Physical Measures for Certain Partially Hyperbolic Attractors on 3-Manifolds

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Abstract

In this work, we study ergodic properties of some attractors “beyond uniform hyperbolicty”, our interest is the existence and finiteness of physical measures. We are going to deal with partially hyperbolic attractors whose central direction has a neutral behavior, the main feature is a condition of transversality between unstable leaves when projected by the stable holonomy.

We prove that partial hyperbolic attractors satisfying conditions of transversality between unstable leaves via the stable holonomy (non-integrability of $E^s \oplus E^u$), neutrality in the central direction and regularity of the stable foliation admits a finite number of physical measures, coinciding with the ergodic u-Gibbs States, whose union of the basins has full Lebesgue measure. Moreover, we describe the construction of a family of robustly nonhyperbolic attractors satisfying these properties.

Dedicatória

Dedicado a:

Luiz Bortolotti, Mauro Turolla, Maria Terezinha Bortolotti,

Luiz Ismael Bortolotti e Thiago Luis Biscaro.

(In memoriam)

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"A imaginação é mais importante que o conhecimento.

O conhecimento é limitado.

A imaginação circunda o mundo."

(Albert Einstein)

Contents

Chapter 1

Introduction

In this work, we study ergodic properties of some attractors "beyond uniform hyperbolicty". Our main interest is the existence and finiteness of physical measures for partially hyperbolic systems.

This study began with Sinai, Ruelle and Bowen for hyperbolic systems and was extended in many works for partially hyperbolic systems, for Henon-like family, for Lorenz-like attractors and others type of systems. In most of these contexts, it was shown that the dynamical system admits a finite number of physical measures whose basins cover a full Lesbesgue measure subset.

It was conjectured by Palis ([13]) that every dynamical system can be approximated by another having finitely many physical measures, whose union of the basins has total Lebesgue measure.

One can consider this Conjecture for the open set of partially hyperbolic systems. In the works of Bonatti-Viana ([4]) for "mostly contracting" and Alves-Bonatti-Viana ([1]) for "mostly expanding" central direction, they prove the existence and finiteness of physical measures for partially hyperbolic systems with some kind of contraction or expansion in the central direction. The case yet not well studied is when the central direction is neutral.

Nevertheless, it is not known if generic partially hyperbolic systems admits physical measures. It is interesting to point for results of genericity for generic partially hyperbolic systems.

In the case of surface endomorphism, Tsujii ([17]) made a generic approach for this question. Actually, he proved that a generic partial hyperbolic endomorphism admits finitely many physical measures whose basins have full Lebesgue measure.

This work is a first stage to extend the analysis of Tsujii for diffeomorphisms.

In the present work, we consider partially hyperbolic attractors with central Lyapunov exponent close to neutral (neither mostly contracting nor mostly expanding) and with a geometrical characteristic of transversality for the strong-unstable leaves relatively to the strong-stable foliation (non joint-integrability of $E^{ss} \oplus E^{uu}$).

We prove existence of physical measures for some open sets of attractors, as the following theorem:

Theorem. *Consider $f_0 : M \to M$ a diffeomorphism of class C^r, $r \geq 2$, a three-dimensional manifold M and Λ a partially hyperbolic attractor for f_0. Suppose that the attractor is dynamically coherent and that are valid the following hypothesis:*

(H1) Transversality between unstable leaves via the stable holonomy (non-integrability of $E^{ss} \oplus E^{uu}$);

(H2) Central direction neutral;

(H3') The foliation $\mathcal{F}^{ss}$ is of Lipschitz class.

Then f_0 admits finite physical measures whose union of their basins has total Lebesgue measure in the basin of attraction of the attractor.

Moreover, if the attractor is robustly dynamically coherent and it is valid:

(H3) $\mathcal{F}^{ss}_f$ varies continuously in the C^1-topology.

Then there exists an open set $\mathcal{U}$ containing f_0 such that the same result holds for every $f \in \mathcal{U}$.

Precise definitions and statements will be given in Chapter 2.

This geometrical characteristic of transversality allows to prove that u-Gibbs States of diffeomorphisms are sent by the stable holonomy into absolutely continuous measures. This step contains the technical part of this work and allows to prove that the ergodic u-Gibbs States are the physical measures.

In the final part of this work, we describe a construction of partially hyperbolic attractors that are robustly nonhyperbolic and satisfies these conditions.

We emphasize that our context includes situations where the central Lyapunov exponent is null and, therefore, we can't use Pesin's theory.

It is expected that some form of the property of transversality is generic among partially hyperbolic systems. Before that, it will be necessary to weaken the exigence of regularity for the stable foliation.

Strategy of the proof

The heart of the proof is an inequality similar to the Doeblin-Fortet inequality (also known as Lasota-Yorke inequality). We will work with a semi-norm for finite measures defined in center-unstable manifolds that is similar to the L^2 norm for the densities of the measures when they are absolutely continuous. We want to see that every u-Gibbs State has certain regularity after projecting by the stable holonomy, this will imply that the ergodic u-Gibbs States are the physical measures.

The transversality hypothesis plays an important role because the density of the u-Gibbs State is good *a priori* only in the unstable direction, then, if the stable projection of unstable discs give many directions, we will obtain certain mass in many directions, guaranting that the projected measure into center-unstable leaves will be absolutely continuous.

Structure of the Work

This work is organized as follows.

In Chapter 2, we give the basic definitions and precise statements of the Theorems.

In Chapter 3, we define the notion of boxes and of semi-norms, that will be used to state the Main Inequality and the technical Lemmas.

In Chapter 4, we prove estimatives for the semi-norm, these estimatives will culminate proving the Main Inequality.

In Chapter 5, we use the Main Inequality to show that every ergodic u-Gibbs State has a good regularity after projecting via the stable holonomy into center-unstables leaves. Then we conclude the existence and finiteness of physical measures and that the union of their basins have full Lebesgue measure.

In Chapter 6, we describe the construction of nonhyperbolic attractors that has central direction close to neutral and satisfies the transversality condition, we also check that they are robustly transitive.

Chapter 2

Definitions and Statements

Our goal in this Chapter is to give the notions that shall be used throughout the text and precise statements of this work.

2.1 Prerequisites

We consider M a compact riemannian manifold with its respective normalized Lebesgue measure m, $f : M \to M$ a differentiable function of class C^r with $r \geq 1$.

2.1.1 Physical Measures and Basins

Definition 2.1. Given a f-invariant measure μ, we define the **basin of** μ as the set $B(\mu)$ of points x such that for every continuous function $\phi : M \to \mathbb{R}$ one has:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \int_M \phi d\mu$$

Definition 2.2. We say that a f-invariant measure μ is a **physical measure** if the Lebesgue measure of $B(\mu)$ is positive.

2.1.2 Partially Hyperbolic Attractors

Definition 2.3. Given an invariant set Λ for f, we say that Λ is an **attractor** (topological) if there exists an open set $U \subset M$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$.

Definition 2.4. Given an attractor Λ, we define the **basin of attraction** as the set $B(\Lambda) = \bigcup_{j\geq 0} f^{-j}(U)$, where U is the open set as in the definition of attractor. This is the set of points whose orbit acummulates on Λ.

Definition 2.5. We say that an attractor Λ is **partially hyperbolic** for f if for every $x \in \Lambda$ there exists constants $\lambda_{ss}^+ < \lambda_c^- < \lambda_c^+ < \lambda_{uu}^-$, $C > 1$, $\lambda_{ss}^+ < 1$, $\lambda_{uu}^- > 1$ and a Df-invariant splitting $T_xM = E_x^{ss} \oplus E_x^c \oplus E_x^{uu}$ where:

$$\begin{aligned}
& ||Df^n v|| < C(\lambda_{ss}^+)^n ||v|| && v \in E_x^{ss} - \{0\} \\
C^{-1}(\lambda_c^-)^n ||v|| < & ||Df^n v|| < C(\lambda_c^+)^n ||v|| && v \in E_x^{c} - \{0\} \\
C^{-1}(\lambda_{uu}^-)^n ||v|| < & ||Df^n v|| && v \in E_x^{uu} - \{0\}
\end{aligned}$$

We call E^{ss} the strong-stable direction and E^{uu} the strong unstable direction. It is a well-known result that the distributions E^{ss} and E^{uu} integrates uniquely into invariant manifolds.

Theorem 2.1. The strong-stable and strong-unstable subbundles E_x^{ss} and E_y^{uu} integrate uniquely into laminations $\mathcal{F}_x^{ss}$ and $\mathcal{F}_y^{uu}$ for every $x \in B(\Lambda)$ and $y \in \Lambda$.

The reader may check [9] for the proof of the above theorem. The existence of $\mathcal{F}^{ss}$ can be guaranteed for every $x \in B(\Lambda)$, but the unstable manifolds are only well defined for points in the attractor since we have to iterate backward in the proof using graph transform.

Remark 2.1. If Λ is a partially hyperbolic attractor then $W^{uu}_x \subset \Lambda$ for every $x \in \Lambda$, so $\Lambda = \bigcup_{x\in\Lambda} W^{uu}(x)$.

The stable foliation $\mathcal{F}^{ss}$ is always α-Holder for some α, actually, the Theorem below guarantees the regularity of this foliation in several cases.

Theorem 2.2. If the application f is of class C^r and satisfies the following bunching condition for a dominated splitting $E_1 \oplus E_2$ and $k \geq 1$:

$$\sup_{x\in\Lambda} ||D_x f_{|E_1}|| \cdot \frac{||D_x f_{|E_2}||^k}{m(D_x f_{|E_2})} < 1 \tag{2.1}$$

Then there exists an invariant foliation $\mathcal{F}_1$ tangent to E_1 of class C^l, where $l = \min\{k, r-1\}$. Moreover, this foliation varies continuously in the C^l topology with the dynamics f.

The proof of this Theorem uses the C^r-Section Theorem and can be consulted in [9] and [15]. Note that the stable holonomy is not usually differentiable, but if is valid (2.1) for $k = 1$ then it is of class C^1.

Partial hyperbolicity is an open property in f, that is, if g is C^r-close to f then the set $\Lambda_g = \bigcap_{n\geq 0} g^n(U)$ is also a partial hyperbolic attractor for g and the constants of partial hyperbolicity can be taken uniform. Moreover, if holds the condition of Theorem 2.2, then the stable foliation of g is C^0-close to the stable foliation of f in the C^l-topology.

Remark 2.2. When Λ is a transitive hyperbolic attractor (ie, $E^c = 0$), the works of Sinai-Ruelle-Bowen guarantee that there exists an unique physical measure μ whose basin $B(\mu)$ has full Lebesgue measure in the basin of attraction of Λ.

2.1.3 u-Gibbs States

The measures that will by important in this work are the called u-Gibbs States, they are measures whose desintegration along strong-unstable leaves corresponds to absolutely continuous measures with respect to the induced Lebesgue measure in each unstable leaf.

Definition 2.6. Consider $x \in \Lambda$, $r > 0$ and Σ a C^1 disk centered at x with dimension $\dim(E^{ss} \oplus E^c)$ and transversal to $\mathcal{F}^{uu}$, we define the **foliated box** $\Pi(x, \Sigma, r) := \bigcup_{z \in \Lambda \cap \Sigma} \gamma^{uu}_{(z,r)}$.

We call a **foliated chart** an application $\Phi_{x,\Sigma,r} : \Pi(x, \Sigma, r) \to I^{uu}_r \times (\Sigma \cap \Lambda)$ that is an homeomorphism into the image and restricted to each γ^{uu} is a diffeomorphism into the horizontal.

Definition 2.7. Consider an invariant Borel finite measure μ, we say that this measure is *absolutely continuous with respect to the Lebesgue measure on unstable leaves* or a **u-Gibbs State** if for every x, Σ and $r > 0$, the desintegration $\{\mu_z\}_{z \in \Sigma \cap \Lambda}$ of the measure $(\Phi_{x,\Sigma,r})_* \mu$ with respect to the partition of $I^{uu}_r \times (\Sigma \cap \Lambda)$ by horizontal lines is formed by absolutely continuous measures with respect to the induced Lebesgue measures $m_{\gamma^{uu}_{(z,r)}}$ for $\hat{\mu}$-a.e z.

It is a well-known result that the u-Gibbs States always exists.

Proposition 2.1 (Pe,Si)**.** Consider a diffeomorphism f of class C^2, a partially hyperbolic attractor Λ, an unstable disk D^{uu} and the restricted Lebesgue measure $m_{D^{uu}}$, define the measures $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f^j_*(m_{D^{uu}})$. Then any accumulation point of μ_n is a u-Gibbs State.

For a study on u-Gibbs States the reader may check the book [3]. We will refer to these measures by "u-Gibbs", here we will state the main properties.

Proposition 2.2. Let $f : M \to M$ be a diffeomorphism of class C^r, $r \geq 2$, and Λ a partially hyperbolic attractor for f, then:

(1) The densities of a u-Gibbs with respect to Lebesgue measure along strong-unstable plaques are positive and bounded from zero and infinity.

(2) The support of every u-Gibbs is W^{uu}-saturated, in particular, is contained in the attractor Λ.

(3) The set of u-Gibbs is non-empty, weak-$*$ compact and convex.

(4) The ergodic components of a u-Gibbs are also u-Gibbs.

(5) Every physical measure suported in Λ is a u-Gibbs.

Proof. See [3], Section 11. □

The last item above says that the u-Gibbs are the correct candidates for the physical measures. Actually, what we will prove in this work is that under certain conditions the ergodic u-Gibbs are the physical measures.

Proposition 2.3. Consider f a diffeomorphism of class C^2, Λ a partially hyperbolic attractor, $E \subset B(\Lambda)$ a measurable set with positive Lebesgue measure and the restricted measure m_E. Then any point of accumulation of the measures $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m_E)$ is a u-Gibbs.

2.2 The Transversality Condition

In this Section we will consider a three-dimensional compact manifold M, a diffeomorphism $f : M \to M$ and Λ a partially hyperbolic attractor with $\dim E^{ss} = \dim E^c = \dim E^{uu} = 1$.

Definition 2.8. We say that Λ is **dynamically coherent** if for every $x \in \Lambda$ there exists uniquely invariant manifolds W_x^{cu} and W_x^{cs} tangent to $E_x^c \oplus E_x^u$ and $E_x^s \oplus E_x^c$.

These invariant manifolds will be called as center-unstable and center-stable manifolds, when the attractor is dynamically coherent we have well defined the invariant center manifolds W^c given by $W^c_x = W^{cu}_x \cap W^{cs}_x$. When $z \notin \Lambda$ is in W^{cu}_x we will denote $W^{cu}(z)$ by W^{cu}_x.

Remark 2.3. It is not known if dynamical coherence is an open property. But is known that it is open if the distribution E^c is of class C^1 (see Theorems 7.1 and 7.4 in [9]).

If the system is dynamically coherent then, by compactness, there exists some constant R_0 such that every center-unstable manifold has an internal radius greater than R_0.

Definition 2.9. Given two unstable curves of finite length γ_1^{uu} and γ_2^{uu}, consider the center-unstable submanifold W_i^{cu} of radius R_0 around the curve γ_i, $i = 1, 2$, we define the **stable distance** between these curves by:

$$
\begin{aligned}
d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) = \min_{\gamma^{ss}} \{ l(\gamma^{ss}) | \gamma^{ss} & \text{ is either a stable segment joining } \gamma_1 \text{ to } W_2^{cu} \\
& \text{or a stable segment joining } \gamma_2 \text{ to } W_1^{cu} \} \\
= \infty & \text{ if does not exist such } \gamma^{ss} \text{ as above.}
\end{aligned}
$$

To the next definitions, we will use foliated charts for the center-unstable manifolds given by the Proposition below.

Proposition 2.4. There exist a finite covering $\{U_i\}_{i \in I}$ of Λ by open sets in M and homeomorphisms $\psi_i : U_i \subset M \to I_i \times D_i \subset \mathbb{R}^3$, where $I_i \subset \mathbb{R}$ and D_i is a ball contained in $\mathbb{R}^2$, such that for every $z \in \Lambda$ exists $a(z)$ with $\psi_i(W^{cu}(z)) \subset \{a(z)\} \times D$ and $\psi_i|_{W^{cu}(z)}$ is a diffeomorphism into the image.

Proof. The center-unstable manifolds form a lamination. So for every point $x \in \Lambda$, considering a transversal section I_x, there exists a neighbourhood U_x

and an homeomorphism $\psi_x : U_x \to I_x \times D$ such that $\psi_x(W^{cu}(z)) \subset \{a(z)\} \times D$ for every $y \in \Lambda$ and every $z \in U_x \cap W^{cu}_y$, and $\psi_x|_{W^{cu}(z)}$ is a diffeomorphism into the image. By compacteness of Λ, we can consider a finite sub-covering and the diffeomorphisms corresponding to this sub-covering. $\square$

Fix $K_1 \geq 1$ such that $K_1^{-1} \leq ||D\psi_i|_{W^{cu}(\cdot)}|| \leq K_1$ and $R_1 > 0$ such that for every $x \in \Lambda$ the set $B^{cu}(x, R_1)$ is contained in some U_i.

Definition 2.10. We say that two continuous curves γ_1 and γ_2 of finite length contained in a subset of the $\mathbb{R}^2$ are **θ-transversals in neighborhoods of radius r in $\mathbb{R}^2$** if:

For every $x_1 \in \gamma_1$, $x_2 \in \gamma_2$ such that $d(x_1, x_2) < r$, there exists cones C_1 and C_2 with vertex at the points x_1 and x_2 such that $\gamma_1 \cap B(x_1, r) \subset C_1$, $\gamma_2 \cap B(x_2, r) \subset C_2$ and $\angle(v_1, v_2) \geq \theta$ for every tangent vectors v_1, v_2 at the points x_1, x_2 contained in the cones C_1, C_2.

When these curves are differentiable we can think on the cones above as having arbitrarily small width around the tangent direction to the curve. But if the curve is not differentiable, then the cone must contain every possible tangent direction to the curve.

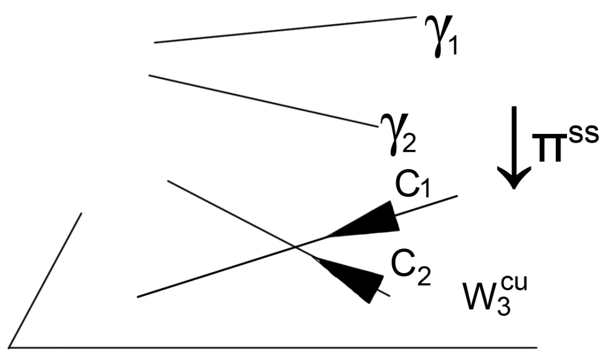
Definition 2.11. Considering $r < \frac{R_1}{2}$, we say that two continuous curves γ_1 and γ_2 of finite length contained in the same center-unstable manifold W^{cu} are **θ-transversals in neighborhoods of radius** r if for every $x_1 \in \gamma_1$ and $x_2 \in \gamma_2$ with $d^{cu}(x_1, x_2) < r$ there exists some $i \in I$ such that the curves $\psi_i(\gamma_1)$ and $\psi_i(\gamma_2)$ are θ-transversals in neighborhouds of radius r in R^2.

Considering two strong-unstable curves that are d^{ss}-close, we will define a notion of transversality between them via the stable holonomy. For it, we will use that each one is contained in one center-unstable manifold that can be sent into the other by the stable holonomy.

Definition 2.12 (Hypothesis (H1))**.** We say that holds the **Transversality Condition** if there exist constants $\epsilon_0 > 0$, $L > 0$ and functions $\theta : (0, \epsilon_0) \to \mathbb{R}^+$ and $r : (0, \epsilon_0) \to \mathbb{R}^+$ such that the following is valid:

Given $\epsilon < \epsilon_0$, strong-unstable curves γ_1 and γ_2 with length smaller than L and W_3^{cu} a center-unstable manifold with $d^{ss}(\gamma_i, W_3^{cu}) < \frac{\epsilon_0}{2}$, $i = 1, 2$, and $d^{ss}(\gamma_1, \gamma_2) > \epsilon$, consider an open set C with product structure of $W^{ss} \times W^{cu}$, with d^{ss} diameter smaller than ϵ_0 and containing γ_1, γ_2, W_3^{cu}. Taking the stable projection $\pi^{ss} : C \to W_3^{cu}$, we ask that the curves $\pi^{ss}(\gamma_1)$ and $\pi^{ss}(\gamma_2)$ are $\theta(\epsilon)$-transversal in neighborhoods of radius $r(\epsilon)$.

Every time we mean the Hypothesis of Transversality, it will be implicit that the manifold is three-dimensional and each subbundle is one-dimensional.



The transversality condition stated above is a form to say in a quantitative way that $E^{ss} \oplus E^{uu}$ is non-integrable. It holds in several cases, for example, for contact Anosov flows ([7], [8], [10]), for algebraic automorphisms on Heisenberg nilmanifols ([?]) and for the dynamics that will be constructed in Chapter 6. As we will see in Proposition 5.4, it is an open condition for dynamics when the strong-stable foliation varies continuously in the C^1 topology.

A similar hypothesis, called uniform non-integrability (UNI), played a fundamental role in the works of Chernov-Dolgopyat-Liverani ([7], [8], [10]) for decay of correlation for contact Anosov flows.

2.3 Main Results

In this section we give the precise statements of this work. First let us give two other main hypothesis, one of neutrality for the central direction and other of regularity of the stable holonomy.

Definition 2.13 (Hypothesis (H2))**.** Consider a partially hyperbolic attractor Λ and constants λ_c^- , λ_c^+ , λ_{uu}^- as in the definition of partial hyperbolicity. We say that Λ has **central direction neutral** if $\lambda_c^- < 1 < \lambda_c^+$ and

$$\frac{\lambda_c^+}{(\lambda_c^-)^2 \cdot \lambda_{uu}^-} < 1$$

This condition of neutrality in the central direction occurs when $Df_{|_{E^c}}$ is close to an isometry. If this condition is valid, then the center-unstable direction is volume-expanding.

Definition 2.14 (Hypothesis (H3))**.** We say holds the Hypothesis (H3) if: the stable foliation $\mathcal{F}_f^{ss}$ is of class C^1 for every f in a neighbourhood of f_0 and the application $f \to \mathcal{F}_f^{ss}$ is continuous in the C^1-topology in f_0.

This conditions guarantees that the stable holonomies of the attractor Λ for f are C^1 close to the ones of Λ_0 for f_0. It follows from Theorem 3 that this hypothesis is valid when the condition (2.1) is satisfied for $k = 1$.

The Results

The precise statements of this work are the following

Theorem A. *Consider $f : M \to M$ a diffeomorphism of class C^r, $r \geq 2$, M a three-dimensional manifold and Λ_0 a partially hyperbolic attractor. Suppose that Λ_0 is dynamically coherent and satisfies the following hypothesis:*

(H1) - Transversality (Uniform non-integrability of $E^{ss} \oplus E^{uu}$);

(H2) - Central direction neutral;

(H3') - The stable holonomy h^{ss} is of class Lipschitz.

Then f admits a finite number of physical measures supported in Λ, coinciding with the ergodic u-Gibbs, whose union of their basins has full measure in $B(\Lambda)$.

Theorem A will be proved in Chapter 5, the technical tools for the proof of this theorem will be developed throughout Chapters 3 and 4.

As a consequence of Theorem A, we have the Corollary below.

Corollary B. *Consider $f_0 : M \to M$ a diffeomorphism of class C^r, $r \geq 2$, M a three-dimensional manifold and Λ_0 a partially hyperbolic attractor. Suppose that Λ_0 is robustly dynamically coherent and satisfies the following hypothesis:*

(H1) - Transversality (Uniform non-integrability of $E^{ss} \oplus E^{uu}$);

(H2) - Central direction neutral;

(H3) - $f \to \mathcal{F}^{ss}_f$ is continuous in the C^1-topology.

Then there exists an open set $\mathcal{U}$ containing f_0 such that every $f \in \mathcal{U}$ admits a finite number of physical measures supported in Λ, coinciding with the ergodic u-Gibbs, whose union of their basins has full measure in $B(\Lambda)$.

In Chapter 6, we will describe the construction of an attractor with central direction neutral satisfying the hypothesis of transversality and of regularity of the stable holonomy, this construction will prove the following.

Theorem C. *There exists $f_0 : M^3 \to M^3$ and a partial hyperbolic attractor Λ_0 that is robustly nonhyperbolic and is robustly dynamically coherent satisfy-*

ing the hypothesis (H1), (H2) and (H3). Moreover, this attractor is robustly transitive.

The proof of Theorem C corresponds to a construction considering a hyperbolic solenoidal attractor and deforming the dynamics in the central direction inside a neighbourhood of a fixed point in a similar way to the construction of Mañé's example ([11]).

Chapter 3

Toolbox

Throughout this Chapter we define the boxes and the semi-norms that will be used in the technical part of the work.

3.1 The Boxes

We will consider subsets of the manifold where it is well defined the stable projection into one fixed center-unstable manifold and every unstable curve that intersects these subsets must cross them.

Definition 3.1. Given $f : M \to M$ a diffeomorphism and a partially hyperbolic attractor Λ that is dynamically coherent, we say that a quadruple $(C, W, \tilde{W}, \pi)$ is a **box** if C is the image of an embedding $h : I^{uu} \times I^a \times I^b \to M$, where I^{uu}, I^a and I^b are intervals, such that:

1) The function h_{z_0} given by $h_{z_0}(x, y) = h(x, y, z_0)$ is an embedding into a surface that coincides with a center-unstable manifold if its image intersects the attractor. The set W is the image of $h(I^{uu} \times I^a \times \{0\})$ and intersects Λ.

2) If $h(x_0, y_0, z_0) \in \Lambda$, then $\gamma(t) = h(t, y_0, z_0)$ is an unstable curve.

3) For every $x, y \in C$, every connected component of $W_x^{cu} \cap C$ and of $W_y^{ss} \cap C$ intersect at most once.

4) The set $\tilde{W}$ is a connected center-unstable manifold with finite diameter containing W and such that $\tilde{W} \cap W_y^{ss} \neq \emptyset$ for every $y \in C$. The application $\pi : C \to \tilde{W}$ sends each $y \in C$ into the point $W_{loc}^{ss}(y) \cap \tilde{W}$.

The first condition says that the set $C \cap \Lambda$ can be seen as a family of center-unstable manifolds, the second states that if an unstable curve intersects C then it crosses C, the third and the fourth condition guarantees that are well defined the strong-stable projection $\pi^{ss} : C \to \tilde{W} \supset W$.

Since every point in the attractor admits arbitrarily small boxes containing it, it is possible consider a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$ such that the sets $\pi_i^{-1}(W_i)$ cover the attractor Λ, ie, $\Lambda \subset \bigcup_i \pi_i^{-1}(W_i)$.

3.2 The Semi-Norm

We will define a semi-norm that estimates a kind of regularity of the projection of measures into a center-unstable manifold, it will be in terms of this semi-norm that we will describe a criteria of absolute continuity for the stable projection of measures. This semi-norm will be used jointly with the boxes of the anterior Section.

Definition 3.2. Given a submanifold $X \subset M$ of dimension 2, the Lebesgue measure m_X in X given by the Riemmanian metric, finite measures μ_1 and μ_2 defined in X and $r > 0$ fixed, we define the following inner-product:

$$\langle \mu_1, \mu_2 \rangle_{X,r} = \frac{1}{r^4} \int_X \mu_1(B(z,r)) \mu_2(B(z,r)) dm_X(z)$$

This allows us to define the **semi-norm** $|| \cdot ||_{X,r}$ by:

$$||\mu||_{X,r} = \sqrt{\langle \mu, \mu \rangle_{X,r}}$$

Let us prove some facts for this semi-norm.

Lemma 3.1. Given a finite family of center-unstable manifolds $\{W_i\}$ with bounded diameter, there exists a constant $C_0 \geq 1$ such that:

$$||\nu||_{W_i,r_2} \leq C_0||\nu||_{W_i,r_1}$$

for every $0 < r_1 \leq r_2 \leq 1$ and every i.

Proof. The union of the closure of these sets is compact, so we can cover with a finite number of foliated charts for the center-unstable manifolds, and these foliated charts are of class C^1. So, there exists a constant C_0 independent of r_1, r_2, x and of i such that it is possible to cover every $B^{cu}(x, r_2)$ in W^{cu} with $\left\lceil C_0 \frac{r_2}{r_1} \right\rceil^2$ disks $B^{cu}(w_k, r_1)$ by choosing the w_k's appropriately, this can be done using these foliated charts that send each W_i into a plane. The Moreover, we have $m^{cu}(B^{cu}(z, r)) \leq C_3 r^2$ for every $z \in \cup_i C_i$ (the area of these balls depends only on the first derivative of the charts when we look to these manifolds as graphs).

The w_k's can be taken as translation of z by terms w_k that do not depends on z, so $dz = dw_k$. Then we have the following estimative:

$$\begin{aligned}
||\mu||^2_{W_i,r_2} &= \frac{1}{r_2^4} \int_{W_i} \mu(B(z, r_2))^2 dm^{cu}(z) \\
&\leq \frac{1}{r_2^4} C_0 \int_{W_i} (\sum_k \mu(B(w_k, r_1))^2) dz \\
&\leq \frac{1}{r_2^4} C_0 \left(\frac{r_2}{r_1}\right)^2 \sum_k \int_{W_i} \mu(B(w_k, r_1))^2 dz \\
&\leq C_0^2 ||\mu||^2_{W_i,r_1}
\end{aligned}$$

□

Also another observation is the following:

Lemma 3.2. Consider ν_n^1 and ν_∞^1 finite measures defined in a center-unstable manifold W such that $\nu_n^1 \xrightarrow{*} \nu_\infty^1$ and $r > 0$ fixed, then

$$\lim_{n\to\infty} ||\nu_n^1||_{W,r} = ||\nu_\infty^1||_{W,r}$$

Moreover, if we consider ν_n^2 and ν_∞^2 finite measures also defined in W such that $\nu_n^2 \xrightarrow{*} \nu_\infty^2$, then

$$\lim_{n\to\infty} \langle \nu_n^1, \nu_n^2 \rangle_{W,r} = \langle \nu_\infty^1, \nu_\infty^2 \rangle_{W,r}$$

Proof. Note that $\nu_\infty^1(\partial B^{cu}(z,r)) = 0$ for m^{cu}-ae z, because:

$$\begin{aligned}\int \nu_\infty^1(\partial B^{cu}(z,r)dm^{cu}(z) &= \int \int_{d(z,w)=r} d\nu_\infty^1(w)dm^{cu}(z)\\ &= \int m^{cu}(\partial B^{cu}(w,r))d\nu_\infty^1(w)\\ &= 0\end{aligned}$$

Define $J_r\nu(x) = \frac{\nu(B(x,r))}{r^2}$, then $J_r\nu_n^1 \to J_r\nu_\infty^1$. Since $||J_r\nu_n^1||_{L^2}$ is uniformly bounded, the lemma follows by the theorem of dominated convergence.

The proof of $\lim_{n\to\infty}\langle \nu_n^1, \nu_n^2 \rangle_{W,r} = \langle \nu_\infty^1, \nu_\infty^2 \rangle_{W,r}$ is similar. □

Let us consider also a norm measuring the total mass of a measure.

Definition 3.3. For finite measures defined in a space X, define the **norm** $|\cdot|$ by:

$$|\mu| = \mu(X)$$

Lemma 3.3. Given a finite measure μ in M and applications $f : M \to M$, $\pi : M \to N$, we have $|(\pi)_* f_*\mu| \leq |(\pi)_*\mu|$.

Proof. Note that $|\pi_* f_*\mu| = \mu(f^{-1}\pi^{-1}(N)) \leq \mu(\pi^{-1}(N)) = |\pi_*\mu|$. □

Definition 3.4. Given a finite family of applications $\pi_i : C_i \subset M \to \tilde{W}_i \supset W_i$, $i = 1, \cdots, s_0$, consider the families $\mathcal{W} = \{W_1, \cdots, W_{s_0}\}$ and $\tilde{\mathcal{W}} = \{\tilde{W}_1, \cdots, \tilde{W}_{s_0}\}$. Given a measure μ in M we define the **semi-norm** $|||\cdot|||$ by:

$$|||\mu|||_{\mathcal{W},r} := \max_{i=1,\cdots,s_0}\{||(\pi_i)_*\mu||_{W_i,r}\}$$

$$|||\mu|||_{\tilde{\mathcal{W}},r} := \max_{i=1,\cdots,s_0}\{||(\pi_i)_*\mu||_{\tilde{W}_i,r}\}$$

Remark 3.1. Given a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$, $i = 1, \cdots, s_0$, with the property that $\{\pi_i^{-1}(W_i)\}$ covers Λ, for measures μ defined in Λ we have an equivalence for $|||\mu|||_{\{W_i\},r}$ and $|||\mu|||_{\{\tilde{W}_i\},r}$:

$$|||\mu|||_{\{\mathcal{W}\},r} \leq |||\mu|||_{\{\tilde{\mathcal{W}}\},r} \leq s_0|||\mu|||_{\{\mathcal{W}\},r}$$

When the boxes or the corresponding center-unstable manifolds are implicit we will denote the semi-norms just by $||\cdot||_r$ and $|||\cdot|||_r$, they estimates the regularity of the stable projection of a measure, the norm $|\cdot|$ measures the size (total mass).

For absolute continue measures defined in a center-unstable surface, the norm $||\cdot||_r$ for small $r > 0$ is related to the L^2 norm of the density.

3.2.1 Criteria of Absolute Continuity

Given a center-unstable manifold W, denote by m^{cu}_W the Lebesgue measure in this submanifold given by the Riemannian metric, when the set W is implicit we will denote just by m^{cu}. Remember that we are considering dynamically coherent attractors with $\dim(E^c \oplus E^{uu}) = 2$.

The main use of these semi-norms is due to the following criteria of absolute continuity of a measure defined in a center-unstable manifold.

Lemma 3.4. Given a center-unstable manifold W, there exists a constant $K > 0$ such that for every Borel finite measure ν defined in W, if

$$\liminf_{r \to 0^+} ||\nu||_{W,r} \leq L$$

for some $L > 0$, then ν is absolutely continuous with respect to m^{cu} and

$$\left|\left|\frac{d\nu}{dm^{cu}}\right|\right|_{L^2(W,m^{cu})} \leq KL$$

Proof. Note that there exist constants $C_1, C_2 > 0$ such that $C_1 \leq \dfrac{m^{cu}(B(x,r))}{r^2} \leq C_2$, this is valid because the center-unstable manifolds form a C^1 lamination and the areas of balls with small radius depends only on the first derivative of these charts.

Define $J_{r_n}\nu(x) = \frac{\nu(B_W^{cu}(x,r_n))}{m^{cu}(B^{cu}(x,r_n))}$. By hypothesis, we can consider a sequence $r_n \to 0^+$ such that $||J_{r_n}\nu||_{L^2(W,m^{cu})}$ is uniformly bounded by $C_1^{-1}L$.

Taking a subsequence we can suppose that $J_{r_n}\nu \to \psi \in L^2$ weakly, so

$$\langle f, \psi \rangle_{L^2} = \lim_{n \to \infty} \int_W f \cdot J_{r_n}\nu \; dm^{cu} = \int_W f d\nu$$

for every continuous function f.

This implies that $\nu = \psi m_W^{cu}$, and that

$$\left|\left|\frac{d\nu}{dm^{cu}}\right|\right|_{L^2} = \lim_{n \to \infty} ||J_{r_n}\nu||_{L^2} \leq KL$$

$\square$

Chapter 4

The Main Inequality

This Chapter is dedicated to state and prove the Main Inequality. It will be used further to estimate the semi-norm $||(\pi_i)_*\mu||_{W_i,r}$ for small parameters r.

Given f and Λ satisfying the hypothesis of Theorem A, consider a finite family of boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$, $i = 1, \cdots, s_0$, such that $\Lambda \subset \underset{i}{\cup}\pi_i^{-1}(W_i)$ and the d^{ss}-diameter of each box is smaller than the ϵ_0 given by the Hypothesis (H1). Fix constants a_1 and a_2 such that for every unstable curve γ^{uu} crossing C_i we have $a_1 \leq l(\gamma^{uu}) \leq a_2$ and consider $K \geq 1$ an upper bound for the Lipschitz constant of every π_i. We will suppose this family fixed once for all.

The Main Inequality of this work is the following.

Proposition 4.1 (Main Inequality)**.** There exist constants $B > 0$ and $\sigma > 1$ such that for every $n \in \mathbb{N}$, there exists $D_n > 0$, $r_n > 0$ and $c_n > 1$ such that for every ergodic u-Gibbs μ and every $r \leq r_n$, we have:

$$|||f_*^n\mu|||_r^2 \leq \frac{B}{\sigma^n}\,|||\mu|||_{c_n r}^2 + D_n\,|\mu|^2$$

This type of inequality is often in the study of the regularity of invariant measures in ergodic theory. There are two norms, one measuring the size and other measuring the regularity of the measure, these inequalities allows to see that the fixed points has a good regularity. This kind of inequality is due to Doeblin-Fortet, and is also called as "Lasota-Yorke type inequality".

Proposition 4.1 will be proved along this Chapter.

4.1 A Lemma on Approximation of u-Gibbs inside the Boxes

For the arguments along this Chapter, it will be useful to approximate the restriction of a u-Gibbs to one box by a finite combination of measures supported on unstable curves crossing the box.

Lemma 4.1. There exists C_1 and α_1 such that the restriction of every ergodic u-Gibbs to a box C_i can be written as $\lim_n \mu_n$ with $\mu_n = \sum_{j=0}^{r_0} \rho_j^n m_{\gamma_j^n}$, where $\log \rho_j^n$ is (C_1, α_1)-Holder and γ_j^n is an unstable curve contained in C_i that crosses C_i.

Proof. For every ergodic u-Gibbs μ, there exist some unstable curve γ^{uu} such that μ can be written as a limit of $\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(m_{|\gamma^{uu}})$.

Given C_1 and α_1, consider the closed space $\mathcal{L}_{C_1,\alpha_1}$ of measures whose logarithm of the density of the conditional measure to the unstable curves are of class (C_1, α_1)-Holder.

Claim 4.1. There exists C_1, α_1 and $n_0 \in \mathbb{N}$ such that the space $\mathcal{L}_{C_1,\alpha_1}$ is invariant under f_*^n for every $n \geq n_0$.

Proof of the Claim. Note that if the conditional measure of μ to an unstable curve γ^{uu} has density ρ, then the conditional measure of $f_*^n \mu$ to the unstable curve $f^n(\gamma^{uu})$ has density $\tilde{\rho}_n$, where $\tilde{\rho}_n(x) = |\det Df^n_{|E^{uu}}(f^{-n}(x))| \cdot \rho(f^{-n}(x)|$

Consider $\tilde{C}$ and $\tilde{\alpha}$ such that the function $z \to \log |\det Df_{|E^{uu}}(z)|$ is $(\tilde{C}, \tilde{\alpha})$-Holder and consider $\hat{C} > 0, \hat{\lambda} \in (0,1)$ such that $d(f^{-n}(x_1), f^{-n}(x_2)) \leq$

$\hat{C}\hat{\lambda}^n d(x_1, x_2)$ for every $x_1 \in W^{uu}_{x_2}$. Take $n_0 \in \mathbb{N}$ such that $\hat{C}\hat{\lambda}^{\alpha_1 n_0} < 1$, $\alpha_1 = \tilde{\alpha}$ and $C_1 := \dfrac{2\tilde{C}\hat{C}}{(1-\hat{\lambda}^{\alpha_1})(1-\hat{C}\hat{\lambda}^{n_0\alpha_1})}$.

If $\log\rho$ is (C_1, α_1)-Holder, then for every $n \geq n_0$

$$\begin{aligned}
\log\tilde{\rho}_n(x_1) - \log\tilde{\rho}_n(x_2) &= \left[\log\rho(f^{-n}(x_1)) - \log\rho(f^{-n}(x_2))\right] \\
&\quad + \left[\log|\det Df^n_{|E^{uu}}(f^{-n}(x_1))| - \log|\det Df^n_{|E^{uu}}(f^{-n}(x_2))|\right] \\
&\leq C_1 d(f^{-n}(x_1), f^{-n}(x_2))^{\alpha_1} \\
&\quad + \sum_{i=0}^{n}\Big(\log|\det Df_{|E^{uu}}(f^{-i}(x_1))| - \log|\det Df_{|E^{uu}}(f^{-i}(x_2))|\Big) \\
&\leq C_1\hat{C}\hat{\lambda}^{n\alpha_1} d(x_1,x_2)^{\alpha_1} + \sum_{i=0}^{n}\tilde{C} d(f^{-i}(x_1), f^{-i}(x_2))^{\tilde{\alpha}} \\
&\leq C_1\hat{C}\hat{\lambda}^{n_0\alpha_1} d(x_1,x_2)^{\alpha_1} + \Big(\sum_{j\geq 0}\tilde{C}\hat{C}\hat{\lambda}^{j\alpha_1}\Big) d(x_1,x_2)^{\alpha_1} \\
&\leq \Big(C_1\hat{C}\hat{\lambda}^{\alpha_1 n_0} + \frac{\tilde{C}\hat{C}}{1-\hat{\lambda}^{\alpha_1}}\Big)\cdot d(x_1,x_2)^{\alpha_1} \\
&< C_1 \cdot d(x_1,x_2)^{\alpha_1}
\end{aligned}$$

What means that $\log\tilde{\rho}_n$ is also (C_1, α_1)-Holder.

□

Consider the measures $\nu_n := \Big(\frac{1}{n}\sum_{j=0}^{n-1} f^j_*(m_{|\gamma^{uu}})\Big)_{|C_i} \to \mu_{|C_i}$. For every $j \in \mathbb{N}$, consider the curve $\tilde{\gamma}^{uu}_j \subset f^j(\gamma^{uu}) \cap C_i$ obtained removing the connected components of $f^j(\gamma^{uu}) \cap C_i$ that do not cross the box C_i, consider the measure $\tilde{\nu}_n := \Big(\frac{1}{n}\sum_{j=0}^{n-1} f^j_*(m_{|f^{-j}\tilde{\gamma}^{uu}_j})\Big)_{|C_i}$ and $\tilde{\tilde{\nu}}_n$ given by $\tilde{\tilde{\nu}}_n = \Big(\frac{1}{n}\sum_{j=n_0}^{n-1} f^j_*(m_{|f^{-j}\tilde{\gamma}^{uu}_j})\Big)_{|C_i}$ if $n > n_0$ and $\tilde{\tilde{\nu}}_n = m_{|\tilde{\gamma}^{uu}_j}$ otherwise.

From Claim 4.1 it follows that $\tilde{\tilde{\nu}}_n \in \mathcal{L}_{C_1,\alpha_1}$ for every $n \in \mathbb{N}$.

Note that $\nu_n - \tilde{\nu}_n \overset{n\to\infty}{\to} 0$ because their difference is a measure supported in the connected components that where removed, but these connected components are at most $2n$, they have length bounded and the density of their conditional measure to these unstable curves converges to zero when $n \to \infty$. Note also that $\tilde{\nu}_n - \tilde{\tilde{\nu}}_n \overset{n\to\infty}{\to} 0$, so $\lim_n \tilde{\tilde{\nu}}_n = \lim_n \tilde{\nu}_n = \lim \nu_n = \mu_{|C_i}$, thus the sequence $\{\tilde{\tilde{\nu}}_n\}_n$ satisfies the conditions that we wanted. □

4.2 Comparing Sizes of Cylinders

Let us denote by $B^{cu}(x,r)$ the center-unstable ball of radius r centered at x. Our interest is to estimate, for u-Gibbs measures μ, the measure of the sets $\pi_i^{-1}B^{cu}(x,r)$, which we will look as cylinders.

Given $n \in \mathbb{N}$, consider the sets $\{R_{i,j,m}\}_{m=1,\cdots,m_0(i,j,n)}$ as the connected components of $f^n(C_i) \cap C_j$, the sets $C_{i,j,m} := f^{-n}R_{i,j,m} \subset C_i$ and the applications $f_{i,j,m} := f^n_{|C_{i,j,m}}$. When the choice of C_i, C_j and n is implicit we will write f_m instead of $f_{i,j,m}$.

Given r we will consider $\delta = 10C^{-1}L^2(\lambda_c^-)^{-n}r$, where C and λ_c^- are the constants of the condition of central direction neutral. The main point in the next lemma is that δ is greater than r.

Lemma 4.2. There exist a constant $B_1 > 0$ such that for every n, C_i and C_j there exists a constant r_1 such that, taking $\delta = 10C^{-1}L^2(\lambda_c^-)^{-n}r$ and $\sigma_{n,r}(x) = (\lambda_c^-)^n \min_{w\in C(x)} ||Df^n_{|E^{uu}_w}||$ with $C(x) = f_m^{-1}\pi_j^{-1}B^{cu}(\pi_j f^n(x), r)$, we have for every ergodic u-Gibbs μ, every $x \in W_i$, every m and every $r < r_1$:

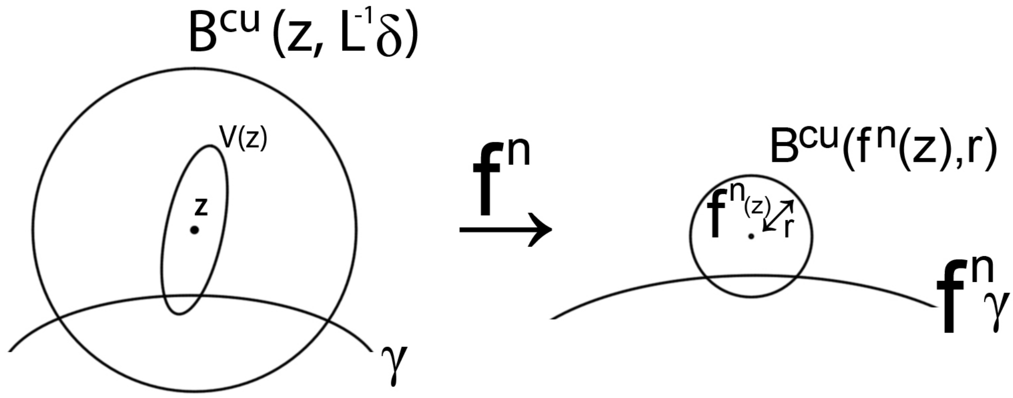
$$\mu\big(f_m^{-1}\pi_j^{-1}B^{cu}(\pi_j f^n(x), r)\big) \leq \Big(\frac{r}{\delta}\Big)^2 \frac{B_1}{\sigma_{n,r}(x)} \cdot \mu\big(\pi_i^{-1}B^{cu}(x,\delta)\big)$$

Proof. First, let us prove this lemma for measures $\mu = m_\gamma$, where γ is an unstable curve contained in C_i that crosses C_i. If $f^n(x) \notin C_j$ then the left-hand side of the inequality is zero, so we can suppose that $f^n(x) \in C_j$.

Consider $z \in W^{ss}(x)$ such that $\gamma \subset W^{cu}(z)$ and define the set $V(z) = W^{cu}(z) \cap f_m^{-1}\pi_j^{-1}B^{cu}(\pi_j f^n(x), r)$, let us consider also the intervals $I = \gamma^{-1}(V(z))$

and $J = \gamma^{-1}(B^{cu}(z, L^{-1}\delta))$. We can note that $V(z) \subset B^{cu}(z, L^{-1}\delta) \subset W^{cu}(z) \cap \pi_i^{-1} B^{cu}(x, \delta)$.

In the center-unstable manifolds containing z and $f^n(z)$ we have the picture below.



In order to bound $\dfrac{m_\gamma(V(z))}{m_\gamma(B^{cu}(z, L^{-1}\delta))}$, let us estimate $\dfrac{m_{\mathbb{R}}(I)}{m_{\mathbb{R}}(J)}$.

Fix $R_2 > 0$ such that for every $r < R_2$ every strong-unstable curve intersects $B^{cu}(\cdot, Lr)$ with length at most $3Lr$. Taking $r < R_2$ we can see that:

$$\begin{aligned}
3Lr &\geq m_{f_m(\gamma)}(f_m\gamma \cap B^{cu}(f^n(z), Lr)) \\
&\geq m_{f_m(\gamma)}(f_m\gamma \cap \pi_j^{-1} B^{cu}(\pi_j f^n(z), r)) \\
&\geq m_\gamma(\gamma \cap f_m^{-1}\pi_i^{-1} B(x, r)) \cdot \min_{w\in\gamma} ||Df^n_{|E^{uu}_w}|| \\
&\geq m_{\mathbb{R}}(I) \cdot \min_{w\in C(z)} ||Df^n_{|E^{uu}_w}||
\end{aligned}$$

Then $m_{\mathbb{R}}(I) \leq \dfrac{3Lr}{\min\limits_{w\in C(z)} ||Df^n_{|E^{uu}_w}||}$.

Suppose also that $r \leq \frac{a_1}{10C^{-1}L^2}$. When $m_\gamma(f_m^{-1}\pi_j^{-1} B^{cu}(f^n(z), r)) = 0$ we have $m_{\mathbb{R}}(I) = 0$ and the inequality of the lemma holds in this case. When $m_\gamma(f_m^{-1}\pi^{-1} B^{cu}(f^n(z), r)) > 0$, the choice of δ $(= 10C^{-1}L^2(\lambda_c^-)^{-n} r)$

guarantees that

$$\begin{aligned} f_m^{-1}\pi_j^{-1}B^{cu}(f^n(z),r) &\subset B^{cu}(z,(\min_{w\in C(z)}||Df^{-n}_{|E^c_w}||)^{-1}Lr) \\ &\subset B^{cu}(z,C^{-1}(\lambda_c^-)^{-n}Lr) \\ &\subset B^{cu}(z,\frac{1}{2}L^{-1}\delta) \end{aligned}$$

Since $m_{\mathbb{R}}(\gamma) \geq a_1 \geq 10\delta$ and $\gamma \cap B^{cu}(z,\frac{1}{10}L^{-1}\delta) \neq \emptyset$, we have that the lenght of $\gamma \cap B^{cu}(z,\delta)$ is greater than $\frac{1}{2}L^{-1}\delta$, so $m_{\mathbb{R}}(J) \geq \frac{1}{2}L^{-1}\delta$.

Thus

$$\frac{m_{\mathbb{R}}(I)}{m_{\mathbb{R}}(J)} \leq \frac{\frac{3LR}{\min||Df^n_{|E^{uu}_w}||}}{\frac{1}{2}L^{-1}\delta} \leq \left(\frac{r}{\delta}\right)^2 \frac{60L^4C}{(\lambda_c^-)^n\cdot \min\limits_{w\in C(z)}||Df^n_{|E^{uu}_w}||}$$

and holds the Lemma for $\sigma_{n,r}(z) = (\lambda_c^-)^n \min\limits_{w\in C(z)}||Df^n_{|E^{uu}_w}|| > 1$.

Now let us prove the Lemma for $\mu = \rho m_\gamma$ with γ as before and $\log\rho$ of class (C_1,α_1)-Holder. We can suppose that ρ is defined in an interval of length smaller than a_2, then for every $I \subset J \subset \mathbb{R}$ it is valid:

$$\frac{\int_I \rho(x)dx}{\int_J \rho(x)dx} \leq e^{2C_1a_2^{\alpha_1}} \cdot \frac{m_{\mathbb{R}}(I)}{m_{\mathbb{R}}(J)}$$

This can be seen fixing $x_0 \in I \cap J$ and noting that $\frac{\rho(x)}{\rho(x_0)} \leq e^{C_1d(x,x_0)^{\alpha_1}} \leq e^{C_1a_2^{\alpha_1}}$ for every $x \in J$. Putting together this observation and the estimative for m_γ, it follows

$$\frac{(\rho m_\gamma)(f_m^{-1}\pi^{-1}B(x,r))}{(\rho m_\gamma)(\pi^{-1}B(\pi f_m^{-1}\pi^{-1}(x),\delta))} \leq \frac{e^{2C_1a_2^{\alpha_1}}\cdot 60L^4C}{\sigma_{n,r}} \cdot \left(\frac{r}{\delta}\right)^2$$

Then the Lemma also holds for measures $\mu = \rho m_\gamma$.

Now, suppose that μ is a finite sum $\mu = \sum_{i=1}^{s_0}\rho_i m_{\gamma_i}$, where ρ_i and γ_i are as before, then the result in this case follows by linearity, since both sides of the inequality are linear in μ.

If μ is an ergodic u-Gibbs, we proceed using Lemma 4.1, the inequality holds for the measures μ_n given by Lemma 4.1 because they are finite sums of measures of the type ρm_γ. Finally, since every ergodic u-Gibbs is the limit of some sequence of measures as above, each term in the inequality passes to the limit, so the inequality also holds for the ergodic u-Gibbs μ. □

4.3 One Inequality for the Semi-Norm

As a consequence of the last Lemma we can state one inequality comparing semi-norms for the iterates of u-Gibbs.

Lemma 4.3. There exists constants $B_2 > 0$ and $\sigma_2 > 1$ such that for every n, C_i and C_j there exist a constant $R_n > 0$ such that, taking $\delta = 10CL(\lambda_c^-)^{-n}r$, for every $r < R_n$ and every ergodic u-Gibbs μ:

$$\sum_m ||(\pi_j)_*(f_m)_*\mu||^2_{W_j,r} \leq \frac{B_2}{\sigma_2^n}||(\pi_i)_*\mu||^2_{\tilde{W}_i,\delta}$$

Proof. Applying Lemma 4.2 to the measure μ, it is possible to check that:

$$\begin{aligned}
||(\pi_j)_*(f_m)_*\mu||^2_{W_i,r} &= \frac{1}{r^4}\int_{W_j} \mu(f_m^{-1}\pi_j^{-1}B^{cu}(y,r))^2 dm^{cu}(y) \\
&\leq \frac{B_1^2}{\delta^4}\int_{W_j} \frac{\mu(\pi_i^{-1}B(\pi_i f_m^{-1}\pi_j^{-1}(y),\delta))^2}{(\lambda_c^-)^{2n}\cdot \min\limits_{w\in C(y)}||Df^n_{|E^{uu}_w}||^2} dm^{cu}(xy
\end{aligned}$$

We will estimate this integral using that $\pi_i \circ f_m^{-1} \circ \pi_j^{-1} : W_j \to \tilde{W}_i$ is well defined, is an homeomorphism into the image, absolutely continuous and has Jacobian bounded as the following expression:

$$L^{-2}\inf_{x\in W^{ss}(y)}|\det Df^n_{|E^{cu}_x}| \leq \text{Jac}\left(\pi_i \circ f_m \circ \pi_j^{-1}(y)\right) \leq L^2 \sup_{x\in W^{ss}(y)}|\det Df^n_{|E^{cu}_x}|$$

Using the change of variables $\tilde{y} = \pi_i \circ f_m^{-1} \circ \pi_j^{-1}(y)$, we can rewrite the last integral as:

$$\frac{B_1^2}{\delta^4}\int\limits_{\tilde W_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B(\tilde y,\delta))^2\cdot \mathrm{Jac}(\pi_i f_m\pi_j^{-1}(y))}{C^{-1}(\lambda_c^-)^{2n}\min\limits_{w\in\pi^{-1}B(\tilde y,L\delta)}||Df^n_{|E^{uu}_w}||^2}dm^{cu}(\tilde y)$$

$$\leq \frac{B_1^2}{\delta^4}L^2C\int\limits_{\tilde W_i\cap R_{i,j,m}}\mu(\pi_i^{-1}B(\tilde y,\delta))^2\cdot\frac{\sup\limits_{y_1\in W^{ss}_{\tilde x}}|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\min\limits_{w\in\pi^{-1}B(\tilde x,L\delta)}||Df^n_{|E^{uu}_w}||^2}dm^{cu}(\tilde y)$$

Using the Hyphotesis (H2) of central direction neutral, it is possible to estimate the term inside the integral.

Claim 4.2. There exists a constant $K_2>0$ such that for every $n\in\mathbb{N}$ there exists $\tilde r_n>0$ such that for every $r<\tilde r_n$ holds:

$$\frac{\sup\limits_{y_1\in W^{ss}_{\tilde x}}|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n}\min\limits_{w\in\pi^{-1}B^{cu}(\tilde x,L\delta)}||Df^n_{|E^{uu}_w}||^2}\leq K_2\left[\frac{(\lambda_c^+)}{(\lambda_c^-)^2(\lambda_u^-)}\right]^n$$

The proof of this Claim will be given in the end of this Section.

In the continuation we will suppose that $r<\tilde r_n$ and we will consider $\sigma_2^{-1}=\frac{\lambda_c^+}{(\lambda_c^-)^2(\lambda_u^-)}<1$.

Then:

$$||(\pi_j)_*(f_m)_*\mu||^2_{W_i,r}\leq\frac{2B_1^2L^2C^3K_2}{\sigma_2^n}\int\limits_{\tilde W_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B^{cu}(\tilde x,\delta))^2}{\delta^4}dm^{cu}(\tilde x)$$

Adding this inequality in m, we get:

$$\begin{aligned}\sum_m||(\pi_j)_*(f_m)_*\mu||^2_{W_i,r}&\leq\frac{B_2}{\sigma_2^n}\sum_m\int\limits_{\tilde W_i\cap R_{i,j,m}}\frac{\mu(\pi_i^{-1}B^{cu}(\tilde x,\delta))^2}{\delta^4}dm^{cu}(\tilde x)\\&=\frac{B_2}{\sigma_2^n}\int\limits_{\tilde W_i}\frac{(\pi_i)_*\mu(B(\tilde x,\delta))^2}{\delta^4}dm^{cu}(\tilde x)\\&=\frac{B_2}{\sigma_2^n}||(\pi_i)_*\mu||^2_{\tilde W_i,\delta}\end{aligned}$$

□

Let us prove the Claim stated during the proof.

Proof of Claim 2. We will prove the Claim looking first to a simpler case and after to the general one.

First, let us consider $y_1 \in W^{ss}_{\tilde{x}}$ and $w = \tilde{x}$, so there exists $\tilde{K}_2 > 0$ such that $\frac{|detDf^n_{|E^{cu}_{y_1}}|}{|detDf^n_{|E^{cu}_{y_2}}|} \leq \tilde{K}_2$ for every $y_1 \in W^{ss}_{y_2}$ and every $n \geq 0$, this is due to the following calculation:

$$\begin{aligned}
\log|\det Df^n_{|E^{cu}_{y_1}}| - \log|\det Df^n_{|E^{cu}_{y_2}}| &= \sum_{j=1}^{n} \log|\det Df_{|E^{cu}_{f^j(y_1)}}| - \log|\det Df_{|E^{cu}_{f^j(y_2)}}| \\
&\leq \sum_{j\geq 0} \tilde{C}_2 d(f^j(y_1), f^j(y_2))^{\alpha_2} \\
&\leq \sum_{j=0}^{+\infty} \tilde{C}_2 l((\lambda^{ss})^{\alpha_2})^j := \tilde{K}_2
\end{aligned}$$

Above was used that the function $x \to \log|\det Df_{|E^{cu}_{x}}|$ is $(\tilde{C}_2, \alpha_2)$-Holder and that the length of W^{ss} is uniformly bounded by l inside the boxes, then:

$$\begin{aligned}
\frac{|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n} \cdot ||Df^n_{|E^{uu}_{\tilde{x}}}||^2} &\leq \tilde{K}_2 \cdot \left[\frac{|\det Df^n_{|E^{cu}_{\tilde{x}}}|}{(\lambda_c^-)^{2n} \cdot ||Df^n_{|E^{uu}_{\tilde{x}}}||^2}\right] \\
&= \frac{\tilde{K}_2 \cdot ||Df^n_{|E^{c}_{\tilde{x}}}|| \cdot ||Df^n_{|E^{uu}_{\tilde{x}}}||}{(\lambda_c^-)^{2n}||Df^n_{|E^{uu}_{\tilde{x}}}||^2} \\
&\leq \frac{C^2\tilde{K}_2(\lambda_c^+)^n}{C^{-1}(\lambda_c^-)^{2n}(\lambda_u^-)^n} \\
&\leq \frac{C^3\tilde{K}_2}{\sigma_2^n}
\end{aligned}$$

Now, let us consider the general case of $y_1 \in W^{ss}_{\tilde{x}}$ and $w \in \pi_j^{-1}B^{cu}(\tilde{x}, L\delta)$. By continuity, there exists $\tilde{r}_n = \tilde{r}_n(w_2)$ such that $\frac{1}{2} \leq \frac{||Df^n_{|E^{uu}_{w_1}}||}{||Df^n_{|E^{uu}_{w_2}}||} \leq 2$ for every $w_1 \in B^{cu}(w_2, r_n(w_2))$. By compacity, $\tilde{r}_n$ can always be taken uniform in w_2. Then we can take $\tilde{r}_n$ small in order that for every $r < \tilde{r}_n$, every $w \in B^{cu}(z, \tilde{r}_n)$ and every $z \in W^{ss}_{\tilde{x}} = W^{ss}_{y_1}$, holds the following from the first case:

$$\frac{|\det Df^n_{|E^{cu}_{y_1}}|}{(\lambda_c^-)^{2n} \cdot ||Df^n_{|E^{uu}_w}||^2} \leq \frac{C^3 \tilde{K}_2 \cdot |\det Df^n_{|E^{cu}_z}|}{(\lambda_c^-)^{2n} \cdot \frac{1}{2}||Df^n_{|E^{uu}_z}||^2} \leq \frac{K_2}{\sigma_2^n}$$

□

4.4 Inequalities using the Transversality

Using the transversality between stable projections of unstable curves, it will be possible to estimate the inner product of measures supported on unstable leaves. First we will state a consequence of the geometric condition of tranversality.

Consider the open set $V = \cup_{i \in I} U_i$ that contains Λ, where the sets U_i are those given by Proposition 2.4.

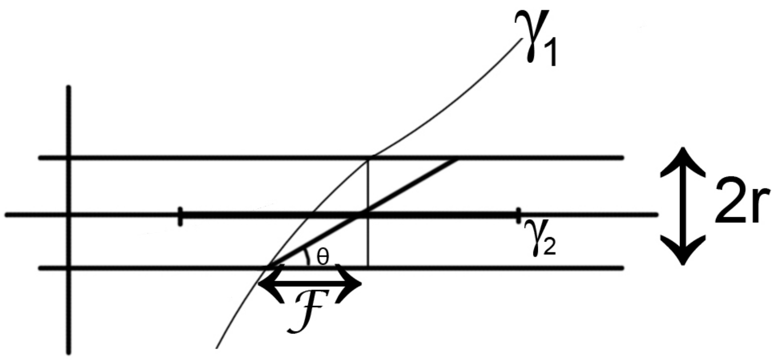
Lemma 4.4. For every $\theta \in (0, \frac{\pi}{2})$ there exist constants r_θ and C_θ for which the following is valid: consider γ_1 and γ_2 Lipschitz curves of finite length contained in some center-unstable manifold contained in V that are θ-transversals in neighbourhoods of radius r_θ, for every $r < \frac{r_\theta}{4}$ consider a connected component E_r of $\{s \mid |d(\gamma_1(s), \gamma_2)| \leq r\}$, then the length of $\gamma_1(E_r)$ is at most $C_\theta r$.

Proof. We will prove this lemma first supposing that the curves γ_1 and γ_2 are contained in $\mathbb{R}^2$, and after considering the case when they are contained in a center-unstable manifold.

If the curves γ_1 and γ_2 are contained in $\mathbb{R}^2$, consider $s \in E_r$ and the point $x = \gamma_1(s)$. For every point $\gamma_2(t)$ we can consider cones C_1 and C_2 as in the definition of θ-transversality and we denote by γ_1^-, γ_1^+, γ_2^-, γ_2^+ the extremal lines of C_1 and C_2.

By transversality in neighbourhoods of radius r, the curves intersect at most once on neighbourhoods of radius smaller than $\frac{r}{2}$. We can also suppose that the curve γ_2^- is contained in the axis x and that $\gamma_1|_{\gamma_1^{-1}E_r}$ is a graph over the axis y.

Consider the following geometric picture



The length of $\tilde{d}$ is at most $\dfrac{2r}{\tan(\theta)} := C_\theta$.
This geometric comparison passes to the metric inequality below:

$$
\begin{aligned}
l(\gamma_1(E_r)) &\leq m_{\mathbb{R}}\big(\{s, d(\gamma_1^+(s), \gamma_2^-) \leq r\}\big) \\
&\leq \frac{2}{\tan(\theta)} r = \tilde{C}_\theta r
\end{aligned}
$$

To the second case, suppose that the curves are contained in the same center-unstable manifold. Since the family of diffeomorphisms $\{\psi_i\}_{i\in I}$ given by Proposition 2.4 have derivative uniformly bounded, we can consider $T = \sup_{i\in I} ||D\psi_i||$, so the length of $\gamma_1(E_r)$ is smaller than the length of $\psi_i(\gamma_1(E_r))$ times T.

Consider r_θ such that $C_\theta r_\theta T < \frac{R_1}{2}$, then $\gamma_1(E_r)$ is contained in some U_{i_0}, $i_0 \in I$. The result follows taking $r < \min\{R_1, r_\theta\}$, noting that in V the distances $d^{cu}(x,y)$ and $d(x,y)$ are equivalent, and that $\{s, d(\gamma_1(s), \gamma_2) \leq r\} \subset \{s, d(\psi_{i_0}\gamma_1(s), \psi_{i_0}\gamma_2) \leq Tr\}$. □

Now, it is possible to estimate the inner product of measures supported in θ-transversal curves.

Lemma 4.5. Given n, C_i and C_j, there exists $\hat{r}_n > 0$ and $A_n > 0$ such that for every ergodic u-Gibbs μ, $r < \hat{r}_n$ and for every $m \neq m'$, holds:

$$\langle (\pi_j)_*(f_m)_*\mu, (\pi_j)_*(f_{m'})_*\mu \rangle_{W_j,r} \leq A_n \cdot |(\pi_j)_*(f_m)_*\mu| \cdot |(\pi_j)_*(f_{m'})_*\mu|$$

Proof. In the proof, we will use the following fact: By finiteness, there exist constants $d_1 > 0$ and $d_2 > 0$ such that for every pair of components $f^n(C_{i,j,m})$ and $f^n(C_{i,j,m'})$, $m \neq m'$, it is valid that either $d^{ss}(R_{i,j,m}, R_{i,j,m'}) > d_1$ or $d^{cu}(R_{i,j,m}, R_{i,j,m'}) > d_2$.

Consider $\theta_n = \theta(d_1)$ and $R_n = r(d_1)$ given by the Hypothesis of Transversality and consider $r < \min\{\frac{R_1}{4}, \frac{d_2}{4}\}$.

We will approximate μ by measures supported in unstable curves, so the proof of this lemma will be done in several cases.

First, consider the measures $\mu_1 = \rho_1 m_{\gamma_1}$ and $\mu_2 = \rho_2 m_{\gamma_2}$, where $\log \rho_i$ is (C_1, α_1)-Holder and γ_i is an unstable curve contained in C_i that crosses C_i, $i = 1, 2$. Consider the Lipschitz curves $\tilde{\gamma}_m = \pi_j \circ f_m(\gamma_1)$ and $\tilde{\gamma}_{m'} = \pi_j \circ f_{m'}(\gamma_2)$, the measures $\nu_m = (\pi_j)_*(f_m)_*\mu_1$ and $\nu_{m'} = (\pi_j)_*(f_{m'})_*\mu_2$ supported on these curves, consider the densities $\rho_m = \frac{d\nu_m}{dm_{\tilde{\gamma}_m}} = (\mathrm{Jac}\, \pi_j^{ss})^{-1}(\mathrm{Jac}\, f_m)^{-1} \cdot \rho_1$ and $\rho_{m'} = \frac{d\nu_{m'}}{dm_{\tilde{\gamma}_{m'}}} = (\mathrm{Jac}\, \pi_j^{ss})^{-1}(\mathrm{Jac}\, f_{m'})^{-1} \cdot \rho_2$.

In this case we will prove that $\langle \nu_m, \nu_{m'} \rangle_{W_j,r} \leq C_{\theta_n} |\nu_m||\nu_{m'}|$.

We know that there exists constants C_2 and α_2 (depending on C_1, α_1, L) such that the densities ρ_m and $\rho_{m'}$ are (C_2, α_2)-Holder functions. Since the curves $\tilde{\gamma}_*$ have length uniformly bounded by $L^{-1}a_1$ and La_2, then we can bound $\rho_*(\gamma_*(x)) \leq e^{C_2(La_2)^{\alpha_2}}\rho_*(\gamma_*(y))$, $* \in \{m, m'\}$.

Consider the function $\mathbb{1}_r$ defined by $\mathbb{1}_r(z_1, z_2) = \begin{cases} 1 & \text{if } d^{cu}(z_1, z_2) < r \\ 0 & \text{otherwise} \end{cases}$

For these functions, we have the following comparisons:

$$\mathbb{1}_r(\tilde{\gamma}_m(s), z) \cdot \mathbb{1}_r(\tilde{\gamma}_{m'}(t), z) \leq \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)) \cdot \mathbb{1}_r(\tilde{\gamma}_m(s), z).$$
$$\mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)) \leq \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}) \cdot \mathbb{1}_{2r}(\tilde{\gamma}_m, \tilde{\gamma}_{m'}(t)).$$

Since the center-unstable manifolds form a C^1 lamination, we have $m^{cu}(B^{cu}(z, r)) \leq C_3 r^2$ for every $z \in \cup_i C_i$ (the area of these balls depends only on the first derivative of the charts when we look to these manifolds as graphs).

If $d^{cu}(R_{i,j,m}, R_{i,j,m'}) < r$ then the inner-product $\langle \nu_m, \nu_{m'} \rangle_{W_j,r}$ is zero and it holds what we want. Else, we have $d^{ss}(R_{i,j,m}, R_{i,j,m'}) > d_1$ and it is possible to use the θ_n-transversality for the projections of unstable curves contained in distinct rectangles. Then we can estimate the desired inner product:

$$\begin{aligned}
\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &= \frac{1}{r^4} \int\limits_{W_j} \nu_m(B(z, r)) \cdot \nu_{m'}(B(z, r)) dm^{cu}(z) \\
&= \frac{1}{r^4} \int\limits_{W_j} \Big(\int\limits_{\tilde{\gamma}_m \cap B(z,r)} \rho_m(\tilde{\gamma}_m(s)) dm_{\tilde{\gamma}_m}(s) \Big) \\
&\qquad \Big(\int\limits_{\tilde{\gamma}_{m'} \cap B(z,r)} \rho_{m'}(\tilde{\gamma}_{m'}(t)) dm_{\tilde{\gamma}_{m'}}(t) \Big) dm^{cu}(z) \\
&= \frac{1}{r^4} \int\limits_{W_j} \int\limits_{\tilde{\gamma}_m} \int\limits_{\tilde{\gamma}_{m'}} \Big(\rho_m(\tilde{\gamma}_m(s)) \rho_{m'}(\tilde{\gamma}_{m'}(t)) \cdot \mathbb{1}_r(\tilde{\gamma}_m(s), z) \mathbb{1}_r(\tilde{\gamma}_{m'}(t), z) \Big) ds dt dz \\
&\leq \frac{1}{r^4} \int\limits_{\tilde{\gamma}_{m'}} \rho_{m'}(t) \Big(\int\limits_{\tilde{\gamma}_{m'}} \rho_m(s) \cdot \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)) \\
&\qquad \Big(\int\limits_{W_j} \mathbb{1}_r(\tilde{\gamma}_m(s), z) dz \Big) ds \Big) dt
\end{aligned}$$

$$\leq \frac{C_3}{r^2} \int\limits_{\tilde{\gamma}_m} \int\limits_{\tilde{\gamma}_{m'}} \Big(\rho_m(\tilde{\gamma}_m(s)) \rho_{m'}(\tilde{\gamma}_{m'}(t)) \cdot \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}(t)) \Big) ds dt$$

$$\leq \frac{C_3 e^{2C_2(La_2)^{\alpha_2}}}{r^2} \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0))$$

$$\Big(\int\limits_{\tilde{\gamma}_m} \mathbb{1}_{2r}(\tilde{\gamma}_m(s), \tilde{\gamma}_{m'}) ds \Big) \Big(\int\limits_{\tilde{\gamma}_{m'}} \mathbb{1}_{2r}(\tilde{\gamma}_m, \tilde{\gamma}_{m'}(t)) dt \Big)$$

To estimate the last expressions in parenthesis, we will use Lemma 4.4. Take $r < R_{\theta_n}$ small such that we can apply Lemma 4 and consider the constant C_{θ_n} given by the lemma. Note also that for these projections of unstable curves, by transversality in neighbourhoods with radius smaller than R_n, there exists an integer $M_n \in \mathbb{N}$ such that each $\tilde{\gamma}_m$ intersects the neighbourhood of radius r of $\tilde{\gamma}_{m'}$ in at most M_n connected components.

Then:

$$\langle \nu_m, \nu_{m'} \rangle_{W_j, r} \leq \frac{C_4}{r^2} \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0)) \cdot (M_n C_{\theta_n} r) \cdot (M_n C_{\theta_n} r)$$

$$= (C_4 (M_n C_{\theta_n})^2) \cdot \rho_m(\gamma_m(s_0)) \rho_{m'}(\gamma_{m'}(t_0))$$

In order to continue, we will check that $\int\limits_{\tilde{\gamma}_m} \rho_m(\gamma_m(s)) ds \geq K_3 \rho_m(\gamma_m(s_0))$ for some constant $K_3 > 0$, this is due to a simple calculation:

$$\int\limits_{\tilde{\gamma}_m} \rho_m(\gamma_m(s)) ds \geq \int\limits_{\tilde{\gamma}_m} e^{-(C_2(La_2)^{\alpha_2})} \rho_m(\gamma_m(s_0)) ds$$

$$= e^{-(C_2(La_2)^{\alpha_2})} \rho_m(\gamma_m(s_0)) l(\tilde{\gamma}_m)$$

$$\geq \left(L^{-1} a_1 e^{-(C_2(La_2)^{\alpha_2})} \right) \rho_m(\gamma_m(s_0))$$

$$= K_3 \rho_m(\gamma_m(s_0)).$$

Analogously, it is valid that $\rho_{m'}(\gamma_{m'}(t_0)) \leq K_3^{-1} \cdot \int\limits_{\tilde{\gamma}_{m'}} \rho_{m'}(\gamma_{m'}(t)) dt$, so we can continue the estimative:

$$\langle \nu_m, \nu_{m'} \rangle_{W_j,r} \leq \big(C_4(M_n C_{\theta_n})^2 (K_3)^2\big) \int\limits_{\tilde{\gamma}_m} \rho_m(\gamma_m(s))ds \int\limits_{\tilde{\gamma}_{m'}} \rho_{m'}(\gamma_{m'}(t))dt$$

$$= A_n \cdot |\nu_m| \cdot |\nu_{m'}|$$

The second case to be considered is when the measure μ is a finite sum $\mu = \sum_{k=1}^{s_0} \rho_k m_{\gamma_k}$, the inequality holds by linearity because we also have θ_n-transversality of the stable projection of unstable curves. Actually, taking $\nu_m = (\pi_j)_*(f_m)_*\mu$ and $\nu_{m'} = (\pi_j)_*(f_{m'})_*\mu$ we can see that:

$$\begin{aligned}
\langle \nu_m, \nu_{m'} \rangle_{W_j,r} &= \langle (\pi_j)_*(f_m)_*(\sum_k \rho_k m_{\gamma_k}), (\pi_j)_*(f_{m'})_*(\sum_{k'} \rho_{k'} m_{\gamma_{k'}}) \rangle_{W_j,r} \\
&= \sum_{k,k'} \langle (\pi_j)_*(f_m)_*(\rho_k m_{\gamma_k}), (\pi_j)_*(f_{m'})_*(\rho_{k'} m_{\gamma_{k'}}) \rangle_{W_j,r} \\
&\leq \sum_{k,k'} A_n \cdot |(\pi_j)_*(f_m)_*(\rho_k m_{\gamma_k})||(\pi_j)_*(f_{m'})_*(\rho_{k'} m_{\gamma_{k'}})| \\
&= A_n \cdot |(\pi_j)_*(f_m)_*\mu||(\pi_j)_*(f_{m'})_*\mu| \\
&= A_n \cdot |\nu_m||\nu_{m'}|
\end{aligned}$$

Finally, let us suppose that μ is an ergodic u-Gibbs. This case follows passing to the limit the inequality, it is done using Lemma 4.1 to approximate μ by probability measures of the type $\sum_i \rho_i m_{\gamma_i}$ and aplying Lemma 3.3 to see that the inner-product of these measures converges to the inner-product of the limit measures. □

4.5 Proof of the Main Inequality

Let us state a Localized Version of the Main Inequality when two boxes C_i, C_j and an integer n are fixed.

Proposition 4.2 (Main Inequality - Localized Version)**.** There exists $\tilde{B} > 0$ and $\sigma > 1$, such that for every $n \in \mathbb{N}$ and C_i, C_j fixed, there exists $\tilde{D}_n > 0$, $r_n > 0$ and $c_n > 1$ such that for every ergodic u-Gibbs μ and every $r < r_n$, it holds:

$$\left|\left|(\pi_j)_*(f_*^n(\mu_{|C_i}))\right|\right|^2_{W_j,r} \leq \frac{\tilde{B}}{\sigma^n} \, ||(\pi_i)_*\mu||^2_{\tilde{W}_i,c_n r} + \tilde{D}_n \, |(\pi_i)_*\mu|^2$$

Proof of the Localized Version. Take r small such that we can apply Lemmas 4.3 and 4.5, then:

$$\begin{aligned}
||(\pi_j)_*(f_*^n(\mu_{|C_i}))||^2_{W_j,r} &= ||(\pi_j)_* \sum_m (f_m)_*\mu||^2_{W_j,r} \\
&= \sum_m ||(\pi_j)_*(f_m)_*\mu||^2_{W_j,r} + \sum_{m \neq m'} \langle (\pi_j)_*(f_m)_*\mu, (\pi_j)_*(f_{m'})_*\mu \rangle_{W_j,r} \\
&\leq \frac{B_2}{\sigma^n} ||(\pi_i)_*\mu||^2_{\tilde{W}_i,c_n r} + \sum_{m \neq m'} A_n |(\pi_j)_*(f_m)_*\mu||(\pi_j)_*(f_{m'})_*\mu| \\
&= \frac{\tilde{B}}{\sigma^n} ||(\pi_i)_*\mu||^2_{\tilde{W}_i,c_n r} + \tilde{D}_n |(\pi_i)_*\mu|^2
\end{aligned}$$

□

Now it is possible to prove the Main Inequality from the Localized Version adding it over the set of all boxes C_i's and C_j's, $1 \leq i, j \leq s_0$.

Proof of the Main Inequality. Note that $|||(f_{n;i,j,m})_*\mu|||_{\{\mathcal{W}\},r} = ||(\pi_j)_*\mu||_{W_j,r}$, that is, the maximum occurs for j. And note also that $f_*^n\mu(E) \leq \sum_{i,j,m} (f_m)_*\mu(E)$

for every measurable set E. Then:

$$\begin{aligned}
|||f^n_*\mu|||^2_{\{\mathcal{W}\},r} &\leq \sum_{i,j} \left|\left|(\pi_j)_*(f^n_*(\mu_{|C_i}))\right|\right|^2_{W_j,r} \\
&\leq \sum_{i,j} \left[\frac{\tilde{B}}{\sigma^n}||(\pi_i)_*\mu||^2_{\tilde{W}_i,c_n r} + \tilde{D}_n|(\pi_i)_*\mu|^2\right] \\
&\leq \sum_{i,j} \left[\frac{\tilde{B}}{\sigma^n}|||\mu|||^2_{\{\tilde{\mathcal{W}}\},c_n r} + \tilde{D}_n|\mu|^2\right] \\
&\leq \frac{\tilde{B}s_0^2}{\sigma^n}|||\mu|||^2_{\{\mathcal{W}\},c_n r} + \tilde{D}_n s_0|\mu|^2 \\
&\leq \frac{B}{\sigma^n}|||\mu|||^2_{\{\mathcal{W}\},c_n r} + D_n|\mu|^2
\end{aligned}$$

□

Chapter 5

Physical Measures

In this Chapter we focus on statistical properties that we can deduce from the Main Inequality. We will show that, under the hypothesis of Theorem A, every ergodic u-Gibbs is a physical measure.

Along this Chapter we will consider f satisfying the hypothesis of Theorem A, the same boxes and semi-norms considered in the Lemmas along Chapter 4.

5.1 Existence of Physical Measures

We will prove that every u-Gibbs projects by the stable holonomy into absolutely continuous measures in the center-unstable manifolds W_i and that this fact implies a positive measure for the basin of these measures.

5.1.1 Absolute Continuity of the Projection of the u-Gibbs

With the Main Inequality proved in Chapter 4, it is possible to deduce the following proposition:

Proposition 5.1. Every ergodic u-Gibbs projects into an absolutely continuous measure in $\tilde{W}_i$ by the stable holonomy π_i.

Moreover, for every ergodic u-Gibbs μ there exist a constant $K > 0$ such that $\left|\left|\frac{d((\pi_i)_*\mu)}{dm^{cu}}\right|\right|_{L^2} \leq K$.

Proof. Given the ergodic u-Gibbs μ, consider B and σ as given by The Main Inequality. Fix N such that $\frac{C}{\sigma^N} < 1$, consider D_N, r_N and c_N as given also by the Main Inequality. Then it is valid for $r < r_N$:

$$|||\mu|||_r^2 = |||f_*^n\mu|||_r^2 \leq \frac{B}{\sigma^N}|||\mu|||_{c_n r}^2 + D_N|\mu|^2$$

We can rewrite, for $r < \frac{r_N}{c_N}$, as the following:

$$|||\mu|||_{c_N^{-1}r}^2 \leq \frac{B}{\sigma^n}|||\mu|||_r^2 + D_N|\mu|^2$$

Define the constant $K_0 := |||\mu|||_{\frac{r_N}{2}}^2$ and iterate j times the inequality:

$$\begin{aligned}
|||\mu|||_{c_N^{-j}\frac{r_N}{2}}^2 &\leq \left(\frac{B}{\sigma^N}\right)|||\mu|||_{c_N^{-(j-1)}\frac{r_N}{2}}^2 + D_N|\mu|^2 \\
&\leq \cdots \\
&\leq \left(\frac{B}{\sigma^N}\right)^j |||\mu|||_{\frac{r_N}{2}}^2 + D_N\left(1 + \frac{B}{\sigma^N} + \cdots + \left(\frac{B}{\sigma^N}\right)^{j-1}\right)|\mu|^2 \\
&\leq 1 \cdot K_0 + D_N \cdot \frac{1}{1 - \left(\frac{B}{\sigma^N}\right)} := K
\end{aligned}$$

So, for every π_i it holds:

$$\liminf_{r \to 0^+} ||(\pi_i)_*\mu||_{W_i,r} \leq \liminf_{j \to \infty} |||\mu|||_{c_N^{-j}\frac{r_N}{2}}^2 \leq K < +\infty$$

By the criteria of absolute continuity for measures (Lemma 3.6),

$$(\pi_i)_*\mu \ll m^{cu} \text{ and } \left|\left|\frac{d((\pi_i)_*\mu)}{dm^{cu}}\right|\right|_{L^2} \leq K$$

□

5.1.2 Conclude that the u-Gibbs is a Physical Measure

Since we have fixed the boxes $\{(C_i, W_i, \tilde{W}_i, \pi_i)\}$, $i = 1, \cdots, s_0$, that covers Λ, for every u-Gibbs μ there exists some i_0 such that $\mu(C_{i_0}) > \frac{1}{2s_0}$, in particular $(\pi_{i_0})_*\mu$ is non-zero.

Proposition 5.2. Consider μ an ergodic u-Gibbs for f, consider i such that $\mu(C_i) > \frac{1}{2s_0}$, suppose that the measure $\nu_i = (\pi_i^{ss})_*\mu$ is absolutely continuous with respect to the Lebesgue measure m^{cu} and that $\left|\left|\frac{d\nu_i}{dm^{cu}}\right|\right|_{L^2} \leq K$.

Then μ is a physical measure. Moreover, there is a constant $C_5 > 0$ depending on K such that $m(B(\mu)) \geq C_5$.

Proof. Since μ is ergodic, we have $\mu(B(\mu)) = 1$. Note that $\pi_i(B(\mu))$ is measurable (Theorem 3.23 in [6] guarantees the measurability of this projection). Then,

$$\nu(\pi_i B(\mu)) = \mu((\pi_i)^{-1}\pi_i B(\mu)) \geq \mu(B(\mu) \cap C_i) > \frac{1}{2s_0}$$

By the absolute continuity of ν, it follows that $m^{cu}(\pi_i B(\mu)) > 0$. Actually, it is possible to prove that $m^{cu}(\pi_i B(\mu)) > (2s_0K)^{-1}$, this is due to the following:

$$\begin{aligned}
(2s_0)^{-1} &< \nu(\pi_i B(\mu)) \\
&= \int \mathbb{1}_{\pi_i B(\mu)} \cdot \frac{d\nu}{dm^{cu}} dm^{cu} \\
&= \langle \mathbb{1}_{\pi_i B(\mu)}, \frac{d\nu}{dm^{cu}} \rangle_{L^2} \\
&\leq ||\mathbb{1}_{\pi_i B(\mu)}||_{L^2} \cdot \left|\left|\frac{d\nu}{dm^{cu}}\right|\right|_{L^2} \\
&\leq m^{cu}(\pi_i B(\mu)) \cdot K
\end{aligned}$$

Using that $B(\mu)$ is $\mathcal{F}^{ss}$-saturated and $\mathcal{F}^{ss}$ is absolutely continuous, we have that $m(B(\mu)) > 0$, so it is a physical measure. Moreover, considering a constant that bounds bellow the Jacobian of h^{ss} we have that $m(B(\mu)) > (2s_0K)^{-1}\,\mathrm{Jac}(h^{ss})^{-1} := C_5$. $\square$

5.2 Proof of Theorem A

By Proposition 5.2, it is possible to see that every ergodic u-Gibbs is a physical measure. To conclude the proof of Theorem A, it remains to prove that there exist at most finite physical measures (Finiteness) and that the union of their basin has full Lebesgue measure in the basin of attraction (Problem of the Basins), this is what will be done in this Section.

5.2.1 Finiteness

Let us show that there exists at most finite ergodic physical measures for f. The finiteness of physical measures will be completely proved at the end of the next step as a consequence of the full Lebesgue measure of the union of the basins of the ergodic physical measures.

Suppose that there are infinitely many ergodic u-Gibbs μ_n, taking a subsequence we can suppose that $\mu_n \to \mu$. Consider i_0 and a subsequence also denoted by μ_n such that $\mu_n(C_{i_0}) > \frac{1}{2s_0}$ for every $n \in \mathbb{N}$.

Applying Proposition 5.2 follows that every $(\pi_{i_0})_*\mu_n$ is absolutely continuous with respect to m^{cu} and that $\frac{d(\pi_{i_0})_*\mu_n}{dm^{cu}} \in L^2$, then $(\pi_{i_0})_*\mu$ is also absolutely continuous, $\frac{d(\pi_{i_0})_*\mu}{dm^{cu}} \in L^2$ and $\left|\left|\frac{d(\pi_{i_0})_*\mu_n}{dm^{cu}}\right|\right|_{L^2} \to \left|\left|\frac{d(\pi_{i_0})_*\mu}{dm^{cu}}\right|\right|_{L^2}$. So the sequence $\frac{d(\pi_{i_0})_*\mu_n}{dm^{cu}}$ has L^2 norm uniformly bounded by some constant $\hat{K} > 0$.

Considering $\hat{C}_5$ as given by Proposition 5.2, it is possible to see that there exist at most $\frac{1}{\hat{C}_5}$ ergodic physical measures with $\left|\left|\dfrac{d\nu}{dm^{cu}}\right|\right|_{L^2} \leq \hat{K}$. Actually, suppose that there are $\mu_1, \cdots, \mu_l$ ergodic physical measures for $l \geq \frac{1}{\hat{C}_5} + 1$. By Proposition 2.1 they are all ergodic u-Gibbs, using Proposition 5.2 we get that the Lebesgue measure of their basin is bounded below by $\hat{C}_5 > 0$. Since these measures are distincts, their basins are disjoint, then:

$$1 \geq m(\bigcup_i B(\mu_i)) = \sum_i m(B(\mu_i)) \geq l \cdot \hat{C}_5 > 1$$

This is a contradiction. Thus there exists a finite number of ergodic physical measures.

5.2.2 The Problem of the Basins

We are now ready to show that the union of the basins of the physical measures have full Lebesgue measure in the whole basin of attraction.

Denote the ergodic u-Gibbs by $\mu_1, \cdots, \mu_l$, consider the sets $X = \cup B(\mu_i)$, $E = B(\Lambda) \backslash \cup B(\mu_i)$ and assume by contradiction that $m(E) > 0$.

Considering the normalized measure m_E, define $\sigma_n = \dfrac{1}{n}\sum_{j=0}^{n-1} f_*^j(m_E)$ and consider σ_∞ an accumulation point of σ_n. By Proposition 2.2, σ_∞ is a u-Gibbs, so it projects by π_j into an absolutely continuous measure with $||\frac{d(\pi_j)_*\sigma_\infty}{dm^{cu}}||_{L^2} \leq K$ for every $j = 1, \cdots, s_0$.

Take j such that $(\pi_j)_*\sigma_\infty$ is non-zero, define the measures $\tilde{\sigma}_{n_k} = (\pi_j)_*\sigma_{n_k}$ and $\tilde{\sigma}_\infty = (\pi_j)_*\sigma_{n_\infty}$. Since $J_r\tilde{\sigma}_{n_k}(x) = \dfrac{\tilde{\sigma}_{n_k}(B(x,r))}{m^{cu}(B(x,r))} \xrightarrow{n_k} \dfrac{\tilde{\sigma}_\infty(B(x,r))}{m^{cu}(B(x,r))} = J_r\tilde{\sigma}_\infty(x)$ for infinitely many $r \to 0^+$ and ae-x, we have that:

$$\left|\left|\frac{d\tilde{\sigma}_{n_k}}{dm^{cu}}\right|\right|_{L^2} \to \left|\left|\frac{d\tilde{\sigma}_\infty}{dm^{cu}}\right|\right|_{L^2} > 0$$

By the invariance of X^c it holds that $supp(\sigma_n) \subset X^c$ for every $n \geq 0$, so when $x \in \pi_j(X)$ we have $\tilde{\sigma}_n(B(x,r)) = 0$ for some r depending on n and x, because $\pi_j^{-1}(x) \subset supp(\sigma_n)^c$ is a compact set contained in an open set, then $\frac{d\tilde{\sigma}_{n_k}}{dm^{cu}}(x) = 0$ for every $x \in \pi_j(X)$. On the other hand, when $x \in \pi_j(X)^c$ we have $\frac{d\tilde{\sigma}_\infty}{dm^{cu}}(x) = 0$, because all the ergodic u-Gibbs give full measure for X, that is, $1 = \tilde{\mu}_\infty(\pi_j(X)) = \int_{\pi_j(X)} \frac{d\tilde{\mu}_\infty}{dm^{cu}}(x)$.

Thus,

$$\begin{aligned}\langle \frac{d\tilde{\sigma}_{n_k}}{dm^{cu}}, \frac{d\tilde{\sigma}_{\infty}}{dm^{cu}} \rangle_{L^2} &= \int_{\tilde{W}_j} \frac{d\tilde{\sigma}_{n_k}}{dm^{cu}} \cdot \frac{d\tilde{\sigma}_{\infty}}{dm^{cu}} dm^{cu} \\ &= \int_{\tilde{W}_j \cap \pi_j(X)} \frac{d\tilde{\sigma}_{n_k}}{dm^{cu}} \cdot \frac{d\tilde{\sigma}_{\infty}}{dm^{cu}} dm^{cu} \\ &= \int_{\tilde{W}_j \cap \pi_j(X)} \frac{d\tilde{\sigma}_{n_k}}{dm^{cu}} d\tilde{\sigma}_{\infty} \\ &= 0\end{aligned}$$

That is a contradiction.

□

5.3 Proof of Corollary B

To obtain Corollary B as a consequence of Theorem A we need to know when the hypothesis of transversality holds robustly. We will see in this Section that the transversality condition is an open property if the stable foliation varies in the C^1 topology.

Proposition 5.3. Suppose that $\mathcal{F}_f^{ss}$ is of class C^1 and that there exist constants a, ϵ_1, θ_2, L and r_2 such that it is valid the following: for every strong-unstable curves γ_1, γ_2 of lengths at most L and every center-unstable manifold W_3^{cu} with $d^{ss}(\gamma_i, W_3^{cu}) \leq \epsilon_1$ and $d^{ss}(\gamma_1, \gamma_2) \in J = [a, Ia]$, where $I = \max_{x \in B(\Lambda)} \{||Df_{|E_x^{ss}}||^{-1}\}$, it holds that the curves $\pi^{ss}\gamma_1$ and $\pi^{ss}\gamma_2$ are θ_2-transversal in neighborhood of radius r_2.

Then it is valid the Hypothesis of Transversality (H1).

Proof. First, since h^{ss} can be seen as a diffeomorphism between center-unstable plaques that are close, let us see that taking ϵ_0 small it is enough to check the transversality only in the projections of the curves into the center-unstable manifold W_2^{cu} that contains γ_2.

Claim 5.1. Consider two open sets $U, V \subset \mathbb{R}^2$ and a diffeomorphism $h : \bar{U} \to \bar{V}$ of class C^1 with $||h - id||_{C^1} \leq \frac{1}{2}$. For every θ, r there exist constants $\tilde{\theta}, \tilde{r}$ such that for every pair of curves γ_1, γ_2 contained in U that are θ-transversal in neighborhoods of radius r, the curves $\tilde{\gamma}_1 = h(\gamma_1)$ and $\tilde{\gamma}_2 = h(\gamma_2)$ are $\tilde{\theta}$-transversal in neighborhoods of radius $\tilde{r}$.

Proof. Consider $\tilde{r}_0$ such that $h(B(x,r)) \supset B(h(x), \tilde{r}_0)$ for every $x \in U$. Note that for unitary vectors v_1 and v_2, $||v_1 - v_2|| = d(v_1, v_2) = 2\sin\left(\frac{\theta}{2}\right)$, where θ is the angle between them. Since $\frac{\sin(z)}{z}$ is bounded, there exists a constant C_1 such that $C_1^{-1}\theta \leq ||v_1 - v_2|| \leq C_1\theta$.

Consider two unitary vectors $\tilde{v}_1$ and $\tilde{v}_2$ tangent to the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ at the points $h(x_1) \in \tilde{\gamma}_1$ and $h(x_2) \in \tilde{\gamma}_2$, for our purposes it is enough to bound below $||\tilde{v}_1 - \tilde{v}_2||$.

By continuity of the derivative, there exist $\tilde{r}_1$ such that $||dh_x - dh_y|| < \frac{C_1^{-1}\theta}{4}$ whenever $d(x,y) \leq \tilde{r}_1$. So we have:

$$\begin{aligned}
||\tilde{v}_1 - \tilde{v}_2|| &= ||dh_{x_1}v_1 - dh_{x_2}v_2|| \\
&\geq ||dh_{x_1} \cdot (v_1 - v_2)|| - ||(dh_{x_1} - dh_{x_2}) \cdot v_2|| \\
&\geq \frac{1}{2}C_1^{-1}\theta - \frac{1}{4}C_1^{-1}\theta \\
&= \frac{1}{2}C_1^{-1}\theta := \tilde{\theta}
\end{aligned}$$

This implies that the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are $\tilde{\theta}$-transversals in neighbourhouds of radius $\tilde{r} = \min\{\tilde{r}_0, \tilde{r}_1\}$.

□

Now, to prove that is valid Hypothesis (H1), consider $\tilde{\epsilon}_0$ small such that it is possible to apply Claim 3 for the stable holonomy between center-unstable plaques whose d^{ss}-distance is smaller than $\tilde{\epsilon}_0$ (we can take a local chart with product structure of $W^{cu} \times W^{ss}$ to say that the holonomy is C^1 close to the vertical projection), then it is enough to check what the transversality between the projection of unstable curves only in the case where $\gamma_2 \subset W_2^{cu}$.

Take $\epsilon_0 = \min\{\epsilon_1, a, \tilde{\epsilon}_0\}$. Given the unstable curves γ_1 and γ_2 with $d^{ss} = d < \epsilon_0$, we can consider the smallest integer $n \in \mathbb{N}$ such that $d(f^{-n}\gamma_1, f^{-n}\gamma_2) \in J = [a, Ia]$ (this is a fundamental domain for iteration of stable distances). Thus, by hypothesis, the projections of these curves into $f^{-n}W_2^{cu}$ are θ_2-transversals in neighborhoods of radius r_2.

Iterate foward n times the projection of these curves and the cones bounding their angle by an angle θ_2, then it is possible to see that there exists τ, depending on the norm of Df restricted to each sub-bundle, such that the angle of the images of the cones containing these projected curves is bounded by $\tau^n\theta_2$ in neighborhouds of radius $(\lambda_c^-)^n r_2$.

If γ_2 is not contained in W_3^{cu}, consider W_2^{cu} that contains γ_2, $\theta = \tau^n\theta_2$ and $r = (\lambda_c^-)^n r_2$. Since the family of derivatives of the stable holonomies is uniformly continuous, we can apply Claim 5.1 to obtain $\theta(\epsilon)$ and $r(\epsilon)$ such that the projections into W_3^{cu} are $\theta(\epsilon)$-transversals in neighborhoods of radius $r(\epsilon)$. □

Now we are able to prove the robustness of the transversality condition when the stable foliation varies in the C^1 topology with the dynamics.

Proposition 5.4. The Hypothesis of Transversality (H1) is an open property under Hypothesis (H3).

Proof. As seen by Proposition 5.3, it is enough to check the transversality for projection of curves when $d^{ss} \in [a, Ia] = J$, where J is a fundamental domain for the size of iterates of stable segments.

Since $x \to E_x^{uu}$ is continuous, we consider a family of unstable cones with small width and fix a constant $\alpha > 0$ that bounds below the angle of each pair of stable projections of unstable cones of this family when $d^{ss} \in J$.

It can be seen that if a curve at x is contained in a cone C, then every curve at x that is C^1 close is also contained in the cone C. Thus the family of cones and the limitation α of the projections can be taken constant if we vary the projection in the C^1 topology. This is the case due to Hypothesis (H3). $\square$

Finally, it follows from these Propositions the Corollary B.

Proof of Corollary B. Given f_0 under the hypothesis of Theorem B, what we need to do is to check that there exists a neighborhood $\mathcal{U}$ of f_0 such that every $f \in \mathcal{U}$ satisfies the hypothesis of Theorem A.

The conditions (H2) and (H3) are clearly open in f, so they also holds for a neighborhood of $\mathcal{U}_1$ of f_0. By hypothesis of robustly dynamical coherence, there exists also a neighborhood $\mathcal{U}_2$ such that every $f \in \mathcal{U}$ is dynamically coherent. From Proposition 5.4, the hypothesis of transversality is open under condition (H3). $\square$

Chapter 6

Attractors with Transversality

In this Chapter we will exhibit the construction of a family of nonhyperbolic attractors with central direction neutral and transversality between unstable leaves via the stable holonomy.

6.1 The Hyperbolic Attractor F_0

Consider $F_0 : S^1 \times [-1,1] \times [-1,1] \to S^1 \times [-1,1] \times [-1,1]$ defined by:

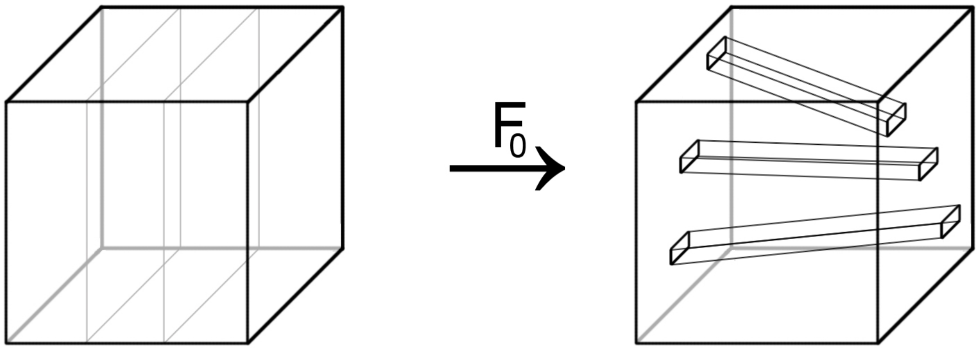
$$F_0(x,y,z) = (3x, \lambda_c y + \alpha(x)x + c_i, \lambda_{ss} z + d_i)$$

Where $\lambda_{ss} < \lambda_c < 1$ and $\lambda_c > \frac{1}{3}$. For $i = 1,2,3$ we consider the rectangles $R_1 = [0, \frac{1}{3}) \times [-1,1]^2$, $R_2 = [\frac{1}{3}, \frac{2}{3}) \times [-1,1]^2$ and $R_3 = [\frac{2}{3}, 1] \times [-1,1]^2$, the function α such that $\alpha(x) = \alpha_i$ if $x \in R_i$, and the constants c_i's and d_i's that are parameters of translation to guarantee that the image of F_0 is well defined.

The dynamics F_0 is a linear model of a hyperbolic attractor, restricted to each rectangle R_i it is a hyperbolic affine transformation inserting $F_0(R_i)$ into $M = S^1 \times [-1,1]^2$ with slope α_i in the y-direction into the xy-plane.

For simplicity we will consider a constant $\alpha \in (0, 1-\lambda^c)$ and choose the parameters $\alpha_1 = \alpha$, $\alpha_2 = 0$, $\alpha_3 = -\alpha$ and $c_i = -\alpha_i$, $i = 1, 2, 3$, then the rectangles are inserted being transversals with respect to the stable (vertical) foliation.

The image of this dynamics can be seen as the picture below.



Consider $\Lambda_0 = \bigcap_{n \geq 0} F_0^n(S^1 \times [-1, 1]^2)$.

Proposition 6.1. The attractor Λ_0 for F_0 satisfy the hypothesis of transversality (H1).

Proof. Since $E_0^{ss} = (0, 0, 1)$, the stable projection of F_0, denoted here by π^{ss}, coincides with the vertical projection $(x, y, z) \to (x, y)$. The center-unstable direction at every point corresponds to the xy-plane.

It is possible to obtain an expression for E^{uu} as a serie of powers of the kind $E^{uu}(p) = \left(1, \sum_j \alpha_j(p) \left(\frac{\lambda_c}{\lambda^{uu}}\right)^j, 0\right)$, where $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$ according to the itinerary of x by the expansion in S^1 by the factor $\lambda_{uu} = 3$. The calculation of this serie is done writting the condition of invariance $DF_0(p) \cdot E_p^{uu} = E_{F_0(p)}^{uu}$ for E_p^{uu} given by a vector $(1, \alpha^{uu}(p), 0)$.

Since $DF_0(p) = \begin{bmatrix} \lambda_{uu} & 0 & 0 \\ \alpha_i(p) & \lambda_c & 0 \\ 0 & 0 & \lambda_{ss} \end{bmatrix}$,
this condition can be written as

$$(\lambda_{uu}, \alpha_i(p) + \alpha^{uu}(p)\lambda_c, 0) = \lambda_{uu}(1, \alpha^{uu}(F_0(p)), 0)$$

So we want to find a function α^{uu} such that:

$$\alpha^{uu}(F_0(p)) = (\lambda_{uu})^{-1}\alpha(p) + \frac{\lambda_c}{\lambda_{uu}}\alpha^{uu}(p)$$

Iterating the equation above j times and considering $\rho := \frac{\lambda_c}{\lambda_{uu}} < \frac{1}{3}$, we have:

$$\alpha^{uu}(F_0^j(p)) = (\lambda_{uu})^{-1}\alpha(F_0^{j-1}(p)) + \rho(\lambda_{uu})^{-1}\alpha(F_0^{j-2}(p)) + \cdots + \rho^j\alpha^{uu}(p)$$

Writting $F_0^j(p) = \tilde{p} \in \Lambda$, and noting that for points in the attractor we can consider infinitely many pre-iterates, we have:

$$\begin{aligned} \alpha^{uu}(\tilde{p}) &= (\lambda_{uu})^{-1}\alpha(F_0^{-1}(\tilde{p})) + \rho(\lambda_{uu})^{-1}\alpha(F_0^{-2}(\tilde{p})) + \cdots + \rho^j\alpha^{uu}(F_0^{-j}(\tilde{p})) \\ &= \sum_{j=0}^{j_0}(\lambda_{uu})^{-1}\rho^j\alpha(F_0^{-(j+1)}(\tilde{p})) + \rho^{j_0}\alpha^{uu}(F_0^{-j_0}(\tilde{p})) \\ &\to \sum_{j\geq 0}(\lambda_{uu})^{-1}\rho^j\alpha(F_0^{-(j+1)}(\tilde{p})) \end{aligned}$$

Let us consider multi-indexes $[k] = (k_0, \cdots, k_{t-1})$ of size t, where $k_j \in \{1, 2, 3\}$ and define the rectangle $R_{[k]}$ as the set of points z such that $f^j(z) \in R_{k_j}$, consider also $f_{[k]}$ as the restriction of f^t to $R_{[k]}$. Since the d^{ss}-diameter of $R_{(k_0,\cdots,k_{t-1})}$ is equals to $(\lambda^{ss})^t$, if $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) > \epsilon$ then these curves are contained in distincts $f_{[k]}(M)$, $f_{[k']}(M)$ with multi-indexes $[k] = (k_0, \cdots, k_{t_0-1})$ and $[k'] = (k'_0, \cdots, k'_{t_0-1})$ of size $t_0 \leq \frac{\log\epsilon}{\log\lambda^{ss}}$ depending on ϵ.

For $p \in f_{[k]}(M)$ and $p' \in f_{[k']}(M)$, taking $j_0 = \inf\{j, k_j \neq k'_j\} \leq t_0$ we have the following:

$$\begin{aligned}
\alpha^{uu}(p) - \alpha^{uu}(p') &= \sum_{j \geq 0} \frac{1}{\lambda_{uu}} \rho^j [\alpha(F^{-(j+1)}(p)) - \alpha(F^{-(j+1)}(p'))] \\
&= \sum_{j \geq j_0} \frac{1}{\lambda_{uu}} \rho^j (\alpha(F^{-(j+1)}(p)) - \alpha(F^{-(j+1)}(p'))) \\
&= \frac{\rho^{j_0}}{\lambda_{uu}} \left(\alpha(F^{-(j_0+1)}(p)) - \alpha(F^{-(j+1)}(p'))\right) + \sum_{j > j_0} \frac{\rho^j}{\lambda_{uu}} (\alpha - \alpha') \\
&\geq \frac{1}{\lambda_{uu}} \left[\rho^{j_0} \alpha - 2\alpha \cdot \frac{\rho^{j_0+1}}{1-\rho}\right] \\
&= \frac{1}{\lambda_{uu}} \rho^{j_0} \alpha \left(1 - \frac{2\rho}{1-\rho}\right) \\
&\geq \frac{1}{\lambda_{uu}} \rho^{\frac{\log \epsilon}{\log \lambda^{ss}}} \alpha \left(\frac{1-3\rho}{1-\rho}\right) := C(\epsilon)
\end{aligned}$$

From the expression for α^{uu} and that $\pi_0^{ss} \cdot E^{uu}(p) = \pi^{ss}(1, \alpha^{uu}(p), 0) = (1, \alpha^{uu}(p))$, it follows that $\angle((1, \alpha^{uu}(p)), (1, \alpha^{uu}(p'))$ is bounded below, because the function α^{uu} is uniformly bounded above and the diference $\alpha^{uu}(p) - \alpha^{uu}(p')$ is bounded below by $C(\epsilon)$, then there exists a function $\theta : (0,1) \to \mathbb{R}^+$ such that

$$\angle\left(\pi^{ss}(\gamma_1^{uu}), \pi^{ss}(\gamma_2^{uu})\right) \geq \theta(\epsilon)$$

Whenever $d^{ss}(\gamma_1^{uu}, \gamma_2^{uu}) > \epsilon$. □

6.2 The Family of Attractors $F_{\mu,n}$

Now let us describe the family that will be considered in order to obtain a nonhyperbolic attractor with the transversality condition.

Considering F_0 as before, note that

$$F_0^n(x, y, z) = (\tau^n(x), \lambda_c^n y + \alpha_n(x)x + c_{[k]}, \lambda_{ss}^n z + d_{[k]})$$

Where $\tau : S^1 \to S^1$ is the expansion in the circle by the factor $\lambda_{uu} = 3$, $\alpha_n(x) = \sum_{j=0}^{n-1} \lambda_c^{n-j-1}\alpha(\tau^j(x)) \cdot \lambda_{uu}^j(x)$ and $c_{[k]}$, $d_{[k]}$ are parameters of translation that depends on the rectangle $R_{[k]}$.

Consider a fixed point q of F_0, an affine function $\psi_0 : B^{cu}(p,\delta) \subset M \to B(0,\delta) \subset \mathbb{R}^2$ that sends $q \to 0$ and $\{E_0^{uu}, E_0^c\} \to \{e_1, e_2\}$, take a bump function $\psi_1 : B(0,\delta) \subset \mathbb{R}^2 \to [0,1]$ of class C^∞, with $\psi_1(x) = \begin{cases} 1 & \text{if } ||x|| \leq \frac{\delta}{3} \\ 0 & \text{if } ||x|| \geq \frac{2\delta}{3} \end{cases}$

This can be done with $||\psi_1||_{C^1} \leq 2\frac{\delta-1}{3}$, $||\psi_1||_{C^0} \leq 1$ and $||\psi_0||_{C^0} \leq \delta$.

Fixing some $\lambda_c^+ > 1$, define $\Phi_{\mu,n} : S^1 \times [-1,1]^2 \to \mathbb{R}$ by

$$\Phi_{\mu,n}(x,y,z) = \mu\psi_1\left(\psi_0(x,y)\right)\left[(\lambda_c^+ - \lambda_c^n)y\right]$$

The family $F_{\mu,n} : S^1 \times [-1,1]^2 \to S^1 \times [-1,1]^2$ is defined by:

$$F_{\mu,n}(x,y,z) = \left(\lambda_{uu}^n, \lambda_c^n y + \alpha_n(x)x + \Phi_{\mu,n}(x,y) + c_{[k]}, \lambda_{ss}^n z + d_{[k]}\right)$$

This family $F_{\mu,n}$ corresponds to a deformation of F_0^n changing the index of the fixed point p when passes through a pitchfork bifurcation. The deformation is done along the central direction, keeping the same central direction and the same stable direction for every parameter μ. We are considering the attractor $\Lambda_{\mu,n} = \cap_{j\geq 0} F_{\mu,n}^j(S^1 \times [-1,1]^2)$.

We will to see that for an appropriate choice of n, it is possible to keep close the strong-unstable direction in order that still holds the transversality for for every parameter $\mu \in [0,1]$.

6.2.1 Keeping Close the Unstable Direction

Proposition 6.2. For every $\epsilon_1 > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(E_{F_{\mu,n}}^{uu}(x), E_{F_0^n}^{uu}(x)) < \epsilon_1$ for every $x \in \Lambda$, every $\mu \in [0,1]$ and every $n \geq n_0$.

Proof. For parameters μ and n, it is possible to obtain the unstable direction $E^{uu}_{\mu}(p) = (1, \alpha^{uu}_{\mu}(p), 0)$ as the solution of:

$$DF_{\mu,n}(p) \cdot (1, \alpha^{uu}(p), 0) = \lambda^{uu}(1, \alpha^{uu}(F_{\mu}(p)), 0)$$

Since $DF_{\mu,n} = \begin{bmatrix} \lambda^n_{uu} & 0 & 0 \\ \alpha_n + \frac{\partial \Phi_{\mu,n}}{\partial x} & \lambda^n_c + \frac{\partial \Phi_{\mu,n}}{\partial y} & \frac{\partial \Phi_{\mu,n}}{\partial z} \\ 0 & 0 & \lambda^n_{ss} \end{bmatrix}$,

the equation becomes:

$$\alpha^{uu}(F_{\mu,n}(p)) = (\lambda_{uu})^{-n}\alpha_n + (\lambda_{uu})^{-n}\frac{\partial \Phi_{\mu,n}}{\partial x} + (\lambda_{uu})^{-n}\left(\lambda^n_c + \frac{\partial \Phi_{\mu,n}}{\partial y}\right)\alpha^{uu}(p)$$

So, the function α^{uu} is obtained as the fixed point of the operator $T_{\mu,n}$ given by:

$$T_{\mu,n}(\alpha)(p) = (\lambda_{uu})^{-n}\alpha_n(F^{-1}_{\mu,n}(p)) + (\lambda_{uu})^{-n}\frac{\partial \Phi_{\mu,n}}{\partial x}(F^{-1}_{\mu,n}(p)) + (\lambda_{uu})^{-n}\left(\lambda^n_c + \frac{\partial \Phi_{\mu,n}}{\partial y}\right)\alpha(F^{-1}_{\mu,n}(p))$$

The fixed point is well defined because the operator $T_{\mu,n}$ is an operator of contraction in the Banach space of continuous functions $\beta : \Lambda \to \mathbb{R}$ endowed with the norm of the supremum, it is a contraction because $(\lambda_{uu})^{-n}\left(\lambda^n_c + \frac{\partial \Phi_{\mu,n}}{\partial y}\right) < 1$. So we have well defined the fixed point $\alpha^{uu}_{\mu,n}$.

We want to prove that $||\alpha^{uu}_{\mu,n} - \alpha^{uu}_{0,n}|| < \epsilon_1$, this will be done proving that the respective operators are close.

Writting $T_{\mu,n}(\beta) = A_{\mu,n} + \lambda_{\mu,n}\beta$, then

$$\begin{aligned} d(\alpha^{uu}_{\mu,n}, \alpha^{uu}_{0,n}) &= d(T_{\mu_1,n}(\alpha^{uu}_{\mu,n}), T_{0,n}(\alpha^{uu}_{0,n})) \\ &\leq d(T_{\mu,n}(\alpha^{uu}_{\mu,n}), T_{\mu,n}(\alpha^{uu}_{0,n})) + d(T_{\mu,n}(\alpha^{uu}_{0,n}), T_{0,n}(\alpha^{uu}_{0,n})) \\ &\leq ||T_{\mu,n}|| \cdot d(\alpha^{uu}_{\mu,n}, \alpha^{uu}_{0,n}) + ||A_{\mu,n} - A_{0,n}|| + ||\lambda_{\mu,n}\alpha^{uu}_{0,n} - \lambda_{0,n}\alpha^{uu}_{0,n}|| \end{aligned}$$

It implies that

$$d(\alpha^{uu}_{\mu,n}, \alpha^{uu}_{0,n}) \le \frac{d(T_{\mu,n}, T_{0,n})}{1 - ||T_{\mu_1,n}||}$$
$$\le \frac{||A_{\mu,n} - A_{0,n}|| + (\lambda_{\mu,n} - \lambda_{0,n})||\alpha_{0,n}||}{1-\eta}$$

Note that $\alpha_n(p)$ depends only in the x-coordinate of p, but the x-coordinate of $F^{-1}_{\mu,n}(p)$ and of $F^{-1}_{0,n}(p)$ are the same, then $\alpha_n(F^{-1}_{\mu,n}(p)) = \alpha_n(F^{-1}_{0,n}(p))$. So,

$$(T_{\mu,n} - T_{0,n})(\beta)(p) = (\lambda_{uu})^{-n}\left[\frac{\partial \Phi_{\mu,n}}{\partial x}(F^{-1}_{\mu,n}(p))\right] +$$
$$(\lambda_{uu})^{-n}\left[\left(\lambda_c^n + \frac{\partial \Phi_{\mu,n}}{\partial y}\right)\beta(F^{-1}_{\mu,n}(p)) - (\lambda_c^n)\,\beta(F^{-1}_{0,n}(p))\right]$$

Each line above can be taken small, since $\lambda_c^n(\lambda_{uu})^{-n} \overset{n\to\infty}{\to} 0$, and $\left|\frac{\partial \Phi_{\mu,n}}{\partial y}\right|$, $\left|\frac{\partial \Phi_{\mu,n}}{\partial y}\right|$ are bounded. Then we can take n large, D_1 and λ_1 small such that:

$$d(\alpha^{uu}_{F_{\mu,n}}, \alpha^{uu}_{F_{0,n}}) \le \frac{D_1 + \lambda_1||\alpha^{uu}||}{1-\eta} < \epsilon_1.$$

□

6.2.2 Robust Transitivity

One important step in the proof of Theorem C corresponds to proving the nonhyperbolicity of the attractors, this will be done checking that these attractors admits hyperbolic periodic points of different indexes and that they are robustly transitive.

Proposition 6.3. There exists an integer n_1 such that the attractor $\Lambda_{n,\mu}$ of $F_{\mu,n}$ is robustly transitive for every $\mu \in [0,1]$ and every $n \ge n_1$.

Proof of Proposition 6.4. We will make an argument similar to the one of Mañé ([11]) to prove robust transversality.

Remember that we did the deformation inside a cylinder $D = B(q, \delta) \times [-1, 1]$ and that outside D the dynamics in the center direction contracts tangent vectors. Let us suppose that $\delta \leq \frac{1}{10\lambda^{uu}} \leq \frac{1}{30}$.

The unstable foliation for F_0 is minimal in Λ_0, then for every ϱ there exists $L > 0$ such that the unstable foliation is $(\frac{L}{2}, \frac{\varrho}{2})$-dense in center-stable manifolds, that is, every center-stable ball of radius $\frac{\varrho}{2}$ intersects every unstable curve γ^{uu} of length greater than $\frac{L}{2}$.

Taking ϵ_1 even small in Proposition 6.2, it is possible to obtain n_1 such that the unstable foliation of $F_{\mu,n}$ is (L, ϱ)-dense in center-stable manifolds for every $n \geq n_1$ (the center-stable foliation is the same for $F_{\mu,n}$ and F_0). Consider a neighbourhood $\mathcal{U}_1$ of $F_{\mu,n}$ such that the same (L, ϱ)-density holds for every f in this open set $\mathcal{U}_1$.

Claim 6.1. For every open set U intersecting Λ there exists a point $x \in U \cap \Lambda$ and an integer $N \in \mathbb{N}$ such that $f^{-n}(x) \notin D$ for $n \geq N$.

Proof of Claim 6.1. Note that for $F_{\mu,n}$ the center-stable foliation is given by the yz-planes, then the set $\mathcal{Z}_{F_{\mu,n}} = \{$ connected components of $W^{cs}_{F_{\mu,n}}(x)\} \cong S^1$ is of class C^1, so $(F_{\mu,n}, \mathcal{F}^{cs})$ is structurally stable (see Theorems 7.1 and 7.4 in [9]). So, for every f close to $F_{\mu,n}$ we have $\mathcal{Z}_f \cong S^1$ and $f : \mathcal{Z}_f \to \mathcal{Z}_f$ is conjugated to the expansion in S^1 by a factor λ^n_{uu}.

This implies that there exists a center-stable leaf that intersects U and and integer $N \in \mathbb{N}$ such that for every $n \geq N$ the n-esim image of this cs-leaf is always contained in the rectangle R_1 that does not intersects D. Taking the intersection of an strong-unstable curve contained in U with this center-stable leaf, we obtain the point that we are looking for. $\square$

Let us prove the transitivity for $f \in \mathcal{U}_1$. Given two open subsets U and V of Λ, we consider a point $x \in U \cap \Lambda$ given by Claim 6.1 and an unstable curve contained in V. Iterate the unstable curve until it has length greater than L and preiterate the center-stable leaf of x contained in U until it has internal radius greater than 2δ. By (L,δ)-density we know that these sets have non-empty intersection, so $U \cap f^n(V)$ for some $n \geq 0$. □

6.3 Proof of Theorem C

Remembering that $\Phi_{\mu,n}(x,y,z) = \mu(\lambda_c^+ - \lambda_c^n)\psi(x,y)y$, we note that

$$\frac{\partial \Phi_{\mu,n}}{\partial z} = 0$$

It means that the stable foliation for every $F_{\mu,n}$ corresponds to the vertical direction. This fact, togheter with Proposition 6.2 allows to check that the attractor satisfies the transversality condition between strong-unstable curves when projected by the strong-stable holonomy.

Proposition 6.4. There exists an integer $n_2 \in \mathbb{N}$ such that the Transversality Condition (H1) holds for the dynamics $F_{\mu,n}$, for every $\mu \in [0,1]$ and every $n \geq n_2$.

Proof. As seen in Proposition 6.1, the transversality holds for $F_{0,n}$ for every $n \in \mathbb{N}$.

Fix $a \in (0, \frac{\lambda_{ss}}{2})$, a fundamental stable domain $J = [a, \lambda_{ss}a]$ and consider $\theta(a)$ that bounds below the angle of projections of unstable curves with $d^{ss} > a$ for F_0. Since the stable projection is the vertical one, for each unstable curve we can consider $\omega > 0$ small such that the center-unstable cone of width ω around the unstable direction is $\frac{\theta(a)}{2}$-transversal to each other center-unstable cone of width ω around other unstable direction with stable distance between them contained in the intervbal J. By compacity this width ω can be taken uniform.

Fix this family of unstable cones of width ω containing the original $E^{uu}_{0,n}$. By Proposition 6.2, there exists $n_2 \in \mathbb{N}$ such that this family contains the unstable direction of F_{μ,n_2} for every $\mu \in [0,1]$ and every $n \geq n_2$.

This implies that

$$\angle(\pi^{ss}(E^{uu}_{\mu,n}(x_1)), \pi^{ss}(E^{uu}_{\mu,n}(x_2))) > \frac{\theta(a)}{2}$$

whenever $d^{ss}(x_1, x_2) > a$.

By Proposition 5.1, this is enough to guarantee the transversality.

□

Now we are able to prove Theorem C.

Proof of Theorem C. Choose $\lambda_c^+ > 1$ such that $\frac{(\lambda_c^+)^2}{3\lambda_c} < 1$, this implies $\frac{(\lambda_c^+)^2}{(3\lambda_c)^n} < 1$ (central direction neutral for $F_{\mu,n}$), which guarantees hypothesis (H2). Take $n_3 \in \mathbb{N}$ such that is valid the Proposition 6.3 for every $F_{\mu,n}$, $\mu \in [0,1]$, $n \geq n_3$, then it follows the Transversality Condition (Hypothesis (H1)) for every $F_{\mu,n}$.

The C^1 regularity follows by chosing $\lambda^{ss} < 1$ with $\lambda^{ss} \cdot \frac{3}{\lambda_c^+} < 1$, this bunching condition implies the C^1 regularity of the stable foliation and also the continuity of this foliation in the C^1-topology, then it holds the Hypothesis (H3).

The attractor is dynamically coherent because the central direction for $F_{\mu,n}$ is of class C^1, actually, $E^c = y$-direction, $E^{cu} = xy$-plane and $E^{cs} = yz$-plane for every $F_{\mu,n}$. By Theorems 7.1 and 7.4 of [9], these laminations are structurally stable, so the system is robustly dynamically coherent.

Taking $N > n_2$ as given by Proposition 6.3, the application $F_{1,N}$ is robustly nonhyperbolic because has fixed points of different indexes and it is robustly transitive, so this is the dynamics that we were looking for. □

Further Questions

Here we point further directions that can advance this work.

1) Write a **generic** condition of transversality between unstable curves via stable holonomy and prove the Main Inequality supposing that holds this generic hypothesis of transversality.

2) Change the condition of Lipschitz stable holonomies by α-Holder stable holonomies. In general, the stable holonomis are just α-Holder.

Two interesting Problems in this direction are:

Problem 1. For pertubations of the time-1 map of the geodesic flow for surfaces with negative curvature, are there physical measures?

Problem 2. For pertubations of a partial hyperbolic automorphism in the three-dimensional Heisenberg manifold, are there physical measures?

For such dynamics we have central direction close to neutral (actually, the dynamics is an isometry in the central direction), and satisfies the transversality condition in an uniform way (uniform non-integrability of $E^s \oplus E^u$).

Toward the Palis' Conjecture for Partially Hyperbolic Systems

Recent advances in this direction are the papers of Alves-Bonatti-Viana ([1], [4]), that deals with partially hyperbolic systems whose behavior in the central direction is mostly contracting or mostly expanding, which corresponds to cases when the central Lypaunov exponent is either positive or negativa for the most of the points. In certain sense, the remaining case is when $\lambda^c = 0$, for this case we expect that generically holds a condition of "non-integrability of $E^s \oplus E^u$" in a non-uniform way. This is true for the case of surface endomorphisms, Tsujii ([17]) proves that generically holds a kind of transversality condition and that this condition implies the existence and finiteness of physical measures.

One problem that would significatively advance in the direction of Palis' Conjecture is to check that for generic diffeomorphisms holds a kind of transvesality between unstable leaves via the stable holonomy and that this transversality implies the existence of physical measures.

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