Survival and Uncertainty through Variational Preferences

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Abstract. The theory of decision under uncertainty has been extensively developed last years and it is yielding relevant consequences in economic theory. It is interesting to investigate if new models for uncertainty aversion are robust when incorporated within classic economic frameworks. Results of [4] indicate that *maxmin* preferences are asymptotically irrelevant in a general equilibrium model. In this paper, *variational* preferences (axiomatized by [10, 11]) are tested in a survival problem based on [3]. This paper shows that to determine survival it is necessary to compare levels of uncertainty aversion and aggregate risk, permitting the survival of an agent with persistent uncertainty aversion.

Date: August 13, 2015.

* *This paper was first presented at IMPA in February 2010. The author is grateful to Aloisio Araujo, José H. Faro, Rodrigo Novinski, Victor F. Martins-da-Rocha and Paulo K. Monteiro for helpful comments and discussion. The author is supported by National Council for Scientific and Technological Development (CNPq) from Brazil.*

Key words and phrases. Ambiguity aversion · Variational preferences · Belief selection.

1. Introduction

The *market selection hypothesis* has long been invoked by economists to justify the assumption that economic agents have rational expectations, i.e., that their beliefs are identical to the probabilistic model that governs the events. The rationale is the following: in an economy populated by heterogeneous agents, those who have such a feature will obtain advantages over others and in the long-term will accumulate more wealth; their decisions will be more important to the economy; asymptotically, such individuals will be the ones to influence prices and dominate the market. However, for this reasoning to work, it is necessary to assume certain hypotheses. To achieve positive results, i.e., those where selection for who makes accurate predictions happen, [12] and [3] suppose that agents have expected utilities and markets are complete. Without market completeness there are negative results as in [3], [1] and [5].

Our focus is on the exclusion of the first assumption, since we want to study the effects of ambiguity aversion. An important work in such a direction is that of [4], whose main result indicates that the influence of *maxmin* agents in complete markets becomes irrelevant when compared with rational expectations individuals. Such a result could make it seem that ambiguity averse preferences are economically unimportant, but this is not true. This kind of preference has been used to improve economic theory in many areas, providing new approaches and solving problems with a realistic appeal (see [8] for a brief survey in finance).

The survival analysis by [12] and [3] shows that if agents behave according to expected utilities, then what matters in determining survival are the intertemporal discount factors and beliefs. [4] analyzes survival of *maxmin* agents, who are ambiguity averse, and concludes that survival for this type of agent is difficult to happen if a rational expectation agent is present, due to the aggregate risk. By considering a general type of ambiguity averse preference, we can reconcile survival of ambiguity averse agents with the presence of aggregate risk.

Next section presents the framework. Preferences are discussed in Section 3. Considerations with respect to Pareto optimality are in Section 4. The Section 5 brings examples of asymptotic behavior of consumption decisions in different situations where optimality conditions are met. The mean results are presented in Section 6, the Section 7 concludes, proofs and auxiliary results are in Appendix.

2. Dynamic Model of General Equilibrium

Consider a dynamic model with discrete time $\mathcal{T} = \{0, 1, \dots\}$. There is a finite set of agents $\mathcal{I} = \{1, \dots, I\}$, which have common information modeled by a filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}})$, where $\Omega := \{\omega_0\} \times \prod_{t\geq 1} \mathcal{S}_t$, with ω_0 the sure state occurring for the first time and $\mathcal{S}_t = \{1, \dots, S_t\}$ the set of possible states occurring at each time $t \geq 1$. A representative element of Ω will be denoted by $\omega = (\omega_0, \omega_1, \dots)$ and time-t history $\omega^t = (\omega_0, \dots, \omega_t) \in \Omega^t := \{\omega_0\} \times \prod_{\tau=1}^{t} \mathcal{S}_\tau$. Let $\mathcal{F}_t$ be the σ-algebra generated by $(t+1)$-dimensional cylinders, i.e., $\mathcal{F}_t = \sigma(\{G_t(\omega); \omega \in \Omega\})$, where $G_t(\omega) := \{\omega^t\} \times \prod_{\tau>t} \mathcal{S}_\tau$.

Let $\mathcal{F}^0 = \cup_{t\in\mathcal{T}} \mathcal{F}_t$ be the algebra of finite-time events and $\mathcal{F} = \sigma(\mathcal{F}^0)$ the σ-algebra generated by $\mathcal{F}^0$. The filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathcal{F})$ represents the informational process known by agents. Process ω_t is governed by probability $\mathbb{P}(\cdot|\omega^{t-1})$ on $\mathcal{S}_t$, which can be understood as the conditional probability given ω^{t-1} in the past. These probabilities generate law $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by constructing the partials $\mathbb{P}(\omega^t) = \mathbb{P}(\omega^{t-1})\mathbb{P}(\omega_t|\omega^{t-1})$ on $\mathcal{F}_t$ for each $t \in \mathcal{T}$, and evoking Kolmogorov's extension theorem (see [13]).

The set of all probabilities on a measurable space $(A, \mathcal{A})$ is denoted by $\Delta(A, \mathcal{A})$, or $\Delta(\Omega)$ instead $\Delta(\Omega, \mathcal{F})$ for simplicity. If $P \in \Delta(\Omega)$, P_t denotes its restriction to $\mathcal{F}_t$, and $P_{t+1}(s|\omega^t) := \frac{P_{t+1}(\omega^t, s)}{P_t(\omega^t)}$ denotes the conditional one-step-ahead probability from P. Note that we can consider $P_{t+1}(\cdot|\omega^t) \in \Delta(G_t(\omega), \mathcal{F}_{t+1})$.

For two probabilities, P and Q, we say that Q is absolutely continuous with respect to P if for $A \in \mathcal{F}$, $P(A) = 0$ implies $Q(A) = 0$, and we denote $Q \ll P$. We say that Q is locally absolutely continuous with respect to P if for $A \in \mathcal{F}^0$,

$P(A) = 0$ implies $Q(A) = 0$, and we denote $Q \overset{loc}{\ll} P$. If $Q \ll P$ and $P \ll Q$ we say that P and Q are equivalents and denote $Q \sim P$. Again for P and Q we denote the total variation distance[1] between P and Q by $\|P - Q\|$.

The acts considered by the agents must be based on their knowledge of the world, hence the consequences of an act at period t will be contingent to events at $\mathcal{F}_t$. The individual choice space is a subset of

$$X = \left\{ (x_t)_{t\in\mathcal{T}};\ x_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sup_{t,\omega} |x_t(\omega)| < \infty \right\},$$

whose dual, which contains the prices, is

$$X^* = \left\{ (p_t)_{t\in\mathcal{T}};\ p_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sum_{t,\omega} |p_t(\omega)| < \infty \right\}^{2},$$

considering the duality pair $\langle x, p\rangle = \sum_{t,\omega} x_t(\omega)p_t(\omega)$ that generates the Mackey topology $\tau(X, X^*)$ on X and the weak topology $\sigma(X^*, X)$ on X^*. It is interesting to note that X could be identified by

$$\left\{ x : \bigcup_{t\in\mathcal{T}} (\{t\} \times \Omega^t) \to \mathbb{R};\ \sup_{t,\omega} |x(t, \omega^t)| < \infty \right\},$$

which in turn is basically ℓ^∞.

3. Variational Preferences

To make his decision, the agent considers at first every belief (probability) in $\Delta(\Omega)$. His utility is determined as if he were playing a game against a malevolent Nature that tries to choose a belief that minimizes agent's expected utility, but Nature has a kind of cost to realize a probability as effective belief. *Variational preferences* were developed by ([10, 11]), and there they explore in detail the behavioral properties of this kind of preference.

[1]For two probabilities on a σ-algebra $\mathcal{G}$ the total variation distance is defined by $\| P - Q \| := \sup_{A\in\mathcal{G}} |P(A) - Q(A)|$

[2]The sum $\sum_{t,\omega} |p_t(\omega)|$ makes sense because for each t there are a finite number of values $p_t(\omega)$ since p_t is $\mathcal{F}_t$-measurable.

Agent i's utility functional is given by

$$V^i(c) = \min_{P \in \Delta(\Omega)} \left\{ \mathbb{E}_P \left[\sum_{t \in \mathcal{T}} \beta^t u_i(c_t) \right] + \Gamma^i(P) \right\}$$

and by its recursive form

$$V_t^i(\omega, c) = u_i(c_t(\omega)) + \min_{P \in \Delta(\Omega, \mathcal{F}_{t+1})} \left\{ \mathbb{E}_P \left[V_{t+1}^i(\omega, c) \right] + \gamma_t^i(\omega, P) \right\}.$$

Where $\beta \in (0,1)$ is the inter-temporal discount factor, common to all agents, $u_i : \mathbb{R}_+ \to \mathbb{R}$ is agent i's utility index, $\Gamma^i : \Delta(\Omega) \to [0, \infty]$ and $\gamma_t^i(\omega, \cdot) : \Delta(\Omega, \mathcal{F}_{t+1}) \to [0, \infty]$ are the ambiguity index and dynamic ambiguity index, respectively.

It is supposed that Γ^i and γ_t^i are convex, lower semi-continuous and with 0 in their image; furthermore, γ_t^i satisfies: fixed P, $\gamma_t^i(\cdot, P)$ is $\mathcal{F}_t$-measurable and fixed ω

$$dom\gamma_t^i(\omega, \cdot) := \{P \in \Delta(\Omega);\ \gamma_t^i(\omega, P) < \infty\} \subset \Delta(G_t(\omega), \mathcal{F}_{t+1}).$$

Conditions on ambiguity indexes ensure that beliefs in $dom\Gamma$ are updated according to Bayes' rule[3]. By recursiveness we need to treat only with one-step-ahead decisions and beliefs, and it simplifies analysis.

Examples of variational preferences are the *maxmin* preferences where

$$\Gamma(P) = \begin{cases} 0; \text{ if } P \in C \\ \infty; \text{ otherwise} \end{cases}$$

and

$$\gamma_t(\omega, P) = \begin{cases} 0; \text{ if } P = Q_{t+1}(\cdot|\omega^t) \text{ for some } Q \in C \\ \infty; \text{ otherwise} \end{cases}$$

where $C \subset \Delta(\Omega)$ is closed, convex and rectangular (for definitions see [7]), the *expected utility* preferences that are *maxmin* with $C = \{Q\}$, and the Q*-multiplier* preferences where

$$\Gamma(P) = \begin{cases} \theta \mathbb{E}_P \left[\log \left(\frac{dP}{dQ} \right) \right]; \text{ if } P \ll Q \\ \infty; \text{ otherwise} \end{cases}$$

[3]For details see [11].

and

$$\gamma_t(\omega, P) = \begin{cases} \theta\beta^{-t}\mathbb{E}_P\left[\log\left(\frac{dP}{dQ_{t+1}(\cdot|\omega^t)}\right)\right];\ \text{if } P \ll Q_{t+1}(\cdot|\omega^t) \\ \infty;\ \text{otherwise} \end{cases}$$

with $\theta > 0$.

While *maxmin* individuals deal with beliefs in an "all or nothing" way, the *multiplier* individual has a "smoother" method of dealing with beliefs. We can see that *variational* preferences are able to encompass several kinds of behavior.

4. Pareto Optimality

Following [3] the analysis will be made from the Pareto optimal allocations, hence, the consequences will be valid for complete markets.

We suppose that each agent is endowed with an initial consumption stream $e^i \in X_+$.

Definition 1. *An allocation $(c^i)_{i\in\mathcal{I}}$ is called* ***Pareto optimal*** *if it is feasible, that is, $\sum_i c^i = \sum_i e^i$, and there is no feasible allocation $(\dot{c}^i)_{i\in\mathcal{I}}$ such that $V^i(\dot{c}^i) \geq V^i(c^i)$ $\forall i$ and $V^{i_0}(\dot{c}^{i_0}) > V^{i_0}(c^{i_0})$ for some $i_0 \in \mathcal{I}$.*

We consider only consumptions in X_{++}, and if $c^* = (c^{1*}, \ldots, c^{I*}) \in X_{++}^{\mathcal{I}}$ is Pareto optimal, there is $(\lambda_1, \ldots, \lambda_I) \gg \mathbf{0}$ such that c^* is the solution for problem

$$\begin{cases} \max\limits_{(c^1,\ldots,c^I)} \sum\limits_i \lambda_i V^i(c^i) \\ s.t.\ \sum\limits_i (c^i - e^i) \leq \mathbf{0}. \end{cases} \tag{1}$$

By the first order conditions [6, 124] for that problem there are constants $\eta_t(\omega) > 0$ such that

$$\lambda_i p_t^i(\omega) = \eta_t(\omega) \tag{2}$$

for some $p^i = (p_t^i) \in \partial V^i(c^{i*})$, for any $i \in \mathcal{I}$.

Next lemma is part of Theorem 18 of [10] and characterizes the superdifferential of a variational utility.

Lemma 1. *The superdifferential of variational utility V has the form*

$$\partial V(c) = \left\{ (\beta^t u'(c_t) dP_t); P \in \arg \min_{Q \in \Delta(\Omega)} \left\{ E_Q \left[\sum_{t \in \mathcal{T}} \beta^t u(c_t) \right] + \Gamma(Q) \right\} \right\},$$

for any $c \in X_{++}$.

If $(c^1, \ldots, c^I)$ is Pareto optimal, by equation (2) and by previous lemma we get for each $i \in \mathcal{I}$ a probability $\mathbb{P}^i \in \arg\min_{Q \in \Delta(\Omega)} \left\{ E_Q \left[\sum_{t \in \mathcal{T}} \beta^t u(c_t) \right] + \Gamma(Q) \right\}$ that is the effective belief of agent i. Such probabilities are related with the fixed allocation and carry all information needed to determine survival[4].

We can derive from (2) some useful relations: for all $t \in \mathcal{T}$, $\omega \in \Omega$ and $i, j \in \mathcal{I}$

$$\lambda_i \beta^t u_i'(c_t^i(\omega)) \mathbb{P}_t^i(\omega) = \lambda_j \beta^t u_j'(c_t^j(\omega))\, \mathbb{P}_t^j(\omega), \tag{3}$$

moreover, we get for every $s \in \mathcal{S}_t$

$$\frac{u_i'(c_t^i(\omega^{t-1}, s))}{u_i'(c_{t-1}^i(\omega^{t-1}))} \mathbb{P}_t^i(s|\omega^{t-1}) = \frac{u_j'(c_t^j(\omega^{t-1}, s))}{u_j'(c_{t-1}^j(\omega^{t-1}))} \mathbb{P}_t^j(s|\omega^{t-1}) \tag{4}$$

and, $\forall r, s \in \mathcal{S}_t$

$$\frac{u_i'(c_t^i(\omega^{t-1}, s))}{u_j'(c_t^j(\omega^{t-1}, s))} \frac{\mathbb{P}_t^i(s|\omega^{t-1})}{\mathbb{P}_t^j(s|\omega^{t-1})} = \frac{u_i'(c_t^i(\omega^{t-1}, r))}{u_j'(c_t^j(\omega^{t-1}, r))} \frac{\mathbb{P}_t^i(r|\omega^{t-1})}{\mathbb{P}_t^j(r|\omega^{t-1})}. \tag{5}$$

By Lemma 1 and recursive form of utilities we get

$$\mathbb{P}_t^i(\cdot|\omega^{t-1}) \in \arg \min_{P \in \Delta(G_t(\omega), \mathcal{F}_{t+1})} \left\{ E_P \left[u(c_t^i) \right] + \gamma_t^i(\omega, P) \right\}.$$

5. Examples

This section presents some representative situations for general results about the survival problem. The context, in terms of uncertainty and endowments, is the same in all cases. There are two states of nature and two agents, $\mathcal{S} = \{1, 2\} = \mathcal{I}$, $\mathbb{P}$ is generated by i.i.d. trials uniformly on $\mathcal{S}$, i.e., $\mathbb{P}_t(1|\omega^{t-1}) = 1/2$, $\forall\ t$. Agent 1 always has expected utility with correct belief and his utility is given by

$$V_1(c^1) = \mathbb{E}_{\mathbb{P}} \left[\sum_{t=0}^{\infty} \left(\frac{1}{2} \right)^t \log c_t^1 \right].$$

[4] Remember that we assume the same inter-temporal discount factor for every agent.

Agent 2 is different in each case, allowing for a comparative analysis. The endowments depend only on the current nature state, $e_t^1(1) = e_t^2(1) = 1/2$, $e_t^1(2) = e_t^2(2) = 1/2 + \delta/2$, with $\delta > 0$.

5.1. *Expected Utility Example.* Beginning with a well known example based on [12] where there are two agents with expected utilities, one of whom has a wrong belief, being driven out of the market by the other one with correct belief. The key to achieve this result is the *law of large numbers*.

Here, agent 2 also has expected utility, but with wrong belief, his utility is given by

$$V_2(c^2) = \mathbb{E}_{\bar{\mathbb{P}}}\left[\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t \log c_t^2\right],$$

with $\bar{\mathbb{P}}_t(1|\omega^t) = 1/2 - \varepsilon$ and $0 < \varepsilon < 1/2$.

By (3) we get

$$\frac{(\frac{1}{2})^t c_t^2(\omega^t)}{(\frac{1}{2}-\varepsilon)^n(\frac{1}{2}+\varepsilon)^{t-n} c_t^1(\omega^t)} = \frac{\lambda_2}{\lambda_1},\ \forall t \in \mathbb{N},$$

where n is the number of times that state 1 occurs.

The law of large numbers gives us $n \approx t/2$, then

$$\frac{(\frac{1}{2})^t}{(\frac{1}{2}-\varepsilon)^n(\frac{1}{2}+\varepsilon)^{t-n}} \approx \frac{(\frac{1}{2})^t}{(\frac{1}{2}-\varepsilon)^{t/2}(\frac{1}{2}+\varepsilon)^{t/2}} = \left(\frac{\frac{1}{4}}{\frac{1}{4}-\varepsilon^2}\right)^{t/2} \xrightarrow{t\to\infty} \infty.$$

Whereas $\frac{\lambda_2}{\lambda_1}$ is a positive constant, $\frac{c_t^2(\omega^t)}{c_t^1(\omega^t)} \to 0$ with probability 1, and by $c_t^1(\omega^t) \leq 1+\delta$ we get $c_t^2(\omega^t) \to 0$ $\mathbb{P}$ a.s.

This example is related to Proposition 2 (1) of [12] and Theorem 3 (ii) of [3]. Note that the only important fact about endowments is their limitation. Below we will show that for survival of an averse ambiguity agent, other features matter.

5.2. *Maxmin Utility Example.* The next example is based on [4] where an agent with maxmin utility cannot survive in the presence of an expected utility with correct belief. In turn, this conclusion strongly depends on the aggregate risk. The ambiguity averse customer acts as if he were an expected utility with wrong belief, and he cannot survive as well as in the previous case. But there is a particularity of maxmin utility that does not occur in the more general model of variational utilities. A maxmin agent deals with his possible beliefs in a homogeneous way, so aggregate risk forces him to take a precautionary attitude that moves away from the one that ensures survival.

In this example the utility of agent 2 is given by

$$V_2(c^2) = \min_{P\in\Delta(\Omega)} \left[\mathbb{E}_P \left(\sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^t \log c_t^2 \right) + \Gamma(P) \right],$$

where $\gamma_t(P_t) = \begin{cases} 0; \text{ if } P_t(1|\omega) \in [1/3, 2/3] \\ \infty; \text{ otherwise.} \end{cases}$

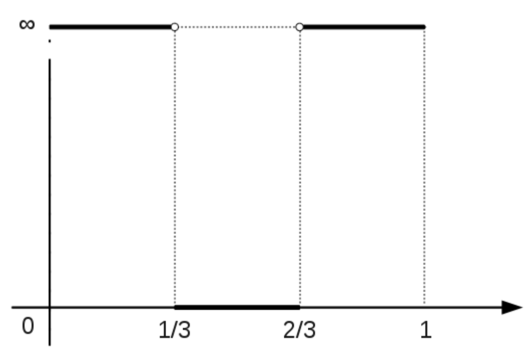

Figure 1: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

By (5) we get

$$\frac{c_t^2(\omega^{t-1},1)}{c_t^1(\omega^{t-1},1)} \frac{\mathbb{P}_t(1|\omega^{t-1})}{\mathbb{P}_t^2(1|\omega^{t-1})} = \frac{c_t^2(\omega^{t-1},2)}{c_t^1(\omega^{t-1},2)} \frac{\mathbb{P}_t(2|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}.$$

and by market clearing

$$\frac{1+\delta - c_t^2(\omega^{t-1},2)}{1 - c_t^2(\omega^{t-1},1)} \frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)} = \frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}. \tag{6}$$

If $c_t^2(\omega^{t-1},1) > c_t^2(\omega^{t-1},2)$ then $\mathbb{P}_t(1|\omega^{t-1}) = 1/3$, because $\mathbb{P}_t^2(\cdot|\omega^{t-1})$ minimizes $\mathbb{E}_P\left[\left(\frac{1}{2}\right)^t (\log c_t^2(\omega^{t-1},\cdot))\right]$ subject to $P(1) \in [1/3, 2/3]$, and by (6) $\frac{1+\delta - c_t^2(\omega^{t-1},2)}{1 - c_t^2(\omega^{t-1},1)} < 1/2$ whence we get

$$1 + \delta - c_t^2(\omega^{t-1},2) < 1/2 - 1/2 c_t^2(\omega^{t-1},1) < 1 - c_t^2(\omega^{t-1},1)$$

so $c_t^2(\omega^{t-1},2) > c_t^2(\omega^{t-1},1)$, a contradiction.

If $c_t^2(\omega^{t-1},1) = c_t^2(\omega^{t-1},2)$, since consumption is positive, from equation (6) we get[5] $\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})} > 1+\delta$. If $c_t^2(\omega^{t-1},2) > c_t^2(\omega^{t-1},1)$, then agent 2 acts like an expected utility assigning probability 2/3 for state 1. In both cases we get $\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})} \geq \min\{1+\delta, 2\}$, so agent 2 does not survive as in the previous example because he always makes inaccurate predictions. Such an example fits Theorem 1 of [4].

5.3. *Motivating Example.* The last example gives an idea of how a variational agent can survive even in a presence of an expected utility with correct belief agent, and with aggregate risk. An individual could be ambiguity averse and survive as long as his ambiguity index is not so small. Such a constraint depends on how big the aggregate risk is.

While agent 1 has expected utility with correct belief, agent 2's utility is given by

$$V_2(c^2) = \min_{P\in\Delta(\Omega)}\left[\mathbb{E}_P\left(\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t \log c_t^2\right) + \Gamma(P)\right],$$

where

$$\gamma_t(P_t) = \begin{cases} (\frac{1}{2} - P_t(1|\omega^{t-1}))\varepsilon; \text{ if } P_t(1|\omega^{t-1}) \leq \frac{1}{2} \\ (P_t(1|\omega^{t-1}) - \frac{1}{2})\varepsilon; \text{ if } P_t(1|\omega^{t-1}) \geq \frac{1}{2} \end{cases}$$

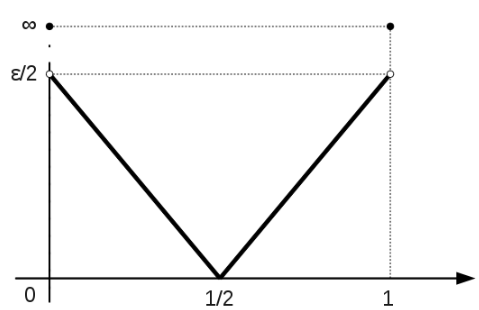

Figure 2: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

[5]Note that $1+\delta < \frac{1+\delta-x}{1-x} < \infty, \forall\, 0 < x < 1$.

Again, as in (6)

$$\frac{1+\delta - c_t^2(\omega^{t-1},2)}{1-c_t^2(\omega^{t-1},1)}\frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)} = \frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}$$

by rearranging this expression

$$\mathbb{P}_t^2(1|\omega^{t-1})\left(\frac{1}{c_t^2(\omega^{t-1},1)}-1\right) = \mathbb{P}_t^2(2|\omega^{t-1})\left(\frac{1+\delta}{c_t^2(\omega^{t-1},2)}-1\right)$$

Consider the possibilities for $\mathbb{P}_t^2(\cdot|\omega^{t-1})$.

If $\mathbb{P}_t^2(1|\omega^{t-1}) < \mathbb{P}_t^2(2|\omega^{t-1})$ we get $\dfrac{c_t^2(\omega^{t-1},2)}{c_t^2(\omega^{t-1},1)} > 1+\delta > 1$, then

$$\begin{aligned}&\mathbb{P}_t^2(1|\omega^{t-1})\log c_t^2(\omega^{t-1},1) + \mathbb{P}_t^2(2|\omega^{t-1})\log c_t^2(\omega^{t-1},2) + \gamma_t(P_t)\\ &> 1/2\log c_t^2(\omega^{t-1},1) + 1/2\log c_t^2(\omega^{t-1},2)\end{aligned}$$

and $\mathbb{P}_t^2$ is not a minimizer.

If $\mathbb{P}_t^2(1|\omega^{t-1}) > \mathbb{P}_t^2(2|\omega^{t-1})$ we get $\dfrac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)} > \dfrac{1}{1+\delta}$. Therefore

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_t^2(\cdot|\omega^{t-1})}[\log c_t^2(\omega^{t-1},\cdot)] + \gamma_t(\omega,P_t) - \mathbb{E}_{\mathbb{P}_t(\cdot|\omega^{t-1})}[\log c_t^2(\omega^{t-1},\cdot)]\\ &= (\mathbb{P}_t^2(1|\omega^{t-1}) - 1/2)\log c_t^2(\omega^{t-1},1) +\\ &+(\mathbb{P}_t^2(2|\omega^{t-1}) - 1/2)\log c_t^2(\omega^{t-1},2) + (\mathbb{P}_t^2(1|\omega^{t-1}) - 1/2)\varepsilon\\ &= (\mathbb{P}_t^2(1|\omega^{t-1}) - 1/2)[\log\left(\frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)}\right) + \varepsilon]\\ &> (\mathbb{P}_t^2(1|\omega^{t-1}) - 1/2)[\log\left(\frac{1}{1+\delta}\right) + \varepsilon]\end{aligned}$$

So if $\varepsilon - \log(1+\delta) > 0$, $P_t = \mathbb{P}_t$ is the only minimizer.

Therefore, agent 2 acts as an expected utility with correct belief if, for example, $\varepsilon = \delta = 1$, which fits into the context of previous examples. The message given to us by these examples is that the relation between survival of an ambiguity averse agent and the presence of aggregate risk could be made in a more precise way than that found in [4]. Theorem 2 is an effort in that direction.

6. Survival

A Pareto optimal allocation $(c_i)_{i\in\mathcal{I}}$ and beliefs $\mathbb{P}^i$ given in (3), for each $i \in \mathcal{I}$, are fixed.

Definition 2. *Agent i survives on the path ω if $\overline{\lim}\, c^i_t(\omega) > 0$. We say that i survives if there is $A \in \mathcal{F}$ with $\mathbb{P}(A) = 1$ such that i survives on all $\omega \in A$.*

Some assumptions are needed to achieve the results.

Assumption 1. *Let $e := \sum_i e^i$. For every $i \in \mathcal{I}$ endowments satisfy $\underline{e} < e^i < e < \bar{e}$, for positive constants $\underline{e}$ and $\bar{e}$.*

Assumption 2. *$u_i' > 0$, $u_i'' < 0$ and $u_i'(x) \xrightarrow{x\to 0} \infty$ for all $i \in \mathcal{I}$.*

Assumption 3. *For all path ω, suppose that $\mathbb{P}_t(\cdot|\omega^{t-1}) > 0$ and*

$$dom\ \gamma^i_t(\omega,\cdot) \subset \Delta^+(G_t(\omega), \mathcal{F}_{t+1}) := \{r \in \Delta(G_t(\omega), \mathcal{F}_{t+1}); r(A) > 0\ \forall A \in \mathcal{F}_{t+1}\backslash\{\emptyset\}\}.$$

Assumption 4. *Agent 1 has expected utility with correct belief.*

Assumptions 1 and 2 guarantee that the solutions to (1) are in X_{++}. Assumption 3 says that every state has a positive chance of occurring any time and after any history; furthermore, relevant beliefs have this same property. Assumption 4 is supposed to test other agents in an unfavorable environment, since they are competing with a well informed agent.

The next lemmata are known results and can be found in [3].

Lemma 2. *Consider $i \neq j$. Agent i does not survive on the event $\left\{\frac{u_i'(c^i_t(\omega))}{u_j'(c^j_t(\omega))} \to \infty\right\}$. If agent i does not survive on ω, then for some $j \in \mathcal{I}$, $\overline{\lim}\, \frac{u_i'(c^i_t(\omega))}{u_j'(c^j_t(\omega))} = \infty$.*

Lemma 3. *Agent i survives $\mathbb{P}^i$ almost surely.*

By the previous lemma, a criterion for survival of an agent i is that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}^i$, Lemma 4 shows that this condition is also necessary.

Lemma 4. *Agent i survives if, and only if,* $\mathbb{P} \ll \mathbb{P}^i$.

To know about survival relies in verifying if two probabilities are equivalent. Next result give us a way to do this. As hypothesis we need local equivalence, what in our case is supplied by Assumption 3

Lemma 5. *If $P, Q \in \Delta(\Omega)$ satisfies $Q_t \ll P_t$ for every t, then are equivalents:*

(i) $\|P(\cdot|\omega^t) - Q(\cdot|\omega^t)\| \xrightarrow{t\to\infty} 0$ Q-***a.s.***;

(ii) $Q \ll P$.

The survival analysis as we can see in [12] and [3] does not take in account the dynamic of aggregate endowments. On the other hand, in [4]'s arguments the asymptotic variation of aggregate endowments plays a crucial role. Next is defined aggregate risk

Definition 3. *Define the functional $\delta : X \times \Omega \to \mathbb{R}$ by*

$$\delta(x, \omega) := \varliminf_{t} \left(\sup\{|x_t(\omega^{t-1}, r) - x_t(\omega^{t-1}, s)|; r, s \in \mathcal{S}_t\} \right).$$

There is ***aggregate risk on the path*** *ω if $\delta(e, \omega) > 0$, if there is aggregate risk $\mathbb{P}$ almost surely we simply say that there is aggregate risk.*

The next definition is analogous to the *strict minimum consensus property* of [4], and the following theorem is a generalization of his Theorem 1 for variational preferences.

Definition 4. *We say that agent i satisfies* ***property P*** *if $\exists\, T \in \mathcal{T}$ and $\varepsilon > 0$ such that $\forall t > T$, if $P \in \Delta(G_t(\omega), \mathcal{F}_{t+1})$ satisfies $\|P(s) - \mathbb{P}_t(s|\omega^{t-1})\| \leq \varepsilon$, then $\gamma_t^i(\omega, P) < \infty$.*

Variational preferences encompass a large spectrum of...

Definition 5. *We say that agent a is more ambiguity averse than agent b if $u_a = u_b$ and $\Gamma_a \leq \Gamma_b$.*

Theorem 1. *Assume that there is aggregate risk. If agent i is more ambiguity averse than a maxmin agent that satisfies property P, then i does not survive.*

The subsequent result can be understood as limiting the scope of maxmin utilities in survival analysis, because a variational agent can survive even believing in "distributions which differ from the truth in all feasible directions"[6].

Theorem 2. *Suppose that* $u_i(0) > -\infty$ *and* $\mathcal{S}_t = \mathcal{S}$ *for all* $t > 0$. *If there is* $T \in \mathcal{T}$ *such that for every* $t > T$, $\gamma^i_{t-1}(\omega, \mathbb{P}_t(\cdot|\omega^{t-1})) = 0$ *and*

$$\gamma^i_{t-1}(\omega, P) \geq S \max\{|u_i(0)|, |u_i(\bar{e})|\} \|P(s) - \mathbb{P}_t(s|\omega^{t-1})\|,$$

then i *survives on* ω.

Lemma 3 and Lemma 5 together compose the main tool to attain survival results. Lemma 3 tell us that an individual always acts to guarantee his survival based on his effective belief, and if its posteriors converge to the truth posteriors then, according to Lemma 5, such an agent survives.

According to the proof of Theorem 2, we can see that relevant one-step-ahead beliefs at time t belongs to set

$$A^i_t(\omega) = \left\{ P \in \Delta(G_t(\omega), \mathcal{F}_{t+1}); \gamma^i_t(\omega, P) \leq S(|u_i(0)| \vee |u_i(\bar{e})|) \|P - \mathbb{P}_{t+1}(\cdot|\omega^t)\| \right\},$$

for each $t \in \mathcal{T}$. So, if $B^i_t = \{P \in \Delta(\Omega); P_{\tau+1}(\cdot|\omega^\tau) \in A^i_\tau(\omega)\ \forall \tau \leq t\}$ the set of relevant beliefs[7] is $B^i = \cap_{t\in\mathcal{T}} B_t$.

In many situations it is natural to suppose that ambiguity aversion vanishes over time. In such a case, dynamic ambiguity indexes will increase with t and sets A^i_t will decrease, as sketched in Figure 3. If sets A^i_t collapse in a point, by the same hypotheses made in Theorem 2, any probability in B^i will be equivalent to $\mathbb{P}$[8]. Therefore, we have conditions on ambiguity indexes that ensure survival. An interesting consequence follows.

[6]This quotation from [4] is part of his explanation about property P that ensures the non survival of maxmin agents.

[7]We refer to belief as relevant when it is a candidate to minimize $E_Q\left[\sum_{t\in\mathcal{T}} \beta^t u_i(c_t)\right] + \Gamma^i(Q)$.

[8]If $\gamma^i_t(\omega, \mathbb{P}_{t+1}(\cdot|\omega^t)) = 0$, then $\mathbb{P}_{t+1}(\cdot|\omega^t) \in A^i_t(\omega)$. We know that $\mathbb{P}^i_{t+1}(\cdot|\omega^t) \in A^i_t(\omega)$, so if the sequence of sets $A^i_t(\omega)$ collapses into a single point we get $\|\mathbb{P}_{t+1}(\cdot|\omega^t) - \mathbb{P}^i_{t+1}(\cdot|\omega^t)\| \to 0$.

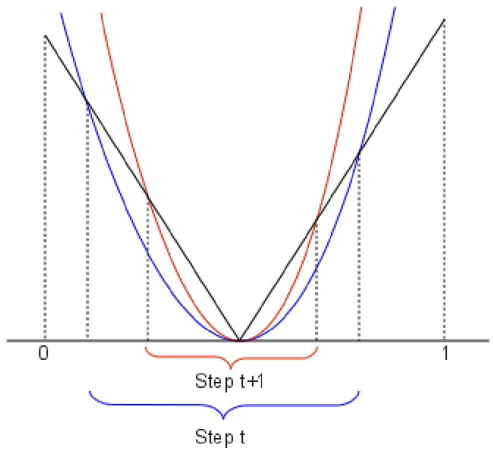

Figure 3: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

Scholium 1. *A multiplier agent* i*, such that* $u_i(0) > -\infty$*, survives if, and only if,* $\mathbb{P} \ll Q^i$ *where* Q^i *is his reference probability.*

In the next proposition we assume that there are only two agents. While agent 1 has an expected utility, agent 2 has a more general variational utility. For agent 2 we consider that two distinct types are possible, a and b. Type a is less ambiguity averse than b, so their utility index are the same and the ambiguity index of a is greater than the ambiguity index of b. Note that if (c^1, c^2) is a Pareto optimal allocation when agent 2 is of type b, then, assuming that $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$, the same allocation is Pareto optimal even when agent 2 is of type a. Proposition 1 gives an inverse relationship between the level of ambiguity aversion and survival.

Proposition 1. *Suppose that* a *is less ambiguity averse than* b *and* $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$*. If type* b *survives, then type* a *also survives.*

7. Conclusion

Survival of individuals behaving according to expected utility depends on intertemporal discount factors and compatibility between beliefs and the truth as shown

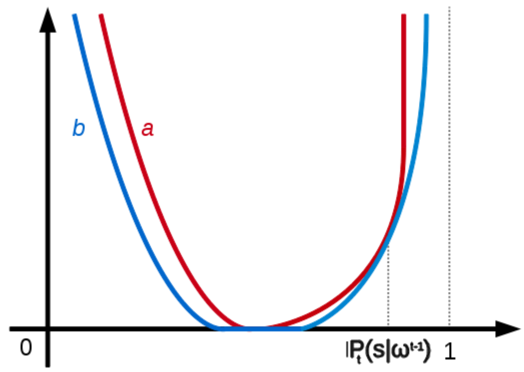

Figure 4: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

in [12] and [3]. To study the influence of ambiguity aversion, the step taken by [4] was to introduce agents with *maxmin* utilities.

Considering $\beta^i = \beta^j \ \forall i, j$ to isolate aversion ambiguity effects, he finds that ambiguity averse agents survive under aggregate risk only in special cases. By introducing *variational* preferences that are more general than *maxmin*, we find that ambiguity averse individuals, with analogous characteristics to those in [4]'s case, can survive under aggregate risk. Moreover, in particular cases it is possible to make finer relations between the level of ambiguity aversion and the magnitude of aggregate risk that lead to survival.

8. Appendix

Proof of Lemma 2: If $\frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))} \to \infty$, then $c_t^i(\omega) \to 0$, by assumptions 1 and 2.

On the other hand, if $c_t^i(\omega) \to 0$, by assumption 1 there is $j \in \mathcal{I}$ such that $c_t^j > \underline{e}/I$ for infinite indexes t. Hence, the denominator of $\frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))}$ is upper bounded, and the result follows by assumption 2.

□

Proof of Lemma 3: Let $j \neq i$ in $\mathcal{I}$. Define the following random variables on $(\Omega, \mathcal{F})$,

$$L_t(\omega) = \frac{\mathbb{P}_t^j(\omega)}{\mathbb{P}_t^i(\omega)}.$$

Will be proven that $\{L_t\}$ is martingale with respect to $(\mathcal{F}_t)$ and $\mathbb{P}^i$. Indeed,

$$\mathbb{E}_{\mathbb{P}^i}[L_{t+1}|\mathcal{F}_t](\omega) = \sum_{s\in\mathcal{S}} \frac{\mathbb{P}_{t+1}^j(\omega^t, s)}{\mathbb{P}_{t+1}^i(\omega^t, s)} \frac{\mathbb{P}^i(\{(\omega^t, s)\} \times \Omega)}{\mathbb{P}^i(\{\omega^t\} \times \Omega)} = \sum_{s\in\mathcal{S}} \mathbb{P}_{t+1}^j(\omega^t, s)/\mathbb{P}_t^i(\omega^t) = L_t(\omega).$$

It is also easy to see that $\mathbb{E}[L_t] = 1$, $\forall t$. Therefore, by martingale convergence (see [13]) (L_t) converges and its limit is finite $\mathbb{P}^i$ almost surely. Finally, by equation (3) and by Lemma 2 agent i survives $\mathbb{P}^i$ almost surely.

□

Proof of Lemma 5: $(i) \Rightarrow (ii)$ follows by [9], on the other hand $(ii) \Rightarrow (i)$ is due to [2].

□

Proof of Lemma 4: If $\mathbb{P} \ll \mathbb{P}^i$ then, by Lemma 3, agent i survives $\mathbb{P}$ almost surely.

Note that, by Assumption 3, $\mathbb{P}_t$ and $\mathbb{P}_t^i$ are equivalents. If i survives then, according to Lemma 2,

$$\mathbb{P}\left(\frac{u_i'(c_t^i(\omega))}{u_1'(c_t^1(\omega))} \nrightarrow \infty\right) = \mathbb{P}\left(L_t(\omega) \nrightarrow \infty\right) = 1,$$

where $L_t(\omega) = \frac{\mathbb{P}_t(\omega)}{\mathbb{P}_t^i(\omega)}$. By the proof of Theorem 1 p. 493 of [13] we get $\mathbb{P}(\exists \lim L_t(\omega)) = 1$, therefore $\mathbb{P}(\lim L_t(\omega) < \infty) = 1$. Finally, by Theorem 2 p. 495 of [13], $\mathbb{P} \ll \mathbb{P}^i$.

□

Proof of Theorem 1: Suppose that i survives.

If $c_t^i(\omega^{t-1}, \cdot)$ is constant for a large enough t, by (4) and Lemma 5 for any j that survives we get[9]

$$\frac{u_i'(c_t^i(\omega^{t-1}, s))}{u_i'(c_{t-1}^i(\omega^{t-1}))} \approx \frac{u_j'(c_t^j(\omega^{t-1}, s))}{u_j'(c_{t-1}^j(\omega^{t-1}))}.$$

[9] $a_t \approx b_t$ means $|a_t - b_t| \xrightarrow{t\to\infty} 1$.

Since $c_t^i(\omega^{t-1},\cdot)$ is constant, $c_t^j(\omega^{t-1},\cdot)$ is asymptotically constant, i.e., $\delta(c^j)=0$. Then $\delta(e)=\delta(\sum_{j\ survives} c^j)=0$, a contradiction.

So, for any $\tau\in\mathcal{T}$ there is $t>\tau$ such that $c_t^i(\omega^{t-1},\cdot)$ is not constant. Then

$$\|\mathbb{P}_t^i(\cdot|\omega^{t-1})-\mathbb{P}_t(\cdot|\omega^{t-1})\|\geq\varepsilon$$

for a sequence $t\nearrow\infty$, and by Lemma 5 again, i does not survive.

□

Proof of Theorem 2: Let $c\in X$ and $\omega\in\Omega$. For any $P\in dom\ \gamma_{t-1}^i(\omega,\cdot)$ we get

$$\begin{aligned}
&\left\{\mathbb{E}_P\left[u_i(c_t(\omega^{t-1},\cdot))\right]+\gamma_{t-1}^i(\omega,P)\right\}-\left\{\mathbb{E}_{\mathbb{P}_t(\cdot|\omega^{t-1})}\left[u_i(c_t(\omega^{t-1},\cdot))\right]+\gamma_{t-1}^i(\omega,\mathbb{P}_t(\cdot|\omega^{t-1})\right\}\\
= &\sum_{s\in\mathcal{S}}u_i(c_t(\omega^{t-1},s))(P(s)-\mathbb{P}_t(s|\omega^{t-1}))+\gamma_{t-1}^i(\omega,P)\\
\geq &\sum_{s\in\mathcal{S}}u_i(c_t(\omega^{t-1},s))(P(s)-\mathbb{P}_t(s|\omega^{t-1}))\\
+ &\ S\max\{|u_i(0)|,|u_i(\bar{e})|\}\|P(s)-\mathbb{P}_t(s|\omega^{t-1})\|\\
\geq &\ 0.
\end{aligned}$$

So $\{\mathbb{P}_t(\cdot|\omega^{t-1})\}=\arg\min_{P\in dom\ \gamma_{t-1}^2(\omega,\cdot)}\left\{\mathbb{E}_P\left[u_2(c_t(\omega^{t-1},\cdot))\right]+\gamma_{t-1}^2(\omega,P)\right\}$.

□

Proof of Scholium 1: The dynamic ambiguity index of agent i has the form

$$\gamma_t(\omega,P)=\begin{cases}\theta\beta^{-t}\mathbb{E}_P\left[\log\left(\frac{dP}{dQ_{t+1}(\cdot|\omega^t)}\right)\right], & \text{if } P\ll Q_{t+1}(\cdot|\omega^t)\\ \infty, & \text{otherwise}\end{cases}$$

We define the sets

$$A_t=A_{t-1}\cap\left\{P\in\Delta(\Omega);\gamma_t^i(\omega,P_{t+1}(\cdot|\omega^t))\leq S\{|u_i(0)|\vee|u_i(\bar{e})|\}\|P_{t+1}(\cdot|\omega^t)-\mathbb{P}_{t+1}(\cdot|\omega^t)\|\right\}$$

As γ_t^i is l.s.c. and $Q^i\in A_t$ for all t the sets are compact and $\cap A_t\neq\varnothing$. Since $\beta^{-t}\to\infty$ any probability in $\cap A_t$ will be a minimizer for

$$\min_{P\in\Delta(\Omega)}\left\{\mathbb{E}_P\left[\sum_{t\in\mathcal{T}}\beta^t u_i(c_t)\right]+\gamma^i(P)\right\}$$

and by Lemma 5 such probabilities are equivalents to Q^i. Therefore, according to Lemma 4 the result follows.

□

Proof of Proposition 1: An agent a with utility V^a is less ambiguity averse than another with utility V^b if $u^a = u^b$ and $\Gamma^a \geq \Gamma^b$. If $\mathbb{P}^b$ minimizes $\mathbb{E}_P[u^b(c^2)] + \Gamma^b(P)$ and $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$, then, $\forall P \in \Delta(\Omega)$

$$\mathbb{E}_{\mathbb{P}^b}[u^a(c^2)] + \Gamma^a(\mathbb{P}^b) = \mathbb{E}_{\mathbb{P}}[u^b(c^2)] + \Gamma^b(\mathbb{P}^b) \leq \mathbb{E}_P[u^b(c^2)] + \Gamma^b(P) \leq \mathbb{E}_P[u^a(c^2)] + \Gamma^a(P).$$

So $\mathbb{P}^b$ minimizes $\mathbb{E}_P[u^a(c^2)] + \Gamma^a(P)$ too, and if b survives then a survives.

□

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