On Conic Eigenvalue Complementarity Problems

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Abstract

The Quadratic Conic Eigenvalue Complementarity Problem (QCEiCP) is investigated without assuming symmetry on the matrices defining the problem. We present a new sufficient condition for existence of solutions of QCEiCP, extending to arbitrary pointed, closed and convex cones a condition known to hold when the cone is the nonnegative orthant.

We also address the Conic Eigenvalue Complementarity Problem (CEiCP) when the matrices are symmetric. We show that this symmetric CEiCP reduces to the computation of a stationary point of an appropriate merit function on a convex subset of the cone. Furthermore, we discuss the use of the so called Spectral Projected Gradient (SPG) algorithm for solving the CEiCP when the cone of interest is the Second Order Cone (SOCEiCP). A new algorithm is designed for the computation of the projections required by the SPG method to deal with the SOCEiCP. Numerical results are included to illustrate the efficiency of the SPG method and the new projection technique in practice.

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1 Introduction

Given matrices $B, C \in \mathbb{R}^{n\times n}$, the *Eigenvalue Complementarity Problem* (to be denoted EiCP(B, C), see e.g. [26] and [27]), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda Bx - Cx, \tag{1}$$

$$w \geq 0, x \geq 0, \tag{2}$$

$$x^t w = 0, \tag{3}$$

$$e^t x = 1, \tag{4}$$

with $e = (1, 1, \dots, 1)^t \in \mathbb{R}^n$. The last normalization constraint has been introduced, without loss of generality, in order to prevent the x component of a solution to vanish. The matrix B is usually assumed to be positive definite. The problem has many applications in engineering (see [1], [24] and [27]), and can be seen as a generalization of the well-known Generalized Eigenvalue Problem, denoted GEiP (see e.g. [15]). Indeed, GEiP consists of solving (1) with $w = 0$, and a solution (λ, x) of GEiP is just an eigenvalue and eigenvector of the matrix $B^{-1}C$ in the usual sense, when B is nonsingular. If a triplet (λ, x, w) solves EiCP, then the scalar λ is called a *complementary eigenvalue* and x is a *complementary eigenvector* associated to λ for the pair (B, C). The condition $x^t w = 0$ and the nonnegative requirements on x and w imply that either $x_i = 0$ or $w_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

It is easy to prove that under strict copositivity of B, EiCP(B, C) always has a solution, because it can be reformulated as the Variational Inequality Problem VIP$(\bar{F}, \Omega)$ with feasible set $\Omega = \{x \in \mathbb{R}^n \colon e^t x = 1, x \geq 0\}$ and operator $\bar{F} : \Omega \to \mathbb{R}^n$ given by

$$\bar{F}(x) = \frac{x^t Cx}{x^t Bx} Bx - Cx, \tag{5}$$

see [19]. Note that $\bar{F}$ is continuous in Ω as a consequence of the strict copositivity of B, and that Ω is convex and compact. It is well known that these two conditions ensure existence of solutions of VIP$(\bar{F}, \Omega)$ [11].

A number of techniques have been proposed for solving the EiCP and its extensions, see e.g. [2], [7], [13], [14], [17], [18], [19], [20], [23], [25], [26], [29] and [30].

Recently an extension of the EiCP has been introduced in [28], where some applications are highlighted. It has been named *Quadratic Eigenvalue Complementarity Problem* (QEiCP), and it differs from EiCP through the existence of an additional quadratic term on λ. Its formal definition follows.

Given $A, B, C \in \mathbb{R}^{n\times n}$, QEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{6}$$

$$w \geq 0, x \geq 0, \tag{7}$$

$$x^t w = 0, \tag{8}$$

$$e^t x = 1, \tag{9}$$

where, as before, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. As in the case of the EiCP, the normalization (9) has been introduced, without loss of generality, for preventing the x component of a solution of the problem from vanishing. Note that QEiCP(A, B, C) reduces to EiCP$(B, -C)$ when $A = 0$. The λ component of a solution of QEiCP(A, B, C) is called a *quadratic complementary eigenvalue* for A, B, C, and the x component a *quadratic complementary eigenvector* for A, B, C associated to λ.

The case of the symmetric QEiCP, i.e., when A, B and C are symmetric matrices and $-C$ is the identity matrix, has been analyzed in [12], where each instance of QEiCP with $n \times n$ matrices is related to an instance of $EiCP$ with $2n \times 2n$ matrices. A new approach for solving the nonsymmetric QEiCP by a similar reduction has been recently studied in [8].

In this paper, we consider a natural generalization of EiCP and QEiCP, proposed in [28] and [29], where the nonnegative orthant of $\mathbb{R}^n$ is replaced by a more general cone in $\mathbb{R}^n$. We state next some basic facts and definitions related to cones in $\mathbb{R}^n$.

We recall that a set $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* when it is closed under multiplication by nonnegative scalars. We are concerned here with convex cones. It is easy to conclude that convex cones are precisely those subsets of $\mathbb{R}^n$ which are closed by linear combinations with nonnegative scalars. In this paper we consider exclusively closed convex cones, i.e. those convex cones which are closed in the standard topology in $\mathbb{R}^n$ (i.e. the topology induced by any norm). We recall that a cone $\mathcal{K}$ is *pointed* if it does not contain lines, or equivalently, if there exists no nonzero $x \in \mathcal{K}$ such that $-x \in \mathcal{K}$. We mention that any cone $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{K}' + L$ where "+" denotes the Minkowski sum, $\mathcal{K}'$ is pointed and L is a linear subspace (L is the *linearity* of $\mathcal{K}$, namely $L = \{x \in \mathcal{K} : -x \in \mathcal{K}\}$, and $\mathcal{K}'$ can be taken as $\mathcal{K}' = \mathcal{K} \cap L^{\perp}$; see, e.g., [16]). Given a cone $\mathcal{K}$, its *dual cone* (or *positive polar cone*) $\mathcal{K}^*$ is defined as $\mathcal{K}^* = \{x \in \mathbb{R}^n : x^t y \geq 0 \ \forall y \in \mathcal{K}\}$. It is elementary to check that $\mathcal{K}$ is pointed if and only if $\mathcal{K}^*$ has nonempty interior.

We proceed now to define the Conic Eigenvalue Complementary Problem. Let $\mathcal{K} \subset \mathbb{R}^n$ be a closed, convex and pointed cone. We fix some point $a \in \mathrm{int}(\mathcal{K}^*)$. Given matrices $B, C \in \mathbb{R}^{n \times n}$, the *Conic Eigenvalue Complementarity Problem*, to be denoted CEiCP(B, C), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda B x - C x, \tag{10}$$

$$x \in \mathcal{K}, \ \ w \in \mathcal{K}^*, \tag{11}$$

$$x^t w = 0, \tag{12}$$

$$a^t x = 1. \tag{13}$$

If (λ, x, w) solves CEiCP(B, C), then λ is said to be a *complementary eigenvalue* and x a *complementary eigenvector*. Since w is fully determined by λ and x, by virtue of (10), we often comit a slight abuse of notation and refer to a pair (λ, x) as a solution of CEiCP, understanding that (11)-(13) hold with w given by (10). As in the case of EiCP, the normalization constraint (13) is included to ensure that complementary eigenvectors are nonzero. It is easy to check that changing the vector $a \in \mathrm{int}(\mathcal{K}^*)$ does not alter the set of complementary eigenvalues, and that each complementary eigenvector is replaced by a positive multiple of itself. Note that when $\mathcal{K} = \mathbb{R}^n_+$ (i.e., the nonnegative orthant of $\mathbb{R}^n$, in which case $\mathcal{K}^* = \mathcal{K}$), and $a = e$, CEiCP(B, C) reduces to EiCP(B, C).

It has been proved in [29] that if $\mathcal{K}$ is closed, convex and pointed, and $x^t Bx \neq 0$ for all nonzero $x \in \mathcal{K}$, then CEiCP(B, C) has solutions. The proof works through the reduction of CEiCP(B, C) to VIP(F, Δ), with F as in (5) and $\Delta = \{x \in \mathcal{K} : a^t x = 1\}$. Pointedness of $\mathcal{K}$ is a key factor in the proof, because it ensures that $\mathrm{int}(\mathcal{K}^*) \neq \emptyset$, and the fact that the vector a in (13) belongs to $\mathrm{int}(\mathcal{K}^*)$ is essential for establishing compactness of Δ, which in turn is a critical ingredient in the proof of existence of solutions of VIP(F, Δ).

Next we define the Quadratic Conic Eigenvalue Complementary Problem. Given $A, B, C \in \mathbb{R}^{n\times n}$, a closed, convex and pointed cone $\mathcal{K} \subset \mathbb{R}^n$ and a vector $a \in \mathrm{int}(\mathcal{K}^*)$, QCEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{14}$$

$$x \in \mathcal{K}, \;\; w \in \mathcal{K}^*, \tag{15}$$

$$x^t w = 0, \tag{16}$$

$$a^t x = 1. \tag{17}$$

If (λ, x, w) solves QCEiCP(B, C), then λ is a *quadratic complementary eigenvalue* and x a *quadratic complementary eigenvector*. In this case we refer to a pair (λ, x) as a solution of QCEiCP, understanding that (15)-(17) hold with w given by (14). As before, the normalization constraint (17) is considered to avoid $x = 0$ to be a solution of the problem. Again, QCEiCP(A, B, C) reduces to QEiCP(A, B, C) when $\mathcal{K} = \mathbb{R}^n_+$.

We start by discussing the issue of existence of solutions of QCEiCP. Contrary to the CEiCP, QCEiCP may lack solutions, even under positive definiteness of A. Indeed if we consider QEICP$(I, 0, I)$ with an arbitrary cone $\mathcal{K}$, then premultiplying (14) by x and using (16), one gets $0 = (\lambda^2 + 1) \|x\|^2$, which has no solution $\lambda \in \mathbb{R}$ and $x \neq 0$. This difference between CEiCP and QCEiCP in terms of existence of solutions mirrors the elementary fact that linear equations in one real variable always have solutions, while quadratic equations may fail to have them.

Thus, the issue of conditions on (A, B, C) ensuring existence of solutions of QCEiCP(A, B, C) is a relevant one. We present in Section 2 a sufficient condition for existence of solutions of QCEiCP(A, B, C), and compare it with the co-regularity and co-hyperbolicity properties introduced by A. Seeger in [28], concluding that both conditions are indeed independent of each other. This new condition extends to the conic case a set of sufficient conditions for existence of solutions of QEiCP introduced in [8].

In Section 3 we show that in the symmetric case (i.e., when both B and C are symmetric) CEiCP reduces to finding a stationary point for the problem of optimizing the so-called Rayleigh

Quotient function on a convex set defined by the cone $\mathcal{K}$ and a special normalization constraint that depends on the cone under study.

We also discuss in this paper the numerical solution of CEiCP when the cone $\mathcal{K}$ is the so called Second Order Cone, defined as follows:

$$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_r, \tag{18}$$

where

$$\mathcal{K}_i = \{x^i \in \mathbb{R}^{n_i} : \left\|\bar{x}^i\right\| \leq x_0^i\} \subset \mathbb{R}^{n_i} \ \ (1 \leq i \leq r), \tag{19}$$

$$\sum_{i=1}^{r} n_i = n.$$

Then any $x \in \mathcal{K}$ takes the form

$$x = (x^1, \ldots, x^r) \in \mathbb{R}^n$$

with

$$x^i = (x_0^i, \bar{x}^i) \in \mathbb{R} \times \mathbb{R}^{n_i - 1}, \ \ (1 \leq i \leq r).$$

It is rather immediate that each $\mathcal{K}_i$ is pointed and *self-dual*, i.e., it satisfies $\mathcal{K}_i = \mathcal{K}_i^*$. As a consequence, the Second Order Cone $\mathcal{K}$ is pointed and satisfies $\mathcal{K} = \mathcal{K}^*$ [3]. In this case CEiCP is called a Second-Order Cone Eigenvalue Complementarity Problem (SOCEiCP) and is denoted by SOCEiCP.

This cone has been chosen because optimization problems whose feasible sets are Second Order Cones are computationally tractable and appear in a large variety of applications, such as filter design, antenna array weight design, truss design, robust estimation and friction in robot grasp. We recommend [3, 6, 21] for Second-Order Cone optimization problems and their applications.

In Section 4 we investigate the numerical solution of the symmetric SOCEiCP, i.e., the case in which the matrices B and C are both symmetric. As stated before, solution of the symmetric SOCEiCP reduces to the computation of a stationary point of a maximization problem whose objective function is the Rayleigh Quotient. As in [17], we propose the Spectral Projected (SPG) algorithm for computing such a stationary point. The efficiency of the algorithm depends on the computation of projections on the feasible (convex) set of the maximization problem. The normalization constraint

$$\sum_{i=1}^{r} x_0^i = 1 \tag{20}$$

is introduced, so that these projections can be computed efficiently by a new algorithm proposed in Section 4. Numerical results with the SPG algorithm, using this new technique for computing projections, are reported, showing the efficiency of this approach for solving the symmetric SOCEiCP.

The paper is organized as follows. The sufficient condition for existence of solutions of QCEiCP is introduced in Section 2. The symmetric case is discussed in Section 3. The SPG algorithm for the SOCEiCP is described in Section 4. Numerical results with this algorithm are reported in Section 5 and some conclusions are presented in the last section of the paper.

2 Existence of solutions of QCEiCP

In this section we present a sufficient condition for the existence of solutions of QCEiCP(A, B, C). We start by recalling the sufficient conditons introduced in [28].

Definition 1. *Consider a cone $\mathcal{K} \subset \mathbb{R}^n$.*

i) A matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{K}$-regular *if $x^t Ax \neq 0$ for all nonzero $x \in \mathcal{K}$.*

ii) A triplet (A, B, C), with $A, B, C \in \mathbb{R}^{n\times n}$ is $\mathcal{K}$-hyperbolic *if*

$$(x^t Bx)^2 \geq 4(x^t Ax)(x^t Cx) \tag{21}$$

for all nonzero $x \in \mathcal{K}$.

Theorem 1. *If $\mathcal{K}$ is a closed, convex and pointed cone, A is $\mathcal{K}$-regular and (A, B, C) is $\mathcal{K}$-hyperbolic, then QCEiCP(A, B, C) has solutions.*

Proof. See Theorem 3.3 in [28]. □

In this paper, we guarantee the existence of solutions of QCEiCP by a different approach based on the relationship between an arbitrary n-dimensional QCEiCP and two specific instances of CEiCP with matrices in $\mathbb{R}^{2n\times 2n}$. A similar relation has been considered in [8] for QEiCP.

Consider now QCEiCP(A, B, C) with $A, B, C \in \mathbb{R}^{n\times n}$ and define $D, G, H \in \mathbb{R}^{2n\times 2n}$ as

$$D = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \tag{22}$$

$$G = \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix}, \tag{23}$$

$$H = \begin{pmatrix} B & -C \\ I & 0 \end{pmatrix}. \tag{24}$$

Given the cone $\mathcal{K} \subset \mathbb{R}^n$, we define the cone $\tilde{\mathcal{K}} \subset \mathbb{R}^{2n}$ as $\tilde{\mathcal{K}} = \mathcal{K} \times \mathcal{K}$. Furthermore, for a given $a \in \mathrm{int}(\mathcal{K}^*)$, we define $\tilde{a} \in \mathbb{R}^{2n}$ as $\tilde{a} = (a, a)$ Note that $\tilde{a}$ belongs to $\mathrm{int}(\tilde{\mathcal{K}})$. Assuming that the cone related to QCEiCP(A, B, C) is $\mathcal{K}$, and the vector in $\mathrm{int}(\mathcal{K})$ appearing in (17) is a, we consider CEiCP(D, G) and CEiCP(D, H) with cone $\tilde{\mathcal{K}}$ and vector $\tilde{a}$.

Next we prove a relation between the solutions of QCEiCP(A, B, C) and those of CEiCP(D, G) and CEiCP(D, H). We emphasize that the following result holds without making any additional hypotheses on A, B, C. We also mention that the proof of Proposition 1(b) differs in a substantial way from the proof of its counterpart for the case of $\mathcal{K} = \mathbb{R}^n_+$, namely Proposition 1 in [8].

Proposition 1. *a) Assume that (λ, x) solves QCEiCP(A, B, C) and consider D, G, H as in* (22)–(24).

i) If $\lambda = 0$ then $(\lambda, z) = (0, z)$ solves both CEiCP(D, G) and CEiCP(D, H), where $z \in \mathbb{R}^{2n}$ is defined as $z = (0, x)$.

ii) If $\lambda > 0$ then (λ, z) solves EiCP(D, G), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1+\lambda)^{-1}(\lambda x, x)$.

iii) If $\lambda < 0$ then the pair $(-\lambda, z)$ solves EiCP(D, H), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1-\lambda)^{-1}(-\lambda x, x)$.

b) Consider D, G, H as in (22)–(24).

i) If (λ, z) solves CEiCP(D, G) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(\lambda, (1+\lambda)x)$ solves QCEiCP(A, B, C)

ii) If (λ, z) solves CEiCP(D, H) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(-\lambda, (1+\lambda)x)$ solves QCEiCP(A, B, C).

Proof. a) For item (i), note that checking whether $(0, x)$ solves QCEiCP(A, B, C) reduces to verifying that $Cx \in \mathcal{K}^*, x \in \mathcal{K}, x^t Cx = 0$, and the same happens when verifying that $(0, (0, x))$ solves either CEiCP(D, G) or CEiCP(D, H). We deal now with item (ii). Note that checking that a pair (λ, z) with $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ solves CEiCP(D, G) is equivalent to verifying:

$$\lambda Au + Bu + Cv \in \mathcal{K}^*, \tag{25}$$

$$\lambda v - u \in \mathcal{K}^*, \tag{26}$$

$$u \in \mathcal{K}, \quad v \in \mathcal{K}, \tag{27}$$

$$u^t(\lambda Au + Bu + Cv) + v^t(\lambda v - u) = 0, \tag{28}$$

$$a^t(u + v) = 1. \tag{29}$$

On the other hand, since (λ, x) solves QCEiCP(A, B, C), we know that

$$\lambda^2 Ax + \lambda Bx + Cx \in \mathcal{K}^*, \tag{30}$$

$$x \in \mathcal{K}, \tag{31}$$

$$x^t(\lambda^2 Ax + \lambda Bx + Cx) = 0, \tag{32}$$

$$a^t x = 1. \tag{33}$$

If we take $u = \frac{\lambda}{1+\lambda}x, v = \frac{1}{1+\lambda}x$, then we have $\lambda v - u = 0$ and (26) holds trivially. The condition (25) follows from (30), and (27) follows from (31) and positivity of λ. The first term of the left hand side of (28) vanishes as a consequence of (32). Since $\lambda v = u$ then the equality (28) holds. Now $a^t(u + v) = (1+\lambda)^{-1}(\lambda a^t x + a^t x) = a^t x = 1$ by (33). Hence the condition (17) is true. For item (iii), note that if (λ, x) solves QCEiCP(A, B, C) then $(-\lambda, x)$ solves QCEiCP$(A, -B, C)$. In such a case, as $-\lambda$ is positive, we can apply item (ii) to QCEiCP$(A, -B, C)$, replacing λ by $-\lambda$ and B by $-B$. This gives the result, taking into account the definitions of z and H.

b) Consider first item (i). We know that (25)–(29) hold with $(u, v) = (y, x)$, and we need to check that

$$(1+\lambda)(\lambda^2 Ax + \lambda Bx + Cx) \in \mathcal{K}^*, \tag{34}$$

$$(1+\lambda)x \in \mathcal{K}, \tag{35}$$

$$(1+\lambda)^2 \left[x^t(\lambda^2 Ax + \lambda Bx + Cx)\right] = 0, \tag{36}$$

$$(1+\lambda)a^t x = 1. \tag{37}$$

If $\lambda \geq 0$ then (35) follows immediately from (27). It is rather elementary to verify that if

$$y = \lambda x, \tag{38}$$

then (34) follows from (25), (36) follows from (32), and (37) follows from (33). Therefore $(\lambda, (1+\lambda)x)$ solves QCEiCP(A, B, C), provided $\lambda \geq 0$.

We prove next that (38) holds. We claim first that $x \neq 0$. Otherwise (26) gives $-y \in \mathcal{K}^*$. Since $y \in \mathcal{K}$ by (27), we get $-y^t y \geq 0$, which implies $y = 0$. Since $x = 0$, we have $a^t(x+y) = 0$, contradicting (29). Consider now (28). Note that each term in the right hand side is nonnegative, because x, y belong to $\mathcal{K}$, and $\lambda Ay + By + Cx, \lambda x - y$ belong to $\mathcal{K}^*$, by (24)–(27). It follows that both terms vanish, and in particular the second one. Hence $0 = x^t(\lambda x - y)$, i.e.

$$\lambda = \frac{x^t y}{\|x\|^2}, \tag{39}$$

taking into account that $x \neq 0$. It follows from (39) that $y \neq 0$, since both x and λ are known to be nonzero. On the other hand, since $\lambda x - y \in \mathcal{K}^*, y \in \mathcal{K}$ by (14), (15), we have

$$\|y\|^2 \leq \lambda x^t y. \tag{40}$$

Substituting (39) in (40), we obtain $\|x\|^2 \|y\|^2 \leq (x^t y)^2$. By using the Cauchy-Schwartz inequality,

$$\|x\| \, \|y\| \leq \left|x^t y\right| \leq \|x\| \, \|y\| \, . \tag{41}$$

It follows from (41) that Cauchy-Schwartz inequality holds with equality. Therefore x and y are colinear, i.e. there exists $\sigma \in \mathbb{R}$ such that $y = \sigma x$. Replacing this equation in (39) and using that fact that $x \neq 0$, we conclude that $\lambda = \sigma$. Hence (38) holds.

Finally, positivity of λ follows also from (38). Since $(x, y) \in \tilde{\mathcal{K}}$, we get that $x \in \mathcal{K}$ and $\lambda x \in \mathcal{K}$, so that $\lambda < 0$ contradicts the pointedness of $\mathcal{K}$.

For item (ii), we apply the same argument as in item (i) to CEiCP(D, H). Since G and H differ just by the sign of B, we conclude that $(\lambda, (1+\lambda)x)$ solves QCEiCP$(A, -B, C)$. It now follows from the definition of QCEiCP(A, B, C) that $(-\lambda, (1+\lambda)x)$ solves it.

□

We comment that our sufficient condition requires only item (b) of Proposition 1. However, item (a) has some interesting consequences, see Remarks 3 and 4 below.

Now we rephrase the result of Proposition 1 in terms of complementary eigenvalues.

Corollary 1. *Consider QCEiCP(A, B, C) with $A, B, C \in \mathbb{R}^{n\times n}$ and the matrices $D, G, H \in \mathbb{R}^{2n\times 2n}$ as defined in* (22)–(24)*. Then,*

i) all quadratic complementary eigenvuales for (A, B, C) are complementary eigenvalues for either (D, G), or (D, H), or both,

ii) all nonzero complementary eigenvalues for (D, G) are positive, and are quadratic complementary eigenvalues for (A, B, C),

iii) all nonzero complementary eigenvalues for (D, H) are positive, and their additive inverses are quadratic complementary eigenvalues for (A, B, C).

Proof. Elementary from Proposition 1. □

Corollary 1 signals a clear path for obtaining a sufficient condition for existence of solutions of QCEiCP(A, B, C). We must first find a sufficient condition for solvability of CEiCP(D, G) or CEiCP(D, H) (which in principle depends only on the matrix in the leading term in (1), namely D in this case, and henceforth just on A, in terms of the data of the QCEiCP), and then impose conditions ensuring that either 0 is a quadratic complementary eigenvalue for (A, B, C), or that 0 is not a complementary eigenvalue of $(D, G), (D, H)$ (which, as mentioned in the proof of Proposition 1(a), depends only upon C).

We present next some classes of matrices needed for our sufficient conditions.

Definition 2. *Consider a cone $\mathcal{K} \subset \mathbb{R}^n$.*

i) A matrix $M \in \mathbb{R}^{n\times n}$ is said to be strictly $\mathcal{K}$-copositive *if $x^t M x > 0$ for all $0 \neq x \in \mathcal{K}$.*

ii) The class $R_0'(\mathcal{K}) \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that $x^t M x = 0$ for all $x \in \mathcal{K}$ such that $Mx \in \mathcal{K}^*$.

iii) The class $S_0'(\mathcal{K}) \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that there exists no nonzero $x \in \mathcal{K}$ such that $Mx \in \mathcal{K}^*$.

We comment that for $\mathcal{K} = \mathbb{R}^n_+$, the complements of classes $R'_0(\mathcal{K}), C'_0(\mathcal{K})$ are the well known classes S_0, R_0 respectively (see e.g. [10]).

Proposition 2. *i) If $M \in \mathbb{R}^{n\times n}$ is strictly $\mathcal{K}$-copositive then CEiCP(M, C) has solutions for any $C \in \mathbb{R}^{n\times n}$.*

ii) If $C \in R'_0(\mathcal{K})$ then 0 is a quadratic complementary eigenvalue for (A, B, C) for any $A, B, C \in \mathbb{R}^{n\times n}$.

iii) If $C \in S'_0(\mathcal{K})$ then 0 is not a complementary eigenvalue for either (D, G) or (D, H) with D, G, H as in (22)–(24).

Proof. Item (i) has been proved in [29], as mentioned in the introduction. Item (ii) is immediate from the definitions of QCEiCP and $R'_0(\mathcal{K})$. For item (iii), assume that 0 is a complementary eigenvalue for (D, G), with associated complementary eivenvector $0 \neq z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. It follows immediately that $By + Cx \in \mathcal{K}^*$, $-y \in \mathcal{K}^*, x \in \mathcal{K}$, $y \in \mathcal{K}$. Hence $y = 0$ and $Cx \in \mathcal{K}^*$. As $z \neq 0$, then $x \neq 0$ and we have a contradiction with the assumption that $C \in S'_0(\mathcal{K})$. The same argument can be used for the case of (D, H). □

Now, all the pieces are in place for stating and proving our existence result for QCEiCP.

Theorem 2. *Consider QCEiCP(A, B, C).*

i) $C \in R'_0(\mathcal{K})$ if and only if 0 is a quadratic complementary eigenvalue for QEiCP(A, B, C).

ii) If $C \in S'_0(\mathcal{K})$ and A is strictly $\mathcal{K}$-copositive, then there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C).

Proof. Item (i) is a consequence of Proposition 2 (ii). To prove item (ii) we first note that strict $\mathcal{K}$-copositivity of A implies strict $\mathcal{K}$-copositivity of D. Hence both CEiCP(D, G) and CEiCP(D, H) have complementary eigenvalues by Proposition 2(i), which are nonzero by Proposition 2 (iii). Hence, they are positive by items (ii) and (iii) of Corollary 1. Therefore there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C). □

In the remainder of this section, we discuss the existence result given in Theorem 2. We start with a corollary, stating that the roles of A and C in item (ii) of Theorem 2 can be reversed.

Corollary 2. *Consider QCEiCP(A, B, C) and assume that $A \in S'_0(\mathcal{K})$ and C is strictly $\mathcal{K}$-copositive. Then there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C).*

Proof. Apply Theorem 2(ii) to QCEiCP(C, B, A) and conclude that it has a solution (λ, x) with $\lambda > 0$, so that

$$w = \lambda^2 Cx + \lambda Bx + Ax \in \mathcal{K}^*, x \in \mathcal{K}, w^t x = 0. \tag{42}$$

Let $\mu = \lambda^{-1}$. Divide the first inequality in (42) by λ^2, and get from (42) $\bar{w} = \mu^2 Ax + \mu Bx + Cx \in \mathcal{K}^*, x \in \mathcal{K}, \bar{w}^t x = 0$, so that (μ, x) solves QCEiCP(A, B, C) and $\mu > 0$. Proceeding in the same fashion with QCEiCP$(C, -B, A)$, get a solution $(\bar{\lambda}, \bar{x})$ of this problem with $\bar{\lambda} > 0$, take $\bar{\mu} = \bar{\lambda}^{-1}$ and conclude that $(\bar{\mu}, \bar{x})$ solves QCEiCP$(A, -B, C)$. Hence $-\bar{\mu}$ is a negative quadratic complementary eigenvalue for (A, B, C). $\square$

We continue with two remarks related to the result in Theorem 2.

Remark 3. When we move from QCEiCP(A, B, C) to CEiCP(D, G), we can settle the issue of existence of solutions for the former except for one "undeterminated" case: when we only know that 0 is a complementary eigenvalue for (D, G). If EiCP(D, G) has no solutions then the same happens to QCEiCP(A, B, C) by Corollary 1(i); if CEiCP(D, G) has a solution (λ, x) with $\lambda \neq 0$ then λ is a quadratic complementary eigenvalue for (A, B, C) by Corollary 1(ii), but the fact that 0 is a complementary eigenvalue for (D, G) entails no conclusion at all about the existence of solutions of QCEiCP(A, B, C). The same considerations hold for CEiCP(D, H).

Remark 4. As another consequence of Corollary 1, if a method for finding all complementary eigenvalues for an arbitrary instance of CEiCP is available, applying it to CEiCP(D, G) and CEiCP(D, H) provides all quadratic complementary eigenvalues of QCEiCP(A, B, C). In fact, all complementary eigenvalues of these two CEiCP's are quadratic complementary eigenvalues for QCEiCP(A, B, C) (with the possible exception of 0, which can be checked separately) by virtue of Corollary 1(ii)–(iii), and no quadratic complementary eigenvalue can be missed, as a consequence of Corollary 1(i).

Finally, we close the section with the comparison between the two sets of sufficient conditions for existence of solutions of QCEiCP(A, B, C) given in Theorems 1 and 2.

For the comparison between the assumptions of Theorem 1 and Theorem 2, we say that a triplet (A, B, C) satisfies (P) when either $C \in S_0'(\mathcal{K})$ and A is strictly $\mathcal{K}$-copositive, or $C \in R_0'(\mathcal{K})$, and that it satisfies (P') when A is $\mathcal{K}$-regular and (A, B, C) is $\mathcal{K}$-hyperbolic.

We mention that if both A and $-C$ are strictly $\mathcal{K}$-copositive, then (P') holds, because in such a case one has $x^t Ax \geq 0$, $x^t Cx \leq 0$ for all $x \in \mathcal{K}$, so that the right hand side in (21) is nonpositive, making this inequality valid.

On the other hand, it is easy to exhibit instances in which (P) holds but (P') does not. Indeed, consider any pointed cone $\mathcal{K}$ which is not a halfline (i.e., it contains at least two linearly independent vectors, say c, d), take $a \in \text{int}(\mathcal{K}^*)$, find a vector $b \in \mathbb{R}^n$ such that $b^t c < 0, b^t d > 0$, and define $C \in \mathbb{R}^{n \times n}$ as $C = ba^t$. We claim that if A is positive definite then the triplet $(A, 0, C)$ satisfies (P) but not (P'). Observe that (21) fails with $x = d$, since

$$(d^t Bd)^2 - 4(d^t Ad)(d^t Cd) = -4(d^t Ad)(a^t d)(b^t d) < 0.$$

On the other hand, $(A, 0, C)$ satisfies (P). Since A is a positive definite matrix then A is $\mathcal{K}$-copositive for all $\mathcal{K}$. To show that $C \in S_0'(\mathcal{K})$ take any nonzero $x \in \mathcal{K}$. Hence $Cx = (a^t x)b$. If $Cx \in \mathcal{K}^*$, then $0 \leq (Cx)^t c = (a^t x)(b^t c) < 0$, as $a^t x > 0$ and $b^t c < 0$ by construction. Hence $Cx \notin \mathcal{K}^*$ and $C \notin S_0'(\mathcal{K})$.

There are also many instances of QCEiCP for which (P') holds but not (P). Take for instance an arbitrary $\mathcal{K}$, $A = C = I$ and $B = 2I$. Validity of (P') for any $\mathcal{K}$ is immediate, but (P) fails, because $I \notin R_0'(\mathcal{K}) \cup S_0'(\mathcal{K})$ for any $\mathcal{K}$. Hence, (P) and (P') are independent of each other for a generic cone $\mathcal{K}$.

Observe also that (P) depends only upon the matrices A and C, while (P') also involves the matrix B.

3 The symmetric CEiCP

It has been proved in [29] that if B is $\mathcal{K}$-regular (as in Definition 1), then the set of solutions of CEiCP(B, C) coincides with the set of solutions of VIP$(\bar{F}, \Delta)$, with $\bar{F}$ as in (5) and $\Delta = \{x \in \mathcal{K} : a^t x = 1\}$. Now, it is well known that if $S \subset \mathbb{R}^n$ is a closed and convex set and $h : S \to \mathbb{R}$ is differentiable on an open set containing S, then a point $\bar{x} \in S$ satisfies the first order optimality condition for the problem of minimizing $h(x)$ subject to $x \in S$ if and only if

$$\nabla h(\bar{x})^t (x - \bar{x}) \geq 0 \ \ \forall x \in S, \tag{43}$$

which is the same as saying that $\bar{x}$ solves VIP$(\nabla h, S)$. Note that the condition (43) means that no direction starting at $\bar{x}$ and pointing to a point in S is a descent direction for h.

Hence, if there exists a function h such that the solutions of VIP$(\bar{F}, \Delta)$ coincide with those of VIP$(\nabla h, \Delta)$, then the solutions of CEiCP(B, C) are precisely the stationary points for the problem of minimizing h on Δ. This is the case when CEiCP(B, C) is *symmetric*, meaning that both B and C are symmetric matrices. Indeed, assume that B is $\mathcal{K}$-regular and consider $h : \mathcal{K} \to \mathbb{R}$ defined as

$$h(x) = -\frac{x^t C x}{x^t B x}. \tag{44}$$

We mention that the quotient in (44) is called the *Rayleigh quotient* for B, C. Note that $\mathcal{K}$-regularity of B implies that h is well defined (and indeed differentiable) in an open set containing Δ, and that its gradient is given by

$$\nabla h(x) = \frac{1}{x^t B x}\left[\frac{x^t C x}{x^t B x} B x - C x\right] = \frac{1}{x^t B x}\bar{F}(x). \tag{45}$$

Now, note that if B is $\mathcal{K}$-regular then either B is $\mathcal{K}$-copositive or $-B$ is $\mathcal{K}$-copositive. If B is $\mathcal{K}$-copositive, then it follows from (45) that $\nabla h(\bar{x})^t (x - \bar{x}) \geq 0$ if and only if $\bar{F}(\bar{x})^t (x - \bar{x}) \geq 0$, so that the solution sets of VIP$(\bar{F}, \Delta)$ and VIP$(\nabla h, \Delta)$ coincide. If $-B$ is $\mathcal{K}$-copositive, then we take

$\bar{h} = -h$, and we conclude in the same way that the solution sets of VIP$(\bar{F}, \Delta)$ and VIP$(\nabla \bar{h}, \Delta)$ coincide. Hence if B is $\mathcal{K}$-regular the solutions of CEiCP(B, C) are the stationary points for the problem of minimizing or maximizing h, given by (44), on $\mathcal{K}$ (where we minimize when B is $\mathcal{K}$-copositive and maximize when $-B$ is $\mathcal{K}$-copositive). We remark that, from a computational viewpoint, computing a stationary point of an optimization problem is in general much easier than finding a solution of a variational inequality problem. We also mention that in the case of $\mathcal{K} = \mathbb{R}^n_+$, the equivalence between solving EiCP and finding a stationary point of the Rayleigh quotient was established in [26].

4 Numerical solution of the symmetric CEiCP with a Second order Cone

In Section 3, we showed that if B and C are symmetric matrices and B is positive definite, then any stationary point $\tilde{x} \neq 0$ of the function h defined by (44) on a convex self-dual cone $\mathcal{K}$ solves the symmetric CEiCP. In this section we consider the Second-Order cone defined by (18) and (19). We start by introducing the normalization constraint (20) that avoids $x = 0$ to be a feasible solution of the corresponding nonlinear program to be solved. Then we consider the maximization of the Rayleigh Quotient function on the set defined by the constraints (18), (19) and (20), that is, the following problem:

$$\begin{array}{ll} \text{NLP:} & \text{Minimize} \quad h(x) = -\dfrac{x^t C x}{x^t B x} \\ & \text{subject to} \quad (18), (19), (20). \end{array} \tag{46}$$

Next, we discuss the use of the so-called Spectral Projected-Gradient (SPG) algorithm for computing a stationary point $\tilde{x}$ of NLP (46). As stated before, $h(\tilde{x})$ and $\tilde{x}$ are a complementary eigenvalue and a complementary eigenvector respectively for the symmetric Second-Order cone (SOCEiCP). The SPG algorithm is a feasible descent method, which means that in each iteration k the current point x_k is feasible, i.e., $x_k \in \mathcal{K}$, and is updated by using a descent direction for the function h and a positive stepsize.

At iteration k, the projected gradient search direction d_k is given by

$$d_k = P_{\mathcal{K}}(x_k - \eta_k \nabla h(x_k)) - x_k, \tag{47}$$

where $\eta_k > 0$, $\nabla h(x_k)$ represents the gradient of h at x_k, and $P_{\mathcal{K}}(y)$ denotes the projection of y on $\mathcal{K}$. If $u_k = x_k - x_{k-1}$ and $v_k = \nabla h(x_k) - \nabla h(x_{k-1})$ satisfy $u_k^t v_k > 0$, the so called Spectral parameter

$$\eta_k = \frac{u_k^t u_k}{u_k^t v_k}$$

should be used. If $u_k^t v_k \leq 0$, then η_k should be a positive real number chosen according to [17]. Now, either $d_k = 0$ and x_k is a stationary point of h at x_k or x_k is updated by $x_{k+1} = x_k + \delta_k d_k$, where the stepsize $\delta_k \in (0,1]$ is computed by the exact line-search technique discussed in [17]. As discussed in [5], the algorithm converges to a stationary point of h under reasonable hypotheses. The steps of the SPG algorithm are described below.

Spectral Projection Algorithm (SPG)

Step 0. Let $\epsilon > 0$ be a tolerance, choose $x_0 \in \mathcal{K}$ and let $k := 0$.

Step 1. Compute d_k according to (47).
If $\|d_k\| < \epsilon$, terminate. The current vector x_k is a stationary point of h on $\mathcal{K}$. Otherwise, compute the stepsize $\delta_k \in (0,1]$ by an exact line-search.

Step 2. Update

$$x_{k+1} := x_k + \delta_k d_k$$

and return to Step 1 with $k := k+1$.

Next, we focus our attention to the choice of the initial point and the computation of the gradient, search direction and the stepsize.

(1) *Initial Point*
The initial point $x_0 = (x^1, \ldots, x^r) \in \mathbb{R}^n$ with $x^i = (x_0^i, \bar{x}^i) \in \mathbb{R} \times \mathbb{R}^{n_i - 1}$, $i = 1, \ldots, r$, has the following components:

$$x_0^i = \frac{1}{r}, \quad \bar{x}^i = \frac{1}{r}\ e^s,$$

where e^s is a vector of the canonical basis and $s = \min\{i, n_i - 1\}$.

(2) *Computation of the gradient* $\nabla h(x)$
The gradient of the (negative) Rayleigh Quotient function h at x is given by (45).

(3) *Computation of the Projected-Gradient Direction* d
The projected gradient search direction at each iteration is given by (47). Due to the choice of the normalization constraint (20), it is possible to design a special purpose efficient algorithm for the computation of the projection that is required for the definition of the search direction. Next, we discuss in detail this new algorithm. Let a point $u = (u^1, \ldots, u^r) \in \mathbb{R}^n$ with $u^i = (u_0^i, \bar{u}^i) \in \mathbb{R} \times \mathbb{R}^{n_i - 1}, i = 1, \ldots, r$, be given. Then the projection of u onto the set $\mathcal{K}$ is the unique solution of the convex optimization problem:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{Minimize}} & \frac{1}{2}\sum_{i=1}^{r} \|x^i - u^i\|^2 \\ \text{subject to} & \|\bar{x}^i\| - x_0^i \leq 0,\ i = 1, \ldots, r, \\ & \sum_{i=1}^{r} x_0^i = 1. \end{array} \tag{48}$$

For finding the optimal solution of problem (48), first fix $x_0^i \geq 0$, $i = 1, \ldots, r$ arbitrarily, and consider the following optimization problem for each i:

$$\begin{array}{ll} \underset{\bar{x}^i \in \mathbb{R}^{n_i - 1}}{\text{Minimize}} & \frac{1}{2}\|x^i - u^i\|^2 \\ \text{subject to} & \|\bar{x}^i\| - x_0^i \leq 0. \end{array}$$

Noticing that $\|x^i - u^i\|^2 = (x_0^i - u_0^i)^2 + \|\bar{x}^i - \bar{u}^i\|^2$, it is not difficult to see that the optimal solution $\bar{x}^i$ of this problem is given by

$$\bar{x}^i = \begin{cases} \bar{u}^i & \text{if } x_0^i \geq \|\bar{u}^i\| \\ \frac{x_0^i}{\|\bar{u}^i\|}\bar{u}^i & \text{if } x_0^i < \|\bar{u}^i\|, \end{cases} \tag{49}$$

and the optimal value is given by

$$\phi_i(x_0^i | u^i) := \begin{cases} \frac{1}{2}(x_0^i - u_0^i)^2 & \text{if } x_0^i \geq \|\bar{u}^i\| \\ \frac{1}{2}(x_0^i - u_0^i)^2 + \frac{1}{2}(x_0^i - \|\bar{u}^i\|)^2 & \text{if } x_0^i < \|\bar{u}^i\|. \end{cases}$$

Thus the optimal solution of problem (48) is obtained by solving the following convex optimization problem with variables $x_0^i \in \mathbb{R}, i = 1, \ldots, r$:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^r \phi_i(x_0^i | u^i) \\ \text{subject to} & \displaystyle\sum_{i=1}^r x_0^i = 1, \\ & x_0^i \geq 0, \ i = 1, \ldots, r. \end{array} \tag{50}$$

In the sequel, for the sake of a simpler notation, we denote $\phi_i(x_0^i)$ for $\phi_i(x_0^i | u^i)$, $i = 1, \ldots, r$. Note that the functions ϕ_i are strongly convex and continuously differentiable. More specifically, the first derivatives of ϕ_i are given by

$$\phi_i'(x_0^i) = \begin{cases} x_0^i - u_0^i & \text{if } x_0^i \geq \|\bar{u}^i\| \\ 2x_0^i - (u_0^i + \|\bar{u}^i\|) & \text{if } x_0^i < \|\bar{u}^i\|. \end{cases} \tag{51}$$

Observe that ϕ_i' is an increasing, piecewise linear and concave function for all i. More specifically, each ϕ_i' has two linear pieces and a single kink, where the right directional derivative is 1 and the left one is 2, which means $\lim_{t \to -\infty} \phi_i'(t) = -\infty$ and $\lim_{t \to \infty} \phi_i'(t) = \infty$.

Since problem (50) is convex, the following KKT conditions are necessary and sufficient for optimality:

$$\phi_i'(x_0^i) - v - w_i = 0, \quad i = 1, \ldots, r, \tag{52}$$

$$\sum_{i=1}^r x_0^i = 1, \tag{53}$$

$$x_0^i \geq 0, \ w_i \geq 0, \ x_0^i w_i = 0, \quad i = 1, \ldots, r, \tag{54}$$

where $v \in \mathbb{R}$ and $w_i \in \mathbb{R}$, $i = 1, \ldots, r$, are Lagrange multipliers.

From (52) and (54), we have

$$w_i = \phi_i'(x_0^i) - v \geq 0, \quad i = 1, \ldots, r,$$

which implies

$$x_0^i \geq (\phi_i')^{-1}(v), \quad i = 1, \ldots, r, \tag{55}$$

where $(\phi_i')^{-1}$ is the inverse function of ϕ_i', which is well-defined by the above-mentioned property of ϕ_i'. In fact, the function $(\phi_i')^{-1}$ has the following explicit representation for each i, cf. (51):

$$(\phi_i')^{-1}(v) = \begin{cases} v + u_0^i & \text{if } v \geq -(u_0^i - \|\bar{u}^i\|) \\ \frac{1}{2}(v + u_0^i + \|\bar{u}^i\|) & \text{if } v < -(u_0^i - \|\bar{u}^i\|). \end{cases}$$

Moreover, from (55) and the complementarity condition (54), we obtain

$$x_0^i = \max(0, (\phi_i')^{-1}(v)), \quad i = 1, \ldots, r, \tag{56}$$

which together with (53) yields the following equation with variable $v \in \mathbb{R}$:

$$\sum_{i=1}^{r} \max(0, (\phi_i')^{-1}(v)) = 1. \tag{57}$$

To proceed further, it will be convenient to define the functions $\psi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, r$, by

$$\psi_i(v) = \max(0, (\phi_i')^{-1}(v))$$

and scalars $\alpha_i, \beta_i, i = 1, \ldots, r$, by

$$\begin{aligned} \alpha_i &:= -(u_0^i + \|\bar{u}^i\|), \\ \beta_i &:= -(u_0^i - \|\bar{u}^i\|). \end{aligned} \tag{58}$$

Note that $\alpha_i \leq \beta_i$ for all i; moreover, $\alpha_i = \beta_i$ if and only if $\bar{u}^i = 0$. Then the functions ψ_i can be represented explicitly as follows:

- If $\alpha_i < \beta_i$, then

$$\psi_i(v) = \begin{cases} v + u_0^i & \text{if } v \geq \beta_i \\ \frac{1}{2}(v + u_0^i + \|\bar{u}^i\|) & \text{if } \alpha_i \leq v < \beta_i \\ 0 & \text{if } v < \alpha_i. \end{cases}$$

- If $\alpha_i = \beta_i$, then

$$\psi_i(v) = \begin{cases} v + u_0^i & \text{if } v \geq \alpha_i \\ 0 & \text{if } v < \alpha_i. \end{cases}$$

In any case, the functions ψ_i are piecewise linear and convex. The subgradients of these functions are given as follows:

- If $\alpha_i < \beta_i$, then

$$\partial\psi_i(v) = \begin{cases} \{1\} & \text{if } v > \beta_i \\ \left[\frac{1}{2}, 1\right] & \text{if } v = \beta_i \\ \{\frac{1}{2}\} & \text{if } \alpha_i < v < \beta_i \\ \left[0, \frac{1}{2}\right] & \text{if } v = \alpha_i \\ \{0\} & \text{if } v < \alpha_i. \end{cases}$$

- If $\alpha_i = \beta_i$, then

$$\partial\psi_i(v) = \begin{cases} \{1\} & \text{if } v > \alpha_i \\ [0, 1] & \text{if } v = \alpha_i \\ \{0\} & \text{if } v < \alpha_i. \end{cases}$$

Now let us define the function $\varphi : \mathbb{R} \to \mathbb{R}$ as:

$$\varphi(v) = \sum_{i=1}^{r} \psi_i(v) - 1.$$

Then the equation (57) can be rewritten as

$$\varphi(v) = 0. \tag{59}$$

It is not difficult to see that $\varphi(v) = -1$ for all $v \leq \alpha$, where

$$\alpha := \min_{1 \leq i \leq r} \alpha_i$$

with α_i given by (58). Moreover, φ is increasing for $v \geq \alpha$, and $\lim_{v\to\infty} \varphi(v) = \infty$. Consequently, equation (59) has a unique solution $v^* \in (\alpha, \infty)$. Once v^* is computed, the optimal solution of problem (50) is obtained from (56) with $v = v^*$. Moreover, the optimal solution of problem (48), i.e., the projection of u onto $\mathcal{K}$, is recovered from (49) with x_0^i so obtained.

A number of algorithms are available for solving the univariate equation (59). Below we present a (generalized) Newton method. Since the function φ is monotonically increasing, piecewise linear and convex, it can easily be shown that the method is finitely convergent to the unique solution v^*.

Newton's method for solving equation (59).

Step 0 Find an initial solution v_0 such that $\varphi(v_0) > 0$. Let $k := 0$.

Step 1 If $\varphi(v_k) = 0$, then terminate. Otherwise, go to Step 2.

Step 2 Choose a subgradient $\xi_k \in \partial\varphi(v_k) = \partial\psi_1(v_k) + \cdots + \partial\psi_r(v_k)$, and compute v_{k+1} by

$$v_{k+1} = v_k - \frac{\varphi(v_k)}{\xi_k}.$$

Let $k := k + 1$ and go to Step 1.

Remark 5. (i) We need to find an initial solution v_0 such that $\varphi(v_0) > 0$. From a practical viewpoint, a small initial value v_0 is preferred, as long as it satisfies $\varphi(v_0) > 0$. Since $\varphi(\alpha) = -1$ and φ is monotonically increasing for $v > \alpha$, we may set $v_0 := \alpha + \hat{\ell}\delta$ for some $\delta > 0$, where $\hat{\ell}$ is the smallest positive integer ℓ such that $\varphi(\alpha + \ell\delta) > 0$.

(ii) In Step 1 we use the stopping criterion $|\varphi(v_k)| < \varepsilon$, with ε a small positive tolerance (in practice $\varepsilon = \sqrt{\bar{\epsilon}}$, where $\bar{\epsilon}$ is the machine precision).

(4) *Computation of the stepsize δ*
The value of the stepsize is obtained with an exact line-search, i.e., it is the solution of the univariate optimization problem

$$\begin{array}{ll} \text{Minimize} & g(\delta) \\ \text{subject to} & 0 \le \delta \le 1, \end{array}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(\delta) = h(x + \delta d)$, for given vectors x and d. According to [17], any solution δ of $g'(\delta) = 0$ associated with the Rayleigh quotient function is a root of the following equation of degree two:

$$a_1 + \delta a_2 + \delta^2 a_3 = 0, \tag{60}$$

where

$$\begin{aligned} a_1 &= (d^t Ax)(x^t Bx) - (d^t Bx)(x^t Ax), \\ a_2 &= (d^t Ad)(x^t Bx) - (d^t Bd)(x^t Ax), \\ a_3 &= (d^t Ad)(x^t Bd) - (d^t Bd)(x^t Ad). \end{aligned}$$

Let s_1 and s_2 be the solutions of equation (60). Noticing that $\varphi'(0) < 0$ and $0 \le \delta \le 1$, we can determine the stepsize as

$$\delta = \begin{cases} 1 & \text{if } a_3 = 0 \text{ or } s_1, s_2 \notin [0,1] \\ s_i & \text{if } s_i \in [0,1], s_j \notin [0,1] \\ s_i & \text{if } s_1, s_2 \in [0,1] \text{ and } \varphi(s_i) \le \varphi(s_j), \varphi(s_i) \le \varphi(1) \\ 1 & \text{if } s_1, s_2 \in [0,1] \text{ and } \varphi(1) \le \varphi(s_i) \ (i = 1, 2). \end{cases}$$

5 Computational Experience

In this section we report some computational experience with the SPG algorithm discussed in the previous section for the solution of symmetric SOCEiCPs. The experiments have been performed on a Pentium IV (Intel) with 3.0 GHz and 2 GBytes of RAM memory, using the operating system Linux. The algorithm was coded in FORTRAN 90 and compiled with the Intel compiler, version 10.0. The algorithm was also implemented in the General Algebraic Modeling System (GAMS) language (Rev 118 Linux/Intel) [9] and the solver MINOS [22] (Version 5.51) was used to solve the problem (46), where the constraints $\left\|\bar{x}^i\right\| \leq x_0^i$ were replaced by $\left\|\bar{x}^i\right\|^2 \leq (x_0^i)^2$. Running times presented in this section are always given in CPU seconds.

In our set of test problems, B is always the identity matrix and $C \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix ($C = EE^t$) or $C = (E + E^t)/2$, where E is randomly generated such that each element is uniformly distributed in the interval $[-1, 1]$. Furthermore, for the SPG algorithm the value of the stopping tolerance has been set to 1.0E-06 and the values of η_{min} and η_{max} have been fixed to 1.0E-05 and 1.0E+05, respectively.

Tables 1 and 2 report the results obtained with the SPG algorithm and its comparison with the solver MINOS for $r = 3, 5$. The notation $(*)$ stands for instances where the solver MINOS was not able to find a solution (solver found the problem unbounded or badly scaled). In these tables It is the total number of iterations, λ is the complementary eigenvalue computed, and T is the total CPU time in seconds required to solve each problem.

The results shown in these tables demonstrate the efficiency and efficacy of the SPG algorithm for solving the symmetric SOCEiCP. The projection technique described in the previous section has performed very well for all the instances. The performance of this projection technique and of the SPG algorithm do not seem to be influenced by an increase of the number r of the Lorenz cones $\mathcal{K}_i$. The SPG algorithm requires in general a number of iterations of order equal to the dimension of the EiCP.

In order to have a better idea of the efficiency of the SPG algorithm, we also solve all the test problems by the well-known code MINOS. The performance of this last method is also illustrated in Tables 1 and 2. It seems that SPG algorithm is in general more efficient than MINOS as the CPU time for SPG method is smaller and the gap between the times of both algorithms tends to increase with the dimension of the EiCP.

6 Conclusions

In this paper, we discuss the existence of a solution to the Quadratic Conic Eigenvalue Complementarity Problem (QCEiCP), where the vectors x and w of complementary variables belong to an arbitrary pointed, closed and convex cone $\mathcal{K}$ and its dual $\mathcal{K}^*$. A sufficient condition for the existence of a solution for QCEiCP is introduced.

It is shown that the symmetric CEiCP reduces to the computation of a stationary point $\tilde{x} \neq 0$

Table 1: Performance of the algorithms for $r = 3$.

					SPG			Minos		
C	n	n_1	n_2	n_3	It	λ	T	It	λ	T
	10	5	3	2	37	9.0559E+00	2.00E-04	57	9.0559E+00	1.30E-02
	20	10	5	5	66	1.4730E+01	5.00E-04	231	1.3680E+01	2.70E-02
	30	15	8	7	19	3.2365E+01	9.00E-04	97	3.2365E+01	2.50E-02
	40	20	10	10	32	3.4207E+01	1.40E-03	369	3.3029E+01	7.60E-02
	50	25	13	12	168	4.6001E+01	4.00E-03	291	4.6001E+01	8.80E-02
	60	30	15	15	105	5.4414E+01	4.60E-03	302	5.4414E+01	1.26E-01
	70	35	18	17	73	6.6755E+01	5.30E-03	399	6.5617E+01	2.08E-01
	80	40	20	20	130	6.9076E+01	8.40E-03	634	6.4299E+01	3.68E-01
	90	45	23	22	99	9.0406E+01	9.70E-03	374	9.0406E+01	3.27E-01
	100	50	25	25	258	9.4997E+01	1.91E-02	799	9.1198E+01	6.84E-01
	200	100	50	50	127	2.0870E+02	5.17E-02	481	2.0870E+02	2.00E+00
	300	150	75	75	134	3.0175E+02	1.18E-01	535	3.0175E+02	5.46E+00
	400	200	100	100	391	3.9770E+02	3.84E-01	815	3.9770E+02	1.39E+01
	500	250	125	125	229	4.9011E+02	4.36E-01	924	4.9011E+02	2.44E+01
	1000	500	250	250	305	9.8749E+02	2.55E+00	1930	9.8749E+02	2.12E+02
	10	5	3	2	33	2.0491E+00	2.00E-04	56	2.3141E+00	1.40E-02
	20	10	5	5	40	2.8138E+00	5.00E-04	190	2.5457E+00	2.40E-02
	30	15	8	7	55	2.7716E+00	1.10E-03	119	2.7716E+00	2.60E-02
	40	20	10	10	97	2.7837E+00	1.90E-03	283	2.7695E+00	6.10E-02
	50	25	13	12	57	4.3995E+00	2.70E-03	141	4.3995E+00	5.50E-02
	60	30	15	15	77	4.6203E+00	4.00E-03	192	4.6203E+00	8.40E-02
	70	35	18	17	61	5.0735E+00	5.00E-03	207	5.0735E+00	1.20E-01
	80	40	20	20	90	5.2576E+00	7.20E-03	195	5.2576E+00	1.46E-01
	90	45	23	22	82	5.5120E+00	8.80E-03	266	5.5120E+00	2.27E-01
	100	50	25	25	220	5.9158E+00	1.72E-02	371	5.8207E+00	3.41E-01
	200	100	50	50	347	8.7372E+00	8.88E-02	390	8.7372E+00	1.38E+00
	300	150	75	75	1598	1.0444E+01	7.06E-01	882	9.1407E+00	5.98E+00
	400	200	100	100	201	1.2547E+01	2.54E-01	674	1.2547E+01	8.92E+00
	500	250	125	125	145	1.3274E+01	3.40E-01	755	1.3274E+01	1.59E+01
	1000	500	250	250	168	1.9215E+01	1.73E+00	1416	1.9215E+01	1.23E+02

of an appropriate merit function on a convex subset of the cone $\mathcal{K}$. The numerical solution of the symmetric CEiCP when $\mathcal{K}$ is the so called Second-Order Cone (SOCEiCP) by the Spectral Projected-Gradient (SPG) algorithm is also investigated. A new technique for computing projections required by the SPG method is introduced. The SPG method and the projection technique seem to perform very well in practice for solving the symmetric SOCEiCP. The solution of the nonsymmetric SOCEiCP is certainly one of our main research interests in the near future.

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Table 2: Performance of the algorithms for $r = 5$.

C	n	n_i, $i = 1, \ldots, 5$	SPG			Minos		
			It	λ	T	It	λ	T
	10	2	28	8.2063E+00	2.00E-04	110	7.9596E+00	1.70E-02
	20	4	22	1.6731E+01	4.00E-04	240	1.2162E+01	2.80E-02
	30	6	24	2.4235E+01	9.00E-04	365	2.3143E+01	4.90E-02
	40	8	46	3.3827E+01	1.60E-03	631	2.7542E+01	1.09E-01
	50	10	56	4.1911E+01	2.60E-03	647	3.8522E+01	1.59E-01
	60	12	72	5.3729E+01	3.90E-03	1000	4.2615E+01	3.20E-01
	70	14	193	6.2644E+01	8.30E-03	1270	5.1216E+01	5.14E-01
	80	16	40	8.4495E+01	5.70E-03	292	7.6975E+01	2.19E-01
	90	18	102	8.3415E+01	9.70E-03	889	7.8461E+01	5.89E-01
	100	20	108	9.4859E+01	1.21E-02	764	8.8754E+01	6.88E-01
	200	40	131	1.9211E+02	5.16E-02	867	1.6195E+02	2.94E+00
	300	60	169	2.9380E+02	1.33E-01		*	
	400	80	302	4.0302E+02	3.23E-01	991	3.7806E+02	1.40E+01
	500	100	208	4.8964E+02	4.14E-01	4630	4.5668E+02	8.87E+01
	1000	200	536	9.6568E+02	4.01E+00		*	
	10	2	31	2.0433E+00	2.00E-04	6.20E+01	1.9823E+00	1.50E-02
	20	4	35	3.2463E+00	5.00E-04	1.04E+02	3.0575E+00	1.80E-02
	30	6	77	2.5276E+00	1.10E-03	3.54E+02	2.3607E+00	4.70E-02
	40	8	144	3.2950E+00	2.40E-03	4.09E+02	3.1474E+00	7.70E-02
	50	10	137	3.7164E+00	3.60E-03	6.49E+02	3.2991E+00	1.53E-01
	60	12	377	4.3943E+00	9.30E-03	2.56E+02	4.0818E+00	1.03E-01
	70	14	74	4.3176E+00	5.30E-03	7.48E+02	4.1019E+00	2.99E-01
	80	16	148	5.1492E+00	9.10E-03	3.94E+02	4.0405E+00	2.39E-01
	90	18	78	5.6044E+00	8.70E-03	1.28E+03	5.3459E+00	8.58E-01
	100	20	147	5.6063E+00	1.54E-02	8.17E+02	4.7623E+00	6.11E-01
	200	40	698	7.7786E+00	1.49E-01		*	
	300	60	689	9.5695E+00	3.40E-01	6.31E+02	8.7583E+00	4.48E+00
	400	80	137	1.1500E+01	2.08E-01		*	
	500	100	1473	1.2762E+01	1.83E+00	1.25E+03	1.1064E+01	2.35E+01
	1000	200	493	1.8729E+01	3.66E+00	1.67E+03	1.7119E+01	1.46E+02

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