A bound for the norms of Tikhonov-regularized solutions and Levenberg-Marquardt steps

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Abstract

We present a new upper bound for the norms of Tikhonov-regularized solutions and Levenberg-Marquardt steps.

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Let X, Y be arbitrary Hilbert spaces and $A : X \to Y$ a bounded linear operator. The problem

$$\min \qquad \frac{1}{2}\|Ax - b\|^2 + \frac{\mu}{2}\|x\|^2 \qquad\qquad x \in X, \tag{1}$$

where $\mu > 0$, occurs in the computation of Tikhonov regularized solution [4] of the system $Ax = b$ as well as in the computation of Levenberg-Marquardt [2, 3] step for finding the least squares solution of $F(x) = 0$, where $F \in C^1(X, Y)$. Recall that Levenberg-Marquardt step at x for such this least square problem is s the minimizer of

$$\frac{1}{2}\|DF(x)s + F(x)\|^2 + \frac{\mu}{2}\|s\|^2.$$

Since the linearization error depends on the size of s, it is useful to have an *a priory* bound for $\|s\|$. In the case of Tikhonov regularized solution of $Ax = b$, it is also interesting to bound x because if $\|x\|$ is larger than expected, this means that the regularized parameter μ is too small.

The solution of (1) is $\bar{x} = (A^*A + \mu I)^{-1}A^*b$. So, the bound $\|\bar{x}\| \leq \|A^*b\|/\mu$ is readily available. Out aim is to provide a new bound for $\bar{x}$, as described in the next lemma.

Lemma 1. *Let $\bar{x}$ be the solution of problem* (1)*, $R(A)$ the range of A, $\overline{R(A)}$ the closure of this subspace, and $P_{\overline{R(A)}}$ the orthogonal projection onto $\overline{R(A)}$. Then*

$$\|\bar{x}\| \leq \frac{\|P_{\overline{R(A)}}(b)\|}{2\sqrt{\mu}} \leq \frac{\|b\|}{2\sqrt{\mu}}.$$

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The second inequality holds trivially, we need only to prove the first one. Existence and unicity of $\bar{x}$ is a well know trivial result. Define

$$b' = P_{\overline{R(A)}}(b), \qquad b'' = b - b'. \tag{2}$$

For any $x \in X$,

$$\|Ax - b\|^2 = \|Ax - b'\|^2 + \|b''\|^2.$$

Therefore, $\bar{x}$ is the minimizer of

$$\min \qquad \frac{1}{2}\|Ax - b'\|^2 + \frac{\mu}{2}\|x\|^2 \qquad\qquad x \in X,$$

so that

$$(A^*A + \mu I)\bar{x} = A^*b'. \tag{3}$$

We will prove a particular case of Lemma 1 and use this result to prove the general case.

Proof for A self-adjoint. Assume that $X = Y$ and A is self-adjoint. It follows from the Spectral Theorem [1] that there exist a measure space $(\Omega, \mathcal{M}, \lambda)$, $\sigma \in L^\infty(\lambda)$, and $U : X \to L^2(\lambda)$ an isometric isomorphism such that

$$A = U^{-1}\Sigma U, \qquad \Sigma : L^2(\lambda) \to L^2(\lambda), \ \ (\Sigma f)(w) = \sigma(w)f(w).$$

Defining $\beta = Ub'$, $h = U\bar{x}$, in view of (3), we have

$$(|\sigma(w)|^2 + \mu)h(w) = \sigma(w)\beta(w), \qquad [\lambda]\text{−a.e.}$$

Therefore, $[\lambda]$-a.e.

$$h = \frac{\sigma}{|\sigma|^2 + \mu}\beta.$$

It is trivial to verify that for any complex number t

$$\left|\frac{t}{|t|^2 + \mu}\right| \le \frac{1}{2\sqrt{\mu}}.$$

It follows from the two above equation that

$$|h| \le \frac{|\beta|}{2\sqrt{\mu}}.$$

Hence

$$\|\bar{x}\| = \|h\|_2 \le \frac{\|\beta\|_2}{2\sqrt{\mu}} = \frac{\|b'\|}{2\sqrt{\mu}}$$

which proves the lemma for the case of A being self-adjoint. $\square$

Proof for the general case: Endow $Z = X \times Y$ with the canonical inner product and norm of Hilbert spaces's Cartesian products

$$\langle (x,y),(x',y')\rangle = \langle x,x'\rangle + \langle y,y'\rangle, \qquad \|(x,y)\| = \sqrt{\|x\|^2+\|y\|^2},$$

and define

$$\mathbf{A} : Z \to Z, \qquad \mathbf{A}(x,y) = (A^*y, Ax); \qquad \mathbf{b} = (0,b).$$

Observe that

$$\|\mathbf{A}(x,y)-\mathbf{b}\|^2 + \mu\|(x,y)\|^2 = (\|A^*y\|^2 + \mu\|y\|^2) + (\|Ax-b\|^2 + \mu\|x\|^2).$$

Therefore, $\bar{z} = (\bar{x}, 0)$ is the solution of

$$\min \qquad \frac{1}{2}\|\mathbf{A}z - \mathbf{b}\|^2 + \frac{\mu}{2}\|z\|^2 \qquad\qquad z \in Z \tag{4}$$

Moreover $\mathbf{A}^* = \mathbf{A}$ and $R(\mathbf{A}) = R(A^*) \times R(A)$. Hence, we can apply Lemma 1, for the case of a self-adjoint operator, to such a problem and the conclusion follows. □

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