Instituto de Matemática Pura e Aplicada

Doctoral Thesis

ON THE ROLE OF AMBIGUITY IN GENERAL EQUILIBRIUM: FINITE AND INFINITE HORIZON ECONOMIES

Juan Pablo Gama-Torres

Rio de Janeiro
September 30, 2014

impa **Instituto de Matemática Pura e Aplicada**

Juan Pablo Gama-Torres

ON THE ROLE OF AMBIGUITY IN GENERAL EQUILIBRIUM: FINITE AND INFINITE HORIZON ECONOMIES

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Advisor: Aloísio Pessoa de Araújo

Rio de Janeiro
2014

To my beloved family,
the best present that I could ever receive.

Acknowledgements

I wish to express my infinite gratitude to Aloísio Araújo and Rodrigo Novinski. For all their help and knowledge that was given to me along these years at IMPA. Their advices have been considered as extremely valuable presents for me and I have tried to follow them the best way I can. To them, my most sincere respect and gratitude.

To Alain Chateauneuf, an excellent professor with an incredibly capacity to help and to worry for others. To Mário Páscoa, an excellent researcher with an extraordinary capacity that has been essential for this work.

I also wish to offer my sincere thanks to the IMPA and its professors for help me to be a better professional.

I want to thank to the seminar of mathematical economics at IMPA for all the ideas that were discussed and for the opportunity of developing my abilities as a speaker. We gratefully acknowledge the financial support from CAPES, CNPq, FAPERJ, FCT/FEDER and NOVA FORUM (Portugal). We are grateful to comments by Harvard-MIT Theory seminar, 10th Annual Cowles Conference on General Equilibrium and its applications at Yale University, and the 13th SAET Conference in Paris.

There have been a lot of people that I have met along these years and I would like to thank Carolina Parra, Erick Lpez, Felipe Macías, Juliana Meza, Rosa María Vargas, Susana Frometa for all their help and friendship.

To Aniel Ojeda and Jyrko Correa. My closest friends that have been next to me along these five years at IMPA especially when I really needed. I also consider myself lucky to have found great friends that have become part of my family. Oliesia Gonzalez, Yanet Larramendi and Fernanda, my niece, that with her energy and happiness, have been transforming us since the day that she was born.

There are two special friends that have been very important to me, even when we have not been together, our friendship is still intact. Vladimir Strokov, we share the same passion for science and the same thoughts about of how to live our lives. Márcio Barros, an incredible person that listen to me when I most needed.

And I also want to thank my family. Everything I do in my life I owe to them. My aunt Dora and my cousin Paola, who have been a big support. Melba, my second mother. My sister, who has always been by my side. My brother, who has supported and cared for me since I was born. To my dad, who with his love, dedication and values, he has wrought in me a passion for work and to live each moment of my life as if it were the last. And finally to my mom, who with her unmeausurable love, kind and passion has been my biggest force. Anything that I could say will not be enough to thank her all the love, support and sacrifice that she has done through these years, especially during the accident.

Abstract

Although implementation of different type of attitudes towards uncertainty in General Equilibrium seems to be completely natural, has not be completely studied. In this work, we presents two form: with ambiguity/risk loving and ambiguity as lack of impatience (Wariness).

We show that the aggregate risk of wealth plays a role in the existence of equilibrium in Arrow-Debreu economies. Moreover, we study properties of the equilibrium allocations such as condition for risk sharing and the price behavior in equilibrium in presence of regulation, and, the decomposition in the risk factor and the ambiguity factor in these prices in some special cases. Our analyses suggest that regulation increases volatility while reduces the social welfare of the economy, however the risk lovers or optimists are those who perceive the larger losses.

We show that, contrary to what happens under purely discounted utility, efficient allocations to wary agents are implemented with a non-vanishing money supply. In fact, the hedging rule of money does not disappear over time and, therefore, the transversality condition allows for consumers to be creditors at infinity. The implementation scheme starts by allocating money and then, at subsequent dates, taxes money balances that deviate from the efficient path. We address also why fiat money does not lose its value when Lucas trees are available and why we might not want to replace money by a tree. And finally, we expose conditions for the existence of stochastic efficient bubbles, which suggest the possibility of crashing under some conditions.

Keywords: General Equilibrium, Ambiguity, Ambiguity Loving, Aggregate Risk, Wariness, Money, Efficient Bubbles, Crashing.

Resumo

Embora a implementação de diferentes atitudes em relação à incerteza no Equilíbrio Geral parece ser completamente natural, não tem sido completamente estudada. Neste trabalho apresentamos duas formas: com propensão à ambigüidade/risco e ambigüidade como falta de impaciência (Wariness).

Mostramos que o risco agregado da riqueza desempenha um papel importante na existncia de equilíbrio nas economias Arrow-Debreu. Também estudamos propriedades das alocações de equilíbrio, como condição para o compartilhamento de risco e o dos preços em equilíbrio na presena de regulamentao, e a decomposição no fator de risco e o fator de ambigüidade dos preços. Nossas análises sugerem que a regulação aumenta a volatilidade, enquanto reduz o bem-estar social da economia, no entanto, os amantes ao risco ou otimistas são aqueles que percebem as perdas maiores.

Mostramos que, ao contrário do que acontece sob utilidade puramente descontada, alocações eficientes para agentes tipo Wary são implementadas com oferta de moeda não nula. A regra de "hedging" da moeda não desaparece ao longo do tempo e, portanto, a condição de transversalidade permite que os consumidores sejam credores no infinito. A implementação começa alocando dinheiro para logo usar impostos em moeda para consumos que se desviam do caminho eficiente. Se trata porque o "fiat money" não perde seu valor quando as árvores Lucas estão disponíveis e por isso que poderia não querer se substituir o dinheiro por uma árvore. E, finalmente, expôr as condições para a existência de bolhas estocásticos e eficientes, o que sugere a possibilidade de "crashing" sob algumas condições.

Palavras-chave: Equilíbrio Geral, Ambigüidade, propensão à Ambigüidade, Risco agregado, Warriness, Moeda, Bolhas eficientes, Crashing.

Contents

Introduction

Since Allais, in 1953 [1], and Ellsberg, in 1961 [22], paradoxes, there has been the desire to find models consistent with the majority of the attitudes towards unknown events that are usually observed in the real world. As a consequence, the Decision Theory works with two different concepts that aim to model this behaviors, *risk*, related to future events that can no be predicted precisely however is known the probability of occurrence, and *uncertainty*, related to an impossibility to know precisely the current situation, and as a consequence, is not possible to know prcisely the probability of future events. The most famous representations are given by Quiggin in 1982, [33], in the case of risk and by Schmeidler in 1989, [37], in the case of uncertainty. Both representations model a distortion of the future events related to the possibility of losses by using Choquet Integrals,

$$\begin{aligned} U(x) &= (C)\int_{\Omega} u(X)d\nu \\ &= \int_{-\infty}^{0} \left(\nu\left[u\circ X{\geq}t\right]-1\right)dt + \int_{0}^{\infty} \nu\left[u\circ X{\geq}t\right]dt \end{aligned}$$

where $\nu : \mathcal{A} \to \mathbb{R}_+$ is a capacity, an increasing function defined in the measurable space $(\Omega, \mathcal{A})$ such that $0 = \nu(\emptyset) < \nu(\Omega) < \infty$.

As a consequence, if we interpret the utility function of some decision makers in the sense of Quiggin, we probably will lead with *Optimistic* or *Pessimistic* agents, since they are distorting the probability of future events giving more weight to gains or losses respectively. On the other hand, if we interpret the utility function in the sense of Schmeidler, we will lead with *Ambiguity Lovers* or *Ambiguity Averse*, since ambiguity is the lack of information that leads to an impossibility to know precisely the current situation of the decision makers.

The study of ambiguity has made possible the realization of several representations that generalized the model defined by Schmeidler as *Variational Preferences*, defined by Maccheroni, Marinacci and Rustichini, [29] or explore ambiguity by different type of representation as the *Smooth Ambiguity* model, defined by Klibanoff, Marinacci and Mukerji, [28].

However, the implementation of ambiguity in the Theory of General Equilibrium is not extensively studied due to a large variety of behaviors that can be explained using ambiguity or risk, and also due to technical problems. Some of them are the lack of convexity of the preferences in the case of ambiguity loving and optimistics, lack of continuity of the preferences, especially in economies with an infinite number of goods or states of nature, and computational problems related to the lack of differentiability for the majority of the preferences.

The convexity of the preferences is a assumption that most of the models require. In general, this assumption is not considered too strong since most of the people avoid uncertain or risky situations when there is a real possibility of large losses, and is also not possible to ensure general condition for the existence of equilibrium. However in the real world, there are some agents, as financial institution and speculators, that are willing to consume or buy risk in the economy. It means that the study of "non-convex" economies could help us to understand the interaction among the agents that can not be explained by the classic theory of general equilibrium.

The problems related to the continuity in economies with an infinite number of goods or states have been studied by Bewley, in 1972 [9], in the case of *Arrow-Debreu* economies, this is the case of economies with only one *Budget-Constraint*

$$\pi(x) \leq \pi(\omega),$$

where π is a continuous linear functional in the topology defined on the set of possible consumptions. However, in sequential economies, the optimality conditions are not the same, and usually it will be necessary to impose additional constraints as *Transversality Condition* to ensure the existence of a solution for the consumer problem, see Aloisio, Novinski and Páscoa, [5]. Nevertheless imposing these type of constraints is not supported by empirical evidence or any type of constraint that is commonly used in the market.

And finally, the lack of differentiability generates computational difficulties since most of the algorithms used require differentiability and the traditional *First Order Conditions* to be implemented. However there is one model which uses a smooth representation of ambiguity, the one mentioned before. The problem with this representation is that the models created to be developed computationally are complex and the consumption problem for a smooth ambiguity agent will increase the computational cost considerably compared to a traditional model like *Expected Utility*.

Contributions and organization

Related to the convexity of the preferences. As it was mentioned before, there is some results related to the existence of equilibrium, the most famous is due to Aumann, in 1966 [8], in which is possible to ensure existence of equilibrium when there is a continuum of agents due to a "convexification" of the economy. Additionally Aloisio, Faro and Novinski guaranteed, in some cases, the existence of equilibrium with nonlinear prices. For a finite number of agents with linear prices is very well known that there is no general condition for the existence of equilibrium. The chapter 1 of the thesis aims to expose sufficient conditions for the existence of equilibrium, conditions that seems to be related to the existence of aggregate risk sufficiently large. More precisely our contributions in this part are:

1. An analysis of sufficient conditions for the existence of equilibrium with Ambiguity lover, optimistics or more general non convex preferences.

2. Similar condition for decision makers with mixed attitude against risk[1].

3. A numerical analysis of volatility when the risk that the ambiguity lovers or risk lovers can absorb is limited.

And related to the problem in economies with an infinite number of goods or states. The implementation of some type of ambiguity in this economies could lead to an additional concern of losses at states with low probability or distant dates. And as it was exposed in Araujo et al. [5], this concern would lead to the existence of *Efficient*[2] *Bubbles* in sequential equilibrium. However, as it was said before, it is necessary to impose an additional constraint to guarantee the existence of optimal solution for the consumer problem. The chapter 2 aims to explore the implementation of efficient allocations in sequential economies with a fiscal policy, which includes taxes and money instead of the transversality conditions exposed before. Also we explore why

[1] A Decision maker that is not completely risk lover or risk averse.

[2] Efficiency means that no agent can increase his/her utility level without reducing the utility level of another agent. An allocation that satisfies this condition is called *Pareto Optima*.

money seems to be quite importat as store of value, it means that it becomes more powerful compared to economies with only *Long-Lived assets* that are no produced by the government as *Stocks*[3].

Additionally an analysis of these efficient bubbles in stochastic economies can be done as in Araujo et al. [5] for the deterministic case, and explore, in this case, the possibility of *Crashing* of bubbles. The chapter 3 aims to establish some condition in which bubbles can exists and crashes at in some states of nature.

Therefore the main contributions of the second part of the thesis are:

1. Implementation of efficient allocation with money and a fiscal policy.
2. Analyze possible implementation with long-lived assets not produced by the government.
3. Analyze the stochastic case and compared to the deterministic case.
4. Analyze an stochastic model in which crashing of efficient bubbles is possible and is also consistent with the literature of decision theory.

[3]Private long-lived assets that pay real good in each date.

Part I

Risk and ambiguity loving

CHAPTER 1

General equilibrium, risk loving, ambiguity and volatility

In the real world, there are some agents, as financial institution and speculators, that are willing to consume or buy risk in the economy. To do so, they can buy from other agents, as regular consumers, part of the risk that these agents own.

The fact that this type of behavior is quite common suggests that the analysis of these types of effects could lead to more precise analysis of situations in which there is a large amount of risk for the risk/ambiguity averse, explaining some phenomena that couldn't be explained by traditional models.

The trade between agents that are willing to buy risk and agents that are willing to exchange risk despite its importance, has not been extensively studied yet in General Equilibrium Theory.

This is a consequence of the difficulties to ensure the existence equilibrium with this choice pattern.[1]

Nevertheless, this type of issues can be solved in some special environments, making possible the existence of equilibria. This is the case of an economy with a continuum of agents (with an atomless measure over the agents) in a finite-dimensional space of consumption (see [8]).

For the case with a finite number of agents, there is some work that tries to adopt an approach to the existence of equilibrium proving the existence of weaker conditions. Araujo et al. (see [4]) performed an extensive analysis

[1] In fact, the difficulties are given by the non-convexity of the preferences, thus most of the traditional tools are not applicable.

of the existence and characterization of Pareto Optima individually rational, and Anderson (see [2]) proved a similar result, for the finite case, as Aumann did about the existence of the core for non-convex preferences for a continuum of agents (see [7]) and Starr proved the existence of ε-equilibrium for economies with an increasing and finite number of agents (see [38]).

In the present chapter we find some conditions under which one guarantees the existence of equilibrium for economies with a finite number of agents. These conditions suggest that it is convenient for the non convex agents, in particular Ambiguity Lovers, to have more aggregate risk in the economy. In fact, we guarantee the existence of a minimum level of risk that ensure equilibrium. Nevertheless this aggregate risk should be increased just only for the convex agents as Ambiguity Averse. The main reason to this is that, to guarantee market clearing, it is necessary to get enough aggregate risk for the Ambiguity averse to be absorbed by the Ambiguity Lovers. And then, if one increases the aggregate risk to the Ambiguity Lovers, the rent obtained by the endowment from the non-desirable states would unbalance the economy leading to failures in market clearing.

The fact that the risk is given to the ambiguity averse Decision Makers implies that at equilibrium, the ambiguity lovers are buying part of the risk that the ambiguity averse have. Leading to an exchange of the risk between the agents, and by doing this, all the agents are improving their utility. This is the reason why our framework helps us to understand that, in fact, this type of behavior can be analyzed in a general equilibrium framework, and also that we can analyze their consequences to the optimal consumption, comonotonicity, characterization of the equilibrium price and volatility in Arrow assets.

We first we introduce this condition for an economy in the Edgeworth box. Analyzing this particular case, we can observe that these conditions are also sufficient for the existence of equilibrium, what helps us to establish uniqueness of the equilibrium in this particular case, and also a very precise characterization of equilibrium prices.

Then we analyze the general case with non completely substitutable goods or states for the agents with convex preferences, one possible case is Ambiguity Averters. And, as a consequence, we will have the same result for three types of preferences that represent ambiguity, *Smooth Ambiguity* (SA) model (see [28]), *Choquet Expected Utility* (CEU) (see [37]), which also analyzes *Rank-Dependent Expected Utility* (RDEU) (see [33] and [34]), and finally *Variational Preference* (VP) (see [29]). Since all Arrow Debreu equilibria are efficient[2], we will study efficient allocation in an implicit form, establishing

[2] As a consequence of the First Welfare Theorem, any equilibrium allocation with locally

a relationship between the perception of ambiguity and the market behavior as in Rigotti et al. (see [32])

And similarly to the previous cases, our result can be extended to other type of behavior when there is risk in the economy given by Friedman Savage (see [24]), in which the agents are risk averse for low consumptions, but become risk lovers when they are able to have big consumptions. These Decision Makers have a behavior which looks like consistent with what happens in the real world, they are willing to specialize in some states but they don't want to specialize completely since they have incentives to have positive consumption in all states.

Comonotonicity has been studied by many authors, including Bühlmann (see [17] and [18]), Chateauneuf et al. (see [19]), Strzalecki et al. (see [39]) and Tsanakas et al. (see [40]), for the convex case these results can be generalized to our framework with some modifications. This work is also related to Bossaerts et al. (see [15]) in which is presented a theoretical and experimental study of asset prices in competitive financial markets in presence of ambiguity.

Even though we find conditions in which we can assure that we have comonotonic behavior. Some of them are related to the fact of having a large enough aggregate risk. This is a consequence of the fact that, in presence of enough aggregate risk, ambiguity lovers can not absorb all the risk that the ambiguity averse have, leading to comonotonic consumptions.

And finally we will carry out an analysis of the equilibrium price in a similar sense as the case of Tsanakas et al. but with a finite number of states of nature. In this case we will prove that the equilibrium price can be characterized in terms of the aggregate risk and the ambiguity aversion, resulting in a generalization to the case with ambiguity lovers of the work that has been done by Bühlmann and Tsanakas et al. And with this type of characterizations we have enough tools to analyze variations in Volatility and Welfare in economies with aggregate risk when there are some exogenous constraints that reduce the risk that the Ambiguity Lovers, or Risk Lovers can absorb.

The chapter is organized as follows: In section 1.1 we start the analysis with the economy in the Edgeworth box with one risk lover and one risk averse. In section 1.2 we study the general case with non completely substitutable goods for the agents with convex preferences as ambiguity/risk averters and some special cases as SA, CEU, VP, and particularly the case with only Risk Aversion and Propension. In section 1.3, we analyze the risk sharing for economies without ambiguity and we extend these results to

non satiable preferences is a Pareto optima.

RDEU Decision Makers, we also make a characterization of the equilibrium price in both cases. In section 1.4, we analyze volatility and welfare when there is regulation. And in section 1.5, we analyze the Friedman Savage case, including existence of equilibrium and an analysis of volatility in presence of regulation.

1.1 Example and first results of Existence of Equilibrium with Aggregate Risk

Before establishing this condition let us give an example to show the relationship that exists between aggregate risk and existence of equilibrium with Risk Lovers.

Example 1. Suppose that each good can be interpreted as a state in the world of an economy with complete markets. Each agent has as utility function $U^i(x_1, x_2) = 1/2\, u^i(x_1) + 1/2\, u^i(x_2)$, where $u^1(x) = \ln x$ and $u^2(x) = x^2$. And let us suppose that we have $\omega^1 = (\omega_1^1, \omega_2^1)$ as the endowment for the agent 1 and $\omega^2 = (\omega_1^2, \omega_2^2)$ as the endowment for the agent 2, and $p = (p_1, 1 - p_1)$ the Arrow-Debreu price.

Since the agent 2 is a Risk Lover, the optimal consumption will satisfy that $x_1^2 = 0$ or $x_2^2 = 0$ as a consequence of Lemma 1 in page 13 .

If $x_1^2 = 0$, it means that the price must satisfy $p_1 \geq 1/2$ and then, with the First Order Conditions (FOC) for the agent 1 and the market clearing equation, $\omega^1 + \omega^2 = x^1 + x^2$, we have that

$$p_1 = \frac{\omega_2^1}{\omega_2^1 + \omega_1 + \omega_1^2}, \tag{1.1}$$

and equilibrium allocation is

$$\begin{aligned} x_1^1 &= \omega_1, \quad x_2^1 = \tfrac{1}{2}\omega_2^1 + \tfrac{\omega_2^1\omega_1^1}{2\left(\omega_1+\omega_1^2\right)}, \\ x_1^2 &= 0, \qquad x_2^2 = \omega_2^2 + \tfrac{\omega_1^2\omega_2^1}{\omega_1+\omega_1^2}, \end{aligned}$$

where $\omega_i = \omega_i^1 + \omega_i^2$.

Using $p_1 \geq 1/2$ and 1.1, we have that the initial endowments must satisfy

$$\omega_2^1 \geq \omega_1 + \omega_1^2. \tag{1.2}$$

And now if $x_2^2 = 0$, it means that the price must satisfy $p \leq 1/2$ and then

$$p_1 = \frac{\omega_2 + \omega_2^2}{\omega_1^1 + \omega_2 + \omega_2^2}, \tag{1.3}$$

and the equilibrium allocation is

$$\begin{aligned} x_1^1 &= \tfrac{1}{2}\left(\omega_1^1 - \omega_2^1\right) + \tfrac{\omega_2^1\left(\omega_1^1+\omega_2+\omega_2^2\right)}{2\left(\omega_2+\omega_2^2\right)}, & x_2^1 &= \omega_2, \\ x_1^2 &= \omega_1^2 - \omega_2^2 + \tfrac{\omega_2^2\left(\omega_1^1+\omega_2+\omega_2^2\right)}{\omega_2+\omega_2^2}, & x_2^2 &= 0. \end{aligned}$$

Using $p \leq 1/2$ and 1.3, we have that the endowments must satisfy

$$\omega_1^1 \geq \omega_2 + \omega_2^2. \tag{1.4}$$

Then if 1.2 or 1.4 are satisfied, is possible to have an equilibrium for the economy[3]. However if they are not, it is easy to check that

- if we suppose $x_1^2 = 0$, the price must satisfy $p_1 < 1/2$, and
- if we suppose $x_2^2 = 0$, the price must satisfy $p_1 > 1/2$.

Which contradicts the condition of the price. Therefore there is no equilibrium for the economy. And as a consequence, the conditions 1.2 and 1.4 are necessary and sufficient for the existence of equilibrium.

From the conditions 1.2 and 1.4 we have that

$$\omega_2 = \omega_2^1 + \omega_2^2 \geq \omega_1 + \left(\omega_1^2 + \omega_2^2\right)$$

or

$$\omega_1 = \omega_1^1 + \omega_1^2 \geq \omega_2 + \left(\omega_1^2 + \omega_2^2\right),$$

which says that a large enough aggregate risk is necessary and sufficient to guarantee the existence of equilibrium, in fact, it must be at least equal to the sum of the endowments of the Risk Lover.

Therefore we can interpreted $\omega_1^2 + \omega_2^2$ as the lowest quantity of risk that the Risk Lover would consume and as a consequence, any additional risk that exists in the economy will ensure the existence of equilibrium.

Intuitively, in presence of a lower quantity of aggregate risk, there are less possible endowment distributions that eliminate the gap between the optimal consumption and the initial endowment in all states.

As can be seen in the Figure 1.1, there is a large difference between economies with large quantity of risk and economies with almost no risk. For example, the first Edgeworth Box (EB) has an aggregate risk[4] of 20%

[3]This equilibrium is unique since conditions 1.2 and 1.4 can not be satisfied simultaneously.

[4]We define aggregate risk as the difference between the aggregate endowments, ω_1 and ω_2.

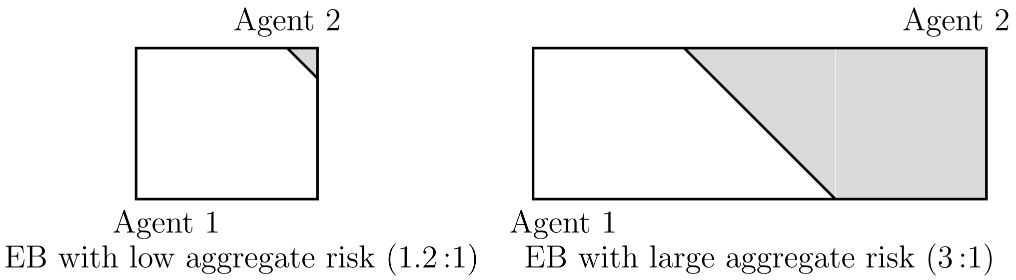

Figure 1.1: Endowment distributions where equilibrium exists (gray region)

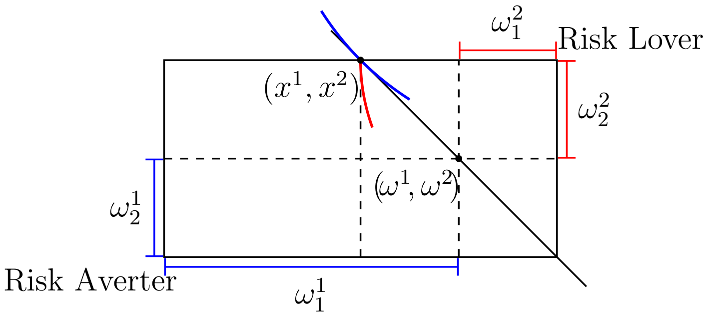

Figure 1.2: Edgeworth box

and as a consequence, the possible endowment distributions in which there is equilibrium are restricted to endowment distributions with an Agent 2 quite poor compared to the Agent 1[5]. And this implies that the endowment distributions in which there is an equilibrium compared to all the possible endowments distribution in this EB is quite low, in the first EB in Figure 1.1 is $1.\overline{6}\%$.

However in presence of large amount of aggregate risk, like in the second EB, the existence of equilibrium is less affected by large endowment distributions given to the risk lover. And this implies that the endowment distributions in which there is an equilibrium compared to all the possible endowments distribution in the second EB is 50%.

This suggests that the existence of equilibrium is strongly related to the aggregate risk that exists in the economy and the wealth that is given to each agent, increasing the possibility of existence if the endowments are in hands of the risk averse.

[5] Agent 1 could be more than 10 times wealthier than Agent 2.

1.2 Existence of Equilibrium with finite number of agents

1.2.1 Model

Let us start by making some comments and notations related to the economies that we will study. Each agent i is characterized by a utility function given by $U^i : \mathbb{R}^S_+ \to \mathbb{R}$ and an Arrow-Debreu constraint given by $px \leq p\omega^i$ where $p \in \Delta^{S-1}_+$ is the price and $\omega^i \in \mathbb{R}^S_+$ is the initial endowment for the agent, where each $s = 1, \dots, S$ will be interpreted as one state of nature.

We will say that $(p, (x^i)_i)$ is an equilibrium (an Arrow-Debreu equilibrium) when x^i is optimal for U^i with the AD-constraint, and also having market clearing, that is $\sum_i \omega^i = \sum_i x^i$.

From now on we will consider an economy with $I + J$ agents with two different type of behaviors. The agents $i = 1, \dots, I$ are of the Type A and the agents $j = I + 1, \dots, I + J$ are of the Type B.

The agents of the type A have:

A1. Utility function, U^i, strictly increasing, concave,

A2. For any s and $\{x^n\}_{n\in\mathbb{N}} \in \mathbb{R}^S_+$ if $x^n_s \to \infty$ and $\{x^n_{s'}\}_{n\in\mathbb{N}}$ is bounded for $s' \neq s$ then

$$\lim_{n\to\infty} \left(\max_{T\in\partial U(x^n)} \frac{T \circ e_s}{T \circ e_{s'}} \right) = 0. \tag{1.5}$$

A 3. For any s, s' and $\{x^n\}_{n\in\mathbb{N}} \in \mathbb{R}^S_+$ such that $\{x^n_s\}_{n\in\mathbb{N}}$, $\{x^n_{s'}\}_{n\in\mathbb{N}}$ are bounded from above and bounded away from zero from below then

$$\liminf_{n\to\infty} \left(\min_{T\in\partial U(x^n)} \frac{T \circ e_s}{T \circ e_{s'}} \right) \in (0, \infty). \tag{1.6}$$

The A2 can be interpreted in terms of the marginal substitution rate between the states s and s' i.e. when the consumption is going to infinity in one state, the marginal demand in that state is going to zero compared to a state with bounded consumption. And A3 can be interpreted as finite and bounded away from zero marginal demand between states with bounded and far away from zero consumptions. Intuitively these two conditions imply that there are no completely substitutable states, or goods, in the economy since the consumption of arbitrarily large in some of them does not null the marginal utility of consuming in the rest of goods or states.

For the agents of Type B:

B1. The utility function is strictly increasing and convex.

The endowments are given by $(\omega_1^i, \ldots, \omega_S^i) \gg 0$. And let us denote

$$\omega_s := \sum_{i=1}^{I+J} \omega_s^i, \ \forall s = 1, \ldots, S.$$

Since the agents of type B have convex utility functions, they have incentives to specialize their consumption as much as possible, however the agents of type A have a behavior such that they will not absorb a large amount of risk in their optimal consumption. And since there is two totally opposites attitudes toward risk in which one of them tends to buy a large amount of risk, it could be desirable for a central planner to reduce the type of specialization that the agents of type B $(I + 1 \leq i \leq I + J)$ would make. One possible form is by forcing them to have a minimal consumption λ_s^i, where $\lambda_s^i \in [0, \omega_s^i]$ for each $s = 1, \ldots, S$ to avoid extreme consumption in equilibrium. Note that if $\lambda_s^i = 0$ $\forall s, i$, we will have a traditional AD economy without any type of additional constraint. Therefore we have

Lemma 1. *Given a price p, all the B agents have an optimal solution:*

$$x_s^i = \begin{cases} \lambda_s^i & \text{for } s \neq s_0 \text{ (minimal consumption)}, \\ \frac{1}{p_{s_0}} \left[p\omega^i - \sum_{s \neq s_0} p_s \lambda_s^i \right] & \text{for some } s_0. \end{cases}$$

This result follows from the fact that all B agents would like to specialize their consumption as much as possible consuming on the boundary of the Budget set, which implies that the any optimal solution has the form described in the Lemma 1.

We can think this type of constraints as a certain type of regulation made by a social planner that is worried about the amount of risk that the agents B are consuming, and therefore is concerned of the possible non existence of equilibrium due to the desire of specialization of these agents, see Lemma 1. If we interpret this agents as gambler or financial institutions, we may think this constraints as a minimal capital requirement imposed by a social planner.

And similar to the example exposed before, we will show that the existence of aggregate risk helps in the matching between the desire of hedging for the agents of type A and the speculation of the agent of type B, however the formers need to have proportionally more wealth in one state to allow the specialization of the latter without violations of market clearing.

Theorem 1. *If the aggregate endowment of agents of type A is sufficiently large in some state s compared to the other states, then there exists an equilibrium for the economy with $p \in \Delta_{++}^{S-1}$.*

The hypotheses A2 and A3 are needed to have some type of independence among states or goods, which would help to the existence of equilibrium when there are agents of type B.

Our result says that in presence of enough wealth in one state for the A agents, we will have that they are willing to transfer this new risk as much as they can to the B agents, and to the latter, this allows them to improve their consumption. Then, even in presence of non convex preferences, we will have that there is a balance between the agents given by the presence of the aggregate risk.

For all the cases below it seems that it is necessary to assume "roughly" speaking that the marginal utility at a given state s tends towards 0 when the consumption is going to infinity in this state.

1.2.2 Implementation with ambiguous Decision Makers and other special cases

In this section we will see that Theorem 1 will imply existence of equilibrium in presence of ambiguous agents. We will analyze mainly the *Smooth Ambiguity* Model, *Choquet Expected Utility* and *Variational Preferences*, and it will be analyzed the special case without ambiguity with common priors, which has special features.

Smooth Ambiguity Decision Maker (SA)

Every decision maker in the sense of SA, see [28], has a probability measure μ^i over the space of probabilities $\Delta_i \subseteq \Delta_+^{S-1}$, that is the space of probabilities that the agent is taking in to account, and we can interpret μ^i as the weight, or importance that each agent is giving to every possible "real probability" or prior that the agent is considering plausible.

We will assume that:

SA1. There exists $\underline{\pi} > 0$ such that for each i and $\pi \in \Delta_i$, $\pi_s \geq \underline{\pi}$.

Which means that each state has a uniform minimum likelihood that is positive. The previous condition is stronger than $\pi_s > 0\ \forall s$, the problem of this condition is that allows the existence of sequences of probability measures such that $\{\pi^n\}_{n\in\mathbb{N}} \in \Delta_i$ such that $\pi_s^n \to_n 0$. This problem can also be eliminated by assuming that Δ_i is a closed set.

The agent preference is represented by U^i which has the form of an SA utility with utility index u^i and distortion ϕ_i for each $i = 1, \ldots, I+J$ then

$$U^i(x) = \int_{\Delta_i} \phi_i \left(\sum_{s=1}^{S} u^i(x_s)\, \pi_s \right) \mu^i(d\pi),$$

where $x \in \mathbb{R}_+^S$, $u^i : \mathbb{R}_+ \to \mathbb{R}$, $\phi_i : \mathbb{R} \to \mathbb{R}$. For $i \leq I$ (Ambiguity Averse),

SA2. u^i and ϕ_i are strictly monotone, concave, C^1 and u^i satisfies $\lim_{x\to\infty} u^{i\prime}(x) = 0$.

And for i such that $I < i \leq I + J$ (Ambiguity Lovers),

SA3. u^i and ϕ_i are strictly monotone and convex.

Proposition 1. *SA1 and SA2 implies A1, A2 and A3.*

Proof. Since we have $\pi_s \geq \underline{\pi} > 0$ and

$$\frac{\partial}{\partial x_{s'}} U^i(x) = \int_{\Delta_i} \phi_i' \left(\sum_{s=1}^{S} u^i\left(x_s\right) \pi_s \right) u^{i\prime}\left(x_{s'}\right) \pi_s \mu^i(d\pi),$$

then we have A1 and

$$\frac{\underline{\pi} u^{i\prime}(x_s)}{\left(1-\underline{\pi}\right) u^{i\prime}(x_{s'})} \leq \frac{\frac{\partial}{\partial x_s} U^i(x)}{\frac{\partial}{\partial x_{s'}} U^i(x)} \leq \frac{\left(1-\underline{\pi}\right) u^{i\prime}(x_s)}{\underline{\pi} u^{i\prime}(x_{s'})}$$

and as a consequence of $\lim_{x\to\infty} u^{i\prime}(x) = 0$, we have A2 and A3. □

Choquet Expected Utility (CEU), Rank-Dependent Expected Utility and Variational Preferences (VP)

For the Ambiguity Averse and Ambiguity Lovers in the sense of CEU, see [37], we have that each agent i considers different capacities $\nu^i : \mathcal{P}\left(\mathcal{S}\right) \to [0,1]$, where $\mathcal{S} = \{1, \ldots, S\}$ such that

CEU1. For each $A \subsetneq B \subseteq \mathcal{S}$, $\nu_i(B) - \nu_i(A) \geq \underline{\pi} > 0$.

The utility function for each agent of this type is given by:

$$U^i(x) = (C) \int_{\mathcal{S}} u^i(x_s)\nu^i(ds) = \int_{-\infty}^{0} \left(\nu^i \left[u^i \circ x {\geq} t\right] - 1\right) dt + \int_{0}^{\infty} \nu^i \left[u^i \circ x {\geq} t\right] dt$$

where $x \in \mathbb{R}_+^S$, $u^i : \mathbb{R}_+ \to \mathbb{R}$ and for $i = 1, \ldots, I$ (Ambiguity Averse or Pessimists),

CEU2. u^i is strictly monotone, concave, C^1 and u^i satisfies $\lim_{x\to\infty} u^{i\prime}(x) = 0$ and ν^i is a convex capacity,

For i such that $I < i \leq I + J$ (Ambiguity Lovers or Optimists),

CEU3. u^i is strictly monotone, convex and $u^i(0) = 0$ and ν^i is a concave capacity.

Proposition 2. *CEU1 and CEU2 satisfies A1, A2 and A3.*

Proof. Since we have $\nu^i(A) - \nu^i(B) \geq \underline{\pi} > 0$ for every $\emptyset \subseteq B \subsetneq A \subseteq \mathcal{S}$, then for every $T \in \partial U^i(x)$ we have A1 and

$$\frac{\underline{\pi} u^{i\prime}(x_s)}{(1-\underline{\pi})\, u^{i\prime}(x_{s'})} \leq \frac{T \circ e_s}{T \circ e_{s'}} \leq \frac{(1-\underline{\pi})\, u^{i\prime}(x_s)}{\underline{\pi} u^{i\prime}(x_{s'})}$$

and as a consequence of $\lim_{x\to\infty} u^{i\prime}(x) = 0$, we have A2 and A3. □

This result can be adapted to Rank-Dependent Expected Utility agents (See [33] and [34]) as:

Each agent i considers different probabilities $\pi_i \in \Delta_+^{S-1}$ such that $\pi_{i,s} \geq \underline{\pi}$ $\forall s$, this probability that each agent considers as the probability of the world. Each agent distorts it with a function f^i, for the pessimist agent f^i would be a convex function, and for the optimist f^i would be a concave one. Also we have that $f^i(o) = 0$, $f^i(1) = 1$.

The utility function for each agent of this type is given by:

$$U^i(x) = (C) \int u^i(x) df^i \circ \pi$$

where for $i = I+1, \ldots, I$, u^i is strictly monotone, concave, C^1 and u^i satisfies $\lim_{x\to\infty} u^{i\prime}(x) = 0$, and for i such that $I + J < i \leq I + J$, u^i is strictly monotone, convex and $u^i(0) = 0$.

Remark 1. If we impose similar conditions as in the Smooth Ambiguity case for the set of priors and for the utility index, as $\lim_{x\to\infty} u^{i\prime}(x) = 0$, the previous result can be extended to ambiguity averse given by Variational Preferences, (see [29]),

$$\min_{\pi \in \Delta^i} \left(\sum_s u(x_s)\, \pi_s + c(\pi) \right).$$

Actually, A2 and A3 will be satisfieds since for each $\pi \in \Delta^i$, we will have a limitation in a similar way as depending only on $u^{i\prime}$ and $\underline{\pi}$, therefore we will have an analogous bounds as in the previous cases.

Risk Averse and Risk Lovers

Now let us analyze the model in which there is no ambiguity, where $I > 0$, $J \geq 0$, and represent the number of risk averters and the number of risk averse, respectively.

Every agent has the same probability $\pi \in \Delta_{++}^{S-1}$, and is represented by a EU representation U^i with utility index u^i, note that this can be a particular case for both types of decision makers (The SA and the CEU). For $i \leq I$, u^i is strictly monotone, concave and C^1, and for $i > I$, u^i is strictly monotone, convex and $u^i(0) = 0$.

The next corollary is a consequence of any of the previous cases and the First Order Conditions.

Corollary 1. *If each risk averse satisfies* $\lim_{x \to 0^+} u'(x) = \infty$, *the equilibrium price is given by the solution of*

$$\begin{aligned} p_s \tilde{x}^i + \textstyle\sum_{\hat{s} \neq s} p_{\hat{s}} {u^{i}}'^{-1} \left(\frac{\pi_s}{\pi_{\hat{s}}} \frac{p_{\hat{s}}}{p_s} {u^i}' \left(\tilde{x}^i \right) \right) &= p \omega^i, \ \forall i = 1, \dots, I, \\ \textstyle\sum_i {u^{i}}'^{-1} \left(\frac{\pi_s}{\pi_{\hat{s}}} \frac{p_{\hat{s}}}{p_s} {u^i}' \left(\tilde{x}^i \right) \right) &= \textstyle\sum_{i=1}^{I+J} \omega_{\hat{s}}^i - \sum_{j=1}^{J} \lambda_{\hat{s}}^{I+j}, \ \forall \hat{s} \neq s, \end{aligned}$$

where $\tilde{x} = \left(\tilde{x}^1, \dots, \tilde{x}^I \right) \gg 0$ *is the consumption of the risk averse in state* s. *And for* $\{\omega^i\}_{i=1}^{I} \gg 0$ *there is always a solution for this system.*

The previous result can be extended to include the case in which the Risk Lover can specialize in different states, however it will need additional condition over the endowment, the number of Risk Lovers and the type of minimal consumption that they have to consume.

Proposition 3. *Given* $\{\omega_s^i\}_{s,i}$ *if there exist* R *states* $1 \leq s_1, \dots, s_R \leq S$ *and* $0 < k < K$, *with* K *sufficiently big such that:*

1. $\pi_{s_1} = \cdots = \pi_{s_R}$,
2. $J = R\tilde{J}$ *with* $\tilde{J} \in \mathbb{N}$ *and* $\omega^{I+j_1} = \omega^{I+j_2}$ *for* $j_1 = \tilde{j}R + l_1$ *and* $j_2 = \tilde{j}R + l_2$ *where* $1 \leq l_1, l_2 \leq R$ *and* $0 \leq \tilde{j} < R$,
3. $\sum_{i \leq I} \omega_{s_r}^i \geq K$ *and* $\sum_{i > I} \omega_{s_r}^i \leq k$ *for all* $r = 1, \dots, R$,
4. $\sum_i \omega_{s'}^i \leq k$ *for* $s_r \neq s'$ $\forall r = 1, \dots, R$,
5. *there exists* $\alpha \in [0, 1]$ *such that* $\lambda_s^i = \alpha \omega_s^i$ *for each* s *and* $i > I$.

Then there is an equilibrium for the economy with $p \in \Delta_{++}^{S-1}$.

Proof. We define a similar game but each $\tilde{j}R+1, \ldots, (\tilde{j}+1)\, R$ would specialize in a different state. And using 1, 2 and 5 we guarantee that in each Nash Equilibrium, the prices must satisfy that $p_{s_1} = \cdots = p_{s_R}$, which concludes the proof. □

It can be easily observed that the conditions exposed before require strong symmetric properties. The Example 7 explores some difficulties of these types of equilibria.

Results in the Edgeworth box

The two agents have Expected Utility (EU) functions.

$$U^i(x) = \pi u^i(x_1) + (1-\pi)\, u^i(x_2)\,,\ \forall i = 1, 2,$$

where $\pi \in (0,1)\,, u^1 \in C^1(0,\infty) \cap C[0,\infty)$ is the utility index for the first agent, which is strictly increasing, concave in $[0,\infty)$ and $\lim_{x\to\infty} u^{1\prime}(x) = 0$, and $u^2 \in C[0,\infty)$ is the utility index for the second agent which is an increasing and convex function satisfying $u^2(0) = 0$.

As can be observed, Theorem 1 can be used in this case, however in the Edgeworth box, it can be proved that there exist also necessary and sufficient conditions for the existence of equilibrium.

Proposition 4. *Under our hypotheses including Inada, there exist* $\underline{\omega}_1^1 \geq 0$ *and* $\underline{\omega}_2^1 \geq 0$ *such that:*

1. *There exists an AD-equilibrium if and only if (a)* $\omega_1^1 \geq \underline{\omega}_1^1$ *and then* $x_2^2 = 0$*, or (b)* $\omega_2^1 \geq \underline{\omega}_2^1$ *and then* $x_1^2 = 0$.

2. *There exists a unique normalized price* $p = (p_1, 1-p_1) \in \Delta_{++}^1$ *for the AD-equilibrium which is the solution of*

$$\omega_1^1 = u^{1\prime(-1)}\left(\left(\frac{p_1}{1-p_1}\right)\left(\frac{1-\pi}{\pi}\right) u^{1\prime}(\omega_2)\right) + \frac{1-p_1}{p_1}\omega_2^2 \tag{1.7}$$

   for (a), and the solution of

$$\omega_2^1 = u^{1\prime(-1)}\left(\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right) u^{1\prime}(\omega_1)\right) + \frac{p_1}{1-p_1}\omega_1^2 \tag{1.8}$$

   for (b).

The proof of Proposition 4 is in the Appendix B.1.

Remark 2. $\underline{\omega}_1^1$ and $\underline{\omega}_2^1$ are the values one would obtain for ω_1^1 and ω_2^1 respectively in 1.7 and in 1.8 taking p as the solution of

$$\pi u^2\left({}^{p\omega^2}/_{p_1}\right) = (1-\pi)u^2\left({}^{p\omega^2}/_{(1-p_1)}\right).$$

It can be noticed that $\underline{\omega}_1^1$ depends on all the other endowments of the economy, similarly to $\underline{\omega}_2^1$.

Remark 3. The previous result can be extended for utility index that does not satisfy Inada. Then condition 1.7 is

$$\omega_1^1 = u^{1'(-1)}\left(\left(\left(\frac{p_1}{1-p_1}\right)\left(\frac{1-\pi}{\pi}\right)u^{1'}(\omega_2)\right)\bigwedge u^{1'}(0)\right) + \frac{1-p_1}{p_1}\omega_2^2 \quad (1.9)$$

and condition 1.8 is

$$\omega_2^1 = u^{1'(-1)}\left(\left(\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)u^{1'}(\omega_1)\right)\bigwedge u^{1'}(0)\right) + \frac{p_1}{1-p_1}\omega_1^2 \quad (1.10)$$

The necessary and sufficient condition of the Proposition 4 does not only characterize the existence of equilibrium, it we will also ensure the uniqueness of the equilibrium, additionally, it helps us to compute the equilibrium price and the optimal consumption for every equilibrium of this economy.

It can be observed in Proposition 4 that to ensure the existence of equilibrium it is necessary a minimum level of endowment to the risk averse, and then, for every economy with a bigger endowment ω_1^1, there is an AD-equilibrium. Therefore if we have economies in which the aggregate risk is relatively large and most of this wealth is in hands of the risk averse[6], equilibrium would exists as a consequence of the exchange of risk with the risk lover. Then, in presence of risk lovers, is desirable to have some level of aggregate risk that can be absorbed by the risk lovers without interfering with market clearing.

Remark 4. The conditions exposed in Proposition 4 could lead to existence of equilibrium even when there is no aggregate risk. However these cases are quite unrealistic since require differences among the agents arbitrarily large. The Example 6 explores this type of equilibrium.

1.3 Analysis of Risk-Sharing in presence of Risk Lovers and Optimists

As mentioned in Chateauneuf et al. (see [19]), to guarantee the risk sharing between risk averse it is necessary that each agent believe in the same probability of the world. To guarantee the risk sharing between ambiguity averse

[6] In the sense of possession of large amount of the aggregate risk.

in a model with finite states of natures we need that each agent believes in the same capacity.

For a continuum of states Tsanakas et al. (see [40]) proved that using Yaari distortions, even with different distortions for the agents, Risk-Sharing occurs when the endowments and the consumption space have non-negative density, they also gave some characterization of the equilibrium price and the optimal portfolio in terms of the aggregate risk aversion and in terms of the aggregate ambiguity aversion.

For non-convex economies the analysis of Risk-Sharing is even more difficult. Our goal will be trying to establish some conditions that help us guarantee that the agents are Sharing their risk.

To analyze properties of Risk-Sharing, let us define a concept which is deeply related to it, that is comonotonicity.

Definition 1. For a consumption plan $x := (x^i)_{i=1}^I$, we say that x is *comonotonic* if x satisfies that, for every $1 \leq i,\ \hat{i} \leq I$ and $1 \leq s,\ \hat{s} \leq S$ we have:

$$\left(x_s^{\hat{i}} - x_{\hat{s}}^{\hat{i}}\right)\left(x_s^i - x_{\hat{s}}^i\right) \geq 0.$$

Our interest in the analysis of this property is because it is useful to compute in a precise form the optimal consumption, the first order conditions and the relationship among the price, the aggregate risk and ambiguity that exists in the economy.

1.3.1 Analysis in the Edgeworth box with EU decision makers with one risk averse and one risk lover

It can be observed that we can not analyze all the possible distributions of endowments because there are several distributions in which the economy does not have any equilibrium. But the Edgeworth box has several advantages, for example: we know that the condition that we expose here (Proposition 4) is necessary and sufficient to guarantee the existence of equilibrium, another advantage is that we have a very clear relationship between the endowments and the equilibrium price. And as a consequence, this model can be analyzed in a very precise way. In fact, we have:

Proposition 5. *If*

1. $\pi \geq 1/2$ *and* $(\omega^i)_i$ *satisfies the condition 1.7 of the Proposition 4 in page 18, or*

2. $\pi \leq 1/2$ *and* $(\omega^i)_i$ *satisfies the condition 1.8 of the Proposition 4.*

Then the agents have comonotonic consumption in equilibrium.

Proof. Let us assume without loss of generality that the equilibrium satisfies the condition 1.7, $\pi \geq 1/2$ and $u^2(0) = 0$. We know, from the way we treated the Edgeworth box, that the second agent, the risk lover, is always consuming all the rent at the first state.

Observing the FOC of the second agent we know that if $p < 1/2$, we have comonotonicity of the agents. Therefore we will analyze the case in which $p \geq 1/2$. And using the optimal condition for the second agent we will have $\pi u^2\left(\frac{(p,1-p)\omega^2}{p}\right) \geq (1-\pi)u^2\left(\frac{(p,1-p)\omega^2}{1-p}\right)$ which can be written as:

$$\begin{aligned} \frac{\pi}{1-\pi} &\geq \frac{u^2\left(\frac{(p,1-p)\omega^2}{1-p}\right)}{u^2\left(\frac{(p,1-p)\omega^2}{p}\right)} \\ &\geq \frac{p}{1-p}. \end{aligned}$$

Using the last inequality that we obtained and the FOC for the second agent we have that $u^{1'}(x_1^1) \leq u^{1'}(x_2^1)$, which clearly implies comonotonicity. □

Additionally we have:

Proposition 6. *If*

1. $\pi < 1/2$ *and* $(\omega^i)_i$ *satisfies the condition 1.7 of the Proposition 4 with* $p \leq \pi$*, or*

2. $\pi > 1/2$ *and* $(\omega^i)_i$ *satisfies the condition 1.8 of the Proposition 4 with* $p \geq \pi$*.*

Then the agents have comonotonic consumption in equilibrium.

Proof. Let assume without loss of generality that the equilibrium satisfies the first case. Using the condition over p we have that $\pi/p \geq 1-\pi/1-p$, which proves the comonotonicity.

□

And for equilibrium where $\pi < 1/2$ and $p > \pi$, we do not have comonotonic consumption. However to obtain these types of equilibria we need some specific characteristics in the economy. In fact some difference in the endowment distribution is needed, however the aggregate risk can not be too large. The Example 8 explores this condition.

1.3.2 Analysis for economies with RDEU Decision Makers

In this case it is possible to ensure comonotonicity under some conditions that can be explained as a generalization of the previous results. Also in this case, it is possible to characterize the equilibrium price in terms of the endowments, risk aversion and ambiguity aversion that exist in the economy.

Proposition 7. *If $(\omega^i)_i$ satisfies the condition of the Theorem 1 in page 13 with no survival consumption for each state s, the consumption is comonotonic if one of the two conditions is satisfied:*

1. *$\pi_s \geq \pi_{\hat{s}}$, $\forall \hat{s} \leq S$ and there exists at least one agent $i \leq I$ such that $f^i(x) = x$.*
2. *$\pi_s < \pi_{\hat{s}}$ for some $\hat{s}$, and K (the one that is given in the proof of theorem 1), is large enough.*

Under the condition of Theorem 1 and Proposition 7, the agents have comonotonic behavior among the agents, however if the conditions of Proposition 7 are not satisfied, the agents will have comonotonic behaviors among the agents with similar type of utility function, that is among the ambiguity lover or optimists, and among the ambiguity averse or pessimists.

In the context of the previous proposition, we now give an explicit formula for utility function that helps us to compute the FOC for each pessimist agent. And then we can use it to find an expression for the equilibrium price.

Proposition 8. *For $\omega_1 > \omega_2 > \cdots > \omega_S$, CARA RDEU decision makers with $\rho^i > 0$ the risk aversion coefficient(for the pessimist agents only) with any type of distortion f^i, and over the conditions of the Proposition 7 for the state 1 with $x^i \gg 0$ $\forall i \leq I$, we have that the equilibrium prices satisfy:*

$$p_s = \frac{e^{-\rho\hat{\omega}_s} e^{\beta_s}}{\sum_{t=1}^{S} e^{-\rho\hat{\omega}_t} e^{\beta_t}}, \tag{1.11}$$

where $\frac{1}{\rho} = \sum_{i=1}^{I} \frac{1}{\rho_i}$, $\beta_s = \rho \sum_{i=1}^{I} \frac{1}{\rho_i} \ln\left(f^i\left(\sum_{t=1}^{s} \pi_t\right) - f^i\left(\sum_{t=1}^{s-1} \pi_t\right)\right)$, $\hat{\omega}_1 := \omega_1 - \sum_{i=I+1}^{J} x_1^i = \omega_1 - \sum_{i=I+1}^{J} \frac{(p_1,\ldots,p_S)\omega^i}{p_1}$ and $\hat{\omega}_s := \omega_s$ for $2 \leq s \leq S$.

Remark 5. We have that:

- ρ can be interpreted as the Risk Aversion coefficient for the Ambiguity Averse or Pessimist, and

- $\{\beta_s\}$ are the coefficients that represent the form in which the Ambiguity Aversion affects the probability of the world and the equilibrium price. In fact, if $f^i(x) = x \ \forall i \leq I$, $\beta_s = \ln \pi_s \ \forall s$

- The previous results can be extended to nonnegative minimal consumption such that $\sum_{i=I+j}^{I+J} \lambda_1^i \leq \sum_{i=I+j}^{I+J} \lambda_2^i \leq \ldots \leq \sum_{i=I+j}^{I+J} \lambda_S^i$, and in this case we have that $\hat{\omega}_s = \omega_s - \sum_{i=I+j}^{I+J} \lambda_s^i$.

Due to the impossibility of knowing a formula that can be easily used to compute the FOC with RDEU agents, we will need to establish some condition in which we can guarantee that the consumption would be comonotonic with the initial endowment distribution. Therefore we will assume this type of properties to make possible the equilibrium price analysis.

Even though the exponential case is a particular case, it helps us to show the relationship among the risk, aggregate risk in terms of the risk aversion coefficient ρ, Ambiguity in terms of the distortion of each pessimist agent and the equilibrium price.

It can be observed that these results are, in a certain way, a generalization of the formulas that were obtained by Bühlmann [17] and [18] without optimist agents or risk lovers and without ambiguity. These formulas are also strongly related to the characterization that was made by Tsanakas see [40] in the continuum case with the same type of distortion.

1.4 Analysis of Volatility in presence of non-convex agents

Now we will interpret the previous AD equilibria as equilibria of an economy with complete financial markets. For simplicity, consider two dates ($t = 0$ and $t = 1$) and two possible states of nature at the second date. We will suppose that the economy is richer at state 1 (i.e., $\omega_1 > \omega_2$)). Let $\pi_1 \in (0, 1)$ be the objective probability[7] of state 1. At first date, there are two real assets: a risk-free asset (will be called bond) with unitary price at $t = 0$ and whose payoff at $t = 1$ is $R > 0$; a risky asset with price $q > 0$ at $t = 0$ and payoff R_s at $t = 1$ if state s occurs. Let us assume that $R_1 > R_2 \geq 0$. So, the risky asset and the aggregate endowment are comonotonic. Each agent i will choose at $t = 0$ a portfolio $\varphi = (\alpha, \beta) \in \mathbb{R}^2$, composed by an amount α of risky asset and an amount β of bond. The agent objective is maximize

[7] It means that all agents have common beliefs of the probability of future events then we can suppose that these beliefs represent a probability of the world and their utility is a expected value of the objective probability.

$V^i(\varphi) := U^i \circ (\omega^i + \varphi A)$, where A is the payoffs matrix $\begin{pmatrix} R_1 & R_2 \\ R & R \end{pmatrix}$, under the following budget constraints: φ must be such that

- Given the risky asset price q,

$$\alpha\, q + \beta = 0. \tag{1.12}$$

- For each s at $t = 1$,

$$\omega_s^i + R_s\alpha + R\beta \geq \lambda_s^i, \tag{1.13}$$

that is, at each state s the private wealth after adding the portfolio payoff is exogenously bounded from below by $\lambda_s^i \in [0, \omega_s^i]$, that will be interpreted as a minimal wealth requisition imposed by the policy maker.

It is related, for instance, to the usual capital requirements for financial institutions. As it depends on s, it could be picked anti-comonotonic with respect to the aggregate wealth as the counter-cyclical buffers on Basel III accord (see ...). And by doing this, is possible to implement and interpret regulation in this paper.

An equilibrium for this economy will be an asset price $\overline{q}$ and a vector of portfolios $(\varphi^i)_{i=1}^{I+J}$ such that

(i) $\forall i$, $\varphi^i = (\alpha^i, \beta^i) \in \arg\max\{V^i(\varphi) : \varphi \in \mathbb{R}^2$ satisfies (1.12) and (1.13) with $q = \overline{q}\}$.

(ii) $\sum_{i=1}^{I+J} \alpha^i = 0$ and $\sum_{i=1}^{I+J} \beta^i = 0$.

Given an equilibrium $((\overline{p}_1, 1-\overline{p}_1), (x^i)_i)$ for the AD economy $\mathcal{E}_{AD} = (U^i, \omega^i, \lambda^i)_i$, it is easy to check that, defining

$$q(\overline{p}_1) = \frac{\overline{p}_1 R_1 + (1-\overline{p}_1) R_2}{R} \tag{1.14}$$

and $\varphi^i(x^i) = (x^i - \omega^i)A^{-1}$, the vector $(q(\overline{p}_1), (\varphi^i(x^i))_i)$ is an equilibrium for the economy with financial markets $\mathcal{E}_{FM} = (A, (V^i, \omega^i, \lambda^i)_i)$. Conversely, if $(\overline{q}, (\varphi^i)_i)$ is an equilibrium for $\mathcal{E}_{FM}$, let γ_1 be the risk neutral probability of state 1 obtained from $\overline{q}$ and A. Then, $((\gamma_1, 1-\gamma_1), x^i(\varphi^i))_i)$ with $x^i(\varphi^i) = \omega^i + \varphi^i A$ is an equilibria for $\mathcal{E}_{AD}$. Thus, there is an one-to-one mapping between the equilibria of these two economies such that (i) it keeps the real allocations and (ii) it converts the normalized equilibrium AD prices into risk neutral probabilities for the respective equilibrium price of the economy with financial markets and vice-versa.

Let us define the volatility of returns at price q by

$$\sigma(q) = \pi_1 \left| \frac{R_1}{q} - \mu(q) \right| + (1 - \pi_1) \left| \frac{R_2}{q} - \mu(q) \right|, \tag{1.15}$$

where $\mu(q) = \pi_1 \frac{R_1}{q} + (1 - \pi_1) \frac{R_2}{q}$. Furthermore, if one parametrizes the risky asset price using Equation (1.14), it will be straightforward to verify that the composed function defined by $\sigma(q(\rho))$ for $\rho \in (0, 1)$ is decreasing. Thus, the volatility of returns decreases with the increases of the respective risk neutral probability of state 1.

1.4.1 Analysis of Volatility, Welfare and Regulation

As it was mentioned in section 1.2, $\lambda_s^i \in [0, \omega_s^i]$ is the type of regulation imposed to the ambiguity lovers ($i \geq I + 1$) by a central planner, to avoid the absorption of large quantities of risk. The idea of this type of regulation is to avoid the extreme specialization by the Risk lovers in the economy reducing their consumption of risk in equilibrium.

Our goal will be to establish a relationship between volatility and regulation[8], and also analyze, in some special cases, the impact of regulation on welfare . The following example illustrates these relationships.

Example 2. Consider an economy with two state of nature ($\pi = 0.5$), three RDEU agents with CARA utility index given by $\rho_1 = 1$, $\rho_2 = 1.5$ and $\rho_3 = -1$, exponential distortion $f^i(z) := \frac{1}{e^{\tau^i}-1}\left(e^{\tau^i z} - 1\right)$ where $\tau^1 = 1$, $\tau^2 = 1.25$ and $\tau^3 = -1$, endowments given by $\omega^1 = (2, 1)$, $\omega^2 = (2, 1)$ and $\omega^3 = (1, 1)$[9], and regulation for the third agent given by $\lambda^3 \in [0, 1]$, which means that if $\lambda^3 = 0$, there is no regulation and if $\lambda^3 = 1$, regulation forces to the consumption without risk given by $(1, 1)$.

From the analysis of the Figure 1.3, changes on the regulation produces variation on welfare. However this variation behaves quite different compared to the regulation in traditional convex economies, since all agents are loosing part of their welfare in presence of more regulation.

From the analysis of Welfare in the economy, we can say that:

- The biggest reduction on the utility level is for the Risk Lovers. This is due to the fact that regulation reduces the quantity of risk that they can absorb in equilibrium, consuming allocation far away from their optimal solution in a economy without regulation.

[8] Regulation in the sense of section 1.2.

[9] This allocation satisfies the condition of the Theorem 1 for any $\lambda_s^i \in [0, \omega_s^i]$.

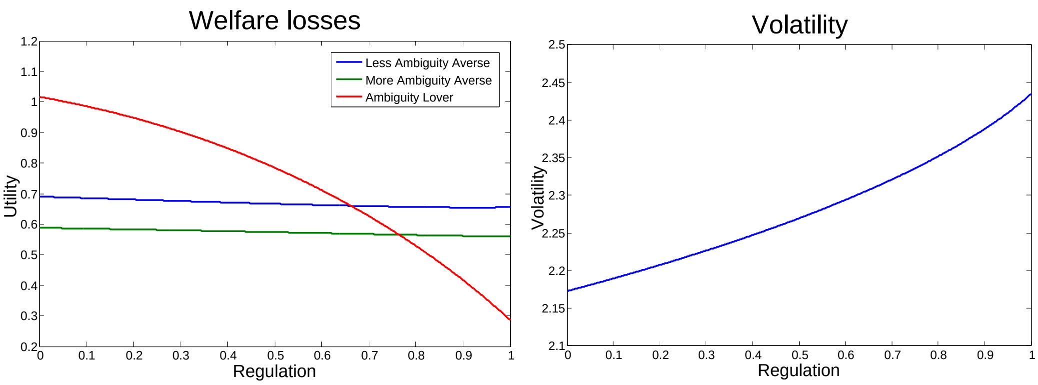

Figure 1.3: Welfare and Volatility $(\sigma(q(p_1)))$ against Regulation (λ^3)

- There exists a reduction on the utility level for the ambiguity averters, however is not as big as the one that affect the ambiguity lovers. This variation is a consequence of the fact that a larger amount of the aggregate risk must be absorbed by the Risk Averters in presence of regulation for the ambiguity lovers, leading to optimal allocations for the ambiguity averse with a bigger amount of risk, which will reduce their utility level.

And as a consequence, for any social welfare function that we use, increments on regulation will reduce the social welfare of the economy, suggesting that regulation which tends to reduce the risk that ambiguity lovers absorb, is not a good tool to increase welfare.

As can be observed empirically, in presence of risk or ambiguity loving, increments of regulation leads to rises of volatility. This is a consequence of the lower level of risk that the risk lover can absorb in presence of regulation.

As it was said before, in presence of ambiguity lovers, a certain amount of aggregate risk is needed in order to allow the trade-off among agents. The existence of it will imply, with assets comonotonic with the aggregate endowment, that a reduction on regulation will increase the price of the risky asset (see equation 1.14) and as a consequence, it will decrease the volatility (see equation 1.15).

However when there is a strong regulation on the ambiguity lovers, the quantity of risk that will be absorbed by them is lower and as a consequence, all the convex agents must absorb a bigger amount of aggregate risk against their will, reducing the price of the risky asset and increasing volatility.

And what can be said in terms of volatility when in economies with more than two ambiguous agents? Since there exists some characterization of the equilibrium prices given by the Proposition 8 in page 22, is posible to know if the volatility is increasing or not looking for prices variations when there are changes on regulation of the Risk Lovers or Optimists.

Proposition 9. *Under the conditions of the Proposition 8 and* $\sum_{j=1}^{J} \omega_s^{I+j} = \sum_{j=1}^{J} \omega_{s'}^{I+j}$ *for all* $1 \leq s, s' \leq S$*,* p_1 *decreases when the regulation increases as* $\lambda_s^i = \alpha \omega_s^i$ *for* $\alpha \in [0, 1]$*.*

As can be seen in the hypotheses of the previous proposition, it is required additional assumptions that guarantee no aggregate risk for the Risk Lovers or Optimists. We conjecture that this assumption can be relaxed to allow some type of aggregate risk for them.

Under the condition of the previous proposition, each equilibrium satisfies that $p_1 < p_s$ for all $s = 1, \ldots, S$ then the fact of p_1 decreases, can be interpreted as an increment of volatility. Therefore, in presence of ambiguity loving, increments of regulation leads to reduction of the comonotonic asset price and as a consequence, an increment of volatility.

This can be interpreted as the fact that the risk lovers or optimists have large incentives to absorb the risk that exists in the economy, increasing the price of these comonotonic assets, which reduces volatility. However, if the regulation does not allow this, the price of these assets will decrease due to its lack of importance by the ambiguity averse, increasing volatility due to equation 1.15. Therefore any regulation will increase the volatility since there exists a bigger amount of risk that must be absorbed by the Risk Averters or Pessimists.

1.5 Existence of equilibrium and Volatility with Friedman-Savage Decision Makers

1.5.1 Conditions for existence of equilibrium

In this section we have two types of agents and a finite number of them, $I+J$, nevertheless we consider Friedman-Savage Decision makers instead of agents with convex utility functions. And we analyze the existence of equilibrium when there are also averse agents as mentioned before.

Each Friedman-Savage (FS) Decision Maker $i = I+1, \ldots, J$ has a subjective probability $\pi^i \in \Delta_{++}^{S-1}$ and a utility index $u^i : \mathbb{R}_+ \to \mathbb{R}$ that is differentiable, strictly increasing and there exists $x_c^i \geq 0$ such that for each $x < x_c^i$, u^i is a concave differentiable function, for $x > x_c^i$, u^i is a convex function and there exists $\tilde{x}^i \geq x_c^i$ such that for each $x \geq \tilde{x}^i$, $u^i(x)$ is a linear function. Then the utility function has the following form

$$U^i(x) := \sum_{s=1}^{S} \pi_s^i u^i (x_s) \,.$$

Their endowments are also given by $(\omega_1^i, \ldots, \omega_S^i) \gg 0$.

This type of agents has a very particular behavior, before the inflection point each FS Decision Maker behaves as a Risk Averse, avoiding extreme consumptions; even though when this agent is able to consume more than that, he will behave as a certain type of Risk Lover, avoiding consumption without risk. This shows a very important behavior in economics, agents that are willing to specify in some states, but are also concerned about avoiding low consumption in all the states. Then this type of agents is worried about the possibility of disappearing in the economy.

It can be noticed that this behavior can not be explained with the previous type of preferences, because without any type of restriction, each prone agent is willing to specialize in one state.

Proposition 10. *Over the same conditions as in Theorem 1, there is an equilibrium for the economy with* $p \in \Delta_{++}^{S-1}$.

Remark 6. If we impose that for each Friedman-Savage Decision Maker we have that $\lim_{x \searrow 0} u^{i\prime}(x) = \infty$, we guarantee that in equilibrium each of them would have positive consumption in each state. But if it is not the case, they could have null consumption in some states even when they have incentives to avoid them.

Remark 7. For the general utility index of Friedman and Savage, it is possible to prove existence of equilibrium for agents that are quite poor or considerably rich such that their consumption will be in the concave part of the utility index. However it is posible to extend the lemma 6 to include all types of FS utility index.

1.5.2 Analysis of Volatility

Since increments on the Friedman-Savage Wealth produce changes in their decision against risk, our goal will be to establish a relationship between risk absorbed by them and their wealth and as a consequence, a relationship between volatility and wealth of the FS decision makers. The following example illustrates these relationships.

Example 3. Consider an economy with two states of nature ($\pi = 0.5$), two EU agents, where $u^1(x) = \ln(x)$ and

$$u^2(x) = \begin{cases} \ln(x) + (1/2)x^2 & \text{if } x \leq 3/2 \\ 13/6(x - 3/2) + 9/8 + \log(3/2) & \text{if } x > 3/2, \end{cases}$$

endowments given by $\omega^1 = (3, 1)$ and $\omega^2 = (2a, a)$ where $a \in [0, 1]$.

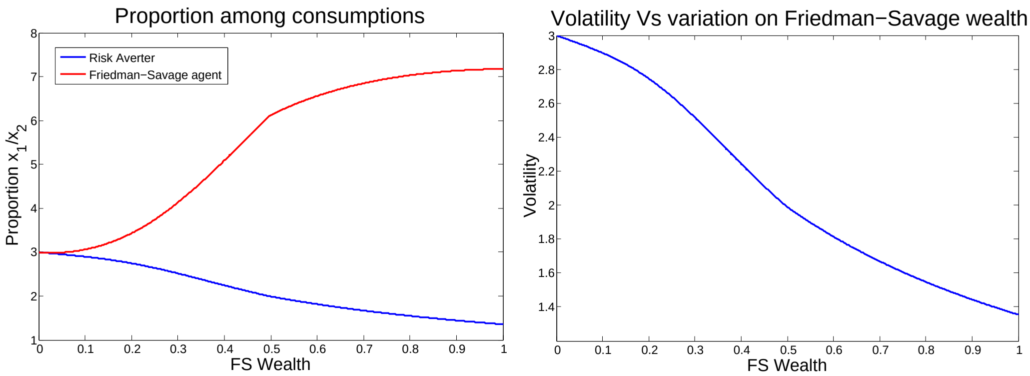

Figure 1.4: Changes in consumption and Volatility against Wealth

From the analysis of the Figure 1.4, increments of the FS wealth leads to reductions of volatility. This is a consequence of increasing desire of this decision maker to specialize when there are increments on his wealth. Under a low wealth, the economy will be like a convex economy since both agents would have convex preferences under the feasible space, however under a large wealth, the FS decision maker will have enough incentives to specialize the consumption in one state.

Since we are using allocations in which there exist equilibrium for any level of regulation, there exists some type of aggregate risk in the economy in order to satisfy market clearing as we said before. Therefore when the Friedman-Savage decision maker is becoming richer, this agent will have incentives to specialize his consumption, absorbing larger quantities of risk, which reduces the volatility of the prices.

1.6 Concluding Remarks

Given the importance of the financial markets, the exchange of aggregate risk between ambiguity lovers and ambiguity averse is an important problem to be studied in order to encompass behaviors that are not usually analyzed in General Equilibrium Theory.

In order to make possible this type of analysis we started by giving conditions in terms of enough aggregate risk for the averse, for a large class of models encompassing *Smooth Ambiguity* (see [28]), *Choquet Expected Utility* (see [37]) and *Variational Preferences* (see [29]), under which we were able to prove the existence of equilibria. We can interpret this condition as: risk is needed in order to enable the trade between the ambiguity lovers and the ambiguity averse and helps with the matching between the desire of hedging for the ambiguity averse and of speculation for the ambiguity lovers.

Interpreting ambiguity in the sense of Choquet Expected Utility (CEU) (see [37]) as a distortion of a probability (RDEU), we are able to analyze the necessity of aggregate risk in presence of optimism in the economy. If we also consider a type of decision maker that is risk lover for big consumptions and risk averse for small consumptions given by Friedman Savage (see [24]), we establish the necessity of aggregate risk to prove existence of equilibrium.

To make a deeper analysis we have to study the equilibrium in order to establish how the ambiguity lovers are affecting the optimal consumption and the equilibrium price. To do so, we establish conditions under which there is risk sharing (comonotonic consumptions) or not. Since Chateauneuf et al. (2000) and Tsanakas (2006) we already know that there is risk sharing between distorted (RDEU) agents. We observed that this is not the case for economies with also ambiguity lovers, therefore we needed a deeper analysis in order to find conditions in which there is risk sharing.

Some of the conditions that we found related to risk sharing were also related to the existence of aggregate risk, suggesting that with enough aggregate risk, the ambiguity lovers can not absorb all the risk of the ambiguity averse, forcing the ambiguity averse to consume part of it. And therefore the agent will share the risk among the states.

Finally, we observe that we generalized the results given by Bühlmann (1980, 1984) and Tsanakas (2006), who found a characterization of the equilibrium price in terms of the aggregate risk aversion coefficient and in terms of the ambiguity aversion to economies with also ambiguity lovers.

The previous results help to analyze changes in Volatility and Social Welfare when there are variations in the regulation imposed to the ambiguity lovers. These analyses suggest that regulation increases volatility while reduces the social welfare of the economy, however the risk lovers or optimists are those who perceive the larger losses.

Part II

Ambiguity in infinite horizon economies

CHAPTER 2

On the efficiency of money when agents are wary

The chapter builds on earlier work (Araujo, Novinski and Pascoa [5]) they did on the sequential implementation of Arrow-Debreu allocations when consumers are not impatient, more precisely, when consumers tend to neglect gains but not loses that occur at arbitrarily distant dates. We referred to this lack of impatience as *wariness* and illustrated it by adding to the usual series of discounted utilities a term dealing with the infimum of lifetime utilities. In the earlier work we showed that efficient allocations can be implemented sequentially using assets that pay dividends provided that the portfolio constraints prevent excessive savings. In the current chapter we dispense with such constraints and look for taxes that discourage excessive savings. We believe this approach is quite novel and illustrates well what can be done differently when the implementing asset is money.

The idea that money plays a crucial reserve role has captured a lot of attention in the literature. Friedman (1953, 1969) put forward the idea that consumers should not economize unnecessarily on money balances as these holdings are "a reserve against future emergencies", allowing consumers to spend more when their earnings are lower and, therefore, hedge against income shocks. The wasteful economizing of cash should be avoided by deflation or by providing money with an explicit real rate of interest. Although Friedman never stated his claim in terms of Pareto efficiency, several authors have explored his ideas using the precise Pareto concept. This proposition has been often associated with the much stronger recommendation of a steady contraction of the money supply. However, the latter seemed to reduce the full impact of the former. An asymptotically null money supply would imply that money could not have a persistent efficient role.

In this chapter we reconcile Friedman's appraisal of the hedging role of money with an optimal non-zero limit for the money supply. We do this by resuming Bewley's (1980) idea that the "devise to give money a value is infinite horizon (together with the need for insurance)", but we take a step forward. Our devise is not simply the use of infinite lived consumers, but consists in taking into account specifically the hedging role of money at infinity, that is, at arbitrarily distant dates. For wary consumers, when the worst outcome is never attained in finite time (there is always a worse outcome sometime ahead), there is a marginal utility at infinity, of raising the infimum of consumption. This creates a demand for a precautionary role of money at infinity, that is, persistently at far away dates. It is therefore not surprising why our efficient monetary equilibrium requires a positive limit for the money supply. Our result is reminiscent of the persistent role of money in Samuelson [35] overlapping generations model, where the young hoard money to sell when old, allowing for an inter-temporal transfer of wealth that could not be done by trading a zero net supply promise (as the old would no longer be alive afterwards to pay back the debt). Such persistent role of money seemed until now impossible to observe in models with immortal agents[1].

In the general model of convex capacities of Schmeidler [37], there is no ambiguous discounted rates, however it satisfies the hypothesis of warriness since the model satisfies upper but not lower Mackey semicontinuity. This implies that there is no time consistency in all possible cases as in Hansen and Sargent [27], nevertheless under the *Recurrent Rare Events Hypothesis* and due to the characterization of the equilibrium, we have time consistency at the equilibrium path.

Even though we focus most of our chapter on a deterministic economy, the model is successful in capturing the persistent precautionary role of money in hedging undesired never ending fluctuations of endowments. The underlying driving feature, consumers' wariness, can actually be interpreted in terms of an uncertainty with regard to the way the future should be discounted, analogously to the literature on ambiguity aversion. Not being sure how to discount the future, the consumer picks for each consumption plan the most penalizing discount process and, therefore, ends up maximizing the minimal utility, over some set of discount factors.

Money positions are only constrained in our model by the usual no-short-

[1] For impatient agents, Bewley (1980, 1983) showed that a non-vanishing money supply, together with interior consumption, had to be inefficient. Levine (1986,1988,1989) confirmed it under Inada's condition and observed that efficiency might prevail under non-interior consumption. See also Woodford [42] and Pascoa, Petrassi and Torres-Martinez [31]

sales constraint. When the infimum of consumption in not attained in finite time and these no-short-sales constraints are non-binding, efficiency requires a zero nominal interest rate, as Friedman required. In other words, the efficient monetary policy must be deflationary. However, this creates a difficulty for implementation: wary agents might get unbounded utility gains by hoarding too much and then taking advantage of deflation to raise that distant infimum of consumption. This problem is a new instance of a difficulty already noticed by Friedman and Bewley, known as the *insatiable demand for precautionary liquidity.* In our context the benefits-costs gap is not a short run gap but rather a (arbitrarily-)long run gap.

A lack of funds at infinity is akin to a long-run arbitrage opportunity, just like for a non-wary, impatient, consumer a Ponzi scheme would constitute a long-run arbitrage. Under impatience, a positive limit in a consumer's deflated real balances would be a waste, whereas a negative limit would constitute a Ponzi scheme that is ruled out as money cannot be short sold. Under wariness, a positive limit for the deflated real value of hoarded funds is not a waste and an improvement strategy, akin to a new type of infinite horizon arbitrage opportunity, becomes available when that limit is lower than the benefit from the hedging effect at infinity.

We introduce taxes that eliminate the gap between the marginal benefit of raising that infimum and the marginal cost of carrying on cash. The tax scheme is non-lump-sum but is impersonal. The accumulated taxes up to date t can be formally seen as a long position on a no-dividends asset, required to accompany the long position on money. The sum of these two long positions constitute the funds put aside up to date t. When the limit of the deflated real value of these hoarded funds falls below what the consumer gains from hedging consumption at infinity, there is a need of funds at infinity (converted in utility terms). We consider taxes that are higher on portfolio plans that would generate a lack of funds at infinity if taxed lump-sum. Efficient aggregate nominal balances may stay constant or tend to a positive level, depending on whether the lump-sum component of taxes is chosen to be null or not. That is, equilibrium money balances do not need to be taxed, only money plans that deviate from it have to be taxed in case of an excessive liquidity demand [2].

It might be asked whether money is essential, that is, whether fiat money could be replaced by another positive net supply asset.[3] In the case of fiat

[2]Real balances grow unboundedly, due to the deflationary evolution in prices. However, deflated real balances tend to a positive constant (under the unique, up to a scalar multiple, non-arbitrage deflator).

[3]We were asked this question at a presentation at the University of Chicago in 2012 and this led us to examine what happens when a Lucas tree is used instead.

money, we have the freedom to choose the initial holdings large enough so that we can implement with taxes and non-negative money positions. In the case of a Lucas tree, there is no such freedom as the initial holdings of the tree determine, through its returns, how the Arrow-Debreu commodity endowments differ from the sequential commodity endowments. Hence, to implement with non-negative tree positions we would need, in general, the help of one-period promises that might be shorted. In such a less appealing implementation, which requires additional financial instruments, the possibility of using the tree to collateralize short sales of the promise should be allowed but, with trees shortrages, this credit could not complete the markets, avoiding the implementation of efficient allocations.

The rest of the chapter is organized as follows. Section 2.1 describes the model and some preliminary results. Section 2.2 introduces the leading example. Section 2.3 presents the result on efficient monetary equilibrium. Section 2.4 proves this result, by implementing efficient allocations first with a no-dividends asset in constant net supply and next by showing that this implementation is equivalent to the implementation with money and taxes (as the positions on that asset can be interpreted as the sum of money balances and accumulated taxes). Section 2.5 addresses the difficulties with replacing fiat money by a Lucas tree.

2.1 Sequential Economy with Fiat Money and Wariness: the Model and Preliminary Results

In this section we describe a deterministic economy with infinitely many dates and a single asset, fiat money, which is used to transfer wealth across dates. Government provides money endowments to the consumers at the initial date and then their money holdings may be taxed at subsequent dates.

2.1.1 Money and the Budget Constraints

There is a finite set of infinite lived agents $\mathcal{I} = \{1, ..., I\}$. The consumption plans that any agent can choose are non-negative bounded sequences $x = (x_t)_{t\in\mathbb{N}}$, where $x_t \geq 0$ stands for consumption of the single good at date t. Thus, the consumption space will be ℓ^{∞}_{+}[4]. We denote by $\omega^i = (\omega^i_t)_{t\in\mathbb{N}} \in \ell^{\infty}_{+}$ the commodity endowments of agent i and suppose that his preferences are

[4]See some properties of the space ℓ^{∞} in the Appendix.

representable by a utility function $U^i : \ell^\infty_+ \to I\!R$ which will be specified in subsection (2.1.2).

Given some initial holdings of money, y^i_0, the purchase of a consumption plan x, can be done by allocating consumer's wealth across time through the choice of a sequence $(y_t)_t$ of *non-negative holdings of fiat money* in order to satisfy the following sequential budget constraints, expressed in units of the consumption good (that is, the single good is the *numéraire* at each date):

$$x_t - \omega^i_t \leq q_t \left[y_{t-1} - y_t - \tau_t(y) \right] \quad \forall\, t \in \mathbb{N}, \tag{2.1}$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of money prices and $(\tau_t(y))_{t\in\mathbb{N}}$ is a taxation profile that depends *only* on the sequential money holdings $y = (y_t)_{t\in\mathbb{N}}$. More precisely, the fiscal policy τ is a function that maps, in an impersonal way, each y into a sequence of time-indexed taxes $\tau(y) = (\tau_t(y))_{t\in\mathbb{N}}$. When τ is constant over all possible choices y, τ is said to be a *lump-sum taxation profile*. Observe also that $\tau_t(y)$ just has an impact at date t when $q_t > 0$.

We suppose that the tax $\tau_t(y)$, even when non-lump-sum, never depends on values that the plan of money holdings takes at any finite set of dates. That is, different money balances trajectories may be taxed differently only if they differ on some infinite subset of dates. It is only the asymptotic implications of different savings strategies that makes them have different fiscal treatment.

Let us denote by $B(q, y^i_0, \omega^i, \tau)$ the set couples $(x, y) \in \ell^\infty_+ \times I\!R^\infty_+$ of consumption and money holdings plans satisfying the sequential budget constraints (2.1). The goal of agent i is to maximize U^i under $B(q, y^i_0, \omega^i, \tau)$.

For a given fiscal policy τ, the *money supply* is endogenous in the sense that the supply M_t at date $t \in \mathbb{N}$ is equal to M_{t-1} net of the date t taxes which depend on what consumers' money balances $(y^1, ..., y^I)$ are. The initial money supply M˙0 is given and equal to $\sum_{i=1}^I y^i_0$. Then, at each date $t \geq 1$

$$M_t(y^1, ..., y^I) = M_{t-1}(y^1, ..., y^I) - \sum_{i=1}^{I} \tau_t(y^i),$$

This condition can be interpreted as the government sequential budget constraint. Equivalently

$$M_t(y^1, ..., y^I) = \sum_{i=1}^{I} \left(y^i_0 - \sum_{s=1}^{t} \tau_s(y^i) \right). \tag{2.2}$$

Clearly, when taxes are lump-sum, the money supply M_t at t does not depend upon how money holdings are allocated across agents.

Definition 2. A vector $(q,(x^i,y^i)_{i\in\mathcal{I}}) \in I\!R^{\infty}_{+} \times (\ell^{\infty}_{+} \times I\!R^{\infty}_{+})^I$ is said to be an *equilibrium* for the economy with initial fiat money holdings $(y^1_0,...,y^I_0)$ and a fiscal policy τ when:

(*a*) $(x^i,y^i) \in \text{argmax}\{U^i(x) : (x,y) \in B(q,y^i_0,\omega^i,\tau)\}$;

(*b*) $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$;

(*c*) $M_t(y^1,...,y^I) = \sum_{i=1}^I y^i_t \ \forall\, t \in \mathbb{N}$.

Notice that, if U^i is monotonous[5] $\forall\, i$ and $(q,(x^i,y^i)_i)$ is an equilibrium, then $x^i_t - \omega^i_t = q_t(y^i_{t-1} - y^i_t - \tau_t(y^i))$ for all t and all i. So, summing over i and using condition (b) of the previous definition, we get $q_t\sum_i(y^i_{t-1} - y^i_t - \tau_t(y^i)) = 0$. If $q_{t_0} > 0$ for some date t_0, it will be true by non-arbitrage that $q_t > 0\ \forall\, t$, so $\sum_i y^i_t = \sum_i(y^i_{t-1} - \tau_t(y^i))$. Solving recursively, $\sum_i y^i_t = \sum_i(y^i_0 - \sum_{s\le t}\tau_s(y^i))$, that is, the monetary market clears. This is a version of Walras' law adapted to the present framework.

Definition 3. An equilibrium $(q,(x^i,y^i))$ is said to be a *monetary equilibrium* if $q \neq 0$.

We introduce next the class of preferences that will be assumed in our main results and examples. It is rich enough to accommodate standard impatience preferences as well as patience driven by wariness. Efficient monetary policies turn out to be quite different depending on whether impatience holds or not, as money supply must be entirely withdrawn under impatience but can be persistently positive otherwise.

2.1.2 Consumer Preferences: Wariness and Ambiguity on Discount Factors

We assume that each agent $i \in \mathcal{I}$ has a utility function U^i of the form

$$U^i(x) = \sum_{t=1}^{\infty} \zeta^i_t u^i\left(x_t\right) + \beta^i \inf_{t\ge1}, u^i(x_t) \tag{2.3}$$

with $\zeta^i \in \ell^1_{++}$, $\beta^i \ge 0$ and u^i increasing, concave and continously differentiable.

When $\beta^i = 0$, U^i is a standard time-separable utility and the agent i is impatient. More precisely, take any plan x. Let us denote by $\mathbb{1}^n$ the sequence

[5]That is, if $h > 0$ and e_t is the sequence whose s-th coordinate is equal to 1 if $s = t$ and equal to zero otherwise, then $U^i(x + he_t) > U^i(x)$ for all $x \in \ell^{\infty}_{+}$.

which is null up to component n and equal to 1 otherwise. The agent is upper semi-impatient at x if for each $\tilde{x}$ such that $x \succ \tilde{x}$ we have $x \succ \tilde{x} + k\mathbb{1}^n$ for $k > 0$ and n large enough; the agent is lower semi-impatient at x if for each $\overline{x}$ such that $\overline{x} \succ x$ we have $\tilde{x} - k\mathbb{1}^n \succ x$ with $k > 0$ for which $\tilde{x} - k\mathbb{1}^n \in \ell^\infty_+$ and n large enough. It is clear that both upper and lower semi-impatience hold at any plan when $\beta^i = 0$. Since preferences are described by (2.3) in the deterministic case, by impatience we mean henceforth that $\beta^i = 0$.

However, when $\beta^i > 0$, the agent is upper but not lower semi-impatient, that is, he tends to overlook gains but not losses at far away dates.[6] In fact, take $x = \mathbb{1}$, the sequence whose terms are all equal to one and $\overline{x} = (1+\varepsilon)\mathbb{1}$, with $\varepsilon > 0$. So $\overline{x} \succ x$ but $x \succ \overline{x} - (1/2)\,\mathbb{1}^n$ for all n large enough and ε small enough, as $\inf\left(\overline{x} - (1/2)\,\mathbb{1}^n\right) = (1/2) + \varepsilon$ whereas $\inf x = 1$.

If $\beta^i > 0$, we say that the agent i is *wary*[7]. This utility function has a nice interpretation in terms of *ambiguity aversion* in the way of discounting the future. Not being sure how to do this, consumers use the worst discounting factor within all that have a certain lower bound at each date (see Dow and Werlang [21]). In fact, the utility given by (2.3) can be written as

$$\inf_{(\eta_t)\,\in\mathcal{D}} \sum_{t=1}^{\infty} \eta_t u\left(x_t\right), \tag{2.4}$$

where $\mathcal{D}$ is the set of all real sequences (η_t) such that $\sum_t \eta_t = 1 + \beta$ and $\eta_t \geq \zeta_t\ \forall t$. This utility is a particular case of the Maxmin Expected Utility model by Schmeidler [37]. The use of this maxmin approach in deterministic dynamic settings had already been suggested also by as a way to represent preferences that are averse to fluctuations in consumption by Gilboa [25].

Let us see how do supporting prices (supergradients[8]) look like for such preferences. This result builds on Bewley [9], which already allowed for AD prices outside of ℓ^1, but goes beyond by finding a condition ensuring that AD prices cannot be in ℓ^1 and by characterizing these prices. The crucial condition is that the infimum of the consumption plan is the limit of some subsequence of consumption and that it is actually never attained in finite time. In this case, the supporting price must have a pure charge component. Let $\underline{x} = \inf_t x_t$.

[6] Upper but not lower impatience had been studied already by Brown and Lewis [16] and Araujo [3].

[7] More generally, a consumer is wary when the preferences are upper but not lower semi-continuous for the Mackey topology $\tau(\ell^\infty, \ell^1)$, as it is the case if $\beta^i > 0$ (see ANP).

[8] Supergradients are the generalization of $\nabla U(x)$ for concave functions, more precisely, $f \in (\ell^\infty)^*$ is a supergradient of U at x if $U(y) \leq U(x) + \langle f, y - x\rangle\ \forall y$.

Lemma 2. *Let $x \in \ell^\infty_+$ so that $\underline{x} > 0$ is a cluster point never attained of x, then any Arrow-Debreu supporting price π for x takes the following value at any $c \in \ell^\infty$*

$$\pi c = \sum_{t \geq 1} \zeta_t \, (u^i)'(x_t) \, c_t \; + \; \beta^i \, (u^i)'(\underline{x}) \, \mathrm{LIM}(c)$$

where LIM *is a bounded linear functional such that* $\mathrm{LIM}(c) \in [\liminf c, \limsup c]$ *and satisfies* $\mathrm{LIM}(x) = \underline{x}$.

(for a proof see Araujo, Novinski and Pascoa [5], where the general characterization, when $\underline{x}$ may be attained, is also given; notice the multiplicity of supporting prices due to the freedom in choosing the *generalized limit* LIM; such multiplicity gives rise to a real indeterminacy of AD equilibria)

The functional mapping each $c \in \ell^\infty$ into $\beta^i \, (u^i)'(\underline{x}) \, \mathrm{LIM}(c)$ is the *pure charge* component of the supporting price.

Remark 8. Although preferences described by (2.3) may fail to be *time-consistent*, it is clear that when the infimum of consumption is not attained time-consistency holds. That is, precisely in the case that matters to us, where we manage to implement a monetary equilibrium, preferences are also time-consistent.

2.1.3 On Friedman's Rule

As a preliminary result, we show next that an efficient monetary equilibrium complies with Friedman's rule requiring a zero nominal interest rate. To see this we derive the Euler conditions.

Euler Conditions

We focus on the case where the optimal consumption plan has an infimum which is never attained in finite time. In this case the utility function (2.3) has partial derivatives $\partial_t U^i(x)$, along the canonical directions e_t, given by $\zeta^i_t (u^i)'(x_t)$[9].

Lemma 3. *Let $x \gg 0$ be an optimal solution to the sequential problem with preferences described by (2.3). Assume $\inf_t x$ is not attained in finite time. Then*

$$q_t \zeta^i_t (u^i)'(x_t) \geq q_{t+1} \zeta^i_{t+1} (u^i)'(x_{t+1}) \tag{2.5}$$

[9] On the contrary, if the infimum were instead attained at $\hat{t}$, then, for the direction $e_{\hat{t}}$, the left hand side derivative might exceed the right hand side derivative (as a reduction in $x_{\hat{t}}$ would lower $\inf_{t \geq 1}, u^i(x_t)$ whereas an increase in $x_{\hat{t}}$ might not affect it).

The nature of the Euler conditions is independent of whether β^i is zero or positive. Notice also that for a consumer i who holds money at date t the Euler equation holds: $q_t\zeta_t^i(u^i)'(x_t) = q_{t+1}\zeta_{t+1}^i(u^i)'(x_{t+1})$.

Remark 9. Friedman's rule prescribing a zero nominal interest rate (sometimes known as the weak rule) follows from this lemma. To see this, notice that agents who hold money at date t would be willing to buy or sell a one-period bond at a null nominal interest rate, if such bond existed. But agents not holding money would be happy with a positive nominal interest rate. In fact, the nominal interest rate i would be such that $q_{t+1}(1+i)\partial_{t+1}U^i(x) = q_t\partial_t U^i(x)$ and the result follows, depending on whether the Euler equation or just (2.5) holds.

Now, suppose that for all i, U^i satisfies Inada condition (that is, for a sequence (x^m) in ℓ_+^∞ such that $x_t^m \to 0$ for some t and x_s^m is bounded away than zero for $s \neq t$, we have $\lim_m \partial_t^+ U^i(x^m) = \infty$). Then, *the nominal interest rate is zero* in any efficient monetary equilibrium $(q, (x^i, y^i))$ with $\inf x^i$ not attained, for any i[10].

To put it in an equivalent way, the inflation rate $(\frac{1/q_{t+1}}{1/q_t} - 1)$ should be equal to the consumers' rate of time preference $(\partial_{t+1}U^i(x)/\partial_t U^i(x))$ minus one. Hence, *efficiency requires deflation*, at least at infinitely many dates (as $(\partial_t U^i(x))_t \in \ell^1$). Actually, deflation occurs always beyond some date when consumption converges to some positive level.

We will examine next a strong rule, that money supply should tend to zero at an efficient monetary equilibrium when $\beta = 0$.

Lump-Sum Tax Policies for Impatient Agents

We start by recalling that what can be said about the efficient money supply when agents are impatient.

Proposition 11. *If for each agent $i \in \mathcal{I}$ we have $\beta^i = 0$ and Inada condition is satisfied, then a lump-sum fiscal policy $(\tau^i)_i$ induces an efficient monetary equilibrium $(q, (x^i, y^i))$, only if $M_t \to 0$.*

In fact, lump-sum taxes do not affect the necessary conditions for individual optimality and the proof follows as in the claim that, under the same assumptions, without taxes, that is, for a constant money supply, monetary equilibrium is inefficient (see Proposition 5 in Pascoa, Petrassi and Torres-Martinez [31]).

[10] In fact, as $\inf x^i$ not attained, $\partial_t U^i(x)$ exist, and, by Inada's condition, consumers' marginal rates of intertemporal substitution should be equal. Hence, as someone must be purchasing money, no one can have a shadow price for the no-short-sales constraint, that is, equalities hold in (2.5).

Remark 10. Proposition 11 is a strong variation upon a claim made by Friedman (1969 [23]), although his claim actually just required a zero nominal interest rate and that, for that purpose, money supply should contract at a rate equal to the equilibrium real interest rate. Bewley (1980,1983) studied thoroughly the case of impatient preferences that do not satisfy Inada conditions. When consumption is non-null, at any date and for any consumer, it can be concluded from his analysis that a constant money supply is inefficient, whereas a money supply decreasing to zero at a constant rate can be made efficient when combined with lump-sum taxes (if the price of money is just required to be different from zero at some date, as shown in the Appendix). Levine (1986) gave interesting examples of efficient non-vanishing money supply for impatient agents with linear utilities, whose parameters suffer stochastic shocks. Corner solutions were crucial for building up large money balances[11]: the Friedman effect (cost of holding real money balances) is dominated by the benefit of hedging the shocks (see Woodford [42], 2.2.2). For the preferences described by (2.3) with $\beta^i > 0$, agents have an incentive to keep large money balances for other hedging purposes, long-run hedging purposes, which affect transversality conditions, and in this case Inada conditions will not prevent efficient monetary equibrium with constant money supply, as the following leading example illustrates.

2.2 The leading example

We consider two-agent economies where endowments suffer shocks that alternate in sign along time. When one consumer gets a positive shock, the other suffers a negative one. Money can be used to hedge against these shocks. Consumers would like to hold money for ever (or at least, along some subsequence) in order find a consumption path in between the upper and the lower endowment paths. That is, consumers would like to raise the infimum of consumption, but there is a trade-off due to the cost of carrying on cash (the forsaken consumption along the sequence).

Take the utility function (2.3) with $u^i(.) = \sqrt{.}$ and $\beta = 6$. Take, for both agents, $\zeta_t = (1/2)^{t-1}\sqrt{1 + 1/t}$. Endowments are $\omega_t^i = 16\frac{t+1}{t} + G_t^i$, where G_t^1 is given by $G_t^1 = 13$ if t is even and $G_t^1 = -11$ if t is odd, and $G_t^2 = -G_t^1$. Recall that the indeterminacy in the generalized limit considered in the AD price leads to a real indeterminacy in AD equilibrium allocations. Take the equilibrium allocation that results from using a Banach limit B[12]. Consider

[11] As in Levine's (1989) later results under differentiable preferences not satisfying Inada.

[12] We say that a generalized limit B is a Banach limit if $B(c) = \lim_n \frac{1}{n}\sum_{t=1}^{n} c_t$ whenever this limit exists.

the allocation $x_t^i = 16\frac{t+1}{t}$ and its supporting price, which (consistently with Lemma (2)) is of the form $\pi c = \sum_{t=1}^{\infty}(\frac{1}{2})^{t+2}c_t + \frac{3}{4}B(c)$. Taking the AD Lagrange multipliers to be one, the pair $((x^i)_i, \pi)$ constitutes an AD equilibrium, as AD budget equations hold since $\pi(G^1) = 0$ follows from $p(G^1) = -3/4$ and $B(G^1) = 1$.

For $y_0^i = 9$, make $q_t = 2^{t+2}$, the inverse of the summable component of π, the deflator $p_t = 2^{-t-2}$. Let z_t be the funds put aside by a consumer at date t, which will be decomposed as a sum of his money balances and the cumulated taxes on his money balances: $z_t = y_t + \sum_{s\leq t}\tau_s^i(y)$.

Now, letting $\alpha = \beta^i u'(\underline{x^i})$, the implementation is achieved (as explained in detail in Section 2.4) with $(z^i)_i$ if we (i) make $\lim_t z_t^i = \alpha \limsup(x^i - W^i)$, that is, the limiting cost of carrying on cash equals the marginal gain of hedging at infinity (given by the second term in the supporting price applied to the net trade, see Lemma (2)) and (ii) require all other plans $\hat{z}$ to satisfy $\lim_t \hat{z}_t^i \geq \alpha \limsup(x(\hat{z}^i) - W^i)$ (a limiting cost of funds not below the marginal gain at infinity).

The latter can be achieved by designing taxes so that the inequality holds at such alternative plans. More precisely, a money holdings plan y must end up paying cumulated taxes $\sum_{t=1}^{\infty}\tau_t^i(y) = \alpha \limsup(x(\hat{y} - W^i) - \lim_t y_y$, which implies (ii).

The former, together with the AD budget equation, determine what z_0^i should be and imply $z_t^i = 9 + \sum_{s=1}^{t} p_s G_s^i$. Then, $\lim z_t^1 = 33/4$ whereas $\lim z_t^2 = 39/4$.

Now, take $\theta = 0$ and $y^i = z^i$ so that equilibrium cash balances are not taxed and money supply remains constant. But we could have taken instead $0 < \sum_{s\leq t}\theta_s < z_t^i$ for $i = 1, 2$ and obtain $y_t^i = z_t^i - \sum_{s\leq t}\theta_s < z_t^i$. For instance, let $\sum_{s\leq t}\theta_s = 9 - 13/12 + \sum_{s=1}^{t} p_s \min\{G_s^1, G_s^2\}$, then $\sum_{t=1}^{\infty}\theta_t = 56/12$, $\lim y_t^1 = 43/12$, $\lim y_t^2 = 61/12$ and the limiting money supply is 8.67. In any case, real money balances $q_t y_t^i$ explode and deflated money balances $p_t q_t y_t^i = y_t^i$ tend to a positive constant.

As $\lim z_t^1$ is different from $\lim z_t^2$ we could not make $\sum_{t=1}^{\infty}\theta_t = \lim z_t^i$ for all i, so that money supply would tend to zero. Impersonal taxes are incompatible with a limiting zero money supply, except in the symmetric case where $\limsup(x^i - \omega^i)$ is the same for all agents, as will be claimed below.

Remark 11. This example can be modified to include the case in which the economy does not necessarily decrease at each date. In fact, the only condition that must be satisfied is that for any date t there exists $T > t$ such that the aggregate endowment at T is lower than in t. To see how the example could be modified, suppose that at even dates consumers' endowments follow increasing sequences and that at odd dates endowments are described as in the example. More precisely, for $t \geq 1$ we have $\omega^i_{2t-1} = 16\frac{t+1}{t} + G^i_{2t-1}$, where G^1_{2t-1} is given by $G^1_{2t-1} = 13$ if t is even and $G^1_{2t-1} = -11$ if t is odd, and $G^2_{2t-1} = -G^1_{2t-1}$. That is, along the odd dates subsequence endowments are oscillating around a decreasing trend. But the even dates subsequence can be chosen to be increasing or constant, say that for $t \geq 1$, we have $\omega^i_{2t} = 32 + G^i_1 + a(t)$, where $a(t)$ is an increasing bounded sequence of positive numbers.

This variant of the example suggests that what is driving the bubble in money is a pattern of endowments showing some subsequence where aggregate endowments fall (this can be achieved through increasing aggregate oscillations, between odd and even dates) together with ideosincratic shocks around that decreasing subsequence (the individual shocks G^i).

The former ensures that in AD allocations the infimum of consumption is never attained, this guarantees that AD prices (and other supporting prices) have a pure charge. The latter ensures that AD net trades do not converge, this guarantees that money has a bubble. In fact, contrary to what happens with assets paying dividends, the bubble is money is not the AD pure charge evaluated at the dividends stream. It is rather the difference between the highest value that a supporting price pure charge (given by Lemma 2) can take at AD net trades and the value taken by the AD pure charge. When net trades converge, these two values coincide and there is no bubble in fiat money (see Sections 4 and 5 below).

2.3 Main Results

We will now show that non-vanishing money supply becomes the rule for implementation of efficient allocations when agents are wary.

Under wariness, efficient monetary equilibrium requires a positive limiting money supply, except in some degenerate equilibria (where consumers' net trades have the same highest cluster point).

We allow for taxes on money holdings, which may decrease money supply over time. However, we will see that equilibrium portfolios do not need to be taxed (or the tax should not completely erode the cash balances). In the case of impatient agents, knowing that taxes will have to be paid later makes consumers hoard but there is no reason to carry on cash to infinity. However, when agents are wary, the incentive to hoard may be too strong. As deflation is necessary for an efficient outcome (as we saw in the previous section), the return from savings may become unbounded and a finite optimum may not exist. The no-short-sales constraint on money does not suffice to guarantee that a finite optimum exists. It is no longer the case that optimality can be achieved among portfolio plans with limiting non-negative deflated positions, as was the case under impatience.

As we will see below, when $\beta^i > 0$ and the infimum of the consumption plan x^i is never attained in finite time, consumers have a marginal benefit at infinity by raising $\inf x^i$. An improvement strategy, akin to a long-run arbitrage, becomes available to wary agents when marginal utility benefits at infinity outweight the limiting cost of carrying on cash. We consider taxes that, while being impersonal, may be non-lump-sum, at least beyond some distant date, and eliminate such improvement opportunities.

2.3.1 On Long-Run Improvement Opportunities

To be more precise about what we mean by long-run improvement opportunities, given any consumer's plan y of money holdings, let $z_t = y_t + \sum_{s\leq t} \tau_s(y)$ be the sum of funds put aside up to date t. Consider an AD equilibrium (x, π) and the portfolio plan z^i that makes (x^i, z^i) satisfy the sequential budget constraints (2.1) for consumer i given y_0^i and q. Take any other plan (X, Z) that verifies the sequential budget constraints (2.1) with equality at (y_0^i, q) and satisfies $X \in \ell_+^\infty$. Given n, let $d(n)$ be the direction in consumption defined by $d(n)_t = 0$ if $t < n$, $d(n)_n = -q_n Z_n$ and $d(n)_t = q_t(Z_{t-1} - Z_t) = X_t - \omega_t^i$ if $t > n$.

Notice that, as far as the non-negativity constraint is concerned, this direction is admissible for positive changes (it is said to be $d(n)$ right-admissible): at any $t \geq n$ we raise z_t^i to $z_t^i + hZ_t$. By moving on the right along this direction, we hoard more at date n and also at subsequent dates for which ω^i was above X, in order to increase consumption at subsequent dates where ω^i was below X. Denoting by $\delta^+ U(x^i, d(n))$ the right hand side derivative[13] of U^i along the direction $d(n)$ evaluated at x^i, we want to rule out $\delta^+ U(x^i, d(n)) > 0$.

Let us see how does the right hand side directional derivative $\delta^+ U(x^i, d(n))$ look like. We can write it as the minimal value taken at $d(n)$ by the supporting prices (supergradients) of U^i at x^i, that is, $\delta^+ U(x^i, d(n)) = \min \{T d(n) : T \in \partial U^i(x^i)\}$.

[13] Given $x \in \ell_+^\infty$ and $v \in \ell^\infty$, $\lim_{h\downarrow 0} \frac{U(x+hv)-U(x)}{h}$ is called the *right-directional derivative* of U at x along (the direction) v and it is denoted by $\delta^+ U(x; v)$. The *left-directional derivative* $\delta^- U(x; v)$ is defined analogously.

Then, by Lemma (2), for some generalized limit LIM, we have

$$\delta^+U(x^i, d(n)) = -\zeta_n(u^i)'(x_n)q_nZ_n + \sum_{t>n}\zeta_t(u^i)'(x_t)q_t(Z_{t-1} - Z_t) \\ + \beta^i(u^i)'(\underline{x})\mathrm{LIM}(X - \omega^i).$$

Now, efficiency requires Euler conditions to hold as equalities for every agent and every date (otherwise agents holding money might have marginal rates of substitution different from those of agents that might not hold money), so $\zeta_t(u^i)'(x^i_t)q_t$ is constant, and it follows that $\sum_{t>n}\zeta_t(u^i)'(x_t)q_t(Z_{t-1} - Z_t) = \zeta_n(u^i)'(x_n)q_nZ_n - \lim\zeta_t(u^i)'(x_t)q_tZ_t$.

Hence, $\delta^+U(x^i, d(n)) = -\lim_t\zeta_t(u^i)'(x_t)q_tZ_t + \beta^i(u^i)'(\underline{x})\mathrm{LIM}(X - \omega^i)$. So, independently of what the generalized limit LIM might be, $\delta^+U(x^i, d(n)) \leq 0$ if $\beta^i(u^i)'(\underline{x}^i)\limsup(X - \omega^i) \leq \lim\zeta^i_t(u^i)'(x^i_t)q_tZ_t$.

We would like to design a fiscal policy that guarantees this condition and, therefore, eliminates the above long-run improvement opportunities.

2.3.2 Taxes that eliminate the marginal benefit - marginal cost gap

Given a plan of money balances y and a sequence of taxes τ, let $x(y,\tau)_t = \omega^i_t + q_t(y_{t-1} - y_t - \tau_t)$ be the associated consumption plan satisfying the sequential budget equations. The tax $\tau_t(y)$ levied at date t upon a plan y of money holdings consists of a fixed, summable, component θ_t, which is the tax that may be imposed on the efficient plans y^i of all agents, and another component that eliminates the above long-run improvement opportunities. Now, the marginal benefit at infinity when taxes are $\tau(y)$ is $\beta^i(u^i)'(\underline{x}(i))\limsup(x(y,\tau(y)) - \omega^i)$, which is less than or equal to analogous marginal benefit when taxes are just θ. Then, as it suffices to show local optimality, we rule out such long-run improvement strategies, near the efficient plan x^i, if $\beta^i(u^i)'(\underline{x}(i))\limsup(x(y,\theta) - \omega^i) \leq \lim_t\zeta^i_t(u^i)'(x^i_t)q_t(y_t + \sum_{s\leq t}\theta_s)$.

To write this in an impersonal way, consider the Lagrange multiplier ρ^i of the AD budget constraint of consumer i, at AD prices π. Now, as seen in Lemma (2), $\rho^i\pi = (\zeta^i_t(u^i)'(x_t)_t)_t + \beta^i(u^i)'(\underline{x}(i))\mathrm{LIM}(.)$, when the infimum of consumption is not attained in finite time. Hence, the AD price can be written as $\pi = p + \alpha\mathrm{LIM}(.)$ such that $\rho^ip = (\zeta^i_t(u^i)'(x_t)_t)_t$ and $\rho^i\alpha = \beta^i(u^i)'(\underline{x}(i))$, for all i.

Now, we saw that $\zeta_t(u^i)'(x^i_t)q_t$ is constant and, therefore, p_tq_t has to be constant and can be made equal to 1, as we have the freedom to choose q_1. The benefits-cost gap is then, up to a scalar multiple (the inverse of ρ^i), given by $\alpha \limsup q_t(y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=1}^{\infty} \theta_s$, where $\alpha = (\rho^i)^{-1}\beta^i(u^i)'(\underline{x}(i))$ for all agents, . This suggests the following tax scheme.

The lump-sum taxes sequence θ_t is chosen as the tax imposed on the efficient plans y^i of all agents and such that $\sum_{t=1}^{\infty} \theta_t < \infty$.

In general, the tax $\tau_t(y)$ levied at date t upon a plan y of money holdings for which $x(y, \theta, i) \in \ell_+^\infty$ consists of the fixed component θ and another component that increases with the "arbitrage" that would be done if the tax were just that fixed part. More precisely,

$$\tau_t(y) = \theta_t + \frac{p_t}{\|p\|_1}\left[{}^{\alpha}/_{\lim_t p_t q_t} \limsup(q_t(y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=1}^{\infty} \theta_s\right]^+ . \tag{2.6}$$

As we claim below, equilibrium money balances do not need to be taxed (we can set $\theta = 0$), unless we use the lump-sum tax θ to withdraw additional initial holdings A that just had the purpose of making $y \geq 0$ (see subsection 2.4.2). For such tax policy we have (2.6) holding with equality[14]. Alternatively, to avoid ever taxing equilibrium money balances, the tax assessment can ignore the cost of carrying on such additional initial holdings A common to all consumers. In this second case, (2.6) might hold with a strict inequality (see the tax formula (2.16) reported in subsection 2.4.2).

It is important to notice also that these non-lump-sum taxes are invariant to changes in money balances at a finite set of dates and, therefore, Euler conditions (2.5) hold. Moreover, we can make taxes lump-sum up to some date T by replacing the coefficient $p_t/\|p\|_1$ by zero for $t \leq T$ and by $p_t/\sum_{s>T} p_s$ otherwise.

2.3.3 Optimal Monetary Policy

Our next result show that the above taxes implement efficient allocations. We focus on efficient allocations that are uniformly bounded away from zero, never attain the infimum in finite time and, at least for some consumer, have a non converging net trade.

Let us formalize our first assumption.

ASSUMPTION H: The consumption plan $(x^i)_i$ of agent i is such that $x^i \ggg 0$, $\underline{x}^i$ is never attained and there is a subsequence S of dates such that $x_t - \omega_t^i > 0$ on S, $\lim_S x = \underline{x}^i$ and $\limsup_S(x^i - \omega^i) = \limsup(x^i - \omega^i)$.

This assumption says that the infimum of consumption, never attained in finite time, can be approached along a subsequence where net trades are

[14]Observe that $[\limsup(q_t(y_{t-1} - y_t - \theta_t) - \lim y_t - \sum_{t=s}^{\infty} \theta_s]^+ < \infty$ as $x(y, \theta, i) \in \ell_+^\infty$. Hence, τ is well defined.

positive and that the highest asymptotic dishoard occurs precisely along such subsequence. That is, the consumer dishoards more when he is raising the infimum of consumption in the face of very low endowments.

Theorem 2. ***(Efficient Monetary Equilibrium)*** *For preferences given by (2.3), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumption H and (ii) for some agent i, $x^i - W^i$ does not converge. Then, there exist initial holdings y_0^i that implement $(x^i)_i$ as a monetary equilibrium with taxes.*

(this theorem is proven in Section 2.4)

Our second theorem says that, in the efficient monetary equilibrium, the money supply cannot go to zero, apart from an exceptional configuration of the AD net trades.

Theorem 3. ***(Non-vanishing Money Supply)*** *Under the assumptions of Theorem 2, impersonal taxes are incompatible with a limiting zero money supply, except in the symmetric case where* $\limsup(x^i - \omega^i)$ *is the same for all agents.*

(this theorem is also proven in Section 2.4)

Remark 12. When some agents are impatient and the others are wary, the implementation of efficient allocations can be done under the same fiscal policy for all agents or by imposing lump-sum taxes on impatient agents and that policy on the others. The Theorem's assumption that $\limsup(x^i - \omega^i) > 0$ should be imposed only on wary agents. For an implementation with non-vanishing money supply, allowed by AD prices outside of ℓ^1, the consumption plan of impatient agents should not be bounded away from zero (see Araujo, Novinski and Pascoa [5] for the case of assets paying dividends).

2.4 The Implementation Argument: Proof of the Theorem and Detailed Example

We prove now Theorem 2 and provide also details on the computation of Example 2.2. We construct an auxiliar economy, with sequential budget constraints as the original economy, but where intertemporal transfers of wealth as achieved by trading a no-dividends asset in constant positive net supply, not subject to taxes. We show that Arrow-Debreu equilibria can be implemented as sequential equilibria for the auxiliar economy and, then, that the latter are in one-to-one correspondence with sequential equilibria with money and taxes. The tax policy ensures that the portfolio constraints of the auxiliar economy are satisfied.

2.4.1 An Auxiliary Economy

The sum of the money position and the accumulated taxes, at date t, are the total funds that were put aside (deviated from current consumption) at this date. We can think of this sum as if it were the long position (z_t) on a no-dividends asset in constant positive net supply, subject to no-short-sales constraints. We refer to this asset as the z-asset. Positions are related by $z_t = y_t + \sum_{s \leq t} \tau^i_s(y)$. This implies that $z^i_{t-1} - z^i_t = y^i_{t-1} - y^i_t - \tau_t(y)$, which suggests defining an auxiliary sequential economy, whose asset is the z-asset, with budget constraints as follows

$$x_t - \omega^i_t \leq q_t(z_{t-1} - z_t) \quad \forall\, t \in \mathbb{N}, \tag{2.7}$$

Let us see how an efficient allocation can be implemented using the z-asset. By the results in Araujo, Novinski and Pascoa [5], for a dividends-less asset to implement an AD allocation $(x^i)_i \ggg 0$, the net trades $x^i - \omega^i$ can not converge for all agents (see Proposition 6) and the implementation can not be done by forcing the sequential choice set $B_P(q, \omega^i, z^i_0)$ to be contained in the AD budget set (as had been done in Theorem 2 of Araujo, Novinski and Pascoa [5]) [15].

So the implementation using the z-asset has to follow a new strategy. A very useful sufficient condition for individual optimality is given as follows. Denote by pch the space of pure charges, that is, the non-summable components of elements in the dual of ℓ^∞ (see Subsection A.3 in the Appendix). Let $x(z)$ be the consumption plan that a portfolio plan z induces so that (x, z) makes (2.7)[16] hold with equality for every t.

Proposition 12. *Let z^* be a feasible portfolio and let $x^* = x(z^*)$. (i) Suppose there exists $T \in \partial U(x^*)$ with $T = \mu + \nu,\ \ \mu \in \ell^1_+$ and $\nu \in \mathrm{pch}_+$ such that, for every t,*

$$\mu_t q_t = \mu_{t+1}\left(R_{t+1} + q_{t+1}\right) \tag{2.8}$$

and

$$\lim \mu_t q_t z^*_t = \nu(x^* - \omega). \tag{2.9}$$

[15] The AD budget set does not have to contain $B_P(q, \omega, z_0)$, even when the latter is convex. This is a consequence of the multiplicy of the generalized limits components of supergradients.

[16] For an asset with bounded and non negative returns R that will be used also in the case of Lucas' Tree

(ii) Suppose also that every feasible portfolio z satisfies the condition

$$\lim_t \mu_t q_t z_t \geq \nu(x(z) - \omega), \tag{2.10}$$

Then z^* is an optimal solution for the problem with constraints (2.7).

Proof. Given a feasible portfolio z, $U(x(z)) - U(x^*) \leq T(x(z) - x^*) = T(x(z) - \omega) + T(\omega - x^*)$. Moreover, $\mu(x(z) - \omega) = \sum_{t=1}^{\infty}[\mu_t(R_t + q_t)z_{t-1} - \mu_t q_t z_t)]$. By (2.8), $\mu(x(z) - \omega) = \mu_1 q_1 z_0 - \mu_1 q_1 z_1 + \sum_{t=2}^{\infty}[\mu_{t-1} q_{t-1} z_{t-1} - \mu_t q_t z_t)] = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t$. Similarly, $\mu(x^* - \omega) = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t^*$. Now by (2.9), $U(x(z)) - U(x^*) \leq \nu(x(z) - \omega) - \lim_t \mu_t q_t z_t$. Now, by (2.10), $U(x(z)) - U(x^*) \leq 0$. □

The constraint in Proposition 12 can use a supergradient which is not, up to a scalar multiple, equal to the AD price. Let us use a supergradient whose pure charge $\tilde{\nu}^i$ takes the highest value on the direction of the net trade[17]. As shown in Appendix (C.2), the following property holds for preferences given by (2.3) under the assumptions of the Theorem:

$$\tilde{\nu}^i(x^i - \omega^i) = (u^i)'(\underline{x}^i) \limsup(x^i - \omega^i) \tag{2.11}$$

This suggests the following portfolio constraint

$$\lim \mu_t^i q_t z_t \geq \alpha^i \limsup(x(z) - \omega^i) \tag{2.12}$$

where $x_t(z) = \omega_t^i + q_t(z_{t-1} - z_t)$ and α^i is the norm $\|\tilde{\nu}^i\|$ for some pure charge $\tilde{\nu}^i$ satisfying (2.11).

Let $B^A(q, y_0^i, \omega^i)$ be the set of plans (x, z) satisfying (2.7) and (2.12).

Definition 4. A vector $(q, (x^i, z^i)_{i\in\mathcal{I}}) \in I\!R_+^{\infty} \times (\ell_+^{\infty} \times I\!R_+^{\infty})^I$ is said to be an *equilibrium for the auxiliary economy* with initial holdings $(z_0^1, ..., z_0^I) = (y_0^1, ..., y_0^I)$ when:

(a) $(x^i, z^i) \in \text{argmax}\{U^i(x) : (x, z) \in B^A(q, y_0^i, \omega^i)\}$;

(b) $\sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i$;

(c) $\sum_{i=1}^I z_t^i = \sum_{i=1}^I y_0^i \;\; \forall\, t \in \mathbb{N}$.

The next example illustrates the use of (2.12).

[17]That is, $\tilde{\nu}^i$ is such that $\delta^- U^i(x^i; x^i - \omega^i) = (\mu^i + \tilde{\nu}^i)(x^i - \omega^i)$, where μ^i is the ℓ^1 component given by $(\partial_t U^i(x))_t$ under the assumptions of the Theorem.

Example 4 (Example details)**.** Let us go back to the Example of Section 2.3. As we saw, Arrow-Debreu equilibrium plans are $x^i_t = 16\frac{t+1}{t}$ and prices π have countably additive component p and pure charge component ν given by $p(y) = \sum_{t=1}^{\infty}(\frac{1}{2})^{t+2}y_t$ and $\nu(y) = \frac{3}{4}B(y)$. Now $\tilde{\nu}^i(x^i - \omega^i) = \alpha^i \limsup(x^i - \omega^i)$ where $\alpha^i = 3/4$. Constraint(2.12) becomes $\lim(1/2)^{t+2}q_t z_t \geq \frac{3}{4}\limsup q_t(z_{t-1} - z_t)$.

By Proposition 12 we should find q such that at $(z^i)_i$ implementing $(x^i)_i$ we have $\lim \mu^i_t q_t z^i_t = \tilde{\nu}^i(x^i - \omega^i)$. Now, $x^i(z)$ belongs to the Arrow-Debreu budget set if and only if

$$\nu(x^i(z) - \omega^i) - \lim_t p_t q_t z^i_t \leq -z^i_0 \lim p_t q_t \tag{2.13}$$

Moreover, (2.13) holds with equality when the Arrow-Debreu budget constraint holds with equality.

On the other hand, the first order condition of the Arrow-Debreu problem requires[18]

$$\exists \rho^i > 0 : \rho^i \pi \in \partial U^i(x^i), \tag{2.14}$$

Then, $\mu^i + \nu^i = \rho^i(p + \nu)$. So, both requirements are met if for any i

$$\tilde{\nu}^i(x^i - \omega^i) - \nu^i(x^i - \omega^i) = z^i_0 \lim \mu^i_t q_t \tag{2.15}$$

Recall that $p_t q_t = p_{t+1}q_{t+1}$ for any $t \geq 1$. As $\rho^i = 1$ for $i = 1, 2$, equations (2.15) can be rewritten as $(3/4)\limsup(-G^1_t) - (3/4)B(-G^1_t) = z^1_0 p_t q_t$ and $(3/4)\limsup(G^1_t) - (3/4)B(G^1_t) = z^2_0 p_t q_t$. Since $\limsup(-G^1_t) = 11$ and $\limsup(G^1_t) = 13$ we must have $z^1_0 = z^2_0$ and $p_t q_t = 9/z^i_0$, Then, $q_t = 2^{t+2}9/z^i_0$ and $p_t q_t z^i_t = 9 + \sum_{s=1}^{t} p_t G^i_s$. No short sales are ever done in equilibrium. This concludes the example.

In general what can be said? If the pure charge component of the AD supporting price of every agent would already satisfy (2.11), there would be no room to find a bubble. This can not happen when some agent has a non-converging net trade and the pure charges of all her supergradients have the same norm (the latter holds for the class of preferences given by (2.3), under the assumptions of the Theorem 2). We have actually the following result, shown in appendix C.2,

Proposition 13. *For preferences given by (2.3), let $((x^i)_i, \pi)$ be an AD equilibrium such that (i) for each agent i assumption H is satisfied at x^i and*

(ii) for some agent i, $x^i - W^i$ does not converge then, there exist initial holdings z^i_0 that implement $(x^i)_i$ as an equilibrium for the auxiliary economy, possibly with short-sales.

[18]See Zeidler [43], p.391, Theorem 47.C

2.4.2 Mapping back into the original sequential economy

Suppose sequential implementation without taxes was achieved with short sales under the constraint (2.12). As usual we normalize the AD prices by setting $\alpha = 1$ and can always take also $p_t q_t = 1$. Take the constraint (2.12) and divide both sides by the Lagrange multiplier ρ^i of the AD budget constraint. We get the requirement $\lim z_t \geq \limsup(x(z) - \omega^i)$ with the equality holding for the equilibrium plan (z^i).

Even if z takes negative values at some dates, we can find money holdings $y_0^i = z_0^i + A$ such that the equilibrium positions z_t^i can be replaced by *non-negative* money balances. To simplify assume that $\theta = 0$. The non-negative plan y^i given by $y_t^i = z_t^i + A$ for $t \geq 0$ with A large enough will be an equilibrium plan if taxes are defined in the following way, still within the class satisfying but possibly with a strict inequality. For any portolio plan y let

$$\tau_t(y) = (p_t/\|p\|_1) \max\{0, \limsup(q_t(y_{t-1} - y_t)) - \lim y_t + A\} \tag{2.16}$$

In fact, (2.9) together with $\limsup(x^i - \omega^i) \geq 0$ (by assumption (i) in the Theorem), imply that z_t^i could be negative just only in a finite number of dates, and as a consequence there exists $A > 0$ such that $z_t^i + A \geq 0$ for all t and all i.

Now, $\sum_{t=1}^{\infty} \tau_t(y) \geq \limsup(q_t(y_{t-1} - y_t)) - \lim y_t + A$. Putting y in one-to-one correspondence with $z = y - A + \tau(y)$, we see that $\sum_{t=1}^{\infty} \tau_t(y) \geq \limsup(x_t(y) - \omega_t^i + q_t\tau_t(y)) - (\lim z + A - \sum_{t=1}^{\infty} \tau_t(y)) + A \geq \limsup(x(y) - \omega^i) - \lim z + \sum_{t=1}^{\infty} \tau_t(y)$. Hence, $\lim z \geq \limsup(x(z) - \omega^i)$. That is, the definition of taxes ensures that any plan y has an image z satisfying constraint (2.10). As we already knew that (2.9) holds, it follows that y^i is optimal, for the initial holding $y_0^i = z_0^i + A$, and no taxes are levied in equilibrium. We saw that $(y^i)_i$ manages to implement, under a no-short-sales constraint, the same efficient allocation that $(z^i)_i$ did.

Alternatively, we can avoid inserting A in the formula for the non-lump-sum tax but need to consider lump-sum taxes θ such that $\sum_{t=1}^{\infty} \theta_t = A$.

Notice that the constant A is not uniquely defined, it just has a known lower bound. Even if initial holdings $(z_0^i)_i$ are compatible with an implementation using non-negative money balances, we can always increase those initial holdings and then, either take away the excess though lump-sum taxes or keep that extra money as long as out-of-equilibrium plans are taxed taking that extra money into account, as we just described.

The tax formula (2.16) has the following nice interpretation. Suppose that in order to implement without short selling an AD allocation we need to give to all agents a common extra initial holding of at least A units of money. Then, a money balances plan y will be taxed whenever the marginal benefit at infinity ($\limsup(x(y) - \omega^i)$, of raising a distant infimum of consumption) exceeds the limiting cost ($\lim y_t - A$) of carrying on cash above that common level A. That is, the cost of carrying on that common minimal initial money holding should be ignored in the tax assessment.

Finally, let us prove Proposition 3. Even if equilibrium money balances were taxed with a lump-sum tax θ, it follows from $\lim y^i - A + \sum_{t=1}^{\infty} \theta_t = \limsup(x^i - \omega^i)$ that the impersonal nature of the taxes is compatible with a zero limiting money supply only in the symmetric case where $\limsup(x^i - \omega^i)$ is the same for all agents.

2.5 Optimal Implementation in other Sequential Economies

2.5.1 Implementation of Efficient Allocations with a Lucas' Tree

Let us replace fiat money by another long lived asset in positive net supply, a Lucas' tree, that is, an asset with returns in the consumption good at each t, given by $(R_t)_{t\in\mathbb{N}}$ satisfying $R_t \leq M \ \forall t$ and $(R_t)_t \neq 0$. This asset cannot be shorted.

The government would now tax in a different form, as it would have to tax in the *numéraire*, the consumption good.

The sequential budget constraint of each agent i is given by:

$$x_t - \omega_t^i \leq q_t \left(y_{t-1} - y_t\right) + R_t y_{t-1} - \tau_t^i(y) \ \forall t \in \mathbb{N},$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of Lucas' tree prices and τ^i is the taxation that depends on the Lucas' tree positions plan y that the agent may choose. Note that, since the taxes are levied in the *numéraire*, the relationship between the portfolio and the taxes has some impact on the agent constraint.

For this economy, the equilibrium is defined analogously to the one considered in the previous sections, adapting the government constraint and the market clearing equations, that must now include the real returns of the Lucas' tree. Notice that, as in Araujo, Novinski and Pascoa [5], AD endowments W^i are now related to sequential endowments ω^i as follows: $W^i = \omega^i + Ry_0^i$.

Let us define an equilibrium for the economy with a Lucas tree and taxes.

Definition 5. A vector $\left(q,(x^i,y^i)_{i\in\mathcal{I}}\right)$ is an equilibrium for the economy with initial Lucas tree holding $(y_0^i)_{i\in\mathcal{I}}$ and fiscal policy τ when $(x^i,y^i)\in \operatorname{argmax}\{U^i(x)\colon (x,y)\in B\left(q,y_0^i,\omega^i,\tau\right)\}$ and, for every date t, we have

1. $\sum_{i\in\mathcal{I}} x_t^i = \sum_{i\in\mathcal{I}} \omega_t^i + R_t \sum_{i\in\mathcal{I}} y_0^i$,
2. $\sum_{i\in\mathcal{I}} y_t^i = \sum_{i\in\mathcal{I}} y_0^i$,

Note that taxes must be zero in equilibrium, due to item 2.. Since taxes are non-negative, the fiscal policy is in fact a punishment to a deviation from the equilibrium path that takes advantage of the long-run improvement opportunities identified above.

We will see next whether taxes can rule out saving strategies that constitute long-run improvements. We observe first that AD allocations can be implemented if the Lucas tree could be shorted.

Proposition 14. *Let be $(x^i)_i$ be an efficient allocation such that for each i, H holds at x^i. (A) Provided that $\liminf_t (R_t) > 0$, there exist initial holdings $(y_0^i)_i$ of the Lucas tree and fiscal policy τ that implement $(x^i)_i$ as an equilibrium with taxes if the Lucas tree could be shorted. (B) If $(R_t)\geq 0$ and for some agent i, $x_t^i - W_t^i$ does not converge, the same result holds.*

Proposition 14 is proven in Appendix C.3.

Remark 13. In general and in the absence of other financial instruments, short sales might not be avoided. If we were to create more Lucas trees (increase y_0^i) to overcome such negative positions (as we did in the case of money), then the commodity endowment of each agent in the sequential economy would be reduced, since $\omega_t^i = W_t^i - R_t z_0^i$, and it may happen that the quantity of Lucas tree required to avoid short sales would make ω_t^i become negative.

Remark 14. To avoid short sales we can either (1) impose an additional condition on the AD net trades, such as

$$\sum_i p_i \left|x^i - W^i\right| \leq \beta \left(u^i\right)'(\underline{x}) \limsup \left(x^i - W^i\right) - \nu^i(x^i - W^i),$$

which says that the net trade oscillations are small relative to what is the positive price of money initial holdings (given by the difference between the value that the two pure charges taken on the net trades) or (2) add a *one-period* asset in zero net supply (an I.O.U. promise) that can be shorted[19] at each date t and in this case taxes would depend on the portfolio formed by the Lucas tree and the *one-period* promise.

[19]With a portfolio constraint to avoid Ponzi schemes.

Notice that we do not allow for the I.O.U. promises to be secured by the Lucas tree. In fact, for such collateralized credit, it is not possible to ensure that the markets are sequentially complete, since the collateral constraint could be binding in presence of a low amount of Lucas tree (and we already know that we do not have the freedom to raise its initial holdings). The next subsection will explore this.

2.5.2 Optimal Implementation with Lucas' Tree as Collateral

The implementation of efficient allocations done by using a Lucas' tree together with an I.O.U. may require quite substantial levels of unsecured debt that are hard to accept in I.O.U. credit. Moreover, in the absence of the efficient taxes (which guarantee individual optimality), the problem of each consumer would not have a finite optimum and Ponzi schemes could be done using the I.O.U.. It is therefore hard to imagine that markets would be organized in such a way, unable to function at all (even with inefficient outcomes) if taxes were not there. Hence, we should check what happens when the Lucas' tree would serve as collateral for the promises.

The sequential budget constraints of agent i are now given by:

$$\begin{aligned} x_t - \omega_t^i + q_t h_t z_t &\leq q_t(y_{t-1} - y_t) + R_t y_{t-1} + r_{t-1} q_{t-1} h_{t-1} z_{t-1} - \tau_t^i(y, z), \\ z_t^- &\leq y_t, \\ y_t &\geq 0, \end{aligned}$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of Lucas' tree prices and τ^i is the taxation that depends both on the Lucas' tree plan y and on the promise z, $r_t = 1+\lambda_t$ where λ_t is the interest rate, $\hat{\alpha} \in (0,1)$ and $h_t \in (0,1]$ is a margin coefficient that prevents default:

$$q_t h_t r_t = \hat{\alpha}(q_{t+1} + R_{t+1}) \,.$$

The equilibrium for this economy is analogous to the previous case, adding now the market clearing condition for the promise z.

Efficient taxes are well-defined but collateralization may lead to sequential market incompleteness and we may no longer be able to ensure non-negativity of position in the Lucas' tree, as the following result asserts:

Proposition 15. *Let be $(x^i)_i$ be an efficient allocation such that for each i, H holds at x^i. Then for $\liminf_t (R_t)$ there exists an initial holding of Lucas' tree asset y_0^i that implements $(x^i)_i$ as an equilibrium with taxes if the Lucas' Tree could be shorted. If $(R_t) \geq 0$ and for some agent i $x_t^i - \omega_t^i$ does not converge, we have the same result.*

Also can be noticed that in equilibrium, since we impose a margin requirement that avoids default, we can observe that in fact the promise becomes redundant. However, both outcomes (with or without the promise) do not have a sound economic interpretation as we could not rule out negative positions in the Lucas' tree.

Proposition 16. *Let $(q, (x^i, y^i, z^i)_i)$ be an equilibrium with taxes $(\tau^i)_i$ as in the previous proposition, then $(q, (x^i, \hat{y}^i, 0)_i)$ is an equilibrium for the economy without the promise, where $\hat{y}^i_t := y^i_t + \alpha z^i_t \geq 0$ for each i.*

Even if more promises could be added to complete sequentially the markets, the apparent success we had in defining efficient taxes on collateral and promises does not prevail when we pass to stochastic economies, as shown next.

2.5.3 Implementation with Lucas trees and unsecured credit

Let us examine what happens in a stochastic economy. Can fiat money, when properly coupled with other spanning instruments, still implement AD allocations? Or does the coexistence with other assets make money lose its role?

We define a stochastic economy that is a natural generalization of the above deterministic model. Define an event tree such that at each date t and at each node s_t there exist 2 successors of s_t, $s_{t,1}$ and $s_{t,2}$, and denote s_t^- as the predecessor of s_t. Let σ be the root of the event tree and $\mathcal{S} := \{s_t : t \in \mathbb{N}\}$. Denote by P_{s_t} the probability for the successors of s_t.

The utility function of each agent i is a generalization of (2.3) given by:

$$U^i(x) := \sum_t \zeta^i_t \mathbb{E}_t \left[u^i(x_t)\right] + \beta_i \inf_t \mathbb{E}_t \left[u^i(x_t)\right] \tag{2.17}$$

where x_t is the consumption of all possible nodes at date t and $\mathbb{E}_t$ is the expected value on $\mathcal{S}_t$, the set of all possible nodes s_t of the date t, with the probability induced by P_{s_t}.

In stochastic economies, wary agents can not be modeled literally as in (2.3), carrying about the worst outcome on the whole event tree[20]. One possible form, that we follow here, is to suppose that agents are worried about the mean losses at each date, as in (2.17). Since agents can not know precisely what will be the state that will occur, their concern about losses at

[20]In fact, that might imply that agents would be worried about some states with arbitrarily low probability.

distant dates is represented in terms of a concern about the expected value at each date t. This means that there is no aversion to uncertainty among the states, but there is an aversion to ambiguity on the discount factors, as in equation 2.3.

Let us start by implementing with Lucas trees and I.O.U.s and then we will drop the I.O.U.s and introduce fiat money.

We now have two Lucas trees in positive net supply and positions $y(j)$, $j = 1, 2$. In the spirit of Remark 14, we allow for trades a in one-period zero-net-supply promises paying an interest rate i_{s_t} in the nodes that immediately follow node s_t. At node s_{t+1} such that $s_{t+1}^{-} = s_t$, the budget constraint and the non-negativity of the Lucas trees constraints are given respectively by:

$$\begin{aligned} x_{s_{t+1}} - \omega_{s_{t+1}} + q_{s_{t+1}} y_{s_{t+1}} + a_{s_{t+1}} + \tau_{s_{t+1}}(y, a) \leq \left(R_{s_{t+1}} + q_{s_{t+1}}\right) y_{s_t} + (1 + i_{s_t}) a_{s_t}, \\ y_{s_{t+1}} \geq 0, \end{aligned} \tag{2.18}$$

where $q = (q_{s_t})_{st \in \mathcal{S}}$, $(R_{s_t})_{st \in \mathcal{S}}$ and $(i_{s_t})_{st \in \mathcal{S}}$ are the Lucas trees prices and returns among the set of nodes and the interest rates respectively, $\mathcal{S}$, and τ is the taxation that depends on y and on a if it is used. And denote $B\left(q, y_0^i, \omega^i, \tau\right)$ as the set of (x, y, a) such that satisfy the budget constraint (2.18).

We define an equilibrium for the economy with Lucas trees and taxes as the natural extension of the Definition 5, with the interest rates i_{s_t} and the promise trades a^i as additional variables, under the condition that the promises' trades clear, $\sum_i a_{s_t}^i = 0$, at each node s_t.

We establish now that we can not implement efficient allocation in a sequential economy with taxes.

Let us reformulate assumption H in the stochastic case.

Assumption H'1: The consumption plan $(x^i)_i$ of agent i is such that $x^i \ggg 0$, $\inf_s \left(\mathbb{E}_s[u^i(x_s^i)]\right) < \mathbb{E}_t[u^i(x_t^i)]$ $\forall t \geq 0$, and there is a subsequence S of dates such that $\mathbb{E}_t[(u^i)'(x_t)(x_t^i - W_t^i)] > 0$ on S and $\limsup_S \mathbb{E}_t[(u^i)'(x_t)(x_t^i - W_t^i)] = \limsup \mathbb{E}_t[(u^i)'(x_t)(x_t^i - W_t^i)]$.

Assumption H'2: $(x^i)_i$ is such that (a)

$$\liminf_{\left\{t\,:\,\mathbb{E}_t\left[u_i'(x_t)\left(x_t^i - W_t^i\right)\right]>0\right\}} \mathbb{E}_t[u^i(x_t^i)] = \inf_s \left(\mathbb{E}_s[u^i(x_s^i)]\right)$$

and (b) $\lim_t \mathbb{E}_t\left[u^{i\prime}\left(x_t^i\right)\right]$ exists for each i[21].

While H'1+H'2(a) are just the extension of assumption H to the stochastic setting, the hypothesis H'2(b) somehow strengthens it.

[21]Part (b) of H'2 can be replaced by the following: there exists $T > 0$ such that for every $t_1,\ t_2 \geq T$ we have that $\zeta_{t_1}^i / \zeta_{t_2}^i = \zeta_{t_1}^j / \zeta_{t_2}^j$ for each pair of agents $i,\ j$.

The following theorem establishes what can be done with taxes both when the trees are traded alone or together with I.O.U.s that are not secured by the trees, in which impersonal taxes ensure efficiency. The idea is that equilibrium plans will not taxed but other plans may be penalized. These taxes will eliminate the usual Ponzi schemes (in the zero-net-supply promises) and any other long-run improvement opportunities.

Theorem 4. ***(Implementability in Unsecured Credit Economies without Money)*** *For preferences given by (2.3), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumptions H'1 and H'2 and (ii) for some agent i, $\mathbb{E}_t\left[u^i\left(x_t^i\right)\left(x_t^i - W_t^i\right)\right]$ does not converge, then, there exist initial holdings of the Lucas trees z_0^i and impersonal taxes that implement $(x^i)_i$ as an equilibrium for the sequential economy, but possibly with trades in the zero-net-supply one period promises (so that short sales of the trees can be avoided).*

Theorem 4 is proven in Appendix C.3.

Remark 15. Theorem 4 says that to implement efficient allocations with Lucas trees and taxes, but without money, the Lucas trees would need to be used together with I.O.U. promises (the latter being shorted so that the former are not). Analogously to what was pointed out in the deterministic case, allowing for secured credit, in the form of these promises being collateralized by the Lucas trees, could lead to market incomplete. However, the resulting dependence on unsecured credit, is a fragility of the implementation, due to the full commitment assumed on debtors, which might not be incentive compatible.

Implementation with Fiat Money

Finally, we observe that in stochastic economics, efficient allocations can always be implemented with fiat money. Taxes will be paid in money and markets can be completed sequentially if another assets are added, say two Lucas trees. Money and the Lucas trees have non-negative positions in equilibrium, thanks to the fact that the initial holdings of money can be adjusted. There is no need to allow for trades in zero-net supply promises. Denoting by $y_{s_t} \in \mathbb{R}_+^2$ the positions in the Lucas trees and by z_{s-t} the money balances in state s_t, we write the consumer budget constraint in this state as follows:

$$\begin{aligned} x_{s_t} - \omega_{s_t} + q_{s_t}^1 y_{s_t} + q_{s_t}^2 z_{s_t} \leq & \left(R_{s_t} + q_{s_t}^1\right) y_{s_t-} + q_{s_t}^2 z_{s_t-} - q_{s_t}^2 \tau_{s_t}(y,z), \\ y_{s_t}, z_{s_t} \geq & \, 0, \end{aligned}$$

where $q_{s_t}^1, R_{s_t} \in \mathbb{R}_+^2$ are the prices and the returns of the Lucas assets trees, $\tau^i(y,z) \in \mathbb{R}_+$ is the taxation that depends on (y,z) and $q_{s_t}^2$ is the price of money. We suppose that $R = (R^1, R^2)$ is such that for each s_t there exists some s_{t+r} successor of s_t such that $R^1_{s_{t+r}} \neq R^2_{s_{t+r}}$. An equilibrium for this economy is defined analogously to the original deterministic monetary case (again for a government cost assumed to be zero), with market clearing for the two Lucas trees as additional conditions.

Theorem 5. ***(Coexistence of Fiat Money and Lucas Trees)***
For preferences given by (2.17), let $(x^i)_i$ be an efficient allocation such that (i) for each i, x^i satisfies assumptions H'1 and H'2 and (ii) for at least one agent i, $\mathbb{E}_t\left[u^{i\prime}(x^i_t)(x^i_t - W^i_t)\right]$ does not converge, then, there exist initial holdings y^i_0, z^i_0 of the fiat money and the Lucas trees that manage to implement $(x^i)_i$ as an equilibrium with taxes, non-negative portfolios $(y^i, z^i)_i$ and a non-vanishing money supply.

Theorem 5 is proven in Appendix C.3.

Remark 16. Under pure discounting and apart from some special cases, fiat money would lose its efficient role (and its positive price) if other long-lived assets were being added to an economy without frictions that might justify the role of money. Wallace [41], among many other of his relevant papers on fiat money, addresses the essentiality of money and comments on the difficult coexistence of money and high-return assets. When impatience is replaced by wariness, our results (Proposition 5) show that, coexistence of money and those assets is compatible with efficient monetary equilibria, in robust cases. While no taxes are being levied on the equilibrium money balances, the threat of taxing off-the-equilibrium plans is crucial.

Remark 17. In stochastic sequential economies as the one that we analyze in this part of the chapter, the study of efficient bubbles and the possibility of their crashing in some parts of the tree are quite interesting things to be analyzed. Since the characterization of them can be done in terms of the pure charge of the AD price and the returns of the assets, if some of the latter becomes zero in a subtree, then the former could crash all along that subtree.

Implementation with Lucas trees and collateralized credit

Consider an economy with two Lucas' trees used as collateral for the I.O.U.s therefore the agent's constraint for each $t \geq 0$ node s_t and s_{t+1} such that $s^-_{t+1} = s_t$ are given by:

$$\begin{aligned} x_{s_{t+1}} - \omega_{s_{t+1}} + q_{s_{t+1}} h_{s_{t+1}} z_{s_{t+1}} + q_{s_{t+1}} y_{s_{t+1}} &\leq \left(R_{s_{t+1}} + q_{s_{t+1}}\right) y_{s_t} + r_{s_t} h_{s_t} q_{s_t} z_{s_t} - \tau^i_{s_{t+1}}(y, z), \\ y_{s_{t+1}} &\geq z^-_{s_{t+1}}, \\ y_{s_{t+1}} &\geq 0, \end{aligned}$$

where $q = (q_t)_{t\in\mathbb{N}}$ is the sequence of Lucas' trees prices and τ^i is the taxation that depends on y and z, $r_t = 1 + \lambda_t$ where λ_t is the interest rate, $\hat{\alpha} \in (0, 1)$ and $h_t \in (0, \infty)$ is such that satisfies

$$\hat{\alpha}^j = \frac{q^j_{s_t} h^j_{s_t} r_{s_t}}{\min_{s^-_{t+1}=s_t} \left\{ R^j_{s_{t+1}} + q^j_{s_{t+1}} \right\}} \in (0, 1), \ \forall s_t, j.$$

Let us define an equilibrium for the economy with Lucas' trees, credit and taxes.

Definition 6. A vector $\left(q, h, r, (x^i, y^i, z^i)_{i\in\mathcal{I}}\right) \in \mathbb{R}^{2\times\mathcal{S}}_+ \times (0,\infty)^{2\times\mathcal{S}} \times (0,\infty)^{\mathcal{S}} \times \left(\ell^\infty_+(\mathcal{S}) \times \mathbb{R}^{2\times\mathcal{S}}_+ \times \mathbb{R}^{2\times\mathcal{S}}\right)^I$ is an equilibrium for the economy with initial Lucas' tree holding $\left(y^1_0, \ldots, y^I_0\right)$ and fiscal policy $\left(\tau^1, \ldots \tau^I\right)$ when:

1. $(x^i, y^i, z^i) \in \operatorname{argmax}\left\{U^i(x) : (x, y, z) \in B\left(q, y^i_0, \omega^i, h, r, \tau^i\right)\right\}$,
2. $\sum_{i=1}^I x^i_{s_t} = \sum_{i=1}^I \omega^i_{s_t} + R_{s_t} \sum_{i\in\mathcal{I}} y^i_0 \ \forall s_t \in \mathcal{S}$,
3. $\sum_{i=1}^I y^i_{s_t} = \sum_{i=1}^I y^i_0 \ \forall t \in \mathbb{N}$,
4. $\sum_{i=1}^I z^i_{s_t} = 0 \ \forall t \in \mathbb{N}$,
5. $\hat{\alpha}^j = \frac{q^j_{s_t} h^j_{s_t} r_{s_t}}{\min_{s^-_{t+1}=s_t}\left\{R^j_{s_{t+1}} + q^j_{s_{t+1}}\right\}}, \ \forall s_t, j.$

It can be observed that in equilibrium there is no default due to the margin requirements. Let us define, in a similar way to what was done in the previous sections, an *efficient equilibrium with taxes.*

We establish, similar to the deterministic case, that we can implement efficient allocations in this type of economies with taxes and Lucas' tree as a collateral if they can be short in equilibrium.

Proposition 17. *For almost any* $(x^i)_i$ *efficient allocation such that for each i,* $x^i \ggg 0$, $\inf_s \left(E_s[u_i(x^i_s)]\right) < E_t[u_i(x^i_{t+1})] \ \forall t \geq 0$, $\limsup \mathbb{E}_t[(u_i)'(x_t)(x^i_t - W^i_t)] \geq 0$, *can not be implemented in a sequential economy with two Lucas' trees with credit and taxes for all type of endowment distributions.*

2.6 Concluding Remarks

We show that sequential implementation of efficient allocations of economies with wary agents is achieved with the help of a fiscal policy that depends on the limiting positions of the agents' portfolios when fiat money is one of the assets that complete the financial markets. Moreover, if we would try to replace this asset by a Lucas tree, some difficulties could be faced. More precisely, under non-negative positions in the tree, to get sequential market completeness we could need also zero-net-supply promises. The amount of unsecured credit that would be required to complete the markets could be quite huge and, presumably, creditors might not be willing to lend it.

If the implementing asset were a long-lived asset with real returns, there are two extensions that might seem to be natural and deserve a comment. One is to allow for the asset to collateralize the short sales of the zero-net-supply promise. The other is to allow for short sales of the long lived asset itself in the way that short sales of shares are actually done in financial markets, by borrowing the shares first rather than doing "'naked"' short sales. In both cases, it is common to observe frictions that lead to inefficiency. In the former, the collateral constraint could be binding. In the latter, we could have a binding constraint linking the short sale of the shares to the amount of shares that were borrowed[22]. For these reasons, in this chapter, by a Lucas tree, we mean the classical notion of a long-lived real asset that can not be shorted and, furthermore, we do not allow it to serve as collateral. In this context, the complementary negative hedging is done through I.O.U. promises.

Fiat money has the merit of dispensing with the problematic role of that unsecured credit (in the form of I.O.U.) in completing the markets. In fact, the initial holdings of money can always be adjusted in order to implement sequentially an efficient allocation using non-negative money balances (alone in a deterministic setting or together with non-negative Lucas tree positions in a stochastic setting). Dispensing with unsecured credit allows us to avoid modeling reputation problems and complex bankruptcy procedures.

Wariness is a lack of impatience that makes consumers care about loses at far away dates. For fiat money to implement sequentially an efficient allocation, the money supply can not go to zero, since wary agents will have a persistent demand for cash to hedge against endowments shocks at far away dates. This optimal positive limit in the money supply is implemented without forcing any money floors or any portfolio constraints at all. We just assume the usual no-short-sales constraint on money together with a tax policy that does not tax the equilibrium plan but taxes plans that lead to excessive savings and, therefore, correct what would be (in Friedman's or Bewley's own words) an insatiable demand for precautionary liquidity in a deflationary context.

[22]Actually, the two cases are often two legs of the same operation and the binding constraint becomes the same. In repo markets, the borrower of shares is a creditor that accepts the shares as collateral for a cash loan.

CHAPTER 3

Crashing of efficient stochastic bubbles with long-lived agents

Rational bubbles have been studied intensively since the late 70's with Blanchard 1979 (see [13]) and Blanchard and Watson 1982 (see [14]) and with Santos and Woodford 1997 (see [36]), was possible a theoretical and systematic study of rational bubbles in an intertemporal competitive equilibrium framework that gives conditions. However the implementation of efficient bubbles in positive net supply with long-lived assets was not extensively studied since requires the study of wariness and it relationship with bubbles. Araujo, Pascoa and Novinski 2012 (see [5]) established a relationship between efficient bubbles and the desire of WARY agents to avoid losses at distant dates. The only problem with these bubbles is that their existence in a deterministic economy means that it will be present at all dates.

In presence of a stochastic economy, is possible to have crashing of efficient bubbles if there is patient agents in the economy. This behavior can be obtained by interpreting this lack of impatience as some type of ambiguous beliefs in the sense of Gilboa and Schmeidler (see [26]), making the agents worried about their losses among the different dates and state of nature.

This chapter aims to establish conditions that characterize the existence of efficient bubbles in positive net supply and their crashing in stochastic economies with patient agents consistent with Gilboa and Schmeidler and Schmeidler (see [37]). These conditions can be considered as generalizations of the ones established in Araujo et al. And as secondary objective we have an analysis of possible increments of volatility in presence of crashing of bubbles in some states of nature.

The chapter is organized as follows: In section 3.1, we describe the stochastic model and is also exposed the type of WARY agents that we will consider during the chapter. In section 3.2, we characterize the consumption of the optimal solution with the AD-price. In section 3.3, we establish the relationship between efficient bubbles and the existence of pure charges in the AD-price, and as a consequence, the relationship between bubbles and the optimal consumption. And in section 3.4, We analyze volatility in presence of crashing of bubble.

3.1 Model

Let us define our environment, there is a enumerable number of periods t, in each period, there exists a finite number of states and finite number of sucessor for each of them. In other words, the set of all the possible states can be seen as tree, and each of the states as a node of the tree.

Let us denote $N \subset \mathbb{N}$ as a finite set that represents all the states that could be successor of each node of the tree, $s^t = (s_1, s_2, \dots, s_t) \in \{1\} \times N^{t-1}$ as a node of the tree in the date t, s^{t-} as the predecessor of s^t, $\sigma := (s_1, \dots, s_t, \dots)$ as an infinite path of the tree, $\{1\} \times N^{\infty}$ as the set of all the infinite paths σ, $\mathcal{N}$ as the σ-algebra induced by $\{\sigma : \sigma^t = s^t\}$ for each $s^t \in \{1\} \times N^{t-1}$ and for each $t \in \mathbb{N}$, and $\mathbb{P}$ as a probability measure in $(\{1\} \times N^{\infty}, \mathcal{N})$ then $\mathbb{P}[\sigma^t = s^t]$ can be interpreted as the probability of the node s^t to occur.

In this economy there are I number of agents and for each agent i the space of consumption is $\mathcal{X} := \{X : \cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t \to \mathbb{R}^+ : \exists K > 0 \text{ such that } X(s^t) \leq K\}$ then the elements of $\mathcal{X}$ can be seen as elements of $L^{\infty}\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ where $\tilde{\mathcal{N}}$ is the discrete σ-algebra in $\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t$ and P is the σ-additive measure induced by $P(\{s^t\}) = \mathbb{P}[\sigma^t = s^t]$.

Since we are in a stochastic economy with infinite number of dates, one form to implement the lack of impatience for losses is by using ambiguity aversion in the sense of Gilboa and Schmeidler, which give us a clear relationship between ambiguity and the importance in the worst consumptions.

The following utility function can be seen is an extension of $\varepsilon-$*Contamination* for the stochastic case, in which the agent consider the worst possible consumption in every possible path, that is,

$$U^i(X) := \int_{N^{\infty}\times\mathbb{N}} u^i \circ X_{\sigma^t} d\mathbb{P} \times \zeta^i(\sigma, t) + \int_{N^{\infty}} \left(\beta^i(\sigma) \inf_t u^i \circ X_{\sigma^t}\right) d\mathbb{P}(\sigma) \quad (3.1)$$

where $u^i : \mathbb{R}_+ \to \mathbb{R}$ is a differentiable, strictly increasing and strictly concave function, ζ_t^i is a measure in $\mathbb{N}$ that can be interpreted as the discounted factors for each state, and β^i is a $\mathcal{N}$-measurable non-negative function. As can be observed in the form of the utility function, the agents worry about their consumption by considering all the possible paths in the tree.

Agents with this behavior worry about all possible paths of consumption taking to account the probability of all paths $\mathbb{P}$ and the discounted factors ζ^i. However they are worried also about the existence of ambiguity among each single path, it means a spencial concern about the worst possible outcome of each path. Differently to Chapter 2, there is ambiguity not just only on the discounted factors, in this case there exists also among all the paths among of the tree.

Can be observed easily that this type of agents are worried about an average of the worst consumptions in each possible infinite path. The fact that the probability of the second integral is exactly $\mathbb{P}$ is a consequence of the implementation of the $\varepsilon-$*Contamination* to this model.

The fact of having a non constant β is a generalization of the Utility function, since the utility function force β to be constant among the paths nevertheless, is completely natural that the agents are not indifferent among the paths of the tree therefore we will assume that $\beta : N^\infty \to \mathbb{R}_+$ is bounded and not necessarily constant for all paths σ.

This type of patience is compatible with the hypotheses imposed by Bewley to guarantee existence of equilibrium in $L^\infty\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\}\times N^t, \tilde{\mathcal{N}}, P\right) \equiv \ell^\infty$, since there is no arbitrage given by the weight of each single path in the infimum, because clearly the infimum of each single path σ with no weight is completely negligible for all agents.

Let us assume also that initial endowments given by $\{\omega^i_{s^t}\}_{s^t}$ in the node s^t are strictly positive, bounded and satisfies $\{\omega^i_{s^t}\}_{s^t} \ggg 0$.

Our goal is to analyze efficient bubble for long-lived assets with non-negative returns, since we know that Mas-Colell et al. (see [30]) and Bewley (see [9]), proved that there exists an AD equilibrium for this economy. In order to do so, we have to deal with lack of impatience and the possibility that the economy has some type of *Long-Run Improvement Opportunities*, see section 2.3.1.

Consider a stochastic sequential economy with one good in each state as *numéraire*, N Long-Lived assets with *no short-sales* and nonnegative returns given by $R_{s^t} \in \mathbb{R}^N_+$ for each node s^t, then the agent constraint is given by

$$\begin{aligned} x_{s^t} - \omega_{s^t} &\leq q_{s^t}\left(z_{s^{t-}} - z_{s^t}\right) + R_{s^t} z_{s^{t-}}, \\ z_{s^t} &\geq 0, \end{aligned}$$

where x_{s^t} is the consumption and ω_{s^t} is the endowment in the *numéraire*, q_{s^t}

are the prices of the assets and z_{s^t} is the assets' allocation in the node s^t.

In order to avoid this type of arbitrage, we must impose a fiscal policy that has to be paid by money produced by the government or we have to impose some type of *P-constraint* that avoids the *Long-Run Improvement Opportunities*, see Araujo et al. [5]. The necessity of this type of constraints is a direct consequence of the lack of impatience, when there is an economy with this type of agents, there would be incentives to save large quantities of wealth in form of assets in order to avoid the worst states in the in the infinity.

Since in this chapter we have more interest in analyze efficient bubbles than to find different ways in which we can implement efficient allocations. We will assume that each agent has some type of *P-constraint* that allows us to implement efficient allocation.

In the chapter 2, efficient allocation can be implemented with money and a fiscal policy without *Lump-Sum* taxes even in stochastic economies, if the markets are complete sequentially. In spite of the fact utility functions that were considered are different than here, it can be extended using the same tools that were discussed. We will prove some of this tools that will help us to understand some properties of the pure charges and their relationship with the existence of efficient bubbles.

3.2 Characterization of the Utility Function Subdifferential

For the type of utility function that we considered, we must analyze the specific form of the subgradient and how can be characterized.

Remark 18. Let $X \ggg 0$, $\underline{X} : N^\infty \to \mathbb{R}$ such that $\underline{X}_\sigma = \inf_t X_{\sigma^t}$.

For U given by 3.1 with $u \in C^1(0,\infty)$, for each $\pi \in \partial U(X)$ we have:

$$\pi Y = \sum_{s^t} u'\left(X_{s^t}\right)\left(\int_{[\sigma:\sigma^t=s^t]} \left(\zeta_t + \alpha\left(\sigma,k\right)\right) d\mathbb{P}(\sigma)\right) Y_{s^t} + \nu_\alpha(Y)$$

where

- for all $\sigma \in N^\infty$, $\alpha\left(\sigma,k\right) \geq 0\ \forall k \geq 1$, $\alpha(\sigma,k) = 0$ if $X_{\sigma^k} > \underline{X}(\sigma)$ and $\sum_{k=1}^{\infty} \alpha\left(\sigma,k\right) \leq 1$,
- ν_α is a pure charge in $\mathcal{X}$ such that $\nu_\alpha(\mathbb{1}_A) = 0$ for each $B \subseteq N$ such that $B \subseteq B_2 := \Big[s^t\ :\ \exists\,\sigma,\ K$ such that $\sigma^t = s^t,\ \inf_s u \circ X_{\sigma^s} = u\left(X_{\sigma^K}\right)$ and $u\left(X_{\sigma^r}\right) > u\left(X_{\sigma^t}\right) \forall r > K\Big]$.

Proof. The proof can be made in a similar form to the deterministic case, see Araujo et al. [5]. However this result can be seen as a corollary to the following proposition, in which we will characterize the subdifferential of each agent depending the consumption in each path of the tree. □

In general is not easy to characterize every element of the subdifferential of stochastic utility functions as the one that we work, however using the known results of the deterministic case and some rule special properties of the subdifferential of non differential functions we have

Proposition 18. *Let $X \ggg 0$, $\underline{X} : N^\infty \to \mathbb{R}$ such that $\underline{X}_\sigma = \inf_t X_{\sigma^t}$. For U given by 3.1 with $u \in C^1(0,\infty)$ and $\pi \in L^{\infty *}\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\}\times N^t, \tilde{\mathcal{N}}, P\right)$. We have that π can be written as*

$$\pi Y = \sum_{s^t}\int_{[\sigma:\sigma^t=s^t]} u'\circ X_{\sigma^t}\zeta_t d\mathbb{P}(\sigma)Y_{s^t} + \int_{A_1}\beta(\sigma)u'\circ\underline{X}_\sigma \mathrm{LIM}_\sigma(Y_\sigma)d\mathbb{P}(\sigma)$$
$$+\sum_{t_1\le\ldots\le t_K}\int_{A^K_{2,t_1,\ldots,t_K}}\beta(\sigma)u'\circ\underline{X}_\sigma\left(\sum_{k=1}^K \alpha_2(\sigma,k)Y_{\sigma^{t_k}}\right)d\mathbb{P}(\sigma)$$
$$+\sum_{t_1\le t_2\le\ldots}\int_{A_{3,t_1,t_2,\ldots}}\beta(\sigma)\left(u'\circ\underline{X}_\sigma\right)\left(\sum_{k=1}^\infty \alpha_3(\sigma,k)Y_{\sigma^{t_k}}+\alpha_3(\sigma,\infty)\mathrm{LIM}_\sigma(Y_\sigma)\beta(\sigma)\right)d\mathbb{P}(\sigma),$$

where

- $A_1 := [\sigma : \inf_t u\circ X_{\sigma^t} < u\left(X_{\sigma^t}\right)\forall t\in\mathbb{N}]$,
- $A^K_{2,t_1,\ldots,t_k} := \Big[\sigma : \inf_t u\circ X_{\sigma^t} = u\left(X_{\sigma^{t_k}}\right)\forall k = 1,\ldots,K \wedge u\left(X_{\sigma^t}\right) > u\left(X_{\sigma^{t_k}}\right)\forall t\neq t_1,\ldots,t_K\Big]$, $A_2 := \bigcup_K\left(\bigcup_{t_1\le\ldots\le t_K} A^K_{2,t_1,\ldots,t_k}\right)$,
- $A_{3,t_1,t_2\ldots} := \Big[\sigma : \inf_t u\circ X_{\sigma^t} = u\left(X_{\sigma^{t_k}}\right)\forall k\ge 1 \wedge u\left(X_{\sigma^t}\right) > u\left(X_{\sigma^{t_k}}\right)\forall t\neq t_1,t_2,\ldots\Big]$, $A_3 := \bigcup_{t_1\le t_2\le\ldots} A_{3,t_1,t_2\ldots}$,
- $\alpha_2(\sigma,k)\ge 0\ \forall k=1,\ldots,K$ *and* $\sum_{k=1}^K\alpha_2(\sigma,k) = 1,\ \forall\sigma\in\{1\}\times N^\infty$,
- $\alpha_2(\sigma,k),\alpha_2(\sigma,\infty)\ \ge\ 0\ \forall k\in\mathbb{N}$ *and* $\sum_{k=1}^\infty\alpha_3(\sigma,k)+\alpha_3(\sigma,\infty) = 1,\ \forall\sigma\in\{1\}\times N^\infty$ *and*
- $\{\mathrm{LIM}_\sigma\}_{\sigma\in A_1\cup A_3}$ *is a collection of generalized limits such that it is a measurable function in* $(\{1\}\times N^\infty, \mathcal{N})$,

if and only if $\pi\in\partial U(X)$.

Given the previous proposition, we know that the existence of pure charges in the AD prices are related to the fact that the infimum is attained in a finite number of states or not.

Now that studied the relationship between the AD-price and the optimal allocation, we can extend this analysis to the existence of bubbles in the stochastic intertemporal economy and analyze also the possibility of crashing.

3.3 Characterization of Efficient Bubbles and the possibility of Crashing

The idea of this section is to establish a relationship between the existence of positive pure charges and the existence of efficient bubbles. In a similar form as it was defined by Santos et al. [36] let us define a bubble in any node s^t as a difference between the price of the asset in this state and, what is called, the *fundamental value* of the asset

$$q_{s^t} - \frac{1}{a_{s^t}} \sum_{r=t+1}^{\infty} \sum_{s^{r,-(r-t)}=s^t} a_{s^r} R_{s^r} \geq 0,$$

where $\{a_{s^t}\}_{s^t}$ are the deflators related to the lack of arbitrage in the economy.

Since we are implementing efficient allocations in a sequential economy with complete markets, the ℓ^1 component of the AD price satisfies the existence of a constant $\alpha > 0$ such that $\mu_{s^t} = \alpha a_{s^t}$ for any s^t. Therefore without loss of generality can be assumed that the deflator are $\{\mu_{s^t}\}_{s^t}$.

Since we implement efficient allocations, can be proved, using the FOC, that

$$\mu_{s^t} q_{s^t} - \sum_{r=t+1}^{\infty} \sum_{s^{r,-(r-t)}=s^t} \mu_{s^r} R_{s^r} = \lim_{r\to\infty} \sum_{s^{r,-(r-t)}=s^t} \mu_{s^r} q_{s^r} \qquad (3.2)$$

then the existence of efficient bubbles in a node s^t depends on the asymptotic behavior of the subtree that is generated by this node. Therefore if the bubble crashes at some node s^t, the bubble can not reappear in the economy in any sucessor of s^t, in other words:

Proposition 19. *For any efficient allocation implemented in a sequential economy with Long-Lived assets. If there exists one node s^t and one asset j in which there is no bubble for the asset price $q^j_{s^t}$ then for any state s^r sucessor of s^t there is no bubble for the asset price $q^j_{s^r}$.*

Then, after the crashing of a bubble of any asset at the node s^t, there is no possible reappearance of the bubble in the subtree generated by s^t.

However it does not mean that there is no bubble in the economy in other subtrees that are not generated by s^t.

Now let us analyze conditions in which is possible the existence of bubbles, in order to do so, we will establish a relationship between bubbles and the existence of pure charges in the subgradient of the utility function.

Since the pure charges are related to the agents' concerns of the worst events, the existence of bubbles in this environment looks like to be strongly related with the pure charges in the subgradient.

As it was the section 2.5.3, in order to implement efficient allocations in a sequential economy we have that

$$\tilde{\nu}^i \left(X^i - \omega^i\right) - \nu^i \left(X^i - \omega^i\right) = \sum_j \left(z^i_{j,0} \left(\mu_1 q_{j,1} - \sum_{s^t} \mu_{s^t} R_{j,s^t} - \nu\left(R_j\right) \right) \right), \tag{3.3}$$

where $\tilde{\nu}$ is the capacity that takes the highest value on the direction of the net trade. Therefore $\mu_1 q_{j,1} = \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ if $x^i_t - \omega^i_t$ converges for any i and $R \ggg 0$, and $\mu_1 q_{j,1} > \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ for at least one asset j and $\mu_1 q_{j,1} \geq \sum_{s^t} \mu_{s^t} R_{j,s^t} + \nu(R_j)$ for every j if $x^i_t - \omega^i_t$ doesn't converge for some i and $R \geq 0$.

If $\partial U(x) \subseteq \ell^1$, can be easily seen that there is no bubble for any asset. If $\partial U(x) \nsubseteq \ell^1$, we guarantee that there exists a bubble for at least one asset, in fact, depending on the prices that were used to implement the allocation, we can have a positive bubble in every asset.

However for nodes different than 1 the analysis can not be done for each asset, we will be able to analyze the existence of bubbles for the hole set of assets only, this means that even in presence of a bubble in a node s^t with $t \neq 1$ we could have no bubble for some assets even in presence of a bubble when is considered the entire set of assets.

For each node s^t, we know that the optimal allocation that we have is efficient even in the subtree generated in s^t, s^t+[1]. Therefore there is $\left\{W^{i,s^t}\right\}_i$ new "endowment allocation" such that $\sum_i W^{i,s^t}_{s^k} = \sum_i W^i_{s^k}$ $\forall s^k$, $\pi W^{i,s^t} = \pi W^i$ where π is the AD price, $W^{i,s^t}_{\sigma^r} = X^i_{\sigma^r}$ for all σ and r such that $r \leq t$ or σ^r is not in the subtree generated in s^t [2], and for the rest for the endowment distributions in the subtree generated in s^t, $W^{i,s^t}_{s^{T+k}} = W^i_{s^{T+k}}$ $\forall k \geq 0$ where T is big enough.

If we let the endowments to be negative in some states of the economy, we can analyze the economy defined by the "endowment allocation" $\left\{W^{i,s^t}\right\}_i$ and its relationship with the original AD economy.

[1]This will be the notation for subtrees generated by the node s^t.

[2]This includes $\left\{s^{t,-(j)}\right\}_{j=1}^{t-1}$ all the predecessors of s^t.

To establish this relationship, let us restrict the AD economy such that the agents maximize their consumption in the subtree generated by s^t. To do that, we fix the consumption in each node that is not in the subtree generated in s^t and then, we analyze the FOC for this restricted AD economy, since the wealth of every agent and the efficient allocation is the same as in the initial AD equilibrium, we have that the initial equilibrium price is in fact an equilibrium price for this restricted economy.

And now let us analyze the stochastic sequential economy. To implement sequentially this allocation, we will maintain the same assets' prices since are given by the Euler equation for each asset. Since we have the same equilibrium price, we will have that the assets' prices do not change in the restricted economies. Now let us define the new endowment distribution $\left\{\omega^{i,s^t}\right\}_i$ as $\omega^{i,s^t} = W^{i,s^t} - \sum_j R_j z^i_{j,s^t}$.

Using the same optimality conditions that were exposed in section 2.5.3 we know that using the pure charge, $\nu_i^{s^t}$, the one that takes the highest value on the net trade $\left\{X^i - \omega^{i,s^t}\right\}$, we can implement the efficient allocation if we impose that

$$\nu_i^{s^t}\left(X^i(z) - \omega^{i,s^t}\right) \leq \lim_r \left(\sum_j \sum_{s^{r,-(r-t)}=s^t} \mu_{s^r} q_{j,s^r} z_{j,s^r}\right)$$

and with equality for X. Then, in order to implement sequentially this allocation, we have that

$$\lim_r \sum_{s^{r,-(r-t)}=s^t} \sum_j \mu_{s^r} q_{j,s^r} = \sum_j z^i_{j,s^t} \nu\left(R_{j,s^t+}\right) + \nu_i^{s^t}\left(X^i - \omega^{i,s^t}\right) - \nu\left(X^i - \omega^{i,s^t}\right) \tag{3.4}$$

and since we have that for each node s^t

$$\mu_{s^t} q_{j,s^t} = \sum_{r>t} \sum_{s^{r,-(r-t)}=s^t} \mu_{s^r} R^j_{s^r} + \lim_r \sum_{s^{r,-(r-t)}=s^t} \sum_j \mu_{s^r} q_{j,s^r},$$

the existence of bubble in the economy at the state s^t is characterized by 3.4. Therefore the existence of efficient bubbles in this type of economies are related to the existence of positive pure charges in the subgradient of the agents.

In 3.4, seems that the right part of the equality could be negative, something that does not make any sense[3], however this term is at least nonnegative due to the relationship between $\nu_i^{s^t}$ and the rest of the pure charges that belongs to the subdifferential of U^i when they are evaluated in the direction of

[3] Since it would mean a negative bubble in the economy.

the new "net trade", $X^i - \omega^{i,s^t}$. Additionally, if we sum 3.4 over the agents we have that

$$\begin{aligned} I \lim_r \sum_{s^{r,-(r-t)}=s^t} \sum_j \mu_{s^r} q_{j,s^r} &= \sum_i \sum_j z^i_{j,s^t} \nu\left(R_{j,s^t+}\right) + \sum_i \nu_i^{s^t}\left(X^i - \omega^{i,s^t}\right) \\ &= \sum_i \sum_j z^i_{j,0} \nu\left(R_{j,s^t+}\right) + \sum_i \nu_i^{s^t}\left(X^i - \omega^{i,s^t}\right) \\ &\geq 0, \end{aligned}$$

due to $\nu_i^{s^t}\left(X^i - \omega^{i,s^t}\right) - \nu\left(X^i - \omega^{i,s^t}\right) \geq 0$ for each i. And as a consequence, we have that:

Proposition 20. *For any efficient allocation with $x \ggg 0$ and any state s^t, if there is one agent such that all paths that contain s^t attain the infimum consumption in a finite number of dates then there is no bubble for any asset at the node s^t.*

This implies that to have a crashing of a bubble at a node s^t is enough to guarantee that the infimum is attained in a finite number of of dates for all the paths that contain s^t and that there is a set of nodes with a positive probability in which the pure charge is positive.

Now let us expose some conditions in which we can guarantee a positive bubble at a node s^t, conditions related to the fact that the infimum is not attained in finite time.

Proposition 21. *For any efficient allocation with $x \ggg 0$ and any state s^t, if there exists one agent such that there is a subset of paths that each of them contain s^t with positive probability (given by $\mathbb{P}$) in which the infimum consumption in the path is a cluster point never attained. Then if there exists at least one asset in positive net supply such that $\nu\left(R^{j,s^t+}\right) > 0$, there is a bubble at the node s^t.*

Similar to the deterministic case, the desire of the WARY agent of increasing their consumption in the worst events in the subtree that contains s^t, produces a pure charge in the AD price that imply existence of bubbles for the set of assets at s^t.

3.4 Analysis of Volatility in presence of Efficient Bubbles and Crashing

If there are bubbles in some subtrees and, at the same time, there is also another one in which there is no bubble, it means that there is one node in which the bubble crashes, and analyzing the successors of that node, is possible to notice the implication of crashing of bubbles in volatility[4].

Under conditions described in proposition 20 and proposition 21 is possible to have bubbles in the economy and also some subtrees in which this bubbles would disappear. Since these bubbles increase the price above the fundamental value of the assets, the crashing of them, naturally, will increase the volatility of the price when the price variations is analyzed considering all successors s^t of s^r, a node before the bubble crashes, for a fixed date $t \geq r$[5].

After the bubble crashes at the node s^t, the fundamental value of the asset and the market price are equal, this means that

$$\max_{\left\{s^r : s^{r,-(r-t)} = s^t\right\}} \mu_{s^r} q_{j,s^r} = \max_{\left\{s^r : s^{r,-(r-t)} = s^t\right\}} \left\{ \sum_{k>r} \sum_{s^{k-(k-r)} = s^r} \mu_{s^k} R^j_{s^k} \right\} \to 0 \text{ when } r \to \infty.$$

However if there is a path σ such that there is always a bubble for the asset j we have that

$$\mu_{\sigma^t} q_{j,\sigma^t} = \sum_{r>t} \sum_{s^{r,-(r-t)} = \sigma^t} \mu_{s^r} R^j_{s^r} + \lim_r \sum_{s^{r,-(r-t)} = \sigma^t} \sum_j \mu_{s^r} q_{j,s^r}, \tag{3.5}$$

and as we know there exists a relationship between the existence of bubble and the pure charges in the subgradient of the agents. Therefore, by proposition 18, we have:

Proposition 22. *For any path σ, the component in 3.5 that defines the bubble of any asset j at the node σ^t tends to zero when $t \to \infty$.*

This means in a intuitive form that the bubble is being distributed among all nodes where the pure charge is positive, reducing its weight in the price of the asset.

Nevertheless it does not mean that the price would be bounded, in fact we have:

Remark 19. For any path σ such that $X_{\sigma^t} > \underline{X}(\sigma)$ for all $t \in \mathbb{N}$ we have that:

$$\lim_{r \to \infty} \frac{\sum_{s^{r,-(r-t)} = \sigma^t} \sum_j \mu_{s^r} q_{j,s^r}}{\mathbb{P}\left([\tilde{\sigma} : \tilde{\sigma}_{s^t} = \sigma^{s^t}]\right)} \to \alpha > 0.$$

And since there is a strong relationship between $\mathbb{P}\left(\left[\tilde{\sigma} : \tilde{\sigma}_{s^t} = \sigma^{s^t}\right]\right)$ and μ_{s^t}, we will have:

[4] We understand by volatility as variation of the assets prices in possible future states.

[5] Mainly in the dates after the bubble crashes, since the will be present in the variation of the prices given by the crashing.

Corollary 2. *Under the conditions exposed in remark 19, the bubble will tend to infinity when $t \to \infty$.*

This means that when t is large, there are big variations of prices between states, due to the existence of bubbles in some nodes s^t but not in everyone.

Therefore when there is a crashing of a bubble in the node s^t, there will be an increasing of volatility when we compare successors of s^{t-} that do not have bubble and successors that do have. This behavior is compatible with the idea that the existence of bubble in some subtrees will increase volatility in the economy.

As can be noticed through this chapter, there is a variety of possibilities for bubbles that can occur in the same example. The following example helps is a clear example in which there is infinite number of crashing, however bubbles occur with positive probability in the tree.

Example 5. Consider an economy with two agents with utility index given by $u^i(x) := \ln(x)$, $\xi^i_t := {}^1/_{2^t}$, $\beta^i = \beta > 0$ for each $i = 1, 2$, $W : \cup_{t\in\mathbb{N}}\{1\} \times N^t \to \mathbb{R}_+$ given by

$$W_{s^t} = \begin{cases} 8 + {}^1/_{2^{t-4}} & \text{if } s^{t,-(t-2)} = (1,2) \text{ and } \exists k \in \mathbb{N} \text{ such that } 2k+1 \leq t \text{ and} \\ & s^{t,(t-2k+1)-}_{2k+1} = 2, \\ 9 & \text{if } s^t = (1,2), (1,2,1,1) \text{ or } s^{t-} = (1,2,2), \\ 10 & \text{if } t = 1 \text{ or } s^{t-} = (1,2), \\ 11 & \text{otherwise if } t \text{ is even}, \\ 12 & \text{otherwise if } t \text{ is odd}, \end{cases}$$

$W^1_{s^t} = W_{s^t} + A_t$ and $W^2_{s^t} = W_{s^t} - A_t$ where A_t is 1 if t is even and $-{}^1/_2$ if t is odd. The probability of each node s^t to occur is symmetric with all the nodes at the t, it means that $\mathbb{P}[\sigma : \sigma^t = s^t] = {}^1/_{2^{t-1}}$ for all $s^t \in \{1\} \times N^{t-1}$. Let us find an AD equilibrium. Given $a : \{1,2\} \to \mathbb{R}_{++}$, the consumption plan $x^i = a(i)\left(W^1 + W^2\right) = 2a\left(W\right)$ is optimal under the budget constraint $\pi x \leq \pi W^i$ when the prices are given by

$$\pi x = \int \frac{x_{\sigma^t}}{2^{t-1}W_{\sigma^t}} d\mathbb{P}(\sigma) + \beta\left({}^{x_{(1)}}/_{20} + {}^{x_{(1,2)}}/_{72} + \sum_{s\geq 2} \frac{x_{\tilde{\sigma}^{2s+1}}}{2^{2s+1}\left(8 + {}^1/_{2^{2t-3}}\right)} + \int_{\bigcup_{t\in\mathbb{N}}[\sigma:\sigma^{2t+1}=(1,2,1,1,\dots,1,2)]} \mathrm{LIM}_\sigma\left(x_\sigma\right) d\mathbb{P}(\sigma)\right)$$

where $\tilde{\sigma}^{2t+1} = (1,2,1,1,\dots,1)$ and $a(i) = {}^{\pi\left(W^i\right)}/_{\pi(W)}$. One possible generalized limit is the Banach limit B, that is an special generalized limit such that $B\left((x_t)_{t\in\mathbb{N}}\right) = \lim_{n\to\infty}\sum_{t=1}^n x_t$ when this limit exists. Under this condition (π, x^1, x^2) is an AD equilibrium.

For the sequential implementation with two Long-Lived assets, one that pays 1 in each state 1 and 0 in each state 2, and the other one that pays 0 in each state 1 and 1 in each state 2, we can fix the prices $q_1 \in \mathbb{R}^2_{++}$ and z^i_0 such that (x^i, z^i) are optimal for the sequential economy as in section 2.5.2, with an additional asset as fiat money if it is necessary.

As a consequence, the sequential prices $(q^1_{s^t}, q^2_{s^t})_{s^t}$ satisfy the conditions exposed before, which implies the relationship between the optimal paths and the existence of bubbles in the economy.

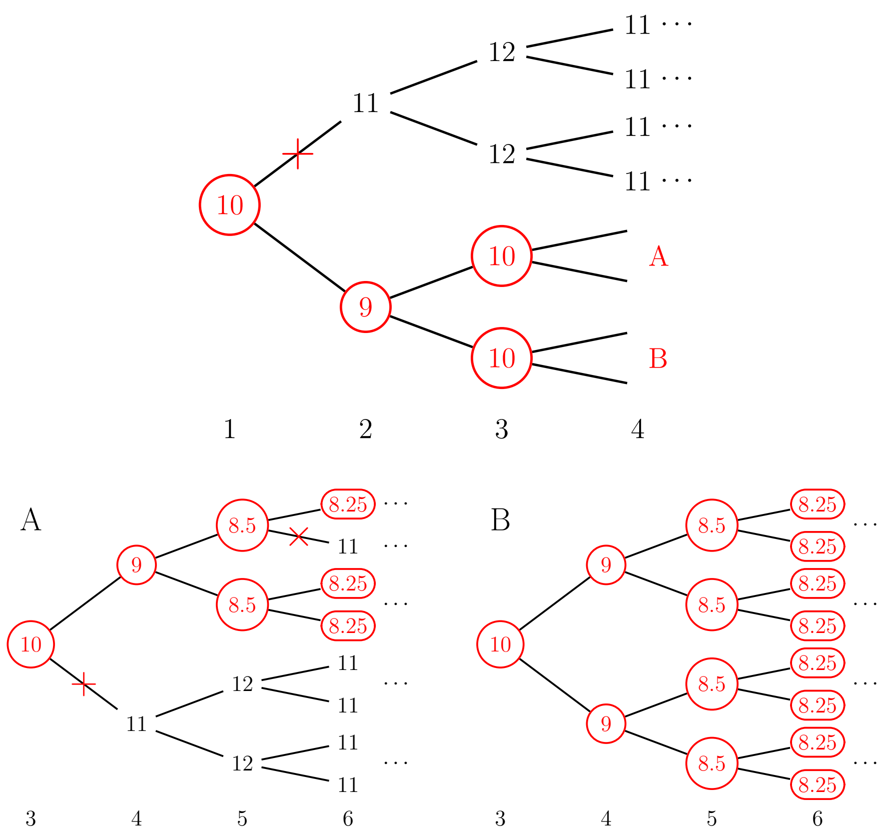

Figure 3.1: Distribution of W, existence of bubbles (with red circles) and crashing (with red $\times$'s)

Therefore, using the relationship between W and the existece of bubble, we have that in $s^2 = (1,1)$, $s^4 = (1,2,1,2)$, $s^6 = (1,2,1,1,1,2)$, $\ldots$, $s^{2t} = (1,2,1,\ldots,1,2)$, $\ldots$, the bubble crashes. And in spite of the infinite crashing of the bubbles, the proportion of paths with positive bubble at any state is positive, in fact, $\mathbb{P}[\sigma : \inf_s X_{\sigma^s} < X_{\sigma^t} \, \forall t \in \mathbb{N}] = \mathbb{P}[\sigma : \inf_s W_{\sigma^s} < W_{\sigma^t} \, \forall t \in \mathbb{N}] = \mathbb{P}[\sigma :$ the bubble in the path σ does not crash$] = {}^1\!/_3$.

This example helps us to notice that in stochastic economies, the existence of bubbles in considerably large set of paths is consistent with infinite number of crashing in the economy. We can say also that the existence of bubbles, as in the deterministic case, is related to the lack of impatience of the agents.

3.5 Concluding Remarks

Due to the lack of crashing of bubbles in deterministic intertemporal economies, is natural to consider economies that allow a large variety of behaviors in term of bubbles. One of these possibilities is to consider stochastic intertemporal economies with lack of impatience. However can be easily observed that depending on the form that agents are worried about distance losses affects the pure charges in the price of the AD economy and the existence of bubbles in the sequential economy.

When the agents are worried about distant losses in each path, is possible to establish a relationship between the existence of positive pure charges in the AD price and the existence of bubbles as in the deterministic case.

To do so, we gave a utility function that is compatible with the model defined by Schmeidler (see [37]) in which we gave a form of modeling impatience in a stochastic framework which is also compatible with the existence of AD equilibrium with infinite many commodities of Bewley (see [9]). We noticed also that with this type of utility function is possible to prove the existence of positive pure charges in the AD price and therefore, the existence of efficient bubbles in the intertemporal economy at the first date. And then, we proved the strong relationship between the existence of bubbles at each node of the tree and the existence of pure charges in the subtree generated by this node, and as a consequence, is possible to establish condition for the crashing of bubbles in some subtrees knowing the behavior of the efficient allocation instead of necessarily observing the equilibrium of the intertemporal economy.

To conclude our analysis of bubbles and crashing, we proved that the existence of crashing of bubbles will increase the variation of the intertemporal prices, which will imply an increment of the volatility.

Finally we gave an example that suggests that the stochastic economies admit behavior considerably more diverse than the one presented in deterministic economies and proves also that the existence of infinite number of crashing is not only possible, is completely compatible with the existence of bubbles in infinite number of subtrees.

APPENDIX A

Basic Concepts and Notations

A.1 AD economies with a finite number of agents without production

Let us suppose that there is a finite number of agents $i \in \{1, \dots, \mathcal{I}\}$ and a finite number of good with the set of consumption given by $\mathcal{X}^i \in \mathbb{R}^N$, utility function $U^i : \mathcal{X}^i \to \mathbb{R}$ and endowment allocation $\omega^i \in \mathcal{X}^i$.

In Arrow-Debreu economies, the budget constraint is given by $px \leq p\omega^i$, then the budget set is defined by $B(p, \omega^i) := \{x \in \mathcal{X}^i : px \leq p\omega^i\}$ for each agent $i = 1, \dots, \mathcal{I}$.

And an equilibrium for this economy is $\big(p, \big(x^1, \dots, x^{\mathcal{I}}\big)\big)$ such that

1. $x^i \in \arg\max_{x \in B(p,\omega^i)} U^i(x)$ for all $i = 1, \dots, \mathcal{I}$.

2. $\sum_{i=1}^{\mathcal{I}} x_n^i = \sum_{i=1}^{\mathcal{I}} \omega_n^i$ for all $n = 1, \dots, N$.

To see sufficient condition for the existence of equilibrium for convex economies with a finite number of agents with a finite number of goods see Arrow and Debreu [6].

A.2 Choquet Integral

Consider S a finite set with the discrete $\sigma-algebra$ $\mathcal{S}$, a *capacity* is a function $\nu : \mathcal{S} \to \mathbb{R}_+$ such that:

1. $\nu(\emptyset) = 0$, $\nu(S) = 1$
2. if $A \subseteq B$, $\nu(A) \leq \nu(B)$.

And for every A, $B \subseteq S$:

- if $\nu(A) + \nu(B) \geq \nu(A \cup B) + \nu(A \cap B)$, ν is a *convex* capacity called also *supermodular*,
- if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$, ν is a *concave* capacity called also *submodular*, and
- if $\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B)$, ν is an additive function, which implies that ν is a probability measure.

We can generalize the concept of integral to capacities as:

$$(C) \int_S U d\nu = \int_{-\infty}^{0} \left(\nu^i \left[U \circ x \geq t\right] - 1\right) dt + \int_0^{\infty} \nu^i \left[U \circ x \geq t\right] dt.$$

it is called *Choquet Integral*. Notice that if ν is additive, $(C) \int_S U d\nu = \int_S U d\nu$.

This integral has been extensively used in decision theory since it was the first model to explain different type of attitudes toward ambiguity (see Schmeidler[37]).

Under some assumptions as in Gilboa and Schmeidler [26], the Coquet integral can be transformed into

$$(C) \int_S U d\nu = \min_{\substack{\pi \geq \nu \\ \pi \text{ prob. measure}}} \int_S U d\nu$$

in case of a convex capacity, and into

$$(C) \int_S U d\nu = \max_{\substack{\pi \leq \nu \\ \pi \text{ prob. measure}}} \int_S U d\nu$$

in case of a concave capacity.

A.3 The Space ℓ^∞

For $x \in \ell^\infty$, $x \geq 0$ if $x_t \geq 0$ for all $t \in \mathbb{N}$, $x > 0$ if $x \geq 0$ and $x \neq 0$, $x \gg 0$ if $x_t > 0$ for all $t \in \mathbb{N}$, and $x \ggg 0$ if exists $a > 0$ such that $x_t \geq a$ for all $t \in \mathbb{N}$. The space ℓ^∞ is the Banach space ba of real bounded sequences equipped with the norm defined by $\|x\| = \sup_t |x_t|$. Its dual is the space ba of bounded finitely additive set functions on $2^{\mathbb{N}}$, also known as charges. Now, ba contains strictly ℓ^1, the Banach space of absolutely convergent real sequences equipped with the norm defined by $\|x\|_1 = \sum_{t=1}^{\infty} |x_t|$, since we can associate each $y \in \ell^1$ with some μ in the subspace ca of countably additive set functions, by setting $\mu(\{t\}) = y_t$.

A charge $\nu \geq 0$ is a *pure charge* when $[\lambda \in ca_+,\ \nu \geq \lambda \Rightarrow \lambda \equiv 0]$. Denote by pch_+ the set of non-negative pure charges on $(\mathbb{N}, 2^{\mathbb{N}})$. By the Yosida-Hewitt Theorem, any $\mu \in ba_+$ can be written in the form $\mu = \pi + \nu$ where $\mu \in ca_+$ and $\nu \in pch_+$. and this decomposition is unique.

Remark 20. If $\nu > 0$ be a pure charge such that $\nu(\mathbb{1}) = 1$, then, $\nu(x) \in [\liminf x, \limsup x]$, for any $x \in \ell^\infty$. In other words, ν is a generalized limit. For a supergradient[1] of a concave function $U : \ell^\infty_+ \to I\!R$ at x, which is an element in the dual space, we can actually say more about the norm of its pure charge component: $\|\nu\|_{ba} \equiv \sup\{\nu(x) : \|x\| \leq 1\} = \nu(\mathbb{1})$ belongs to $[\lim_n \delta^+ U(x; \mathbb{1}^n), \lim_n \delta^- U(x; \mathbb{1}^n)]$. The set of all supergradients of U at x is called the superdifferential of U at x and is denoted by $\partial U(x)$.

Let us see an additional property for pure charge components of a supergradient.

Let us analyze the distortion coefficient α in 2.12.

Lemma 4. Let $T = \mu + \nu \in \partial U(x)$ such that $(\mu, \nu) \in$ ca $\times$ pch. There are a generalized limit LIM *and a positive constant $\alpha \in [\lim_n \delta^+ U(x; \mathbb{1}_{E_n}), \lim_n \delta^- U(x; \mathbb{1}_{E_n})]$ such that $\nu(x) = \alpha\, \mathrm{LIM}(x) \ \ \forall\, x \in \ell^\infty$.*

Proof. We just need to show that α belongs to the mentioned interval. Given $n \in \mathbb{N}$, it is true that $\delta^+ U(x; \mathbb{1}_{E_n}) \leq T(\mathbb{1}_{E_n}) \leq \delta^- U(x; \mathbb{1}_{E_n})$. Moreover, $T(\mathbb{1}_{E_n}) = \sum_{t>n} \mu_t + \nu(\mathbb{1}_{E_n})$. Since, $\forall n$, $\nu(\mathbb{1}_{E_n}) = \nu(\mathbb{1}) = \alpha$ and $\lim_n \sum_{t>n} \mu_t = 0$, we get $\lim_n \delta^+ U(x; \mathbb{1}_{E_n}) \leq \alpha \leq \lim_n \delta^- U(x; \mathbb{1}_{E_n})$. □

Notice that the constant α in the statement of this lemma is actually the norm of the pure charge: $\alpha = \|\nu\|_{ba} = \sup\{\nu(x) : \|x\| \leq 1\} = \nu(\mathbb{1})$.

[1] Any T such that $U(x+h) - U(x) \leq Th$, for any $h \in \ell^\infty$.

A.3.1 General Characterization of Supergradients for the Utility Function (2.3)

If T is a supporting price of U^i at x^i, then $T(a) = \sum_{t+1}^{\infty} u'(x_t^i)(\zeta_t^i + \gamma_t \beta^i)a_t + \sigma\beta^i u'(\underline{x^i})\text{LIM}^T(a)$, where (i) $\gamma_t \geq 0$, (ii) $\gamma_t = 0$, if $x_t > \underline{x}$, (iii) $\sigma \geq 0$ is zero when $\underline{x^i}$ is not a cluster point of the sequence x^i and (iv) $\sum_{t=1}^{\infty} \gamma_t + \sigma = 1$. For a proof see Araujo, Novinski and Pascoa [5].

A.3.2 Rational Bubbles

In the general equilibrium framework, the existence of state prices is related to non existence of arbitrage in the economy, it means that there is no possibility to obtain positive gains in future states of you do not invest a positive amount of wealth.

Since these state prices establish a clear relationship among states and dates, are commonly used as the weight at each state to compute the *fundamental value* of the assets. In the second part of the thesis, we see that the state prices are, in fact, the marginal rate of substitution at each date and at each state (in stochastic economies).

Therefore, if the price of one asset is larger that its fundamental value, it will be a bubble for it in the node that we are analyzing.

In finite economies, it is impossible to have rational bubble in presence of complete markets. In incomplete markets, there are more that one possible state price, then it could be possible to have bubbles for one state price, however there exists one possible state price in which tere is no bubble.

In infinite economies, it is possible to have bubbles due to inefficiency of the economy. To have efficient bubbles, we must have agents that are not patients for distant losses as in Araujo, Novinski and Pascoa [5].

APPENDIX B

Appendix of Part I

B.1 Proofs of existence of equilibrium

B.1.1 Proof of Proposition 4

Proof. For any $p_1 \in (0,1)$, since the agent 2 has a strictly convex and strictly increasing utility index, the optimal solution must be on the frontier of the budget set therefore the possible optimal solution are $\left(\frac{p\omega^2}{p_1}, 0\right)$ and $\left(0, \frac{p\omega^2}{1-p_1}\right)$, see lemma 1.

From now on we will analyze the case in which the agent 2 specializes in the first state. Consider $\overline{p}_1 \in (0,1)$ and $\overline{p} = (\overline{p}_1, 1-\overline{p}_1)$[1] such that

$$\pi u^2\left(\frac{\overline{p}\omega^2}{\overline{p}_1}\right) = (1-\pi)\, u^2\left(\frac{\overline{p}\omega^2}{1-\overline{p}_1}\right) \tag{B.1}$$

and $\underline{\omega}_1^1 > 0$ given by 1.7 where $p_1 = \overline{p}_1$. Since $\omega(p) = {u^{1}}'^{(-1)}\Big(\Big(\frac{p}{1-p}\Big)\big(\frac{1-\pi}{\pi}\big){u^{1}}'(\omega_2)\Big) + \frac{1-p}{p}\omega_2^2$ is strictly decreasing for $p \in (0,1)$, the equation 1.7 has a unique p_1 for each ω_1^1.

Now let us prove that for each $\omega_1^1 \geq \underline{\omega}_1^1$ there is an equilibrium when the price $(p_1, 1-p_1)$ is the solution of 1.7.

If $\omega_1^1 \geq \underline{\omega}_1^1$, then $p_1 \leq \overline{p}_1$ so $\pi u^2\left(\frac{p\omega^2}{p_1}\right) \geq (1-\pi)\, u^2\left(\frac{p\omega^2}{1-p_1}\right)$.

[1]The uniqueness of $\overline{p}_1$ is a consequence of u^2 being strictly increasing, $u^2(0) = 0$, $u^2(x) \to \infty$ when $x \to \infty$, and that the left part of B.1 is decreasing and right part is increasing with $\overline{p}_1$.

This implies that the optimal consumption of the agent 2 is $\left(\frac{p\omega^2}{p_1},0\right)$ given p.

Now, for the first agent we have

$$\pi u^{1\prime}\left(x_1^1\right) = p_1\mu,$$
$$\left(1-\pi\right) u^{1\prime}\left(x_2^1\right) = \left(1-p_1\right)\mu,$$

and

$$p_1 x_1^1 + \left(1-p_1\right) x_2^1 = p_1\omega_1^1 + \left(1-p_1\right)\omega_2^1, \tag{B.2}$$

for $\mu > 0$. If we define $\left(x_1^1, x_2^1\right) = \left(u^{1\prime(-1)}\left(\left(\frac{p}{1-p}\right)\left(\frac{1-\pi}{\pi}\right)u^{1\prime}\left(\omega_2\right)\right), \omega_2\right)$, is not hard to see that for $\mu = \frac{1-\pi}{1-p}u^{1\prime}(\omega_2)$, (x_1^1, x_2^1) is a solution for the FOC exposed before.

And since we have $x_2^1+x_2^2 = \omega_2$ and $x_1^1+x_1^2 = u^{1\prime(-1)}\left(\left(\frac{p}{1-p}\right)\left(\frac{1-\pi}{\pi}\right)u^{1\prime}\left(\omega_2\right)\right)+\omega_1^2 + \frac{1-p}{p}\omega_2^2 = \omega_1^1 + \omega_1^2$, which concludes the proof of existence of equilibrium when $x_1^2 \neq 0$. The proof is analogous when $x_2^2 \neq 0$.

Suppose that there is an equilibrium $(\hat{p},\, \hat{x})$ for the economy in which $\hat{x}_1^2 \neq 0$, then $\hat{x}_2^1 = \omega_2$. Since we know that $\hat{p}$ satisfies

$$\pi u^2\left(\frac{\hat{p}\omega^2}{\hat{p}_1}\right) - (1-\pi)u^2\left(\frac{\hat{p}\omega^2}{1-\hat{p}_1}\right) \geq 0,$$

and using that the function f defined by

$$f(p) := \pi u^2\left(\frac{\left(p, 1-p\right)\omega^2}{p}\right) - (1-\pi)u^2\left(\frac{\left(p, 1-p\right)\omega^2}{1-p}\right)$$

is strictly decreasing and that $\overline{p}_1$ is the greatest $p \in (0,1)$ such that the inequality above is satisfied, we have that $\hat{p} \leq \overline{p}$. And as a consequence of the FOC of the agent 1, the equation 1.7 is implied for $p_1 = \hat{p}_1$, and since $\hat{p}_1 \leq \overline{p}_1$, $\omega_1^1 \geq \underline{\omega}_1^1$. Which concludes the proof. □

B.1.2 Proof of Remark 3

Proof. It is enough to show that

$$\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)u^{1\prime}\left(\frac{p_1\omega_1^1 + \left(1-p_1\right)\omega_2^1}{p_1}\right) < u^{1\prime}\left(0\right) \tag{B.3}$$

is satisfied for any $p_1 \leq \overline{p}_1$ where ω_1^1 satisfies the equation 1.9 to guarantee that the agent 1 has positive consumption in each state then can be used the previous proof to the case without Inada. The left part of the inequality can be written as:

$$\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)u^{1'}\left(\frac{p\omega^1}{p_1}\right)$$
$$=\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)u^{1'}\left(u^{1'(-1)}\left(\left(\left(\frac{p_1}{1-p_1}\right)\left(\frac{1-\pi}{\pi}\right)u^{1'}(\omega_2)\right)\wedge u^{1'}(0)\right)+\frac{1-p_1}{p_1}\omega_2\right).$$

Using the fact that u_1' is strictly decreasing and $\omega_2 > 0$ we have:

$$\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)u^{1'}\left(\frac{p\omega^1}{p_1}\right)$$
$$<\left(\frac{1-p_1}{p_1}\right)\left(\frac{\pi}{1-\pi}\right)\left(\left(\left(\frac{p_1}{1-p_1}\right)\left(\frac{1-\pi}{\pi}\right)u^{1'}(\omega_2)\right)\wedge u^{1'}(0)\right)$$
$$< u^{1'}(0).$$

Which concludes the proof.

□

B.1.3 Proof of Theorem 1

Proof. Without loss of generality we can analyze the case in which we have $\sum_{i\leq I}\omega_s^i$ big enough for $s=1$. Now let us define $\beta\in\Delta_+^{I-1}$, $\{\underline{\omega}_1^i\}_{i\leq I}\gg 0$ and $\omega_1^i := \underline{\omega}_1^i+\beta_i K$. Our goal is to show that for every $\{(\omega_2^i,\ldots,\omega_S^i)\}_{i=1}^{I+J}\gg 0$, $\beta\in\Delta_+^{I-1}$ and $\{\underline{\omega}_1^i\}_{i\leq I}$ there exists $\underline{K}\geq 0$ such that if $K\geq\underline{K}$ there is an equilibrium for the economy for $\{\omega_1^i\}_{i\leq I}$ defined as before.

In order to prove it, we will define a fictitious economy and use the existence of Nash Equilibrium for non-cooperative games.

Let $\{\omega_i^i\}_{i\leq I} > 0$, define $\omega := \max_s\left\{\sum_{i=1}^{I+J}\omega_s^i\right\}$ and a non-cooperative game with $I+J+1$ players, in which:

- For $1\leq i\leq I$ we have:
  - **Utility:** $V^i(x) := U^i(x)$.
  - **Set of actions:** $x\in X_i := \mathbb{R}_+^S\cap\overline{B_0}(2\omega)$.
  - **Constraint:** $B_i(p) := \{x\in X_i\ :\ px\leq p\omega^i\}$.
- For $i = I+1,\ldots,I+J$ we have:
  - **Utility:** $V^i(x) := \sqrt{x_1}$.

  - **Set of actions:** $x \in X_i := \{(x_1, \lambda_2^i, \dots, \lambda_S^i) \ : \ 0 \leq x_1 \leq 2\omega\}$.
  - **Constraint:** $B_i(p) := \{x \in X_i \ : \ px \leq p\omega^i\}$.

- And the $I + J + 1$ player is the market defined as usual.

Notice that, there exists an analogy between the game and the economy defined before, the first I agents (and players) are, in fact defined by the same boundary constraint, for bounded consumption, and they also are defined by the same utility functions.

For $i \in [\![I+1, I+J]\!]$ players of the game are totally different compared to the prone agents in the economy, the main reason is to be able to guarantee existence of a Nash equilibrium for the game. Since our goal is to guarantee existence of equilibrium for the original economy, we will establish a relationship between the equilibrium consumption for the players of the game and the optimal consumption of the prone in the economy.

As is already known by the General Equilibrium Theory, the $I + J + 1$ player helps us to establish the equilibrium price $p \in \Delta_+^{S-1}$ in the economy.

Let us denote $\left((x^i)_{i=1}^{I+J}, p\right)$ the Nash equilibrium for the previous game. As it was proved by Arrow and Debreu see [6], we have that since each agent has a strictly increasing utility function, $p_s > 0$ in each state s and

$$\sum_i x^i = \sum_i \omega^i.$$

And now, in order to prove that the Nash Equilibrium for the restricted game is, in fact, an equilibrium for the economy, we need to prove that each agent is maximizing in the entire AD constraint instead of maximizing in $\{x : px \leq p\omega^i \wedge |x_s| \leq 2\omega \ \forall s\}$ for the averse and $\{(x, \underline{x}_2^i, \dots, \underline{x}_S^i \ : \ 0 \leq x \leq 2\omega)\}$ for the prone.

Since $\sum_i \omega_s^i \leq \max_s \{\sum_i \omega_s^i\} = \omega < 2\omega$, x^i is an optimal consumption in the initial economy for each $i \in [\![1, I]\!]$. In order to prove the same result for the prone, we will need to analyze the behavior of these types of Nash equilibria when there are some modifications in the initial endowment.

And now, let $\beta \in \Delta_+^{I-1}$ and $\{\underline{\omega}_1^i\}_{i \leq I} \geq 0$ and $\{K_n\} \in \mathbb{R}_+$ an increasing sequence such that $K_n \to \infty$. And defining ω_n^i as before (using the sequence $\{K_n\}$), we can define the non-cooperative game given by these endowments and applying the existence of Nash equilibrium for these games we have that is a sequence $\left\{\left((x_n^i)_{i=1}^{I+J}, p_n\right)\right\}$ of equilibria.

Our goal is to prove that for n big enough, the optimal consumption for the prone will be optimal given the price p_n. In order to do so, we need to have incentives to make them specialize in the first state. This suggests that the price in the first state should be small enough compared to the other states.

Lemma 5. *Under the same assumptions, we have* $p_{1,n} \to 0$ *and there exist* $\underline{p} > 0$ *such that* $p_{s,n} \geq \underline{p}$ *for every* $s \neq 1$.

Proof of Lemma 5. The idea of the proof is to use the FOC for the averse in order to establish relationships between prices and the consumption in different states of nature, this would help us to analyze the asymptotic behavior of the price when we are increasing the endowment of the first state.

We can suppose that $p_n \to \hat{p} \in \Delta_+^{S-1}$. To prove this lemma we will separate it in some cases:

1. Let us show first that $\hat{p}_1 = 0$. So assume that $\hat{p}_1 > 0$ and let us show that we obtain a contradiction.

   As a consequence of $K_n \to \infty$, there exists at least one agent $\left(\bar{i} \leq I\right)$ such that $p_n \omega_n^{\bar{i}}$ is going to infinity, and since we have market clearing in each state, the agent's consumption in the first state will tend to infinity, and knowing that $\hat{p}_1 > 0$, we will have that $x_{1,n}^{\bar{i}} > 0$. Then the FOC for the state 1 is satisfied with equality. And now using the FOC for this agent and for n big enough we will have that there is $T_n^{\bar{i}} \in \partial U^{\bar{i}} \left(x_n^{\bar{i}}\right)$ such that:

$$\frac{T_n^{\bar{i}} \circ e_1}{T_n^{\bar{i}} \circ e_s} \geq \frac{p_{1,n}}{p_{s,n}} \geq \hat{p}_1 - \varepsilon > 0$$

   for $0 < \varepsilon < p_s$. And as a consequence of 2 we have that $x^{\bar{i}_{s,n}} \to \infty$ which contradicts market clearing.

   The intuition for this part of the proof is that if the price of the state 1 is not going to 0, since we have ambiguity averse DM they will try to consume an allocation with low variations, and as a consequence, the agent will consume quantities arbitrarily big in states different than the state 1 which is a contradiction.

2. Let us show now that $\hat{p}_s > 0 \; \forall s \geq 2$. So let us assume that there exists at least one $2 \leq s \leq S$ such that $\hat{p}_s = 0$.

   Now let us take $\hat{s}$ such that $\hat{p}_{\hat{s}} > 0$, using market clearing we already know that there exists an agent $i_n \leq I$ such that $x_{\hat{s},n}^{i_n} \geq {\left(\omega_{\hat{s}} - \sum_{j=1}^{J} \underline{x}_{\hat{s}}^{I+j}\right)}/{I+J} > 0$. Then the FOC for $x_{\hat{s},n}^{i_n}$ is satisfied with equality, and using the FOC for this agent we have that for each n:

$$\frac{T_n^{i_n} \circ e_s}{T_n^{i_n} \circ e_{\hat{s}}} \leq \frac{p_{s,n}}{p_{\hat{s},n}}.$$

Since $p_{s,n}/p_{\hat{s},n} \to 0$ and 3, then there exists i such that for a subsequence n_k we have that the only possible case is that $x^i_{s,n_k} \to \infty$, which is a contradiction with market clearing.

The proof of the second case has a different intuition, it says that if there exists other state price going to 0, it means that each agent will consume more in this state than the others that have positive limit price, that's why, when n is going to infinity, they will try to consume quantities that would contradict the market clearing condition for this state.

□

And now with this lemma we can prove the second required condition and as a consequence, the theorem. As mentioned before, in order to prove the theorem we have to prove that $(x^i_n)_{i=I+1}^{I+J}$ is optimal for the prone when the price is given by p_n for n big enough. Since we have that for each agent $i = I+j \geq I+1$ has a convex utility function, the possible optimal solutions has the following form

$$\left\{x^{1,I+j},\dots,x^{S,I+j}\right\} = \left\{\left(\frac{p_n\omega^{I+j} - \sum_{s\geq 2} p_{n,s}\lambda_s^{I+j}}{p_{n,1}}, \lambda_2^{I+j},\dots,\lambda_S^{I+j}\right),\dots,\right.$$
$$\left.\left(\lambda_1^{I+j},\dots,\lambda_{S-1}^{I+j}, \frac{p_n\omega^{I+j} - \sum_{s\leq S} p_{n,s}\lambda_s^{I+j}}{p_{n,S}}\right)\right\},$$

then it is enough to compare the points that generates the constraint in order to find an optimal solution.[2]

Now if we use the lemma 5, we have that $p_{n,s} \to \hat{p}_s > 0$ and $p_{n,1} \to \hat{p}_1 = 0$, enable to consume, for $i = I+1,\dots,I+J$ in the game, quantities that tend to infinity when $n \to \infty$ in the first state. And since $\left(\omega_1^{I+j},\dots,\omega_S^{I+j}\right)$ are constant for each n and each $j = 1,\dots,J$; if the prone decides to consume as much as they can in a state different from the state 1, the maximum consumption would be bounded by

$$\frac{\max_s \omega_s^{I+j}}{\min_{s\neq 1} p_{n,s}};$$

and as mentioned before, the fact that $p_n\omega^{I+j}$ is always limited, we will do that the previous expression is bounded when $n \to \infty$. And now using the fact that U_{I+j} is a convex function and strictly increasing in every state for

[2] This is due to the fact that we are maximizing a convex function in a convex set.

each $j \leq J$, $\lambda_s^{I+j} \leq \omega_s^{I+j}$ we have that $U^{I+j}\left(x^{1,I+j}\right) \geq U^{I+j}\left(x^{s,I+j}\right)$, for all $s \geq 2$, $j = 1, \dots, J$, then we will have that for each prone it is optimal to consume just only at the first state for n big enough. □

B.1.4 Proof of Proposition 10

Proof. As in the previous cases, without loss of generality let us analyze the case in which $s = 1$.

Our approach is similar to the other theorems, nevertheless there are some parts that have to be modified in order to complete the proof. The first thing that has to be adapted is the players in the fictitious economy (or game) that would represent the Friedman-Savage Decision Makers in the initial economy.

Let us define a game that for the agents $i = 1, \dots, I$ and for $i = I + J + 1$ as in the previous cases. And for $i = I + 1, \dots, I + J$ we have:

- **Utility:** $\tilde{U}^i(x) := \sum_s \tilde{u}_s^i(x_s)$.
- **Set of actions:** $x \in X_i := \{(x_1, x_2, \dots, x_S) \ : \ 0 \leq x_s \leq 2\omega \ \forall s = 1, \dots, S\}$.
- **Constraint:** $B_i(p) := \{x \in X_i \ : \ px \leq p\omega^i\}$.

Where

$$\tilde{u}_s^i(x) := \begin{cases} u^i(x) & \text{if } x \leq x_c^i, \\ v^i(x) & \text{if } x > x_c^i, \end{cases}$$

for $s = 2, \dots, S$,

$$\tilde{u}_1^i(x) := u^{i\prime}\left(\tilde{x}^i\right)\left(x - \tilde{x}^i\right) + u^i\left(\tilde{x}^i\right)$$

for $s = 1$ and $v^i : [x_c^i, \infty) \to \mathbb{R}$ is a concave and differentiable function that satisfies that $\lim_{x\to\infty} v^{i\prime}(x) = 0$, $v^i\left(x_c^i\right) = u^i\left(x_c^i\right)$ and $\lim_{h\searrow 0} \frac{v^i\left(x_c^i+h\right)-v^i\left(x_c^i\right)}{h} = u^{i\prime}\left(x_c^i\right)$, hence $\tilde{u}_s^i$ is a concave, increasing and differentiable function. Therefore there exists a Nash equilibrium for the game $\left(\left(\tilde{x}^i\right)_{i=1}^{I+J}, p\right)$.

As we analyzed before, let us prove the property about the price when $n \to \infty$.

Lemma 6. *Under the same assumptions, we have* $p_{1,n} \to 0$ *and there exist* $\underline{p} > 0$ *such that* $p_{s,n} \geq \underline{p}$ *for every* $s \neq 1$.

Proof of Lemma 6. We can suppose, as before, that $\left(p_n \to \hat{p} \in \Delta_+^{S-1}\right)$. As in the lemma 5, we can prove the lemma analyzing two different cases: The first case is $\hat{p}_1 = 0$, which is a result of the first part of the lemma 5.

And now to prove that $\hat{p}_s > 0\ \forall s = 2, \ldots, S$, let us take $\hat{s}$ such that $\hat{p}_{\hat{s}} > 0$. If for each n exists an averse such that their consumption in the state $\hat{s}$ is positive, we can apply what was done in the previous lemma.

If there is no at least one agent satisfying this, the consumption of at least one $i = I+1, \ldots, I+J$ (the adapted FS Decision Makers for the game) is positive in the state $\hat{s}$, since this type of player in the game has differentiable and concave utility function, we can apply first order conditions, and since $\{\pi_s^i\}_{s\leq S} \geq \underline{\pi} > 0$ and $\lim_{x\to\infty} (\tilde{u}^i)'_s(x) = 0$ for each $s \neq 2$, we will have the same result as in the previous lemma. □

Let us now conclude the proof of the Theorem, since we have the same behavior of the equilibrium "prices", our goal is to prove that the consumption of the players $i = I+1, \ldots, I+J$ in the game (or fictitious economy) is in fact optimal for the FS Decision Makers. In order to prove this result, let us prove a preliminary lemma.

Lemma 7. For any type of Friedman-Savage Decision Maker with strictly increasing utility index (u^i) and for any $p \in \Delta_{++}^{S-1}$, there exists an optimal consumption that has at most one state with a consumption bigger than x_c.

Proof of lemma 7. Let us suppose that there exists an optimal consumption, $(\hat{x}_s)_{s=1}^S$, in which the agent is consuming more than the inflection point in two states, w.l.o.g. these two states are state 1 and state 2.

Since for $x \geq x_c$ the utility index is convex, we have that $\pi_1 u(x_1) + \pi_2 u(x_2)$ is convex for $(x_1, x_2) \geq (x_c, x_c)$. And also, since $(\hat{x}_1, \hat{x}_2)$ is in the convex set generated by $(\hat{x}_1^1, \hat{x}_2^1) := (\hat{x}_1 + p_2(\hat{x}_2 - x_c)/p_1, x_c)$ and $(\hat{x}_1^2, \hat{x}_2^2) := (x_c, \hat{x}_2 + p_1(\hat{x}_1 - x_c)/p_2)$, that satisfy each component is bigger than x_c, and then, using the convexity of the utility index for $x \geq x_c$, we have

$$\max\left\{\pi_1 u\left(\hat{x}_1^1\right) + \pi_2 u\left(\hat{x}_2^1\right), \pi_1 u\left(\hat{x}_1^2\right) + \pi_2 u\left(\hat{x}_2^2\right)\right\} \geq \pi_1 u(\hat{x}_1) + \pi_2 u(\hat{x}_2)$$

which is equivalent to

$$\max\left\{\pi_1 u\left(\hat{x}_1^1\right) + \pi_2 u\left(\hat{x}_2^1\right), \pi_1 u\left(\hat{x}_1^2\right) + \pi_2 u\left(\hat{x}_2^2\right)\right\} + \sum_{s=3}^{S} \pi_s u(\hat{x}_s) \geq \sum_{s=1}^{S} \pi_s u(\hat{x}_s)$$

then $(\hat{x}_1^1, \hat{x}_2^1, \hat{x}_3, \ldots, \hat{x}_S)$ or $(\hat{x}_1^2, \hat{x}_2^2, \hat{x}_3, \ldots, \hat{x}_S)$ are also an optimal solution, which proves the case in which there is an optimal solution with two states consuming more than the inflection point.

For the general case, in which exists an optimal solution where $2 \leq \tilde{s} \leq S$ states has a consumption bigger than the inflection point, can be proved by induction in the quantities of states that the consumption is bigger than x_c in the optimal solution $(\hat{x}_s)_{s=1}^S$. □

As a consequence of the lemma 6, when the aggregate endowment for the agents $i = 1, \ldots, I$ is increasing, we will have that the price $p_{1,n} \to 0$ and $p_{s,n} \to \hat{p}_s > 0$, then the consumption in the states $s \neq 1$ can not tend to infinity. And since the optimal consumption has at most one state above x_c due to the lemma 7, for n big enough, the FS Decision Makers will have incentives to consume more than x_c in just only the first state.

With this, let us carry out the analysis of the optimal consumption of each player $i = I+1, \ldots, I+J$ in the game, or fictitious economy. Since the utility function for these agents is differentiable and concave, the optimal solution, x_n^i, satisfies the FOC and as a consequence, if $x_{s,n}^i > 0$ for $s \neq 1$, we have that

$$\frac{\pi_1 (\tilde{u}_1^i)' \left(x_{1,n}^i\right)}{\pi_s (\tilde{u}_s^i)' \left(x_{s,n}^i\right)} = \frac{p_{1,n}}{p_{s,n}} = \frac{\pi_1 u^{i'} \left(x_{1,n}^i\right)}{\pi_s u^{i'} \left(x_{s,n}^i\right)},$$

then the optimal consumption satisfies that $x_{s,n}^i \to 0$ for $s \neq 1$ when $n \to \infty$, therefore for n big enough any agent i is consuming above the inflection point in at most one state, the state 1.

And finally, let us prove that for n big enough, the optimal consumption for each player $i = I+1, \ldots, I+J$ in the game, or fictitious economy, it is in fact optimal for the FS Decision Maker i of the initial economy. Let us suppose that there is i such that $x_n^i \neq \tilde{x}_n^i$, where $\tilde{x}_n^i$ is an optimal consumption for the FS Decision Maker in the initial economy and x_n^i is the optimal consumption in the game, or fictitious economy given by the Nash Equilibrium. We know that $\lim_{n\to\infty} \tilde{x}_{1,n}^i = \infty$ and $\lim_{n\to\infty} x_{1,n}^i = \infty$[3] then for n big enough, we have that $x_{1,n}^i, \tilde{x}_{1,n}^i \geq \tilde{x}^i$ and $x_{s,n}^i, \tilde{x}_{s,n}^i \leq x_c^i$ for $s = 2, \ldots, S$, and, as a consequence, we have:

$$U^i\left(x_n^i\right) = \tilde{U}^i\left(x_n^i\right) \text{ and } U^i\left(\tilde{x}_n^i\right) = \tilde{U}^i\left(\tilde{x}_n^i\right).$$

To conclude, let us notice that x_n^i is in the constraint of the FD Decision Maker i in the initial economy, $\{x : p_n x \leq p_n \omega^i\}$, and $\tilde{x}_n^i$ is in the constraint of the game, or fictitious economy, $\left\{x : p_n x \leq p_n \omega^i \text{ and } 0 \leq x_s \leq 2 \max_s \left\{\sum_{i=1}^{I+J} \omega_{n,s}^i\right\}\right\}$ where $\omega_{n,s}^i = \omega_s^i$ for $i = I+1, \ldots, I+J$, $s = 1, \ldots, S$, and $i = 1, \ldots, I$, $s = 2, \ldots, S$. And then we have that

$$U^i\left(x_n^i\right) = U^i\left(\tilde{x}_n^i\right)$$

and, as a consequence, x_n^i is also optimal for the FS Decision Maker i, which concludes the theorem. □

[3]Since $p_{1,n}$ is the only price that tends to 0 when $n \to \infty$.

B.2 Additional examples

The following example shows that in some cases there is an equilibrium even with no aggregate risk. To get such a situation it is necessary to have differences in the endowment distribution of the agents and big differences among the probabilities of the agents.

Example 6. Consider two agents, two states of nature with complete markets (Edgeworth Box). Let $(\omega_1^2, \omega_2^2) = (1, 2)$, $\pi \in (1/2, 1)$, $\alpha := \sqrt{\frac{\pi}{1-\pi}}$, $u^1(x) := \ln x$ and $u^2(x) := x^2$. Therefore by proposition 4, there exists an equilibrium with initial endowments of agent 1 given by $(\omega_1^1, \omega_2^1) = (\omega_1^1, \underline{\omega}_2^1) = \left(\frac{2+\alpha^2}{\alpha-1} + 1, \frac{2+\alpha^2}{\alpha-1}\right)$. This equilibrium is characterized by $p = \overline{p} = \frac{\alpha}{\alpha+1}$ and

$$\begin{array}{ll} x_1^1 = \frac{2+\alpha^2}{\alpha-1} + 2, & x_2^1 = \frac{2+\alpha}{\alpha-1}, \\ x_1^2 = 0, & x_2^2 = \alpha + 2. \end{array}$$

As we mentioned before, the fact that in this case we have existence of equilibrium with no aggregate risk is because there exists a big variation among the agents and the states, making the risk averse be considerably richer than the risk lover. In fact, for $\pi = 3/4$ the wealth of Agent 1 is $2 + 2\sqrt{3} \approx 5.464$ times larger than the wealth of the Agent 2. Notice that, when π goes to $1/2$ the difference goes to infinity. This shows that this type of equilibria requires that there is almost no wealth for the risk lover or an arbitrarily large wealth for the risk averse and a very specific endowment distribution. And this makes the example a small class of economies compared to the set of economies in which there is an equilibrium.

The following example illustrates the conditions exposed in Proposition 3.

Example 7. Let us consider two states of nature and three agents, one risk averse and two risk lovers defined by $u^1(x) := \ln x$, $u^2(x) := x^2$ and $u^3(x) := x^2$ with $\pi := 1/2$ the probability of the first state for every agent. Their endowments are $\omega^1 = (2, 2)$, $\omega_s^i := 1\ \forall i = 2,\ 3,\ \forall s = 1,\ 2$.

Then we have that $p = 1/2$, and, $x^1 = (2,\ 2)$, $x^2 = (2,\ 0)$ and $x^3 = (0,\ 2)$ is an equilibrium for the economy. You also can note that in this example there is more than one equilibrium, and all of them have the property that the risk lovers consume in different states.

As can be absorbed in Proposition 3, the conditions for existence of equilibrium where the risk lovers specify their consumption in different states require strong conditions of symmetry of the endowments. Therefore little variations of any endowment in the previous economy would lead to nonexistence of equilibrium for the economy.

The following example explores conditions for no comonotonic consumption.

Example 8. In the Edgeworth box with $\pi = 1/4$ and the agents defined by $u^1(x) := \ln x$, $u^2(x) := x^2$. Let $p = 7/24$, $\omega_2^1,\ \omega_1^2,\ \omega_2^2 > 0$ and $\omega_1^1 := \frac{1-p}{(1-\pi)p}\left(\omega_2^1 + \omega_2^2\right)$. It is not hard to prove that the economy has an equilibrium and that $(p,\ 1-p)$ is the equilibrium price.

The optimal consumption is defined by:

$$\begin{aligned} x_1^1 &= \tfrac{6}{7} p\omega^1, \\ x_2^1 &= \tfrac{18}{17} p\omega^1, \\ x_1^2 &= \tfrac{(p,\ 1-p)\omega^2}{p} \\ x_2^2 &= 0. \end{aligned}$$

In which we do not have comonotonicity, in fact the consumption is anti-comonotonic between the agents.

Note that, in economies with two states, there exist two different types of behaviors related to risk sharing among agents under the condition of Section 1.3, comonotonicity and anticomonotonicity.

For more than two states, if the consumption is not comonotonic, it does not imply that it will be anticomonotonic in equilibrium. However can be noticed that the agents will have comonotonic behaviors among the agents with similar type of utility function.

B.3 Some proofs of sections 1.3 and 1.4

Proof of Proposition 7. To prove the first part of the proposition, let us notice that can be proved $\frac{\pi_1}{p_1} \geq \frac{\pi_s}{p_s}$ in a similar way as in proposition 5, and because of the existence of a pessimistic agent with the identity as distortion and the comonotonicity for all the pessimists when they distort the same probability (see [19]), we will have the first part of the proposition.

To prove the second part of the proposition, it is enough to notice that if K tends to infinity, we will have that each agent is consuming arbitrarily big quantities in the first state, this is the reason why it is obvious that there exists K large enough such that it makes the the agents have comonotonic consumption in equilibrium. □

Proof of Proposition 8. Using the Theorem 1 and the proposition 7, we can compute in a precise way the utility function of each pessimist agent. And since we suppose that $\omega_1 > \omega_2 > \cdots > \omega_S$ and we have that each Ambiguity Lover will consume just only in the first state. Then the endowment $(\hat{\omega}_s)$ available for the Ambiguity Averse will be:

$$\begin{aligned} \hat{\omega}_1 &:= \omega_1 - \textstyle\sum_{i=I+1}^{J} \frac{(p_1,\ldots,p_S)\omega^i}{p_1}, \\ \hat{\omega}_s &:= \omega_s, \qquad \forall s \in \{2,\ldots,S\}. \end{aligned}$$

And therefore $\hat{\omega}_1 > \hat{\omega}_2 > \cdots > \hat{\omega}_S$ and, as a consequence, of the comonotonicity of every agent utility function at the optimal consumption can be written as:

$$U^i(x^i) = u^i\left(x_S^i\right) + \sum_{s=1}^{S-1}\left(u^i\left(x_s^i\right) - u^i\left(x_{s+1}^i\right)\right)\left(f^i\left(\sum_{r=1}^{s}\pi_r\right)\right).$$

And now let us define a new economy with aggregate endowment given by $(\hat{\omega}_s)$, and redistribute the agents endowment (from ω^i to $\hat{\omega}^i$) in such a way that for the equilibrium price of the initial economy $p \in \Delta_{S-1}^{++}$, each Ambiguity Averse would maintain the rent $(p\omega^i)$ and $\sum_{i=1}^{I}\hat{\omega}_s^i = \hat{\omega}_s$. Then using the FOC, we can obtain a characterization of the equilibrium price in term of f^i, ρ^i and $\{\hat{\omega}_s\}_{s=1}^S$ which proves the proposition. □

Proof of Proposition 9. Let us consider the equilibrium price as a function in terms of the regulation α, $p_1(\alpha)$, and using 1.11 we have that

$$\begin{aligned} &p_1'(\alpha)\left(\sum_{t=1}^{S} e^{-\rho\hat{\omega}_t}e^{\beta_t}\right)^2 \\ &= -\rho e^{-\rho\left(\omega_1-(1-\alpha)\sum_{i=I+1}^{J}\frac{(p_1,\ldots,p_S)\omega^i}{p_1}\right)} e^{\beta_1}\left[\left(\sum_{j=1}^{J}\frac{(p_1,\ldots,p_S)\omega^{I+j}}{p_1}\right.\right. \\ &\qquad \left. - (1-\alpha)\sum_{s=2}^{S}\sum_{j=1}^{J}\omega_s^{I+j}\frac{p_s'(\alpha)p_1 - p_1'(\alpha)p_s}{p_1^2}\right)\left(\sum_{s=2}^{S} e^{-\rho\hat{\omega}_s}e^{\beta_s}\right) \\ &\qquad \left. + \sum_{s=2}^{S}\sum_{j=1}^{J} e^{-\rho\hat{\omega}_s}e^{\beta_s}\omega_s^{I+j}\right], \end{aligned}$$

since $\sum_{j=1}^{J} \omega_s^{I+j} = \sum_{j=1}^{J} \omega_{s'}^{I+j}$ for all $1 \leq s, s' \leq S$, we have that

$$p_1'(\alpha) \left(\sum_{t=1}^{S} e^{-\rho\hat{\omega}_t} e^{\beta_t} \right)^2 = -\rho e^{-\rho\left(\omega_1 - (1-\alpha)\sum_{i=I+1}^{J} \frac{(p_1,\dots,p_S)\omega^i}{p_1}\right)} e^{\beta_1} \left(\sum_{s=2}^{S} e^{-\rho\hat{\omega}_s} e^{\beta_s} \right) \left(\sum_{j=1}^{J} \omega_s^{I+j} \right) \left[\frac{1}{p_1} - (1-\alpha) \frac{\sum_{s\neq 1} p_s'(\alpha) p_1 - p_1'(\alpha) \sum_{s\neq 1} p_s}{p_1^2} + 1 \right].$$

Since $\sum_s p_s = 1$, we have that

$$p_1'(\alpha) \left(\sum_{t=1}^{S} e^{-\rho\hat{\omega}_t} e^{\beta_t} \right)^2 e^{\rho\hat{\omega}_1} e^{-\beta_1} \left(\sum_{s=2}^{S} e^{-\rho\hat{\omega}_s} e^{\beta_s} \right)^{-1} (-\rho)^{-1} \left(\sum_{j=1}^{J} \omega_s^{I+j} \right)^{-1}$$
$$= \frac{1}{p_1} - (1-\alpha) \frac{-p_1'(\alpha) p_1 - p_1'(\alpha)\left(1 - p_1\right)}{p_1^2} + 1$$
$$= \frac{p_1(\alpha) + (1-\alpha) p_1'(\alpha) + \left(p_1(\alpha)\right)^2}{\left(p_1(\alpha)\right)^2}.$$

Now if $p_1'(\alpha) > 0$, we have that $p_1(\alpha) + (1-\alpha)p_1'(\alpha) + \left(p_1(\alpha)\right)^2 < 0$, then $p_1'(\alpha) < \frac{-p_1(\alpha)(1+p_1(\alpha))}{1-\alpha} < 0$ which is a contradiction. □

APPENDIX C

Appendix Part II

C.1 On Bewley (1980, 1983) results

Bewley (1980, 1983) assumed instantaneous utilities that are continously differentiable also on the boundary of the positive orthant. The discount factor is of the form $\zeta_t^i = (\bar{\zeta}^i)^t$. For the sake of comparison with our model, we comment on a deterministic version of his model. Two cases can be addressed using his analysis: the case of constant money supply and the case of a money supply decreasing to zero at a constant rate r. Bewley's (1983) budget constraint, adapted to our single commodity case, is written as $x_t - \omega_t^i \leq q_t\,[(1+r)y_{t-1} - y_t - r\tau_t(y)]$, where r is a nominal interest rate paid on money and $\sum_i \tau_t^i = \sum_i y_0^i$. Making $r = 0$, we get (see equation (5) and Theorem 2 in Bewley (1980)) that for an equilibrium, with constant money supply and nonnull consumption, at every date by every consumer, to have q uniformly bounded from above and away from zero it would have to be inefficient.

Equivalently, Bewley's budget constraint can be written in deflationary form as $x_t - \omega_t^i \leq \bar{q}_t\,[\bar{y}_{t-1} - \bar{y}_t - r\bar{\tau}_t(y)]$, where $\bar{y}_t = (1+r)^{-t+1}y_t$, $\bar{\tau}_t = (1+r)^{-t+1}\tau_t$ and $\bar{q}_t = (1+r)^{t-1}\bar{q}_t$. In this form, money supply $\sum_i \bar{y}_t^i$ goes to zero at the rate r. Assuming $(\bar{\zeta}^i)^{-1} - 1 > r$ (as implied by the conditions ensuring existence of equilibrium in the Theorem in Bewley (1983)), for an equilibrium, with nonnull consumption, at every date by every consumer, to have q uniformly bounded away from zero it would have to be inefficient (see Theorem 2 and equation (5) in Bewley (1980)).

However, if we just require q to be different from zero at some date, the implementation of efficient allocations among impatient agents can be done with a vanishing money supply and lump-sum taxes, with $\sum_t \tau_t^i = y_0^i$ and $q_t = 1/p_t$, where $p \in \ell^1$ is the Arrow-Debreu equilibrium price.

C.2 On Fiat Money and the Marginal Utility in the Direction of Net Trades

We show here that for a utility function U of the form given by (2.3), if z^* is an optimal portfolio plan in $B^A(q, y_0^i, \omega^i)$ (defined in Subsection 2.4.1) such that, at $x^* := x(z^*) \ggg 0$, we have $\inf x^*$ not attained and $\lim_s x_s^* = \inf_s x_s^*$, then

$$\delta^- U(x^*)(x^*; x^* - \omega^i) = \mu(x^* - \omega^i) + \alpha \limsup(x^* - \omega^i)$$

for $\alpha > 0$ equal to the norm of the pure charge component of a supergradient of U at x^* (see Remark 20), where μ is given by $\mu_t = \zeta_t u'(x_t^*)$.

We will estimate $\lim_{r\to 0} \frac{1}{r}[U \circ x(z^* + rz^*) - U \circ x(z^*)]$. Consider the direction $\Delta \in \ell^\infty$ given by $\Delta_t = q_t z_{t-1}^* - q_t z_t^*$. Notice that $\lim_{r\to 0} \frac{1}{r} \sum_{t\geq 1} \zeta_t [u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t\geq 1} \zeta_t \lim_{r\to 0} \frac{1}{r}[u(x_t^* + r\Delta_t) - u(x_t^*)] = \sum_{t\geq 1} \zeta_t u'(x_t^*)\Delta_t$. So, what we still need to do is to estimate $\lim_{r\uparrow 0} \frac{1}{r}\beta[\inf_t u(x_t^* + r\Delta_t) - \inf_s u(x_s^*)]$, which is $\delta^- \inf_t u(x^*, \Delta)$, the left-derivative of the function $\inf_t u(.)$ along the direction Δ evaluated at x^*.

Observe that there exists $\chi > 0$ such that $\forall r \in (-\chi, 0)$ the following holds: $(1 + r)z^* > 0$ is a non-negative plan, $x(z^* + rz^*)$ satisfies (2.7) and $x(z^* + rz^*) = x^* + r(x^* - \omega) \ggg 0$.

Claim. $\lim_{r\uparrow 0} \frac{1}{r}[\inf_t u(x_t^* + r\Delta_t) - \inf_t u(x_t^*)] = u'(\underline{x}^*) \limsup_t \Delta_t$

Proof. Let us denote $\underline{x}^* = \inf x^*$. There exists $\lim_{r\uparrow 0} \frac{1}{r}[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)]$ since $\inf(.) : \ell^\infty \to I\!R$ is a concave function.

Fixed $r \in (-\chi, 0)$ and given $\epsilon > 0$, it is valid for all τ large enough that $(1/r)[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)] + \epsilon = (-1/r)[u(\underline{x}^*) - \epsilon r - \inf_t u(x_t^* + r\Delta_t)] \geq (-1/r)[u(x_\tau^*) - u(x_\tau^* + r\Delta_\tau)] \geq u'(x_\tau^*)\Delta_\tau$. Making $\tau \to \infty$ we get $(1/r)[\inf_t u(x_t^* + r\Delta_t) - u(\underline{x}^*)] + \epsilon \geq \limsup_t u'(x_t^*)\Delta_t = u'(\underline{x}^*) \limsup_t \Delta_t$, for an arbitrary $\epsilon > 0$.

To prove the reverse inequality, notice that, under the hypothesis,

$$\delta U(x^*; \mathbb{1}^n) = \sum_{t>n} \zeta^t u'(x_t^*) + \beta u'(\underline{x}^*)$$

and, therefore, any supergradient has a pure charge component with norm $\beta u'(\underline{x}^*)$ by Remark 20. Hence, for any supergradient T of U at x^* we have $T(\Delta) = \sum_{t\geq 1} \zeta_t u'(x^*_t)\Delta_t + \beta u'(\underline{x}^*)\text{LIM}(\Delta)$, for some generalized limit LIM. So, $\delta^- \inf_t u(x^*, \Delta) \leq u'(\underline{x}^*) \limsup_t \Delta_t$ □

Even if x^*_t doesn't converge to $\inf x^*$, the previous claim stills holds if we have $\liminf_{\{t:x^i_t - W^i_t>0\}} x^i_t = \underline{x}$ and fact that $\limsup_{\{t:x^i_t - W^i_t>0\}} \Delta_t = \limsup \Delta_t$. We close this subsection with the proof of Proposition 13.

Proof of Proposition 13. It suffices to find q and $(z^i_0)_i$ so that (2.13) and $\lim \mu^i_t q_t z^i_t = \tilde{\nu}^i(x^i - \omega^i)$ hold at $(z^i)_i$ implementing $(x^i)_i$ (see Proposition 12) . These hold if (2.15) holds for any i (as by (2.14), $\mu^i + \nu^i = \rho^i(p+\nu)$). Let us see that (2.15) (whose left hand sides are nonnegative) has a solution $b \equiv \lim p_t q_t > 0$, for some $(z^i_0)_i > 0$ (allowing us to make $q_t = b(p_t)^{-1}$).

We just have to rule out that $\tilde{\nu}^i(x^i - \omega^i) = \nu^i(x^i - \omega^i)$ for all i, which, implies that $\|\nu^i\|_{ba} = \alpha^i$. Now, $\nu^i = \rho^i\nu$, $\nu = \alpha\text{LIM}$ and AD prices can be normalized so that $\alpha = 1$. Hence, $\alpha^i = \rho^i$ and we get $\nu(x^i - \omega^i) = \limsup(x^i - \omega^i)$, for any i. Adding across agents, $0 = \sum_i \limsup(x^i - \omega^i)$. Say it is agent 1 whose net trade $x^1 - \omega^1$ does not converge. Now, $\limsup(x^1 - \omega^1) = -\sum_{i\neq 1}\limsup(x^i - \omega^i) = \sum_{i\neq 1}\liminf(\omega^i - x^i) \leq \liminf(x^1 - \omega^1)$, a contradiction. □

C.3 Proofs of Section 2.5

Proof of Proposition 14. Lets pick up an AD equilibria for endowments given by W^i, our goal is to implement it using an auxiliar economy without taxes but with some type of *P-constraint* in order to avoid *Long-Run Arbitrage.*

Consider $B^A(q, y^i_0, \omega^i)$ the set of plans (x, z) satisfying $z \geq 0$ and:

$$x_t - \omega^i_t \leq q_t(z_{t-1} - z_t) + R_t z_{t-1} \ \forall t \in \mathbb{N}.$$

We can define the equilibrium for this economy in a similar form as it was made in the auxiliar economy of the previous case, making the adjustments in the market clearing equation of the Lucas' tree.

Using the same proposition for optimality for each agent, we can see that we will have the same sufficient condition for optimality, and analogous to the previous model, the supergradient that would satisfy this optimality condition is a super-gradient whose pure charge $\tilde{\nu}^i$ takes the highest on the direction of the net trade and assuming that $\liminf(x^i - \omega^i) > 0$, we have that the direction is

$$\tilde{\nu}^i\left(x^i - \omega^i\right) = \left(u^i\right)'\left(\underline{x}^i\right)\limsup\left(x^i - \omega^i\right).$$

Therefore it suggests the following portfolio constraint

$$\lim \mu_t q_t z_t \geq \alpha^i \limsup \left(x(z) - \omega^i\right)$$

where $x_t(z) = \omega_t^i + q_t\left(z_{t-1} - z_t\right) + R_t z_{t-1}$ and $\alpha^i = \|\tilde{\nu}^i\|_{ba}$ for some $\tilde{\nu}^i$ satisfying the previous portfolio constraint.

By the proposition we should find q such that at $(z^i)_i$ implements $(x^i)_i$ having $\lim_t \mu_t^i q_t z_t^i = \tilde{\nu}^i\left(x^i - \omega^i\right)$.

And $x^i(z)$ belongs to the Arrow-Debreu budget set if and only if

$$\nu\left(x^i(z) - \omega^i\right) - \lim_t p_t q_t z_t^i = z_0^i\left(\nu(R) - \lim p_t q_t\right)$$

when the AD budget constraint hold with equality.

Then this condition can be written as

$$\tilde{\nu}^i\left(x^i - \omega^i\right) - \nu^i\left(x^i - \omega^i\right) = z_0^i\left(\mu_1 q_1 - \sum_{i=1}^{\infty} \mu_i R_i - \nu(R)\right).$$

If $x^i - \omega^i$ converge for every agent, we choose q_1 such that $\mu_1 q_1 = \nu(R) + \sum_{i=1}^{\infty} \mu_i R_i$ and $z_0^i \geq 0$ such that $M z_0^i < \underline{W}^i$ where $\underline{W}^i = \inf_t \{W_t^i\}$.

If there exists some agent such that $x^i - \omega^i$ does not converge, we choose big enough q_1 such that $\mu_1 q_1 > \nu(R) + \sum_{i=1}^{\infty} \mu_i R_i$ and z_0^i such that satisfies the condition above and $M z_0^i < \underline{W}^i$.

Now let us define some constants that would be necessary to guarantee the existence of the implementation with taxes.

Let us define $\gamma := \prod_{i=1}^{\infty}\left(\frac{R_i}{q_i} + 1\right)$, $(\tilde{q}_t)_t = (1/q_t)_t \in \ell^1$, $\underline{\beta} := \mu_1 q_1 - \sum_{i=1}^{\infty} \mu_i R_i$, $\tilde{\alpha} := \alpha/\underline{\beta}$ and

$$\tau_t(y) := \left(\frac{\underline{\beta}^{-1}}{\tilde{\alpha} + \|\tilde{q}\|_1 \gamma \underline{\beta}^{-1}}\right)\left(\left(\tilde{\alpha} \limsup (q_t(y_{t-1} - y_t) + R_t y_{t-1}) - \lim_t y_t\right) \vee 0\right)$$

And the relationship between y and z is given by:

$$z_t - y_t = \sum_{i=1}^{t} \frac{\prod_{j=0}^{t-i-1}\left(q_{t-j} + R_{t-j}\right)}{\prod_{j=0}^{t-i}\left(q_{t-j}\right)} \tau_i(y) = \sum_{i=1}^{t} \frac{\tau_i(y)}{q_i}\left(\prod_{j=0}^{t-j-1}\left(1 + \frac{R_{t-j}}{q_{t-j}}\right)\right)$$

And making the proper substitutions we have:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\beta \tau_i(y)}{q_i} \gamma &\geq \underline{\beta}\left(\tilde{\alpha} \limsup \left(x(y) - \omega^i\right) - \lim_t y_t\right) \\ &= \alpha \limsup \left(x(y) - \omega^i\right) - \underline{\beta} \lim_t y_t \\ &= \alpha \limsup \left(x(y) - \omega^i\right) - \lim_t \mu_t q_t y_t\end{aligned}$$

with equality with $y = y^i$, where y^i is the asset portfolio which implements the AD allocation with taxes τ. □

Proof of Proposition 15. Lets pick up an AD equilibria for endowments given by W^i, our goal is to implement it using an auxiliar economy without taxes but with some type of *P-constraint* in order to avoid *Long-Run Arbitrage.*

Consider $B^A(q, y_0^i, \omega^i, h, r)$ the set of plans $(x, \tilde{y}, \tilde{z})$ satisfying

$$\begin{aligned} x_t - \omega_t^i + q_t h_t \tilde{z}_t &\leq q_t(\tilde{y}_{t-1} - \tilde{y}_t) + R_t \tilde{y}_{t-1} + r_{t-1} q_{t-1} h_{t-1} \tilde{z}_{t-1}, \forall t \in \mathbb{N}, \\ \tilde{z}^- &\geq \tilde{y}. \end{aligned}$$

Let us mention that the equilibrium for this economy would be analogous to the previous auxiliar economies, making the proper adjustments in order to add the constraints related to the promises.

In order to guarantee that the AD allocation can be implemented in a sequential economy, we will need an useful sufficient condition for the optimality for these type of economies, that are, in a certain way, an extension to the case with just only one asset.

Proposition 23. *Let $(\tilde{y}^*, \tilde{z}^*)$ be a feasible portfolio and let $x^* = x(\tilde{y}^*, \tilde{z}^*)$. Suppose there exists $T \in \partial U(x^*)$ with $T = \mu + \nu$, $\mu \in \ell_+^1$ and $\nu \in pch_+$ such that, for t,*

$$\begin{aligned} \mu_t q_t &= \mu_{t+1}(q_{t+1} + R_{t+1}), \\ \mu_{t+1} r_t &= \mu_t, \end{aligned}$$

and

$$\lim_t (\mu_t q_t \tilde{y}_t^* + \mu_t q_t h_t \tilde{z}_t^*) = \nu(x^* - \omega).$$

And suppose also every feasible portfolio z satisfies the condition

$$\lim_t (\mu_t q_t \tilde{y}_t + \mu_t q_t h_t \tilde{z}_t) \geq \nu(x(\tilde{y}, \tilde{z}) - \omega).$$

Then $(\tilde{y}^*, \tilde{z}^*)$ is an optimal solution for the consumption problem with sequential constraints.

Analogous to the previous cases, the supergradient that would satisfy this optimality condition is a super-gradient whose pure charge $\tilde{\nu}^i$ takes the highest on the direction of the net trade and assuming that $\liminf(x^i - W^i) > 0$, we have that this direction is

$$\tilde{\nu}^i(x^i - \omega^i) = (u^i)'(\underline{x}^i) \limsup(x^i - \omega^i).$$

This suggest the following portfolio constraint

$$\lim_t (\mu_t q_t \tilde{y}_t + \mu_t q_t h_t \tilde{z}_t) \geq \alpha^i \limsup(x(\tilde{y}, \tilde{z}) - \omega^i)$$

where $x_t(\tilde{y}, \tilde{z}) = \omega_t^i + q_t(\tilde{y}_{t-1} - \tilde{y}_t) + R_t \tilde{y}_{t-1} + r_{t-1} q_{t-1} h_{t-1} \tilde{z}_{t-1} - q_t h_t \tilde{z}_t$ and $\alpha^i = \|\tilde{\nu}^i\|_{ba}$ for some $\tilde{\nu}^i$ satisfying the previous portfolio constraint.

By the proposition we should find q such that at $(\tilde{y}^i, \tilde{z}^i)_i$ implementing $(x^i)_i$ having $\lim_t \left(\mu_t q_t \tilde{y}^i_t + \mu_t q_t h_t \tilde{z}^i_t\right) \geq \tilde{\nu}^i \left(x^i - \omega^i\right)$.

And $x^i\left(\tilde{y}^*, \tilde{z}^*\right)$ belongs to the Arrow-Debreu budget set if and only if

$$\nu\left(x\left(\tilde{y}^i, \tilde{z}^i\right) - \omega^i\right) - \lim_t \left(\mu_t q_t \tilde{y}^i_t + \mu_t q_t h_t \tilde{z}^i_t\right) = z^i_0 \left(\nu(R) - \lim p_t q_t\right)$$

when the AD budget constraint hold with equality.

Then this condition can be written as

$$\tilde{\nu}^i\left(x^i - \omega^i\right) - \nu^i\left(x^i - \omega^i\right) = z^i_0 \left(\mu_1 q_1 - \sum_{i=1}^{\infty} \mu_i R_i - \nu(R)\right).$$

If $x^i - \omega^i$ converge for every agent we choose q_1 such that $\mu_1 q_1 = \nu(R) + \sum_{i=1}^{\infty} \mu_i R_i$ and $z^i_0 \geq 0$ such that $M z^i_0 < \underline{W}^i$ where $\underline{W}^i = \inf_t \{W^i_t\}$.

If there exists some agent such that $x^i - \omega^i$ does not converge we choose big enough q_1 such that $\mu_1 q_1 > \nu(R) + \sum_{i=1}^{\infty} \mu_i R_i$ and z^i_0 such that satisfies the condition above and $M z^i_0 < \underline{W}^i$.

Now let us define some constants that would be necessary to guarantee the existence of the implementation with taxes.

Let us define $\gamma := \prod_{i=1}^{\infty} \left(\frac{R_{i+1} + q_{i+1} - q_i h_i r_i}{q_{i+1}(1-h_{i+1})}\right)$, $(\tilde{q}_t)_t = (1/q_t)_t \in \ell^1$, $\underline{\beta} := \mu_1 q_1 - \sum_{i=1}^{\infty} \mu_i R_i$, $\tilde{\alpha} := \alpha/\underline{\beta}$, $\delta = \left(\frac{\underline{\beta}^{-1}}{\tilde{\alpha} + \|\tilde{q}\|_1 \gamma \underline{\beta}^{-1}}\right)$ and

$\tau_t(y,z)$:

$$= \delta\Big(\Big(\tilde{\alpha} \limsup (q_t (y_{t-1} - y_t) + R_t y_{t-1} + r_{t-1} q_{t-1} h_{t-1} z_{t-1} - q_t h_t z_t) - \lim_t (\hat{\alpha} z_t + y_t)\Big) \vee 0\Big).$$

And the relationship between y and $\tilde{y}$, and between z and between $\tilde{z}$ is given by:

$$\tilde{y}_t - y_t = z_t - \tilde{z}_t = \sum_{i=1}^{t} \frac{\prod_{j=0}^{t-i-1} \left(q_{t-j} + R_{t-j} - q_{t-j-1} h_{t-j-1} r_{t-j-1}\right)}{\prod_{j=0}^{t-i} \left(q_{t-j}\left(1 - h_{t-j}\right)\right)} \tau_i(y,z).$$

And making the proper substitutions we have:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\underline{\beta} \tau_i(y,z)}{q_i(1-h_i)} \gamma &\geq (1-\hat{\alpha})^{-1} \underline{\beta} \left(\tilde{\alpha} \limsup \left(x(y,z) - \omega^i\right) - \lim_t \left(y_t + \hat{\alpha} z_t\right)\right) \\ &= (1-\hat{\alpha})^{-1} \Big(\alpha \limsup \left(x(y,z) - \omega^i\right) - \lim_t \left(\underline{\beta} y_t + \hat{\alpha} \underline{\beta} z_t\right)\Big) \\ &= \frac{\alpha \limsup \left(x(y,z) - \omega^i\right) - \lim_t \left(\mu_t q_t y_t + \mu_t q_t h_t z_t\right)}{(1-\hat{\alpha})}\end{aligned}$$

with equality when $y = y^i$ and $z = z^i$, where (y^i, z^i) is the portfolio. And for $(\tilde{y}, \tilde{z})$ making the substitutions we have:

$$\lim_t \left(\mu_t q_t \tilde{y}_t + \mu_t q_t h_t \tilde{z}_t\right) \geq \alpha^i \limsup \left(x(\tilde{y}, \tilde{z}) - \omega^i\right)$$

which implements the AD allocation with taxes τ. □

Proof of Theorem 5. To start our analysis let us exposed sufficient conditions for a consumption plan to be optimal and an expression of the supergradient for the optimal plan.

Proposition 24. Consider a consumption plain $x^* \ggg 0$ such that

$$\mathbb{E}_t\left[u\left(x_t^*\right)\right] > \inf_{s\geq 1} \mathbb{E}_s\left[u\left(x_s^*\right)\right] \quad \forall t \geq 0$$

and $\mathbb{E}_t\left[u\left(x_s^*\right)\right] \to \inf_{s\geq 1} \mathbb{E}_s\left[u\left(x_s^*\right)\right]$. $\pi \in \partial U\left(x^*\right)$ if and only if it is given by

$$\pi(x) = \sum_{t\geq 0} \delta^t \mathbb{E}_t\left[u'(x_t^*)\cdot x_t\right] + \beta\,\nu\left(\left(\mathbb{E}_t\left[u'(x_t^*)\cdot x_t\right]\right)_{t\geq 0}\right)$$

where $\nu \in pch(\ell^\infty)$ such that $\|\nu\|= 1$.

Proof. It is enough to show that, given $x \in \ell^\infty(\mathcal{S})$,

$$\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x_t^*)] \;\leq\; \nu\left(\left(\mathbb{E}_t[u'(x_t^*)\cdot(x_t - x_t^*)]\right)_{t\geq 0}\right).$$

Given $\varepsilon > 0$, we have, for $t_1 > 0$ large enough,

$$\begin{aligned}\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x_t^*)] - \varepsilon \;&<\; \mathbb{E}_{t_1}[u(x_{t_1})] - \mathbb{E}_{t_1}[u(x_{t_1}^*)] \\ &\leq\; \mathbb{E}_{t_1}[u'(x_{t_1}^*)\cdot(x_{t_1} - x_{t_1}^*)].\end{aligned}$$

Making $t_1 \to \infty$, we get $\inf_t \mathbb{E}_t[u(x_t)] - \inf_t \mathbb{E}_t[u(x_t^*)] - \varepsilon \;\leq\; \liminf_t \mathbb{E}_t[u'(x_t^*)\cdot(x_t - x_t^*)]$. As $\|\nu\|= 1$ implies $\nu(z) \geq \liminf z\ \forall z \in \ell^\infty$ and the constant ε is arbitrary.

To prove the other part of the proposition, let us use some results of nonsmooth analysis (see [20]) and also express the utility function as a composition of two different functions ϕ:

$$\begin{aligned}\phi : \ell^\infty(\mathcal{S}) &\to \ell^\infty \\ x &\mapsto \left(\phi_t(x)\right)_{t\in\mathbb{N}} := \left(\mathbb{E}_t\left[u\left(x_t\right)\right]\right)_{t\in\mathbb{N}}\end{aligned}$$

and V

$$\begin{aligned}V : \ell^\infty &\to \mathbb{R} \\ y &\mapsto V(y) := \textstyle\sum_{t\geq 1} \delta_t y_t + \beta \inf_t y_t\end{aligned}$$

as $U(x) = V \circ \phi(x)$. Since U is concave and Lipschitz[1] close to x^* (this is a consequence of $x^* \ggg 0$), we have that $\partial_c U\left(x^*\right) = \partial U\left(x^*\right)$, see page 36 proposition 2.2.7, where $\partial_c F(y)$ is the Clarke subdifferential, see page 10. Also notice that for V we have the same property.

[1] In the sup-norm

And since ϕ is Lipschitz close to x^*, we have that ϕ is strictly differentiable (see page 30 proposition 2.2.4). And as a consequence of the Chain Rule, see page 45 proposition 2.3.10, we have that

$$\partial U(x) \subseteq \partial V(\phi(x^*)) \circ \phi'(x^*)$$

which concludes the proof. □

Now let us expose the sufficient condition for optimality.

Proposition 25. *Let* $(\tilde{y}^*, \tilde{z}^*)$ *be a feasible portfolio and let* $x^* = x(\tilde{y}^*, \tilde{z}^*)$. *Suppose there exists* $T \in \partial U(x^*)$ *with* $T = \mu + \nu$, $\mu \in \ell^1_+$ *and* $\nu \in pch_+$ *such that, for* t,

$$\begin{aligned} \mu_{s_t} q^{1,j}_{s_t} &= \textstyle\sum_{s^-_{t+1}=s_t} \mu_{s_{t+1}} \left(R^j_{s_t} + q^{1,j}_{s_t}\right) \ \forall s_t, j = 1, 2, \\ \mu_{s_t} q^2_{s_t} &= \textstyle\sum_{s^-_{t+1}=s_t} \mu_{s_{t+1}} q^2_{s_t} \ \forall s_t \end{aligned}$$

and

$$\lim_t \left(\sum_{s_t} \left[\mu_{s_t} q^1_{s_t} \tilde{y}^*_{s_t} + \mu_{s_t} q^2_{s_t} \tilde{z}^*_{s_t} \right] \right) = \nu\left(x^* - \omega\right).$$

And suppose also every feasible portfolio z *satisfies the condition*

$$\lim_t \left(\sum_{s_t} \left[\mu_{s_t} q^1_{s_t} \tilde{y}_{s_t} + \mu_{s_t} q^2_{s_t} \tilde{z}_{s_t} \right] \right) \geq \nu\left(x\left(\tilde{y}, \tilde{z}\right) - \omega\right).$$

Then $(\tilde{y}^*, \tilde{z}^*)$ *is an optimal solution for the consumption problem with sequential constraints.*

Given $z^i_0 \geq 0$ and $q_1 \gg 0$, using the Euler Conditions that we mentioned before and choosing $\tilde{z}^i_{s_t} = z^i_0$ for each s_t and for each i, there exists $\tilde{y} = (\tilde{y}^1, \tilde{y}^2)$ such that $(\tilde{y}^i, \tilde{z}^i)$ implement the efficient allocation in the auxiliar economy. To do so, the supergradient that we will consider is the one that the pure charge $(\tilde{\nu}^i)$ assumes the highest value in the direction of the net trade $x^i - W^i$.

Now, in order to stablish a relationship between the constraints in the auxiliar economy and the AD constraint, we will analyze in a similar way as the previous cases.

Then $x^i(\tilde{y}, \tilde{z})$ belongs to the AD budget set with equality if and only if

$$\begin{aligned} &\nu\left(x^i\left(\tilde{y}, \tilde{z}\right) - \omega\right) - \lim_t \left(\sum_{s_t} \left(\mu_t q^1_t \tilde{y}_t + \mu_t q^2_t \tilde{z}_t \right) \right) \\ &= - \left(\tilde{y}^i_0 \lim \sum_{s_t} \mu_{s_t} q^1_{s_t} + \tilde{z}^i_0 \lim \sum_{s_t} p_t q^2_t \right), \end{aligned}$$

and now using the transversality condition of the previous proposition for the efficient allocation we have that, in order to implement the efficient allocation we will have that:

$$(\rho^i)^{-1}\tilde{\nu}^i\left(x^i-\omega^i\right)-\nu\left(x^i-\omega^i\right)=\tilde{y}_0^i \lim\sum_{s_t}\mu_{s_t}q^1_{s_t}+\tilde{z}_0^i\lim\sum_{s_t}{\mu^i}_{s_t}q^2_{s_t}$$

As a consequence of this, the efficient allocation can be implemented in the auxiliar economy with an additional constraint for each agent in order to avoid *Long-Run Arbitrage*. And now to implement this equilibrium in the initial economy with taxes, we need to implement the transversality condition using taxes.

Since $\tilde{\nu}^i$ must be a capacity that in the direction of the net trade takes the highest value, the proposition 24 says that the pure charge that we take in the proposition 25 must satisfy in the net trade

$$\limsup_{\{\{t_r\}_r:\lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})]=\inf_t \mathbb{E}_t[u_i(x_t)]\}}\mathbb{E}_t[(u_i'(x_t^i)(x_t(\tilde{y},\tilde{z})-\omega_t^i))].$$

Then choosing the capacity given by:

$$\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega^i_{s_t}\right)=\limsup_{\{\{t_r\}_r:\lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})]=\inf_t \mathbb{E}_t[u_i(x_t)]\}}\mathbb{E}_t\left[u_i'(x_t^i)\left(x_t\left(\tilde{y}^i,\tilde{z}^i\right)-\omega_t^i\right)\right] \tag{C.1}$$

where $x_{s_t}(\tilde{y},\tilde{z})=W^i_{s_t}+q^1_{s_t}\left(\tilde{y}_{s_{t-1}}-\tilde{y}_{s_t}\right)+R_{s_t}\tilde{y}_{s_{t-1}}+q^2_{s_t}\left(\tilde{z}_{s_{t-1}}-\tilde{z}_{s_t}\right)$, can be defined the personal taxes as

$$\begin{aligned}&\tau^i_{s_t}(y,z)\\&=\frac{p_t}{\|p\|_1}\max\Bigg\{0,\limsup_{\{\{t_r\}_r:\lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})]=\inf_t \mathbb{E}_t[u_i(x_t)]\}}\mathbb{E}_t\Bigg[u_i'(x_t^i)\Bigg(q^1_{s_t}\left(y_{s_{t-1}}-y_{s_t}\right)\\&\quad+R_{s_t}y_{s_{t-1}}+q^2_{s_t}\left(z_{s_{t-1}}-z_{s_t}\right)\Bigg)\Bigg]-\lim_t\left(\sum_{s_t}\left[\mu_{s_t}q^1_{s_t}y_{s_t}+\mu_{s_t}q^2_{s_t}z_{s_t}\right]\right)+A\Bigg\}\end{aligned}$$

where $p_t=\sum_{s_t}p_{s_t}$.

Since we have

$$\liminf_{\left\{t:\mathbb{E}_t\left[u_i'(x_t)\left(x_t^i-W_t^i\right)\right]>0\right\}}\mathbb{E}_t[u_i(x_t^i)]=\inf_s\left(\mathbb{E}_s[u_i(x_s^i)]\right),$$

the pure charge $\tilde{\nu}^i$ in the direction of the net trade satisfies

$$\begin{aligned}\nu^i\left(x_t(\tilde{y}^i,\tilde{z}^i)-\omega_t^i\right)&=\limsup_{\{\{t_r\}_r:\lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})]=\inf_t \mathbb{E}_t[u_i(x_t)]\}}\mathbb{E}_t[(u_i'(x_t^i)(x_t(\tilde{y}^i,\tilde{z}^i)-\omega_t^i))]\\&=\limsup\mathbb{E}_t[(u_i'(x_t^i)(x_t(\tilde{y}^i,\tilde{z}^i)-\omega_t^i))],\end{aligned}$$

and now let us analyze each case. To prove the first one, let us notice that using the FOC for each i we have $\frac{u_i'\left(x_{s_t}^i\right)}{\mathbb{E}_t\left[u_i'\left(x_t^i\right)\right]} = \frac{p_{s_t}}{\mathbb{E}_t[p_t]}$ and that $\lim_t \mathbb{E}_t\left[u_i'\left(x_t^i\right)\right]$ exists, we have that C.1 can be written as

$$\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega_{s_t}^i\right) = \alpha^i \limsup \mathbb{E}_t\left[\frac{p_t}{\mathbb{E}_t\left[p_t\right]}\left(x_t\left(\tilde{y},\tilde{z}\right)-\omega_t^i\right)\right]$$

where $\alpha^i := \lim_t \mathbb{E}_t\left[u_i'\left(x_t^i\right)\right]$, similarly as in the other cases, we have that without loss of generality we can have that $\alpha^i = \alpha^j = \alpha$ for each i, j therefore the taxes can be defined as

$$\tau_{s_t}(y,z) = \frac{p_{s_t}}{\|p\|_1}\max\left\{0, \alpha \limsup \mathbb{E}_t\left[\frac{p_t}{\mathbb{E}_t\left[p_t\right]}\left(q_{s_t}^1\left(y_{s_{t-1}}-y_{s_t}\right)+R_{s_t}y_{s_{t-1}}\right.\right.\right.$$
$$\left.\left.\left.+\, q_{s_t}^2\left(z_{s_{t-1}}-z_{s_t}\right)\right)\right] - \lim_t\left(\sum_{s_t}\left[\mu_{s_t}q_{s_t}^1 y_t + \mu_{s_t}q_{s_t}^2 z_t\right]\right)+A\right\}.$$

And now let us prove the second case. If we define $\alpha^i := \frac{u_i'\left(x_{s_T}^i\right)}{p_{s_T}}$ for each i and any s_T state of T, we will have that C.1 can be written as

$$\tilde{\nu}^i\left(x(\tilde{y}^i,\tilde{z}^i)-\omega_{s_t}^i\right) = \alpha^i \limsup \mathbb{E}_t\left[p_t\left(x_t\left(\tilde{y}^i,\tilde{z}^i\right)-\omega_t^i\right)\right],$$

and then, similarly as in the other cases, we have that without loss of generality we can have that $\alpha^i = \alpha^j = \alpha$ therefore the taxes will be

$$\tau_{s_t}(y,z) = \frac{p_{s_t}}{\|p\|_1}\max\left\{0, \alpha \limsup \mathbb{E}_t\left[p_t\left(q_{s_t}^1\left(y_{s_{t-1}}-y_{s_t}\right)+R_{s_t}y_{s_{t-1}}\right.\right.\right.$$
$$\left.\left.\left.+\, q_{s_t}^2\left(z_{s_{t-1}}-z_{s_t}\right)\right)\right] - \lim_t\left(\sum_{s_t}\left[\mu_{s_t}q_{s_t}^1 y_t + \mu_{s_t}q_{s_t}^2 z_t\right]\right)+A\right\}.$$

□

Proof of Proposition 5. Let $(\pi,(x^i)_i)$ an Arrow-Debreu equilibrium, the idea is to use a similar technique that it was used before. Since we have two assets and two states of nature in each date, we can choose an specific allocation of money in each state in an auxiliar economy without taxes. Also let us assume that $\mu_1 q_1 = 1$.

To do so, it is necessary to have a proposition that helps us with sufficient conditions for optimality in sequential economies with the three assets with possitive price for each of them in each state.

Proposition 26. Let $(\tilde{y}^*, \tilde{z}^*)$ *be a feasible portfolio and let* $x^* = x(\tilde{y}^*, \tilde{z}^*)$. *Suppose there exists* $T \in \partial U(x^*)$ *with* $T = \mu + \nu$, $\mu \in \ell^1_+$ *and* $\nu \in pch_+$ *such that, for t,*

$$\mu_{s_t} q_{s_t} = \textstyle\sum_{s_{t+1}^- = s_t} \mu_{s_{t+1}} (R_{s_t} + q_{s_t}) \;\; \forall s_t,$$
$$\left(\textstyle\sum_{s_{t+1}^- = s_t} \mu_{s_{t+1}}\right) r_{s_{t+1}} = \mu_{s_t} \;\; \forall s_t$$

and

$$\lim_t \left(\sum_{s_t} \left[\mu_{s_t} q_{s_t} \tilde{y}^*_{s_t} + \mu_{s_t} q_{s_t} h_{s_t} \tilde{z}^*_{s_t} \right] \right) = \nu (x^* - \omega).$$

And suppose also every feasible portfolio z satisfies the condition

$$\lim_t \left(\sum_{s_t} \left[\mu_{s_t} q_{s_t} \tilde{y}_{s_t} + \mu_{s_t} q_{s_t} h_{s_t} \tilde{z}_{s_t} \right] \right) \geq \nu (x(\tilde{y}, \tilde{z}) - \omega).$$

Then $(\tilde{y}^*, \tilde{z}^*)$ *is an optimal solution for the consumption problem with sequential constraints.*

Therefore to implement with taxes is necessary to use the taxes in order to satisfy the last two equations using the relationship between both portfolios (with taxes (y, z) and without taxes $(\tilde{y}, \tilde{z})$) given by

$$\tilde{y}^1_{s_t} - y^1_{s_t} = z^1_{s_t} - \tilde{z}^1_{s_t} = \sum_{i=1}^{t} \frac{\prod_{k=0}^{t-i-1} \left(q^1_{s_{t-k}} + R^1_{s_{t-k}} - q^1_{s_{t-k-1}} h^1_{s_{t-k-1}} r_{s_{t-k-1}} \right)}{\prod_{k=0}^{t-i} \left(q^1_{s_{t-k}} \left(1 - h^1_{s_{t-k}} \right) \right)} \tau_i(y, z).$$

And then implementation depends mainly on the convergence of $\sum_t C_{s_t}$ for any path $(s_t)_{t\in\mathbb{N}}$ of the tree when:

$$C_{\hat{s}_{t+2}} := \frac{\sum_{s_{t+2}^- = \hat{s}_{t+1}} \left(\mu_{s_{t+2}} \left(q^1_{s_{t+2}} - q^1_{\hat{s}_{t+2}} \frac{1 - h^1_{\hat{s}_{t+2}}}{\left(\alpha^1_{\hat{s}_{t+1}}\right)^{-1} - 1} \right) \right) \left(\left(\alpha^1_{\hat{s}_{t+1}} \right)^{-1} - 1 \right) h^1_{\hat{s}_{t+1}}}{\sum_{s_{t+2}^- = \hat{s}_{t+1}} \mu_{s_{t+2}} q^1_{\hat{s}_{t+2}} \left(1 - h^1_{\hat{s}_{t+2}} \right)}$$

where $\alpha^1_{s_t} = \frac{q^1_{s_t} h^1_{s_t} r_{s_t}}{R^1_{s_{t+1}} + q^1_{s_{t+1}}}$, we can suppose without loss of generality that $\hat{s}_{t+1}$ is the first successor of $\hat{s}_t$, and then $C_{\hat{s}_{t+2}}$ can be written as

$$C_{\hat{s}_{t+2}} := \frac{\sum_{s_{t+2}^- = \hat{s}_{t+1}} \left(\mu_{s_{t+2}} \left(q^1_{s_{t+2}} - q^1_{\hat{s}_{t+2}} \frac{1 - h^1_{\hat{s}_{t+2}}}{\left(\alpha^1_{\hat{s}_{t+1}}\right)^{-1} - 1} \right) \right) \left(\left(\alpha^1_{\hat{s}_{t+1}} \right)^{-1} - 1 \right) h^1_{\hat{s}_{t+1}}}{\left(\mu_{\hat{s}_{t+1,1}} + \mu_{\hat{s}_{t+1,2}} \right) q^1_{\hat{s}_{t+2}} \left(1 - h^1_{\hat{s}_{t+2}} \right)},$$

then the convergence of the series depends on

$$\frac{\left|q_{\hat{s}_{t+1,1}} - q_{\hat{s}_{t+1,2}}\right|}{\min_{j=1,2} q_{\hat{s}_{t+1,j}}} \to 0 \quad a.s.$$

But can be seen that the previous condition can not necessarily satisfied in all the cases, in presence of big differences between states, this condition can not be satisfied always. Therefore there is impossible to define the taxes in order to satisfy the *P-constraint* to guarantee the implementation of the efficient allocation.

□

Proof of Proposition 4. The proof is similar to the proposition 5 in terms of the optimal conditions that must be satisfied and similar to the proposition 14 related to the type of constants that are required to implement the efficient allocation. Therefore the personal taxes[2] can be defined as:

$$\tau^i_{s_t}(y) = \frac{1}{\alpha^i \underline{\beta} + \gamma} \max\left\{0, \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})] = \inf_t \mathbb{E}_t[u_i(x_t)]\}} \mathbb{E}_t\left[u'_i(x^i_t)\left(q_{s_t}\left(y_{s_{t-1}} - y_{s_t}\right) + R_{s_t} y_{s_{t-1}}\right)\right] - \lim_t \left(\sum_{s_t} \mu_{s_t} q_{s_t} y_{s_t}\right)\right\}$$

where $\alpha^i = \limsup_{\{\{t_r\}_r \,:\, \lim_r \mathbb{E}_{t_r}[u_i(x_{t_r})] = \inf_t \mathbb{E}_t[u_i(x_t)]\}} \mathbb{E}_t\left[u'_i\left(x^i_t\right)\right]$, $\underline{\beta} = \mu_{1,1} q_{1,1} - \sum_{s_t} R_{1,s_t}$, and $\gamma = \sum_{s_j}\left[\Pi^{\infty}_{i=j+1}\left(1 + \frac{R_{1,s_i}}{q_{1,s_i}}\right)/q_{1,s_j}\left(\lim_{t\to\infty} \sum_{\tilde{s}\,:\,\tilde{s}^{-(t-j)}_t = s_j} \mu_{1,s_t} q_{1,s_t}\right)\right]$.

And if I.O.U.s are added in order to complete markets, we can define optimality conditions, similarly to Lemma 25, which allow us to define a fiscal policy τ to avoid the *long-run improvement opportunities*, under a no short sales constraint on the Lucas trees.

□

C.4 Proofs of Chapter 3

Proof of Proposition 18. The idea of the proof is to separate each set A_1, A_2 and A_3 to analyze them separately, and then apply the dominated convergence theorem to have the result. For each set A_1, $A^K_{2,t_1,\ldots,t_K}$, $A_{3,t_1,t_2\ldots}$ and σ belonging to any of the previous sets, we have that the analysis that can be done in the path σ is analogous to the deterministic case with a ε-*contamination* utility function (see Araujo et al. [5]), therefore the results will be true for each path σ that belongs to any of the sets described before.

[2]This taxes are similar to the type of taxes needed in Remark 14

Since the collection of all sets that have been described before is $2-2$ disjoint and non enumerable, there is a enumerable subcollection with positive measure. Therefore we can rewrite the utility function in terms of these enumerable subcollection only and apply the deterministic case in each path that belongs to any of this subsets of the collection. And finally if we apply the dominated convergence theorem (for a collection of generalized limits that are measurables in $(\{1\} \times N^{\infty}, \mathcal{N})$). Concluding one part of the proof.

To prove the other part, notice that what we have done is to prove that the integral in σ of elements of the subdifferential of the utility function in each path σ, are in the subdifferential of the utility function. And can be easily observed that in order to prove second part is enough to prove that the subdifferential is contained in the composition between the integral and the subdifferential for each path.

And now using some results of non differential analysis in Banach spaces we have that, under the condition that we exposed before, the definition of subgradient and Clarke subgradient are equivalents (see Clarke [20], page 36 Proposition 2.2.7). Also we have that under our hypothesis the Clarke subdifferential of the utility function is contained in the integral in σ of elements in the Clarke subgradients of $\int_{\mathbb{N}} u \circ X_{\sigma^t} d\zeta_i(t) + \beta(\sigma) \inf_t u \circ X_{\sigma^t}$ for each σ (see Clarke [20], page 76 Proposition 2.7.2). Which concludes the proof of the proposition. □

Proof of Proposition 22. Since:

- the bubble in the economy is characterized by the pure charges that exists in the subgradient of the agent, more precisely given by 3.4,
- the pure charges that exist in the subgradient of the agents are integral in σ of a collection of generalized limits and
- the probability of each path, $\mathbb{P}(\{\sigma\})$, is zero;

we have that $\lim_r \sum_{s^r,-(r-t)=\sigma^t} \sum_j \mu_{s^r} q_{j,s^r} \rightarrow_{t\rightarrow\infty} 0$ □

Bibliography

[1] M. Allais, *Le comportement de l'homme rationnel devant le risque : critique des postulats et axiomes de l'école américaine*, Econometrica, 21, 503-546, 1953.

[2] R.M. Anderson, *Strong core theorems with non-convex preferences*, Econometrica, Vol. 53 (6), 1283-1294, 1985.

[3] A.Araujo *Lack of Pareto Optimal Allocations in Economics with Infinitely Many Commodities: the Need for Impatience.* Econometrica **146** 455-461, 1985.

[4] A. Araujo, J. M. Bonnisseau, A. Chateauneuf and R. Novinski, *Optimal Risk Sharing with Optimistic and Pessimistic Decision Makers*, working paper.

[5] A.Araujo, R.Novinski and M.Pascoa, *General Equilibrium, Wariness and Efficient Bubbles*, Journal of Economic Theory **53** 785-811, 2011.

[6] K. Arrow, G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica, Vol. 22 (3), 265-290, 1954.

[7] R. Aumann, *Markets with a Continuum of Traders*, Econometrica, Vol. 32 (1-2), 39-50, 1964.

[8] R. Aumann, *Existence of Competitive Equilibria in Markets with a Continuum of Traders*, Econometrica, Vol. 34 (1), 1-17, 1966.

[9] T.Bewley, *Existence of Equilibrium in Economies with Infinitely Many Commodities*, Journal of Economic Theory **4** 514-40, 1972.

[10] T.Bewley (1980). *The Optimum Quantity of Money*, Models of Monetary Economics, ed. by J. Kareken and N.Wallace. Federal Reserve Bank, Minneapolis, 1980.

[11] T.Bewley, *A Difficulty with the Optimum Quantity of Money*, Econometrica **51**, 1485-1504, 1983.

[12] K.Bhaskara Rao and M.Bhaskara Rao, *Theory of Charges*. Academic Press, New York, 1983.

[13] O. Blanchard, *Speculative bubbles, crashes and rational expectations*, *Economics Letters* **3** 387-389, 1979.

[14] O. Blanchard and M. Watson, *Bubbles, rational expectations and speculative markets*. Crisis in economic and financial structure: bubbles, bursts, and shocks, Lexington, 1982.

[15] P. Bossaerts, P. Ghirardato, S. Guarnaschelli and W. Zame, *Ambiguity in Asset Markets: Theory and Experiment*, Review of Financial Studies 23 (4), 1325-1359, 2012.

[16] D.Brown and L.Lewis, *Myopic Economic Agents,* Econometrica, 49, 359-368, 1981.

[17] H. Bühlmann, *An economic Premium Principle*, ASTIN Bulletin 11 (1), 52-60, 1980.

[18] H. Bühlmann, *The General Economic Premium Principle*, ASTIN Bulletin 14 (1), 13-21, 1984.

[19] S. Chateauneuf, R. A. Dana and J. M. Tallon, *Optimal risk-sharing rules and equilibria with Choquet-Expected utility*, Journal of Mathematical Economics 34, 191-214, 2000.

[20] F. H. Clarke, *Optimization and Nonsmooth Analysis*. Society for Industrial and Applied Mathematics, Philadelphia, 1990.

[21] J. Dow and S. Werlang (1992). *Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio*, Econometrica **60**, 197-204, 1992.

[22] D. Ellsberg, *Risk, ambiguity and the Savage axioms*, Quartely Journal of Economics, 75, 643-669, 1961.

[23] M.Friedman *The Optimum Quantity of Money*, in The Optimum Quantity of Money and Other Essays. Aldine, Chicago, 1969.

[24] M. Friedman and L. J. Savage, *The Utility Analysis of Choices Involving Risk*, The Journal of Political Economy 56 (1), 279-304, 1948.

[25] I.Gilboa *Expectation and Variation in Multi-period Decisions*, *Econometrica* **57** 1153-69, 1989.

[26] I.Gilboa and D.Schmeidler, *Maxmin Expected Utility with a Non-Unique Prior*, Journal of Mathematical Economics **110** 605-39, 1989.

[27] L. Hansen and T. Sargent, *Robustness*. Princeton University Press, New Jersey, 2008.

[28] P. Klibanoff, M. Marinacci and S. Mukerji, *A smooth model of decision making under ambiguity*, Econometrica, 73 (6), 1849-1892, 2005.

[29] F. Maccheroni, M. Marinacci and A. Rustichini, *Ambiguity Aversion, Robustness, and the Variational Representation of Preferences*, Econometrica, 74 (6), 1447-1498, 2006.

[30] A. Mas-Colell and W. Zame, *Equilibrium theory in infinite dimensional spaces*, Handbook of Mathematical Economics **4**, 1835-1898, 1991.

[31] M.Pascoa, M.Petrassi and J.Torres-Martinez, *Fiat Money and the Value of Binding Portfolio Constraints*, Economic Theory **46**, 189-209, 2010.

[32] L. Rigotti, C. Shannon and T. Strzalecki, *Subjective Beliefs and Ex Ante Trade*, Econometrica 76 (5), 1176-1190, 2008.

[33] J. Quiggin, *A Theory of Anticipated Utility*, Journal of Economic Behavior and Organization, 3, 323-343, 1982.

[34] J. Quiggin, *Generalized Expected Utility Theory: The Rank-dependent Model*, Kluwer Academic Publisher, Boston, 1993.

[35] P. Samuelson, *An Exact Consumption-Loan Model of Interest with or without the SocialContrivance of Money*, The Journal of Political Economy **66** 467-82, 1958.

[36] M. Santos and M. Woodford, *Rational asset pricing bubbles*, Econometrica **65** 1, 19-57, 1997.

[37] D. Schmeidler, *Subjective Probability and Expected Utility without Additivity*, Econometrica, 57 (3), 571-587, 1989.

[38] R. M. Starr, *Quasi-Equilibria in Markets with Non-Convex Preferences*, Econometrica, 37 (1), 25-38, 1969.

[39] T. Strzalecki and J. Werner, *Efficient Allocations under Ambiguity*, Journal of Economic Theory, 146 (3), 1173-1194, 2011.

[40] A. Tsanakas and N. Christofides, *Risk Exchange with Distorted Probabilities*, ASTIN Bulletin, 2006.

[41] N. Wallace, *Optimal Money-Creation in "Pure-Currency" Economies: a Conjecture,* Quarterly Journal of Economics 129, 259-274, 2014.

[42] M. Woodford, *The Optimum Quantity of Money*, in B. Friedman and F. Hahn (eds.), Handbook of Monetary Economics, vol. II, ch. 20. North Holland, Amsterdam, 1990.

[43] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* Vol. III, Springer, New York, 1984.