Instituto Nacional de Matemática Pura e Aplicada

Doctoral Thesis

Low Degree Rational Curves on Varieties

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August 2015

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Thesis presented to the Post-graduate Program in Mathematics at Instituto Nacional de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctoral in Philosophy in Mathematics.

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July 2015

Abstract

In this thesis we study the space of lines and conics on special varieties, and we apply this study to address relevant problems concerning the geometry of Fano varieties. In the first part of this thesis we study general codimension 2 linear sections of the Grassmannians $\mathbb{G}(1,4)$ and $\mathbb{G}(1,5)$ (in the Plücker embedding). We give a description of their spaces of lines passing through any given point. As an application we show that these Fano manifolds are not weakly 2-Fano, completing the classification of weakly 2-Fano manifolds of high index, initiated by Araujo and Castravet. In the second part, we study conic-connected manifolds. We prove that the space $W_{x,y}$ of conics on a conic-connected manifold X passing through two general points $x, y \in X$ is smooth, and we define a natural polarization on this space. Relating this study with the study of minimal pointed rational curves by de Jong and Starr, we give a formula for the canonical bundle of $W_{x,y}$ in terms of the second Chern character of X and the first Chern class of our polarization. We conclude that $W_{x,y}$ is Fano if X is weakly 2-Fano.

Acknowledgments

First and foremost I thank my advisor Carolina Araujo for her guidance, attention and for her incredible patience in teaching me as many times as I needed. I thank her for the trust and all mathematics I learned from her.

I thank my parents, who always excelled for my education.

To my fellows at IMPA, I am thankful for all the time spent together learning, struggling and laughing. I would have never made it without them.

I thank my committee members, Karl-Otto Stöhr, Alex Massarenti, Israel Vainsencher, Parham Salehyan and Oliver Lorscheid, for their comments and suggestions.

I thank IMPA for the hospitality and support.

I thank CNPq for the financial support.

To my parents, Antonio and Eleni.

Contents

Chapter 1

Introduction

Rational curves of low degree on projective varieties is one of the oldest subjects in Algebraic Geometry. It goes back to the nineteenth century with the study of lines on hypersurfaces in the projective space. For example, in 1849 A. Cayley presented in [Cay49] G. Salmon's proof that a general cubic surface contains 27 lines. In the same year, Salmon generalized this fact for any smooth cubic surface. In 1904, G. Fano presented his work [Fan04] on the variety of lines on a general complex cubic hypersurface of dimension 3.

In this thesis we study the space of lines and conics on special varieties, and we apply this study to address relevant problems concerning the geometry of *Fano* varieties. A smooth complex projective variety X is called Fano if it has ample anti-canonical class, $-K_X > 0$. Fano varieties play a major role in birational geometry. For example, from the point of view of the Minimal Model Program they are one of the building blocks of all complex projective varieties (together with the varieties X with $K_X = 0$, and $K_X > 0$).

Many Fano varieties are covered by lines. This is the case for those with high *index*. The index of a Fano variety X is the largest natural number i_X such that $-K_X = i_X H$, for some Cartier divisor H. For Fano varieties X covered by lines the study of the variety H_x of lines on X passing through a general point $x \in X$ has shown to be a powerful tool for the study of X. This was very much recognized after the work [Mor79] of S. Mori, with a systematized study of H_x. In some cases it is important to know H_x for all points $x \in X$, not just general points. The first result of this thesis is a description of H_x for every x in two linear sections of Grassmannians. Let X be a general codimension 2 linear section (in the Plücker embedding) of $\mathbb{G}(1,4)$ or $\mathbb{G}(1,5)$. First, we prove that the automorphism group of X acts with finitely many orbits and in each case there are 4 orbits. Then we identify the space of lines passing through points in each orbit. For $\mathbb{G}(1,4)$, the possibilities are the twisted cubic, the union of a line and a conic, the union of 3 lines, and the union of a double line and a line (see Theorem 3.3.1 for details). For $\mathbb{G}(1,5)$, the possibilities are the quartic scroll $S_{2,2}$, the quartic scroll $S_{1,3}$, the union of a cubic scroll and a plane, and the union of a quadric surface and two planes (see Theorem 3.3.2 for details). The main motivation for this work is that these varieties occur in the classification of Fano manifolds of dimension n and index $i \geq n-2$. As a consequence, we complete the Araujo-Castravet classification of *weakly* 2-*Fano* manifolds of high index, showing that the codimension two linear sections of $\mathbb{G}(1,4)$ and $\mathbb{G}(1,5)$ are not weakly 2-Fano (see Corollary 3.4.1 for details). A Fano variety X is called weakly 2-Fano if it has non-negative second Chern character, $\mathrm{ch}_2(X) \geq 0$.

Next we investigate another special class of Fano varieties, namely *conic-connected* varieties. An embedded smooth variety $X \subset \mathbb{P}^N$ is called conic-connected if for any two general points $x, y \in X$ there is an irreducible conic on X passing through x, y. Complex conic-connected varieties were classified in [IR10] by P. Ionescu and F. Russo (see Theorem 4.1.1). Let X be a complex conic-connected variety and let $x, y \in X$ be general points. We define $W_{x,y}$ to be the space of conics on X passing through x, y (this is viewed as a subvariety of a certain Kontsevich moduli space of stable maps; see Section 4.2 for details). It comes with universal family morphisms

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mu} & X \\ s_x \uparrow \;\; \downarrow \pi \;\; \uparrow s_y & & \\ W_{x,y} & & \end{array} .$$

We define a natural polarization on $W_{x,y}$ as follows. Consider the morphism

$$\tau_x : W_{x,y} \longrightarrow \mathbb{P}\left(T_x X\right)$$

which maps a conic to its tangent direction at x. We show that this is a finite morphism and define a polarization on $W_{x,y}$ to be $\mathcal{M}_{x,y} = \tau_x^* \mathcal{O}_{\mathbb{P}(T_x X)}(1)$ (see Lemma 4.2.2 for details). We propose to use the polarized variety $(W_{x,y}, \mathcal{M}_{x,y})$ to study conic-connected varieties. As a first step we give a formula for the canonical class $K_{W_{x,y}}$ of $W_{x,y}$:

$$K_{W_{x,y}} = -\pi_* \mu^* \operatorname{ch}_2(X) - 2\operatorname{c}_1(\mathcal{M}_{x,y}).$$

(see Subsection 4.3.3 for details). In particular, we see that if X is weakly 2-Fano, then $W_{x,y}$ is Fano.

Chapter 2

Preliminaries

In this chapter we gather the prerequisites for our study of rational curves on varieties. In Sections 2.1–2.5 we define several spaces parameterizing rational curves on varieties. We present in Section 2.4 the main results concerning deformation of rational curves on varieties, and in Section 2.6 we focus on the varieties that we will work with, namely Fano varieties and rationally connected varieties. Finally, in Section 2.7 we study Grassmannian varieties and Schubert Calculus, which will be essential in Chapter 3. We omit the proofs of most of the results in this chapter. References will be given in each section.

2.1 Hilbert Scheme

Studying the geometry of an algebraic variety frequently involves understanding some of its subvarieties. For that reason, it is very useful to consider some space parameterizing such subvarieties. Among the several notions of parameter space, here we define the Hilbert scheme, which parameterizes closed subvarieties of a projective variety. We refer to [Kol96, Section I.1] for generalities and proofs.

Let X be a scheme over a field k. The *Hilbert functor*

$$\mathit{Hilb}(X) : (\text{schemes over } k) \longrightarrow (\text{sets})$$

from the category of schemes over k to the category of sets is defined by

$$\mathit{Hilb}(X)(S) = \left\{ \begin{array}{l} \text{subschemes } Z \subset X \times S \\ \text{which are proper and flat over } S \end{array} \right\}.$$

Theorem 2.1.1 ([Kol96, Thm. I.1.4])**.** Let X be a projective scheme over a field k. Then the functor $\mathit{Hilb}(X)$ is representable. More precisely, there exist a scheme $\mathrm{Hilb}(X)$ over k, called *Hilbert scheme*, and a closed subscheme $\mathcal{U}_{\mathrm{Hilb}(X)} \subset X \times \mathrm{Hilb}(X)$ which is flat over $\mathrm{Hilb}(X)$ together with universal family morphisms

$$\begin{array}{ccc} \mathcal{U}_{\mathrm{Hilb}(X)} & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_{\mathrm{Hilb}(X)}}\downarrow & & \\ \mathrm{Hilb}(X) & & \end{array},$$

satisfying the following property: for every scheme S over k and every closed subscheme $Z \subset X \times S$ which is flat over S, there exists a unique morphism $S \to \mathrm{Hilb}(X)$ such that

$Z \cong \mathcal{U}_{\mathrm{Hilb}(X)} \times_{\mathrm{Hilb}(X)} S$. The scheme Hilb$(X)$ has countably many connected components, each of them is a projective scheme over k.

In particular, we have a bijection between the set of closed subschemes of X and closed points of Hilb(X). For a closed subscheme Z of X, we will denote by $[Z]$ the corresponding point in Hilb(X). In order to give some local properties of the scheme Hilb(X), assume that X is smooth. We have the following results:

(i) $T_{[Z]} \operatorname{Hilb}(X) \cong H^0(Z, \mathcal{N}_{Z/X})$, where $\mathcal{N}_{Z/X}$ denotes the normal sheaf of Z in X;

(ii) if Z is a locally complete intersection, then

$$\dim_{[Z]} \operatorname{Hilb}(X) \geq h^0(Z, \mathcal{N}_{Z/X}) - h^1(Z, \mathcal{N}_{Z/X});$$

(iii) if Z is a locally complete intersection, and if $H^1(Z, \mathcal{N}_{Z/X}) = 0$, then Hilb$(X)$ is smooth at $[Z]$.

2.2 Hom **Scheme**

As an application of the Hilbert scheme, we construct the scheme Hom(Y, X), which parameterizes morphisms between two projective schemes Y and X. A morphism can be identified with its graph, and thus it is natural to construct the scheme Hom(Y, X) as a subscheme of Hilb$(Y \times X)$. We refer to [Kol96, Section I.1] for generalities and proofs.

Let Y and X be schemes over a field k. Consider the functor

$$Hom(Y, X) : (\text{schemes over } k) \longrightarrow (\text{sets})$$

from the category of schemes over k to the category of sets, defined by

$$Hom(Y, X)(S) = \{S\text{-morphisms } Y \times S \to X \times S\}.$$

Theorem 2.2.1 ([Kol96, Thm. I.1.10])**.** Let Y and X be projective schemes over a field k. Then the functor $Hom(Y, X)$ is representable. More precisely, there exists a locally Noetherian subscheme Hom(Y, X) of Hilb$(Y \times X)$ together with universal family morphisms

$$\begin{array}{ccc} Y \times \operatorname{Hom}(Y, X) & \xrightarrow{F} & X, \\ \downarrow{\scriptstyle \pi} & & \\ \operatorname{Hom}(Y, X) & & \end{array}$$

satisfying the following property: for every scheme S over k and every S-morphism $F_S : Y \times S \to X \times S$, there exist a unique morphism $\varphi : S \to \operatorname{Hom}(Y, X)$ such that $F_S(y, s) = (F(y, \varphi(s)), s)$, for every $(y, s) \in Y \times S$.

In particular, there exists a bijection between the set of morphisms from Y to X and the closed points of Hom(Y, X). For a morphism $f : Y \to X$, we will denote by $[f]$ the corresponding point in Hom(Y, X).

Let $g : B \to X$ be a morphism from a fixed closed subscheme B of Y. It is useful to consider morphisms $f : Y \to X$ that restricts to g on B. For that reason, consider the functor

$$Hom(Y, X; g) : (\text{schemes over } k) \longrightarrow (\text{sets})$$

from the category of schemes over k to the category of sets, defined by

$$Hom(Y,X;g)(S) = \left\{ \begin{array}{l} S\text{-morphisms } f : Y \times S \to X \times S \\ \text{such that } f|_{B\times S} = g \times id_S \end{array} \right\}.$$

Let

$$R : \mathrm{Hom}(Y,X) \longrightarrow \mathrm{Hom}(B,Y)$$

be the restriction morphism. Then the functor $Hom(Y,X;g)$ is represented by the subscheme $\mathrm{Hom}(Y,X;g) := R^{-1}(g)$ of $\mathrm{Hom}(Y,X)$. In order to give some local properties of the scheme $\mathrm{Hom}(Y,X;g)$ at a point $[f]$ (we allow B and g to be empty), assume that X is smooth along the image of f. Then we have the following results:

(i) $T_{[f]}\,\mathrm{Hom}(Y,X;g) \cong H^0(Y, f^*T_X \otimes \mathcal{I}_B)$, where $\mathcal{I}_B$ denotes the ideal sheaf of B on Y;

(ii) $\mathrm{Hom}(Y,X;g)$ can be defined by $h^1(Y, f^*T_X \otimes \mathcal{I}_B)$ equations in a smooth variety of dimension $h^0(Y, f^*T_X \otimes \mathcal{I}_B)$; in particular,

$$\dim_{[f]} \mathrm{Hom}(Y,X;g) \geq h^0(Y, f^*T_X \otimes \mathcal{I}_B) - h^1(Y, f^*T_X \otimes \mathcal{I}_B),$$

and if $H^1(Y, f^*T_X \otimes \mathcal{I}_B) = 0$, then $\mathrm{Hom}(Y,X;g)$ is smooth at $[f]$.

2.3 Chow Scheme

Here we define the Chow scheme, which parameterizes effective cycles on a projective scheme. We refer to [Kol96, Section I.3] for precise definitions, generalities and proofs.

Let X be a scheme over a field k of characteristic zero. The *Chow functor*

$$Chow(X) : (\text{schemes over } k) \longrightarrow (\text{sets})$$

from the category of schemes over k to the category of sets is defined by

$$Chow(X)(S) = \left\{ \begin{array}{l} \text{families of effective, proper,} \\ \text{algebraic cycles of } X \times S \text{ over } S \end{array} \right\}.$$

Theorem 2.3.1 ([Kol96, Thm. I.3.21])**.** Let X be a projective scheme over a field k. Then the functor $Chow(X)$ is representable. More precisely, there exist a scheme $\mathrm{Chow}(X)$ over k, called *Chow scheme*, and a subscheme $\mathcal{U}_{\mathrm{Chow}(X)} \subset X \times \mathrm{Chow}(X)$ together with universal family morphisms

$$\begin{array}{ccc} \mathcal{U}_{\mathrm{Chow}(X)} & \xrightarrow{\ \eta\ } & X\ , \\ {\scriptstyle u}\downarrow & & \\ \mathrm{Chow}(X) & & \end{array}$$

satisfying the following property: for every normal scheme S over k and every family $\mathcal{C} \to S$ of effective cycles on X, there exists a unique morphism $S \to \mathrm{Chow}(X)$ such that $\mathcal{C}$ is the pullback of $\mathcal{U}_{\mathrm{Chow}(X)}$ to $X \times S$. Every connected component V of $\mathrm{Chow}(X)$ is a reduced projective scheme over k and $\mathcal{U}_{\mathrm{Chow}(X)} \times_{\mathrm{Chow}(X)} V$ is an effective cycle on $X \times V$.

In general, the schemes $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ may look different at the points corresponding to a closed subscheme Z of X. However, if Z is reduced and pure dimensional, and if $\mathrm{Hilb}(X)$ is smooth at $[Z]$, then $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ are locally isomorphic at $[Z]$.

2.4 Rational Curves on Varieties

In the previous sections we defined several spaces parameterizing subvarieties of a given variety. The general results for these spaces are used to study rational curves on varieties. In this case everything takes a particularly simple form. In fact, general deformation theory tells us that the tangent spaces to Hilb or Hom are given by the cohomology group H^0 of certain sheaves and the obstructions are the cohomology group H^1 of the same sheaves. In the curve case, $H^0 - H^1$ is the Euler characteristic, which can usually be computed easily using Riemann-Roch's Theorem. In Subsection 2.4.3 we study objects for which the obstruction H^1 is zero.

2.4.1 Parameterizing Rational Curves

Let X be a projective variety over a field k. One way to study rational curves on X is by looking at the parameter space $\mathrm{Hom}(\mathbb{P}^1, X)$, defined in the Section 2.2. However, that parameter space is too big if we are interested in studying rational curves on X as closed subschemes or 1-cycles on X. In fact, a morphism $f : \mathbb{P}^1 \to X$ and the composition of f with any automorphism of $\mathbb{P}^1$ have the same image. In this subsection we define the scheme $\mathrm{RatCurves}(X)$, a refined parameter space for rational curves on X. For details and proofs, we refer to [Kol96, Section II.2].

Let X be a projective variety over a field k. Consider the scheme $\mathrm{Hom}(\mathbb{P}^1, X)$. Let V be an irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$. Let $\widetilde{V}$ be the normalization of the open subscheme of V parameterizing morphisms that are birational onto their images. By the universal property of the scheme $\mathrm{Chow}(X)$, there exists a natural morphism

$$\begin{aligned} \varphi : \widetilde{V} &\longrightarrow \mathrm{Chow}(X) \\ [f] &\longmapsto f_*[\mathbb{P}^1]. \end{aligned}$$

The automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ of $\mathbb{P}^1$ acts naturally on $\widetilde{V}$, and the morphism φ is invariant under this action. Let $\overline{W}_V \subset \mathrm{Chow}(X)$ be the closure of the image of φ. Let W_V be the open subset of $\overline{W}_V$ parameterizing irreducible reduced 1-cycles, and let H_V be its normalization. We define

$$\mathrm{RatCurves}(X) = \bigcup_V H_V,$$

where V runs through all irreducible components of $\mathrm{Hom}(\mathbb{P}^1, X)$.

Let V be an irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$. Let $\mathcal{U}$ be the normalization of the universal family over H_V. We have the commutative diagram

$$\begin{array}{ccccc} \mathbb{P}^1 \times \widetilde{V} & \xrightarrow{\Phi} & \mathcal{U} & \xrightarrow{\eta} & X, \\ \downarrow & & \downarrow \pi & & \\ \widetilde{V} & \xrightarrow{\varphi} & H_V & & \end{array} \qquad F = \eta \circ \Phi : \mathbb{P}^1 \times \widetilde{V} \to X$$

where π is a $\mathbb{P}^1$-bundle, Φ and φ are smooth of relative dimension 3 with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1)$.

We can carry out the above construction starting with $\mathrm{Hom}(\mathbb{P}^1, X; o \mapsto x)$ instead of $\mathrm{Hom}(\mathbb{P}^1, X)$. We denote by $\mathrm{RatCurves}(X, x)$ the scheme obtained in this way. For each irreducible component V_x of $\mathrm{Hom}(\mathbb{P}^1, X; o \mapsto x)$, there exist an irreducible component H_x of $\mathrm{RatCurves}(X, x)$ and morphisms

$$\begin{array}{ccccc} \mathbb{P}^1 \times \widetilde{V}_x & \xrightarrow{\Phi_x} & \mathcal{U}_x & \xrightarrow{\eta_x} & X \, , \\ \downarrow & & \downarrow{\scriptstyle \pi_x} & & \\ \widetilde{V}_x & \xrightarrow{\varphi_x} & H_x & & \end{array} \qquad F_x = \eta_x \circ \Phi_x : \mathbb{P}^1 \times \widetilde{V}_x \to X$$

where $\mathcal{U}_x$ is the normalization of the universal family over H_x, π_x is a $\mathbb{P}^1$-bundle, Φ_x and φ_x are smooth of relative dimension 2 with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1, o)$.

2.4.2 Bend and Break Lemmas

We present here the *bend and break* technique, a powerful tool developed by S. Mori in [Mor79] to prove Hartshorne's conjecture, characterizing projective spaces as the only smooth projective varieties over an algebraically closed field with ample tangent bundle. The tool provides means of producing rational curves on varieties. The idea is that if a curve deforms in a projective variety while passing through a fixed point, then it must at some point break up with at least one rational component. The proofs are simple and can be found in [Kol96, Section II.5], [KM98, Section 1.1] and [Deb01, Section 3.1].

Lemma 2.4.1 (Bend-and-Break I, [Deb01, Prop. 3.1])**.** Let X be a projective variety over an algebraically closed field. Let C be a smooth curve and let $p \in C$ be a point. Let $f : C \to X$ be a morphism and assume that $\dim_{[f]} \mathrm{Hom}(C, X; f|_{\{p\}}) \geq 1$. Then there exist a morphism $f' : C \to X$ and a connected, non-zero, effective, rational 1-cycle Z on X passing through $f(p)$ such that

$$f_*[C] \sim f'_*[C] + Z.$$

Lemma 2.4.2 (Bend-and-Break II, [Deb01, Prop. 3.2])**.** Let X be a projective variety over an algebraically closed field. Let $f : \mathbb{P}^1 \to X$ be a morphism and assume that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X; f|_{\{0,\infty\}}) \geq 2$. Then the cycle $f_*[\mathbb{P}^1]$ is numerically equivalent to a connected, non-integral, effective, rational 1-cycle passing through $f(0)$ and $f(\infty)$.

2.4.3 Free Rational Curves

Let X be a smooth variety of dimension n and let $f : \mathbb{P}^1 \to X$ be a morphism. Since X is smooth, its tangent sheaf T_X is locally free of rank n; hence, so is f^*T_X. By Grothendieck's Theorem, every vector bundle on $\mathbb{P}^1$ decomposes as a sum of line bundles (see [Har77, Ex. V.2.6]). Therefore, for suitable integers $a_1, \ldots, a_n$,

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \tag{2.1}$$

where we assume $a_1 \geq \cdots \geq a_n$.

Definition 2.4.1. Let $r \geq 0$ be an integer. We will say that a morphism $f : \mathbb{P}^1 \to X$ is *r-free* if $f^*T_X(-r)$ is generated by its global sections, or equivalently, if $a_n \geq r$. We will say "*free*" instead of "0-free", and "*very free*" instead of "1-free".

Note that, by Serre Duality,

$$H^0\left(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(-a_i + r - 1)\right) \cong H^1\left(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i - r + 1)\right).$$

Therefore, a morphism $f : \mathbb{P}^1 \to X$ is r-free if and only if $H^1(\mathbb{P}^1, f^*T_X(-r-1)) = 0$. Also, by the Semi-continuity Theorem, the set of r-free morphisms on X is an open subset of $\mathrm{Hom}(\mathbb{P}^1, X)$, possibly empty, which is smooth (see Section 2.2).

Example 2.4.1. Let $f : \mathbb{P}^1 \to X$ be a constant morphism with image $x \in X$. Then

$$f^*T_X = f^{-1}T_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{\mathbb{P}^1} \cong T_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\mathbb{P}^1} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}.$$

Therefore every constant morphism $f : \mathbb{P}^1 \to X$ is free.

Because of the previous example, from now on, we will consider only non-constant morphisms $f : \mathbb{P}^1 \to X$, that is, rational curves on X. In this case, the differential

$$df : T_{\mathbb{P}^1} \to f^*T_X$$

is zero at only finitely many points of $\mathbb{P}^1$. This means that df is not identically zero at a general point. Since $T_{\mathbb{P}^1}$ is a line bundle, its stalk at any point is just a free module of rank 1 over the local ring. Thus, non-zero implies injective. Therefore, f^*T_X contains $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$.

Example 2.4.2. Let $f : \mathbb{P}^1 \to X$ be a very free rational curve on X. Then

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n), \quad \text{with } a_n \geq 1.$$

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a ramified finite morphism of degree $r > 0$. Then, the composition $g = f \circ \phi$ is such that

$$g^*T_X = \phi^* f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(ra_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(ra_n), \quad \text{with } ra_n \geq r.$$

Therefore, if a smooth variety contains a very free rational curve, then it contains an r-free rational curve, for every $r > 0$.

Example 2.4.3. Let $f : \mathbb{P}^1 \to X$ be a free rational curve on a smooth variety X with canonical divisor K_X. Then

$$-K_X \cdot f_*\mathbb{P}^1 = \deg(f^*T_X) = a_1 + \cdots + a_n \geq 2.$$

Therefore, there are no free rational curves on a smooth variety whose canonical divisor is nef.

Example 2.4.4. Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a rational curve of degree d on the projective space $\mathbb{P}^n$. Consider the dual of the Euler's exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0.$$

By taking the pullback under f we obtain

$$\bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(d) \to f^*T_{\mathbb{P}^n} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \to 0.$$

Therefore, every rational curve $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree d is d-free.

As we can see from the remarks and examples above, r-free rational curves enjoy many nice properties. The next result says that, the freer a rational curve is, the more it can be moved while keeping points fixed.

Proposition 2.4.1 ([Deb01, Prop. 4.8]). Let X be a smooth projective variety. Let $f : \mathbb{P}^1 \to X$ be an r-free rational curve on X. Let B be a finite subscheme of $\mathbb{P}^1$ of length b, and let $s > 0$ be an integer such that $s + b \leq r + 1$. Then the evaluation morphism

$$\begin{aligned} \mathrm{ev} : (\mathbb{P}^1)^s \times \mathrm{Hom}(\mathbb{P}^1, X; f|_B) &\longrightarrow X^s \\ (t_1, \ldots, t_s, [g]) &\longmapsto (g(t_1), \ldots, g(t_s)) \end{aligned}$$

is smooth at $(t_1, \ldots, t_s, f)$ if $\{t_1, \ldots, t_s\} \cap B = \emptyset$.

Proof. The differential of ev at $(t_1, \ldots, t_s, f)$ is given by

$$\begin{aligned} \textstyle\bigoplus_{i=1}^s T_{\mathbb{P}^1, t_i} \oplus H^0(\mathbb{P}^1, f^*T_X(-B)) &\longrightarrow \textstyle\bigoplus_{i=1}^s T_{X, f(t_i)} \\ (u_1, \ldots, u_s, \sigma) &\longmapsto (d_{t_1} f(u_1) + \sigma(t_1), \ldots, d_{t_s} f(u_s) + \sigma(t_s)) \end{aligned}$$

(see [Kol96, Prop. 3.4]). This map is surjective if the evaluation map

$$\begin{aligned} H^0(\mathbb{P}^1, f^*T_X(-B)) \longrightarrow H^0(\mathbb{P}^1, f^*T_X) &\longrightarrow \textstyle\bigoplus_{i=1}^s T_{X, f(t_i)} \\ \sigma &\longmapsto (\sigma(t_1), \ldots, \sigma(t_s)) \end{aligned}$$

is surjective. With the notation in (2.1), this is in turn the case if the map

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j - b)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_j)) \longrightarrow \textstyle\bigoplus_{i=1}^s k_{t_i}$$

is surjective, for every $j = 1, \ldots, n$. This holds, because $\{t_1, \ldots, t_s\} \cap B = \emptyset$ and $a_j - b \geq s - 1$. On the other hand, since $a_j - b \geq -1$, we have $H^1(\mathbb{P}^1, f^*T_X(-B)) = 0$; hence $\mathrm{Hom}(\mathbb{P}^1, X; f|_B)$ is smooth at $[f]$. Therefore, ev is smooth at $(t_1, \ldots, t_s, f)$. □

There exists a partial converse of this result. The proof is very similar to the previous one, and for that reason, we only state it.

Proposition 2.4.2 ([Deb01, Prop. 4.9]). Let X be a smooth projective variety over a field of characteristic zero. Let $f : \mathbb{P}^1 \to X$ be a rational curve on X. Let B be a finite subscheme of $\mathbb{P}^1$ of length b, and let $s > 0$ be an integer. If the differential of the evaluation morphism

$$\begin{aligned} \mathrm{ev} : (\mathbb{P}^1)^s \times \mathrm{Hom}(\mathbb{P}^1, X; f|_B) &\longrightarrow X^s \\ (t_1, \ldots, t_s, [g]) &\longmapsto (g(t_1), \ldots, g(t_s)) \end{aligned}$$

is surjective at some point of $\mathbb{P}^1 \times \{f\}$, then f is $\min\{2, b + s - 1\}$-free.

The next result says that a rational curve through a very general point of a smooth variety is free.

Proposition 2.4.3 ([Deb01, Prop. 4.20]). Let X be a smooth quasi-projective variety over a field of characteristic zero. Let B be a subscheme of X of length $b \leq 2$ (we allow B to be empty). Then there exists a subset X_B^{free} of X, which is the intersection of countably many dense open subsets of X, such that every rational curve on X containing B and whose image meets X_B^{free} is b-free. If we restrict to rational curves on X with bounded degree, then X_B^{free} is the intersection of finitely many dense open subsets of X.

The following lemma will be useful in the future.

Lemma 2.4.3. Let X be a smooth projective variety over a field of characteristic zero, and let $d > 0$ be an integer. Assume that through s general points $x_1, \dots, x_s \in X$ there passes a rational curve C on X of degree d. Then $h^1(C, N_{C/X}(-\sum_{i=1}^{s} x_i)) = 0$.

Proof. Let $f : \mathbb{P}^1 \xrightarrow{\sim} C \subset X$ be a morphism representing a degree d rational curve C on X passing through s general points $x_1, \dots, x_s \in X$. By the remark after Example 2.4.1, we can write $f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, and $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$. To prove that $h^1(C, N_{C/X}(-\sum_{i=1}^{s} x_i)) = 0$ we have to prove that $a_i \geq s-1$, for every $i = 2, \dots, s$. Considere the evaluation morphism

$$\begin{aligned} \mathrm{ev}_d : (\mathbb{P}^1)^s \times \mathrm{Hom}_d(\mathbb{P}^1, X) &\longrightarrow X^s \\ (t_1, \dots, t_s, [g]) &\longmapsto (g(t_1), \dots, g(t_s)). \end{aligned}$$

By hypothesis this morphism is dominant, and by Proposition 2.4.3 and the results in Section 2.2 the scheme $\mathrm{Hom}_d(\mathbb{P}^1, X)$ is generically smooth. By Generic Smoothness, the differential of ev_d at $(t_1, \dots, t_s, [f])$

$$\begin{aligned} \bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \oplus H^0\left(\mathbb{P}^1, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)\right) &\longrightarrow \bigoplus_{i=1}^{s} T_{X, f(t_i)} \\ (u_1, \dots, u_s, \sigma) &\longmapsto (d_{t_1} f(u_1) + \sigma(t_1), \dots, d_{t_s} f(u_s) + \sigma(t_s)) \end{aligned}$$

is surjective. This implies the evaluation map

$$\bigoplus_{i=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) \longrightarrow \bigoplus_{i=1}^{s} \left(T_{X, f(t_i)} / \mathrm{im}(d_{t_i} f)\right) \cong \{0\} \oplus k^s \oplus \cdots \oplus k^s$$

is surjective as well. Therefore, $a_i \geq s-1$, for every $i = 2, \dots, s$, as we wanted. □

2.4.4 Smoothing of Morphisms of Curves

In Subsection 2.4.2 we saw ways to deform smooth curves into reducible ones. In this subsection, we are interested in the reverse process, that is, we are interested in smoothing a union of rational curves. For this subject we refer to [Kol96, Section II.7], [Deb01, Section 4.6] and [AK03].

Definition 2.4.2. A *(rational) smoothing of a curve* C is a flat morphism $q : \mathcal{C} \to (T, o)$, where (T, o) is a connected, smooth, pointed curve such that

(i) $q^{-1}(o) \cong C$;

(ii) $q^{-1}(t) \cong \mathbb{P}^1$, for every $t \in T \setminus \{o\}$.

More generally, a *(rational) smoothing of a morphism* $f : C \to X$ is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & X \\ {\scriptstyle q}\downarrow & & \\ (T, o) & & \end{array},$$

where $q : \mathcal{C} \to (T, o)$ is a rational smoothing of C and $F : C \to X$ is a morphism such that $F|_{q^{-1}(o)} = f$ under the isomorphism $q^{-1}(o) \cong C$. For $t \in T \setminus \{o\}$, the restriction $F|_{q^{-1}(t)} : q^{-1}(t) \to X$ is called a *nearby smoothing* of f. Let $p_1, \dots, p_r$ be distinct smooth points of C. A *smoothing keeping* $f(p_1), \dots, f(p_r)$ *fixed* is a smoothing of f with r sections $s_1, \dots, s_r : T \to \mathcal{C}$ such that

(i) $s_i(o) = p_i$, for every $i = 1, \ldots, r$;

(ii) $F \circ s_i(T) = f(p_i)$, for every $i = 1, \ldots, r$.

Definition 2.4.3. A *(rational) tree* is a projective, connected, reduced, at worst nodal curve C satisfying the following equivalent properties:

(i) $\chi(C, \mathcal{O}_C) = 1$;

(ii) the irreducible components of C are all isomorphic to $\mathbb{P}^1$, and there exist $\#\operatorname{Sing}(C) + 1$ of them;

(iii) the irreducible components of C are all isomorphic to $\mathbb{P}^1$, and they can be listed as $C_1, \ldots, C_m$ in such a way that, for every $i = 1, \ldots, m-1$, the component C_{i+1} intersects $C_1 \cup \cdots \cup C_i$ transversely in a single smooth point.

The irreducible components of a rational tree are called *twigs*.

Given a rational tree C, there always exists a rational smoothing of C. The proof is by induction on the number m of irreducible components of C. If C is irreducible, that is, $C \cong \mathbb{P}^1$, then the projection (to anyone of the factors)

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow (\mathbb{P}^1, o)$$

is a rational smoothing of C. Assume that the claim holds for every rational tree C' with $m - 1 \geq 1$ irreducible components. Let C be a rational tree with $m \geq 2$ irreducible components $C_1, \ldots, C_m$. The subcurve $C' = C_1 \cup \cdots \cup C_{m-1}$ is a rational tree, and by the induction hypothesis, there exists a rational smoothing of C'

$$\mathcal{C}' \longrightarrow (T, o).$$

Let $\{p\} = C_{m-1} \cap C_m$ and let $\pi : \mathcal{C} \to \mathcal{C}'$ be the blow-up of $\mathcal{C}'$ at the point p. Then

$$\mathcal{C} \xrightarrow{\pi} \mathcal{C}' \longrightarrow (T, o)$$

is a rational smoothing of C.

The proposition below says that, under certain conditions, a morphism from a rational tree can be smoothable keeping points fixed.

Proposition 2.4.4 ([Deb01, Prop. 4.24])**.** Let X be a smooth projective variety, and let $f : C \to X$ be a morphism from a rational tree C with irreducible components $C_1, \ldots, C_m$. Let $p_1, \ldots, p_r$ be distinct smooth points of C, with r_i of them on C_i. Assume that the restriction $f|_{C_1}$ is $(r_1 - 1)$-free and, for each $i = 2, \ldots, m$, the restriction $f|_{C_i}$ is r_i-free. Then f is smoothable into an $(r-1)$-free rational curve keeping $f(p_1), \ldots, f(p_r)$ fixed.

2.5 Kontsevich Moduli Space of Stable Maps

In this section we introduce the Kontsevich moduli space, another parameter space for rational curves on a projective variety. The advantages of adopting this space is that it is compact, much more manageable and many recent works have been developed concerning its Picard group, effective cone and virtual canonical bundle (at least for the projective space). The downside is that, because of its compactness, we are forced to consider many objects in the boundary which we never wanted to consider in the first place. For this topic we refer to the already classical notes [FP97], and [KV07].

2.5.1 Stable Maps and the Kontsevich Functor

We start with the basic definitions.

Definition 2.5.1. An n-pointed, genus g, *quasi-stable* curve $(C, p_1, \ldots, p_n)$ is a projective, connected, reduced, at worst nodal curve C of arithmetic genus g with n distinct, non-singular, marked points $p_1, \ldots, p_n$. Let S be an algebraic scheme over $\mathbb{C}$. A *family* of n-pointed, genus g, quasi-stable curves over S is a flat projective map $\pi : \mathcal{C} \to S$ with n sections $s_1, \ldots, s_n$ such that each geometric fiber $(\mathcal{C}_s, s_1(s), \ldots, s_n(s))$ is an n-pointed, genus g, quasi-stable curve. Let X be an algebraic scheme over $\mathbb{C}$. A *family of maps* over S from n-pointed genus g curves to X consists of the data $(\pi : \mathcal{C} \to S, \{s_i\}_{1 \le i \le n}, \mu : \mathcal{C} \to X)$, where

(i) $\pi : \mathcal{C} \to S$ is a family of n-pointed, genus g, quasi-stable curves with n sections $s_1, \ldots, s_n$;

(ii) $\mu : \mathcal{C} \to X$ is a morphism.

Two families of maps over S, $(\pi : \mathcal{C} \to S, \{s_i\}, \mu)$ and $(\pi' : \mathcal{C}' \to S, \{s_i'\}, \mu')$, are *isomorphic* if there exists a scheme isomorphism $\tau : \mathcal{C} \to \mathcal{C}'$ satisfying $\pi = \pi' \circ \tau$, $s_i' = \tau \circ s_i$ and $\mu = \mu' \circ \tau$. When $\pi : C \to \mathrm{Spec}(\mathbb{C})$ is the structure map, $(\pi : C \to \mathrm{Spec}(\mathbb{C}), \{s_i\}, \mu)$ will be simply denoted by $(C, \{p_i\}, \mu)$, where $p_1, \ldots, p_n$ are the distinct, non-singular, marked points on C. Let $(C, \{p_i\}, \mu)$ be a map from an n-pointed quasi-stable curve to X. The *special points* of an irreducible component $E \subset C$ are the marked points on E and the component intersections of C that lie on E. The map $(C, \{p_i\}, \mu)$ is *stable* if the following conditions hold for every component $E \subset C$:

(i) If $E \cong \mathbb{P}^1$ and E is contracted by μ, then E contains at least three special points;

(ii) If E has arithmetic genus 1 and E is contracted by μ, then E contains at least one special point.

A family of pointed maps $(\pi : \mathcal{C} \to S, \{s_i\}, \mu)$ is *stable* if the pointed map on each geometric fiber of π is stable. Let β be a curve class on X. We will say that a map $\mu : C \to X$ *represents* β if the μ-push-forward of the fundamental class $[C]$ is equal to β. The *Kontsevich functor*

$$\overline{\mathcal{M}}_{g,n}(X, \beta) : (\text{alg. schemes over } \mathbb{C}) \longrightarrow (\text{sets})$$

from the category of complex algebraic schemes to the category of sets is defined by

$$\overline{\mathcal{M}}_{g,n}(X, \beta)(S) = \left\{ \begin{array}{l} \text{isomorphism classes of stable families over } S \\ \text{of maps from } n\text{-pointed genus } g \text{ curves to } X \\ \text{representing the class } \beta \end{array} \right\}.$$

The first result concerning the representability of the Kontsevich functor is the following:

Theorem 2.5.1 ([FP97, Thm. 1])**.** Let X be a complex, projective, algebraic scheme. Then $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is coarsely represented for any non-negative integers g, n and curve class β. More precisely, there exists a complex projective scheme $\overline{M}_{g,n}(X, \beta)$ with a natural transformation of functors

$$\phi : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \mathcal{H}om(\,-\,, \overline{M}_{g,n}(X, \beta))$$

satisfying the following properties:

(i) $\phi(\text{Spec}(\mathbb{C})) : \overline{\mathcal{M}}_{g,n}(X,\beta)(\text{Spec}(\mathbb{C})) \to \mathcal{H}om(\text{Spec}(\mathbb{C}), \overline{M}_{g,n}(X,\beta))$ is a set bijection;

(ii) If Z is a scheme and $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{H}om(\,-\,, Z)$ is a natural transformation of functors, then there exists a unique morphism of schemes $\gamma : \overline{M}_{g,n}(X,\beta) \to Z$ such that $\psi = \widetilde{\gamma} \circ \phi$, where $\widetilde{\gamma}$ is the natural transformation induced by γ.

Example 2.5.1. The moduli space of stable maps to a point coincides with the moduli space of curves, that is, $\overline{M}_{g,n}(\mathbb{P}^0, 0) \cong \overline{M}_{g,n}$.

Example 2.5.2. The moduli space of degree zero stable maps to a complex algebraic scheme X is described by the isomorphism $\overline{M}_{g,n}(X,0) \cong \overline{M}_{g,n} \times X$, because a degree zero map from a connected curve is determined by specifying a point of X.

Example 2.5.3. The moduli space of degree one maps to $\mathbb{P}^N$ is isomorphic to the Grassmannian of lines in $\mathbb{P}^N$, that is, $\overline{M}_{0,0}(\mathbb{P}^N, 1) \cong \mathbb{G}(1,N)$.

Example 2.5.4. The moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics, or equivalently, it is isomorphic to the blow up of the Hilbert scheme of conics in $\mathbb{P}^2$ along the Veronese surface of double lines.

Let $(C, \{p_i\}, \mu)$ be a map from an n-pointed quasi-stable curve to X. An automorphism of the map is an automorphism τ of C satisfying $\tau(p_i) = p_i$ and $\mu = \mu \circ \tau$. We will denote by $\overline{M}^*_{g,n}(X,\beta)$ the open locus of $\overline{M}_{g,n}(X,\beta)$ parameterizing stable maps with no non-trivial automorphisms.

We recall that a smooth variety X is *convex* if $H^1(\mathbb{P}^1, f^*T_X) = 0$, for every map $f : \mathbb{P}^1 \to X$. The projective space $\mathbb{P}^N$ is a convex variety.

Theorem 2.5.2 ([FP97, Thm. 2])**.** Let X be a complex, smooth, projective variety. Then

(i) $\overline{M}_{0,n}(X,\beta)$ is smooth at the points parameterizing automorphism-free stable maps whose restriction to each irreducible component is a free map; its dimension at these points is
$$\dim(X) - K_X \cdot \beta + n - 3;$$

(ii) if X is convex, then $\overline{M}^*_{0,n}(X,\beta)$ is a smooth fine moduli space for automorphism-free stable maps, and it is equipped with a universal family.

Proof. When X is a convex variety, these results are explicitly stated and proved in [FP97, Theorem 2]. To see that (i) holds, we simply note that Lemma 10 and Lemma 11 in [FP97] are immediately satisfied by a stable map with the assumed properties; hence, all the arguments used to compute the dimension in the convex case holds for X. □

We will denote by U^{free} the open locus of $\overline{M}_{0,n}(X,\beta)$ parameterizing stable maps whose restriction to each irreducible component is a free map.

2.5.2 The Boundary

Here we summarize the main results concerning the boundary of the Kontsevich moduli space $\overline{M}_{0,n}(X,\beta)$.

Definition 2.5.2. Let X be a projective algebraic scheme over $\mathbb{C}$. The *boundary* of the moduli space $\overline{M}_{g,n}(X,\beta)$ is the locus corresponding to stable maps whose domain curves are reducible.

Definition 2.5.3. Let X be a projective, algebraic scheme over $\mathbb{C}$. A curve class β on X is *effective* if it is represented by some stable map to X.

If $n = 0$, then the boundary of $\overline{M}_{0,0}(X, \beta)$ decomposes into a union of irreducible components which are in bijective correspondence with partitions $\beta_1 + \beta_2 = \beta$, with β_1 and β_2 effective curve classes. For general n, the boundary of $\overline{M}_{0,n}(X, \beta)$ decomposes into a union of irreducible components in bijective correspondence with data of weighted partitions $(A, B; \beta_1, \beta_2)$, where

(i) $A \cup B$ is a partition of $[n] = \{1, 2, \dots, n\}$;

(ii) $\beta_1 + \beta_2 = \beta$, with β_1 and β_2 effective curve classes;

(iii) If $\beta_1 = 0$ (resp. $\beta_2 = 0$), then $\#A \geq 2$ (resp. $\#B \geq 2$).

The divisor $D(A, B; \beta_1, \beta_2)$ corresponding to the data of weighted partitions $(A, B; \beta_1, \beta_2)$ is the locus of stable maps $f : C_A \cup C_B \to X$ satisfying the following conditions:

(i) $C = C_A \cup C_B$, with C_A and C_B genus 0, quasi-stable curves meeting in a single point;

(ii) C_A (resp. C_B) has $\#A$ (resp. $\#B$) marked points;

(iii) The map $\mu_A = \mu|_A$ (resp. $\mu_B = \mu|_B$) represents β_1 (resp. β_2).

Theorem 2.5.3 ([FP97, Thm. 3])**.** Let X be a complex, smooth, projective, convex variety. Then the boundary of $\overline{M}_{0,n}(X, \beta)$ is a normal crossing divisor up to a finite group quotient.

2.5.3 Evaluation Morphisms

Let X be a complex, projective, algebraic scheme, and consider the Kontsevich moduli space $\overline{M}_{g,n}(X, \beta)$ of stable maps with n marked points. For each $i = 1, \dots, n$, we have the evaluation morphism

$$\begin{aligned} \mathrm{ev}_i : \qquad \overline{M}_{g,n}(X, \beta) &\longrightarrow X \\ [C, p_1, \dots, p_n, f] &\longmapsto f(p_i), \end{aligned}$$

which evaluates each stable map at its i-th marked point. Taking the product of all of these evaluation morphisms we obtain the total evaluation morphism

$$\begin{aligned} \mathrm{ev} : \qquad \overline{M}_{g,n}(X, \beta) &\longrightarrow X \times \cdots \times X \\ [C, p_1, \dots, p_n, f] &\longmapsto (f(p_1), \dots, f(p_n)). \end{aligned}$$

Although simple, these evaluation morphisms allow us to relate the geometry of X to the geometry of $\overline{M}_{g,n}(X, \beta)$.

Lemma 2.5.1 ([dS06b, Lemma 5.1])**.** Let X be a complex, smooth, projective variety, and let β be a curve class on X. Assume that every point in a general fiber of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,n}(X, \beta) \to X^n$$

parameterizes an automorphism-free stable map whose irreducible components are all free. Then a non-empty general fiber of ev is smooth of expected dimension

$$-K_X \cdot \beta - (n-1)\dim(X) + n - 3$$

and the intersection with the boundary Δ is a simple normal crossing divisor.

Proof. The open locus U^{free} parameterizing stable maps whose domains are unions of free rational curves is smooth. Consider the restriction $\text{ev}|_{U^{\text{free}}} : U^{\text{free}} \to X^n$. By Generic Smoothness applied to this restriction, the intersection of U^{free} with a general fiber of ev is smooth. But this intersection is the general fiber itself if it is contained in U^{free}. The second part of the lemma follows from Theorem 2.5.3 and 2.4.4 (see [dS06b, Lemma 5.1]). □

2.5.4 Forgetful Morphisms

Let X be a complex, projective, algebraic scheme, and consider the Kontsevich moduli space $\overline{M}_{g,n+k}(X,\beta)$ of stable maps with $n+k$ marked points. For each subset $\{i_1,\dots,i_k\} \subset \{1,\dots,n+k\}$, we have the forgetful morphism

$$\varphi_{i_1,\dots,i_k} : \overline{M}_{g,n+k}(X,\beta) \longrightarrow \overline{M}_{g,n}(X,\beta),$$

which forgets the marked points $p_{i_1},\dots,p_{i_k}$ of each stable map $[C,p_1,\dots,p_{n+k},f] \in \overline{M}_{g,n+k}(X,\beta)$. To have this a well defined morphism, curves that become unstable by the absence of the suppressed marked points must be contracted (note that this happens only for twigs of degree 0, and hence, the new stable map obtained also represents the curve class β).

Example 2.5.5. Consider the forgetful morphism

$$\begin{aligned} \varphi_{n+1} : \qquad \overline{M}_{g,n+1}(X,\beta) &\longrightarrow \overline{M}_{g,n}(X,\beta) \\ [C,p_1,\dots,p_n,p_{n+1}] &\longmapsto [C,p_1,\dots,p_n,f], \end{aligned}$$

which forgets the $(n+1)$-th marked point. Figure 2.1 below shows the images of two morphisms with source curves that become unstable by the absence of the suppressed marked point p_{n+1}.

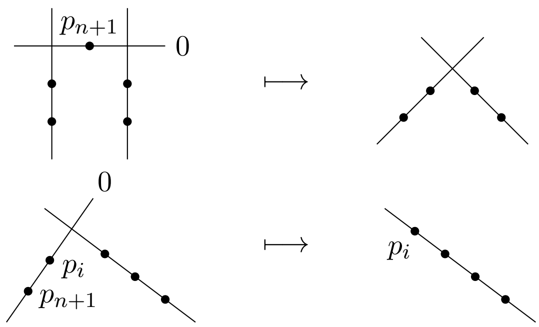

Figure 2.1: Forgetting the marked point p_{n+1}.

We can have the inverse situation, when we want to mark more points on a source curve of a stable map that becomes unstable with this new marked point. In this situation, we have to add degree 0 twigs to the curve. We call this process of *stabilization of a stable map*, with a point marked twice or with a marked node. In Figure 2.1, the curves at the left-hand side are the stabilizations of the curves at the right-hand side, respectively with a marked node and with the point p_i marked twice.

2.5.5 The Universal Family over $\overline{M}^*_{0,n}(X,\beta)$.

Let X be a complex, smooth, projective, convex variety. As stated in Theorem 2.5.2, the open set $\overline{M}^*_{0,n}(X,\beta)$ of automorphism-free stable maps is a fine moduli space for the functor $\overline{\mathcal{M}}^*_{0,n}(X,\beta)$. Let us describe its universal family. Consider the forgetful morphism

$$\begin{aligned}\varphi_{n+1}: \qquad \overline{M}_{0,n+1}(X,\beta) &\longrightarrow \overline{M}_{0,n}(X,\beta)\\ [C,p_1,\dots,p_n,p_{n+1},f] &\longmapsto [C,p_1,\dots,p_n,f],\end{aligned}$$

which forgets the $(n+1)$-th marked point. Let $[C,p_1,\dots,p_n,f]\in\overline{M}^*_{0,n}(X,\beta)$, and denote by $F_{[C,f]}$ the fiber of φ_{n+1} over this stable map. It is clear that $F_{[C,f]}\subset\overline{M}^*_{0,n+1}(X,\beta)$. Note that we have a natural bijection between C and the fiber $F_{[C,f]}$. Indeed, for each non-marked smooth point $p_{n+1}\in C$ associate the stable map $[C,p_1,\dots,p_n,p_{n+1},f]$; for each marked point $p_i\in C$, $1\le i\le n$, associate the stabilization of f with the point p_i marked twice; and for each node $q\in C$ associate the stabilization of f with the node q marked. As a matter of fact, this correspondence is an isomorphism. In order to have a morphism from C to $\overline{M}_{0,n+1}(X,\beta)$, by the coarse representability of $\overline{M}_{0,n}(X,\beta)$, it is sufficient to provide a family over C of $(n+1)$-pointed stable maps representing the class β. Before providing such a family, consider the evaluation morphism

$$\begin{aligned}\mathrm{ev}_{n+1}:\overline{M}_{0,n+1}(X,\beta) &\longrightarrow X\\ [C,p_1,\dots,p_{n+1},f] &\longmapsto f(p_{n+1}),\end{aligned}$$

which evaluates each stable map at its $(n+1)$-th marked point. If we identify the fiber $F_{[C,f]}$ with C, then the restriction of ev_{n+1} to C coincides with the map f itself. Hence, φ_{n+1} and ev_{n+1} restricted to the open set of automorphism-free stable maps are the candidates for the universal family. Going back to the question of the isomorphism between $F_{[C,f]}$ and C, note that we have a family over C of maps from n-pointed genus 0 curves to X representing the class β

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{\ \mu\ } & X\\ \pi_1\big\downarrow\ \big\uparrow s_i,\ i=1,\dots,n & & \\ C & & \end{array},$$

where $\mathcal{C}=C\times C$, the morphism π_1 is the first projection, s_i is the section given by $s_i(p)=(p,p_i)$ and $\mu(p,q)=f(q)$. Now, a result due to F. F. Knudsen, [Knu83, Theorem 2.4], assures that this family with the diagonal section $\Delta(p)=(p,p)$ gives rise to a family over C of $(n+1)$-pointed stable curves. The idea is the following: consider the exact sequence

$$0\to\mathcal{O}_{\mathcal{C}}\xrightarrow{\delta}\mathcal{I}^\vee\oplus\mathcal{O}_{\mathcal{C}}(s_1+\cdots+s_n)\xrightarrow{p}\mathcal{K}\to 0, \tag{2.2}$$

where $\mathcal{I}$ is the defining ideal of the image of the section Δ and δ is the diagonal morphism, given by $\delta(t)=(t,t)$. Define $\mathcal{C}^s=\mathbf{Proj}_{\mathcal{C}}(\mathrm{Sym}\,\mathcal{K})$. The union of the images of the sections $s_1,\dots,s_n$ and Δ support sheaves $\mathcal{L}_s=\mathcal{K}/p(\mathcal{I}^\vee)$ and $\mathcal{L}_\Delta=\mathcal{K}/p(\mathcal{O}_{\mathcal{C}}(s_1+\cdots+s_n))$, and there exist surjections $s_i^*\mathcal{K}\to s_i^*\mathcal{L}_s$ and $\Delta^*\mathcal{K}\to\Delta^*\mathcal{L}_\Delta$. The important fact is that (2.2) commutes with base-change and that the sheaves $s_i^*\mathcal{L}_s$ and $\Delta^*\mathcal{L}_\Delta$ are invertible. Hence these surjections define liftings of the sections making $\mathcal{C}^s$ together with the lifted sections a family of $(n+1)$-pointed stable curves. Composing the projection morphism $\pi:\mathcal{C}^s\to\mathcal{C}$ with π_1 and with μ, we obtain the desired family of $(n+1)$-pointed stable maps. The corresponding morphism from C to $\overline{M}_{0,n+1}(X,\beta)$ is an isomorphism onto its image $F_{[C,f]}$.

Therefore, the considerations above show that $(\varphi_{n+1},\{s_i\}_{1\le i\le n},\mathrm{ev}_{n+1})$, restricted to the open set of automorphism-free stable maps, play the role of the universal family over $\overline{M}^*_{0,n}(X,\beta)$.

2.6 Fano Varieties

In this subsection we study smooth projective varieties with ample anti-canonical sheaf. Varieties with this property were first studied by G. Fano, and for that reason are called Fano varieties. Although they are varieties satisfying a very restrictive condition, Fano varieties appear naturally as important examples of varieties. Also, Fano varieties plays an important role in the birational classification of algebraic varieties. We recall that a minimal model in the sense of Mori is a normal projective $\mathbb{Q}$-Gorenstein variety with a numerically effective canonical divisor. According to the Mori Minimal Model Program, which is completely carried out in dimension up to 3 and partially in higher dimensions, every irreducible algebraic variety X over an algebraically closed field of characteristic zero is birationally equivalent either to a minimal model (if X has Kodaira dimension $\kappa(X) \geq 0$) or to a fibration over a variety of smaller dimension with rational singularities whose general fiber is a Fano variety (if X is covered by rational curves). For a treatment on the Minimal Model Program, see [Mat02] and [KM98]. References for Fano varieties are [IP99], [Kol96] and [Deb01].

2.6.1 Definition and Examples

We begin with the formal definition of smooth Fano varieties:

Definition 2.6.1. We say that a smooth projective variety X is a *Fano* variety if its anti-canonical sheaf $\omega_X^\vee$ is ample, or equivalently, if its anti-canonical class $-K_X$ has positive intersection with every non-zero 1-cycle in the Mori cone $\overline{NE}(X)$.

Kollár, Miyaoka and Mori proved in [KMM92a] and [KMM92b] that, fixed the dimension n, there exist only finitely many smooth Fano varieties of dimension n up to deformation. In this sense, smooth Fano varieties are quite rare. Despite this, it is not difficult to find examples of Fano varieties; in fact, many of them are quite elementary.

Example 2.6.1. The anti-canonical class of a projective space $\mathbb{P}^n$ is given by $-K_{\mathbb{P}^n} = (n+1)\cdot H$, where H denotes the hyperplane class of $\mathbb{P}^n$. Therefore, projective spaces are Fano varieties.

Example 2.6.2. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of k hypersurfaces of degrees $d_1, \ldots, d_k$. By the Adjunction Formula, the anti-canonical class of X is given by $-K_X = \left(n+1-\sum_{i=1}^k d_i\right)\cdot H|_X$, where H denotes the hyperplane class of $\mathbb{P}^n$. Therefore, X is Fano if and only if $\sum_{i=1}^k d_i \leq n$.

Example 2.6.3. Let $\mathbb{G} = \mathbb{G}(k,N)$ be the Grassmannian of k-planes in a projective space $\mathbb{P}^N$ embedded in $\mathbb{P}^{\binom{N+1}{k+1}}$ under the Plücker embedding. Then $\mathbb{G}$ has Picard group $\mathrm{Pic}(\mathbb{G}) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{G}}(1)]$ and its anti-canonical class is given by $-K_{\mathbb{G}} = (N+1)\cdot H|_{\mathbb{G}}$, where H denotes the hyperplane class of $\mathbb{P}^{\binom{N+1}{k+1}}$. Therefore, Grassmannians are Fano varieties.

Fano varieties have rich geometry. An example of that is the next theorem, which says that a smooth Fano variety X over an algebraically closed field is covered by rational curves with bounded $(-K_X)$-degree.

Theorem 2.6.1 ([Mor79, Thm. 6])**.** Let X be an n-dimensional smooth Fano variety over an algebraically closed field. Then through any point of X there exists a rational curve on X with $(-K_X)$-degree at most $n+1$.

Varieties X with the property that through any given point of X passes a rational curve on X are called uniruled varieties. The formal definition is below.

Definition 2.6.2. Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero. We say that X is *uniruled* if for any point $x \in X$ there exists a rational curve on C passing through x.

Like Fano manifolds, uniruled varieties have a rich geometry and they are very much studied. The following proposition is an example.

Proposition 2.6.1 ([Deb01, Cor. 4.11])**.** Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero. Then X is uniruled if and only if there exists a free rational curve on X passing through a general point of X.

Another theorem concerning the geometry of smooth Fano varieties is the following.

Theorem 2.6.2 ([KMM92a, Thm. 0.1])**.** Let X be a smooth Fano variety over an uncountable algebraically closed field of characteristic zero. Then given two points $x, y \in X$ there exists a rational curve on X connecting x and y.

Smooth varieties satisfying the property in the above theorem are called rationally connected varieties. The formal definition below is in fact the summary of many theorems.

Definition 2.6.3. Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero. We say that X is *rationally connected* if it satisfies the following equivalent conditions:

(i) for every $x_1, x_2 \in X$ there exists a chain of rational curve on X connecting x_1 and x_2;

(ii) for every $x_1, x_2 \in X$ there exists a rational curve on X passing though x_1 and x_2;

(iii) for any general points $x_1, x_2 \in X$ there exists a rational curve on X passing though x_1 and x_2;

(iv) for every $x_1, x_2 \in X$ there exists a free rational curve on X passing through x_1 and x_2;

(v) there exists a very free rational curve on X.

The conditions (i)-(iv) can be restated replacing the pair of points $x_1, x_2 \in X$ by any finitely many points $x_1, \dots, x_m \in X$.

Like Fano manifolds, rationally connected varieties have a rich geometry and they are very much studied. We gather in the proposition below some properties that a rationally connected variety satisfies.

Proposition 2.6.2 ([Deb01, Cor. 4.18])**.** Let X be a rationally connected variety over an uncountable algebraically closed field k of characteristic zero. Then X satisfies the following properties:

(i) $H^0\left(X, \left(\Omega_X^p\right)^{\otimes m}\right) = 0$, for every integers $m, p > 0$. In particular, $\chi(X, \mathcal{O}_X) = 1$;

(ii) there exists a very free rational curve passing through any finite subset of X;

(iii) if $k = \mathbb{C}$, then X is simply connected.

2.6.2 Classification of Fano Manifolds

Let X be a smooth projective curve of genus g. The anti-canonical class $-K_X$ of X has degree $2-2g$. For $-K_X$ to be ample, this number must be positive, that is, X must have genus $g=0$. Hence $X \cong \mathbb{P}^1$, because X is smooth. Therefore, the projective line $\mathbb{P}^1$ is the only Fano curve.

The situation becomes more complicated for surfaces. The classification of Fano surfaces, also known as *del Pezzo* surfaces, is a classical result. They are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-up of $\mathbb{P}^2$ at n points in general position, with $1 \leq n \leq 8$.

The classification of Fano threefolds with Picard number equals 1 was established by V. A. Iskovskih in [Isk77] and [Isk78]. Iskovskih proved that there exist 17 deformation types of these. The classification of Fano threefolds with higher Picard number was established by S. Mori and S. Mukai in [MM82] and [MM03]. They proved that there exist 88 deformation types of these.

In higher dimensions, there do not exist complete classification. However, if we fix some invariants of the Fano manifold we can obtain partial results. Two important invariants are the index and the degree, which we define as follows.

Definition 2.6.4. Let X be an n-dimensional Fano manifold. The *index* i_X of X is defined by

$$i_X = \max\{m \in \mathbb{N} \mid -K_X = mH, \text{ for some Cartier divisor } H\}.$$

Assume that X has Picard number equals 1. Let L be the ample generator of $\mathrm{Pic}(X)$. The *degree* d_X of X is defined as $d_X = \mathrm{c}_1(L)^n$.

The following result, due to S. Kobayashi and T. Ochiai, gives an upper bound for the index of a Fano manifold and characterizes those whose index are the highest ones.

Theorem 2.6.3 ([KO73]). Let X be an n-dimensional Fano manifold over a field of characteristic zero. Then

(i) $i_X \leq n+1$;

(ii) $i_X = n+1$ if and only if $X \cong \mathbb{P}^n$;

(iii) $i_X = n$ if and only if $X \cong Q \subset \mathbb{P}^{n+1}$, a quadric hypersurface.

The following result is due to J. A. Wiśniewski.

Theorem 2.6.4 ([Wiś91]). Let X be an n-dimensional Fano manifold with index $i_X \geq \frac{n+1}{2}$. Then X satisfies one of the following conditions:

(i) X has Picard number $\rho(X) = 1$;

(ii) $X \cong \mathbb{P}^{\frac{n}{2}} \times \mathbb{P}^{\frac{n}{2}}$, and n is even;

(iii) $X \cong \mathbb{P}^{\frac{n-1}{2}} \times Q^{\frac{n+1}{2}}$, and n is odd;

(iv) $X \cong \mathbb{P}\left(T_{\mathbb{P}^{\frac{n+1}{2}}}\right)$, and n is odd;

(v) $X \cong \mathbb{P}_{\mathbb{P}^{\frac{n+1}{2}}}\left(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n-1}{2}}\right)$, and n is odd.

An n-dimensional Fano manifold X with index $i_X = n-1$ is called *del Pezzo* manifold. These manifolds were classified by T. Fujita in [Fuj82a] and [Fuj82b]:

Theorem 2.6.5. Let X be an n-dimensional del Pezzo manifold with $n \geq 3$.

(1) Assume that X has Picard number $\rho(X) = 1$. Then the degree d_X of X satisfies $1 \leq d_X \leq 5$. Moreover, for each $d = 1, \ldots, 4$ and each $n \geq 3$, and for $d = 5$ and each $n = 3, \ldots, 6$, there exists a unique deformation class of n-dimensional del Pezzo manifolds Y_d with $\rho(Y_d) = 1$ and $d_{Y_d} = d$. They have the following description:

   (i) Y_1 is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$;

   (ii) $Y_2 \to \mathbb{P}^n$ is a double cover branched along a quartic hypersurface in $\mathbb{P}^n$; alternatively, Y_2 is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(2, 1, \ldots, 1)$;

   (iii) Y_3 is a cubic hypersurface in $\mathbb{P}^{n+1}$;

   (iv) Y_4 is the intersection of two quadrics in $\mathbb{P}^{n+2}$;

   (v) Y_5 is a linear section of the Grassmannian $\mathbb{G}(1, 4)$ embedded in $\mathbb{P}^9$ under the Plücker embedding.

(2) Assume that X has Picard number $\rho(X) > 1$. Then X is isomorphic to one of the following:

   (i) $\mathbb{P}^2 \times \mathbb{P}^2$, and $n = 4$;

   (ii) $\mathbb{P}(T_{\mathbb{P}^2})$, and $n = 3$;

   (iii) $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$, and $n = 3$;

   (iv) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and $n = 3$.

An n-dimensional Fano manifold X with index $i_X = n - 2$ is called *Mukai* manifold. The classification of such manifolds was first announced in [Muk89] by S. Mukai. The complete list can be found in [IP99] and [AC13]. We only note that, among all Fano varieties appearing in the classification of Mukai manifolds, we have the general codimension 2 linear section of the Grassmannian $\mathbb{G}(1, 5)$ embedded by the Plücker embedding. This linear section of $\mathbb{G}(1, 5)$ and the general codimension 2 linear section of $\mathbb{G}(1, 4)$ will be studied in Chapter 3.

2.6.3 2-Fano Manifolds

In 2003, T. Graber, J. Harris and J. Starr proved the following beautiful theorem:

Theorem 2.6.6 ([GHS03, Thm. 1])**.** Let $f : X \to B$ be a proper morphism of complex varieties with B a smooth curve. If the general fiber of f is rationally connected, then f has a section.

This theorem says that if K is the function field of a curve over $\mathbb{C}$, then any rationally connected variety X defined over K has a K-point. This generalizes the following C. Tsen's Theorem in the case of function fields of curves.

Theorem 2.6.7. Let K be a field of transcendence degree r over an algebraically closed field k. Let $Y \subset \mathbb{P}^N_K$ be a hypersurface of degree d. If $d^r \leq N$, then Y has a K-point.

A hypersurface of degree d in $\mathbb{P}^N$ is Fano or rationally connected if and only if $d \leq N$. Hence, for $r = 1$, the Graber-Harris-Starr's Theorem replaces in Tsen's Theorem the condition of Y being a hypersurface of degree $d \leq N$ with the condition of Y being rationally connected. This means much because rationally connected varieties form the largest class of varieties for which such statement holds true for $r = 1$. This motivated a search for the suitable geometric conditions on Y that generalizes Tsen's Theorem for function fields of higher dimensional varieties. In [dHS11], A. J. de Jong, X. He and J. M. Starr established a generalization of Tsen's Theorem for function fields os surfaces. They replaced the condition of Y being a hypersurface of degree d, $d^2 \leq N$, with the condition of X satisfying a notion of *rationally simply connectedness*. Roughly speaking, a rationally connected variety is rationally simply connected if the space of based 2-pointed rational curves is rationally connected. In order to find geometric conditions that imply rationally simply connectedness, in [dS06c] and [dS07] de Jong and Starr introduced *2-Fano manifolds*. We will define this notion after some notation.

Let X be a smooth, complex, projective variety, and let $k \geq 1$ be an integer. Let $A_k(X)$ be the group of k-cycles on X modulo rational equivalence, and let $A^k(X)$ be the k-th graded piece of the Chow ring $A^*(X)$ of X. Let $N_k(X)$ be the quotient of $A_k(X)$ by numerical equivalence, and let $N^k(X)$ be the quotient of $A^k(X)$ by numerical equivalence. The Abelian groups $N_k(X)$ and $N^k(X)$ are finitely generated, and intersection product induces a perfect pairing

$$N^k(X) \times N_k(X) \longrightarrow \mathbb{Z}.$$

We will denote $N_k(X)_{\mathbb{R}} = N_k(X) \otimes \mathbb{R}$ and $N^k(X)_{\mathbb{R}} = N^k(X) \otimes \mathbb{R}$. We will denote by $\overline{NE}_k(X) \subset N_k(X)_{\mathbb{R}}$ the closure of the convex cone generated by classes of effective k-cycles.

Let $\alpha \in N^k(X)_{\mathbb{R}}$. We say that α is *nef* if $\alpha \cdot \beta \geq 0$, for every $\beta \in \overline{NE}_k(X)_{\mathbb{R}}$. We write $\alpha \geq 0$ for α nef.

Definition 2.6.5. Let X be a smooth, complex, projective variety with second Chern character $\mathrm{ch}_2(X)$. We say that X is *2-Fano* (respectively *weakly 2-Fano*) if X is Fano and $\mathrm{ch}_2(X) \cdot \alpha > 0$ (respectively $\mathrm{ch}_2(X) \cdot \alpha \geq 0$), for every $\alpha \in \overline{NE}_2(X)_{\mathbb{R}} \setminus \{0\}$.

In [AC13], C. Araujo and A.-M. Castravet gave a complete classification for 2-Fano manifolds and an almost complete classification for weakly 2-Fano manifolds, both cases for manifolds of dimension $n \geq 3$ and index at least $n - 2$. The only cases left open were the general codimension 2 linear sections of Grassmannians $\mathbb{G}(1, 4)$ and $\mathbb{G}(1, 5)$. In Chapter 3 we will complete this classification proving that these manifolds are not weakly 2-Fano.

2.7 Grassmannians

This section is devoted to Grassmannians, which are varieties parameterizing $(k + 1)$-dimensional vector subspaces of a given finite-dimensional vector space, or equivalently, k-planes in a projective space. Grassmannians are widely studied and they appear very often in classification theorems. We cite [Har92] and [Sha13a] as references for Grassmannians.

2.7.1 Plücker Embedding

Let V be an $(N + 1)$-dimensional vector space and let $k \geq 0$ be an integer such that $k \leq N$. The *Grassmannian* $\mathbb{G}(k, \mathbb{P}(V))$ is the set of all k-planes in the projective space $\mathbb{P}(V)$.

Notation. A k-plane in the projective space $\mathbb{P}(V)$ is the projectivization $\mathbb{P}(W)$ of a $(k+1)$-dimensional vector subspace $W \subset V$. We will denote by $[\mathbb{P}(W)]$ the corresponding point in the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$. Often we will work over the field of complex numbers, so it will be convenient to denote by $\mathbb{G}(k, N)$ the Grassmannian of k-planes in the projective space $\mathbb{P}^N = \mathbb{P}(\mathbb{C}^{N+1})$.

Example 2.7.1. For $V = \mathbb{C}^{N+1}$, the Grassmannian $\mathbb{G}(0, \mathbb{P}^N)$ is the set of all points in $\mathbb{P}^N$, that is, $\mathbb{G}(0, N) = \mathbb{P}^N$, and $\mathbb{G}(1, \mathbb{P}^3)$ is the set of all lines in $\mathbb{P}^3$.

We can give to the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$ the structure of a projective variety as follows. Let $\mathbb{P}(W)$ be a k-plane in $\mathbb{P}(V)$ which is the projectivization of a $(k+1)$-dimensional vector subspace $W \subset V$. Choose a basis $\{w_1, \dots, w_{k+1}\}$ for W and consider the $(k+1) \times (N+1)$-matrix of maximal rank

$$\begin{pmatrix} w_1 \\ \vdots \\ w_{k+1} \end{pmatrix} = \begin{pmatrix} w_{1,1} & \cdots & w_{1,N+1} \\ \vdots & \ddots & \vdots \\ w_{k+1,0} & \cdots & w_{k+1,N+1} \end{pmatrix}.$$

Denote by $W_{i_1 \cdots i_{k+1}}$ the $(i_1, \dots, i_{k+1})$-minor of this matrix, that is, the determinant of the submatrix formed by the columns $1 \leq i_1 < \cdots < i_{k+1} \leq N+1$. Then we have a well defined map

$$\begin{aligned} \psi : \mathbb{G}(k, \mathbb{P}(V)) &\longrightarrow \mathbb{P}^{\binom{N+1}{k+1}-1} \\ [\mathbb{P}(W)] &\longmapsto (\cdots : W_{i_1 \cdots i_{k+1}} : \cdots). \end{aligned}$$

Indeed, if $\{w'_1, \dots, w'_{k+1}\}$ is another basis for W, then there exists an invertible matrix $M \in \mathrm{GL}(k+1, \mathbb{C})$ such that $(w_{ij}) = M(w'_{ij})$. Hence, the minors of the matrices (w_{ij}) and (w'_{ij}) differ from one another by the multiplication of $\det(M)$, a non-zero number. Moreover, since the matrix (w_{ij}) has maximal rank, at least one of the minors $W_{i_1 \cdots i_{k+1}}$ is non-zero. Therefore, ψ is well-defined. Furthermore, one can prove that ψ is one-to-one and its image is a smooth algebraic variety which is the intersection of quadric hypersurfaces (see, for example, [Har92, Lecture 6] or [Sha13a, Subsection 1.4.1]). This map ψ is known as the *Plücker embedding.*

There exists another way to represent the Plücker embedding, making less use of homogeneous coordinates. In this way, the Plücker embedding is given by

$$\begin{aligned} \psi : \mathbb{G}(k, \mathbb{P}(V)) &\longrightarrow \mathbb{P}\left(\bigwedge^{k+1} V\right) \\ [\mathbb{P}(W)] &\longmapsto \mathbb{P}(w_1 \wedge \cdots \wedge w_{k+1}). \end{aligned}$$

Let $\mathcal{O}_{\mathbb{G}} \otimes V$ be the trivial vector bundle (of rank $N+1$) on the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$. The *tautological* or *universal bundle* of the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$ is the vector sub-bundle $\mathcal{U}_{\mathbb{G}}$ of $\mathcal{O}_{\mathbb{G}} \otimes V$ (of rank $k+1$) whose fiber over a point $[\mathbb{P}(W)] \in \mathbb{G}(k, \mathbb{P}(V))$ is the vector subspace $W \subset V$ itself, that is,

$$\begin{aligned} f : \mathcal{U}_{\mathbb{G}} = \{(w, [\mathbb{P}(W)]) \in V \times \mathbb{G}(k, \mathbb{P}(V)) \mid w \in W\} &\longrightarrow \mathbb{G}(k, \mathbb{P}(V)) \\ (w, [\mathbb{P}(W)]) &\longmapsto [\mathbb{P}(W)]. \end{aligned}$$

The quotient $\mathcal{Q}_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}} \otimes V / \mathcal{U}_{\mathbb{G}}$, called *tautological* or *universal quotient bundle* on the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$, is the vector bundle (of rank $N-k$) whose fiber over a point

$[\mathbb{P}(W)]$ is the vector space V/W. The exact sequence of vector bundles on $\mathbb{G}(k, \mathbb{P}(V))$

$$0 \to \mathcal{U}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}} \otimes V \to \mathcal{Q}_{\mathbb{G}} \to 0$$

is referred as the *tautological* or *universal exact sequence* on the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$. Note that, the line bundle $\mathcal{O}(1)$ on $\mathbb{P}\left(\bigwedge^{k+1} V\right)$ restricts to $\bigwedge^{k+1} \mathcal{U}^\vee$ on the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$, because $\mathcal{O}(-1)$ is the tautological vector sub-bundle of $\mathbb{P}\left(\bigwedge^{k+1} V\right)$, as we can see from the example below.

Example 2.7.2. The tautological bundle of the projective space $\mathbb{P}^N = \mathbb{G}(0, N)$ is the line bundle $\mathcal{U}$ whose fiber over a point $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N$ is the line defined by x, that is,

$$\begin{aligned} f : \mathcal{U} = \{(z, x) \in \mathbb{C}^{N+1} \times \mathbb{P}^N \mid z = \lambda(x_0, \ldots, x_N) \text{ for some } \lambda \in \mathbb{C}\} &\longrightarrow \mathbb{P}^N \\ (z, x) &\longmapsto x. \end{aligned}$$

If we cover the projective space $\mathbb{P}^N$ with the $N+1$ affine open subsets

$$U_i := \{(x_0 : \cdots : x_N) \in \mathbb{P}^N \mid x_i \neq 0\}, \quad \text{for } i = 0, \ldots, N,$$

then the trivializations of $\mathcal{U}$ over this cover are

$$\begin{aligned} \varphi_i : f^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C}, \quad \text{for } i = 0, \ldots, N. \\ (z, x) &\longmapsto (x, z_i) \end{aligned}$$

Denote by U_{ij} the intersection of U_i and U_j. The transition function for $\mathcal{U}$ relative to the trivializations φ_i and φ_j is

$$\begin{aligned} \varphi_{ij} : U_{ij} \times \mathbb{C} \xrightarrow{\varphi_i^{-1}} f^{-1}(U_{ij}) &\xrightarrow{\varphi_j} U_{ij} \times \mathbb{C} \\ (x, t) \longmapsto\quad &\quad (x, (x_i/x_j)t). \end{aligned}$$

Therefore, the tautological bundle $\mathcal{U}$ of $\mathbb{P}^N$ is represented by co-cycles $\{g_{ij}\}_{i,j} \in H^1(\mathbb{P}^N, \mathcal{O}^*_{\mathbb{P}^N})$, where

$$\begin{aligned} g_{ij} : U_{ij} &\longrightarrow \mathbb{C}^* \\ x &\longmapsto x_i/x_j. \end{aligned}$$

On the other hand, if H is a hyperplane in $\mathbb{P}^N$, say $H : x_0 = 0$, then we have isomorphisms

$$\mathcal{O}_{\mathbb{P}^N}(H)|_{U_i} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^N}|_{U_i}, \quad \text{for } i = 0, \ldots, N,$$

given by the multiplication by x_0/x_i. Hence, $\mathcal{O}_{\mathbb{P}^N}(H)$ is represented by the co-cycles $\{g_{ij}^{-1}\}_{i,j} \in H^1(\mathbb{P}^N, \mathcal{O}^*_{\mathbb{P}^N})$, and therefore $f : \mathcal{U} \to \mathbb{P}^N$ is the line bundle associated to the invertible sheaf $\mathcal{O}_{\mathbb{P}^N}(-1)$.

It is not difficult to prove that the tangent bundle $\mathcal{T}_{\mathbb{G}}$ of the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$ is described by

$$\mathcal{T}_{\mathbb{G}} \cong \mathcal{H}om\,(\mathcal{U}_{\mathbb{G}}, \mathcal{Q}_{\mathbb{G}}) \cong \mathcal{U}_{\mathbb{G}}^\vee \otimes \mathcal{Q}_{\mathbb{G}}$$

(see [Sha13b, Example 6.24]), and thus the cotangent bundle is described by

$$\Omega_{\mathbb{G}} \cong \mathcal{U}_{\mathbb{G}} \otimes \mathcal{Q}_{\mathbb{G}}^\vee.$$

In particular, taking the highest exterior power, we can describe the canonical bundle $\omega_{\mathbb{G}}$ of the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$ by

$$\omega_{\mathbb{G}} \cong \left(\bigwedge^{k+1} \mathcal{U}_{\mathbb{G}}\right)^{\otimes(N-k)} \otimes \left(\bigwedge^{N-k} \mathcal{Q}_{\mathbb{G}}^{\vee}\right)^{k+1}.$$

But, by the tautological exact sequence, $\bigwedge^{k+1} \mathcal{U}_{\mathbb{G}}$ and $\bigwedge^{N-k} \mathcal{Q}_{\mathbb{G}}$ are dual vector bundles. Since $\bigwedge^{k+1} \mathcal{U}_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-1)$, we conclude that

$$\omega_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-N-1).$$

One can prove more: the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$ has Picard group

$$\mathrm{Pic}(\mathbb{G}(k, \mathbb{P}(V))) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{G}}(1)].$$

We end this section describing lines on the Grassmannian $\mathbb{G}(k, \mathbb{P}(V))$; we will conclude that $\mathbb{G}(k, \mathbb{P}(V))$ is covered by lines.

Example 2.7.3. Let U be a k-dimensional vector space and let V be a $(k+2)$-dimensional vector space such that $U \subset V$. We claim that

$$L_{U,V} = \{[\mathbb{P}(W)] \in \mathbb{G}(k, N) \mid U \subset W \subset V\} \cong \mathbb{P}(V/U)$$

is a line on $\mathbb{G}(k, N)$. Moreover, every line on $\mathbb{G}(k, N)$ is determined by the choice of vector spaces U and V satisfying the properties above. Indeed, let $\{u_1, \ldots, u_k\}$ be a basis for U; since $U \subset V$, we can complete a basis to V, say $\{u_1, \ldots, u_k, v_{k+1}, v_{k+2}\}$. Then, for every point $[\mathbb{P}(W)] \in L_{U,V}$, the corresponding $(k+1)$-dimensional vector subspace W has a basis $\{u_1, \ldots, u_k, \lambda v_{k+1} + \mu v_{k+2}\}$, for some $(\lambda : \mu) \in \mathbb{P}^1$. Using the linearity of the determinant map (in the $(k+1)$-th coordinate), we conclude that

$$[\mathbb{P}(W)] = (\cdots : W_{i_1 \cdots i_{k+1}} : \cdots) = (\cdots : \lambda W^{k+1}_{i_1 \cdots i_{k+1}} + \mu W^{k+2}_{i_1 \cdots i_{k+1}} : \cdots),$$

where $W^{k+1}_{i_1 \cdots i_{k+1}}$ is the $(i_1 \cdots i_{k+1})$-minor of the matrix consisting of rows $u_1, \ldots, u_k, v_{k+1}$ and $W^{k+2}_{i_1 \cdots i_{k+1}}$ is the $(i_1 \cdots i_{k+1})$-minor of the matrix consisting of rows $u_1, \ldots, u_k, v_{k+2}$. Therefore, it shows that $L_{U,V}$ is a line on the Grassmannian $\mathbb{G}(k, N)$.

Conversely, let L be a line on $\mathbb{G}(k, N)$, passing through two distinct points $[\mathbb{P}(W)]$ and $[\mathbb{P}(W')]$. Using the equations defining the Grassmannian variety $\mathbb{G}(k, N)$ and by usual computations, one can see that the condition $L \subset \mathbb{G}(k, N)$ implies that L is given as claimed.

2.7.2 Schubert Calculus

Enumerative Geometry is an important part of the Algebraic Geometry. The goal is to count the number of objects (points, curves, etc.) satisfying certain incidence conditions. A very useful tool for this task is the Schubert Calculus, which refers to the calculus of enumerative geometry concerning linear subspaces. For this subject we refer to [Har92, Section 1.5] and the nice I. Coskun's notes [Cos06].

Definition 2.7.1. Let $F_\bullet$ be a complete flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_{N+1} = \mathbb{C}^{N+1}$$

of the vector space $\mathbb{C}^{N+1}$. Given a partition $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{k+1})$ satisfying

$$N - k \geq \lambda_1 \geq \cdots \geq \lambda_{k+1} \geq 0,$$

the *Schubert variety* $\Sigma_{\underline{\lambda}}(F_\bullet) = \Sigma_{\lambda_1,\ldots,\lambda_{k+1}}(F_\bullet)$ of type $\underline{\lambda}$ with respect to the flag $F_\bullet$ is defined by

$$\Sigma_{\lambda_1,\ldots,\lambda_{k+1}}(F_\bullet) = \{[\mathbb{P}(W)] \in \mathbb{G}(k,N) \mid \dim(W \cap F_{N-k+i-\lambda_i}) \geq i, \text{ for } i = 1, \ldots, k+1.\}\,.$$

Note that this definition does not depend on the choice of the flag. Indeed, let $E_\bullet$ and $F_\bullet$ be two complete flags of the vector space $\mathbb{C}^{N+1}$. Because of the transitive action of $\mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ on $\mathbb{G}(k,N)$, there exists an automorphism of $\mathbb{C}^{N+1}$ mapping the flag $E_\bullet$ to the flag $F_\bullet$ and, consequently, the Schubert varieties $\Sigma_{\underline{\lambda}}(E_\bullet)$ and $\Sigma_{\underline{\lambda}}(F_\bullet)$ are isomorphic. For that reason, we simply denote the Schubert variety of type $\underline{\lambda}$ by $\Sigma_{\lambda_1,\ldots,\lambda_{k+1}}$.

The dimension of the Schubert variety $\Sigma_{\lambda_1,\ldots,\lambda_{k+1}}$ is $(k+1)(N-k) - \sum_{i=1}^{k+1} \lambda_i$.

It can be proved that the Schubert varieties give a cell-decomposition of the complex Grassmannian $\mathbb{G}(k,N)$ with only even dimensional cells. It follows that the classes of the Schubert varieties generate the homology of $\mathbb{G}(k,N)$. Applying Poincaré Duality we obtain the following fundamental theorem about the cohomology of $\mathbb{G}(k,N)$. We will denote the cohomology class that corresponds to the Schubert variety $\Sigma_{\lambda_1,\ldots,\lambda_{k+1}}$ by $\sigma_{\lambda_1,\ldots,\lambda_{k+1}}$. We omit the indices that are zero.

Theorem 2.7.1 (Chow Basis Theorem)**.** The Schubert classes $\sigma_{\underline{\lambda}}$ freely generate the Chow group $A_*(\mathbb{G}(k,N))$.

Example 2.7.4. Consider the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. A complete flag on $\mathbb{P}^3$ corresponds to a choice of a point $p \in \mathbb{P}^3$, a line $\ell \subset \mathbb{P}^3$ containing p and a plane $P \subset \mathbb{P}^3$ containing ℓ. In this case, the Schubert variety Σ_1 parameterizes lines that intersect ℓ. Σ_2 parameterizes lines that contain p. $\Sigma_{1,1}$ parameterizes lines that are contained in P. $\Sigma_{2,1}$ parameterizes lines that are contained in P and contain p.

Example 2.7.5. How many lines are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$? The answer to this question is given by the self-intersection of the class $[\Omega]$, where Ω is the set of lines contained in a quadric hypersurface in $\mathbb{P}^4$. The dimension of Ω is 3. Since the Schubert cycles σ_3 and $\sigma_{2,1}$ generate the group of 3-cycles on X, we can write $[\Omega] = a\sigma_3 + b\sigma_{2,1}$, with $a, b \in \mathbb{Z}$. The cycle σ_3 is self-dual. Hence $a = [\Omega]\sigma_3$, and it corresponds to the number of lines contained in a quadric hypersurface and passing through a fixed point. As long as the point is not contained in the quadric hypersurface, we have $a = 0$. The cycle $\sigma_{2,1}$ is also self-dual. Hence $b = [\Omega]\sigma_3$, and it corresponds to the number of lines contained in the intersection of a quadric hypersurface with a $\mathbb{P}^3$ and intersecting a $\mathbb{P}^1 \subset \mathbb{P}^3$. We claim that $b = 4$. Indeed, the intersection of the quadric hypersurface with the $\mathbb{P}^3$ is a quadric surface. The lines have to be contained in this surface and must pass through the two points of intersection of the $\mathbb{P}^1$ with the quadric surface. There exist 4 such lines. Therefore, there exist $[\Omega]^2 = 16$ lines contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$.

In the previous example (and many others found in the literature) we pretend that all intersections are transverse. This is indeed the case, but it requires some attention. We can either explicitly calculate the tangent spaces to check that the intersections are transverse or appeal to a general theorem that guarantees the result. We state that theorem for the sake of completeness.

Theorem 2.7.2 (Kleiman's Transversality)**.** Assume the ground field is algebraically closed of characteristic zero. Let X be an integral algebraic scheme, on which an integral algebraic group G acts. Let $f : Y \to X$ and $g : Z \to X$ be two morphisms of integral algebraic schemes. For each rational element of $g \in G$, denote by gY the X-scheme given by $y \mapsto gf(y)$. Then there exists a dense open subset U of G such that for every rational element $g \in U$, the fiber product $(gY) \times_X Z$ is either empty or equidimensional of the expected dimension

$$\dim(Y) + \dim(Z) - \dim(X).$$

Furthermore, if Y and Z are regular, for a dense open set this fiber product is regular.

The dual of the Schubert cycle $\sigma_{\lambda_1,\ldots,\lambda_{k+1}}$, denoted by $\sigma^*_{\lambda_1,\ldots,\lambda_{k+1}}$, is the cycle $\sigma_{N-k-\lambda_{k+1},\ldots,N-k-\lambda_1}$. If $\sigma_{\underline{\lambda}}$ and $\sigma_{\underline{\mu}}$ are Schubert cycles of complementary dimensions, then Duality Theorem (see [Ful98, Ex. 14.7.4]) assures that

$$\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\mu}} = \begin{cases} 1, & \text{if } \sigma_{\underline{\mu}} = \sigma^*_{\underline{\lambda}}; \\ 0, & \text{otherwise.} \end{cases}$$

Other useful formulas, such as Pieri's Formula and Giambelli's Formula are given below. We will say that a Schubert cycle $\sigma_{\underline{\lambda}}$ is *special* if $\underline{\lambda} = (h, 0, \ldots, 0)$ and we will denote such a cycle by σ_h.

Theorem 2.7.3 (Pieri's Formula)**.** Let σ_h be a special Schubert cycle and $\sigma_{\underline{\lambda}}$ be any Schubert cycle. Then

$$\sigma_h \cdot \sigma_{\underline{\lambda}} = \sum_{\underline{\mu}} \sigma_{\underline{\mu}},$$

where the sum is over all partitions $|\underline{\mu}|$ such that $|\underline{\mu}| = h + |\underline{\lambda}|$ and

$$N - k \geq \mu_1 \geq \lambda_1 \geq \cdots \geq \mu_{k+1} \geq \lambda_{k+1}.$$

Theorem 2.7.4 (Giambelli's Formula)**.** Any Schubert cycle $\sigma_{\lambda_1,\ldots,\lambda_{k+1}}$ can be expressed as a linear combination of products of special Schubert cycles as follows

$$\sigma_{\lambda_1,\ldots,\lambda_{k+1}} = \det \begin{pmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \cdots & \sigma_{\lambda_1+k} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \cdots & \sigma_{\lambda_2+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\lambda_{k+1}-k} & \sigma_{\lambda_{k+1}-k+1} & \cdots & \sigma_{\lambda_{k+1}} \end{pmatrix}.$$

The next result gives a description of the cohomology ring of the Grassmannian $\mathbb{G}(k, N)$ in terms of the Chern classes of the tautological bundle $\mathcal{U}_{\mathbb{G}}$ and the universal quotient bundle $\mathcal{Q}_{\mathbb{G}}$ on $\mathbb{G}(k, N)$.

Theorem 2.7.5. As a ring, the cohomology ring of the Grassmannian $\mathbb{G}(k, N)$ is isomorphic to

$$\mathbb{R}[\mathrm{c}_1(\mathcal{U}_{\mathbb{G}}), \ldots, \mathrm{c}_{k+1}(\mathcal{U}_{\mathbb{G}}), \mathrm{c}_1(\mathcal{Q}_{\mathbb{G}}), \ldots, \mathrm{c}_{N-k}(\mathcal{Q}_{\mathbb{G}})]/(\mathrm{c}(\mathcal{U}_{\mathbb{G}})\,\mathrm{c}(\mathcal{Q}_{\mathbb{G}}) = 1).$$

Moreover, the Chern classes of the tautological quotient bundle $\mathcal{Q}$ generate the cohomology ring.

Proposition 2.7.1. The Chern classes of the tautological bundle $\mathcal{U}_{\mathbb{G}}$ on the Grassmannian $\mathbb{G}(k, N)$ are given by

$$\mathrm{c}_i(\mathcal{U}_{\mathbb{G}}) = (-1)^i \sigma_{1,\ldots,1},$$

where the Schubert cycle has length i. The Chern classes of the tautological quotient bundle $\mathcal{Q}_{\mathbb{G}}$ on $\mathbb{G}(k, N)$ are given by

$$\mathrm{c}_i(\mathcal{Q}_{\mathbb{G}}) = \sigma_i.$$

Chapter 3

Lines on Varieties

In some sense a straight line is the simplest example of embedded rational curve. For that reason, it is natural to begin the study of rational curves on projective varieties by considering their varieties of lines, sometimes satisfying some additional condition. Our interest is in lines on projective varieties passing through a general point of the variety. In Section 3.1 we present some basic facts about these varieties of lines and then we compute some examples in the case of hypersurfaces, Grassmannians and linear sections of Grassmannians. In this last case we focus attention on linear sections of codimension 2 of the Grassmannians $\mathbb{G}(1,4)$ and $\mathbb{G}(1,5)$. In order to give a more satisfactory description of their varieties of lines through a fixed point, in Section 3.2 we study their automorphism groups, which were determined by J. Piontkowski and A. Van de Ven in [PVdV99]. We will see that these automorphism groups act with finitely many orbits and we will describe such orbits. Then, in Section 3.3 we describe the variety of lines on these linear sections passing through a fixed point in each orbit of the actions. Finally, as an application of these results, in Section 3.4 we prove that these linear sections of Grassmannians are not weakly 2-Fano, completing the classification of weakly 2-Fano manifolds of high index, initiated by C. Araujo and A.-M. Castravet in [AC13].

3.1 Fano Variety of Lines

Let $X \subset \mathbb{P}^N$ be an n-dimensional variety. We will denote by $F(X)$ the Hilbert scheme $\mathrm{Hilb}_{t+1}(X)$ of lines on X. This scheme was studied by G. Fano, and for that reason is called the *Fano variety of lines* on X. We stress that $F(X)$ is not always a Fano variety in the sense of Definition 2.6.1. The Fano variety of lines $F(X)$ can be viewed as the image of the morphism

$$\begin{aligned} \mathrm{Hom}_1(\mathbb{P}^1, X) &\longrightarrow \mathbb{G}(1,N) \\ [f] &\longmapsto [f(\mathbb{P}^1)], \end{aligned}$$

where $\mathrm{Hom}_1(\mathbb{P}^1, X)$ denotes the irreducible component of the scheme $\mathrm{Hom}(\mathbb{P}^1, X)$ parameterizing morphisms $f : \mathbb{P}^1 \to X$ of degree 1, and $\mathbb{G}(1,N)$ denotes the Grassmannian of lines in the projective space $\mathbb{P}^N$. The induced morphism

$$\rho : \mathrm{Hom}_1(\mathbb{P}^1, X) \longrightarrow F(X)$$

is simply the quotient by the automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ of $\mathbb{P}^1$. Assume that X is smooth along a line ℓ on X parameterized by $[f] \in \mathrm{Hom}_1(\mathbb{P}^1, X)$. By the results in

Section 2.2, the differential map of ρ at $[f]$ fits into an exact sequence

$$0 \to H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, f^*T_X) \xrightarrow{d_{[f]}\rho} H^0(\mathbb{P}^1, f^*\mathcal{N}_{\ell/X}) \to 0.$$

Since f is an isomorphism onto its image, the same exact sequence can be considered on the line ℓ. Therefore,

$$T_{[\ell]}F(X) \cong H^0(\ell, \mathcal{N}_{\ell/X}),$$

as we already expected from the results in Section 2.3.

We can carry out the above description starting with $\mathrm{Hom}_1(\mathbb{P}^1, X; o \mapsto x)$ instead of $\mathrm{Hom}_1(\mathbb{P}^1, X)$. The quotient of $\mathrm{Hom}_1(\mathbb{P}^1, X; o \mapsto x)$ by the automorphism group $\mathrm{Aut}(\mathbb{P}^1; x)$ of $\mathbb{P}^1$ fixing the point x parameterizes lines on X passing through the point $x \in X$. This scheme will be denoted by $F(X; x)$ and we refer to it as the Fano varieties of lines on X passing through x. The tangent space to $F(X; x)$ at a point $[\ell]$ is described by the isomorphism

$$T_{[\ell]}F(X; x) \cong H^0(\ell, \mathcal{N}_{\ell/X}(-x)).$$

Next we present some basic facts concerning the Fano varieties of lines $F(X)$ and $F(X; x)$. The following result, proved by W. Barth and A. Van de Ven, considers the case that X is a hypersurface.

Theorem 3.1.1 ([BVdV79, Thm. 8])**.** Let $X_d \subset \mathbb{P}^N$ be a hypersurface of degree d.

(i) If $d \leq 2N - 3$, then $F(X_d) \neq \emptyset$;

(ii) If $d \leq 2N - 3$ and X_d is general, then $F(X_d)$ is smooth of dimension $2N - 3$;

(iii) If $d \leq 2N - 4$, then $F(X_d)$ is connected, except in the case of a smooth quadric hypersurface $X_2 \subset \mathbb{P}^3$.

Proposition 3.1.1 ([Deb01, Prop. 2.13])**.** Let $X \subset \mathbb{P}^N$ be a variety defined by equations of degrees $d_1, \ldots, d_k$. Set $|d| = d_1 + \cdots + d_k$.

(i) $\dim F(X; x) \geq N - 1 - |d|$, for every point $x \in X$; in particular, if $|d| \leq N - 1$ then $F(X; x) \neq \emptyset$;

(ii) Assume $|d| \leq N - 1$. If X is general (and therefore, a complete intersection) and ℓ is a general line on X, then

$$\mathcal{N}_{\ell/X} \cong \mathcal{O}_\ell(1)^{N-1-|d|} \oplus \mathcal{O}_\ell^{|d|-k}.$$

Sketch of proof of (i). Let X be given by the zeroes of homogeneous polynomials $G_1, \ldots, G_k$ of degrees $d_1, \ldots, d_k$. Let $x \in X$. Fix a hyperplane $H \subset \mathbb{P}^N$ such that $x \notin \mathbb{P}^N$. The lines in $\mathbb{P}^N$ passing through x are parameterized by $H \cong \mathbb{P}^{N-1}$. Let $y \in H$. The line passing through these points,

$$L_{x,y} = \{(\lambda x_0 + \mu y_0 : \cdots : \lambda x_N + \mu y_N) \in \mathbb{P}^N \mid (\lambda : \mu) \in \mathbb{P}^1\},$$

is contained in X if and only if

$$G_i(\lambda x_0 + \mu y_0, \ldots, \lambda x_N + \mu y_N) = 0,$$

for every $(\lambda : \mu) \in \mathbb{P}^1$ and every $i = 1, \ldots, k$. Each of these equations is homogeneous polynomial of degree d_i in the variables λ, μ, with $d_i + 1$ coefficients. Note that the coefficient of the term λ^{d_i} of each of these equations is zero, since $x \in X$. Therefore, $\dim F(X;x) \geq N-1-|d|$. $\square$

Example 3.1.1 (Lines on a Cubic Hypersurface Through a Point)**.** Let X_3 be a cubic hypersurface in a projective space $\mathbb{P}^N$, with $N \geq 4$, given by the polynomial equation

$$\sum_{i_0+\cdots+i_N=3} c_{(i)} x_0^{i_0} \cdots x_N^{i_N} = 0.$$

Let us describe the Fano variety $F(X_3;r)$ of lines on X_3 passing through a general point $r \in X_3$. Under a projective change of coordinates, we can assume that $r = (1:0:\cdots:0)$, that is, $c_{30\cdots 0} = 0$. The lines in $\mathbb{P}^N$ passing through r are parameterized by the hyperplane $H \cong \mathbb{P}^{N-1}$ given by the equation $x_0 = 0$. Let $s = (0 : s_1 : \cdots : s_N)$ be a point of H. The line passing through these points,

$$L_{r,s} = \{(\lambda : \mu s_1 : \cdots : \mu s_N) \in \mathbb{P}^N \mid (\lambda : \mu) \in \mathbb{P}^1\},$$

is contained in X_3 if and only if

$$\begin{aligned} &\left(\sum_{\substack{i_1+\cdots+i_N=1 \\ i_0=2}} c_{(i)} s_1^{i_1} \cdots s_N^{i_N} \right) \lambda^2 \mu \\ + &\left(\sum_{\substack{i_1+\cdots+i_N=2 \\ i_0=1}} c_{(i)} s_1^{i_1} \cdots s_N^{i_N} + 2s_0 \sum_{\substack{i_1+\cdots+i_N=1 \\ i_0=2}} c_{(i)} s_1^{i_1} \cdots s_N^{i_N} \right) \lambda \mu^2 \\ &+ \left(\sum_{i_1+\cdots+i_N=2} c_{(i)} s_1^{i_1} \cdots s_N^{i_N} \right) \mu^3 = 0, \end{aligned}$$

for every $(\lambda : \mu) \in \mathbb{P}^1$. Therefore, $F(X_3;r)$ is isomorphic to a complete intersection of type $(1,2,3)$ in $\mathbb{P}^{N-1}$. Note that, the vanishing condition of the coefficient of the monomial $\lambda^2\mu$ means that s is in the tangent space of X_3 at the point r. The vanishing condition of the coefficient of the monomial μ^3 means that s is a point of X_3.

For the next examples we will consider the Grassmannian $\mathbb{G}(k,N)$ of k-planes in a (complex) projective space $\mathbb{P}^N$ embedded into $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{N+1})$ by the Plücker embedding

$$\begin{aligned} \mathbb{G}(k,N) &\longrightarrow \mathbb{P}\left(\textstyle\bigwedge^{k+1}\mathbb{C}^{N+1}\right) \\ \mathbb{P}(\operatorname{span}\{u_1,\ldots,u_{k+1}\}) &\longmapsto \mathbb{P}(u_1 \wedge \cdots \wedge u_{k+1}). \end{aligned}$$

Example 3.1.2 (Lines on a Grassmannian Through a Point)**.** Let $\mathbb{G} = \mathbb{G}(k,N)$ be the Grassmannian of k-planes in a projective space $\mathbb{P}^N$ embedded in $\mathbb{P}^{\binom{N+1}{k+1}}$ under the Plücker embedding. Let us compute the Fano variety $F(\mathbb{G};x)$ of lines on $\mathbb{G}$ passing through a fixed point $x \in \mathbb{G}$. Let us say that the point $x \in \mathbb{G}$ corresponds to a $(k+1)$-dimensional vector subspace W. We know from Example 2.7.3 that a line on $\mathbb{G}$ passing through $x = [\mathbb{P}(W)]$ is determined by the choice of a k-dimensional vector subspace U and a $(k+2)$-dimensional vector subspace V such that $U \subset W \subset V$. Such a line, denoted by $L_{U,V}$, is given by

$$L_{U,V} = \{[\mathbb{P}(W')] \in \mathbb{G} \mid U \subset W' \subset V\}.$$

Therefore, the Fano variety $F(\mathbb{G}; x)$ can be identified in the following way:

$$F(\mathbb{G}; x) = \left\{ \begin{array}{l} \text{lines on } \mathbb{G} \text{ passing} \\ \text{through } x = [\mathbb{P}(W)] \end{array} \right\} \longleftrightarrow \mathbb{P}(W)^\vee \times \mathbb{P}(\mathbb{C}^{N+1}/W) \cong \mathbb{P}^k \times \mathbb{P}^{N-k-1}$$
$$L_{U,V} \longmapsto (\mathbb{P}(U), \mathbb{P}(V/W)).$$

Example 3.1.3 (Lines on a Linear Section of Grassmannian Through a Point)**.** We will assume notation as in Example 3.1.2. Let H^l be a linear subspace of codimension l of $\mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{N+1}\right)$. Consider the linear section of Grassmannian $X = \mathbb{G}(k, N) \cap H^l$. The Fano variety of lines $F(X; x)$ of lines on X passing through a point $x \in X$ is a subvariety of the Fano variety $F(\mathbb{G}; x)$. The Fano variety $F(X; x)$ is an intersection of l divisors D_i of type $(1, 1)$ in $F(\mathbb{G}; x) \cong \mathbb{P}^k \times \mathbb{P}^{N-k-1}$, obtained from the condition that the lines are on each hyperplane defining H^l. If H^l and x are general, then using some geometric invariant theory one can prove that $F(X; x)$ is smooth (this is true for every complex, projective, connected variety X; see [Mor79, Lemma 9]). For the cases treated in the next sections, $F(X; x)$ will be given explicitly. Thus, we can check the smoothness of $F(X; x)$, for x general, using Jacobi's Criterion.

For varieties X covered by lines the study of the Fano variety $F(X; x)$ of lines on X passing through a general point $x \in X$ has shown to be a powerful tool for the study of the variety X. However, in some cases it is important to know $F(X; x)$ for all points $x \in X$, not just general points. The work presented in this chapter is an example. In the next sections we investigate general codimension 2 linear sections of the Grassmannians $\mathbb{G}(1, 4)$ and $\mathbb{G}(1, 5)$. We give descriptions of their varieties of lines passing through any given point of them. With such descriptions we are able to show that this linear sections of Grassmannians are not weakly 2-Fano, completing the Araujo-Castravet classification of weakly 2-Fano manifolds of high index.

More generally, for the study of uniruled varieties X (see Definition 2.6.2) we consider a minimal family of rational curves on X through a general point $x \in X$. For completeness, but very briefly, we discuss this tool. Let X be a smooth, complex, projective, uniruled variety. Let $x \in X$ be a general point. Let $\mathrm{RatCurves}(X, x)$ be the scheme parameterizing rational curves on X passing through x (see Subsection 2.4.1 for definition). An irreducible component H_x of $\mathrm{RatCurves}(X, x)$ is called a *family of rational curves through* x; if H_x is proper, then H_x is called a *minimal family of rational curves through* x. There always exists such a minimal family: one can take H_x to be an irreducible component of $\mathrm{RatCurves}(X, x)$ parameterizing rational curves through x having minimal degree with respect to some fixed ample line bundle. Let H_x be a minimal family of rational curves through x. It comes with universal family morphisms

$$\begin{array}{ccc} U_x & \xrightarrow{\mathrm{ev}_x} & X, \\ {\scriptstyle \pi_x}\big\downarrow \big\uparrow{\scriptstyle \sigma_x} & & \\ H_x & & \end{array}$$

where π_x is a $\mathbb{P}^1$-bundle and σ_x is the unique section of π_x such that $\mathrm{ev}_x(\sigma_x(H_x)) = \{x\}$. A natural polarization L_x is defined on H_x as follows. There exists an inclusion of sheaves

$$\sigma_x^* T_{\pi_x} \hookrightarrow \sigma_x^* \mathrm{ev}_x^* T_X \cong T_x X \otimes \mathcal{O}_{H_x}.$$

By [Keb02, Theorem 3.3 and Theorem 3.4], the quotient is locally free and defines a finite morphism

$$\tau_x : H_x \longrightarrow \mathbb{P}(T_xX),$$

which maps a curve that is smooth at x to its tangent direction at x. Then we define $L_x = \tau_x^*\mathcal{O}_{\mathbb{P}(T_xX)}(1)$. The polarized variety (H_x, L_x) is called a *polarized minimal family of rational curves through* x. It is a powerful tool for the study of uniruled varieties. The morphism τ_x has been studied extensively in a series of papers by J.-M. Hwang and N. Mok (see, for example, [Hwa01] and [HM04]).

3.2 Automorphism Groups of Linear Sections of the Grassmannians $\mathbb{G}(1, N)$

In this section we present some results concerning the automorphism groups of linear sections of Grassmannians of lines in a projective space. These results were proven by J. Piontkowski and A. Van de Ven in [PVdV99], to which we refer for proofs and details.

Before we begin, we introduce the following notation:

Notation. Given a subvariety Y of a variety X, we will denote by $\mathrm{Aut}(Y, X)$ the group of automorphisms of X that induce automorphisms of Y, that is,

$$\mathrm{Aut}(Y, X) = \{\varphi \in \mathrm{Aut}(X) \mid \varphi(Y) \subset Y\}.$$

Throughout this section we will consider the Grassmannian $\mathbb{G}(k, N)$ of k-planes in a (complex) projective space $\mathbb{P}^N$ embedded in $\mathbb{P}\left(\bigwedge^{k+1}\mathbb{C}^{N+1}\right)$ under the Plücker embedding. We will follow the notation in Subsection 2.7.1: given a k-plane $\mathbb{P}(W)$ in $\mathbb{P}^N$, we will denote by $[\mathbb{P}(W)]$ the corresponding point in the Grassmannian $\mathbb{G}(k, N)$.

The following is a well known theorem about the automorphism group of Grassmannians.

Theorem 3.2.1 ([Har92, Thm. 10.19])**.** For $N > 2k + 1$,

$$\mathrm{Aut}(\mathbb{G}(k, N)) \cong \mathrm{Aut}(\mathbb{G}(k, N), \mathbb{P}(\textstyle\bigwedge^{k+1}\mathbb{C}^{N+1})) \cong \mathbb{P}\mathrm{GL}(N + 1, \mathbb{C}),$$

and for $N = 2k + 1$,

$$\mathrm{Aut}(\mathbb{G}(k, 2k + 1)) \cong \mathrm{Aut}(\mathbb{G}(k, 2k + 1), \mathbb{P}(\textstyle\bigwedge^{k+1}\mathbb{C}^{2(k+1)})) \supset \mathbb{P}\mathrm{GL}(2(k + 1), \mathbb{C}),$$

where the inclusion has index 2.

Denote by H^l a general linear subspace of codimension l of $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{N+1})$. For general linear sections of Grassmannians, J. Piontkowski and A. Van de Ven in [PVdV99] determined the automorphism groups of $\mathbb{G}(1, N) \cap H^2$. Their first result is the following:

Theorem 3.2.2 ([PVdV99, Thm. 1.2 and Cor. 1.3])**.** For $N \geq 4$ and a general linear subspace $H^l \subset \mathbb{P}(\bigwedge^2\mathbb{C}^{N+1})$ of codimension $l \leq 2N - 5$,

$$\mathrm{Aut}(\mathbb{G}(1, N) \cap H^l) \cong \mathrm{Aut}(\mathbb{G}(1, N) \cap H^l, H^l).$$

If $l \leq N - 2$, then

$$\mathrm{Aut}(\mathbb{G}(1, N) \cap H^l) \cong \mathrm{Aut}(\mathbb{G}(1, N), \mathbb{P}(\textstyle\bigwedge^2\mathbb{C}^{N+1})) \cap \mathrm{Aut}(H^l, \mathbb{P}(\textstyle\bigwedge^2\mathbb{C}^{N+1})).$$

When $N \geq 4$ and $l \leq N-2$, the theorem says that the automorphisms of $\mathbb{G}(1,N) \cap H^l$ are the automorphisms in $\mathrm{Aut}(\mathbb{G}(1,N), \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})) \cong \mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ such that their induced action on the dual space $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^\vee$ fixes $(H^l)^\vee$. If $\{e_1, \dots, e_{N+1}\}$ is a basis of $\mathbb{C}^{N+1}$ and $E_{ij} \in \mathrm{Mat}(N+1,\mathbb{C})$ the matrix with the entry (i,j) equal to 1 and otherwise equal to 0, then

$$\begin{aligned} \left(\textstyle\bigwedge^2 \mathbb{C}^{N+1}\right)^\vee &\longrightarrow \mathrm{Antisym}(N+1,\mathbb{C}) \\ \textstyle\sum_{i,j} \lambda_{i,j} (e_i \wedge e_j)^\vee &\longmapsto \tfrac{1}{2} \textstyle\sum_{i,j} \lambda_{i,j} (E_{i,j} - E_{j,i}) \end{aligned}$$

is an isomorphism of vector spaces; that allows us to identify $\mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})^\vee$ with $\mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C}))$. A point $[\mathbb{P}(\mathrm{span}\{p,q\})] \in \mathbb{G}(1,N)$ is contained in a hyperplane $H = \mathbb{P}(A)^\vee$, with $A \in \mathrm{Antisym}(N+1,\mathbb{C})$, if and only if, $pA^tq = 0$. The action of an automorphism $\mathbb{P}(T) \in \mathbb{P}\mathrm{GL}(N+1,\mathbb{C})$ on $\mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C}))$ is given by

$$\begin{aligned} \mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C})) &\longrightarrow \mathbb{P}(\mathrm{Antisym}(N+1,\mathbb{C})) \\ \mathbb{P}(A) &\longmapsto \mathbb{P}({}^tT^{-1} A T^{-1}). \end{aligned}$$

Therefore, a linear subspace $H^l \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{N+1})$ of codimension l, dually given by $\mathbb{P}(\mathrm{span}\{A_1, \dots, A_l\})^\vee$, is preserved under T if and only if,

$$ {}^tT^{-1} A_i T^{-1} \in \mathrm{span}\{A_1, \dots, A_l\}, \quad \text{for all } i = 1, \dots, l. \tag{3.1}$$

The second step in the task of determining the automorphism groups is done separately for the different cases using the above description.

Consider the case $l = 1$. The automorphism group of $Y = \mathbb{G}(1,2n-1) \cap H$, where $H = \mathbb{P}(A)^\vee$ is a general hyperplane of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$, is isomorphic to the group $\mathrm{Sp}(2n,\mathbb{C})/\{\pm I\}$, where $\mathrm{Sp}(2n,\mathbb{C})$ denotes the symplectic group associated to A. Its action on Y is homogeneous (see [PVdV99, Prop. 2.1]). The automorphism group of $X = \mathbb{G}(1,2n) \cap H$, where $H = \mathbb{P}(A)^\vee$ is a general hyperplane of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$, act on X with two orbits (see [PVdV99, Prop. 5.3] for details).

3.2.1 The Case $\mathbb{G}(1,2n) \cap H^2$

Now let $L = H^2 = H_1 \cap H_2 \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$ be a linear subspace of codimension 2 given by the intersection of two distinct hyperplanes $H_1 = \mathbb{P}(A)^\vee, H_2 = \mathbb{P}(B)^\vee$, with $A, B \in \mathrm{Antisym}(2n+1,\mathbb{C})$. As before, L is dual to the line $L^\vee = \mathbb{P}(\lambda A - \mu B) \subset \mathbb{P}(\mathrm{Antisym}(2n+1,\mathbb{C}))$. It can be shown that the dual Grassmannian of $\mathbb{G}(1,2n)$ is given by

$$\mathbb{G}(1,2n)^\vee = \{\mathbb{P}(C) \in \mathbb{P}(\mathrm{Antisym}(2n+1,\mathbb{C})) \mid \mathrm{rank}(C) \leq 2n-2\}$$

and that it is a subvariety of codimension 3 in $\mathbb{P}(\mathrm{Antisym}(2n+1,\mathbb{C}))$ (see [PVdV99, Corollary 1.7]). We will say that $L = H^2$ is *general* if $L^\vee$ does not intersect $\mathbb{G}(1,2n)^\vee$. Since antisymmetric matrices have even rank, the ones $\lambda A - \mu B$ corresponding to the hyperplanes $H_{(\lambda:\mu)} = \mathbb{P}(\lambda A - \mu B)^\vee$ have all corank 1. The corresponding point $c(\lambda : \mu) \in \mathbb{P}^{2n}$ of the 1-dimensional kernel of $\lambda A - \mu B$ is called *center* of $H_{(\lambda:\mu)}$. The map

$$\begin{aligned} c: \quad \mathbb{P}^1 &\longrightarrow \mathbb{P}^{2n} \\ (\lambda : \mu) &\longmapsto c(\lambda : \mu) = \mathbb{P}(\ker(\lambda A - \mu B)) \end{aligned}$$

is a parametrization of a rational normal curve C of degree n, called *center curve*. Denoting by the same symbol $c(\lambda:\mu)$ a generator of the kernel of $\lambda A-\mu B$, we have $\lambda c(\lambda:\mu)A=\mu c(\lambda:\mu)B$. Then we have a well defined map

$$\begin{aligned} h: \quad \mathbb{P}^1 &\longrightarrow (\mathbb{P}^{2n})^\vee \\ (\lambda:\mu) &\longmapsto h(\lambda:\mu)=\mathbb{P}\left(\ker\begin{pmatrix} c(\lambda:\mu)A \\ c(\lambda:\mu)B \end{pmatrix}\right), \end{aligned}$$

which is a parametrization of a rational normal curve E of degree $n-1$ in the space of hyperplanes containing the center curve.

Any automorphism $\mathbb{P}(T)\in \mathrm{Aut}(\mathbb{G}(1,2n)\cap H^2)\subset \mathbb{P}\mathrm{GL}(2n+1,\mathbb{C})$ maps the center curve onto itself and also the projective space $P\cong\mathbb{P}^n$ spanned by the center curve onto itself. Hence $\mathbb{P}(T)$ induces an automorphism on the rational normal curve E in the dual projective space $(\mathbb{P}^{2n}/P)^\vee$. The group of automorphisms of $\mathbb{P}^n$ fixing a rational normal curve of degree n is isomorphic to $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ (see, for example, [Har92, Example 10.12]). In other words, we know how to describe $\mathbb{P}(T)$ when restricted to C and E. With such a description and using (3.1) Piontkowski and Van de Ven obtain:

Theorem 3.2.3 ([PVdV99, Thm. 6.6])**.** The automorphism group of the intersection of $\mathbb{G}(1,2n)$ with a general linear subspace of codimension 2 of $\mathbb{P}(\bigwedge^2\mathbb{C}^{2n+1})$ is isomorphic to the subgroup of $\mathbb{P}\mathrm{GL}(2n+1,\mathbb{C})$ that consists of the elements

$$\begin{pmatrix} \alpha I_n & 0 \\ S & I_{n+1} \end{pmatrix}\cdot\begin{pmatrix} {}^t t_n^{-1} & 0 \\ 0 & t_{n+1} \end{pmatrix},$$

where $\alpha\in\mathbb{C}^*$, $S\in \mathrm{Mat}((n+1)\times n,\mathbb{C})$ with $s_{ij}=s_{kl}$ for $i+j=k+l$, $t_n\in\mathrm{Aut}(H,\mathbb{P}^{n-1})$ and $t_{n+1}\in\mathrm{Aut}(C,\mathbb{P}^n)$.

The Orbits of $\mathrm{Aut}(\mathbb{G}(1,4)\cap H^2)$**.** For the particular case $X=\mathbb{G}(1,4)\cap H^2$, Piontkowski and Van de Ven proved that $\mathrm{Aut}(X)$ acts on X with finitely many orbits and they described such orbits with very geometrical conditions. First they showed that, up to changing coordinates, the antisymmetric matrices $A,B\in\mathrm{Antisym}(5,\mathbb{C})$ corresponding to hyperplanes $H_1=\mathbb{P}(A)^\vee$ and $H_2=\mathbb{P}(B)^\vee$ defining H^2 can be written as

$$A=\begin{pmatrix} 0&0&-1&0&0\\ 0&0&0&-1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix} 0&0&0&-1&0\\ 0&0&0&0&-1\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix} \tag{3.2}$$

(see [PVdV99, Prop. 6.4]). In this basis, computations are clearer. Note that, in this basis the center curve C is given by $c(\lambda:\mu)=(0:0:\mu^2:\mu\lambda:\lambda^2)$.

Proposition 3.2.1 ([PVdV99, Prop. 6.8])**.** The action of $\mathrm{Aut}(\mathbb{G}(1,4)\cap H^2)$ on $X=\mathbb{G}(1,4)\cap H^2$ has four orbits:

- $o_1=\{x=[l]\in X\mid l \text{ is tangent to the center conic } C\}$;
- $o_2=\{x=[l]\in X\mid l \text{ is secant to the center conic } C\}$;
- $o_3=\{x=[l]\in X\mid l \text{ intersects the center conic but does not lie in } P\}$, where P denotes the plane spanned by C;

- $o_4 = \{x = [l] \in X \mid l$ does not intersect the plane $P\}$.

There do not exist lines in X that intersect the plane P but not the conic C. Also, lines in $\mathbb{P}^4$ contained in P are contained in X.

3.2.2 The Case $\mathbb{G}(1, 2n-1) \cap H^2$.

We begin this subsection by stating the theorem that describes the automorphism group of the intersection of the Grassmannian $\mathbb{G}(1, 2n-1)$ with a general linear subspace of codimension 2 of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$. Using such a description we are able to compute the orbits of the action of the automorphism group on that linear section of the Grassmannian.

Theorem 3.2.4 ([PVdV99, Thm. 3.5])**.** For $n \geq 3$ the automorphism group of the intersection of $\mathbb{G}(1, 2n-1)$ with a general linear subspace of codimension 2 of $\mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ is isomorphic to the subgroup of $\mathbb{P}\mathrm{GL}(2n, \mathbb{C})$ that consists of elements

$$P_\sigma \cdot \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$$

where $t_1, \ldots, t_n \in \mathrm{SL}(2, \mathbb{C})$ and P_σ is the identity for $n \geq 5$ and otherwise defined by $P_\sigma(e_{2i}) = e_{2\sigma(i)}, P_\sigma(e_{2i-1}) = e_{2\sigma(i)-1}$,

$$\text{for } \sigma \in \begin{cases} S(n), & \text{if } n = 3; \\ \{(1\ 2\ 3\ 4), (2\ 1\ 4\ 3), (3\ 4\ 1\ 2), (4\ 3\ 2\ 1)\}, & \text{if } n = 4. \end{cases}$$

Let us explain the generality condition on H^2 assumed in the theorem. A linear subspace $L = H^2 = H_1 \cap H_2 \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ of codimension 2, given by the intersection of two distinct hyperplanes $H_1 = \mathbb{P}(A)^\vee, H_2 = \mathbb{P}(B)^\vee$, with $A, B \in \mathrm{Antisym}(2n, \mathbb{C})$, is dual to the line $L^\vee = \mathbb{P}(\lambda A - \mu B) \subset \mathbb{P}(\mathrm{Antisym}(2n, \mathbb{C}))$. It can be shown that the dual Grassmannian of $\mathbb{G}(1, 2n-1)$ is given by

$$\mathbb{G}(1, 2n-1)^\vee = \{\mathbb{P}(C) \in \mathbb{P}(\mathrm{Antisym}(2n, \mathbb{C})) \mid \mathrm{rank}(C) \leq 2n-2\}$$

and it is an irreducible hypersurface of degree n (see [PVdV99, Corollary 1.7]). Therefore the line $L^\vee$ intersects $\mathbb{G}(1, 2n-1)^\vee$ in at most n distinct points. We will say that $L = H^2$ is *general* if $L^\vee$ and $\mathbb{G}(1, 2n-1)^\vee$ have n distinct points in common, which we denote by $\mathbb{P}(\lambda_i A - \mu_i B), i = 1, \ldots, n$. The corresponding hyperplanes $H_i = \mathbb{P}(\lambda_i A - \mu_i B)^\vee$, are tangent to the Grassmannian $\mathbb{G}(1, 2n-1)$ at the points $[l_i]$, where $l_i = \mathbb{P}(\ker(\lambda_i A - \mu_i B)) \subset \mathbb{P}^{2n-1}$ are called *exceptional lines.*

The Orbits of $\mathrm{Aut}(\mathbb{G}(1, 5) \cap H^2)$. J. Piontkowski and A. Van de Ven in [PVdV99] do not compute the orbits of the action of the automorphism group of $X = \mathbb{G}(1, 5) \cap H^2$. But, following similar argument as in the case of the linear section $\mathbb{G}(1, 4) \cap H^2$ we can compute the orbits of $\mathrm{Aut}(X)$.

For $j = 1, 2, 3$, let V_j be the 3-plane in $\mathbb{P}^5$ generated by the exceptional lines $l_i's$ with

$i \neq j$. Let $V = \cup_{i=1}^3 V_i$. Up to changing coordinates we can write

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.3}$$

(see [Don77] or [PVdV99, Proposition 3.2]). The exceptional lines are $l_1 = \mathbb{P}(\text{span}\{e_1, e_2\})$, $l_2 = \mathbb{P}(\text{span}\{e_3, e_4\})$ and $l_3 = \mathbb{P}(\text{span}\{e_5, e_6\})$.

Lemma 3.2.1. The automorphism group $\text{Aut}(\mathbb{G}(1,5) \cap H^2) \subset \mathbb{P}\text{GL}(6, \mathbb{C})$ acts transitively on $\mathbb{P}^5 \setminus V$.

Proof. It is sufficient to prove the existence of an automorphism in $\text{Aut}(\mathbb{G}(1,5) \cap H^2)$ mapping $p = (1 : 0 : 1 : 0 : 1 : 0)$ to a given point $q = (q_1 : q_2 : q_3 : q_4 : q_5 : q_6) \in \mathbb{P}^5 \setminus V$. Any block diagonal matrix $T = \text{diag}(t_1, t_2, t_3) \in \text{Mat}(6, \mathbb{C})$ with

$$t_1 = \begin{pmatrix} q_1 & a_{12} \\ q_2 & a_{22} \end{pmatrix}, t_2 = \begin{pmatrix} q_3 & a_{34} \\ q_4 & a_{44} \end{pmatrix}, t_3 = \begin{pmatrix} q_5 & a_{56} \\ q_6 & a_{66} \end{pmatrix} \in \text{Mat}(2, \mathbb{C})$$

defines a projective transformation $\mathbb{P}(T)$ of $\mathbb{P}^5$ such that $\mathbb{P}(T)(p) = q$. Since $q \notin V$, that is, $(q_1, q_2), (q_3, q_4), (q_5, q_6) \neq (0, 0)$, we can choose the $a_{ij}'s$ satisfying $\det(t_1) = \det(t_2) = \det(t_3) = 1$. Then we have $\mathbb{P}(T) \in \text{Aut}(\mathbb{G}(1,5) \cap H^2)$. $\square$

Lemma 3.2.2. The automorphism group $\text{Aut}(\mathbb{G}(1,5) \cap H^2) \subset \mathbb{P}\text{GL}(6, \mathbb{C})$ acts transitively on $V \setminus \cup_{i=1}^3 l_i$.

Proof. Let $p, q \in V \setminus \cup_{i=1}^3 l_i$. By a permutation P_σ on the exceptional lines we can suppose $p = (0 : 0 : p_3 : p_4 : p_5 : p_6), q = (0 : 0 : q_3 : q_4 : q_5 : q_6) \in V_1 \setminus \cup_{i=1}^3 l_i$. Let $T = \text{diag}(t_1, t_2, t_3) \in \text{Mat}(6, \mathbb{C})$ be a block diagonal matrix with

$$t_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, t_2 = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}, t_3 = \begin{pmatrix} a_{55} & a_{56} \\ a_{65} & a_{66} \end{pmatrix} \in \text{Mat}(2, \mathbb{C}).$$

The induced projective transformation $\mathbb{P}(T)$ satisfies $\mathbb{P}(T)(p) = q$ if

$$(p_3 a_{33} + p_4 a_{34}) e_3 + (p_3 a_{43} + p_4 a_{44}) e_4 + (p_5 a_{55} + p_6 a_{56}) e_5 + (p_5 a_{65} + p_6 a_{66}) e_6 = \textstyle\sum_{i=3}^6 q_i e_i.$$

Since $p, q \notin \cup_{i=1}^3 l_i$ we can choose the $a_{ij}'s$ satisfying the condition above and the three additional ones $\det(t_1) = \det(t_2) = \det(t_3) = 1$. $\square$

Lemma 3.2.3. Let $x = [l] \in \mathbb{G}(1,5) \cap H^2$. If the line l intersects one V_i, then it intersects the other two.

Proof. By a permutation P_σ on the exceptional lines we can suppose that $l \cap V_1 \neq \emptyset$. Then $l = \mathbb{P}(\text{span}\{p, q\})$, where $p = (0, 0, p_3, p_4, p_5, p_6)$ and $q = (q_1, q_2, q_3, q_4, q_5, q_6)$, with $p_5 q_6 - p_6 q_5 = 0$ and $p_3 q_4 - p_4 q_3 = 0$. Denote by M_{ij}, $1 \leq i < j \leq 6$, the (i, j)-minor of the matrix

$$\begin{pmatrix} 0 & 0 & p_3 & p_4 & p_5 & p_6 \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix}.$$

Note that $M_{34} = M_{56} = 0$. We have

$$\begin{aligned} q_3p - p_3q &= (-p_3q_1, -p_3q_2, q_3p_4 - p_4q_3, q_3p_4 - p_3q_4, q_3p_6 - p_3q_5, q_3p_6 - p_3q_6) \\ &= (M_{13}, M_{23}, 0, 0, -M_{35}, -M_{36}) \\ q_4p - p_4q &= (-p_4q_1, -p_4q_2, q_4p_3 - p_4q_3, q_4p_4 - p_4q_4, q_4p_5 - p_4q_5, q_4p_6 - p_4q_6) \\ &= (M_{14}, M_{24}, 0, 0, -M_{45}, -M_{46}) \\ q_5p - p_5q &= (-p_5q_1, -p_5q_2, q_5p_3 - p_5q_3, q_5p_4 - p_5q_4, q_5p_5 - p_5q_5, q_5p_6 - p_5q_6) \\ &= (M_{15}, M_{25}, M_{36}, M_{46}, 0, 0) \\ q_6p - p_6q &= (-p_6q_1, -p_6q_2, q_6p_3 - p_6q_3, q_6p_4 - p_6q_4, q_6p_5 - p_6q_5, q_6p_6 - p_6q_6) \\ &= (M_{16}, M_{26}, M_{35}, M_{45}, 0, 0). \end{aligned}$$

Since p and q are linearly independent, it is clear that l intersects V_2 and V_3. □

Proposition 3.2.2. The action of $\mathrm{Aut}(\mathbb{G}(1,5) \cap H^2)$ on $X = \mathbb{G}(1,5) \cap H^2$ has four orbits:

- $o_1 = \{x = [l] \in X \mid l \text{ intersects two exceptional lines}\}$;
- $o_2 = \{x = [l] \in X \mid l \text{ intersects only one exceptional line}\}$;
- $o_3 = \{x = [l] \in X \mid l \text{ intersects } V \setminus \cup_{i=1}^3 l_i\}$;
- $o_4 = \{x = [l] \in X \mid l \text{ does not intersect } V\}$.

Note that none of the exceptional lines lies in X and that no line in $\mathbb{P}^5$ intersects the three exceptional lines. A line in $\mathbb{P}^5$ intersecting two excepcional lines is contained in X.

Proof. Since any automorphism permutes the exceptional lines, it is clear by the geometric description that the four kinds of lines described lie in different orbits.

Let $x = [l], x' = [l'] \in o_1$. By a permutation P_σ on the exceptional lines we reduce to particular cases. The first one is when there is only one exceptional line intersecting both l and l', say $l = \mathbb{P}(\mathrm{span}\{re_1 + se_2, te_3 + ue_4\})$ and $l' = \mathbb{P}(\mathrm{span}\{r'e_1 + s'e_2, t'e_5 + u'e_6\})$. Let $T = \mathrm{diag}(t_1, t_2, t_3) \in \mathrm{Mat}(6, \mathbb{C})$ be a block diagonal matrix with

$$t_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, t_2 = \begin{pmatrix} a_{35} & a_{36} \\ a_{45} & a_{46} \end{pmatrix}, t_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{C}).$$

The induced projective transformation $\mathbb{P}(P_{(2\ 3)} \cdot T)$ satisfies $\mathbb{P}(P_{(2\ 3)} \cdot T)(re_1 + se_2) = r'e_1 + s'e_2$ and $\mathbb{P}(P_{(2\ 3)} \cdot T)(te_3 + ue_4) = t'e_5 + u'e_6$ if

$$\begin{aligned} (ra_{11} + sa_{12})e_1 + (ra_{21} + sa_{22})e_2 &= r'e_1 + s'e_2 \\ (ta_{53} + ua_{54})e_5 + (ta_{63} + ua_{64})e_6 &= t'e_5 + u'e_6. \end{aligned}$$

Since $(r, s), (t, u), (r', s'), (t', u') \neq (0, 0)$ we can choose the $a_{ij}'s$ satisfying the conditions above and the two additional ones $\det(t_1) = \det(t_2) = 1$. With these values of $a_{ij}'s$, $\mathbb{P}(P_{(2\ 3)} \cdot T)(x) = x'$. The second case, when two exceptional lines intersect both l and l', can be treated similarly.

Similarly to the previous case, it is shown that o_2 is an orbit.

Let $x = [l], x' = [l'] \in o_3$. Since l and l' do not intersect any of the exceptional lines l_i, by Lemma 3.2.3 and its proof we can suppose $l = \mathbb{P}(\text{span}\{p = (0, 0, p_3, p_4, p_5, p_6), q = (q_1, q_2, 0, 0, q_5, q_6)\})$ with $p_5 q_6 - p_6 q_5 = 0$ and $l' = \mathbb{P}(\text{span}\{p' = (0, 0, p'_3, p'_4, p'_5, p'_6), q' = (q'_1, q'_2, 0, 0, q'_5, q'_6)\})$ with $p'_5 q'_6 - p'_6 q'_5 = 0$. By Lemma 3.2.2 the group $\text{Aut}(X)$ acts transitively on $\mathbb{P}^5 \setminus V$, so we can suppose $p = p' = (0, 0, 1, 0, 1, 0)$, $q_6 = q'_6 = 0$ and $q_5 = q'_5 = 1$. Hence $l = \mathbb{P}(\text{span}\{p = (0, 0, 1, 0, 1, 0), q = (q_1, q_2, 0, 0, 1, 0)\})$ and $l' = \mathbb{P}(\text{span}\{p' = (0, 0, 1, 0, 1, 0), q' = (q'_1, q'_2, 0, 0, 1, 0)\})$ with $(q_1 : q_2), (q'_1 : q'_2) \in \mathbb{P}^1$. Let $T = \text{diag}(t_1, t_2, t_3) \in \text{Mat}(6, \mathbb{C})$ be a block diagonal matrix with

$$t_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, t_2 = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}, t_3 = \begin{pmatrix} a_{55} & a_{56} \\ a_{65} & a_{66} \end{pmatrix} \in \text{Mat}(2, \mathbb{C}).$$

The induced projective transformation $\mathbb{P}(T)$ satisfies $\mathbb{P}(T)(p) = p$ and $\mathbb{P}(T)(q) = q'$ if

$$\begin{aligned} a_{33}e_3 + a_{43}e_4 + a_{55}e_5 + a_{65}e_6 &= e_3 + e_5 \\ (a_{11}q_1 + a_{12}q_2)e_1 + (a_{21}q_1 + a_{22}q_2)e_2 + a_{55}e_5 + a_{65}e_6 &= q'_1 e_1 + q'_2 e_2 + e_5. \end{aligned}$$

Hence we put $a_{33} = 1, a_{43} = 0, a_{55} = 1, a_{65} = 0, a_{44} = a_{66} = 1$. Since $(q_1, q_2), (q'_1, q'_2) \neq (0, 0)$ we can choose the rest of the $a_{ij}'s$ satisfying the last equation above and the additional one $\det(t_1) = 1$. With these values of $a_{ij}'s$, $\mathbb{P}(T)(x) = x'$.

To show that o_4 is an orbit it is sufficient to find an automorphism in $\text{Aut}(X)$ mapping the line $l_0 = [\mathbb{P}(\text{span}\{(1, 0, 1, 0, 1, 0), (1, 1, 1, -2, 1, 1)\})]$ to a given $x = [l] \subset \mathbb{P}^5 \setminus V$. By Lemma 3.2.1 the group $\text{Aut}(X)$ acts transitively on $\mathbb{P}^5 \setminus V$, so we can suppose $l = \mathbb{P}(\text{span}\{(1, 0, 1, 0, 1, 0), (q_1, q_2, q_3, -2q_2, q_5, q_2)\})$. Note that $q_2 \neq 0$, otherwise $l \cap V_3 \neq \emptyset$. We can put $q_2 = 1$. The block diagonal matrix $T = \text{diag}(t_1, t_2, t_3) \in \text{Mat}(6, \mathbb{C})$ with

$$t_1 = \begin{pmatrix} 1 & q_1 - 1 \\ 0 & 1 \end{pmatrix}, t_2 = \begin{pmatrix} 1 & (1 - q_3)/2 \\ 0 & 1 \end{pmatrix}, t_3 = \begin{pmatrix} 1 & q_5 - 1 \\ 0 & 1 \end{pmatrix} \in \text{Mat}(2, \mathbb{C})$$

defines an automorphism $\mathbb{P}(T)$ of X such that $\mathbb{P}(T)(x_0) = x$. □

3.3 Varieties of Lines on Linear Sections of Grassmannians

Let $X = \mathbb{G}(1, N) \cap H^2$ be a general linear section of codimension 2 of $\mathbb{G}(1, N)$, with $N \in \{4, 5\}$. In the previous section we saw that the automorphism group $\text{Aut}(X)$ acts on X with finitely many orbits. Moreover, each of these orbits was described by geometrical conditions that allow us now to describe the variety Z_x of lines on X passing through a fixed point $x \in X$, for x in each of the orbits. Such descriptions for Z_x are given in Theorem 3.3.1 and Theorem 3.3.2 below. As an application of these theorems, in the next subsection we show that these linear sections of Grassmannians are not weakly 2-Fano, completing the classification of weakly 2-Fano manifolds of high index.

Notation. For shortness, we will denote by H_x the variety of lines on $\mathbb{G}(1, N)$ passing through a point $x \in \mathbb{G}(1, N)$ (see Example 3.1.2). We will denote by Z_x the variety of lines on the linear section $X = \mathbb{G}(1, N) \cap H^2$ passing through a point $x \in X$ (see Example 3.1.3).

To make computations easier, in the proof of the following theorem we will work with the normal form (3.2).

Theorem 3.3.1. Let $X = \mathbb{G}(1,4) \cap H^2$ be a general linear section of codimension 2 of $\mathbb{G}(1,4)$, and let $Z_x \subset \mathbb{P}^1 \times \mathbb{P}^2$ be the variety of lines on X passing through a point $x \in X$. Then Z_x has pure dimension 1, and its numerical class in $N_1(\mathbb{P}^1 \times \mathbb{P}^2)$ is $[Z_x] \equiv 2[L_1] + [L_2]$, where $[L_1]$ is the class of a line in a fiber of the first projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $[L_2]$ is the class of a fiber of the second projection. For x in each orbit of the action of $\mathrm{Aut}(X)$ on X we have the following description of Z_x:

- for $x \in o_1$, Z_x has two irreducible components, all rational. One of them has numerical class $[L_1]$ (and multiplicity 2), and the other one has numerical class $[L_2]$;
- for $x \in o_2$, Z_x has three irreducible components, all rational. Two of them has numerical class $[L_1]$, and the other one has numerical class $[L_2]$;
- for $x \in o_3$, Z_x has two irreducible components, all rational. One of them has numerical class $[L_1]$ and the other one has numerical class $[L_1] + [L_2]$;
- for $x \in o_4$, $Z_x \cong \mathbb{P}^1$.

Proof. We know from Example 3.1.3 that the subvariety Z_x is the intersection of two divisors D_1 and D_2 in $H_x \cong \mathbb{P}^1 \times \mathbb{P}^2$, both of type $(1,1)$. We will see in the next paragraphs that this is a complete intersection. This means that, if $[\alpha]$ and $[\beta]$ denote, respectively, the numerical classes of the pullbacks of the hyperplanes classes of $\mathbb{P}^1$ and $\mathbb{P}^2$ by the projections, then

$$[Z_x] \equiv ([\alpha] + [\beta])^2 \equiv 2[\alpha][\beta] + [\beta]^2 \equiv 2[L_1] + [L_2],$$

where $[L_1] \equiv [\alpha][\beta]$ is the class of a line in a fiber of the first projection and $[L_2] \equiv [\beta]^2$ the class of a fiber of the second projection.

Let us write down the equations defining the subvariety Z_x. If $x = [l]$, with $l = \mathbb{P}(\mathrm{span}\{p,q\})$, then any line $L_{U,V}$ on $\mathbb{G}(1,4)$ passing through x is determined by two vector subspaces U and V given by

$$U = \mathrm{span}\{rp + sq\} \subset \mathrm{span}\{p,q\} \subset V = \mathrm{span}\{p,q,v\},$$

where $(r:s) \in \mathbb{P}^1$ and $v \in \mathbb{P}(\mathbb{C}^5/\mathrm{span}\{p,q\})$. Such a line $L_{U,V}$ is on X if and only if, every point $x' = [l'] \in L_{U,V}$, with $l' = \mathbb{P}(\mathrm{span}\{rp+sq, r'p+s'q+t'v\})$, is contained in X, that is,

$$\begin{cases} (rp+sq)A^t(r'p+s'q+t'v) = 0 \\ (rp+sq)B^t(r'p+s'q+t'v) = 0 \end{cases} \sim \begin{cases} (rp+sq)A^t v = 0 \\ (rp+sq)B^t v = 0. \end{cases} \tag{3.4}$$

The equivalence holds because since $x \in X$, we have $pA^tq = pB^tq = 0$; and since A and B are antisymmetric matrices, we also have $uA^tu = uB^tu$ for any $u \in \mathbb{C}^5$. The subvariety Z_x is defined by the equations (3.4).

Let $x = [l] \in o_1$, with l a line tangent to the center conic C. We can work with a particular x. We choose $l = \mathbb{P}(\mathrm{span}\{p = (0,0,1,0,0), q = (0,0,0,1,0)\})$ (recall that $C \subset P \cong \mathbb{P}^4_{(0:0:x_3:x_4:x_5)}$ is given by $x_4^2 - x_3x_5 = 0$, so the tangent line to C at $c(0:1) = \mathbb{P}(p)$ is given by $x_5 = 0$). In this case, the system of equations (3.4) is

$$\begin{cases} rv_1 + sv_2 = 0 \\ sv_1 = 0. \end{cases}$$

The matrix associated to that system has only one nonzero minor, namely $M_{12} = -s^2$. Therefore, for $(r, s) = (1, 0)$ the subspace of solutions to that system is 4-dimensional (containing $\text{span}\{p, q\}$). Hence, there is an irreducible component of Z_x with numerical class $[L_1]$ (and multiplicity 2). Clearly, the 3-dimensional vector subspace $\text{span}\{e_3, e_4, e_5\}$ is solution for the system for every $(r, s) \in \mathbb{C}^2$. This means that there is an irreducible component of Z_x with numerical class $[L_2]$.

Let $x = [l] \in o_2$, with $l = \mathbb{P}(\text{span}\{p = (0, 0, 1, 0, 0), q = (0, 0, 0, 0, 1)\})$ a secant line to the center conic C through the points $c(0 : 1) = \mathbb{P}(p)$ and $c(1 : 0) = \mathbb{P}(q)$. The system of equations (3.4) is

$$\begin{cases} rv_1 = 0 \\ sv_2 = 0. \end{cases}$$

The only nonzero minor of that system $M_{12} = rs$ says that for $(r, s) = (1, 0)$ and $(r, s) = (0, 1)$ the solution space for that system is 4-dimensional (containing $\text{span}\{p, q\}$). Therefore, there are two irreducible components of Z_x with numerical class $[L_1]$. The 3-dimensional vector subspace $\text{span}\{e_3, e_4, e_5\}$ is the solution for the system that does not depend on $(r, s) \in \mathbb{C}^2$. Hence, there is an irreducible component of Z_x with numerical class $[L_2]$.

Let $x = [l] \in o_3$, with $l = \mathbb{P}(\text{span}\{p = (0, 0, 1, 0, 0), q = (q_i)\})$ a line intersecting the center conic C at $c(0 : 1) = \mathbb{P}(p)$, but not contained in the plane P spanned by C. The condition $x \in X$ implies $q_1 = 0$, and consequently $q_2 \neq 0$. We have the system of equations

$$\begin{cases} (r + sq_3)v_1 + sq_4v_2 - sq_2v_4 = 0 \\ \qquad sq_4v_1 + sq_5v_2 - sq_2v_5 = 0. \end{cases}$$

Looking at the second equation and the minor $M_{45} = s^2q_2$ we see that only for $(r, s) = (1, 0)$ the solution space for that system is 4-dimensional (containing $\text{span}\{p, q\}$). Therefore, there is an irreducible component of Z_x with numerical class $[L_1]$. Now, note that any vector $v = (v_i) \in \mathbb{C}^5$ that does not depend on (r, s) and is a solution for the system is contained in $\text{span}\{p, q\}$, which implies that the other irreducible component of Z_x must have numerical class $[L_1] + [L_2]$. Indeed, for $(r, s) = (1, 0)$ we get $v_1 = 0$ and for $(r, s) = (0, 1)$ we get $v_4 = q_4v_2/q_2$ and $v_5 = q_5v_2/q_2$. Hence, $v = (v_3 - v_2q_3/q_2)p + (v_2/q_2)q$.

Let $x \in o_4$. Since x is general, Z_x is smooth. By the Adjunction Formula (see [Har77, Proposition II.8.20]) the canonical divisor K_{Z_x} of Z_x is

$$\begin{aligned} K_{Z_x} &= (K_{\mathbb{P}^1 \times \mathbb{P}^2} + D_1 + D_2)|_{Z_x} \\ &= (-2[\alpha] - 3[\beta] + 2([\alpha] + [\beta])|_{Z_x} \\ &= -[\beta] \cdot (2[\alpha][\beta] + [\beta]^2) \\ &= -2[P], \end{aligned}$$

where $[P]$ denotes the numerical class of a point. In particular, $\deg(K_{Z_x}) = -2$. But, as a well known consequence of Riemann-Roch Theorem, $\deg K_{Z_x} = 2g - 2$, where g denotes the genus of Z_x. Therefore, $g = 0$, and $Z_x \cong \mathbb{P}^1$. □

To make computations easier, in the proof of the following theorem we will work with the normal form (3.3).

Theorem 3.3.2. Let $Y = \mathbb{G}(1,5) \cap H^2$ be a general linear section of codimension 2 of $\mathbb{G}(1,5)$, and let $Z_x \subset \mathbb{P}^1 \times \mathbb{P}^3$ be the variety of lines on Y passing through a point $x \in Y$. Then Z_x has pure dimension 2, and its numerical class in $N_2(\mathbb{P}^1 \times \mathbb{P}^3)$ is $[Z_x] \equiv 2[P] + [L]$, where $[P]$ is the class of a plane in a fiber of the first projection $\mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^1$ and $[L]$ the class of the inverse image under the second projection of a line in $\mathbb{P}^3$. For x in each orbit of the action of $\mathrm{Aut}(Y)$ on Y we have the following description of Z_x:

- for $x \in o_1$, Z_x has three irreducible components. Two of them have numerical class $[P]$, and the other one has numerical class $[L]$;
- for $x \in o_2$, Z_x has two irreducible components. One of them has numerical class $[P]$, and the other one has numerical class $[P] + [L]$;
- for $x \in o_3$, Z_x is isomorphic to the blowup of a quadric cone in $\mathbb{P}^3$ at the vertex, or equivalently, isomorphic to the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$;
- for $x \in o_4$, Z_x is isomorphic to a smooth quadric in $\mathbb{P}^3$.

Proof. We know from Example 3.1.3 that the subvariety Z_x is the intersection of two general divisors D_1 and D_2 in $H_x \cong \mathbb{P}^1 \times \mathbb{P}^3$, both of type $(1,1)$. We will see in the next paragraphs that this is a complete intersection. Therefore, if $[\alpha]$ and $[\beta]$ denote, respectively, the numerical classes of the pullbacks of the hyperplanes classes of $\mathbb{P}^1$ and $\mathbb{P}^3$ by the projections, then

$$[Z_x] \equiv ([\alpha] + [\beta])^2 \equiv 2[\alpha][\beta] + [\beta]^2 \equiv 2[P] + [L],$$

where $[P] \equiv [\alpha][\beta]$ is the class of a plane P in a fiber of the first projection and $[L] \equiv [\beta]^2$ the class of the inverse image under the second projection of a line in $\mathbb{P}^3$.

Let $x = [l]$, with $l = \mathbb{P}(\mathrm{span}\{p,q\})$. Any line $L_{U,V}$ on $\mathbb{G}(1,5)$ passing through x is determined by two vector subspaces U and V given by

$$U = \mathrm{span}\{rp + sq\} \subset \mathrm{span}\{p,q\} \subset V = \mathrm{span}\{p,q,v\},$$

where $(r:s) \in \mathbb{P}^1$ and $v \in \mathbb{P}(\mathbb{C}^6/\mathrm{span}\{p,q\})$. Such a line $L_{U,V}$ is on X if and only if, every point $x' = [l'] \in L_{U,V}$, with $l' = \mathbb{P}(\mathrm{span}\{rp + sq, r'p + s'q + t'v\})$ is contained in Y, that is,

$$\begin{cases} (rp+sq)A^t(r'p+s'q+t'v) = 0 \\ (rp+sq)B^t(r'p+s'q+t'v) = 0 \end{cases} \sim \begin{cases} (rp+sq)A^t v = 0 \\ (rp+sq)B^t v = 0. \end{cases} \tag{3.5}$$

These are the equations defining the subvariety Z_x.

Let $x = [l] \in o_1$, with $l = \mathbb{P}(\mathrm{span}\{p = (1,0,0,0,0,0), q = (0,0,1,0,0,0)\})$. Then, the system of equations (3.5) becomes

$$\begin{cases} -rv_2 - sv_4 = 0 \\ -rv_2 = 0. \end{cases}$$

The only nonzero minor of that system is $M_{24} = -rs$, which says that for $(r,s) = (1,0)$ and $(r,s) = (0,1)$ the solution subspace for that system is 5-dimensional (and contains $\mathrm{span}\{p,q\}$). Therefore, there are two irreducible components of Z_x with numerical class $[P]$. Clearly the 4-dimensional vector subspace $\mathrm{span}\{p,q,e_5,e_6\}$ is solution for the system and does not depend on $(r,s) \in \mathbb{C}^2$. Hence, there is an irreducible component of Z_x with numerical class $[L]$.

Let $x = [l] \in o_2$, with $l = \mathbb{P}(\operatorname{span}\{p = (1,0,0,0,0,0), q = (0,0,1,0,1,0)\})$. Now, the system of equations (3.5) is

$$\begin{cases} -rv_2 - sv_4 - sv_6 = 0 \\ -rv_2 + sv_6 = 0. \end{cases}$$

The nonzero minors of that system $M_{24} = -rs, M_{26} = -2rs$ and $M_{46} = -s^2$ say that $(r, s) = (1, 0)$ is the only value for which the solution subspace for that system is 5-dimensional. Therefore, there is a unique irreducible component of Z_x with numerical class $[P]$. Since any vector which is solution for the system and does not depend on (r, s) is in the 3-dimensional vector subspace $\operatorname{span}\{p, q, e_5\}$, we conclude that Z_x does not have irreducible components with numerical class $[L]$. Therefore, the other irreducible component of Z_x has numerical class $[P] + [L]$.

Let $x = [l] \in o_3$, with $l = \mathbb{P}(\operatorname{span}\{p = (0,0,1,0,1,0), q = (1,0,0,0,1,0)\})$. We have the system

$$\begin{cases} -sv_2 - rv_4 - (r+s)v_6 = 0 \\ -sv_2 + (r+s)v_6 = 0 \end{cases} \sim \begin{cases} -rv_4 - 2(r+s)v_6 = 0 \\ -sv_2 + (r+s)v_6 = 0. \end{cases}$$

Consider $\operatorname{span}\{p, q, e_1, e_2, e_4, e_6\}$ basis for $\mathbb{C}^6$. Any vector $v = (v_i) \in \mathbb{C}^6$ in that basis is written as $v = v_3p + (v_5 - v_3)q + (v_1 - v_5 + v_3)e_1 + v_2e_2 + v_4e_4 + v_6e_6$. Hence, the homogeneous coordinates of v in $\mathbb{P}(\mathbb{C}^6/\operatorname{span}\{p, q\}) \cong \mathbb{P}^3$ are $(v_1 - v_5 + v_3 : v_2 : v_4 : v_6) =: (t_i)$. With these homogeneous coordinates, Z_x is given by the equations

$$\begin{cases} -rt_2 - 2(r+s)t_3 = 0 \\ -st_1 + (r+s)t_3 = 0. \end{cases} \tag{3.6}$$

Using Jacobi's Criterion, we can see that Z_x is smooth. Denote by $\pi_2 : \mathbb{P}^1_{(r:s)} \times \mathbb{P}^3_{(t_i)} \to \mathbb{P}^3_{(t_i)}$ the second projection. A point $(t_i) \in \mathbb{P}^3$ is in the image of Z_x under π_2 if and only if there is a point $(r : s) \in \mathbb{P}^1$ such that $((r : s), (t_i))$ satisfies (3.6), or equivalently,

$$\det \begin{pmatrix} -t_2 - 2t_3 & -2t_3 \\ t_3 & t_3 - t_1 \end{pmatrix} = t_1t_2 + 2t_1t_3 - t_2t_3 = 0.$$

This is the equation of a quadric cone Q with vertex $o = (1 : 0 : 0 : 0)$. It is easy to see that the restriction of π_2 to $\pi_2^{-1}(Q \setminus \{o\})$ is an isomorphism onto $Q \setminus \{o\}$, and $\pi_2^{-1}(o) = \mathbb{P}^1 \times \{o\} \cong \mathbb{P}^1$. Therefore Z_x is isomorphic to the blowup of Q at the vertex o, or equivalently, isomorphic to the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ (see, for example, [Bea96, Ex. IV.18(1)]).

Let $x = [l] \in o_4$, with $\ell = \mathbb{P}(\operatorname{span}\{p = (1,0,1,0,1,0), q = (0,1,0,-2,0,1)\})$. We have the system

$$\begin{cases} -2s(v_3 - v_5) - r(v_4 + 2v_6) = 0 \\ s(v_1 - v_5) - r(v_2 - v_6) = 0. \end{cases}$$

Consider $\operatorname{span}\{p, q, e_1, e_2, e_3, e_4\}$ basis for $\mathbb{C}^6$. Any vector $v = (v_i) \in \mathbb{C}^6$ in that basis is written as $v = v_5p + v_6q + (v_1 - v_5)e_1 + (v_2 - v_6)e_2 + (v_3 - v_5)e_3 + (v_4 + 2v_6)e_4$. Hence, the homogeneous coordinates of v in $\mathbb{P}(\mathbb{C}^6/\operatorname{span}\{p, q\}) \cong \mathbb{P}^3$ are $(v_1 - v_5 : v_2 - v_6 : v_3 - v_5 :$

$v_4 + 2v_6) =: (t_i)$. With these homogeneous coordinates, Z_x is given by the equations

$$\begin{cases} -2st_2 - rt_3 = 0 \\ \quad st_0 - rt_1 = 0. \end{cases}$$

Using Jacobi's Criterion, we can see that Z_x is smooth. By Adjunction Formula and similar computations as in the proof of Theorem 3.3.1 we can find the anticanonical class $-K_{Z_x}$ of Z_x in $N_1(\mathbb{P}^1 \times \mathbb{P}^3)$; it is

$$-K_{Z_x} = 2 \cdot [\beta]|_{Z_x}.$$

Now, note that there is no curve in Z_x contracted by the second projection $\mathbb{P}^1_{(r:s)} \times \mathbb{P}^3_{(t_i)} \to \mathbb{P}^3_{(t_i)}$. This implies that $-K_{Z_x}$ is ample, with index $i_{Z_x} = 2$. By Kobayashi-Ochiai's Theorem (see Theorem 2.6.3), Z_x is isomorphic to a smooth quadric in $\mathbb{P}^3$. □

3.4 Application: Weakly 2-Fano Manifolds

As an application of the results from the previous sections, we complete the classification of weakly 2-Fano manifolds, initiated in [AC13]. Recall that a smooth, complex, projective variety X with second Chern character $\mathrm{ch}_2(X)$ is *weakly 2-Fano* if

$$\mathrm{ch}_2(X) \cdot [S] \geq 0,$$

for all surface $S \subset X$. Weakly 2-Fano manifolds were introduced by de Jong and Starr in [dS06c] and further studied by Araujo and Castravet in [AC12] and [AC13]. This notion is related to the problem of finding sections of fibrations over surfaces, and with the notion of rational simple connectedness introduced by de Jong and Starr. In [AC13], Araujo and Castravet gave an almost complete classification of weakly 2-Fano manifolds of dimension $n \geq 3$ and index at least $n-2$. The only cases left open were the general linear sections $\mathbb{G}(1,4) \cap H^2$ and $\mathbb{G}(1,5) \cap H^2$. Here we will prove that these manifolds are not weakly 2-Fano.

Let $X = \mathbb{G}(1,N) \cap H^2$ be a general linear section of codimension 2 of the Grassmannian $\mathbb{G}(1,N)$, with $N \in \{4,5\}$. By [AC13, Prop. 31] we have

$$\mathrm{ch}_2(X) = \binom{N-3}{2} \sigma_2|_X - \binom{N-3}{2} \sigma_{1,1}|_X, \tag{3.7}$$

where σ_2 and $\sigma_{1,1}$ are Schubert cycles, generators of the $(2N-4)$-th graded piece of the Chow ring of $\mathbb{G}(1,N)$. The strategy to prove that X is not weakly 2-Fano is to find a surface $S \subset X$ with class $\sigma^*_{1,1}$ in $\mathbb{G}(1,N)$. By Duality Theorem (see Subsection 2.7.2) we will have

$$\mathrm{ch}_2(X) \cdot [S] = \mathrm{ch}_2(X) \cdot \sigma^*_{1,1} = -\binom{N-3}{2} < 0, \quad \text{for } N = 4,5.$$

Corollary 3.4.1. The general linear sections of codimension 2 of Grassmannians $\mathbb{G}(1,4) \cap H^2$ and $\mathbb{G}(1,5) \cap H^2$ under the Plücker embedding are not weakly 2-Fano.

Proof. Let $X = \mathbb{G}(1,4) \cap H^2$. Take $x = [l] \in X$, with $l = \mathbb{P}(W)$, in the first or second orbit of the action of $\mathrm{Aut}(X)$ on X (see Proposition 3.2.1). By Theorem 3.3.1, the variety Z_x of lines passing through x and contained in X has an irreducible component C with numerical class $[L_2]$ in $\mathbb{P}(W) \times \mathbb{P}(\mathbb{C}^5/W)$. Geometrically, it means that there exists a 3-dimensional vector subspace $V \supset W$ such that for all 1-dimensional vector subspace $U \subset W$, the line $L_{U,V} = \{[\mathbb{P}(W')] \in \mathbb{G}(1,4) \mid U \subset W' \subset V\}$ is contained in X. Consider the universal family morphisms

$$\begin{array}{ccc} \mathcal{U}_x & \xrightarrow{e_x} & X \\ \Big\downarrow{\scriptstyle \pi_x} & & \\ Z_x & & \end{array} \tag{3.8}$$

We claim that the surface S defined by

$$S = e_x(\pi_x^{-1}(C)) = \bigcup_U L_{U,V} \subset X$$

has Schubert class $\sigma_{1,1}^* = \sigma_{2,2}$ in $\mathbb{G}(1,4)$. To see that, take a complete flag of $\mathbb{C}^5$ of the form

$$F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset F_3 := V \subset F_4 \subset F_5 = \mathbb{C}^5.$$

Note that $S = \{[\mathbb{P}(W')] \in \mathbb{G}(1,4) \mid W' \subset F_3 = V\}$. Indeed, if $[\mathbb{P}(W')] \neq [\mathbb{P}(W)]$ is such that $W' \subset V$, then $[\mathbb{P}(W')] \in L_{W \cap W',V} \subset Y$. The reverse inclusion of sets is trivial. Hence,

$$\begin{aligned} S &= \{[\mathbb{P}(W')] \in \mathbb{G}(1,4) \mid W' \subset F_3 = V\} \\ &= \{[\mathbb{P}(W')] \in \mathbb{G}(1,4) \mid \dim(W' \cap F_3) \geq 2\} \\ &= \{[\mathbb{P}(W')] \in \mathbb{G}(1,4) \mid \dim(W' \cap F_{(4-1)+i-2}) \geq i, \text{ for } i = 1,2\} \\ &= \Sigma_{2,2}. \end{aligned}$$

Thus S has Schubert class $\sigma_{2,2}$ and $\mathrm{ch}_2(X) \cdot [S] = -\frac{1}{2} < 0$. Therefore, $X = \mathbb{G}(1,4) \cap H^2$ is not weakly 2-Fano.

Now let $Y = \mathbb{G}(1,5) \cap H^2$. Take $x = [l] \in Y$, with $l = \mathbb{P}(W)$, in the first orbit of the action of $\mathrm{Aut}(Y)$ on Y (see Proposition 3.2.2). By Theorem 3.3.2, the variety Z_x of lines passing through x and contained in Y has an irreducible component with numerical class $[L]$ in $\mathbb{P}(W) \times \mathbb{P}(\mathbb{C}^6/W)$. Geometrically, it means that there exists a 4-dimensional vector subspace $V \supset W$ such that for all 1-dimensional vector subspace $U \subset W$, the line $L_{U,V} = \{[\mathbb{P}(W')] \in \mathbb{G}(1,5) \mid U \subset W' \subset V\}$ is contained in Y. In particular, there exists an irreducible curve $C \subset Z_x$ parameterizing lines $L_{U,V'}$ contained in Y, where V' is a fixed 3-dimensional vector subspace such that $W \subset V' \subset V$ and $U \subset W$ is any 1-dimensional vector subspace. Consider the universal family morphisms analogue to (3.8). The surface S defined by $S = e_x(\pi_x^{-1}(C)) \subset Y$ has Schubert class $\sigma_{1,1}^* = \sigma_{3,3}$ in $\mathbb{G}(1,5)$, and $\mathrm{ch}_2(X) \cdot [S] = -1 < 0$. Therefore, $Y = \mathbb{G}(1,5) \cap H^2$ is not weakly 2-Fano. □

Chapter 4

Conics on Varieties

In this chapter we investigate another special class of Fano manifolds, namely conic-connected manifolds. In Section 4.1 we define conic-connected manifolds and we present the Ionescu-Russo classification of conic-connected manifolds. In Section 4.2 we define the space $W_{x,y}$ of conics on a conic-connected manifold X passing through general points $x, y \in X$. We prove that this space is smooth and we provide several examples. Then, in Subsection 4.2.2 we define a natural polarization $\mathcal{M}_{x,y}$ on $W_{x,y}$. In Section 4.3 we relate this polarization with the polarization defined by de Jong and Starr for the space of minimal pointed rational curves. As a consequence, in Subsection 4.3.3 we give a formula for the canonical bundle of $W_{x,y}$ in terms of the second Chern character of X and the first Chern class of our polarization. We conclude that $W_{x,y}$ is Fano if X is weakly 2-Fano.

4.1 Conic-connected Varieties

As we saw in Chapter 2, many birational properties of a variety can be detected by studying rational curves on it. A birational class of varieties fairly studied are the rationally connected varieties, introduced by J. Kollár, Y. Miyaoka and S. Mori in [KMM92a]. Recall that a smooth, complex, projective variety X is rationally connected if any two general points of X are connected by a rational curve on X. A way to measure the complexity of such X is studying a family of rational curves of minimal degree on X connecting two general points. The first case, that of varieties containing a line joining two general points on them, is completely classified by Lemma 4.1.1.

First we introduce the following notation:

Notation. We will always work over the field $\mathbb{C}$ of complex numbers. Given a smooth complex projective variety $X \subset \mathbb{P}^N$, we will denote by $\mathcal{O}_X(1)$ the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ on X. By the degree of a curve on X we mean its $\mathcal{O}_X(1)$-degree.

Lemma 4.1.1. A projective variety $X \subset \mathbb{P}^N$ is a linear variety if and only if for any two general points $x, y \in X$ the line on $\mathbb{P}^N$ joining x and y is contained in X.

Proof. If X is a linear variety, that is, a projective space linearly embedded, then it is clear that for any two points of X the line joining these two points is contained in X. Conversely, assume that there exists an open dense subset $U \subset X$ such that for any two points of U the line in $\mathbb{P}^N$ joining these two points is contained in X. Take $x \in U$ a smooth point of X, and consider T_xX the Zariski tangent space of X at x embedded in $\mathbb{P}^N$. Every line passing through x and any other point $y \in U$ is contained in T_xX. In particular,

$U \subset T_xX$. Therefore $X = \overline{U} \subset T_xX$, and thus $X = T_xX$, since $\dim(X) = \dim(T_xX)$ and T_xX is irreducible. □

The next lemma generalizes the previous one, and characterizes projective spaces and quadric hypersurfaces as the only smooth projective varieties with low degree rational curves passing through too many general points.

Lemma 4.1.2 ([dS06b, Lemma 5.5])**.** Let $X \subset \mathbb{P}^N$ be a smooth projective variety and assume that $-K_X$ is an integral multiple of $c_1(\mathcal{O}_X(1))$. Let $n \geq 2$ be an integer and assume that through n general points of X there passes a rational curve of degree less than n. Then $(X, \mathcal{O}_X(1))$ is isomorphic to either $(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ or a quadric hypersurface $(Q, \mathcal{O}_{\mathbb{P}^{d+1}}(1)|_Q)$, where $d = \dim(X)$.

Proof. Given n general points $x_1, \dots, x_n \in X$, let C be a rational curve on X of degree less than n passing through $x_1, \dots, x_n$. By Lemma 2.4.3

$$h^1(C, N_{C/X}(-\textstyle\sum_{i=1}^n x_i)) = 0,$$

and therefore

$$\chi(C, N_{C/X}(-\textstyle\sum_{i=1}^n x_i)) \geq 0.$$

By Riemann-Roch's Theorem, we have

$$\begin{aligned} 0 \leq \chi(C, N_{C/X}(-\textstyle\sum_{i=1}^n x_i)) &= \chi(C, T_X|_C(-\textstyle\sum_{i=1}^n x_i)) - \chi(C, T_C(-\textstyle\sum_{i=1}^n x_i)) \\ &= -K_X \cdot C + (1-n)\dim(X) - (2-n+1) \\ &= -K_X \cdot C - 2 - (n-1)(\dim(X) - 1), \end{aligned}$$

that is,

$$-K_X \cdot C \geq 2 + (n-1)(\dim(X) - 1).$$

Denote by i_X the index of X. Since $-K_X$ is an integral multiple of $c_1(\mathcal{O}_X(1))$, we have $-K_X = i_X\, c_1(\mathcal{O}_X(1))$. Intersecting with the curve C, using the inequality obtained above and the hypothesis that $c_1(\mathcal{O}_X(1)) \cdot C \leq n-1$, we conclude that

$$i_X \geq \frac{2}{n-1} + \dim(X) - 1.$$

Therefore, by Kobayashi-Ochiai's Theorem (see Theorem 2.6.3), if $n = 2$ then $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$, and if $n \geq 3$, then $(X, \mathcal{O}_X(1))$ is isomorphic either to $(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ or to a quadric hypersurface $(Q, \mathcal{O}_{\mathbb{P}^{d+1}}(1)|_Q)$. □

The next case is that of conic-connected varieties, defined below.

Definition 4.1.1. A projective variety $X \subset \mathbb{P}^N$ is called *conic-connected* if for any two general points $x, y \in X$ there exists an irreducible conic contained on X passing through $x, y \in X$.

Conic-connected varieties have been studied by many authors; for example, in [IR10] and [MMT11]. In [IR10] P. Ionescu and F. Russo give the following classification of conic-connected manifolds:

Theorem 4.1.1 ([IR10, Thm. 2.2])**.** Let $X \subset \mathbb{P}^N$ be an n-dimensional conic-connected manifold. Assume that X is linearly normal and non-degenerate. Then either X is a Fano manifold with Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$ and index $i_X \geq \frac{n+1}{2}$, or it is isomorphic to one of the following:

(i) $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$, the image of $\mathbb{P}^n$ by the Veronese embedding of degree 2;

(ii) the projection of $\nu_2(\mathbb{P}^n)$ from the linear space $\langle \nu_2(\mathbb{P}^s) \rangle$, where $\mathbb{P}^s \subset \mathbb{P}^n$ is a linear subspace with $0 \leq s \leq n-2$; equivalently, $X \cong \mathrm{Bl}_{\mathbb{P}^s}(\mathbb{P}^n)$ embedded in $\mathbb{P}^N$, with $N = \frac{n(n+3)}{2} - \binom{s+2}{2}$, by the linear system of quadric hypersurfaces of $\mathbb{P}^n$ passing through $\mathbb{P}^s$; alternatively, $X \cong \mathbb{P}_{\mathbb{P}^r}(\mathscr{E})$ with $\mathscr{E} \cong \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus n-r} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$ and $1 \leq r \leq n-1$, embedded in $\mathbb{P}^N$ by $|\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)|$.

(iii) $\mathbb{P}^r \times \mathbb{P}^s \subset \mathbb{P}^{rs+r+s}$ Segre embedded, where $r, s \geq 1$ are integers such that $r + s = n$;

(iv) a hyperplane section of $\mathbb{P}^r \times \mathbb{P}^s \subset \mathbb{P}^{rs+r+s}$ Segre embedded, where $n \geq 3$ and $r, s \geq 2$ are integers such that $r + s = n + 1$.

4.2 Space of Conics Through Two Points

In this section we introduce tools for our study of conic-connected manifolds. We begin by defining the space of conics through two general points of a conic-connected manifold.

Let $X \subset \mathbb{P}^N$ be a conic-connected manifold. Typically, the space of conics on X passing through two general points $x, y \in X$ is not compact. We compactify it with the Kontsevich moduli space of stable maps. Let β be a class of a conic on X passing through x, y. Recall from Section 2.5 that the Kontsevich moduli space $\overline{M}_{0,2}(X, \beta)$ parameterizes data $[C, p, q, f]$, where

(i) C is a projective, connected, reduced, at worst nodal curve C of arithmetic genus 0,

(ii) p and q are distinct, non-singular, marked points on C,

(iii) $f : C \to X$ is a morphism such that: (1) $f_*[C] = \beta$; (2) if E is an irreducible component of C contracted by f then E contains at least one special point.

There exists an evaluation morphism

$$\begin{aligned} \mathrm{ev} : \overline{M}_{0,2}(X, \beta) &\longrightarrow X \times X \\ [C, p, q, f] &\longmapsto (f(p), f(q)). \end{aligned}$$

The fiber of ev over the general point $(x, y) \in X \times X$ parameterizes conics on X (with class β) passing through the two general points $x, y \in X$.

Proposition 4.2.1. Let $X \subset \mathbb{P}^N$ be a conic-connected manifold, and let β be a class of a conic passing through general points $x, y \in X$. Let $W_{x,y}$ be the fiber over $(x, y) \in X \times X$ of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,2}(X, \beta) \longrightarrow X \times X.$$

Assume that X is not a linear variety. Then every point in $W_{x,y}$ parameterizes an automorphism-free stable map. Moreover, $W_{x,y}$ is smooth of expected dimension

$$-K_X \cdot \beta - \dim(X) - 1$$

and it intersects the boundary Δ in a simple normal crossing divisor.

Proof. Let $[C,p,q,f] \in W_{x,y}$. Since $X \subset \mathbb{P}^N$ is not a linear variety, by Lemma 4.1.1, the line in $\mathbb{P}^N$ joining $x, y \in X$ is not contained in X. Hence, every irreducible component of C has degree 1 over its image. Therefore, $[C,p,q,f]$ is automorphism-free.

For the second part, note that, by Lemma 2.4.3 the restrictions of f to the irreducible components of C are free. This is because $f : C \to X$ has fixed class β and passes through general points $x, y \in X$. By Lemma 2.5.1 the result follows. □

4.2.1 Examples

Here we compute several examples of spaces of conics on a conic-connected variety passing through two general points on it.

Let $X \subset \mathbb{P}^N$ be a conic-connected manifold, and let β be a class of a conic passing through general points $x, y \in X$. Let $W_{x,y}$ be the fiber over $(x,y) \in X \times X$ of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,2}(X,\beta) \longrightarrow X \times X.$$

There exists a natural evaluation morphism

$$\begin{aligned} W_{x,y} &\longrightarrow \mathrm{Hilb}(X) \\ [C,p,q,f] &\longmapsto [f(C)]. \end{aligned}$$

The image of this morphism, denoted by $\mathrm{Hilb}_{2t+1}(X; x, y)$, is also a parameter space for conics on X passing through $x, y \in X$. The spaces of conics $W_{x,y}$ and $\mathrm{Hilb}_{2t+1}(X; x, y)$ can be different. This happens when there exists a stable map in $W_{x,y}$ with automorphisms. For example, for $X = \mathbb{P}^N$, there exist double coverings of the line in $\mathbb{P}^N$ joining the points x, y. By Proposition 4.2.1 this is the only case when $W_{x,y}$ parameterizes some stable map with automorphisms.

Example 4.2.1. Consider the projective space $\mathbb{P}^N$ with $N \geq 2$. Let us describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(\mathbb{P}^N; p, q)$ of conics in $\mathbb{P}^N$ passing through two fixed points $p, q \in \mathbb{P}^N$. Since $\mathbb{P}^N$ is a bihomogenous variety (that is, given two pairs of points in $\mathbb{P}^N$, there exists an automorphism of $\mathbb{P}^N$ mapping one pair to the other one), we can assume that the fixed points are $p = (1 : 0 : \cdots : 0)$ and $q = (0 : 1 : 0 : \cdots : 0)$. Let $\Sigma_{N-2,N-2} \subset \mathbb{G}(2,N)$ be the Schubert variety of planes in $\mathbb{P}^N$ containing these points. Note that $\Sigma_{N-2,N-2} \cong \mathbb{P}^{N-2}$. Indeed, every plane in $\mathbb{P}^N$ parameterized by $\Sigma_{N-2,N-2}$ has a unique representative W of the matricial form

$$W = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & a_N \end{pmatrix}, \quad \text{with } (a_2 : \cdots : a_N) \in \mathbb{P}^{N-2}.$$

For such planes we will always take the representatives of this form. The plane in $\mathbb{P}^N$ corresponding to $[W] \in \Sigma_{N-2,N-2}$ will be denoted by

$$\Pi_{(a_2:\cdots:a_N)} := \left\{ (x : y : za_2 : \cdots : za_N) \in \mathbb{P}^N \mid (x : y : z) \in \mathbb{P}^2 \right\}.$$

A conic $C \subset \Pi_{(a_2:\cdots:a_N)}$, given by the zeroes of a homogeneous polynomial of degree 2

$$c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2 \in \mathrm{Sym}^2(W^\vee),$$

passes through $p = (1 : 0 : \cdots : 0)$ and $q = (0 : 1 : 0 : \cdots : 0)$ if and only if $c_0 =$

$c_3 = 0$. Hence, conics $C \subset \Pi_{(a_2:\cdots:a_N)}$ passing through p and q are parameterized by the projectivization of a 4-dimensional sub-vector space $V_{(a_2:\cdots:a_N)} \subset \mathrm{Sym}^2(W^\vee)$. Therefore, $\mathrm{Hilb}(\mathbb{P}^N; p, q)$ must be the projective bundle associated to some sub-vector bundle $\mathcal{E}$ of

$$\mathrm{Sym}^2\left(\mathcal{U}|^\vee_{\Sigma_{N-2,N-2}}\right),$$

where $\mathcal{U}$ denotes the tautological bundle on $\mathbb{G}(2, N)$ (see 2.7). Let us describe this vector bundle $\mathcal{E}$. Cover the Schubert variety $\Sigma_{N-2,N-2}$ with the affine open subsets

$$U_i := \{[W] \in \Sigma_{N-2,N-2} \mid a_i \neq 0\}, \quad \text{for } i = 2, \dots, N,$$

and denote by U_{ij} the intersection of U_i and U_j. The trivializations of $\mathcal{U}|_{\Sigma_{N-2,N-2}}$ over this cover are

$$\begin{aligned} \varphi_i : f^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C}^3, \qquad \text{for } i = 2, \dots, N, \\ (w, [W]) &\longmapsto ([W], x, y, z) \end{aligned}$$

where x, y and z are the coordinates of $w \in W$ with respect to the basis

$$\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, a_2/a_i, \dots, a_N/a_i)\}.$$

By computing the transition functions relative to these trivializations, we see that $\mathcal{U}|_{\Sigma_{N-2,N-2}}$ is represented by the co-cycles

$$\begin{aligned} h_{ij} : \qquad U_{ij} &\longrightarrow \mathrm{GL}_3(\mathbb{C}) \\ (a_2 : \cdots : a_N) &\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_j/a_i \end{pmatrix}, \end{aligned}$$

which correspond to the locally free sheaf $\mathcal{O}^{\oplus 2}_{\mathbb{P}^{N-2}} \oplus \mathcal{O}_{\mathbb{P}^{N-2}}(-1)$. Now, consider the vector bundle

$$\begin{aligned} \pi : \mathcal{E} := \left\{(C, [W]) \in \mathbb{C}^6 \times \Sigma_{N-2,N-2} \mid C \in V_{(a_2:\cdots:a_N)}\right\} &\longrightarrow \Sigma_{N-2,N-2} \\ (C, [W]) &\longmapsto [W]. \end{aligned}$$

The trivializations of this vector bundle are

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C}^4 \\ (C, [W]) &\longmapsto ([W], c_1, c_2, c_4, c_5), \end{aligned}$$

where c_1, c_2, c_4, c_5 are coordinates with respect to the basis $\{xy, xz, yz, z^2\}$ of $V_{(a_2:\cdots:a_N)}$. By computing the transition functions relative to these trivializations, we see that $\mathcal{E}$ is represented by the co-cycles

$$\begin{aligned} k_{ij} : \qquad U_{ij} &\longrightarrow \mathrm{GL}_4(\mathbb{C}) \\ (a_2 : \cdots : a_N) &\longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_i/a_j & 0 & 0 \\ 0 & 0 & a_i/a_j & 0 \\ 0 & 0 & 0 & (a_i/a_j)^2 \end{pmatrix}, \end{aligned}$$

which corresponds to the locally free sheaf $\mathcal{O}_{\mathbb{P}^{N-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{N-2}}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{N-2}}$. Therefore, $\mathbb{P}_{\mathbb{P}^{N-2}}\left(\mathcal{O}_{\mathbb{P}^{N-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{N-2}}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{N-2}}\right)$ is the Hilbert scheme of conics in $\mathbb{P}^N$ passing through two fixed points.

Let us compute the locus $\Delta \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^N; p, q)$ corresponding to reducible conics. A conic $C \subset \Pi_{(a_2:\cdots:a_N)}$, given by the zeroes of a homogneous polynomial

$$c_1xy + c_2xz + c_4yz + c_5z^2,$$

is reducible if and only if

$$\det \begin{pmatrix} 0 & c_1 & c_2 \\ c_1 & 0 & c_4 \\ c_2 & c_4 & 2c_5 \end{pmatrix} = 0,$$

that is, $c_1(c_2c_4 - c_1c_5) = 0$. Therefore, we see that Δ is the union of two irreducible components Δ_1 and Δ_2. The component Δ_1, corresponding to the vanishing of c_1, parameterizes reducible conics with the line joining p and q as an irreducible component; it is the projective bundle associated to the sub-vector bundle $\mathcal{O}_{\mathbb{P}^{N-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{N-2}}(1)^{\oplus 2}$ of $\mathcal{E}$. The component Δ_2, corresponding to the vanishing of $c_2c_4 - c_1c_5$, is the conic-bundle over $\mathbb{P}^{N-2}$ whose fiber is the conic given by this same polynomial equation.

Example 4.2.2. Let X_2 be a smooth quadric hypersurface of $\mathbb{P}^N$, with $N \geq 2$. Let us describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(X_2; p, q)$ of conics in X_2 passing through two general fixed points $p, q \in X_2$. Under a projective change of coordinates, we can assume that X_2 is given by the polynomial equation

$$x_0^2 + \cdots + x_N^2 = 0.$$

Under a permutation of coordinates, we can assume that $p = (p_0 : \cdots : p_N)$ and $q = (q_0 : \cdots : q_N)$ with $p_0q_1 - p_1q_0 \neq 0$. Let $\Sigma_{N-2,N-2} \subset \mathbb{G}(2, N)$ be the Schubert variety of planes in $\mathbb{P}^N$ containing these points. As we have seen in Example 4.2.1, $\Sigma_{N-2,N-2} \cong \mathbb{P}^{N-2}$. Indeed, every plane in $\mathbb{P}^N$ parameterized by $\Sigma_{N-2,N-2}$ has a unique representative W of the the matricial form

$$W = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_N \\ q_0 & q_1 & q_2 & \cdots & q_N \\ 0 & 0 & a_2 & \cdots & a_N \end{pmatrix}, \quad \text{with } (a_2 : \cdots : a_N) \in \mathbb{P}^{N-2}.$$

For such planes we will always take the representative of this form. Also, the plane in $\mathbb{P}^N$ corresponding to $[W] \in \Sigma_{N-2,N-2}$ will be denoted by

$$\Pi_{(a_2:\cdots:a_N)} = \left\{ (xp_i + yq_i + za_i)_{i=0}^N \in \mathbb{P}^N \mid (x : y : z) \in \mathbb{P}^2 \right\},$$

where by convention we put $a_0 = a_1 = 0$. In the homogeneous coordinates x, y, z of $\Pi_{(a_2:\cdots:a_N)}$, the intersection $X_2 \cap \Pi_{(a_2:\cdots:a_N)}$ of X_2 is given by the polynomial equation

$$2\left(\sum_{i=0}^N p_iq_i\right)xy + 2\left(\sum_{i=0}^N p_ia_i\right)xz + 2\left(\sum_{i=0}^N q_ia_i\right)yz + \left(\sum_{i=0}^N a_i^2\right)z^2 = 0.$$

Since we are assuming that p and q are general points of X_2, we have $\sum_{i=0}^N p_iq_i \neq 0$; hence, the intersection $X_2 \cap \Pi_{(a_0:\cdots:a_N)}$ is a single conic $C_{(a_2:\cdots:a_N)}$. Therefore, $\mathrm{Hilb}_{2t+1}(X_2; p, q)$ is isomorphic to $\mathbb{P}^{N-2}$.

Let us compute the locus $\Delta \subset \mathrm{Hilb}_{2t+1}(X_2; p, q)$ corresponding to reducible conics. The conic $C_{(a_0:\cdots:a_N)}$ is reducible if and only if

$$\det \begin{pmatrix} 0 & \sum_{i=0}^N p_i q_i & \sum_{i=0}^N p_i a_i \\ \sum_{i=0}^N p_i q_i & 0 & \sum_{i=0}^N q_i a_i \\ \sum_{i=0}^N p_i a_i & \sum_{i=0}^N q_i a_i & \sum_{i=0}^N a_i^2 \end{pmatrix} = 0,$$

that is,

$$\sum_{i=0}^N \left(\sum_{j=0}^N (2\delta_{ij} - 1) p_j q_j \right) a_i^2 + 2 \sum_{0 \le i < j \le N} (p_i q_j + p_j q_i) a_i a_j = 0,$$

where δ_{ij} denotes the Kronecker's delta. Therefore Δ is the quadric hypersurface in $\mathbb{P}^{N-2}$ given by the above polynomial equation. The locus Δ could also be obtained in a more geometrical way: reducible conics on X_2 correspond to intersections with planes that are tangent to X_2.

Example 4.2.3. Let X_3 be a smooth cubic hypersurface of $\mathbb{P}^N$, with $N \geq 4$. Let us describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(X_3; p, q)$ of conics in X_3 passing through two general fixed points $p, q \in X_3$. The line $L_{p,q}$ passing through p and q intersects X_3 at a third point, which we will denote by r. Since p and q are general points, so is r. A conic $C \subset X_3$ passing through p and q generates a plane. The intersection of this plane and X_3 is the union of the conic C and a line through r. Conversely, a line $L \subset X_3$ passing through r and the line $L_{p,q}$ determine a plane. The intersection of this plane and X_3 is the union of the line L and a conic through p and q. Therefore $\mathrm{Hilb}_{2t+1}(X_3; p, q)$ is isomorphic to the Fano variety $F(X_3; r)$ of lines contained in X_3 passing through a general point $r \in X_3$. We saw in Example 3.1.1 that the Fano variety $F(X_3; r)$ is a complete intersection of type $(1, 2, 3)$ in $\mathbb{P}^{N-1}$.

The locus $\Delta \subset \mathrm{Hilb}_{2t+1}(X_3; p, q) \cong F(X_3; r)$ corresponding to reducible conics is obtained with similar computations to the Example 4.2.2. It is a complete intersection of type $(1, 2, 2, 3)$ in $\mathbb{P}^{N-1}$.

Example 4.2.4. Let $r, s \geq 1$ be integers and let $S : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$ be the Segre embedding. Let us describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(S(\mathbb{P}^r \times \mathbb{P}^s); S(p), S(q))$ of conics in the Segre variety $S(\mathbb{P}^r \times \mathbb{P}^s)$ passing through two general fixed points $S(p), S(q) \in S(\mathbb{P}^r \times \mathbb{P}^s)$. Let $\pi_1 : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^r$ and $\pi_2 : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^s$ be the projections from $\mathbb{P}^r \times \mathbb{P}^s$ to its factors. From the generality assumption, it follows that the points p and q are not in a same fiber of either π_1 or π_2. Then a conic $C \subset S(\mathbb{P}^r \times \mathbb{P}^s)$ passing through $S(p)$ and $S(q)$, as well as its inverse image $S^{-1}(C)$, has numerical class of type $(1, 1)$ in $N_1(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z} \oplus \mathbb{Z}$. Write $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Since $S^{-1}(C)$ has numerical class of type $(1, 1)$, the image L_1 of $S^{-1}(C)$ under π_1 is the line through p_1 and q_1, and the image L_2 of $S^{-1}(C)$ under π_2 is the line through p_2 and q_2. Thus

$$S^{-1}(C) \subset \pi_1^{-1}(L_1) \cap \pi_2^{-1}(L_2) \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Under the Segre embedding, this $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to a smooth quadric hypersurface of a $\mathbb{P}^3 \subset \mathbb{P}^{rs+r+s}$. Therefore, by Example 4.2.2, $\mathrm{Hilb}_{2t+1}(S(\mathbb{P}^r \times \mathbb{P}^s); S(p), S(q)) \cong \mathbb{P}^1$ and the locus $\Delta \subset \mathbb{P}^1$ corresponding to reducible curves is the union of two points.

Example 4.2.5. Assume the notation as in Example 4.2.4. Let H be a general hyperplane of $\mathbb{P}^{rs+r+s}$. Let us describe the Hilbert scheme $\mathrm{Hilb}_{2t+1}(S(\mathbb{P}^r \times \mathbb{P}^s) \cap H; S(p), S(q))$ of conics in $S(\mathbb{P}^r \times \mathbb{P}^s) \cap H$ passing through two fixed general points $S(p), S(q) \in S(\mathbb{P}^r \times \mathbb{P}^s) \cap H$. We have seen that conics in $S(\mathbb{P}^r \times \mathbb{P}^s)$ passing through two fixed general points $S(p), S(q) \in S(\mathbb{P}^r \times \mathbb{P}^s)$ are contained in a smooth quadric hypersurface of a $\mathbb{P}^3 \subset \mathbb{P}^{rs+r+s}$. The hyperplane H cuts this quadric into a conic, which must be the original conic. Therefore $\mathrm{Hilb}_{2t+1}(S(\mathbb{P}^r \times \mathbb{P}^s) \cap H; S(p), S(q))$ is a single point and $\Delta = \emptyset$.

Example 4.2.6. Let r be an integer such that $1 \leq r \leq n-1$. Set $\mathcal{E} = \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus n-r} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$. Consider the projective bundle $\mathbb{P}(\mathcal{E})$ embedded by the complete linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ into the projective space $\mathbb{P}^N$, where $N = (n-r)(n-r) + \binom{r+2}{2}$. Denote this embedding by $i : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$, and by X its image. Let us describe the space of conics in X passing through two fixed general points $i(p), i(q) \in X$. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^r$ be the projection morphism. Denote by ξ the divisor class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and by h the divisor class of the pullback $\pi^*(\mathcal{O}_{\mathbb{P}^r}(1))$. Also, denote by f the curve class of a line contained in a fiber of π, and by l the curve class of the image of a line under the section corresponding to the surjection of sheaves $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^r}(1)$. We have the following intersection numbers

$$f \cdot \xi = 1, \quad f \cdot h = 0, \quad l \cdot \xi = 1, \quad l \cdot h = 1.$$

Recall that the Mori cone $\overline{NE}(\mathbb{P}(\mathcal{E}))$ of effective curves is generated by f and l. Hence, if C is a conic contained in X, then

$$[i^{-1}(C)] \equiv af + bl, \quad \text{for certain integers } a, b \geq 0.$$

We have

$$2 = [C] \cdot [H] = [i^*C] \cdot \xi = a + b,$$

where $[H]$ denotes the hyperplane class of $\mathbb{P}^N$. If C passes through points $i(p)$ and $i(q)$, with p or q not contained in the image of the section, then $a \neq 0$. And if p and q are not contained in a same fiber of the projection morphism π, then $b \neq 0$. In this general situation, the image of $i^{-1}(C)$ under π is a line; thus we are reduced to the case $r = 1$. In this case it is clear that there exists a unique such conic C.

Here is another way to see this example. View X as the blown-up $\mathrm{Bl}_{\mathbb{P}^s}(\mathbb{P}^n)$ embedded in $\mathbb{P}^N$, with $N = \frac{n(n+3)}{2} - \binom{s+2}{2}$, by the linear system of quadric hypersurfaces of $\mathbb{P}^n$ containing $\mathbb{P}^s$, with $0 \leq s \leq n-2$. Then the conic C is the strict transform of the unique line in $\mathbb{P}^n$ passing through the image of the two points. The generality condition gives that this line does not meet the blown-up linear space $\mathbb{P}^s$.

4.2.2 Polarization on the Space of Conics Through Two Points

When studying a uniruled variety, it is very useful to consider a polarized minimal family of rational curves passing through a general point of it (see the final comments in Subsection 3.1). For example, in [AC12], C. Araujo and A.-M. Castravet studied Fano manifolds whose Chern characters satisfy some positivity conditions considering polarized minimal families of rational curves through a general point. With this study the authors found out new examples of higher Fano manifolds, and provided conditions for these manifolds to be covered by subvarieties isomorphic to $\mathbb{P}^2$ and $\mathbb{P}^3$. Analogously, it seems that when studying conic-connected manifolds one should consider a polarized variety $(W_{x,y}, \mathcal{M}_{x,y})$, where $W_{x,y}$ is an irreducible component of the space of conics on X passing through general points $x, y \in X$, and $\mathcal{M}_{x,y}$ is a natural polarization on $W_{x,y}$, which we define below.

Let $X \subset \mathbb{P}^N$ be a conic-connected manifold, and let β be a class of a conic passing through general points $x, y \in X$. Assume that X is not a linear variety. Let $W_{x,y}$ be an irreducible component of the fiber over $(x, y) \in X \times X$ of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,2}(X, \beta) \longrightarrow X \times X.$$

By Proposition 4.2.1 the space $W_{x,y}$ of conics on X (with class β) passing through $x, y \in X$ is automorphism-free and smooth. Therefore, there exists a universal family of stable maps over $W_{x,y}$,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \mu\ } & X\ , \\ {\scriptstyle s_x}\ \uparrow\ \downarrow{\scriptstyle \pi}\ \uparrow\ {\scriptstyle s_y} & & \\ W_{x,y} & & \end{array}$$

where s_x and s_y are sections of π such that $\mu(s_x(W_{x,y})) = \{x\}$ and $\mu(s_y(W_{x,y})) = \{y\}$. We will denote by the same symbol both s_x and its image in $\mathcal{C}$, and analogously for s_y. Recall that, by the functorial property, the fiber F of the morphism π over a stable map $[C, p, q, f] \in W_{x,y}$ is isomorphic to C, and under this isomorphism we have $\mu|_F = f$ (see Subsection 2.5.5). Consider the differentials of the universal family morphisms over $W_{x,y}$,

$$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ T_\pi & & \\ \downarrow & & \\ T_{\mathcal{C}} & \xrightarrow{\ d\mu\ } & \mu^* T_X. \\ {\scriptstyle d\pi}\downarrow & & \\ \pi^* T_{W_{x,y}} & & \end{array}$$

The smooth locus of the morphism π is the open subset $\mathcal{U}$ of $\mathcal{C}$ consisting only of smooth points of its fibers. Hence the restriction

$$\pi|_{\mathcal{U}} : \mathcal{U} \longrightarrow W_{x,y}$$

is a smooth morphism of relative dimension 1, and then the restriction $(T_\pi)|_{\mathcal{U}}$ is a locally free sheaf of rank 1. By stability condition, source curves of stable maps parameterized by $W_{x,y}$ do not have irreducible components contracted by their maps. Furthermore, since we are assuming that X is not a linear variety, none of the stable maps in $W_{x,y}$ is a double covering. Thus the restriction of the differential $(\,\pi|_{\mathcal{U}})$ to $(T_\pi)|_{\mathcal{U}}$ is non-zero, and therefore we have the inclusion of sheaves

$$d\,(\,\pi|_{\mathcal{U}}) : (T_\pi)|_{\mathcal{U}} \hookrightarrow (\mu|_{\mathcal{U}})^* \, T_X.$$

The section s_x is such that $s_x(W_{x,y}) \subset \mathcal{U}$. Thus, pulling back under s_x we obtain the inclusion of sheaves

$$s_x^* T_\pi \hookrightarrow s_x^* \mu^* T_X \cong T_x X \otimes \mathcal{O}_{W_{x,y}}.$$

The quotient is locally free and defines a morphism

$$\tau_x : W_{x,y} \longrightarrow \mathbb{P}(T_x X)$$

which maps a stable map $[C, p, q, f] \in W_{x,y}$ to the tangent direction at x to the image conic $f(C)$.

Lemma 4.2.1. The image of the morphism $\tau_x : W_{x,y} \to \mathbb{P}(T_xX)$ does not contain the direction of the line in $\mathbb{P}^N$ joining x, y.

Proof. Denote by v the direction of the line in $\mathbb{P}^N$ joining x, y. Suppose by contradiction that there exists a conic C on X passing through $x, y \in X$ with tangent direction v at x. Then, by Bézout's Theorem, the conic C must be reducible. Hence, X contains the line in $\mathbb{P}^N$ joining x, y. But, by Lemma 4.1.1, this contradicts the assumption that X is not a linear variety. Therefore the result holds. □

Lemma 4.2.2. The morphism $\tau_x : W_{x,y} \to \mathbb{P}(T_xX)$ is a finite morphism.

Proof. Since a projective morphism with finite fibers is a finite morphism, it is sufficient to prove that τ_x has finite fibers. Suppose by contradiction that there exists a 1-dimensional family of conics on X passing through $x, y \in X$ with the same tangent direction $v \in \mathbb{P}(T_xX)$ at x. By Lemma 4.2.1, v and the line in $\mathbb{P}^N$ joining x and y generate a plane P, which must contains the 1-dimensional family of conics. This family of conics cover P, and therefore $P \subset X$. By Lemma 4.1.1, this contradicts the assumption that X is not a linear variety. Therefore the result holds. □

Since the pullback of an ample sheaf under a finite morphism of noetherian schemes is ample,

$$\mathcal{M}_{x,y} = \tau_x^* \mathcal{O}_{\mathbb{P}(T_xX)}(1)$$

is an ample line bundle on $W_{x,y}$.

In order to study conic-connected manifolds, we propose to work with the polarized variety $(W_{x,y}, \mathcal{M}_{x,y})$. In the next section we will relate this polarization with that one studied by A. J. de Jong and J. M. Starr in [dS06b].

4.3 Minimal Pointed Rational Curves

In the unpublished notes [dS06b], A. J. de Jong and J. M. Starr study a special class of rationally connected varieties. They study rationally connected varieties X with the property that through n general points $x_1, \ldots, x_n \in X$ there passes a rational curve of degree exactly n. As tools, they consider the space of rational curves of degree n on X passing through the general points $x_1, \ldots, x_n \in X$, and define a natural polarization on this space. In this section we review part of the theory developed by de Jong and Starr for the study of these varieties. We will see that, in the case $n = 2$, the polarization defined by de Jong and Starr coincides with our polarization defined in Subsection 4.2.2.

Although many of the results presented here have first been announced in [dS06b], some of them were stated without precision and without proofs. Our work here is to rewrite the definitions and provide complete proofs for these results with the appropriate hypothesis. We give all credits to the authors.

Let $X \subset \mathbb{P}^N$ be a smooth variety, and let β be a curve class on X of degree n. By the space of rational curves on X (with class β) passing through general points $x_1, \ldots, x_n \in X$

we mean the fiber over $(x_1, \ldots, x_n) \in X^n$ of the evaluation morphism

$$\begin{aligned} \text{ev}: \qquad \overline{M}_{0,n}(X,\beta) &\longrightarrow X^n \\ [C, p_1, \ldots, p_n, f] &\longmapsto (f(p_1), \ldots, f(p_n)). \end{aligned}$$

4.3.1 Minimal Curve Classes

Let $X \subset \mathbb{P}^N$ be a smooth variety. We will consider rational curves on X having minimal curve class among those passing through n general points of X. A curve satisfying this property is called *minimal pointed rational curve.* This notion of minimality of a curve class on X was introduced by de Jong and Starr; we define this notion below.

Definition 4.3.1. Let $X \subset \mathbb{P}^N$ be a smooth variety. A curve class β on X is *n-dominant* if the evaluation morphism

$$\text{ev}: \overline{M}_{0,n}(X,\beta) \longrightarrow X^n$$

is dominant. An n-dominant curve class β on X is *n-minimal* if for every partition

$$n = n_1 + \cdots + n_r$$

and for every collection $(\beta_1, \ldots, \beta_r)$ of n_i-dominant curve classes β_i satisfying

$$\sum_{i=1}^{r} \beta_i \leq \beta,$$

in fact occurs the equality

$$\sum_{i=1}^{r} \beta_i = \beta.$$

Lemma 4.3.1 ([dS06b, Lemma 5.3])**.** Let $X \subset \mathbb{P}^N$ be a smooth variety, and let β be a curve class on X. If β is n-minimal, then every point in a general fiber of the evaluation morphism

$$\text{ev}: \overline{M}_{0,n}(X,\beta) \longrightarrow X^n$$

parameterizes a curve whose irreducible components are all free. Moreover, every point in a general fiber parameterizes an automorphism-free stable map. Therefore, a general fiber of ev is smooth of expected dimension

$$-K_X \cdot \beta - (n-1)\dim(X) + n - 3$$

and it intersects the boundary Δ in a simple normal crossing divisor.

Proof. Let $[C, p_1, \ldots, p_n, f]$ be a point in a fiber of ev over a general point $(x_1, \ldots, x_n) \in X^n$. Let D be the union of all irreducible components of C which are free curves. Denote by $D_1, \ldots, D_r$ the connected components of D containing at least one of the marked points $p_1, \ldots, p_n$. Let E be the union of all irreducible components of C that are not contained in $D_1, \ldots, D_r$. For each $i = 1, \ldots, n$, every irreducible curve on X passing through x_i is free. Therefore every point p_i is contained in one of the connected components $D_1, \ldots, D_r$. Let C_j be an irreducible component of C contracted to x_i. Every irreducible component of C intersecting C_j is either contracted (and thus trivially free) or else mapped to an irreducible curve on X passing through x_i (and thus free). Therefore, by the stability condition, every connected component D_k contains at least one irreducible component that is not contracted.

Denote by β_i the curve class of $f_*[D_i]$. Let

$$n_i = \#\{p_j \mid p_j \in D_i\}.$$

Since every non-empty subset of $\{x_1, \dots, x_n\}$ is general, β_i is n_i-dominant. Clearly

$$\sum_{i=1}^{r} \beta_i \leq \beta.$$

In fact

$$\sum_{i=1}^{r} \beta_i = \beta,$$

because β is n-minimal. In particular, every irreducible component of E is contracted, and thus free. Therefore, every irreducible component of C is a free curve.

To prove that $[C, p_1, \dots, p_n, f]$ is automorphism-free, it is sufficient to prove that every non-contracted component C_i of C has degree 1 over its image. If the degree is greater than 1, then C_i can be replaced by a curve with smaller curve class also meeting all marked points contained in C_i, namely, the normalization of the image of C_i (maybe we also need to attach some contracted components if any special point is mapped to the same point on the image). Thus C can be replaced by an n-dominating curve with smaller curve class. This contradicts the n-minimality of β. Therefore, $[C, p_1, \dots, p_n, f]$ is an automorphism-free stable map.

The final statement now follows from Lemma 2.5.1. □

Example 4.3.1. Let $X \subset \mathbb{P}^N$ be a conic-connected manifold. Assume that X is not a linear variety. Let β be the class of a conic on X passing through general points $x, y \in X$. Then β is 2-minimal. Therefore, by Lemma 4.3.1, the space of conics on X (with class β) passing through $x, y \in X$ is automorphism-free and smooth of expected dimension. This concurs with Proposition 4.2.1.

4.3.2 Polarization on the Space of Minimal Pointed Rational Curves

In this subsection we define a polarization on the space of minimal pointed rational curves on a smooth projective variety. This polarization was introduced by A. J. de Jong and J. M. Starr in [dS06b].

Definition 4.3.2. Let $X \subset \mathbb{P}^N$ be a smooth variety, and let β be a curve class on X of degree n, with $1 \leq n \leq N$. Let $\underline{x} = (x_1, \dots, x_n) \in X^n \subset (\mathbb{P}^N)^n$ be an n-tuple of linearly general points, and denote by P the $(n-1)$-plane in $\mathbb{P}^N$ generated by $x_1, \dots, x_n$. The *linearly non-degenerate locus* $U_{\underline{x}}$ of $\underline{x}$ is the maximal open substack of the corresponding fiber of

$$\mathrm{ev} : \overline{M}_{0,n}(X, \beta) \longrightarrow X^n$$

parameterizing stable maps for which none of the irreducible components are mapped into the $(n-1)$-plane P.

This condition is equivalent to asking that none of the non-contracted irreducible components are mapped into P. Indeed, suppose by contradiction that there exists a stable map $(C, p_1, \dots, p_n, f)$ with none of the non-contracted irreducible components mapped into P, but with an irreducible component E contracted to a point $y \in P$ (note that y may or may not be one of the points $x_1, \dots, x_n$). By stability condition, E contains at least three special points. Since $x_1, \ldots, x_n$ are distinct, E contains at most one marked point. Hence E contains at least two special points which are intersection points of E with others irreducible components of C. Therefore $f(C)$ is singular at y; the image curve $f(C)$ and P have at least $n+1$ points in common, counted with multiplicity. This is a contradiction to Bézout's Theorem, because $n = \deg(f) \geq \deg(f(C))$.

Lemma 4.3.2. Let $X \subset \mathbb{P}^N$ be a smooth variety, and let β be a curve class on X of degree n, with $1 \leq n \leq N$. Let $\underline{x} = (x_1, \ldots, x_n) \in X^n \subset (\mathbb{P}^N)^n$ be an n-tuple of linearly general points. Denote by $W_{\underline{x}}$ the fiber over $\underline{x}$ of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,n}(X, \beta) \longrightarrow X^n.$$

If the linearly non-degenerate locus $U_{\underline{x}}$ of $\underline{x}$ is the whole corresponding stack of $W_{\underline{x}}$, then $W_{\underline{x}}$ is automorphism-free.

Proof. Let $(C, p_1, \ldots, p_n, f)$ be a stable map in the corresponding stack of $W_{\underline{x}}$. By hypothesis none of the non-contracted irreducible components of C are mapped into the $(n-1)$-plane P generated by $x_1, \ldots, x_n$. To prove that $(C, p_1, \ldots, p_n, f)$ is automorphism-free, it is sufficient to prove that for every non-contracted irreducible component E of C, $f|_E$ has degree 1 over its image. If for some non-contracted irreducible component E of C the degree of $f|_E$ over its image has degree greater than 1, then $\deg(f|_E) > \deg(f(E))$. Hence $n = \deg(f) > \deg(f(C))$. Since the n distinct points $x_1, \ldots, x_n$ are in the intersection of $f(C)$ with P, by Bézout's Theorem P contains an irreducible component of $f(C)$, a contradiction. □

Example 4.3.2. Let $X \subset \mathbb{P}^N$ be a conic-connected manifold, and let β be a class of a conic on X passing through general points $x, y \in X$. Assume that X is not a linear variety. Then the linearly non-degenerate locus of $(x, y) \in X^2$ is the whole stack of the corresponding fiber $W_{x,y}$ of the evaluation morphism $\mathrm{ev} : \overline{M}_{0,2}(X, \beta) \to X^2$.

It is convenient to gather in a single paragraph all the necessary hypothesis to define a polarization on the space of minimal pointed rational curves. Often we will refer to these hypothesis.

Hypothesis 4.3.1. Let $X \subset \mathbb{P}^N$ be a smooth variety, and let β be an n-minimal curve class on X of degree n, with $1 \leq n \leq N$. Assume that X is not a linear variety. Furthermore, assume that for $n \geq 3$ the variety X is not a quadric hypersurface. Let $\underline{x} = (x_1, \ldots, x_n) \in X^n \subset (\mathbb{P}^N)^n$ be an n-tuple of linearly general points. Let $W_{\underline{x}}$ be an irreducible component of the fiber over $\underline{x} \in X^n$ of the evaluation morphism

$$\mathrm{ev} : \overline{M}_{0,n}(X, \beta) \longrightarrow X^n.$$

Assume that the linearly non-degenerate locus $U_{\underline{x}}$ of $\underline{x}$ is the whole corresponding stack of $W_{\underline{x}}$. By Lemma 4.3.1 every point in $W_{\underline{x}}$ parameterizes an automorphism-free stable map. Moreover, $W_{\underline{x}}$ is smooth of expected dimension

$$-K_X \cdot \beta - (n-1)\dim(X) + n - 3$$

and it intersects the boundary Δ in a simple normal crossing divisor. Therefore, there exists a universal family of stable maps over $W_{\underline{x}}$,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \mu\ } & X \subset \mathbb{P}^N \ , \\ {\scriptstyle \pi}\big\downarrow \big\uparrow {\scriptstyle s_i,\ i=1,\dots,n} & & \\ W_{\underline{x}} & & \end{array} \qquad (4.1)$$

where $s_1, \dots, s_n$ are sections of π such that $\mu(s_1(W_{\underline{x}})) = \{x_1\}, \dots, \mu(s_n(W_{\underline{x}})) = \{x_n\}$. We will denote by the same symbol both s_1 and its image in $\mathcal{C}$, and analogously for $s_2, \dots, s_n$. By the functorial property, the fiber F of the morphism π over a stable map $[C, p_1, \dots, p_n, f] \in W_{\underline{x}}$ is isomorphic to C, and under this isomorphism we have $\mu|_F = f$ (see Subsection 2.5.5).

Under Hypothesis 4.3.1, given a stable map $[C, p_1, \dots, p_n, f] \in W_{\underline{x}}$, the image curve $f(C)$ has degree n and it passes through the general points $x_1, \dots, x_n \in X$. Since the linearly non-degenerate locus $U_{\underline{x}}$ is the whole corresponding stack of $W_{\underline{x}}$, the image curve $f(C)$ generates an n-plane in $\mathbb{P}^N$, which contains the $(n-1)$-plane P generated by $x_1, \dots, x_n$. Therefore, we have a well defined point

$$[f(C)] \in \mathbb{P}\left(\mathbb{C}^{N+1}/\operatorname{span}\{x_1, \dots, x_n\}\right) \cong \mathbb{P}^{N-n}.$$

In the next paragraphs we will prove that this correspondence is, indeed, a morphism from $W_{\underline{x}}$ to $\mathbb{P}^{N-n}$.

We begin by proving the following claim:

Claim 4.3.1. The sheaf $\pi_* \mu^* \mathcal{O}_{\mathbb{P}^N}(1)$ on $W_{\underline{x}}$ is locally free of rank $n+1$. The fiber over a stable map $[C, p_1, \dots, p_n, f] \in W_{\underline{x}}$ of the corresponding vector bundle is isomorphic to $H^0(C, f^* \mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. By Grauert's Theorem is sufficient to prove that, for every fiber F of π, we have

$$h^0(F, (\mu^* \mathcal{O}_{\mathbb{P}^N}(1))|_F) = n + 1.$$

We will prove this by induction on n. Let F be the fiber of π over a stable map $[C, p_1, \dots, p_n, f] \in W_{\underline{x}}$. If C is irreducible, that is, $C \cong \mathbb{P}^1$, then

$$h^0(F, (\mu^* \mathcal{O}_{\mathbb{P}^N}(1))|_F) = h^0(\mathbb{P}^1, f^* \mathcal{O}_{\mathbb{P}^N}(1)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg(f))) = n + 1.$$

Assume that C is reducible, with $r \geq 2$ irreducible components. Since C is a tree, we can write $C = E \cup D$ with $E \cong \mathbb{P}^1$ and $E \cap D = \{r\}$. Note that f does not contract E, otherwise E would contain at least two marked points p_i and p_j being mapped to the same point $x_i = x_j$, a contradiction. Set $\beta_1 = f_*[E]$ and $\beta_2 = f_*[D]$; it is clear that $\beta_1 + \beta_2 = \beta$. Since we are assuming that X is not a linear variety and for $n \geq 3$ the variety X is not a quadric hypersurface, by Lemma 4.1.2 the image curves $f(E)$ and $f(D)$ pass through $\deg(\beta_1)$ and $\deg(\beta_2)$ points, respectively. Moreover, these degrees are non-zero (because E is not a contracted component) and less than n (because $\deg(\beta_1) + \deg(\beta_2) = \deg(\beta)$). In other words, β_1 is $\deg(\beta_1)$-minimal and β_2 is $\deg(\beta_2)$-minimal. Consider the exact sequence

$$0 \to \mathcal{O}_E(-r) \to \mathcal{O}_C \to \mathcal{O}_D \to 0.$$

After tensorizing by $f^*\mathcal{O}_{\mathbb{P}^N}(1)$ we obtain

$$0 \to (f|_E)^*\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_E(-r) \to f^*\mathcal{O}_{\mathbb{P}^N}(1) \to (f|_D)^*\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_D \to 0.$$

Then we have the long exact sequence of cohomology

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg(f|_E) - 1)) \to H^0(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(D, (f|_D)^*\mathcal{O}_{\mathbb{P}^N}(1)) \to$$
$$\to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg(f|_E) - 1)) \to \cdots$$

Applying the induction hypothesis to D and using the fact that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg(f|_E) - 1)) = \{0\}$, we conclude that

$$\begin{aligned} h^0(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)) &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg(f|_E) - 1)) + h^0(D, (f|_D)^*\mathcal{O}_{\mathbb{P}^N}(1)) \\ &= \deg(f|_E) + \deg(f|_D) + 1 \\ &= n + 1, \end{aligned}$$

as we wanted. □

From now on, we will denote

$$\mathcal{E} = \pi_*\mu^*\mathcal{O}_{\mathbb{P}^N}(1).$$

We have an evaluation morphism (of locally free sheaves)

$$\pi^*\mathcal{E} \longrightarrow \mu^*\mathcal{O}_{\mathbb{P}^N}(1),$$

whose morphism between the fibers over a point $c \in \mathcal{C}$, with $\pi(c) = [C, p_1, \dots, p_n, f]$, is given by

$$\begin{aligned} H^0(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)) &\longrightarrow \mathcal{O}_{\mathbb{P}^N}(1) \otimes k(\mu(c)) \\ s &\longmapsto s(c). \end{aligned}$$

For each $i = 1, \dots, n$, we have $\pi \circ s_i = id$ and $\mu(s_i(W_{\underline{x}})) = \{x_i\}$. Hence, pulling back to $W_{\underline{x}}$ under s_i gives the evaluation morphism at the point p_i

$$\mathcal{E} \longrightarrow (\mathcal{O}_{\mathbb{P}^N}(1) \otimes k(x_i)) \otimes \mathcal{O}_{W_{\underline{x}}},$$

whose morphism between the fibers over a stable map $[C, p_1, \dots, p_n, f] \in W_{\underline{x}}$ is given by

$$\begin{aligned} H^0(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)) &\longrightarrow \mathcal{O}_{\mathbb{P}^N}(1) \otimes k(x_i) \\ s &\longmapsto s(p_i). \end{aligned}$$

Taking the product of all of these morphisms, we obtain a total evaluation morphism

$$\lambda : \mathcal{E} \longrightarrow \left(\bigoplus_{i=1}^n (\mathcal{O}_{\mathbb{P}^N}(1) \otimes k(x_i))\right) \otimes \mathcal{O}_{W_{\underline{x}}}.$$

For shortness, we will denote

$$\mathcal{F} = \left(\bigoplus_{i=1}^n (\mathcal{O}_{\mathbb{P}^N}(1) \otimes k(x_i))\right) \otimes \mathcal{O}_{W_{\underline{x}}}.$$

Notation. We will denote by $\mathcal{L}_{\underline{x}}$ the kernel of the sheaf homomorphism $\lambda : \mathcal{E} \to \mathcal{F}$.

Claim 4.3.2. The sheaf $\mathcal{L}_{\underline{x}}$ is invertible and generated by $N - n + 1$ global sections.

Proof. The sheaf homomorphism λ factors the evaluation morphism

$$\begin{aligned} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_{W_{\underline{x}}} &\longrightarrow \mathcal{E} \xrightarrow{\lambda} \mathcal{F} \\ H &\longmapsto (H(p_i))_{i=1}^n \end{aligned}$$

Clearly, this homomorphism is surjective, because $x_1, \ldots, x_n$ are general points of X (recall that X is non-degenerate in $\mathbb{P}^N$). Hence, λ is surjective, and therefore $\mathcal{L}_{\underline{x}}$ is invertible. In fact, λ is also surjective in global sections, which implies

$$\begin{aligned} h^0(W_{\underline{x}}, \mathcal{L}_{\underline{x}}) &= h^0(W_{\underline{x}}, \mathcal{E}) - h^0(W_{\underline{x}}, \mathcal{F} \otimes \mathcal{O}_{W_{\underline{x}}}) \\ &= h^0(W_{\underline{x}}, \mathcal{E}) - n. \end{aligned}$$

We claim that $h^0(W_{\underline{x}}, \mathcal{E}) = N+1$, and thus $h^0(W_{\underline{x}}, \mathcal{L}_{\underline{x}}) = N - n + 1$. Indeed, since we are assuming that the curve class β is n-minimal, the morphism μ dominates X, and since X is non-degenerate in $\mathbb{P}^N$, the $N+1$ global sections $\mu^*(X_0), \ldots, \mu^*(X_N)$ are linearly independent and generate $H^0(\mathcal{C}, \mu^*\mathcal{O}_{\mathbb{P}^N}(1)) = H^0(W_{\underline{x}}, \mathcal{E})$. Therefore, $\mathcal{L}_{\underline{x}}$ is generated by $N - n + 1$ global sections. □

By the above claim, the sheaf $\mathcal{L}_{\underline{x}}$ is base-point-free, and therefore defines a morphism from $W_{\underline{x}}$ to $\mathbb{P}^{N-n}$ given by

$$\begin{aligned} \phi : \qquad W_{\underline{x}} &\longrightarrow \mathbb{P}\left(\mathbb{C}^{N+1}/\operatorname{span}\{x_1, \ldots, x_n\}\right) \cong \mathbb{P}^{N-n} \\ [C, p_1, \ldots, p_n, f] &\longmapsto [f(C)]. \end{aligned}$$

Lemma 4.3.3. Assume Hypothesis 4.3.1 and additionally that $n - 1 \leq \operatorname{codim}(X)$. Then the line bundle $\mathcal{L}_{\underline{x}}$ on $W_{\underline{x}}$ is ample.

Proof. We begin by recalling that the pullback of an ample sheaf under a finite morphism of noetherian schemes is ample. On the other hand, a projective morphism with finite fibers is a finite morphism. Therefore, to prove that $\mathcal{L}_{\underline{x}}$ is an ample sheaf, it is sufficient to prove that the morphism ϕ has finite fibers. Let $y \in \mathbb{P}^{N-n}$ be a point in the image of the morphism ϕ. Let T be the corresponding n-plane in $\mathbb{P}^N$ containing $x_1, \ldots, x_n$. A stable map $[C, p_1, \ldots, p_n, f]$ is contained in the pre-image of y if and only if the image curve $f(C)$ is contained in the intersection of X and T. Suppose that $\dim(T \cap X) = 1$, that is, $T \cap X$ is a curve. Thus, the curve C have non-contracted irreducible components consisting of the irreducible components of $T \cap X$. Since $T \cap X$ has a finitely many irreducible components, we conclude that the fiber is finite.

To prove that $\dim(T \cap X) = 1$, it is sufficient to prove that $\dim(P \cap X) = 0$, where P denotes the $(n-1)$-plane generated by the general points $x_1, \ldots, x_n$. Recall the Trisecant Lemma: let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible variety over a field of characteristic zero. Given n general points $x_1, \ldots, x_n \in X$ such that $n < \operatorname{codim}(X)$, the $(n-1)$-plane P generated by these points is not $(n+1)$-secant, that is, $P \cap X = \{x_1, \ldots, x_n\}$. Therefore, when $n - 1 < \operatorname{codim}(X)$ the claim follows immediately from the Trisecant Lemma. The case $n - 1 = \operatorname{codim}(X)$ needs more.

Assume that $n - 1 = \operatorname{codim}(X)$. Suppose by contradiction that $\dim(P \cap X) = 1$. We claim that this only happens when X is a linear variety or a quadric hypersurface;

and this contradicts Hypothesis 4.3.1. Let E be a 1-dimensional irreducible component of $P \cap X$. Denote by P' the $(n-2)$-plane generated by the general points $x_1, \ldots, x_{n-1}$. By Bézout's Theorem, the intersection of E and P' is non-empty, and by the Trisecant Lemma, $P' \cap X = \{x_1, \ldots, x_{n-1}\}$. Hence, E pass through at least one of the points $x_1, \ldots, x_{n-1}$, say x_1. Denote by P'' the $(n-2)$-plane generated by the general points $x_2, \ldots, x_n$. By Bézout's Theorem, the intersection of E and P'' is non-empty, and by the Trisecant Lemma, $P'' \cap X = \{x_2, \ldots, x_n\}$. Hence, E passes through at least one of the points $x_2, \ldots, x_n$, say x_2. If $\deg(E) = 1$, then by Lemma 4.1.1 X is a linear variety, a contradiction. Thus, $\deg(E) \geq 2$. Again by Bézout's Theorem, P'' intersects X at a second point, say x_3. If $\deg(E) = 2$, then by Bézout's Theorem, E is contained in the plane generated by x_1, x_2, x_3. Now we recall that a non-degenerate irreducible curve $C \subset \mathbb{P}^d$ of degree d is a normal rational curve. Then, by Lemma 4.1.2, X is a linear variety or a quadric hypersurface, a contradiction. Repeating this reasoning we conclude that we can not have $\dim(P \cap X) = 1$. Therefore, $\dim(P \cap X) = 0$, as we desired. □

Let X be a conic-connected manifold. Assume that X is not a linear variety. We end this subsection showing that, the polarizations $\mathcal{M}_{x,y}$ and $\mathcal{L}_{x,y}$ on the space $W_{x,y}$ of conics on X passing through general points $x, y \in X$ are isomorphic.

Proposition 4.3.1. Let $X \subset \mathbb{P}^N$ be a conic-connected manifold, and let β be a class of a conic on X passing through general points $x, y \in X$. Let $W_{x,y}$ be an irreducible component of the space of conics on X (with class β) passing through x, y. Assume that X is not a linear variety. Then the polarizations $\mathcal{M}_{x,y}$ and $\mathcal{L}_{x,y}$ on $W_{x,y}$ are isomorphic.

Proof. Denote by ℓ the line in $\mathbb{P}^N$ joining the points $x, y \in X$. We have a commutative diagram

$$\begin{array}{ccc} W_{x,y} & \xrightarrow{\ \phi\ } & \mathbb{P}\left(\mathbb{C}^{N+1}/\operatorname{span}\{x,y\}\right) \cong \mathbb{P}^{N-2}, \\ & {\scriptstyle\tau_x}\searrow & \uparrow{\scriptstyle \pi_\ell} \\ & & \mathbb{P}(T_{X,x}) \end{array}$$

where π_ℓ is the projection from the line ℓ. By Lemma 4.2.1 the image of τ_x is contained in the open U where π_ℓ is defined. Thus $\pi_\ell^* \mathcal{O}_{\mathbb{P}^{N-2}}(1) \cong \mathcal{O}_{\mathbb{P}(T_x X)}(1)\big|_U$, and therefore $\mathcal{M}_{x,y} \cong \mathcal{L}_{x,y}$. □

4.3.3 Classes Computations

Here we will compute some cycles relations on the space of rational curves $W_{\underline{x}}$.

Notation. Assume Hypothesis 4.3.1 and its notation. We will denote by

$$L = \mathrm{c}_1(\mathcal{L}_{\underline{x}}),$$

the corresponding class divisor of the polarization $\mathcal{L}_{\underline{x}}$ on $W_{\underline{x}}$.

Lemma 4.3.4 ([dS06b, Lemma 6.4])**.** With Hypothesis 4.3.1 and its notation, we have the following identities of cycles on $W_{\underline{x}}$:

(i) $s_i \cdot \mu^*(\alpha) = 0$, for every $\alpha \in A^k(W_{\underline{x}})$, $k \geq 1$;

(ii) $\mu^* \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^N}(1)) = \sum_{i=1}^n s_i + \pi^* L$;

(iii) $s_i^2 = -s_i \cdot \pi^* L$, for $i = 1, \ldots, n$;

(iv) $\pi_*(s_i \cdot s_i) = -L$, for $i = 1, \ldots, n$;

(v) $\pi_* \mu^* \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^N}(1))^2 = nL$.

Proof. Since $\dim W_{\underline{x}} \geq 1$ and $\mu(s_i(W_{\underline{x}})) = \{x_i\}$, the identity (i) follows immediately from Projection Formula. Consider the evaluation morphism

$$\pi^* \pi_* \mu^* \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow \mu^* \mathcal{O}_{\mathbb{P}^N}(1).$$

Since $\mathcal{L}_{\underline{x}}$ is a subsheaf of $\pi_* \mu^* \mathcal{O}_{\mathbb{P}^N}(1)$, we can restrict this morphism to $\mathcal{L}_{\underline{x}}$. This gives us a morphism of invertible sheaves

$$\pi^* \mathcal{L}_{\underline{x}} \longrightarrow \mu^* \mathcal{O}_{\mathbb{P}^N}(1),$$

or equivalently, a global section of the twist $\pi^* \mathcal{L}_{\underline{x}}^\vee \otimes \mu^* \mathcal{O}_{\mathbb{P}^N}(1)$, which we denote by

$$\sigma : \mathcal{O}_{\mathcal{C}} \longrightarrow \pi^* \mathcal{L}_{\underline{x}}^\vee \otimes \mu^* \mathcal{O}_{\mathbb{P}^N}(1).$$

The zero locus of this section is

$$\text{Zero locus}(\sigma) = \mu^{-1}(P),$$

where P denotes the $(n-1)$-plane in $\mathbb{P}^N$ generated by $x_1, \ldots, x_n$. Since $\mu(s_i(W_{\underline{x}})) = \{x_i\}$, we have

$$\cup_{i=1}^n s_i(W_{\underline{x}}) \subset \mu^{-1}(P).$$

Suppose that this inclusion is strict. Then there exists a stable map $[C, p_1, \ldots, p_n, f] \in W_{\underline{x}}$ passing through a point $y \in P$ different from $x_1, \ldots, x_n$. We are assuming that the image curve $f(C)$ generates an n-plane, which contains P as a hyperplane. Since $x_1, \ldots, x_n$ and y are distinct points in the intersection of $f(C)$ with P, by Bézout's Theorem $f(C)$ have an irreducible component contained in P. This is a contradiction, because the linearly non-degenerate locus $U_{\underline{x}} = W_x$ parameterizes stable maps for which none of the irreducible components are mapped into P. Therefore

$$\text{Zero locus}(\sigma) = \bigcup_{i=1}^m s_i(U_{\underline{x}})$$

as closed subschemes of $\mathcal{C}$. The global section σ induces an isomorphism of invertible sheaves

$$\mathcal{O}_{\mathcal{C}}(s_1 + \cdots + s_n) \cong \pi^* \mathcal{L}_{\underline{x}}^\vee \otimes \mu^* \mathcal{O}_{\mathbb{P}^N}(1).$$

Taking the first Chern class and using its elementary properties, we obtain (ii). Intersecting (ii) with s_i and using (i) and the Projection Formula, we obtain (iii). Pushing forward the identity (iii) under π and using the Projection Formula, we obtain (iv). Finally, using the identities (ii) and (iii) we obtain the identity (v):

$$\begin{aligned}
\pi_*\mu^* \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^N}(1))^2 &= \pi_* \left(\sum_{i=1}^n s_i + \pi^* L \right) \\
&= \pi_* \left(\sum_{i=1}^n s_i^2 + (\pi^* L)^2 + 2 \sum_{i=1}^n s_i \cdot \pi^* L \right) \\
&= \pi_* \left((\pi^* L)^2 + \sum_{i=1}^n s_i \cdot \pi^* L \right) \\
&= \pi_* \left(\pi^* L \cdot \pi^* L \right) + \sum_{i=1}^n \pi_*(s_i \cdot \pi^* L) \\
&= nL,
\end{aligned}$$

where the last equality holds because of the Projection Formula. □

Now we state the formula for the first Chern class of the space of rational curves $W_{\underline{x}}$. This formula was computed by A. J. de Jong and J. M. Starr in [dS06b, Lemma 6.5] and it is given in terms of the first and second Chern classes of X. This is a consequence of a much more difficult computation: the virtual canonical class of the Kontsevich moduli space $\overline{M}_{0,n}(X, \beta)$. For the special case $X = \mathbb{P}^N$, this virtual canonical class and other divisor class relations were computed by R. Pandharipande in [Pan97] and [Pan99]. With a completely different method from that used by Pandharipande, de Jong and Starr in [dS06a] computed the virtual canonical class and other divisor class relations on $\overline{M}_{0,n}(X, \beta)$ for the general case. The authors make use of a *perfect obstruction theory* for $\overline{M}_{0,n}(X, \beta)$, defined by K. Behrend and B. Fantechi in [BF97]. This is a perfect complex $E^\bullet$ of amplitude $[-1, 0]$ together with a map to the cotangent complex $\phi : E^\bullet \to L^\bullet_{\overline{M}_{0,n}(X,\beta)}$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. In many cases, ϕ is a quasi-isomorphism, and then the dualizing sheaf on $\overline{M}_{0,n}(X, \beta)$ is the determinant $\det(E^\bullet)$. For this reason, $\det(E^\bullet)$ is called the *virtual canonical bundle*. The proofs reduce to local computations and the use of Grothendieck-Riemann-Roch Theorem for the universal family over the Artin stack of all prestable curves of genus 0. Since such proofs require too much effort on subject not addressed in this thesis, we are content in just stating the result that we are interested.

Lemma 4.3.5 ([dS06b, Lemma 6.5])**.** Assume Hypothesis 4.3.1 and additionally that X has Picard number equal to 1. Let α be a curve class on X of degree 1. Then the first Chern class of $W_{\underline{x}}$ is given by the formula

$$\mathrm{c}_1(W_{\underline{x}}) = \pi_*\mu^* \left(\mathrm{ch}_2(X) - \frac{(n-2)\,\mathrm{c}_1(X) \cdot \alpha + 2n}{2n} \mathrm{c}_1(\mathcal{O}(1))^2 \right) + 2\Delta. \tag{4.2}$$

In particular, for $n = 1$,

$$\mathrm{c}_1(W_{\underline{x}}) = \pi_*\mu^* \left(\mathrm{ch}_2(X) + \frac{\mathrm{c}_1(X) \cdot \alpha - 2}{2} \mathrm{c}_1(\mathcal{O}(1))^2 \right), \tag{4.3}$$

and for $n = 2$,

$$\mathrm{c}_1(W_{\underline{x}}) = \pi_*\mu^* \left(\mathrm{ch}_2(X) - \mathrm{c}_1(\mathcal{O}(1))^2 \right) + 2\Delta. \tag{4.4}$$

As we can see in the previous lemma, the linear map $\pi_*\mu^*$ plays an important role in the formula for the canonical bundle of the space of rational curves $W_{\underline{x}}$. In order to study this map, we introduce the following definition (see Subsection 2.6.3 for notation).

Definition 4.3.3. Assume Hypothesis 4.3.1. For every positive integer k, we define linear maps

$$\begin{aligned} T^k : N^k(X)_{\mathbb{R}} &\longrightarrow N^{k-1}(W_{\underline{x}})_{\mathbb{R}}, \\ \alpha &\longmapsto \pi_*\mu^*\alpha \end{aligned} \qquad \begin{aligned} T_k : N_k(W_{\underline{x}})_{\mathbb{R}} &\longrightarrow N_{k+1}(X)_{\mathbb{R}} \\ \beta &\longmapsto \mu_*\pi^*\beta. \end{aligned}$$

Note that, by the Projection Formula,

$$T^{k+1}(\alpha)\cdot\beta = \alpha\cdot T_k(\beta), \tag{4.5}$$

for every $\alpha \in N^{k+1}(X)_{\mathbb{R}}$ and $\beta \in N_k(W_{\underline{x}})_{\mathbb{R}}$.

Following C. Araujo and A.-M. Castravet in [AC12], and using the identities in Lemma 4.3.4, we can prove that the linear maps T^k and T_k preserve nice properties.

Lemma 4.3.6. Let $X \subset \mathbb{P}^N$ be a smooth variety with Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$.

(i) If D is an $\mathbb{R}$-divisor on X, then

$$T^k(D^k) = \frac{d^k}{n^{k-1}}L^{k-1},$$

where $d = \deg(f^*D)$, for any stable map $[C, p_1, \ldots, p_n, f] \in W_{\underline{x}}$;

(ii) T_k maps $\overline{NE}_k(W_{\underline{x}}) \setminus \{0\}$ into $\overline{NE}_{k+1}(X) \setminus \{0\}$;

(iii) T^k preserves the property of being nef.

Proof. Since $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$, every $\mathbb{R}$-divisor D on X is written as $D = a\,\mathrm{c}_1(\mathcal{O}_X(1))$, with $a \in \mathbb{R}$. By Lemma 4.3.4

$$\mu^*D = a\mu^*\,\mathrm{c}_1(\mathcal{O}_X(1)) = a\left(\sum_{i=1}^n s_i + \pi^*L\right).$$

Now let F be the fiber of π over a stable map $[C, p_1, \ldots, p_n, f] \in W_{\underline{x}}$. We know that $F \cong C$, and under this isomorphism $\mu|_F = f$. Restricting the above identity to the fiber F we conclude that

$$\begin{aligned} f^*(A) &= a\left(\sum_{i=1}^n s_i\cdot F + \pi^*\,\mathrm{c}_1(\mathcal{L})\cdot F\right) \\ &= a\sum_{i=1}^n p_i. \end{aligned}$$

In particular, taking the degree, we have $a = \deg(f^*(A))/n$. By Lemma 4.3.4 and Projection Formula,

$$\begin{aligned}
T^k(D^k) &= a^k \pi_* \left(\sum_{i=1}^{n} s_i + \pi^* L \right)^k \\
&= a^k \pi_* \left[\sum_{j=0}^{k} \binom{k}{j} \left(\sum_{i=1}^{n} s_i \right)^{k-j} \cdot \pi^* L^j \right] \\
&= a^k \pi_* \left[\sum_{j=0}^{k-1} \binom{k}{j} \left(\sum_{i=1}^{n} s_i^{k-j} \cdot \pi^* L^j \right) + \pi^* L^k \right] \\
&= a^k \pi_* \left[\sum_{j=0}^{k-1} \binom{k}{j} \left(\sum_{i=1}^{n} (-1)^{k-j-1} s_i \cdot \pi^* L^{k-1} \right) + \pi^* L^k \right] \\
&= a^k \pi_* \left[\left(\sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-j-1} \right) \left(\sum_{i=1}^{n} s_i \cdot \pi^* L^{k-1} \right) + \pi^* L^k \right] \\
&= a^k \pi_* \left[\left(\sum_{i=1}^{n} s_i \cdot \pi^* L^{k-1} \right) + \pi^* L^k \right] \\
&= a^k n L^{k-1} \\
&= \frac{d^k}{n^{k-1}} L^{k-1},
\end{aligned}$$

and then we obtain (i).

It is clear that T_k maps effective cycles to effective cycles, and therefore, it maps $\overline{NE}_k(W_{\underline{x}})$ into $\overline{NE}_{k+1}(X)$, by continuity. Let $\beta \in \overline{NE}_k(W_{\underline{x}}) \setminus \{0\}$. Take A an ample divisor on $W_{\underline{x}}$. In view of (4.5), to prove that $T_k(\beta) \neq 0$, it is sufficient to prove that $T^{k+1}(A^{k+1}) \cdot \beta > 0$. By item (i),

$$T^{k+1}(A^{k+1}) \cdot \beta = \frac{d^{k+1}}{n^k} L^k \cdot \beta > 0,$$

because L is an ample divisor by Lemma 4.3.3, and thus $L^k \cdot \beta > 0$. Therefore, (ii) holds.

In view of 4.5, $T^{k+1}(\alpha) \cdot \beta = \alpha \cdot T_k(\beta)$, for every $\alpha \in N^{k+1}(X)_{\mathbb{R}}$ and $\beta \in N_k(W_{\underline{x}})_{\mathbb{R}}$. Together with (i) and (ii) above, this implies that T^k preserves the property of nef, as claimed in (iii). □

Final remark. Let $X \subset \mathbb{P}^N$ be a conic-connected manifold of Picard number 1, and let β be a class of a conic on X passing through general points $x, y \in X$. Assume that X is not a linear variety. Let $W_{x,y}$ be an irreducible component of the space of conics on X (with class β) passing through x, y. By Lemma 4.3.5, the anti-canonical class of $W_{x,y}$ is given by the formula

$$-K_{W_{x,y}} = \pi_* \mu^* \operatorname{ch}_2(X) - \pi_* \mu^* \operatorname{c}_1(\mathcal{O}(1))^2 + 2\Delta.$$

By Lemma 4.3.4 and Proposition 4.3.1, we know that

$$\pi_* \mu^* \operatorname{ch}_2(X) = 2L = 2M,$$

where M is the corresponding divisor to the polarization $\mathcal{M}_{x,y}$. It was also proved by de Jong and Starr in [dS06b, Lemma 6.4] that

$$2L = \Delta.$$

Hence, we have

$$-K_{W_{x,y}} = \pi_* \mu^* \operatorname{ch}_2(X) + 2M. \tag{4.6}$$

By Lemma 4.3.6, the linear map $T^2 = \pi_* \mu^*$ preserves the property of being nef. Therefore, if X is weakly 2-Fano, then $W_{x,y}$ is Fano (see Subsection 2.6.3 for the definition of weakly 2-Fano manifold).

The reason to obtain the formula (4.6) in terms of the polarization $\mathcal{M}_{x,y}$ is because the tangent morphism τ_x has been studied extensively in a series of papers by J.-M. Hwang and N. Mok (see, for example, [Hwa01] and [HM04]), and also studied by other authors (see, for example, [Keb02]). We hope that this point of view is useful in the study of weakly 2-Fano conic-connected manifolds.

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