STATISTICAL PROPERTIES OF THE MAXIMAL ENTROPY MEASURE FOR PARTIALLY HYPERBOLIC ATTRACTORS

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Abstract. We show the existence and uniqueness of the maximal entropy probability measure for partially hyperbolic diffeomorphisms which are semi-conjugate to nonuniformly expanding maps. And especially, we obtain good statistical properties for such measures. More precisely, using the theory of projective metric on cones we prove exponential decay of correlations for Hölder continuous observables and the central limit theorem for the maximal entropy probability measure. Furthermore, for systems derived from solenoid-like we also prove the statistical stability for the maximal entropy probability measure that we constructed.

1. Introduction

The thermodynamical formalism from the statistical mechanics was introduced in Dynamical Systems by the former works of Sinai, Ruelle and Bowen for uniformly hyperbolic maps and Hölder potentials, in the beginning of the 70's. Beyond the uniformly hyperbolic context, the theory is still quite incomplete. Several contribution do exist, for example [BK98, BF09, Yur03, OV08, SV09, BF09, Sar99, Cas02, VV10, CV13].

In the recent years, the thermodynamical formalism of a class of partial hyperbolic diffeomorphisms introduced by Alves, Bonatti, Viana [ABV00] and Castro [Cas98] has been developed under some conditions that resemble or may lead to some mostly expanding or mostly contracting assumption in the central direction.

In the non-invertible setting this has been studied by Castro, Oliveira, Varandas and Viana [OV08, VV10, CV13]. Given a compact metric space M and a local homeomorphism $f : M \to M$ in with Lipschitz inverse branches that admit some expanding and some possibly contracting domains of invertibility it was proved in [VV10] that for every Hölder continuous potential ϕ satisfying a small variation condition there are finitely many ergodic equilibrium states for f with respect to ϕ. Furthermore, the equilibrium states are absolutely continuous with respect to some conformal measure and there exists a unique equilibrium state provided that the dynamical system is topologically exact. Later on, using a functional analytic approach by means of projective metrics techniques to the study of the spectral properties of Ruelle-Perron-Frobenius operators on the space of $C^{r+\alpha}$ observables ($r \in \mathbb{N}, \alpha > 0$), Castro and Varandas [CV13] presented a more general proof for the uniqueness of equilibrium states for this class of maps and deduced many statistical

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properties as exponential decay of correlations, Central Limit Theorem, and also both statistical and spectral stabilities.

In this paper our motivation is to contribute to the study of the thermodynamical formalism of a large class of partially hyperbolic diffeomorphisms. For that purpose we will consider partially hyperbolic diffeomorphisms which are semiconjugate to the class of local diffeomorphisms discussed above. This class includes many examples of partially hyperbolic diffeomorphisms that arise as local bifurcations of Axiom A diffeomorphisms and will be mostly expanding with respect to some conformal measure. Let us mention that SRB measures for large classes of partially hyperbolic diffeomorphisms have been constructed by [Car93, ABV00, BV00, Cas98] and, more recently, existence and uniqueness of maximal entropy measures have been proved by Buzzi, Fisher, Sambarino, Vasquez [BFSV12] for derived from Anosov diffeomorphisms, by Buzzi, Fisher [BF13] for wide class of deformations of Anosov diffeomorphisms that include the examples by Bonatti and Viana of robustly transitive non-partiallly hyperbolic diffeomorphisms, and by Ures [Ur12] for partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ homotopic to a hyperbolic automorphism. In most of these cases the approach is to established a semiconjugacy between the dynamical system and some uniformly hyperbolic one and prove that the points that remain in a non-hyperbolic region do not contribute much for the topological entropy. The drawback is that this method is not enough to deduce some good statistical properties for the original dynamical system. To illustrate this fact let us mention that in the case of nonuniformly expanding maps the Ruelle-Perron-Frobenius transfer operator acts in the space of Hölder continuous functions and the dominant eigenvector of its adjoint operator leads to the measure of maximal entropy, while in the invertible context any invariant measure is an eigenvector for the adjoint operator. For that reason the cone method used in [CV13] could not be applied here. So, to deduce exponential decay of correlations for the original dynamical systems we introduce a suitable Banach space and prove that the transfer operator does preserve some cone of functions. The construction of such cone of functions is done by constructing a family of probability measures on stable leaves that is equidistributed and holonomy invariant. A very laborious work is to prove the invariance of such suitable cone of functions by the transfer operator and that the image of this by the transfer operador has finite diameter in the projective metrics, which implies that transfer operator is a contraction with respect to the projective metrics. From that and the duality properties of transfer and Koopman operators we derive the exponential decay of correlations and the Central Limit Theorem as a consequence.

This paper is organized as follows. In the initial sections (up to section 4), we give precise definitions of the family of partially hyperbolic diffeomorphisms that we consider and state the main results. Some robust class of examples is also discussed. In sections 5 and 6, we establish the existence and uniqueness of equlibrium states. and, restricting to the skew-products and derived from solenoid case, in section 7, we also prove statistical stability of the equilibrium states, meaning that the measure varies continuously in the weak* topology with the dynamics and the potential. In the remaining sections, we prove that the maximal entropy measure satisfies good statistical properties, namely exponential decay of correlations and the Central Limit Theorem in the space of Hölder continuous observables.

2. Context and Statements

Let N be a connected compact Riemannian manifold, and let $g : N \to N$ be a *local homeomorphism* with Lipschitz inverse branches. For that, we mean there exists $L(x) \geq 0$ such that, for all $x \in N$ has a neighborhood $U_x \ni x$ such that $g_x := g|_{U_x} : U_x \to g(U_x)$ is invertible and

$$d(g_x^{-1}(y), g_x^{-1}(z)) \leq L(x)\; d(y,z), \quad \forall y, z \in g(U_x). \tag{2.1}$$

Let us denote by $\deg(g)$ the degree of g, which coincides with the number of preimages of any $x \in N$ by g. We also assume that there exist $0 < \lambda_u < 1$ and an open region $\Omega \subset N$ such that

(H1) $L(x) \leq L$ for $x \in \Omega$ e $L(x) < \lambda_u$ for $x \notin \Omega$, for some L close to 1.
(H2) There exists a covering $\mathcal{U}$ of N by injective domain of g, such that Ω can be covered by $q < \deg(g)$ elements of $\mathcal{U}$.

Let M be a compact invariant manifold, and $f : M \to M$ a diffeomorphism onto its image. Suppose there exists a continuous and sujective $\Pi : M \to N$ such that

$$\Pi \circ f = g \circ \Pi. \tag{2.2}$$

Given $y \in N$, set $M_y = \Pi^{-1}(y)$. Therefore, $M = \bigcup_{y \in N} M_y$. Note that $f(M_y) \subset M_{g(y)}$, and also suppose that there exists $0 < \lambda_s < 1$ such that

$$d(f(z), f(w)) \leq \lambda_s d(z,w) \tag{2.3}$$

for all $z, w \in M_y$.

As the maximizing entropy measure is f-invariant, by Poincaré's Recurrence Theorem such measure is supported in the attractor

$$\Lambda := \bigcap_{n=0}^{\infty} f^n(M).$$

Note that Λ is compact and invariant by f. So, it is sufficient to study the dynamics of f restricted to Λ.

Given $x, y \in M$, write $\hat{x} := \Pi(x)$, $\hat{y} := \Pi(y)$. We assume that there exist holonomies $\pi_{\hat{x},\hat{y}} : M_{\hat{x}} \cap \Lambda \to M_{\hat{y}} \cap \Lambda$ satisfying

$$\frac{1}{C}\left[d_N(\hat{x},\hat{y}) + d_M(\pi_{\hat{x},\hat{y}}(x), y)\right] \leq d_M(x,y) \leq C\left[d_N(\hat{x},\hat{y}) + d_M(\pi_{\hat{x},\hat{y}}(x), y)\right] \tag{2.4}$$

for some constant $C > 0$, and d_M, d_N to be the metrics of M,N, respectively. For simplicity we shall write d for any of such metrics.

We suppose such holonomies are invariant by f, that is,

$$f\left(\pi_{\hat{x},\hat{y}}(z)\right) = \pi_{g(\hat{x}),g(\hat{y})}\left(f(z)\right) \tag{2.5}$$

for all $z \in M_{\hat{x}} \cap \Lambda$.

3. Examples

(1) The most simple family of examples is a skew-product obtained from a map $g : N \to N$ as in [CV13] (this means that g can be taken in a robust class

of nonuniformly expanding maps that, in particular, includes all expanding maps) and an endomorphism $\Phi : N \times K \to K$, by the formula

$$\begin{array}{rll} f: & N \times K & \to N \times K \\ & (x,y) & \mapsto (g(x), \Phi(x,y)) \end{array}$$

such that f is a diffeomorphism onto its image, and for each $x \in N$, $\Phi(x,\cdot) : K \to K$ is a λ_s-contraction. In such case, Π is the canonical projection in the first condinate, and $N \times K = \bigcup_{x \in N} K_x$, where $K_x = \{x\} \times K$ forall $x \in N$.

(2) As a subexample, we may take the solenoid generated in the solid torus $S^1 \times D$. We define f by

$$\begin{array}{rll} f: & S^1 \times D & \to S^1 \times D \\ & (\theta, z) & \mapsto (g(\theta), \varphi(\theta) + A(z)) \end{array}$$

where g is the Manneville-Pomeau map given by

$$g(\theta) = \begin{cases} \theta(1 + 2^\alpha \theta^\alpha) & , \text{ if } 0 \le \theta \le \frac{1}{2} \\ (\theta - 1)(1 + 2^\alpha(1-\theta)^\alpha) + 1 & , \text{ if } \frac{1}{2} < \theta \le 1 \end{cases}$$

where $\alpha \in (0,1)$, φ is a local diffeomorphism and A is a contraction.

(3) One can modify the examples above in order to obtain robust (containing an open set) classes of examples. These are examples derived from solenoid-like systems. For sake of simplicity, we will give a construction in dimension four, which can be easily adapted to similar higher dimensional examples.

Let us begin with a solenoid-like C^2-skew-product hyperbolic diffeomorphism $f_0 : T^2 \times D \to T^2 \times D$ similar to the examples 1 and 2 above. We suppose that

$$\begin{array}{rll} f_0: & T^2 \times D & \to T^2 \times D \\ & (x,y) & \mapsto (g_0(x), \Phi_0(x,y)) \end{array}$$

is such that g_0 is an expanding map.

We suppose that the norm of Df_0 along the stable subbundle and the norm of Df_0^{-1} along the unstable bundle are bounded by a constant $\lambda_0 < 1/3$. Let p be a fixed point of f_0 and let $\delta > 0$ be a small constant. Denote $V_0 = B(p, \delta/2)$. Then, in the same manner as in [Cas02], we deform f_0^{-1} inside V_0 by a isotopy obtaining a continuous family of maps $f_t, 0 < t < 2$ in such a way that

i) The continuation p_{f_t} of the fixed point p goes through some generic bifurcation such as a flip bifurcation or a Hopf bifurcation. Points of different indexes appear in a transitive attractor for values of t between 1 and 2 (staying all the time inside V_0). For $t = 1$ we have the first moment of the Hopf (or flip) bifurcation, with f_1 conjugated to f_0. We suppose that the derivative $Df_1|_{E^{cu}}$ does not contract vectors. In the case of Hopf bifurcation, we suppose that $Df_t|_{E^{cu}}(p_{f_t})$ exhibits complex eigenvalues, for all t;

ii) In the process, there always exist a strong- stable cone field C^{ss} (cf. [Vi97] for definitions) and a center-unstable cone field C^{cu}, defined everywhere, such that C^{cu} contains the unstable direction of the initial map f_0; We also suppose that there exists a continuation of the torus

$T^2 \times \{0\}$ which is f_0-invariant and normally hyperbolic. So, for each $t \in [1,2]$ there exists a f_t-invariant manifold T_t that is the normally hyperbolic continuation of $T^2 \times \{0\}$.

iii) Moreover, the width of the cone fields C^{ss} and C^{cu} are bounded by a small constant $\alpha > 0$.

iv) There exist a constant $\sigma > 1$ and a neighbourhood $V_1 \subset V_0 \cap W^s(p)$, such that $J^c = \|detDf_t^{-1}|_{E^{cu}}\| > \sigma$ outside V_1;

v) The maps f_t^{-1} is $\delta - C^0$ close to f_0^{-1} outside V_0 so that $\|(Df_1^{-1}|_{E^{cu}})\| < \lambda_0 < 1/3$ outside V_0.

Note that the properties stated in conditions i) through v), which are valid for $f_t, 0 \leq t \leq 2$, are also valid for a whole C^1-neighbourhood $\mathcal{U}$ of the set of diffeomorphisms $\{f_t, 0 \leq t \leq 2\}$. In particular, by [HPS77] conditions i) through iii) imply that any $f \in \mathcal{U}$ has an invariant central foliation, since the central cone field enables us to define a graph transform associated to it, with domain in the space of foliations tangent to C^{cu}, which is not empty, since the unstable foliation of f_0 is tangent to it. On the other hand, all $f \in \mathcal{U}$ also exhibits a strong stable foliation varying continuously with the diffeomorphism.

As a consequence of lemma 6.1 of [BV00] there is a C^1-neighbourhood $\mathcal{U}_1 \subset \mathcal{U}$ of the set $\{f_t, 1 < t \leq 2\}$ such that for all $f \in \mathcal{U}_1$, $\Lambda = T^n$ is a partially hyperbolic attractor, which is not hyperbolic, because it is transitive and contains points with different indexes.

One can embed $T^2 \times D$ as a subset of T^4. So, it is easy to extend f_t above to T^4 in a manner that each f_t is hyperbolic (and structurally stable) outside $T^2 \times D$. So, we will assume each f_t defined in T^4 in such way.

Now take f in some small ball $B = B(f_1, \delta'), \delta' < \delta/2$. Suppose also that δ' is sufficiently small such that all diffeomorphism in $B(f_1, \delta') \subset \mathcal{U}$ is partially hyperbolic. So, if $\delta' > 0$ is small, $B(f_1, \delta')$ is an open set of diffeomorphisms of T^4 satisfying the conditions in section 2.

Corollary 3.1. *There exists an open set of non-hyperbolic diffeomorphisms $f : T^4 \to T^4$ satisfying conditions expressed by equations 2.2 through 2.5.*

Proof. Just take the open set of diffeomorphisms $\mathcal{U}_2 = \mathcal{U}_1 \cap B(f_1, \delta')$, δ' as in the proposition above. Conditions in equations 2.2-2.5 fit for every diffeomorphism in a ball $B(f_1, \delta')$. □

4. Definitions and main results

We recall the definition of topological entropy due to Bowen, using (n, ϵ)-separable sets. A compacto set K contained in a metric space (X, d) is (n, ϵ)-separable if

$$\forall x, y \in K, x \neq y, \max \left\{ d(f^j(x), d(f^j(y)); j = 0, \cdots, n-1 \right\} > \epsilon$$

We denote by $S(n, \epsilon, K)$ the greatest cardinality of a (n, ϵ)-separate subset of K. The *relative entropy* of f with respect to a (not necessarily invariant) compact $K \subset X$, is given by

$$h(f, K) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \epsilon, K).$$

For a uniformly continuous map $f : X \to X$, (X not necessarily compact), the *topological entropy* is defined by

$$h(f) := \sup \{h(f, K); K \text{ compact }\}$$

In our context $X = \Lambda$ is a compact set, and f is automaticaly uniformly continuous.. We also have by [W93] that $h(f) = h(f, X)$ does not depend on the metrics.

For an invariant measure μ, we also recall the definition by Shannon [Sh48] of its metric entropy h_μ. Given a probability space $(X, \mathcal{B}, \mu)$ such that $\mu \in \mathcal{M}^1_f(X)$, we define the entropy of a finite of a finite partition $\mathcal{P}$ of X by:

$$h_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

Then the entropy of a partition with respect to f is

$$h_\mu(f, \mathcal{P}) := \lim_{n\to\infty} \frac{1}{n} h_\mu(\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{n-1}(\mathcal{P})).$$

and the metric entropy of f with respect to μ is given by

$$h_\mu(f) := \sup_{\mathcal{P}} \{h_\mu(f, \mathcal{P})\}.$$

The *variational principle* stablishes, that for a continuous map f on a compact metric space X, the equation

$$h(f) = \sup \left\{h_\mu(f); \mu \in \mathcal{M}^1_f(X)\right\}$$

holds. We say that an invariant probability μ is a *maximal entropy measure* for f if $h(f) = h_\mu(f)$. We now state the main results in this work:

Theorem A. ***Existence and Uniqueness of Maximal Entropy measure)*** *Let $f : \Lambda \to \Lambda$ a diffeomorphism as in section 2 (that is, the conditions given by equations 2.2 through 2.5). Then, there exists a unique maximal entropy measure μ for f.*

As a by-product of the proof we also obtain

Corollary 4.1. ***(Statistical Stability in the Derived from Solenoid-like case.)*** *Let f_n be a sequence of derived from solenoid-like diffeomorphisms such as in example 3 and call μ_n the maximal entropy probability measure for f_n. If $f_n \to f$ in the C^1-topology, then μ_n converges to the maximal entropy probability measure for f in the weak-* topology.*

We say that a measure ν has exponential decay of correlations for Hölder continuous observables, if there exists some $0 < \tau < 1$ such that for α-Hölder continuous φ, ψ there exists $K(\varphi, \psi) > 0$ satisfying

$$\left| \int (\varphi \circ f^n) \psi d\nu - \int \varphi d\nu \int \psi d\nu \right| \le K(\varphi, \psi) \cdot \tau^n, \quad \text{for all } n \ge 1.$$

Using the theory of projective metrics over invariant cones, we prove:

Theorem B. ***(Exponential Decay of Correlations)*** *The maximal measure entropy μ for $f : \Lambda \to \Lambda$ has exponential decay of correlations for Hölder continuous observables.*

For the maximal entropy measure μ the following theorem also holds:

Theorem C. ***(Central Limit Theorem)***

Let μ be the maximal entropy measure for $f:\Lambda\to\Lambda$, as in (2.2) and let φ be a Hölder continuous function. If

$$\sigma_\varphi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty}\int \phi\cdot(\phi\circ f^j)\,d\mu, \qquad \text{with} \qquad \phi=\varphi-\int\varphi\,d\mu,$$

then $\sigma_\varphi<\infty$ e $\sigma_\varphi=0$ if, and only if, $\varphi=u\circ f-u$ for some $u\in L^1(\mu)$. Moreover, if $\sigma_\varphi>0$ then, for all interval $A\subset\mathbb{R}$

$$\lim_{n\to\infty}\mu\left(x\in M:\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\varphi(f^j(x))-\int\varphi d\mu\right)\in A\right)=\frac{1}{\sigma_\varphi\sqrt{2\pi}}\int_A e^{-\frac{t^2}{2\sigma_\varphi^2}}dt$$

holds.

5. Construction of the Maximal Entropy Measure

Due to the contraction in the stable foliation, the dynamics of distinct orbits of $f:M\to M$ will be determined by the dynamical behavior of the map $g:N\to N$. As seen in [CV13], such map g has only a unique maximal entropy measure, which we will denote by ν.

We start the construction of the maximal entropy measure for f by definining it on measurable sets of the form $\Pi^{-1}(A)$, where A is a Borelian set of N.

Since Π is a semiconjugation, by [W93] one obtain that,

$$h(f)\geq h(g).$$

Moreover, due to Bowen [Bow71] it follows that

$$h(f)\leq h(g)+\sup\{h(f,\Pi^{-1}(y));y\in N\}$$

We now prove that $h(f,\Pi^{-1}(y))=0$ for all $y\in N$. Indeed, since $f:M_y\to M_{g(y)}$ is a λ_s-contraction, given $\epsilon>0$, the only (n,ϵ)-separate subsets restricted to M_y are unitary subsets. As $\Pi^{-1}(y)$ can be writen as a union of $m(\epsilon)\in\mathbb{N}$ balls of ϵ-diameter, we conclude that the cardinality of any (n,ϵ)-separate subset of $\Pi^{-1}(y)$ is at most $m(\epsilon)$. By the definition entropy due to Bowen, this implies $h(f,\Pi^{-1}(y))=0$ for all $y\in N$. Therefore, $h(f)\leq h(g)$, and so $h(f)=h(g)$.

This allows us to construct the maximal entropy measure for f from the one for g. In fact, denote by ν the unique maximal entropy measure built in [CV13]. Due to the variational principle and the fact of $h(f)=h(g)$, it follows that $h_\nu(g)$, is greater than, or equal to the metric entropy of any $f-$invariant probability. So, for the proof of existence part of the statement, it is sufficient to obtain an f-invariant probablity μ, whose metric entropy with respect to f is greater or equal than $h_\nu(g)=h(g)$.

For that purpose, let $\Pi_\Lambda=\Pi|_\Lambda$. Let $\mathcal{A}_N$ be the Borel σ-algebra on N. Clearly, $\mathcal{A}_0:=\Pi_\Lambda^{-1}(\mathcal{A}_N)$ is a σ-algebra on Λ. Since f is a bijection in Λ and $\Pi_\Lambda\circ f=g\circ\Pi_\Lambda$, we have

$$A=\Pi_\Lambda^{-1}(B)=f\circ\Pi_\Lambda^{-1}\circ g^{-1}(B).$$

As $g^{-1}(B)$ belongs to $\mathcal{A}_N$, it follows that $\mathcal{A}_0\subset f(\mathcal{A}_0)$ and therefore $\mathcal{A}_n:=f^n(\mathcal{A}_0)$ is a sequence of σ-algebras such that $\mathcal{A}_0\subset\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n\subset\cdots$. Define $\mu_n:\mathcal{A}_n\to[0,1]$ by $\mu_n(f^n(A_0))=\nu(\Pi_\Lambda(A_0))$, for all $A_0\in\mathcal{A}_0$. Note that μ_n is an f-invariant probability for all $n\in\mathbb{N}$. In fact, given $A=f^n(A_0)$, where $A_0=\Pi_\Lambda^{-1}(B)$ and

$B \in \mathcal{A}_N$, due to the g-invariance of ν and the surjection of maps g and Π_Λ, we have:

$$\begin{aligned}
\mu_n(f^{-1}(A)) &= \mu_n(f^{-1}(f^n(A_0))) = \mu_n(f^n(f^{-1}(A_0))) \\
&= \nu(\Pi_\Lambda(f^{-1}(A_0))) = \nu(\Pi_\Lambda(f^{-1} \circ \Pi_\Lambda^{-1}(B))) \\
&= \nu(\Pi_\Lambda(\Pi_\Lambda^{-1} \circ g^{-1}(B))) = \nu(g^{-1}(B)) \\
&= \nu(B) = \nu(\Pi_\Lambda(A_0)) \\
&= \mu_n(f^n(A_0)) = \mu_n(A)
\end{aligned}$$

Now, as $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, $\mathcal{A} := \bigcup_{n=0}^{\infty} \mathcal{A}_n$ is an algebra in Λ.

Then we define $\mu : \mathcal{A} \to [0,1]$ the probability such that $\mu(A) = \mu_n(A)$ if $A \in \mathcal{A}_n$. By the standard measure theory arguments(see [Mane]), μ is σ-aditive.

Moreover, μ is an f-invariant probability, as μ_n are f-invariant probabilities. It rests to prove that the smallest $\sigma-$algebra that contains $\mathcal{A}$ is the Borel $\sigma-$algebra.

For that, it is sufficient to see that $\mathcal{A}$ contains a sequence of partitions whose diameter goes to zero.

This is because $f : M_y \to M_{g(y)}$ is a λ_s-contraction.

In fact, for each $n \in \mathbb{N}$, by the continuity of g^n, there exists $\delta(n) > 0$ such that $d(z,w) < \delta(n)$ implies $d(g^n(z), g^n(w)) < \lambda_s^n$, for all $z, w \in N$. Taking $\mathcal{P}_n^0$ a partition of N whose diameter is less than $\delta(n)$, we define a sequence of partitions of Λ by

$$\mathcal{P}_n := f^n\left(\Pi_\Lambda^{-1}\left(\mathcal{P}_n^0\right)\right) \tag{5.1}$$

Clearly, $\text{diam}(\mathcal{P}_n) \to 0$ as $n \to +\infty$. Indeed, given $\bar{x}, \bar{y}$ in the same element of $\mathcal{P}_n$, writing $\bar{x} = f^n(x)$ and $\bar{y} = f^n(y)$ we have $\hat{x} = \Pi(x), \hat{y} = \Pi(y) \in P_n^0$. Therefore, noting that $g^n(\hat{x}) = g^n(\Pi(x)) = \Pi(f^n(x)) = \tilde{x}$ e $g^n(\hat{y}) = g^n(\Pi(y)) = \Pi(f^n(y)) = \tilde{y}$ we obtain

$$\begin{aligned}
d(f^n(x), f^n(y)) &\leq C\left[d(\tilde{x},\tilde{y}) + d(\pi_{\tilde{x},\tilde{y}} \circ f^n(x), f^n(y))\right] \\
&= C\left[d(g^n \circ \Pi(x), g^n \circ \Pi(y)) + d(f^n(\pi_{\hat{x},\hat{y}}(x)), f^n(y))\right] \\
&\leq C\left[\lambda_s^n + \lambda_s^n d(\pi_{\hat{x},\hat{y}}(x), y)\right] \\
&\leq C\left[1 + diam(M)\right]\lambda_s^n.
\end{aligned}$$

By a slight abuse of notation, we also write μ for its natural extension to the Borel σ-algebra of M.

Now we prove that μ is a maximizing entropy measure for f, by proving that $h_\mu(f) \geq h_\nu(g)$. Denote by $B_\epsilon^n(g, y_0)$ a (n,ϵ)-dynamical ball of g around $y_0 \in N$, that is, the set of points $y \in N$, such that $d(g^j(y), g^j(y_0)) < \epsilon, \forall j \in \{0, \cdots, n-1\}$. Due Brin-Katok Theorem, ν-a.e. point $y \in N$,

$$h_\nu(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\nu\left(B_\epsilon^n(g,y)\right)}$$

holds.

Take now $B_\epsilon^n(f,x)$ the (n,ϵ) dynamical ball of f restricted to Λ at $x \in \Lambda$. By the uniform continuity of Π, given $\epsilon > 0$ there exists $0 < \delta < \epsilon$ such that $\Pi(B_\delta(w)) \subset B_\epsilon(\Pi(w))$ for all $w \in M$. Note that $B_\delta^n(f,x) \subset \Pi_\Lambda^{-1}\left(B_\epsilon^n(g,y)\right)$ for all $x \in \Pi_\Lambda^{-1}(y)$.

In fact, given $z \in B^n_\delta(f,x)$ we shall prove that $\Pi(z) \in B^n_\epsilon(g,y)$. As $\Pi(x) = y$ we have for all $j \in \{0,\cdots,n-1\}$

$$d(g^j \circ \Pi(z), g^j(y)) \quad = \quad d(g^j \circ \Pi(z), g^j \circ \Pi(x)) = d(\Pi \circ f^j(z), \Pi \circ f^j(x)) < \epsilon.$$

Therefore

$$\mu\left(B^n_\delta(f,x)\right) \leq \mu\left(\Pi^{-1}_\Lambda\left(B^n_\epsilon(g,y)\right)\right) = \nu\left(B^n_\epsilon(g,y)\right)$$

and since $\delta \to 0$ as $\epsilon \to 0$ we obtain

$$h_\nu(g) \leq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu\left(B^n_\delta(f,x)\right)}$$

for $\mu-$a.e. $x \in \Lambda$. So,

$$\begin{aligned} h_\mu(f) &= \int_\Lambda \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu\left(B^n_\delta(f,x)\right)} d\mu \\ &\geq \int_\Lambda h_\nu(g) d\mu \\ &= h_\nu(g) \end{aligned}$$

and we conclude that $h_\mu(f) \geq h_\nu(g) = h(g) = h(f)$, which is the equivalent to say that μ is maximal entropy measure for f.

6. Uniqueness of Maximal Entropy Measure

Now we prove the uniqueness of maximal entropy measure for f built in the last section. For such purpose, we use the uniqueness of the maximal entropy measure for g, provided by [CV13]. Suppose that μ_1 is another invariant maximal entropy measure for f, different to μ. Let $\nu_1 := (\Pi_\Lambda)_* \mu_1$, the push-forward of μ_1.

We claim that since μ_1 is different to μ, it follows that ν_1 is different to ν. Indeed, since $\mu_1 \neq \mu$, $\mu_1(A) \neq \mu(A)$ for some $A \in \mathcal{A} = \mathcal{A}_0 \cup f(\mathcal{A}_0) \cup \cdots \cup f^n(\mathcal{A}_0) \cup \cdots$. The fact that such algebras on $\mathcal{P}(\Lambda)$ are nested implies that exist $A_0 \in \mathcal{A}_0$ and $n \in \mathbb{N}$ such that $f^n(A_0) = A$. By the definition of $\mathcal{A}_0$, there exists $B_0 \in \mathcal{A}_N$ such that $\Pi^{-1}_\Lambda(B_0) = A_0$. We now observe that, on one hand,

$$\nu_1(B_0) = (\Pi_\Lambda)_* \mu_1(B_0) = \mu_1(\Pi^{-1}_\Lambda(B_0)) = \mu_1(A_0) = \mu_1(f^n(A_0)) = \mu_1(A)$$

and on the other hand,

$$\nu(B_0) = \nu(\Pi_\Lambda(A_0)) = \mu(A_0) = \mu(f^n(A_0)) = \mu(A).$$

So, $\nu_1 \neq \nu$. By the f-invariance of μ_1 it follow that ν_1 is g-invariant.

Let us prove that ν_1 is a maximal entropy measure for g, which is a contradiction, since by [CV13], such probability is unique. For that, it is sufficient to prove that $h_{\nu_1}(g) \geq h_{\mu_1}(f)$, since $h_{\mu_1}(f) = h(f) = h(g)$.

In fact, we may suppose that the sequence $\mathcal{P}_n = f^n\left(\Pi^{-1}_\Lambda\left(\mathcal{P}^0_n\right)\right)$, in 5.1, is such that $\mathcal{P}_0 \leq \mathcal{P}_1 \leq \cdots \leq \mathcal{P}_n \leq \cdots$ and as $\displaystyle\bigcup_{n=0}^{\infty} \mathcal{P}_n$ generates the Borel σ-algebra of Λ, we obtain

$$h_{\mu_1}(f) = \sup_n \left\{h_{\mu_1}(f, P_n)\right\}.$$

Therefore, for all $\epsilon > 0$ there exists $n \in N$ such that

$$h_{\mu_1}(f, P_n) \geq h_{\mu_1}(f) - \epsilon.$$

However, if follows from the definition of ν_1 that for all $n \in \mathbb{N}$

$$h_{\nu_1}(g, P_n^0) = h_{\mu_1}(f, \Pi_\Lambda^{-1}\left(P_n^0\right)).$$

Indeed, for a partition $\mathcal{P}$ we have

$$h_{\nu_1}(g, \mathcal{P}) = \lim_{m\to\infty} \frac{1}{m} h_{\nu_1}\left(\mathcal{P} \vee g^{-1}(\mathcal{P}) \vee \cdots \vee g^{-(m-1)}(\mathcal{P})\right)$$

Due to the definition of ν_1 and the semiconjugation between f and g we obtain

$$\begin{aligned}
\nu_1\left(\bigvee_{j=0}^{m-1} g^{-j}(P_{i_j})\right) &= \mu_1\left(\Pi_\Lambda^{-1}\left(\bigvee_{j=0}^{m-1} g^{-j}(P_{i_j})\right)\right) \\
&= \mu_1\left(\bigvee_{j=0}^{m-1} \Pi_\Lambda^{-1}\left(g^{-j}(P_{i_j})\right)\right) \\
&= \mu_1\left(\bigvee_{j=0}^{m-1} f^{-j}\left(\Pi_\Lambda^{-1}(P_{i_j})\right)\right)
\end{aligned}$$

which guarantees $h_{\nu_1}\left(\bigvee_{j=0}^{m-1} g^{-j}(\mathcal{P})\right) = h_{\mu_1}\left(\bigvee_{j=0}^{m-1} f^{-j}\left(\Pi_\Lambda^{-1}(\mathcal{P})\right)\right)$ and so, we have

$$h_{\nu_1}(g, \mathcal{P}) = h_{\mu_1}(f, \Pi_\Lambda^{-1}\left(\mathcal{P}\right)).$$

From the f-invariance of μ_1 it follows that

$$h_{\mu_1}(f, \Pi_\Lambda^{-1}\left(\mathcal{P}_n^0\right)) = h_{\mu_1}(f, \mathcal{P}_n)$$

because $P_{n_j} \in \mathcal{P}_n$ if and only if there exist $P_{n_j}^0 \in \mathcal{P}_n^0$ such that $P_{n_j} = f^n(\Pi_\Lambda^{-1}(P_{n_j}^0))$. Therefore

$$\begin{aligned}
\mu_1\left(\bigvee_{j=0}^{m-1} f^{-j}(P_{n_j})\right) &= \mu_1\left(\bigvee_{j=0}^{m-1} f^{-j}\left(f^n\left(\Pi_\Lambda^{-1}(P_{n_j}^0)\right)\right)\right) \\
&= \mu_1\left(\bigvee_{j=0}^{m-1} f^{n}\left(f^{-j}\left(\Pi_\Lambda^{-1}(P_{n_j}^0)\right)\right)\right) \\
&= \mu_1\left(f^n\left(\bigvee_{j=0}^{m-1} f^{-j}\left(\Pi_\Lambda^{-1}(P_{n_j}^0)\right)\right)\right) \\
&= \mu_1\left(\bigvee_{j=0}^{m-1} f^{-j}\left(\Pi_\Lambda^{-1}(P_{n_j}^0)\right)\right).
\end{aligned}$$

We then obtain that for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\begin{aligned}
h_{\nu_1}(g) &\geq h_{\nu_1}\left(g, P_n^0\right) \\
&= h_{\mu_1}\left(f, \Pi_\Lambda^{-1} P_n^0\right) \\
&= h_{\mu_1}\left(f, P_n\right) \\
&\geq h_{\mu_1}(f) - \epsilon
\end{aligned}$$

and this proves that $h_{\nu_1}(g) \geq h_{\mu_1}(f)$, and the uniqueness of the maximal entropy measure.

7. Statistical Stability

Now we prove the statistical stability for the maximizing probability measure μ. That is, given $f_n \to f$ in the C^1-topology, then $\mu_n \to \mu$ in weak-$*$ topology, where μ_n (respectively, μ) is the maximizing entropy measure for f_n (respectively, f).

Let us fix such f, and consider the collection $\mathcal{C}$ whose elements are open subsets $A \subset M$ whose frontier are μ-zero sets with the form $A = \cup_{x\in B} M_x$, for some ball $B \subset N$ with ν-zero frontier. Also denote by $\hat{\mathcal{C}} \supset \mathcal{C}$ the collection whose elements are nonnegative interate of some element of $\mathcal{C}$. Observe that, if we fix $k \in \mathbb{N}$, $f^k(\cup_{x\in N} M_x)$ is a neighborhood for the attractors Λ_n where μ_n are supported, for all sufficiently big n. Note that $\hat{\mathcal{C}}$ is a neighborhood basis for Λ.

The key ingredient for the proof is the lemma:

Lemma 7.1. *Let $\hat{A} \in \hat{\mathcal{C}}$. Then $\mu_n(\hat{A}) \to \mu(\hat{A})$ as $n \to +\infty$.*

Proof. Given $\hat{A} = f^k(A)$, with $A = \cup_{x\in B} M_x$. We start with the case $k = 0$, that is, first we prove that $\mu_n(A) \to \mu(A)$ as $n \to +\infty$.

Set $A_n := \Pi_n^{-1}(B)$. Therefore, $\mu_n(A_n) = \nu_n(B)$, where ν_n is the maximizing measure g_n as in [CV13]. We also have $\mu(A) = \nu(B)$, where ν is the entropy maximizing probability associated to g, as in [CV13].

Given $\epsilon > 0$, take $B^+ \supset B \supset B^-$, $\nu-$zero frontier such that

$$\nu(B^+) - \epsilon/3 < \nu(B) < \nu(B^-) + \epsilon/3,$$

Let us also assume that $A_n^\pm := \Pi_n^{-1}(B^\pm)$, with $\mu-$zero frontier such that there exists n_2 that forall $n \geq n_2$ $A_n^+ \supset A \supset A_n^-$ and

$$\mu(A_n^+) - \epsilon/3 < \mu(A) < \mu(A_n^-) + \epsilon/3,$$

hold.

Such sets exist by the C^0-convergence of (strong stable/center-unstable) foliations for f_n to the respective foliations for f.

On the one hand, $\exists n_1 \geq n_2$ such that

$$\mu(A) - \mu_n(A) \leq \mu(A) - \mu_n(A) \leq \mu(A) - \mu_n(A_n^-) = \nu(B) - \nu_n(B^-) \leq \frac{2\epsilon}{3},$$

for all $n \geq n_1$, as $\nu_n(B^-) \to \nu(B^-)$ by the statistical stability for g proved in [CV13].

In the same manner, we prove the other inequality, implying there exists $n_0 \geq n_1$ such that

$$|\mu(A) - \mu_n(A)| < \epsilon, \forall n \geq n_0.$$

The same arguments also are valid for the case $k > 0$.

This finishes the lemma.

□

Theorem 7.2. *Given $\varphi : M \to \mathbb{R}$ a continuous function, then $\int_M \varphi d\mu_n \to \int_M \varphi d\mu$.*

Proof. Let $\epsilon > 0$ given, and the $\delta > 0$ we obtain by the uniform continuity of φ associated to $\epsilon/9$. Take a covering $\cup_{j=1}^k C_j$, $C_j \in \mathcal{C}$ de Λ, with diameter less then $\delta/3$. There is also n_0 such that $\cup_{j=1}^k C_j \supset \Lambda_n$, $\forall n \geq n_0$. In particular, $\mu_n(M \setminus \cup_{j=1}^k C_j) = 0$, $\forall n \geq n_0$.

Consider a partition of unity $\{\psi_j, j = 1, \ldots, k\}$ associated to $\cup_{j=1}^{k} C_j$.

For each C_j, take $x_j \in C_j$ and set

$$\hat{\varphi} := \sum_{j=1}^{k} \varphi(x_j)\psi_j.$$

Therefore, $\|\varphi - \hat{\varphi}\|_\infty < \epsilon/3$.

Now, take $n_1 \geq n_0$ such that

$$|(\mu_n - \mu)(C_j)| < \frac{\epsilon}{3k\|\varphi\|_\infty}, \forall n \geq n_1.$$

So, we conclude that

$$\left|\int_M \varphi d\mu_n - \int_M \varphi d\mu\right| \leq \left|\int_M \varphi d\mu_n - \int_M \hat{\varphi} d\mu_n\right| + \left|\int_M \hat{\varphi} d\mu_n - \int_M \hat{\varphi} d\mu\right| + \left|\int_M \varphi d\mu - \int_M \hat{\varphi} d\mu\right|$$

$$\leq \|\varphi - \hat{\varphi}\|_\infty + \sum_{j=1}^{k} \|\varphi\|_\infty |\mu_n(C_j) - \mu(C_j)| + \|\varphi - \hat{\varphi}\|_\infty < \epsilon, \forall n \geq n_0.$$

□

8. Cones and Projective Metrics

We recall here some necessary results in Projective Metrics defined in Cones whose proofs can be found in [Li95,Ba00,Vi95].

Given a linear space E we say that $C \subset E \backslash \{0\}$ is a convex cone if

$$t > 0 \text{ e } v \in C \Rightarrow t \cdot v \in C.$$

and

$$t_1, t_2 > 0 \text{ e } v_1, v_2 \in C \Rightarrow t_1 \cdot v_1 + t_2 \cdot v_2 \in C.$$

We define $\overline{C}$ to be the set of points $w \in E$ such that there exists $v \in C$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$, going to zero, such that $w + t_n \cdot v \in C$ forall $n \in \mathbb{N}$. We will only consider the so called *projective cones*, such that

$$\overline{C} \cap \left(-\overline{C}\right) = \{0\}.$$

We then define

$$\alpha(v, w) = \sup\{t > 0; w - t \cdot v \in C\}$$

and

$$\beta(v, w) = \inf\{s > 0; s \cdot v - w \in C\}.$$

We convention $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. The projective metrics associated to C is given by

$$\theta(v, w) = \log \frac{\beta(v, w)}{\alpha(v, w)}.$$

Indeed,

Proposition 8.1. *Given a projective cone C then $\theta(\cdot, \cdot) : \overline{C} \times \overline{C} \to [0, +\infty]$ is a metrics in the projective space of C, that is,*

- *$\theta(v, w) = \theta(w, v)$.*
- *$\theta(u, w) \leq \theta(u, v) + \theta(v, w)$.*
- *$\theta(v, w) = 0$ iff there exists $t > 0$ such that $v = t \cdot w$.*

The proof of the following essential result can be found in [Vi97, Proposition 2.3].

Theorem 8.2. *Let E_1 and E_2 be linear spaces and let $C_1 \subset E_1$ and $C_2 \subset E_2$ be projective cones. If $L : E_1 \to E_2$ is a linear operador such that $L(C_1) \subset C_2$ and*

$$D = \sup \{\theta_2(L(v), L(w)); v, w \in C_1\} < \infty$$

then

$$\theta_2(L(v), L(w)) \leq \left(1 - e^{-D}\right) \theta_1(v, w),$$

for all $v, w \in C_1$.

9. Ruelle-Perron-Frobenius Operator and Invariant Cones

We recall that the main goal of this work is to deduce good statistical properties of the maximal entropy probability measure associated to the dynamics f. The technique presented use the Ruelle-Perron-Frobenius operator(for simplicity called transfer operator) and its duality with the Koopman operator, $U(\varphi) = \varphi \circ f$, to obtain the exponential decay of correlations and consequently the central limit theorem.

However, this technique may also be useful to prove exponential decay of correlations and consequently the central limit theorem for more general equilibrium states, not just particularly for measures of maximum entropy. We recall that given a map $f : \Lambda \to \Lambda$, and a fixed potential $\phi : \Lambda \to \mathbb{R}$, we say that a measure η is an equilibrium state for f with respect to ϕ if

$$h_\eta(f) + \int \phi d\eta = \sup \left\{ h_\mu(f) + \int \phi d\mu; \mu \text{ is an } f\text{-invariant probability} \right\}.$$

That is, the variational principle tells us that η carries out the topological pressure $P(f, \phi)$. The reader can easily see that in the case where the potential ϕ is a constant, obtain an equilibrium state is equivalent to obtain a maximum entropy measure. What we do in this section is to obtain some preliminar results, for more general potentials than constant potentials, namely, low variation potentials. That is, we assume that $\sup \phi - \inf \phi < \varepsilon$ for some small enough ε. Moreover such potential must belong to the following cone:

$$\left|e^\phi\right|_\alpha \leq \varepsilon \inf e^\phi \tag{9.1}$$

where $\left|e^\phi\right|_\alpha = \inf \left\{C > 0; |e^\phi(x) - e^\phi(y)| \leq Cd(x,y)^\alpha, \forall x, y \in \Lambda\right\}$. Let E is the space of continuous functions $\varphi : \Lambda \to \mathbb{R}$. Define the Ruelle-Perron-Frobenius operator $\mathcal{L} : E \to E$ given

$$\mathcal{L}(\varphi)(y) = \varphi(f^{-1}(y)) e^{\phi(f^{-1}(y))}$$

where ϕ satisfies the above conditions.

Our inspiration is the work developed in [CV13], where the exponential decay of correlations and other good statistical and regularity properties are proven for the unique equilibrium state in a nonuniformly expanding context. Castro-Varandas defined suitable cones for the Ruelle-Perron-Frobenius (or transfer) operator$\mathcal{L}$, proving the invariance and the finite diameter for the image of such cones by $\mathcal{L}$.

More precisely, the basic cone used by [CV13] is the cone of Hölder continuous, positive functions φ such that $|\varphi|_\alpha \leq \kappa \inf \varphi$. The invariance of such cone by f is due some increase in the regularity given by the contraction of some inverse branch of f. In our context, however, we always have backward expansion in stable directions for the points into each strong stable manifold $\Pi^{-1}(y)$ instead of contraction. Since for the case of entropy (potential $\phi \equiv 0$) the transfer operator $\mathcal{L}$, is just the composition of each observable φ with f^{-1}, it is obvious that the Hölder constants of $\mathcal{L}(\varphi)$, can not better, if one take a cone as in [CV13].

In order to avoid this undesirable effect in stable directions, we will analyse the action of $\mathcal{L}$ in some kind of averages taken in each stable leaf restricted to the attractor Λ. We will write the lowercase letter γ to denote a stable leaf (instersected with Λ) and $\mathcal{F}^s$ will denote the stable foliation.

Fixed $y \in N$, let y_j such that $g(y_j) = y$, where $j \in \{1, \cdots, deg(g)\}$. Writing $\gamma = \Pi_\Lambda^{-1}(y)$ and $\gamma_j = \Pi_\Lambda^{-1}(y_j)$, it follows that $f(\gamma_j) \subset \gamma$, since $\Pi \circ f(x) = g \circ \Pi(x) = g(y_j) = y$, $\forall x \in \gamma_j$.

Let $\mathbf{p}$ be the degree of g. Let us construct a family of measures $\{\mu_\gamma\}_{\gamma \in \mathcal{F}^s}$ supported in Λ, such that for all $\hat{\gamma}$, where $f^n(\hat{\gamma}) \subset \gamma$, we have $\mu_\gamma(f^n(\hat{\gamma})) = \dfrac{1}{p^n}$. In particular $\mu_\gamma(\gamma) = 1$. Furthermore, for all γ_j, with $f(\gamma_j) \subset \gamma$ we will obtain

$$\int_{f(\gamma_j)} \psi d\mu_\gamma = \frac{1}{p} \int_{\gamma_j} \psi \circ f d\mu_{\gamma_j}.$$

The construction of such family of measures is rather natural. Fix $\gamma = \Pi_\Lambda^{-1}(y)$ and $n \in \mathbb{N}$, $n > 0$. By setting $\gamma_j := \Pi_\Lambda^{-1}(y_j)$, where $y_j \in g^{-n}(y)$, one can write $\gamma = \dot{\bigcup}_{j=1}^{p^n} f^n(\gamma_j)$, since f^n is a bijection in Λ and $\Pi \circ f^n = g^n \circ \Pi$. Therefore, $\{f^n(\gamma_j)\}_{j=1}^{p^n}$ is a sequence of partitions in γ. As $\gamma_j = \Pi_\Lambda^{-1}(y_j)$ and $f^n : M_{y_j} \to M_{g^n(y_j)}$ is a λ_s^n-contraction it follows that the diameter of $\{f^n(\gamma_j)\}_{j=1}^{p^n}$ goes to zero. So, we just define μ_γ in the elements of such partition by mass distribution

$$\mu_\gamma(f^n(\gamma_j)) = \frac{1}{p^n}$$

and extend μ_γ by approximation to any Borelian $A \subset \Lambda$.

If $\gamma_j = \Pi_\Lambda^{-1}(x_j)$, $x_j \in g^{-1}(x)$, then

$$\mu_\gamma(A) = \mu_\gamma(A \cap \gamma) = \mu_\gamma\left(A \cap \bigcup_{j=1}^{p} f(\gamma_j)\right) = \mu_\gamma\left(\bigcup_{j=1}^{p} (A \cap f(\gamma_j))\right) = \sum_{j=1}^{p} \mu_\gamma(A \cap f(\gamma_j))$$

Seting $\mu_{\gamma_j}(A) := p \cdot \mu_\gamma(f(A \cap \gamma_j))$ we obtain $\mu_\gamma(A \cap f(\gamma_j)) = \dfrac{1}{p}\mu_{\gamma_j}(f^{-1}(A))$ and so

$$\mu_\gamma(A) = \frac{1}{p} \sum_{j=1}^{p} \mu_{\gamma_j}(f^{-1}(A)).$$

We conclude that for any measurable set A, its indicator function χ_A satisfies

$$\int_{f(\gamma_j)} \chi_A d\mu_\gamma = \frac{1}{p} \int_{\gamma_j} \chi_A \circ f d\mu_{\gamma_j}$$

By Lebesgue Dominated Convergence Theorem, for any $g : \Lambda \to \mathbb{R}$ continuous we have

$$\int_{f(\gamma_j)} g d\mu_\gamma = \frac{1}{p} \int_{\gamma_j} g \circ f d\mu_{\gamma_j}. \tag{9.2}$$

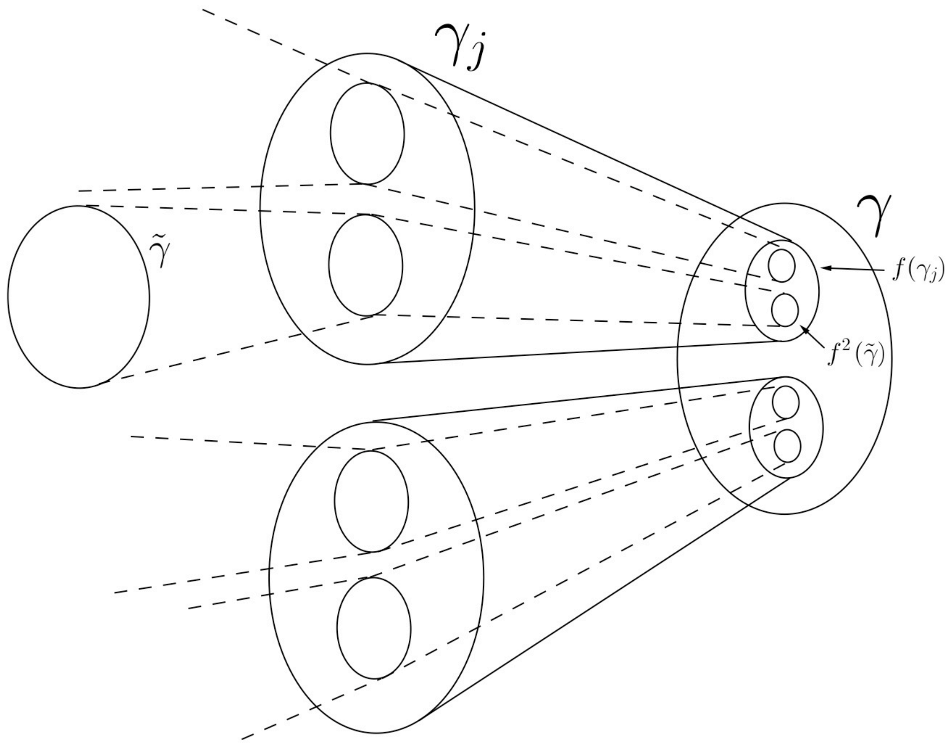

FIGURE 1. Mass distribution

Note also that for all $\hat{\gamma}$, $f^n(\hat{\gamma}) \subset \gamma$, we have $\mu_\gamma(f^n(\hat{\gamma})) = \dfrac{1}{p^n}$. So it follows that for all $\widetilde{\gamma}$ such that $f^n(\widetilde{\gamma}) \subset \gamma_j$ and $f(\gamma_j) \subset \gamma$

$$\begin{aligned} \mu_{\gamma_j}(f^n(\widetilde{\gamma})) &= p\mu_\gamma(f(f^n(\widetilde{\gamma}) \cap \gamma_j)) \\ &= p\mu_\gamma\left(f^{n+1}(\widetilde{\gamma})\right) \\ &= \frac{p}{p^{n+1}} = \frac{1}{p^n} \end{aligned}$$

holds.

That is, μ_{γ_j} is the mass distribution measure constructed for γ_j.

Moreover, for $y \in N$ and y_j such that $g(y_j) = y$, $j \in \{1, \cdots, p\}$ if we consider $\gamma = \Pi_\Lambda^{-1}(y)$ and $\gamma_j = \Pi_\Lambda^{-1}(y_j)$, $f(\gamma_j) \subset \gamma$, then $\gamma = \dot{\bigcup}_{j=1}^p f(\gamma_j)$. Therefore, for all measurable bounded function $\psi : \gamma \to \mathbb{R}$ it follows that

$$\int_\gamma \psi d\mu_\gamma = \sum_{j=1}^p \int_{f(\gamma_j)} \psi d\mu_\gamma.$$

For $\rho : \gamma \to \mathbb{R}$, we conclude that

$$\int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma = \sum_{j=1}^p \int_{f(\gamma_j)} \mathcal{L}(\varphi)\rho d\mu_\gamma = \sum_{j=1}^p \frac{1}{p} \int_{\gamma_j} \varphi \cdot e^\phi \cdot \rho \circ f d\mu_{\gamma_j}$$

defining $\rho_j := \dfrac{1}{p}\rho \circ f e^{\phi}$, we have

$$\int_{\gamma} \mathcal{L}(\varphi)\rho d\mu_{\gamma} = \sum_{j=1}^{p} \int_{\gamma_j} \varphi \rho_j d\mu_{\gamma_j}.$$

We will study the action of the transfer operator in the strong stable leaves via its action on the integrals of densities in a suitable cones of functions which are defined in each strong stable leaf. More precisely, for each $\gamma \in \mathcal{F}^s$ we define the auxiliary cone of Hölder continuous functions

$$\mathcal{D}(\gamma, \kappa) := \{\rho : \gamma \to \rho > 0 \text{ and } |\rho|_{\alpha} < \kappa \inf \rho\},$$

with $|\rho|_{\alpha} = \inf\{C > 0; |\rho(x) - \rho(y)| \leq C d(x, y)^{\alpha}, \forall x, y \in \gamma\}$.

Note that for ρ in a cone $\mathcal{D}(\gamma, \kappa)$ we have $\sup \rho \leq \inf \rho\,(1 + \kappa \cdot diam M^{\alpha})$.

The next lemma is about the invariance of the auxiliary cones under the action of the transfer operator.

Lemma 9.1. *There exist sufficiently small $0 < \lambda < 1$ and $\kappa > 0$, such that the following itens hold:*

1. *If $\rho \in \mathcal{D}(\gamma, \kappa)$ then $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$ for all $j \in \{1, \ldots, p\}$.*
2. *For all $\gamma \in \mathcal{F}^s_{loc}$, if $\rho, \hat{\rho} \in \mathcal{D}(\gamma, \lambda\kappa)$ then $\theta(\rho, \hat{\rho}) \leq 2\log\left(\dfrac{1+\lambda}{1-\lambda}\right)$.*
3. *If $\rho', \rho'' \in \mathcal{D}(\gamma, \kappa)$ then there exists $\Lambda_1 = 1 - \left(\dfrac{1-\lambda}{1+\lambda}\right)^2$ such that $\theta_j(\rho'_j, \rho''_j) \leq \Lambda_1 \theta(\rho', \rho'')$ for all $j \in \{1, \ldots, p\}$;*

where θ_j e θ are the projective metrics associated to $\mathcal{D}(\gamma_j, \kappa)$ and $\mathcal{D}(\gamma, \kappa)$, respectively.

Proof. (1) In our context we suppose $\sup \phi - \inf \phi < \varepsilon$ and $\left|e^{\phi}\right|_{\alpha} < \varepsilon \inf e^{\phi}$. Therefore

$$\begin{aligned}
\frac{|\rho_j|_{\alpha}}{\inf\{\rho_j\}} &= \frac{\left|\dfrac{1}{p}\rho \circ f \cdot e^{\phi}\right|_{\alpha}}{\inf\left\{\dfrac{1}{p}\rho \circ f \cdot e^{\phi}\right\}} \\
&= \frac{|\rho \circ f \cdot e^{\phi}|_{\alpha}}{\inf\{\rho \circ f \cdot e^{\phi}\}} \\
&\leq \frac{|\rho \circ f|_{\alpha} \cdot e^{\sup \phi} + \sup\{\rho \circ f\} \cdot |e^{\phi}|_{\alpha}}{\inf \rho \cdot e^{\inf \phi}} \\
&\leq \frac{\lambda_s^{\alpha} \kappa \inf \rho \cdot e^{\sup \phi}}{\inf \rho \cdot e^{\inf \phi}} + \frac{(1 + \kappa \cdot diam M^{\alpha}) \inf \rho \cdot |e^{\phi}|_{\alpha}}{\inf \rho \cdot e^{\inf \phi}} \\
&\leq \lambda_s^{\alpha} \kappa e^{\varepsilon} + (1 + \kappa \cdot diam M^{\alpha})\varepsilon \\
&= (\lambda_s^{\alpha} e^{\varepsilon} + diam M^{\alpha} \epsilon)\kappa + \varepsilon
\end{aligned}$$

In order to guarantee a $0 < \lambda < 1$ such that

$$(\lambda_s^{\alpha} e^{\varepsilon} + diam M^{\alpha} \varepsilon)\kappa + \varepsilon < \lambda\kappa$$

it is sufficient to obtain

$$\frac{(\lambda_s^\alpha e^\varepsilon + diamM^\alpha\varepsilon)\kappa + \varepsilon}{\kappa} < \lambda < \ 1.$$

For that we just need

$$\frac{(\lambda_s^\alpha e^\varepsilon + diamM^\alpha\varepsilon)\kappa + \varepsilon}{\kappa} < 1$$

or, equivalently,

$$\kappa > \frac{\varepsilon}{1 - (\lambda_s^\alpha e^\varepsilon + diamM^\alpha\varepsilon)}. \tag{9.3}$$

Note that λ and κ can be chosen sufficiently small for since we have chosen in our hipothesis $\varepsilon > 0$ and $0 < \lambda_s < 1$ suitably small.

(2) By a triangular argument, it is sufficient to bound $\theta(1,\rho)$ for $\rho \in \mathcal{D}(\gamma, \lambda\kappa)$. There is no loss of generality in assuming that $\inf\rho = 1$. So, for $t = 1 - \lambda$ we have

$$\frac{|\rho - t|_\alpha}{\inf(\rho - t)} = \frac{|\rho|_\alpha}{\inf\rho - t} < \frac{\lambda\kappa}{1 - t} = \frac{\lambda\kappa}{\lambda} = \kappa.$$

Since $\inf\rho = 1$ it follows that $\rho - t \geq \inf\rho - (1 - \lambda) = \lambda > 0$ which guarantees $\alpha(1, \rho) \geq 1 - \lambda$. On the other hand, by setting $s = 1 + \lambda$ we obtain

$$\frac{|s - \rho|_\alpha}{\inf(s - \rho)} = \frac{|\rho|_\alpha}{s - \inf\rho} < \frac{\lambda\kappa}{s - 1} = \frac{\lambda\kappa}{\lambda} = \kappa.$$

As $\sup\rho \leq \inf\rho\,(1 + \kappa diamM^\alpha) = 1 + \kappa diamM^\alpha$ escolhendo κ tal que $\lambda > \kappa diamM^\alpha$, segue que $s - \rho = 1 + \lambda - \rho \geq 1 + \lambda - \sup\rho \geq 1 + \lambda - (1 + \kappa diamM^\alpha) > 0$ portanto $\beta(1, \rho) \leq 1 + \lambda$. Logo $\theta(\rho, \hat\rho) \leq 2\log\left(\dfrac{1+\lambda}{1-\lambda}\right)$.

Finally, in order to prove (3) it is sufficient to note that by item (1) we have $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$ for all $j \in \{1, \ldots, p\}$ and by item (2) the diameter $\mathcal{D}(\gamma_j, \lambda\kappa)$ in $\mathcal{D}(\gamma_j, \kappa)$ is, at most, $2\log\left(\dfrac{1+\lambda}{1-\lambda}\right)$. Therefore, the result goes on by theorem 8.2, considering $\Delta = 2\log\left(\dfrac{1+\lambda}{1-\lambda}\right)$ and the linear map

$$\rho \mapsto \frac{1}{p}\rho \circ f e^\phi$$

we have $\theta_j(\rho'_j, \rho''_j) \leq \Lambda_1\theta(\rho', \rho'')$ where

$$\Lambda_1 = 1 - e^{-\Delta} = 1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2$$

□

For the definition of the main cone on which we will apply the transfer operator we need to define a notion of distance between two strong stable leaves γ e $\widetilde{\gamma}$ in $\mathcal{F}^s$. Given $x, y \in N$ let $\gamma = \Pi_\Lambda^{-1}(x)$ and $\widetilde{\gamma} = \Pi_\Lambda^{-1}(y)$. Suppose $\pi = \pi_{x,y} : \widetilde{\gamma} \to \gamma$ satisfies

$$\int_\gamma \varphi d\mu_\gamma = \int_{\widetilde{\gamma}} \varphi \circ \pi d\mu_{\widetilde{\gamma}}$$

for all continuous function φ and define the distance $d(\gamma, \widetilde{\gamma}) = \sup\{d(\pi(p), p); p \in \widetilde{\gamma}\}$.

Now let us define our main cone. Denote by $\mathcal{D}_1(\gamma)$ the set of densities $\rho \in \mathcal{D}(\gamma, \kappa)$ such that $\int_\gamma \rho d\mu_\gamma = 1$. Given $b > 0$, $c > 0$ and κ as in lemma 9.1, let $C(b, c, \alpha)$ be the cone of functions $\varphi \in E$ satisfying for all $\gamma \in \mathcal{F}^s$ the following:

(A): For all $\rho \in \mathcal{D}(\gamma, \kappa)$:

$$\int_\gamma \varphi \rho d\mu_\gamma > 0$$

(B): For all $\rho', \rho'' \in \mathcal{D}_1(\gamma)$:

$$\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right| < b\theta\left(\rho', \rho''\right) \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \varphi \rho d\mu_\gamma \right\}$$

(C): Given any $\widetilde{\gamma}$ sufficiently close to γ:

$$\left| \int_\gamma \varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}} \right| < cd(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma \left\{ \int_\gamma \varphi d\mu_\gamma \right\}$$

We then prove:

Lemma 9.2. *$C\left(b, c, \alpha\right)$ is a projective cone.*

Proof. We start by the convexity of $C\left(b, c, \alpha\right)$. Given $\varphi, \psi \in C\left(b, c, \alpha\right)$ and $s, t > 0$ we have

(A): $\int_\gamma (s\varphi + t\psi) d\mu_\gamma = s \int_\gamma \varphi d\mu_\gamma + t \int_\gamma \psi d\mu_\gamma > 0.$

(B): By hipohesis,

$$\frac{\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \varphi \rho d\mu_\gamma \right\}} < b$$

and

$$\frac{\left| \int_\gamma \psi \rho' d\mu_\gamma - \int_\gamma \psi \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \psi \rho d\mu_\gamma \right\}} < b.$$

Then, it follows

$$\inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \left(s\varphi + t\psi\right) \rho d\mu_\gamma \right\} \geq s \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \varphi \rho d\mu_\gamma \right\} + t \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \psi \rho d\mu_\gamma \right\}.$$

Therefore, $$\frac{\left| \int_\gamma (s\varphi + t\psi) \rho' d\mu_\gamma - \int_\gamma (s\varphi + t\psi) \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \left(s\varphi + t\psi\right) \rho d\mu_\gamma \right\}} < b.$$

(C): Analogous to (B).

In order to prove that $\overline{C\left(b, c, \alpha\right)} \cap -\overline{C\left(b, c, \alpha\right)} = 0$, take $\varphi \in \overline{C\left(b, c, \alpha\right)} \cap -\overline{C\left(b, c, \alpha\right)}$. If $\varphi \in \overline{C\left(b, c, \alpha\right)}$, there exists $\psi \in C\left(b, c, \alpha\right)$ and a sequence $(t_n)_{n \in N} \searrow 0$ such that $\varphi + t_n \psi \in C\left(b, c, \alpha\right)$ for all $n \in N$. In particular, given $\gamma \in \mathcal{F}^s$ and $\rho \in D(\gamma, \kappa)$, we have $\int_\gamma \varphi\rho d\mu_\gamma > -t_n \int_\gamma \psi\rho d\mu_\gamma$, $\forall t_n > 0$. As $t_n \to 0$ and $\int_\gamma \psi\rho d\mu_\gamma > 0$, if follows that $\int_\gamma \varphi\rho d\mu_\gamma \geq 0$. On the other hand, if $\varphi \in -\overline{C(b,c,\alpha)}$ then $\varphi = -\overline{\varphi}$ where $\overline{\varphi} \in \overline{C(b,c,\alpha)}$ and so, $\int_\gamma \varphi\rho d\mu_\gamma = -\int_\gamma \overline{\varphi}\rho d\mu_\gamma \leq 0$. Therefore, $\int_\gamma \varphi\rho d\mu_\gamma = 0$ for all $\gamma \in \mathcal{F}^s$ and $\rho \in D(\gamma,\kappa)$. All that rests is to prove

$$\int_\gamma \varphi\rho d\mu_\gamma = 0, \forall \gamma \in \mathcal{F}^s \text{ and } \rho \in \mathcal{D}(\gamma,\kappa) \Rightarrow \varphi = 0 \text{ in } \Lambda.$$

Indeed, fixed γ, given any Hölder continuous function $\psi : \gamma \to \mathbb{R}$ we can write $\psi = \psi^+ - \psi^-$, with ψ^+, ψ^- belonging in $\mathcal{D}(\gamma,\kappa)$. For that, set $\psi^\pm = \frac{1}{2}(|\psi| \pm \psi) + B$ for a sufficiently large B. By linearity, we have $\int_\gamma \varphi\psi d\mu_\gamma = 0$. As all bounded function can be aproximated in $L^1(\mu_\gamma)$ by Hölder functions, it followd that $\int_\gamma \varphi\psi d\mu_\gamma = 0$, for all bounded $\psi : \gamma \to \mathbb{R}$. By taking $\psi = \varphi|_A$, A is a Borel subset of Λ restrited to γ, we obtain $\int_\gamma \varphi^2|_A d\mu_\gamma = 0$ and so $\varphi|_A = 0$ for μ_γ-a.e point in A. As A and γ are arbitrary, we conclude that $\varphi = 0$ in Λ.

$\square$

Proposition 9.3. *Let $\phi \equiv 0$. There exists $0 < \sigma < 1$ such that $\mathcal{L}(C(b,c,\alpha)) \subset C(\sigma b, \sigma c, \alpha)$ for sufficiently large b,c > 0.*

Proof. Invariance of condition (A): Let $\varphi \in C(b,c,\alpha)$. We know that $\int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma = \sum_{j=1}^{p} \int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}$ and by lemma 9.1 $\rho_j \in \mathcal{D}(\gamma_j,\kappa)$. Therefore, $\int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma > 0$.

Invariance of condition (B): Denoting $\dfrac{\rho_j}{\int_{\gamma_j} \rho_j d\mu_{\gamma_j}}$ by $\hat{\rho}_j$ we can write

$$\begin{aligned}
\inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{ \int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma \right\} &\geq \sum_{j=1}^{p} \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j} \right\} \\
&= \sum_{j=1}^{p} \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \varphi\hat{\rho}_j d\mu_{\gamma_j} \int_{\gamma_j} \rho_j d\mu_{\gamma_j} \right\} \\
&\geq \sum_{j=1}^{p} \inf_{\rho\in\mathcal{D}_1(\gamma_j)} \left\{ \int_{\gamma_j} \varphi\rho d\mu_{\gamma_j} \right\} \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \rho_j d\mu_{\gamma_j} \right\}
\end{aligned}$$

Given $\rho^{,}, \rho^{,,} \in \mathcal{D}_1(\gamma)$ writing $\rho^{,}_j / \int_{\gamma_j} \rho^{,}_j d\mu_{\gamma_j}$ and $\rho^{,,}_j / \int_{\gamma_j} \rho^{,,}_j d\mu_{\gamma_j}$ for $\overline{\rho}_j$ and $\overline{\overline{\rho}}_j$, respectively, follows that

$$\left| \int_\gamma \mathcal{L}(\varphi)\rho^{,} d\mu_\gamma - \int_\gamma \mathcal{L}(\varphi)\rho^{,,} d\mu_\gamma \right| \leq \sum_{j=1}^{p} \left| \int_{\gamma_j} \varphi \overline{\rho}_j d\mu_{\gamma_j} - \int_{\gamma_j} \varphi \overline{\overline{\rho}}_j d\mu_{\gamma_j} \right| \int_{\gamma_j} \rho^{,}_j d\mu_{\gamma_j} + \sum_{j=1}^{p} \int_{\gamma_j} \varphi \overline{\overline{\rho}}_j d\mu_{\gamma_j} \left| \int_{\gamma_j} \rho^{,}_j d\mu_{\gamma_j} - \int_{\gamma_j} \rho^{,,}_j d\mu_{\gamma_j} \right|. \tag{9.4}$$

By hypothesis, φ is in the cone and by lemma 9.1, we have

$$\begin{aligned} \left| \int_{\gamma_j} \varphi \overline{\rho}_j d\mu_{\gamma_j} - \int_{\gamma_j} \varphi \overline{\overline{\rho}}_j d\mu_{\gamma_j} \right| &\leq b\theta_j \left(\overline{\rho}_j, \overline{\overline{\rho}}_j\right) \inf_{\rho \in \mathcal{D}_1(\gamma_j)} \left\{ \int_{\gamma_j} \varphi \rho d\mu_{\gamma_j} \right\} \\ &= b\theta_j \left(\rho^{,}_j, \rho^{,,}_j\right) \inf_{\rho \in \mathcal{D}_1(\gamma_j)} \left\{ \int_{\gamma_j} \varphi \rho d\mu_{\gamma_j} \right\} \qquad (9.5) \\ &\leq b\Lambda_1 \theta \left(\rho^{,}, \rho^{,,}\right) \inf_{\rho \in \mathcal{D}_1(\gamma_j)} \left\{ \int_{\gamma_j} \varphi \rho d\mu_{\gamma_j} \right\}. \end{aligned}$$

For all $\hat{\rho} \in \mathcal{D}_1(\gamma)$ we obtain the following estimative

$$\frac{\int_{\gamma_j} (\hat{\rho})_j d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \rho_j d\mu_{\gamma_j} \right\}} \leq (1 + \kappa diam M^\alpha)^2 \tag{9.6}$$

In fact, given $\delta > 0$ there exists $\widetilde{\rho} \in \mathcal{D}_1(\gamma)$ such that $\int_{\gamma_j} (\widetilde{\rho})_j d\mu_{\gamma_j} \leq (1 + \delta) \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \rho_j d\mu_{\gamma_j} \right\}$. Moreover, as $\hat{\rho}$ and $\widetilde{\rho}$ are normalized, we necessarily have $\inf \hat{\rho} \leq 1$ and $\sup \widetilde{\rho} \geq 1$. Therefore,

$$\frac{(\hat{\rho})_j}{(\widetilde{\rho})_j} = \frac{\frac{1}{p}\hat{\rho} \circ f e^{\phi}}{\frac{1}{p}\widetilde{\rho} \circ f e^{\phi}} \leq \frac{\sup \hat{\rho}}{\inf \widetilde{\rho}} \leq \frac{(1 + \kappa diam M^\alpha) \inf \hat{\rho}}{(1 + \kappa diam M^\alpha)^{-1} \sup \widetilde{\rho}} = (1 + \kappa diam M^\alpha)^2 .$$

And so $\int_{\gamma_j} (\hat{\rho})_j d\mu_{\gamma_j} \leq (1 + \kappa diam M^\alpha)^2 \int_{\gamma_j} (\widetilde{\rho})_j d\mu_{\gamma_j}$, we obtain for all $\delta > 0$

$$\begin{aligned} \frac{\int_{\gamma_j} (\hat{\rho})_j d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_{\gamma_j} \rho_j d\mu_{\gamma_j} \right\}} &\leq \frac{(1 + \delta)(1 + \kappa diam M^\alpha)^2 \int_{\gamma_j} (\widetilde{\rho})_j d\mu_{\gamma_j}}{\int_{\gamma_j} (\widetilde{\rho})_j d\mu_{\gamma_j}} \\ &\leq (1 + \delta)(1 + \kappa diam M^\alpha)^2 \end{aligned}$$

giving the estimative we wish.

Now, for fixed j, we obtain

$$\begin{aligned}\frac{\left|\int_{\gamma_j}\varphi\overline{\rho}_j d\mu_{\gamma_j}-\int_{\gamma_j}\varphi\overline{\overline{\rho}}_j d\mu_{\gamma_j}\right|\int_{\gamma_j}\rho^{\prime}_j d\mu_{\gamma_j}}{\inf\limits_{\rho\in\mathcal{D}_1(\gamma_j)}\left\{\int_{\gamma_j}\varphi\rho d\mu_{\gamma_j}\right\}\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\rho_j d\mu_{\gamma_j}\right\}\theta(\rho^{\prime},\rho^{\prime\prime})} &\leq \frac{b\Lambda_1\theta\left(\rho^{\prime},\rho^{\prime\prime}\right)\int_{\gamma_j}\rho^{\prime}_j d\mu_{\gamma_j}}{\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\rho_j d\mu_{\gamma_j}\right\}\theta(\rho^{\prime},\rho^{\prime\prime})}\\ &\leq \left(1+\kappa diam M^{\alpha}\right)^2\Lambda_1 b\end{aligned} \tag{9.7}$$

Let us analyse the second parcel of 9.4. First, note that for all $\hat{\rho}\in\mathcal{D}_1(\gamma)$, denoting $(\hat{\rho})_j/\int_{\gamma_j}(\hat{\rho})_j d\mu_{\gamma_j}$ by $\bar{\hat{\rho}}_j$, we claim that

$$\frac{\int_{\gamma_j}\varphi\bar{\hat{\rho}}_j d\mu_{\gamma_j}}{\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\varphi\overline{\rho}_j d\mu_{\gamma_j}\right\}}< b\log\left(\frac{1+\lambda}{1-\lambda}\right)^2+1$$

In fact, analogously to what was done in 9.6, it is sufficient to to note that, since φ is in the cone, we have

$$\frac{\int_{\gamma_j}\varphi\bar{\hat{\rho}}_j d\mu_{\gamma_j}}{\int_{\gamma_j}\varphi\overline{\rho}_j d\mu_{\gamma_j}}< b\theta(\bar{\hat{\rho}}_j,\overline{\rho}_j)+1=b\theta((\hat{\rho})_j,\rho_j)+1$$

By 9.1, we conclude the proof of our claim.

Now, we stablish the other necessary estimative:

$$\frac{\left|\int_{\gamma_j}\rho^{\prime}_j d\mu_{\gamma_j}-\int_{\gamma_j}\rho^{\prime\prime}_j d\mu_{\gamma_j}\right|}{\theta\left(\rho^{\prime},\rho^{\prime\prime}\right)\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\rho_j d\mu_{\gamma_j}\right\}}\leq 2\left(1+\kappa diam M^{\alpha}\right)^2.$$

In order to prove this last estimative we observe that

$$\frac{\rho^{\prime}_j}{\rho^{\prime\prime}_j}\leq\frac{\sup\rho^{\prime}}{\inf\rho^{\prime\prime}}\leq\frac{\sup\rho^{\prime}/\inf\rho^{\prime}}{\inf\rho^{\prime\prime}/\sup\rho^{\prime\prime}}=e^{\theta_+(\rho^{\prime},\rho^{\prime\prime})}\leq e^{\theta(\rho^{\prime},\rho^{\prime\prime})}$$

Therefore, by assuming without loss of generality that $\int_{\gamma_j}\rho^{\prime}_j d\mu_{\gamma_j}\geq\int_{\gamma_j}\rho^{\prime\prime}_j d\mu_{\gamma_j}$ we obtain

$$\frac{\left|\int_{\gamma_j}\rho^{\prime}_j d\mu_{\gamma_j}-\int_{\gamma_j}\rho^{\prime\prime}_j d\mu_{\gamma_j}\right|}{\theta\left(\rho^{\prime},\rho^{\prime\prime}\right)\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\rho_j d\mu_{\gamma_j}\right\}}\leq\frac{\left(e^{\theta\left(\rho^{\prime},\rho^{\prime\prime}\right)}-1\right)\int_{\gamma_j}\rho^{\prime\prime}_j d\mu_{\gamma_j}}{\theta\left(\rho^{\prime},\rho^{\prime\prime}\right)\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_{\gamma_j}\rho_j d\mu_{\gamma_j}\right\}}$$

for $\theta(\rho', \rho'') \leq 1$ it follows $\dfrac{e^{\theta(\rho', \rho'')} - 1}{\theta(\rho', \rho'')} < 2$ and so we obtain our estimative.

If $\theta(\rho', \rho'') \geq 1$ we also have that

$$\frac{\left|\int_{\gamma_j} \rho'_j d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j d\mu_{\gamma_j}\right|}{\theta(\rho', \rho'') \inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \rho_j d\mu_{\gamma_j}\right\}} \leq \frac{\left|\int_{\gamma_j} \rho'_j d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j d\mu_{\gamma_j}\right|}{\inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \rho_j d\mu_{\gamma_j}\right\}} \leq 2\left(1 + \kappa diam M^\alpha\right)^2$$

and again for fixed j and by writing $M(\kappa, \alpha)$ for $(1 + \kappa diam M^\alpha)^2$,

$$\begin{aligned}
\frac{\int_{\gamma_j} \varphi\bar{\bar{\rho}}_j d\mu_{\gamma_j} \left|\int_{\gamma_j} \rho'_j d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j d\mu_{\gamma_j}\right|}{\inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}\right\} \inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \rho_j d\mu_{\gamma_j}\right\} \theta(\rho', \rho'')} &\leq \frac{\int_{\gamma_j} \varphi\bar{\bar{\rho}}_j d\mu_{\gamma_j}}{\inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}\right\}} 2M(\kappa, \alpha) \\
&\leq \left(b \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 + 1\right) 2M(\kappa, \alpha) \\
&\leq 2M(\kappa, \alpha) \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 b + 2M(\kappa, \alpha)
\end{aligned} \tag{9.8}$$

The inequalities 9.7 and 9.8 does not depend on j, so

$$\begin{aligned}
\frac{\left|\int_{\gamma} \mathcal{L}(\varphi)\rho' d\mu_\gamma - \int_{\gamma} \mathcal{L}(\varphi)\rho'' d\mu_\gamma\right|}{\inf\limits_{\rho \in \mathcal{D}_1(\gamma)} \left\{\int_{\gamma} \mathcal{L}(\varphi)\rho d\mu_\gamma\right\} \theta(\rho', \rho'')} &\leq M(\kappa, \alpha)\Lambda_1 b + 2M(\kappa, \alpha) \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 b + 2M(\kappa, \alpha) \\
&= \left(\Lambda_1 + 2\log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right) M(\kappa, \alpha) b + 2M(\kappa, \alpha)
\end{aligned}$$

We need that the term which multiplies b above to be less than 1. Recall that by lemma (9.1), $\Lambda_1 = 1 - \left(\dfrac{1-\lambda}{1+\lambda}\right)^2$. So, we need to guarantee that

$$\left(1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2 + 2\log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right)\left(1 + \kappa diam M^\alpha\right)^2 < 1$$

Also by lemma (9.1), we can choose κ, such that $\kappa diam M^\alpha < \lambda$, λ to be fixed. So let us find a bound $0 < \lambda < 1$ such that

$$\left(1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2 + 2\log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right)(1+\lambda)^2 < 1.$$

It is possible because (9.1), $0 < \lambda < 1$ can be taken sufficiently small depending on the contraction rate in the strong stable directions. So, there exists $0 < \tilde{\sigma}_1 < 1$

such that

$$\left(\Lambda_1 + 2\log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right) M(\kappa,\alpha) < \tilde{\sigma}_1.$$

Since $M(\kappa,\alpha)$ does not depend on b, for sufficiently large b we can obtain $\sigma_1 < 1$ such that

$$\frac{\left|\int_\gamma \mathcal{L}(\varphi)\rho' d\mu_\gamma - \int_\gamma \mathcal{L}(\varphi)\rho'' d\mu_\gamma\right|}{\inf_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma\right\}\theta(\rho',\rho'')} \leq \sigma_1 b$$

This prove the strict invariance of condition (B).

Invariance of condition (C): This is the place we need $\phi \equiv 0$. This implies that

$$\inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi d\mu_\gamma\right\} \geq e^\phi \inf_\gamma\left\{\int_\gamma \varphi d\mu_\gamma\right\}.$$

For g as in 2.1, every $y \in N$ has a pre-image in the region Ω such that $L(y) \leq \lambda_u$. So, for $\gamma = \Pi^{-1}(y)$ and $\widetilde{\gamma} = \Pi^{-1}(\tilde{y})$ sufficiently close to γ such that $\tilde{y}_i$ is pre-image $\tilde{y}$, close to y_i, stay in U_{y_i} we obtain that $d(\gamma_i,\widetilde{\gamma}_i) \leq \lambda_u d(\gamma,\widetilde{\gamma})$.

In fact, let $x \in \widetilde{\gamma}_i$ realizing the distance $d(\gamma_i,\widetilde{\gamma}_i)$. By a slight abuse of notation, we write d for the product distance equivalent to the original metrics. So,

$$\begin{aligned} d(\gamma_i,\widetilde{\gamma}_i) &= d(x,\pi_{\tilde{y}_i,y_i}(x)) = d(\tilde{y}_i,y_i) \leq \lambda_u d(g(\tilde{y}_i),g(y_i)) \\ &= \lambda_u d(\tilde{y},y) = \lambda_u\left[d(\tilde{y},y) + d(\pi_{\tilde{y},y}(f(x)),\pi_{\tilde{y},y}(f(x)))\right] \\ &= \lambda_u d(f(x),\pi_{\tilde{y},y}(f(x))) \leq \lambda_u d(\widetilde{\gamma},\gamma) \end{aligned}$$

Analogously, in the other cases we have $d(\gamma_j,\widetilde{\gamma}_j) \leq Ld(\gamma,\widetilde{\gamma})$. Furthermore, we can assume with no loss of generality, that $d(\gamma_1,\widetilde{\gamma}_1) \leq \tilde{\lambda}_u d(\gamma,\widetilde{\gamma})$, and for other pre-images we have $d(\gamma_j,\widetilde{\gamma}_j) \leq \tilde{L}d(\gamma,\widetilde{\gamma})$. It follows that

$$\begin{aligned} \left|\int_\gamma \mathcal{L}\varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \mathcal{L}\varphi d\mu_{\widetilde{\gamma}}\right| &\leq \frac{e^\phi}{p}\sum_{j=1}^p\left|\int_{\gamma_j}\varphi d\mu_{\gamma_j} - \int_{\widetilde{\gamma}_j}\varphi d\mu_{\widetilde{\gamma}_j}\right| \\ &\leq \frac{e^\phi C}{p}\inf_\gamma\left\{\int_\gamma \varphi d\mu_\gamma\right\}\sum_{j=1}^p d(\gamma_j,\widetilde{\gamma}_j)^\alpha \\ &\leq \frac{\tilde{\lambda}_u^\alpha + (p-1)\tilde{L}^\alpha}{p} Cd(\gamma,\widetilde{\gamma})^\alpha \inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi d\mu_\gamma\right\} \end{aligned}$$

We should obtain $\dfrac{\tilde{\lambda}_u^\alpha + (p-1)\tilde{L}^\alpha}{p} < 1$. This is equivalent to

$$\tilde{L}^\alpha < \frac{p - \tilde{\lambda}_u^\alpha}{p-1}.$$

Due to the fact that $\tilde{L} \geq 1$, we have

$$\frac{p - \tilde{\lambda}_u^\alpha}{p-1} \geq 1$$

because $\tilde{\lambda}_u < 1$. Therefore, there exists $0 < \sigma_2 < 1$ such that

$$\left|\int_\gamma \mathcal{L}\varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \mathcal{L}\varphi d\mu_{\widetilde{\gamma}}\right| < \sigma_2 Cd(\gamma,\widetilde{\gamma})^\alpha \inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi d\mu_\gamma\right\}$$

which proves (C). By setting $\sigma = \max\{\sigma_1, \sigma_2\}$, we finish the proof of the proposition. $\square$

10. Finite Diameter of the Main Cone

In this section, we prove the strict invariance of the main cone $C\,(b, c, \alpha)$ by the Ruelle-Perron-Frobenius operator $\mathcal{L}$. First, let us calculate the projective metrics Θ. Recall that $\alpha(\varphi, \psi) = \sup\{t > 0; \psi - t\varphi \in C\,(b, c, \alpha)\}$. By (A), for all $\gamma \in \mathcal{F}^s_{loc}$ and $\rho \in \mathcal{D}\,(\gamma)$ we have $\displaystyle\int_\gamma (\psi - t\varphi)\rho d\mu_\gamma > 0$, that is,

$$t < \frac{\displaystyle\int_\gamma \psi\rho d\mu_\gamma}{\displaystyle\int_\gamma \varphi\rho d\mu_\gamma}.$$

By condition (B), one obtains

$$\left|\int_\gamma (\psi - t\varphi)\rho' d\mu_\gamma - \int_\gamma (\psi - t\varphi)\rho'' d\mu_\gamma\right| < b\theta\,(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_\gamma (\psi - t\varphi)\rho d\mu_\gamma\right\}$$

and so, for all ρ', ρ'', and $\hat{\rho}$ in $\mathcal{D}_1(\gamma)$ we have

$$t < \frac{\displaystyle\int_\gamma \psi\rho' d\mu_\gamma - \int_\gamma \psi\rho'' d\mu_\gamma + b\theta(\rho', \rho'')\int_\gamma \psi\hat{\rho} d\mu_\gamma}{\displaystyle\int_\gamma \varphi\rho' d\mu_\gamma - \int_\gamma \varphi\rho'' d\mu_\gamma + b\theta(\rho', \rho'')\int_\gamma \varphi\hat{\rho} d\mu_\gamma}$$

and

$$t < \frac{\displaystyle\int_\gamma \psi\rho'' d\mu_\gamma - \int_\gamma \psi\rho' d\mu_\gamma + b\theta(\rho', \rho'')\int_\gamma \psi\hat{\rho} d\mu_\gamma}{\displaystyle\int_\gamma \varphi\rho'' d\mu_\gamma - \int_\gamma \varphi\rho' d\mu_\gamma + b\theta(\rho', \rho'')\int_\gamma \varphi\hat{\rho} d\mu_\gamma}.$$

By condition (C),

$$\left|\int_\gamma (\psi - t\varphi) d\mu_\gamma - \int_{\widetilde{\gamma}} (\psi - t\varphi) d\mu_{\widetilde{\gamma}}\right| < cd(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma \left\{\int_\gamma (\psi - t\varphi)\, d\mu_\gamma\right\}$$

therefore, for all $\gamma, \hat{\gamma} \in \mathcal{F}^s_{loc}$ and $\widetilde{\gamma}$ sufficiently close to γ we have

$$t < \frac{\displaystyle\int_{\widetilde{\gamma}} \psi d\mu_{\widetilde{\gamma}} - \int_\gamma \psi d\mu_\gamma + cd(\gamma, \widetilde{\gamma})\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}{\displaystyle\int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}} - \int_\gamma \varphi d\mu_\gamma + cd(\gamma, \widetilde{\gamma})\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}$$

and

$$t < \frac{\displaystyle\int_\gamma \psi d\mu_\gamma - \int_{\widetilde{\gamma}} \psi d\mu_{\widetilde{\gamma}} + cd(\gamma, \widetilde{\gamma})\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}{\displaystyle\int_\gamma \varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}} + cd(\gamma, \widetilde{\gamma})\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}.$$

By defining

$$\xi(\gamma,\rho',\rho'',\hat{\rho},\varphi,\psi)=\frac{\left(\int_\gamma \psi\rho'' d\mu_\gamma-\int_\gamma \psi\rho' d\mu_\gamma\right)/\int_\gamma \psi\hat{\rho}d\mu_\gamma+b\theta(\rho',\rho'')}{\left(\int_\gamma \varphi\rho'' d\mu_\gamma-\int_\gamma \varphi\rho' d\mu_\gamma\right)/\int_\gamma \varphi\hat{\rho}d\mu_\gamma+b\theta(\rho',\rho'')}$$

and

$$\eta(\gamma,\widetilde{\gamma},\hat{\gamma},\hat{\rho},\varphi,\psi)=\frac{\left(\int_\gamma \psi d\mu_\gamma-\int_{\widetilde{\gamma}} \psi d\mu_{\widetilde{\gamma}}\right)/\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}+cd(\gamma,\widetilde{\gamma})}{\left(\int_\gamma \varphi d\mu_\gamma-\int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}}\right)/\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}+cd(\gamma,\widetilde{\gamma})}.$$

we can write

$$\alpha(\varphi,\psi)=\inf\left\{\frac{\int_\gamma \psi\rho d\mu_\gamma}{\int_\gamma \varphi\rho d\mu_\gamma},\frac{\int_\gamma \psi\hat{\rho} d\mu_\gamma}{\int_\gamma \varphi\hat{\rho} d\mu_\gamma}\xi(\gamma,\rho',\rho'',\hat{\rho},\varphi,\psi),\frac{\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}\eta(\gamma,\widetilde{\gamma},\hat{\gamma},\hat{\rho},\varphi,\psi)\right\}$$

as $\beta(\varphi,\psi)=\alpha(\psi,\varphi)^{-1}$ we obtain

$$\beta(\varphi,\psi)=\sup\left\{\frac{\int_\gamma \varphi\rho d\mu_\gamma}{\int_\gamma \psi\rho d\mu_\gamma},\frac{\int_\gamma \varphi\hat{\rho} d\mu_\gamma}{\int_\gamma \psi\hat{\rho} d\mu_\gamma}\xi(\gamma,\rho',\rho'',\hat{\rho},\psi,\varphi),\frac{\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}\eta(\gamma,\widetilde{\gamma},\hat{\gamma},\hat{\rho},\psi,\varphi)\right\}$$

Now, we prove that the Θ-diameter of $\mathcal{L}\left(C(b,c,\alpha)\right)$ is finite.

Proposition 10.1. *For all sufficiently large $b>0$, $c>0$ and for $\alpha\in(0,1]$ we have*

$$\Delta:=\sup\left\{\Theta\left(\mathcal{L}\varphi,\mathcal{L}\psi\right);\varphi,\psi\in C(b,c,\alpha)\right\}<\infty.$$

Proof. Given $\varphi,\psi\in C(\sigma b,\sigma c,\alpha)$, note that

$$\frac{1-\sigma}{1+\sigma}<\xi(\gamma,\rho',\rho'',\hat{\rho},\psi,\varphi)<\frac{1+\sigma}{1-\sigma}$$

and

$$\frac{1-\sigma}{1+\sigma}<\eta(\gamma,\widetilde{\gamma},\hat{\gamma},\hat{\rho},\psi,\varphi)<\frac{1+\sigma}{1-\sigma}$$

Indeed, given $\rho',\rho'',\hat{\rho}\in\mathcal{D}_1(\gamma)$

$$\frac{\int_\gamma \varphi\rho'' d\mu_\gamma-\int_\gamma \varphi\rho' d\mu_\gamma}{\int_\gamma \varphi\hat{\rho}d\mu_\gamma}\le\frac{\sigma b\theta(\rho',\rho'')\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_\gamma \varphi\rho d\mu_\gamma\right\}}{\int_\gamma \varphi\hat{\rho}d\mu_\gamma}\le\sigma b\theta(\rho',\rho'')$$

and

$$\frac{\int_\gamma \varphi\rho'' d\mu_\gamma-\int_\gamma \varphi\rho' d\mu_\gamma}{\int_\gamma \varphi\hat{\rho}d\mu_\gamma}\ge\frac{-\sigma b\theta(\rho',\rho'')\inf\limits_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_\gamma \varphi\rho d\mu_\gamma\right\}}{\int_\gamma \varphi\hat{\rho}d\mu_\gamma}\ge-\sigma b\theta(\rho',\rho'')$$

holds.

The same is valid for ψ and as $\sigma < 1$ we conclude that

$$\frac{1-\sigma}{1+\sigma} < \frac{\left(\int_\gamma \psi\rho" d\mu_\gamma - \int_\gamma \psi\rho' d\mu_\gamma\right) / \int_\gamma \psi\hat{\rho} d\mu_\gamma + b\theta(\rho', \rho")}{\left(\int_\gamma \varphi\rho" d\mu_\gamma - \int_\gamma \varphi\rho' d\mu_\gamma\right) / \int_\gamma \varphi\hat{\rho} d\mu_\gamma + b\theta(\rho', \rho")} < \frac{1+\sigma}{1-\sigma}$$

That is,

$$\frac{1-\sigma}{1+\sigma} < \xi(\gamma, \rho', \rho", \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}.$$

In a similar way, we prove that

$$\frac{1-\sigma}{1+\sigma} < \eta(\gamma, \widetilde{\gamma}, \hat{\gamma}, \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}$$

Denoting by Θ_+ the projective metrics associated to the cone defined just by condition (A),

$$\Theta_+(\varphi, \psi) = \sup_{\gamma, \rho \in \mathcal{D}(\gamma), \hat{\gamma}, \hat{\rho} \in \mathcal{D}(\hat{\gamma})} \left\{ \frac{\int_\gamma \varphi\rho d\mu_\gamma \int_{\hat{\gamma}} \psi\hat{\rho} d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi\hat{\rho} d\mu_{\hat{\gamma}} \int_\gamma \psi\rho d\mu_\gamma} \right\}$$

it followd by the expression of Θ that

$$\Theta(\varphi, \psi) < \Theta_+(\varphi, \psi) + log\left(\frac{1+\sigma}{1-\sigma}\right)^2.$$

So, we just need to prove that the Θ_+-diameter of $\mathcal{L}(C(b, c, \alpha))$ is finite. By a triangular argument, it is sufficient to show that $\{\Theta_+(\mathcal{L}\varphi, 1); \varphi \in C(b, c, \alpha)\}$ is finite. For that, we just need to find an upper bound for

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho} d\mu_{\hat{\gamma}}}{\int_\gamma \mathcal{L}\varphi\rho d\mu_\gamma}$$

for all $\varphi \in C(b, c, \alpha)$, $\rho \in \mathcal{D}_1(\gamma)$ and $\hat{\rho} \in \mathcal{D}_1(\hat{\gamma})$. First, note that

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho} d\mu_{\hat{\gamma}}}{\int_\gamma \mathcal{L}\varphi\rho d\mu_\gamma} = \frac{\sum_{j=1}^{p} \int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\sum_{j=1}^{p} \int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}}$$

and we reduce our problem to bound

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}} = \frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}} \frac{\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi d\mu_{\gamma_j}} \frac{\int_{\gamma_j} \varphi d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}}$$

Denoting $\dfrac{\rho_j}{\displaystyle\int_{\gamma_j} \rho_j d\mu_{\gamma_j}}$ and $\dfrac{(\hat{\rho})_j}{\displaystyle\int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j}}$ by $\overline{\rho}_j$ e $\overline{\overline{\rho}}_j$, respectively, applying (B) and lemma 9.1, we obtain

$$\begin{aligned}
\frac{\displaystyle\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}} &= \frac{\displaystyle\int_{\hat{\gamma}_j} \varphi\overline{\overline{\rho}}_j d\mu_{\hat{\gamma}_j} \int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}} \\
&\leq \left(1 + b\theta_j\left(\overline{\overline{\rho}}_j, 1\right)\right) \int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j} \\
&\leq \left(1 + b\log\left(\frac{1+\lambda}{1-\lambda}\right)\right) \int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j}
\end{aligned}$$

and

$$\frac{\displaystyle\int_{\gamma_j} \varphi d\mu_{\gamma_j}}{\displaystyle\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}} = \frac{\displaystyle\int_{\gamma_j} \varphi d\mu_{\gamma_j}}{\displaystyle\int_{\gamma_j} \varphi\overline{\rho}_j d\mu_{\gamma_j} \int_{\gamma_j} \rho_j d\mu_{\gamma_j}} \leq \frac{\left(1 + b\theta_j\left(1, \overline{\rho}_j\right)\right)}{\displaystyle\int_{\gamma_j} \rho_j d\mu_{\gamma_j}} \leq \frac{\left(1 + b\log\left(\frac{1+\lambda}{1-\lambda}\right)\right)}{\displaystyle\int_{\gamma_j} \rho_j d\mu_{\gamma_j}}$$

We know that $(\hat{\rho})_j = \dfrac{1}{p}\hat{\rho} \circ f \cdot e^{\phi}$ e $\rho_j = \dfrac{1}{p}\rho \circ f \cdot e^{\phi}$. Since ρ and $\hat{\rho}$ are normalized, it follows that $(\hat{\rho})_j \leq \dfrac{1}{p}(1 + \kappa diam(M)^{\alpha})e^{\phi}$ and $\rho_j \geq \dfrac{1}{p}(1 + \kappa diam(M)^{\alpha})^{-1}e^{\phi}$. Therefore

$$\frac{\displaystyle\int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} \rho_j d\mu_{\gamma_j}} < (1 + \kappa diam(M)^{\alpha})^2 \frac{\displaystyle\int_{\hat{\gamma}_j} e^{\phi} d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} e^{\phi} d\mu_{\gamma_j}},$$

On the other hand, $\left|e^{\phi}\right|_{\alpha} < \varepsilon \inf e^{\phi}$ and so $\sup e^{\phi} < (1 + \varepsilon diam(M)^{\alpha}) \inf e^{\phi}$. We then obtain

$$\frac{\displaystyle\int_{\hat{\gamma}_j} e^{\phi} d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} e^{\phi} d\mu_{\gamma_j}} < 1 + \varepsilon diam(M)^{\alpha}$$

and by consequence

$$\frac{\displaystyle\int_{\hat{\gamma}_j} (\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} \rho_j d\mu_{\gamma_j}} < (1 + \kappa diam(M)^{\alpha})^2(1 + \varepsilon diam(M)^{\alpha}).$$

Moreover for γ e $\hat{\gamma}$ such that we can apply (C) we have

$$\frac{\displaystyle\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} \varphi d\mu_{\gamma_j}} \leq 1 + cd\left(\hat{\gamma}_j, \gamma_j\right)^{\alpha} \leq 1 + c.diam(M)^{\alpha}$$

implying that

$$\frac{\displaystyle\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}} < \left(1 + b\log\left(\frac{1+\lambda}{1-\lambda}\right)\right)^2 (1 + \max\{\kappa, c, \varepsilon\} diam(M)^{\alpha})^4,$$

finishing the proof of the proposition. □

11. Exponential Decay of Correlations

In this section, we prove the Exponential Decay of Correlation for Hölder continuous.

In our context, the transfer operator is just $\mathcal{L}(\varphi) = \varphi \circ f^{-1}$ acting in the space of continuous observables.

The adjoint operador of $\mathcal{L}$ is

$$\int \tilde{\mathcal{L}}\varphi d\mu = \int \varphi d\mathcal{L}^* \mu.$$

for all continuous φ and all probability measure μ. Instead of the nonuniformly expanding case, any invariant probability is an eigenmeasure of the transfer operator's adjoint:

Proposition 11.1. *If f is invertible, then $\mathcal{L}^*(\mu) = \mu$ if and only if μ is f-invariant.*

Proof. Let φ be a continuous function. If $\mathcal{L}^*(\mu) = \mu$ then

$$\int \varphi \circ f^{-1} d\mu = \int \mathcal{L}(\varphi) d\mu = \int \varphi d\mathcal{L}^*(\mu) = \int \varphi d\mu,$$

Now, givem an f-invariant measure μ, we have

$$\int \varphi d\mathcal{L}^*(\mu) = \int \tilde{\mathcal{L}}(\varphi) d\mu = \int \varphi \circ f^{-1} d\mu = \int \varphi d\mu$$

□

Other important relation obtained from the f-invariance of a measure μ is that

$$\int (\varphi \circ f^n)\, \psi d\mu = \int \varphi \mathcal{L}^n(\psi) d\mu \tag{11.1}$$

Indeed, as $\tilde{\mathcal{L}}(\varphi) = \varphi \circ f^{-1}$ we have

$$\int (\varphi \circ f)\, \psi d\mu = \int \varphi \circ f \circ f^{-1} \psi \circ f^{-1} d\mu = \int \varphi \mathcal{L}(\psi) d\mu$$

and by induction,

$$\int (\varphi \circ f^n)\, \psi d\mu = \int \varphi \mathcal{L}^n(\psi) d\mu.$$

The exponential decay of correlations of the maximizing entropy measure will be a consequence of the strict invariance of the Main Cone that we proved in the last section, and the following

Lemma 11.2. *For all $\varphi \in C^{\alpha}(M)$ there exists $K(\varphi) > 0$ such that $\varphi + K(\varphi) \in C(b, c, \alpha)$.*

Proof. First we prove that there exists $K_3 = K_3(\varphi) > 0$ such that $\varphi + K_3$ satisfies the condition (C) in the definition of cone $C(b, c, \alpha)$. The projections between stable leaves guarantees that

$$\int_\gamma \varphi d\mu_\gamma = \int_{\widetilde{\gamma}} \varphi \circ \pi d\mu_{\widetilde{\gamma}}$$

Given $\varphi \in C^\alpha(M)$ we have

$$\frac{\varphi(x) - \varphi(\pi(x))}{d(\gamma, \widetilde{\gamma})} \leq \frac{\varphi(x) - \varphi(\pi(x))}{d(\pi(x), x)} \leq |\varphi|_\alpha$$

So

$$\sup_{\gamma, \widetilde{\gamma}} \left\{ \frac{\left| \int_\gamma \varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}} \right|}{d(\gamma, \widetilde{\gamma})} \right\} \leq |\varphi|_\alpha < \infty$$

On the other hand, for $K > 0$, all we have $\inf_\gamma \left\{ \int_\gamma (\varphi + K) \, d\mu_\gamma \right\} = \inf_\gamma \left\{ \int_\gamma \varphi d\mu_\gamma \right\} +$ K. It is sufficient to choose $K_3 = K_3(\varphi) > 0$ such that

$$c \inf_\gamma \left\{ \int_\gamma (\varphi + K_3) \, d\mu_\gamma \right\} > |\varphi|_\alpha$$

In order to see that there exists $K_2 = K_2(\varphi)$ such that $\varphi + K_2$ satisfies the condition (B), just note that

$$\sup_{\rho', \rho'' \in \mathcal{D}_1(\gamma)} \left\{ \frac{\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} \right\} < \infty.$$

Indeed, as $\rho', \rho'' \in \mathcal{D}_1(\gamma)$ we have $\frac{\rho'}{\rho''} \leq e^{\theta(\rho', \rho'')}$ and so, for all bounded φ

$$\begin{aligned}
\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right| = & \left| \int_\gamma \left(\frac{\rho'}{\rho''} - 1 \right) \varphi \rho'' d\mu_\gamma \right| \leq \int_\gamma \left| \frac{\rho'}{\rho''} - 1 \right| |\varphi| \, \rho'' d\mu_\gamma \\
\leq & \sup \left| \frac{\rho'}{\rho''} - 1 \right| \sup \varphi \sup \rho'' = \left| \sup \frac{\rho'}{\rho''} - 1 \right| \sup \varphi \sup \rho'' \\
\leq & \left| e^{\theta(\rho', \rho'')} - 1 \right| \sup \varphi \sup \rho''
\end{aligned}$$

Let B such that $\sup(\varphi + B) = 1$. It follows that

$$\begin{aligned}
\frac{\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} &= \frac{\left| \int_\gamma (\varphi + B) \, \rho' d\mu_\gamma - \int_\gamma (\varphi + B) \, \rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} \\
&\leq \frac{\left(e^{\theta(\rho', \rho'')} - 1 \right) \sup \rho''}{\theta(\rho', \rho'')}.
\end{aligned}$$

If $\theta(\rho', \rho'') < 1$ then $\dfrac{e^{\theta(\rho',\rho'')} - 1}{\theta(\rho', \rho'')} < 2$ and as $\rho'' \in \mathcal{D}_1(\gamma)$ we have

$$\frac{\left| \int_\gamma \varphi\rho' d\mu_\gamma - \int_\gamma \varphi\rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} \leq 2(1 + \kappa diam(M)^\alpha)$$

Now, if $\theta(\rho', \rho'') \geq 1$ we obtain

$$\begin{aligned}
\frac{\left| \int_\gamma \varphi\rho' d\mu_\gamma - \int_\gamma \varphi\rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} &\leq \left| \int_\gamma (\varphi + B)\rho' d\mu_\gamma - \int_\gamma (\varphi + B)\rho'' d\mu_\gamma \right| \\
&\leq \int_\gamma |(\varphi + B)(\rho' - \rho'')| d\mu_\gamma \\
&\leq \sup(\varphi + B)(\sup \rho' + \sup \rho'') \\
&\leq 2(1 + \kappa diam(M)^\alpha)
\end{aligned}$$

and this implies

$$\sup_{\rho',\rho'' \in \mathcal{D}_1(\gamma)} \left\{ \frac{\left| \int_\gamma \varphi\rho' d\mu_\gamma - \int_\gamma \varphi\rho'' d\mu_\gamma \right|}{\theta(\rho', \rho'')} \right\} < \infty.$$

The choice of $K_2 = K_2(\varphi)$ is similar of what we have done for (C). On condition (A) , since φ is continuous with compact domain, there exists $K_1 = K_1(\varphi)$ such that $\varphi + K_1 > 0$ and so $\int_\gamma (\varphi + K_1)\rho d\mu_\gamma > 0$, $\forall \gamma \in F^s_{loc}$ and $\rho \in \mathcal{D}(\gamma)$. We complete the proof by taking $K(\varphi) = \max\{K_1, K_2, K_3\}$.

□

Now, denote by $\mu_\gamma \times \nu$ the measure given by

$$\mu_\gamma \times \nu(\varphi) := \int \int_\gamma \varphi d\mu_\gamma d\nu(\gamma).$$

By unicity of the maximal entropy probability measure, we notice that $\mu = \mu_\gamma \times \nu$, where ν is the maximal entropy probability measure for g. Indeed, let us first show that $\mu_\gamma \times \nu$ is an f-invariant probability. In fact, for all $x \in M$, given $\gamma = \Pi_\Lambda^{-1}(x)$ and $\gamma_j = \Pi_\Lambda^{-1}(x_j)$, with $f(\gamma_j) \subset \gamma$ and $g(x_j) = x$ we have $\mu_\gamma(A) = \dfrac{1}{p}\displaystyle\sum_{j=1}^{p} \mu_{\gamma_j}\left(f^{-1}(A)\right)$. By Castro-Varandas[CV13], ν is an eigenmeasure of the adjoint $\mathcal{L}^*_{g,\phi}$ given by

$$\mathcal{L}_{g,\phi}(\varphi)(x) := \sum_{g(x_j)=x} e^{\phi(x_j))}\varphi(x_j),$$

for constant potential ϕ. More precisely, if r is the spectral radius of $\mathcal{L}^*_{g,\phi}$, which is equal to the degree of g, then $\mathcal{L}^*_{g,\phi}(\nu) = r\nu$. By normalizing $\mathcal{L}^*_{g,\phi}$ by $r = p$, we obtain for any continuous ϕ

$$\int \varphi(x) d\nu = \frac{1}{r} \int \varphi(x) d\mathcal{L}_{g,\phi}^{*} \nu = \frac{1}{r} \int \mathcal{L}_{g,\phi}(\varphi)(x) d\nu = \int \frac{1}{p} \sum_{j=1}^{p} \varphi(x_j) d\nu.$$

Therefore, for $A \in \mathcal{A}_0$ we deduce

$$\begin{aligned}
(\mu_\gamma \times \nu)(f^{-1}(A)) &= \mu_\gamma \times \nu(\chi_{f^{-1}(A)}) = \int \int_\gamma \chi_{f^{-1}(A)} d\mu_\gamma d\nu \\
&= \int \mu_\gamma \left(f^{-1}(A)\right) d\nu = \int \frac{1}{p} \sum_{j=1}^{p} \mu_{\gamma_j} \left(f^{-1}(A)\right) d\nu \\
&= \int \mu_\gamma(A) d\nu = \int \int_\gamma \chi_A d\mu_\gamma d\nu \\
&= \mu_\gamma \times \nu(A)
\end{aligned}$$

As we have shown in previous sections, this implies the same equality for any borelian A.

Furthermore $\mu_\gamma \times \nu(A) = \mu(A)$. Indeed, let $A = \Pi_\Lambda^{-1}(A_N)$, with $A_N \in \mathcal{A}_N$.

On the one hand, we have that

$$\mu(\Pi_\Lambda^{-1}(A_N)) = \nu\left(\Pi_\Lambda\left(\Pi_\Lambda^{-1}(A_N)\right)\right) = \nu\left(A_N\right) = \int_N \chi_{A_N} d\nu$$

and on the other hand,

$$\mu_\gamma \times \nu\left(\Pi_\Lambda^{-1}(A_N)\right) = \int \int_\gamma \chi_{\Pi_\Lambda^{-1}(A_N)} d\mu_\gamma d\nu$$

As $\chi_{\Pi_\Lambda^{-1}(A_N)}(x) = \chi_{A_N}(\Pi_\Lambda(x))$ and for all γ there exists $x_0 \in N$ such that $\gamma = \Pi_\Lambda^{-1}(x_0)$. So

$$\int_\gamma \chi_{\Pi_\Lambda^{-1}(A_N)}(x) d\mu_\gamma = \int_\gamma \chi_{A_N}(\Pi_\Lambda(x)) d\mu_\gamma = \int_\gamma \chi_{A_N}(x_0) d\mu_\gamma = \chi_{A_N}(x_0)$$

and then, $\mu_\gamma \times \nu(A) = \mu(A)$ for all $A \in \mathcal{A}_0$. Now, given $A \in \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$, as $\mathcal{A}_n = f^n(\mathcal{A}_0)$, we have that there exist $n \in \mathbb{N}$ and $A_0 \in \mathcal{A}_0$ such that $A = f^n(A_0)$. Therefore

$$\mu_\gamma \times \nu(A) = \mu_\gamma \times \nu(f^n(A_0)) = \mu_\gamma \times \nu(A_0).$$

Since μ is f-invariant, $\mu(A) = \mu(f^n(A_0)) = \mu(A_0)$, we conclude that $\mu = \mu_\gamma \times \nu$.

Teorema B. *The measure μ has exponential decay of de correlations for Hölder continuous observables.*

Proof. We should prove that for α-Hölder observables φ, ψ , there exist $0 < \tau < 1$ and $K(\varphi, \psi) > 0$ such that

$$\left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \forall n \geq 1.$$

By (11.1) this is equivalent to prove

$$\left| \int \varphi \tilde{\mathcal{L}}^n (\psi) \, d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \quad \text{para todo } n \geq 1.$$

We start with the case $\varphi_{|\gamma} \in \mathcal{D}(\gamma)$, $\forall \gamma \in \mathcal{F}_{loc}^s$ and $\psi \in C(b, c, \alpha)$. We also assume $\int \varphi d\mu \neq 0$ and $\int \psi d\mu = 1$.

Recall that $\mathcal{L}(1) = 1 \circ f = 1$. Since $\varphi_{|\gamma} \in \mathcal{D}(\gamma)$ for all $\gamma \in \mathcal{F}^s_{loc}$ by (A) we have

$$\frac{\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma}{\int_\gamma \varphi d\mu_\gamma} \leq \beta_+ \left(\tilde{\mathcal{L}}^n(\psi), 1\right)$$

Since ψ is normalized we have $\int \tilde{\mathcal{L}}^n(\psi) d\mu = \int \psi d\mu = 1$. As $\mu = \mu_\gamma \times \nu$

$$\int \left(\int_\gamma \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma \right) d\nu = \int \tilde{\mathcal{L}}^n(\psi) d\mu = 1$$

and so there exists $\hat{\gamma}$ such that $\int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) d\mu_{\hat{\gamma}} \leq 1$. We conclude that

$$\alpha_+ \left(\tilde{\mathcal{L}}^n(\psi), 1\right) \leq \frac{\int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} d\mu_{\hat{\gamma}}} = \int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) d\mu_{\hat{\gamma}} \leq 1$$

and for all $\gamma \in \mathcal{F}^s_{loc}$

$$\frac{\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma}{\int_\gamma \varphi d\mu_\gamma} \leq \frac{\beta_+ \left(\tilde{\mathcal{L}}^n(\psi), 1\right)}{\alpha_+ \left(\tilde{\mathcal{L}}^n(\psi), 1\right)} \leq e^{\Theta_+\left(\tilde{\mathcal{L}}^n(\psi),1\right)} \leq e^{\Theta\left(\tilde{\mathcal{L}}^n(\psi),1\right)}.$$

By proposition 9.3 and by proposition 10.1, since the cone $C(\sigma b, \sigma c, \alpha)$ has Θ-diameter less or equal than Δ, it follows from proposition 8.2 that $\exists\, 0 < \tau < 1$ such that $\forall \varphi, \psi \in C(b, c, \alpha)$ we have $\Theta(\tilde{\mathcal{L}}^n(\varphi), \tilde{\mathcal{L}}^n(\psi)) \leq \Delta \tau^{n-1}$. In consequence,

$$\frac{\int \varphi \tilde{\mathcal{L}}^n(\psi) d\mu}{\int \varphi d\mu} = \frac{\int \int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma d\nu}{\int \int_\gamma \varphi d\mu_\gamma d\nu} \leq e^{\Theta\left(\tilde{\mathcal{L}}^n(\psi),1\right)} \leq e^{\Delta \tau^{n-1}}.$$

Note now that $\lim_{n\to\infty} \dfrac{e^{\Delta\tau^{n-1}} - 1}{\tau^n} = \dfrac{\Delta}{\tau}$. So, there exists $\tilde{\Delta} > 0$ such that $e^{\Delta\tau^{n-1}} - 1 < \tilde{\Delta}\tau^n$, for all $n \in \mathbb{N}$. This implies that

$$\left|\int \varphi d\mu\right| \left| \frac{\int \varphi \tilde{\mathcal{L}}^n(\psi) d\mu}{\int \varphi d\mu} - 1 \right| \leq \left|\int \varphi d\mu\right| \left(e^{\Delta\tau^{n-1}} - 1\right) \leq \left|\int \varphi d\mu\right| \tilde{\Delta}\tau^n$$

If $\int \psi d\mu \neq 1$ then

$$\begin{aligned} \left| \int \varphi \tilde{\mathcal{L}}^n(\psi)\, d\mu - \int \varphi d\mu \int \psi d\mu \right| &= \left|\int \psi d\mu\right| \left| \int \varphi \tilde{\mathcal{L}}^n \left(\frac{\psi}{\int \psi d\mu} \right) d\mu - \int \varphi d\mu \right| \\ &\leq \left|\int \psi d\mu\right| \left|\int \varphi d\mu\right| \tilde{\Delta}\tau^n \end{aligned}$$

for all $n \geq 1$.

By lemma 11.2 given an α-Hölder continuous function ψ, there exists $K(\psi) > 0$, such that $\psi + K(\psi) \in C(b, c, \alpha)$. Therefore $\psi = \psi + K(\psi) - K(\psi)$ and noting that $\int \varphi \mathcal{L}^n(K(\psi)) d\mu = \int \varphi d\mu \int K(\psi) d\mu$ we obtain

$$\begin{aligned}\left|\int \varphi \tilde{\mathcal{L}}^n(\psi)\, d\mu - \int \varphi d\mu \int \psi d\mu\right| &= \left|\int \varphi \tilde{\mathcal{L}}^n(\psi + K(\psi))\, d\mu - \int \varphi d\mu \int (\psi + K(\psi)) d\mu\right| \\ &\leq \left(\left|\int \psi d\mu\right| + K(\psi)\right)\left|\int \varphi d\mu\right| \tilde{\Delta}\tau^n\end{aligned}$$

Now, given an α-Hölder φ, note that there exists $K(\varphi) \in \mathbb{R}$ such that $\varphi_{|\gamma} + K(\varphi) + B \in \mathcal{D}(\gamma)$ for all $\gamma \in \mathcal{F}^s_{loc}$ and $\int \varphi + K(\varphi) + B d\mu > 0$, for all $B > 0$. Indeed,

$$\left|\varphi_{|\gamma} + K(\varphi)\right|_\alpha < \kappa \inf\left\{\varphi_{|\gamma} + K(\varphi)\right\}$$

if, and only if,

$$K(\varphi) > \frac{\left|\varphi_{|\gamma}\right|_\alpha}{\kappa} - \inf\left\{\varphi_{|\gamma}\right\}$$

Set $K(\varphi) = \sup_{\gamma \in \mathcal{F}^s_{loc}} \left\{\left|\frac{\left|\varphi_{|\gamma}\right|_\alpha}{\kappa}\right|\right\} - \inf \varphi$. Observe that $K(\varphi) \leq \frac{|\varphi|_\alpha}{\kappa} - \inf \varphi < \infty$. As $\varphi_{|\gamma} + K(\varphi) \geq \frac{\left|\varphi_{|\gamma}\right|_\alpha}{\kappa} \geq 0$ for all $\gamma \in \mathcal{F}^s_{loc}$, it follows that $\varphi_{|\gamma} + K(\varphi) + B \in \mathcal{D}(\gamma)$ and $\int (\varphi + K(\varphi)) d\mu + B > 0$, $\forall B > 0$. Analogously to the last case

$$\left|\int \varphi \tilde{\mathcal{L}}^n(\psi)\, d\mu - \int \varphi d\mu \int \psi d\mu\right| \leq \left(\left|\int \psi d\mu\right| + K(\psi)\right)\left(\left|\int \varphi d\mu\right| + K(\varphi) + B\right) \tilde{\Delta}\tau^n$$

and since B is any positive number

$$\left|\int \varphi \tilde{\mathcal{L}}^n(\psi)\, d\mu - \int \varphi d\mu \int \psi d\mu\right| \leq \left(\left|\int \psi d\mu\right| + K(\psi)\right)\left(\left|\int \varphi d\mu\right| + K(\varphi)\right) \tilde{\Delta}\tau^n$$

.

Since $\left|\int \varphi d\mu\right| - \inf \varphi \geq 0$, we have $\left|\int \varphi d\mu\right| + K(\varphi) \geq 0$. By taking

$$K(\varphi, \psi) := \left(\left|\int \psi d\mu\right| + K(\psi)\right)\left(\left|\int \varphi d\mu\right| + K(\varphi)\right) \tilde{\Delta},$$

we conclude the proof of the Theorem.

□

12. Central Limit Theorem

Let $\mathcal{G}$ be the Borel σ-algebra of M and let $\mathcal{G}_n := f^{-n}(\mathcal{G})$ be a nonincreasing family of σ-algebras. A function $\xi : M \to \mathbb{R}$ is $\mathcal{G}_n$-measurable if, and only if, $\xi = \xi_n \circ f^n$ for some $\mathcal{G}$- measurable ξ_n. Let $L^2(\mathcal{G}_n) = \{\xi \in L^2(\mu); \xi$ is $\mathcal{G}_n$-measurable $\}$. Note that $\mathcal{L}^2(\mathcal{G}_{n+1}) \subset \mathcal{L}^2(\mathcal{G}_n)$ for each $n \geq 0$. Given $\varphi \in L^2(\mu)$, we will denote by $\mathbb{E}(\varphi|\mathcal{G}_n)$ the L^2-orthogonal projection of φ on $L^2(\mathcal{G}_n)$.

We will apply the following adaption of a result due to Gordin, whose proof can be found in [Vi97]:

Theorem 12.1. *[Gordin.] Let $(M,\mathcal{F},\mu)$ be a probability space, and let $\phi \in L^2(\mu)$ be such that $\int \phi d\mu = 0$. Assume that $f : M \to M$ is an invertible bimeasurable map and that μ is an f-ergodic invariant probability. Let $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_n := f^{-n}(\mathcal{F}_0)$, $n \in \mathbb{Z}$, is a nonincreasing family of σ-algebras. Define*

$$\sigma_\phi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty} \phi \cdot (\phi \circ f^j)\, d\mu.$$

If

$$\sum_{n=0}^{\infty} \|\mathbb{E}(\phi|\mathcal{F}_n)\|_2 < \infty \;\; e \;\; \sum_{n=0}^{\infty} \|\phi - \mathbb{E}(\phi|\mathcal{F}_{-n})\|_2 < \infty$$

then $\sigma_\phi < \infty$ e $\sigma_\phi = 0$ if, and only if $\phi = u \circ f - u$ for some $u \in L^1(\mu)$. Moreover, if $\sigma_\phi > 0$ then for any interval $A \subset \mathbb{R}$

$$\mu\left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \big(\phi(f^j(x))\big) \in A \right) \to \frac{1}{\sigma_\phi \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma_\phi^2}} dt,$$

as $n \to \infty$.

Let $\mathcal{F}_0$ the σ-algebra whose elements are Borelian subsets of Λ which are union local stable leaves (intersected with Λ). Not that, if φ F_0-mensurable then φ is constant along local stable leaves.

We start by proving a statement of exponential decay of correlation concerning to function in $L^1(F_0)$.

Proposition 12.2. *Let $\varphi \in L^1(F_0)$ and ψ be a α-Hölder continuous function. Then, there exist constants $0 < \tau < 1$ and $C(\psi) > 0$ such that*

$$\left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le C(\psi) \int |\varphi|\, d\mu \cdot \tau^n$$

for all $n \ge 1$.

Proof. Since φ is F_0-measurable, it is constant restricted to local stable leaves, so, $|\varphi|_\gamma|_\alpha = 0$, $\forall \gamma \in \mathcal{F}^s_{loc}$. Suppose $\varphi \ge 0$ and let $K(\varphi)$ and $K(\psi)$ as in the proof of Th. B. Therefore

$$K(\varphi) = \sup_{\gamma \in \mathcal{F}^s_{loc}} \left\{ \left| \frac{|\varphi|_\gamma|_\alpha}{\kappa} \right| \right\} - \inf \varphi = -\inf \varphi$$

Since $\left| \int \varphi d\mu \right| - \inf \varphi \le \int |\varphi|\, d\mu$, just as in the proof of Th. B, it follows that

$$\left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le \left(\left| \int \psi d\mu \right| + K(\psi) \right) \int |\varphi|\, d\mu \cdot \tau^n.$$

Now, we can write $\varphi = \varphi^+ - \varphi^-$ where $\varphi^\pm = \frac{1}{2}(|\varphi| \pm \varphi)$. Noting that $\int |\varphi^\pm|\, d\mu \le \int |\varphi|\, d\mu$ from linearity of the integral we obtain

$$\left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le C(\psi) \int |\varphi|\, d\mu \cdot \tau^n$$

with $C(\psi) := 2\left(\left| \int \psi d\mu \right| + K(\psi) \right)$. □

As a consequence of the proposition we are able to prove:

Lemma 12.3. *For every Hölder continuous function φ with $\int \varphi d\mu = 0$ there is $R = R(\varphi)$ such that $\|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 \leq R\tau^n$ for all $n \geq 0$.*

Proof. Due to the last proposition, if $\psi \in L^1(F_0)$ and $\int \psi d\mu \leq 1$, then

$$\left| \int (\psi \circ f^n)\varphi d\mu - \int \psi d\mu \int \varphi d\mu \right| \leq C(\varphi) \cdot \tau^n.$$

As $\|\psi\|_1 \leq \|\psi\|_2$ and $\int \varphi d\mu = 0$ we have

$$\begin{aligned} \|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 &= \sup\left\{\int \xi\varphi d\mu; \xi \in L^2(\mathcal{F}_n), \|\xi\|_2 = 1\right\} \\ &= \sup\left\{\int (\psi \circ f^n)\,\varphi d\mu; \psi \in L^2(F_0), \|\psi\|_2 = 1\right\} \\ &\leq R\,(\varphi)\,\tau^n \end{aligned}$$

□

Now, we can prove:

Teorema C. ***(Central Limit Theorem)***
Let μ be the maximal entropy probability for $f : \Lambda \to \Lambda$, as in (2.2). Given a Hölder continuous function φ and

$$\sigma_\varphi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty} \int \phi \cdot (\phi \circ f^j)\, d\mu, \qquad \textit{with} \qquad \phi = \varphi - \int \varphi\, d\mu.$$

Then $\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ if, and only if, $\varphi = u \circ f - u$ for some $u \in L^1(\mu)$. Moreover, if $\sigma_\varphi > 0$ then for all interval $A \subset \mathbb{R}$

$$\lim_{n\to\infty} \mu\left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu\right) \in A\right) = \frac{1}{\sigma_\varphi\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma_\varphi^2}} dt.$$

Proof. By the last lemma, $\sum_{n=0}^{\infty} \|\mathbb{E}(\phi|\mathcal{F}_n)\|_2 < \infty$, so the first condition for Gordin's Theorem holds. The second condition follows from the Hölder continuity of φ. In fact, $\mathbb{E}(\phi, \mathcal{F}_{-n})$ is constant in each n-image $\eta = f^n(\gamma)$ of a stable leaf γ and

$$\inf(\phi|_\gamma) \leq \mathbb{E}(\phi, \mathcal{F}_{-n}) \leq \sup(\phi|_\gamma).$$

Since the diameter of η is less $C_s\lambda_s^n$ for some constant C_s which does not depend on γ, $\lambda_s \in (0, 1)$, and ϕ is (A, α)-Hölder for some constant $A > 0$, we obtain that

$$\|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_2 \leq \|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_0 \leq AC_s^\alpha \lambda_s^{\alpha n}.$$

which guarantees $\sum_{n=0}^{\infty} \|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_2 < \infty$. The result then follows as a consequence of Gordin's Theorem. □

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