$\mathcal{F}$-EXPANSIVITY FOR BOREL MEASURES

HELMUTH VILLAVICENCIO FERNÁNDEZ

Abstract. We introduce the notion of $\mathcal{F}$-*expansive measure* by making the dynamical ball in [4] to depend on a given subset $\mathcal{F}$ of the set of all the reparametrizations $\mathcal{H}$. We prove that these measures satisfy some interesting properties resembling the expansive ones. These include the equivalence with expansivity when $\mathcal{F} = \mathcal{H}$, the vanishing along the orbits, the absence of singularities in the support, the $\mathcal{F}$-expansivity with respect to time t-maps, the invariance under equivalence and the characterization for suspensions. We also analyze the support of the $\mathcal{F}$-expansive measures and prove that there exists a dense subset of measures (in the set of $\mathcal{F}$-expansive measures) all of them with a common support. Finally, we extend to flows the recent result for homeomorphisms in [12].

1. Introduction

The notion of *expansive homeomorphism* has been important in the development of the theory of dynamical systems. Since the introduction of this concept by Utz [16] an extensive literature about it has been developed. This concept was subsequently extended to flows by Bowen and Walters [2]. Basically, the idea behind Bowen-Walters definition is that points which are far away in the topology induced by the flow can be separated at the same time with the help of a continuous time lag. Afterwards, Keynes and Sears [8] restricted the reparametrizations in the Bowen-Walters definition to subsets $\mathcal{F}$ giving rise to the concept of $\mathcal{F}$-*expansive transformation group*. The recent appearance of the *expansive measure* [13] extended the expansivity of homeomorphisms to Borel probability measures considering the behavior of the dynamical ball respect to the measure. Further steps were given in [4] with the concept of expansive measures for flows or in [6] and [11] with the notions of *asymptotic* and *weak expansive* measures [6], [11].

In light of these results, it is natural to consider a notion of expansivity for measures by restricting the reparametrizations as in [8]. We obtain the notion of $\mathcal{F}$-*expansive measure* for flows in which $\mathcal{F}$ is a given subset of the set of reparametrizations $\mathcal{H}$.

We prove that these measures satisfy some interesting properties resembling the expansive ones. These include the equivalence with expansivity when $\mathcal{F}$=$\mathcal{H}$, the vanishing along the orbits, the absence of singularities in the support, the $\mathcal{F}$-expansivity with respect to time t-maps, the invariance under equivalence and the characterization for suspensions. We also analyze the support of the $\mathcal{F}$-expansive

2010 *Mathematics Subject Classification.* Primary 54H20; Secondary 37C10.

Key words and phrases. Expansive measure; Expansive flow; Support of a measure; Metric space.

Partially supported by Fondecyt (C.G. 217–2014).

measures and prove that there exists a dense subset of measures (in the set of $\mathcal{F}$-expansive measures) all of them with a common support. Finally, we extend to flows the recent result for homeomorphisms in [12].

2. Statement of the results

Hereafter (X, d) will denote a compact metric space. The closed and open ball operations will be denoted by $B[x, \delta]$ and $B(x, \delta)$ respectively. The closure and boundary operations will be denoted by $\overline{(\cdot)}$ and $\partial(\cdot)$ respectively. A *flow* of X is a map $\phi : \mathbb{R} \times X \to X$ satisfying $\phi(0, x) = x$ and $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $t, s \in \mathbb{R}$ and $x \in X$. A flow is continuous if it is continuous with respect to the product metric of $\mathbb{R} \times X$. Given $A \subset X$ and $I \subset \mathbb{R}$ we define $\phi_I(A) = \{\phi_t(x) : (t, x) \in I \times A\}$. If A consists of a single point x, then we write $\phi_I(x)$ instead of $\phi_I(\{x\})$. If $x \in X$ satisfies $\phi_{\mathbb{R}}(x) = \{x\}$, then we say that x is a *singularity* of ϕ. Denote by $Sing(\phi)$ the set of singularities of ϕ.

The Borel σ-algebra of X is the σ-algebra $\mathcal{B}(X)$ generated by the open subsets of X. A *Borel probability measure* is a σ-additive measure μ defined in $\mathcal{B}(X)$ such that $\mu(X) = 1$. For any subset $B \subset X$ we write $\mu(B) = 0$, if $\mu(A) = 0$ for every Borel set $A \subset B$. Denote by $\mathcal{H}$ the set of continuous maps $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$. Given a flow ϕ of X, $x \in X$ and $\delta > 0$ we define the dynamical ball as

$$\Gamma_\delta^\phi(x) = \bigcup_{h \in \mathcal{H}} \bigcap_{t \in \mathbb{R}} \phi_{-h(t)}(B[\phi_t(x), \delta]).$$

Note that this ball is not always a closed set of X. The following is a straightforward reformulation of the notion of expansive flow due to Bowen and Walters [2].

Definition 2.1. *A flow ϕ is* expansive *if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\Gamma_\delta^\phi(x) \subset \phi_{[-\epsilon, \epsilon]}(x), \quad \textit{for all } x \in X.$$

Next, we recall the definition of expansive measure for flows [4].

Definition 2.2. *A Borel probability measure μ is* expansive *for a flow ϕ if there exists $\delta > 0$ such that*

$$\mu(\Gamma_\delta^\phi(x)) = 0, \quad \textit{for all } x \in X.$$

To motivate our main definition we recall the following generalization of expansive flow introduced by H.B. Keynes and Sears in [8]. They introduced the idea of restriction of the time lag, and gave one definition of expansiveness weaker than Bowen-Walters. More precisely: Given a flow ϕ of X, $x \in X$, $\delta > 0$ and a subset $\mathcal{F}$ of $\mathcal{H}$, we define the $\mathcal{F}$-dependent dynamical ball as

$$\Gamma_\delta^\phi(x, \mathcal{F}) = \bigcup_{h \in \mathcal{F}} \bigcap_{t \in \mathbb{R}} \phi_{-h(t)}(B[\phi_t(x), \delta]),$$

and the following definition holds.

Definition 2.3. *Given a subset $\mathcal{F} \subset \mathcal{H}$ we say that a flow ϕ is $\mathcal{F}$-*expansive *if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\Gamma_\delta^\phi(x, \mathcal{F}) \subset \phi_{[-\epsilon, \epsilon]}(x), \quad \textit{for all } x \in X.$$

Clearly the $\mathcal{H}$-expansive flows are precisely the expansive flows in the sense of Definition 2.1. To illustrate further the above definition we present the following example.

Example 2.4. *An Anosov flow ϕ on a compact Riemannian manifold is $\{id\}$-expansive.*

Proof. Given $\epsilon > 0$ we have by Theorem 3.4 in [14] that there exists $\delta > 0$ such that for any x, y in X with $y \notin \phi_{\mathbb{R}}(x)$ there exists $t \in \mathbb{R}$ for which $d(\phi_t(x), \phi_t(y)) > \delta$. It follows that if $\delta > 0$ is small enough, then $\Gamma_\delta^\phi(x, \{id\}) \subset \phi_{(-\epsilon,\epsilon)}(x)$ holds for every x in X. □

Motivated by the definition of expansive measure for flows we define the main object of study of this work.

Definition 2.5. *Given a subset $\mathcal{F} \subset \mathcal{H}$ we say that a Borel probability measure μ of X is $\mathcal{F}$-*expansive *for a flow ϕ if there exists $\delta > 0$ such that*

$$\mu(\Gamma_\delta^\phi(x, \mathcal{F})) = 0, \quad \textit{for all } x \in X.$$

It is apparent that the $\mathcal{H}$-expansive measures of a given flow are precisely the expansive measures of that flow. In what follows we will obtain some properties of the $\mathcal{F}$-expansive measures. For this, we endow $\mathcal{H}$ with the supremum metric

$$\widehat{d}(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}.$$

Under this distance, we obtain that $\mathcal{H}$ is what is called an ∞-*metric space* in the sense that it allows an infinite distance between certain points (see [3], [5]).

If ϕ is a continuous flow of a compact metric space X, there exists a natural map $\phi^* : \mathcal{H} \to C(\mathbb{R}, H(X))$, where $H(X)$ is the self-homeomorphisms of X with the topology of pointwise convergence, given by $\phi^*(f)(t) = \phi_{f(t)}$. Additionally, ϕ^* is continuous whenever $C(\mathbb{R}, H(X))$ have the topology generated by the base of neighborhoods

$$N(h, x_1, \cdots, x_m, \delta) = \bigcap_{i=1}^{m} \{g : d(g(t)(x_i), h(t)(x_i)) < \delta \text{ for every } t \in \mathbb{R}\},$$

where $h \in C(\mathbb{R}, H(X))$, $\{x_1, \cdots, x_m\} \subset X$ and $\delta > 0$.

With these definitions, we can state the following result.

Theorem 2.6. *Let ϕ be a continuous flow on a compact metric space X, let μ be a Borel probability measure on X and let $\mathcal{F}$ be a subset of $\mathcal{H}$. Then, μ is $\mathcal{F}$-expansive if and only if μ is $(\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})$-expansive.*

As a consequence of the above theorem we obtain the following equivalence.

Corollary 2.7. *Let ϕ be a continuous flow on a compact metric space X, let μ be a Borel probability measure on X and let $\mathcal{F}$ be a subset of $\mathcal{H}$. Then, μ is $\mathcal{F}$-expansive if and only if μ is $\overline{\mathcal{F}}$-expansive.*

The following is a simple consequence of Corollary 2.7.

Corollary 2.8. *If μ is $\mathcal{F}$-expansive and $g \in \mathcal{H}$ is such that for every $x \in X$ and for every $\delta > 0$ there exists $f \in \mathcal{F}$ with $d(\phi_{g(t)}(x), \phi_{f(t)}(x)) \le \delta$ for all $t \in \mathbb{R}$, then μ is $(\mathcal{F} \cup \{g\})$-expansive.*

For the next result, we shall use the following standard topological concept. A subset of a topological space Y is a G_δ subset of Y if it is the intersection of countably many open subsets of Y.

Given $f \in \mathcal{H}$, we define $\mathcal{B}_f = \{h \in \mathcal{H} : \widehat{d}(f,h) < \infty\}$ and $d_f = \widehat{d}_{|\mathcal{B}_f}$. It follows that $\mathcal{H}$ can be written as a union of metric spaces $(\mathcal{B}_f, d_f)$. Note that in the ∞-metric a subset of $\mathcal{H}$ is compact if and only if it is a union of a finite number of compact subsets each one belonging to some $(\mathcal{B}_f, d_f)$ (p. 15 in [3]).

Theorem 2.9. *Let ϕ be a continuous flow on a compact metric space X. If $\mathcal{F}$ is a compact subset of $\mathcal{H}$, then*

(1) *For every $x \in X$ and each $\delta > 0$ the $\mathcal{F}$-dependent dynamical ball, $\Gamma^{\phi}_{\delta}(x, \mathcal{F})$, is a G_δ set of X.*
(2) *Given μ a Borel probability measure on X, then μ is $\mathcal{F}$-expansive if and only if μ is $\{f\}$-expansive for every $f \in \mathcal{F}$.*

To state our next result we will need more notations. Let ϕ be a flow of X. The *time t-map* $\phi_t : X \to X$ defined by $\phi_t(x) = \phi(t,x)$ is a homeomorphism of X for all $t \in \mathbb{R}$. So, the flow ϕ can be interpreted as a family of homeomorphisms $\Phi = \{\phi_t\}_{t\in\mathbb{R}}$ such that $\phi_0 = id$ and $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$. We call $\phi_{\mathbb{R}}(x)$ the orbit of $x \in X$ under ϕ. By a *periodic point* of ϕ we mean a point $x \in X$ for which there is a minimal $t > 0$ satisfying $\phi_t(x) = x$. This minimal t is the so-called *period*. Denote by $Per(\phi)$ the set of periodic points of ϕ.

We denote by $\mathcal{M}(X)$ the set of all Borel probability measures of X. We say that a Borel probability measure μ *vanishes along the orbits* of ϕ whenever $\mu(\phi_{\mathbb{R}}(x)) = 0$ for all $x \in X$. We say that μ is *nonatomic* if $\mu(\{x\}) = 0$ for all $x \in X$. Every measure vanishing along the orbits is clearly nonatomic, but not conversely (take for instance the Borel measure supported on a periodic orbit). The *support* of $\mu \in \mathcal{M}(X)$ is the set $\text{supp}(\mu)$ of points $x \in X$ such that for any neighborhood U of x, $\mu(U) > 0$. It follows that $\text{supp}(\mu)$ is a nonempty compact subset of X.

Given another metric space Y and a Borel measure map $f : X \to Y$ we define the pullback measure $f_*\mu = \mu \circ f^{-1}$ on Y whenever $\mu \in \mathcal{M}(X)$.

An *equivalence* between continuous flows ϕ on X and ψ on another metric space Y is a homeomorphism $f : X \to Y$ carrying the orbits of ϕ onto orbits of ψ. In this case we say that the flows are *equivalent*. We denote by $\mathcal{K}$ and $\mathcal{B}_0$ the subsets of $\mathcal{H}$ consisting of increasing homeomorphisms and bounded functions respectively. Given a subset $\mathcal{F} \subset \mathcal{H}$ we write $\mathcal{K}\mathcal{F}\mathcal{K} \subset \mathcal{F}$ if $g_1 \circ f \circ g_2 \in \mathcal{F}$ whenever $g_1, g_2 \in \mathcal{K}$ and $f \in \mathcal{F}$.

Definition 2.10. *A subset $\mathcal{F}$ of $\mathcal{H}$ is called regular for the flow ϕ if for every $\delta > 0$, we have that*

$$x \in \Gamma^{\phi}_{\delta}(x, \mathcal{F}), \text{ for all } x \in X. \tag{1}$$

Clearly, the regularity condition implies that the dynamical ball contains the basis point. Also, if $id \in \mathcal{F}$ then $\mathcal{F}$ is regular for every flow ϕ. The example below proves a sort of converse for this result.

Example 2.11. *Let ϕ be a flow continuous without singularities on a compact metric space X. If the subset $\mathcal{F}$ is regular for the flow ϕ, then $id \in \overline{\mathcal{F}}$.*

Proof. If $id \notin \overline{\mathcal{F}}$ then $\widehat{d}(id, \mathcal{F}) > 0$. We can choose $0 < \lambda < \widehat{d}(id, \mathcal{F})$ small enough, then by Lemma 3.2, exists $\gamma > 0$ such that $d(\phi_\lambda(w), z) > \gamma$ whenever $d(w,z) < \gamma$. Since $\mathcal{F}$ is regular, given $x \in X$ there exists $g \in \mathcal{F}$ such that $d(\phi_t(x), \phi_{g(t)}(x)) < \gamma$. Moreover, there is $t_0 \in \mathbb{R}$ such that $g(t_0) - t_0 = \lambda$, thus by Lemma 3.2 we have $d(\phi_{t_0}(x), \phi_{g(t_0)}(x)) = d(\phi_{t_0}(x), \phi_{g(t_0)-t_0}(\phi_{t_0}(x))) > \gamma$, which contradicts the regularity condition (1). $\square$

Let $f : X \to X$ be a homeomorphism and $\tau : X \to (0, +\infty)$ be a continuous function. Consider the quotient space $Y^{\tau,f} = \{(x,t) : 0 \leq t \leq \tau(x), x \in X\}/\sim$, where $(x, \tau(x)) \sim (f(x), 0)$ for all $x \in X$. The *suspension flow over f with height function τ* is the flow $\Phi = \{\phi_t\}_{t\in\mathbb{R}}$ on $Y^{\tau,f}$ defined by $\phi_t^{\tau,f}(x,s) = (x, s+t)$ whenever $s + t \in [0, \tau(x)]$ (see [2], [7]).

Replacing d by the the equivalent metric $\frac{d}{diam(X)}$ if necessary, we can assume that $diam(X) = 1$. Then, there is a natural metric $d^{\tau,f}$ on $Y^{\tau,f}$ making it a compact metric space (this is the so-called *Bowen-Walters metric* [2]). Moreover, there exists an injective map $T^{\tau,f} : \mathcal{M}(X) \to \mathcal{M}(Y^{\tau,f})$ such that $T^{\tau,f}(\mu) = \frac{1}{\mu(\tau)}(\mu \times m)|_{Y^{\tau,f}}$ where $\mu(\tau) = \int_X \tau(x) d\mu(x)$ and m is the Lebesgue measure. So, for every continuous function $h : Y^{\tau,f} \to \mathbb{R}$ one has

$$\int_{Y^{\tau,f}} h(y) dT^{\tau,f}(\mu) = \frac{1}{\mu(\tau)} \int_X \int_0^{\tau(x)} h(\phi_t^{\tau,f}(x,0)) dt \; d\mu(x).$$

Every suspension of f is conjugate to the suspension of f under the constant function 1. A homeomorphism from $Y^{1,f}$ to $Y^{\tau,f}$ that conjugates the flows is given by the map $(x,t) \mapsto (x, t\tau(x))$. For this reason we will concentrate on suspensions under the function 1.

Next, we recall the definition of expansive measure for homeomorphisms [13].

Definition 2.12. *We say that a Borel probability measure μ of X is expansive for a homeomorphism $f : X \to X$ if there exists $\delta > 0$ such that $\mu(\Gamma_\delta^f(x)) = 0$ for every $x \in X$, where*

$$\Gamma_\delta^f(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta, \text{ for all } n \in \mathbb{Z}\}.$$

With these definitions, we can state our following result motivated by Theorems 2.1, 2.2 and 2.4 in [4].

Theorem 2.13. *The following properties hold for every continuous flow ϕ on a compact metric space, every Borel probability measure μ and every subset $\mathcal{F} \subset \mathcal{H}$:*

1. *If $\mathcal{F}$ is regular for ϕ and μ is $\mathcal{F}$-expansive, then μ vanishes along orbits.*
2. *If ϕ is $\mathcal{F}$-expansive, then every Borel measure vanishing along the orbits of ϕ is $\mathcal{F}$-expansive for ϕ.*
3. *If μ is $\mathcal{F}$-expansive and $\mathcal{F} \cap \mathcal{B}_0 \neq \emptyset$, then $supp(\mu) \cap Sing(\phi) = \emptyset$.*
4. *If $\mathcal{K}\mathcal{F}\mathcal{K} \subset \mathcal{F}$ and f is an equivalence between ϕ and ψ, then μ is $\mathcal{F}$-expansive if and only if $f_*\mu$ is $\mathcal{F}$-expansive.*
5. *If $\mathcal{F}$ is regular for ϕ and μ is $\mathcal{F}$-expansive, then μ is an expansive measure of the homeomorphism ϕ_T for all $T \in \mathbb{R}$.*
6. *If $\mathcal{F}$ is regular and $T^{1,f}(\mu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$, then μ is expansive for f.*
7. *If μ is expansive for f, then $T^{1,f}(\mu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$.*

We have the related example below.

Example 2.14. *In the noncompact case, the converse of Item (3) of Theorem 2.13 is false. Consider the flow defined by the ODE $(\dot{x}, \dot{y}) = (x, y)$. We have $(0,0) \in supp(Leb) \cap Sing(\phi)$ where Leb is the Lebesgue measure. In addition, Leb is $\mathcal{K}$-expansive and $\mathcal{K} \cap \mathcal{B}_0 = \emptyset$.*

Henceforth we will study the topological behavior of the $\mathcal{F}$-expansive measures of ϕ. The set $\mathcal{M}(X)$ of all Borel probability measures of X is a compact metrizable convex space and its topology is the *weak* topology* defined by the convergence $\mu_n \to \mu$ if and only if $\int \phi d\mu_n \to \int \phi d\mu$ for every continuous map $\phi : X \to \mathbb{R}$. Every approximation of a Borel probability measure will be considered under this topology. We say that a measure μ is *fully supported* if supp(μ)=X. We denote by $\mathcal{M}_{ex}(X, \phi, \mathcal{F})$ the set of $\mathcal{F}$-expansive measures of ϕ. This set is a *convex cone* in $\mathcal{M}(X)$, that is, $\alpha\mu + \nu \in \mathcal{M}_{ex}(X, \phi, \mathcal{F})$ whenever $\alpha \in \mathbb{R}_+$ and $\mu, \nu \in \mathcal{M}_{ex}(X, \phi, \mathcal{F})$.

We shall use the following standard topological concepts. A subset Z of a topological space Y is said to be *nowhere dense* in Y if the closure of Z in Y has empty interior in Y, and *meagre* if it is the union of countably many nowhere dense subsets of Y.

A topological space Y is a *Baire space* if the intersection of each countable family of open and dense subsets in Y is dense in Y. A set $A \subset Y$ is a *Baire subset* of Y if A is a Baire space with respect to the topology induced by Y.

The following example can be seen as a motivation for the next theorem.

Example 2.15. *Let ϕ be a flow of a compact metric space without isolated points X and let $\mathcal{F} \subset \mathcal{H}$ be regular for ϕ. If ϕ is $\mathcal{F}$-expansive, then the set of $\mathcal{F}$-expansive measures of ϕ is a Baire subset of the set of nonatomic Borel probability measures of X.*

Proof. If ϕ and $\mathcal{F}$ are as in the statement, then items (1) and (2) of Theorem 2.13 imply that the set of $\mathcal{F}$-expansive measures of ϕ coincides with the Borel probability measures vanishing along the orbits of ϕ. Let us prove that the latter set is a Baire subset of the set of nonatomic Borel probability measures. By Theorem 1 in [9], we have that the set $\mathcal{M}_{non}(X)$ of nonatomic Borel probability measures of X is a Baire subset of $\mathcal{M}(X)$. If $\mathcal{M}^{\phi}_{non}(X)$ denote the set of nonatomic Borel probability measures vanishing along the orbits, then it suffices to show that this set is a G_δ subset of $\mathcal{M}_{non}(X)$ (see [17]). For each $\lambda, \epsilon > 0$ we define

$$\Lambda(\lambda, \epsilon) = \{\mu \in \mathcal{M}_{non}(X) : \mu(\phi_{[-\lambda,\lambda]}(x)) \geq \epsilon \text{ for some } x \in X\}.$$

It follows that

$$\mathcal{M}^{\phi}_{non}(X) = \bigcap_{(k,m)\in\mathbb{N}\times\mathbb{N}} \left(\mathcal{M}_{non}(X) \setminus \Lambda\left(k, \frac{1}{m}\right)\right).$$

It remains to show that $\Lambda(\lambda, \epsilon)$ is closed. Let $\mu_n \in \Lambda(\lambda, \epsilon)$ be a sequence with property that $\mu_n \to \mu$ for some $\mu \in \mathcal{M}_{non}(X)$. Choose a sequence $x_n \in X$ such that

$$\epsilon \leq \mu_n(\phi_{[-\lambda,\lambda]}(x_n)), \text{ for every } n \in \mathbb{N}.$$

By compactness we can assume that $x_n \to x$ for some $x \in X$. Fix an open neighborhood U of $\phi_{[-\lambda,\lambda]}(x)$ such that $\mu(\partial U) = 0$. Suppose that there exists a subsequence $n_j \to \infty$ such that $\phi_{[-\lambda,\lambda]}(x_{n_j}) \not\subset U$ for each $j \in \mathbb{N}$. Then, we can select a sequence $w_j \in \phi_{[-\lambda,\lambda]}(x_{n_j}) \setminus U$ and so we obtain a sequence $t_j \in [-\lambda, \lambda]$ such that $w_j = \phi_{t_j}(x_{n_j})$. We can suppose that $t_j \to t$ and $w_j \to w$ where $w = \phi_t(x)$. Thus, $w \in U$ which is a contradiction. Then, $\phi_{[-\lambda,\lambda]}(x_n) \subset U$ for all n large. Since $\mu_n \to \mu$, we obtain

$$\epsilon \leq \limsup_{x\to\infty} \mu_n(\phi_{[-\lambda,\lambda]}(x_n)) \leq \lim_{x\to\infty} \mu_n(U) = \mu(U).$$

Then, $\epsilon \leq \mu(\phi_{[-\lambda,\lambda]}(x))$ and so it follows that $\mu \in \Lambda(\lambda, \epsilon)$. ∎

Motivated by the above example, we give two sufficient conditions to guarantee that the set of $\mathcal{F}$-expansive measures of the flow is a Baire subset of $\mathcal{M}(X)$.

Theorem 2.16. *The set of $\mathcal{F}$-expansive measures of a continuous flow on a compact metric space X is a Baire subset of $\mathcal{M}(X)$ in any of these cases:*

1. *If $\mathcal{F}$ is a compact subset of $\mathcal{H}$ or*
2. *If $\mathcal{F} = \mathcal{H}$ and the flow ϕ has no singularities.*

The following corollary is immediate.

Corollary 2.17. *If ϕ is a continuous flow without singularities on a compact metric space X, then the set of expansive measures of ϕ is a Baire subset of $\mathcal{M}(X)$.*

The following is a generalization of the definition of the measure-expansive center defined recently in [12].

Definition 2.18. *The $\mathcal{F}$-measure-expansive center of a flow ϕ, denoted by $E(\phi, \mathcal{F})$, is the union of the support of all the $\mathcal{F}$-expansive measures of ϕ.*

With this definition, we will obtain the followings results generalizing [12].

Theorem 2.19. *Let ϕ be a flow on a compact metric space X and let $\mathcal{F} \subset \mathcal{H}$. Suppose that the $\mathcal{F}$-expansive measures form a Baire subset of $\mathcal{M}(X)$, then the set of $\mathcal{F}$-expansive measures is not empty if and only if every $\mathcal{F}$-expansive measure can be approximated by an $\mathcal{F}$-expansive measure whose support is equal to the $\mathcal{F}$-measure-expansive center of ϕ.*

Corollary 2.20. *Let ϕ be a flow on a compact metric space X and let $\mathcal{F} \subset \mathcal{H}$. Suppose that the $\mathcal{F}$-expansive measures form a Baire subset of $\mathcal{M}(X)$, then the $\mathcal{F}$-expansive measures are dense in $\mathcal{M}(X)$ if and only if the fully supported $\mathcal{F}$-expansive measures are dense in $\mathcal{M}(X)$.*

Corollary 2.21. *Let ϕ be a flow on a compact metric space X and let $\mathcal{F} \subset \mathcal{H}$ be regular for ϕ. If the $\mathcal{F}$-expansive measures form a Baire dense subset of $\mathcal{M}(X)$, then X has no isolated points.*

Finally, we obtain the following result (originally proved in [12]).

Theorem 2.22. *A homeomorphism of a compact metric space has an expansive measure if and only if every expansive measure of it can be approximated by an expansive measure with invariant support.*

3. Preliminaries

In this section we prove some preparatory results.

Lemma 3.1. *The following properties hold for any continuous flow ϕ on a compact metric space X and any Borel probability measure μ on X:*

1. *If $\mathcal{F}$ is a subset of $\mathcal{H}$ and μ is $\mathcal{F}$-expansive, then μ is $\mathcal{F}_0$-expansive for all subset $\mathcal{F}_0 \subset \mathcal{F}$.*
2. *If μ is $\mathcal{F}_i$-expansive, where $\mathcal{F}_i \subset \mathcal{H}$ for every $i = 1, \cdots, k$, then μ is $(\bigcup_{i=1}^{k} \mathcal{F}_i)$-expansive.*

Proof. Item (1) follows from the definition. Given an $\mathcal{F}$-expansive measure μ for the flow ϕ, for every subset $\mathcal{F}_0 \subset \mathcal{F}$ we have $\Gamma^{\phi}_{\delta}(x, \mathcal{F}_0) \subset \Gamma^{\phi}_{\delta}(x, \mathcal{F})$ for all $x \in X$. Thus, the proof follows directly from monotony of the measure.

To prove (2), let $\delta_i > 0$ be an $\mathcal{F}_i$-expansitivy constant of μ. Take $\alpha = \min_{1 \le i \le k} \delta_i > 0$. Clearly

$$\Gamma^{\phi}_{\alpha}(x, \bigcup_{i=1}^{k} \mathcal{F}_i) \subset \bigcup_{i=1}^{k} \Gamma^{\phi}_{\delta_i}(x, \mathcal{F}_i), \text{ for all } x \in X.$$

Since $\mu(\Gamma^{\phi}_{\delta_i}(x, \mathcal{F}_i)) = 0$ for all $i \in \{1, \cdots, k\}$, we get by subadditivity that

$$\mu\left(\Gamma^{\phi}_{\alpha}(x, \bigcup_{i=1}^{k} \mathcal{F}_i)\right) = 0 \text{ for every } x \in X.$$

□

The following lemma is contained in [2] and we include its proof for the sake of completeness.

Lemma 3.2. *Let ϕ be a continuous flow on a compact metric space X. If the flow ϕ has no singularities, then there exists $T_0 > 0$ such that for all λ satisfying $0 < \lambda < T_0$ there exists $\gamma > 0$ with the property that $d(\phi_{\pm\lambda}(x), y) > \gamma$ provided that $x, y \in X$ and $d(x, y) < \gamma$.*

Proof. If the flow ϕ has no periodic orbits, let $T_0 = 1$ and if the flow ϕ does have some periodic orbits let T_0 be the smallest period of ϕ. Then $T_0 > 0$. If the Lemma is false there are $0 < \lambda < T_0$ and sequences $x_n, y_n \in X$, with $d(x_n, y_n) < \frac{1}{n}$, such that $d(\phi_\lambda(x_n), y_n) \le \frac{1}{n}$ or $d(\phi_{-\lambda}(x_n), y_n) \le \frac{1}{n}$. By compactness we can suppose $x_n \to z$, and therefore, $y_n \to z$ where $z \in X$ and $d(\phi_\lambda(x_n), y_n) \le \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus, we obtain

$$d(\phi_\lambda(x_n), y_n) \to d(\phi_\lambda(z), z) = 0.$$

It follows that $\phi_\lambda z = z$ with $0 < \lambda < T_0$, which is a contradiction. □

In [15], Thomas makes a variant of the dynamical ball to define the notion of strongly h-expansiveness. So by combining the ideas of Keynes, Sears and Thomas we introduce a new dynamical ball with some interesting properties. More precisely, given a flow ϕ of X, $x \in X$, $\delta > 0$ and a subset $\mathcal{F} \subset \mathcal{H}$, we define the strongly $\mathcal{F}$-dependent dynamical ball as

$$\widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{F}) = \bigcap_{r>0} \bigcap_{\gamma>\delta} \bigcup_{h \in \mathcal{F}} \bigcap_{|t| \le r} \phi_{-h(t)}(B[\phi_t(x), \gamma]). \tag{2}$$

We will show later that, in the case without singularities, $\widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{F})$ is closed in X. Clearly $\Gamma^{\phi}_{\delta}(x, \mathcal{F}) \subset \widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{F})$ for every $x \in X$. The lemma below proves a sort of converse for this result.

Lemma 3.3. *If the flow ϕ has no singularities, then for every subset $\mathcal{F} \subset \mathcal{H}$ and every $\delta > 0$ there exists $\delta' \in (0, \delta)$ such that*

$$\widetilde{\Gamma}^{\phi}_{\delta'}(x, \mathcal{F}) \subset \Gamma^{\phi}_{\delta}(x) \quad \textit{for all } x \in X.$$

Proof. By Lemma 3.1 item (1), is suffices to prove the result for $\mathcal{F} = \mathcal{H}$. Fix $\delta > 0$ and T_0 as in Lemma 3.2. There exists $0 < \lambda < T_0$ with the property that

$$d(\phi_t(x), x) < \frac{\delta}{2} \text{ for every } x \in X \text{ whenever } |t| < \lambda. \tag{3}$$

By Lemma 3.2 for this $\lambda > 0$ there exists $\gamma > 0$ such that

$$d(\phi_\lambda(x), y) > \gamma \text{ whenever } d(x, y) < \gamma. \tag{4}$$

Fix $m \in \mathbb{N}$ with $\delta < \gamma m$ and take $\delta' = \frac{\delta}{3m} > 0$. Given $z \in \widetilde{\Gamma}^{\phi}_{\delta'}(x, \mathcal{H})$ then for all $k \in \mathbb{N}$ there is $h_k \in \mathcal{H}$ such that

$$d(\phi_t(x), \phi_{h_k(t)}(z)) < \frac{3\delta'}{2} \quad \text{for each } |t| \leq k. \tag{5}$$

It follows that for all $-k \leq t \leq k$ we have

$$d(\phi_{h_{k+1}(t)}(z), \phi_{h_k(t)}(z)) \leq d(\phi_t(x), \phi_{h_{k+1}(t)}(z)) + d(\phi_t(x), \phi_{h_k(t)}(z)) < 3\delta' < \gamma.$$

Therefore

$$d(\phi_{h_{k+1}(t)-h_k(t)}(\phi_{h_k(t)}(z)), \phi_{h_k(t)}(z)) = d(\phi_{h_{k+1}(t)}(z), \phi_{h_k(t)}(z)) < \gamma.$$

By (4) and since $(h_{k+1} - h_k)(0) = 0$ we obtain $|h_{k+1}(t) - h_k(t)| < \lambda$ for every $-k \leq t \leq k$. Now we define a function $h : \mathbb{R} \to \mathbb{R}$ inductively. Define $h = h_1$ on $[-1, 1]$. As we know $|h_2(1) - h_1(1)| < \lambda$, so there exists a continuous function h on $[1, 2]$ such that $h(1) = h_1(1)$ and $h(2) = h_2(2)$ with $|h(t) - h_2(t)| < \lambda$ for each $t \in [1, 2]$. Also we have $|h_2(-1) - h_1(-1)| < \lambda$. There exists also a continuous function (call it h as well) on $[-2, -1]$ such that $h(-1) = h_1(-1)$ and $h(-2) = h_2(-2)$ with $|h(t) - h_2(t)| < \lambda$ for all $t \in [-2, -1]$. If we carry on in the same way, then we have such a continuous function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. That is, $h \in \mathcal{H}$. Now pick $t \in \mathbb{R}$, say, first $t > 0$. We have two cases:

Case 1: $t \in [0, 1]$.

In this case, by the inequality (5) we obtain

$$d(\phi_t(x), \phi_{h(t)}(z)) = d(\phi_t(x), \phi_{h_1(t)}(z)) < \frac{3\delta'}{2} = \frac{\delta}{2m} < \delta.$$

Case 2: $t \in [k, k+1]$, for some $k \geq 1$.

In this case, since $|h(t) - h_{k+1}(t)| < \lambda$, by condition (3) it follows that

$$d(\phi_{h(t)}(z), \phi_{h_{k+1}(t)}(z)) < \frac{\delta}{2},$$

and finally, by (5), we have

$$d(\phi_t(x), \phi_{h(t)}(z)) \leq d(\phi_t(x), \phi_{h_{k+1}(t)}(z)) + d(\phi_{h(t)}(z), \phi_{h_{k+1}(t)}(z)) < \frac{3\delta'}{2} + \frac{\delta}{2},$$

therefore

$$d(\phi_t(x), \phi_{h(t)}(z)) < \frac{\delta}{2m} + \frac{\delta}{2} \leq \delta.$$

Thus, $z \in \Gamma^{\phi}_{\delta}(x)$. □

The following corollary shows that, in the non-singular case, the study of the expansive measures can be made with the dynamical ball defined in (2).

Corollary 3.4. *If the flow ϕ has no singularities, then for every $\delta > 0$ there exists $\delta' \in (0, \delta)$ such that*

$$\widetilde{\Gamma}^{\phi}_{\delta'}(x, \mathcal{H}) \subset \Gamma^{\phi}_{\delta}(x) \subset \widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{H}) \quad \textit{for all } x \in X.$$

In the compact case we have the following equivalence.

Lemma 3.5. *Let ϕ be a continuous flow on a compact metric space X and let $\delta > 0$. If $\mathcal{F}$ is a compact subset of $\mathcal{H}$, then*

$$\widetilde{\Gamma}^{\phi}_{\delta}(x,\mathcal{F}) = \Gamma^{\phi}_{\delta}(x,\mathcal{F}) \text{ for all } x \in X.$$

Proof. Fix $x \in X$. Since $\Gamma^{\phi}_{\delta}(x,\mathcal{F}) \subset \widetilde{\Gamma}^{\phi}_{\delta}(x,\mathcal{F})$, we show the converse inclusion. Let $z \in \widetilde{\Gamma}^{\phi}_{\delta}(x,\mathcal{F})$. Then, for each $r > 0$ and $\gamma > \delta$ there exists $h \in \mathcal{F}$ with the property that

$$d(\phi_t(x), \phi_{h(t)}(z)) \leq \gamma, \text{ for every } -r \leq t \leq r.$$

Thus, given $m \in \mathbb{N}$, there exists $h_m \in \mathcal{F}$ such that

$$d(\phi_t(x), \phi_{h_m(t)}(z)) \leq \delta + \frac{1}{m}, \text{ for every } -m \leq t \leq m. \tag{6}$$

By compactness of $\mathcal{F}$ we can assume that there exists $f \in \mathcal{F}$ such that $h_m \in \mathcal{B}_f$ for all $m \in \mathbb{N}$ and $h_m \to h$ for some $h \in \mathcal{B}_f \cap \mathcal{F}$. Let $t \in \mathbb{R}$, there is $m_0 \in \mathbb{N}$ such that $-m_0 \leq t \leq m_0$ and by (6) we have $d(\phi_t(x), \phi_{h_{m_0}(t)}(z)) \leq \delta + \frac{1}{m_0}$. Then

$$d(\phi_t(x), \phi_{h_m(t)}(z)) \leq \delta + \frac{1}{m}, \text{ for every } m \geq m_0.$$

Letting $m \to \infty$, we obtain $d(\phi_t(x), \phi_{h(t)}(z)) \leq \delta$. It follows that $z \in \Gamma^{\phi}_{\delta}(x,\mathcal{F})$. □

The next thing we have to do is investigate the topological nature of the dynamical ball (2) in the compact case. Given $(r,\delta) \in \mathbb{R}^2_+$ and given a subset $\mathcal{F}$ of $\mathcal{H}$, we consider the $(r,\delta,\phi,\mathcal{F})$-*open ball*

$$B^{\phi}_r(x,\delta,\mathcal{F}) = \bigcup_{h\in\mathcal{F}} \bigcap_{|t|\leq r} \phi_{-h(t)}(B(\phi_t(x),\delta)),$$

and the $(r,\delta,\phi,\mathcal{F})$-*closed ball*

$$B^{\phi}_r[x,\delta,\mathcal{F}] = \bigcup_{h\in\mathcal{F}} \bigcap_{|t|\leq r} \phi_{-h(t)}(B[\phi_t(x),\delta]).$$

Using these definitions, we can state the following lemma.

Lemma 3.6. *Let ϕ be a continuous flow on a compact metric space X. If $\mathcal{F}$ is a compact subset of $\mathcal{H}$, then the following properties are true for all $(r,\delta) \in \mathbb{R}^2_+$:*

1. *The $(r,\delta,\phi,\mathcal{F})$-open ball is an open set in X.*
2. *The $(r,\delta,\phi,\mathcal{F})$-closed ball is a G_δ set in X.*

Proof. To prove (1), choose $z \in B^{\phi}_r(x,\delta,\mathcal{F})$, there exists $h \in \mathcal{F}$ and $\epsilon > 0$ such that

$$\max_{|t|\leq r}\{d(\phi_t(x), \phi_{h(t)}(z))\} \leq \epsilon < \delta.$$

For $\delta - \epsilon > 0$, take $\gamma > 0$, with the property that $d(\phi_{h(t)}(z), \phi_{h(t)}(y)) < \delta - \epsilon$ for all $|t| \leq r$ whenever $d(z,y) < \gamma$. Thus, if $d(z,y) < \gamma$ then for every $-r \leq t \leq r$ we have

$$d(\phi_t(x), \phi_{h(t)}(y)) \leq d(\phi_t(x), \phi_{h(t)}(z)) + d(\phi_{h(t)}(z), \phi_{h(t)}(y)) < \delta.$$

It follows that $B(z,\gamma) \subset B^{\phi}_r(x,\delta,\mathcal{F})$.

To prove (2) it suffices to prove that for all $x \in X$

$$B^{\phi}_r[x,\delta,\mathcal{F}] = \bigcap_{n=1}^{\infty} B^{\phi}_r\left(x, \delta + \frac{1}{n}, \mathcal{F}\right).$$

Let $z \in \bigcap_{n=1}^{\infty} B_r^{\phi}(x, \delta + \frac{1}{n}, \mathcal{F})$. Then, for every $n \in \mathbb{N}$ there exists $h_n \in \mathcal{F}$ such that

$$d(\phi_t(x), \phi_{h_n(t)}(z)) < \delta + \frac{1}{n}, \text{ for every } -r \leq t \leq r. \tag{7}$$

Since $\mathcal{F}$ is compact, we can suppose that there exists $f \in \mathcal{F}$ with $h_n \in \mathcal{B}_f$ for each $n \in \mathbb{N}$. Again by compactness we can assume that $h_n \to h$ for some $h \in \mathcal{B}_f \cap \mathcal{F}$. Fix $t \in [0, r]$, and letting $n \to \infty$ in (7) we obtain

$$d(\phi_t(x), \phi_{h(t)}(z)) \leq \delta.$$

That is, $z \in B_r^{\phi}[x, \delta, \mathcal{F}]$. The reciprocal inclusion is trivial. □

Corollary 3.7. *Let ϕ be a continuous flow on a compact metric space X. If $\mathcal{F}$ is a compact subset of $\mathcal{H}$, then given $x \in X$ and $\delta > 0$ the $\mathcal{F}$-dependent dynamical ball, $\Gamma_\delta^{\phi}(x, \mathcal{F})$, is a G_δ set in X.*

Proof. By definition of strongly $\mathcal{F}$-dependent dynamical ball and Lemma 3.5 we obtain

$$\Gamma_\delta^{\phi}(x, \mathcal{F}) = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} B_k^{\phi}\left[x, \delta + \frac{1}{n}, \mathcal{F}\right].$$

Also, by Lemma 3.6 item (2), the sets $B_k^{\phi}\left[x, \delta + \frac{1}{n}, \mathcal{F}\right]$ are G_δ sets in X for every $(k, n) \in \mathbb{N}^2$. Then, the $\mathcal{F}$-dependent dynamical ball $\Gamma_\delta^{\phi}(x, \mathcal{F})$ is a G_δ set in X. □

We have the next lemma.

Lemma 3.8. *Let ϕ be a continuous flow of a compact metric space X. If $\mathcal{F} \subset \mathcal{H}$ is regular for ϕ and for any $x \in X$ and every $\delta > 0$ there are $\gamma > 0$ and $y \in X$ such that $d(\phi_t(x), \phi_t(y)) \leq \delta$ for all $t \in \mathbb{R}$ whenever $d(x, y) \leq \gamma$, then $y \in \Gamma_\delta^{\phi}(x, \mathcal{F})$.*

Proof. Fix $x \in X$ and $\delta > 0$. Let $y \in B[x, \gamma]$ where $\gamma > 0$ is such that

$$d(\phi_t(x), \phi_t(y)) \leq \frac{\delta}{2} \text{ for all } t \in \mathbb{R}. \tag{8}$$

By regularity condition of $\mathcal{F}$ there exists $h \in \mathcal{F}$ with the property that

$$d(\phi_t(y), \phi_{h(t)}(y)) \leq \frac{\delta}{2} \text{ for all } t \in \mathbb{R}. \tag{9}$$

Then, from (8) and (9) we obtain

$$d(\phi_t(x), \phi_{h(t)}(y)) \leq d(\phi_t(x), \phi_t(y)) + d(\phi_t(y), \phi_{h(t)}(y)) \leq \delta \text{ for each } t \in \mathbb{R}.$$

So, $y \in \Gamma_\delta^{\phi}(x, \mathcal{F})$. □

The following result is an adaptation of Lemma 3.8 in [4] for the $\mathcal{F}$-dependent dynamical ball. If $f : X \to Y$ is an equivalence between the flows ϕ on X and ψ on Y respectively, then for every $x \in X$ there exists $h_x \in \mathcal{K}$ satisfying

$$f^{-1}(\psi(t, f(x))) = \phi(h_x(t), x) \text{ for each } t \in \mathbb{R}.$$

Lemma 3.9. *Let $\mathcal{F} \subset \mathcal{H}$ such that $\mathcal{K}\mathcal{F}\mathcal{K} \subset \mathcal{F}$ and f be an equivalence between continuous flows ϕ on X and ψ on Y, where X and Y are compact metric spaces. Then for all $\delta > 0$ there exists $\alpha > 0$ with $f^{-1}(\Gamma_\alpha^{\psi}(z, \mathcal{F})) \subset \Gamma_\delta^{\phi}(f^{-1}(z), \mathcal{F})$, for all $z \in Y$.*

Proof. Let $\delta > 0$. By compactness we have that f^{-1} is uniformly continuous, so, there exists $\beta > 0$ with the property that $d(f^{-1}(z), f^{-1}(w)) \leq \delta$ whenever $d(z,w) \leq \beta$ with $z, w \in Y$. Choose $0 < \alpha < \beta$. Given $z, w \in Y$ such that $w \in \Gamma^{\psi}_{\alpha}(z, \mathcal{F})$, there exists $h \in \mathcal{F}$ such that

$$d(\psi_t(z), \psi_{h(t)}(w)) \leq \alpha \text{ for every } t \in \mathbb{R}.$$

By uniform continuity

$$d(f^{-1}(\psi_t(z)), f^{-1}(\psi_{h(t)}(w))) \leq \delta \text{ for every } t \in \mathbb{R}.$$

Then $d(\phi_t(f^{-1}(z)), \phi_{\widehat{h}(t)}(f^{-1}(w))) \leq \delta$ for all $t \in \mathbb{R}$, where $\widehat{h} = h_{f^{-1}(w)} \circ h \circ h^{-1}_{f^{-1}(z)}$ and $h \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{H}$ satisfies $\mathcal{K}\mathcal{F}\mathcal{K} \subset \mathcal{F}$, then $\widehat{h} \in \mathcal{F}$. So $f^{-1}(w) \in \Gamma^{\phi}_{\delta}(f^{-1}(z), \mathcal{F})$. □

The following result is a variant of Lemma 12 in [15].

Lemma 3.10. *Let ϕ be a continuous flow without singularities on a compact metric space X. For each $\lambda > 0$ small enough, there exists $\epsilon > 0$ such that for every $x, y \in X$ and for every interval $[T_1, T_2]$ containing the origin and for every $\alpha \in \mathcal{H}$, the following holds: if $d(\phi_t(x), \phi_{\alpha(t)}(y)) \leq \epsilon$ for all $t \in [T_1, T_2]$, then $|\alpha(t) - t| < \lambda$ for $|t| \leq 2$ in $[T_1, T_2]$ and $|\alpha(t) - t| < |t|\lambda$ for $|t| > 2$ in $[T_1, T_2]$.*

Proof. Without loss of generality we assume that $T_1 = 0$. Fix $0 < \lambda < T_0$. We choose $\gamma > 0$ satisfying the hypothesis of Lemma 3.2. Given $0 < \epsilon < \gamma$ with the property that

$$d(\phi_t(x), \phi_t(y)) < \gamma \text{ for all } 0 \leq t \leq 2 \text{ whenever } d(x,y) \leq \epsilon. \tag{10}$$

Let $\alpha \in \mathcal{H}$ be such that $d(\phi_t(x), \phi_{\alpha(t)}(y)) \leq \epsilon$ for all $t \in [0,2]$. We claim that $|\alpha(t) - t| < \lambda$ for all $t \in [0,2]$. Indeed, otherwise there exists $t_0 \in [0,2]$ such that the continuous function $g(t) = |\alpha(t) - t|$ satisfies $g(t_0) = \lambda$. Without loss of generality we consider the case $\alpha(t_0) > t_0$. Since $d(x,y) \leq \epsilon$ by condition (10) we have that $d(\phi_{t_0}(x), \phi_{t_0}(y)) < \gamma$, and so, by Lemma 3.2 we have

$$\gamma < d(\phi_{t_0}(x), \phi_{\lambda}(\phi_{t_0}(y))) = d(\phi_{t_0}(x), \phi_{\alpha(t_0)-t_0}(\phi_{t_0}(y))) = d(\phi_{t_0}(x), \phi_{\alpha(t_0)}(y)),$$

which contradicts the hypothesis. Since $g(0) = 0$, it follows that $g(t) < \lambda$ for every $t \in [0,2]$. This proves our claim. For the case $t \in [2,4]$, suppose $d(\phi_t(x), \phi_{\alpha(t)}(y)) \leq \epsilon$. Then letting $u = t - 2$, we get

$$d(\phi_u(\phi_2(x)), \phi_{\alpha(u+2)-\alpha(2)}(\phi_{\alpha(2)}(y))) = d(\phi_{u+2}(x), \phi_{\alpha(u+2)}(y)) = d(\phi_t(x), \phi_{\alpha(t)}(y)) \leq \epsilon$$

By defining $G : u \in [0,2] \mapsto \alpha(u+2) - \alpha(2)$ we have $G(0) = 0$ and also

$$d(\phi_u(\phi_2(x)), \phi_{G(u)}(\phi_{\alpha(2)}(y))) \leq \epsilon \text{ for all } 0 \leq u \leq 2.$$

By repeating the above argument we obtain that $|G(u) - u| < \lambda$ for every $u \in [0,2]$, that is, for each $t \in [2,4]$

$$\lambda \geq |G(t-2) - (t-2)| = |\alpha(t) - \alpha(2) - t + 2| \geq |\alpha(t) - t| - |\alpha(2) - 2|,$$

and it follows that $|\alpha(t) - t| \leq 2\lambda$. Using a similar argument one can show inductively that for every $n \geq 1$:

$$|\alpha(t) - t| \leq n\lambda, \text{ whenever } 2n - 2 \leq t \leq 2n.$$

Finally, for each $t > 2$ in $[0, T_2]$ we have

$$|\alpha(t) - t| \leq n\lambda = \frac{n}{t} t\lambda \leq t\lambda.$$

□

Lemma 3.11. *Let ϕ be a continuous flow without singularities on a compact metric space X. There exists $\epsilon > 0$ such that for every $x \in X$, $r > 0$ and each pair of sequences h_n in $\mathcal{H}$ and y_n in X with $y_n \to y$, where $y \in X$, the following holds: if $d(\phi_t(x), \phi_{h_n(t)}(y_n)) \leq \epsilon$ for all $(n,t) \in \mathbb{N} \times [-r,r]$, then for each $\delta > 0$ there exists an $M \in \mathbb{N}$ satisfying*

$$d(\phi_{h_n(t)}(y_n), \phi_{h_n(t)}(y)) \leq \delta \quad \textit{for every } -r \leq t \leq r \textit{ and } n \geq M.$$

Proof. Given $0 < \lambda < T_0$ we can choose $\epsilon > 0$ satisfying Lemma 3.10 with respect to λ. If the result is not true, then there are subsequences y_{n_k}, h_{n_k} and t_k such that $-r \leq t_k \leq r$ with the property that

$$d(\phi_{h_{n_k}(t_k)}(y_{n_k}), \phi_{h_{n_k}(t_k)}(y)) > \delta \quad \text{for every } k \in \mathbb{N}. \tag{11}$$

By Lemma 3.10 for each $k \in \mathbb{N}$ we have

$$|h_{n_k}(t_k) - t_k| < \lambda \max\{|t_k|, 1\}.$$

Since $-r \leq t_k \leq r$, then there are $a_r, b_r \in \mathbb{R}$ such that $a_r \leq h_{n_k}(t_k) \leq b_r$ for every $k \in \mathbb{N}$. Thus, we can assume that $h_{n_k}(t_k) \to t_0$ where $t_0 \in [a_r, b_r]$. Letting $k \to \infty$ in (11) we obtain a contradiction. □

Next we explore the topological properties of the dynamical ball defined in (2). Denote by 2^X_c the space of all compact subsets of X endowed with the *Hausdorff distance* d_H [10]. The space $(2^X_c, d_H)$ is itself a compact metric space. A set-valued map $\Psi : X \to 2^X_c$ is said upper-semicontinuous if for every $x \in X$ and any open $V \subset X$ containing $\Psi(x)$, there exists a neighborhood U of x in X such that V contains $\Psi(w)$ for all $w \in U$. With these definitions we obtain the following result.

Lemma 3.12. *If the flow ϕ has no singularities, then there exists $\delta_0 > 0$ such that the following properties hold for every $\mathcal{F} \subset \mathcal{H}$, every $\delta \in (0, \delta_0)$:*

1. *For every $x \in X$ the strongly $\mathcal{F}$-dependent dynamical ball, $\widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{F})$, is compact.*
2. *The set-valued map*

$$\begin{array}{rccc} \Phi: & X & \longrightarrow & 2^X_c \\ & x & \mapsto & \widetilde{\Gamma}^{\phi}_{\delta}(x, \mathcal{F}), \end{array}$$

is upper-semicontinuous.

Proof. Given $0 < \lambda < T_0$ we can choose $\delta_0 > 0$ satisfying Lemma 3.11 with respect to λ. Let $0 < \delta < \delta_0$. In order to prove item (1) it is sufficient to prove that for every $x \in X$ and $r > 0$, the set $\bigcap_{\gamma > \delta} B^{\phi}_r[x, \gamma, \mathcal{F}]$ is closed in X. Fix $(r, x) \in \mathbb{R}_+ \times X$. Let y_n be any sequence in $\bigcap_{\gamma > \delta} B^{\phi}_r[x, \gamma, \mathcal{F}]$ and assume that y_n converges to y in X. Given $\gamma > 0$ such that $\delta < \gamma < \delta_0$ take $\delta < \beta < \gamma$. Then there exists a sequence h_n in $\mathcal{F}$ such that

$$d(\phi_t(x), \phi_{h_n(t)}(y_n)) \leq \beta \quad \text{for each } |t| \leq r. \tag{12}$$

Since $\gamma - \beta > 0$, using Lemma 3.11, there is an $M \in \mathbb{N}$ satisfying

$$d(\phi_{h_n(t)}(y_n), \phi_{h_n(t)}(y)) \leq \gamma - \beta \quad \text{for every } |t| \leq r \text{ and } n \geq M. \tag{13}$$

By (12) and (13) for all $-r \leq t \leq r$ and $n \geq M$ we have

$$d(\phi_t(x), \phi_{h_n(t)}(y)) \leq d(\phi_t(x), \phi_{h_n(t)}(y_n)) + d(\phi_{h_n(t)}(y_n), \phi_{h_n(t)}(y)) \leq \gamma.$$

Then $y \in B_r^{\phi}[x,\gamma,\mathcal{F}]$ and since $\gamma > \delta$ was chosen arbitrarily, the result follows.

To prove (2), by item (1), the set-valued map

$$\begin{array}{rccc}\Phi: & X & \longrightarrow & 2_c^X \\ & x & \mapsto & \widetilde{\Gamma}_{\delta}^{\phi}(x,\mathcal{F}),\end{array}$$

is well defined. Fix $x \in X$. If Φ is not upper-semicontinuous in x, then there exists an open neighborhood V of $\Phi(x)$ and a sequence x_n converging to x such that $\Phi(x_n) \not\subset V$ for all $n \in \mathbb{N}$. Then, we can select a sequence $z_n \in \Phi(x_n) \setminus V = \widetilde{\Gamma}_{\delta}^{\phi}(x_n,\mathcal{F}) \setminus V$. Given $m \in \mathbb{N}$ and $r > 0$, there exists a sequence $g_n \in \mathcal{F}$ such that

$$d(\phi_t(x_n), \phi_{g_n(t)}(z_n)) \leq \delta + \frac{1}{3m}, \text{ for all } |t| \leq r. \tag{14}$$

By compactness we can assume that $z_n \to z$ for some $z \in X$. Since V is open, $z \notin V$. Also, there exists $K \in \mathbb{N}$ such that

$$d(\phi_t(x_n), \phi_t(x)) \leq \frac{1}{3m} \text{ for every } |t| \leq r \text{ and } n \geq K. \tag{15}$$

Then by (14) and (15) for every $-r \leq t \leq r$ and $n \geq K$ we obtain

$$d(\phi_t(x), \phi_{g_n(t)}(z_n)) \leq d(\phi_t(x), \phi_t(x_n)) + d(\phi_t(x_n), \phi_{g_n(t)}(z_n)) \leq \delta + \frac{2}{3m}. \tag{16}$$

If m is chosen such that $\delta + \frac{2}{3m} < \delta_0$, then by Lemma 3.11 there is an $M \in \mathbb{N}$ satisfying

$$d(\phi_{g_n(t)}(z_n), \phi_{g_n(t)}(z)) \leq \frac{1}{3m} \text{ for every } |t| \leq r \text{ and } n \geq M. \tag{17}$$

Then by (16) and (17) we obtain for each $-r \leq t \leq r$ and $j \in \mathbb{N}$ large enough

$$d(\phi_t(x), \phi_{g_j(t)}(z)) \leq d(\phi_t(x), \phi_{g_j(t)}(z_j)) + d(\phi_{g_j(t)}(z_j), \phi_{g_j(t)}(z)) \leq \delta + \frac{1}{m}.$$

It follows that $z \in \Phi(x) = \widetilde{\Gamma}_{\delta}^{\phi}(x,\mathcal{F}) \subset V$. Then, $z \in V$ which is a contradiction. □

The following corollary is then a direct consequence of Lemmas 3.5 and 3.12.

Corollary 3.13. *If the flow ϕ has no singularities and $\mathcal{F} \subset \mathcal{H}$ is compact, then there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$ the dynamical ball $\Gamma_{\delta}^{\phi}(x,\mathcal{F})$ is compact for all $x \in X$.*

4. Proofs

Proof of Theorem 2.6. Since $\mathcal{F} \subset (\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})$, by Lemma 3.1, each Borel probability measure $\Big((\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})\Big)$-expansive is $\mathcal{F}$-expansive.

Conversely, let $\delta > 0$ be the expansivity constant of μ. It is enough to show that

$$\Gamma_{\frac{\delta}{2}}^{\phi}(x, (\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})) \subset \Gamma_{\delta}^{\phi}(x,\mathcal{F}) \text{ for all } x \in X.$$

Let $z, x \in X$ be such that $z \in \Gamma_{\frac{\delta}{2}}^{\phi}(x, (\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})}))$. Then, there is $h \in (\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})$ with the property that

$$d(\phi_t(x), \phi_{h(t)}(z)) \leq \frac{\delta}{2} \text{ for all } t \in \mathbb{R}. \tag{18}$$

Since $\phi^*(h) \in \overline{\phi^*(\mathcal{F})}$, there exists $f \in \mathcal{F}$ such that $\phi^*(f) \in N(\phi^*(h), z, \frac{\delta}{2})$. It follows that

$$d(\phi_{h(t)}(z), \phi_{f(t)}(z)) \leq \frac{\delta}{2} \text{ for all } t \in \mathbb{R}. \tag{19}$$

Therefore, from (18) and (19) we have

$$d(\phi_t(x), \phi_{f(t)}(z)) \leq d(\phi_t(x), \phi_{h(t)}(z)) + d(\phi_{h(t)}(z), \phi_{f(t)}(z)) \leq \delta \text{ for every } t \in \mathbb{R}.$$

Thus, we can conclude that $z \in \Gamma^{\phi}_{\delta}(x, \mathcal{F})$. □

Proof of Corollary 2.7. By continuity of ϕ^*, we obtain $\phi^*(\overline{\mathcal{F}}) \subset \overline{\phi^*(\mathcal{F})}$. Then, we have the following inclusion $\overline{\mathcal{F}} \subset (\phi^*)^{-1}(\overline{\phi^*(\mathcal{F})})$. Thus, by Lemma 3.1 and Theorem 2.6, if μ is $\mathcal{F}$-expansive, then μ is $\overline{\mathcal{F}}$-expansive. □

Proof of Theorem 2.9. To prove Item (1), it is sufficient to apply Corollary 3.7.

To prove Item (2) if μ is $\mathcal{F}$-expansive, by Lemma 3.1, then μ is $\{f\}$-expansive for every $f \in \mathcal{F}$. Conversely, let $\delta > 0$ be the expansivity constant of μ. Given $f \in \mathcal{F}$, by compactness argument we can show that there exists $\epsilon > 0$ such that $\phi_{(-\epsilon,\epsilon)}(x) \subset B(x, \frac{\delta}{2})$ for every $x \in X$. We define $\mathcal{U}_f = \{g \in \mathcal{H} : \widehat{d}(f, g) < \epsilon\}$ the which is an open subset in $(\mathcal{B}_f, d_f)$. We claim that

$$\Gamma^{\phi}_{\frac{\delta}{2}}(x, \mathcal{U}_f) \subset \Gamma^{\phi}_{\delta}(x, \{f\}) \text{ for all } x \in X. \tag{20}$$

Let $z, x \in X$ be such that $z \in \Gamma^{\phi}_{\frac{\delta}{2}}(x, \mathcal{U}_f)$. Then, there is $g \in \mathcal{U}_f$ such that

$$d(\phi_t(x), \phi_{g(t)}(z)) \leq \frac{\delta}{2} \text{ for each } t \in \mathbb{R}. \tag{21}$$

Fix $t \in \mathbb{R}$. Since $g \in \mathcal{U}_f$ and $\phi_{(-\epsilon,\epsilon)}(\phi_{f(t)}(z)) \subset B(\phi_{f(t)}(z), \frac{\delta}{2})$, we have

$$d(\phi_{f(t)}(z), \phi_{g(t)}(z)) = d(\phi_{f(t)}(z), \phi_{g(t)-f(t)}(\phi_{f(t)}(z))) < \frac{\delta}{2}. \tag{22}$$

From (21) and (22), we obtain

$$d(\phi_t(x), \phi_{f(t)}(z)) \leq d(\phi_t(x), \phi_{g(t)}(z)) + d(\phi_{f(t)}(z), \phi_{g(t)}(z)) < \delta \text{ for every } t \in \mathbb{R}.$$

That is, $z \in \Gamma^{\phi}_{\delta}(x, \{f\})$. Thus, for each $f \in \mathcal{F}$ there is an open neighborhood $\mathcal{U}_f$ of f such that (20) holds. By compactness, choose $f_1, \cdots, f_m \in \mathcal{F}$ such that $\mathcal{F} \subset \bigcup_{i=1}^{m} \mathcal{U}_{f_i}$ and by Lemma 3.1, μ is $\mathcal{F}$-expansive. □

Proof of Theorem 2.13. To prove Item (1), by the definition of $\mathcal{F}$-expansiveness of μ, there exists $\delta > 0$. By c.f. p. 506 in [1], there exists $\alpha > 0$ such that if

$$y \in \phi_{(-\alpha,\alpha)}(x), \text{ then } d(\phi_t(x), \phi_t(y)) < \frac{\delta}{2} \text{ for all } (x, t) \in X \times \mathbb{R}. \tag{23}$$

Let $y \in \phi_{(-\alpha,\alpha)}(x)$. Since $\mathcal{F}$ is regular, then by (23) and Lemma 3.8 it follows that $y \in \Gamma^{\phi}_{\delta}(x, \mathcal{F})$. That is, $\phi_{(-\alpha,\alpha)}(x) \subset \Gamma^{\phi}_{\delta}(x, \mathcal{F})$ for all $x \in X$. Let $x \in X$. Then $\phi_{\mathbb{R}}(x)$ is separable since X is compact. Then there exists a sequence $\{x_n\} \subset \phi_{\mathbb{R}}(x)$ dense in $\phi_{\mathbb{R}}(x)$ and $\{\phi_{(-\alpha,\alpha)}(x_n) : n \in \mathbb{N}\}$ covers $\phi_{\mathbb{R}}(x)$ so that

$$\mu(\phi_{\mathbb{R}}(x)) \leq \sum_{n \in \mathbb{N}} \mu(\phi_{\mathbb{R}}(x_n)) = 0.$$

Item (2) follows from the definition of $\mathcal{F}$-expansiveness and the monotony of the measure.

We now prove (3). Since μ is expansive, there is a $\delta > 0$ such that for every $\sigma \in Sing(\phi)$ we have $\mu(\Gamma^{\phi}_{\delta}(\sigma, \mathcal{F})) = 0$. Given $h \in \mathcal{F} \cap \mathcal{B}_0$, let $\lambda > 0$ be such that $|h(t)| \leq \lambda$ for every $t \in \mathbb{R}$. Fix $\sigma_0 \in Sing(\phi)$. There exists $\gamma > 0$ such that

$$d(\phi_s(y), \sigma_0) \leq \delta \text{ for every } |s| \leq \lambda, \text{whenever } d(\sigma_0, y) \leq \gamma.$$

So, if $y \in B[\sigma_0, \gamma]$, then $d(\phi_{h(t)}(y), \sigma_0) \leq \delta$ for every $t \in \mathbb{R}$. That is, $B[\sigma_0, \gamma] \subset \Gamma^{\phi}_{\delta}(\sigma_0, \mathcal{F})$. Therefore, $\mu(B[\sigma_0, \gamma]) = 0$. It follows that $\sigma \notin supp(\mu)$.

To prove Item (4) let $f : X \to Y$ be an equivalence between continuous flows ϕ on X and ψ on Y. By Lemma 3.9, for all $\delta > 0$ there is $\alpha > 0$ such that $f^{-1}(\Gamma^{\psi}_{\alpha}(z, \mathcal{F})) \subset \Gamma^{\phi}_{\delta}(f^{-1}(z), \mathcal{F})$. Let $\delta > 0$ be the expansivity constant of μ. Let $z \in Y$ and let B be a Borel set such that $B \subset \Gamma^{\psi}_{\alpha}(z, \mathcal{F})$. By Lemma 3.9, $f^{-1}(B) \subset \Gamma^{\phi}_{\delta}(f^{-1}(z), \mathcal{F})$ so that $f_*\mu(B) = \mu(f^{-1}(B)) = 0$ since $\mu(\Gamma^{\phi}_{\delta}(f^{-1}(z), \mathcal{F})) = 0$. That is, $f_*\mu(\Gamma^{\psi}_{\alpha}(z, \mathcal{F})) = 0$. The converse is analogous (just replace f by f^{-1}).

To prove Item (5) suppose $T > 0$. For every $\delta > 0$ there exists $\alpha > 0$ such that

$$d(\phi_t(z), \phi_t(w)) \leq \delta \text{ for all } t \in [0, T] \text{ whenever } z, w \in X \text{ and } d(z, w) \leq \alpha.$$

Let $x, y \in X$ with $y \in \Gamma^{\phi_T}_{\alpha}(x)$. Given $t \in \mathbb{R}$ there exists a unique $m \in \mathbb{Z}$ such that $t \in [mT, (m+1)T]$. Then

$$d(\phi_t(x), \phi_t(y)) = d(\phi_{t-mT}(\phi_{mT}(x)), \phi_{t-mT}(\phi_{mT}(y))) \leq \delta.$$

From $d(\phi_{mT}(x), \phi_{mT}(y)) \leq \alpha$ and $t - mT \in [0, T]$, it follows that $d(\phi_t(x), \phi_t(y)) \leq \delta$. By the regularity condition of $\mathcal{F}$ and Lemma 3.8 we obtain that $y \in \Gamma^{\phi}_{\delta}(x, \mathcal{F})$. Thus $\Gamma^{\phi_T}_{\alpha}(x) \subset \Gamma^{\phi}_{\delta}(x, \mathcal{F})$ and the proof follows.

To prove Item (6) if $T^{1,f}(\mu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$, by Theorem 2.13 item (5), $T^{1,f}(\mu)$ is also expansive for the homeomorphism $\phi^{1,f}_1$. Since $\phi^{1,f}_1 = f \times id$ for all $(x, t) \in Y^{1,f}$ we have that $T^{1,f}(\mu)$ is expansive for $f \times id : Y^{1,f} \to Y^{1,f}$. Let $\delta > 0$ be the expansivity constant of $T^{1,f}(\mu)$ with the property that $0 < \delta < \frac{1}{2}$. By definition of Bowen-Walters metric, we conclude that for all $x \in X$, there are $t_1, \cdots, t_{k(x)} \in [0, 1)$ satisfying $[0, 1) = \bigcup_{1 \leq j \leq k(x)} [t_j, t_{j+1})$ and

$$\Gamma^{f}_{\frac{\delta}{2}}(x) \times [0, 1) \subset \bigcup_{j=1}^{r=k(x)} \Gamma^{f \times id}_{\delta}(x, t^*_j),$$

where t^*_j is the midpoint of $[t_j, t_{j+1})$. Then, by the expansiveness of $T^{1,f}(\mu)$, for each $x \in X$ we have

$$\mu(\Gamma^{f}_{\frac{\delta}{2}}(x)) \leq \sum_{j=1}^{j=k(x)} \int_{\Gamma^{f \times id}_{\delta}(x, t^*_j)} dT^{1,f}(\mu) = \sum_{j=1}^{j=k(x)} T^{1,f}(\mu)(\Gamma^{f \times id}_{\delta}(x, t^*_j)) = 0.$$

It follows that μ is expansive for f.

Finally, To prove Item (7), we see that by Theorem 2.4 in [4], $T^{1,f}(\mu)$ is expansive for $\phi^{1,f}$. By Lemma 3.1 and since $\mathcal{F} \subset \mathcal{H}$ it follows that $T^{1,f}(\mu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$. □

Proof of Theorem 2.16. For each $\delta, \epsilon > 0$ we define

$$C(\delta, \epsilon, \mathcal{F}) = \{\mu \in \mathcal{M}(X) : \mu(\Gamma^{\phi}_{\delta}(x, \mathcal{F})) \geq \epsilon \text{ for some } x \in X\}.$$

It follows that

$$\mathcal{M}_{ex}(X,\phi,\mathcal{F}) = \bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty}\left(\mathcal{M}(X)\setminus C\left(\frac{1}{n},\frac{1}{m},\mathcal{F}\right)\right).$$

If we prove that $C(\delta,\epsilon,\mathcal{F})$ is closed in $\mathcal{M}(X)$ for all $\delta,\epsilon>0$, then $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ is a $G_{\delta\sigma}$ subset of $\mathcal{M}(X)$, that is, the union of countably many G_δ subsets of $\mathcal{M}(X)$. Thus, by Corollary 6 in [12], $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ is a Baire subset of $\mathcal{M}(X)$.

To prove that $C(\delta,\epsilon,\mathcal{F})$ is closed, take a sequence $\mu_n\in C(\delta,\epsilon,\mathcal{F})$ such that $\mu_n\to\mu$ for some $\mu\in\mathcal{M}(X)$. Choose a sequence $x_n\in X$ such that

$$\epsilon\leq\mu_n(\Gamma_\delta^\phi(x_n,\mathcal{F})),\ \text{for all}\ \ n\in\mathbb{N}.$$

Suppose that $x_n\to x$ for some $x\in X$. Fix an open neighborhood U of $\Gamma_\delta^\phi(x,\mathcal{F})$. Now we analyze the following two cases:

Case 1: $\mathcal{F}$ is a compact subset of $\mathcal{H}$.

In this case suppose there exists a subsequence $n_k\to\infty$ such that $\Gamma_\delta^\phi(x_{n_k},\mathcal{F})\not\subset U$ for all $k\in\mathbb{N}$. Then, we can select a sequence $z_k\in\Gamma_\delta^\phi(x_{n_k},\mathcal{F})\setminus U$ and so, by definition of dynamical ball, we obtain a sequence $g_k\in\mathcal{F}$ such that

$$d(\phi_t(x_{n_k}),\phi_{g_k(t)}(z_k))\leq\delta,\ \text{for each}\ t\in\mathbb{R}. \tag{24}$$

Since $\mathcal{F}$ is compact, we can assume $z_k\to z$ and $g_k\to g$ for some $z\in X$ and $g\in\mathcal{F}$. As U is open, $z\notin U$. Fixing $t\in\mathbb{R}$ on (24) and letting $k\to\infty$ we obtain

$$d(\phi_t(x),\phi_{g(t)}(z))\leq\delta.$$

Hence we obtain that $z\in\Gamma_\delta^\phi(x,\mathcal{F})$. Then $z\in U$, which is a contradiction.

Case 2: The flow ϕ has no singularities.

In this case, by Corollary 3.4 we can work with the $\mathcal{H}$-dependent dynamical ball, $\widetilde{\Gamma}_\delta^\phi(x,\mathcal{H})$. Then, by Lemma 3.12 item (2), the function Φ is upper semicontinuous and so $\Phi(x_n)\subset U$ holds for n large.

Therefore, in both cases, we have that $\Gamma_\delta^\phi(x_n,\mathcal{F})\subset U$ holds for n large. Since $\mu_n\to\mu$ we obtain

$$\epsilon\leq\limsup_{x\to\infty}\mu_n(\Gamma_\delta^\phi(x_n,\mathcal{F}))\leq\limsup_{x\to\infty}\mu_n(\overline{U})\leq\mu(\overline{U}).$$

We can choose U such that $\mu(\partial U)=0$. Then $\epsilon\leq\mu(\Gamma_\delta^\phi(x,\mathcal{F}))$. It follows that $C(\delta,\epsilon,\mathcal{F})$ is closed in $\mathcal{M}(X)$ for all $\delta,\epsilon>0$. □

Proof of Theorem 2.19. Let ϕ be a continuous flow with $\mathcal{F}$-expansive measures of a compact metric space X. By Corollary 1 p.71 in [10], the set of discontinuities $\mathcal{D}$ of the set-valued map $\Psi:\mathcal{M}_{ex}(X,\phi,\mathcal{F})\to 2_c^X$ defined by $\Psi(\mu)=\text{supp}(\mu)$ is meagre in $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$. Then, the set $\mathcal{R}=\mathcal{M}_{ex}(X,\phi,\mathcal{F})\setminus\mathcal{D}$ is dense in $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$. Given $\mu\in\mathcal{R}$ and $\nu\in\mathcal{M}_{ex}(X,\phi,\mathcal{F})$, define the measure μ_n with the property that $\mu_n=(1-\frac{1}{n})\mu+\frac{1}{n}\nu$ for each $n\in\mathbb{N}$. Then $\mu_n\in\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ and $\mu_n\to\mu$ as $n\to\infty$. Since $\mu\notin\mathcal{D}$, Ψ is continuous at μ and so $\Psi(\mu_n)=\text{supp}(\mu)\cup\text{supp}(\nu)$ converges to $\Psi(\mu)=\text{supp}(\mu)$. Therefore, $\text{supp}(\nu)\subset\text{supp}(\mu)$. It follows that $E(\phi,\mathcal{F})=\text{supp}(\mu)$. Thus, there exists a dense subset $\mathcal{R}$ of $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ whose supports are all equal to $E(\phi,\mathcal{F})$. □

Proof of Corollary 2.20. Suppose that the set of $\mathcal{F}$-expansive measures $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ is dense in $\mathcal{M}(X)$. By Lemma 10 in [12], $E(\phi,\mathcal{F})=X$. The Theorem 2.19 provides a dense subset $\mathcal{R}$ of $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ such that $\operatorname{supp}(\mu)=E(\phi,\mathcal{F})=X$ for all $\mu\in\mathcal{R}$. Since $\mathcal{M}_{ex}(X,\phi,\mathcal{F})$ is dense in $\mathcal{M}(X)$, we obtain that $\mathcal{R}$ is dense in $\mathcal{M}(X)$. □

Proof of Corollary 2.21. By Corollary 2.20 there exists an $\mathcal{F}$-expansive measure w for the flow such that $\operatorname{supp}(w)=X$. If z is an isolated point of X, then $\{z\}$ is a neighborhood of z and so $w(\{z\})>0$. By the regularity condition of $\mathcal{F}$, we have $z\in\Gamma^{\phi}_{\delta}(z,\mathcal{F})$, and therefore $w(\{z\})=0$, which leads to a contradiction. □

Proof of Theorem 2.22. Let $\mathcal{M}_{ex}(X,f)$ be the set of expansive measures of the homeomorphism f. Then, if $\mu\in\mathcal{M}_{ex}(X,f)$ it follows of Theorem 2.13 item (7) that $T^{1,f}(\mu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$. Since the flow $\phi^{1,f}$ has no singularities, by Theorems 2.16 and 2.19, the set $\mathcal{M}_{ex}(Y^{1,f},\phi^{1,f},\mathcal{F})$ has a dense subset $\{w_k\}$ of $\mathcal{F}$-expansive measures with support equal to the $\mathcal{F}$-measure-expansive center.

Given $k\in\mathbb{N}$, we define $\mu_k\in\mathcal{M}(X)$ such that $\mu_k=(\pi\circ i^{-1})_*w_k$ where $i: X\times[0,1)\to Y^{1,f}$ is the inclusion map and $\pi: X\times[0,1)\to X$ is the first projection. Taking $\mathcal{F}$ regular, we can repeat the argument of Theorem 2.13 item (6) to have that u_k is expansive for f.

We claim that the set of measures $\{\mu_k\}$ is dense in $\mathcal{M}_{ex}(X,f)$. Fix $\nu\in\mathcal{M}_{ex}(X,f)$. Then, by Theorem 2.13 item (7), $T^{1,f}(\nu)$ is $\mathcal{F}$-expansive for $\phi^{1,f}$. Thus, there is a sequence $\{w_k\}_{k\in\mathbb{N}}$ such that $w_k\to T^{1,f}(\nu)$. Let $\psi: X\to\mathbb{R}$ be a continuous function. Define $h: Y^{1,f}\to\mathbb{R}$ such that $h(x,t)=\psi(x)$ whenever $0\le t<1$. By weak convergence we obtain

$$\int_X \psi(x)d\mu_k(x)=\int_{Y^{1,f}} h(x,t)dw_k(x,t)\to\int_{Y^{1,f}} hdT^{1,f}(\nu)=\int_X\psi(x)d\nu(x).$$

Thus, $\mu_k\to\nu$, that is, the sequence $\{\mu_k\}$ is dense in $\mathcal{M}_{ex}(X,f)$. To complete the proof, given $\eta\in\mathcal{M}_{ex}(X,f)$ and $k\in\mathbb{N}$, then $\operatorname{supp}(w_k)=E(\phi^{1,f},\mathcal{F})$. Therefore

$$\operatorname{supp}(\eta)\times[0,1)\subset\operatorname{supp}(T^{1,f}(\eta))\subset E(\phi^{1,f},\mathcal{F})=\operatorname{supp}(w_k)\subset\operatorname{supp}(\mu_k)\times[0,1).$$

Hence, $\operatorname{supp}(\eta)\subset\operatorname{supp}(\mu_k)$ and thus, $\operatorname{supp}(\mu_k)=E(f)$. □

References

1. Artigue, A., Expansive flows on surfaces, *Discrete Contin. Dyn. Syst.* 33 (2013), no.2, 505–525.
2. Bowen, R., Walters, P., Expansive one-parameter flows, *J. Differential Equations*, 12 (1972), 180–193.
3. Bugaro, D., Bugaro, Y., Ivanov, S., *A course in metric geometry*, Providence, RI: American Mathematical Society, 2001. 415 p. (Graduate studies in Mathematics; v. 33).
4. Carrasco-Olivera, D., Morales, C.A., Expansive measures for flows, *J. Differential Equations*, Volume 256, Issue 7, 1 April 2014, Pages 2246–2260.
5. Dydak, J., Hoffland, C.S., An alternative definition of coarse structures, *Topology and its Applications*, Volume 155, Issue 9, 15 April 2008, Pages 1013–1021.
6. Fakhari, Abbas, Morales, C.A., Tajbakhsh, Khosro, Asymptotic measure expansive diffeomorphisms, *J. Math. Anal. Appl.* 435 (2016), no. 2, 1682–1687.
7. Iommi, G., Jordan, T., Todd, M., Recurrence and transience for suspension flows, *Israel J. Math.* 209 (2015), no. 2, 547–592.
8. Keynes, H.B., Sears, M., $\mathcal{F}$-expansive transformation groups, *General Topology and its Applications*, Volume 10, Issue 1, February 1979, Pages 67–85.
9. Knowles, J.D., On the existence of non-atomic measures, *Mathematika*, 14 (1967), 62–67.
10. Kuratowski, K., *Topology. Vol. II*, New edition, revised and augmented. Translated from the French by A. Kirkor Academic Press, New York-London; Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw 1968.
11. Lee, K., Oh, J., Weak measure expansive flows, *J. Differential Equations* 260 (2016), no. 2, 1078–1090.

12. Morales, C.A., On supports of expansive measures, Preprint arXiv:1601.03618v1 [math.DS] 14 Jan 2016.
13. Morales, C.A., Measure-expansive systems, Preprint IMPA Série D (2011).
14. Norton, V., O'Brien, T., Anosov flows and expansiveness, *Proc. Amer. Math. Soc.* 40 (1973), 625–628.
15. Thomas, R.F., Entropy of expansive flows, *Ergodic Theory Dynam. Systems* 7 (1987), no. 4, 611–625.
16. Utz, W.R., Unstable homeomorphisms, *Proc. Amer. Math. Soc.* 1 (1950), no. 6, 769–774.
17. Willard, S., *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1970.

Instituto de Matemática y Ciencias Afines. Calle los Biológos 245–Urb. San César–Primera Etapa, la Molina, Lima 12, Perú.
E-mail address: hvillavicencio@imca.edu.pe.