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Doctoral Thesis

C^r-DENSITY OF (NON-UNIFORM) HYPERBOLICITY IN PARTIALLY HYPERBOLIC SYMPLECTIC DIFFEOMORPHISMS

Karina Daniela Marin

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Advisor: Marcelo Viana

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Abstract

A volume-preserving diffeomorphism is said to be hyperbolic, in the non-uniform sense, if its Lyapunov exponents are different from zero at almost every point. This work is concerned with the classical question: *which diffeomorphisms may be approximated by non-uniformly hyperbolic ones?* More specifically, *is that always the case for a partially hyperbolic diffeomorphism?*

While the latter question is completely solved (in the affirmative) in the C^1 case, only a few results are known for the C^r topology with $r > 1$. We use the Invariance Principle of Avila and Viana to give a partial answer in the symplectic setting. Indeed, we prove that every partially hyperbolic symplectic diffeomorphism with 2-dimensional center bundle, and satisfying certain pinching and bunching conditions, can be C^r-approximated by non-uniformly hyperbolic diffeomorphisms.

Contents

1 Introduction

In the theory of Dynamical Systems, hyperbolicity is a core concept whose roots may be traced back to Hadamard and Perron and which was first formalized by Smale [S] in the 1960's. It implies several features that are most effective to describe the system's dynamical behavior.

While Smale's uniform hyperbolicity was soon realized to be a fairly restrictive property, a more flexible version was proposed by Pesin [P] about a decade later: one speaks of *non-uniform hyperbolicity* when all the Lyapunov exponents are different from zero, almost everywhere with respect to some preferred invariant measure (for instance, a volume measure).

While being more general, non-uniform hyperbolicity still has many important consequences, most notably: the stable manifold theorem (Pesin [P]), the abundance of periodic points and Smale horseshoes (Katok [K]) and the fact that the fractal dimension of invariant measures is well defined (Barreira, Pesin and Schmelling [BPS]). Thus, the question of how general non-uniform hyperbolicity is, naturally arises, and indeed, it goes back to Pesin's original work.

However, the set of non-uniform hyperbolic systems is usually *not* dense. Herman (see the presentation of Yoccoz [Y]) constructed open subsets of C^r, with large r, volume-preserving diffeomorphisms admitting invariant subsets with positive volume consisting of codimension-1 quasi-periodic tori: on such subsets all the Lyapunov exponents vanish identically. Other examples with a similar flavor were found by Cheng and Sun [CS] and Xia [X].

Before that, in the early 1980's, Mañé [Ma] observed that every area-preserving diffeomorphism that is not Anosov can be C^1-approximated by diffeomorphisms with zero Lyapunov exponents. His arguments were completed by Bochi [B1] and were extended to arbitrary dimension by Bochi and Viana [BV1, B2]. In particular, Bochi [B2] proved that every partially hyperbolic symplectic diffeomorphism can be C^1-approximated by partially hyperbolic diffeomorphisms whose center Lyapunov exponents vanish.

By the end of last century, Alves, Bonatti and Viana were studying the ergodic properties of partially hyperbolic diffeomorphisms. In [BV2, ABV] they proved that under some amount of hyperbolicity along the center bundle ('mostly contracting' or 'mostly expanding' center direction) the diffeomorphism admits finitely many physical measures and the union of their basins contains almost every point in the manifold.

This again raised the question of how frequent non-uniform hyperbolicity is, this time focusing on the partially hyperbolic setting. *Can one always approximate the diffeomorphism by another whose center Lyapunov exponents are non-zero?* This question was the origin of a whole research program, focusing first on linear cocycles and dealing more recently also with non-linear systems. We refer the reader to the book of Viana [V] for a detailed survey of some of the progress attained so far.

Our own results are based on methods that were developed in these 15 years or so and may be viewed as the fulfillment of that program in the context of symplectic diffeomorphisms with 2-dimensional center. We proved (all the keywords will be recalled in the next section):

Theorem A. *Let $f : M \longrightarrow M$ be a partially hyperbolic symplectic C^r diffeomorphism on a compact manifold M. Assume that f is accessible, center-bunched and pinched, the set of periodic points is non-empty, and the center bundle E^c is 2-dimensional. Then, f can be C^r-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.*

Let us stress that our perturbation holds in the C^r topology, for any $r \in [2, +\infty)$. The case $r = 1$ is very special and much better understood.

The first result along these lines was due to Shub and Wilkinson [SW1], who proved that certain partially hyperbolic skew-products with circle center leaves can be perturbed to make the center Lyapunov exponent different from zero. Their approach relates the issue of non-uniform hyperbolicity into the analysis of the center foliation and its measure-theoretical properties, a connection that has been much deepened and clarified in the recent work of Avila, Viana and Wilkinson [AVW].

Baraviera and Bonatti [BB] extended the approach of Shub and Wilkinson to prove that any stably ergodic partially hyperbolic diffeomorphism can be C^1-approximated by another for which the *sum* of the center Lyapunov exponents is non-zero. This fact, together with the results mentioned above of Bochi and Viana [BV1], were used by Bochi, Fayad and Pujals [BFP], to prove that every $C^{1+\alpha}$ stably ergodic diffeomorphism can be C^1-approximated by non-uniformly hyperbolic ones. More recently, Avila, Crovisier and Wilkinson [ACW] proved a general theorem that implies that every partially hyperbolic volume-preserving diffeomorphism can be C^1-approximated by non-uniformly hyperbolic ones, thus solving the question completely in the C^1 case.

Perturbative results in the C^r topology, $r > 1$, are notoriously more difficult and, in fact, there is good evidence suggesting that the conclusions may also be very different. In this regard, we refer the reader to the discussions in Chapter 12 of Bonatti, Díaz and Viana [BDV], Chapter 10 of Viana [V] and Theorem A of [AV1].

An important tool in our approach is the Invariance Principle, which was first developed by Bonatti, Gomez-Mont, Viana [BGV] for linear cocycles over hyperbolic systems, and was extended to general (diffeomorphism) cocycles by Avila, Viana [AV1] and by Avila, Santamaria, Viana [ASV]: in the first paper the base dynamics is still assumed to be hyperbolic, whereas in the second one it is taken to be partially hyperbolic and volume-preserving.

The Invariance Principle asserts that for the Lyapunov exponents to vanish the system must exhibit rather rigid (holonomy invariant) features. Often, one can successfully exploit those features to describe the system in a rather explicit way. One fine example is the main result of Avila, Viana and Wilkinson [AVW]: small perturbations of the time-1 map of the geodesic flow on a surface with negative curvature either are non-uniformly hyperbolic or embed into a smooth flow.

Another fine application was made by Avila and Viana [AV1], who exhibited partially hyperbolic diffeomorphisms for which the Lyapunov exponents can not vanish because structure arising from the Invariance Principle, namely invariant line fields, is incompatible with the topology of the center leaves (which are assumed to be surfaces of genus $g > 1$).

Perhaps the main novelty in this work is that we are able to use the Invariance Principle in a perturbative way, to prove that the Lyapunov exponents can be made non-zero. At a more technical level, another main novelty resides in our handling of the accessibility property, namely the way su-paths and their holonomies vary, under perturbations of the diffeomorphism; see Section 5 and 6.

As an example of the reach of Theorem A, let us state the following result that is related to Question 1b) in [SW1]. Let $f : \mathbb{T}^{2d} \longrightarrow \mathbb{T}^{2d}$ be a Anosov symplectic C^r diffeomorphism and $g_\lambda : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ denote the *standard map* on the 2-torus.

Corollary. *If λ is close enough to zero, then $f \times g_\lambda$ can be C^r-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.*

2 Preliminaries and Statements

A diffeomorphism $f : M \longrightarrow M$ of a compact manifold M is *partially hyperbolic* if there exists a nontrivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

invariant under the derivative map Df, a Riemannian metric $\|\cdot\|$ on M, and positive continuous functions μ, $\hat{\mu}$, ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with

$$\mu < \nu < 1 < \widehat{\nu}^{-1} < \widehat{\mu}^{-1} \quad \text{and} \quad \nu < \gamma < \widehat{\gamma}^{-1} < \widehat{\nu}^{-1},$$

such that for any unit vector $v \in T_pM$,

$$\begin{aligned} \mu(p) < \|Df_p(v)\| < \nu(p) \qquad & \text{if } v \in E^s(p), \\ \gamma(p) < \|Df_p(v)\| < \widehat{\gamma}(p)^{-1} \quad & \text{if } v \in E^c(p), \\ \widehat{\nu}(p)^{-1} < \|Df_p(v)\| < \widehat{\mu}(p)^{-1} \quad & \text{if } v \in E^u(p). \end{aligned} \tag{1}$$

The stable and unstable bundles E^s and E^u are uniquely integrable and their integral manifolds form two tranverse (continuous) foliations W^s and W^u, whose leaves are immersed submanifolds of the same class of differenciability as f. These foliations are called the *strong-stable* and *strong-unstable* foliations. They are invariant under f, in the sense that

$$f(W^s(x)) = W^s(f(x)) \qquad \text{and} \qquad f(W^u(x)) = W^u(f(x)),$$

where $W^s(x)$ and $W^u(x)$ denote the leaves of W^s and W^u, respectively, passing through any $x \in M$.

We consider in M the distance associated to such a Riemannian structure. The Lebesgue class is the measure class of the volume induced by this (or any other) Riemannian metric on M.

We say f is *volume-preserving* if it preserves some probability measure in the Lebesgue class of M. For $r \geq 2$, denote by $PH^r_\mu(M)$ the set of partially hyperbolic volume-preserving C^r diffeomorphisms. If M is a symplectic manifold and ω is the symplectic form, then $PH^r_\omega(M)$ denote the set of partially hyperbolic C^rdiffeomorphisms preserving ω.

Given two points $x, y \in M$, x is *accessible* from y if there exist a C^1 path that connects x to y, tangent at every point to $E^u \cup E^s$. We call this type of path, *su-path*. This is a equivalence relation and we say that f is *accessible* if M is the unique accessible class.

From the works in [A] and [HPS], we know that for every partially hyperbolic C^2 diffeomorphism, the subbundles in the splitting of the tangent bundle are Hölder. Moreover, the Hölder exponent depend on some relation between the functions in Equation (1). From [PSW1], we have analogous results for the W^s and W^u holonomies.

For $\alpha > 0$, we are going to define a pinching condition that will imply that E^c, and the W^s and W^u holonomies are α-Hölder:

Definition 2.1 (α-pinched)**.** *Let f be a partially hyperbolic diffeomorphism. We say that f is α-pinched if the functions in Equation (1) satisfy,*

$$\begin{aligned} \nu < \gamma\mu^{\alpha} \quad &\text{and} \quad \widehat{\nu} < \widehat{\gamma}\widehat{\mu}^{\alpha}, \\ \nu < \gamma\widehat{\mu}^{\alpha} \quad &\text{and} \quad \widehat{\nu} < \widehat{\gamma}\mu^{\alpha}. \end{aligned}$$

We say that a partially hyperbolic diffeomorphism is *center bunched* if

$$\nu < \gamma\widehat{\gamma} \quad \text{and} \quad \widehat{\nu} < \gamma\widehat{\gamma}.$$

For technical reasons that will be clear later this notion is not enough for our work and we need to define a similar condition.

Definition 2.2 (α-bunched)**.** *For $\alpha > 0$, if f is a partially hyperbolic diffeomorphism, we say that f is α-bunched if the functions in Equation (1) satisfy,*

$$\nu^{\alpha} < \gamma\widehat{\gamma} \qquad \text{and} \qquad \widehat{\nu}^{\alpha} < \gamma\widehat{\gamma}.$$

Observe that for any $\alpha < 1$, α-bunched implies center bunched.

At this point we are ready to give the precise definition of the set of partially hyperbolic systems where Theorem A holds. From now on, M will denote a compact manifold and $* \in \{\mu, \omega\}$ where μ denotes some probability measure in the Lebesgue class and ω denotes a symplectic form.

Definition 2.3. *If $r \geq 2$, we will denote $B^r_*(M)$, the subset of $PH^r_*(M)$ where f is accessible, α-pinched and α-bunched for some $\alpha > 0$, and the center bundle E^c is 2-dimensional.*

We want to remark two properties of the set $B^r_*(M)$. Avila and Viana [AV2] proved, under the hypothesis of 2-dimensional center bundle, that accessibility is an open property, this implies that $B^r_*(M)$ is an open set. We will give the precise statement and some ideas of the proof of this result in Section 6. Moreover, Theorem 0.1 in [BW] implies that every $f \in B^r_*(M)$ is ergodic.

If f is a volume-preserving C^1 diffeomorphism, by Oseledets Theorem for μ almost every point $x \in M$, there exist $k(x) \in \mathbb{N}$, real numbers $\lambda_1(f,x) > \cdots > \lambda_{k(x)}(f,x)$ and a splitting $T_xM = E^1_x \oplus \cdots \oplus E^{k(x)}_x$ of the tangent bundle at x, all depending measurably on the point x, such that

$$\lim_{n\to\pm\infty} \frac{1}{n}\log\|Df^n_x(v)\| = \lambda_j(f,x) \quad \text{for all } v \in E^j_x \setminus \{0\}.$$

The real numbers $\lambda_j(f,x)$ are the Lyapunov exponents of x. We say that f is *non-uniformly hyperbolic* if the set of points with non-zero Lyapunov exponents has full measure.

If $f \in B^r_*(M)$, the Oseledets Theorem can be applied and because of the ergodicity, the functions k and λ_j are constants almost everywhere. Moreover, the Oseledets splitting is a measurable refinement of the original splitting and it is possible to talk of Lyapunov exponents of E^c. They are called *center Lyapunov exponents* and will be denoted by λ^c_1 and λ^c_2.

In the symplectic case $\lambda^c_1 = \lambda^c_2$ is equivalent to $\lambda^c_1 = \lambda^c_2 = 0$, because the Lyapunov exponents of symplectic diffeomorphisms have the following symmetry property: If $dim\ M = 2d$, then

$$\lambda_j(f,x) = -\lambda_{2d-j+1}(f,x) \quad \text{for all } 1 \leq j \leq d.$$

Now we have all the necessary definitions to give the precise statement of the main Theorem.

Theorem A. *Let $f \in B^r_\omega(M)$ and assume the set of periodic points of f is non-empty, then f can be C^r-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.*

Remark 2.4. *Observe that the hypothesis of existence of a periodic point can be replace with the hypothesis of f having a periodic compact C^r center leaf. In this case, we can find a diffeomorphism arbitrarily C^r-close to f having a periodic point. See [XZ].*

The proof of Theorem A relies in two principal cases determined by the periodic point being hyperbolic or elliptic. The hyperbolic case has a generalization to the volume-preserving setting with the appropriate modifications in the hypotheses. This Theorem will be state as Theorem B.

Definition 2.5. *Let f be a partially hyperbolic diffeomorphism and p a periodic point with $n_0 = per(p)$. We say that p is a* pinching periodic point *if $Df^{n_0}|E^c(p)$ has two real eigenvalues whit different norms.*

Recall $* \in \{\mu, \omega\}$.

Theorem B. *Let $f \in B^r_*(M)$ and assume f has a pinching periodic point, then f can be C^r-approximated by volume-preserving (symplectic) diffeomorphisms whose center Lyapunov exponents are different. In particular, f can be C^r-approximated by diffeomorphisms with some center Lyapunov exponent non-zero.*

Invariance Principle

As already mentioned, one important tool in our proof is the Invariance Principle. Here we give some preliminaries and state it in the form of [ASV].

Let f be a partially hyperbolic diffeomorphism and $\pi : \mathcal{V} \longrightarrow M$ a continuous vector bundle with fiber $N = \mathbb{R}^k$ for some k. A *linear cocycle* over $f : M \longrightarrow M$ is a continuous transformation, $F : \mathcal{V} \longrightarrow \mathcal{V}$, satisfying $\pi \circ F = f \circ \pi$ and acting by linear isomorphisms, $F_x : \mathcal{V}_x \longrightarrow \mathcal{V}_{f(x)}$, on the fibers. By Fustenberg, Kesten [FK], the extremal Lyapunov exponents

$$\lambda_+(F,x) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\| \quad \text{and} \quad \lambda_-(F,x) = \lim_{n\to\infty} \frac{1}{n} \log \left\|(F_x^n)^{-1}\right\|^{-1},$$

exist at μ-almost every $x \in M$, relative to any f-invariant probability measure μ. If (f,μ) is ergodic, then they are constant on a full μ-measure set. It is clear that $\lambda_-(F,x) \leq \lambda_+(F,x)$ whenever they are defined.

The *projective bundle* associated to a vector bundle $\pi : \mathcal{V} \longrightarrow M$ is the continuous fiber bundle $\pi : \mathbb{P}(\mathcal{V}) \longrightarrow M$ whose fibers are the projective quotients of the fibers of $\mathcal{V}$. This is a fiber bundle with smooth leaves modeled on $N = \mathbb{P}(\mathbb{R}^k)$.

The *projective cocycle* associated to a linear cocycle $F : \mathcal{V} \longrightarrow \mathcal{V}$ is the smooth cocycle $\mathbb{P}(F) : \mathbb{P}(\mathcal{V}) \longrightarrow \mathbb{P}(\mathcal{V})$ whose action $\mathbb{P}(F_x) : \mathbb{P}(\mathcal{V}_x) \longrightarrow \mathbb{P}(\mathcal{V}_{f(x)})$ on the fibers is given by the projectivization of F_x.

Observe that if μ is a f-invariant probability measure in the Lebesgue class of M, then a $\mathbb{P}(F)$-invariant probability measure m that project down to μ always exist in this setting, because the projective cocycle is continuous and the domain is compact. Moreover,

$$\lambda_+(\mathbb{P}(F),x,\xi) \leq \lambda_+(F,x) - \lambda_-(F,x) \quad \text{and}$$
$$\lambda_-(\mathbb{P}(F),x,\xi) \geq \lambda_-(F,x) - \lambda_+(F,x).$$

Let $R > 0$ be fixed, then the *local strong-stable leaf* $W^s_{loc}(x)$ of a point $x \in M$ is the neighborhood of radius R around x inside $W^s(x)$. The *local strong-unstable leaf* is defined analogously. Since we are working in the context of [ASV], the choice of R here will be the same than in Section 5 of their paper.

Definition 2.6. *We call* invariant stable holonomy *for* $\mathbb{P}(F)$ *a family* h^s *of homeomorphisms* $h^s_{x,y} : \mathbb{P}(\mathcal{V}_x) \longrightarrow \mathbb{P}(\mathcal{V}_y)$*, defined for all* x *and* y *in the same strong-stable leaf of* f *and satisfying*

(a) $h^s_{y,z} \circ h^s_{x,y} = h^s_{x,z}$ *and* $h^s_{x,x} = Id$*;*

(b) $\mathbb{P}(F_y) \circ h^s_{x,y} = h^s_{f(x),f(y)} \circ \mathbb{P}(F_x)$

(c) $(x,y,\xi) \mapsto h^s_{x,y}(\xi)$ *is continuous when* (x,y) *varies in the set of pairs of points in the same local strong-stable leaf;*

(d) *there are* $C > 0$ *and* $\eta > 0$ *such that* $h^s_{x,y}$ *is* (C, η)*-Hölder continuous for every* x *and* y *in the same local strong-stable leaf.*

Invariant unstable holonomy *is defined analogously, for pairs of points in the same strong-unstable leaf.*

Let m be a probability measure in $\mathbb{P}(\mathcal{V})$ such that $\pi_* m = \mu$, then there exist a disintegration of m into conditional probabilities $\{m_x : x \in M\}$ along the fibers which is essentially unique, that is, a measurable family of probability measures such that $m_x(\mathbb{P}(\mathcal{V}_x)) = 1$ for almost every $x \in M$ and

$$m(U) = \int m_x(U \cap \mathbb{P}(\mathcal{V}_x))d\mu(x),$$

for every measurable set $U \subset \mathbb{P}(\mathcal{V})$. See [Rok].

Definition 2.7. *A disintegration* $\{m_x : x \in M\}$ *is* s-invariant *if*

$$(h^s_{x,y})_* m_x = m_y \quad \text{for every } x \text{ and } y \text{ in the same strong-stable leaf.}$$

The definition of u-invariant *is analogous and we say the disintegration is* bi-invariant *if it is both s-invariant and u-invariant.*

Invariance Principle (Theorem B of [ASV])**.** *Let* $f : M \longrightarrow M$ *be a* C^2 *partially hyperbolic, volume-preserving, center bunched diffeomorphism and* μ *be an invariant probability in the Lebesgue class. Let* F *be a linear cocycle such that* $\mathbb{P}(F)$ *admits holonomies and suppose that* $\lambda_-(F, x) = \lambda_+(F, x)$ *at* μ*-almost every point.*

Then every $\mathbb{P}(F)$*-invariant probability* m *on the projective fiber bundle* $\mathbb{P}(\mathcal{V})$ *with* $\pi_* m = \mu$ *admits a disintegration* $\{m_x : x \in M\}$ *along the fibers such that*

(a) *the disintegration is bi-invariant over a full measure bi-saturated set* $M_F \subset M$*;*

(b) *if* f *is accessible then* $M_F = M$ *and the conditional probabilities* m_x *depend continuously on the base point* $x \in M$*, relative to the weak* topology.*

3 Toy Model and Structure of the Proof

In this section, we present a toy model that give the necessary intuition to understand the ideas behind the proof of Theorem B and explain the several steps needed for it.

Given $f \in B^r_*(M)$, the linear cocycle $F = Df|E^c$ will be called *the center derivative cocycle* defined by f. In Section 4, we prove that we can apply the Invariance Principle to this cocycle when $\lambda^c_1 = \lambda^c_2$. For this, we prove the existence of holonomies for $\mathbb{P}(F)$ and how they vary under the perturbation of the diffeomorphism.

For the toy model suppose $f \in B^r_*(M)$, $\lambda^c_1 = \lambda^c_2$ and there exist $p \in Per(f)$, a hyperbolic fixed point, with $z \in W^{ss}(p) \cap W^{uu}(p)$. Then, we can apply the Invariance Principle and conclude that there exist $a, b \in \mathbb{P}(E^c_p)$ such that for any $\mathbb{P}(F)$-invariant probability, m, we have $supp\ m_p \subset \{a, b\}$. Moreover, if $h^s(f)$ and $h^u(f)$ denote the holonomies along the stable and unstable strong manifolds, then $(h^s(f))_* m_p = m_z$ and $(h^u(f))_* m_p = m_z$.

We will make a perturbation supported in a neighborhood of z, $B_\delta(z)$, which has the property that all iterates of z do not belong to it. The perturbation, g, is chosen to satisfy $h^s(g) = R_\beta \circ h^s(f)$ and $h^u(g) = h^u(f)$, where R_β denotes a rotation of angle $\beta > 0$. This implies that g can not satisfy the Invariance Principle, therefore $\lambda^c_1(g) \neq \lambda^c_2(g)$.

The first step in order to generalize these arguments is to find a su-path from p to itself with some kind of slow recurrence. This is done in Section 5 where we give details of how to construct the perturbation. In the toy model the su-path has no recurrence and it is stable under the perturbation of f. This is not the case anymore and we need estimations about how the su-path is changing with the perturbation.

In order to conclude the argument we need to understand some relation between the disintegration given by the Invariance Principle for f and for the perturbation. This difficulty is solved using the hyperbolicity of p and the results in Section 6 and 7. In Section 6, we state the results from [AV2] and prove a corollary about some kind of continuity for su-paths under the variation of the diffeomorphism. In Section 7, we study the disintegration given by the Invariance Principle for a particular case.

Finally, in Sections 8 we combine all this results to give the proof of Theorem B. The proof of Theorem A will be given in Section 9 and, as already mentioned, will be divide in two cases: the elliptic and the hyperbolic cases. In Section 10, we apply Theorem A to partially hyperbolic diffeomorphisms of the torus and give the proof of the Corollary stated above.

4 Derivative Cocycle

In order to prove Theorem B we need to be able to apply the Invariance Principle to the center derivative cocycle, $F = Df|E^c$. That is, to prove that $\mathbb{P}(F)$ admits holonomies. Since $f \in B^r_*(M)$, there exist $\alpha > 0$ such that f is α-pinched, then E^c is α-Hölder and the cocycle is a $C^{0,\alpha}$ cocycle. Moreover, its Lyapunov exponents coincide with the center Lyapunov exponents of f. These properties together with the hypothesis of α-bunched are enough to prove the existence of holonomies using the results in Section 3 of [ASV]. Here we provide a new proof that allow us to give estimations about how these holonomies change under the variation of the diffeomorphism. These are new results that we had to prove in order to be able to work in a perturbative way.

Although the statements are for $f \in B^r_*(M)$, the only necessary hypotheses are the α-pinched and α-bunched conditions.

The next Proposition prove the existence of a family of maps for F, $H^s_{x,y}$, with certain properties that will imply that $\mathbb{P}(H^s_{x,y})$ define an invariant stable holonomy for $\mathbb{P}(F)$. Analogously, we will find an invariant unstable holonomy.

Proposition 4.1. *Fix $f \in B^r_*(M)$ and denote $F = Df|E^c$. Then, for any pair of points x, y in the same leaf of the strong-stable foliation W^s, there exist a linear homeomorphism $H^s_{x,y} : E^c_x \to E^c_y$ satisfying:*

(a) $F_y \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ F_x$;

(b) $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ *and* $H^s_{x,x} = Id$.

For the proof of this Proposition, we will need the following definitions and Lemma 4.2 given below.

For $n \in \mathbb{N}$, let

$$F^n(x) = F(f^{n-1}(x)) \circ \cdots \circ F(x),$$

and for any continuous function $\tau : M \longrightarrow \mathbb{R}^+$ denote

$$\tau^n(x) = \tau(x)\tau(f(x)) \cdots \tau(f^{n-1}(x)).$$

Since M is compact, we can define a distance in TM in the following way: For every $x, y \in M$ close enough, denote $\pi_{x,y} : T_xM \longrightarrow T_yM$ the parallel transport along ζ, where ζ is the geodesic satisfying $dist(x, y) = \text{length}(\zeta)$. Then, given two points (x, v) and (y, w) in TM define

$$d((x, v), (y, w)) = dist(x, y) + \|\pi_{x,y}(v) - w\|.$$

To simplify the notation we are going to write

$$d((x,v),(y,w)) = d(v,w) \quad \text{and} \quad \pi^n_{x,y} = \pi_{f^n(x),f^n(y)}.$$

Since f is C^2, there exist $C_0 > 0$ such that $\forall\,(x,v),\ (y,w) \in TM$,

$$d(Df(x,v), Df(y,w)) < C_0\, d(v,w).$$

We also need a definition for the distance between two subspaces of TM. Let V be a vector space with inner product and let E_1 and E_2 be subspaces. Then, define $dist(E_1,E_2) = \max\{\delta_1,\delta_2\}$ where

$$\delta_1 = \sup_{x\in E_1,\|x\|=1} \inf_{y\in E_2} \|x-y\|,$$

and δ_2 is defined analogously changing the places of E_1 and E_2. If P_E denote the orthogonal projection to the subspace E, then

$$\inf_{y\in E} \|x-y\| = \|x - P_E(x)\|.$$

Therefore $\delta_1 = \|(Id - P_{E_2})P_{E_1}\|$ and we have an analogous identity for δ_2.

If x and y are close enough, given E_x and E_y subspaces of T_xM and T_yM respectively, define

$$dist(E_x,E_y) = dist(\pi_{x,y}(E_x),E_y) = dist(E_x,\pi_{y,x}(E_y)).$$

Since E^c is α-Hölder, there exist $C_1 > 0$ such that

$$dist(E^c_x,E^c_y) < C_1\, dist(x,y)^\alpha.$$

Moreover, the constant C_1 can be taken uniform in a C^2 neighborhood of f, see [W].

Lemma 4.2. *Fix $f \in B^r_*(M)$ and denote $F = Df|E^c$. Then, there exist $C_2 > 0$ and $m < 1$ such that for all $x \in M$, $y,z \in W^s_{loc}(x)$ and $n \geq 1$,*

$$\prod_{j=0}^{n-1} \left\|F(f^j(y))\right\| \left\|F(f^j(z))^{-1}\right\| \leq C_2\,\nu^n(x)^{-\alpha m}.$$

Proof. If $f \in B^r_*(M)$, then f is α-bunched, that is Definition 2.2. Moreover, since the functions are continuous and M is compact, there exist $m < 1$ such that

$$\nu^{\alpha m} < \gamma\widehat{\gamma} \qquad \text{and} \qquad \widehat{\nu}^{\alpha m} < \gamma\widehat{\gamma}.$$

Since E^c is α-Hölder, there exist $C > 0$ such that

$$\begin{aligned}\left\|F(f^j(y))\right\| / \left\|F(f^j(x))\right\| &\leq \exp(C\, dist(f^j(x), f^j(y))^\alpha) \\ &\leq \exp(C\, \nu^j(x)^\alpha dist(x,y)^\alpha).\end{aligned}$$

Analogously for $\left\|F(f^j(z))^{-1}\right\| / \left\|F(f^j(x))^{-1}\right\|$. Now since $\nu < 1$ we can take C_2 a bound for

$$\exp\left(C \sum_{j=0}^{n-1} \nu^j(x)^\alpha (dist(x,y)^\alpha + dist(z,x)^\alpha)\right).$$

Then,

$$\begin{aligned}\prod_{j=0}^{n-1} \left\|F(f^j(y))\right\| \left\|F(f^j(z))^{-1}\right\| &\leq C_2 \prod_{j=0}^{n-1} \left\|F(f^j(x))\right\| \left\|F(f^j(x))^{-1}\right\| \\ &\leq C_2 \prod_{j=0}^{n-1} \widehat{\gamma}(f^j(x))^{-1} \gamma(f^j(x))^{-1} \\ &\leq C_2 \prod_{j=0}^{n-1} \nu(f^j(x))^{-\alpha m}.\end{aligned}$$

□

Now we are ready to prove Proposition 4.1.

Proof. Fix $f \in B_*^r(M)$ and let $F = Df|E^c$. If $x, y \in M$ with $y \in W^s_{loc}(x)$, define

$$A_n(x,y) = F^n(y)^{-1} \circ P_{E^c(f^n(y))} \circ \pi^n_{x,y} \circ F^n(x),$$

for every $n \in \mathbb{N}$. Recall P_E denotes the orthogonal projection to the subspace E.

We are going to prove that the limit exists when n goes to infinity and then define

$$H^s_{x,y} = \lim_{n \to \infty} A_n(x,y).$$

Observe that

$$A_{n+j}(x,y) = F^j(y) \circ A_n(f^j(x), f^j(y)) \circ F^j(x),$$

then we can apply this identity to prove the general case where $y \in W^s(x)$ and to prove (a).

We want to see that $A_n(x,y)$ is a Cauchy sequence, then we need to estimate for every $n \in \mathbb{N}$,

$$\left\|A_{n+1}(x,y) - A_n(x,y)\right\|.$$

Using the definition of $A_n(x,y)$ we get an expression of the form:

$$\left\|F^n(y)^{-1}\right\| \|F^n(x)\| \|B(x,y,n)\|,$$

where B depends on the $n+1$ and n terms.

Let $C_3 = \sup_{z\in M}\{\|F(z)\|, \left\|F(z)^{-1}\right\|\}$. We have the following estimations in order to bound $B(x,y,n)$,

$$\begin{aligned}
&\left\|(Id - P_{E^c(f^{n+1}(y))}) \circ \pi^{n+1}_{x,y} \circ F(f^n(x))\right\| \\
&\leq \left\|(Id - P_{E^c(f^{n+1}(y))}) \circ \pi^{n+1}_{x,y} \circ P_{E^c(f^{n+1}(x))}\right\| \|F(f^n(x))\| \\
&\leq C_3\, dist(E^c(f^{n+1}(x)), E^c(f^{n+1}(x))) \\
&\leq C_3\, C_1\, dist(f^{n+1}(x), f^{n+1}(y))^\alpha \\
&\leq C_3\, C_1\, \nu^n(x)^\alpha\, dist(x,y)^\alpha.
\end{aligned}$$

For any $v \in E^c(f^n(x))$ with $\|v\| = 1$, define $w = P_{E^c(f^n(y))} \circ \pi^n_{x,y}(v)$. Then,

$$\begin{aligned}
&\left\|\pi^{n+1}_{x,y} \circ F(f^n(x))(v) - F(f^n(y))(w)\right\| \\
&\leq d\,(F(f^n(x))(v), F(f^n(y))(w)) \\
&\leq C_0\,(dist(f^n(x), f^n(y)) + \left\|\pi^n_{x,y}(v) - w\right\|) \\
&\leq C_0\,[\nu^n(x)\, dist(x,y) + dist(E^c(f^n(x)), E^c(f^n(y)))] \\
&\leq (C_0 + C_1)\, \nu^n(x)^\alpha\, dist(x,y)^\alpha.
\end{aligned}$$

This two inequalities implies that there exist $\hat{C}_0 > 0$ such that

$$\|B(x,y,n)\| \leq \hat{C}_0\, \nu^n(x)^\alpha\, dist(x,y)^\alpha.$$

Then, the difference $\|A_{n+1}(x,y) - A_n(x,y)\|$ is bounded by

$$\hat{C}_0\, \left\|F^n(y)^{-1}\right\| \|F^n(x)\|\, \nu^n(x)^\alpha\, dist(x,y)^\alpha.$$

Using Lemma 4.2, we have that this expression is bounded by

$$C_2\, \hat{C}_0\, \nu^n(x)^{(1-m)\alpha}\, dist(x,y)^\alpha.$$

Since $\nu < 1$, this implies that $A_n(x,y)$ is a Cauchy sequence and then the limit exists.

The following estimations prove that $\|A_n(x,y) - A_n(z,y) \circ A_n(x,z)\|$ is going to zero when n goes to ∞. This is enough to have property (b).

We need to estimate $\left\|\pi^n_{z,y} \circ \pi^n_{x,z} - \pi^n_{x,y}\right\|$, for this fix $v \in E^c(f^n(x))$ with $\|v\| = 1$, then

$$\begin{aligned}
&\left\|\pi^n_{z,y} \circ \pi^n_{x,z}(v) - \pi^n_{x,y}(v)\right\| \\
&\leq d(\pi^n_{x,z}(v), \pi^n_{x,y}(v)) \\
&\leq d(\pi^n_{x,z}(v), v) + d(v, \pi^n_{x,y}(v)) \\
&\leq dist(f^n(z), f^n(x)) + dist(f^n(x), f^n(y)).
\end{aligned}$$

This finish the proof of the Proposition. $\square$

Remark 4.3. *For every $y \in W^s_{loc}(x)$ we have*

$$H^s_{x,y} = \sum_{j=0}^{\infty}(A_{n+1}(x,y) - A_n(x,y)) + A_0(x,y).$$

Let $\hat{C}_1 > 0$ be a bound for $\sum_{j=0}^{\infty} \nu^n(x)^{(1-m)\alpha}$ and let $C = C_2\, \hat{C}_0\, \hat{C}_1$, then

$$\begin{aligned}\left\|H^s_{x,y} - A_0(x,y)\right\| &\leq \sum_{j=0}^{\infty} \|A_{n+1}(x,y) - A_n(x,y)\| \\ &\leq C_2\, \hat{C}_0\, dist(x,y)^\alpha \sum_{j=0}^{\infty} \nu^n(x)^{(1-m)\alpha} \\ &\leq C\, dist(x,y)^\alpha.\end{aligned} \tag{2}$$

Then, we have proved that there exist $C > 0$ such that for every $y \in W^s_{loc}(x)$,

$$\left\|H^s_{x,y}\right\| \leq 1 + C\, dist(x,y)^\alpha.$$

Moreover, the constant C depends only on f.

Observe that all the estimations in the proof of the Proposition and in the Remark can be taken uniform in a C^2 neighborhood of f. In the sequel, always that we refer to some g close to f, we are thinking that g is in this neighborhood.

Now that we have proved the existence of $H^s_{x,y}$, we can define $h^s_{x,y} = \mathbb{P}(H^s_{x,y})$. To show that the family $h^s_{x,y}$ is an invariant stable holonomy for $\mathbb{P}(F)$, we need to prove (c) in Definition 2.6.

We are going to prove a much stronger result, that will imply (c), but provides also an estimation about how $H^s_{x,y}$ change under the variation of the diffeomorphism. More precisely,

Proposition 4.4. *Fix $f \in B^r_*(M)$, $x \in M$, $y \in W^s_{loc}(x,f)$ and $a \in E^c(x,f)$. For every $\epsilon > 0$ there exist $\delta > 0$ and a neighborhood of f in the C^2-topology, $\mathcal{V}(f)$, such that for every $g \in \mathcal{V}(f)$, every $w, z \in M$ with $w \in W^s_{loc}(z,g)$, $dist(x,z) < \delta$ and $dist(y,w) < \delta$ and every $b \in E^c(z,g)$ with $d(a,b) < \delta$, we have*

$$d(H^s_{x,y}(f)(a), H^s_{z,w}(g)(b)) < \epsilon.$$

Proof. Let $F = Df|E^c(f)$ and $G = Dg|E^c(g)$. Similar estimations that the ones in Equation (2), provides a $C > 0$ such that for any $n \geq 1$,

$$\left\|H^s_{x,y}(f) - A_n(f,x,y)\right\| \leq C\, \nu^n(x)^{\alpha(1-m)}$$

and

$$\left\| H^s_{z,w}(g) - A_n(g,z,w) \right\| \leq C\ \nu^n(z)^{\alpha(1-m)}.$$

So, in order to prove the Proposition, we only need to estimate

$$d(A_n(f,x,y)(a), A_n(g,z,w)(b)).$$

Define

$$c = P_{E^c(f^n(y),f)} \circ \pi^n_{x,y} \circ F^n(x)(a) \quad \text{and} \quad d = P_{E^c(g^n(w),g)} \circ \pi^n_{z,w} \circ G^n(z)(b).$$

Then,

$$d(A_n(f,x,y)(a), A_n(g,z,w)(b)) = d(F^n(y)^{-1}(c), G^n(w)^{-1}(d)).$$

This distance is bounded by some expression involving the distance between f and g, the distance between the n-th first iterates of the points y, w, the distance between the center bundles and the following term,

$$\prod_{j=0}^{n-1} \left\| F(f^j(y))^{-1} \right\| d(c,d).$$

Let $\widetilde{a} = F^n(x)(a)$ and $\widetilde{b} = G^n(z)(b)$. From the definition of c and d, we can bound the distance between them by an expression involving the distance of $f^n(y)$ and $g^n(w)$, the distance between the center bundles in these points and

$$d(P_{E^c(f^n(y),f)} \circ \pi^n_{x,y}(\tilde{a}), \tilde{a}) + d(\tilde{b}, \pi^n_{z,w}(\tilde{b})) + d(\tilde{a}, \tilde{b}). \tag{3}$$

The first term is bounded by

$$d(E^c(f^n(x),f), E^c(f^n(y),f))\ \|\tilde{a}\| \leq C_1\ \|\tilde{a}\|\, dist(f^n(x), f^n(y))^\alpha.$$

The second term is bounded by

$$dist(g^n(z), g^n(w))^\alpha.$$

And the last term is equal to $d(F^n(x)(a), G^n(z)(b))$. Like before, this distance is bounded by some expression involving the distance between f and g, the distance between the first n-th iterates of the points x, z, the distance between the center bundles and $d(a,b)$.

So, in order to finish the proof of the continuity we need to prove that the following expressions goes to zero as n goes to ∞,

$$\prod_{j=0}^{n-1} \left\| F(f^j(y))^{-1} \right\| \|F^n(x)\|\, \nu^n(x)^\alpha$$

and

$$\prod_{j=0}^{n-1} \left\| F(f^j(y))^{-1} \right\| \nu^n(z)^\alpha.$$

For the first expression we only need to apply Lemma 4.2. We are going to need some new definitions for the second one.

For the functions in Equation (1) and $p \in M$ define

$$\nu(p,r) = \sup_{q \in B^r(p)} \nu(q) \quad \text{and} \quad \gamma(p,r) = \inf_{q \in B^r(p)} \gamma(q).$$

Then by continuity of the functions and compactness of M, there exist $r_0 > 0$ and $\theta < 1$ such that $\nu(p, r_0) < \theta\, \gamma(p, r_0)$ for every $p \in M$.

Suppose $dist(x, y) < r_0$, then $dist(f^j(x), f^j(y)) < r_0$ for every $j \geq 1$. If $dist(g^j(z), f^j(x)) < r_0$ for every $0 \leq j \leq n$, then

$$\begin{aligned}
\prod_{j=0}^{n-1} \left\| F(f^j(y))^{-1} \right\| \nu^n(z)^\alpha &\leq \prod_{j=0}^{n-1} \gamma(f^j(y))^{-1} \nu(g^j(z)) \\
&\leq \prod_{j=0}^{n-1} \gamma(f^j(x), r_0)^{-1} \nu(f^j(x), r_0) \\
&\leq \theta^n.
\end{aligned}$$

Then, we fix n big enough such that all the terms going to zero are small enough and define $\delta > 0$ and $\mathcal{V}(f)$ in order to have the other terms also small enough. This proves the Proposition. □

By Proposition 4.1 and Proposition 4.4, we have proved that $h^s_{x,y} = \mathbb{P}(H^s_{x,y})$ is an invariant stable holonomy for $\mathbb{P}(F)$. We are going to refer to both $h^s_{x,y}$ and $H^s_{x,y}$ as stable holonomies.

Observe that Proposition 4.4 implies the continuity of stable holonomies in compact parts of the strong-stable manifold, that is, the application

$$(f, x, y) \mapsto H^s_{x,y}(f),$$

is continuous on $W^s_N(f) = \{(g, x, y) : g \in \mathcal{V}(f) \text{ and } g^N(y) \in W^s_{loc}(g^N(x))\}$, for every $N \geq 1$.

There are analogous Propositions and properties for the invariant unstable holonomy, $h^u_{x,y}$. Locally, they will be defined by the projectivization of

$$H^u_{x,y} = \lim_{n \to -\infty} F^n(y)^{-1} \circ P_{E^c(f^n(y))} \circ \pi^n_{x,y} \circ F^n(x),$$

where

$$F^{-n}(x) = F^{-1}(f^{-n+1}(x)) \circ \cdots \circ F^{-1}(x),$$

for every $n \in \mathbb{N}$.

Given a su-path for f, we can define the holonomy associated to the su-path by the composition of the stable and unstable holonomies defined by the nodes. Let $\zeta = [z_0, z_1, .., z_n]$ be a su-path and denote $H_{z_i} = H^*_{z_{i-1},z_i}$ with $* \in \{s, u\}$, then $H_\zeta = H_{z_n} \circ \dots H_{z_1}$. The following Corollary gives an estimation about how this holonomy change under the variation of f and the su-path.

Corollary 4.5. *If g is close enough to f and $\zeta_f = [x_0, ..., x_n]$ and $\zeta_g = [y_0, ..., y_n]$ are two su-paths for f and g respectively, $a \in E^c(x_0, f)$, $b \in E^c(y_0, g)$, then*

$$d(H_{\zeta_f}(a), H_{\zeta_g}(b)) \leq \sum_{i=0}^{n-1} \prod_{j=i+2}^{n} \left\|H_{x_j}\right\| \psi(H_{x_{i+1}}) + \prod_{j=1}^{n} \left\|H_{x_j}\right\| d(a, b),$$

where

$$\psi(H_{x_{i+1}}) = d(H_{x_{i+1}}(a_i), H_{y_{i+1}}(\pi_{x_i,y_i}(a_i)))$$

and

$$a_i = H_{x_i} \circ \cdots \circ H_{x_1}(a).$$

By Remark 4.3, if $x_{j-1} \in W^*_{loc}(x_j)$, then $\left\|H_{x_j}\right\| < 1 + C\, dist(x_{j-1}, x_j)^\alpha$ where $C > 0$ does not depend on the points. Then, if $\zeta = [x_0, ..., x_n]$ is a su-path with $x_{j-1} \in W^*_{loc}(x_j)$ and $dist(x_{j-1}, x_j) < L$ for every $j \in [1, ..n]$, we have

$$\prod_{j=1}^{n} \left\|H_{x_j}\right\| < (1 + C\, L^\alpha)^n.$$

This proves that we can find a bound for $\prod_{j=1}^{n} \left\|H_{x_j}\right\|$ depending only on the number of legs of the su-path and the distance between the nodes. This will be important in Section 7.

5 Perturbation

As explained in the toy model the first steps for the proof of Theorem B can be summarized in the following way: We need to find a special su-path for f, make a local perturbation and estimate how the dynamics change. These are going to be the goals of this section.

Fixed $r \in [2, \infty)$, we are going to define the (C^r) Whitney topology in the volume-preserving and symplectic case specifying basic neighborhoods, η^r_* with $* \in \{\mu, \omega\}$.

Let μ be a volume form and pick two finite open coverings $\mathcal{U} = \{(U_i, \phi_i) : i = 1, ..., k\}$, $\mathcal{V} = \{(V_i, \psi_i) : i = 1, ..., k\}$ of M by C^r conservative coordinates charts such that $f(\overline{U_i}) \subset V_i$ for all i. This means that we are using [Mo] to find $\phi_i : U_i \longrightarrow \mathbb{R}^d$ and $\psi_i : V_i \longrightarrow \mathbb{R}^d$, C^r diffeomorphisms with $\mu = \phi_i^*(du_1 \wedge \cdots \wedge du_d) = \psi_i^*(du_1 \wedge \cdots \wedge du_d)$ where $(u_1, \ldots, u_d)$ are coordinates in $\mathbb{R}^d$.

Let $\epsilon > 0$. Define $\eta^r_\mu(f, \mathcal{U}, \mathcal{V}, \epsilon)$ to be the set of diffeomorphisms $g \in Diff^r_\mu(M)$ such that

(a) $g(\overline{U_i}) \subset V_i$ for all i and

(b) $\left\|\partial^\rho \psi_i g \phi_i^{-1}(x) - \partial^\rho \psi_i f \phi_i^{-1}(x)\right\| < \epsilon$ for $x \in \phi(U_i)$, $|\rho| \le r$, for all i.

Here $\rho = (\rho_1, ..., \rho_r)$ is a multi-index of non-negative integers, $|\rho| = \rho_1 + ... + \rho_r$, and ∂^ρ denotes the corresponding partial derivative.

For the symplectic case, pick two finite open coverings by C^r symplectic charts. That is, use Darboux's Theorem to find $\phi_i : U_i \longrightarrow \mathbb{R}^{2d}$ and $\psi_i : V_i \longrightarrow \mathbb{R}^{2d}$, C^r diffeomorphisms with $\omega = \phi_i^*(du \wedge dv) = \psi_i^*(du \wedge dv)$ where (u, v) are coordinates in $\mathbb{R}^{2d}$. Then, $\eta^r_\omega(f, \mathcal{U}, \mathcal{V}, \epsilon)$ is defined analogously.

We will write $\eta^r_*(f, \mathcal{U}, \mathcal{V}, \epsilon)$ with $* \in \{\mu, \omega\}$.

Lemma 5.1. *Fix $r \in [2, \infty)$. Let f be a partially hyperbolic volume-preserving (symplectic) diffeomorphism with dim $E^c = 2$. Then, there exist $\epsilon_0 > 0$, $\delta_0 > 0$ and $C_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, $0 < \delta < \delta_0$ and $z \in M$, there exist $g \in \eta^r_*(f, \mathcal{U}, \mathcal{V}, \epsilon)$ such that*

(a) *$g(x) = f(x)$ if $x \notin B_\delta(z)$,*

(b) *$g(z) = f(z)$ and*

(c) *$Dg_z = Df_z \circ A_\beta$ with $sin\, \beta = C_0\, \delta^{r-1} \epsilon$,* *where A_β is the linear map from TM_z to TM_z given by*

$$\begin{pmatrix} Id_u & 0 & 0 \\ 0 & R_\beta & 0 \\ 0 & 0 & Id_s \end{pmatrix}$$

whit $Id_{**} : E_z^{**} \longrightarrow E_z^{**}$ being the identity map for $** \in \{s, u\}$ and R_β the rotation of angle β in some (symplectic) base for $E^c(z)$, $\{e_1, f_1\}$.

Proof. Define another finite cover $\mathcal{U}'$ such that $U_i \subset \overline{U_i'}$, $f(\overline{U_i'}) \subset V_i$ and $\phi_i'|U_i = \phi_i$. Then there exist $C_1 > 0$ depending on f, such that if

$$h \in \eta_*^r(id, \mathcal{U}, \mathcal{U}', \epsilon)$$

then for $g = f \circ h$, we have

$$g \in \eta_*^r(f, \mathcal{U}, \mathcal{V}, C_1\, \epsilon).$$

This reduces the problem to find h with the right properties near the identity and then define $g = f \circ h$.

Fix i such that $z \in U_i$. By composition with a translation, we can suppose that $\phi_i(z) = 0$. From now on, we denote $U = U_i$ and $\phi = \phi_i$.

Let $\lambda : \mathbb{R} \longrightarrow [0, 1]$ be a C^∞ function such that $\lambda(z) = 1$ if $z \leq 1/2$ and $\lambda(z) = 0$ if $z \geq 1$. Let

$$\sigma = \sup_{x\in\mathbb{R}}\{1, |\lambda'(x)|, \ldots, |\lambda^{(r+1)}(x)|\}.$$

Fix $\epsilon > 0$, $\delta > 0$ and $\beta > 0$ an suppose δ is smaller than the Lebesgue number for $\mathcal{U}$.

At this point we need to consider the volume-preserving and the symplectic cases separately.

Volume-Preserving Perturbation

Define $\tilde{E}^* = D\phi_z(E_z^*)$ for $* \in \{u, c, s\}$, then

$$\tilde{E}^u \oplus \tilde{E}^c \oplus \tilde{E}^s = \mathbb{R}^d.$$

Let us consider the inner product associated to this decomposition:

$$< u, v > = < u^u, v^u >_* + < u^c, v^c >_* + < u^s, v^s >_*,$$

where $<, >_*$ is the usual inner product in $\mathbb{R}^d$. Then there exist $c_1, c_2 > 0$ such that $c_1\ \|u\| \leq \|u\|_* \leq c_2\ \|u\|$.

Define

$$\psi(v) = \beta\, \lambda(\|v\|/\delta)$$

and $\tilde{h} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by

$$\tilde{h}(v^u + v^c + v^s) = v^u + R_{\psi(v)}(v^c) + v^s.$$

Here for every $\theta \in [0, \pi/2]$, R_θ denotes the rotation of angle θ for the base $D\phi_z(e_1)$, $D\phi_z(f_1)$.

Then we have the following properties:

(a) $\tilde{h}$ preserves volume,

(b) $\tilde{h}(0) = 0$ and $D\tilde{h}_0(v) = v^u + R_\beta(v^c) + v^s$,

(c) $\tilde{h}(v) = v$ if $\|v\| \geq \delta$,

(d) $\left\|\tilde{h} - id\right\|_r \leq C_2\, \sigma\, \delta^{-(r-1)} \sin\beta$ for some $C_2 > 0$.

Define,

$$h(x) = \begin{cases} \phi^{-1} \circ \tilde{h} \circ \phi(x) & \text{if } x \in U \\ x & \text{if } x \notin U. \end{cases}$$

Then, there exist K_1 and K_2 (depending only on the charts and the Riemannian metric on M) such that $h \in \eta^r_\mu(id, \mathcal{U}, \mathcal{U}', K_1\, \delta^{-(r-1)} \sin\beta)$ and $h(x) = x$ if $dist(x, z) \geq K_2\, \delta$.

So, if $\sin\beta < (C_1\, K_1)^{-1} \epsilon \delta^{r-1}$, we have $h \in \eta^r_\mu(id, \mathcal{U}, \mathcal{U}', C_1^{-1}\epsilon)$ and $h(x) = x$ if $dist(x, z) \geq K_2\, \delta$. This proves the Lemma in the volume-preserving setting.

Symplectic Perturbation

We are going to recall some elementary facts about generating functions before doing the proof in the symplectic case. See [N] for more details.

Let $(u_1, \ldots, u_d, v_1, \ldots, v_d)$ be coordinates of $\mathbb{R}^d$ and

$$f(u, v) = (\xi(u, v), \eta(u, v))$$

be a C^r symplectic diffeomorphism defined in a simply connected neighborhood V of $(0, 0)$ with $r > 0$. Then $\sum_{i=1}^d d\xi_i \wedge d\eta_i = \sum_{i=1}^d du_i \wedge dv_i$. Suppose $\frac{\partial \eta}{\partial v}(u, v)$ is non-singular at each point of V. Then we have a new C^r system of coordinates on a neighborhood of $(0, \eta(0, 0))$ given by $(u_1, \ldots, u_d, \eta_1, \ldots, \eta_d)$, and $v = v(u, \eta)$. Since the 1-form $\alpha = \sum_{i=1}^d (\xi_i d\eta_i + v_i du_i)$ is closed, there exist $S = S(u, \eta)$ a C^{r+1} real valued function unique up to a constant satisfying $\frac{\partial S}{\partial \eta_i} = \xi_i$ and $\frac{\partial S}{\partial u_i} = v_i$. The function S is called a generating function for f.

Conversely, given a C^{r+1} real valued function $S = S(u, \eta)$ defined in a neighborhood of $(0, \eta(0, 0))$ such that the matrix $\frac{\partial^2 S}{\partial u_i \partial \eta_j}$ is non-singular for every (u, η), define $\xi_i(u, \eta) = \frac{\partial S}{\partial \eta_i}(u, \eta)$ and $v_i(u, \eta) = \frac{\partial S}{\partial u_i}(u, \eta)$. We can solve η in terms of (u, v) and obtain a symplectic diffeomorphism $f(u, v) = (\xi(u, \eta(u, v)), \eta(u, v))$.

Remember we fixed some chart such that $z \in U$ and $\phi(z) = 0$. Choose a symplectic base for T_zM, $\{e_1, e_2, \ldots, e_d, f_1, \ldots, f_d\}$ where $\{e_1, f_1\}$ is the chosen base for E^c_z. Let $A : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}$ be the symplectic map defined by $D\phi_z(e_i) = u_i$ and $D\phi(f_i) = v_i$.

Let $S_1(u, \eta)$ be the generating function for the symplectic map $B(u, v) = R_\beta(u_1, v_1) + (0, u_2, \ldots, u_d, 0, v_2, \ldots, v_d)$.

Define

$$S_0(u, \eta) = \sum_{i=1}^{d} u_i \eta_i,$$

and

$$S(u, \eta) = \begin{cases} \lambda(\frac{\|(u,\eta)\|}{\delta}) S_1(u, \eta) + \left[1 - \lambda(\frac{\|(u,\eta)\|}{\delta})\right] S_0(u, \eta) & \text{if } \|(u, \eta)\| < \delta, \\ S_0(u, \eta) & \text{if } \|(u, \eta)\| \geq \delta. \end{cases}$$

Then, S is C^{r+1} and there exist $C_2 > 0$ such that

$$\|S - S_0\|_{r+1} < C_2\, \sigma\, \delta^{-(r-1)} \sin \beta.$$

Let $\tilde{h}$ be the C^r symplectic diffeomorphism defined by S, then

(a) $\tilde{h}((0, 0)) = (0, 0)$ and $D\tilde{h}_0(u, v) = B(u, v)$,

(b) $\tilde{h}((u, v)) = (u, v)$ if $\|(u, v)\| \geq \delta$,

(c) $\left\|\tilde{h} - id\right\|_r \leq C_1\, \sigma\, \delta^{-(r-1)} \sin \beta$.

Define,

$$h(x) = \begin{cases} \phi^{-1} \circ A^{-1} \circ \tilde{h} \circ A \circ \phi(x) & \text{if } x \in U \\ x & \text{if } x \notin U. \end{cases}$$

Now, like before, there exist K_1 and K_2 such that

$$h \in \eta^r_\omega(id, \mathcal{U}, \mathcal{U}', K_1\, \delta^{-(r-1)} \sin \beta),$$

and $h(x) = x$ if $dist(x, z) \geq K_2\, \delta$.

So, if $\sin \beta < (C_1\, K_1)^{-1} \epsilon \delta^{r-1}$, we have $h \in \eta^r_\omega(id, \mathcal{U}, \mathcal{U}', C_1^{-1}\epsilon)$ and $h(x) = x$ if $dist(x, z) \geq \delta$. This finish the proof. □

The following Proposition is stated and proved as Proposition 8.2 in [ASV].

Proposition 5.2. *Let f be a partially hyperbolic accessible C^2 diffeomorphism. Then, for every $x \in M$, there exist a su-loop, $\zeta = [z_0, ..., z_n]$ with $x = z_0 = z_n$, $l \in \{0, ..., n\}$ and $c > 0$ such that*

$$dist(f^j(z_i), z_l) \geq \frac{c}{1 + j^2},$$

for every $j \in \mathbb{Z}$ and every $i \in \{0, ..., n\}$ except $j = 0$ and $i = l$.

Given $\widehat{\delta} > 0$ we can suppose that the su-loop in the Proposition above was construct to satisfy $dist(z_i, z_{i+1}) < \widehat{\delta}$ for every $i \in \{0, ..., n-1\}$. This is a technical observation that we need to consider in the estimations below.

Let $* \in \{\mu, \omega\}$, $f \in B^r_*(M)$ and p a periodic point for f. Apply Proposition 5.2 to f and p and let $z = z_l$. We are going to construct a sequence of perturbations for f, like in Lemma 5.1, supported in this point.

Define

$$\delta_k = \frac{c}{1 + (qk)^2},$$

for every $k \geq 1$, where $c > 0$ is given by Proposition 5.2 and $q > 0$ is a fixed (technical) constant depending on f. More precisely, it depends on the functions in Equation (1), the α for which f is α-pinched and α-bunched, the period of p and the number of nodes in the su-path.

Recall there exist $\widehat{\epsilon}_0 > 0$ such that all the estimation in Section 4 for the holonomies are uniform for every g $\widehat{\epsilon}_0$-close to f. Then, using Lemma 5.1 we have the following:

Lemma 5.3. *There exist $C_0 > 0$ and $k_0 \in \mathbb{N}$ such that for any $\epsilon > 0$ smaller that ϵ_0 and $\widehat{\epsilon}_0$ and $k \geq k_0$, there exist $f_k \in B^r_*(M) \cap \eta^r_*(f, \mathcal{U}, \mathcal{V}, \epsilon)$ with $* \in \{\mu, \omega\}$ such that*

(a) $f_k(z) = f(z)$,

(b) $Df_k(z) = Df(z) \circ A_{\beta_k}$ *with* $\sin \beta_k = C_0\, \delta_k^{r-1}\, \epsilon$,

(c) $f_k(x) = f(x)$ *if* $x \notin B_{\delta_k}(z)$ *and*

(d) $f_k \to f$ *in the C^1 topology when* $k \to \infty$.

Estimations

The property of the point $z = z_l$ given by Proposition 5.2 give information about how much time the nodes stay outside of the support of the perturbation. We will use this information to estimate how the dynamics is changing. Some results similar to these appear in [DW].

Let $* \in \{\mu, \omega\}$, $f \in B_*^r(M)$ and p a periodic point for f. Let $\zeta = [z_0, ..., z_n]$ be the su-path given by Proposition 5.2. Fix $\epsilon > 0$ and consider k_0 and f_k with $k \geq k_0$ given by Lemma 5.3.

For the functions on Equation (1), define

$$\nu(x,r) = \sup_{y \in B^r(x)} \nu(y) \qquad \gamma(x,r) = \inf_{y \in B^r(x)} \gamma(y),$$

$$\widehat{\nu}(x,r) = \sup_{y \in B^r(x)} \nu(y) \quad \text{and} \quad \widehat{\gamma}(x,r) = \inf_{y \in B^r(x)} \gamma(y).$$

Then, by continuity of the functions and compactness of M, there exist $r_0 > 0$ and $\tau_0 < 1$ such that

$$\nu(x, r_0) < \tau_0 \, \gamma(x, r_0) \quad \text{and} \quad \widehat{\nu}(x, r_0) < \tau_0 \, \widehat{\gamma}(x, r_0),$$

for every $x \in M$.

Lemma 5.4. *Fix $\tau \in (\tau_0, 1)$. There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that for every $x \in M$, $y \in W^s_{loc}(x)$ and g ϵ_1-close to f in the C^1-topology, if there exist $k' \in \mathbb{N}$ with $f^j(x) = g^j(x)$ and $f^j(y) = g^j(y)$ for every $1 \leq j \leq k'$, then there exist $w \in W^s(x, g) \cap B(y, C_1 \tau^{k'})$.*

Proof. Since f is partially hyperbolic, there exist K^{cu} a cone family around $E^u \oplus E^c$ such that

(a) $Df(K^{cu}(x)) \subset K^{cu}(f(x))$ for every $x \in M$,

(b) K^{cu} is uniformly transverse to E^s,

(c) For every $v \in K^{cu}(x)$,

$$\|Df_x(v)\| \geq \tau \, \nu(x, r_0) \, \|v\|,$$

for every $x \in M$.

For every g C^1-close enough to f all the above are still valid.

Let V be a topological disk of dimension $u + c$ passing through y such that $TV \subset K^{cu}$. Since $f^{k'}(x) = g^{k'}(x)$ and $f^{k'}(y) = g^{k'}(y)$, we have

$$dist(g^{k'}(x), g^{k'}(y)) < \nu(x, r_0)^{k'}.$$

Then, there exist $C_1 = C_1(f)$, depending only on f, and $w' \in W^s(g^{k'}(x), g) \cap g^{k'}(V)$ such that

$$dist(g^{k'}(x), w') < C_1 \, \nu(x, r_0)^{k'}.$$

Define $w = g^{-k'}(w')$, then $w \in W^s(x, g)$ and

$$dist(y, w) < C_1 \, \tau^{k'}.$$

□

There is an analogous statement of this Lemma for W^u. We apply these two results to f_k and obtain the following.

Lemma 5.5. *If $\zeta = [z_0, ..., z_n]$ is the su-path given by Proposition 5.2 and f_k is given by Lemma 5.3, then there exist $C_1 > 0$, $\tau \in (0,1)$ and $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$ there exist points $w_i^k \in M$ with*

$$w_i^k \in W^*(z_{i-1}, f_k) \cap B(z_i, C_1\, \tau^{qk}),$$

for every $i \in \{1, ..., n\}$ where

$$W^*(z_{i-1}, f_k) = \begin{cases} W^s(z_{i-1}, f_k) & \text{if} \quad z_i \in W^s(z_{i-1}, f), \\ W^u(z_{i-1}, f_k) & \text{if} \quad z_i \in W^u(z_{i-1}, f). \end{cases}$$

We can suppose that f_k and f are α-pinched for the same α. By [PSW1], the α-pinched condition implies that the W^s and W^u holonomies are α-Hölder. Moreover, there exist a C^2 neighborhood of f, $\mathcal{V}(f)$, such that the holonomies for every $g \in \mathcal{V}(f)$ are α-Hölder with uniform Hölder constant. See [W].

For every $k \geq k_1$, define $z_1^k = w_1^k$ by Lemma 5.5, then

$$dist(z_1^k, z_1) < C_1\, \tau^{qk}.$$

Suppose $z_3 \in W^s(z_2, f)$. If k is big enough, z_1, z_1^k and w_2^k are all in the same W^s foliation box U. Let $\Sigma(x)$ be a smooth foliation by admissible transversals defined in U. Then, define z_2^k to be the only point of intersection of $W^s(z_1^k, f_k)$ with $\Sigma(w_2^k)$, then there exist $\widehat{C}_1 = \widehat{C}_1(f)$ such that

$$dist(w_2^k, z_2^k) < \widehat{C}_1\ dist(z_1^k, z_1)^\alpha.$$

Then,

$$dist(z_2^k, z_2) < dist(z_2, w_2^k) + dist(w_2^k, z_2^k) < C_1\, \tau^{qk} + \widehat{C}_1\, C_1^\alpha\, \tau^{qk\alpha}.$$

If $z_3 \in W^u(z_2, f)$, we proceed in the same way using a foliation box for W^u.

Repeating the argument for all the other nodes of ζ we have the following:

Proposition 5.6. *If $\zeta = [z_0, ..., z_n]$ is the su-path given by Proposition 5.2 and f_k is given by Lemma 5.3, then there exist $C_2 > 0$ and $k_2 \in \mathbb{N}$ such that for every $k \geq k_2$ there exist $\zeta_k = [z_0^k, ..., z_n^k]$ su-path for f_k such that $z_0^k = z_0 = p$ and*

$$dist(z_i, z_i^k) < C_2\, \tau^{mk},$$

for $i \in \{1, ..., n\}$, where $m = q\alpha^n$.

Now we study how the splitting in the tangent bundle is changing under the variation of the diffeomorphism. Let V be a vector space with inner product and let E_1 and E_2 be subspaces. Define $\angle\,(E_1, E_2) = \max\{\delta_1, \delta_2\}$ where

$$\delta_1 = \sup_{x \in E_1, x \neq 0} \inf_{y \in E_2, y \neq 0} \angle\, x, y,$$

and δ_2 is defined analogously, changing the places of E_1 and E_2.

The relation of this definition with the distance of subspaces defined in Section 4 is given by

$$\sin \angle\,(E_1, E_2) = dist(E_1, E_2).$$

By Equation (1), there exist $C_3 > 0$ and $\theta_0 > 0$ such that for every F^{u+c} and F^{c+s} distributions of dimension $u+c$ and $c+s$ respectively, such that

$$\max\{\angle(E^{u+c}, F^{u+c}), \angle(E^{c+s}, F^{c+s})\} \leq \theta_0, \tag{4}$$

we have

$$\angle(Df_x^j(F_x^{u+c}), Df_x^j(E_x^{u+c})) \leq C_3\, \rho^j, \tag{5}$$

and

$$\angle(Df_x^{-j}(F_x^{c+s}), Df_x^{-j}(E_x^{c+s})) \leq C_3\, \rho^j. \tag{6}$$

for every $x \in M$ and $j \geq 0$, where

$$\rho = \max_{x \in M} \left(\max\left\{\nu(x)/\gamma(x), \widehat{\nu}(x)/\widehat{\gamma}(x)\right\}\right).$$

Lemma 5.7. *There exist $C_3 > 0$, $\rho \in (0,1)$ and $\epsilon_3 > 0$ such that if g is ϵ_3-close to f in the C^1-topology, $g = f$ outside some compact set I and there exist $k' \in \mathbb{N}$ such that $f^j(x)$ do not enter I for every j with $|j| \leq k'$, then*

$$\angle(E^{u+c}(x, g), E^{u+c}(x, f) \leq C_3\, \rho^{k'} \tag{7}$$

and

$$\angle(E^{c+s}(x, g), E^{c+s}(x, f)) \leq C_3\, \rho^{k'} \tag{8}$$

Proof. Since $f^j(x) \notin I$, we have $f^j(x) = g^j(x)$ and $Df(f^j(x)) = Dg(g^j(x))$ for every j with $|j| \leq k'$. If g is close enough to f, then inequality in Equation (4) holds for $F^{u+c} = E^{u+c}(g)$ and $F^{c+s} = E^{c+s}(g)$ and

$$\begin{aligned}
&\angle(E^{u+c}(g, x), E^{u+c}(f, x)) \\
&= \angle(Dg^{k'}(E^{u+c}(g, g^{-k'}(x))), Df^{k'}(E^{u+c}(f, f^{-k'}(x)))) \\
&= \angle(Df^{k'}(E^{u}(g, f^{-k'}(x))), Df^{k'}(E^{u}(f, f^{-k'}(x)))).
\end{aligned}$$

Then, by Equation (5) we prove Equation (7). The stable case is analogous: we use Equation (6) to prove Equation (8).

□

This provides a result for the center bundle of f_k.

Proposition 5.8. *If $\zeta = [z_0, ..., z_n]$ is the su-path given by Proposition 5.2 and f_k is given by Lemma 5.3, then there exist $C_4 > 0$, $\rho \in (0,1)$ and $k_4 \in \mathbb{N}$ such that for every $k \geq k_4$ and $i \in [1, ..., n]$, we have*

$$\angle(E^c(z_i, f_k), E^c(z_i, f)) \leq C_4\, \rho^{(q-1)k}. \tag{9}$$

Proof. If $i \neq l$, that is, z_i is not the point where we did the perturbation, then by the definition of f_k we have that $f^j(z_i)$ do not enter the closure of the neighborhood of perturbation for every $j \in \mathbb{Z}$ with $|j| \leq (q-1)k$. So we can apply the Lemma above for E^{u+c} and E^{c+s}. Then, we can conclude the Proposition because E^c is the intersection of this two transversal bundles.

Equation (9) is also true for $z_l = z$ because, although $Df_k(z) \neq Df(z)$, they are related by A_{β_k} and this application leaves invariant the subbundles in the splitting $E^u_z \oplus E^c_z \oplus E^s_z$, so we can do a similar proof and conclude the Proposition like for the other nodes. $\square$

Summarizing the results in this section we have the following: For $f \in B^r_*(M)$ and p a periodic point, Proposition 5.2 gives a *su*-path from p to p, $\zeta = [z_0, ..., z_n]$, with slow recurrence. Recall we denote $z_l = z$. This allows us to find a sequence f_k given by Lemma 5.3 and satisfying Proposition 5.6 and 5.8.

Let

$$\zeta_1 = [z_0, ..., z_l] \quad \text{and} \quad \zeta_2 = [z_n, ..., z_l].$$

In the notation of Proposition 5.6, define

$$z_k = z^k_l, \quad p_k = z^k_n,$$

$$\zeta^k_1 = [p, ..., z_k] \quad \text{and} \quad \zeta^k_2 = [p_k, ..., z_k].$$

In the following, for $i \in \{1, 2\}$, H_{ζ_i} will denote the holonomy defined by ζ_i for $F = Df|E^c(f)$ and $H_{\zeta^k_i}$ the holonomy defined by ζ^k_i for $F_k = Df_k|E^c(f_k)$.

Using Corollary 4.5 combined with Propositions 5.6 and 5.8 we can estimate the variation in the holonomies for f_k:

Corollary 5.9. *There exist $C > 0$, $\lambda \in (0,1)$ and $K \in \mathbb{N}$ such that for every $k \geq K$, $a \in E^c(p)$ and $a_k \in E^c(p_k, f_k)$ we have*

$$d(R^{-1}_{\beta_k} \circ H_{\zeta_1}(a), H_{\zeta^k_1}(a)) \leq C\, \lambda^k,$$

and

$$d(H_{\zeta_2}(a), H_{\zeta^k_2}(a_k)) \leq C\, \lambda^k + C\, d(a, a_k).$$

6 Accessibility

We obtain a continuity property for su-paths under the variation of the diffeomorphism using the results and techniques in [AV2]. In order to clarify the presentation we state here the results that we are going to need.

In this section, all the maps will be C^1 and we will always consider the C^1-topology. If f is a partially hyperbolic diffeomorphism, we denote $u = dim\ E^u$, $s = dim\ E^s$.

Recall that given two points $x, y \in M$, x is *accessible* from y if there exist a C^1 path that connects x to y, tangent at every point to $E^u \cup E^s$. The equivalence classes defined by this relation are called *accessibility classes.* We say that f is *accessible* if M is the unique accessible class.

One of the main results in [AV2] implies the following Theorem, that we already mentioned in Section 2.

Theorem 6.1. *If f is a partially hyperbolic accessible diffeomorphism and the center bundle E^c is 2-dimensional, then f is stably accessible.*

In the sequel, we state the principal results in [AV2] which will allow us to explain the proof of Theorem 6.1 and prove the main results in this Section. The next Theorem provides a kind of parametrization for accessibility classes.

Theorem 6.2. *For every partially hyperbolic diffeomorphism $f : M \longrightarrow M$, there exist $k \geq 1$, a neighborhood of f, $\mathcal{V}(f)$, and a sequence $P_l : \mathcal{V}(f) \times M \times \mathbb{R}^{k(u+s)l} \longrightarrow M$ of continuous maps such that, for every $(g, z, v) \in \mathcal{V}(f) \times M \times \mathbb{R}^{k(u+s)l}$,*

(a) *$P_m(g, P_l(g, z, v), w) = P_{m+l}(g, z, (v, w))$ for every $w \in \mathbb{R}^{k(u+s)m}$;*

(b) *$\xi \mapsto P_l(g, \xi, v)$ is a homeomorphism from M to M and $P_l(g, *, 0) = id$;*

(c) *$\bigcup_{l \geq 0} P_l(\{(g, z)\} \times \mathbb{R}^{k(u+s)l})$ is the g-accessibility class of z.*

Using this Theorem, Avila and Viana introduce a class of paths, called *deformation paths*, contained in accessibility classes and having a useful property of persistence under the variation of the diffeomorphism and the base point. More precisely,

Definition 6.3. *A deformation path based on (f, z) is a map $\gamma : [0, 1] \longrightarrow M$ such that there exist $l \geq 1$ and a continuous map $\Gamma \mapsto \mathbb{R}^{k(u+s)l}$ satisfying $\gamma(t) = P_l(f, z, \Gamma(t))$.*

The continuity of the sequence of maps P_l given by Theorem 6.2 implies the following Corollary.

Corollary 6.4. *If γ is a deformation path based on (f,z), then for every g close to f and any w close to z, there exists a deformation path based on (g,w) that is uniformly close to γ.*

The main technical step in the proof of Theorem 6.1 is a result of approximation of general paths in accessibility classes by the deformation paths defined above. We state a simpler version of this result that is sufficient for our purposes.

Theorem 6.5. *If f is accessible, then for every $z \in M$ the set of deformation paths based on (f,z) is dense on $C^0([0,1],M)$.*

The final ingredient is what it is called Intersection Property, and it is in this result where the hypothesis of $dim\ E^c = 2$ is necessary.

Theorem 6.6 (Intersection Property)**.** *Let f be a partially hyperbolic diffeomorphism with 2-dimensional center. Let D be a 2-dimensional disk transverse to $E^s \oplus E^u$ and η_u, η_s be smooth paths in D intersecting transversely at some point. Then, for every C^1 diffeomorphism g close to f and any continuous paths γ_u, γ_s uniformly close to η_u, η_s, there are points x_u, x_s in the images of γ_u, γ_s such that $W^u(x_u,g)$ intersects $W^s(x_s,g)$.*

The proof of the Intersection Property follows by considering a local change of coordinates near the transverse intersection of η_u and η_s and applying the following Lemma.

Lemma 6.7. *Let $d = u + 2 + s$. There exist $\epsilon > 0$ with the following property. Let W^u and W^s be foliations with C^1 leaves in $\mathbb{R}^d$, tangent to continuous distributions E^u and E^s of u and s-dimensional planes. Assume that E^u_x is ϵ-close to $\mathbb{R}^u \times \{0^{2+s}\}$ and E^s_x is ϵ-close to $\{0^{u+2}\} \times \mathbb{R}^s$ for every x in the unit ball B^d of $\mathbb{R}^d$. Let $\gamma_u, \gamma_s : [-1,1] \longrightarrow \mathbb{R}^d$ be continuous paths ϵ-close to the paths $\eta_u, \eta_s : [-1,1] \longrightarrow \mathbb{R}^d$ given by $\eta_u(t) = (0,t,0,0)$ and $\eta_s(t) = (0,0,t,0)$. Then there exist $t_u, t_s \in (-1,1)$ such that $W^u(\gamma_u(t_u))$ intersects $W^s(\gamma_s(t_s))$.*

Proof. Let $\rho_u(t) = \gamma_u(t) - \eta_u(t)$ and $\widehat{\rho}_u$ be a continuous extension to $\mathbb{R}$ with compact support. Let ϕ_u be the only continuous map $\phi_u : [-1/4,1/4]^{u+1} \longrightarrow \mathbb{R}^{2+s}$ such that $\phi_u(0,t) = 0$ and $x \mapsto (x, \phi_u(x,t)) + \gamma_u(t)$ is a C^1 map from $[-1/4,1/4]^u$ to $W^u(\gamma_u(t))$, for every $t \in [-1/4,1/4]$. Let $\widehat{\phi}_u$ be a continuous extension of ϕ_u to $\mathbb{R}^{u+1}$ with compact support. Define for $(x,t) \in \mathbb{R}^{u+1}$,

$$\Phi_u(x,t) = (x, \widehat{\phi}_u(x,t)) + \eta_u(t) + \widehat{\rho}_u(t).$$

This map admits a continuous extension

$$\Phi_u : S^u \times S^1 \longrightarrow S^u \times S^1 \times S^1 \times S^s$$

and by the hypotheses if ϵ is small enough, Φ is homotopic to the map $(x,t) \mapsto (x,t,0,0)$. Analogously for γ_s and W^s we obtain Φ_s homotopic to the map $(t,x) \mapsto (0,0,t,x)$. Then the intersection number of Φ_u and Φ_s is 1 and this implies that exist (x_u,t_u,t_s,x_s) such that $\Phi_u(x_u,t_u) = \Phi_s(t_s,x_s)$. Again if ϵ is small enough we can suppose that $(x_u,t_u,t_s,x_s) \in [-1/4,1/4]^d$. Then, by the definition of Φ_u and Φ_s we conclude the proof of the Lemma. □

Let us see how the above mentioned results conclude the proof of Theorem 6.1.

Proof. Let f be a partially hyperbolic accessible diffeomorphism. It suffices to prove that for every $x,y \in M$ there exist a neighborhood of f, $\mathcal{V}(f)$, and a neighborhood of y, $U(y)$, such that for every $g \in \mathcal{V}(f)$ and $z \in U(y)$, z is in the g-accessibility class of x.

Fix a small 2-disk D, η_u and η_s like in the hypotheses of Theorem 6.6. By Theorem 6.5, there exists a deformation path based on (f,x) which is uniformly close to η_u and there exist a deformation path based on (f,y) which is uniformly close to η_s. Then, by Corollary 6.4, for each g close enough to f and z close enough to y, there exists

(a) a deformation path γ_u based on (g,x) which is still close to η_u and

(b) a deformation path γ_s based on (g,z) which is still close to η_s.

Applying Theorem 6.6, we find points x_u, x_s in the images of γ_u, γ_s such that $W^u_g(x_u)$ intersects $W^s_g(x_s)$. Since, x_u is in the g-accessibility class of x and x_s is in the g-accessibility class of z, the conclusion follows. □

By Theorem 6.1 we have that for every f accessible and $x,y \in M$, if g close enough to f, then there exist some su-path for g from x to y. Moreover, the proof gives more information, it provides a way to find the su-path for g. The following results uses that information to prove relations between the su-paths for f and for g. More precisely, we prove a relation between su-paths for f and for a sequence $f_k \to f$.

Proposition 6.8. *Let f be a partially hyperbolic accessible diffeomorphism with 2-dimensional center bundle. For every $x,y \in M$, $y_k \to y$ and every sequence $f_k \to f$ in the C^1-topology, there exist a subsequence k_j, su-paths for f_{k_j} denoted by ζ_{k_j} and a su-path for f denoted by ζ satisfying the following:*

(a) $\zeta_{k_j} = [z^j_0, ..., z^j_n]$ *joins x to y_{k_j},*

(b) $\zeta = [z_0, ..., z_n]$ *joins x to y and*

(c) *for every* $\epsilon > 0$ *there exist* $K \in \mathbb{N}$ *such that for every* $k_j \geq K$,

$$dist(z_i, z_i^j) < \epsilon$$

for every $i \in [0, ..., n]$.

Proof. We proceed in the same way that in the proof of Theorem 6.1.

Fix a small 2-disk D, η_u and η_s like in the hypotheses of Theorem 6.6. By Theorem 6.5, there exists a deformation path based on (f, x) which is uniformly close to η_u and there exist a deformation path based on (f, y) which is uniformly close to η_s. Then, by Corollary 6.4, for each k big enough, there exists

(a) a deformation path γ_u^k based on (f_k, x) which is still close to η_u and

(b) a deformation path γ_s^k based on (f_k, y_k) which is still close to η_s.

Theorem 6.6 implies that there exist $t_u^k, t_s^k \in [0, 1]$ and $w_k \in M$ such that

$$w_k \in W^u(\gamma_u^k(t_u^k), f_k) \cap W^s(\gamma_s^k(t_s^k), f_k).$$

Then, for every k big enough, we have a su-path for f_k, joining x to y_k, denoted by ζ_k and defined by the nodes of $\gamma_u^k(t_u^k)$, the intersection point w_k and the nodes of $\gamma_s^k(t_s^k)$.

By some change of coordinates, using the notation of Lemma 6.7, we have functions Φ_u^k and Φ_s^k and points x_u^k and x_s^k such that

$$w_k = \Phi_u^k(t_u^k, x_u^k) = \Phi_s^k(x_s^k, t_s^k).$$

By compactness, there exist a subsequence k_j and t_u, t_s, x_u and x_s such that

$$t_*^{k_j} \to t_* \quad \text{and} \quad x_*^{k_j} \to x_*,$$

for $* \in \{s, u\}$.

Since $f_k \to f$ and W^s and W^u are continuous under the variation of the diffeomorphism, there exist $w \in M$ such that

$$w = \Phi_u(t_u, x_u) = \Phi_s(x_s, t_s),$$

this implies that

$$w \in W^u(\gamma_u(t_u), f) \cap W^s(\gamma_s(t_s), f).$$

Then, denote ζ the su-path for f joining x to y and defined by the nodes of $\gamma_u(t_u)$, the intersection point w and the nodes of $\gamma_s(t_s)$.

Finally, by the construction of the su-paths and, again, Corollary 6.4, we have that for every $\epsilon > 0$ the distance between the nodes of ζ and ζ_{k_j} are ϵ-close if k_j is big enough. □

Moreover, the *su*-paths in the above Proposition can be chosen in a uniform way. That is, with a uniform number of legs and a uniform bound for the distance between the nodes.

Corollary 6.9. *Let f be a partially hyperbolic accessible diffeomorphism with 2-dimensional center bundle. Then, there exist $L > 0$ and $N > 0$ such that for every $x, y \in M$, $y_k \to y$ and every sequence $f_k \to f$ in the C^1-topology, the su-paths defined by Proposition 6.8 can be taken to have at most N legs and distance between the nodes bounded by L.*

Proof. Observe that in order to prove this Corollary it is sufficient to prove the following claim.

Claim. *Fix $\eta \in C^0([0,1], M)$ and $\epsilon > 0$. Then, there exist $L > 0$ and $N > 0$ such that for every $x \in M$ there exist a deformation path based on (f, x), denoted γ, which is ϵ-close to η and satisfies that for every $t \in [0,1]$, the su-path defined by $\gamma(t)$ has at most N legs and the distance between the nodes is bounded by L.*

By Theorem 6.5, for every $x \in M$ there exist a deformation path based on (f, x), that is ϵ-close to η, then the claim follows from the persistence of the deformations under the variation of the base point and the compactness of M. □

This Corollary, together with Remark 4.3 and Corollary 4.5 give the following result.

Corollary 6.10. *For $f \in B^r_*(M)$, there exist $C > 0$ such that for every $x, y \in M$, $y_k \to y$ and every sequence $f_k \to f$ in the C^1-topology, the su-paths given by Proposition 6.8, ζ_{k_j} and ζ, can be taken to satisfy the following estimation for the holonomies defined by them,*

$$d(H_\zeta(a), H_{\zeta_{k_j}}(b)) \leq \psi(k_j) + C\, d(a, b),$$

where $\psi(k_j)$ goes to zero as k_j goes to ∞.

Observe that there are analogous estimations for $h_\zeta = \mathbb{P}(H_\zeta)$ and $h_{\zeta_{k_j}} = \mathbb{P}(H_{\zeta_{k_j}})$. We are going to use this result to prove the main Proposition in Section 7 and we are going to apply it to the sequence of perturbations constructed in Section 5 to conclude the proof of Theorem B in Section 8.

7 Disintegration

Let $f \in B^r_*(M)$ and p be a pinching periodic point for f. That is, $Df^{n_0}|E^c(p)$ has two real eigenvalues whit different norms, where $n_0 = per(p)$.

Then, there exist $C_1 > 0$, $\theta_0 > 0$, $\rho \in (0,1)$ and one-dimensional subspaces of E^c_p, E_1, E_2, such that for every F_1 and F_2 one-dimensional subspaces of E^c_p with

$$\max\{\angle(E_1, F_1), \angle(E_2, F_2)\} < \theta_0, \tag{10}$$

and for every $j \geq 0$, we have

$$\angle(Df^{jn_0}(E_1), Df^{jn_0}(F_1)) \leq C_1\, \rho^j \tag{11}$$

and

$$\angle(Df^{-jn_0}(E_2), Df^{-jn_0}(F_2)) \leq C_1\, \rho^j. \tag{12}$$

Suppose $\lambda^c_1(f) = \lambda^c_2(f)$, then for every $\mathbb{P}(F)$-invariant probability measure, m, the Invariance Principle gives a disintegration, $\{m_x : x \in M\}$ invariant by holonomies and continuous with the weak* topology. The continuity of m_x and the invariance of m implies that $\mathbb{P}(F(x))_* m_x = m_{f(x)}$ for every $x \in M$.

Then, if $a, b \in \mathbb{P}(E^c_p)$ are defined by $a = [E_1]$ and $b = [E_2]$, we have

$$supp\ m_p \subset \{a, b\}.$$

Fix $\epsilon > 0$ small enough and consider k_0 and the sequence of perturbations f_k defined by Lemma 5.3. Suppose for every $k \geq k_0$, $\lambda^c_1(f_k) = \lambda^c_2(f_k)$. We will denote $F_k = Df_k|E^c(f_k)$ the center derivative cocycle and $\mathbb{P}(F_k)$ its projectivization. We can suppose that k_0 is big enough to have $\mathcal{O}(p) \notin B_{\delta_k}(z)$ for every $k \geq k_0$. Then, for any $\mathbb{P}(F_k)$-invariant probability, m^k, we have

$$supp\ m^k_p \subset \{a, b\}.$$

For every $k \geq k_0$ fix some m^k. Then, there exist a subsequence k_j and a measure m in $\mathbb{P}(TM)$ such that $m^{k_j} \to m$ in the weak* topology. The limit measure m has the following properties:

(a) $supp\ m \subset \mathbb{P}(E^c(f))$,

(b) m project down to μ,

(c) m is $\mathbb{P}(F)$-invariant.

Denote $m^{k_j}_p$ and m_p the element of the disintegration given by the Invariance Principle at p for m^{k_j} and m respectively. We have the following relation.

Proposition 7.1. *If $|supp\, m_p| = 1$, then there exist a subsequence of k_j, that we continue to denote k_j, and $K \in \mathbb{N}$ such that*

$$supp\ m_p \subset supp\ m_p^{k_j},$$

for every $k_j \geq K$.

Proof. Suppose that $supp\ m_p = \{a\}$. The case $supp\ m_p = \{b\}$ is analogous.

Consider $C > 0$ given by Corollary 6.10 and fix some $0 < \delta < d(a, b)/4C$.

Define the function $\sigma : M \longrightarrow \mathbb{P}(\mathcal{V})$ by $\sigma(x) = (x, supp\ m_x)$ and the set

$$T_\delta = \{(x, v) \in \mathbb{P}(TM) : (x, v) \in B_\delta(\sigma(x))\}.$$

The Invariance Principle implies that the function σ is continuous and therefore T_δ is an open set. Moreover, by definition, $m(T_\delta) = 1$. This two properties implies that

$$m^{k_j}(T_\delta) = \int m_x^{k_j}(T_\delta \cap \mathbb{P}(E_x^c(k_j)))d\mu(x) \to 1.$$

Then, there exist a subsequence of k_j, that we continue to denote k_j, $x \in M$ and $k_1 \in \mathbb{N}$ such that $T_\delta \cap supp\ m_x^{k_j} \neq \emptyset$ for every $k_j \geq k_1$.

We apply Proposition 6.8 to f_{k_j}, x and $y_{k_j} = p$. Then, we have a new subsequence, that we continue to denote k_j, su-paths ζ_{k_j} for f_{k_j} and ζ for f, all joining x to p. Moreover, they have the property that we can make the distance between the nodes arbitrarily small, letting $k_j \to \infty$.

Denote h the holonomy defined by ζ for $\mathbb{P}(F)$ and h_{k_j} the holonomy defined by ζ_{k_j} for $\mathbb{P}(F_{k_j})$. By Corollary 6.10, there exist $C > 0$ and a function $\psi(k_j)$ going to zero as $k_j \to \infty$, such that for every $a' \in \mathbb{P}(E_x^c)$ and every $b' \in \mathbb{P}(E_x^c(f_{k_j}))$, we have

$$d(h(a'), h_{k_j}(b')) < \psi(k_j) + C\, d(a', b').$$

Since the disintegration given by the Invariance Principle is invariant by holonomies, we have $supp\ m_x = h^{-1}(a)$. Moreover, $supp\ m_x^{k_j} \cap T_\delta \neq \emptyset$ for every $k_j \geq k_1$, then there exist $a'_{k_j} \in supp\, m_x^{k_j}$ with $d(a'_{k_j}, h^{-1}(a)) < \delta$.

Define $a_{k_j} = h_{k_j}(a'_{k_j})$, then $a_{k_j} \in supp\ m_p^{k_j}$ and for k_j big enough,

$$d(a, a_{k_j}) = d(h(h^{-1}(a)), h_{k_j}(a'_{k_j})) < \psi(k_j) + C\, d(h^{-1}(a), a'_{k_j}) < d(a, b)/2.$$

Since $supp\, m_p^{k_j} \subset \{a, b\}$, this implies $a_{k_j} = a$ and finish the proof. □

8 Proof of Theorem B

In the following we recall the statement of Theorem B and give the details of its proof.

Definition 8.1. *Let f be a partially hyperbolic diffeomorphism and p a periodic point with $n_0 = per(p)$. We say that p is a* pinching periodic point *if $Df^{n_0}|E^c(p)$ has two real eigenvalues whit different norms.*

Theorem B. *Let $* \in \{\mu, \omega\}$, $f \in B^r_*(M)$ and assume f has a pinching periodic point, then f can be C^r-approximated by volume-preserving (symplectic) diffeomorphisms whose center Lyapunov exponents are different.*

Proof. Let $f \in B^r_*(M)$ have a pinching periodic point, p, with $n_0 = per(p)$, and suppose $\lambda^c_1 = \lambda^c_2$. As before, consider $F = Df|E^c$.

We prove in the previous section that there exist $a, b \in \mathbb{P}(E^c_p)$ such that for every $\mathbb{P}(F)$-invariant probability, m, the element of the disintegration given by the Invariance Principle at p satisfies, $supp\, m_p \subset \{a, b\}$.

Consider the su-path given by Proposition 5.2 and denote $z = z_l$. Fix $\epsilon > 0$ small enough and let $C_0 > 0$, $k_0 \in \mathbb{N}$ and f_k be defined by Lemma 5.3, that is,

(a) $f_k(z) = f(z)$,

(b) $Df_k(z) = Df(z) \circ A_{\beta_k}$ with $\sin \beta_k = C_0\, \delta_k^{r-1}\, \epsilon$,

(c) $f_k(x) = f(x)$ if $x \notin B_{\delta_k}(z)$,

(d) $f_k \to f$ in the C^1 topology when $k \to \infty$ and

(e) $f_k \in \eta^r_*(f, \epsilon)$ with $* \in \{\mu, \omega\}$.

Suppose $\lambda^c_1(f_k) = \lambda^c_2(f_k)$ for every $k \geq k_0$ and denote $F_k = Df_k|E^c_k$. Then, for every $k \geq k_0$ and any $\mathbb{P}(F_k)$-invariant probability, m^k, we have $supp\, m^k_p \subset \{a, b\}$.

It is enough to prove the Theorem in the following two cases:

(i) $\forall\, k \geq k_0$ there exist a $\mathbb{P}(F_k)$-invariant probability, m^k, such that $supp\, m^k_p = \{a, b\}$.

(ii) $\forall\, \mathbb{P}(F)$-invariant probability, m, we have $|supp\, m_p| = 1$.

In order to see this suppose the Theorem is true for both cases. Then, if we are not in case (i) there exist $k' \in \mathbb{N}$ such that $f_{k'} \in B^r_*(M)$, p is a pinching periodic point for $f_{k'}$, $\lambda^c_1(f_{k'}) = \lambda^c_2(f_{k'})$ and $f_{k'}$ is in case (ii).

Then, we can find a diffeomorphism ϵ-close to $f_{k'}$ and having different center Lyapunov exponents. This implies, there exists a diffeomorphism having different center Lyapunov exponents which is $2\,\epsilon$-close to f which proves Theorem B.

Since the proof of case (i) and the proof of case (ii) are very similar, we are going to explain both of them simultaneously. We will need to take subsequences of f_k several times, but in order to simplify the notation we will continue to denote them as f_k. We are also going to use the same symbol to denote both a nonzero vector in E^c_x and the corresponding element of $\mathbb{P}(E^c_x)$ for $x \in M$.

If we are in case (i), there exist m^k satisfying $supp\, m^k_p = \{a, b\}$, then we define m to be the limit of m^k as we did in Section 6. If we are in case (ii), we choose any $\mathbb{P}(F_k)$-invariant measure, m^k, define m as the limit of them and apply Proposition 7.1. From now on, it is understood that, if we are in the case (ii), we are working with the subsequence given by this Proposition.

Suppose $a \in supp\, m_p$, the other case is analogous. Proposition 5.6 defines $p_k = z^k_n$ such that $dist(p, p_k) < C\,\tau^{mk}$ with $C > 0$ and $\tau \in (0, 1)$. Then, for $q_k = f_k^{-n_0 k}(p_k)$, there exist $L > 0$ such that

$$dist(p, q_k) \leq L^{-n_0 k} dist(p, p_k) \leq C\,(L^{-n_0}\tau^m)^k.$$

Here $m = q\alpha^n$ and we can suppose q in the perturbation was chosen in order to have this expression going to zero exponentially fast.

Claim. *Let $\theta_0 > 0$ be the constant for p defined in Equation (10). Then, there exist $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$, there exist $d_k \in supp\, m^k_{q_k}$ with $d(a, d_k) < \theta_0$.*

Proof. Since $q_k \to p$, we can apply Theorem 6.8 for f_k, $x = p$ and $y_k = q_k$. Then, there exist su-paths for f_k denoted by ζ_k joining p to q_k and a su-path for f denoted by ζ joining p to p. Moreover, they satisfy Corollary 6.10.

Denote by h the holonomy defined by ζ for $\mathbb{P}(F)$ and h_k the holonomy defined by ζ_k for $\mathbb{P}(F_k)$.

If we are in case (i), we have to possibilities: $h(a) = a$ or $h(a) = b$. Then, define

$$d_k = \begin{cases} h_k(a) & \text{if } h(a) = a, \\ h_k(b) & \text{if } h(a) = b. \end{cases}$$

Since $supp\, m^k_p = \{a, b\}$, in any case we have $d_k \in supp\, m^k_{q_k}$.

If we are in case (ii), then $h(a) = a$ and we define $d_k = h_k(a)$. Proposition 7.1 implies that $d_k \in supp\, m^k_{q_k}$.

Then, by Corollary 6.10 there exist $\psi(k) \to 0$ as $k \to \infty$, such that

$$d(a, d_k) \leq \psi(k).$$

Therefore, choose k_1 big enough such that $\psi(k) < \theta_0$. $\square$

As before, P_E will denote the orthogonal projection to E and $\pi_{x,y}$ the parallel transport between x and y.

Define

$$a_k = \left[P_{E^c_p} \circ \pi_{q_k,p}(d_k) \right],$$

where $[*]$ denote the class in the projective space, and

$$c_k = \mathbb{P}(F_k^{n_0}(q_k))(d_k).$$

We have the following consequences:

$$a_k \in \mathbb{P}(E^c_p), \quad \angle(a, a_k) < \theta_0 \quad \text{and} \quad c_k \in supp\ m^k_{p_k}.$$

Then, there exist $C_2 > 0$ such that

$$\begin{aligned} d(a, c_k) &= d(\mathbb{P}(F^{n_0 k}(p))(a), \mathbb{P}(F_k^{n_0 k}(q_k))(d_k)) \\ &\leq d(\mathbb{P}(F^{n_0 k}(p))(a), \mathbb{P}(F^{n_0 k}(p))(a_k)) \\ &+ d(\mathbb{P}(F_k^{n_0 k}(p))(a_k), \mathbb{P}(F_k^{n_0 k}(q_k))(d_k)) \\ &\leq C_1\, \rho^k + C_2^{n_0 k}\, dist(p, q_k) \\ &\leq C_1\, \rho^k + C\, (C_2^{n_0} L^{-n_0} \tau^m)^k. \end{aligned} \tag{13}$$

The estimation in the first term is a consequence of Equation (7). Since $m = q\alpha^n$, we suppose again that q was chosen to have the expression on the second term going exponentially fast to zero.

If $\zeta = [z_0, ..z_n]$ is the su-path for p given by Propostition 5.2, denote $z = z_l$,

$$\zeta_1 = [z_0, ..., z] \quad \text{and} \quad \zeta_2 = [z, ..., z_n].$$

Then, H_{ζ_i} and h_{ζ_i} denote the holonomies defined by ζ_i for F and $\mathbb{P}(F)$ respectively, with $i \in \{1, 2\}$.

To finish the proof we need to consider two different cases. If we are in case (i), we can have $supp\, m_p = \{a, b\}$ or $supp\, m_p = \{a\}$. If we are in case (ii), since we are supposing that $a \in supp\, m_p$ we only can have $supp\, m_p = \{a\}$. However, we are going to suppose that $supp\, m_p = \{a, b\}$, since this case impose more restrictions than the other. We are also going to suppose that there exist $c, d \in \mathbb{P}(E^c_z)$ such that $c = h_{\zeta_1}(a) = h_{\zeta_2}(a)$ and $d = h_{\zeta_1}(b) = h_{\zeta_2}(b)$. The other cases are analogous.

In the notation of Proposition 5.6 we have

$$z_k = z^k \qquad p_k = z_n^k,$$

and *su*-paths for f_k,

$$\zeta_1^k = [p, ..., z_k] \quad \text{and} \quad \zeta_2^k = [p_k, ..., z_k].$$

Denote $H_{\zeta_i^k}$ and $h_{\zeta_i^k}$ the holonomies defined by ζ_i^k for F_k and $\mathbb{P}(F_k)$ respectively, with $i \in \{1, 2\}$.

Define $\Phi_k : E^c_{z_k}(f_k) \longrightarrow E^c_z$, by $\Phi_k = P_{E^c_z} \circ \pi_{z_k,z}$. Then, for k big enough Φ_k is an isomorphism. By Corollary 5.9 and Equation (13), there exist $C > 0$, $\lambda \in (0, 1)$ and $K \in \mathbb{N}$ such that for every $k \geq K$,

(a) $\left\| R^{-1}_{\beta_k}(c) - \Phi_k(H_{\zeta_1^k}(a)) \right\| < C\,\lambda^k$,

(b) $\left\| R^{-1}_{\beta_k}(d) - \Phi_k(H_{\zeta_1^k}(b)) \right\| < C\,\lambda^k$ and

(c) $\left\| c - \Phi_k(H_{\zeta_2^k}(c_k)) \right\| < C\,\lambda^k$.

Since

$$\frac{\lambda^k}{\sin^2 \beta_k} \to 0,$$

as $k \to \infty$, and Φ_k is an isomorphism, we have that for k big enough, the one-dimensional subspaces generated by $H_{\zeta_1^k}(a)$, $H_{\zeta_1^k}(b)$ and $H_{\zeta_2^k}(c_k)$ are all different. In the projective level this means,

$$h_{\zeta_1^k}(a), h_{\zeta_1^k}(b) \neq h_{\zeta_2^k}(c_k).$$

On the other hand, since $supp\; m_p^k \subset \{a, b\}$, the invariance by holonomies given by the Invariance Principle implies

$$supp\; m_{z_k}^k \subset \{h_{\zeta_1^k}(a), h_{\zeta_1^k}(b)\}.$$

Moreover, since $c_k \in supp\; m_{p_k}^k$, then $h_{\zeta_2^k}(c_k) \in supp\; m_{z_k}^k$.

We arrive to this contradiction because we were assuming that the Invariance Principle could be applied for every f_k with $k \geq k_0$, then there exist $k \in \mathbb{N}$ such that $\lambda_1^c(f_k) \neq \lambda_2^c(f_k)$, with f_k in a ϵ C^r-neighborhood of f. Since ϵ was chosen arbitrarily, this proves Theorem B. □

9 Proof of Theorem A

Theorem A. *Let $f \in B^r_\omega(M)$ and assume the set of periodic points of f is non-empty, then f can be C^r-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.*

The following observations are going to prove that we can reduce the proof to two cases.

Suppose M is symplectic manifold and $dim\ M = 2d$. Given a periodic point p of a symplectic diffeomorphism f, define the *principal eigenvalues* to be those d eigenvalues with norm greater that one or with norm one and imaginary part greater than zero and the half of the eigenvalues equals to 1 or -1. If the principal eigenvalues are multiplicative independent over the integers, that is $\prod \lambda_i^{p_i} = 1$ with $p_i \in \mathbb{Z}$ implies $p_i = 0$ for every $i \in \{0, ..., d\}$, we say that p is *elementary.*

Let $f \in B^r_\omega(M)$ and p a periodic point of f, then by the results in [Rob], by making a perturbation if necessary, we can suppose that p is elementary. This implies that the eigenvalues of $Df^{n_0}|E^c_p$, where $n_0 = per(p)$, satisfy one of the followings:

(i) there exist $0 < \rho < 1$ such that the eigenvalues are ρ and ρ^{-1}, or

(ii) there exist $x, y \in \mathbb{R}$ such that $x + iy$ and $x - iy$ are the eigenvalues, $x^2 + y^2 = 1$ and they are not a root of unity.

We are going to call option (i) the hyperbolic case and option (ii) the elliptic case. Then, it is sufficient to prove Theorem A under the hypothesis of p being in one of these cases.

Hyperbolic Case

Fix $f \in B^r_\omega(M)$ and suppose p is a periodic point for f satisfying (i). Then, p is a pinching periodic point and we can apply Theorem B to find a symplectic diffeomorphism, g, C^r-arbitrarily close to f, with $\lambda^c_1(g) \neq \lambda^c_2(g)$. By the symmetry of the Lyapunov exponents for symplectic diffeomorphisms, we have that they are non-zero and therefore we proved Theorem A in this case.

Elliptic Case

We are going to use the following result that appear in [N].

Definition 9.1. *We say that a periodic point p of period n_0 is* quase-elliptic *if there exist $1 \leq k \leq d$ such that $Df_p^{n_0}$ has $2k$ non-real eigenvalues of norm one and its remaining eigenvalues have norm different from one.*

Theorem 9.2. *There exist a residual set $R \subset Diff_\omega^r(M)$, $1 \leq r \leq \infty$, such that if $f \in R$, then each quasi-elliptic periodic point of f is the limit of transversal homoclinic points.*

Fix $f \in B_\omega^r(M)$ and suppose p is a periodic point of f satisfying (ii). Then, p is a quasi-elliptic periodic point. By Theorem 9.2, we can suppose that besides of p, there exist q a hyperbolic periodic point of f. At this point we could apply Theorem B to conclude the proof. However, we are going to show that the coexistence of hyperbolic and elliptic periodic points is an obstruction for the Invariance Principle. An argument similar to the one we give here can be found in Remark 2.9 of [ASV].

Define $F = Df|E^c$ and suppose $\lambda_1^c = \lambda_2^c$. Then, we can apply the Invariance Principle for the projective cocycle. Like already mentioned, this implies that for every $\mathbb{P}(F)$-invariant probability measure, m, there exist a disintegration $\{m_x : x \in M\}$ such that $\mathbb{P}(F(x))_* m_x = m_{f(x)}$.

Since f is accessible, there exist a su-path ζ joining q to p. Let h_ζ denote the holonomy defined by ζ for $\mathbb{P}(F)$.

We are going to show that the coexistence of hyperbolic and elliptic periodic points is an obstruction for the Invariance Principle. An argument similar to the one we give here can be found in Remark 2.9 of [ASV].

Denote $n_0 = per(p)$, $l_0 = per(q)$ and, m_p and m_q the elements of the disintegration given by the Invariant Principle at p and q for some m. Then,

$$\mathbb{P}(F^{n_0}(p))_* m_p = m_p, \qquad \mathbb{P}(F^{l_0}(q))_* m_q = m_q \quad \text{and} \quad (h_\zeta)_* m_q = m_p.$$

By the second equality and the fact that q is hyperbolic, there exist two points in $\mathbb{P}(E^c(q))$, a and b, such that $supp\, m_q \subset \{a, b\}$. By the third equality we have that the support of m_p contains at most two points, and then by the first equality we have that $\mathbb{P}(F^{n_0}(p))$ has a periodic point of period 1 or 2. This contradicts the fact of p being a elliptic periodic point satisfying (ii).

Then we have that the center Lyapunov exponents of f must be different almost everywhere. Then again, by the symmetry property, both center Lyapunov exponents must be different from zero. This finish the proof of Theorem A.

□

10 Applications

In this section we prove some results using Theorem A. We will construct examples of partially hyperbolic symplectic diffeomorphism that can be C^r-approximated by diffeomorphisms in $B^r_\omega(M)$ having a periodic point. Then, by Theorem A, we will be able to approximate these examples by non-uniformly hyperbolic systems.

Let $r \geq 2$. Remember, $B^r_\omega(M)$ is the subset of $PH^r_\omega(M)$ where f is accessible, α-pinched and α-bunched for some $\alpha > 0$, and the center bundle E^c is 2-dimensional.

Theorem A. *Let $f \in B^r_\omega(M)$ and assume the set of periodic points of f is non-empty, then f can be C^r-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.*

For $d \geq 1$, let $\mathbb{T}^{2d}$ denote the $2d$-torus.

Corollary 1. *Let $f : \mathbb{T}^{2d} \longrightarrow \mathbb{T}^{2d}$ be a C^r Anosov symplectic diffeomorphism and $g : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ a symplectic linear map with eigenvalues of norm one. Then, $f \times g$ can be C^r-approximated by non-uniformly hyperbolic diffeomorphisms.*

Corollary 2. *Let $g : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be a C^r symplectic diffeomorphism. Then, for every $d \geq 1$ there exist $f : \mathbb{T}^{2d} \longrightarrow \mathbb{T}^{2d}$ a C^r Anosov symplectic diffeomorphism such that $f \times g$ can be C^r-approximated by non-uniformly hyperbolic diffeomorphisms.*

Since the arguments are the same, we are going to discuss the two proofs together.

In the first Corollary, $f \times g$ is a partially hyperbolic diffeomorphism that is α-pinched and α-bunched for some $\alpha > 0$. In the second one, we can find $f : \mathbb{T}^{2d} \longrightarrow \mathbb{T}^{2d}$ Anosov such that $f \times g$ has the same properties. Moreover, we can suppose that $f \times g$ has a periodic point in both cases. For this, we may have to perturb g in the second case if necessary.

The results in [SW2] imply that for every $\epsilon > 0$, there exist a partially hyperbolic accessible diffeomorphism h that is ϵ-close to $f \times g$ in the C^r-topology. Moreover, we can suppose that h coincides with $f \times g$ in the orbit of some periodic point and therefore has a periodic point.

If ϵ is small enough we have $h \in B^r_\omega(M)$. Then, we can apply Theorem A to C^r-approximate h by non-uniformly hyperbolic diffeomorphisms. This finish the proof.

Let λ be a real parameter. The *standard map* g_λ of the 2-torus is defined by

$$g_\lambda(z, w) = (z + w, w + \lambda \sin(2\pi(z + w))),$$

and it preserves the symplectic form in $\mathbb{T}^2$. By KAM theory, for all values of λ near zero, g_λ has a positive measure set of invariant circles. Moreover, there exist a neighborhood of g_λ such that any diffeomorphisms in this neighborhood has a positive measure subset where both Lyapunov exponents are zero. However, if we add some transverse hyperbolicity we are able to remove the zero Lyapunov exponents.

Let $f : \mathbb{T}^{2d} \longrightarrow \mathbb{T}^{2d}$ be a C^r Anosov symplectic diffeomorphism.

Corollary 3. *If λ is close enough to zero, $f \times g_\lambda$ can be C^r-approximated by non-uniformly hyperbolic diffeomorphisms.*

Proof. The argument is the same as before, we need to prove that $f \times g_\lambda$ can be approximated by diffeomorphisms in $B^r_\omega(M)$ having a periodic point. The only observation we need to make is that $f \times g_\lambda$ is α-pinched and α-bunched for λ close to zero because $f \times g_0$ is. The rest of the proof follows using [SW2] and Theorem A. $\square$

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