquad ((\pi, \omega) \in S). \tag{2.12}$$

Equation (2.9) specifies that, for $\pi \in \Omega$ and $e \in E$, the edge e is acquired by π (if it does not already contain it) at rate 1; any edge so acquired is added also to ω if it does not already contain it. (A configuration ψ contains the edge e if $\psi(e) = 1$.) Equation (2.10) specifies that, for $\omega \in \Omega$ and $e \in E$ with $w(e) = 1$, the edge e is removed from ω (and also from π if $\pi(e) = 1$) at the rate given in (2.10). For e with $\pi(e) = 1$, there is an additional rate given in (2.11) at which e is removed from π but not from ω. This additional rate is indeed non-negative, since the required inequality

$$\mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\pi^e)\mu_2(\omega_e) \qquad \text{whenever } \pi \leq \omega \tag{2.13}$$

follows from (2.4) with $\omega_1 = \pi^e$ and $\omega_2 = \omega_e$

Let $(Y_t, Z_t)_{t \geq 0}$ be a Markov chain on S with generator H and set $(Y_0, Z_0) = (0, 1)$, where 0 (respectively, 1) is the state of all zeros (respectively, ones). We write $\mathbb{P}$ for the appropriate probability measure. Since all transitions retain the ordering of the two components of the state, we may assume that the chain satisfies $\mathbb{P}(Y_t \leq Z_t,\ \forall\ t) = 1$. By examination of (2.9) – (2.11) we see that $Y = (Y_t : t \geq 0)$ is a Markov chain with generator given by (2.6) with $\mu = \mu_1$ and that $Z = (Z_t : t \geq 0)$ arises similarly with $\mu = \mu_2$. In the case of Y (a similar argument holds for Z), for $\pi \in \Omega$ and $e \in E$,

$$\begin{aligned}
&\mathbb{P}[Y_{t+h} = \pi^e |\ Y_t = \pi_e] \\
&= \sum_{\omega \in \Omega} \mathbb{P}[Y_{t+h} = \pi^e |\ (Y_t, Z_t) = (\pi_e, \omega)]\mathbb{P}[Z_t = \omega |\ Y_t = \pi_e] && (2.14) \\
&= \sum_{\omega \in \Omega} [h + o(h)]\mathbb{P}[Z_t = \omega |\ Y_t = \pi_e] && (2.15) \\
&= h + o(h). && (2.16)
\end{aligned}$$

Similarly, with J_e the event that e is open,

$$\begin{aligned}
&\mathbb{P}[Y_{t+h} = \pi_e |\ Y_t = \pi^e] \\
&= \sum_{\omega \in J_e, \omega' \in \Omega} \mathbb{P}[(Y_{t+h}, Z_{t+h}) = (\pi_e, \omega') |\ (Y_t, Z_t) = (\pi^e, \omega^e)]\mathbb{P}[Z_t = \omega^e |\ Y_t = \pi^e] && (2.17) \\
&= \sum_{\omega \in J_e} [\{H(\pi^e, \omega^e; \pi_e, \omega_e) + H(\pi^e, \omega^e; \pi_e, \omega^e)\}h + o(h)]\mathbb{P}[Z_t = \omega^e |\ Y_t = \pi^e)] && (2.18) \\
&= \sum_{\omega \in J_e} \Big[\frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} h + o(h)\Big]\mathbb{P}(Z_t = \omega^e |\ Y_t = \pi^e) \qquad \text{by (2.10 and (2.11))} && (2.19) \\
&= \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} h + o(h). && (2.20)
\end{aligned}$$

Let κ be an invariant measure for the paired chain $(Y_t, Z_t)_{t \geq 0}$. Since Y and Z have (respective) unique invariant measures μ_1 and μ_2, the marginals of κ are μ_1 and μ_2. Since $\mathbb{P}[Y_t \leq Z_t,\ \forall\ t] = 1$,

$$\kappa(S) = \kappa(\{(\pi, \omega) : \pi \leq \omega\}) = 1, \tag{2.21}$$

and κ is the required coupling of μ_1 and μ_2.

Let $(\pi, \omega) \in S$ be chosen according to the measure κ. Then

$$\mathbb{E}_{\mu_1}[X] = \mathbb{E}_\kappa[X(\pi)] \leq \mathbb{E}_\kappa[X(\omega)] = \mathbb{E}_{\mu_2}[X], \tag{2.22}$$

for any increasing function X. Therefore $\mu_1 \leq_{st} \mu_2$. □

Theorem 2.3. *Let μ_1, μ_2 be a pair of strictly positive probability measures on $(\Omega, \mathcal{F})$ such that*

$$\mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\omega^e)\mu_2(\omega_e) \qquad (\omega \in \Omega,\ e \in E). \tag{2.23}$$

If, in addition, either μ_1 or μ_2 satisfies

$$\mu(\omega^{ef})\mu(\omega_{ef}) \geq \mu(\omega_f^e)\mu(\omega_e^f) \qquad (\omega \in \Omega,\ e, f \in E), \tag{2.24}$$

then (2.4) holds.

Proof. Let μ be strictly positive probability measure satisfying (2.24). We show first that μ satisfies (2.4) with $\mu_1 = \mu_2 = \mu$, that is

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2). \tag{2.25}$$

We will prove this by induction on the Hamming distance $H(\omega_1, \omega_2)$. The Hamming distance between two configurations is given by

$$H(\omega_1, \omega_2) = \sum_{e \in E} |\omega_1(e) - \omega_2(e)|, \qquad (\omega_1, \omega_2 \in \Omega). \tag{2.26}$$

Inequality (2.25) is trivial when: either $H(\omega_1, \omega_2) = 1$, or the ω_i are ordered (in that either $\omega_1 \leq \omega_2$, or vice-versa). The only non-trivial case with $H(\omega_1, \omega_2) = 2$ is of the form: $\omega_1 = \omega_f^e,\ \omega_2 = \omega_e^f$ where $e,\ f$ are distinct edges. This is handled by assumption (2.24).

Let $h \geq 3$ and suppose that (2.25) holds for all pairs ω_1, ω_2 satisfying $H(\omega_1, \omega_2) < h$. Let $\omega_1, \omega_2 \in \Omega$ be such $H(\omega_1, \omega_2) = h$, and furthermore such that neither $\omega_1 \leq \omega_2$ nor $\omega_1 \geq \omega_2$. There exist integers a, b such that $a, b \geq 1$ and $a + b = h$, and disjoint subsets $A, B \subseteq E$ with cardinalities a and b respectively, such that:

$$\begin{aligned} if\ e \in A, &\quad (\omega_1(e), \omega_2(e)) = (1, 0), &(2.27)\\ if\ e \in B, &\quad (\omega_1(e), \omega_2(e)) = (0, 1), &(2.28)\\ if\ e \in E \backslash (A \cup B), &\quad \omega_1(e) = \omega_2(e). &(2.29) \end{aligned}$$

We fix an ordering $(e_i : i = 1, 2, \cdots, |E|)$ of the set E in which edges in A are indexed $1, 2, \cdots, a$, and edges in B are indexed $a+1, a+2, \cdots, a+b$. A configuration ω may be written as a word $\omega(e_1) \cdot \omega(e_2) \cdot ... \cdot \omega(e_{|E|})$; we write 0^x for a sub-word of length x every entry of which is 0, with a similar meaning for 1^y. Since the entries of the configurations $\omega_1, \omega_2, \omega_1 \vee \omega_2, \omega_1 \wedge \omega_2$ are constant off $A \cup B$, we omit explicit reference to these values. Thus, for example, $\omega_1 = 1^a \cdot 0^b$ and $\omega_2 = 0^a \cdot 1^b$.

Since $h = a + b \geq 3$, either $a \geq 2$ or $b \geq 2$; it suffices by symmetry to assume $a \geq 2$. By the induction hypothesis,

$$\mu(1^{a+b})\mu(0^{a-1} \cdot 1 \cdot 0^b) \geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1^{b+1}) \tag{2.30}$$

$$\text{since} \quad H(1^a \cdot 0^b, 0^{a-1} \cdot 1^{b+1}) = h - 1, \tag{2.31}$$

$$\mu(0^{a-1} \cdot 1^{b+1})\mu(0^{a+b}) \geq \mu(0^{a-1} \cdot 1 \cdot 0^b)\mu(0^a \cdot 1^b) \tag{2.32}$$

$$\text{since} \quad H(0^{a-1} \cdot 1 \cdot 0^b, 0^a \cdot 1^b) = b + 1 < h \ , \tag{2.33}$$

whence

$$\mu(1^{a+b})\mu(0^{a-1} \cdot 1 \cdot 0^b) \geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1^{b+1})\mu(0^{a+b}) \tag{2.34}$$

$$\geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1 \cdot 0^b)\mu(0^a \cdot 1^b). \tag{2.35}$$

Therefore,

$$\mu(1^{a+b})\mu(0^{a+b}) \geq \mu(1^a \cdot 0^b)\mu(0^a \cdot 1^b), \tag{2.36}$$

and the induction step is complete.

We identify a configuration $\omega \in \Omega$ with the set of indices $\eta(\omega)$ at which ω takes the value 1. Let $\xi_1, \xi_2 \in \Omega$ and write $A_k = \eta(\xi_k)$. Let $B = A_1 \backslash A_2 = \{b_1, \cdots b_r\}$ and write $B_s = \{b_1, \cdots b_s\}$ for $s \geq 1$. Assume $\xi_1 \neq \xi_2$ and without loss of generality that $r \geq 1$. By (2.23),

$$\frac{\mu_2(\xi_1 \vee \xi_2)}{\mu_2(\xi_2)} = \frac{\mu_2(A_2 \cup B_r)}{\mu_2(A_2 \cup B_{r-1})} \cdot \frac{\mu_2(A_2 \cup B_{r-1})}{\mu_2(A_2 \cup B_{r-2})} \cdots \frac{\mu_2(A_2 \cup B_1)}{\mu_2(A_2)} \tag{2.37}$$

$$\geq \frac{\mu_1(A_2 \cup B_r)}{\mu_1(A_2 \cup B_{r-1})} \cdot \frac{\mu_1(A_2 \cup B_{r-1})}{\mu_1(A_2 \cup B_{r-2})} \cdots \frac{\mu_1(A_2 \cup B_1)}{\mu_1(A_2)} \tag{2.38}$$

$$= \frac{\mu_1(\xi_1 \vee \xi_2)}{\mu_1(\xi_2)}. \tag{2.39}$$

If μ_1 satisfies (2.24), then it satisfies (2.25) and (2.4) follows with $\xi_i = \omega_i, \ i \in \{1, 2\}$. □

Theorem 2.4. *A pair μ_1, μ_2 of positive probability measures on $(\Omega, \mathcal{F})$ satisfies (2.4) if, and only if, the one-point conditional probabilities satisfy:*

$$\begin{aligned} &\mu_2(\omega(e) = 1 | \ \omega(f) = \zeta(f) \ for \ all \ f \in E \backslash \{e\}) \\ &\geq \mu_1(\omega(e) = 1 | \ \omega(f) = \xi(f) \ for \ all \ f \in E \backslash \{e\}), \end{aligned} \tag{2.40}$$

for all $e \in E$ and all pairs $\xi, \zeta \in \Omega$ satisfying $\xi \leq \zeta$.

Proof. Notice that inequality (2.40) is equivalent to

$$\frac{\mu_2(\zeta^e)}{[\mu_2(\zeta^e)+\mu_2(\zeta_e)]} \geq \frac{\mu_1(\xi^e)}{[\mu_1(\xi^e)+\mu_1(\xi_e)]}, \tag{2.41}$$

or, equivalently,

$$\mu_2(\zeta^e)\mu_1(\xi_e) \geq \mu_1(\xi^e)\mu_2(\zeta_e). \tag{2.42}$$

Assume (2.42) holds. By using the same argument and notation at the end of the last proof, we have that

$$\frac{\mu_2(\xi_1 \vee \xi_2)}{\mu_2(\xi_2)} = \frac{\mu_2(A_2 \cup B_r)}{\mu_2(A_2 \cup B_{r-1})} \cdot \frac{\mu_2(A_2 \cup B_{r-1})}{\mu_2(A_2 \cup B_{r-2})} \cdots \frac{\mu_2(A_2 \cup B_1)}{\mu_2(A_2)} \tag{2.43}$$

$$\geq \frac{\mu_1((A_1 \cap A_2) \cup B_r)}{\mu_1((A_1 \cap A_2) \cup B_{r-1})} \cdot \frac{\mu_1((A_1 \cap A_2) \cup B_{r-1})}{\mu_1((A_1 \cap A_2) \cup B_{r-2})} \cdots \frac{\mu_1((A_1 \cap A_2) \cup B_1)}{\mu_1(A_1 \cap A_2)} \tag{2.44}$$

$$= \frac{\mu_1(\xi_1)}{\mu_1(\xi_1 \wedge \xi_2)}. \tag{2.45}$$

Conversely, if (2.4) holds, then so does (2.42) for $\xi \leq \zeta$. □

2.2 Positive association

A probability measure μ on Ω is said to have the *FKG lattice property* if it satisfies the so-called *FKG lattice condition*:

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \qquad (\omega_1, \omega_2 \in \Omega). \tag{2.46}$$

Theorem 2.5 (FKG inequality)**.** *Let μ be a positive probability measure on Ω satisfying the FKG lattice condition. Then*

$$\mathbb{E}_\mu[XY] \geq \mathbb{E}_\mu[X]\mathbb{E}_\mu[Y] \tag{2.47}$$

for increasing functions $X, Y : \Omega \to \mathbb{R}$.

Proof. Assume that μ satisfies the FKG lattice condition and let X and Y be increasing functions. Let $a > 0$ and $Y' = Y + a$. Since

$$\mathbb{E}_\mu[XY'] - \mathbb{E}_\mu[X]\mathbb{E}_\mu[Y'] = \mathbb{E}_\mu[XY] - \mathbb{E}[X]\mathbb{E}[Y], \tag{2.48}$$

it suffices to prove (2.47) with Y replaced by Y'. We may pick a sufficiently large that $Y'(\omega) > 0$, for all $\omega \in \Omega$. Thus, it suffices to prove (2.47) under the additional hypothesis that Y is positive; so we assume henceforth that this holds. Define the positive probability measures μ_1 and μ_2 on $(\Omega, \mathcal{F})$ by $\mu_1 = \mu$ and

$$\mu_2(\omega) = \frac{Y(\omega)\mu(\omega)}{\sum_{\omega' \in \Omega} Y(\omega')\mu(\omega')} \qquad (\omega \in \Omega). \tag{2.49}$$

Since Y is increasing,

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \tag{2.50}$$

follows from the FKG lattice condition. By the Holley inequality (Theorem 2.2), $\mathbb{E}_{\mu_2}[X] \geq \mathbb{E}_{\mu_1}[X]$, which is to say that

$$\frac{\sum_{\omega \in \Omega} X(\omega)Y(\omega)\mu(\omega)}{\sum_{\omega' \in \Omega} Y(\omega')\mu(\omega')} \geq \sum_{\omega \in \Omega} X(\omega)\mu(\omega). \tag{2.51}$$

□

Any probability measure μ satisfying (2.47) is said to have the property of *positive association*.

Let $X = (X_1, \cdots, X_r)$ be a vector of random variables taking values in $\{0,1\}^r$. We speak of X as being positively associated if its law on $\{0,1\}^r$ is positively associated. Let $Y = h(X)$ where $h : \{0,1\}^r \to \{0,1\}^s$ is a non-decreasing function. Then the vector Y is positively associated whenever X is positively associated. Let A, B be increasing subsets of $\{0,1\}^s$. Then

$$\begin{aligned}
\mathbb{P}[Y \in A \cap B] &= \mathbb{P}[X \in h^{-1}(A) \cap h^{-1}(B)] && (2.52)\\
&\geq \mathbb{P}[X \in h^{-1}(A)]\mathbb{P}[X \in h^{-1}(B)] && (2.53)\\
&= \mathbb{P}[Y \in A]\mathbb{P}[Y \in B], && (2.54)
\end{aligned}$$

since $h^{-1}(A)$ and $h^{-1}(B)$ are increasing subsets of $\{0,1\}^r$.

A pair $\omega_1, \omega_2 \in \Omega$ is called comparable, if either $\omega_1 \geq \omega_2$ or $\omega_1 \geq \omega_2$, and incomparable, otherwise.

Theorem 2.6. *A positive probability measure μ on $(\Omega, \mathcal{F})$ satisfies the FKG lattice condition if, and only if, this condition (2.46) holds for all incomparable pairs $\omega_1, \omega_2 \in \Omega$ with $H(\omega_1, \omega_2) = 2$.*

Proof. It follows from Theorem 2.3. □

The FKG lattice condition is sufficient but not necessary for positive association. It is equivalent for strictly positive measures to a stronger property called strong positive association. For $F \subseteq E$ and $\xi \in \Omega$, we write $\Omega_F = \{0,1\}^F$ and

$$\Omega_F^\xi = \{\omega \in \Omega : \omega(e) = \xi(e),\ \forall\ e \in E \backslash F\}, \tag{2.55}$$

the set of configurations that agree with ξ on the complement of F. Let μ be a probability measure on $(\Omega, \mathcal{F})$ and let F, ξ be such that $\mu(\Omega_F^\xi) > 0$. We define the conditional probability measure μ_F^ξ on Ω_F by

$$\mu_F^\xi(\omega_F) = \mu(\omega_F |\ \Omega_F^\xi) = \frac{\mu(\omega_F \times \xi)}{\mu(\Omega_F^\xi)} \qquad (\omega_F \in \Omega_F), \tag{2.56}$$

where $\omega_F \times \xi$ denotes the configuration that agrees with ω_F on F and with ξ on its complement. We say that μ is *strongly positively-associated* if: for all $F \subseteq E$ and all $\xi \in \Omega$ such that $\mu(\Omega_F^\xi) > 0$, the measure μ_F^ξ is positively associated.

We call μ monotonic if for all $F \subseteq E$, all increasing subsets A of Ω_F and all $\xi, \zeta \in \Omega$ such that $\mu(\Omega_F^\xi), \mu(\Omega_F^\zeta) > 0$,

$$\mu_F^\xi(A) \le \mu_F^\zeta(A) \qquad \text{whenever } \xi \le \zeta. \tag{2.57}$$

That is, μ is monotonic if, for all $F \subseteq E$,

$$\mu_F^\xi \le_{st} \mu_F^\zeta \qquad \text{whenever } \xi \le \zeta. \tag{2.58}$$

We call *1-monotonic* if (2.58) holds for all singleton sets F. That is, μ is 1-monotonic if, and only if, for all $f \in E$, $\mu(J_f |\ \Omega_f^\xi)$ is a non-decreasing function of ξ. Here, J_f denotes the event that f is open.

Theorem 2.7. *Let μ be a positive probability measure on $(\Omega, \mathcal{F})$. The following are equivalent.*

(a) μ is strongly positively-associated.

(b) μ satisfies the FKG lattice condition.

(c) μ is monotonic.

(d) μ is 1-monotonic.

Proof. $(a) \Leftrightarrow (b)$. We prove first that (a) implies (b). By Theorem 2.6, it suffices to prove (2.46) for two incomparable configurations ω_1, ω_2 that disagree on exactly two edges. Let e, f be distinct members of E and take e and f to be the first two bits in a given ordering (permutation) of E. We adopt the notation used in the proof of Theorem 2.3. Thus we write $\omega_1 = 0 \cdot 1 \cdot \omega$ and $\omega_2 = 1 \cdot 0 \cdot \omega$ for some word ω of length $|E| - 2$. By strong positive-association, $\alpha(xy) = \mu(x \cdot y \cdot \omega)$ satisfies (take $\Omega_F = \{0,1\}^{\{e,f\}}, \xi = \omega, X = 1_{\{(0,1),(1,1)\}}$ and $Y = 1_{\{(1,0),(1,1)\}}$)

$$\alpha(11)[\alpha(00)+\alpha(01)+\alpha(10)+\alpha(11)] \geq [\alpha(01)+\alpha(11)][\alpha(10)+\alpha(11)], \quad (2.59)$$

which may be simplified to obtain, as required, that

$$\alpha(11)\alpha(00) \geq \alpha(01)\alpha(10). \quad (2.60)$$

We prove next that (b) implies (a). Suppose (b) holds and let $F \subseteq E$, $\xi \in \Omega$. Since $(\omega_1)_F \times \xi$ and $(\omega_2)_F \times \xi$ possibly differ from each other just at variables in F, $(\omega_1 \vee \omega_2)_F \times \xi = ((\omega_1)_F \times \xi) \vee ((\omega_2)_F \times \xi)$ (similarly for $\wedge$). It follows from (2.56) that

$$\mu_F^\xi(\omega_1 \vee \omega_2)\mu_F^\xi(\omega_1 \wedge \omega_2) \geq \mu_F^\xi(\omega_1)\mu_F^\xi(\omega_2) \qquad (\omega_1, \omega_2 \in \Omega_F). \quad (2.61)$$

By Theorem 2.5, μ_F^ξ is positively correlated.

$(b) \Rightarrow (c)$ By Theorem 2.2 (Holley inequality), it suffices to prove for $\omega_F, \rho_F \in \Omega_F$ that

$$\mu_F^\zeta(\omega_F \vee \rho_F)\mu_F^\xi(\omega_F \wedge \rho_F) \geq \mu_F^\zeta(\omega_F)\mu_F^\xi(\rho_F) \quad (2.62)$$

whenever $\xi \leq \zeta$. This is, by (2.56), an immediate consequence of the FKG lattice property applied to the pair $\omega_F \times \zeta, \rho_F \times \xi$.

$(c) \Rightarrow (d)$. This is trivial.

$(d) \Rightarrow (b)$. Let μ be 1-monotonic. By Theorem 2.4, the pair μ, μ satisfies (2.4), which is to say that μ satisfies the FKG lattice condition. $\square$

2.3 Influence for monotonic measures

Let $A \in \mathcal{F}$ be an increasing event and write 1_A for its indicator function. The *conditional influence* on A of the edge $e \in E$ is defined by

$$I_A(e) = \mu(A|\ J_e = 1) - \mu(A|\ J_e = 0), \quad (2.63)$$

where $J = (J_e : e \in E)$ denotes the identity function on Ω. (J_e denotes both the event $\{\omega \in \Omega : \omega(e) = 1\}$ and its indicator function.) The conditional influence is not generally equal to the (absolute) influence of Chapter 1,

$$I_A(e) = \mu(1_A(\omega^e) \neq 1_A(\omega_e)).$$

Theorem 2.8 (Influence)**.** *Let A be an increasing subset of $\Omega = \{0,1\}^E$. Let μ be a positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic. There exist $e \in E$ and a constant $c \in (0, \infty)$ such that*

$$I_A(e) \geq c \min\{\mu(A), 1 - \mu(A)\} \frac{\log N}{N}. \tag{2.64}$$

Proof. The idea is to encode μ in terms of Lebesgue measure λ on the Euclidean cube $[0,1]^N$ and then to apply Theorem 1.7 (BKKKL).

Give an ordering to the set E so that $E = \{e_1, \cdots, e_N\}$. Let $x = (x_1, \cdots, x_N) \in [0,1]^N$ and $f(x) = (f_1(x), \cdots, f_N(x)) \in \mathbb{R}^N$ be given recursively as follows. The first coordinate $f_1(x)$ is defined by:

$$\text{with } a_1 = \mu(J_1), \text{ let } f_1(x) = \begin{cases} 1 & if\ x_1 > 1 - a_1 \\ 0 & otherwise. \end{cases} \tag{2.65}$$

Suppose we know the values $f_i(x)$ for $i = 1, \cdots, k-1$. Let

$$a_k = \mu(J_k = 1 | J_i = f_i(x)\ for\ i = 1, \cdots, k-1), \tag{2.66}$$

and define

$$f_k(x) = \begin{cases} 1 & if\ x_k > 1 - a_k \\ 0 & otherwise \end{cases} \tag{2.67}$$

Now we show that the function $f : [0,1]^N \to \{0,1\}^N$ is non-decreasing. Let $x \leq x'$ and write $a_k = a_k(x)$ and $a'_k = a_k(x')$ for the values in (2.65)-(2.66) corresponding to the vectors x and x'. Clearly $a_1 = a'_1$, so that $f_1(x) \leq f_1(x')$. Since μ is monotonic, $a_2 \leq a'_2$ (J_k is increasing, for all $k \in E$), implying that $f_2(x) \leq f_2(x')$. Continuing inductively, we find that $f_k(x) \leq f_k(x')$ for all k, which is to say that $f(x) \leq f(x')$.

Let $A \in \mathcal{F}$ be an increasing event and let B be the increasing subset of $[0,1]^N$ given by $B = f^{-1}(A)$. Notice the following facts concerning the definition of f.

(a) For given x, each a_k depends only on $x_1, \cdots, x_{k-1}$.

(b) Since μ is strictly positive, the a_k satisfy $0 < a_k < 1$ for all $x \in [0,1]^N$ and $k \in E$.

(c) For any $x \in [0,1]^N$ and $k \in E$, the values $f_k(x), f_{k+1}(x), \cdots, f_N(x)$ depend on $x_1, \cdots, x_{k-1}$ only through the values $f_1(x), \cdots, f_{k-1}(x)$.

(d) The function f and the event B depend on the ordering of the set E.

Let $U = (U_i : i = 1, \cdots, N)$ be the identity function on $[0,1]^N$, so that U has law λ. By the definition of f, $f(U)$ has law μ. Hence,

$$\mu(A) = \lambda(f(U) \in A) = \lambda(U \in f^{-1}(A)) = \lambda(B). \tag{2.68}$$

Let

$$K_B(i) = \lambda(B | \, U_i = 1) - \lambda(B | \, U_i = 0), \tag{2.69}$$

where the conditional probabilities are interpreted as

$$\lambda(B | \, U_i = u) = \lim_{\epsilon \downarrow 0} \lambda(B | \, U_i \in (u - \epsilon, u + \epsilon)) \tag{2.70}$$

By Theorem 1.7, there exists a constant $c < \infty$, independent of the choice of N and A, such that there exists $i \in [N]$ with

$$K_B(i) \geq c \min \{\lambda(B), 1 - \lambda(B)\} \frac{\log N}{N}. \tag{2.71}$$

We choose i accordingly. We claim that

$$I_A(e_j) \geq K_B(j) \qquad \text{for } j \in [N]. \tag{2.72}$$

By (2.68) and (2.71), it suffices to prove (2.72). We prove first that

$$I_A(e_1) \geq K_B(1). \tag{2.73}$$

By (b) and (c) above,

$$\begin{aligned} I_A(e_1) &= \mu(A \mid J_1 = 1) - \mu(A \mid J_1 = 0) && (2.74) \\ &= \lambda(B \mid f_1(U) = 1) - \lambda(B \mid f_1(U) = 0) && (2.75) \\ &= \lambda(B \mid U_1 > 1 - a_1) - \lambda(B \mid U_1 \leq 1 - a_1) && (2.76) \\ &= \lambda(B \mid U_1 = 1) - \lambda(B \mid U_1 = 0) && (2.77) \\ &= K_B(1). && (2.78) \end{aligned}$$

We turn to (2.73) with $j \geq 2$. We reorder the set E to bring the index j to the front. That is, we let F be the reordered index set $F = (k_1, \cdots, k_N) = (j, 1, \cdots, j-1, j+1, \cdots, N)$. Let $g = (g_{k_r} : r = 1, \cdots, N)$ denote the associated function given by (2.65)-(2.67) subject to the new ordering, and let $C = g^{-1}(A)$. We claim that

$$K_C(k_1) \geq K_B(j). \tag{2.79}$$

By (2.74)-(2.78) with E replaced by F, $K_C(k_1) = I_A(j)$, and (2.72) follows. It remains to prove (2.79); we use monotonicity again for this. It suffices to prove that

$$\lambda(C \mid U_j = 1) \geq \lambda(B \mid U_j = 1), \tag{2.80}$$

together with the reversed inequality given $U_j = 0$. Let

$$\overline{U} = (U_1, \cdots, U_{j-1}, 1, U_{j+1}, \cdots, U_N). \tag{2.81}$$

The 0/1-vector $f(\overline{U}) = (f_i(\overline{U}) : i = 1, \cdots, N)$, constructed sequentially by considering the indices $1, \cdots, N$ in turn. At stage k, we declare $f_k(\overline{U})$ equal to 1 if U_k exceeds a certain function a_k of the variables $f_i(\overline{U})$, $1 \leq i < k$. By the monotonicity of μ, this function is non-increasing in these variables. Notice that (i) $f_j(\overline{U} = 1)$, and (ii) given this fact, it is more likely than before that the variables $f_k(\overline{U})$, $j < k \leq N$, will take the value 1. The values $f_k(\overline{U}), 1 \leq k < j$ are unaffected by the value of U_j.

Consider now the 0/1-vector $g(\overline{U}) = (g_{k_r}(\overline{U}) : r = 1, \cdots N)$, constructed in the same manner as above but with the new ordering F of the index set E. First we examine index $k_1 (= j)$ and we automatically declare $g_{k_1}(\overline{U}) = 1$ (since $U_j = 1$). We then construct $g_{k_r}(\overline{U})$, $r = 2, 3, \cdots, N$, in sequence. Since the a_k are non-decreasing in the variables constructed so far,

$$g_{k_r}(\overline{U}) \geq f_{k_r}(\overline{U}) \qquad (r = 2, 3, \cdots, N). \tag{2.82}$$

Therefore, $g(\overline{U}) \geq f(\overline{U})$, and hence

$$\lambda(C \mid U_j = 1) = \lambda(g(\overline{U}) \in A) \geq \lambda(f(\overline{U}) \in A) = \lambda(B \mid U_j = 1). \tag{2.83}$$

Inequality (2.80) has been proved. The same argument implies the reversed inequality obtained from (2.80) by changing the conditioning to $U_j = 0$. Inequality (2.79) follows and the proof is complete. □

2.4 Sharp thresholds for increasing events

Let μ be a probability measure on $(\Omega, \mathcal{F})$. For $p \in (0,1)$, let μ_p be the probability measure given by

$$\mu_p(\omega) = \frac{1}{Z_p}\mu(\omega)\Big\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\} \qquad (\omega \in \Omega), \tag{2.84}$$

where Z_p is the normalizing constant

$$Z_p = \sum_{\omega\in\Omega}\mu(\omega)\Big\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\}. \tag{2.85}$$

Thus, $\mu = \mu_{\frac{1}{2}}$ and each μ_p is positive if, and only if, μ is positive. Since

$$\begin{aligned}&\prod_{e\in E} p^{(\omega_1\vee\omega_2)(e)+(\omega_1\wedge\omega_2)(e)}(1-p)^{(\omega_1\vee\omega_2)(e)+(\omega_1\wedge\omega_2)(e)}\\ &= \prod_{e\in E} p^{\omega_1(e)+\omega_2(e)}(1-p)^{\omega_1(e)+\omega_2(e)}\end{aligned} \tag{2.86}$$

each μ_p satisfies the FKG lattice condition if, and only if, μ satisfies this condition; and it follows from Theorem 2.7 that, for positive μ, μ is monotonic if, and only if, each μ_p is monotonic.

Theorem 2.9. *For a random variable $X : \Omega \to \mathbb{R}$,*

$$\frac{d}{dp}\mathbb{E}_{\mu_p}[X] = \frac{1}{p(1-p)} cov_p[|\eta|, X] \qquad (p \in (0,1)], \tag{2.87}$$

where cov_p denotes covariance with respect to the probability measure μ_p and $\eta(\omega)$ is the set of ω-open edges.

Notice that, since $|\eta|(\omega) = \sum_{e\in E} J_e(\omega)$ $(\omega \in \Omega)$,

$$cov_p[|\eta|, X] = \sum_{e\in E} cov_p[J_e, X]. \tag{2.88}$$

Proof. Write

$$\nu_p(\omega) = p^{|\eta(\omega)|}(1-p)^{N-|\eta(\omega)|}\mu(\omega) \qquad (\omega \in \Omega), \tag{2.89}$$

so that

$$\mathbb{E}_{\mu_p}[X] = \frac{1}{Z_p} \sum_{\omega \in \Omega} X(\omega)\nu_p(\omega). \tag{2.90}$$

By differentiating (2.90), we obtain

$$\frac{d}{dp}\mathbb{E}_{\mu_p}[X] = \frac{1}{Z_p} \sum_{\omega \in \Omega} \Big(\frac{|\eta(\omega)|}{p} - \frac{N - |\eta(\omega)|}{1-p} \Big) X(\omega)\nu_p(\omega) - \frac{Z_p'}{Z_p}\mathbb{E}_{\mu_p}[X], \tag{2.91}$$

where $Z_p' = \frac{dZ_p}{dp}$. Setting $X = 1$, we find that

$$0 = \frac{1}{p(1-p)}\mathbb{E}_{\mu_p}[|\eta| - pN] - \frac{Z_p'}{Z_p}, \tag{2.92}$$

whence

$$p(1-p)\frac{d}{dp}\mathbb{E}_{\mu_p}[X] = \mathbb{E}_{\mu_p}[(|\eta| - pN)X] - \mathbb{E}_{\mu_p}[|\eta| - pN]\mathbb{E}_{\mu_p}[X] \tag{2.93}$$

$$= \mathbb{E}_{\mu_p}[|\eta|X] - \mathbb{E}_{\mu_p}[|\eta|]\mathbb{E}_{\mu_p}[X] \tag{2.94}$$

$$= cov_p[|\eta|, X] \tag{2.95}$$

□

Let Π be the group of permutations of $|E|$. Any $\pi \in \Pi$ acts on Ω by $\pi\omega = (\omega(\pi_e) : e \in E)$. We say that a subgroup $\mathcal{A}$ of Π acts transitively on E if, for all pairs $j, k \in E$, there exists $\alpha \in \mathcal{A}$ with $\alpha_j = k$. Let $\mathcal{A}$ be a subgroup of Π. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called $\mathcal{A}$-invariant if $\phi(\omega) = \phi(\alpha\omega)$ for all $\alpha \in \mathcal{A}$. An event $A \in \mathcal{F}$ is called $\mathcal{A}$-invariant if $A = \alpha A$ for all $\alpha \in \mathcal{A}$. Thus, for any subgroup $\mathcal{A}$, μ is $\mathcal{A}$-invariant if and only if each μ_p is $\mathcal{A}$-invariant.

Theorem 2.10 (Sharp threshold)**.** *Let $A \in \mathcal{F}$ be an increasing event and μ be a positive probability measure on $(\Omega, \mathcal{F})$ that is monotonic, Suppose there exists a subgroup $\mathcal{A}$ of Π acting transitively on E such that μ and A are $\mathcal{A}$-invariant. Then there exists a constant $c \in (0, \infty)$ such that*

$$\frac{d\mathbb{E}_{\mu_p}[A]}{dp} \geq \frac{cm_p}{p(1-p)} \min\{\mu_p(A), 1 - \mu_p(A)\} \log N \qquad (p \in (0,1)),$$

where $m_p = \mu_p(J_e)(1 - \mu_p(J_e))$.

Let $I_{p,A}(e) = \mu_p(A|J_e = 1) - \mu_p(A|J_e = 0)$.

Lemma 2.1. *Let $A \in \mathcal{F}$. Suppose there exists a subgroup $\mathcal{A}$ of Π acting transitively on E such that μ and A are $\mathcal{A}$-invariant. Then $I_{p,A}(e) = I_{p,A}(f)$ for all $e, f \in E$ and all $p \in (0,1)$.*

Proof. Since μ is $\mathcal{A}$-invariant, so is μ_p for every p. Let $e, f \in [E]$ and find $\alpha \in \mathcal{A}$ such that $\alpha_e = f$, under the given conditions,

$$\mu_p(A|J_f) = \sum_{\omega \in A} \mu_p(\omega) 1_{J_f}(\omega) = \sum_{\omega \in A} \mu_p(\alpha\omega) 1_{J_e}(\alpha\omega)$$

$$= \sum_{\omega' \in A} \mu_p(\omega') 1_{J_e}(\omega') = \mu_p(A|J_e).$$

We deduce with $A = \Omega$ that $\mu_p(J_f) = \mu_p(J_e)$. On dividing we obtain that $\mu_p(A|J_f) = \mu_p(A|J_e)$. A similar equality holds with J_k replaced by $\overline{J_k}$, and the lemma follows. □

Proof of Theorem 2.10. By Lemma 2.1, $I_{p,A}(e) = I_{p,A}(f)$ for all $e, f \in E$. Since A is increasing and μ_p is monotonic, each $I_{p,A}(e)$ is non-negative, and therefore

$$cov_p(1_{J_e}, 1_A) = \mathbb{E}_{\mu_p}[1_{J_e} 1_A] - \mathbb{E}_{\mu_p}[1_{J_e}]\mathbb{E}_{\mu_p}[1_A]$$

$$= \mathbb{E}_{\mu_p}[1_{J_e})](1 - \mathbb{E}_{\mu_p}[1_{J_e}]) I_{p,A}(e) \geq m_p I_{p,A}(e) \qquad (e \in E).$$

Summing over the set of variables E as in Theorem 2.10, and by noting that $cov_p(|\eta| + X) = \sum_{e \in E} cov_p(1_{J_e}, X)$, we deduce the result by Theorem 2.8 applied to the monotonic measure μ_p. □

2.5 Probability measures on the Euclidean cube

The method of the proof of Theorem 2.8 may also be applied to probability measures on the Euclidean cube $[0,1]^N$ that are absolutely continuous with respect to the Lebesgue measure. Any such measure μ has a density function ρ, that is

$$\mu(A) = \int_A \rho(x) \lambda(dx),$$

for Lebesgue measurable subsets A of $[0,1]^N$, with λ denoting Lebesque measure.

Let $N \geq 1$ and write $\Omega = [0,1]^N$. Let $\rho : \Omega \to [0,\infty)$ be Lebesgue measurable. We call ρ a *density function* if

$$\int_\Omega \rho(x)\lambda(dx) = 1,$$

and in this case we denote by μ_ρ the corresponding probability measure,

$$\mu_\rho(A) = \int_A \rho(x)\lambda(dx).$$

We call ρ *positive* if it is strictly positive function on Ω and we say it satisfies the (*continuous*) FKG lattice condition if

$$\rho(x \vee y)\rho(x \wedge y) \geq \rho(x)\rho(y) \qquad (for\ all\ x,\ y \in \Omega), \tag{2.96}$$

where the operations $\vee$, $\wedge$ are defined as the coordinate-wise maximum and minimum, respectively.

Let ρ be a density function. We call μ_ρ *positively associated* if

$$\mu_\rho(A \cap B) \geq \mu_\rho(A)\mu_\rho(B),$$

for all increasing subsets of Ω.

Let $I = \{1, 2, ..., N\}$. For $J \subseteq I$, let $\Omega_J = [0,1]^J$ and

$$\Omega_J^\xi = \{x \in \Omega : x_j = \xi_j\ for\ j \in I \backslash J\} \qquad (\xi \in \Omega). \tag{2.97}$$

The Lebesgue σ-algebra of Ω_J is denoted by $\mathcal{F}_J$. Let ρ be a positive density function. We define the conditional probability measure $\mu_{\rho,J}^\xi$ on $(\Omega_J, \mathcal{F}_J)$ by

$$\mu_{\rho,J}^\xi(E) = \int_E \rho_J^\xi(x)\lambda(d(x_j : j \in J)) \qquad (E \in \mathcal{F}_J), \tag{2.98}$$

where ρ_J^ξ is the conditional density function

$$\rho_J^\xi(x) = \frac{1}{Z_J^\xi}\rho(x)1\!\!1_{\Omega_J^\xi}(x), \quad Z_J^\xi = \int_{\Omega_J^\xi} \rho(x)\lambda(d(x_j : j \in J)).$$

We sometimes write $\mu_\rho(E|(\xi_i : i \in I\backslash J))$ for $\mu_{\rho,J}^\xi(E)$ and we recall the standard fact that $\mu_\rho(\cdot|(\xi_i : i \in I\backslash J))$ is a version of the conditional expectation given the σ-algebra $\mathcal{F}_{I\backslash J}$.

We say that ρ is *strongly positively associated* if for all $J \subseteq I$ and all $\xi \in \Omega$, the measure $\mu^{\xi}_{\rho,J}$ is positively associated. We call ρ *monotonic* if for all $J \subseteq I$, all increasing subsets A of Ω_J and all $\xi, \zeta \in \Omega$,

$$\mu^{\xi}_{\rho,J}(A) \leq \mu^{\zeta}_{\rho,J}(A) \qquad \textit{whenever } \xi \leq \zeta, \tag{2.99}$$

which is to say that, for all $J \subseteq I$,

$$\mu^{\xi}_{\rho,J} \leq_{st} \mu^{\zeta}_{\rho,J} \qquad \textit{whenever } \xi \leq \zeta. \tag{2.100}$$

Now, a basic result concerning stochastic ordering:

Theorem 2.11. *[17, Theorem 3] Let $N \geq 1$ and let f_1 and f_2 be density functions on $\Omega = [0,1]^N$. If*

$$f_1(x \vee y) f_2(x \wedge y) \geq f_1(x) f_2(y) \qquad \textit{for all } x, y \in [0,1]^N,$$

then $\mu_2 = \mu_{f_2} \leq_{st} \mu_{f_1} = \mu_1$.

We will change the statement of the Theorem and consider the following proposition.

Proposition 2.1. *As above, suppose f_1, f_2 satisfy*

$$f_1(x \vee y) f_2(x \wedge y) \geq f_1(x) f_2(y) \qquad \textit{for all } x, y \in [0,1]^N,$$

Then there exists a probability measure ν on $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ such that

$$\nu(A \times \Omega) = \mu_1(A) \qquad (\textit{for all } A \in \mathcal{F}), \tag{2.101}$$

$$\nu(\Omega \times B) = \mu_2(B) \qquad (\textit{for all } B \in \mathcal{F}), \tag{2.102}$$

$$\nu\{(x,y) \in \Omega \times \Omega : x \geq y\} = 1. \tag{2.103}$$

Theorem 2.11 is an immediate consequence of Proposition 2.1, since if $X : \Omega \to \mathbb{R}$ is an increasing function and $E = \{(x,y) \in \Omega \times \Omega : x \geq y\}$, then

$$\int_{\Omega} X \mu_1(dx) - \int_{\Omega} X \mu_2(dx) = \int_{\Omega \times \Omega} (X(x) - X(y)) \nu(d(x,y))$$

$$= \int_{E} (X(x) - X(y)) \nu(d(x,y)) \geq 0,$$

because $X(x) \geq X(y)$ if $(x,y) \in E$.

Proof. The Proposition 1 is proved by induction on $|E| = N$. Suppose for the moment that $N \geq 2$, let $k \in E$ and put $E' = E \backslash \{k\}$. Let π_1, π_2 denote the projection of μ_1, μ_2, respectively, onto $\Omega_{I'}$. Then we have $\pi_i = g_i \lambda$ $(i = 1, 2)$, where $g_i : \Omega_{E'} \to \mathbb{R}$ are given by,

$$g_i(x) = \int_{[0,1]} f_i(x, w) dw,$$

Lemma 2.2. *Suppose that for all $x, y \in \Omega$*

$$f_1(x \vee y) f_2(x \wedge y) \geq f_1(x) f_2(y).$$

Then for all $x', y' \in \Omega'$ we have

$$g_1(x' \vee y') g_2(x' \wedge y') \geq g_1(x') g_2(y').$$

Proof. Let $K = \{(w, z) \in [0, 1] \times [0, 1] : w > z\}$, $L = \{(w, z) \in [0, 1] \times [0, 1] : w = z\}$, $M = \{(w, z) \in [0, 1] \times [0, 1] : w < z\}$. Then

$$g_1(x' \vee y') g_2(x' \wedge y') = \iint_{K \cup L \cup M} f_1(x' \vee y', w) f_2(x' \wedge y', z) dwdz. \quad (2.104)$$

$$= \iint_L f_1(x' \vee y', w) f_2(x' \wedge y', z) dwdz$$

$$+ \iint_K \{f_1(x' \vee y', w) f_2(x' \wedge y', z) + f_1(x' \vee y', z) f_2(x' \wedge y', w)\} dwdz. \quad (2.105)$$

Similarly,

$$g_1(x') g_2(y') = \iint_L f_1(x', w) f_2(y', z) dwdz$$

$$+ \iint_K \{f_1(x', w) f_2(y', z) + f_1(x', z) f_2(y', w)\} dwdz.$$

But by hypothesis we have

$$f_1(x' \vee y', w) f_2(x' \wedge y', w) \geq f_1(x', w) f_2(y', w)$$

and thus we can ignore the terms involving integrations over L. It remains to show that

$$f_1(x' \vee y', w) f_2(x' \wedge y', z) + f_1(x' \vee y', z) f_2(x' \wedge y', w)$$

$$\geq f_1(x', w)f_2(y', z) + f_1(x', z)f_2(y', w) \qquad \text{whenever } w > z.$$

Let us write

$$\begin{aligned} a &= f_1(x' \vee y', w)f_2(x' \wedge y', z), \\ b &= f_1(x' \vee y', z)f_2(x' \wedge y', w), \\ c &= f_1(x', w)f_2(y', z), \\ d &= f_1(x', z)f_2(y', w). \end{aligned}$$

Using the hypothesis, one may see that if $w > z$, then $a \geq c$, $a \geq d$ and $ab \geq cd$. We want to show that $a + b \geq c + d$; this follows from the next claim.

Claim 2.1. *Let a, b, c, d be non-negative real number with $a \geq c$, $a \geq d$ and $ab \geq cd$. Then $a + b \geq c + d$.*

If $a = 0$ then $c = d = 0$ and the result is true; thus we can assume that $a > 0$. Now $(a - c)(a - d) \geq 0$ gives $aa + cd \geq ac + ad$ and since $cd \geq ab$ we get $aa + ab \geq ac + ad$. By dividing by a, we have the result. □

Let α be a non-negative measure on $([0, 1], \mathcal{F})$, the Lebesgue σ-algebra on $[0, 1]$. Let h_1, h_2 be the densities with respect to α of probability measures γ_1, γ_2 on $([0, 1], \mathcal{F})$, and let $\bar{\alpha}$ be the measure on $([0, 1] \times [0, 1], \mathcal{F} \times \mathcal{F})$ got by projecting α onto the diagonal of $[0, 1] \times [0, 1]$; thus if $B \in \mathcal{F} \times \mathcal{F}$ then

$$\bar{\alpha}(B) = \alpha\{y \in Y : (y, y) \in B\}.$$

Define a probability measure δ on $([0, 1] \times [0, 1], \mathcal{F} \times \mathcal{F})$ by

$$\delta(x, y) = \min\{h_1(x), h_2(y)\}\bar{\alpha} + [\iint h_2'(z)d\alpha(z)]^{-1}h_1'(x)h_2'(y)\alpha \times \alpha,$$

where $h_1'(x) = [h_1(x) - h_2(x)]^+$, $h_2' = [h_2(y) - h_1(y)]^+$. ($h^+ = \max\{0, h\}$.) Note that since $h_1' + h_2 = h_2' + h_1$, we have

$$\int h_2'(z)d\alpha(z) = \int h_1'(z)d\alpha(z),$$

thus if $\int h_2'(z)d\alpha(z) = 0$ then $h_1 = h_2 = 0$ and we will leave out the second term in the definition of δ.

Claim 2.2. *Let α as above. Then*

$$\delta(A \times Y) = \gamma_1(A) \qquad \textit{(for all } A \in \mathcal{B}\textit{)} \tag{2.106}$$
$$\delta(Y \times B) = \gamma_2(B) \qquad \textit{(for all } B \in \mathcal{B}\textit{)} \tag{2.107}$$

It follows from a simple calculation.

Claim 2.3. *Suppose for all $x, y \in Y$ with $x \geq y$ we have*

$$h_1(x)h_2(y) \geq h_1(y)h_2(x).$$

Then $\delta\{(x, y) \in Y \times Y : x \geq y\} = 1$.

It is sufficient to show that $h_1'(x)h_2'(x) = 0$, unless $x \geq y$. Thus suppose there exist x, y with $x > y$ and $h_1'(y)h_2'(x) > 0$. Then $h_1(y) > h_2(y)$, $h_2(x) > h_1(x)$, and hence

$$h_1(x)h_2(y) < h_1(y)h_2(x)$$

which contradicts the hypothesis of the lemma.

Together Claims 2.2 and 2.3 give us Proposition 2.1 for the case $N = 1$; the explicit expression for δ will enable us to complete the proof in general. Let $q : [0, 1] \to \mathbb{R}$ with $q \geq 0$ and $\int q(w)dw = 1$. Define, for $i = 1, 2$,

$$F_i(x', w) = \begin{cases} \frac{f_i(x', w)}{\int f_i(x', w)} dw & if \ \int f_i(x', y)dy > 0 \\ q(w) & otherwise. \end{cases} \tag{2.108}$$

Thus F_1 (respectively, F_2) is a version of the Radon-Nikodym derivative of μ_1 (respectively, μ_2) with respect to $\pi_1 \times \lambda_1$ (respectively, $\pi_2 \times \lambda_1$), where λ_1 denotes the Lebesgue measure on $[0, 1]$.

Define $Q, R : [0, 1]^{E'} \times [0, 1]^{E'} \times [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$Q(x', y', w, z) = \min\{F_1(x', w), F_2(y', z)\} \tag{2.109}$$
$$R(x', y', w, z) = [S(x', y')]^{-1}[F_1(x', w) - F_2(y', w)]^+[F_2(y', z) - F_1(x', z)]^+, \tag{2.110}$$

where $S(x', y') = \int[F_2(y', z) - F_1(x', z)]^+dz$. As in the definition of δ, $S(x, x') = 0$ if, and only if, $F_1(x', w) = F_2(y', w)$ (for λ_1-a.e. w) and in this case we define $R(x', y', w, z) = 0$. Let $\overline{\lambda_1}$ be the measure on $([0, 1] \times [0, 1], \mathcal{F} \times \mathcal{F}_1)$ got by projecting λ_1 onto the diagonal of $[0, 1] \times [0, 1]$ and define the probability measure ν on $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ by

$$\nu = Q\nu' \times \overline{\lambda_1} + R\nu' \times \lambda_1 \times \lambda_1 \tag{2.111}$$

Claim 2.4. *ν satisfies (2.100) and (2.101), in Proposition 1.*

This is a straightforward calculation.

Claim 2.5. *ν satisfies (2.102), in Proposition 1.*

For $i = 1, 2$, let $B_i = \{x' \in [0,1]^{E'} : \int f_i(x', w)dw = 0\}$. If $x' \in B_1$, $y' \in B_2$ and $x' \geq y'$ then

$$F_1(x', w)F_2(y', z) \geq F_1(x', z)F_2(y', w) \tag{2.112}$$

, whenever $w \geq z$. As in the Claim 2.3, $R(x', y', w, z) = 0$, unless $w \geq z$. Therefore, it just remains to show that $\nu(B_1 \times [0,1] \times [0,1]^{E'}) = \nu([0,1]^{E'} \times B_2 \times [0,1]) = 0$. But

$$\nu(B_1 \times [0,1] \times [0,1]^{E'}) = \mu_1(B_1 \times [0,1]) = \int_{B_1} \int_{[0,1]} f_1(x', w)dwd\lambda_{N-1} = 0, \tag{2.113}$$

and similarly $\nu([0,1]^N \times B_2 \times [0,1]) = 0$.

□

If ρ satisfies the FKG lattice condition and A is an increasing event, then

$$\not\Vdash_A(x \vee y)\rho(x \vee y)\rho(x \wedge y) \geq \not\Vdash_A(x)\rho(x)\rho(y),$$

whence, by Theorem 2.11,

$$\mu_\rho(A)\mu_\rho(B) \leq \mu_\rho(A \cap B)$$

for all increasing A, B. Therefore, μ_ρ is positively associated. (Here consider $\frac{\not\Vdash_A \rho}{\mu_\rho(A)}$ as g with $\mu_\rho(A) > 0$, μ_ρ as f in Theorem 1 and notice that $\mu_g(B) = \mu_\rho(A \cap B)$, for measurable subsets $B \subseteq [0,1]^N$.)

Henceforth, we restrict ourselves to *positive* density functions. Arguments similar to the above are valid with ρ (assumed positive) replaced by the conditional density function ρ_J^ξ, and thus one arrives at the following:

Theorem 2.12. *Let $N \geq 1$, and let ρ be a positive density function on $\Omega = [0,1]^N$ satisfying the FKG lattice condition (2.96). Then ρ is strongly positively associated and monotonic.*

We turn now to a continuous version of the Influence Theorem. Let $N \geq 1$ and let ρ be a monotonic positive density on $\Omega = [0,1]^N$. Let $U = (U_1, \ldots, U_N)$ be the identity function on $[0,1]^N$. For an increasing subset A of Ω, let the *conditional influences* be

$$I_A(i) = \mu_\rho(A|U_i = 1) - \mu_\rho(A|U_i = 0) \qquad (i \in I). \tag{2.114}$$

Theorem 2.13 (Influence). *Let A be an increasing subset of $\Omega = [0,1]^N$, $N \geq 1$. Let ρ be a positive density function on Ω that is monotonic. There exist $i \in I$ and a constant $c \in (0,\infty)$ such that*

$$I_A(i) \geq c \min\{\mu(A), 1 - \mu(A)\} \frac{\log N}{N}. \tag{2.115}$$

Proof. First we construct an increasing event B such that $\lambda(B) = \mu(A)$, by way of a function $f : \Omega \to \Omega$. Let $x = (x_i : 1 \leq i \leq N) \in \Omega$ and write $f(x) = (f_1(x), \ldots, f_N(x))$. The first coordinate $f_1(x)$ depends on x_1 only and is defined by

$$\mu_\rho(U_1 > f_1(x)) = 1 - x_1.$$

Since the density function ρ is strictly positive, $f_1(x)$ is a continuous and strictly increasing function of x_1 (hence it is a homeomorphism between $[0,1]$ and the image of f_1). Notice that the law of $f_1(U)$ under λ is the same as that of U_1 under μ_ρ. Indeed, by considering the pushforward measures, the strict monotonicity of f_1 implies that, for all $x \in \Omega$,

$$f_1(U)_*(\lambda)(f_1(x), 1] = \lambda(U^{-1} \circ f^{-1}(f_1(x), 1]) = \lambda(x_1, 1] = 1 - x_1,$$

$$U_{1*}(\mu_\rho)(f_1(x), 1] = \mu_\rho(U_1^{-1}(f_1(x), 1]) = \mu_\rho(U_1 \geq f_1(x)) = 1 - x_1.$$

Having defined f_1, we define f_2 in terms of x_1, x_2 only by

$$\mu_\rho(U_2 > f_2(x)|U_1 = f_1(x)) = 1 - x_2.$$

The left-hand side is defined according to (2.98). It is a standard fact that $\mu_\rho(\cdot|U_1 = f_1)$ is a version of the conditional expectation $\mathbb{E}_{\mu_\rho}(\cdot|\sigma(U_1))$, where $\sigma(U_1)$ denotes the σ-algebra generated by U_1. As above, the pair $(f_1(U), f_2(U))$ has the same law under λ as does the pair (U_1, U_2) under μ_ρ. Since ρ is positive and monotonic, for each given $x_1 \in (0,1)$, $f(x)$ is a continuous and strictly increasing function of x_2.

We continue inductively. Suppose we know $f_i(x)$ for $1 \leq i < k$. Then $f_k(x)$ depends on $x_1, \ldots, x_k$ and is given by

$$\mu_\rho(U_k > f_k | U_i = f_i(x) \ for \ i \leq i < k)) = 1 - x_k.$$

Analogously, by monotonicity, f is strictly increasing and the law of $f(U)$ under λ is the same as the law of U under μ_ρ. We set $B = f^{-1}(A)$.

Let

$$J_B(i) = \lambda(B|U_i = 1) - \lambda(B|U_i = 0) \qquad (i \in I).$$

By Theorem 1.7 (BKKKL), there exists a constant $c \in (0, \infty)$, independent of the choice of N and A, such that there exists $i \in I$ with

$$J_B(i) \geq c \min\{\lambda(B), 1 - \lambda(b)\} \frac{\log N}{N}.$$

Since f is continuous and strictly increasing,

$$\mu_\rho(A|U_1 = b) = \lambda(B|f_1(U_1) = b) = \lambda(B|U_1 = b) \qquad (b = 0, 1), \tag{2.116}$$

implying that $I_A(1) = J_B(1)$. It remains to show that $I_A(j) \geq J_B(j)$ for $j \in I$. Let $j \in I, j \neq 1$. We reorder the coordinate set as $K = \{k_1, k_2, \ldots, k_N\} = \{j, 1, \ldots, j-1, j+1, \ldots, N\}$ and construct a continuous increasing function g as above, but subject to the new ordering. Let $C = g^{-1}(A)$. We claim that

$$J_C(k_1) \geq J_B(j). \tag{2.117}$$

Thus, by (2.115) , $J_C(k_1) = I_A(j)$ and $I_A(j) \geq J_B(j)$, $j \in I$ follows. It remains to prove the claim. It suffices to prove that

$$\mu_\rho(A|U_j = 1) = \lambda(C|U_{k_1} = 1) \geq \lambda(B|U_j = 1), \tag{2.118}$$

a similar argument being valid with 1 replaced by 0 and the inequality reversed.

Conditioned on $[U_j = 1]$, $g_1(U) \geq f_1(U)$, for f_1 and g_1 depend only on U_1 and U_j, respectively, and f (hence, also g) are strictly increasing functions. Under the same conditioning, let $1 \leq r < j$, and assume it has already been proved that $f_i(x) \leq g_i(x)$ for $x \in \Omega$ and $1 \leq i < r$. We claim that, for $x \in \Omega$,

$$\mu_\rho(U_r > \xi | U_i = f_i(x) \ for \ 1 \leq i < r) \tag{2.119}$$

$$\leq \mu_\rho(U_r > \xi | U_j = 1, U_i = g_i(x) \ for \ 1 \leq i < r)) \qquad \xi \in [0, 1].$$

By monotonicity,

$$\mu_{\rho,J}(\cdot|U_j = u, U_i = f_i(x)\ for\ 1 \le i < r) \tag{2.120}$$
$$\le_{st} \mu_{\rho,J}(\cdot|U_j = 1, U_i = g_i(x)\ for\ 1 \le i < r), \qquad u \in [0,1].$$

The left-hand side of (2.120) is a version of conditional expectation of the conditional measure $\mu_{\rho,J}(\cdot|U_i = f_i(x)\ for\ 1 \le i < r)$, given $\sigma(U_j)$. By averaging over the value of u in (2.120), we obtain (2.119). Therefore, $f_r(x) \le g_r(x)$, $x \in \Omega$, and we have $f \le g$. Hence,

$$\lambda(C|U_j = 1) = \lambda(g(U) \in A|U_j = 1) \ge \lambda(f(U) \in A|U_j = 1) = \lambda(B|U_j = 1).$$

Inequality (2.118) has been proved and (2.117) follows, which completes the proof. □

Unlike the discrete setting, Theorem 2.13 does not imply a sharp-threshold. Any density function ρ on $[0,1]^N$ may be used to generate a parametric family $(\rho_p : 0 < p < 1)$ of densities given by

$$\rho_p(x) = \frac{1}{Z_{\rho,p}}\rho(x)\prod_{i=1}^{N} p^{x_i}(1-p)^{1-x_i} \qquad (x = (x_1, \dots, x_N) \in [0,1]^N),$$

and we write $\mu_p = \mu_{\rho,p}$. Let A be an increasing subset of $[0,1]^N$. The proof of Theorem 2.9 may be adapted to this setting (by using $X = 1_A$ and replacing $|\eta|$ by $\sum_{i=1}^{N} U_i$) to obtain that

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)}\sum_{i=1}^{N} cov_p(U_i, 1_A),$$

where $U = (U_1, \dots, U_N)$ is the identity function on $[0,1]^N$, and cov_p denotes covariance with respect to μ_p.

Let ρ be a nonzero constant function, so that μ_p is Lebesgue measure. As above, let $p \in (0,1)$ and let $Y_1, \dots, Y_N$ be independent random variables taking values in [0,1] with common density function

$$\rho_p(x) = \begin{cases} \frac{\log p/(1-p)}{2p-1}p^x(1-p)^{1-x}, & if\ p \ne \frac{1}{2}, x \in (0,1), \\ 1, & if\ p = \frac{1}{2}, x \in (0,1). \end{cases}$$

Notice that the joint density function, $\rho_p(x) = \prod_{i=1}^N \rho_p(x_i), x = (x_1, \ldots, x_N) \in [0,1]^N$, satisfies the FKG lattice condition and therefore is monotonic: for $x, y \in [0,1]^N$,

$$\rho_p(x \vee y)\rho_p(x \wedge y) \geq [\frac{\log p/(1-p)}{2p-1}]^{2N} \prod_{i=1}^N p^{x_i+y_i}(1-p)^{2-x_i-y_i}$$

$$= \rho_p(x)\rho_p(y).$$

Let $A = (N_{-1}, 1]^N$. Then,

$$\mu_p(A) = \int_A \rho_p(x)\lambda(dx) = \begin{cases} [\frac{\log[p/(1-p)]}{2p-1} \int_{\frac{1}{N}}^1 p^x(1-p)^{1-x}dx]^N, & if \ p \neq \frac{1}{2}, \\ (1-\frac{1}{N})^N, & if \ p = \frac{1}{2}. \end{cases}$$

By writing $\pi = \frac{p}{1-p}$, for $p \neq \frac{1}{2}$ and setting $u = \pi^x$, we have that $x(\log(\pi)) = \log(u)$, and $dx = (\log(\pi))^{-1}u^{-1}du$. Thus

$$\int_{\frac{1}{N}}^1 \pi^x dx = \int_{\pi^{1/N}}^{\pi} du = \pi - \pi^{1/N},$$

and by noting that $\pi - 1 = (2p-1)/(1-p)$, one may see that

$$\mu_p(A) = \begin{cases} (1 - \frac{\pi^{1/N}-1}{\pi-1})^N, & if \ p \neq \frac{1}{2} \\ (1-\frac{1}{N})^N, & if \ p = \frac{1}{2}, \end{cases}$$

Therefore, as $N \to \infty$,

$$\mu_p(A) \to \begin{cases} \pi^{-1/(\pi-1)}, & if \ p \neq \frac{1}{2}, \\ \exp^{-1}, & if \ p = \frac{1}{2}. \end{cases}$$

In addition,

$$cov_{1/2}(U_i, 1_A) = \frac{1}{N}(1-\frac{1}{N})^{N-1} \sim \frac{\exp^{-1}}{N}.$$

Theorem 2.13 may be applied to the event A, but there is no sharp threshold for $\mu_p(A)$. This situation diverges from that of the discrete setting at the point where a lower bound for the conditional influence $I_A(i)$ is used to calculate a lower bound for the covariance $cov_p(U_i, 1_A)$.

Chapter 3

The Random Cluster model

3.1 Introduction

Let $G = (V, E)$ be a finite graph, usually assumed to have neither loops nor multiple edges (otherwise, the property is stressed). An edge e having endvertices x and y is written as $e = \langle x, y \rangle$. As in the previous Chapter, we consider as state space the set $\Omega = \{0, 1\}^E$ of which are 0/1-vectors $\omega = (\omega(e) : e \in E)$. We call the edge e *open* (in ω) if $\omega(e) = 1$, and *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges. There is a one-one correspondence between vectors $\omega \in \Omega$ and subsets $F \subseteq E$, given by $F = \eta(\omega)$. Let $k(\omega)$ be the number of connected components (or open clusters) of the graph $(V, \eta(\omega))$, and note that $k(\omega)$ includes a count of isolated vertices, that is, of vertices incident to no open edge.

A *random cluster measure* on G has two parameters, an edge-weight p and a cluster-weight q, satisfying $p \in [0, 1]$ and $q \in (0, \infty)$, and is defined as the measure $\phi_{p,q}$ on the measurable pair $(\Omega, \mathcal{F})$ given by

$$\phi_{p,q}(\omega) = \frac{1}{Z(p,q)} \Big\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big\} q^{k(\omega)} \qquad (\omega \in \Omega) \tag{3.1}$$

where the *partition function* or *normalizing constant*, $Z(p, q)$ is given by

$$Z(p,q) = \sum_{\omega \in \Omega} \Big\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big\} q^{k(\omega)}. \tag{3.2}$$

Sometimes $\phi_{p,q}$ is written as $\phi_{G,p,q}$, when the choice of graph G is to be stressed.

This measure differs from product measure due to the term $q^{k(\omega)}$. Note the difference between the cases $q \le 1$ and $q \ge 1$: the former favours fewer clusters, whereas the latter favours a larger number of clusters. When $q = 1$ (we write $\phi_{G,p}$ or ϕ_p), edges are open/closed independently of one another. This special case corresponds to bond percolation and random graphs. Perhaps the most important values of q are the integers, since the random-cluster model with $q \in \{2, 3, \cdots\}$ corresponds to the Potts model with q local states.

3.2 Conditional probabilities

For $e = \langle x, y \rangle \in E$, the expression $G \backslash e$ (respectively, $G.e$) denotes the graph obtained from G by deleting (respectively, contracting) the edge e. We write $\Omega_{\langle e \rangle} = \{0, 1\}^{E \backslash \{e\}}$ and, for $\omega \in \Omega$, we define $\omega_{\langle e \rangle} \in \Omega_{\langle e \rangle}$ by

$$\omega_{\langle e \rangle}(f) = \omega(f) \qquad (f \in E, f \neq e). \tag{3.3}$$

Let K_e ($e = \langle x, y \rangle$) denote the event that x and y are joined by an open path not using e.

Theorem 3.1 (Conditional probabilities). *Let $p \in (0, 1)$, $q \in (0, \infty)$.*

(a) We have for $e \in E$ that

$$\phi_{G,p,q}(\omega \mid \omega(e) = j) = \begin{cases} \phi_{G \backslash e,p,q}(\omega_{\langle e \rangle}) & if\ j = 0, \\ \\ \phi_{G.e,p,q}(\omega_{\langle e \rangle}) & if\ j = 1, \end{cases} \tag{3.4}$$

and

$$\phi_{G,p,q}(\omega(e) = 1 \mid \omega_{\langle e \rangle}) = \begin{cases} p & if\ \omega_{\langle e \rangle} \in K_e, \\ \frac{p}{p+q(1-p)} & if\ \omega_{\langle e \rangle} \notin K_e. \end{cases} \tag{3.5}$$

(b) Conversely, if ϕ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (3.5) for all $\omega \in \Omega$ and $e \in E$, then $\phi = \phi_{G,p,q}$.

Proof. (a) By expanding the conditional probability,

$$\phi_{G,p,q}(\omega \mid \omega(e) = j) = \begin{cases} \phi_{G,p,q}(\omega_e)/\phi_{G,p,q}(\overline{J_e}) & if\ j = 0, \\ \\ \phi_{G,p,q}(\omega^e)/\phi_{G,p,q}(J_e) & if\ j = 1, \end{cases} \qquad (\omega \in \Omega) \tag{3.6}$$

where $J_e = \{\omega \in \Omega : \omega(e) = 1\}$, and ω_e, ω^e are given by

$$\omega^e(f) = \begin{cases} \omega(f) & if\ f \neq e, \\ 1 & if\ f = e, \end{cases} \qquad (f \in E) \tag{3.7}$$

$$\omega_e(f) = \begin{cases} \omega(f) & if\ f \neq e, \\ 0 & if\ f = e, \end{cases} \qquad (f \in E). \tag{3.8}$$

Similarly,

$$\phi_{G,p,q}(\omega(e) = 1 \mid \omega_{\langle e \rangle}) = \frac{\phi_{G,p,q}(\omega^e)}{\phi_{G,p,q}(\omega^e) + \omega_{G,p,q}(\omega_e)} \tag{3.9}$$

$$= \frac{[p/(1-p)]^{|\eta(\omega^e)|} q^{k(\omega^e)}}{[p/(1-p)]^{|\eta(\omega^e)|} q^{k(\omega^e)} + [p/(1-p)]^{|\eta(\omega_e)|} q^{k(\omega_e)}} \tag{3.10}$$

$$= \begin{cases} \frac{p/(1-p)}{[p/(1-p)]+1} & if\ \omega_e \in K_e, \\ \\ \frac{p/(1-p)}{[p/(1-p)]+q} & if\ \omega_e \notin K_e, \end{cases} \tag{3.11}$$

for $|\eta(\omega^e)| - |\eta(\omega_e)| = 1$. Regarding the difference $k(\omega_e) - k(\omega^e)$, notice that by closing the edge e, if $\omega_e \in K_e$, then the number of open clusters remains the same; otherwise, this quantity increases by one.

(b) The claim follows from the fact that a strictly positive probability measure ϕ is specified uniquely by the conditional probabilities $\phi(\omega(e) = 1 \mid \omega_{\langle e \rangle})$, $\omega \in \Omega$, $e \in E$. Indeed, let ϕ and ψ be two such probability measures which agree on conditionings as above (if $E = \{e\}$, then $\phi = \psi$). This condition implies that $\phi(\omega^f_{\langle e \rangle} | \omega_{\langle e,f \rangle}) = \psi(\omega^f_{\langle e \rangle} | \omega_{\langle e,f \rangle})$, and immediately that $\phi(\omega_{\langle e \rangle f} | \omega_{\langle e,f \rangle}) = \psi(\omega_{\langle e \rangle f} | \omega_{\langle e,f \rangle})$. Or more generally, for any ordering of edges in E, $e_1, \cdots e_N$, $\phi(\omega_{\langle e_1,\ldots,e_{k-1} \rangle}(e_k) = j_k | \omega_{\langle e_1,\ldots,e_k \rangle}) = \psi(\omega_{\langle e_1,\ldots,e_{k-1} \rangle}(e_k) = j_k | \omega_{\langle e_1,\ldots,e_k \rangle})$, for $j_k = 0, 1$ and $2 \leq k \leq N-1$. Thus, since for any $\omega \in \Omega$,

$$\begin{aligned} \phi(\omega) =& \phi(\omega(e_1) = j_1 | \omega_{\langle e_1 \rangle}) \phi(\omega_{\langle e_1 \rangle}(e_2) = j_2 | \omega_{\langle e_1,e_2 \rangle}) \ldots \\ & \phi(\omega_{\langle e_1,\ldots,e_{N-2} \rangle}(e_{N-1}) = j_{N-1} | \omega_{\langle e_1,\ldots,e_{N-1} \rangle}) \phi(\omega_{\langle e_1,\ldots,e_{N-1} \rangle}), \end{aligned} \tag{3.12}$$

for $j_1, \cdots, j_{N-1} \in \{0, 1\}$, we conclude that $\phi = \psi$.

□

The effect of conditioning on the state open or closed of an edge e is to replace the measure $\phi_{G,p,q}$ by the random-cluster measure on the respective graph $G\backslash e$ or $G.e$. In addition, the conditional probability that e is open, given the configuration elsewhere, depends only on whether or not K_e occurs, and is then given by the stated formula. By (3.5),

$$0 < \phi_{G,p,q}(\omega(e) = 1 \mid \omega_{\langle e\rangle}) < 1 \qquad (e \in E,\ p \in (0,1), q \in (0,\infty)) \tag{3.13}$$

Thus, given $\omega_{\langle e\rangle}$, each of the two possible states of e occurs with a strictly positive probability. This fact is known as the *finite-energy property*, and is related to the property of so-called *insertion tolerance*.

Let $\xi \in \Omega$, $F \subseteq E$, and let Ω_F^ξ be the subset of Ω containing all configurations ψ satisfying $\psi(e) = \xi(e)$ for all $e \notin F$. We define the random-cluster measure $\phi_{F,p,q}^\xi$ (on the finite graph (V_F, F) with boundary condition ξ) on $(\Omega, \mathcal{F})$ by

$$\phi_{F,p,q}^\xi = \begin{cases} \frac{1}{Z_F^\xi(p,q)}\Big\{\prod_{e\in F} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\} q^{k(\omega,F)} & if\ \omega \in \Omega_F^\xi, \\ 0 & otherwise, \end{cases} \tag{3.14}$$

where $k(\omega, F)$ is the number of components of the graph $(G, \eta(\omega))$ that intersect the set of endvertices of F, and

$$Z_F^\xi(p,q) = \sum_{\omega\in\Omega_F^\xi} \Big\{\prod_{e\in F} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\} q^{k(\omega,F)}. \tag{3.15}$$

Note that $\phi_{F,p,q}^\xi(\Omega_F^\xi) = 1$.

Now, we introduce some notation. For $W \subseteq V$, let E_W denote the set of edges of G having both endvertices in W. We write $\mathcal{F}_W$ (respectively, $\mathcal{T}_W$) for the smallest σ-field of $\mathcal{F}$ with respect to which each of the random variables $\omega(e)$, $e \in E_W$ (respectively, $e \notin E_W$), is measurable. The notation $\mathcal{F}_F, \mathcal{T}_F$ is to be interpreted similarly for $F \subseteq E$. The intersection of the $\mathcal{T}_F$ over all finite sets F is called the *tail* σ-field and is denoted by $\mathcal{T}$. Sets in $\mathcal{T}$ are called *tail events*.

Theorem 3.2. *Let $p \in [0,1]$, $q \in (0,\infty)$, and $F \subseteq E$. Let X be a random variable that is $\mathcal{F}_F$-measurable. Then*

$$\mathbb{E}_{\phi_{G,p,q}}[X|\mathcal{T}_F](\xi) = \mathbb{E}_{\phi_{F,p,q}^\xi}[X] \tag{3.16}$$

Proof. This holds by repeated application of (3.4), with one application for each edge not belonging in F. □

In other words, given the states of edges not belonging to F, the conditional measure on F is a random-cluster measure subject to the retention of open connections of ξ using edges not belonging to F.

3.3 Positive association and comparison inequalities

Let $\phi_{p,q}$ denote the random-cluster measure on G with parameters p and q. We will see that $\phi_{p,q}$ satisfies the FKG lattice condition whenever $q \geq 1$, and we arrive thus at the following conclusion.

Theorem 3.3. *Let $p \in (0,1)$ and $q \in [1,\infty)$.*

a The random-cluster measure $\phi_{p,q}$ is strictly positive and satisfies the FKG lattice condition.

b The random-cluster measure $\phi_{p,q}$ is strongly positively-associated, and in particular

$$\mathbb{E}_{\phi_{p,q}}(XY) \geq \mathbb{E}_{\phi_{p,q}}(X)\mathbb{E}_{\phi_{p,q}}(Y) \qquad for\ increasing\ X, Y : \Omega \to \mathbb{R}, \tag{3.17}$$

$$\phi_{p,q}(A \cap B) \geq \phi_{p,q}(A)\phi_{p,q}(B) \qquad for\ increasing\ A, B \in \mathcal{F}. \tag{3.18}$$

Proof. Let $p \in (0,1)$ and $q \in [1,\infty)$. Part (b) follows from (a) and Theorem 2.7. It is elementary that $\phi_{p,q}$ is strictly positive. We now check as required that $\phi_{p,q}$ satisfies the FKG lattice condition. Since the set $\eta(\omega)$ of open edges in a configuration ω satisfies

$$|\eta(\omega_1 \wedge \omega_2)| + |\eta(\omega_1 \vee \omega_2)| = |\eta(\omega_1)| + |\eta(\omega_2)| \qquad (\omega_1, \omega_1 \in \Omega), \tag{3.19}$$

it suffices, on taking logarithms, to prove that

$$k(\omega_1 \wedge \omega_2) + k(\omega_1 \vee \omega_2) \geq k(\omega_1) + k(\omega_2) \qquad (\omega_1, \omega_1 \in \Omega). \tag{3.20}$$

By Theorem 2.6, we may restrict our attention to incomparable pairs ω_1, ω_2 that differ on exactly two edges. There must then exist distinct edges $e, f \in E$ and a configuration $\omega \in \Omega$ such that $\omega_1 = \omega_f^e$, $\omega_2 = \omega_e^f$. As in the proof of Theorem 2.7, we omit reference to the states of edges other than e and f, and we write $\omega_1 = 10$ and $\omega_2 = 01$. Let D_f be the indicator function of the event that the endvertices of f are connected by no open path of $E \backslash \{f\}$. Since D_f is a decreasing random variable, we have that $D_f(10) \leq D_f(00)$. Therefore,

$$k(10) - k(11) = D_f(10) \leq D_f(00) = k(00) - k(01), \tag{3.21}$$

which implies (20). □

In general, $\phi_{p,q}$ is not positively associated when $q \in (0,1)$, as illustrated in the following example. Let G be the graph containing just two vertices and having exactly two parallel edges e and f joining these vertices. It is a straightforward computation that

$$\phi_{p,q}(J_e \cap J_f) - \phi_{p,q}(J_e)\phi_{p,q}(J_f) = \frac{p^2 q^2 (q-1)(1-p)^2}{Z(p,q)^2}, \tag{3.22}$$

where J_g is the event that g is open. This is strictly negative if $0 < p, q < 1$.

Now, restricting to the case $G = (V, E)$ is a finite graph, we present the comparison inequalities.

Theorem 3.4.

$$\phi_{p_1,q_1} \leq_{st} \phi_{p_2,q_2} \qquad \textit{if } q_1 \geq q_2, q_1 \geq 1 \textit{ and } p_1 \leq p_2. \tag{3.23}$$

Proof. We may assume that $p_1, p_2 \in (0,1)$, since the other cases are straightforward. Let $X : \Omega \to \mathbb{R}$ be increasing. Then

$$\mathbb{E}_{\phi_{p_2,q_2}}[X] \tag{3.24}$$

$$= \frac{1}{Z(p_2,q_2)} \sum_{\omega \in \Omega} X(\omega) p_2^{|\eta(\omega)|} (1-p)^{|E \backslash \eta(\omega)|} q_2^{k(\omega)} \tag{3.25}$$

$$= \Big(\frac{1-p_2}{1-p_1}\Big)^{|E|} \frac{1}{Z(p_2,q_2)} \sum_{\omega \in \Omega} X(\omega) Y(\omega) p_1^{|\eta(\omega)|} (1-p_1)^{|E \backslash \eta(\omega)|} q_1^{k(\omega)} \tag{3.26}$$

$$= \Big(\frac{1-p_2}{1-p_1}\Big)^{|E|} \frac{Z(p_1,q_1)}{Z(p_2,q_2)} \mathbb{E}_{\phi_{p_1,q_1}}[XY] \tag{3.27}$$

$$\tag{3.28}$$

where

$$Y(\omega) = \left(\frac{q_2}{q_1}\right)^{k(\omega)} \left(\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right)^{|\eta(\omega)|}. \tag{3.29}$$

Setting $X = 1$, we obtain

$$\mathbb{E}_{\phi_{p_2,q_2}}[1] = 1 = \left(\frac{1-p_2}{1-p_1}\right)^{|E|} \frac{Z(p_1,q_1)}{Z(p_2,q_2)} \mathbb{E}_{\phi_{p_1,q_1}}[Y], \tag{3.30}$$

whence, on dividing,

$$\mathbb{E}_{\phi_{p_2,q_2}}[X] = \frac{\phi_{p_1,q_1}[XY]}{\phi_{p_1,q_1}[Y]}. \tag{3.31}$$

Assume now that the conditions of (3.23) hold. Since $k(\omega)$ is a decreasing function and $|\eta(\omega)|$ is increasing, we have that Y is increasing. Since $q_1 \geq 1$, ϕ_{p_1,q_1} is positively associated, so

$$\mathbb{E}_{\phi_{p_1,q_1}}[XY] \geq \mathbb{E}_{\phi_{p_1,q_1}}[X]\mathbb{E}_{\phi_{p_1,q_1}}[Y], \tag{3.32}$$

and (3.31) yields $\mathbb{E}_{\phi_{p_2,q_2}}[X] \geq \mathbb{E}_{\phi_{p_1,q_1}}[X]$. Claim (3.23) follows.

□

3.4 Differential formulae and sharp thresholds

One way of estimating the probability of an event A is via an estimate of its derivative $d\phi_{p,q}(A)/dp$. For $\omega \in \Omega$, let $|\eta| = |\eta(\omega)| = \sum_{e\in E} \omega(e)$ be the number of open edges of ω as usual, and $k = k(\omega)$ the number of open clusters.

Theorem 3.5. *Let $p \in (0,1)$, $q \in (0,\infty)$, and let $\phi_{p,q}$ be the corresponding random-cluster measure on a finite graph $G = (V,E)$. We have that*

$$\frac{d}{dp}\mathbb{E}_{\phi_{p,q}}(X) = \frac{1}{p(1-p)} cov_{p,q}(|\eta|, X), \tag{3.33}$$

$$\frac{d}{dq}\mathbb{E}_{\phi_{p,q}}(X) = \frac{1}{q} cov_{p,q}(k, X), \tag{3.34}$$

for any random variable $X : \Omega \to \mathbb{R}$, where $cov_{p,q}$ denotes covariance with respect to $\phi_{p,q}$.

Proof. The first formula was proved for Theorem 2.9 and the second is obtained in a similar fashion. □

In most applications, we set $X = 1_A$, the indicator function of some given event A, and we obtain that

$$\frac{d}{dp}\phi_{p,q}(A) = \frac{\mathbb{E}_{\phi_{p,q}}(1_A|\eta|) - \phi_{p,q}(A)\mathbb{E}_{\phi_{p,q}}(|\eta|)}{p(1-p)} \tag{3.35}$$

with a similar formula for the derivative with respect to q.

Now we present two examples of Theorem 4 which result in monotonicities valid for all $q \in (0,\infty)$. Let $h : \mathbb{R} \to \mathbb{R}$ be non-decreasing. On setting $X = h(|\eta|)$, we have from (3.33) that

$$\frac{d}{dp}\mathbb{E}_{\phi_{p,q}}(X) = \frac{1}{p(1-p)} cov_{p,q}(|\eta|, h(|\eta|)) \geq 0. \tag{3.36}$$

In the special case $h(x) = x$, we deduce that the mean number of open edges is a non-decreasing function of p, for all $q \in (0,\infty)$. Similarly, by (3.34), for non-decreasing h,

$$\frac{d}{dq}\mathbb{E}_{\phi_{p,q}}(h(k)) = \frac{1}{q} cov_{p,q}(k, h(k)) \geq 0. \tag{3.37}$$

This time we take $h = -1_{(-\infty,1]}$, so that $-h$ is the indicator function of the event that the open graph $(V, \eta(\omega))$ is connected. We deduce that the probability of connectedness is a decreasing function of q on the interval $(0,\infty)$.

Let $q \in [1,\infty)$. Since $\phi_{p,q}$ satisfies the FKG lattice condition, it is monotonic. Let $\mathcal{A}$ be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of the graph $G = (V, E)$. We call E $\mathcal{A}$-*transitive* if $\mathcal{A}$ acts transitively on E.

Theorem 3.6 (Sharp threshold)**.** *Let $A \in \mathcal{F}$ be an increasing event, and suppose there exists a subgroup $\mathcal{A}$ of $Aut(G)$ such that E is $\mathcal{A}$-transitive and A is $\mathcal{A}$-invariant. Then, for $p \in (0,1)$ and $q \in [1,\infty)$, there exists an absolute constant $c \in (0,\infty)$ such that*

$$\frac{d}{dp}\phi_{p,q}(A) \geq C \min\{\phi_{p,q}(A), 1 - \phi_{p,q}(A)\} \log |E|, \tag{3.38}$$

where

$$C = c \min\left\{1, \frac{q}{\{p + q(1-p)\}^2}\right\} \tag{3.39}$$

Proof. With $\mathcal{A}$ as in the theorem, $\phi_{p,q}$ is $\mathcal{A}$-invariant since $\mathcal{A} \subseteq \mathrm{Aut}(G)$. The claim is a consequence of Theorem (2.10) on noting from (3.5) that

$$\frac{\phi_{p,q}(J_e)\phi_{p,q}(\overline{J_e})}{p(1-p)} \geq \min\left\{1, \frac{q}{\{p+q(1-p)\}^2}\right\} \qquad (e \in E). \tag{3.40}$$

□

Since $q \geq 1$, (3.38) implies that

$$\frac{d}{dp}\phi_{p,q}(A) \geq \frac{c}{q}\min\{\phi_{p,q}(A), 1-\phi_{p,q}(A)\}\log|E|, \tag{3.41}$$

an inequality that may be integrated directly. Let $p_1 = p_1(A,q) \in (0,1)$ be chosen such that $\phi_{p_1,q}(A) \geq \frac{1}{2}$. Note that $\phi_{p,q}(A) \geq \frac{1}{2}$ for $p \in (p_1, p_2)$ (by comparison inequalities). Then

$$-\frac{d}{dp}\log\left(1-\phi_{p,q}(A)\right) \geq \frac{c}{q}\log|E| \qquad (p \in (p_1, 1)) \tag{3.42}$$

and hence, by integration,

$$\phi_{p,q}(A) \geq 1 - \frac{1}{2}|E|^{-c(p-p_1)/q} \qquad (p \in (p_1,1),\ q \in [1,\infty)) \tag{3.43}$$

whenever the conditions of Theorem 3.6 are satisfied. If in addition $p_1 \geq \sqrt{q}/(1+\sqrt{q})$, then $C = c$, and hence

$$\phi_{p,q}(A) \geq 1 - \frac{1}{2}|E|^{-c(p-p_1)} \qquad (p \in (p_1,1)), \tag{3.44}$$

under the condition $\phi_{p_1,q}(A) \geq \frac{1}{2}$. As an application of this inequality, we derive, in the next section, a lower bound for the probability of an open crossing of a rectangle of $\mathbb{Z}^2$.

Now, we present an extension of the sharp-threshold theorem for monotonic probability measures applied to increasing events (Theorem 2.10) with no assumption of symmetry. In what follows, μ is a positive measure on $\Omega = \{0,1\}^E$, $|E| = N$, ($\mu(\omega) > 0$, $\omega \in \Omega$) satisfying the FKG lattice condition, μ_p, for $p \in (0,1)$, is the probability measure given by

$$\mu_p(\omega) = \frac{1}{Z_p}\Big\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\}\mu(\omega) \qquad (\omega \in \Omega),$$

and $J_{A,p}(e) = \mu_p(A|J_e = 1) - \mu_p(A|J_e = 0)$ is the conditional influence of the element $e \in E$ on the event A.

Theorem 3.7. *There exists a constant $c > 0$ such that for any increasing event $A \neq \emptyset,\ \Omega$,*

$$\frac{d}{dp}\mu_p(A) \geq \frac{c\xi_p}{p(1-p)}\mu_p(A)(1-\mu_p(A))\log[1/(2m_{A,p})], \qquad (3.45)$$

where $m_{A,p} = max_{e\in E} J_{A,p}(s)$ and $\xi_p = min_{e\in E}[\mu_p(J_e)(1-\mu_p(J_e))]$.

Proof. It is proved (in Theorem 2.9) that

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)}\sum_{e\in E}\mu_p(e)(1-\mu_p(e))J_{A,p}(e). \qquad (3.46)$$

Let $K = [0,1]^E$ be the "continuous" cube, endowed with Lebesgue measure λ, and let B be an increasing subset of K. The influence $I_B(e)$ of an element e is given as

$$I_B(e) = \lambda(1_B(\psi^e) \neq 1_B(\psi_e)), \qquad (3.47)$$

where ψ^e (respectively, ψ_e) is the member of K obtained from $\psi \in K$ by setting $\psi(e) = 1$ (respectively, $\psi(e) = 0$). We know that there exists a constant $c > 0$, independent of all other quantities, such that for any increasing event $B \subseteq K$,

$$\sum_{e\in E} I_B(e) \geq c\lambda(B)(1-\lambda(B))\log[1/(2m_B)], \qquad (3.48)$$

where $m_B = max_{e\in E} I_B(e)$.

It is shown in the proof of the Theorem 2.8 that there exists an increasing subset B of K such that $\mu_p(A) = \lambda(B)$, and $J_{A,p}(e) \geq I_B(e)$ for all $e \in E$ (see (2.72)). Inequality (3.45) follows by (3.46) and (3.48). □

Corollary 3.1. *In the notation of Theorem 3.7,*

$$\mu_{p_1}(A)[1-\mu_{p_2}(A)] \geq \kappa^{B(p_2-p_1)} \qquad (0 < p_1 \leq p_2 < 1), \qquad (3.49)$$

where

$$B = \inf_{p\in(p_1,p_2)}\left\{\frac{c\xi_p}{p(1-p)}\right\}, \qquad \kappa = 2\sup_{p\in(p_1,p_2),e\in E} J_{A,p}(e). \qquad (3.50)$$

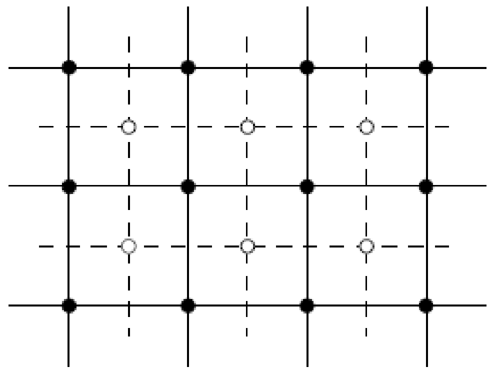

Figure 3.1: The planar dual of the square lattice [10].

Proof. By (3.45),

$$\left(\frac{1}{\mu_p(A)} + \frac{1}{1-\mu_p(A)}\right)\mu_p'(A) \geq B\log(\kappa^{-1}) \qquad (p_1 < p < p_2), \tag{3.51}$$

whence, on integrating over (p_1, p_2),

$$\frac{\mu_{p_2}(A)}{1-\mu_{p_2}(A)} \Big/ \frac{\mu_{p_1}(A)}{1-\mu_{p_1}(A)} \geq \kappa^{-B(p_2-p_1)}. \tag{3.52}$$

The claim follows. □

3.5 Planar duality

Let $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ be the integers and $\mathbb{Z}^2$ the set of all 2-vectors $x = (x_1, x_2)$ of integers. We turn $\mathbb{Z}^2$ into a graph by placing an edge between any two vertices x, y with $|x - y| = 1$, where

$$|z| = |z_1| + |z_2|, \qquad (z \in \mathbb{Z}). \tag{3.53}$$

We write $\mathbb{E}^2$ for the set of such edges and $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ for the ensuing graph.

A graph is callled *planar* if it may be embedded in $\mathbb{R}^2$ in such a way that two edges intersect only at a common endvertex. Let $G = (V, E)$ be a planar (finite or infinite) graph embedded in $\mathbb{R}^2$. We obtain its dual graph $G_d = (V_d, E_d)$ as follows. We place a dual vertex within each face of G, including any infinite face of G if such exist. For each $e \in E$ we place a dual

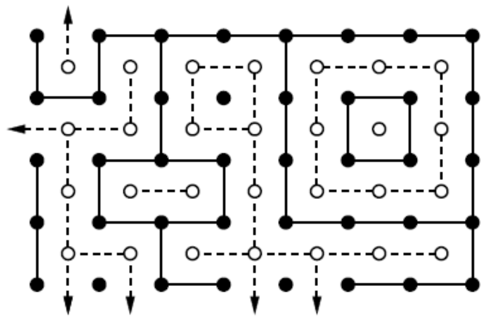

Figure 3.2: A primal configuration ω, with solid lines and vertices, and its dual configuration ω_d, with dashed lines and hollow vertices [10].

edge $e_d = \langle x_d, y_d \rangle$ joining the two dual vertices lying in the two faces of G abutting e; if these two faces are the same, then $x_d = y_d$ and e_d is a loop. Thus, V_d is in one-one correspondence with the set of faces of G, and E_d is in one-one correspondece with E. It is clear that the dual $\mathbb{L}_d^2$ of the square lattice $\mathbb{L}^2$ is isomorphic to $\mathbb{L}^2$. See Figure 3.1.

Suppose that G is finite. A configuration $\omega \in \Omega = \{0,1\}^E$ gives rise to a dual configuration $\omega_d \in \Omega_d = \{0,1\}^{E_d}$ given by $\omega_d(e_d) = 1 - \omega(e)$. That is, e_d is declared open if and only if e is closed. As before, to each configuration ω_d there corresponds the set $\eta(\omega_d) = \{e_d \in E_d : \omega_d(e_d) = 1\}$ of its 'open edges', so that $\eta(\omega_d)$ is in one-one correspondence with $E \backslash \eta(\omega)$. Let $f(\omega_d)$ be the number of faces of the graph $(V_d, \eta(\omega_d))$, including the unique infinite face. Note that each face of the dual graph corresponds to a unique component of the primal graph lying 'just within' (see Figure 3.2). The faces of $(V_d, \eta(\omega_d))$ are in one-one correspondence with the components of $(V, \eta(\omega))$; therefore

$$f(\omega_d) = k(\omega). \qquad (3.54)$$

We shall make use of Euler's formula, namely

$$k(\omega) = |V| - |\eta(\omega)| + f(\omega) - 1 \qquad (\omega \in \Omega) \qquad (3.55)$$

and we note also for later use that

$$|\eta(\omega)| + |\eta(\omega_d)| = |E|. \qquad (3.56)$$

Let $q \in (0, \infty)$ and $p \in (0,1)$. The random-cluster measure on G is given by

$$\phi_{G,p,q}(\omega) \propto (\frac{p}{1-p})^{|\eta(\omega)|} q^{k(\omega)} \qquad (\omega \in \Omega), \qquad (3.57)$$

where the constant of proportionality depends on G, p and q. Therefore,

$$\phi_{G,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{-|\eta(\omega_d)|} q^{f(\omega_d)} \qquad (\textit{by } (3.54) \textit{ and } (3.56)) \tag{3.58}$$

$$\propto \left(\frac{q(1-p)}{p}\right)^{|\eta(\omega_d)|} q^{k(\omega_d)} \quad (\textit{by } (3.55) \textit{ applied to } \omega_d) \tag{3.59}$$

$$\propto \phi_{G,p_d,q}(\omega_d), \tag{3.60}$$

where the dual parameter p_d is given by

$$\frac{p_d}{1-p_d} = \frac{q(1-p)}{p}. \tag{3.61}$$

Note that the value of p_d satisfies $(p_d)_d = p$. Since (3.58) involves probability measures, we deduce that

$$\phi_{G,p,q}(\omega) = \phi_{G_d,p_d,q}(\omega_d) \qquad (\omega \in \Omega). \tag{3.62}$$

It will later be convenient to work with the edge-parameter

$$x = \frac{q^{-\frac{1}{2}}p}{1-p}, \tag{3.63}$$

for which the primal/dual transformation (3.61) becomes

$$x x_d = 1. \tag{3.64}$$

The unique fixed point of the mapping $p \mapsto p_d$ is easily seen from (3.61) to be the *self-dual point* $p_{sd}(q)$ given by

$$p_{sd} = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{3.65}$$

We note that

$$\phi_{G,p_{sd}(q),q}(\omega) \propto \ q^{\frac{1}{2}|\eta(\omega)|+k(\omega)} \propto q^{\frac{1}{2}(k(\omega_d)+k(\omega))}, \tag{3.66}$$

by (3.54)-(3.55). This representation at the self-dual point $p_{sd}(q)$ highlights the duality of measures.

When we keep track of the constants of proportionality in (3.58), we find that the partition function

$$Z_G(p,q) = \sum_{\omega \in \Omega} p^{|\eta(\omega)|}(1-p)^{|E \setminus \eta(\omega)|} q^{k(\omega)} \tag{3.67}$$

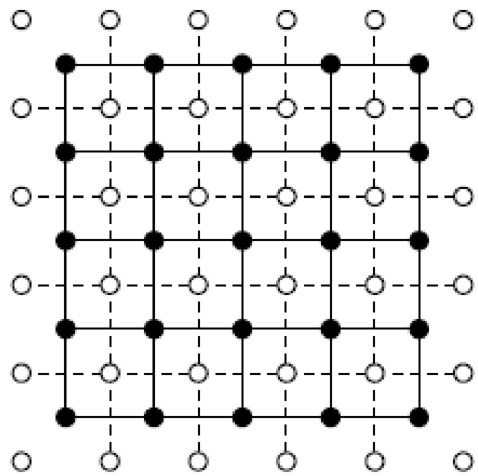

Figure 3.3: The dual of the box $\Lambda(n) = [-n, n]^2$ [10].

satisfies the duality relation

$$Z_G(p, q) = q^{|V|-1}\Big(\frac{1-p}{p_d}\Big)^{|E|} Z_{G_d}(p_d, q). \tag{3.68}$$

Therefore,

$$Z_G(p_{sd}(q), q) = q^{|V|-1-\frac{1}{2}|E|} Z_{G_d}(p_{sd}(q), q). \tag{3.69}$$

Now, we return to $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$. Given a finite graph G, we will focus on two boundary conditions: the *wired* boundary condition, denoted by $\phi_{G,p,q}$, is specified by the fact that all the vertices on the boundary are pairwise wired; and the *free* boundary condition, denoted by $\phi_{G,p,q}$, is specified by no wiring between sites. Let $\Lambda(n) = [-n, n]^2$, viewed as a subgraph of $\mathbb{L}^2$, and note that its dual graph $\Lambda(n)_d$ may be obtained from the box $[-n-1, n]^2 + (\frac{1}{2}, \frac{1}{2})$ by identifying all boundary vertices (see Figure 3.3). By (3.62), and with a small adjustment on the boundary of $\Lambda(n)_d$,

$$\phi^0_{\Lambda(n),p,q}(\omega) = \phi^1_{\Lambda(n)_d,p_d,q}(\omega_d) \tag{3.70}$$

for configurations ω on $\Lambda(n)$. Let A be a cylinder event of $\Omega = \{0, 1\}^{\mathbb{E}^2}$, and write A_d for the dual event of $\Lambda_d = \{0, 1\}^{\mathbb{E}^2_d}$, that is, $A_d = \{\omega_d \in \Omega_d : \omega \in A\}$. On letting $n \to \infty$ in (3.70), we obtain by the Thermodynamic Limit Theorem [10, Theorem 4.19(a)] that

$$\phi^0_{p,q}(A) = \overline{\phi}^1_{p_d,q}(A_d), \tag{3.71}$$

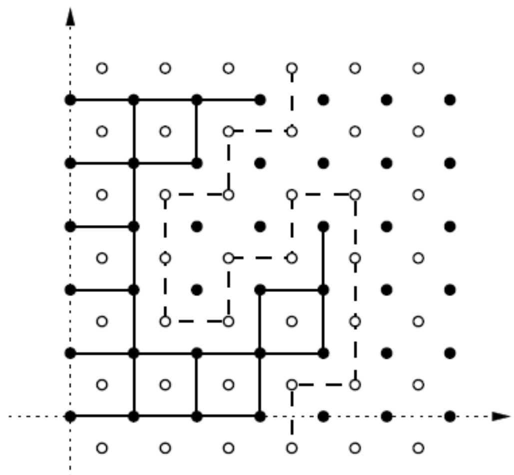

Figure 3.4: The box $R(5)$ and its dual $R(5)_d$ [10].

where the notation $\overline{\phi}$ is used to indicate the random-cluster measure on the dual configuration space Ω_d. By a similar argument,

$$\phi^1_{p,q}(A) = \overline{\phi}^0_{p_d,q}(A_d) \tag{3.72}$$

We summarize the above in a theorem.

Theorem 3.8. *Consider the square lattice $\mathbb{L}^2$, and let $q \in [1, \infty)$. For any cylinder event A,*

$$\phi^b_{p,q}(A) = \overline{\phi}^{1-b}_{p_d,q}(A_d) \qquad (b = 0, 1), \tag{3.73}$$

where $A_d = \{\omega_d \in \Omega_d : \omega \in A\}$.

3.6 Box-crossings in the Random Cluster model

There is a key application of duality to the existence of open crossings of a box. Let $R(n) = [0, n+1] \times [0, n]$ ($n \geq 1$, $[0, n] = \{0, 1, 2, \cdots, n\}$) and let $R(n)_d$ be its dual box $[0, n] \times [-1, n] + (\frac{1}{2}, \frac{1}{2})$. Let A_n be the event that there exists an open path of $R(n)$ joining some vertex on its left side to some vertex on its right side. It is standard that $(A_n)_d$ is the event that there exists no open dual crossing from the top to the bottom of $R(n)_d$. See Figure 3.4.

Indeed, denote by B_n the event that there exists an open path of $R(n)_d$ joining a vertex on the top side of $R(n)_d$ to a vertex on its bottom side. Notice that $A_n \cap B_n = \emptyset$, since if both A_n and B_n occur, then there exists an open path in $R(n)$ which crosses an open path in $R(n)_d$. Where these two paths cross, there is an open edge of $\mathbb{L}^2$ which is crossed by an open edge of $\mathbb{L}^2_d$, and this is impossible. On the other hand, either A_n or B_n must occur. Suppose that A_n does not occur, and let D be the set of all vertices of $R(n)$ which are attainable from the left side of $R(n)$ along open paths; we turn D into a graph by adding all open edges of $R(n)$ joining pairs of vertices in D. It is straightforward that there exists an open path of $\mathbb{L}^2_d$ crossing $R(n)_d$ from top to bottom, and which crosses only edges of $R(n)$ contained in the edge boundary of D. Thus, B_n occurs whenever A_n does not occur.

Theorem 3.9. *Let $q \in [1, \infty)$. We have that*

$$\phi^0_{p_{sd}(q),q}(A_n) + \phi^1_{p_{sd}(q),q}(A_n) = 1 \qquad (n \geq 1). \tag{3.74}$$

Proof. Apply Theorem 3.8 with $b = 0$ to the event $A = A_n$, and use the fact that $\overline{\phi}^1_{p,q}((A_n)_d) = \phi^1_{p,q}(\overline{A_n}) = 1 - \phi^1_{p,q}(A_n)$. □

Now consider the square $S(n) = [0, n]^2$ viewed as a subgraph of $\mathbb{L}^2$. We identify certain pairs of vertices on the boundary of $S(n)$ in order to make it symmetric. More specifically, we identify any pair of the form $(0, m), (n, m)$ and of the form $(m, 0), (m, n)$, for $0 \leq m \leq n$, and we merge any parallel edges that ensue. Let $T_n = (V_n, E_n)$ denote the resulting toroidal graph. Let $\mathcal{A}_n$ be the automorphism group of the graph T_n and note that $\mathcal{A}_n$ acts transitively on E_n. The configuration space of the random-cluster model on T_n is denoted by $\Omega(n) = \{0, 1\}^{E_n}$.

Let $p \in (0, 1)$ and $q \in [1, \infty)$. Write $\phi_{n,p}$ for the random-cluster measure on T_n with parameters p and q and note that $\phi_{n,p}$ is $\mathcal{A}_n$-invariant. We note that the dual of T_n is isomorphic to T_n, and the random-cluster measure on T_n is self-dual when $p = p_{sd}$ (by (3.62)).

Let $\omega \in \Omega(n)$. Any translate in T_n of a rectangle of the form $[0, r] \times [0, s]$ is said to be of size $r \times s$. When $r \neq s$, such a translate is said to be transversed *long-ways* (respectively, transversed *short-ways*) if the two shorter sides (respectively, longer sides) of the rectangle are joined within the rectangle by an open path of ω.

Let $\alpha \in (1, \infty)$ and let $SW_{n,\alpha}$ denote the event that the rectangle $H_{n,\alpha} = [0, \lceil n\alpha \rceil] \times [0, \lfloor n/\alpha \rfloor]$ is crossed short-ways. One would normally take $\alpha - 1$ to be small and n to be large in the next theorem.

Theorem 3.10. *[11] Let $\alpha \in (1,\infty), k, n \geq 2, q \in [1,\infty)$ and $p_{sd} < p < 1$. Suppose that $n/(n-1) \leq \alpha < \min\{k,n\}$. We have that*

$$\phi_{kn,p}(SW_{n,\alpha}) \geq 1 - \exp^{-g(p-p_{sd})}, \tag{3.75}$$

where

$$g = g(k,n,\alpha) = \frac{2c}{M}\log(kn) \tag{3.76}$$

and

$$M = 2\Big(1 + \frac{k}{\alpha - 1}\Big)\Big(1 + \frac{k\alpha}{\alpha - 1}\Big). \tag{3.77}$$

Note that M is of order $2k^2\alpha/(\alpha-1)^2$ for large k, n. For $p > p_{sd}$, one may make $\phi_{kn,p}(SW_{n,\alpha})$ large by holding k fixed and sending $n \to \infty$.

Proof. Assume the given conditions. Let $R(n) = [0, n+1] \times [0, n]$, viewed as a subgraph of T_{kn}, and let LW_n be the event that $R(n)$ is transversed long-ways. By a standard duality argument,

$$\phi_{kn,p_{sd}}(LW_n) = \frac{1}{2} \qquad (k \geq 2,\ n \geq 1). \tag{3.78}$$

Let A_n be the event that there exists in T_{kn} some translate of the square $S(n) = [0,n] \times [0,n]$ that possesses either an open top-bottom crossing or an open left-right crossing. The event A_n is $\mathcal{A}_n$-invariant, and

$$\phi_{kn,p_{sd}}(A_n) \geq \phi_{kn,p_{sd}}(LW_n) = \frac{1}{2}. \tag{3.79}$$

We apply (3.34) to the event A_n, with $p_1 = p_{sd}$ and with $N = 2(kn)^2$ being the number of edges in T_{kn}. This yields that

$$\begin{aligned}\phi_{kn,p}(A_n) &\geq 1 - \frac{1}{2}[2(kn)^2]^{-c(p-p_{sd})} \\ &\geq 1 - (kn)^{-2c(p-p_{sd})} \qquad (p_{sd} < p < 1).\end{aligned} \tag{3.80}$$

The event A_n is defined on the whole of the torus. Let $\alpha = \lceil n\alpha \rceil$, $b = \lfloor n/\alpha \rfloor$, and let $H_{n,\alpha} = [0,a] \times [0,b]$ and $V_{n,\alpha} = [0,b] \times [0,a]$. Let $h_{n,\alpha}, v_{n,\alpha}$ be the sets of vertices in T_{kn} given by

$$h_{n,\alpha} = \Big\{\big(l_1(a-n), l_2(n-b)\big) \in V_{kn} : 0 \leq l_1 < \frac{kn}{a-n}, 0 \leq l_2 < \frac{kn}{n-b}\Big\}, \tag{3.81}$$

$$\nu_{n,\alpha} = \Big\{ \big(l_1(n-b), l_2(a-n)\big) \in V_{kn} : 0 \le l_1 < \frac{kn}{n-b}, 0 \le l_2 < \frac{kn}{a-n} \Big\}, \quad (3.82)$$

where the l_i are integers. That $n-b \ge 1$ follows by assumption $\alpha \ge n/(n-1)$. Consider the set $\mathcal{H} = H_{n,\alpha} + h_{n,\alpha}$ of translates of $H_{n,\alpha}$ by vectors in $h_{n,\alpha}$, and also the set $\mathcal{V} = V_{n,\alpha} + \nu_{n,\alpha}$. If A_n occurs, then some rectangle in $\mathcal{H} \cup \mathcal{V}$ is traversed short-ways. By positive association and symmetry,

$$\begin{aligned} \phi_{kn,p}(\overline{A_n}) &\ge \phi_{kn,p} \qquad (no\ member\ of\ \mathcal{H} \cup \mathcal{V}\ is\ traversed\ short-ways) \\ &\ge \{1 - \phi_{kn,p}(SW_{n,\alpha})\}^R \qquad (3.83) \end{aligned}$$

where $SW_{n,\alpha}$ is the event that H_n is traversed short-ways, and

$$R = |h_{n,\alpha}| + |\nu_{n,\alpha}| \le 2\Big\lceil \frac{kn}{a-n} \Big\rceil . \Big\lceil \frac{kn}{n-b} \Big\rceil . \quad (3.84)$$

After taking into account rounding effects, we find that $R \le M$. Inequality (3.75) follows from (3.80), (3.83) e (3.84). □

Consider the square lattice $\mathbb{Z}^2$ with edge-set $\mathbb{E}$, and let $\Omega = \{0,1\}^{\mathbb{E}}$. Let $\Lambda = \Lambda_n = [-n,n]^2$ be a finite box of $\mathbb{Z}^2$, with edge-set $\mathbb{E}_\Lambda$. For $b \in \{0,1\}$ define

$$\Omega_\Lambda^b = \{\omega \in \Omega : \omega(e) = b\ for\ e \notin \mathbb{E}_\Lambda\}. \quad (3.85)$$

On Ω_Λ^b we define a random-cluster $\phi_{\Lambda,p,q}^b$ as follows. For $p \in [0,1]$ and $q \in [1,\infty)$, let

$$\phi_{\Lambda,p,q}^b(\omega) = \frac{1}{Z_{\Lambda,p,q}^b} \Big\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \Big\} q^{k(\omega,\Lambda)}, \qquad (\omega \in \Omega_\Lambda^b), \quad (3.86)$$

where $k(\omega,\Lambda)$ is the number of clusters of $(\mathbb{Z}^2, \eta(\omega))$ that intersect Λ.

For $A, B \subseteq \mathbb{Z}^2$, we write $A \leftrightarrow B$ if there exists an open path joining some $a \in A$ to some $b \in B$. We write $x \leftrightarrow \infty$ if the vertex x is the endpoint of some infinite open path. The percolation probabilities are given as

$$\theta^b(p,q) = \phi_{p,q}^b(0 \leftrightarrow \infty) \qquad (b = 0,1). \quad (3.87)$$

Since each θ^b is nondecreasing in p, one may define the critical point by

$$p_c(q) = \sup\{p : \theta^1(p,q) = 0\}. \quad (3.88)$$

It is known that $\phi^0_{p,q} = \phi^1_{p,q}$ if $p \neq p_{sd}(q)$, and we write $\phi_{p,q}$ for the common value. In particular, $\phi^0(p,q) = \phi^1(p,q)$ for $p \neq p_c(q)$. It is conjectured that $\phi^0_{p,q} = \phi^1_{p,q}$ when $p = p_c(q)$ and $q \leq 4$.

Let $B_k = [0,k] \times [0,k-1]$, and let H_k be the event that B_k possesses an open left-right crossing. That is, H_k is the event that B_k contains an open path having one endvertex on its left side and one on its right-hand side.

Theorem 3.11. *[12] Let $q \geq 1$. We have that*

$$\phi_{p,q}(H_k) \leq 2\rho_k^{p_{sd}-p} \qquad (0 < p < p_{sd}(q)), \tag{3.89}$$

$$\phi_{p,q}(H_k) \geq 1 - 2\nu_k^{p-p_{sd}} \qquad (p_{sd}(q) < p < 1), \tag{3.90}$$

for $k \geq 1$, where

$$\rho_k = [2q\eta_k/p]^{c/q}, \qquad \nu_k = [2q\eta_k/p_d]^{c/q} \tag{3.91}$$

and

$$\eta_k = \phi^0_{p_{sd}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad as\ k \to \infty \tag{3.92}$$

Here, c is an absolute positive constant.

Let $B_{k,m} = [0,k] \times [0,m]$ and let $H_{k,m}$ be the event that there exists an open left-right crossing of $B_{k,m}$

Theorem 3.12. *[12] Let $q \geq 1$. We have that*

$$\phi_{p_1,q}(H_{k,m})[1 - \phi_{p_2,q}(H_{k,m})] \leq \rho_k^{p_2-p_1} \qquad (0 < p_1 < p_2 \leq p_{sd}(q)), \tag{3.93}$$

$$\phi_{p_1,q}(H_{k,m})[1 - \phi_{p_2,q}(H_{k,m})] \leq \nu_{m+1}^{p_2-p_1} \qquad (p_{sd}(q) \leq p_1 < p_2 < 1), \tag{3.94}$$

for $k, m \geq 1$, where ρ_k (resp., ν_k) is with $p = p_1$ (resp., $p = p_2$), and $\phi_{p_{sd}(q),q}$ is to be interpreted as $\phi^0_{p_{sd}(q),q}$.

We shall apply Theorem 3.6 to a random-cluster measure $\phi_{p,q}$ with $q \geq 1$. By using Theorem 3.1, we obtain

$$\frac{p}{q} \leq \frac{p}{p + q(1-p)} \leq \phi_{p,q}(e) \leq p \tag{3.95}$$

whence

$$\phi_{p,q}(e)[1 - \phi_{p,q}(e)] \geq \frac{p(1-p)}{q}. \tag{3.96}$$

We may thus take $B = \frac{c}{q}$ in Corollary 3.1.

Let $q \geq 1, 1 \leq k, m < n$, and consider the random-cluster measure $\phi_{n,p}^b = \phi_{\Lambda_n,p,q}^b$ on the box Λ_n. For $e \in \mathbb{E}^2$, write $J_{k,m,n}^b(e)$ for the (conditional) influence of e on the event $H_{k,m}$ under the measure $\phi_{n,p}^b$. We set $J_{k,m,n}^b(e) = 0$ for $e \notin \mathbb{E}_{\Lambda_n}$.

At this point, we introduce and explain the boundary coupling property, which will be used in the proof of Theorem 3.11. Recall that a coupling of two measures ϕ_1 and ϕ_2 on Ω is a measure μ on $\Omega \times \Omega$ with marginals ϕ_1 and μ_2 (in order). If ϕ_1 and ϕ_2 are conditional distributions of some ϕ given boundary conditions ξ_1 and ξ_2, we also say μ is a coupling of ξ_1 and ξ_2 under ϕ.

We say that a measure ϕ on Ω has the *boundary coupling property* if for every finite Γ and every boundary condition ξ on $\overline{\Gamma}$, there exists a coupling μ of ξ and 1 (the configuration that assigns 1 to each site) under ϕ with the property that

$$\mu\{(\omega, \omega') \in \Omega \times \Omega : \omega(e) = \omega'(e) \text{ } for \text{ } all \tag{3.97}$$

$$e \in \overline{C_{\partial_E(\Gamma)}(\omega)} \cap \overline{C_{\partial_E(\Gamma)}(\omega')} = 1\}, \tag{3.98}$$

where $C_{\partial_E(\Gamma)}(\omega)$ denotes the boundary cluster in ω, that is, the union of the connected components of the edges of $\partial_E(\Gamma)$ in the configuration ω.

Lemma 3.1. *[1, Lemma 2.3] For a measure ϕ on Ω, suppose that for every finite Γ,*

(a) for every pair of boundary conditions ξ_1, ξ_2 on $\overline{\Gamma}$ with $\xi_1 \leq \xi_2$, and every $e \in \Gamma$,

$$\phi_\Gamma^{\xi_1}(\omega(e) = 1) \leq \phi_\Gamma^{\xi_2}(\omega'(e) = 1); \tag{3.99}$$

(b) for every boundary condition ξ on $\overline{\Gamma}$ with $\xi(e) = 0$ for all $e \in \partial_E(\Gamma)$,

$$\phi_\Gamma^{\xi} = \phi(\cdot|\xi(x) = 0 \text{ } for \text{ } all \text{ } e \in \partial_E(\Gamma)) \tag{3.100}$$

Then ϕ has the boundary coupling property with respect to $b = 1$.

Proof. Order the sites of $\Gamma = \{e_1, \cdots, e_m\}$ in such a way that e precedes f in the ordering if $d(e, \overline{\Gamma}) < d(f, \overline{\Gamma})$ (for example, spiraling inwards if Γ is a cube). We select the pairs $(\omega_{e_1}, \omega_{e_2})$ one at a time, as follows. Let $S_0 = \emptyset$ and suppose some set of edges S_n has been selected, and the corresponding values $(\omega(e), \omega'(e))$ chosen, by time n. Suppose also that $\omega(e) \geq \omega'(e)$ for all $e \in S_n$. At time $n+1$, we let i be the least index, if any, such that *edge e_i has not been selected and some site adjacent to e_i is connected to $\partial_E \Gamma$ in ω' by an open path of previously selected edges*. We then have $(\xi_1)_{\overline{\Gamma}} \times \omega_{S_n} \leq (\xi_2)_{\overline{\Gamma}} \times \omega'_{S_n}$, and from (a),

$$\begin{aligned}\phi_\Gamma^{\xi_1}(\omega(e_i) = 1 | \omega(e),\ e \in S_n) &= \phi_{\Gamma \setminus S_n}^{((\xi_1)_{\overline{\Gamma}} \times \omega_{S_n})}(\omega(e_i) = 1) \\ &\leq \phi_{\Gamma \setminus S_n}^{(\xi_2)_{\overline{\Gamma}} \times \omega'_{S_n}}(\omega(e_i) = 1) = \phi_\Gamma^{\xi_2}(\omega(e_i) = 1 | \omega'(e),\ e \in \Gamma). \qquad (3.101)\end{aligned}$$

Let p and p' denote the probabilities on the left and right sides of (3.101), respectively. Then let $(\omega(e_i), \omega'(e_i))$ be $(0, 0)$ with probability $1 - p'$, $(0, 1)$ with probability $p' - p$ and $(1, 1)$ with probability p. Let τ be the first time at which there are no longer any edges satisfying the property stressed in italic. Then S_τ is necessarily the cluster $C_{\partial_E(\Gamma)}(\omega')$, so $\omega(e) = \omega'(e) = 0$ for all $e \in \partial_E(C_{\partial_E(\Gamma)}(\omega'))$. Then by (b), the inequality (3.101) becomes an equality from time τ onward. This means the coupling constructed satisfies

$$\omega_e = \omega'_e \ for\ all\ e \in \overline{C(\partial_E, \omega')}, \qquad (3.102)$$

which establishes the boundary coupling property. □

Lemma 3.2. *Let $q \geq 1$. We have that*

$$\sup_{e \in \mathbb{E}^2} J^0_{k,m,n}(e) \leq \frac{q}{p}\eta_k \qquad (0 < p \leq p_{sd}(q), 1 \leq k, m < n) \qquad (3.103)$$

$$\sup_{e \in \mathbb{E}^2} J^1_{k,m,n} \leq \frac{q}{p_d}\eta_{m+1} \qquad (p_{sd}(q) \leq p < 1, 1 \leq k, m < n) \qquad (3.104)$$

where p_d satisfies

$$\frac{p_d}{1 - p_d} = \frac{q(1-p)}{p} \qquad (3.105)$$

and

$$\eta_k = \phi^0_{p_{sd}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad k \to \infty. \qquad (3.106)$$

Proof. For any configuration $\omega \in \Omega$ and vertex z, let $C_z(\omega)$ be the open cluster at z, that is, the set of all vertices joined to z by open paths.

Suppose first that $0 < p \leq p_{sd}(q)$, and let $e = \langle x, y \rangle$ be an edge of Λ_n. We couple the two conditional measures $\phi^0_{n,p}(\cdot | \omega(e) = b), b = 0, 1$, in the following manner. Let Ω_n be the configuration space of the edges in Λ_n, and let $T = \{(\pi, \omega) \in \Omega_n^2 : \pi \le \omega\}$ be the set of all ordered pairs of configurations. Since $\phi^0_{n,p}$ is strongly positively associated (see (2.56)-(2.57)), and by Theorems 3.1 and 3.2, $\phi^0_{n,p}$ satisfies the hypothesis of Lemma 3.1. Then there exists a measure μ^e on T such that:

(a) the first marginal of μ^e is $\phi^0_{n,p}(\cdot|1_e = 0)$;

(b) the second marginal of μ^e is $\phi^0_{n,p}(\cdot|1_e = 1)$;

(c) for any subset γ of Λ_n, conditional on the event $\{(\pi, \omega) : C_x(\omega) = \gamma\}$, the configuration π and ω are μ^e-almost-surely equal on all edges having no endvertex in γ.

We claim that

$$J^0_{k,m,n}(e) \ge \phi^0_{n,p}(D_x|1_e = 1), \tag{3.107}$$

where D_x is the event that C_x intersects both the left and right sides of $B_{k,m}$. This is proved as follows. By the conditional influence formula,

$$J^0_{k,m,n}(e) = \mu^e(\omega \in H_{k,m}, \pi \notin H_{k,m}) \tag{3.108}$$

$$\le \mu^e(\omega \in H_{k,m} \cap D_x) \tag{3.109}$$

$$\le \mu^e(\omega \in D_x) = \phi^0_{n,p}(D_x|1_e = 1), \tag{3.110}$$

since, when $\omega \notin D_x$, either both or neither of ω, π belong to $H_{k,m}$. By (26),

$$J^0_{k,m,n}(e) \ge \frac{\phi^0_{n,p}(D_x)}{\phi^0_{n,p}(1_e)}. \tag{3.111}$$

On D_x, the radius of the open cluster at x is at least $\frac{1}{2}k$. Since $\phi^0_{n,p} \le_{st} \phi_{p,q}$ and $\phi_{p,q}$ is translation-invariant,

$$\phi^0_{n,p} \le \phi_{p,q}(x \leftrightarrow x + \partial\Lambda_{k/2}) = \phi_{p,q}(0 \leftrightarrow \partial\Lambda_{k/2}). \tag{3.112}$$

By $\theta^0(p_{sd}(q), q) = 0$ $(q \ge 1)$ (see [10, Theorem 6.17(a)]),

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_{k/2}) \le \phi^0_{p_{sd}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad \text{as } k \to \infty, \tag{3.113}$$

and, by (3.95) and (3.111), the conclusion of the lemma is proved when $p \le p_{sd}(q)$. , we work with the dual open paths. Each edge $e_d = \langle x, y\rangle$ of the dual lattice traverses some edge $e = \langle x, y\rangle$ Suppose next that $p_{sd}(q) \leq p < 1$. Instead of working with the open paths, we work with the dual open paths. Each edge $e_d = \langle u, v\rangle$ of the dual lattice traverses some edge $e = \langle x, y\rangle$ of the primal, and, for each configuration ω, we define the dual configuration ω_d by $\omega_d(e_d) = 1 - \omega(e)$. Thus, the dual edge e_d is open if and only if e is closed. With ω distributed according to $\phi^1_{n,p}$, ω_d has as law the random-cluster measure, denoted $\phi_{n,p_d,d}$, on the dual of Λ_n with free boundary condition [10, Equation (6.12)]. The event $H_{k,m}$ occurs if and only if there is no dual open path traversing the dual of $B_{k,m}$ from top to bottom. We may therefore apply the above argument to the dual process, obtaining thus that

$$J^1_{k,m,n}(e) \leq \frac{\phi_{n,p_d,d}(V_u)}{\phi_{n,p_d,d}(1_e)}, \tag{3.114}$$

where V_u is the event that C_u intersects both the top and bottom sides of the dual of $B_{k,m}$.

On the event V_u, the radius of the open cluster at u is at least $\frac{1}{2}(m+1)$. Since $\phi_{n,p_d,d} \leq_{st} \phi_{p_d,q}$,

$$\phi_{n,p_d,d}(V_u) \leq \phi_{p_d,q}(u \leftrightarrow u + \partial\Lambda_{(m+1)/2}) = \phi_{p_d,q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}). \tag{3.115}$$

As above, by $p < p_{sd}(q)$ if and only if $p_d > p_{sd}(q)$,

$$\phi_{p_d,q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}) \leq \phi^0_{p_{sd}(q),q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}) = \eta_{m+1}, \tag{3.116}$$

and this completes the proof when $p \geq p_{sd}(q)$. □

Proof of Theorem 3.12. This follows immediately from Corollary 3.1 by taking $B = \frac{c}{q}$ and Lemma 3.2. □

Proof of Theorem 3.11. By planar duality,

$$\phi^0_{p,q}(H_k) = 1 - \phi^1_{p_d,q}(H_k), \tag{3.117}$$

where p, p_d are related by

$$\frac{p_d}{1-p_d} = \frac{q(1-p)}{p}. \tag{3.118}$$

Since $\phi^0_{p_{sd}(q),q} \leq_{st} \phi^1_{p_{sd}(q),q}$,

$$\phi^0_{p_{sd}(q),q}(H_k) \leq \frac{1}{2} \leq \phi^1_{p_{sd}(q),q}(H_k), \tag{3.119}$$

and Theorem 3.11 follows from Theorem 3.12. □

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