Level-δ and stable limit linear series on singular curves.

Autor: Pedro Hernandez Rizzo

Orientador: Prof. Dr. Eduardo Esteves

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Banca Examinadora:

- Prof. Dr. Eduardo Esteves-IMPA (Orientador).
- Prof. Dra. Carolina Araujo-IMPA.
- Prof. Dr. Dave Anderson- IMPA.
- Prof. Dr. Filippo Viviani-ROMA TRE.
- Prof. Dr. Alex Abreu- UFF.
- Prof. Dr. Karl Otto-Störh-IMPA (Suplente).
- Prof. Dr. Marco Pacini- UFF (Suplente).

Sumário

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Resumo

Para uma curva X de tipo compacto com duas componentes suaves Y e Z que se intersectam transversalmente em um único nó P, definimos o functor de series lineares limites estáveis. Provamos é grosseiramente representado por um espaço moduli de mapas estáveis invariantes pela ação do toro no sentido de Kontsevich: mapas de curvas de gênero 0 ao espaço de series lineares limites generalizados sobre X y que satisfazem certas condições esperadas em homología.

Também provamos que o espaço de series lineares limites estáveis sobre X possui uma cobertura natural por certos abertos de series lineares limites de nível delta sobre X, denominados series lineares limites exatos de nível delta, o qual é um conceito introduzido neste trabalho e que generaliza alguns conceitos introduzidos por Osserman e, por sua vez, por Eisenbud e Harris.

A partir da relação entre series lineares limites exatos de nível delta e as fibras do mapa de Abel, generalizando os resultados obtidos por Osserman e Esteves, provamos que o espaço de series lineares limites estáveis é o candidato natural que resolve o mapa determinado por enviar series lineares limites exatos à la Osserman a subesquemas bem comportados ao respeito de deformações de curvas suaves a curva X na fibra do mapa de Abel.

Palavras chaves: Series lineares limites, series lineares limites exatas de nível delta, series lineares limites estáveis, mapas de Abel.

Abstract

Abel Theorem on a smooth curve C, in modern terms, is the isomorphism

$$\mathbb{P}\colon G^r_d(C) \longrightarrow \mathrm{Hilb}^{\binom{r+t}{r}}_{A_d}.$$

Esteves and Osserman ([18]) produced a rational map

$$\mathbb{P}\colon G^r_d(X) \dashrightarrow \mathrm{Hilb}^{\binom{r+t+s}{r}}_{A_d}$$

defined on $G^{r,\mathrm{exact}}_d(X)$, the exact locus of Osserman's variety. The goal of this thesis is to resolve this map. Moreover, we produce a meaningful resolution by adding to the boundary of $G^{r,\mathrm{exact}}_d(X)$ what we call "stable limit linear series".

The search for this resolution produced as collateral results the construction of varieties of what we term level-δ limit linear series. The integer $\delta > 0$ is the singularity degree of the total space of the smoothing at the node P, the only point where the total space fails to be regular. In a nutshell, whereas $G^r_d(X)$ is the appropriate space to describe limits of linear series on smooth curves degenerating to X along regular (one-parameter) smoothings, the space of level-δ limit linear series, $G^r_{d,\delta}(X)$, is the appropriate space to describe limits along nonregular smoothings.

Keywords: Limit linear series, exact level-δ limit linear series, stable limit linear series, Abel maps.

1 Introduction

1.1 Goal

The aim of this thesis is to present two new ideas for dealing with limit linear series on curves of compact type:

- Level-δ limit linear series.
- Stable limit linear series.

These new ideas can be viewed as developments of the theory started by Osserman [37], [38], who in turn developed on ideas by Eisenbud and Harris [8], [9], and pursued by Esteves and Osserman [18].

As in [37], to avoid combinatorics, we work with the "toy case" of a nodal curve X with only two components Y and Z, which are smooth and meet transversally a unique point P.

Osserman [37] constructed a variety $G^r_d(X)$ parametrizing what he called limit linear series of degree d and rank r. It behaves functorially better than the similar variety constructed by Eisenbud and Harris [9]. It contains as an open subset the locus of refined limit linear series, the better-behaved type of limits used in the approach of Eisenbud and Harris. Furthermore, the refined limit linear series are instances of what Osserman calls exact limit linear series, which are parameterized by a larger open subset $G^{r,\text{exact}}_d(X) \subseteq G^r_d(X)$.

Again, the locus of exact limit linear series is better behaved than the whole $G^r_d(X)$. For instance, if X is general, $G^{r,\text{exact}}_d(X)$ is smooth. Moreover, it was shown by Esteves and Osserman [18] that exact limit linear series correspond to fibers of Abel maps.

Indeed, a modern interpretation of the celebrated Abel Theorem is that it establishes an isomorphism between the variety of linear series of degree d and rank r on a smooth curve C and the relative Hilbert scheme of subschemes of fibers of the degree-d Abel map A_d with Hilbert polynomial $P(t) = \binom{r+t}{r}$. Esteves and Osserman produced a rational map

$$\alpha\colon G^r_d(X) \dashrightarrow \mathrm{Hilb}^{\binom{r+s+t}{r}}_{A_d}$$

defined on $G^{r,\text{exact}}_d(X)$. It is the goal of this thesis to resolve this map. Not only do we attain this goal, but also we produce a meaningful resolution, by adding to the boundary of $G^{r,\text{exact}}_d(X)$ what we term stable limit linear series.

Nevertheless, the search for this resolution produced as collateral results the construction of varieties of what we term level-δ limit linear series. In a nutshell, whereas $G^r_d(X)$ is the appropriate space to describe limits of linear series on smooth curves degenerating to X along regular (one-parameter) smoothings, the space of level-δ limit linear series, $G^r_{d,\delta}(X)$, is the appropriate space to describe limits along nonregular smoothings. The integer $\delta > 0$ is the singularity degree of the total space of the smoothing at the node P, the only point where the total space fails to be regular. Limits of linear series along regular smoothings give rise to exact limit linear series; limits of linear series along smoothings with singularity degree δ give rise to exact level-δ limit linear series.

The construction of the $G^r_{d,\delta}(X)$ and the notion of exactness is very similar to that given by Osserman to $G^r_d(X)$. Also, we show in this thesis that there is a rational map

$$\alpha_\delta\colon G^r_{d,\delta}(X) \dashrightarrow \mathrm{Hilb}_{A_d}^{\binom{r+s+t}{r}},$$

similar to α, defined on the open locus $G^{r,\mathrm{exact}}_{d,\delta}(X) \subseteq G^r_{d,\delta}(X)$ parameterizing exact level-δ limit linear series. Moreover, for each δ there are forgetful maps $\rho_\delta\colon G^r_{d,\delta}(X) \to G^r_d(X)$, which are isomorphisms over the locus of exact limit linear series. More precisely,

$$\rho_\delta^{-1}(G^{r,\mathrm{exact}}_d(X)) \subseteq G^{r,\mathrm{exact}}_{d,\delta}(X), \quad \text{and} \quad \rho_\delta\colon \rho_\delta^{-1}(G^{r,\mathrm{exact}}_d(X)) \to G^{r,\mathrm{exact}}_d(X)$$

is a bijection. Moreover, $\alpha_\delta = \alpha\rho_\delta$.

Besides, the maps ρ_δ are surjective and, strikingly,

$$\rho_\delta(G^{r,\mathrm{exact}}_{d,2}(X)) = G^r_d(X).$$

It might seem natural to expect that for a high enough δ, the map α_δ would be a resolution of α. However, α_δ fails to be defined everywhere by the very same reason that α is not. Furthermore, though α_δ is defined on the larger $G^{r,\mathrm{exact}}_{d,\delta}(X)$, it looks like, as seen in simple examples, that this space gets larger and larger with δ, meaning that the number of its connected components grows to infinity with δ.

It was clear that we needed a way of establishing an equivalence relation on the various $G^{r,\mathrm{exact}}_{d,\delta}(X)$ to cut down their sizes. This is exactly what the notation of stable limit linear series gives. The idea is as simple as it is beautiful. It came with the realization that the notion of exactness imposed by Osserman is exactly the condition necessary for the coincidence of limits of two consecutive linear series in the data encoded by Osserman's limit linear series. So, we may gather the discrete data defining exact limit linear series in a single continuous data, a family of linear series parameterized by a chain of rational smooth curves!.

In the case of Osserman's exact limit linear series of degree d and rank r, the chain consists of $d+1$ rational curves. In the case of a level-δ limit linear series of the same degree and rank, the same observation can be made, and the chain consists now of $d\delta+1$ rational curves. Clearly, the growth of $G^{r,\mathrm{exact}}_{d,\delta}(X)$ was associated to the number of these curves. It was here that the concept of stabilization came about. Families of linear series over chains in the same or in different levels would be identified by requiring that their stabilizations would be the same! Here is the origin of the notion of stable limit linear series.

In this thesis we define a moduli functor for stable limit linear series, and show that this functor is coarsely represented by a Kontsevich moduli space of stable maps from genus-0 curves, which we denote by $G^r_d(X)^{\mathrm{st}}$. We show that $G^r_d(X)^{\mathrm{st}}$ contains Osserman's $G^{r,\mathrm{exact}}_d(X)$ as an open subset, and we construct a map

$$\mathbb{P}^{\mathrm{st}}\colon G^r_d(X)^{\mathrm{st}} \longrightarrow \mathrm{Hilb}^{\binom{r+s+t}{r}}_{A_d}$$

which is a resolution of α, our goal.

1.2 More details...

The classical theory of linear series on smooth curves is a useful tool for our understanding of properties and invariants of the curve itself. Specifically, two of the main applications of linear series on smooth curves are the study of morphisms to projective spaces and the study of invariants of divisors such as Weierstrass points.

On the other hand, the theory of linear series is completly determined by the Abel maps and their fibers, due to the remarkable Abel-Jacobi Theorem. Actually, the Abel-Jacobi Theorem can be interpreted as an isomorphism between $G^r_d(X)$ the projective space of linear series on a smooth curve X and the relative Hilbert scheme $\mathrm{Hilb}^{P(t),H}_{A_d}$ of subschemes in the fiber of the degree d Abel-Jacobi map

$$\begin{array}{rcccc} A_d & : & \mathrm{S}^d(X) & \longrightarrow & J_X \\ & & \sum_{i=1}^d q_i & \mapsto & [\mathcal{O}_X(\sum_{i=1}^d q_i - dp)]. \end{array}$$

Here $\mathrm{S}^d(X)$ is the symmetric product of X, $P(t) = \binom{r+t}{r}$ is the Hilbert polynomial of (flat degenerations) the projective space $\mathbb{P}^r$, $H := P + \mathrm{S}^{d-1}(X) \subset \mathrm{S}^d(X)$ is the relative ample divisor parametrizing families of effective divisors whose support contains the fix (any) point p and J_X is the Jacobian variety parametrizing isomorphic classes of line bundles of degree 0 on X.

Motivated by the premise "most the problems of interest about curves are, or can be, formulated in terms of (families of) linear series" (see, [9] pag. 339) Eisenbud and Harris introduced in eighties (see [9]) the theory of limit linear series as a powerful tool for handling degenerations of linear series on smooth curves to singular curves of compact type. The remarkable applications of this theory to understanding smooth curves, such as new and simplified arguments for the proofs of Brill–Noether and Gieseker–Petri Theorems, results related to Weierstrass points and to the moduli space of curves, poses find a satisfactory compactification of the spaces of limit linear series on singular curves, at least those of compact type.

However, the Eisenbud–Harris approach to answer this question is partial, in the following sense. Eisenbud and Harris make a distinction between refined and crude their limit linear series on a singular curve X of compact type, according to compatibility conditions on ramification conditions at the nodes. Precisely, if X is a curve of compact type then a limit linear series $L = \{(L_Y, V_Y)\}$ of degree d and rank r is a collection of linear series of degree d and rank r on each smooth irreducible component Y of X satisfying: if Y and Z are components of X meeting at the node p, then for each $i = 0, \ldots, r$ we have that

$$\epsilon_i^{L_Y}(p) + \epsilon_{r-i}^{L_Z}(p) \geq d, \tag{1.1}$$

where $\{\epsilon_i^{L_Y}\}$ is the vanishing sequence of L_Y at p of the linear series on the component Y. A refined limit series is a limit linear series such that all the inequalities above are equalities. Otherwise, the limit linear series is called crude. Thus, for any $\mathcal{X} \to B$ a smoothing of X the scheme $G_d^{r,\mathrm{EH}}(\mathcal{X}/B; (q_1, \epsilon^1), \ldots, (q_s, \epsilon^s))$ parametrizing linear series on the general fiber X_η and refined limit linear series on the special fiber X, under certain ramifications sequences conditions on smooth sections q_i, is quasi-projective on B. This lack of properness caused by disregarding crude limit linear series can be interpreted as saying that there exist degenerations of linear series whose limit linear series is crude on X.

Attempts to generalize the Eisenbud–Harris theory to curves not of compact type are sparse in the literature. See for instance [14] and [17] on nodal reducible curves and limit canonical system on stables curves with two components, respectively.

In the spirit of Eisenbud and Harris Theory, "but more functorial in nature, and involving a substantially new approach which appears better suited to generalization to higher-dimensional varieties and higher-rank bundles" ([37], p. 1165) Osserman constructed a moduli space ([37]) parametrizing a new concept of limit linear series on a curve X of compact type with (only) two components. Roughly speaking, Osserman's moduli space parameterizes limit linear series considering "all posible degrees".

More precisely, given a regular smoothing $\mathcal{X}/B$ of X, where $B = \mathrm{Spec}(\mathbb{C}[[t]])$, let $(\mathcal{L}_\eta, V_\eta)$ be a linear series on the general smooth fiber X_η. If $\mathcal{L}$ is an extension of $\mathcal{L}_\eta$ to $\mathcal{X}$, then $\mathcal{L}(iY) \cong \mathcal{L}(-iZ)$ are extensions too, for any $i \in \mathbb{Z}$, where Y and Z are the irreducible components of X. Fixing the degree d of $\mathcal{L}$, there exists a unique extension of $\mathcal{L}_\eta$ to $\mathcal{X}$ such that $\mathcal{L}$ has degree d when restricted to Y and degree 0 when restricted to Z. In this case, the extensions $\mathcal{L}(-iZ)$ for $i = 0, \ldots, d$ has degree $d - i$ on Y and degree i on Z. Eisenbud and Harris approach is that, for many of its applications, it suffices to consider the "extreme" degrees extensions $\mathcal{L}_0 := \mathcal{L}$ and $\mathcal{L}_d := \mathcal{L}(-dZ)$ having degree d and 0 on Y (resp., 0 and d on Z). Thus they defined the limit linear series on X as the pairs $(L, V) := \{(L_Y, V_Y), (L_Z, V_Z)\}$ satisfying (1.1), with $L_Y := \mathcal{L}_0|_Y$ and $L_Z := \mathcal{L}_d|_Z$.

The main insight of Osserman's approach (see [37]) is to consider all the extensions $\mathcal{L}_i := \mathcal{L}(-iZ)$ for $i = 0, \ldots, d$ linked by the natural morphisms $\varphi^i : \mathcal{L}_i \to \mathcal{L}_{i+1}$ and the reverse direction $\varphi_i : \mathcal{L}_{i+1} \to \mathcal{L}_i$, satisfying $\varphi_i\varphi^i = \varphi^i\varphi_i = t\mathrm{Id}$ with t a uniformizer of B. This construction is compatible with base change, therefore allows to define a moduli space $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$, which is proper over B and parametrizes linear series on each smooth general fiber X_η and a new notion of limit linear series over X. In fact, on the (singular) curve X we have a projective scheme $G_d^{r,\mathrm{Oss}}(X)$ parametrizing $d+2-$tuples $(L, V_0, \dots, V_d)$, where $V_i \subset H^0(X, L_i)$, L is an invertible sheaf on X of degree d on Y and degree 0 on Z so that we obtain the collection of invertible sheaves on X with $L_i|_Y \cong L(-ip)$ and $L_i|_Z \cong L(ip)$. Furthermore, this data satifies the linked conditions $\varphi^i(V_i) \subseteq V_{i+1}$, $\varphi_i(V_{i+1}) \subseteq V_i$ and $\varphi_i\varphi^i = \varphi^i\varphi_i = 0$, with φ^i and φ_i the induced maps over the spaces of sections.

The projection from $G_d^{r,\mathrm{Oss}}(X)$ to $G_d^r(Y) \times G_d^r(Z)$ has image the Eisenbud–Harris scheme $G_d^{r,\mathrm{EH}}(X)$ of Eisenbud–Harris limit linear series on X. This surjective map establishes an isomorphism between the refined Eisenbud–Harris limit linear series and an open set of $G_d^{r,\mathrm{Oss}}(X)$ that are called refined, too. The refined Osserman limit linear series are properly contained in the open subspace of exact limit linear series, i.e., those limit linear series with the following compatibility conditions on the maps φ^i and φ_i:

$$\mathrm{Im}(\varphi^i) = \mathrm{Ker}(\varphi_i) \quad \text{and} \quad \mathrm{Im}(\varphi_i) = \mathrm{Ker}(\varphi^i) \quad \forall i = 0, \dots, d. \tag{1.2}$$

The exact limit linear series are closely related to degenerations, in the following sense. Linear series degenerating along of a regular smoothing to X yields an exact limit linear series. This last claim is related to the Eisenbud–Harris question of finding a compactification of space of limit linear series. Osserman's compactification gives a partial solution this question, since only when X is a general curve of compact type like above, the smoothable limit linear series are dense ([34]).

We might think that we have more chances of compactifying the space of linear series over $\overline{M}_g$ looking at degenerations of effective divisors, or more generally, degenerations of Abel maps motivated by its close relation in the case of the smooth curves. Abel maps of degree 1 on stable curves were constructed in [33] and recently Coelho and Pacini [4] have constructed Abel maps of any degrees for curves of compact type. However, when comparing this latter construction with Eisenbud–Harris theory the relationship between two concepts, via subschemes of the fiber of Abel map, it seems far from being thoroughly understood. This is due to limitations on both sides. On one hand, the fibers of the Abel maps are not well behaved, for instance, they are in general not equidimensional. On the other hand, concepts as complete limit linear series are not all obvious to define from of a limit linear series.

Esteves and Osserman [18] have been investigating the relationship between limit linear series a la Osserman and fibers of the Abel maps for curves X of compact type with two smooth components meeting at a unique node. We can summarize their results as follows. Given a limit linear series $\mathfrak{g} = (L, V_0, \dots, V_d) \in G_d^{r,\mathrm{Oss}}(X)$ where L is an invertible sheaf on X of degree d on Y and degree 0 on Z, for Y and Z smooth components of X. Denote by $\mathfrak{g}_i := (L_i, V_i)$ the pairs where L_i is the unique invertible sheaf on X such that $L_i|_Y \cong L(-ip)$ and $L_i|_Z \cong L(ip)$ with p

the node on X, and $V_i \subset \Gamma(X, L_i)$. Esteves–Osserman define the map

$$\begin{array}{rccc} \alpha & : & G_d^{r,\mathrm{Oss}}(X) & \longrightarrow & \mathrm{Hilb}_{A_d} \\ & & \mathfrak{g} & \mapsto & \mathbb{P}(\mathfrak{g}) = \bigcup_{i=0}^{d} \mathbb{P}(\mathfrak{g}_i) \subset A_d^{-1}([L]), \end{array} \tag{1.3}$$

where $\mathbb{P}(\mathfrak{g}_i) = \overline{\{\mathrm{div}(s)|\, s \in V_i,\, s|_Y \neq 0,\, s|_Z \neq 0\}} \subset S^d(X)$, and $A_d : S^d(X) \to \mathrm{Pic}^d(X)$ is the degree-d Abel map on X. Now, they prove that on set of exact limit linear series (see (1.2)) the map α in (1.3) is well behaved, in the following sense: if $\mathfrak{g}$ is exact then $\mathbb{P}(\mathfrak{g})$ is Cohen–Macaulay of pure dimension r and Hilbert polynomial $P_{\mathbb{P}(\mathfrak{g})}(s,t) = \binom{r+s+t}{r}$ (of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$) (see [18] Theorem 4.3). Furthermore, if $\mathfrak{g}$ arise from a regular smoothing then $\mathbb{P}(\mathfrak{g})$ is the flat limit of the corresponding $\mathbb{P}^r$ on nearby fibers (see [18] Theorem 5.2).

In this way, it is natural to ask: can be extended the map α for nonexact limit linear series? The answer is no. A natural process could resolve the map. How to do it? It is clear that, as is known in the literature (see, for instance, [27], [17]), we need to understand better nonregular smoothings. We address this in the third chapter of this Thesis. Before sketching our results, we comments about the tools needed to reach them.

To study limits of special Weierstrass points, Cumino–Esteves–Gatto [5] use *twists* as an important tool for understanding limit linear series along families whose total space is nonregular, i.e., families of smooth curves degenerating to curve X, with a unique $A_{\delta-1}$-singularity at the node p, for some integer $\delta \geq 1$. Twists are sheaves of rank 1 and torsion-free on X introduced in [15]. Fixing an invertible sheaf L on X of (total) degree d, we obtain using twist a collection of $d\delta+1$ rank 1 and torsion-free sheaves (see 3.5, Chapter 3), yields as the admissible extensions of semi-stable model of X (see, [3] §5). In this way, we regard the "linkage" this collection in the natural way of "restriction and inclusion" made by Osserman to construct the space $G_d^{r,\mathrm{Oss}}(X)$.

Thus, combining the two approaches above on one side the Cumino–Esteves–Gatto ideas and, on the other side, Osserman's construction of moduli space of limit linear series, we introduce a new definition of limit linear series (briefly, lls) on the curve X, namely, level-δ limit linear series (see Definition 3.8), understanding them as limit of linear series degenerating along nonregular smoothings. It follows that there exists a projective space $G_{d,\delta}^r(X)$ parametrizing level-δ lls, which is related to Osserman's space $G_{d,1}^r(X) = G_d^{r,\mathrm{Oss}}(X)$ via a surjective proper map $\rho_\delta : G_{d,\delta}^r \to G_d^{r,\mathrm{Oss}}$ (see Proposition 3.9), such that Osserman's exact lls are lifting uniquely for any δ and contained in the open subspace of exact level-δ lls (see Proposition 3.16(1)), the natural generalization of Osserman's exact lls. Furthermore, given a nonexact lls $\mathfrak{g}$, there exist a δ and an exact level-δ lifting $\widetilde{\mathfrak{g}}$, i.e., $\widetilde{\mathfrak{g}} \in G_{d,\delta}^r(X)$ is an exact level-δ lls such that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$ (see, Proposition 3.16(2)).

The relation of our space $G_{d,\delta}^r(X)$ with the map α can be summarized as follows. Given a level-δ lls $\mathfrak{g} = (L, V_0, \ldots, V_{i\delta+j}, \ldots, V_{d\delta})$, we can define in a similar fashion to Esteves–Osserman, the subscheme $\mathbb{P}(\mathfrak{g}) \subset A_d^{-1}([L])$. It follows that $\mathbb{P}(\mathfrak{g})$ has the "correct" Hilbert polynomial if and only if is exact level-δ lls. (see Theorem 3.19). Precisely, if $\mathfrak{g} \in G_d^{r,\mathrm{Oss}}(X)$ is a nonexact lls then exist δ and an exact lifting $\widetilde{\mathfrak{g}} \in G^{r,\mathrm{exact}}_{d,\delta}(X)$ such that $\mathbb{P}(\mathfrak{g}) \subset \mathbb{P}(\widetilde{\mathfrak{g}})$ is contained properly and $\mathbb{P}(\widetilde{\mathfrak{g}})$ has Hilbert polynomial $P(s,t) = \binom{r+s+t}{r}$.

Considering the results above, and observing that all our results depend on a certain choice of δ, a natural question is: will be the process of completing the subschemes related to lls stops for some δ? Do we need all the δ's? In fact, no. The answer is that above level-$r+2$, does not have more relevant geometric information, i.e., we can always complete the subscheme with exact level-δ's below level-$r+2$ (see Theorem 3.21).

Although, we have found good candidates to limit linear series as limits of linear series along to any direction to X in $\overline{M}_g$ (see [27] p. 146), or better, exacts level-δ lls along to any smoothing of X, we continue with some limitations. First, the exact level-δ lls form an open subspace(!!), which do not expect to be a compactification. Second, the dependence of exact points of the parameter δ and principally the absence of control, i.e., absence of some relation between distinct levels that allows to control the exact points (see Remark 3.18). Thus,

- We would like to find a more general concept such that embraces all exact points at distinct levels.
- We would like to find a moduli space parametrizing elements with this concept that allows to identify exact lls in distinct levels but with the same subscheme in the fiber of Abel map.
- We would like this moduli space be (coarsely) represented by a projective scheme which contains all (open) subschemes of exact points of all levels. Roughly speaking, a compactification of the space of all limits.

These are problems that motivate Chapter 4 of this Thesis. Before we give details of our construction, we begin with a useful observation. We know that, for any invertible sheaf L of degree d on the curve X satisfies $\mathrm{Ext}^1(L_i|_Z, L_{i+1}|_Y) \cong \mathbb{C}$ and $\mathrm{Ext}^1(L_{i+1}|_Y, L_i|_Z) \cong \mathbb{C}$, for each $i-0,\dots,d-1$. Notice that, the trivial extension $L_i|_Z \oplus L_{i+1}|_Y$ correspond to the sheaves in the "middle" of our construction of level-δ spaces. A natural question is: what happens with the spaces of sections in the limit when the extensions L_i and L_{i+1} degenerate to the trivial extension $L_i|_Z \oplus L_{i+1}|_Y$? The interesting case of the study of limits of these degenerations is when we consider a limit linear series a la Osserman $\mathfrak{g}$, more exactly, an exact lls. In fact, if $\mathfrak{g}$ is an exact limit linear series then the limit through degenerations of extensions are equals (see Lemma 4.16). Notice that the degeneration to zero in $\mathrm{Ext}^1(L_i|_Z, L_{i+1}|_Y) \cong \mathbb{C}$ can be interpreted as degeneration to infinity in $\mathrm{Ext}^1(L_{i+1}|_Y, L_i|_Z) \cong \mathbb{C}$. Thus, we obtain a chain of $d+1$ projective lines parametrizing an exact Osserman's lls.

To formalize the last ideas, first we construct a moduli space $H^r_d(X)$ of *families of generalized linear series* of degree d and dimension r along a chain the projective lines T (see section 4.1.1). In short words, the scheme $H^r_d(X)$ parameterizes linear series $(\mathcal{I}, V)$, where $\mathcal{I}$ is any torsion-free, rank-1 sheaf on X of degree d whose restrictions to Y and Z, modulo torsion, have degrees ranging from -1 to d, and V is any vector subspace of $H^0(X, \mathcal{I})$ of dimension $r+1$. The main tool to construct $H^r_d(X)$ is the construction of a sheaf $\mathcal{F}$ on $X \times T/T$ which is "locally constant" (see subsection 4.1.1), i.e., $\mathcal{F}|_{X\times T_i^*} \cong \mathcal{O}_X(-i,i) \otimes \mathcal{O}_{T_i^*}$ for each $i = 0, \ldots, d$ and $\mathcal{F}|_{X\times N_i} \cong \mathcal{O}_Y(-iP) \oplus \mathcal{O}_Z((i-1)P)$ for $i = 0, \ldots, d+1$. Here, $\mathcal{O}_X(-i,i)$ is the unique invertible sheaf on X whose restriction to Y is $\mathcal{O}_Y(-ip)$ and to Z is $\mathcal{O}_Z(ip)$, and N_i the unique point where two components $T_i \cong \mathbb{P}^1$ and $T_{i+1} \cong \mathbb{P}^1$ of T intersecting transversally.

Second, we observe that an exact lls $\mathfrak{g}$ a la Osserman could be interpreted as a map $f_{\mathfrak{g}} : T \to H^r_d(X)$ from the chain T to the moduli scheme $H^r_d(X)$ (see Proposition 4.14). More generally, any $\widetilde{\mathfrak{g}}$ exact level-δ lls corresponds to a map $h_{\widetilde{\mathfrak{g}}} : S \to H^r_d(X)$, with the source curve a chain of projective lines (see Proposition 4.18). This motivates the construction of the moduli space $\mathfrak{G}^r_d(X)$ of *families of stable limit linear series* (see, subsection 4.2.1). Now our problem of representation becomes equivalent to another problem: find an interpretation of families of stable limit linear series in terms of a moduli space (coarsely) represented by a projective scheme.

Now the last question leads us to study features of the maps to $H^r_d(X)$. We emphasize that a map $h : S \to H^r_d(X)$ is related to a stable limit linear series if:

- It has the same "type" of a Osserman exact limit linear series. Here "type" means for us, the source curve S being a chain that contains $d+1$ components isomorphism to the chain T. The "translation" of Osserman "type" is the fundamental (effective) class $f_{\mathfrak{g}*}[T] := \beta \in H_2(H^r_d(X), \mathbb{Z})$ (see Lemma 4.23).

- It has the "locally constant" property. This inherited property of the sheaf $\mathcal{F}$, is related to certain torus action and its invariants subspaces on the product $G \times T$, where the image of each map h can be embedding. Here G is the absolute Grassmanian of subspaces of dimension $r+1$ of $H^0(L|_Y) \oplus H^0(L|_Z(dp))$ (see subsections 4.1.2, 4.2.1 and Lemma 4.12).

- It satisfies a stability condition with the main intention avoid any redundancy. For instance, the stable map defined by an Osserman exact represent the Osserman exact in the "all levels" (see Proposition 3.16(1)).

According to the conditions above, we regard a suitable moduli space: $\overline{\mathcal{M}}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$ of stable maps fixed by a torus action of (arithmetic) genus zero to $H^r_d(X)$ whose images lie in β, with $\beta \in H_2(H^r_d(X), \mathbb{Z})$. Thus the main results of the Chapter 4 of thesis are:

- There exists a projective, coarse moduli space $G^r_d(X)^{\mathrm{st}}$ (see Theorem 4.32).

- The map $[f : S \to H^r_d(X)] \in G^r_d(X)^{\mathrm{st}}$, has a representative such that the source curve is a chain. So, the $\mathbb{C}^*-$action and the class β determines the type of our stable maps (see Theorem 4.25). Intuitively, as in discussion above, this moduli space is a "good candidate" for the desired compactification.

- There exists a functorial equivalence between $\overline{\mathcal{M}}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$ and the functor $\mathfrak{G}^r_d(X)$ that define the families of stable limit linear series (see Proposition 4.30).

- For all $\delta \geq 1$, there exists a map

$$\Psi_\delta : G^{r,\mathrm{exact}}_{d,\delta}(X) \longrightarrow G^r_d(X)^{\mathrm{st}}.$$

  whose union over δ is equal to $G^r_d(X)^{\mathrm{st}}$ (see Theorem 4.36).

In particular, our projective scheme $G^r_d(X)^{\mathrm{st}}$ is a "good candidate" for resolving the (set-theoretically) map (see Corollary 4.38)

$$\begin{array}{cccccc} \alpha & : & G^{r,\mathrm{Oss}}_d(X) & \dashrightarrow & \mathrm{Hilb}^{P,H}_{A_d} \\ & & \mathfrak{g} & \mapsto & \alpha(\mathfrak{g}) = \mathbb{P}(\mathfrak{g}), \end{array}$$

studied by Esteves–Osserman (see [18]).

2 Abel maps and limit linear series: An overview

2.1 Generalities on limit linear series.

2.1.1 Linear series on (families of) smooth projective curves.

Let X be a smooth, connected and complex projective curve of genus g. A *linear series* on X of *degree* d and *rank* r is a pair $\mathfrak{g} := (L, V)$, where L is a line bundle on X with $d := \deg(L)$ and $V \subseteq H^0(X, L)$ is a vector subspace of sections of L with $r = \dim(V) - 1$.
It is well known that linear series on a smooth curve X is a useful tool to study maps of X into projective spaces. Specifically, each non degenerate morphism from X to a projective space $\mathbb{P}^r$ of degree d corresponds, up to automorphism of $\mathbb{P}^r$, to pair (L, V) as above.

One of the basic applications of linear series is related to the study of invariants of smooth curves. For this, we need to introduce the theory of *ramification points* of $\mathfrak{g}$ on X. Given $p \in X$, we say that an integer ϵ is an *order* of the linear series $\mathfrak{g} = (L, V)$ at p if there is a nonzero section of L in V vanishing at p with order ϵ. If two sections of L have the same order, a certain linear combination of them will be zero or have higher order. Thus, there are exactly $r + 1$ orders of $\mathfrak{g}$ at p: $\epsilon_0(p) < \epsilon_1(p) < \cdots < \epsilon_r(p)$. Furthermore, notice that $i \leq \epsilon_i(p) \leq d$. We call $\mathrm{wt}(p) := \sum\limits_{i=0}^{r} (\epsilon_i(p) - i)$ the *ramification weight* of $\mathfrak{g}$ at p. Observe that, $0 \leq \mathrm{wt}(p) \leq (r+1)(d-r)$. If $\mathrm{wt}(p) > 0$ then we say that p is a *ramification point* of $\mathfrak{g}$. A known and important example of ramification points happens when $r = g - 1$ and $d = 2g - 2$. In this case, the only linear series $\mathfrak{g}$ on X is the *canonical series* $\mathfrak{g} = (\omega_X, H^0(X, \omega_X))$, whose ramification points are precisely the *Weierstrass points* of X. Now, given the *ramification cycle* of $\mathfrak{g}$, $[W(\mathfrak{g})] := \sum\limits_{p \in X} \mathrm{wt}(p)[p]$, we have the *Plucker Formula*, which computes the degree of $[W(\mathfrak{g})]$

$$\deg\left([W(\mathfrak{g})]\right) = \sum_{p \in X} \mathrm{wt}(p) = (r + 1)(d + r(g - 1)).$$

As we may see, the theory of linear series is a useful tool that allows us to better understand the curve X. Indeed, in the words by Eisenbud and Harris "most problems of interest about curves are, or can be, formulated in terms of linear series." (see, [9] p. 339). One of these interesting set of problems is the Brill–Noether Theory, whose most fundamental question is: *For which values of r and d does a general smooth curve of genus g possess a linear system of degree d and* *rank* r. A rather complete answer is known as the Brill–Noether Theorem:

Theorem 2.1. *A general smooth, connected, complex projective curve of genus* $g \geq 2$ *has a linear system of degree* d *and rank* r *if and only if* $\rho(g,d,r) := (r+1)(d-r) - gr \geq 0$*; and if so, then* $\rho(g,d,r)$ *is the (pure) dimension of the projective moduli space* $G^r_d(X)$ *of linear series on* X *of rank* r *and degree* d*.*

We make some general remarks concerning the proof of the Brill–Noether Theorem. The "if" part was proved independently by Kempf (see [29]) and Kleiman-Laksov (see [30] and [31]). The arguments in their proofs are actually valid for any smooth, connected, projective curve. In this way, we may re-write the "if" part of the Brill–Noether Theorem: *If* $\rho(g,d,r) \geq 0$*, then for all smooth, connected, projective curves* X *of genus* g*, the space* $G^r_d(X)$ *is non-empty with every component of dimension at least* $\rho(g,d,r)$*.*
The arguments in the proof of the "only if" part illustrate our understanding of the interactions of the Brill–Noether Theory with the moduli theory of curves, in the sense that the analysis of desired properties of smooth curves relies on the analysis of degenerations to singular curves. This is the key point of the techniques developed by Eisenbud and Harris. The first complete proof of the Brill–Noether Theorem was given by Griffiths and Harris (see [26]), based in arguments considered by Severi. Later, Eisenbud and Harris (see [9] Theorem 4.5), following work by Gieseker ([24]), simplified and generalize the Theorem, introducing the notion of limit linear series. We will discuss it in the next section.

On the other hand, the proof of the existence of the projective moduli space

$$G^r_d(X) := \{(L,V) \,|\, L \in \mathcal{P} := \mathrm{Pic}^d(X),\, V \subset H^0(X,L),\, \dim(V) = r+1\}$$

is not difficult. In fact, we can summarize the construction as follows. It is the zero scheme of a bundle map ν on the relative Grassmannian $G := \mathrm{Grass}_{\mathcal{P}}(r+1, p_{2*}\mathcal{M})$. Here, $\mathcal{M} := \mathcal{L} \otimes p_1^*\mathcal{O}_X(np)$ is the invertible sheaf on $X \times \mathcal{P}$, where $\mathcal{L}$ a universal bundle on $X \times \mathcal{P}$, $p_1 : X \times \mathcal{P} \to X$, $p_2 : X \times \mathcal{P} \to \mathcal{P}$ are the canonical projections, and $n >> 0$. The integer n is chosen such that $R^1 p_{2*}\mathcal{M} = 0$, or equivalently, $p_{2*}\mathcal{M}$ is locally free. It is enough to choose, by the Riemann–Roch Theorem, $n \geq 2g-1-d$, since $n+d$ is the relative degree of $\mathcal{M}$. Now, from the diagram below

$$\begin{tikzcd} & X \times G \arrow[r, "q_2"] \arrow[d, "(\mathrm{Id},\pi)"'] & G \arrow[d, "\pi"] \\ X & X \times \mathcal{P} \arrow[l, "p_1"] \arrow[r, "p_2"] & \mathcal{P} \end{tikzcd}$$

we conclude that $G^r_d(X)$ is precisely the vanishing locus of the composition

$$\nu : \mathcal{V} \hookrightarrow \pi^* p_{2*}\mathcal{M} = q_{2*}(Id,\pi)^*\mathcal{M} \longrightarrow q_{2*}(Id,\pi)^*\mathcal{M}|_{np \times \mathcal{P}},$$

with the first map being tautological, with $\mathcal{V}$ the universal sub-bundle on G, and the second map is obtained by restriction. For more details, see [2] chapter IV, §3. Finally, since by definition $\mathcal{V}$ is locally free of rank $r+1$ and $\mathcal{M}|_{np\times\mathcal{P}}$ is locally free of rank n, we obtain that the codimension of each component of $G^r_d(X)$ in G is at most $(r+1)n$. It follows that $G^r_d(X)$ is a projective scheme, such that each component has dimension at least:

$$\dim(G)-(r+1)n = g+(r+1)(n+d+1-g-(r+1))-(r+1)n = (r+1)(d-r)-gr = \rho(g,d,r).$$

The rest of the section is dedicated to some relevant results about $G^r_d(X)$, as consequences of the Brill–Noether Theory. For more details we refer the reader to [2] Chapter V.

Theorem 2.2 (Fulton–Lazarsfeld)**. Connectedness Theorem.** *Let X be a smooth projective and connected curve of genus g and assume that $\rho(g,d,r)\geq 1$. Then $G^r_d(X)$ is connected.*

Theorem 2.3 (Gieseker)**. Smoothness Theorem.** *Let X be a general smooth projective and connected curve of genus g. Then $G^r_d(X)$ is smooth of dimension $\rho(g,d,r)$.*

An immediate consequence of combining the results by Gieseker and Fulton–Lazarsfeld, in the general case, is that: *if $\rho(g,d,r)\geq 1$ then $G^r_d(X)$ is irreducible.*

2.1.2 The limit linear series Space I: The Einsenbud–Harris approach.

The technique of limit linear series was introduced by Eisenbud and Harris in the eighties. They were able to obtain remarkable results from their techniques: results about the geometry of the moduli space of curves (see for instance [7]), about existence of Weierstrass points (see for instance [10]), generalizations of the Brill–Noether Theorem ([9]), enumeration of linear series (see for instance [11]), among others. We may describe the technique as the analysis of the geometric properties via degenerations to reducible curves of limits of linear series, with the goal of deducing something about the geometry of the linear series on smooth curves, i.e., general members of the family. Below, we give a brief survey of the principal results of this theory, avoiding some details.

First, observe that as defined linear series make sense for singular curves. Second, although this technique treats the more general case of singular curves of compact type, we restrict our attention to curves $X = Y\cup Z$, with Y and Z smooth components intersecting at a unique node P. The reason to this focus is that our main results, whose principal support come from the Osserman Theory about limit linear series, are restricted to this simple case.
Therefore, let us suppose that $\mathcal{X}/B$, choosing for simplicity $B := \mathrm{Spec}(\mathbb{C}[[t]])$, is a flat and projective family of curves, where the total space X is regular, the generic fiber $\mathcal{X}_\eta$ is smooth, and the special fiber X is as above. This family of curves is called of *regular smoothig of the curve* X. Since X is regular, it holds that:

- Y and Z are Cartier divisors of X, and every Cartier divisor of X supported in X is a $\mathbb{Z}-$linear combination of Y and Z. Also, since $\mathcal{O}_{\mathcal{X}}\cong\mathcal{O}_{\mathcal{X}}(Y+Z)$, we have that $\mathcal{O}_{\mathcal{X}}(iY)\cong$

$\mathcal{O}_{\mathcal{X}}(-iZ)$ for any $i \in \mathbb{Z}$, i.e., the each linear combination is reduced to the multiple of one component.

- For each invertible sheaf $\mathcal{L}_\eta$ on $\mathcal{X}_\eta$ there exists an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$, called of the *extension* of $\mathcal{L}_\eta$, such that $\mathcal{L}|_{X_\eta} \cong \mathcal{L}_\eta$. This extension is not unique. In fact, it is easy to check that $\mathcal{L}(iY) := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(iY)$, with $i \in \mathbb{Z}$, are all the extensions of L_η to $\mathcal{X}$.

The existence of distinct extensions to each $\mathcal{L}_\eta$ on $\mathcal{X}$, means the relative Picard scheme $\mathrm{Pic}(\mathcal{X}/B)$ is universally closed but not separated. Notice that this does not happen for vector subspaces: If $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$ and $\mathcal{V}_\eta \subset H^0(\mathcal{X}_\eta, \mathcal{L}|_{\mathcal{X}_\eta})$ is a vector subspace of dimension $r+1$, then there exists a unique extension $\mathcal{V} \subset H^0(\mathcal{X}, \mathcal{L})$, which is $\mathcal{V} = \mathcal{V}_\eta \cap H^0(\mathcal{X}, \mathcal{L})$.

If we specify the degree of $\mathcal{L}$ on Y and Z, it is well known that the Picard scheme is separated, and hence proper. Consequently, for each linear series $(\mathcal{V}_\eta, \mathcal{L}_\eta)$ of degree d and rank r, there exists a collection of extensions $(\mathcal{L}_i, \mathcal{V}_i)$ on $\mathcal{X}$, with $\mathcal{L}_i$ characterized by the conditions that has degree $d-i$ when restricted to Y and degree i when restricted to Z, or bi-degree $(d-i, i)$.

The main idea behind the Eisenbud–Harris Theory is to consider just the "extremal degree" linear series, $(\mathcal{L}_0, \mathcal{V}_0)$ and $(\mathcal{L}_d, \mathcal{V}_d)$, i.e., those of bi-degrees $(d, 0)$ and $(0, d)$, respectively. This is due to the fact that in all of their applications these linear series were enough. In fact, they considered the restrictions $(L_Y, V_Y) := (\mathcal{L}_0|_Y, \mathcal{V}_0 \otimes k(0))$ and $(L_Z, V_Z) := (\mathcal{L}_d|_Z, \mathcal{V}_d \otimes k(0))$. Notice that, identifying V_Y and V_Z with its image into $H^0(Y, L_Y)$ and $H^0(Z, L_Z)$ respectively, we have a sequence of inclusions

$$V_Y \subseteq H^0(X, \mathcal{L}_0|_X) \subseteq H^0(Y, L_Y) \quad \text{and} \quad V_Z \subseteq H^0(X, \mathcal{L}_d|_X) \subseteq H^0(Z, L_Z).$$

In other words, we don't lose geometric information by restricting to the components. So, given a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on a family of curves $\mathcal{X}/B$ we obtain as "limit linear series" a pair $\{(L_Y, V_Y), (L_Z, V_Z)\}$ on the "limit" curve X. We emphasize that is possible to construct pairs on X as "limits" of linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on $\mathcal{X}_\eta$, which aren't like the last pair above construct by Eisenbud and Harris (see for instance [14], Theorem 1). One of the first results by Eisenbud and Harris is the following inequalities

Proposition 2.4 ([9] proposition 2.1)**.** *If (L_Y, V_Y) and (L_Z, V_Z) arise as the limit of the linear series (L_η, V_η) on $\mathcal{X}_\eta$, and ϵ_i^Y, ϵ_i^Z are the orders of vanishing at P of (L_Y, V_Y) and (L_Z, V_Z), respectively, then holds that, for each $i = 0, \ldots, r$,*

$$\epsilon_i^Y + \epsilon_{r-i}^Z \geq d. \tag{2.1}$$

Thus, they were motivated to give the definition:

Definition 2.5 ([9] p. 346)**.** Let X the curve with two smooth components Y and Z meeting at a unique node P. The pair $\mathfrak{g} := \{(L_Y, V_Y), (L_Z, V_Z)\}$ is called a limit linear series (briefly lls) on X if it satisfies (2.1).

They call the lls **Refined** the ramification conditions (2.1) are equalities for all i. Otherwise, they call the lls **Crude**. The reason for the distinction is that refined lls on X have a similar behavior to linear series on smooth curves.

Having made the distinction, their results focus on lls refined, like the proposition below, about their characterization with respect to specialization of ramification points, as a consequence of the Plucker formula in this context:

Proposition 2.6 ([9] proposition 2.5)**.** *Under the hypotheses of proposition 2.4, $\mathfrak{g}$ is a refined lls if and only if no ramifications points of (L_η, V_η) specialize to the node P of X.*

Under these circumstances, they prove how from $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ there arises a refined lls on X or (possibly) on a semi-stable model X'. X' is derived from X by replacing the node P with a chain of rational smooth curves. So, the new family $\mathcal{X}'/B'$, with the special fiber X' and generic fiber $\mathcal{X}'_\eta = \mathcal{X}_\eta$, is obtained from $\mathcal{X}/B$ by making a finite base change and resolving the resulting singularities of the total space $\mathcal{X}$.

Theorem 2.7 ([9] Theorem 2.6)**.** *Let $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ be a linear series on $\mathcal{X}_\eta$. Up to a finite base change and finitely many blowups of the total space at nodes of the special fiber, the limit of $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on X is a refined lls.*

To sum up, limits of linear series on smooth curves yield limit linear series on X. A natural question is: are all limit linear series $\mathfrak{g}$ on X *smoothable*? The answer is, in general, no (see for instance, [9] example 3.2 p. 353). To say that $\mathfrak{g}$ *can be smoothed* means that there are a family $\mathcal{X}/B$ as above, and a linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on $\mathcal{X}_\eta$ whose limit is the given $\mathfrak{g}$ on X.

Eisenbud and Harris are able to construct a moduli space of refined limit linear series on a family of curves as above. Since this moduli space just parametrizes refined limit linear series it is not proper in general, as crude limit linear series are disregarded. We emphasize that the following theorem is an adapted version to our situation of one main results of Eisenbud and Harris (cf. [9] Theorem 3.3). Before stating the theorem, we fix some notation. Given two non negative integers r and d, a *ramification sequence of type* (r, d) is a sequence of integers $\epsilon = (\epsilon_0, \ldots, \epsilon_r)$ with $0 \leq \epsilon_0 \leq \ldots \leq \epsilon_r \leq d - r$. Suppose that q is a smooth point of X and is contained in the component Z. We say that a limit linear series $\mathfrak{g} = \{(L_Y, V_Y), (L_Z, V_Z)\}$ on X satisfies the ramification condition (q, ϵ) if the ramification sequence of $\mathfrak{g}$ at q, i.e., the ramification sequence of L_Z at q, is termwise $\geq (\epsilon_0, \ldots, \epsilon_r)$.

If $\epsilon^1, \ldots, \epsilon^s$ are ramification sequences of type (r, d) we set:

$$\rho(g, r, d; \epsilon^1, \ldots, \epsilon^s) := (r+1)(d-r) - rg - \sum_{i,j} \epsilon_i^j.$$

Following the words of Eisenbud and Harris ([9], p. 354): "this is the "expected dimension" of the family of limit series on X satisfaying ramification conditions (q_i, ϵ^i) for fixed $q_1, \ldots, q_s \in X$".

Theorem 2.8. *Let $\mathcal{X}/B$ be a regular smoothing of X, with $q_1, \ldots, q_s : B \to \mathcal{X}$ smooth sections, and let $\epsilon^1, \ldots, \epsilon^s$ be ramification sequences of type (r, d). There exists a scheme $G := G_d^{r,\mathrm{EH}}(\mathcal{X}/B; (q_1, \epsilon^1), \ldots, (q_s, \epsilon^s))$ quasi-projective over B, compatible with base change, parametrizing linear series on the smooth fiber $\mathcal{X}_\eta$ of $\mathcal{X}$, and refined limit linear series on X, all of degree d and rank r, satisfying the ramification conditions $(q_1(b), \epsilon^1), \ldots, (q_s(b), \epsilon^s)$, for any $b \in B$. Every component of G has dimension $\geq \rho + 1$. If*

$$\sum_{i,j} \epsilon_i^j = (r+1)d + \binom{r+1}{2}(2g-2),$$

or no X has crude limit linear series with the established ramification conditions, then G is proper over B.

Thus, Eisenbud and Harris obtain the smoothing result below

Corollary 2.9 ([9], corollary 3.5)**.** *Under the hypotheses of Theorem 2.8, if $G_d^{r,\mathrm{EH}}(X)$ has dimension exactly ρ, then all refined limit linear series can be smoothed to linear series on nearby fibers.*

Finally, we introduce the Eisenbud–Harris limit linear series scheme on X:

$$G_d^{r,\mathrm{EH}}(X) := \{((L_Y, V_Y), (L_Z, V_Z)) \,|\, \text{satisfying (2.1)}\} \tag{2.2}$$

is a projective subscheme of the product $G_d^r(Y) \times G_d^r(Z)$. In fact, $G_d^{r,\mathrm{EH}}(X)$ is the closed subscheme of the product defined as the union of the schemes defined by pairs of linear series satisfying at least the imposed vanishing conditions at the node. Clearly, $G_d^{r,\mathrm{EH}}(X)$ includes the crude and refined limit linear series. The set of refined limit linear series is an open subset. We emphasize that there are degenerations of linear series which are not refined.

2.1.3 The limit linear series Space II: The Osserman approach.

Concerning the construction of the Eisenbud–Harris quasi-projective scheme $G_d^{r,\mathrm{EH}}(\mathcal{X}/B)$, there are two problems. The first is the absence of "naturality" of the construction, in the sense that it does not represent any natural functor. The second is the exclusion in the relative case, of the crude limit linear series, which prevents the properness of $G_d^{r,\mathrm{EH}}(\mathcal{X}/B)$.

Observe that, even though $G_d^{r,\mathrm{EH}}(X)$ parametrizes the crude and refined limit linear series, it is not at all obvious that it has a functorial description.

We may say that the Osserman approach is motivated by these two problems. Roughly speaking, Osserman's construction of the moduli space of limit linear series, besides considering the "extremal" degrees as in the Eisenbud–Harris approach, considers attach all possible degrees.

We will discuss below the construction of moduli space of limit linear series à la Osserman, as well as its limitations. Its limitations in a sense that will be explained in the next sections, will motivate our new approach.

Given a regular smoothing of X, $\pi : \mathcal{X} \to B$, we have seen above that for any linear series $(\mathcal{L}_\eta, \mathcal{V}_\eta)$ on $\mathcal{X}_\eta$ of degree d and rank r, there exists a unique extension to X, $(\mathcal{L}_i, \mathcal{V}_i)$, satisfying that $\mathcal{L}_i$ has degree $d - i$ when restricted to Y and degree i when restricted to Z, and $\mathcal{V}_i = \mathcal{V}_\eta \cap H^0(\mathcal{X}, \mathcal{L}_i)$. Since $\mathcal{L}_{i+1} = \mathcal{L}_i(Y)$, we have a natural induced map $\mathcal{L}_i \to \mathcal{L}_{i+1}$. On the other hand, fixing the choice of an isomorphism $\mathcal{O}_\mathcal{X}(Y + Z) \cong \mathcal{O}_\mathcal{X}$, we obtain a map in the reverse direction $\mathcal{L}_{i+1} = \mathcal{L}_i(Y) \cong \mathcal{L}_i(-Z) \to \mathcal{L}_i$. If t is uniformizer of B, we have that the compositions, $\mathcal{L}_i \to \mathcal{L}_{i+1} \to \mathcal{L}_i$ and $\mathcal{L}_{i+1} \to \mathcal{L}_i \to \mathcal{L}_{i+1}$, are equals to multiplication by t. For the spaces $\mathcal{V}_i$, since $\mathcal{V}_i \cap H^0(\mathcal{X}, \mathcal{L}(-(i+1)Z) = \mathcal{V}_{i+1}$, the induced maps on the global sections $H^0(\mathcal{X}, \mathcal{L}_{i+1}) \to H^0(\mathcal{X}, \mathcal{L}_i)$ and $H^0(\mathcal{X}, \mathcal{L}_i \to H^0(\mathcal{X}, \mathcal{L}_{i+1})$, map $\mathcal{V}_{i+1}$ to $\mathcal{V}_i$ and $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$. Denote by $\varphi_i : \mathcal{V}_{i+1} \to \mathcal{V}_i$ and $\varphi^i : \mathcal{V}_i \to \mathcal{V}_{i+1}$ these maps.

The Osserman moduli space $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$ parametrizes choices of an invertible sheaf $\mathcal{L} =: \mathcal{L}_0$ of bi-degree $(d, 0)$, together with $(r+1)-$dimensional subspaces $\mathcal{V}_i$ of global sections of the invertible sheaves $\mathcal{L}_i := \mathcal{L}_0(iY)$, respectively, which are linked by the maps φ_i and φ^i above. Osserman imposes ramification points along smooth sections, similarly to what Eisenbud–Harris did and proves that

Theorem 2.10 (cf. [37], Theorem 5.3)**.** *Given a regular smoothing $\mathcal{X}/B$ of X, smooth sections p_i and ramifications sequences $\{\alpha_i(p_j)\}$, the functor $\mathfrak{G}_d^r(\mathcal{X}/B)$ of limit series on $\mathcal{X}/B$ having ramification index at least $\alpha_i(p_j)$ at each p_j is compatible with base change, and representable by a scheme $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$ projective over B. Every component of $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$ has dimension at least $\rho(g, r, d; \alpha_i(p_j)) + dimB$, with $\rho(g, r, d; \alpha_i(p_j)) = (r+1)(d-r) - rg - \sum_{i,j} \alpha_i(p_j)$.*

If the dimension of a fiber of $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$ is exactly $\rho(g, r, d; \alpha_i(p_j))$, then every limit linear series on that fiber can be smoothed to linear series on nearby fibers.

We will to describe what $G_d^{r,\mathrm{Oss}}(\mathcal{X}/B)$ parametrizes on its fibers over B. If $b \neq 0$, then $\mathcal{X}_b$ is a smooth curve. Since, as it is easy to check, the maps $\mathcal{L}_i|_{\mathcal{X}_b} \to \mathcal{L}_{i+1}|_{\mathcal{X}_b}$ are all isomorphisms, each V_i is uniquely determined by V_0. Thus, we recover the classical space G_d^r of $\mathcal{X}_b$.

As for the curve X, we have that each invertible sheaf $L_i := \mathcal{L}_i|_X$ on X is determined by its restrictions to Y and Z. It is clear that $L_i|_Z = \mathcal{L}_0(iY)|_Z \cong L_0|_Z(ip)$ and similarly $L_i|_Y = \mathcal{L}_0|_Y(-iZ) \cong L_0|_Y(-ip)$. The maps $L_i \to L_{i+1}$ are defined as the canonical inclusion on Z, and the zero map on Y, and viceversa for the maps $L_{i+1} \to L_i$. In this way, denoting by V_i the image of $\mathcal{V}_i$ by the maps $H^0(X, \mathcal{L}_i) \to H^0(X, L_i)$, the statement that V_i is mapped into V_{i+1} (resp. V_{i+1} is mapped into V_i) is equivalent to that the spaces $V_i|_Z$ (resp. $V_i|_Y$) can be regarded as an increasing (resp. decreasing) filtration of $V_Z := V_d|_Z$ (resp. $V_Y := V_0|_Y$). However, in the words by Osserman ([39], p. 12): "V_i includes, in general, strictly more information than $V_i|_Z$ and $V_i|_Y$, as there are choices about how sections on Y and Z can be glued if they both vanish at p. This additional information will mean in particular that our space is not the same as the Eisenbud-Harris space".

From now on, we restrict our attention to the curve X. Given an invertible sheaf L on X, there are natural short exact sequences,

$$\begin{gathered}0 \longrightarrow L|_Y(-p) \longrightarrow L \longrightarrow L|_Z \longrightarrow 0\ .\\ \\ 0 \longrightarrow L|_Z(-p) \longrightarrow L \longrightarrow L|_Y \longrightarrow 0\end{gathered} \tag{2.3}$$

For each integer i, we define L_i as the invertible sheaf on X determined by the restrictions $L|_Y(-ip)$ and $L|_Z(ip)$. There exist natural maps $\varphi^i : L_i \to L_{i+1}$ and $\varphi_i : L_{i+1} \to L_i$, defined by composing

$$\begin{tikzcd} 0 \arrow[r] & L_{i+1}|_Z(-p) \arrow[r] \arrow[d, equal] & L_{i+1} \arrow[r] \arrow[d, shift left, "\varphi_i"] & L_{i+1}|_Y \arrow[r] \arrow[d, equal] & 0 \\ 0 & L_i|_Z \arrow[l] & L_i \arrow[l] \arrow[u, shift left, "\varphi^i"] & L_i|_Y(-p) \arrow[l] & 0 \arrow[l] \end{tikzcd} \tag{2.4}$$

Notice that $\varphi^i\varphi_i = \varphi_i\varphi^i = 0$ for every i.

Definition 2.11. A limit linear series (briefly, lls) on X of degree d and dimension r, for fixed integers d and r with $r \leq d$, is a collection, denoted $(L, V_0, \dots, V_d)$, consisting of an invertible sheaf L on degree d when restricted on Y and degree 0 when restricted on Z, and vectors subspaces $V_i \subseteq \Gamma(X, L_i)$ of dimension $r+1$, for each $i = 0, \dots, d$, such that $\varphi^i(V_i) \subseteq V_{i+1}$ and $\varphi_i(V_{i+1}) \subseteq V_i$ for each i.

Its clear that there exists a forgetful morphism

$$G_d^{r,\mathrm{Oss}}(X) := \{\mathfrak{g} := (L, V_0, \dots, V_d)|\ V_i \underset{\varphi_i}{\overset{\varphi^i}{\rightleftarrows}} V_{i+1}\ \} \longrightarrow G_d^r(Y) \times G_d^r(Z)$$

defined by $\mathfrak{g} \mapsto ((L_0|_Y, V_0|_Y), (L_d|_Z, V_d|_Z))$. Thus, Osserman is able to obtain the following comparison to Eisenbud–Harris space (2.2):

Theorem 2.12 ([39], Theorem 3.2.1)**.** *The morphism above induces a set-theoretic surjection $G_d^{r,\mathrm{Oss}}(X) \xrightarrow{\rho_0} G_d^{r,\mathrm{EH}}(X)$, which is an isomorphism over the open subscheme corresponding to refined limit series.*

A point of $G_d^{r,\mathrm{Oss}}(X)$ is called refined (resp. crude) if it maps to a refined (resp. crude) point of $G_d^{r,\mathrm{EH}}(X)$ by ρ_0. In particular, Osserman recovers the results of specialization and smoothing obtained by Eisenbud and Harris. Furthermore, a careful analysis of the dimension of the fibers of the map ρ_0 and a classical inductive reasoning over limit linear series, is carried out by Osserman to prove:

Theorem 2.13 ([37], Proposition 6.6, corollary 6.8 or [40] Theorem 5.3)**.** *The natural map above*

$$\rho_0 : G_d^{r,\mathrm{Oss}}(X) \longrightarrow G_d^r(Y) \times G_d^r(Z)$$

has set-theoretic image consisting precisely of $G_d^{r,\mathrm{E-H}}(X)$. This map is an isomorphism when restricted to the open subscheme of $G_d^{r,\mathrm{Oss}}(X)$ mapping to refined Eisenbud–Harris limit series.

Now, our interest is focused on the intrinsic properties of $G_d^{r,\mathrm{Oss}}(X)$. For this, we need some definitions.

Given a lls $(L, V_0, \ldots, V_d)$, associated to each V_i we have the short exact sequences:

$$
\begin{gathered}
0 \longrightarrow V_i^{Y,0} \longrightarrow V_i \longrightarrow V_i|_Y \longrightarrow 0 \\
\\
0 \longrightarrow V_i^{Z,0} \longrightarrow V_i \longrightarrow V_i|_Z \longrightarrow 0
\end{gathered}
\tag{2.5}
$$

where $V_i^{Y,0}$ denotes the subspace of sections of V_i that vanish on Y and $V_i|_Y$ denotes the image in $\Gamma(Y, L_i|_Y)$ of V_i by restriction. Similar conclusions can be drawn with Y replaced by Z. Besides, the map $\varphi^i : V_i \longrightarrow V_{i+1}$ has kernel $V_i^{Z,0}$ and image contained in $V_{i+1}^{Y,0}$, whereas $\varphi_i : V_{i+1} \longrightarrow V_i$ has kernel $V_{i+1}^{Y,0}$ and image contained in $V_i^{Z,0}$.

Definition 2.14. A lls $(L, V_0, \ldots, V_d)$ is called **exact** if, for every i,

$$\mathrm{Im}(\varphi^i : V_i \longrightarrow V_{i+1}) = V_{i+1}^{Y,0} = \mathrm{Ker}(\varphi_i : V_{i+1} \longrightarrow V_i)$$

$$\mathrm{Im}(\varphi_i : V_{i+1} \longrightarrow V_i) = V_i^{Z,0} = \mathrm{Ker}(\varphi^i : V_i \longrightarrow V_{i+1}).$$

The exactness condition can be translated in numerical terms: A lls $(L, V_0, \ldots, V_d)$ is exact if and only if $\mathrm{rank}(\varphi^i)+\mathrm{rank}(\varphi_i) = r$ for every i. So, since for any lls we have $\mathrm{rank}(\varphi^i)+\mathrm{rank}(\varphi_i) \leq r$, the exact points form (by semi-continuity) an open subset of $G_d^{r,\mathrm{Oss}}(X)$.

Examples of exact lls are the refined lls, as it is easy to check. However, the converse does not hold. An other important property of the set of exact points in the space $G_d^{r,\mathrm{Oss}}(X)$ in the case of a general curve is

Theorem 2.15 ([34])**.** *If X is a general curve then the exact points are dense in $G_d^{r,\mathrm{Oss}}(X)$.*

Let $\mathcal{X}/B$ be a regular smoothing of X. Let $\mathcal{L}_\eta$ be an invertible sheaf of degree d on the generic smooth fiber $\mathcal{X}_\eta$. We have seen above that there exists an extension $\mathcal{L}$ on $\mathcal{X}$ such that $L := \mathcal{L}|_X$ has degrees d on Y and 0 on Z. Fix this extension $\mathcal{L}$, and set $L_i := \mathcal{L}(iY)|_X \cong \mathcal{L}(-iZ)|_X$, for each $i = 0, \ldots, d$. Recall that the $\mathcal{L}_i := \mathcal{L}(iY)$ are all extensions of $\mathcal{L}_\eta$ to $\mathcal{X}$.
Let V_η be a vector subspace of $H^0(\mathcal{X}_\eta, \mathcal{L}_\eta)$ of dimension $r+1$. Regarding V_η as a subspace of $H^0(\mathcal{X}_\eta, \mathcal{L}(iY)|_{\mathcal{X}_\eta})$ for each $i = 0, \ldots, d$, set $\mathcal{V}_i := H^0(\mathcal{X}, \mathcal{L}(iY)) \cap V_\eta$. We denote by $V_i \subset H^0(X, L_i)$ the image of the restriction of $\mathcal{V}_i$ to the special fiber, in other words, the image by the natural induced map on the global sections $\mathcal{V}_i \hookrightarrow H^0(\mathcal{X}, \mathcal{L}_i) \to H^0(X, L_i)$. It follows that

Proposition 2.16. *The collection $(L, V_0, \ldots, V_d)$ is an exact lls.*

Proof 2.17. First, from of the "product" maps,

$$\mathcal{L}_i \xrightarrow{\otimes Z} \mathcal{L}_{i-1} \qquad \mathcal{L}_{i-1} \xrightarrow{\otimes Y} \mathcal{L}_i. \tag{2.6}$$

we have the short exact sequence $0 \longrightarrow \mathcal{L}_i \xrightarrow{\otimes Z} \mathcal{L}_{i-1} \longrightarrow L_{i-1}|_Z \longrightarrow 0$ and similarly for Y. So, from the induced maps of restrictions to X we obtain the short exact sequence $0 \to L_i|_Y \to L_{i-1} \to L_{i-1}|_Z \to 0$ of the diagram below

$$\begin{tikzcd} L_i \arrow[r] \arrow[d, two heads] & L_{i-1} \arrow[r] & L_{i-1}|_Z \arrow[r] & 0 . \\ L_i|_Y \arrow[r, two heads, "\cong"] & L_{i-1}|_Y(-p) \arrow[u, hook] & & \end{tikzcd} \tag{2.7}$$

Second, by the natural induced maps below, and chasing in diagrams,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V}_i \arrow[r] \arrow[d, hook] & \mathcal{V}_{i-1} \arrow[r] \arrow[d, hook] & H^0(X, L_{i-1}|_Z) \arrow[d, equal] \\ 0 \arrow[r] & H^0(X, \mathcal{L}_i) \arrow[r] \arrow[d] & H^0(X, \mathcal{L}_{i-1}) \arrow[r] \arrow[d] & H^0(X, L_{i-1}|_Z) \arrow[d, equal] \\ & H^0(X, L_i) \arrow[r] & H^0(X, L_{i-1}) \arrow[r] & H^0(X, L_{i-1}|_Z) \\ & V_i \arrow[u, hook] \arrow[r, hook, "\varphi_{i-1}"] & V_{i-1} \arrow[u, hook] & \end{tikzcd} \tag{2.8}$$

We conclude, if we denote by φ_{i-1} and φ^{i-1} the induced maps on the global sections by the products above, then they satisfy by (2.7) and (2.8) that $\varphi_{i-1}\varphi^{i-1}$ and $\varphi_{i-1}(V_i) \subseteq V_{i-1}$, respectively. A similar reasoning applies to obtain $\varphi^{i-1}\varphi_{i-1} = 0$ and $\varphi^{i-1}(V_{i-1}) \subset V_i$.

On the other hand, from the two natural sequences below

$$0 \to \mathcal{V}_{i-1} \to \mathcal{V}_i \to H^0(X, L_i|_Y)$$

$$H^0(X, L_{i-1}) \to H^0(X, L_{i-1}|_Z) \hookrightarrow H^0(X, L_i)$$

we obtain that

$$\mathrm{Ker}(H^0(X, L_{i-1}) \to H^0(X, L_{i-1}|_Z)) = \mathrm{Ker}(H^0(X, L_{i-1}) \to H^0(X, L_i))$$

and consequently $\mathrm{Ker}(V_{i-1} \to H^0(X, L_{i-1}|_Z)) = \mathrm{Ker}(V_{i-1} \xrightarrow{\varphi^{i-1}} V_i)$.

Thus, by chasing in diagrams again,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_i \arrow[r] \arrow[d, two heads] & \mathcal{V}_{i-1} \arrow[r] \arrow[d, two heads] & H^0(X, L_{i-1}|_Z) \arrow[dd, equal] \\
& V_i \arrow[r, "\varphi_{i-1}"] \arrow[d, hook] & V_{i-1} \arrow[d, hook] & \\
& H^0(X, L_i) \arrow[r] & H^0(X, L_{i-1}) \arrow[r] & H^0(X, L_{i-1}|_Z)
\end{tikzcd}$$

we conclude that $\mathrm{Im}(\varphi_{i-1}) = \mathrm{Ker}(\varphi^{i-1})$, which is the desired conclusion.

Finally, one of the key properties of exact limit linear series is the following.

Lemma 2.18 ([37], Lemma A.12)**.** *If $(L, V_0, \ldots, V_d)$ is an exact LLS on X, then there exist integers $0 \leq i_0 \leq i_1 \leq \cdots \leq i_r \leq d$ and sections $s_0, \ldots, s_r$ with $s_j \in V_{i_j}$ such that for each $i = 0, \ldots, d$, the s_j with $i_j = i$ form a basis of $V_i/V_i^{Y,0} \oplus V_i^{Z,0}$ and the iterated images of all the s_j form a basis for V_i.*

In conclusion, we may assert that the exact points are the natural candidates to generalize in Ossermans moduli space of lls the refined lls a la Eisenbud and Harris. In fact, on one hand every smoothing of a linear series along regular smoothing correspond to an exact point in $G_d^{r,\mathrm{Oss}}(X)$ and, on the other hand the lemma 2.18 yields that all relevant information of the ramification values on the node $P \in X$ over the exact points, are completly determined for certain degrees. More explicitly, a refined lls in Osserman's space, which by definition is isomorphic to the refined lls in Eisenbud–Harris space by ρ_0 at the Theorem 2.13, is determined for "extremal degree" vector subspaces V_0 and V_d. Thus, the lemma 2.18 asserts that for each exact lls there exist a "collection of degrees" such that the vector subspaces of sections $\{V_{i_j}\}$ determine completly the rest of sections in the lls. In addition, since the refined points are included properly in the set of exact points, we have a structural difference between the compactifications $G_d^{r,\mathrm{Oss}}(X)$ and $G_d^{r,\mathrm{EH}}(X)$ (see, for instance Example 3.1 in the chapter 3).

We emphasize, as it follows from 2.13 and 2.15, there exist points in $G_d^{r,\mathrm{Oss}}(X)$ that can not be smoothed. In other words, Osserman space of limit linear series does not resolve the question put by Eisenbud and Harris about smoothing linear series in [12].

2.2 Abel Maps and their relation to limit linear series.

Concerning the classical theory of linear series on smooth curves, there are two relevant concepts, which are intimately related to the fibers of the Abel(-Jacobi) map: complete linear series and families of effective divisors. Specifically, given a linear series $\mathfrak{g} := (L, V)$, with degree d and rank $r+1$ on the smooth curve X, we may associate to each $s \in V \setminus \{0\}$ its *zero divisor on* X, $\mathrm{div}(s)$. This defines an effective divisor on X of degree d, equivalently, a point of the symmetric product

$X^{(d)}$. Since $s \in V \setminus \{0\}$ is uniquely determined by $\mathrm{div}(s)$ up to scaling, we have that the family of effective divisors associated to $\mathfrak{g}$ forms a r-dimensional projective subspace of $X^{(d)}$. Actually, the inclusion $\mathbb{P}(V) \hookrightarrow X^{(d)}$ factors through $\mathbb{P}(H^0(X, L))$, the projective space associated to the complete linear series on X. We can view the space $\mathbb{P}(H^0(X, L))$ as a fiber of the d Abel map of X. Explicitly, $A_d^{-1}(L(-dp)) = \mathbb{P}(H^0(X, L))$, where

$$\begin{array}{cccc} A_d & : & X^{(d)} & \longrightarrow & J_X \\ & & \sum_{i=1}^{d} q_i & \mapsto & [\mathcal{O}_X(\sum_{i=1}^{d} q_i - dp)], \end{array}$$

is the degree d Abel map of X, where J_X is the Jacobian of X, parametrizing line bundles of degree 0 on X, and $p \in X$ is any base point. For each $\mathfrak{g}$ we denote the image at the inclusion $\mathbb{P}(V) \hookrightarrow X^{(d)}$ by $\mathbb{P}(\mathfrak{g})$.

The approach above with respect to A_d can be viewed more technically. For this, consider the Hilbert polynomial $p(t) := \binom{r+t}{r}$ and the relative ample divisor $H := p + X^{(d-1)} \subset X^{(d)}$, parametrizing families of effective divisors whose support contains the point p. Thus, we obtain a projective scheme, called the relative Hilbert scheme of the map A_d:

$$\mathrm{Hilb}_{A_d}^{\binom{r+t}{r}, H} := \{Y \subset A_d^{-1}(L) / \, P_Y(t) = \binom{r+t}{r}\},$$

which parametrizes the $r-$dimensional projective subspaces of the fibers of the Abel map A_d. So, the assignment $\mathfrak{g} \mapsto \mathbb{P}(\mathfrak{g})$ can be translated to the map:

$$\begin{array}{cccc} \Phi & : & G_d^r(X) & \longrightarrow & \mathrm{Hilb}_{A_d}^{\binom{r+t}{r}, H} \\ & & \mathfrak{g} & \mapsto & \mathbb{P}(\mathfrak{g}). \end{array}$$

The Abel Theorem arising from the study by Abel of sums of integrals along nonrational curves, can now be interpreted as

Theorem 2.19 (**Abel**. [19], Chapter 9)**.** *The map Φ is an isomorphism.*

In particular, the assignment $\mathfrak{g} \mapsto \mathbb{P}(\mathfrak{g})$ behaves well in families. Moreover, if we have a family of linear series on a family of smooth curves, we obtain a flat family of closed subschemes of the fibers of the Abel map A_d.
What about this relation on singular curves? As it was mentioned before, our study focuses on curves of compact type X, with two smooth components Y and Z meeting at the disconnecting node p. On one hand, we have seen Osserman's construction of $G_d^{r,\mathrm{Oss}}(X)$, although there is no evident concept of complete limit linear series nor of families of effective divisors associated to a limit linear series. On the other hand, in ([4]) Coelho and Pacini construct the Abel maps

$$\begin{array}{cccc} A_d & : & X^{(d)} & \longrightarrow & \mathrm{Pic}^d(X) \\ & & \sum_{i=1}^{d} q_i & \mapsto & [\mathcal{O}_X(\sum_{i=1}^{d} q_i)], \end{array}$$

where $\mathrm{Pic}^d(X)$ is the Picard scheme of lines bundles over X of bi-degree (d_1, d_2) such that $d_1 + d_2 = d$. However, the fibers of A_d are not well behaved. For instance, they are in general not even equidimensional. So fibers of the Abel map A_d do not constitute a flat family, not even for $d >> 0$, and cannot be seen as flat limits of fibers of Abel maps of the smooth curves degenerating to X.

2.2.1 The Esteves–Osserman results

Esteves and Osserman obtain in ([18]), an important result on the relationship between limit linear series and fibers of Abel maps in the last context. Before going into details, we make some general remarks about their construction.

First, for their purposes, they consider the Abel map:

$$A_d : X^{(d)} \longrightarrow \mathrm{Pic}^d(X)/\sim$$

where $\mathrm{Pic}^d(X)/\sim$ parametrizes all line bundles of total degree d up to twisting up and down by p. Precisely, two lines bundles L_1 and L_2 on X are said to be equivalent if there exists an integer j such that $L_1|_Y \cong L_2|_Y(-jp)$ and $L_1|_Z \cong L_2|_Z(jp)$. Recall that $\mathrm{Pic}^d(X) \cong \mathrm{Pic}^{d-i}(Y) \times \mathrm{Pic}^i(Z)$ for any integer i. Thus, the map A_d is given as follows: If $D = D_Y + D_Z$ is an effective divisor of degree d on X, where D_Y and D_Z are effective divisors supported on Y and Z respectively, then $A_d(D)$ is the class of the line bundle whose restrictions to Y and Z are, respectively, $\mathcal{O}_Y(D_Y)$ and $\mathcal{O}_Z(D_Z)$. Observe that the only ambiguity is if D_Y or D_Z contains the node P, resulting different decompositions of D. But, this is exactly the meaning of the twist. So, $A_d(D)$ does not depend on how D is decomposed.

Second, observe that if $(L, V) := ((L_Y, V_Y), (L_Z, V_Z))$ is a lls a la Eisenbud–Harris, it is not immediate how to associate to a section in some vector space $V's$ an effective divisor on X, or better, a subscheme of $X^{(d)}$. A reasonable attempt is as follows: For each i, and each $s_Y \in V_Y$ and $s_Z \in V_Z$, vanishing to order at least i and $d-i$ at p, respectively, we can associate $\mathrm{div}(s_Y) - ip + \mathrm{div}(s_Z) - (d-i)p$. However, the subscheme of $X^{(d)}$ parametrizing these divisors is not well behaved. Precisely, if (L, V) is a lls refined, i.e., we can find basis of V_Y and V_Z in correspondence to each other such that corresponding to s_Y and s_Z vanish to order exactly i and $d-i$ at p, respectively, the behavior will be good. Otherwise, in the crude case, we may get non-equidimensional subschemes.

Now, Esteves-Osserman approach is to associate to a lls $\mathfrak{g} := (L, V_0, \ldots, V_d)$ on X a subscheme of the fiber $A_d^{-1}(L)$, which is defined as follows:

$$\mathbb{P}(\mathfrak{g}) := \overline{\{\mathrm{div}(s|_Y) + \mathrm{div}(s|_Z) |\, s \in V_i \setminus V_i^{Y,0} \cup V_i^{Z,0},\ i = 0, \ldots, d\}}.$$

(Recall that for any V_i we have that the two short exact sequences (2.5), which define $V_i^{Y,0}$ and $V_i^{Z,0}$). They prove that the $\mathbb{P}(\mathfrak{g})$ are well behaved when $\mathfrak{g}$ is a exact point of $G_d^{r,\mathrm{Oss}}(X)$.

Specifically,

Theorem 2.20 ([18], Theorem 4.3)**.** *If $\mathfrak{g} = (L, V_0, \ldots, V_d)$ is an exact limit linear series on X of degree d and rank r, then $\mathbb{P}(\mathfrak{g})$ is reduced, connected and Cohen–Macaulay of pure dimension r, has bivariate Hilbert polynomial $P(s,t) = \binom{r+s+t}{r}$ and is a flat degeneration of $\mathbb{P}^r$.*

They also show that $\mathbb{P}(\mathfrak{g})$ is a flat limit when $\mathfrak{g}$ is a degeneration. More precisely, recall that given a regular smoothing $\mathcal{X}/B$ of X, and linear series on the generic fiber, it is possible to construct an exact lls $\mathfrak{g} = (L, V_0, \ldots, V_d)$, which is the limit of the linear series on the generic fiber.

Theorem 2.21 ([18], Theorem 5.2)**.** *Let $\mathcal{X}/B$ be a regular smoothing of X and (L_η, V_η) a linear series of rank r and degree d on the generic fiber. Let $\mathfrak{g}$ be the limit linear series that is limit of (L_η, V_η). Then $\mathbb{P}(V_\eta)$, viewed as a subscheme of the fiber of the relative symmetric product $S^d(\mathcal{X}/B)$ over η, has closure intersecting $S^d(X)$ in $\mathbb{P}(\mathfrak{g})$.*

3 The level-δ limit linear series and its moduli space

The aim of this section is to bring together two constructions as an attempt to understand limit linear series on nodal curves with two smooth components meeting at a unique node as limits of linear series on smooth curves along families of curves whose total space is not regular. The first of these constructions is Osserman's space of limit linear series. The second is the "twisting" used by Cumino, Esteves and Gatto as an important tool for studying degenerations of linear series on families whose total space is not regular.

Our study is motivated by the relationship between limit linear series and Abel fibers. More precisely, we generalize the (set-theoretic) assignment $\mathfrak{g} \mapsto \mathbb{P}(\mathfrak{g})$ found by Esteves and Osserman, from exact limit linear series $\mathfrak{g}$ to equidimensional, Cohen–Macaulay subschemes $\mathbb{P}(\mathfrak{g})$ of fibers of Abel maps which are flat degenerations of $\mathbb{P}^r$.

In the first subsection we review certain standard facts on smoothings and twists. In the second subsection, we introduce the notion of level-δ limit linear series and proceed with the construction of a projective scheme parametrizing level-δ limit linear series. Everything, as in Osserman's works, will be done for a nodal curve with two smooth components meeting at a unique node.

3.1 Smoothings and Twists

Let X be a nodal curve with two irreducible smooth components Y and Z intersecting (transversally) at a single point P. A *smoothing* of X is a flat, projective map $\pi_\delta \colon \mathcal{X} \to B$ to $B := \mathrm{Spec}(\mathbb{C}[[t]])$ with smooth generic fiber X_η, and special fiber X_0 isomorphic to X. Notice that X_η is defined over the field of Laurent series $\mathbb{C}((t))$ and is not only smooth, but also geometrically connected, by semicontinuity. We will identify X_0 with X.

The total space $\mathcal{X}$ is regular except possibly at the node P. However, since the general fiber is smooth, there are a positive integer δ and a $\mathbb{C}[[t]]-$algebra isomorphism (see, [16] pp. 92–93):

$$\widehat{\mathcal{O}}_{\mathcal{X},P} \cong \frac{\mathbb{C}[[t,y,z]]}{(yz - t^\delta)}. \tag{3.1}$$

The integer δ is called the *singularity degree of* π_δ *at* P. (Also, we say that the singularity of $\mathcal{X}$ at P is of type $A_{\delta-1}$.) It will profoundly affect our constructions, for which reason we decided to make it stand out as an index of the map. We say that π_δ is a *regular smoothing* if its singularity type at P is 1, in other words, if $\mathcal{X}$ is regular.

In general, the components Y, Z are Weil divisors, but not necessarily Cartier divisors on $\mathcal{X}$. However, as will be shown below, there exist natural effective Cartier divisors on $\mathcal{X}$, whose associated 1-cycles are $\delta[Y]$ and $\delta[Z]$, respectively.

Lemma 3.1. *Let $\pi_\delta\colon \mathcal{X} \to B$ be a smoothing of X. The schematic closure of the effective Cartier divisor on $\mathcal{X} - P$ with local equations 1 on $\mathcal{X} - Y$ (resp. $\mathcal{X} - Z$) and t^δ on $\mathcal{X} - Z$ (resp. $\mathcal{X} - Y$) is an effective Cartier divisor of $\mathcal{X}$.*

Proof 3.2. We need only show that the closure is given by a nonzero divisor at the node P. We will deal with the first case, the second being completely analogous. Fix an isomorphism of the form (3.1), and set $A := \mathbb{C}[[t, y, z]]/(yz - t^\delta)$. Let $I \subset A$ be the ideal defining the closure at P. Then Y is given, say, by the ideal (y, t), and thus $\sqrt{I} = (y, t)$, i.e., the unique associated prime of I is (y, t). On the other hand, $I_z = t^\delta A_z$. So, since $z \notin (y, t)$, we have $I = I_z \cap A$. We claim that $I = (y)$. Indeed, if $g \in I_z \cap A$ then there are an integer $r \geq 0$ and $h \in A$ such that $z^r g = t^\delta h$. If $r = 0$ then $g \in (t^\delta) \subseteq (y)$. Otherwise, $z^{r-1} g = yh$. Moreover, since y and z form a regular sequence in A, it follows that $g \in (y)$. Conversely, $y = t^\delta / z \in I$.

We let δY (resp. δZ) denote the effective Cartier divisor of $\mathcal{X}$ whose existence is asserted by the above lemma.

Proposition 3.3. *Let $\pi_\delta\colon \mathcal{X} \to B$ be a smoothing of X. Then*

(a) Y (resp. Z) is a Cartier divisor of $\mathcal{X}$ if and only if $\delta = 1$.

(b) (Local intersection multiplicity) $\delta Y \cdot Z = \delta Z \cdot Y = 1$.

(c) $\mathcal{O}_{\mathcal{X}}(\delta Y)|_X \cong \mathcal{O}_X(Y)$ and $\mathcal{O}_{\mathcal{X}}(\delta Z)|_X \cong \mathcal{O}_X(Z)$.

Proof 3.4. The first two statements follow from the proof of Lemma 3.1. Indeed, fixing an isomorphism of the form (3.1), we have that Y (resp. Z) is given at P by, say, (y, t) (resp. (z, t)) and δY (resp. δZ) by y (resp. z). Of course, (y, t) (resp. (z, t)) is principal if and only if $\delta = 1$, proving (a). Furthermore,

$$\delta Y \cdot Z = \delta Z \cdot Y = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[[t, y, z]]}{(t, y, z)} \right) = 1,$$

proving (b). As for (c), note first that $\mathcal{O}_{\mathcal{X}}(-\delta Y)|_Z \cong \mathcal{O}_Z(-P)$. Indeed, the natural short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}(-\delta Y)|_Z \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{\delta Y \cap Z} \longrightarrow 0$$

corresponds, by (b), to the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}(-\delta Y)|_Z \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

From this and the Five Lemma, by the comparing the above exact sequence with the natural exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-P) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

we obtain the isomorphism. Second, from the flatness of π_δ and Lemma 3.1, it follows that

$$\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}(X) \cong \mathcal{O}_{\mathcal{X}}(\delta Y + \delta Z) \cong \mathcal{O}_{\mathcal{X}}(\delta Y) \otimes \mathcal{O}_{\mathcal{X}}(\delta Z).$$

Thus $\mathcal{O}_{\mathcal{X}}(-\delta Y) \cong \mathcal{O}_{\mathcal{X}}(\delta Z)$. Since $\mathcal{O}_{\mathcal{X}}(-\delta Z)|_Y \cong \mathcal{O}_Y(-P)$ by analogy, we get $\mathcal{O}_{\mathcal{X}}(-\delta Y)|_Y \cong \mathcal{O}_Y(P)$. Thus $\mathcal{O}_{\mathcal{X}}(-\delta Y)|_X \cong \mathcal{O}_X(-Y)$, from which follows the first isomorphism in (c). The second is derived analogously.

Remark 3.5. Let $\pi\colon \mathcal{X} \to B$ be a **regular** smoothing of X. (There is always one; see [6].) Let $\Delta : B \longrightarrow B$ the map given by sending t to t^δ. To differentiate source from target, we will denote the source of Δ by B_δ. According to [16] pp. 92–93, the fibered product $\mathcal{X}_\delta := \mathcal{X} \times_B B_\delta$ has general fiber over B_δ isomorphic to the base extension $X_\eta \times k$, and special fiber isomorphic to X. Here, as before, X_η denotes the general fiber of π, and k is a finite field extension of degree δ of $\mathbb{C}((t))$, the field of Laurent series over which X_η is defined. The projection $\pi_\delta : \mathcal{X}_\delta \to B_\delta$ is flat and proper, but $\mathcal{X}_\delta$ fails to be regular if $\delta > 1$; namely, $\mathcal{X}_\delta$ fails to be regular at P. In fact, after base change in the isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X},P} \cong \frac{\mathbb{C}[[t,y,z]]}{(yz-t)}. \tag{3.2}$$

we obtain an isomorphism as in (3.1).

Let $\pi_\delta\colon \mathcal{X} \to B$ be a smoothing of X and $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$. Set $\mathcal{L}_1 := \mathcal{L}(-\delta Z) := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-\delta Z)$. If $\mathcal{L}$ has degrees k on Y and $d-k$ on Z, then $\mathcal{L}_1$ is an invertible sheaf of degrees $k-1$ on Y and $d-k+1$ on Z. Indeed, it is enough to observe from Lemma 3.3 that $\mathcal{O}_{\mathcal{X}}(-\delta Z)$ has degrees -1 on Y and 1 on Z. For each integer $i \geq 0$, we may thus generate recursively a sequence

$$\mathcal{L}_i := \mathcal{L}(-i\delta Z) := \mathcal{L}_{i-1} \otimes \mathcal{O}_{\mathcal{X}}(-\delta Z)$$

of invertible sheaves on $\mathcal{X}$, having bidegree $(k-i, d-k+i)$ on X, i.e., degree $k-i$ on Y and degree $d-k+i$ on Z.

From now on, suppose that $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$ with degree d on Y and degree 0 on Z. From $\mathcal{L}_0 := \mathcal{L}$, we define a sequence of invertible sheaves on the curve X:

$$L_i := \mathcal{L}_i|_X, \quad \text{where } \mathcal{L}_i := \mathcal{L}(-i\delta Z) \text{ for } i = 0, 1 \dots$$

Here each L_i has bidegree $(d-i, i)$ on X. More precisely, it follows from Lemma 3.3 that $L_{i+1}|_Y = L_i|_Y(-P)$ and $L_{i+1}|_Z = L_i|_Z(P)$ for each $i = 0, 1, \dots$.

Our next topic is *twists*. Twists were introduced by Esteves in ([15], section 3). Nevertheless, it was only in [5], Section 3, that they were applied, by Cumino, Esteves and Gatto, to the study of limit linear series on families whose total space is not regular. Though the theory is more general, for our purposes we continue considering only the case of nodal curve X with two smooth components Y and Z meet at a unique node P.

Let $\pi_\delta\colon \mathcal{X} \to B$ be a smoothing of X of singularity degree δ at P. Set $\mathcal{I}_Z^{(0)} := \mathcal{O}_{\mathcal{X}}$, and for each integer $i > 0$, define the ***i*-th twist by** Z of $\mathcal{O}_{\mathcal{X}}$ by:

$$\mathcal{I}_Z^{(i)} := \ker\left(\mathcal{I}_Z^{(i-1)} \longrightarrow \frac{\mathcal{I}_Z^{(i-1)}|_Z}{\text{torsion}}\right).$$

Clearly, $\mathcal{I}_Z^{(1)}$ is simply the sheaf of ideals $\mathcal{I}_{Z|\mathcal{X}}$ of Z in $\mathcal{X}$. And, in general, $\mathcal{I}_Z^{(i)}$ is a sheaf of ideals which is locally principal away from P. More precisely, $\mathcal{I}_Z^{(i)} = \mathcal{I}_{Z|\mathcal{X}}^i$ on $\mathcal{X} - P$, by [5], p. 13. In particular, for each i, the sheaf $\mathcal{I}_Z^{(i)}$ has relative *rank 1* and is relatively *torsion-free* over B. Furthermore, we have by [5], Prop. 3.1, p. 13, the short exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_Z^{(i+1)}|_Y}{\text{torsion}} \longrightarrow \mathcal{I}_Z^{(i)}|_X \longrightarrow \frac{\mathcal{I}_Z^{(i)}|_Z}{\text{torsion}} \longrightarrow 0, \tag{3.3}$$

and the isomorphisms

$$\frac{\mathcal{I}_Z^{(i+1)}|_Y}{\text{torsion}} \cong \mathcal{O}_Y(-(q+1)P) \quad \text{and} \quad \frac{\mathcal{I}_Z^{(i)}|_Z}{\text{torsion}} \cong \mathcal{O}_Z(qP) \tag{3.4}$$

where q is the quotient of the Euclidean division of i by δ.

Besides, by [5], Section 3.1, pp. 13–15, we have that $\mathcal{I}_Z^{(i\delta)}$ is invertible for each i. Explicitly, $\mathcal{I}_Z^{(i\delta)} \cong \mathcal{O}_{\mathcal{X}}(-i(\delta Z))$ for each i. Otherwise,

Lemma 3.6. *$\mathcal{I}_Z^{(i\delta+j)}|_X$ is not simple for any i and any j with $1 \le j \le \delta - 1$,*

Proof 3.7. By [5], Subsection 3.1, we have the filtration

$$\mathcal{I}_Z^{((i+1)\delta)} \subset \mathcal{I}_Z^{(i\delta+\delta-1)} \subset \cdots \subset \mathcal{I}_Z^{(i\delta+1)} \subset \mathcal{I}_Z^{(i\delta)}$$

for each i and, in addition, under the isomorphism (3.1), using that, say, $\widehat{\mathcal{I}}_{Z|\mathcal{X},P} = (z,t)$, we have that $\widehat{\mathcal{I}}_{Z,P}^{(i\delta+j)} = (z^i, z^{i-1}t^j)$ for each $i \ge 0$ and $j = 1, \ldots, \delta-1$. In particular, $\mathcal{I}_Z^{(i\delta+j)}|_X$ is not invertible at the node P. Thus, since $\mathcal{I}_Z^{(i\delta+j)}|_X$ is torsion-free and has rank 1, it must decompose, i.e.,

$$\mathcal{I}_Z^{(i\delta+j)}|_X \cong \left(\mathcal{I}_Z^{(i\delta+j)}|_X\right)_Y \oplus \left(\mathcal{I}_Z^{(i\delta+j)}|_X\right)_Z$$

for each i and $j = 1, \ldots, \delta-1$, where $\left(\mathcal{I}_Z^{(i\delta+j)}|_X\right)_Y := \frac{\left(\mathcal{I}_Z^{(i\delta+j)}|_X\right)|_Y}{\text{torsion}}$.

Thus, $\mathcal{I}_Z^{(i\delta+j)}|_X$ is not simple.

Clearly, we may apply the construction and the reasoning above with Z replaced by Y.

We thus obtain from a given invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ with bidegree $(d, 0)$ on X, the following data on the curve X:

- A collection of $d+1$ invertible sheaves on X, namely, $L_{i\delta} := \mathcal{L}(-i(\delta Z))|_X$ with bidegree $(d-i,i)$ for $i=0,\dots,d$.
- A collection of $d(\delta-1)$ rank-1, torsion-free sheaves on X, namely, $L_{i\delta+j} := \mathcal{L}\otimes\mathcal{I}_Z^{i\delta+j}|_X$ isomorphic to $L_{(i+1)\delta}|_Y \oplus L_{i\delta}|_Z$ for $i=0,\dots,d-1$ and $j=1,\dots,\delta-1$.

Using the exact sequences (3.3) and the isomorphisms (3.4), we have short exact sequences linking the above sheaves as follows:

$$
\begin{gathered}
0 \longrightarrow \frac{L_{i\delta+1}|_Y}{(\text{torsion})} \longrightarrow L_{i\delta} \longrightarrow L_{i\delta}|_Z \longrightarrow 0 \\[2em]
0 \longrightarrow \frac{L_{i\delta+2}|_Y}{(\text{torsion})} \longrightarrow L_{i\delta+1} \longrightarrow \frac{L_{i\delta+1}|_Z}{(\text{torsion})} \longrightarrow 0 \\[2em]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\[2em]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\[2em]
0 \longrightarrow L_{(i+1)\delta}|_Y \longrightarrow L_{i\delta+\delta-1} \longrightarrow \frac{L_{i\delta+\delta-1}|_Z}{(\text{torsion})} \longrightarrow 0 \\[2em]
0 \longrightarrow \frac{L_{(i+1)\delta+1}|_Y}{(\text{torsion})} \longrightarrow L_{(i+1)\delta} \longrightarrow L_{(i+1)\delta}|_Z \longrightarrow 0
\end{gathered}
\tag{3.5}
$$

and isomorphisms

$$
\begin{aligned}
\frac{L_{i\delta+j}|_Y}{(\text{torsion})} &\cong L_{i\delta}|_Y(-P) \cong L|_Y(-(i+1)p) \cong L_{(i+1)\delta}|_Y \text{ for } j=1,\dots,\delta, \\
L_{i\delta+j} &\cong L_{(i+1)\delta}|_Y \oplus L_{i\delta}|_Z \text{ for } j=1,\dots,\delta-1, \\
\frac{L_{i\delta+j}|_Z}{(\text{torsion})} &\cong L_{i\delta}|_Z \cong L|_Z(iP) \text{ for } j=0,\dots,\delta-1,
\end{aligned}
\tag{3.6}
$$

where $L := \mathcal{L}|_X$.

Now, having in mind the isomorphisms above, we can compose maps coming from distinct exact sequences to get $2d\delta$ natural maps :

$$\varphi^{i\delta+j} : L_{i\delta+j} \longrightarrow L_{i\delta+j+1} \quad \text{and} \quad \varphi_{i\delta+j} : L_{i\delta+j+1} \longrightarrow L_{i\delta+j}$$

for each $i=0,\dots,d-1$ and $j=0,\dots,\delta-1$. These maps are described below, where the diagrams commute:

- Maps in the "first extreme:"

$$
\begin{tikzcd}
 & & L_{i\delta+1} \arrow[d, equal] & & \\
 & & L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y \arrow[d, shift left, "\varphi_{i\delta}"] \arrow[dr, "{(0,1)}"] & & \\
0 & L_{i\delta}|_Z \arrow[l] \arrow[ur, "{(1,0)}"] & L_{i\delta} \arrow[l] \arrow[u, shift left, "\varphi^{i\delta}"] & L_{(i+1)\delta}|_Y \arrow[l] & 0, \arrow[l]
\end{tikzcd}
\tag{3.7}
$$

  where $(1,0)$ and $(0,1)$ represent the natural inclusion and the natural projection, respectively;

- Maps in the "middle," for $j = 1, \dots, \delta - 2$:

$$
\begin{tikzcd}
L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y \cong L_{i\delta+j} \arrow[d, shift left, "\varphi^{i\delta+j} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"] \\
L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y \cong L_{i\delta+j+1}. \arrow[u, shift left, "\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \varphi_{i\delta+j}"]
\end{tikzcd}
\tag{3.8}
$$

- Finally, maps in the "second extreme:"

$$
\begin{tikzcd}
0 \arrow[r] & L_{i\delta}|_Z \arrow[r] & L_{(i+1)\delta} \arrow[r] \arrow[d, shift left, "\varphi_{(i+1)\delta-1}"] & L_{(i+1)\delta}|_Y \arrow[r] \arrow[dl, "{(0,1)}"] & 0 \\
 & & L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y \arrow[ul, "{(1,0)}"] \arrow[u, shift left, "\varphi^{(i+1)\delta-1}"] \arrow[d, equal] & & \\
 & & L_{i\delta+\delta-1}. & &
\end{tikzcd}
\tag{3.9}
$$

Notice that $\varphi^k \varphi_k = \varphi_k \varphi^k = 0$ for each $k = 0, \dots, d\delta - 1$.

3.2 Level-δ limit linear series and their moduli space

Recall that X is a nodal curve with two smooth irreducible components Y and Z meeting at a unique node P. Let d be a positive integer. Let $\mathrm{Pic}^d(X)$ denote the Picard scheme of X, parametrizing invertible sheaves of (total) degree d on X. We know that it decomposes as the disjoint union of the Picard schemes $\mathrm{Pic}^{d-i,i}(X)$ parameterizing invertible sheaves of bidegree $(d-i, i)$ on X. There exist isomorphisms $\mathrm{Pic}^{d-i,i}(X) \cong \mathrm{Pic}^{d-i}(Y) \times \mathrm{Pic}^i(Z)$ given by restriction. In other words, any invertible sheaf L on X is determined by its restrictions $L|_Y$ and $L|_Z$. Furthermore, L sits on short exact sequences like (2.3) and (2.4).

Let d and δ be positive integers and L an invertible sheaf on X. Define the invertible sheaves $L_{i\delta}$ on X for $i = 0, \dots, d$, whose restrictions to Y and Z are $L|_Y(-iP)$ and $L|_Z(iP)$. In particular, since $L_{(i+1)\delta}|_Y = L_{i\delta}|_Y(-P)$ and $L_{i\delta}|_Z = L_{(i+1)\delta}|_Z(-P)$, we have the short exact sequences below for each $i = 0, \ldots, d-1$:

$$\begin{gathered} 0 \longrightarrow L_{(i+1)\delta}|_Y \longrightarrow L_{i\delta} \longrightarrow L_{i\delta}|_Z \longrightarrow 0, \\ \\ 0 \longrightarrow L_{i\delta}|_Z \longrightarrow L_{(i+1)\delta} \longrightarrow L_{(i+1)\delta}|_Y \longrightarrow 0. \end{gathered} \tag{3.10}$$

For our purposes, and inspired by the constructions in the last subsection, we define as well the rank-1, torsion-free sheaves on X, for each $i = 0, \ldots, d-1$ and $j = 1, \ldots, \delta - 1$, given by $L_{i\delta+j} := L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y$, whose connection to the sequences in (3.10) is established by the maps

$$\begin{tikzcd} 0 \arrow[r] & L_{(i+1)\delta}|_Y \arrow[r] \arrow[d, equal] & L_{i\delta} \arrow[r] \arrow[d, shift left, "\varphi^{i\delta}"] & L_{i\delta}|_Z \arrow[r] \arrow[d, equal] & 0 \\ 0 & L_{(i+1)\delta}|_Y \arrow[l] & L_{i\delta+1} \arrow[l, "(0,1)"'] \arrow[u, shift left, "\varphi_{i\delta}"] & L_{i\delta}|_Z \arrow[l, "(1,0)"'] & 0, \arrow[l] \end{tikzcd} \tag{3.11}$$

$$\begin{tikzcd} L_{i\delta+j} = L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y \arrow[d, shift left, "\varphi^{i\delta+j} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"] \\ L_{i\delta+j+1} = L_{i\delta}|_Z \oplus L_{(i+1)\delta}|_Y, \arrow[u, shift left, "\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \varphi_{i\delta+j}"] \end{tikzcd} \tag{3.12}$$

$$\begin{tikzcd} 0 & L_{i\delta}|_Z \arrow[l] \arrow[d, equal] & L_{i\delta+\delta-1} \arrow[l, "(1,0)"'] \arrow[d, shift right, "\varphi^{(i+1)\delta-1}"'] & L_{(i+1)\delta}|_Y \arrow[l, "(0,1)"'] \arrow[d, equal] & 0 \arrow[l] \\ 0 \arrow[r] & L_{i\delta}|_Z \arrow[r] & L_{(i+1)\delta} \arrow[r] \arrow[u, shift right, "\varphi_{(i+1)\delta-1}"'] & L_{(i+1)\delta}|_Y \arrow[r] & 0. \end{tikzcd} \tag{3.13}$$

It can be noted again that $\varphi^k \varphi_k = \varphi_k \varphi^k = 0$ for each $k = 0, \ldots, d\delta - 1$.

We introduce now the notion of level-δ limit linear series on the curve X. By abuse of notation, we denote by the same φ^k and φ_k the induced linear maps on global sections of the corresponding invertible sheaves on X.

Definition 3.8. Fix integers $d > 0$, $\delta > 0$ and $r \geq 0$. A *level-δ limit linear series* on X of degree d and dimension r is a $(d\delta + 2)$-tuple $(L, V_0, \ldots, V_k, \ldots, V_{d\delta})$ of data on X, where L is an invertible sheaf of degree d on Y and degree 0 on Z, and where $V_k \subseteq \Gamma(X, L_k)$ is a vector subspace of dimension $r+1$ for each $k = 0, \ldots, d\delta$, such that $\varphi^k(V_k) \subseteq V_{k+1}$ and $\varphi_k(V_{k+1}) \subseteq V_k$ for each $k = 0, \ldots, d\delta - 1$.

In the same manner as in Osserman's case, we denote by $V_k^{Z,0}$ (resp. $V_k^{Y,0}$) the subspace of V_k of sections that vanish on Z (resp. on Y). Analogously, we define $V_k|_Z$ (resp. $V_k|_Y$) as the subspace of restrictions to Z (resp. Y) of the sections of V_k. Notice that $V_k^{Z,0} = \mathrm{Ker}(\varphi^k)$ and $V_k|_Z = \mathrm{Im}(\varphi^k)$ for $k = 0, \ldots, d\delta - 1$, while $V_k^{Y,0} = \mathrm{Ker}(\varphi_{k-1})$ and $V_k|_Y = \mathrm{Im}(\varphi^{k-1})$ for $k = 1, \ldots, d\delta$. (By degree considerations, $V_0^{Y,0} = 0$ and $V_{d\delta}^{Z,0} = 0$, while $V_0|_Y = V_0$ and

$V_{d\delta}|_Z = V_{d\delta}$.) Thus, we have exact sequences,

$$0 \longrightarrow V_k^{Y,0} \longrightarrow V_k \longrightarrow V_k|_Y \longrightarrow 0,$$

$$0 \longrightarrow V_k^{Z,0} \longrightarrow V_k \longrightarrow V_k|_Z \longrightarrow 0$$

for each $k = 0, \dots, d\delta$, and, since $\varphi^k \varphi_k = \varphi_k \varphi^k = 0$ for each $k = 0, \dots, d\delta - 1$, inclusions

$$\varphi^k(V_k) = V_k|_Z \subseteq V_{k+1}^{Y,0} \subseteq V_{k+1}, \quad \text{and} \quad \varphi_k(V_{k+1}) = V_{k+1}|_Y \subseteq V_k^{Z,0} \subseteq V_k$$

for each $k = 0, \dots, d\delta - 1$.

In particular, $\mathrm{Im}(\varphi^k) \cap \mathrm{Ker}(\varphi^{k+1}) = 0$ and $\mathrm{Im}(\varphi_{k+1}) \cap \mathrm{Ker}(\varphi_k) = 0$ for $k = 0, \dots, d\delta - 2$, which implies that we have inclusions:

$$\begin{aligned} &V_0|_Z \subseteq V_1|_Z \subseteq \cdots \subseteq V_{d\delta-1}|_Z \subseteq V_{d\delta}|_Z, \\ &V_0|_Y \supseteq V_1|_Y \supseteq \cdots \supseteq V_{d\delta-1}|_Y \supseteq V_{d\delta}|_Y. \end{aligned} \tag{3.14}$$

Set $J := \mathrm{Pic}^{(d,0)}(X)$. From now on, denote by $q_1 \colon X \times J \to X$ and $q_2 \colon X \times J \to J$ the projections. The following proposition claims the existence of a projective scheme parametrizing level-δ limit linear series on the curve X.

Proposition 3.9. *Fix integers $d > 0$, $r \geq 0$ and $\delta > 0$. Let X be a curve with two smooth irreducible components Y and Z meeting transversally at a point P. Let g be its (arithmetic) genus, and L an invertible sheaf on X of degrees d on Y and 0 on Z. Then there exists a projective scheme*

$$G^r_{d,\delta}(X) := \{(L, V_0, \dots, V_{d\delta}) \,|\, \varphi^k(V_k) \subseteq V_{k+1} \text{ and } \varphi_k(V_{k+1}) \subseteq V_k \text{ for } k = 0, \dots, d\delta - 1\}$$

parametrizing level-δ limit linear series of degree d and dimension r on X. Furthermore, there exists a proper and surjective map $\rho_\delta : G^r_{d,\delta}(X) \longrightarrow G^r_{d,1}(X)$, where $G^r_{d,1}(X)$ is identified with Osserman's scheme of limit linear series $G^{r,\mathrm{Oss}}_d(X)$.

Proof 3.10. The proof will be divided in 3 steps. The first two, the construction of $G^r_{d,\delta}(X)$, follow the argument given to [37], Thm. 5.3, p. 1178.

First step: Construction of the linked Grassmannian LG.

Let's remind that, the linked Grassmannian is a closed subscheme of a product of Grassmannians, whence a projective variety, parametrizing collections of subbundles of fixed vector bundles, linked together via maps between the fixed vector bundles; see ([37], Appendix). In our case, the Grassmannians are $d\delta + 1$ relative Grassmannians over J.

More precisely, let $\mathcal{L}$ be a universal invertible sheaf on $X \times J$. For each $k = 0, \dots, d\delta$, let

$$\mathcal{L}_k := \mathcal{L} \otimes q_1^*(\mathcal{O}_X)_k,$$

where the $(\mathcal{O}_X)_k$ are the sheaves obtained by applying the construction on sheaves described above. In particular, there are maps as in (3.11), (3.12) and (3.13), with L replaced by $\mathcal{O}_X$, that induce maps

$$\varphi^k\colon \mathcal{L}_k \longrightarrow \mathcal{L}_{k+1} \text{ and } \varphi_k\colon \mathcal{L}_{k+1} \longrightarrow \mathcal{L}_k$$

for $k = 0, \dots, d\delta - 1$ such that

1. $\varphi^k\varphi_k = \varphi_k\varphi^k = 0$ for $k = 0, \dots, d\delta - 1$,
2. $\mathrm{Im}(\varphi^k) \cap \mathrm{Ker}(\varphi^{k+1}) = 0$ for $k = 0, \dots, d\delta - 2$,
3. $\mathrm{Im}(\varphi_{k+1}) \cap \mathrm{Ker}(\varphi_k) = 0$ for $k = 0, \dots, d\delta - 2$.

Let D be a sufficently ample divisor of X supported away from P, and put $D' := q_1^*(D)$. It is enough to choose D ample enough that $h^1(X, \mathcal{L}_k|_{X\times Q}(D) = 0$ for each $k = 0, \dots, d\delta$ and each $Q \in J$. Then $\mathcal{E}_k := q_{2*}(\mathcal{L}_k(D'))$ is a locally free sheaf for each $k = 0, \dots, d\delta$. Define

$$G_k := \mathrm{Grass}_J(r + 1, \mathcal{E}_k)$$

for each $k = 0, \dots, d\delta$. Of course, the φ_k induce maps between the $\mathcal{E}_k$ in a natural way; these will also be denoted by the same φ_k. These satisfy the same properties listed above.

We define LG as the relative linked Grassmannian of subbundles of rank $r + 1$ of the $d\delta + 1$ locally free sheaves $\mathcal{E}_k$, linked by the φ_k, over J. Thus LG is a closed subscheme of $G_0\times\cdots\times G_{d\delta}$.

Second step: Construction of $G^r_{d,\delta}(X)$.

For each $k = 0, \dots, d\delta$, let $p_k\colon LG \to J$ be the composition of the projection $LG \to G_k$ with the structure map $p'_k\colon G_k \to J$. Let $\mathcal{F}'_k \subseteq (p'_k)^*(\mathcal{E}_k)$ be the universal rank $(r+1)$-subbundle and $\mathcal{F}_k \subseteq (p_k)^*(\mathcal{E}_k)$ its pullback to LG. Set $\mathcal{H}_k := q_{2*}(\mathcal{L}_k(D')|_{D'})$; it is a locally free sheaf of rank $\deg(D)$, and there is a natural map $v_{k,D}\colon \mathcal{E}_k \to \mathcal{H}_k$. We define $G^r_{d,\delta}(X)$ as the maximum closed subscheme of LG where all the compositions

$$\mathcal{F}_k \longrightarrow (p_k)^*(\mathcal{E}_k) \longrightarrow (p_k)^*(\mathcal{H}_k).$$

It follows that $\mathcal{F}_k \subseteq \widetilde{q}_{2*}(r_k^*(\mathcal{L}_k)$, where $\widetilde{q}_2\colon X \times G^r_{d,\delta}(X) \to G^r_{d,\delta}(X)$ is the projection and $r_k = (\mathrm{id}_X, p_k|_{G^r_{d,\delta}(X)})\colon X \times G^r_{d,\delta}(X) \to X \times J$.

Part three: The map ρ_δ.

Let's recall that Osserman's linking maps in (2.4) are defined in one direction as restriction to Z composed with extension by 0 over Y, and in the reverse direction as restriction to Y composed with extension by 0 over Z. Symbolically, in the index ascending direction, we have the composition $\iota_Y \circ |_Z$, and in the descending direction, the composition $\iota_Z \circ |_Y$. The same notation can be employed to describe our linking maps φ_k above:

$$\varphi^{i\delta+1} = (1,0)\circ|_Z;\ \varphi^{i\delta+2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\cdots;\varphi^{(i+1)\delta-2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ \varphi^{(i+1)\delta-1} = \iota_Y\circ(1,0).$$

$$\varphi_{i\delta+1} = \iota_Z \circ (0,1);\ \varphi_{i\delta+2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \cdots ; \varphi_{(i+1)\delta-2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ \varphi_{(i+1)\delta-1} = (0,1) \circ |_Y.$$

Thus, it is clear that if we define $\hat{\varphi}^i \colon \mathcal{L}_i \to \mathcal{L}_{i+1}$ and $\hat{\varphi}_i \colon \mathcal{L}_{i+1} \to \mathcal{L}_i$ as the compositions $\varphi^{(i+1)\delta-1} \circ \cdots \circ \varphi^{i\delta+1}$ and $\varphi_{i\delta+1} \circ \cdots \circ \varphi_{(i+1)\delta-1}$, respectively, then we have the linking maps used by Osserman. Thus, we have defined a map $\rho \colon LG \to LG^{\mathrm{Oss}}$ between our linked Grassmannian LG and Osserman's LG^{Oss}; this one appearing implicitly in the proof given in [37]. The restriction to $G^r_{d,\delta}(X)$ defines a proper map whose image is contained in $G^{r,\mathrm{Oss}}_d(X)$. Set-theoretically, we have a forgetful map:

$$\begin{array}{cccc} \rho_\delta \ : & G^r_{d,\delta}(X) & \longrightarrow & G^{r,\mathrm{Oss}}_d(X) \\ & (L, V_0, \ldots, V_{i\delta+j}, \ldots, V_{d\delta}) & \mapsto & (L, V_0, \ldots, V_{i\delta}, \ldots, V_{d\delta}). \end{array}$$

The remaining claim to be proved is that ρ_δ is surjective. For this we need the following lemma, which is an adapted version to our situation of Lemma A.7 in [37], p. 1196:

Lemma 3.11. *Given a level-δ limit linear series $(L, V_0, \ldots, V_k, \ldots, V_{d\delta})$, for each $k = 0, \ldots, d\delta$ there is a direct sum decomposition*

$$V_k = \varphi^{k-1}(V_{k-1}) \oplus C'_k \oplus V_k^{Z,0} \oplus C''_k, \tag{3.15}$$

where $V_k^{Y,0} = \varphi^{k-1}(V_{k-1}) \oplus C'_k$.

The proof of the above lemma is elementary and can be found in [37].

So, let $(L, V_0, \ldots, V_k, \ldots, V_{d\delta})$ be a level-δ limit linear series, and consider the decompositions of the V_k given by Lemma 3.11. Notice that the lemma describes a decomposition of V_k only with respect to φ^k. Analogously, using the maps φ_k we obtain a second decomposition of V_k that can be described by

$$V_k = \varphi_k(V_{k+1}) \oplus D'_k \oplus V_k^{Y,0} \oplus D''_k \tag{3.16}$$

where $V_k^{Z,0} = \varphi_k(V_{k+1}) \oplus D'_k$.

A useful formula compares the dimensions of the C'_k with those of the D'_k:

$$\begin{aligned} \dim C'_{k+1} &= \dim V^{Y,0}_{k+1} - \operatorname{rk} \varphi^k \\ &= \dim \operatorname{Ker} \varphi_k - \operatorname{rk} \varphi^k \\ &= \Big(\dim \operatorname{Ker} \varphi_k + \operatorname{rk} \varphi_k\Big) - \Big(\dim \operatorname{Ker} \varphi^k + \operatorname{rk} \varphi^k\Big) + \Big(\dim \operatorname{Ker} \varphi^k - \operatorname{rk} \varphi_k\Big) \\ &= \Big(\dim \operatorname{Ker} \varphi^k - \operatorname{rk} \varphi_k\Big) \\ &= \dim V^{Z,0}_k - \operatorname{rk} \varphi_k \\ &= \dim D'_k. \end{aligned} \tag{3.17}$$

On the other hand, notice that the subspaces C''_k of Lemma 3.11 satisfy $C''_k \cap (V_k^{Z,0} \oplus V_k^{Y,0}) = 0$. Thus φ^k gives an isomorphism between C''_k and $C''_{k,Z} := \varphi^k(C''_k) \subseteq V_{k+1}$, and φ_k gives an isomorphism between C''_{k+1} and $C''_{k+1,Y} := \varphi_k(C''_{k+1})$. Likewise for the D''_k. In this sense, we call the sections of the spaces C''_k (resp. D''_k) *linked*.

Now, to the proof of the surjectivity of ρ_δ. Let $\mathfrak{g} = (L, V_0, \ldots, V_d) \in G_d^{r,\mathrm{Oss}}(X)$. Let $\varphi_i \colon L_{i+1} \to L_i$ and $\varphi^i \colon L_i \to L_{i+1}$ denote the linking maps. We need to show that there exists $\widetilde{\mathfrak{g}} \in G^r_{d,\delta}(X)$ such that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$. To avoid confusion, we will denote $\widetilde{\mathfrak{g}} := (L, V_0, \ldots, V_{i+j/\delta}, \ldots, V_d)$, for spaces $V_{i+j/\delta}$ to be constructed below, and let $\widetilde{\varphi}_k \colon L_{k+1} \to L_k$ and $\widetilde{\varphi}^k \colon L_k \to L_{k+1}$ denote the linking maps, for each $k \in (1/\delta)\mathbb{Z}$ with $0 \le k < d$. Notice, as before, that

$$\begin{aligned} \varphi_i =& \widetilde{\varphi}_i \circ \widetilde{\varphi}_{i+1/\delta} \circ \cdots \circ \widetilde{\varphi}_{i+1-1/\delta}, \\ \varphi^i =& \widetilde{\varphi}^{i+1-1/\delta} \circ \cdots \circ \widetilde{\varphi}^{i+1/\delta} \circ \widetilde{\varphi}^i. \end{aligned} \tag{3.18}$$

Consider decompositions of the V_i, for $i = 0, \ldots, d$ as in (3.15) and (3.16). From (3.17) we know that $\dim C'_{i+1} = \dim D'_i$ for each $i = 0, \ldots, d-1$. Thus, it suffices to let $\widetilde{\mathfrak{g}} := (L, V_0, \ldots, V_{i+j/\delta}, \ldots, V_d)$, where

$$V_{i+j/\delta} := \left(V_i^{Y,0} \oplus D''_{i,Z} \oplus \overline{D}_i\right) \oplus \left(\overline{C}_{i+1} \oplus C''_{i+1,Y} \oplus V_{i+1}^{Z,0}\right) \subseteq H^0(Z, L_i|_Z) \oplus H^0(Y, L_{i+1}|_Y)$$

for each $i = 0, \ldots, d-1$ and $j = 1, \ldots, \delta-1$. Here we choose subspaces $\overline{D}_i \subseteq C'_{i+1}$ and $\overline{C}_{i+1} \subseteq D'_i$ such that

$$\dim \overline{D}_i + \dim \overline{C}_{i+1} = \dim C'_{i+1} = \dim D'_i.$$

Since $D'_i \supseteq \overline{C}_{i+1}$, we have

$$\begin{aligned} &\widetilde{\varphi}^i(V_i) = V_i^{Y,0} \oplus D''_{i,Z} \subseteq V_i^{Y,0} \oplus D''_{i,Z} \oplus \overline{D}_i = \operatorname{Ker} \widetilde{\varphi}_i, \\ &\operatorname{Ker} \widetilde{\varphi}^i = \operatorname{Ker} \varphi^i = V_i^{Z,0} = D'_i \oplus \varphi_i(V_{i+1}) \supseteq \overline{C}_{i+1} \oplus C''_{i+1,Y} \oplus V_{i+1}^{Z,0} = \widetilde{\varphi}_i(V_{i+1/\delta}). \end{aligned} \tag{3.19}$$

Analogously, since $C'_{i+1} \supseteq \overline{D}_i$, we have

$$\begin{aligned} \widetilde{\varphi}_{i+1-1/\delta}(V_{i+1}) &= C''_{i+1,Y} \oplus V_{i+1}^{Z,0} \subseteq \overline{C}_{i+1} \oplus C''_{i+1,Y} \oplus V_{i+1}^{Z,0} = \operatorname{Ker} \widetilde{\varphi}^{i+1-1/\delta}, \\ \operatorname{Ker} \widetilde{\varphi}_{i+1-1/\delta} &= \operatorname{Ker} \varphi_i = V_{i+1}^{Y,0} = \varphi^i(V_i) \oplus C'_{i+1} \supseteq V_i^{Y,0} \oplus D''_{i,Z} \oplus \overline{D}_i = \widetilde{\varphi}^{i+1-1/\delta}(V_{i+1}). \end{aligned} \tag{3.20}$$

Finally, all the $V_{i+j/\delta}$ are equal, for fixed i and $j = 1, \ldots, \delta-1$. Thus, it is easy to check that for each $i = 0, \ldots, d-1$ and each $j = 1, \ldots, \delta-2$,

$$\widetilde{\varphi}^{i+j/\delta}(V_{i+j/\delta}) = \operatorname{Ker} \widetilde{\varphi}_{i+j/\delta} \quad \text{and} \quad \widetilde{\varphi}_{i+j/\delta}(V_{i+(j+1)/\delta}) = \operatorname{Ker} \widetilde{\varphi}^{i+j/\delta}.$$

In conclusion, $\widetilde{\mathfrak{g}} \in G^r_{d,\delta}(X)$ and $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$, which completes the proof of the proposition.

Our next goal is to study the behavior of the map ρ_δ over the open set of Osserman's exact limit linear series. As a matter of fact, we have a similar notion for level-δ limit linear series:

Definition 3.12. A *level-δ limit linear series* $\mathfrak{g} = (L, V_0, \ldots, V_k, \ldots, V_{d\delta})$ with linking maps φ_k and φ^k is called *exact* if any (and thus all) of the following equivalent conditions holds for each

$k = 0, \dots, d\delta - 1$:

1. $V_k|_Z = V_{k+1}^{Y,0}$ and $V_{k+1}|_Y = V_k^{Z,0}$.
2. $\varphi^k(V_k) = \operatorname{Ker} \varphi_k|_{V_{k+1}}$ and $\varphi_k(V_{k+1}) = \operatorname{Ker} \varphi^k|_{V_k}$
3. The induced complex

$$V_k \xrightarrow{\varphi^k} V_{k+1} \xrightarrow{\varphi_k} V_k \xrightarrow{\varphi^k} V_{k+1}$$

is an exact sequence.

Define the subspace of exact level-δ limit linear series of $G^r_{d,\delta}(X)$:

$$G^{r,\text{exact}}_{d,\delta}(X) := \left\{ (L, V_0, \dots, V_{d\delta}) \,|\, V_k|_Z = V_{k+1}^{Y,0} \text{ and } V_{k+1}|_Y = V_k^{Z,0} \text{ for } k = 0, \dots, d\delta - 1 \right\}.$$

Since exactness is an open condition, $G^{r,\text{exact}}_{d,\delta}(X)$ is an open subspace of $G^r_{d,\delta}(X)$. In fact, it suffices to observe that $\operatorname{rk} \varphi^k + \operatorname{rk} \varphi_k \leq r + 1$ for each $k = 0, \dots, d\delta - 1$, with equality for every $k = 0, \dots, d\delta - 1$ if and only if the limit linear series is exact. Thus, exactness is an open condition by semicontinuity (see [28], Chapter 3, §12, p. 281).

We continue with some elementary properties of exact level-δ limit linear series. By Lemma 3.11, if $\mathfrak{g} = (L, V_0, \dots, V_k, \dots, V_{d\delta})$ is a level-δ limit linear series, then there exists $V_k^L \subset V_k$ (possibly $V_k^L = 0$) such that $V_k = V_k^L \oplus V_k^{Y,0} \oplus V_k^{Z,0}$ for each $k = 0, \dots, d\delta$. The subspaces of sections V_k^L are called *linked*. Thus, we obtain sequences of nonnegative integers

$$\{P_k\} := \{\dim(V_k^{Y,0})\}, \quad \{Q_k\} := \{\dim(V_k^{Z,0})\}, \quad \{M_k\} := \{\dim(V_k^L)\}$$

satisfying:

$$\begin{aligned} P_k &\leq P_k + M_k \leq P_{k+1}; \\ Q_{k+1} &\leq Q_{k+1} + M_{k+1} \leq Q_k; \\ P_k &+ Q_k + M_k = r + 1 \end{aligned} \tag{3.21}$$

for each $k = 0, \dots, d\delta$.

Furthermore,

Lemma 3.13. *Let $\mathfrak{g} := (L, V_0, \dots, V_k, \dots, V_{d\delta}) \in G^r_{d,\delta}(X)$ be a level-δ limit linear series. Then $\mathfrak{g}$ is exact if and only if any (and thus all) of the following equivalent conditions holds:*

1. $P_{k+1} = P_k + M_k \quad \forall k.$
2. $Q_{k-1} = Q_k + M_k \quad \forall k.$
3. $\sum_{k=0}^{d\delta} M_k = r + 1.$

Proof 3.14. First we show the equivalence between (1), (2) and (3).
(1) ⇔ (2): It is an easy consequence of the equalities:

$$P_k + Q_k + M_k = r + 1 \quad \forall k.$$

(1) ⇔ (3): Clearly,

$$\sum_{k=0}^{d\delta} M_k = M_{d\delta} + \sum_{k=0}^{d\delta-1} M_k \leq M_{d\delta} + \sum_{k=0}^{d\delta-1} (P_{k+1} - P_k) = M_{d\delta} + P_{d\delta} - P_0.$$

Now, since the restriction maps $V_0 \to V_0|_Y$ and $V_{d\delta} \to V_{d\delta}|_Z$ are isomorphisms, it follows that $P_0 = Q_{d\delta} = 0$ and thus $Q_0 + M_0 = r + 1 = P_{d\delta} + M_{d\delta}$. We obtain an inequality

$$\sum_{k=0}^{d\delta} M_k \leq r + 1,$$

which is strict if and only if $P_{k+1} - P_k > M_k$ for some k.

Now, we will show that exactness is equivalent to $P_{k+1} = P_k + M_k \quad \forall k$. $\mathfrak{g}$ is exact if and only if $\mathrm{rk}(\varphi^k) + \mathrm{rk}(\varphi_k) = r + 1 \quad \forall k$, equivalently, if and only if $P_{k+1} + Q_k = r + 1 \quad \forall k$. However, from (3.21) we have $P_k + Q_k + M_k = r + 1 \quad \forall k$. Thus, $P_{k+1} + Q_k = r + 1 \quad \forall k$ if and only if $P_{k+1} = P_k + M_k \quad \forall k$

To each $\mathfrak{g} \in G^r_{d,\delta}(X)$ we assign the set

$$S_{\mathfrak{g}} := \{k \,|\, M_k \neq 0\} \subseteq \{0, \ldots, d\delta\}.$$

Clearly, $\sum\limits_{k=0}^{d\delta} M_k = \sum\limits_{i\in S} M_k$. Thus $|S| := \#S \leq r + 1$. If $\mathfrak{g}$ is exact, also $|S| \geq 1$.

Remark 3.15.

1. We would like to illustrate the conditions of linkage and exactness for the "middle" maps. For each $i = 0, \ldots, d-1$ and $j = 1, \ldots, \delta - 2$, let $k := i\delta + j$, and put $A := H^0(Y, L_{i+1}|_Y)$ and $B := H^0(Z, L_i|_Z)$, and $\pi_1 := \varphi_k$ and $\pi_2 := \varphi^k$. We may view π_1 and π_2 as projections, $\pi_1 : A \oplus B \to A$, $\pi_2 : A \oplus B \to B$. Letting $V_1 := V_k \subseteq A \oplus B$ and $V_2 := V_{k+1} \subseteq A \oplus B$ we have:
   a) **Linkage**
      i. $\pi_1(V_2) \oplus 0 \subseteq V_1$,
      ii. $0 \oplus \pi_2(V_1) \subseteq V_2$.
   b) **Exactness**
      i. $\pi_1(V_2) = \pi_1(V_1 \cap (A \oplus 0))$,
      ii. $\pi_2(V_1) = \pi_2(V_2 \cap (0 \oplus B))$.

Actually, exactness is reduced to the inclusions "$\supseteq$", once the reverse inclusions are assumed (linkage).

2. Lemma 3.13 implies that $\mathfrak{g} \in G^r_{d,\delta}(X)$ is exact if and only if for each $k = 0, \ldots, d\delta$, the iterated images of bases of the spaces V^L_ℓ for all $\ell \in S_\mathfrak{g}$ form a basis of V_k.

3. For each$S \subset \{0, \ldots, d\delta\}$, define

$$G^{r,\text{exact}}_{d,\delta}(X; S) := \left\{ \mathfrak{g} \in G^{r,\text{exact}}_{d,\delta}(X) \mid \sum_{k \in S} M_k = r + 1 \right\}.$$

By Lemma 3.13, the $G^{r,\text{exact}}_{d,\delta}(X; S)$ cover $G^{r,\text{exact}}_{d,\delta}(X)$. Furthermore,

$$G^{r,\text{exact}}_{d,\delta}(X; S) \bigcap G^{r,\text{exact}}_{d,\delta}(X; T) = \emptyset \quad \text{if } S \neq T.$$

As a consequence, we have stratifications:

$$G^{r,\text{exact}}_{d,\delta}(X; \ell) := \coprod_{|S| = \ell} G^{r,\text{exact}}_{d,\delta}(X; S) \quad G^{r,\text{exact}}_{d,\delta}(X) = \coprod_{1 \leq \ell \leq r+1} G^{r,\text{exact}}_{d,\delta}(X; \ell). \tag{3.22}$$

For instance, the refined locus in Osserman's space $G^{r,\text{Oss}}_d(X)$, which is isomorphic to the refined locus in the corresponding Eisenbud–Harris space, corresponds to $G^{r,\text{exact}}_{d,1}(X; r+1)$, since in this case $0 \leq M_k \leq 1$ for each $k = 0, \ldots, d$; see [37], Def. 6.5, p. 1184 and Cor. 6.8, p. 1189.

The next proposition gives two important properties of the map ρ_δ defined in Proposition 3.9. These properties are fundamental for understanding the relation between level-δ limit linear series and fibers of Abel maps. Roughly speaking, they say that Osserman's exact points are uniquely determined in any level-δ space and that, for each Osserman non-exact limit linear series, there exists a level δ in which it becomes exact. In fact, $\delta = 2$ is enough!

Proposition 3.16. *The following properties hold:*

1. $\rho_\delta^{-1}(G^{r,\text{exact}}_d(X)) \subseteq G^{r,\text{exact}}_{d,\delta}(X)$ *and the restricted map*

$$\rho_\delta \colon \rho_\delta^{-1}(G^{r,\text{exact}}_d(X)) \longrightarrow G^{r,\text{exact}}_d(X)$$

is a bijection.

2. *The restricted map* $\rho_\delta : G^{r,\text{exact}}_{d,2}(X) \longrightarrow G^{r,Oss}_d(X)$ *is surjective.*

Proof 3.17. We prove Statement 1. Let $\mathfrak{g} := (L, V_0, \ldots, V_d) \in G^{r,\text{exact}}_{d,1}(X)$ be an exact limit linear series and $\widetilde{\mathfrak{g}} := (L, V_0, \ldots, V_{i+j/\delta}, \ldots, V_d) \in G^r_{d,\delta}(X)$ lifting it. Let φ^i and φ_i denote the linking maps for $\mathfrak{g}$, and $\widetilde{\varphi}^k$ and $\widetilde{\varphi}_k$ those of $\widetilde{\mathfrak{g}}$. Since $\mathfrak{g}$ is exact, we must have $\mathrm{rk}\,\varphi^i + \mathrm{rk}\,\varphi_i = r+1$ for each $i = 0, \dots, d$. Now, the linkage conditions for $\widetilde{\mathfrak{g}}$ imply that

$$\widetilde{\varphi}^i(V_i) \subseteq V^{Y,0}_{i+j/\delta} \text{ and } \widetilde{\varphi}_{i+1-1/\delta}(V_{i+1}) \subseteq V^{Z,0}_{i+j/\delta} \text{ for each } i = 0, \dots, d-1 \text{ and } j = 1, \dots, \delta - 1. \tag{3.23}$$

Since

$$\mathrm{rk}\,\widetilde{\varphi}^i + \mathrm{rk}\,\widetilde{\varphi}_{i+1-1/\delta} = \mathrm{rk}\,\varphi^i + \mathrm{rk}\,\varphi_i = r+1,$$

it follows that we have equalities in (3.23). In particular,

$$V_{i+j/\delta} = V^{Y,0}_{i+j/\delta} \oplus V^{Z,0}_{i+j/\delta} = V_i|_Z \oplus V_{i+1}|_Y$$

for each $i = 0, \dots, d-1$ and $j = 1, \dots, \delta - 1$, determining uniquely the lifting $\widetilde{\mathfrak{g}}$. Furthermore, having equalities in (3.23), and since the $V_{i+j/\delta}$ do not depend of the choice of j, it follows that $\widetilde{\mathfrak{g}}$ is exact.

As for the second statement, let now $\mathfrak{g} \in G^{r,\mathrm{Oss}}_d(X)$ be a non-exact limit linear series. We would like to find $\widetilde{\mathfrak{g}} := (L, V_0, \dots, V_{i+j/2}, \dots, V_d) \in G^r_{d,2}(X)$ lifting $\mathfrak{g}$. As before, let φ^i and φ_i denote the linking maps for $\mathfrak{g}$. If $\mathrm{rk}\,\varphi^i + \mathrm{rk}\,\varphi_i = r+1$, we may simply define

$$V_{i+1/2} := V_i|_Z \oplus V_{i+1}|_Y$$

for each $j = 1, \dots, \delta - 1$. The argument used above guarantees the exactness of $\widetilde{\mathfrak{g}}$ at $k = i, i+1/2$.

So, let i be such that $\mathrm{rk}\,\varphi^i + \mathrm{rk}\,\varphi_i < r+1$, or equivalently, $\dim \mathrm{Ker}\,\varphi^i - \mathrm{rk}\,\varphi_i > 0$. Bringing back the notation used in the proof of Proposition 3.9; instead of finding just any $\widetilde{\mathfrak{g}}$ we have to find one that is exact. Recall (3.17):

$$\dim D'_i = \dim \mathrm{Ker}\,\varphi^i - \mathrm{rk}\,\varphi_i = \dim C'_{i+1} = \dim \mathrm{Ker}\,\varphi_k - \mathrm{rk}\,\varphi^i,$$

where D'_i and C'_{i+1} subspaces of $V^{Z,0}_i$ and $V^{Y,0}_{i+1}$, respectively, such that

$$V^{Z,0}_i = D'_i \oplus \varphi_i(V_{i+1}) \text{ and } V^{Y,0}_{i+1} = \varphi^i(V_i) \oplus C'_{i+1}. \tag{3.24}$$

Define

$$V_{i+1/2} := \varphi^i(V_i) \oplus DC_{i,i+1} \oplus \varphi_i(V_{i+1}), \tag{3.25}$$

where $DC_{i,i+1}$ is the subspace of $D'_i \oplus C'_{i+1}$ obtained as the graph of any isomorphism $D'_i \cong C'_{i+1}$. In particular, $\dim DC_{i,i+1} = \dim D'_i = \dim C'_{i+1}$. Clearly, by definition, $\widetilde{\mathfrak{g}}$ is a limit linear series. It follows from (3.24) that $\widetilde{\mathfrak{g}}$ is exact at $k = i, i+1/2$.

Remark 3.18.

1. It is clear that the $\widetilde{\mathfrak{g}}$ given in the proof of Proposition 3.16 is neither unique nor canonical, as it depends on the choice of an isomorphism $D'_i \cong C'_{i+1}$.

2. Given $\mathfrak{g} \in G^r_d(X)$ it is possible to find $\widetilde{\mathfrak{g}} \in G^{r,\text{exact}}_{d,\delta}(X)$ in any level $\delta \geq 2$ such that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$. Indeed, simply repeat the space used, letting

$$V_{i+j/\delta} := \varphi^i(V_i) \oplus DC_{i,i+1} \oplus \varphi_i(V_{i+1})$$

for every $j = 1, \ldots, \delta - 1$.

3. The are natural forgetful maps

$$\rho_{\delta_1,\delta_2} \colon G^r_{d,\delta_1}(X) \to G^r_{d,\delta_2}(X)$$

for each $\delta_1, \delta_2 > 0$, if $\delta_2 | \delta_1$. As in the proof of Proposition 3.16 (1), we can show that $\rho^{-1}_{\delta_1,\delta_2}(G^{r,\text{exact}}_{d,\delta_2}(X)) \subseteq G^{r,\text{exact}}_{d,\delta_1}(X)$ and that the restricted map

$$\rho_{\delta_1,\delta_2} \colon \rho^{-1}_{\delta_1,\delta_2}(G^{r,\text{exact}}_{d,\delta_2}(X)) \longrightarrow G^{r,\text{exact}}_{d,\delta_2}(X)$$

is a bijection.

Example 3.1. Let $Y := \mathbb{P}^1 =: Z$. Let y be an affine coordinate for Y and z for Z. Let X be the union of Y and Z meeting transversally at 0. Set $d := 2$, $r := 1$. Our principal interest is the description of the exact limit linear series of degree 2 and rank r in level 2.

In this case, there is a unique invertible sheaf of degree 2 on Y (resp. Z), namely $\mathcal{O}(2)$. We may view its space of sections as the space of meromorphic functions with order of pole at ∞ at most 2. In other words, we write the sections as a polynomial on y (resp. z) of degree at most 2. The invertible sheaves on X that will interest us are just three: $L_0 = \mathcal{O}(2,0)$, $L_1 = \mathcal{O}(1,1)$, $L_2 = \mathcal{O}(0,2)$. Their sections will be obtained by identifying polynomials on y with degree at most 2 with polynomials on z of degree at most 2. Specifically, we will require that the i-th coefficient of the polynomial on y agrees with the $(2-i)$-th coefficient of the polynomial on z. In case of $L_0 = \mathcal{O}(2,0)$, its sections will be identified with a pair $(a_0 + a_1 y + a_2 y^2, a_0 z^2)$ consisting of a polynomial on y and a polynomial on z vanishing at 0 with order at least 2 such that the constant coefficient of the polynomial on y is equal to the degree-2 coefficient of the polynomial on z. Likewise, the sections of $L_1 = \mathcal{O}(1,1)$ have the form $(a_1 y + a_2 y^2, a_1 z + b_2 z^2)$, and the sections of $L_2 = \mathcal{O}(0,2)$ have the form $(a_2 y^2, a_2 + b_1 z + b_2 z^2)$.

We will view a limit linear series $(L, V_0, \ldots, V_{2\delta})$ as a collection of $2\delta + 1 = 5$ subspaces of dimension 2 of $H^0(Y, L|_Y) \oplus H^0(Z, L|_Z(2P))$, that is, subspaces of $\Gamma^3_Y \oplus \Gamma^3_Z$, where Γ^3_Y (resp. Γ^3_Z) is the 3-dimensional space of polynomials of degree at most 2 on y (resp. z). In matrix terms, such spaces will be given as 6-by-2 matrices.

For instance, and to fix notation,

$$\begin{pmatrix} 0 & 0 \\ a & b \\ c & d \\ 0 & 0 \\ a & b \\ e & f \end{pmatrix}$$

will denote the subspace of $\Gamma_Y^3 \oplus \Gamma_Z^3$ generated by $(ay+cy^2, az+ez^2)$ and $(by+dy^2, by+fz^2)$. It lies in $H^0(\mathcal{O}(1,1))$.

First, we describe the points on Osserman's space $G_2^1(X)$. It is a 2-dimensional space stratified as follows:

Exact Points: Form four 2-dimensional strata.

$$W_1^o : \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & b_1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad W_2^o : \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 1 & 0 \\ a_2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ b_2 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$W_3^o : \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ b_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ a_3 & b_3 \end{pmatrix} \qquad W_4^o : \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a_4 & b_4 \\ c_4 & d_4 \\ 0 & 0 \\ a_4 & b_4 \\ e_4 & f_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where a_i, b_i, c_4, d_4, e_4, $f_4 \in \mathbb{C}$, with $\begin{vmatrix} a_4 & b_4 \\ c_4 & d_4 \end{vmatrix} \neq 0$, $\begin{vmatrix} a_4 & b_4 \\ e_4 & f_4 \end{vmatrix} \neq 0$.

Non-Exact Points: Form four 1-dimensional strata and one 0-dimensional stratum.

$$L^o_{1,2}\quad \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 1 & 0 \\ a_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad L^o_{2,3}: \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ b_2 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$L^o_{1,4}: \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ c_4 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad L^o_{3,4}: \begin{array}{ccc} V_0 & V_1 & V_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ e_4 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

and the point

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let W_i and $L_{i,j}$ denote the closures of W_i^o and $L_{i,j}^0$ in $G_2^1(X)$. Then $L_{i,j} = W_i \cap W_j$ for each i and j. In addition, $\cap_i W_i$ is the 0-dimensional stratum. Also, $W_1^o \cup W_2^o \cup W_3^o$ is mapped isomorphically to the locus of refined points in the corresponding Eisenbud–Harris space and the lines $L_{1,2}$ and $L_{2,3}$ to their locus of crude points; see [37] §6, p. 1183 or [39], Example, p.14. As for $G_{2,2}^1(X)$, we may follow the proof of Proposition 3.16 to obtain that the exact points form nine 2-dimensional strata:

$$\widetilde{W}_1^o: \begin{array}{ccccc} V_0 & V_1 & V_2 & V_3 & V_4 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & b_1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\widetilde{W}_2^o : \quad \begin{pmatrix} 1 & 0 \\ a_2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ b_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{W}_3^o : \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ b_3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ b_3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ a_3 & b_3 \end{pmatrix}$$

$$\begin{array}{cccccc} & V_0 & V_1 & V_2 & V_3 & V_4 \\ \widetilde{W}_4^o : & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ a_4 & b_4 \\ c_4 & d_4 \\ 0 & 0 \\ a_4 & b_4 \\ e_4 & f_4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\widetilde{L}_{1,2}^o \quad \begin{pmatrix} 1 & 0 \\ a_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \alpha_1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{L}_{2,3}^o : \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ b_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{L}^o_{1,4}: \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ c_4 & \alpha_3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ c_4 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{L}^o_{3,4}: \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ e_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ e_4 & \alpha_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W^o_5: \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \alpha_5 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \alpha_6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

lying over each of the indicated nine strata of $G^1_2(X)$, the last one lying over the point.

As we can see, the exact points of $G^1_{2,2}(X)$ live on many more components than those of $G^1_2(X)$. In general, the number of connected components of the locus of exact points of $G^1_{d,\delta}(X)$ grows to infinity as δ grows.

3.3 The Abel maps and level-δ limit linear series.

The main goal in this section is to establish a relation between the spaces $G^{r,\mathrm{exact}}_{d,\delta}(X)$ and fibers of the Abel map A_d of degree d, as defined in [18] §3. Furthermore, it will be shown that the necessary and sufficient condition for the subscheme $\mathbb{P}(\mathfrak{g}) \subseteq A_d^{-1}(L)$ assigned in [18] §4 to each $\mathfrak{g} = (L, V_0, \dots, V_d) \in G^r_d(X)$ to have the "correct" Hilbert polynomial, i.e., the Hilbert polynomial of the diagonal of $\mathbb{P}^r \times \mathbb{P}^r$, is the exactness of $\mathfrak{g}$.

The degree-d Abel map $A_d\colon X^{(d)} \to J$ associates to each Weil divisor D of the form $D = D_1 + D_2$, where D_1 is supported in Y and D_2 in Z, the invertible sheaf $\mathcal{O}_X(D)$ defined as that having restrictions $\mathcal{O}_Y(D_1 + d_2P)$ and $\mathcal{O}_Z(D_2 - d_2P)$, where $d_2 := \deg(D_2)$. By its very definition, $\mathcal{O}_X(D)$ does not depend on the decomposition $D = D_1 + D_2$; see [18], §3.

Let r, d, δ be nonnegative integers. Let $\mathfrak{g} = (L, V_0, \dots, V_d) \in G^r_d(X)$. For each $i = 0, \dots, d$, set $\Gamma^i_Y := H^0(Y, L_{d-i}|_Y)$ and $\Gamma^i_Z := H^0(Z, L_i|_Z)$. Notice that $\mathbb{P}(\Gamma^i_Y) \times \mathbb{P}(\Gamma^{d-i}_Z) \subseteq A_d^{-1}(L)$ in a natural way, by letting the pair (D_1, D_2) consisting of divisors D_1 and D_2 on Y and Z such that

$L_{d-i}|_Y \cong \mathcal{O}_Y(D_1)$ and $L_i|_Z \cong \mathcal{O}_Z(D_2)$ going to $D_1 + D_2$. Clearly,

$$A_d^{-1}(L) = \mathbb{P}(\Gamma_Y^0) \times \mathbb{P}(\Gamma_Z^d) \cup \cdots \cup \mathbb{P}(\Gamma_Y^d) \times \mathbb{P}(\Gamma_Z^0).$$

Also, since

$$\mathbb{P}(\Gamma_Y^0) \subseteq \mathbb{P}(\Gamma_Y^1) \subseteq \cdots \subseteq \mathbb{P}(\Gamma_Y^{d-1}) \subseteq \mathbb{P}(\Gamma_Y^d),$$

and

$$\mathbb{P}(\Gamma_Z^0) \subseteq \mathbb{P}(\Gamma_Z^1) \subseteq \cdots \subseteq \mathbb{P}(\Gamma_Z^{d-1}) \subseteq \mathbb{P}(\Gamma_Z^d),$$

we may view $A_d^{-1}(L) \subseteq \mathbb{P}(\Gamma_Y^d) \times \mathbb{P}(\Gamma_Z^d)$ and $\mathbb{P}(\Gamma_Y^i) \times \mathbb{P}(\Gamma_Z^j) \subseteq A_d^{-1}(L)$ for any i and j such that $0 \leq i \leq i + j \leq d$.

Recall from [18], §4, that $\mathbb{P}(\mathfrak{g}) := \bigcup_i \mathbb{P}(\mathfrak{g}_i)$, where $\mathfrak{g}_i := (L_i, V_i)$ and

$$\mathbb{P}(\mathfrak{g}_i) \subseteq \mathbb{P}(V_i|_Y) \times \mathbb{P}(V_i|_Z) \subseteq \mathbb{P}(\Gamma_Y^{d-i}) \times \mathbb{P}(\Gamma_Z^i) \subseteq A_d^{-1}(L) \subseteq X^{(d)}$$

is the reduced subscheme given set-theoretically by

$$\mathbb{P}(\mathfrak{g}_i) := \overline{\left\{\operatorname{div}(s|_Y) + \operatorname{div}(s|_Z) \mid s \in V_i - (V_i^{Y,0} \cup V_i^{Z,0})\right\}}. \tag{3.26}$$

The scheme $\mathbb{P}(\mathfrak{g}_i)$ is empty if and only if $V_i = V_i^{Y,0} \cup V_i^{(Z,0)}$

In a similar fashion, we may define for each $\delta > 0$ and each $\mathfrak{g} = (L, V_0, \ldots, V_{d\delta}) \in G_{d\delta}^r(X)$ a subscheme $\mathbb{P}(\mathfrak{g}) \subseteq X^{(d)}$ by $\mathbb{P}(\mathfrak{g}) := \bigcup_k \mathbb{P}(\mathfrak{g}_k)$, where $\mathfrak{g}_k := (L_k, V_k)$ and

$$\mathbb{P}(\mathfrak{g}_k) \subseteq \mathbb{P}(V_k|_Y) \times \mathbb{P}(V_k|_Z) \subseteq \mathbb{P}(\Gamma_Y^{d-i-1}) \times \mathbb{P}(\Gamma_Z^i) \subseteq A_d^{-1}(L) \subseteq X^{(d)}$$

is the reduced subscheme given set-theoretically by

$$\mathbb{P}(\mathfrak{g}_k) := \overline{\left\{\operatorname{div}(s|_Y) + \operatorname{div}(s|_Z) \mid s \in V_k - (V_k^{Y,0} \cup V_k^{Z,0})\right\}}. \tag{3.27}$$

Here, $k = i\delta + j$ with $0 \leq j < \delta$.

Another way of viewing the $\mathbb{P}(\mathfrak{g}_k)$, and consequently $\mathbb{P}(\mathfrak{g})$, inside of $\mathbb{P}(\Gamma_Y^d) \times \mathbb{P}(\Gamma_Z^d)$ is to use the linking maps φ_k and φ^k. In this way,

$$\mathbb{P}(\mathfrak{g}) \subseteq \mathbb{P}(V_0|_Y) \times \mathbb{P}(V_d|_Z) \subseteq \mathbb{P}(\Gamma_Y^d) \times \mathbb{P}(\Gamma_Z^d)$$

is given by

$$\mathbb{P}(\mathfrak{g}_k) = \overline{\{(\varphi^{d\delta-1,k}(s), \varphi_{0,k-1}(s)) \mid s \in V_k - (\operatorname{Ker}\varphi^{d\delta-1,k} \cup \operatorname{Ker}\varphi_{0,k-1})\}}, \tag{3.28}$$

where

$$\begin{aligned}\varphi^{d\delta-1,k} &:= \varphi^{d\delta-1}\circ\cdots\circ\varphi^{k}\colon V_k\to V_d \text{ for } k=0,\dots,d\delta,\\ \varphi_{0,k-1} &:= \varphi_0\circ\cdots\circ\varphi_{k-1}: V_k\to V_0 \text{ for } i=1,\dots,d.\end{aligned} \tag{3.29}$$

Indeed, on one hand, $\mathrm{Ker}(\varphi^{d\delta-1,k})=\mathrm{Ker}(\varphi^k)=V_k^{Z,0}$ and $\mathrm{Ker}(\varphi_{0,k-1})=\mathrm{Ker}(\varphi_{k-1})=V_k^{Y,0}$. On the other hand, $\mathrm{Im}(\varphi^k)\subseteq V_{k+1}^{Y,0}$, and the restricted map $\varphi^k|_{V_k^{Y,0}}: V_k^{Y,0}\to V_{k+1}^{Y,0}$ is injective for every k, since $V_k^{Y,0}\cap V_k^{Z,0}=0$. A similar analysis applies to the φ_k. Thus we conclude that (3.27) and (3.28) give the same closed subset of $\mathbb{P}(\Gamma_Y^d)\times\mathbb{P}(\Gamma_Z^d)$.

Theorem 3.19. *Let r,d,δ be nonnegative integers.*

1. *For each $\mathfrak{g}\in G_{d,\delta}^{r,exact}(X)$ there is a naturally associated subscheme $\mathbb{P}(\mathfrak{g})\subseteq A_d^{-1}(L)$ defined like above, which is Cohen-Macaulay, connected of pure dimension r and, as a subscheme of $\mathbb{P}^r\times\mathbb{P}^r$, has bivariate Hilbert polynomial $P(s,t)=\binom{s+t+r}{r}$.*
2. *If $\mathfrak{g}\in G_d^{r,\mathrm{exact}}(X)$ then for any $\delta>0$ and any $\widetilde{\mathfrak{g}}\in G_{d,\delta}^{r,exact}(X)$ such that $\rho_\delta(\widetilde{\mathfrak{g}})=\mathfrak{g}$, we have $\mathbb{P}(\mathfrak{g})=\mathbb{P}(\widetilde{\mathfrak{g}})$.*
3. *For each $\mathfrak{g}\in G_d^{r,\mathrm{Oss}}(X)\setminus G_d^{r,\mathrm{exact}}(X)$ and each $\widetilde{\mathfrak{g}}\in G_{d,\delta}^{r,\mathrm{exact}}(X)$ such that $\rho_\delta(\widetilde{\mathfrak{g}})=\mathfrak{g}$, we have $\mathbb{P}(\mathfrak{g})\subsetneqq\mathbb{P}(\widetilde{\mathfrak{g}})$.*

In particular, $\mathfrak{g}\in G_{d,1}^r(X)=G_d^r(X)$ is exact if and only if $\mathbb{P}(\mathfrak{g})\subset\mathbb{P}^r\times\mathbb{P}^r$ has bivariate Hilbert polynomial $P(s,t)=\binom{s+t+r}{r}$.

Proof 3.20. For the first part, note that Lemma 2.18 has an analogous version for higher δ, which can be derived from our Lemma 3.13. Thus the proof of the first statement follows word-for-word the proof of [18], Thm. 4.3 (our Theorem 2.20), using exactly the same sequence of results with the notation adapted to our situation. We observe that this is possible since the sequence of the results used by Esteves–Osserman relies only on the properties of linkage and exactness of the maps φ^i and φ_i, which are satisfied as well in our case.

For the second statement, notice that a consequence of the proof of Proposition 3.16(1) is that, if $\widetilde{\mathfrak{g}}=(L,V_0,\dots,V_{i+j/\delta},\dots,V_d)\in G_{d,\delta}^{r,\mathrm{exact}}(X)$ lies over an exact $\mathfrak{g}\in G_d^{r,\mathrm{exact}}(X)$, then

$$V_{i+j/\delta}=V_{i+1}^{Y,0}\oplus V_i^{Z,0}=V_{i+j/\delta}^{Y,0}\oplus V_{i+j/\delta}^{Z,0}$$

for each $i=0,\dots,d-1$ and $j=1,\dots,\delta-1$. As it follows from [18], Rmk. 4.9, adapted to our situation, that $\mathbb{P}(\widetilde{\mathfrak{g}})$ is the union of those $\mathbb{P}(\widetilde{\mathfrak{g}}_k)$ for which $V_k\neq V_k^{Y,0}\oplus V_k^{Z,0}$, it follows that $\mathbb{P}(\widetilde{\mathfrak{g}})=\bigcup_i\mathbb{P}(\widetilde{\mathfrak{g}}_i)$. Since $\mathbb{P}(\widetilde{\mathfrak{g}}_i)=\mathbb{P}(\mathfrak{g}_i)$ for each i, it follows that $\mathbb{P}(\mathfrak{g})=\mathbb{P}(\widetilde{\mathfrak{g}})$.

For the third statement, suppose that $\mathfrak{g}\in G_{d,1}^r(X)$ is non-exact. By Proposition 3.16(2) there exist $\delta>0$ and $\widetilde{\mathfrak{g}}\in G_{d,\delta}^{r,\mathrm{exact}}(X)$, such that $\rho_\delta(\widetilde{\mathfrak{g}})=\mathfrak{g}$. Keep the notation used above. The above reasoning gives

$$\mathbb{P}(\widetilde{\mathfrak{g}})=\bigcup_i\mathbb{P}(\mathfrak{g}_i)\bigcup\bigcup_{i,j}\mathbb{P}(\widetilde{\mathfrak{g}}_{i+j/\delta})\supseteq\bigcup_i\mathbb{P}(\mathfrak{g}_i)=\mathbb{P}(\mathfrak{g}). \tag{3.30}$$

We show now that $\mathbb{P}(\mathfrak{g})$ is a proper closed subscheme of $\mathbb{P}(\widetilde{\mathfrak{g}})$. We have already observed that $\mathbb{P}(\widetilde{\mathfrak{g}})$ is the union of those $\mathbb{P}(\widetilde{\mathfrak{g}}_k)$ for which $V_k \neq V_k^{Y,0} \oplus V_k^{Z,0}$. In other words,

$$\mathbb{P}(\widetilde{\mathfrak{g}}) = \bigcup_{k \in S_{\widetilde{\mathfrak{g}}}} \mathbb{P}(\widetilde{\mathfrak{g}}_k). \tag{3.31}$$

Moreover, each $\mathbb{P}(\widetilde{\mathfrak{g}}_k)$ for $k \in S_{\widetilde{\mathfrak{g}}}$ has dimension r by [18], Lemmas 4.4 and 4.8. Finally, though not explicitly stated in [18], it can be deduced that these $\mathbb{P}(\widetilde{\mathfrak{g}}_k)$ are distinct, that is, (3.31) is the decomposition of $\mathbb{P}(\widetilde{\mathfrak{g}})$ in irreducible components. Now, since $\mathfrak{g}$ is not exact, we have that there is $k \in S_{\widetilde{\mathfrak{g}}}$ which is not an integer, and thus accounts for a component $\mathbb{P}(\widetilde{\mathfrak{g}}_k)$ of $\mathbb{P}(\widetilde{\mathfrak{g}})$ not entirely contained in $\mathbb{P}(\mathfrak{g})$.

Finally, if $\mathfrak{g} \in G^r_{d,\delta}(X)$ is exact then $\mathbb{P}(\mathfrak{g}) \subset \mathbb{P}^r \times \mathbb{P}^r$ has bivariate Hilbert polynomial $P(s,t) = \binom{s+t+r}{r}$, as stated in [18], Thm. 4.3, or by our Statement 1. Conversely, suppose $\mathbb{P}(\mathfrak{g}) \subset \mathbb{P}^r \times \mathbb{P}^r$ has bivariate Hilbert polynomial $P(s,t) = \binom{s+t+r}{r}$. Let $\delta > 0$ and $\widetilde{\mathfrak{g}} \in G^{r,\text{exact}}_{d,\delta}(X)$ such that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$, whose existence is guaranteed by Proposition 3.16. Then $\mathbb{P}(\mathfrak{g}) \subsetneqq \mathbb{P}(\widetilde{\mathfrak{g}})$ by Statements 2 and 3, proved above. Since $\widetilde{\mathfrak{g}}$ is exact, if follows from Statement 1 that $\mathbb{P}(\widetilde{\mathfrak{g}}) \subset \mathbb{P}^r \times \mathbb{P}^r$ has bivariate Hilbert polynomial $P(s,t) = \binom{s+t+r}{r}$, the same as $\mathbb{P}(\mathfrak{g})$. Since $\mathbb{P}(\mathfrak{g}) \subsetneqq \mathbb{P}(\widetilde{\mathfrak{g}})$, it follows that $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(\widetilde{\mathfrak{g}})$, and hence, by Statement 2, that $\mathfrak{g}$ is exact.

Theorem 3.21. *Given any $\delta' > r+2$ suppose that $\rho_{\delta'}(\mathfrak{g}') = \mathfrak{g}$, where $\mathfrak{g}$ is non exact lls in $G^r_{d,1}(X)$ and $\mathfrak{g}' \in G^{r,exact}_{d,\delta'}(X)$. Then exist $2 \leq \delta \leq r+2$ and $\widetilde{\mathfrak{g}} \in G^{r,\text{exact}}_{d,\delta}(X)$ such that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$ and $\mathbb{P}(\mathfrak{g}') = \mathbb{P}(\widetilde{\mathfrak{g}})$.*

Proof 3.22. Our proof starts with the observation that for any $\delta > 0$, from the Lemma 3.13 and Remark 3.15(3), (3.22) we have a partition of the set the exact points in the level-δ, for $S \subset \{0, \ldots, d\delta\}$:

$$G^{r,\text{exact}}_{d,\delta}(X;S) := \left\{ \mathfrak{g} | \sum_{i\delta+j \in S} M_{i\delta+j} = r+1 \right\}.$$

Also its easy to check $G^{r,\text{exact}}_{d,\delta}(X;S) \bigcap G^{r,\text{exact}}_{d,\delta}(X;T) = \phi$ for $S \neq T$ and as a consequence the stratifications:

$$\begin{aligned} G^{r,\text{exact}}_{d,\delta}(X;l) &:= \coprod_{|S|=l} G^{r,\text{exact}}_{d,\delta}(X;S) \\ G^{r,\text{exact}}_{d,\delta}(X) &= \coprod_{1 \leq l \leq r+1} G^{r,\text{exact}}_{d,\delta}(X;l). \end{aligned}$$

By assumption, exists a lifting exact $\mathfrak{g}'$, i.e., s.t. $\rho_{\delta'}(\mathfrak{g}') = \mathfrak{g}$, in the level-$\delta' > r+2$ for $\mathfrak{g}$ no exact in the level-1. According to partition above exist $S' \subset \{0, \ldots, d\delta'\}$ and l such that $\mathfrak{g}' \in G^{r,\text{exact}}_{d,\delta'}(X; S', l)$. So, a trivial verification shows that $l = l_{0,1} + \cdots + l_{d-1,d} + \#\{\{0, \ldots, d\} \cap S'\}$, where $l_{i,i+1} = \#\{\text{elements of } S' \text{ between } V_i \text{ and } V_{i+1}\} - \#\{\{i, i+1\} \cap S'\}$. Notice that, $0 \leq l_{i,i+1} \leq r+1$. Thus, we define $\delta := \max_i l_{i,i+1} + 1$ and $S := \{\{0, \ldots, d\} \cap S'\} \bigcup \{i\delta + j | M_{i\delta'+k_j} \neq$

$0\} \subset \{0, \dots, d\delta\}$, where the set is ordered by $\{k_1 < k_2 < \cdots < k_{l_{i,i+1}}\}$. Clearly, $2 \leq \delta \leq r+2$ and $|S| = l$. It follows that, our construction starts by choose $\widetilde{\mathfrak{g}} \in G^{r,\text{exact}}_{d,\delta}(X; S, l)$ such that $V_{i\delta+j} = V_{i\delta'+k_j}$ for all $i\delta+j \in S \backslash \{\{0, \dots, d\} \cap S'\}$ and without change on $D := \{\{0, \dots, d\} \cap S'\}$. Naturally, the linked maps between $V_{i\delta+j}$ will be defined by the composed of maps below:

$$\begin{aligned} \varphi^{i\delta} :=& \varphi^{i\delta'} \circ \cdots \circ \varphi^{i\delta'+k_j-1} : V_{i\delta} \to V_{i\delta+1} \\ \varphi^{i\delta+j-1} :=& \varphi^{i\delta'+k_j} \circ \cdots \circ \varphi^{i\delta'+k_{j+1}-1} : V_{i\delta+j} \longrightarrow V_{i\delta+j+1} \end{aligned} \tag{3.32}$$

and so on. Similar constructions apply to $\varphi_{i\delta+j}$. Clearly, the latter assertions implies that $\rho_\delta(\widetilde{\mathfrak{g}}) = \mathfrak{g}$ and by assumption and construction we conclude that $\widetilde{\mathfrak{g}}$ is an exact level-δlls.

Our next claim is that $\mathbb{P}(\mathfrak{g}') = \mathbb{P}(\widetilde{\mathfrak{g}})$. In fact, following the ideas in the proof of the theorem 3.19(2) and the construction of $\widetilde{\mathfrak{g}}$, we obtain that

$$\begin{aligned} \mathbb{P}(\widetilde{\mathfrak{g}}) &= \bigcup_{i\delta+j \in S} \mathbb{P}(\mathfrak{g}_{i\delta+j}) \\ &= \mathbb{P}(\mathfrak{g}) \bigcup \bigcup_{i\delta+j \in S \backslash D} \mathbb{P}(\mathfrak{g}_{i\delta+j}) \\ &= \mathbb{P}(\mathfrak{g}) \bigcup \bigcup_{i\delta+k_j \in S' \backslash D} \mathbb{P}(\mathfrak{g}_{i\delta+k_j}) \\ &= \mathbb{P}(\mathfrak{g}'), \end{aligned}$$

which proves the theorem.

Remark 3.23. So, the level-δ are helpful in understanding the relative Hilbert scheme associated to the fibers of Abel maps. In particular, we have a rational (set-theoretically) map for any $\delta \geq 2$:

$$\begin{array}{cccc} \alpha_\delta & : & G^r_{d,\delta}(X) & \dashrightarrow & \mathrm{Hilb}^{P,H}_{A_d} \\ & & \mathfrak{g} & \mapsto & \alpha_\delta(\mathfrak{g}) = \mathbb{P}(\mathfrak{g}). \end{array}$$

where, $H := n(R_1 + X^{(d-1)}) + m(R_2 + X^{(d-1)})$, $R_1 \in Y - P$, $R_2 \in Z - P$, with $m, n > 0$, is the (relative) ample divisor of A_d and $P := P(s,t) = \binom{s+t+r}{r}$ is the Hilbert polynomial. Naturally, the map α_δ factorizing via ρ_δ at map α

$$\begin{array}{cccc} \alpha & : & G^{r,\text{Oss}}_d(X) & \dashrightarrow & \mathrm{Hilb}^{P,H}_{A_d} \\ & & \mathfrak{g} & \mapsto & \alpha(\mathfrak{g}) = \mathbb{P}(\mathfrak{g}). \end{array}$$

studied by Esteves–Osserman (see [18]). In this sense, we paraphrase the theorem 3.21 as follows: the $\delta-$levels above to $r+2$ not provide relevant geometric information.

Example 3.2. Continuing with the study of $G^1_{2,2}(X)$ of the example 3.1, we explain which subscheme correspond to $\widetilde{\mathfrak{g}} \in G^{1,\text{exact}}_{2,2}(X)$ into the product $\mathbb{P}^1 \times \mathbb{P}^1$. First, study the case of $\widetilde{\mathfrak{g}} \in \rho_2^{-1}(G^{1,\text{exact}}_{2,1}(X)) \cong G^{1,\text{exact}}_2(X)$. If $\widetilde{\mathfrak{g}} \in W^o_1 \bigcup W^o_2 \bigcup W^o_3$, i.e., of refined type then a general description of $\mathbb{P}(\widetilde{\mathfrak{g}}) = \mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_i|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_j|_Z)$ for some $i, j = 0, 1, 2$ and $P_0 \subset \mathbb{P}(V_j|_Z)$,

$P_1 \subset \mathbb{P}(V_i|_Y)$ two points. Explicitly,

- For any $\mathfrak{g} \in W_1^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_0|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_1|_Z)$.
- For any $\mathfrak{g} \in W_2^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_0|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_2|_Z)$.
- For any $\mathfrak{g} \in W_3^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_1|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_2|_Z)$.

Now, for each $\mathfrak{g} \in W_4^o$ we have $\mathbb{P}(\mathfrak{g}) = \Delta \subset \mathbb{P}(V_1|_Y) \times \mathbb{P}(V_1|_Z)$ is the diagonal.

Secondly, for each $\widetilde{\mathfrak{g}} \in G_{2,2}^{1,\mathrm{exact}}(X) \setminus \rho_2^{-1}(G_{2,1}^{1,\mathrm{exact}}(X))$ we have, in a similar way of the Ossermann refined case,

1. For any $\mathfrak{g} \in \widetilde{L_{1,2}}^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_0|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_{3/2}|_Z)$.
2. For any $\mathfrak{g} \in \widetilde{L_{2,3}}^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_{1/2}|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_2|_Z)$.
3. For any $\mathfrak{g} \in \widetilde{L_{1,4}}^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_{1/2}|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_1|_Z)$.
4. For any $\mathfrak{g} \in \widetilde{L_{3,4}}^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_1|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_{3/2}|_Z)$.
5. For any $\mathfrak{g} \in W_5^o$ we have $\mathbb{P}(\mathfrak{g}) = \mathbb{P}(V_{1/2}|_Y) \times P_1 \bigcup P_0 \times \mathbb{P}(V_{3/2}|_Z)$.

4 Stable limit linear series

4.1 Preliminary notions

Our purpose in this section is to introduce the concept of a *generalized linear series* and the projective scheme parametrizing them. As it was stated before, we will consider only the case of a nodal curve X made up of two smooth irreducible components Y and Z intersecting transversally at a unique point P. Our principal motivation is to put together the exact limit linear series in all levels in a single projective parameter space. Of course, identifications must be made, and they will be made through the notion of stabilization. In short, the moduli space of stable limit linear series is the fixed locus by a certain torus action on the moduli space of genus zero stable maps with appropriate homology class to a certain relative Grassmannian. Details will be given below.

4.1.1 Generalized linear series and their moduli space

Fix integers d and r with $r \geq 0$. Recall that X is the curve made up of two smooth irreducible components Y and Z intersecting transversally at a unique point P.

Definition 4.1. Let H be an algebraic scheme. A *family of (generalized) linear series* of degree d and dimension r over X along H consists of the following data:

1. a relatively torsion-free, rank-1, degree-d sheaf $\mathcal{F}$ on $X \times H/H$,
2. a rank-$(r+1)$ locally free subsheaf $\mathcal{V} \subseteq p_{2*}\mathcal{F}$, where $p_2 \colon X \times H \to H$ is the second projection, such that, for every $t \in H$, the induced linear map $\mathcal{V}_t \to H^0(X, \mathcal{F}_t)$ is injective.

As we will see below, the relevant case for us is when H is a chain of projective lines, as those continuous limit linear series will be the ones corresponding to level-δ limit linear series. The construction below will thus be directed to this case.

First, we need to produce a scheme that parameterizes all the (generalized) linear series $(\mathcal{F}_t, \mathcal{V}_t)$ that show up in the continuous limit linear series we are interested in. This is the scheme $H_d^r(X)$ below. Its construction will follow closely the standard construction of the scheme of linear series on a smooth curve, as it can be found in [2], §3 Theorem 3.6, p. 184, for instance.

Construction of the scheme $H_d^r(X)$.

Let T be a chain of $d+1$ projective lines. More precisely, T has exactly $d+1$ irreducible components $T_0, \ldots, T_d$ which are smooth and can be so ordered that $T_i \cap T_j \neq \emptyset$ if and only if

$|i-j| \leq 1$. Furthermore, T_{i-1} intersects T_i transversally at a unique point, henceforth denoted by N_i, for $i=1,\ldots,d$. In addition, we mark two points on the smooth locus of T, the first on T_0, denoted N_0, the second on T_d, denoted N_{d+1}. Set $\Delta_T := \{N_0,\ldots,N_{d+1}\}$ and $T^* := T - \Delta_T$. Then T^* is the disjoint union of $d+1$ connected components T_i^*, where $T_i^* := T^* \cap T_i$ for $i=0,\ldots,d$.

We consider a certain coherent sheaf $\mathcal{F}$ on $X \times T$ whose construction we will give later. Here we give only its properties. It is a relatively torsion-free, rank-1 sheaf on $X \times T/T$ of relative degree d over T whose fibers over T are as follows: $\mathcal{F}|_{X\times T_i^*} \cong \mathcal{O}_X(-i,i) \otimes \mathcal{O}_{T_i^*}$ for each $i=0,\ldots,d$ and $\mathcal{F}|_{X\times N_i} \cong \mathcal{O}_Y(-iP) \oplus \mathcal{O}_Z((i-1)P)$ for $i=0,\ldots,d+1$. Here, $\mathcal{O}_X(-i,i)$ is the unique invertible sheaf on X whose restriction to Y is $\mathcal{O}_Y(-ip)$ and and to Z is $\mathcal{O}_Z(ip)$. Notice that $\mathcal{F}$ is "locally constant" over T^*, but not globally constant. We say that $\mathcal{F}$ is the (truncated) *family of twisters* of X.

Let $\mathcal{L}$ now be a universal sheaf on $X \times \mathrm{Pic}_X^{(d,0)}$, where we recall that $\mathrm{Pic}_X^{(d,0)}$ is the connected component of the Picard scheme of X parameterizing invertible sheaves on X of degree d on Y and 0 on Z. Consider the projections

$$\begin{tikzcd} & X \times \mathrm{Pic}_X^{(d,0)} \times T \arrow[dl, "p"'] \arrow[d, "\mu"] \arrow[dr, "q"] & \\ X \times T & \mathrm{Pic}_X^{(d,0)} \times T & X \times \mathrm{Pic}_X^{(d,0)}. \end{tikzcd} \tag{4.1}$$

Define $\mathcal{L} \boxtimes \mathcal{F} := p^*(\mathcal{F}) \otimes q^*(\mathcal{L})$ and put $\mathcal{W} := \mu_*(\mathcal{L} \boxtimes \mathcal{F})$. Then put

$$H_d^r(X) := \mathrm{Grass}_{\mathrm{Pic}_X^{(d,0)} \times T}(r+1, \mathcal{W}).$$

The scheme $H_d^r(X)$ parameterizes linear series $(\mathcal{I}, V)$, where $\mathcal{I}$ is any torsion-free, rank-1 sheaf on X of degree d whose restrictions to Y and Z, modulo torsion, have degrees ranging from -1 to d, and V is any vector subspace of $H^0(X,\mathcal{I})$ of dimension $r+1$.

The above definition of $H_d^r(X)$ is satisfactory if $\mathcal{W}$ is locally free. If not, here is what we do. Let D be an ample effective divisor of X supported away from P. Let D' denote its pullback to $X \times \mathrm{Pic}_X^{(d,0)} \times T$ under the projection. Then, for each integer $n \geq 0$, we have the natural short exact sequence

$$0 \longrightarrow \mathcal{L} \boxtimes \mathcal{F} \longrightarrow \mathcal{L} \boxtimes \mathcal{F}(nD') \longrightarrow \mathcal{L} \boxtimes \mathcal{F}(nD')|_{nD'} \longrightarrow 0, \tag{4.2}$$

from which we get an exact sequence on $\mathrm{Pic}_X^{(d,0)} \times T$:

$$0 \longrightarrow \mathcal{W} \longrightarrow \mu_*(\mathcal{L} \boxtimes \mathcal{F}(nD')) \longrightarrow \mu_*(\mathcal{L} \boxtimes \mathcal{F}(nD')|_{nD'}). \tag{4.3}$$

Set $\mathcal{E} := \mu_*(\mathcal{L} \boxtimes \mathcal{F}(nD'))$ and $\mathcal{E}' := \mu_*(\mathcal{L} \boxtimes \mathcal{F}(nD')|_{nD'})$. If $n >> 0$, the higher direct images of $\mathcal{L} \boxtimes \mathcal{F}(nD')$ vanish, and thus $\mathcal{E}$ is locally free. Furthermore, since nD' is finite, it follows that also $\mathcal{E}'$ is locally free, of rank $n\deg(D)$. Let $\alpha\colon \mathcal{E}\to\mathcal{E}'$ denote the map between them.

Let $G := \mathrm{Grass}_{\mathrm{Pic}_X^{(d,0)}\times T}(r+1,\mathcal{E})$. Denote by $\nu : G \to \mathrm{Pic}_X^{(d,0)}\times T$ the structural map and by $\mathcal{V}\subseteq \nu^*(\mathcal{E})$ the universal subsheaf. Then $H_d^r(X)$ is defined to be the zero scheme of the map of bundles $\nu^*\alpha|_{\mathcal{V}}\colon \mathcal{V}\to\nu^*\mathcal{E}'$.

Put $H := H_d^r(X)$ and $\lambda := \nu|_H$. Furthermore, form the Cartesian diagram

$$\begin{tikzcd} X\times H \arrow[r, "\lambda_X"] \arrow[d, "\mu_H"'] & X\times \mathrm{Pic}_X^{(d,0)}\times T \arrow[d, "\mu"] \\ H \arrow[r, "\lambda"] & \mathrm{Pic}_X^{(d,0)}\times T. \end{tikzcd} \tag{4.4}$$

Since the formation of $\mathcal{E}$ and $\mathcal{E}'$ commutes with base change, it follows that

$$\mathrm{Ker}(\lambda^*\alpha) = \mu_{H*}(\lambda_X^*(\mathcal{L}\boxtimes\mathcal{F})).$$

Thus $\mathcal{V}|_H\subseteq \mu_{H*}(\lambda_X^*(\mathcal{L}\boxtimes\mathcal{F}))$. Moreover, by the same reasoning, as the formation of $\mathcal{E}$ and $\mathcal{E}'$ commutes with *any* base change, $(\lambda_X^*(\mathcal{L}\boxtimes\mathcal{F}),\mathcal{V}|_H)$ is a family of linear series of degree d and dimension r parameterized by H.

Construction of the sheaf $\mathcal{F}$.

Here we will present two different constructions for the sheaf $\mathcal{F}$.

The first construction is through a degeneration argument. Actually, examples of this type of construction have already appeared before in this thesis, when we observed that the sheaves $\mathcal{O}_X(-i,i)$ are the restrictions to X of the sheaves $\mathcal{O}_{\mathcal{X}}(iY)$, where $\mathcal{X}\to B$ is any regular smoothing of X. In fact, the construction below gathers together the constructions of the $\mathcal{O}_X(-i,i)$ and their degenerations.

More precisely, let $\widehat{T}$ be the chain obtained from T by adding an extra rational curve at each end: one, denoted T_{-1}, intersecting T_0 transversally at N_0, the other, denoted T_{d+1}, intersecting T_d transversally at N_{d+1}. View $T\subset\widehat{T}$.

Let $\pi\colon\mathcal{X}\to B$ be a regular smoothing of X and $\tau:\mathcal{T}\to B$ a regular smoothing of $\widehat{T}$. Form the threefold $\mathcal{X}\times_B\mathcal{T}$. It can be regarded as a smoothing of the surface $X\times\widehat{T}$. However, it fails to be regular exactly at the pairs (P,N_i) for $i=0,\dots,d+1$. Following the ideas in [3], with an obvious adaptation to our case, we resolve the singularities of $\mathcal{X}\times_B\mathcal{T}$ at (P,N_i) by blowing it up along $Y\times T_{i-1}$. The effect of this blowup is:

- The inverse image of (P,N_i) is a smooth rational curve, denoted E_i, along which the blowup is regular.
- The strict transforms of $Y\times T_{i-1}$ and $Z\times T_i$ in the blowup contain E_i, while those of $Y\times T_i$ and $Z\times T_{i-1}$ intersect E_i transversally at unique and distinct points.

- The composition of the blowup with any of the projections of the fibered product $\mathcal{X} \times_B \mathcal{T}$ onto its factors is flat.

Blow up $\mathcal{X} \times_B \mathcal{T}$ repeatedly along the (strict transforms of) $Y \times T_{-1}$, $Y \times T_1$, $\dots$, $Y \times T_d$. Let $\widetilde{\mathcal{X}}$ denote the resulting space and $\mathcal{Y}_i$ denote the strict transform in $\widetilde{\mathcal{X}}$ of $Y \times T_i$ for each $i = -1, \dots, d+1$. If follows that $\widetilde{\mathcal{X}}$ is regular and the $\mathcal{Y}_i$ are Cartier divisors of $\widetilde{\mathcal{X}}$. Let $\psi\colon \widetilde{\mathcal{X}} \to \mathcal{X} \times_B \mathcal{T}$ denote the blowup map. In the sense of [3], p. 32, the map ψ is a semistable modification of any of the projections of the fibered product $\mathcal{X} \times_B \mathcal{T}$ onto its factors

Let $E_i := \psi^{-1}(P, N_i)$ for each $i = 0, \dots, d+1$. Then the E_i are rational smooth curves. Furthermore, $\mathcal{Y}_j \cdot E_i = 0$ if $j \neq i-1, i$, whereas $\mathcal{Y}_{i-1} \cdot E_i = -1$ and $\mathcal{Y}_i \cdot E_i = 1$. Set

$$\mathcal{G} = \mathcal{O}_{\widetilde{\mathcal{X}}}(-\mathcal{Y}_{-1} + \mathcal{Y}_1 + 2\mathcal{Y}_2 + \cdots + (d+1)\mathcal{Y}_{d+1}).$$

Then $\mathcal{G}|_{E_i} \cong \mathcal{O}_{E_i}(1)$ for each $i = 0, \dots, d+1$. Since $\mathcal{G}$ is ψ-admissible, in the sense of [3], p. 32, it follows from *loc. cit.*, Prop. 5.2, that $\psi_*\mathcal{G}$ is relatively torsion-free, rank-1 sheaf of degree 0 on $\mathcal{X} \times_B \mathcal{T}/\mathcal{T}$, whose formation commutes with base change. Finally, set

$$\mathcal{F} := \psi_*\mathcal{G}|_{X \times T}.$$

Since the formation of the direct image $\psi_*\mathcal{G}$ commutes with base change, it is not difficult to see that $\mathcal{F}$ is as prescribed in the construction of $H^r_d(X)$. Indeed, since ψ is an isomorphism over $X \times T_i^*$ for each $i = 0, \dots, d$, it follows that

$$\mathcal{F}|_{X \times T_i^*} = \mathcal{O}_{\mathcal{X} \times_B \mathcal{T}}(iY \times_B \mathcal{T})|_{X \times T_i^*} \cong \mathcal{O}_{\mathcal{X}}(iY)|_X \otimes \mathcal{O}_{T_i^*} = \mathcal{O}_X(-i, i) \otimes \mathcal{O}_{T_i^*}.$$

On the other hand, for each $i = 0, d+1$, the fiber $\psi^{-1}(X \times N_i)$ is the curve obtained from X by splitting apart Y and Z and connecting them by E_i. Let $X_i := \psi^{-1}(X \times N_i)$ and put $\psi_i := \psi|_{X_i}\, X_i \to X$, where we identify $X \times N_i = X$. Let Y_i and Z_i denote the irreducible subcurve of X_i mapping to $Y \times N_i$ and $Z \times N_i$, respectively. Since

$$\mathcal{G}|_{Y_i} \cong \mathcal{O}_{Y_i}(-iP), \quad \mathcal{G}|_{E_i} \cong \mathcal{O}_{E_i}(1) \quad \text{and} \quad \mathcal{G}|_{Z_i} \cong \mathcal{O}_{Z_i}((i-1)P),$$

it follows that

$$\mathcal{F}|_{X \times N_i} = \psi_{i*}(\mathcal{G}|_{X_i}) \cong \mathcal{O}_Y(-iP) \oplus \mathcal{O}_Z((i-1)P).$$

A second construction of $\mathcal{F}$, useful in our computations, is achieved by patching together degenerations of extensions. It follows from the observation that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_Z(iP), \mathcal{O}_Y(jP)) \cong \mathbb{C} \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_Y(lP), \mathcal{O}_Z(mP))$$

for any integers i, j, l, m. Furthermore, the middle sheaves for the nontrivial extensions of $\mathcal{O}_Z(iP)$ by $\mathcal{O}_Y(jP)$ are invertible sheaves on X whose restrictions to Y and Z are $\mathcal{O}_Y((j+1)P)$ and

$\mathcal{O}_Z(iP)$, whereas those for the nontrivial extensions of $\mathcal{O}_Y(lP)$ by $\mathcal{O}_Z(mP)$ are invertible sheaves on X whose restrictions to Y and Z are $\mathcal{O}_Y(lP)$ and $\mathcal{O}_Z((m+1)P)$.

Alternatively, fix isomorphisms $\mathcal{O}_Y(P)|_P \cong \mathcal{O}_P$ and $\mathcal{O}_Z(P)|_P \cong \mathcal{O}_P$, and from these construct isomorphisms $\mathcal{O}_Y(iP)|_P \cong \mathcal{O}_P$ and $\mathcal{O}_Z(jP)|_P \cong \mathcal{O}_P$ for each integers i and j. For each $i = 1, \dots, d$, consider the following composition of maps of sheaves on X:

$$\tau^i \colon \mathcal{O}_Y(-iP) \oplus \mathcal{O}_Z(iP) \longrightarrow \mathcal{O}_Y(-iP)|_P \oplus \mathcal{O}_Z(iP)|_P \xrightarrow{\cong} \mathcal{O}_P^2,$$

where the first map is the sum of two restriction maps and the second is the sum of two of the isomorphisms we mentioned above. Besides, for each $i = 1, \dots, d$, let $T_i := \mathbb{P}^1$, and let $\xi_i \colon \mathcal{O}_{T_i}^2 \to \mathcal{O}_{T_i}(1)$ be the tautological quotient. Let $\tau^i_{T_i}$ be the pullback of τ^i to $X \times T_i$. Identify $P \times T_i = T_i$, and consider the composition $\xi_i \tau^i_{T_i}$. It is a surjection; let $\mathcal{F}_i$ denote its kernel.

Then $\mathcal{F}_i$ is a relatively torsion-free, rank-1 sheaf of degree 0 on $X \times T_i/T_i$. Over the point $0_i \in T_i$ where $\xi_i|_{\mathcal{O}_P \oplus 0}$ is zero, the fiber of $\mathcal{F}_i$ is $\mathcal{O}_Y(-iP) \oplus \mathcal{O}_Z((i-1)P)$, whereas over the point $\infty_i \in T_i$ where $\xi_i|_{0 \oplus \mathcal{O}_P}$ is zero, the fiber of $\mathcal{F}_i$ is $\mathcal{O}_Y(-(i+1)P) \oplus \mathcal{O}_Z(iP)$. Elsewhere, the fiber of $\mathcal{F}_i$ is the invertible sheaf $\mathcal{O}_X(-i, i)$. We may thus patch together the families $\mathcal{F}_i$, by identifying ∞_i with 0_{i+1} for $i = 0, \dots, d-1$. Thus we obtain the chain T of $d+1$ rational curves $T_0, \dots, T_d$, as in the construction of $H^r_d(X)$, and a coherent sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}|_{X \times T_i} = \mathcal{F}_i$. It is clear now that $\mathcal{F}$ is as prescribed in the construction of $H^r_d(X)$.

Embedding $H^r_d(X)$ in a trivial Grassmann bundle.

From the second construction of $\mathcal{F}$, using that $\mathcal{O}_Y(-iP) \subseteq \mathcal{O}_Y$ for each $i \geq 0$ and $\mathcal{O}_Z(iP) \subseteq \mathcal{O}_Z(dP)$ for each $i \leq d$, it follows that $\mathcal{F} \subseteq (\mathcal{O}_Y \oplus \mathcal{O}_Z(dP)) \otimes \mathcal{O}_T$ with T-flat cokernel. Thus, there is a natural embedding

$$H^r_d(X) \hookrightarrow \mathrm{Grass}_{\mathrm{Pic}^{(d,0)}_X \times T}(r+1, \mathcal{W}'), \text{ where } \mathcal{W}' := \mu_* q^*(\mathcal{L}|_{Y \times \mathrm{Pic}^{(d,0)}_X} \oplus \mathcal{L}|_{Z \times \mathrm{Pic}^{(d,0)}_X}(dP \times \mathrm{Pic}^{(d,0)}_X)),$$

and q and μ are as in the construction of $H^r_d(X)$.

The ambient space need not be a Grassmann bundle, as $\mathcal{W}'$ need not be a locally free sheaf. However, $\mathcal{W}'$ is the pullback of a sheaf on $\mathrm{Pic}^{(d,0)}_X$, namely

$$\mathcal{W}'' := p_{2*}(\mathcal{L}|_{Y \times \mathrm{Pic}^{(d,0)}_X} \oplus \mathcal{L}|_{Z \times \mathrm{Pic}^{(d,0)}_X}(dP \times \mathrm{Pic}^{(d,0)}_X)),$$

where $p_2 \colon X \times \mathrm{Pic}^{(d,0)}_X \to \mathrm{Pic}^{(d,0)}_X$ is the second projection. So

$$\mathrm{Grass}_{\mathrm{Pic}^{(d,0)}_X \times T}(r+1, \mathcal{W}') = \mathrm{Grass}_{\mathrm{Pic}^{(d,0)}_X}(r+1, \mathcal{W}'') \times T.$$

Furthermore, let D be an ample effective divisor of X supported away from P. By the

Riemann–Roch Theorem, for $n >> 0$, there are (noncanonical) embeddings

$$\mathcal{L}|_{Y\times \mathrm{Pic}_X^{(d,0)}} \hookrightarrow \mathcal{O}_{X\times \mathrm{Pic}_X^{(d,0)}}(D\times \mathrm{Pic}_X^{(d,0)})|_{Y\times \mathrm{Pic}_X^{(d,0)}},$$
$$\mathcal{L}|_{Z\times \mathrm{Pic}_X^{(d,0)}}(dP\times \mathrm{Pic}_X^{(d,0)}) \hookrightarrow \mathcal{O}_{X\times \mathrm{Pic}_X^{(d,0)}}(D\times \mathrm{Pic}_X^{(d,0)})|_{Z\times \mathrm{Pic}_X^{(d,0)}}$$

whose cokernels are flat over $\mathrm{Pic}_X^{(d,0)}$. Thus, setting $W''' := W_1 \oplus W_2$, where $W_1 := H^0(Y, \mathcal{O}_Y(D\cap Y))$ and $W_2 := H^0(Z, \mathcal{O}_Z(D\cap Z))$, we get a (noncanonical) embedding $\mathcal{W}'' \subseteq W''' \otimes \mathcal{O}_{\mathrm{Pic}_X^{(d,0)}}$, and thus a (noncanonical) embedding

$$\mathrm{Grass}_{\mathrm{Pic}_X^{(d,0)}}(r+1, \mathcal{W}'') \hookrightarrow \mathrm{Grass}(r+1, W_1\oplus W_2)\times \mathrm{Pic}_X^{(d,0)}.$$

The bottom line is that we have an embedding

$$H_d^r(X) \hookrightarrow \mathrm{Grass}(r+1, W_1\oplus W_2)\times \mathrm{Pic}_X^{(d,0)}\times T.$$

4.1.2 Stable maps, torus actions and Grassmann bundles.

Loosely speaking, a stable limit linear series on X will be represented by a stable map from a chain S of smooth rational curves to the space $H_d^r(X)$ we constructed in Section 4.1.1. Not just any such map, but only those fixed by a certain torus action and whose image lives in a certain homology class in $H_2(H_d^r(X), \mathbb{Z})$. In this section we will explain all the notions involved in the statements above. Obviously, there exists a wide literature on spaces of stable maps and fixed spaces of schemes under torus actions, as well as on Grassmann bundles. For a more general treatments, we refer the reader to [22], [32], [36] for stable maps and torus actions, and to [21], [23] for Grassmann bundles. For the convenience of the reader, we present without proofs the relevant material from the above references, adapted to our situation, thus making our exposition as self-contained as possible.

Before we proceed, we describe the layout of this section. In the first subsection, we present the moduli space of stable maps of curves of genus zero to any projective scheme. The second subsection is reserved for the definition, classical results, and relevant examples of torus actions on moduli spaces of stable maps. Finally, the third subsection is dedicated to the study of the Chow ring of Grassmann bundles.

Stable maps from nodal curves and their moduli spaces.

A *nodal curve* is a (projective, reduced and connected) curve C whose irreducible components intersect each other and self-intersect transversally or, equivalently, whose singularities are ordinary double points; nodes, for short. Its (arithmetic) genus is 0 if and only if the components are rational, smooth, and form a tree, or equivalently, intersect in such a way that the number of intersection points is smaller (by one) than the number of components.

A *family* of nodal curves parametrized by a scheme B is a flat, projective map $\pi\colon \mathcal{C} \to B$ such that $\mathcal{C}_b$ is nodal curve for each geometric point b of B. A *map* from nodal curve C to a scheme G is a map $\mu\colon C \to G$. A *family of maps* parametrized by a scheme B from nodal curves to a scheme G is defined by the tuple of data

$$(\pi\colon \mathcal{C} \to B, \mu\colon \mathcal{C} \to G),$$

consisting of a family nodal curves π and a morphism μ. Two families of maps to G parametrized by the same B,

$$(\pi\colon \mathcal{C} \to B, \mu) \text{ and } \left(\pi' : \mathcal{C}' \to B, \mu'\right),$$

are called *isomorphic* if $m = n$ and there exists an isomorphism $\tau : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $\pi = \pi' \circ \tau$ and $\mu = \mu' \circ \tau$, that is, the diagrams below commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[d, "\tau"'] & B \\ \mathcal{C}' \arrow[ur, "\pi'"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mu"] \arrow[d, "\tau"'] & G \\ \mathcal{C}' \arrow[ur, "\mu'"'] & \end{tikzcd}$$

The *special points* of an irreducible component of a nodal curve $(C, p_1, \ldots, p_n)$ are the nodes that lie on it. A map $\mu\colon C \to G$ from curve C to a scheme G is called *stable* if each rational, smooth component of C that is mapped to a point by μ contains at least three special points. For a given class $\beta \in H_2(G, \mathbb{Z})$, the map μ is said to *represent* β if $\mu_*[C] = \beta$. When β is represented, it is called *effective.* A family (π, μ) of maps from nodal curves to G parameterized by a scheme B is called *stable* if the induced map over each geometric point of B is stable.

Let G be a scheme and $\beta \in H_2(G, \mathbb{Z})$. Define the contravariant functor:

$$\begin{array}{ccccc} \overline{\mathcal{M}}_0(G, \beta) & : & \{\text{Schemes}\} & \longrightarrow & \{\text{Sets}\} \\ & & B & \mapsto & \overline{\mathcal{M}}_0(G, \beta)(B), \end{array}$$

where $\overline{\mathcal{M}}_0(G, \beta)(B)$ is the set of isomorphism classes of stable families of maps from genus-0 nodal curves to G parameterized by B and representing the class β.

We are now in position to present results on stable maps. As they are not the main focus of this article, the results will be presented without proofs. Assume from now on that G is a nonsingular projective variety, and assume that G is *convex*, that is, $H^1\left(\mathbb{P}^1, \mu^*(T_G)\right) = 0$ for every morphism $\mu : \mathbb{P}^1 \to G$. This is the case, for instance, when T_G is generated by global sections. Examples of convex varieties are thus projective spaces, Grassmannians, flag varieties and their products. The proof of the following result may be found in [22], Thms. 1 and 2, p. 11–12.

Theorem 4.2. *Let G be a projective scheme and $\beta \in H_2(G, \mathbb{Z})$. Then there exists a projective scheme $\overline{M}_0(G, \beta)$ coarsely representing $\overline{\mathcal{M}}_0(G, \beta)$. If in addition G is a convex variety then*

$\overline{M}_0(G, \beta)$ *is a normal projective variety, which is locally a quotient of a nonsingular variety by a finite group.*

While we will just need the existence of a coarse moduli space, it is appropriate that we say a few words about the existence of a fine moduli space. It turns out, as it is typical, that the existence of automorphisms forces us to search for a fine moduli space outside the category of schemes, in the category of *stacks*. Though the category of stacks is often the appropriate one in the study of moduli problems, to introduce and deal with it is beyond the scope of this article. At any rate, in our case, we have (see [32])

Theorem 4.3. *Let G be a convex projective variety and $\beta \in H_2(G, \mathbb{Z})$. Then the functor $\overline{\mathcal{M}}_0(G, \beta)$ is (finely) represented by a complete, nonsingular Deligne–Mumford stack.*

In any case, in one language or in the other, the points parametrized by $\overline{M}_0(G, \beta)$ correspond to isomorphism classes of stable maps of the same type β.

The following result will be extremely useful in our proof of the existence of a coarse moduli space for stable limit linear series. It is a natural consequence of [22], Thm. 1, p. 11, and Lemma 8, p. 26.

Proposition 4.4. *Let G be a projective scheme and $G' \subseteq G$ a closed subscheme. Let $i\colon G' \to G$ denote the inclusion and $\beta \in H_2(G', \mathbb{Z})$. Then there exists a natural closed embedding*

$$\overline{\mathcal{M}}_0(G', \beta) \hookrightarrow \overline{\mathcal{M}}_0(G, i_*\beta).$$

Torus actions on moduli spaces of stable maps.

In this section we will study certain actions of the one-dimensional torus $\mathbb{T} := \mathbb{C}^*$ on spaces of stable maps from nodal curves to Grassmannians. More precisely, we will study special linear actions of $\mathbb{T}$ on $\mathbb{C}^m$, for any m, the induced actions on Grassmannians $G := \operatorname{Grass}(r, m)$, for any r, and the resulting actions on $\overline{M}_0(G, d)$, the coarse moduli space of stable maps of degree d from nodal curves to G, for any d and n. Our main goal is to describe the stable maps that are represented by fixed points under such actions. From the analysis made here we will derive important consequences for our understanding of stable maps into $H^r_d(X)$, the spaces of linear series introduced at the beginning of the chapter.

Recall that G is a nonsingular projective variety with Picard group $\operatorname{Pic}(G) = \mathbb{Z}$. Thus $H_2(G, \mathbb{Z})$ is free of rank 1. We say that an element of $H_2(G, \mathbb{Z})$ has degree d if it can be expressed as d times the effective generator of $H_2(G, \mathbb{Z})$, or equivalently, if its product with the ample generator of $\operatorname{Pic}(G)$ is d. A map $\mu\colon \mathbb{P}^1 \to G$ is said to be of degree d if $\mu_*[\mathbb{P}^1]$ has degree d, or equivalently, if the pullback of the ample generator of $\operatorname{Pic}(G)$ to $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(d)$.

Thus, a map $\mu\colon C \to G$ from a genus-0 projective nodal curve C gives a stable map of degree d if and only if it satisfies the following two conditions:

- For each component C_i of C, the degree of the restriction $\mu|_{C_i}$ being denoted by d_i, we have $\sum_i d_i = d$.
- If $\mu|_{C_i}$ is constant, or equivalently, $d_i = 0$, then C_i must contain at least three special points.

Our next step is to present the actions of $\mathbb{T}$ on $\mathbb{C}^m$ that will be useful for us. The first is very simple: Write $m = m_1 + m_2$ and consider the canonical decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}$; then $c *_1 (v_1, v_2) := (cv_1, v_2)$ for each $c \in \mathbb{T}$ and $v_i \in \mathbb{C}^{m_i}$. The second action is also easy to describe: Given distinct nonnegative integers $\ell_1, \ldots, \ell_m$, put $c*_2(z_1, \ldots, z_m) := (c^{\ell_1} z_1, \ldots, c^{\ell_m} z_m)$ for each $c \in \mathbb{T}$ and $z_i \in \mathbb{C}$. Since the actions are linear, given an r-dimensional subspace $V \subseteq \mathbb{C}^m$, it follows that $c *_i V$ is an r-dimensional subspace of $\mathbb{C}^m$ for $i = 1, 2$. In other words, there is an induced action of $\mathbb{T}$ on G in both cases.

In terms of matrices, the first action is represented by a matrix of the form

$$A_c^1 := \begin{pmatrix} c & 0 & \cdots & \cdots & 0 \\ 0 & c & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}, \tag{4.5}$$

while the second is represented by a matrix of the form

$$A_c^2 := \begin{pmatrix} c^{\ell_1} & 0 & \cdots & \cdots & 0 \\ 0 & c^{\ell_2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & c^{\ell_{m-1}} & 0 \\ 0 & 0 & \cdots & \cdots & c^{\ell_m} \end{pmatrix}. \tag{4.6}$$

We now focus on the study of the fixed points by these actions on G. Naturally, this question is related to the study of the invariant subspaces of the matrices A_c^i.

In the first case, the eigenvalues of the matrix A_c^1 are c and 1, with arithmetic multiplicity m_1 and m_2, respectively. Therefore, by [25] Thm. 2.15 , we get that $V \in G$ is fixed by $\mathbb{T}$, that is, $c *_1 V = V$ for every $c \in \mathbb{T}$, if and only if $V = V_1 \oplus V_2$ where $V_1 \subseteq \mathbb{C}^{m_1}$ and $V_2 \subseteq \mathbb{C}^{m_2}$. Clearly, if r_i denotes the dimension of V_i for $i = 1, 2$, then $r = r_1 + r_2$.

As for the second case, the eigenvalues of A_c^2 are $c^{\ell_1}, \ldots, c^{\ell_2}$, and they are all distinct if c does not belong to the finite set of all the $(\ell_i - \ell_j)$-th roots of unity for $i \neq j$. In this case, by [25] Example 2.1.1, we get that $V \in G$ is fixed by $\mathbb{T}$, that is, $c *_2 V = V$ for every $c \in \mathbb{T}$, if and only if V is spanned by a set of r vectors of the canonical basis of $\mathbb{C}^m$.

Given a linear action $*$ of $\mathbb{T}$ on $\mathbb{C}^m$, the induced action of $\mathbb{T}$ on $\overline{M}_0(G, d)$ is also easy to define. Just notice that, given a stable map (C, μ), and $c \in \mathbb{T}$, the map $c * \mu$ assigning $Q \in C$

to $\{c * v \,|\, v \in \mu(Q)\}$ gives a stable map $(C, c * \mu)$. Clearly, isomorphic stable maps give rise to isomorphic stable maps under this action. Hence, we have a well-defined action of $\mathbb{T}$ on $\overline{M}_0(G,d)$: given a point $(C,\mu) \in \overline{M}_0(G,d)$ and $c \in \mathbb{T}$, we have

$$c * (C, \mu) := (C, c * \mu).$$

Our next goal is to describe the points in $\overline{M}_0(G,d)^{\mathbb{T}}$, that is, the torus fixed stable maps in the first case. A similar conclusion can be drawn in the second case.

First, we make some general remarks on the action of $\mathbb{T}$ on G, under the assumption that $r \leq m_1$ and $r \leq m_2$. In this case, as a consequence of the study of the direct sum on Grassmanians in ([43]), Section 2, the fixed point locus is the union of certain products of Grassmanians: $G_{r_1,r_2} := G_{r_1} \times G_{r_2}$ where $G_{r_1} := \mathrm{Grass}(r_1, m_1)$ and $G_{r_2} := \mathrm{Grass}(r_2, m_2)$, for $r_1 + r_2 = r$, where G_{r_1,r_2} is embedded in G in the natural way.

In particular, the images by a torus fixed map $\mu : T \to G$ of the nodes and the contracted components are fixed points in one of the G_{r_1,r_2}.

Clearly, (C,μ) is a fixed map if and only if its restrictions $(C_i, \mu|_{C_i})$ are all fixed. Now, if C_i is not contracted by μ, then $\mu(C_i)$ is either entirely contained in some G_{r_1,r_2} or, otherwise, as an application of the Localization Theorem (see [36], Lemma 6, p. 12 or [1], Prop. 6, p. 8), is an invariant curve joining two fixed points lying on distinct G_{r_1,r_2}.

In conclusion, for the first action the image of a torus fixed stable map $\mu : T \to G$ is an invariant curve in G, the contracted components and nodes sent to fixed points, and the non-contracted components sent to curves either entirely contained in some G_{r_1,r_2} or linking two fixed points lying on distinct G_{r_1,r_2}.

An analogous reasoning applies for the second action. In this case, however, since the ℓ_i are distinct, there are only a finite number of fixed points, and thus the noncontracted components are sent to curves linking two fixed points lying on distinct G_{r_1,r_2}.

Remark 4.5. Though we are considering only diagonal actions, no generality is lost, as all actions of $\mathbb{T}$ on $\mathbb{C}^m$ can be diagonalized; see, for instance, [13], Ch. 6, Prop. 1.6, p. 6. More precisely, given an action $*$ of $\mathbb{T}$ on $\mathbb{C}^m$, there exist a basis $v_1, \ldots, v_m$ of $\mathbb{C}^m$ and integers $\ell_1, \ldots, \ell_m$ such that, for each $c \in \mathbb{T}$ and $x_i \in \mathbb{C}$,

$$c * (\sum_i x_i v_i) = \sum_i c^{\ell_i} x_i v_i.$$

The Chow Group of Grassmann Bundles.

In this section, our main goal is to present the fundamental tools to define the class $\beta \in H_2(H^r_d(X), \mathbb{Z})$ we are interested in. More precisely, it will be the class of a certain section $\lambda \colon T \to H^r_d(X)$ of the composition

$$H^r_d(X) \xrightarrow{\nu} \mathrm{Pic}^{(d,0)}_X \times T \xrightarrow{q_2} T$$

of the structure map ν with the projection q_2, a section corresponding to an Osserman exact limit linear series in a way we will make precise later. We will view λ as a translation from a discrete to a continuous point of view.

Whatever λ is, notice that the composition $q_1\nu\lambda$ is constant, as T is a union of rational curves and $\mathrm{Pic}_X^{(d,0)}$ is an Abelian variety. Thus λ factors through $H_d^r(X,L)$, the fiber of $q_1\lambda$ over a certain $L \in \mathrm{Pic}_X^{(d,0)}$.

As we have seen in Subsection 4.1.1 that $H_d^r(X)$ can be embedded in a natural (though not canonical) way in a Grassmann bundle over $\mathrm{Pic}_X^{(d,0)} \times T$, we get a natural embedding of $H_d^r(X,L)$ in a Grassmann bundle G over T. Since $A_1(G) \cong H_2(G,\mathbb{Z})$ in this case, we will focus on presenting the basic facts on the Chow group $A_*(G)$. We begin by recalling fundamental facts about Chow rings, Chern classes and Grassmann bundles. Our main references are [21] Chs. 3, 10 and 14 and [23] Chs. 1 and 2.

The Chow group of a scheme G is

$$A_*(G) := \bigoplus_{\ell \geq 0} A_\ell(G),$$

where $A_\ell(G)$ is the group of ℓ-dimensional cycles modulo rational equivalence. In particular, when G is purely n-dimensional, $A_n(G)$ is the free Abelian group on the set of irreducible components of G.

If G is irreducible and nonsingular of dimension n, define the *Chow group of cycles of G of codimension ℓ* as $A^\ell(G) := A_{n-\ell}(G)$, and put $A^*(G) := \oplus A^\ell(G)$. In this case, $A^*(G)$ admits the structure of a graded ring via the *intersection product*:

$$\begin{array}{cccc} \cdot \ : & A^\ell(G) \times A^m(G) & \longrightarrow & A^{\ell+m}(G) \\ & (\alpha,\beta) & \mapsto & \alpha\cdot\beta. \end{array}$$

It would take us too much astray to give the definition of $\cdot$. Let us just say that, justifying the name, if E and F are subvarieties of G such that their intersect is proper, that is, $E \cap F$ is of pure codimension $\mathrm{codim}(E) + \mathrm{codim}(F)$, then

$$[E]\cdot[F] = [E \cap F] = \sum_H m_H[H],$$

where the sum runs over the irreducible components H of $E \cap F$, and the m_H are the *geometric multiplicities* of H in $E \cap F$; see for instance [21] Ex. 8.1.11 or [23], p. 32.

As usual, we may view $A_*(G)$ as a module over $A^*(G)$ if G is nonsingular, that is, the elements of $A^*(G)$ induce endomorphisms of the group $A_*(G)$. In any case, even if G is singular, certain endomorphisms arise from vector bundles. More precisely, given a vector bundle E over G of constant rank r, its *Chern classes* $c_i(E)$, and thus the *total Chern class*

$$c(E) = 1 + c_1(E) + \cdots + c_r(E),$$

are endomorphisms of $A_*(G)$, denoted $\alpha \mapsto c_i(E) \cap \alpha$, satisfying the following properties (see [21] Thm. 3.2):

1. (Vanishing) $c_0(E) = 1$ and $c_i(E) = 0$ for $i < 0$ or $i > r$.
2. (Projection formula) For each proper map $f : H \to G$,

$$f_* \left(c_i(f^*E) \cap \alpha\right) = c_i(E) \cap f_*\alpha.$$

3. (Flat pullback) For each flat map $f : H \to G$,

$$c_i(f^*E) \cap f^*\alpha = f^* \left(c_i(E) \cap \alpha\right).$$

4. If D is a Cartier divisor of G then $c_1(\mathcal{O}_G(D)) \cap [G] = [D]$.
5. (Whitney sum) For each exact sequence of vector bundles over G

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E_2 \longrightarrow 0,$$

the total Chern class satisfies $c(E_0) = c(E_1)c(E_2)$; equivalently, $c_\ell(E_0) = \sum_{i+j=\ell} c_i(E_1)c_j(E_2)$ for each integer ℓ.

When G is nonsingular and irreducible, we identify $c_i(E)$ with an element of $A^i(G)$, namely $c_i(E) \cap [G]$. In this case, we have an isomorphism $\mathrm{Pic}(G) \cong A^1(G)$ defined by $L \mapsto c_1(L)$.

We are now in a position to give a rough introduction to the intersection theory of Grassmann bundles. Let E be a vector bundle of rank d over a scheme U, and r an integer such that $0 < r < d$. Let $G := \mathrm{Grass}(r, E)$ the Grassmann bundle of r-planes in the fibers of E, and $\pi\colon G \to U$ the structure map. From the universal short exact sequence on G:

$$0 \longrightarrow S \longrightarrow \pi^*(E) \longrightarrow Q \longrightarrow 0,$$

where S is the universal rank-r subbundle and Q is the universal rank-$(d-r)$ quotient bundle, define $c_i := c_i(Q - \pi^*E)$. More precisely, c_i is the degree-i part of the quotient:

$$c(Q-\pi^*E) := \frac{c(Q)}{c(\pi^*(E))} = 1+(c(Q)-c(\pi^*E))+(c_2(Q)-c_1(Q)c_1(\pi^*E)+c_1^2(\pi^*E)-c_2(\pi^*E))+\cdots.$$

A partition $\lambda := (\lambda_1, \ldots, \lambda_r)$ is a sequence of r integers such that $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Given λ, define the Schur polynomial

$$\Delta_\lambda := \Delta_\lambda(c) := \det \begin{pmatrix} c_{\lambda_1} & \cdots & c_{\lambda_r+r-1} \\ \vdots & \vdots & \vdots \\ c_{\lambda_1-r+1} & \cdots & c_{\lambda_r} \end{pmatrix} = \det(c_{\lambda_j+j-i}(Q - \pi^*E))$$

The Schur polynomials have the following properties:

1. (Vanishing, [21] Lemma 14.5.1) If q is an integer such that $C_i = 0$ for all $i > q$ and $\lambda_q > 0$ then $\Delta_\lambda = 0$.

2. (Pieri's formula, [21] Prop. 14.6.1) For each partition λ and integer $m \geq 0$, we have $\Delta_\lambda \cdot c_m = \sum_\mu \Delta_\mu$, where $\mu = (\mu_1, \ldots, \mu_r)$ runs over all partitions satisfying $|\mu| = |\lambda| + m$ and $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq \lambda_r$.

3. (Product formula, [21] Prop. 14.6.2) For each two partitions λ and μ, it obtains that $\Delta_\lambda \cdot \Delta_\mu = \sum_\eta N_{\lambda,\mu,\eta} \Delta_\eta$, where the sum runs over all partitions η with $|\eta| = |\mu| + |\lambda|$, and the $N_{\lambda,\mu,\eta}$ are given by the Littlewood–Richardson rule.

4. (Duality theorem, [21] Prop. 14.6.3) For each two partitions λ and μ such that $|\lambda| + |\mu| \leq r + 1(d - r)$, and each $\alpha \in A_*(U)$, we have

$$\pi_* (\Delta_\lambda \cdot \Delta_\mu \cap \pi^* \alpha) = \begin{cases} \alpha & \text{if} \quad \lambda_i + \mu_{r-i+1} = d - r \quad \text{for} \quad i = 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases}$$

The next result will be useful in the description of our class β. For a proof, see [21] Prop. 14.6.5.

Lemma 4.6. (Basis Theorem).
For each $\ell \geq 0$, there is a canonical isomorphism

$$A_\ell(G) \cong \bigoplus_\lambda A_{\ell - r(d-r) + |\lambda|}(U),$$

where λ runs over all partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $d - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Moreover, each element in $A_\ell(G)$ has a unique expression in the form $\sum_\lambda \Delta_\lambda \cap \pi^*(\alpha_\lambda)$ with $\alpha_\lambda \in A_{\ell - r(d-r)+|\lambda|}(U)$.

In particular, when U is nonsingular, $A^*(G)$ is the algebra over $A^*(U)$ with generators $a_1, \ldots, a_r$, $b_1, \ldots, b_{d-r}$, where $a_i = c_i(S)$ and $b_i = c_i(Q)$, and relations $\sum_{i=0}^{\ell} a_i b_{\ell-i} = c_\ell(E)$ for $\ell = 1, \ldots, d$; see [21] Ex. 14.6.6, p. 270.

Our main analysis is made on Grassmannians, that is, Grassman bundles over a point. In this case, the above formulas make part of what we call the *Schubert calculus.* Changing the notation, $G := \mathrm{Grass}(d+1, \mathbb{C}^{n+1})$, or, in projective terms, $G = G_d(\mathbb{P}^n)$, the Grassmann variety of d-planes in $\mathbb{P}^n$. It is a smooth and irreducible projective variety of dimension $(d+1)(n-d)$. The universal quotient bundle Q has rank $n - d$. In this case, the classes $\sigma_i := c_i = c_i(Q)$ and $\{\lambda_0, \ldots, \lambda_d\} := \Delta_\lambda = \Delta_\lambda(\sigma)$, for $i = 0, \ldots, n-d$ and partitions $\lambda = (\lambda_0, \ldots, \lambda_d)$ with $n - d \geq \lambda_0 \geq \cdots \geq \lambda_d \geq 0$, are called *special Schubert classes* and *Schubert classes*, respectively. (Indeed, $\sigma_i = \{i, 0 \ldots, 0\}$.) It follows from the general theory presented above that the Schubert classes form a free $\mathbb{Z}-$basis of $A^*(G)$. Their product is determined by the product formula.

The celebrated *Giambelli's formula* in this context is

$$\{\lambda_0, \ldots, \lambda_d\} \cap [G] = (a_0, \ldots, a_d) = [\Omega(A_0, \ldots, A_d)], \tag{4.7}$$

where $a_i := n - d + i - \lambda_i$ for $i = 0, \ldots, d$, and $\Omega(A_0, \ldots, A_d)$ is an associated *Schubert variety*, the $\sum_i (a_i - i)$−dimensional subvariety of G defined by a flag of linear subspaces $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_d \subseteqq \mathbb{P}^n$ with $a_i = \dim(A_i)$ for $i = 0, \ldots, d$ as follows:

$$\Omega(A_0, \ldots, A_d) := \{L \in G;\, \dim(L \cap A_i) \geq i,\, 0 \leq i \leq d\}.$$

The class $[\Omega(A_0, \ldots, A_d)]$ of $\Omega(A_0, \ldots, A_d)$ depends only on the a_i.

With this notation, $\sigma_i = (n-d-i, n-d+1, \ldots, n)$ for $i = 0, \ldots, n-d$. In addition, for each $\alpha \in A_\ell(G)$, the expression of α in terms of Schubert classes of dimension ℓ is

$$\alpha = \sum \alpha_{a_0,\ldots,a_d}(a_0, \ldots, a_d), \quad \text{where} \quad \alpha_{a_0,\ldots,a_d} = \int_G \alpha \cdot (n - a_0, \ldots, n - a_d).$$

Example 4.1. Let $T = T_0 \cup \cdots \cup T_d$ be the chain of $d+1$ projective lines. In contrast to the nonsingular case, $A_0(T) \neq \mathrm{Pic}(T)$. Indeed, $\mathrm{Pic}(T) = \mathbb{Z}^{d+1}$, as each line bundle over T is determined by its restrictions to the T_i, and the restrictions by their degrees. On the other hand, $A_0(T) = \mathbb{Z}$. Indeed, if N_i is the point of intersection between T_{i-1} and T_i, then $[Q] = [N_i]$ for each $Q \in T_{i-1} \cup T_i$.

Let $\mathrm{Grass}_T(r+1, V) = \mathrm{Grass}(r+1, V) \times T =: G \times T$, for a vector space V of dimension $n+1 \geq r+1$. By the Künneth Formula (see [21], Ex. 1.10.2, p. 25), there is a natural surjection:

$$\begin{array}{cccc} \times \;:\; & \bigoplus_{i+j=1} A_i(G) \otimes A_j(T) & \longrightarrow & A_1(G \times T) \\ & (\alpha, \beta) & \mapsto & \alpha \times \beta. \end{array}$$

Since $A_1(T) = \bigoplus_i \mathbb{Z} \cdot [T_i]$ and $A_1(G) \cong \mathbb{Z} \cdot (0, 1, \ldots, r, r+1)$, we conclude that any $\gamma \in A_1(G \times T)$ can be expressed as

$$\gamma = \sum a_i([\mathrm{pt}] \times [T_i]) + b((0, 1, \ldots, r, r+1) \times [\mathrm{pt}]),$$

for some $a_i, b \in \mathbb{Z}$.

The integer b can be determined as follows. Considerer the Plücker embedding of G, determined by the line bundle $\wedge^{n-r} Q$; then σ_1 becomes the class of a hyperplane section. Under this embedding, $b = \deg(p_{1*}\gamma) := \int_G (p_{1*}\gamma) \cdot \sigma_1$, where $p_1 \colon G \times T \to G$ is the projection. In fact, the expression for b follows from the equality (see [21] Ex. 14.7.4, p. 272):

$$\int_G (d - r - 1, d - r + 1, \ldots, d, d + 1) \cdot (0, 1, \ldots, r, r + 1) = 1.$$

Remark 4.7. Recall that we are interested in describing a certain class $\beta \in H_2(H^r_d(X))$, which will actually be described in $H_2(G \times T)$ for a certain Grassmannian G, where T is the chain $T_0, \ldots, T_d$ of projective lines. However, $A_1(G \times T) \cong H_2(G \times T)$. (Hence, the above example describes the group we are interested in.) First notice that the cycle map $cl_{\mathbb{P}^1} : A_1(\mathbb{P}^1) \to H_2(\mathbb{P}^1)$ is an isomorphism; see [21] Section 19.1. (Actually, $\mathbb{P}^1$ is the unique projective curve for which this holds; see [21] Ex. 19.1.11, p. 378.) Thus, since $A_1(T) = \bigoplus_i A_1(T_i)$, we conclude that $A_1(T) \cong H_2(T)$ via the cycle map. Second, by [21], Ex. 19.1.11(d), p. 378, we have that the cycle map of a Grassmann, or more generally a flag, bundle is an isomorphism if and only if the cycle map of the base is so, which is precise the case of $G \times T$.

4.2 Stable limit linear series.

In this section we define what stable limit linear series are, and construct their moduli space as that of torus fixed points in a certain moduli space of genus-0 stable maps to $H^r_d(X)$. After this, we explain the connection with level-δ limit linear series.

We sketch now briefly the contents of this section. In the first subsection, we introduce the notion of a stable limit linear series and the construction of its functor. This new definition is apparently artificial, although will make sense later on.

In the second subsection, we will be concerned with the study the stable maps from genus zero curves to the scheme $H^r_d(X)$, which parameterizes generalized linear series (briefly, gls) on X along of a chain T of the its relation to a family of gls. Loosely speaking, this relation is the translation from the discrete concept: of level-δ exact points, to the continuous concept: of fixed stable maps by of the torus action.

Here the Ossermann exact points has attached a special homological classes which will determine, in some sense, the type all level-δ exact. Finally, in the third subsection we present the equivalence between the two functors and will be proved that the functor the fixed stable maps is coarsely represented by a projective scheme $\overline{M}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$.

4.2.1 The functor of stable limit linear series.

Follow the notation used in Sections 4.1.2 and 4.1.1. We will define a special type of families of linear series over X.

Definition 4.8. Let B be an algebraic scheme and T a chain of $d+1$ projective lines. A *family of chain maps to T parametrized by B*, denoted $(\pi\colon \mathcal{S} \to B, \mu : \mathcal{S} \to T)$, consists of:

1. a family of chains $(\pi\colon \mathcal{S} \to B, 0, \infty)$, that is, a family of curves $\pi\colon \mathcal{S} \to B$ such that the fiber $\mathcal{S}_b$ over each geometric point $b \in B$ is a chain of projective lines.

2. a morphism $\mu : \mathcal{S} \to T$ such that *it contracts to T*, i.e., for every $b \in B$, we have $\mu_{b*}[\mathcal{S}_b] = [T]$.

Consider the family of twisters $\mathcal{F}$ on $X \times T/T$ constructed in Remark 4.1.1. Let B be an algebraic scheme, and $\mathcal{C} := (\pi\colon \mathcal{S} \to B, \mu\colon \mathcal{S} \to T)$ be a family of chain maps to T parametrized by B. Let $\mathcal{L}$ be an invertible sheaf on $X \times B$ of (relative) multidegree $(d, 0)$ over B. From the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\mu"] \arrow[d, "\pi"'] & T \\ B, & \end{tikzcd}$$

we have the diagram below:

$$\begin{tikzcd} & X \times \mathcal{S} \arrow[dl, "(\mathrm{id},\pi)"'] \arrow[d, "p_{\mathcal{S}}"] \arrow[dr, "(\mathrm{id},\mu)"] & \\ X \times B & \mathcal{S} & X \times T, \end{tikzcd} \tag{4.8}$$

where $p_{\mathcal{S}}$ is the projection. We will call the sheaf

$$\mathcal{L}(\mathcal{C}) := \mathcal{L} \boxtimes \mathcal{F} = (\mathrm{id}, \pi)^* \mathcal{L} \otimes (\mathrm{id}, \mu)^* \mathcal{F}$$

on $X \times \mathcal{S}$ the *family of twists of $\mathcal{L}$ along the family of chain maps $\mathcal{C}$.*

Definition 4.9. A *family of generalized linear series* of degree d and dimension r over X consists of the following data:

1. a family of chain maps $\mathcal{C} := (\pi\colon \mathcal{S} \to B, \mu\colon \mathcal{S} \to T)$;
2. an invertible sheaf $\mathcal{L}$ on $X \times B$ of relative multidegree $(d, 0)$ over B;
3. a family $\mathcal{V} \subseteq p_{\mathcal{S}*}\mathcal{L}(\mathcal{C})$ of linear series of dimension r over X along $\mathcal{S}$ of sections of $\mathcal{L}(\mathcal{C})$.

Two families $(\mathcal{L}, \mathcal{V})$ and $(\mathcal{L}', \mathcal{V}')$ along the same family of chain maps $\mathcal{C}$ are said to be *equivalent* if there is an invertible sheaf $\mathcal{Q}$ on $\mathcal{S}$ and an isomorphism $\mathcal{L}(\mathcal{C}) \cong \mathcal{L}'(\mathcal{C}) \otimes p_{\mathcal{S}}^* \mathcal{Q}$ inducing an isomorphism $\mathcal{V} \cong \mathcal{V}' \otimes \mathcal{Q}$.

When B is a point, the family becomes a single generalized linear series. And, from the definition, a family $(\mathcal{L}, \mathcal{V})$ along a family of chain maps $(\pi\colon \mathcal{S} \to B, \mu\colon \mathcal{S} \to T)$ gives rise to a generalized linear series $(\mathcal{L}_b, \mathcal{V}_b)$ along the chain map $(\mathcal{S}_b, , \mu_b\colon \mathcal{S}_b \to T)$ for each geometric point $b \in B$.

A family of generalized linear series $(\mathcal{L}, \mathcal{V})$ along a chain map $(\mathcal{S}, \mu\colon \mathcal{S} \to T)$ is called *everywhere nonconstant* if, for every component C of $\mathcal{S}$ contracted to a point $t \in T$, the vector subspaces $\mathcal{V}_s \subseteq H^0(X, \mathcal{L} \otimes \mathcal{F}_t)$ vary as $s \in C$.

There is one extra property we need to address, namely we would like our family of generalized linear series to be as "locally constant" as the family of twisters $\mathcal{F}$ is.
Indeed, recall that for any i and any $t \in T_i^*$, we have that $\mathcal{F}|_{X \times t} \cong \mathcal{O}_X(-i, i)$ and $\mathcal{F}|_{X \times T_i^*} \cong$

$\mathcal{F}|_{X\times t}\otimes\mathcal{O}_{T_i^*}$. Moreover, recall that, choosing coordinates $(a:b)$ for T_i such that N_i corresponds to $(0:1)$ and N_{i+1} to $(1:0)$, we have that $\mathcal{F}_{X\times\{(a:b)\}} = \mathrm{Ker}(\rho(a,b))$, where $\rho(a:b)$ is the composition below:

$$\begin{tikzcd} \mathcal{O}_Y(-iP)\oplus\mathcal{O}_Z(iP) \arrow[r] \arrow[rr, bend left, "\rho(a,b)"] & \mathcal{O}_P\oplus\mathcal{O}_P \arrow[r, "(a,b)"'] & \mathcal{O}_P \end{tikzcd}.$$

Thus, the natural inclusions $\mathcal{O}_Y(-iP)\oplus\mathcal{O}_Z(iP)\subseteq\mathcal{O}_Y\oplus\mathcal{O}_Z(dP)$ for each $i=0,\dots,d$ give rise to a natural inclusion of sheaves on $X\times T$:

$$\mathcal{F}\subseteq(\mathcal{O}_Y\oplus\mathcal{O}_Z(dP))\otimes\mathcal{O}_T\subseteq\mathcal{O}_{Y\times T}\oplus\mathcal{O}_{Z\times T}(dP\times T). \tag{4.9}$$

Consequently, if L is an invertible sheaf on X with degree d on Y and 0 on Z, we have that

$$L\boxtimes\mathcal{F}:=p^*L\otimes\mathcal{F}\subseteq(L_Y\oplus L_Z(dp))\otimes\mathcal{O}_T, \tag{4.10}$$

where $p\colon X\times T\to T$ is the projection. More precisely, for each i and $t\in T_i$, we have $L\otimes\mathcal{F}|_{X\times t}\subseteq L_i|_Y\oplus L_i|_Z$, with $L_i|_Y:=L|_Y(-iP)$ and $L_i|_Z:=L|_Z(iP)$.

Now, there is a natural action of the torus $\mathbb{T}$ on each T_i, fixing N_i and N_{i+1}, namely $c*(a:b)=(ca:b)$. (As we will be interested in the orbits of lifts of this action to a certain space over T, the action described above or $c*(a:b)=(a:cb)$ is the same for us.) We wish to lift this action to one on $\mathcal{F}$ and consequently, and most important for us, on the spaces of sections of families of twists of invertible sheaves over X. The next paragraphs illustrate how the lifting works. All ideas share the spirit of G-linearizations; see [20], Section 3, p. 30.

The actions of $\mathbb{T}$ on each T_i, letting N_i and N_{i+1} fixed, can be assembled together in an action $\sigma\colon\mathbb{T}\times T\to T$. We lift it trivially to the action $\sigma_X:=(\mathrm{id}_X\times\sigma)$ on $X\times T$. We claim that there is an isomorphism $\sigma_X^*\mathcal{F}\cong p_{2,X}^*\mathcal{F}$, where $p_{2,X}\colon\mathbb{T}\times X\times T\to X\times T$ is the projection. In other words, for each $t\in T$ and $c\in\mathbb{T}$, there is a natural isomorphism $\mathcal{F}_{c*t}\cong\mathcal{F}_t$. Indeed, if $t=(a:b)\in T_i$, then $\mathcal{F}_{c*t}=\mathrm{Ker}(\rho(ca,b))$, whereas $\mathcal{F}_t=\mathrm{Ker}(\rho(a,b))$. Since $\rho(ca,b)=\rho(a,b)(c,1)$, it follows that $\mathcal{F}_t=(c,1)\mathcal{F}_{c*t}$, where $(c,1)$ is the endomorphism of $\mathcal{O}_Y\oplus\mathcal{O}_Z(dP)$ indicated. Furthermore, since

$$\mathcal{F}_t=(c_1,1)\mathcal{F}_{c_1*t}=(c_1,1)(c_2,1)\mathcal{F}_{c_2c_1*t}=(c_1c_2,1)\mathcal{F}_{c_1c_2*t},$$

the cocycle condition is satisfied; see [20], §3 Def. 1.6.

We are now in a position to furnish the last property the stable limit linear series we are interested in must satisfy. Let $\mathcal{C}:=(\pi\colon\mathcal{S}\to B,\mu:\mathcal{S}\to T)$ be a family of chain maps to T parametrized by an algebraic scheme B. An *automorphism* of $\mathcal{C}$ is an automorphism $g:\mathcal{S}\to\mathcal{S}$ such that $\mu\circ g=f\circ\mu$ for an automorphism $f\colon T\to T$ fixing N_0 and N_{d+1}.

Notice that any automorphism $f\colon T\to T$, fixing N_0 and N_{d+1}, fixes all the nodes N_i, and restricts to an automorphism of T_i for each i. Thus, for each i there is $c\in\mathbb{T}$ such that $f(t)=c*t$ for each $t\in T_i$.

Definition 4.10. Let $(\mathcal{L}, \mathcal{V})$ be a family of continuous limit linear series along a family of chain maps $\mathcal{C} = (\pi : \mathcal{S} \to B, \mu : \mathcal{S} \to T)$. We say that $(\mathcal{L}, \mathcal{V})$ is *locally constant* if for each automorphism $g \colon \mathcal{S} \to \mathcal{S}$ of $\mathcal{C}$ and each $s \in \mathcal{S}$, letting $b := \pi(s)$ and $t := \mu(s)$, and $c \in \mathbb{T}$ such that $\mu(g(s)) = c * t$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{V}_{g(s)} \arrow[r, hook] \arrow[d, "\cong"] & H^0(X, \mathcal{L}_b \otimes \mathcal{F}|_{c*t}) \arrow[r, hook] \arrow[d, "\cong"] & H^0(X, \mathcal{L}_b|_Y \oplus \mathcal{L}_b|_Z(dp)) \arrow[d, "{(c,1)}"] \\ \mathcal{V}_s \arrow[r, hook] & H^0(X, \mathcal{L}_b \otimes \mathcal{F}|_t) \arrow[r, hook] & H^0(X, \mathcal{L}_b|_Y \oplus \mathcal{L}_b|_Z(dp)) \end{tikzcd} .$$

Furthermore, $(\mathcal{L}, \mathcal{V})$ is called *a family of stable limit linear series* if it is locally constant but everywhere nonconstant.

We can finally define the functor we are interested in:

Definition 4.11. Let X be the curve obtained as the union of two smooth curves Y and Z meeting transversally at a unique point P. Let T be the chain of $d+1$ rational smooth curves $T_0, \ldots, T_d$. Let $\mathcal{F}$ be the sheaf of twisters on $X \times T/T$. Define the contravariant functor of *stable limit linear series of degree d and dimension r on X*:

$$\begin{array}{rcccc} \mathfrak{G}^r_d(X) & : & \{\text{Schemes}\} & \longrightarrow & \{\text{Sets}\} \\ & & B & \mapsto & \mathfrak{G}^r_d(X)(B) \end{array}$$

as the functor that associates to each scheme B the set $\mathfrak{G}^r_d(X)(B)$ of the following data:

1. an invertible sheaf $\mathcal{L}$ on $X \times B$ of relative multidegree $(d, 0)$ over B;
2. a family $(\pi : \mathcal{S} \to B, \mu : \mathcal{S} \to T)$ of chain maps to T parametrized by B.
3. a locally free subsheaf $\mathcal{V} \subseteq \mathcal{L}(\mathcal{C})$ of constant rank $r+1$ of the family $\mathcal{L}(\mathcal{C}) = \mathcal{L} \boxtimes \mathcal{F}$ of twists of $\mathcal{L}$ along $\mathcal{C}$,

such that $(\mathcal{L}, \mathcal{V})$ is a family of stable limit linear series.

We will see in the next subsections that stable limit linear series can be interpreted as torus fixed stable maps from genus-0 curves to $H^r_d(X)$ whose images lie on a certain homology class β. This will give us the representability of $\mathfrak{G}^r_d(X)$.

4.2.2 Stable Maps to $H^r_d(X)$ and their moduli space.

Our main objective is to define a suitable functor $\overline{\mathcal{M}}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$. Specifically, we will define a certain class $\beta \in H_2(H^r_d(X), \mathbb{Z})$ and study the stable maps invariant by a certain torus action from genus-0 curves to $H^r_d(X)$ whose images lie in β.

Recall that $H^r_d(X) = \mathrm{Grass}_{\mathrm{Pic}^{(d,0)}_X \times T}(r+1, u_*(\mathcal{L} \boxtimes \mathcal{F}))$, where:

1. $\mathrm{Pic}_X^{(d,0)}$ is the component of the Picard scheme of X parameterizing invertible sheaves of degree d on Y and 0 on Z;
2. $\mathcal{L}$ is the Poincaré sheaf $X \times \mathrm{Pic}_X^{(d,0)}$;
3. $\mathcal{F}$ is the sheaf of twisters on $X \times T/T$;
4. $\mathcal{L} \boxtimes \mathcal{F} := q^*\mathcal{L} \otimes p^*\mathcal{F}$, where $p : X \times \mathrm{Pic}_X^{(d,0)} \times T \to X \times T$ and $q : X \times \mathrm{Pic}_X^{(d,0)} \times T \to X \times \mathrm{Pic}_X^{(d,0)}$ are the projections;
5. and $u : X \times \mathrm{Pic}_X^{(d,0)} \times T \to \mathrm{Pic}_X^{(d,0)} \times T$ is the projection.

Let $\pi\colon H_d^r(X) \to \mathrm{Pic}_X^{(d,0)} \times T$ denote the structure map.

Our first results shows that all morphisms from a nodal genus-0 curve to $H_d^r(X)$ factors through the subscheme $H_d^r(X, L)$ for a certain $L \in \mathrm{Pic}_X^{(d,0)}$, where $H_d^r(X, L) := \pi^{-1}(\{L\} \times T)$.

Lemma 4.12. *Let S be a genus-0 nodal curve and $f : S \to H_d^r(X)$ a morphism. Then there exists $L \in Pic_X^{(d,0)}$ such that f factors thorugh $H_d^r(X, L)$.*

Proof 4.13. The statement follows from the fact that any morphism from $\mathbb{P}^1$ to an Abelian variety is constant, by [35], Prop. 3.9, p. 19, for instance.

Our next proposition asserts that each Osserman exact limit linear series $\mathfrak{g} = (L, V_0, \dots, V_d)$ of degree d and dimension r defines a section of $H_d^r(X, L)/T$. More generally, we will see that an exact level-δ limit linear series of degree d and dimension r gives rise to a map from a chain S of $d\delta + 1$ smooth rational curves to $H_d^r(X)$ whose composition with the map to T contracts all the components of S but the $(i\delta + 1)$-th, for $i = 0, \dots, d$. Loosely speaking, the "discrete" notion of limit linear series that has been the standard so far will be replaced by a "continuous" one.

Proposition 4.14. *To each Osserman exact limit linear series $\mathfrak{g} = (L, V_0, \dots, V_d)$ over X, there corresponds a section $f_{\mathfrak{g}}\colon T \to H_d^r(X, L)$ of $H_d^r(X, L)/T$ such that, for every i and $t \in T_i^*$, we have $f_{\mathfrak{g}}(t) = (L_i, V_i)$, where L_i is the invertible sheaf on X whose restrictions to Y and Z are $L|_Y(-iP)$ and $L|_Z(iP)$.*

Proof 4.15. Recall that, from the construction of $\mathcal{F}$, for each $i = 0, \dots, d$ we have

$$L \boxtimes \mathcal{F}|_{X \times \{(a:b)\}} = \mathrm{Ker}\left(\psi_i(a, b)\colon L_i|_Y \oplus L_i|_Z \to \mathcal{O}_P\right),$$

where L_i is an invertible sheaf on X satisfying $L_i|_Y = L|_Y(-iP)$ and $L_i|_Z = L|_Z(iP)$, and $\psi(a, b)$ is the composition of fixed isomorphisms $L|Y(-iP)|_P \cong \mathcal{O}_P$ and $L|_Z(iP)|_P \cong \mathcal{O}_P$ with $(a, b)\colon \mathcal{O}_P^2 \to \mathcal{O}_P$. It follows that

$$L \boxtimes \mathcal{F}|_{X \times T_i^*} \cong L_i \otimes \mathcal{O}_{T_i^*}, \tag{4.11}$$

whereas $L \boxtimes \mathcal{F}|_{X\times\{(0:1)\}} \cong L_i|_Y \oplus L_{i-1}|_Z$ and $L \boxtimes \mathcal{F}|_{X\times\{(1:0)\}} \cong L_{i+1}|_Y \oplus L_i|_Z$. In particular, $L \boxtimes \mathcal{F}|_{X\times T_i} \subset (L_i|_Y \oplus L_i|_Z) \otimes \mathcal{O}_{T_i}$.

Thus, given $\mathfrak{g}$ we may construct a locally free subsheaf $\mathcal{V}_i^*$ of $u_*(L\boxtimes\mathcal{F})|_{T_i^*}$ of rank $r+1$, where $u\colon X \times T \to T$ is the projection, as the image of

$$V_i \otimes \mathcal{O}_{T_i^*} \subseteq H^0(X, L_i) \otimes \mathcal{O}_{T_i^*} \cong u_*(L_i \otimes \mathcal{O}_T)|_{T_i^*}$$

under the isomorphism induced by (4.11).

Another way of putting this is by viewing V_i as a subspace of $H^0(Y, L_i|_Y) \oplus H^0(Z, L_i|_Z)$ and $u_*(L \boxtimes \mathcal{F})|_{T_i^*}$ as a subsheaf of $(H^0(Y, L_i|_Y) \oplus H^0(Z, L_i|_Z)) \otimes \mathcal{O}_{T_i^*}$. Then $\mathcal{V}_i^* \subseteq (H^0(Y, L_i|_Y) \oplus H^0(Z, L_i|_Z)) \otimes \mathcal{O}_{T_i^*}$ in such a way that $\mathcal{V}_i^*|_{(a:b)} = (b, a)V_i$ for each $(a : b) \in T_i^*$.

The next lemma describes what happens with the $\mathcal{V}_i^*$ at the boundary of T_i^* in T_i.

Lemma 4.16.

$$\lim_{a\to 0}(1, a)V_i = V_i|_Y \oplus V_i^{Y,0} \tag{4.12}$$

and

$$\lim_{b\to 0}(b, 1)V_i = V_i^{Z,0} \oplus V_i|_Z \tag{4.13}$$

Proof 4.17 (proof Lemma 4.16)**.** Recall that we have the decomposition $V_i = V_i^{Z,0} \oplus V_i^{Y,0} \oplus V_i^L$, where $V_i^{Z,0} \subseteq V_i|_Y$ and $V_i^{Y,0} \subseteq V_i|_Z$ are, respectively, the kernels of restriction maps $V_i \to V_i|_Z$ and $V_i \to V_i|_Y$, and $V_i^L \subset V_i|_Y \oplus V_i|_Z \subseteq V_Y(-ip) \oplus V_Z(-(d-i)p)$ is the space of linked vectors, the sections glued under conditions of vanishing at the node P. Then $(b, a)V_i = (b, a)V_i^{Z,0} \oplus (b, a) \cdot V_i^{Y,0} \oplus (b, a)V_i^L$. Now, $(b, a)V_i^{Z,0} = bV_i^{Z,0} = V_i^{Z,0}$ and $(b, a)V_i^{Y,0} = aV_i^{Y,0} = V_i^{Y,0}$ for $ab \neq 0$, and thus $\lim_{a\to 0}(1, a)V_i \supset V_i^{Z,0}$ and $\lim_{a\to 0}(1, a)V_i \supset V_i^{Y,0}$. On the other hand, $\lim_{a\to 0}(1, a)V_i^L = (1, 0)V_i^L$. Since $(1, 0)V_i^L \oplus V_i^{Z,0} = V_i|_Y$, Equation (4.12) follows. A similar argument establishes (4.13)

We return now to the proof of Proposition 4.14. By hypothesis, $\mathfrak{g}$ is exact. Thus, it follows from Lemma 4.16 that

$$\lim_{b\to 0}(b, 1)V_i = \lim_{a\to 0}(1, a)V_{i+1}.$$

In fact, exactness of $\mathfrak{g}$ means exactness of

$$V_i \xrightarrow{\varphi^i} V_{i+1} \xrightarrow{\varphi_i} V_i \xrightarrow{\varphi^i} V_{i+1},$$

which is equivalent to the equalities $V_{i+1}^{Y,0} = V_i|_Z$ and $V_i^{Z,0} = V_{i+1}|_Y$. In other words,

$$\lim_{t\to N_{i+1}} \mathcal{V}_i^*|_t = \lim_{t\to N_{i+1}} \mathcal{V}_{i+1}^*|_t.$$

We may thus put together all the extensions $\mathcal{V}_i$ of $\mathcal{V}_i^*$ as subsheaves of $(H^0(Y, L_i|_Y) \oplus H^0(Z, L_i|_Z)) \otimes \mathcal{O}_{T_i}$, and hence of $u_*(L \boxtimes \mathcal{F})|_{T_i}$ to get a locally free subsheaf $\mathcal{V}$ of $u_*(L \boxtimes \mathcal{F})$ in such a way that $(L \boxtimes \mathcal{F}, \mathcal{V})$ is a family of linear series parameterized by T.

Finally, it follows from the construction of $H_d^r(X)$ that $(L \boxtimes \mathcal{F}, \mathcal{V})$ corresponds to a morphism $f_{\mathfrak{g}} : T \to H_d^r(X, L)$, which, by its very construction, is a section of $H_d^r(X, L)/T$ such that $f_{\mathfrak{g}}(t) = (L_i, V_i)$ for each $t \in T_i^*$ and each $i = 0, \dots, d$.

Proposition 4.18. *To each level-δ exact limit linear series $\mathfrak{g} = (L, V_0, \dots, V_{i\delta+j}, \dots, V_{d\delta})$ there corresponds a morphism $f_{\mathfrak{g}} \colon S \to H_d^r(X, L)$, where S is a chain $S_0, \dots, S_{i\delta+j}, \dots, S_{d\delta}$ of rational smooth curves, such that*

1. *$(S, \mu \colon S \to T)$ is a chain map to T, contracting each $S_{i\delta+j}$ for $j > 0$, where μ is the composition of $f_{\mathfrak{g}}$ with the natural map $H_d^r(X, L) \to T$;*
2. *For each i and j, and each $s \in S_{i\delta+j}^*$, we have $f_{\mathfrak{g}}(s) = (L_{i\delta+j}, V_{i\delta+j})$, where $L_{i\delta+j}$ is the invertible sheaf on X whose restrictions to Y and Z are $L|_Y(-iP)$ and $L|_Z(iP)$ if $j = 0$ or the sheaf $L|_Y(-iP) \oplus L|_Z((i-1)P)$ if $j > 0$.*

Proof 4.19. The proof is similar to that of Proposition 4.14.

Proposition 4.20. *Under the hypotheses of Proposition 4.14 the map $f_{\mathfrak{g}} \colon T \to H_d^r(X, L)$ corresponds to a stable limit linear series $(L, \mathcal{V})$ along the trivial chain map $(T, id \colon T \to T)$. Conversely, if $(L, \mathcal{V})$ is a nonconstant, locally constant generalized linear series along the trivial chain map $(T, id \colon T \to T)$, then it gives rise to an Osserman exact limit linear series.*

Proof 4.21. Indeed, by the proof of Proposition 4.14, the map $f_{\mathfrak{g}} \colon T \to H_d^r(X, L)$ corresponds to a generalized linear series $(L, \mathcal{V})$ along the trivial chain map $(T, \mathrm{id} \colon T \to T)$. Since $f_{\mathfrak{g}}$ is a section of $H_d^r(X, L) \to T$, the generalized linear series is everywhere nonconstant. Moreover, it is locally constant, thus stable.

Conversely, choose $t_i \in T_i^*$ for each $i = 0, \dots, d$, and set

$$V_i := \mathcal{V}|_{t_i} \subseteq H^0(X, L \otimes \mathcal{F}|_{t_i}).$$

Notice that $L \otimes \mathcal{F}|_{t_i}) \cong L_i$ for each $i = 0, \dots, d$, where L_i is the invertible sheaf on X whose restrictions to Y and Z are $L|_Y(-iP)$ and $L|_Z(iP)$.
Furthermore, since $(L, \mathcal{V})$ is locally constant, putting coordinates on each T_i, for $i = 0, \dots, d$, such that N_i, t_i and N_{i+1} correspond to $(0:1)$, $(1:1)$ and $(1:0)$, respectively, it follows that $\mathcal{V}|_{(a:b)} = (b, a)V_i$ for each $a, b \in \mathbb{C}^*$. Thus, since the limit of $\mathcal{V}|_t$ as t tends to N_{i+1} is the same, whether $t \in T_i$ or $t \in T_{i+1}$, it follows from Lemma 4.16 that

$$V_{i+1}|_Y \oplus V_{i+1}^{Y,0} = V_i^{Z,0} \oplus V_i|_Z,$$

or equivalently, $V_i^{Z,0} = V_{i+1}|_Y$ and $V_{i+1}^{Y,0} = V_i|_Z$. Thus $\mathfrak{g} := (L, V_0, \dots, V_d)$ is an Osserman exact limit linear series.

Remark 4.22. Regarding Proposition 4.18, the map $f_{\mathfrak{g}} \colon S \to H_d^r(X, L)$ corresponds to a generalized linear series $(L, \mathcal{V})$ along the chain map $(S, \mu \colon S \to T)$, which is also locally constant, but may fail to be everywhere nonconstant.

We want now to define $\beta \in H_2(H^r_d(X))$ as the class of the image of a map $f_\mathfrak{g}$ corresponding to an Osserman exact limit linear series, if one such limit linear series exists. By the very definition of $f_\mathfrak{g}$, it is clear that $f_\mathfrak{g} \in \overline{M}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$.

The aim of the following results is to show that every $f \in \overline{M}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$, where the class β corresponds to the Osserman "type," and the invariance to the property of being "locally constant," corresponds to a stable limit linear series over X.

Our first step is to describe the class $\beta \in H_2(H^r_d(X), \mathbb{Z})$.

So, let $\beta := f_{\mathfrak{g}*}[T]$, where $f_\mathfrak{g} \colon T \to H^r_d(X)$ is the map arising from an Osserman exact limit linear series $\mathfrak{g} = (L, V_0, \dots, V_d)$. Then $f_\mathfrak{g}(T) \subseteq H^r_d(X, L)$. Now, $f_\mathfrak{g}$ corresponds to a family of linear series of the form $(L \boxtimes \mathcal{F}, \mathcal{V})$. Let $u \colon X \times T \to T$ denote the projection. Then

$$\mathcal{V} \subseteq u_*(L \boxtimes \mathcal{F}) \subseteq \big(H^0(L|_Y) \oplus H^0(L|_Z(dP))\big) \otimes \mathcal{O}_T.$$

Letting $W := H^0(L|_Y) \oplus H^0(L|_Z(dP))$ and $G := \mathrm{Grass}(r+1, W)$, we get an embedding,

$$H^r_d(X, L) = \mathrm{Grass}_T(r+1, u_*(L \boxtimes \mathcal{F})) \overset{\iota}{\hookrightarrow} \mathrm{Grass}_T(r+1, W \otimes \mathcal{O}_T) = G \times T.$$

By Example 4.1 and Remark 4.7, letting p_1 and p_2 denote the projections of $G \times T$ onto the indicated factors, we have that $p_{2*}\iota_*\beta = \sum a_i[T_i]$, and $p_{1*}\iota_*\beta = b\gamma$, where the a_i and b are integers, and γ is the dual class to σ_1 on G, that is $\sigma_1 \cdot \gamma = 1$.

Lemma 4.23. *Let $\beta := f_{\mathfrak{g}*}[T]$, where $\mathfrak{g}$ is an Osserman exact limit linear series. Then*

1. $p_{2*}\iota_*\beta = [T]$.
2. $p_{1*}\iota_*\beta = (r+1)\gamma$.

Proof 4.24. For the first statement, recall that $f_\mathfrak{g}$ factors through $H^r_d(X, L)$ for a certain L, and that $\pi_L f_\mathfrak{g} = \mathrm{id}_T$, where $\pi_L \colon H^r_d(X, L) \to T$ is the natural map, thus $\pi_L = p_2\iota$. Thus

$$p_{2*}\iota_*\beta = p_{2*}\iota_* f_{\mathfrak{g}*}[T] = [T].$$

As for the second statement, set $f_i := f_\mathfrak{g}|_{T_i}$ and $\beta_i := f_{i*}[T_i]$. Writing $b_i := \deg(p_{1*}\iota_*\beta_i)$, it remains to show that $\sum b_i = r+1$.

Claim: $b_i = \dim(V_i^L)$ for $i = 0, \dots, d$

The second statement follows from the claim and Lemma 3.13(3), Chapter 3, as that lemma says that if $\mathfrak{g}$ is an exact lls then $\sum_i \dim(V_i^L) = r+1$.

Proof (Claim): Write $\mathfrak{g} = (L, V_0, \dots, V_d)$. Choose coordinates $(a : b)$ on each T_i such that N_i corresponds to $(0:1)$ and N_{i+1} to $(1:0)$. Then $p_1\iota f_i$ is the map that sends $(a:b)$, for $ab \neq 0$, to $(b,a)V_i$, where $V_i \in G$ under the embedding

$$V_i \subset V_i|_Y \oplus V_i|_Z \subset V_Y(-iP) \oplus V_Z(-(d-i)P) \subset H^0(X, L_Y) \oplus H^0(X, L_Z(dP)).$$

Choose a basis of $W := H^0(X, L_Y) \oplus H^0(X, L_Z(dP))$, respecting the decomposition. For each subset I of this basis, let $n_{I,Y}$ be the number of vectors in $H^0(X, L|_Y)$ and $n_{I,Z}$ the number of those in $H^0(X, L_Z(dP))$. For each I with $|I| = r+1$, let p_I be the Plücker coordinate of V and $p_I(c)$ that of $(c,1)V$. Then $p_I(c) = c^{n_{I,Y}} p_I$.

Now, from our study of invariant curves by our torus action in Subsection 4.1.2, we have two cases:

- $p_1 \iota f_i(T_i)$ consists of torus fixed points
- $p_1 \iota f_i(T_i)$ is an invariant curve connecting two torus fixed points.

In the first case, by Lemma 4.16, we conclude that $V_i = V_i^{Z,0} \oplus V_i|_Z = V_i^{Y,0} \oplus V_i|_Y$, which is only possible when $V_i = V_i^{Y,0} \oplus V_i^{Z,0}$, whence $\dim(V_i^L) = 0$. So $(b,a)V_i = bV_i^{Y,0} \oplus aV_i^{Z,0} = V_i$. It follows that $p_1 \iota f_i$ is constant, or equivalently, that $b_i = 0$.

In the second case, let $V_{i,0}$ and $V_{i,\infty}$ be the two fixed points, such that the rational curve $(p_1 \iota f_i)(T_i)$ passes through them. Then $(p_1 \iota f_i)(T_i)$ can be identified with the projective fixed line Γ_i in G that links $V_{i,0}$ and $V_{i,\infty}$. Clearly, by Lemma 4.16, $V_{i,0} = V_i^{Z,0} \oplus V_i|_Z$ and $V_{i,\infty} = V_i^{Y,0} \oplus V_i|_Y$, and these are also the unique branch points of $p_1 \iota f_i \colon T_i \to \Gamma_i$. Hence, the degree of $d_i := (p_1 \iota f_i)(T_i)$ is equal to $n_{I_0,Y} = n_{I_\infty,Y}$ the sum of ramification index on $V_{i,0}$ and $V_{i,\infty}$, respectively. Hence,

$$n_{I_0,Y} = \dim \frac{V_i|_Z}{V_i^{Y,0}} = \dim V_i|_Z - \dim V_i^{Y,0} = \dim V_i^L = \dim V_i|_Y - \dim V_i^{Z,0} = \dim \frac{V_i|_Y}{V_i^{Z,0}} = n_{I_\infty,Y},$$

where the equalities follow from the exactness of $\mathfrak{g}$, since $V_i|_Z = V_{i+1}^{Y,0} \cong V_i^{Y,0} \oplus V_i^L$ and $V_i|_Y = V_{i-1}^{Z,0} \cong V_i^{Z,0} \oplus V_i^L$.

To follow, it is only necessary to recall from $\beta \in H_2(H_d^r(X), \mathbb{Z})$ the equalities in Lemma 4.23, and the fact that $\nu_* \beta = 0$, where $\nu \colon H_d^r(X) \to \mathrm{Pic}_X^{(d,0)}$ is the natural map.

Our next goal is to prove that each point in $\overline{M}_{0,2}(H_d^r(X), \beta)^{\mathbb{C}^*}$ is represented by a stable map $f \colon S \to H_d^r(X)$ where S is necessarily a chain. So, forcing $\mathbb{C}^*$-invariance and the class β determines strongly the type of our stables maps. Intuitively, by the discussion so far, this moduli space is a "good candidate" for the moduli space parametrizing stable limit linear series.

Theorem 4.25. *Any point of $\overline{\mathcal{M}}_0(H_d^r(X), \beta)^{\mathbb{C}^*}$ is represented by a map $f \colon S \to H_d^r(X)$ where S is a chain of smooth rational curves contracting to T under the natural map $H_d^r(X) \to T$.*

Proof 4.26. Let us recall first the various objects and notions involved in the above statement. According to Lemma 4.12, we may replace $H_d^r(X)$ by $H := \mathrm{Grass}_T(r+1, q_*(L \boxtimes \mathcal{F}))$, where L is an invertible sheaf of multidegree $(d,0)$ on X, and $q \colon X \times T \to T$ is the projection. Since $L \boxtimes \mathcal{F} \hookrightarrow (L_Y \oplus L_Z) \otimes \mathcal{O}_T$, with $L_Y := L|_Y$ and $L_Z := L(dP)|_Z$, we have a natural inclusion $\iota \colon H \hookrightarrow G \times T$, where $G := \mathrm{Grass}(r+1, W)$, for $W := H^0(L_Y) \oplus H^0(L_Z)$. More precisely,

$$L \boxtimes \mathcal{F}|_{X \times T_i} \hookrightarrow (L_Y(-iP) \oplus L_Z((i-d)P)) \otimes \mathcal{O}_{T_i},$$

and thus

$$\iota(H) \subseteq \bigcup_{i=0}^{d} G_i \times T_i,$$

where $G_i := \mathrm{Grass}(r+1, W_i)$, for $W_i := H^0(L_Y(-iP)) \oplus H^0(L_Z((i-d)P))$.

Thus, we may view $\mathcal{M}_0(H,\beta)$ as the closed subfunctor of $\mathcal{M}_0(G \times T, \iota_*\beta)$ parameterizing stable maps factoring through H. By hypothesis,

$$p_{1*}\iota_*\beta = (r+1)\gamma \quad p_{2*}\iota_*\beta = [T], \tag{4.14}$$

where p_1, p_2 are the projections of $G \times T$ onto the indicated factors, and γ is the positive generator of $H_2(G,\mathbb{Z})$.

Now, the torus action on H can be described as:

$$\begin{array}{ccc} \mathbb{C}^* \times H & \longrightarrow & H \\ (c,(V,t)) & \mapsto & (V^c, c \star t), \end{array} \tag{4.15}$$

where, for $t \in T_i$, the space V^c is defined as the image of $V \subset W_i$ under the action:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & W_i \arrow[r, equal] \arrow[d] & H^0(L_Y(-iP)) \oplus H^0(L_Z((i-d)P)) \arrow[d, "{(c,1)}"] . \\ V^c := c(V) \arrow[r, hook] & W_i \arrow[r, equal] & H^0(L_Y(-iP)) \oplus H^0(L_Z((i-d)P)) \end{tikzcd} \tag{4.16}$$

Since $c_*\beta = \beta$, it induces naturally an action on the space of stable maps:

$$\begin{array}{ccc} \mathbb{C}^* \times \mathcal{M}_0(H,\beta) & \longrightarrow & \mathcal{M}_0(H,\beta) \\ (c,[f: S \to H]) & \mapsto & [f^c: S \to H \xrightarrow{c} H]. \end{array} \tag{4.17}$$

Recall that $\mathcal{M}_0(H,\beta)$ parameterizes classes of maps modulo automorphisms of the source. Therefore, to say that $[f\colon S \to H]$ is in $\mathcal{M}_0(H,\beta)^{\mathbb{C}^*}$ means that there is an automorphism g_c of S such that $f = f^c g_c$ for each $c \in \mathbb{C}^*$.

Let $[f\colon S \to H] \in \mathcal{M}_0(H,\beta)^{\mathbb{C}^*}$. Using the inclusion $\iota\colon H \hookrightarrow G \times T$, we can write $f = (f_1, f_2) : S \to G \times T$.

The proof will be divided in 3 steps.

First Claim: *S has $d+1$ components $S_0, \ldots, S_d$ isomorphic to $T_0, \ldots, T_d$ under f_2. For each $i = 0, \ldots, d+1$, let $N_{i-1,\infty}$ (resp. $N_{i,0}$) be the point on S_{i-1} (resp. S_i) mapped to N_i under f_2. Let $S_T := S_0 \cup \cdots \cup S_d$ and put $S'_T := \overline{S - S_T}$. Then S'_T consists of at most $d+2$ connected components $S_{-1/2}, S_{1/2}, \ldots, S_{d+1/2}$, where $S_{i-1/2}$, if nonempty, is collapsed by f_2 and intersects the rest of S only at $N_{i-1,\infty}$ and $N_{i,0}$. For each $i = 1, \ldots, d$, we have that $S_{i-1/2}$ is nonempty if and only if $N_{i-1,\infty} \neq N_{i,0}$, in which case there is a chain $S^0_{i-1/2} \subset S_{i-1/2}$ of smooth rational curves connecting $N_{i-1,\infty}$ with $N_{i,0}$. Let S^0 be the union of S_T and the $S^0_{i-1/2}$ for $i = 1, \ldots, d$.*

Then S^0 is a chain.

Proof 4.27 (proof first claim)**.** By the sake of preciseness, we put $S_{-1} := \emptyset$ and $S_{d+1} := \emptyset$. Since, by our choice of β, we have $f_*[S] = [T]$, we conclude that there exist irreducible components $S_0, \ldots, S_d$ of S mapping to $T_0, \ldots, T_d$ via f_2 with degree 1, the remaining components being collapsed. Since $[f]$ is fixed by the torus action, and the action moves the points on T_i^* for each i, it follows that each collapsed irreducible component collapses to a point among $N_0, \ldots, N_{d+1}$.

For each $i = 0, \ldots, d+1$, let $S_{i-1/2} \subseteq S'_T$ be the union of the connected components of S'_T which are mapped to N_i under f_2. As $f_2|_{S_i}$ is a map of degree 1 onto T_i for each $i = 0, \ldots, d$, there is a unique point on S_T, lying exclusively on S_0 (resp. S_d) mapped by f_2 to N_0 (resp. N_{d+1}). Then $S_{-1/2}$ and $S_{d+1/2}$ are connected, if nonempty. In addition, for each $i = 1, \ldots, d$, there are at most two points on S_T that map to N_i, one on S_{i-1} and one on S_i, and they coincide if and only if S_{i-1} intersects S_i, in which case the intersection only consists of that point. Clearly, $S_{i-1/2}$ is empty if and only if S_{i-1} intersects S_i and consists of at most two connected components otherwise, one intersecting each point of S_T mapped to N_i. However, since S is connected, in this case $S_{i-1/2}$ is connected.

The remaining statements are clear since, as S is nodal of genus 0, so are the $S_{i-1/2}$.

Second Claim: *Let C be an irreducible component of the chain S^0. Let 0 and ∞ denote its special points. Then $f_1(0) = V_{1,0} \oplus V_{2,0}$ and $f_1(\infty) = V_{1,\infty} \oplus V_{2,\infty}$, where $V_{1,0}$ and $V_{1,\infty}$ are subspaces of $H^0(L_Y)$ and $V_{2,0}$ and $V_{2,\infty}$ are subspaces of $H^0(L_Z)$. Furthermore, $\dim V_{1,0} - \dim V_{1,\infty} = \dim V_{2,\infty} - \dim V_{2,0}$, and the difference is nonzero if and only $f_1(C)$ contains a nonfixed point of G. In this case, the degree of $f_1|_C \colon C \to G$ is a nonzero multiple of the absolute value of the difference.*

Proof 4.28 (proof second claim)**.** Since $[f]$ is fixed by the torus action, it follows that $f_1(C)$ is invariant by the action of $\mathbb{C}^*$ on G and that $f_1(0)$ and $f_1(\infty)$ are fixed points by this action. Then, as a consequence of our study of torus action in Subsection 4.1.2, the images $f_1(0)$ and $f_1(\infty)$ are as claimed, and either $f_1(C)$ consists of fixed points or is a fixed curve with a nonfixed point. In the first case, since the fixed points are of the form $V_1 \oplus V_2$, for spaces $V_1 \subseteq H^0(L_Y)$ and $V_2 \subseteq H^0(L_Z)$, it follows that $f_1(s) = [V_1(s) \oplus V_2(s)]$ for $V_1(s) \subseteq H^0(L_Y)$ and $V_2(s) \subseteq H^0(L_Z)$ varying algebraically with $s \in C$. Since $V_i(s)$ depends algebraically on s, for $i = 1, 2$, it follows that

$$\dim V_{1,0} - \dim V_{1,\infty} = 0 = \dim V_{2,\infty} - \dim V_{2,0}.$$

In the second case, let $V \in f_1(C)$ be a nonfixed point. Then

$$f_1(C) = \overline{\{(c,1)V /\, c \in \mathbb{C}^*\}}.$$

In this case, as in the proof of Lemma 4.23, there are two fixed points in $f_1(C)$, the limits of $(c,1)V$ as c tends to 0 and ∞. Letting $V|_Y$ denote the image of the projection $V \to H^0(L_Y)$, and V^Y its kernel, and $V|_Z$ the image of the projection $V \to H^0(L_Z)$, and V^Z its kernel, the limits are $V|_Y \oplus V^Y$ and $V^Z \oplus V|_Z$. Of course, $V^Y \subsetneq V|_Z$ and $V^Z \subsetneq V|_Y$, where the inclusions are strict because V is nonfixed. Also,

$$\dim V|_Z - \dim V^Y = \dim V - \dim V^Z - \dim V^Y = \dim V|_Y - \dim V^Z,$$

is the degree of $f_1(C)$ in G under the Plücker embedding.

There are two cases now, either $f_1(0) = V|_Y \oplus V^Y$ and $f_1(\infty) = V^Z \oplus V|_Z$, or the other way around. In any case, the claim is valid.

Claim Three: $S^0 = S$.

Proof 4.29 (proof claim three)**.** Let $C_1, \dots, C_m$ be the irreducible components of S^0, ordered in such a way that $C_1 = S_0$, $C_d = S_d$ and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. For each $i = 1, \dots, m$, let $Q_{i,0}$ and $Q_{i,\infty}$ denote the special points of S^0 on C_i, with $Q_{i,0}$ on C_{i-1} if $i > 1$.

Since $f_1(Q_{i,0})$ and $f_1(Q_{i,\infty})$ are fixed points of G for $i = 1, \dots, m$, we may write

$$f_1(Q_{i,0}) = V^Y_{i,0} \oplus V^Z_{i,0} \quad \text{and} \quad f_1(Q_{i,\infty}) = V^Y_{i,\infty} \oplus V^Z_{i,\infty}$$

for certain subspaces $V^Y_{i,0}$ and $V^Y_{i,\infty}$ of $H^0(L_Y)$ and $V^Z_{i,0}$ and $V^Z_{i,\infty}$ of $H^0(L_Z)$. It follows from Claim 2 that the degree of $f_1(S^0)$ is at least

$$\sum_{i=1}^{m} |\dim V^Y_{i,0} - \dim V^Y_{i,\infty}|.$$

Now, $f_1(Q_{1,0})$ is a subspace of $H^0(L_Y) \oplus H^0(L_Z(-(d+1)P)$. Since L_Z has degree d, it follows that $V^Y_{1,0}$ has dimension $r+1$. Analogously, we have that $V^Y_{m,\infty}$ has dimension 0. Since $V^Y_{i,\infty} = V^Y_{i+1,0}$ for $i = 1, \dots, m-1$, it follows that

$$\sum_{i=1}^{m} |\dim V^Y_{i,0} - \dim V^Y_{i,\infty}| \geq r + 1.$$

But, by our choice of β, the degree of the map $f_1 \colon S \to G$ is exactly $r+1$!

A number of consequences follow: First, all the connected components of $\overline{S - S^0}$ are collapsed by f_1 to points. Since they are also collapsed by f_2, and f is stable, they must be stable curves. But there is no stable 1-pointed genus-0 curve. So $S = S^0$. Furthermore, either $f_1(C_i)$ is a (fixed) point, or $f_1(C_i)$ is a curve containing a nonfixed point. The first case can only occur if C_i is one of the S_j. If the second occurs, then f_1 maps C_i isomorphically to $f_1(C_i)$ and $f_1(C_i)$ has (nonzero) degree $\dim V^Y_{i,0} - \dim V^Y_{i,\infty}$.

4.2.3 Equivalence of functors and coarse representation.

This subsection will be divided in two parts. In the first part, it will be established an isomorphism between the functor of stable limit linear series of degree d and dimension r, denoted

$\mathfrak{G}^r_d(X)$ and defined in Subsection 4.2.1, and the functor of torus fixed stable maps from genus-0 curves to $H^r_d(X)$ with homology class β, denoted $\overline{\mathcal{M}}_0\left(H^r_d(X),\beta\right)^{\mathbb{C}^*}$ and defined in the last subsection. The second part is devoted to showing that $\overline{\mathcal{M}}_0\left(H^r_d,\beta\right)^{\mathbb{C}^*}$ is coarsely represented by a projective scheme.

We begin recalling the definitions of the functors $\mathfrak{G}^r_d(X)$ and $\overline{\mathcal{M}}_0\left(H^r_d(X),\beta\right)^{\mathbb{C}^*}$. The first was defined as the contravariant functor

$$\begin{array}{rcccc} \mathfrak{G}^r_d(X) & : & \{\text{Schemes}\} & \longrightarrow & \{\text{Sets}\} \\ & & B & \mapsto & \mathfrak{G}^r_d(X)(B), \end{array}$$

that associates to each scheme B the set $\mathfrak{G}^r_d(X)(B)$ the following data:

1. an invertible sheaf $\mathcal{L}$ on $X \times B$ of relative multidegree $(d,0)$ over B;
2. a family $\mathcal{C} = (\pi : \mathcal{S} \to B, \mu : \mathcal{S} \to T)$ of chain maps to T parametrized by B.
3. a locally free subsheaf $\mathcal{V} \subseteq \mathcal{L}(\mathcal{C})$ of constant rank $r+1$ of the family $\mathcal{L}(\mathcal{C}) = \mathcal{L} \boxtimes \mathcal{F}$ of twists of $\mathcal{L}$ along $\mathcal{C}$,

such that $(\mathcal{L},\mathcal{V})$ is a family of stable limit linear series.

On the other hand, $\overline{\mathcal{M}}_0\left(H^r_d(X),\beta\right)^{\mathbb{C}^*}$ was defined as the contravariant functor

$$\begin{array}{rcccc} \overline{\mathcal{M}}_0(H^r_d(X),\beta)^{\mathbb{C}^*} & : & \{\text{Schemes}\} & \longrightarrow & \{\text{Sets}\} \\ & & B & \mapsto & \overline{\mathcal{M}}_0(H^r_d(X),\beta)(B), \end{array}$$

that associates to each scheme B the set $\overline{\mathcal{M}}_0(H^r_d(X),\beta)^{\mathbb{C}^*}(B)$ of isomorphism classes of families over B of stable maps from a family of genus-0 curves $\mathcal{S}/B$ to $H^r_d(X)$, whose images represent the class β, and which are fixed by the torus action; see Subsection 4.1.2. The class β is a class satisfying the two conditions displayed in Lemma 4.23, plus the fact that $\nu_*\beta = 0$, where $\nu\colon H^r_d(X) \to \mathrm{Pic}^{(d,0)}_X$ is the natural map.

Recall that $H^r_d(X) := \mathrm{Grass}_{\mathrm{Pic}^{(d,0)}_X \times T}(r+1,\mathcal{W})$ is the scheme parameterizing linear series $(\mathcal{I},V)$, where $\mathcal{I}$ is any torsion-free, rank-1 sheaf on X of degree d whose restrictions to Y and Z, modulo torsion, have degrees ranging from -1 to d, and V is any vector subspace of $H^0(X,\mathcal{I})$ of dimension $r+1$; see Section 4.1.

Proposition 4.30. *The functors $\overline{\mathcal{M}}_0(H^r_d(X),\beta)^{\mathbb{C}^*}$ and $\mathfrak{G}^r_d(X)$ are isomorphic.*

Proof 4.31. In fact, for each scheme B the bijection between the sets $\overline{\mathcal{M}}_0(H^r_d(X),\beta)^{\mathbb{C}^*}(B)$ and $\mathfrak{G}^r_d(X)(B)$ is defined as follows: For each element in the first set, let $(\pi\colon \mathcal{S} \to B, f\colon \mathcal{S} \to H^r_d(X))$ be one of its representatives. By Theorem 4.25, the map π defines a family of chains of rational smooth curves. Composing μ with the natural map $H^r_d(X) \to T$, we get a map $\mu\colon \mathcal{S} \to T$ such that, by the properties of β, the pair $\mathcal{C} = (\pi : \mathcal{S} \to B, \mu : \mathcal{S} \to T)$ is a family of chain maps to T parametrized by B. Finally, $H^r_d(X)$ comes with a family of linear series on $X \times H^r_d(X)/H^r_d(X)$.

Pulling it back to $X \times \mathcal{S}/S$, we get a family $(\mathcal{L}(\mathcal{C}), \mathcal{V})$ of linear series of sections of an invertible sheaf which is precisely $\mathcal{L}(\mathcal{C})$, where $\mathcal{L}$ is the pullback to $X \times \mathcal{S}$ of the universal invertible sheaf on $X \times \mathrm{Pic}_X^{(d,0)}$ under the map induced by the composition of f with the natural map $\nu\colon H_d^r(X) \to \mathrm{Pic}_X^{(d,0)}$. Finally, the stability of $(\pi\colon \mathcal{S} \to B, f\colon \mathcal{S} \to H_d^r(X))$ is equivalent to the fact that $(\mathcal{L}, \mathcal{V})$ is everywhere nonconstant, and the fact that $(\pi\colon \mathcal{S} \to B, f\colon \mathcal{S} \to H_d^r(X))$ is torus fixed is equivalent to the fact that $(\mathcal{L}, \mathcal{V})$ is locally constant. Thus $(\mathcal{L}, \mathcal{V})$ is stable, and hence defines an element of $\mathfrak{G}_d^r(X)(B)$.

Conversely, let $\mathcal{C} = (\pi : \mathcal{S} \to B, \mu : \mathcal{S} \to T)$ and $(\mathcal{L}(\mathcal{C}), \mathcal{V})$ be a family of stable limit linear series. On one hand, by the existence of a family of generalized linear series $(\mathcal{L}, \mathcal{V})$ and the Universal property of $H_d^r(X)$ we obtain that a unique map $f : \mathcal{S} \to H_d^r(X)$. On the other hand, the fact that $(\mathcal{L}(\mathcal{C}), \mathcal{V})$ is locally constant and everywhere nonconstant is equivalent to the fact of the map $f : \mathcal{S} \to H_d^r(X)$ is a torus fixed and stable. Thus, the representative of $(\pi\colon \mathcal{S} \to B, f\colon \mathcal{S} \to H_d^r(X))$ is the searched element of $\overline{\mathcal{M}}_0(H_d^r(X), \beta)^{\mathbb{C}^*}(B)$.

Theorem 4.32. *There exists a projective scheme $\overline{M}_0(H_d^r(X), \beta)^{\mathbb{C}^*}$ coarsely representing the functor $\overline{\mathcal{M}}_0(H_d^r(X), \beta)^{\mathbb{C}^*}$.*

Proof 4.33. First, since the algebraic scheme $H_d^r(X)$ is projective, by Theorem 4.2, we have that $\overline{\mathcal{M}}_0(H_d^r(X), \beta)$ is coarsely represented by a projective scheme, $\mathfrak{Z} := \overline{M}_0(H_d^r(X), \beta)$. From the embedding $H_d^r(X) \overset{\iota}{\hookrightarrow} G \times T \times \mathrm{Pic}_X^{(d,0)}$ described in Subsection 4.1.1, where

$$G := \mathrm{Grass}(r+1, W_1 \oplus W_2),$$

we obtain, by Proposition 4.4, a closed embedding

$$\mathfrak{Z} \hookrightarrow \overline{M}_0(G \times T \times \mathrm{Pic}_X^{(d,0)}, \iota_* \beta). \tag{4.18}$$

Now, Lemma 4.12 yields a natural isomorphism

$$\overline{M}_0(G \times T \times \mathrm{Pic}_X^{(d,0)}, \iota_* \beta) \cong \overline{M}_0(G \times T, \beta') \times \mathrm{Pic}_X^{(d,0)}, \tag{4.19}$$

where β' is the direct image of $\iota_* \beta$ under the projection, since $\mathrm{Pic}_X^{(d,0)} = \mathrm{Pic}^{(d-i)}(Y) \times \mathrm{Pic}^{(i)}(Z)$ is an Abelian variety.

On the other hand, we may embed $G \times T \hookrightarrow G \times \mathbb{P}^{d+1}$ as a $\mathbb{C}^*$-invariant closed subscheme, where we define the torus action on $G \times \mathbb{P}^{d+1}$ by $c * (V_1, V_2) := (A_c^1 V_1, A_c^2 V_2)$. Here V_1 denotes a $(r+1)$-dimensional subspace of $W_1 \oplus W_2$ and V_2 a one-dimensional subspace of $\mathbb{C}^{d+2}$. Besides,

A_c^1 and A_c^2 are the $\mathbb{C}$-linear transformations represented by the matrices, respectively:

$$A_c^1 := \begin{pmatrix} c & 0 & \cdots & \cdots & 0 \\ 0 & c & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \qquad A_c^2 := \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & c & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & c^d & 0 \\ 0 & 0 & \cdots & \cdots & c^{d+1} \end{pmatrix}$$

Finally, $T \subset \mathbb{P}^{d+1}$ is the chain of $d+1$ lines $T_0, \ldots, T_d$, where T_i is the line represented by the 2-dimensional subspace of $\mathbb{C}^{d+2}$ generated by e_i and e_{i+1}, where $e_0, \ldots, e_{d+1}$ is the canonical basis of $\mathbb{C}^{d+2}$; see Subsection 4.1.2. A consequence of this embedding is the lemma below, which will be proved later:

Lemma 4.34. *Every point $z \in \mathfrak{Z}$ has a torus invariant affine open neighborhood.*

Now, for each scheme B, the set $F(B) := \overline{\mathcal{M}}_0(H_d^r(X), \beta)^{\mathbb{C}^*}(B)$ is identified as the subset of $\mathrm{Hom}_{\mathfrak{Z}}(B)$ of maps $f : B \to \mathfrak{Z}$ such that the following diagram

$$\begin{array}{ccc} \mathbb{C}^* \times \mathfrak{Z} & \xrightarrow{\sigma} & \mathfrak{Z} \\ {\scriptstyle (\mathrm{id},f)}\uparrow & & \uparrow{\scriptstyle f} \\ \mathbb{C}^* \times B & \xrightarrow{p_2} & B, \end{array} \tag{4.20}$$

is commutative, where σ denotes the action. Thus, to prove that F is coarsely representable, by [41] Prop. E.18, p. 382, it suffices to show that the functor F satisfies the following two properties:

1. F is a sheaf (in the Zariski topology).

2. F admits a covering by representable open functors.

The first property is easily checked. In fact, we need to prove that the sequence of sets on top below is exact, for each open covering $B = \cup_i B_i$:

$$\begin{array}{ccccccc} & & F(B) & \longrightarrow & \Pi_i F(B_i) & \rightrightarrows & \Pi_{i,j} F(B_i \cap B_j) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathfrak{Z}}(B) & \longrightarrow & \Pi_i \mathrm{Hom}_{\mathfrak{Z}}(B_i) & \rightrightarrows & \Pi_{i,j} \mathrm{Hom}_{\mathfrak{Z}}(B_i \cap B_j) \end{array} .$$

(The commutativity of the above diagram follows from the definition of F.) Now, given $(f_i \colon B_i \to \mathfrak{Z}) \in \Pi_i F(B_i)$ in the kernel of the second map at the top of the above diagram, since $\mathrm{Hom}_{\mathfrak{Z}}(\cdot)$

is a sheaf, it follows that there exists a unique $f\colon B \to \mathfrak{Z}$ such that $f|_{B_i} = f_i$. By assumption,

$$\begin{tikzcd} \mathbb{C}^* \times \mathfrak{Z} \arrow[r, "\sigma"] & \mathfrak{Z} \\ \mathbb{C}^* \times B_i \arrow[r, "p_2"] \arrow[u, "(\mathrm{id}, f_i)"] & B_i \arrow[u, "f_i"'] \end{tikzcd} ,$$

commutes, that is, $f \circ p_2|_{\mathbb{C}^* \times B_i} = f_i \circ p_2 = \sigma(\mathrm{id}, f_i) = \sigma(\mathrm{id}, f)|_{\mathbb{C}^* \times B_i}$ for each i, which implies that $f \circ p_2 = \sigma(\mathrm{id}, f)$, i.e., $f \in F(B)$.

As for the second property, since F is a sheaf, by [41] Lemma E.19, p. 382 , we may restrict ourselves to the category of affine schemes. On the other hand, by Lemma 4.34, we may replace $\mathfrak{Z}$ by an affine scheme. Thus, it suffices to prove, for a $\mathbb{C}$-algebra A over which there is a co-action $\sigma\colon A \to A[t, t^{-1}]$, that there is a universal quotient $h\colon A \to B$:

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "h"'] & A[t, t^{-1}] \arrow[d, "h'"] \\ B \arrow[r, "\iota"] & B[t, t^{-1}], \end{tikzcd} \tag{4.21}$$

where ι is the natural inclusion and h' is the algebra homomorphism extending h, that is, such that $h'(t) = t$. Equivalently, we would like to show that there exists an ideal $I \subset A$ such that

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "q"'] & A[t, t^{-1}] \arrow[d, "q'"] \\ A/I \arrow[r, "\iota"] & A/I[t, t^{-1}], \end{tikzcd} \tag{4.22}$$

commutes, and such that for any morphism $h\colon A \to B$ making (4.21) commute, we have that $I \subset \mathrm{Ker}(h)$.

In fact, from the commutativity of Diagram (4.22) we must have

$$\begin{tikzcd} a \arrow[r] \arrow[d] & \sum\limits_i a_i t^i \arrow[d] \\ \overline{a} \arrow[r, "\iota"] & \sum\limits_i \overline{a}_i t^i \end{tikzcd} , \tag{4.23}$$

and since ι is the natural inclusion, the sum $\sum\limits_i \overline{a}_i t^i$ should be equal to $\overline{a}_0 = \overline{a}$. Thus, it suffices to let

$$I := (a_i \mid i \neq 0,\, a \in A) \subset A,$$

the ideal generated by the nonconstant coefficients of $\sigma(a)$. It is easy to check that for any h making (4.21) commute, we have $I \subset \mathrm{Ker}(h)$. This finishes the proof of the theorem.

Proof 4.35. (of Lemma 4.34) Using the isomorphism (4.19), and the fact that the $\mathbb{C}^*$-action on $\mathrm{Pic}^{(0,d)}$ is trivial, we may replace $\mathfrak{Z}$ by $\overline{M}_0(G \times T, \beta')$. Furthermore, using the embedding of $G \times T$ in $G \times \mathbb{P}^{d+1}$, we may replace the former by the latter. Now, since $G \times \mathbb{P}^{d+1}$ is a convex variety, $\overline{M}_0(G \times \mathbb{P}^{d+1})$ is normal. Thus the lemma is a consequence of [42], Cor. 2.

4.3 Comparison with $G^{r,\mathrm{exact}}_{d,\delta}(X)$.

From the Theorem 4.32, we have the set bijection

$$\phi(\mathrm{Spec}(\mathbb{C})) : \overline{\mathcal{M}}_0(H^r_d(X), \beta)^{\mathbb{C}^*}(\mathrm{Spec}(\mathbb{C})) \longrightarrow \mathrm{Hom}\left(\mathrm{Spec}(\mathbb{C}), \overline{M}_0(H^r_d(X), \beta)^{\mathbb{C}^*}\right), \qquad (4.24)$$

where ϕ is a natural transformation of the functors determined by $G^r_d(X)^{\mathrm{St}} := \overline{M}_0(H^r_d(X), \beta)^{\mathbb{C}^*}$.

Our next result, may be paraphrased saying that the projective (coarse) moduli space $\mathfrak{Z}$ parametrizes the limit linear series in all levels on the curve X. Precisely,

Theorem 4.36. *For all $\delta \geq 1$, exists a natural map*

$$\Psi_\delta : G^{r,exact}_{d,\delta}(X) \longrightarrow G^r_d(X)^{St},$$

whose union run on δ of the left side is equal to (set-theoretically) $G^r_d(X)^{St}$.

Proof 4.37. According to Proposition 4.18 and the proof of Theorem 4.25, each $\mathfrak{g} \in G^{r,\mathrm{Exact}}_{d,\delta}(X)$ determines a unique class of $[f^{\mathfrak{g}}] \in G^r_d(X)^{\mathrm{st}}$, for any δ. More precisely, from Remark 3.15(3) of (3) we know that $G^{r,\mathrm{Exact}}_{d,\delta}(X)$ admits a covering by:

$$G^{r,\mathrm{Exact}}_{d,\delta}(X; U) := \left\{ \mathfrak{g} \in G^{r,\mathrm{exact}}_{d,\delta}(X) | \sum_{k \in U} M_k = r+1 \right\}.$$

According to Proposition 4.18 and the proof of Theorem 4.25, each point in $G^{r,\mathrm{exact}}_{d,\delta}(X; U)$ correspond to a unique point of $[f^{\mathfrak{g}}] \in G^r_d(X)^{\mathrm{st}}$, whose chain source is indexed by U and corresponds to non-collapsed components by $f^{\mathfrak{g}}_2$ and $f^{\mathfrak{g}}_1$, which are the compositions of $f^{\mathfrak{g}}$ with the natural projections to $p_2 : H^r_d(X) \to T$ and $p_1 : H^r_d(X) \to \mathrm{Grass}(r+1, W)$.

Conversely, to each point of $[f] \in G^r_d(X)^{\mathrm{st}}$ we associate $\mathfrak{g}_f \in G^{r,\mathrm{Exact}}_{d,\delta}(X)$ for some δ, by adding the necessary $\mathbb{P}^1$ to get a chain of length $d\delta + 1$. Recall that the class β and the torus action determines the type of our stable maps.

It follows that, associated to each stable limit linear series $[f] \in G^r_d(X)^{\mathrm{st}}$ we have a subscheme in the fiber of the degree d Abel map A_d as defined in 3: $\mathbb{P}(\mathfrak{g}_f) \in A^{-1}_d(L)$. In particular,

Corollary 4.38. *For any $\mathfrak{g} = (L, V_0, \ldots, V_d) \in G^{r,\mathrm{Oss}}_d(X)$ there exist a stable limit linear series $[f_{\widetilde{\mathfrak{g}}}] \in G^r_d(X)^{st}$ such that $\widetilde{\mathfrak{g}} \in G^{r,Exact}_{d,\delta}(X)$ and $\rho_{1,\delta}(\widetilde{\mathfrak{g}}) = \mathfrak{g}$ for some δ and whose subscheme associated in the fiber of the Abel map $\mathbb{P}(\widetilde{\mathfrak{g}}_f) \in A^{-1}_d(L)$ has "correct" Hilbert polynomial, i.e., the Hilbert polynomial $P(s,t) = \binom{s+t+r}{r}$ of the diagonal of $\mathbb{P}^r \times \mathbb{P}^r$.*

Proof 4.39. In fact, by Proposition 3.16(2) in (3) there exists an exact level-δ lls $\widetilde{\mathfrak{g}} \in G^{r,\mathrm{Exact}}_{d,\delta}(X;U)$ for some $U \neq \phi$. By Theorem 4.36 above $\widetilde{\mathfrak{g}}$ corresponds to $[f_{\widetilde{\mathfrak{g}}}]$ a unique stable limit linear series on X, which by the comments above, exactness and Theorem 3.19 in (3) the subscheme $\mathbb{P}(\widetilde{\mathfrak{g}}_f)$ has Hilbert polynomial $P(s,t) = \binom{s+t+r}{r}$.

Notice that, since $\mathbb{P}(\mathfrak{g}_f) = \bigcup \mathbb{P}(\mathfrak{g}_{f,k})$ we have that each class in $G^r_d(X)^{\mathrm{st}}$ determines a unique subscheme in the fiber of the Abel map A_d. In this sense, the projective scheme $G^r_d(X)^{\mathrm{st}}$ is a "good candidate" for resolving the (set-theoretically) map

$$\begin{array}{rcccc} \mathbb{P} & : & G^{r,\mathrm{Oss}}_d(X) & \dashrightarrow & \mathrm{Hilb}^{\binom{r+s+t}{r}}_{A_d} \\ & & \mathfrak{g} & \mapsto & \mathbb{P}(\mathfrak{g}). \end{array}$$

studied by Esteves–Osserman (see [18]).

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