CHARACTERISTIC SHOCKS FOR FLOW IN POROUS MEDIA

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ABSTRACT. We utilize the wave curve method to prove the existence and give a characterization of a certain bifurcation locus in the saturation triangle. Such structure arises, for instance, when water and gas are injected in a mature reservoir either to dislodge oil or to sequestrate CO_2. The proof takes advantage of a certain wave curve to ensure that the waves in the flow are a rarefaction preceded by a shock, which is in turn preceded by a constant two-phase state (*i.e.*, it lies at the boundary of the saturation triangle). For convex permeability models of Corey type, the analysis reveals further details, such as the number of possible two-phase states that correspond to the above mentioned shock, whatever the left state of the latter is within the saturation triangle.

1. Introduction

We are interested in solving problems of injection of water and gas in oil recovery, as well as the injection of gas alone into porous media in order to sequestrate CO_2. In both problems, shocks connecting three-phase states (*i.e.* interior to the saturation triangle) to two-phase states (*i.e.* at the boundary of the saturation triangle) are important. It is important to find the boundary separating admissible from non-admissible shocks. In this work we present theorems establishing the existence and uniqueness of such a separation curve.

We are interested in injection problems leading to flow in porous media of three phases that do not exchange mass. Such flows are modeled by systems of two conservation laws; a survey of the mathematical theory for such flows may be found in [1, 3, 15] and references therein.

In the interior of the saturation triangle, convex relative permeability Corey models such as those considered in this work lose strict hyperbolicity only at an "umbilic point", *i.e.*, a point where the two characteristic speeds coincide (and there are two distinct eigenvectors), but the speeds are distinct around this point. In previois works, the location and characterization of the umbilic point has already established, see *e.g.* [4, 5, 16].

However, loss of strict hyperbolicity gives rise to rich structures for solutions of Riemann problems (*cf.* [8, 9, 10, 11]). In this work, we characterize one of the structures that is fundamental in these constructions, which has been named "extension of the physical boundaries". These estimates can shorten the uniqueness proof in [2].

2. Mathematical model

Consider the flow of a mixture of three fluid phases (which, for concreteness, are called water, gas and oil) in a thin, horizontal cylinder of porous rock. Let $s_{\mathrm{w}}(x,t)$, $s_{\mathrm{g}}(x,t)$ and $s_{\mathrm{o}}(x,t)$ denote the respective saturations at distance x along the cylinder, at time t. Because

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$s_{\mathrm{w}}+s_{\mathrm{g}}+s_{\mathrm{o}}=1$ and $0 \le s_{\mathrm{w}},\ s_{\mathrm{g}},\ s_{\mathrm{o}} \le 1$, the state space for the model is the saturation triangle Δ; see *e.g.* Fig. 3.2. In our analysis, we choose s_{w} and s_{g} as the two independent variables, thus $S := (s_{\mathrm{w}},\ s_{\mathrm{g}})^T$; the vertices of Δ are $\mathrm{W} = (1,\ 0)^T$, $\mathrm{G} = (0,\ 1)^T$ and $\mathrm{O} = (0,\ 0)^T$.

2.1. **Conservation laws.** Three-phase flow in 1D at constant injection rate is governed by the non-dimensionalized system $\partial S/\partial t + \partial F(S)/\partial x = 0$, or

$$\frac{\partial s_{\mathrm{w}}}{\partial t} + \frac{\partial f_{\mathrm{w}}(s_{\mathrm{w}},\ s_{\mathrm{g}})}{\partial x} = 0, \qquad \frac{\partial s_{\mathrm{g}}}{\partial t} + \frac{\partial f_{\mathrm{g}}(s_{\mathrm{w}},\ s_{\mathrm{g}})}{\partial x} = 0, \tag{1}$$

representing conservation of water and gas. (Of course, as the satuarion S is the vector $(s_{\mathrm{w}},\ s_{\mathrm{g}})^T$, the flux $F(S)$ is given by the vector $(f_{\mathrm{w}}(S),\ f_{\mathrm{g}}(S))^T$.) The flux functions $f_{\mathrm{w}}(s_{\mathrm{w}},\ s_{\mathrm{g}})$ and $f_{\mathrm{g}}(s_{\mathrm{w}},\ s_{\mathrm{g}})$ are determined by the relative permeabilities of the three phases. For simplicity we assume that the relative permeabilities are strictly positive within the saturation triangle. From Darcy's law the fluxes are

$$f_\alpha(S) = \frac{\mathrm{m}_\alpha(S)}{\mathrm{m}(S)}, \quad \text{for} \quad \alpha = \mathrm{w},\ \mathrm{g},\ \mathrm{o}, \quad \text{where} \quad \mathrm{m} := \mathrm{m}_{\mathrm{w}} + \mathrm{m}_{\mathrm{g}} + \mathrm{m}_{\mathrm{o}}$$

is the total mobility; m_{w}, m_{g}, m_{o} represent the mobility of each phase.

A Corey type model is defined by a set of mobilities $\mathrm{m}_\alpha(s_\alpha)$ that are nondecreasing continuous functions of their own saturation s_α. In this work we focus on convex Corey models:

Definition 1. A Corey model is said to be **convex** when the mobilities are $\mathcal{C}^1[0,\ 1] \cap \mathcal{C}^2(0,\ 1)$ functions satisfying:

(1) $\mathrm{m}_\alpha(s_\alpha) > 0$ for $s_\alpha \in (0,\ 1]$ and $\mathrm{m}_\alpha(0) = 0$,
(2) $\mathrm{m}'_\alpha(s_\alpha) > 0$ for $s_\alpha \in (0,\ 1]$ and $\mathrm{m}'_\alpha(0) = 0$,
(3) $\mathrm{m}''_\alpha(s_\alpha) \ge 0$ for $s_\alpha \in (0,\ 1)$,
(4) no pair of the quantities $\mathrm{m}''_{\mathrm{w}}(s_{\mathrm{w}})$, $\mathrm{m}''_{\mathrm{g}}(s_{\mathrm{g}})$, $\mathrm{m}''_{\mathrm{o}}(s_{\mathrm{o}})$ vanish simultaneously for any state in the interior of the saturation triangle ($0 < s_{\mathrm{w}},\ s_{\mathrm{g}},\ s_{\mathrm{o}} < 1$).

2.1.1. *Basic solutions.* System (1) has solutions that propagate as waves. The Jacobian matrix of the fluxes is the key for rarefaction waves. The characteristic speeds are the two eigenvalues of the Jacobian derivative matrix

$$\mathrm{J}(S) := \frac{\partial(f_{\mathrm{w}},\ f_{\mathrm{g}})}{\partial(s_{\mathrm{w}},\ s_{\mathrm{g}})}(S) = \frac{\partial F}{\partial S}(S),$$

provided that these eigenvalues are real, in which case the smaller one is called the slow-family characteristic speed $\lambda_{\mathrm{s}}(s_{\mathrm{w}},\ s_{\mathrm{g}})$ and the larger one is called the fast-family characteristic speed $\lambda_{\mathrm{f}}(s_{\mathrm{w}},\ s_{\mathrm{g}})$. For the Corey model, both eigenvalues are real and nonnegative for each state in the saturation triangle.

The *self-similarity* of solutions of a Riemann problem implies that if $u(x,\ t)$ is a solution at a given time t, then $u(\alpha x,\ \alpha t)$ is also a solution for any $\alpha > 0$. Centered rarefaction and shock waves are self-similar. In this work we analyze particuliarities of the second type, *i.e.*, the jump discontinuities produced by shocks.

The Hugoniot locus of a state S^o, denoted by $\mathcal{H}(S^o)$, is given by all the states S that satisfy the Rankine-Hugoniot (RH) condition:

$$F(S) - F(S^o) = \sigma(S - S^o), \tag{2}$$

where $\sigma = \sigma(S^o, S)$ is the propagation speed of the discontinuity between S and S^o, and the fluxes $F(S)$ and saturations S are given as before. (Notice that S belongs to $\mathcal{H}(S^o)$ if and only if S^o belongs to $\mathcal{H}(S)$.)

Since the Hugoniot locus for a state on an edge of the saturation triangle always contains the edge, it is important to identify the part of this locus that lies in the interior of the saturation triangle, so we are led to define for any state S on an edge of the triangle the subset $\mathcal{H}_{int}(\mathrm{S})$ of $\mathcal{H}(S)$ consisting of points interior to the triangle. In the case of convex Corey models, taking S as a vertex of the saturation triangle, the Hugoniot locus is given by $\mathcal{H}_{int}(\mathrm{S})$ as a curve interior to the triangle except at S and at the state on the opposite edge where the curve reaches the boundary of the saturation triangle, and the two edges that meet at S (*cf.* [4, 5]).

We recall two versions of the very useful triple shock rule [4, 5]. Denoting the shock speed $\sigma(S_i, S_j)$ by σ_{ij}: *Triple Shock Rule I*: consider states S_1, S_2 belonging to $\mathcal{H}(S_0)$; if $\sigma_{01} = \sigma_{02}$ holds, then S_1 belongs to $\mathcal{H}(S_2)$ and $\sigma_{01} = \sigma_{02} = \sigma_{12}$ holds. *Triple Shock Rule II*: let S_0, S_1, S_2 be noncollinear states such that S_1, S_2 belong to $\mathcal{H}(S_0)$ and S_1 belongs to $\mathcal{H}(S_2)$; then $\sigma_{01} = \sigma_{02} = \sigma_{12}$ holds.

The following definitions are inspired by the Oleĭnik-Welge's construction (see [4, 6, 13, 14, 17]) for a single conservation equation.

Definition 2. A state S is said to be an *extension of a state S^o relative to the characteristic family i* (slow or fast) if (i) S lies in $\mathcal{H}(S^o)$ and (ii) $\sigma(S, S^o) = \lambda_i(S)$. Typically the extension of a state consists of a number of states. We denote by $\mathcal{E}_i(S^o)$ the locus of extension states of the state S^o. An *interior boundary extension relative to the characteristic family i for the edge WO* is the union of loci $\mathcal{E}_i(S)$ for all $S \in WO$. We denote the slow locus by $\mathcal{I}^{\mathrm{s}}_{\mathrm{WO}}$ and the fast by $\mathcal{I}^{\mathrm{f}}_{\mathrm{WO}}$; analogous definitions hold for the other edges. Conversely, an *exterior boundary extension relative to the characteristic family i for the edge WO* is the locus of all states the i-th extension of which are states on WO. We denote the slow locus by $\mathcal{E}^{\mathrm{s}}_{\mathrm{WO}}$ and the fast by $\mathcal{E}^{\mathrm{f}}_{\mathrm{WO}}$; analogous definitions hold for the other edges.

A system is called strictly hyperbolic at S if the characteristic speeds satisfy the inequality $\lambda_{\mathrm{s}}(S) < \lambda_{\mathrm{f}}(S)$; they are well studied [12, 14]. In three-phase flow models there are points where the characteristic speeds coincide, which are called coincidence points. Furthermore, in Corey models there are isolated coincidence points where the Jacobian matrix is a multiple of the identity, *i.e.*, umbilic points. The classification for power-law convex Corey models is in [16].

The vertices of the saturation triangle are also umbilic points, [18].

Property 3. ([4, 5]) *For convex Corey models, the speed of the shocks between the interior umbilic point to the vertices of the triangle are equal to 1.*

3. Structures in the saturation triangle for convex Corey models

In this section we focus on properties of the shock extension of boundaries, the shocks are characteristic either at the interior point or at the edges of the triangle. We characterize their construction and show that for convex Corey models the extension is determined uniquely by the average particle (or "interstitial") speed of one of the shock phase-states, namely the phase that is missing on the side of the triangle in question. Moreover, we provide an estimate on the number of states on each edge of the triangle belonging to the Hugoniot locus of any state S in the interior of the saturation triangle.

3.1. **Boundary extensions.**
In this section we focus on extensions of the side WO, see Definition 2; extensions of the other edges have similar properties. Take a state $E := (e,\, 0)^T$ on this boundary, and examine relation (2) for phase g; noticing that $f_{\mathrm{g}}(E)$ is zero, the interior point S must satisfy $f_{\mathrm{g}}(S) = \sigma s_{\mathrm{g}}$. Besides this equality, the shock speed σ equals either $\lambda_{\mathrm{s}}(S)$ or $\lambda_{\mathrm{f}}(S)$ on the interior extension $\mathcal{I}^{\mathrm{s}}_{\mathrm{WO}}$ or $\mathcal{I}^{\mathrm{f}}_{\mathrm{WO}}$ of WO, see Definition 2. As $\lambda_{\mathrm{s}}(S) = \lambda_{\mathrm{f}}(S)$ holds only at the umbilic point, generically the two above mentioned loci are disconnected. As the family involved is irrelevant in the proof of the following theorem, we take $\lambda(S)$ to denote either $\lambda_{\mathrm{s}}(S)$ or $\lambda_{\mathrm{f}}(S)$.

Theorem 4. *Consider a convex Corey model. A point S in the saturation triangle belongs to $\mathcal{I}_{\mathrm{WO}}$ if and only if $f_{\mathrm{g}}(S)/s_{\mathrm{g}} = \lambda(S)$ holds. (Analogous statements hold for the other boundaries.)*

Proof. By the very Definition 2 and the RH condition (2) a point $S \in \mathcal{I}_{\mathrm{WO}}$ satisfies $f_{\mathrm{g}}/s_{\mathrm{g}} = \lambda$ at S. Now let us prove the reciprocal: if $S = (s_{\mathrm{w}},\, s_{\mathrm{g}})^T$ satisfies $f_{\mathrm{g}}/s_{\mathrm{g}} = \lambda$ at S, then S belongs to the extension of the boundary WO.

Let S be a point in the saturation triangle satisfying $f_{\mathrm{g}}(S)/s_{\mathrm{g}} = \lambda(S)$, which is the second RH condition (2.b) between S and a point E on WO with $\sigma(E,\, S) = \lambda(S)$. In order to prove that S belongs to $\mathcal{I}_{\mathrm{WO}}$, it is sufficient to exhibit a point $E := (e,\, 0)^T$ for $e \in [0,\, 1]$ satisfying

$$f(e) = f_{\mathrm{w}}(S) + \lambda(S)(e - s_{\mathrm{w}}), \tag{3}$$

where on the LHS of (3) we have $f_{\mathrm{w}}(E)$ on WO written as the two-phase fractional flow function $f(e) = \mathrm{m_w}(e)/(\mathrm{m_w}(e) + \mathrm{m_o}(1-e))$ that depends solely on e. On the RHS we have a straight line with slope $\lambda(S)$ that depends on the same e; for convenience we denote this line in the independent variable e as

$$r(e) := f_{\mathrm{w}}(S) + \lambda(S)(e - s_{\mathrm{w}}), \qquad e \in [0,\, 1]; \tag{4}$$

recalling that S is fixed, notice that $f_{\mathrm{w}}(S)$, $\lambda(S)$ and s_{w} are constant. Equation (3) aims at finding the intersection of $f(e)$ with $r(e)$.

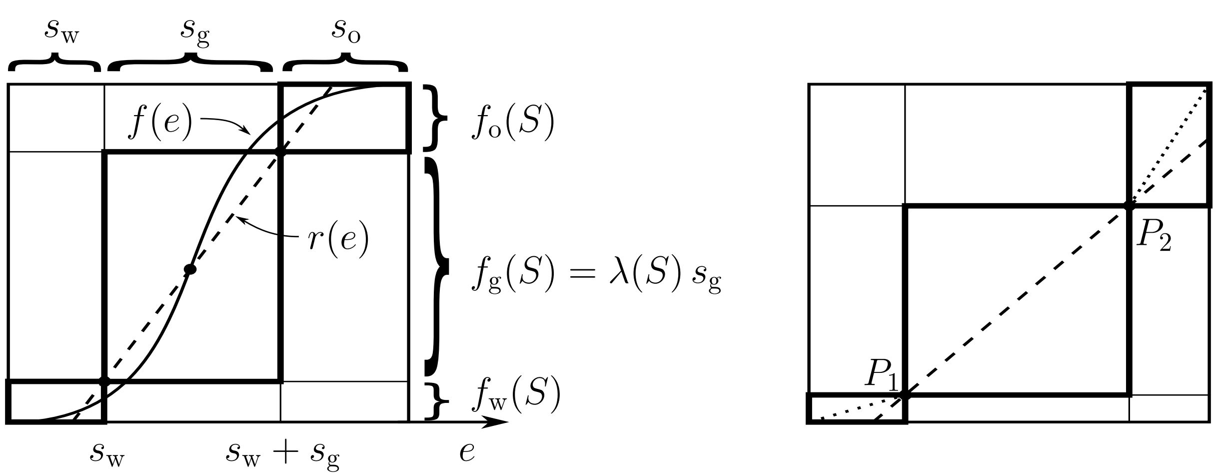

FIGURE 3.1. (a) In the unit square the solid curve is $f(e)$ and the dashed line is $r(e)$. Their intersection represents the solution; it is pictured as a dot. The three solid rectangles are $s_{\mathrm{w}} \times f_{\mathrm{w}}(S)$, $s_{\mathrm{g}} \times f_{\mathrm{g}}(S)$, $s_{\mathrm{o}} \times f_{\mathrm{o}}(S)$ (from left to right). (b) The same unit square when $f_{\mathrm{w}}(s)/s_{\mathrm{w}} < f_{\mathrm{g}}(s)/s_{\mathrm{g}} < f_{\mathrm{o}}(s)/s_{\mathrm{o}}$ hold. $P_1 = (s_{\mathrm{w}},\, f_{\mathrm{w}}(S))$ and $P_2 = (1 - s_{\mathrm{o}},\, 1 - f_{\mathrm{o}}(S))$, case *II* in Fig. 3.2.

From (4), notice that $r(s_w) = f_w(S)$ and $r(s_w + s_g) = f_w(S) + f_g(S)$ hold, hence the schematic plot in Fig. 3.1.a. The internal rectangles with thick dark sides along the diagonal of the unit square have width s_α and height $f_\alpha(S)$. Because the slope $\lambda(S) = f_g(S)/s_g$ is positive, the intersections of $r(e)$ with the border of the unit square determine four configurations, see Fig. 3.2:

I) "left-top", when $f_o(S)/s_o < f_g(S)/s_g < f_w(S)/s_w$ holds (Fig. 3.2-*I*);
II) "bottom-right", when $f_w(S)/s_w < f_g(S)/s_g < f_o(S)/s_o$ holds (Figs. 3.1.b, 3.2-*II*);
III) "bottom-top", when $f_w(S)/s_w,\ f_o(S)/s_o < f_g(S)/s_g$ holds (Figs. 3.1.a, 3.2-*III*);
IV) "left-right", when $f_g(S)/s_g < f_w(S)/s_w,\ f_o(S)/s_o$ holds (Fig. 3.2-*IV*).

Notice that the origin and (1, 1) belong to the graph of the two-phase fractional flow function $f(e)$, so that there exists e satisfying (3) for S in cases *III* or *IV*, the shaded regions in Fig. 3.2. We need to verify (3) for cases *I* and *II*.

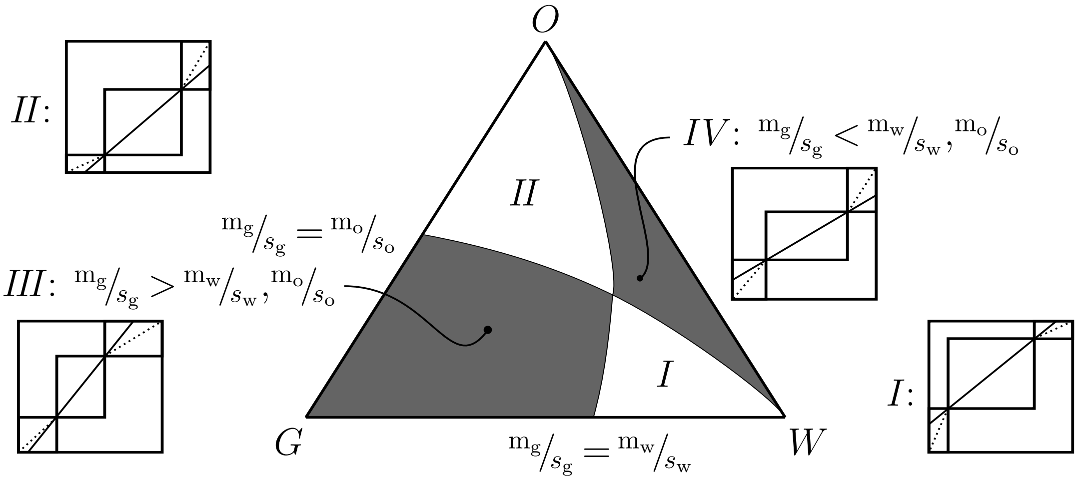

FIGURE 3.2. Internal Hugoniot $\mathcal{H}_{int}(O)$ and $\mathcal{H}_{int}(W)$ split the saturation triangle by comparing the $\mathrm{m}_\alpha/s_\alpha$ ratios. In the shaded regions $\mathrm{m_g}/s_g$ is the extremal ratio, in regions *I* and *II*, $\mathrm{m_g}/s_g$ is the intermediate ratio. (*I*: see Fig. 3.1.a. *II*: see Fig. 3.1.b.) The small boxes represent the four cases; solid lines have slope $\lambda(S)$, dotted lines have slopes $f_w(S)/s_w$ and $f_o(S)/s_o$.

Since all inequalities involving $f_\alpha(S)/s_\alpha$ in all cases have the same denominator $\mathrm{m}(S)$, we simplify the comparison by eliminating common denominators and using analogous relations involving $\mathrm{m}_\alpha(S)/s_\alpha = \mathrm{m}_\alpha(s_\alpha)/s_\alpha$ instead of $f_\alpha(S)/s_\alpha$. Therefore, in the two white regions in the triangle of Fig. 3.2, we have

$$\text{In } I: \quad \frac{\mathrm{m_o}(S)}{s_o} < \frac{\mathrm{m_g}(S)}{s_g} < \frac{\mathrm{m_w}(S)}{s_w},$$
$$\text{In } II: \quad \frac{\mathrm{m_w}(S)}{s_w} < \frac{\mathrm{m_g}(S)}{s_g} < \frac{\mathrm{m_o}(S)}{s_o}.$$

Consider a state S in region *II* satisfying the desired condition $f_g(S)/s_g = \lambda(S)$. (It is easier to follow the argument geometrically in Fig. 3.3.) The slope $\mathrm{m_o}(s_o)/s_o$ is larger than $\mathrm{m_g}(s_g)/s_g$, thus with a direct aid of convex properties we have $\mathrm{m_g}(s_g) + \mathrm{m_o}(s_o) < (s_g + s_o)\mathrm{m_o}(s_o)/s_o < \mathrm{m_o}(s_o + s_g)$, see [4, 6]. Then adding $\mathrm{m_w}(s_w)$ to the outer inequalities,

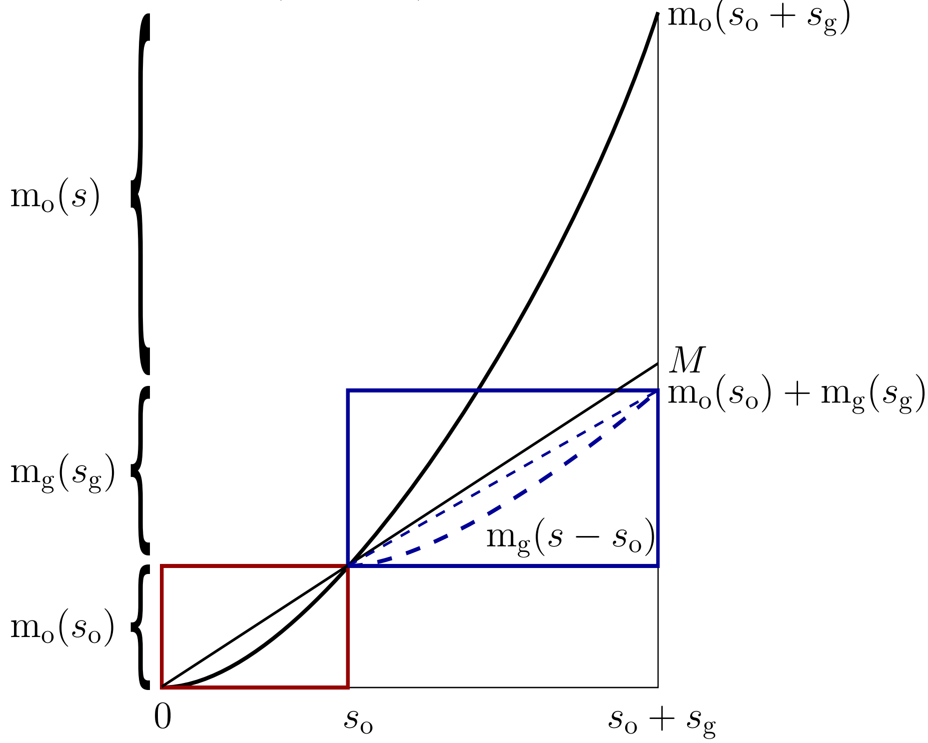

FIGURE 3.3. The solid curve is $\mathrm{m_o}(s)$ *versus* s, the dashed curve is $\mathrm{m_g}(s)$ starting at $(s_\mathrm{o},\, \mathrm{m_o}(s_\mathrm{o}))$. At the left heights in the auxiliary rectangles, at the right total heights. The straight solid line has slope $\mathrm{m_o}(s_\mathrm{o})/s_\mathrm{o}$ where M has coordinates $(s_\mathrm{o}+s_\mathrm{g},\, (s_\mathrm{g}+s_\mathrm{o})\mathrm{m_o}(s_\mathrm{o})/s_\mathrm{o})$, the dashed line has slope $\mathrm{m_g}(s_\mathrm{g})/s_\mathrm{g}$.

and taking the reciprocal we obtain

$$f_\mathrm{w}(S) = \frac{\mathrm{m_w}(s_\mathrm{w})}{\mathrm{m_w}(s_\mathrm{w}) + \mathrm{m_g}(s_\mathrm{g}) + \mathrm{m_o}(s_\mathrm{o})} > \frac{\mathrm{m_w}(s_\mathrm{w})}{\mathrm{m_w}(s_\mathrm{w}) + \mathrm{m_o}(1-s_\mathrm{w})} = f(s_\mathrm{w}). \qquad (5)$$

As (4) implies that $r(s_\mathrm{w}) = f_\mathrm{w}(S)$, which is larger than $f(s)$ at $s = s_\mathrm{w}$, we have proved that there exists at least a value e larger than s_w at which the graphs of $r(e)$ and $f(e)$ intersect. (Inequality (5) also shows that there exists at least a value e smaller than s_w leading to an analogous conclusion.) Then $E = (e, 0)^T$ on WO belongs to $\mathcal{H}(S)$ when S is within region *II*. The proof for S in region *I* is similar. The proof for S in regions *III* and *IV* follows by inspecting the diagrams in Fig. 3.2 and noting that the graph of $f(s)$ is a continuous curve passing through (0, 0) and (1, 1); thus the theorem is proved. □

Remark 5. *We may extract some further information from Theorem 4 proof. In the shaded regions in Fig. 3.2, the straight line $r(e)$ intersects $f(e)$ at least once ($f(e)$ is a continuous curve passing through* (0, 0) *and* (1, 1)*). On the other hand, for states in the extension boundary belonging to region I or II, such a configuration possesses at least two intersections.*

For a state S belonging to $\mathcal{I}^\mathrm{s}_\mathrm{WO}$ (or $\mathcal{I}^\mathrm{f}_\mathrm{WO}$), let E be a state in WO such that S belongs to $\mathcal{H}(E)$. Notice that the "number" of such states E can increase when $r(e)$ intersects the graph of the two-phase flow $f(e)$ at more points; such intersections occur in pairs. Thus there is an odd number of solutions for S in the shaded regions and an even number of solutions for S in regions I or II. The curves separating these regions are the internal Hugoniot curves $\mathcal{H}_{int}(W)$ and $\mathcal{H}_{int}(O)$, where the parity of number of solutions changes.

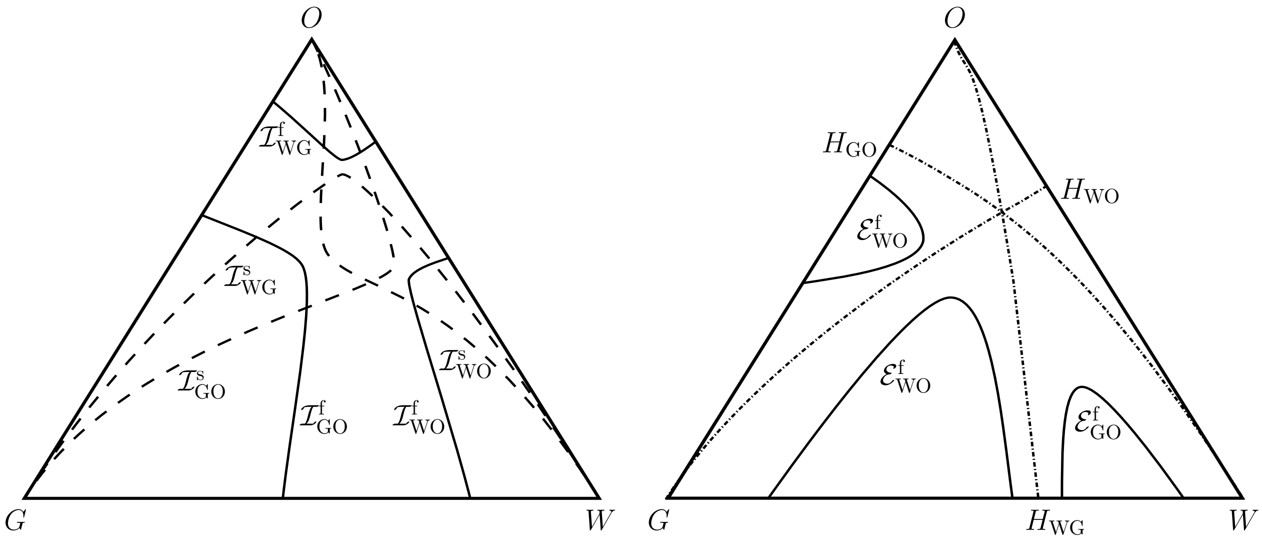

FIGURE 3.4. Extensions of the boundary for the slow family (solid) and the internal Hugoniot locus from the pure oil saturation (dashed). The location of the interior umbilic point is enclosed by the boundary extensions $\mathcal{I}^{\mathrm{s}}_{\mathrm{WO}}$, $\mathcal{I}^{\mathrm{s}}_{\mathrm{GO}}$ and $\mathcal{I}^{\mathrm{s}}_{\mathrm{WG}}$.

The following result is a consequence of the Triple Shock Rule.

Remark 6. *Let S be a state on the extension of the boundary GO, the Hugoniot locus of which, $\mathcal{H}(S)$, possesses two states E^1 and E^2 belonging to GO. As the edges of the saturation triangle are invariant manifolds for the PDE system* (1)*, the Triple Shock Rule II implies that the shock speeds $\sigma(S,\, E^1)$, $\sigma(S,\, E^2)$, and $\sigma(E^1,\, E^2)$ are equal to $\lambda(S)$. Thus S is an extension state of both E^1 and E^2.*

Claim 7. *For any state S in the interior of the saturation triangle there is at least a state $E = (e,\, 0)^T$ on the edge WO such that E belongs to the Hugoniot locus of S; hence $E \in \mathcal{H}(S)$. Moreover, the parity of the number of solutions varies in the four regions described in Fig. 3.2 as follows: an even number of states E for S in regions I and II, and an odd number of states E for S in regions III and IV. The change of the parity occurs for states S on $\mathcal{H}_{int}(O)$, $\mathcal{H}_{int}(W)$ or over the fast exterior extension boundary of the edge WO, $\mathcal{E}^{\mathrm{f}}_{\mathrm{WO}}$; characteristic at the edge.*

Proof. As in the proof of Theorem 4 for any state S in the saturation triangle we need to exhibit a state E or a value $e \in [0,\, 1]$ satisfying

$$f(e) = f_{\mathrm{w}}(S) + \sigma(e - s_{\mathrm{w}}), \tag{6}$$

with $f(e)$ defined as in (3) and the shock speed $\sigma = \sigma(S,\, E)$ given from the RH condition (2.b) as $f_{\mathrm{g}}(S)/s_{\mathrm{g}}$. In this case, let us define the straight line at the RHS of (6) as $r(e)$ given by

$$r(e) := f_{\mathrm{w}}(S) + \sigma(e - s_{\mathrm{w}}), \qquad e \in [0,\, 1];$$

recalling that S is fixed, notice that $f_{\mathrm{w}}(S)$, σ and s_{w} are constant. Equation (6) aims at finding the intersection of $f(e)$ with $r(e)$.

Actually the separation in the four regions given in Fig. 3.4 follows from the relations satisfied by the internal Hugoniot loci of W and O. Indeed, the two-phase flux $f(e)$ is continuous and passes through (0, 0) and (1, 1) and the straight line satisfies:

$$
\begin{aligned}
r(0) &= f_{\mathrm{w}}(S) - s_{\mathrm{w}}\frac{f_{\mathrm{g}}(S)}{s_{\mathrm{g}}} \begin{cases} > 0, & \text{for} \quad f_{\mathrm{w}}(S)/s_{\mathrm{w}} > f_{\mathrm{g}}(S)/s_{\mathrm{g}} \\ = 0, & \text{for} \quad f_{\mathrm{w}}(S)/s_{\mathrm{w}} = f_{\mathrm{g}}(S)/s_{\mathrm{g}}\ , \\ < 0, & \text{for} \quad f_{\mathrm{w}}(S)/s_{\mathrm{w}} < f_{\mathrm{g}}(S)/s_{\mathrm{g}} \end{cases} \qquad (7)\\
r(1) &= f_{\mathrm{w}}(S) + \frac{f_{\mathrm{g}}(S)}{s_{\mathrm{g}}}(s_{\mathrm{g}} + s_{\mathrm{o}}) \\
&= f_{\mathrm{w}}(S) + f_{\mathrm{g}}(S) + s_{\mathrm{o}}\frac{f_{\mathrm{g}}(S)}{s_{\mathrm{g}}} \begin{cases} > 1, & \text{for} \quad f_{\mathrm{g}}(S)/s_{\mathrm{g}} > f_{\mathrm{o}}(S)/s_{\mathrm{o}} \\ = 1, & \text{for} \quad f_{\mathrm{g}}(S)/s_{\mathrm{g}} = f_{\mathrm{o}}(S)/s_{\mathrm{o}}\ . \\ < 1, & \text{for} \quad f_{\mathrm{g}}(S)/s_{\mathrm{g}} < f_{\mathrm{o}}(S)/s_{\mathrm{o}} \end{cases} \qquad (8)
\end{aligned}
$$

Notice that a state S in region *I* satisfies (7.a) and (8.a), in region *II* satisfies (7.c) and (8.c), in region *III* satisfies (7.c) and (8.a) and in region *IV* satisfies (7.a) and (8.c). Therefore for states in region *III* and *IV*, we guarantee the existence of E on edge WO. (As the two-phase flux is strictly increasing, in region *IV* we have a unique value e satisfying (6).)

For any state in region *I*, as $r(0) > 0$ holds, it suffices to show that the relation $r(s_{\mathrm{w}} + s_{\mathrm{g}}) < f(s_{\mathrm{w}} + s_{\mathrm{g}})$ holds. As in this region $\mathrm{m_w}(s_{\mathrm{w}})/s_{\mathrm{w}} > \mathrm{m_g}(s_{\mathrm{g}})/s_{\mathrm{g}}$ holds, it follows from convexity that $\mathrm{m_w}(s_{\mathrm{w}}) + \mathrm{m_g}(s_{\mathrm{g}}) < \mathrm{m_w}(s_{\mathrm{w}} + s_{\mathrm{g}})$ holds. (See Fig. 3.3 with o as w.) Now, multiplying this relation by $\mathrm{m_o}(s_{\mathrm{o}})$, adding $\mathrm{m_w}(s_{\mathrm{w}} + s_{\mathrm{g}})(\mathrm{m_w}(s_{\mathrm{w}}) + \mathrm{m_g}(s_{\mathrm{g}}))$ on both sides, factoring $\mathrm{m_w}(s_{\mathrm{w}}) + \mathrm{m_g}(s_{\mathrm{g}})$ at LHS and $\mathrm{m_w}(s_{\mathrm{w}} + s_{\mathrm{g}})$ at RHS, and crossing the correct total mobilities, one obtains

$$
f_{\mathrm{w}}(s_{\mathrm{w}}) + f_{\mathrm{g}}(s_{\mathrm{g}}) = \frac{\mathrm{m_w}(s_{\mathrm{w}}) + \mathrm{m_g}(s_{\mathrm{g}})}{\mathrm{m_w}(s_{\mathrm{w}}) + \mathrm{m_g}(s_{\mathrm{g}}) + \mathrm{m_o}(s_{\mathrm{o}})} < \frac{\mathrm{m_w}(s_{\mathrm{w}} + s_{\mathrm{g}})}{\mathrm{m_w}(s_{\mathrm{w}} + s_{\mathrm{g}}) + \mathrm{m_o}(s_{\mathrm{o}})} = f(s_{\mathrm{w}} + s_{\mathrm{g}}).
$$

Since $r(s_{\mathrm{w}} + s_{\mathrm{g}}) = f_{\mathrm{w}}(S) + f_{\mathrm{g}}(S)$, we have proved that for a state S in region I, there exists at least two states E on the edge WO. The proof for S in region *II* is analogous by showing that $r(s_{\mathrm{w}}) > f(s_{\mathrm{w}})$ holds.

From relations (7.b) and (8.b) it is clear that over the internal Hugoniot loci, the state W belongs to $\mathcal{H}(S)$ for S in $\mathcal{H}_{int}(W)$, so the vertices are the new states and parity changes from region *I* to *III* or from region *II* to *IV*. Nonetheless, in region *III* the straight line $r(e)$ crosses the unitary box in the configuration "bottom-top", so an odd number of solutions must exist. As the passing from region *I* or *II* shows that the vertices W or O are solutions, we notice that the change of parity can only occur when $r(e)$ is tangent to $f(e)$; therefore σ equals $f'(e)$ and the shock speed matches the characteristic speed at the two-phase flux. We notice that λ_{s} is zero at the boundary, thus the characteristic speed is $\lambda_{\mathrm{f}}(E)$ and the state S in the interior of the saturation triangle belongs to $\mathcal{E}^{\mathrm{f}}_{\mathrm{WO}}$. □

Corollary 8. *Actually, when the flux function restricted to the boundaries is "S-shaped" and it has a unique inflexion point, Claim 7 provides the exact number of states on the edge WO belonging to $\mathcal{H}(S)$ as follows. Refer to Fig. 3.2; there is a single state on WO for S in region IV, a pair of states on WO for S in regions I and II. For S in region III extra care is needed, as there may be exterior boundary extensions $\mathcal{E}^{\mathrm{f}}_{\mathrm{WO}}$, $\mathcal{E}^{\mathrm{f}}_{\mathrm{GO}}$ (see Fig. 3.4). When S is inside of region III, there are three states on WO; once the state S crosses one of such extensions, the number of states on WO reduces to one. Moreover, notice that for S on $\mathcal{E}^{\mathrm{f}}_{\mathrm{WO}}$ and $\mathcal{E}^{\mathrm{f}}_{\mathrm{GO}}$ one of the two states is characteristic at the edge WO. Notice also that the new state on the edge WO for S crossing $\mathcal{H}_{int}(\mathrm{O})$, $\mathcal{H}_{int}(\mathrm{W})$ is a vertex.*

Remark 9. *A proof that convex permeability Corey models have S-shaped flux function is given in* [7] *for the class of power-law permeabilities.*

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