**Instituto Nacional de Matemática Pura e Aplicada**

# On autoduality for tree like curves, Abel maps for stable curves and translations for the compactified Jacobian

Flávio Ferreira da Rocha

Advisor: Eduardo Esteves

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

Rio de Janeiro

August 27, 2013

*Em memória de meu pai Francisco Rocha*

# Agradecimentos

Ao Altíssimo e Soberano Deus criador de todas as coisas, pela inestimável ajuda. Aos meu familiares, pelo apoio incondicional. Ao professor Eduardo Esteves, pela orientação. Ao professor Amílcar Pacheco, por me direcionar ao IMPA. Aos amigos de longa data como Manoel, Vítor, Débora e Rômulo, como também aos de curta, como Pedro, Robertinho, Alessandro, Ruben e Juan, cuja amizade tornara mais agradável esta jornada. Aos professores Karl-Otto Stöhr, Oliver Lorscheid, Marco Pacini, Margarida Melo, Carolina Araujo e Juliana Coelho, por participarem da banca examinadora. Ao IMPA e ao CNPq, pelo apoio na realização deste trabalho.

*"Meaningless! Meaningless!" says the Teacher. "Everything is meaningless!" Not only was the Teacher wise, but he also imparted knowledge to the people. He pondered and searched out and set in order many proverbs. The Teacher searched to find just the right words, and what he wrote was upright and true. The words of the wise are like goads, their collected sayings like firmly embedded nails given by one shepherd. Be warned, my son, of anything in addition to them. Of making many books there is no end, and much study wearies the body. Now all has been heard; here is the conclusion of the matter: Fear God and keep his commandments, for this is the duty of all mankind. For God will bring every deed into judgment, including every hidden thing, whether it is good or evil.*

*Ecclesiastes 12: 8-14.*

# Contents

# Introduction

This thesis is about compactified Jacobians of reducible nodal curves, and addresses mainly three issues: the existence of an autoduality theorem for reducible curves, the construction of degree-1 and degree-0 Abel maps in a natural way and the existence of a theory of translations for compactified Jacobians.

Concerning the first issue, let $\mathscr{L}$ be an invertible sheaf of degree 1 on a projective, connected, reduced curve $C$. The classical *Autoduality Theorem* says that: If $C$ is smooth, the Abel map $A : C \to J_C^0$, given by $P \mapsto \mathscr{L} \otimes \mathscr{I}_P$, where $\mathscr{I}_P$ is the sheaf of ideals of $P$, is well-defined and induces an isomorphism $A^* : \mathrm{Pic}^0(J_C^0) \to J_C^0$ which is independent of the choice of $\mathscr{L}$; see [Mu65], Prop. 6.9, p. 118. We say that $J_C^0$, the *Jacobian* of $C$, is *autodual*.

In a more general setting, when $C$ has singularities, the Autoduality Theorem was first proved in [EGK] for irreducible curves with at most double points, where $J_C^0$ was replaced by its natural compactification $\bar{J}_C^0$, the moduli space of degree-0 torsion-free rank-1 sheaves on $C$, constructed by D'Souza [D'S] and Altman and Kleiman [AK80]. Later, Arinkin extended the theorem to irreducible curves with at most planar singularities; see [A07], Thm. 1.3, p. 1217. (Also, the autoduality isomorphism extends to compactifications, as proved in [EK] and [A10].)

Facing the existence of these autoduality theorems for irreducible curves, and given the importance of certain reducible curves, like the stable ones, it is natural to wonder if autoduality also holds for reducible curves. In this thesis we show autoduality when $C$ has at most planar singularities and is *treelike*, that is, its irreducible components meet transversely at disconnecting points of $C$.

Compactified Jacobians of reducible curves are less understood and more complex. For instance, they may contain more than one copy of $J_C^0$. Nevertheless, Abel maps $A : C \to \bar{J}_C^0$ have been constructed when $C$ is Gorenstein; see [CCE], which extends [CE]. The map $A$ and the compactified Jacobians $\bar{J}_C^0$ are easier to describe when $C$ is treelike

Assume from now on until a new order that $C$ is treelike, and let $C_1, ..., C_n$

be its irreducible components. For each $n$-tuple $\underline{d} = (d_1, ..., d_n) \in \mathbb{Z}^n$, let $\bar{J}_C^{\underline{d}}$ be the scheme parameterizing simple torsion-free rank-1 sheaves on $C$ of multidegree $\underline{d}$, and $\bar{J}_{C_i}^{d_i}$ the scheme parameterizing torsion-free rank-1 sheaves on $C_i$ of degree $d_i$, for each $i = 1, \dots, n$. Let $J_C^{\underline{d}} \subseteq \bar{J}_C^{\underline{d}}$ and $J_{C_i}^{d_i} \subseteq \bar{J}_{C_i}^{d_i}$ be the open subschemes parameterizing invertible sheaves. Since $C$ is treelike, restricting to components of $C$, we have a natural isomorphism (see [E09], Prop. 3.2, p. 172):

$$\bar{\pi}^{\underline{d}} = (\bar{\pi}_1^{\underline{d}}, \dots, \bar{\pi}_n^{\underline{d}}) : \bar{J}_C^{\underline{d}} \longrightarrow \bar{J}_{C_1}^{d_1} \times \cdots \times \bar{J}_{C_n}^{d_n}.$$

As for Abel maps, they come in all sorts, but there is a common property of those constructed so far that we explore. We say that a map $A : C \to \bar{J}_C^{\underline{d}}$ is a *decomposable Abel map* if $\bar{\pi}_j^{\underline{d}} A|_{C_i}$ is constant for $i \neq j$ and $A_i := \bar{\pi}_i^{\underline{d}} A|_{C_i}$ is an Abel map for each integer $i = 1, \dots, n$, that is, there is a line bundle $\mathscr{L}_i$ over $C_i$ such that $A_i$ associates $P$ to $\mathscr{L}_i \otimes \mathscr{I}_{P|C_i}$. We say that $(\mathscr{L}_1, \dots, \mathscr{L}_n)$ defines $A$.

Finally, we prove the Autoduality Theorem: *If $C$ is a treelike curve with planar singularities, and $A : C \to \bar{J}_C^{\underline{d}}$ is a decomposable Abel map, the pullback $A^* : \mathrm{Pic}^0(\bar{J}_C^{\underline{d}}) \to J_C^{\underline{0}}$ is an isomorphism which is independent of the $n$-tuple $(\mathscr{L}_1, \dots, \mathscr{L}_n)$ defining $A$.*

The proof relies on the fact that the various pullbacks constitute a commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}^0(\bar{J}_C^{\underline{d}}) & \xrightarrow{A^*} & J_C^{\underline{0}} \\
\downarrow & & \downarrow \pi^{\underline{0}} \\
\mathrm{Pic}^0(\bar{J}_{C_1}^{d_1}) \times \cdots \times \mathrm{Pic}^0(\bar{J}_{C_n}^{d_n}) & \xrightarrow{(A_1^*, \dots, A_n^*)} & J_{C_1}^0 \times \cdots \times J_{C_n}^0
\end{array}$$

where $\pi^{\underline{0}}$ is the restriction of $\bar{\pi}^{\underline{0}}$ to $J_C^{\underline{0}}$, and the left vertical map is induced by the isomorphism $\bar{\pi}^{\underline{d}}$; see Proposition 50. Both vertical maps are isomorphisms. By Arinkin's Autoduality Theorem, so is the bottom map. Hence, the commutativity of the diagram yields that $A^*$ is also an isomorphism.

A generalization of the result obtained in this note has been recently made available at [MRV] by Melo, Rapagnetta and Viviani, with an appendix by López-Martín. They state autoduality for any curve with planar singularities. Their methods are technically more involved, following the approach laid out by Arinkin, and their proof of autoduality is rather long, whereas our methods and proof are quite simple. Given the interest in the subject from people with different backgrounds, we believe that a simple approach as ours, albeit for a special case, may be useful.

The second issue is that of the construction of Abel maps in a natural way. Let us be more precise. Let $C$ be a projective, reduced, connected curve. For each integer $d \geq 1$, let $C^{(d)}$ be the $d$-th symmetric product of $C$, that is, the quotient of $C^d$ by the action of the $d$-th symmetric group, and $J_C^d$ its Jacobian, parameterizing isomorphism classes of invertible sheaves of degree $d$ on $C$.

If $C$ is smooth, for each integer $d \geq 1$ there is a natural map

$$A_C^d : C^d \to J_C^d,\ (P_1, ..., P_d) \mapsto [\mathfrak{m}_{P_1}^* \otimes \cdots \otimes \mathfrak{m}_{P_d}^*], \tag{1}$$

where $\mathfrak{m}_{P_i}^*$ denotes the dual sheaf of the ideal sheaf of $P_i$ in $C$, which factors through a map

$$A_C^{(d)} : C^{(d)} \to J_C^d,$$

called the *degree-d Abel map* of $C$.

Abel maps of smooth curves are important because they encode much of the geometry of $C$, since their fibers are the projectivized complete linear systems of $C$: For each invertible sheaf $\mathscr{L}$ of degree $d$ on $C$, the fiber ${A_C^{(d)}}^{-1}(\mathscr{L})$ is identified with $\mathbb{P}(H^0(C, \mathscr{L}))$. Therefore, in order to study linear systems of $C$, and thus the projective geometry of $C$, we may study the fibers of Abel maps.

Given the importance of Abel maps of smooth curves, there is a reasonable interest in constructing Abel maps for singular curves. But if $C$ is singular, the Abel map (1) is not defined at all points of $C^{(d)}$. More precisely, if $Q \in C$ is not a smooth point, the sheaf $A^{(d)}(Q, P_2, ..., P_n)$ is not invertible, that is, it is not in $J_C^d$. So, we face the question of how to construct Abel maps of $C$ in a natural way.

First, we need to find a "good target" for the Abel map, which leads us to the problem of how to find a good compactification for the Jacobian. This problem goes back at least to the work of Igusa's [Igu56] and the notes by Mumford and Mayer [Mu64], [May70].

In his thesis [D'S], D'Souza worked as well on the compactification problem, however, Altman and Kleiman [AK80] were who gave a good solution for the case of families of geometrically integral curves. Their relative compactification parametrizes torsion-free rank-1 sheaves on the fibers, and it admits a universal sheaf after an étale base change.

Once they obtained a good compactification for Jacobians of integral curves, Altman and Kleiman constructed Abel maps for these curves as follows: For each $d \geq 1$, they constructed a well-defined map

$$A_C(d) : \mathrm{Hilb}^d(C) \to \bar{J}_C^d,\ [Y] \mapsto [\mathcal{I}_{Y|C}^*],$$

where $\mathrm{Hilb}^d(C)$ is the Hilbert scheme of $C$ parameterizing length-$d$ subschemes of $C$, and $\bar{J}_C^d$ is the compactified Jacobian parameterizing torsion-free rank-1 sheaves of degree $d$ on $C$. Also, for each $[Y] \in \mathrm{Hilb}^d(C)$, $\mathcal{I}_{Y|C}$ denotes the ideal sheaf of $Y$ on $C$. We notice that if $C$ is smooth, $\mathrm{Hilb}^d(C) = C^{(d)}$ and $A_C(d) = A_C^{(d)}$.

Now, notice that all these compactifications were only carried out for integral curves. For reducible nodal curves defined over an algebraically closed field, Oda–Seshadri [OS] and Seshadri [Ses82] produced some compactifications. However, these compactifications are not applicable to families of reduced curves.

In her Ph.D. thesis [Ca94], Caporaso constructed a compactification for the relative generalized Jacobians of families of stable curves. One year later, also in his Ph.D. thesis, Pandharipande [Pan] produced an equivalent construction, valid for higher ranks as well. At nearly the same time, Simpson [Sim] constructed moduli spaces of coherent sheaves over any family of projective varieties, in particular for families of curves.

Later, given a family of curves $f : \mathcal{C} \to T$, Esteves considered the space constructed by Altman and Kleiman $\bar{J}_{\mathcal{C}/T}$ in [AK80], parameterizing torsion-free rank-1 simple sheaves on the fibers of $f : \mathcal{C} \to T$. He showed that $\bar{J}_{\mathcal{C}/T}$ is universally closed over $T$, and consequently one can regard it as a compactification of the relative Jacobian of $f : \mathcal{C} \to T$; see [E01]. Esteves's compactification admits a universal sheaf, in contrast to Caporaso's, Pandharipande's and Simpson's compactifications.

However, even with all these compactifications, it is still not easy to choose a good target for Abel maps of reducible curves. Indeed, assume $C$ is reducible. Then, depending on the singularities of $C$ , the Abel map

$$A_C : C \to \bar{J}_C^1,\ P \mapsto [\mathfrak{m}_P^*],$$

where $\bar{J}_C^1$ is Esteves' compactified Jacobian parameterizing torsion-free rank-1 simple sheaves of degree 1 on $C$, may not be well defined at all points of $C$. To be precise, if $P \in C$ is a separating node, $A_C(P) = \mathfrak{m}_P^*$ is not a simple sheaf. If $C$ is a stable curve with separating nodes and $\overline{P_C^1}$ is Caporaso's compactification, [Ca94], p. 638, one can show that it is not possible either to define the Abel map as

$$A_C : C \to \overline{P_C^1},\ P \mapsto [\mathfrak{m}_P^*]$$

in a natural way.

So, Caporaso and Esteves [CE] constructed *twisted degree-1 Abel map* for reducible curves, where the obstruction is overcome by using a special type of invertible sheaves, named *twisters*. In this thesis, in contrast to Caporaso and Esteves, we show that by putting Simpson's compactified Jacobians as the target of Abel maps target, it is possible to define degree-1 and degree-0 Abel maps by simple formulas, like that for $A_C$ above, avoiding the use of twisters.

More precisely, let $L$ be a very ample invertible sheaf on a stable curve $C$. Simpson's compactified Jacobian $\bar{J}_{L,d}(C)$ parametrizes torsion-free rank-1 *slope-semistable* sheaves of degree $d$ with respect to $L$ on $C$. More precisely, it coarsely represents the $k$-functor $\bar{\mathbf{J}}_{L,d}(C)$ which associates to each $k$-scheme $S$, the set of torsion-free rank-1 slope-semistable sheaves on $C' := C \times S \to S$ with respect to $L \otimes \mathscr{O}_{C'}$. Then, if $C$ is stable and $Q \in C$ is a smooth point of $C$, we construct natural very ample invertible sheaves $M$ and $N$ on $C$ such that both maps

$$A_1 : C \to \bar{J}_{M,1},\ P \mapsto [\mathfrak{m}_P^*]$$

and

$$A_0 : C \to \bar{J}_{N,0},\ P \mapsto [\mathfrak{m}_P \otimes \mathscr{O}_C(P)]$$

are well defined. Furthermore we show that these very ample invertible sheaves $M$ and $N$ are so natural that they may be defined on families of stable curves, so that we are able to define degree-1 and degree-0 Abel maps for such families.

Classically, Abel maps should satisfy two properties. The first is their modular meaning. More plainly, for a smooth curve $C$ the $d$-th Abel map $A_C^{(d)}$ is the moduli map of an invertible sheaf on $C^d \times C$. For example, our degree-1 Abel map $A_1 : C \to \bar{J}_{M,1},\ P \mapsto [\mathfrak{m}_P^*]$, has the following modular meaning: consider a projection $C \times C \to C$ as a family of curves, let $\mathcal{I}$ be the ideal sheaf of the diagonal $\Delta \subset C \times C$ and let $\mathcal{I}^*$ be its dual sheaf. The sheaf $\mathcal{I}^*$ on $C \times C$ defines $A_1$.

The second is the continuous variation in families. For example, given a one-parameter family of smooth curves degenerating to a singular one, it is expected that the $d$-th Abel maps of the smooth fibers specialize to the $d$-th Abel map of the singular fiber. We also show that our Abel maps varies continuously in families of stable curves, as we have already said, those very ample invertible sheaves $M$ and $N$ may be defined on such families of curves.

For our Abel maps, showing the continuous variation in families is the difficult part, whereas the modular meaning is easier.

Furthermore, we construct as well a special type of very ample invertible sheaves on $C$ such that, if $F$ is one of them, then the Abel map

$$A : C \to \bar{J}_{F,1},\ P \mapsto [\mathfrak{m}_P^*]$$

is well defined, and it does not depend of the choice of $F$ as if $G$ is another special one, then $\bar{J}_{F,1} \cong \bar{J}_{G,1}$. These compactified Jacobians are called *de-*

*generated*. However, we do not know to answer if these special very ample invertible sheaves may be defined on families of stable curves.

The last issue which we deal with, it is that of translations in Esteves' compactifications. Let us be more precise. If $C$ is an integral curve, for each invertible sheaf $\mathscr{L}$ on $C$, the translation

$$T_{\mathscr{L}} : \bar{J}_C \to \bar{J}_C,\ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L}$$

is well-defined. Here, $\bar{J}_C$ is Altman–Kleiman's compactified Jacobian parameterizing torsion-free rank-1 sheaves on $C$ [AK80]. On the other hand, if $C$ is reducible, Esteves' compactification $\bar{J}_C$ is a nonseparated and not of finite type space. So, in order to investigate it better, Esteves used continuous polarizations to divide it in smaller pieces. Given a smooth point $P \in C$ and a polarization $E$ on $C$, one of these pieces is $\bar{J}_E^P$, the compactified Jacobian parameterizing $P-$*quasistable* sheaves with respect to $E$; see [E01], Thm. A (3), p. 3047. These spaces are projective schemes.

In this thesis we study two curves for which it is possible to put, in a non trivial way, an "action" of $J_C$ on $\bar{J}_E^P$ such that given any smooth point $Q \in C$ and any polarization $F$ on $C$, for each $\mathscr{L} \in J_C^{|F|-|E|}$, we have

$$\mathscr{L} \cdot \bar{J}_E^P \cong \bar{J}_F^Q,$$

that is, the "action" of $J_C$ takes one of these special pieces of $\bar{J}_C$ to the other. This desirable behavior does not hold in general.

Actually, this "action" of $J_C$ which we put over $\bar{J}_E^P$ is adjusted by the special class of invertible sheaves formed by twisters. As the reader will note, the difficulty of finding such actions is linked to the fact that $\bar{J}_C$ is not a separated space, and thus degenerations of sheaves in $\bar{J}_C$ may have different limits. On the other hand it is this pathology that allows us to produce such "actions". These two examples will hopefully be enticing for those that would like to think about a possible theory of translations for compactified Jacobians.

# Chapter 1

# Compactified Jacobians

A *curve* is a projective, connected and reduced scheme of pure dimension 1 over a fixed algebraically closed field $k$.

A *family* of curves is a flat, projective morphism of schemes $f : \mathcal{C} \to T$ whose fibers are curves. By a sheaf on $f : \mathcal{C} \to T$ we mean a $T$-flat coherent sheaf on $\mathcal{C}$.

Let $C$ be a curve and $J_C$ its Jacobian, that is, the scheme parameterizing isomorphism classes of invertible sheaves on $C$. More precisely, $J_C$ represents the contravariant Jacobian functor $\mathbf{J}_C$ from the category of $k$-schemes to sets, defined on a $k$-scheme $T$ by

$$\mathbf{J}_C(T) = \{\text{invertible sheaves on } C \times T/T\}/\sim$$

where we say that two invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$ on $C \times T$ are equivalent, that is, $\mathscr{L}_1 \sim \mathscr{L}_2$, if there is an invertible sheaf $\mathscr{M}$ on $T$ such that $\mathscr{L}_1 \cong \mathscr{L}_2 \otimes p^*\mathscr{M}$, where $p : X \times T \to T$ is the projection.

The connected components of $J_C$ are projective over $k$ if $C$ is smooth, but otherwise there may be components that are only quasi-projective over $k$, that is, there may exist families of invertible sheaves that do not degenerate to invertible sheaves. For example, suppose that $C$ is irreducible with a unique singularity, a node $P$. For each $Q \in C$, let $\mathfrak{m}_Q$ be the ideal sheaf of $Q$ on $C$. So, for each point $Q \in C - P$, the sheaf $\mathfrak{m}_Q$ is invertible while $\mathfrak{m}_P$ is not. Thus, when $Q$ tends to $P$, the invertible sheaf $\mathfrak{m}_Q$ tends to the non-invertible sheaf $\mathfrak{m}_P$.

In the case of a family of curves $f : \mathcal{C} \to T$, we have the *relative Jacobian functor*, the contravariant functor $\mathbf{J}_{\mathcal{C}/T}$ from the category of $T$-schemes to sets, defined on a $T$-scheme $S$ by

$$\mathbf{J}_{\mathcal{C}/T}(S) = \{\text{invertible sheaves on } \mathcal{C} \times_T S/S\}/\sim$$

where we say that two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{C} \times_T S$ are equivalent, that is, $\mathcal{L}_1 \sim \mathcal{L}_2$, if there is an invertible sheaf $\mathscr{M}$ on $S$ such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes p^*\mathscr{M}$, where $p : \mathcal{C} \times_T S \to S$ is the projection. It is known that Mumford showed that if the irreducible components of each fiber of $f : \mathcal{C} \to T$ are geometrically irreducible, then $\mathbf{J}_{\mathcal{C}/T}$ is represented by a $T$-scheme $J_{\mathcal{C}/T}$, called the relative Jacobian of $f : \mathcal{C} \to T$; see [FGAE], Thm. 9.4.8, p. 263, for a sketch of a proof of this fact given by Kleiman. However, the connected components of $J_{\mathcal{C}/T}$ may fail to be proper over $T$ if some curves of the family $f : \mathcal{C} \to T$ are not smooth.

So, a problem arises: How to find a good compactification for $J_C$? Furthermore, how to find a good relative compactification for the relative Jacobian over a family of curves?

In this chapter we introduce the solutions given by Altman–Kleiman, Caporaso and Simpson to this problem. Before this, let us introduce some basic notions and facts about algebraic spaces and Geometric Invariant Theory.

## 1.1 Preliminary subjects

### 1.1.1 Algebraic spaces

We recommend [SP] for more details about algebraic spaces.

**Definition 1.** A *family of morphisms with fixed target* in a category $\mathscr{C}$ is composed of the following data:

1. an object $U \in \mathrm{Ob}(\mathscr{C})$,
2. a set $I$,
3. for each $i \in I$, a morphism $U_i \to U$ of $\mathscr{C}$ with target $U$.

We use the notation $\{U_i \to U\}_{i \in I}$ to indicate such a family.

**Definition 2.** A *site* consists of the following data: a category $\mathscr{C}$ and a set $\mathrm{Cov}(\mathscr{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$, called *coverings of* $\mathscr{C}$, satisfying the following axioms:
1) If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

2) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathscr{C})$ and for each $i \in I$ we have $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathscr{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathscr{C})$.

3) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathscr{C})$ and $V \to U$ is a morphism of $\mathscr{C}$, then $U_i \times_U V$ exists for every $i \in I$ and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathscr{C})$.

**Definition 3.** Let $\mathscr{C}$ be a category. A *presheaf* on $\mathscr{C}$ is a contravariant functor $\mathcal{F} : \mathscr{C} \to \text{Sets}$.

Let $\text{Cov}(\mathscr{C})$ be a set of families of morphisms with fixed target satisfying the above axioms. We say that a presheaf $\mathcal{F}$ on the site $(\mathscr{C}, \text{Cov}(\mathscr{C}))$ is a *sheaf* if for each covering $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathscr{C})$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{1.1}$$

is exact.

Loosely speaking, this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for each pair $(i, j) \in I \times I$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

**Definition 4.** An *étale covering* of a scheme $U$ is a family of morphisms of schemes with fixed target $\{U_i \to U\}_{i\in I}$ where each $U_i \to U$ is étale and such that $U = \bigcup \text{Im}(U_i \to U)$.

Let $\mathfrak{Sch}$ denote the category of schemes. Let $\mathfrak{Sch}_{\text{et}}$ denote the site constituted by the category $\mathfrak{Sch}$ and by the set $\text{Cov}(\mathfrak{Sch})$ of coverings of $\mathfrak{Sch}$ such that each $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathfrak{Sch})$ is an étale covering of $U$.

**Theorem 5** (Grothendieck). *For every scheme $X$, the functor of points of $X$, $h_X := \text{Mor}(-, X)$, is a sheaf on the site $\mathfrak{Sch}_{et}$.*

*Proof.* See [Vis], Thm. 2.55, p. 36. □

This theorem and Yoneda's Lemma show that the category $\mathfrak{Sch}$ is a full subcategory of the sheaves over $\mathfrak{Sch}_{\text{et}}$. Thus, for each scheme $X$, we let $X$ denote as well the sheaf $h_X$ on $\mathfrak{Sch}_{\text{et}}$.

**Definition 6.** We say a sheaf $\mathcal{F}$ on $\mathfrak{Sch}_{\text{et}}$ is *representable* if there is a scheme $X$ and an isomorphism of functors $\mathcal{F}(-) \cong \text{Mor}(-, X)$.

**Definition 7.** Let $\mathcal{G}$ be a presheaf on $\mathfrak{Sch}_{\text{et}}$. We call the sheafification of $\mathcal{G}$ on $\mathfrak{Sch}_{\text{et}}$ its *associated sheaf in the étale topology.*

The process of sheafification of $\mathcal{G}$ on $\mathfrak{Sch}_{\text{et}}$ is similar to that of a presheaf on a scheme and will not be described here. To finish this section we present the definition of an algebraic space.

**Definition 8.** An algebraic space $\mathcal{A}$ is a sheaf on $\mathfrak{Sch}_{\text{et}}$ such that

1. For each two schemes $X$ and $Y$ and each two morphisms of sheaves $x : X \to \mathcal{A}$ and $y : Y \to \mathcal{A}$, the sheaf $X \times_{\mathcal{A}} Y$ is represented by a scheme.

2. There are a scheme $A$ and a surjective étale morphism $a : A \to \mathcal{A}$ (that is, for each morphism $z : Z \to \mathcal{A}$, where $Z$ is a scheme, the projection $A \times_{\mathcal{A}} Z \to Z$ is a surjective étale morphism of schemes).

### 1.1.2 Geometric Invariant Theory

Let $Z$ be a projective scheme over an algebraically closed field $k$ endowed with an action of a reductive algebraic group $G$, that is, an algebraic group $G$ over the same field such that the unipotent radical of $G$ is trivial.

Consider an embedding of $Z$ in some projective space $\mathbb{P}(V)$. Then $Z = \text{Proj}(R)$, where $R$ is a graded ring, finitely generated over $k$. If the action of $G$ on $Z$ can be lifted to a linear action on $V$, we say that $G$ *acts linearly* with respect to the embedding.

If $G$ acts linearly, of course, $G$ will act on $R$. In this case, let $R^G$ denote the subring of elements of $R$ which are invariant under the action of $G$. From a foundational theorem of Geometric Invariant Theory, since $G$ is reductive, $R^G$ is a finitely generated graded algebra over $k$.

Now, consider the inclusion $R^G \subseteq R$ and the associated rational map

$$\pi : \text{Proj}(R) = Z \dashrightarrow \text{Proj}(R^G).$$

Let $Z_R^{SS} := \{z \in Z : \exists \text{ a homogeneous nonconstant } f \in R^G \text{ with } f(z) \neq 0\}$, that is, $Z_R^{SS}$ is precisely the locus where $\pi$ is regular, and

$$Z_R^S := \{z \in Z_R^{SS} : \overline{O_G(z)} \cap Z_R^{SS} = O_G(z) \text{ and } \dim(O_G(z)) \text{ is maximum among the dimension of all } G\text{-orbits in } Z_R^{SS}\},$$

where $O_G(z)$ denotes the orbit of $z$. The points on $Z_R^{SS}$ are called (*GIT-*)*semistable* whereas those on $Z_R^S$ are called (*GIT-*)*stable*.

**Definition 9.** We say that a morphism $f : Z \to W$ is *G-invariant* if $f(g \cdot z) = f(z)$ for each $g \in G$ and $z \in Z$.

**Definition 10.** We say that a morphism $\pi : Z \to W$ is a *categorical quotient* of $Z$ by $G$ if:

1. $\pi$ is invariant, and

2. $\pi$ satisfies the universal property: every $G$-invariant morphism $\rho : Z \to Y$ factors uniquely through $\pi$.

**Theorem 11** (Fundamental Theorem of GIT)**.** *Let $G$ be a reductive group acting linearly on a projective scheme $Z = \mathrm{Proj}(R)$. Then $Q := \mathrm{Proj}(R^G)$ is a projective scheme and the natural morphism*

$$\pi : Z_R^{SS} \to Q$$

*satisfies the following properties:*

1. *(Universality) If there is a scheme $Q'$ with a $G$-invariant morphism $\pi' : Z_R^{SS} \to Q'$, then there is an unique morphism $\rho : Q \to Q'$ such that $\pi' = \rho \circ \pi$.*

2. *For each $x, y \in Z_R^{SS}$, $\pi(x) = \pi(y)$ if and only if $\overline{O_G(x)} \cap \overline{O_G(y)} \cap Z_R^{SS} \neq \varnothing$.*

Notice that, from property 2 above, for each $x, y \in Z_R^S$, $\pi(x) = \pi(y)$ if and only if $O_G(x) = O_G(y)$.

## 1.2 Altman–Kleiman's compactification

Let $C$ be a curve. Let $C_1, ..., C_n$ be the irreducible components of $C$ and $\eta_i$ the generic points of the $C_i$. We say that a coherent sheaf $\mathscr{L}$ on $C$ is *torsion-free* if the map

$$\mathscr{L} \longrightarrow \prod_i \mathscr{L}_{\eta_i},$$

where $\mathscr{L}_{\eta_i}$ is the skyscraper sheaf of $\mathscr{L}$ on $\eta_i$, has trivial kernel.
We say that $\mathscr{L}$ is *rank*-1 if $\mathscr{L}$ has generic rank 1 on each irreducible component of $C$.
We say that $\mathscr{L}$ is *simple* if $\mathrm{Hom}(\mathscr{L}, \mathscr{L}) = k$. We notice that an invertible sheaf on $C$ is torsion free, rank-1 and simple.

Let $f : \mathcal{C} \to T$ be a family of curves. We say that a sheaf $\mathcal{L}$ on $f : \mathcal{C} \to T$ is torsion-free (resp. rank-1, resp. simple) on $f : \mathcal{C} \to T$ if for each geometric point $t \in T$ the restriction of $\mathcal{L}$ to $\mathcal{C}_t$, denoted by $\mathcal{L}_t$, is a torsion-free (resp. rank-1, resp. simple) on $\mathcal{C}_t$.

The *relative compactified Jacobian functor* for the family $f : \mathcal{C} \to T$ is the contravariant functor $\bar{\mathbf{J}}_{\mathcal{C}/T}$ from the category of $T$-schemes to sets, defined on a $T$-scheme $S$ by

$$\bar{\mathbf{J}}_{\mathcal{C}/T}(S) = \{\text{torsion-free rank-1 simple sheaves on } \mathcal{C} \times_T S/S\}/\!\sim$$

where $\mathcal{L}_1 \sim \mathcal{L}_2$ if there is an invertible sheaf $\mathscr{M}$ on $S$ such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes p^*\mathscr{M}$, where $p : X \times_T S \to S$ is the projection.

As we can see, the relative Jacobian functor defined before, $\mathbf{J}_{\mathcal{C}/T}$, is an open subfunctor of $\bar{\mathbf{J}}_{\mathcal{C}/T}$. Therefore, if $\bar{\mathbf{J}}_{\mathcal{C}/T}$ were representable by a universally closed scheme over $T$, this would "solve the problem", because this would tell us that invertible sheaves degenerate to torsion-free rank-1 simple sheaves. But, in general, $\bar{\mathbf{J}}_{\mathcal{C}/T}$ is not a representable functor.

However, Altman and Kleiman showed that its *associated sheaf in the étale topology*, which we also denote by $\bar{\mathbf{J}}_{\mathcal{C}/T}$, is always representable, say by $\bar{J}_{\mathcal{C}/T}$, in a larger category, the category of algebraic spaces; see Theorem 12 below. Furthermore, they showed that if the curves of the family $f : \mathcal{C} \to T$ are geometrically integral, then the connected components $\bar{J}_{\mathcal{C}/T}$ are proper over $T$, "thus solving the problem". Later Esteves extended this compactification to any family $f : \mathcal{C} \to T$ by showing that $\bar{J}_{\mathcal{C}/T}$ "contains enough degenerations" over $T$, that is, $\bar{J}_{\mathcal{C}/T}$ meets the existence condition of the valuative criterion of properness, without necessarily meeting the uniqueness condition; see [E01], Thm. 32, p. 3068. In another words, $\bar{J}_{\mathcal{C}/T}$ is universally closed over $T$.

**Theorem 12** ( [AK80], Thm. 7.4, p. 99)**.** *Let $f : \mathcal{C} \to T$ be a family of curves. Then, $\bar{\mathbf{J}}_{\mathcal{C}/T}$ is represented by an algebraic space.*

As we have already observed, Theorem 5, p. 14, showed by Grothendieck, and Yoneda's Lemma tell us that the category of schemes is a subcategory of the category of sheaves over $\mathfrak{Sch}_{\mathrm{et}}$, so that we may view, any scheme as an algebraic space. However, since the converse is not true, it makes sense to inquire whether there are conditions on the family $f : \mathcal{C} \to T$ for which $\bar{J}_{\mathcal{C}/T}$ is a scheme. In [AK80], Thm. 3, p. 948, Altman and Kleiman showed that if every geometric fiber of the family $f : \mathcal{C} \to T$ is integral, each connected component of $\bar{J}_{\mathcal{C}/T}$ is a proper scheme over $T$, and consequently $\bar{J}_{\mathcal{C}/T}$ is a scheme.

On the other hand, Esteves showed in [E01], Thm. B, p. 3048, that if there are sections $\sigma_1, ..., \sigma_n$ through the smooth locus of $f : \mathcal{C} \to T$, such that for each geometric point $t \in T$, each irreducible component of the fiber $\mathcal{C}_t$ is geometrically integral and contains $\sigma_i(t)$ for some $i$, then $\bar{J}_{\mathcal{C}/T}$ is a scheme. Moreover, in the same paper [E01], Lemma 18, p. 3061, Esteves showed that it is always possible, after a suitable étale base change, to obtain such sections, that is, Esteves shows that after a suitable base change $\bar{J}_{\mathcal{C}/T}$ becomes a scheme.

But although the representability of $\bar{\mathbf{J}}_{\mathcal{C}/T}$ by an algebraic space (or a scheme) $\bar{J}_{\mathcal{C}/T}$ solves the problem of compactification, in the sense that invertible sheaves degenerate to torsion-free rank-1 simple sheaves, the algebraic space is "too big". On one hand this is good, because this allows a family of torsion-free rank-1 simple sheaves to have a *limit*, but on the other hand, the limit may not be unique.

**Definition 13.** Let $\mathscr{L}$ be a torsion-free rank-1 sheaf on $C$. We define the *degree* of $\mathscr{L}$ to be

$$\deg(\mathscr{L}) := \chi(\mathscr{L}) - \chi(\mathscr{O}_C),$$

where $\chi(-)$ is the Euler characteristic.

Let $Y$ be a subcurve of $C$, that is, $Y$ is the reduced union of irreducible components of $C$. Let $\mathscr{L}_Y := \mathscr{L}|_Y/\text{torsion}$, where $\mathscr{L}|_Y$ is the restriction of $\mathscr{L}$ to $Y$. We let

$$\deg_Y(\mathscr{L}) := \chi(\mathscr{L}_Y) - \chi(\mathscr{O}_Y),$$

and call it the degree of $\mathscr{L}$ on $Y$.

Let $h : \mathcal{C} \to T$ be a family of curves where $T$ is the spectrum of a discrete valuation ring with special point $\vartheta$ and generic point $\eta$. Let $\mathcal{C}_\eta$ (resp. $\mathcal{C}_\vartheta$) be the generic (resp. special) fiber of the family $h : \mathcal{C} \to T$ and $\mathscr{I}_\eta$ a sheaf on $\mathcal{C}_\eta$.

The next example illustrates a situation where there are infinitely many different limits of the trivial sheaf $\mathscr{O}_{\mathcal{C}_\eta}$.

**Definition 14.** We say that a sheaf $\mathscr{I}$ on $\mathcal{C}$ is an *extension* of $\mathscr{I}_\eta$ if $\mathscr{I}|_{\mathcal{C}_\eta} = \mathscr{I}_\eta$; in this case, we say that $\mathscr{I}_\vartheta := \mathscr{I}|_{\mathcal{C}_\vartheta}$ is the *limit* of $\mathscr{I}_\eta$ on $\mathcal{C}_\vartheta$.

**Example 15.** Suppose $\mathcal{C}_\eta$ is smooth and $\mathcal{C}_\vartheta$ is the union of two irreducible curves $C_1$ and $C_2$ such that they intersect transversely at a unique point $P$. It is plain that $\mathscr{O}_\mathcal{C}$ is a extension of $\mathscr{O}_{\mathcal{C}_\eta}$. Assume $\mathcal{C}$ is regular. Then each integral subscheme of codimention 1 of $\mathcal{C}$ is a Cartier divisor of $\mathcal{C}$, in particular so are $C_1$ and $\mathcal{C}_2$. Thus, $\mathscr{O}_\mathcal{C}(nC_1)$ is an invertible sheaf on $\mathcal{C}$ for each $n \in \mathbb{Z}$. Furthermore,

$$\mathscr{O}_\mathcal{C}(nC_1)|_{\mathcal{C}_\eta} = \mathscr{O}_\mathcal{C}|_{\mathcal{C}_\eta} = \mathscr{O}_{\mathcal{C}_\eta}$$

for each $n \in \mathbb{Z}$, that is, $\mathscr{O}_\mathcal{C}(nC_1)$ is an extension of $\mathscr{O}_{\mathcal{C}_\eta}$.

But on the other hand, we claim that $\mathscr{O}_\mathcal{C}(mC_1)|_{\mathcal{C}_\vartheta} \neq \mathscr{O}_\mathcal{C}(nC_1)|_{\mathcal{C}_\vartheta}$ if $m \neq n$. Indeed, for each $n \in \mathbb{N}$, we have

$$\deg(\mathscr{O}_\mathcal{C}(nC_1)|_{\mathcal{C}_\vartheta}) = \deg(\mathscr{O}_{C_2}(nP)) = n.$$

Then if $m$ and $n$ are different integers, we have

$$\mathscr{O}_\mathcal{C}(mC_1)|_{\mathcal{C}_\eta} = \mathscr{O}_\mathcal{C}(nC_1)|_{\mathcal{C}_\eta} = \mathscr{O}_\mathcal{C}|_{\mathcal{C}_\eta} = \mathscr{O}_{\mathcal{C}_\eta},$$

but

$$\mathscr{O}_{\mathcal{C}}(mC_1)|_{\mathcal{C}_\vartheta} \neq \mathscr{O}_{\mathcal{C}}(nC_1)|_{\mathcal{C}_\vartheta},$$

because

$$m = \deg(\mathscr{O}_{\mathcal{C}}(mC_1)|_{\mathcal{C}_\vartheta}) \neq \deg(\mathscr{O}_{\mathcal{C}}(nC_1)|_{\mathcal{C}_\vartheta}) = n,$$

showing our claim.

Thus, we showed that there is an infinity of different limits of the trivial sheaf $\mathscr{O}_{\mathcal{C}_\eta}$.

Due to its large "size", $\bar{J}_{\mathcal{C}/T}$ was studied through smaller and simpler pieces. In order to obtain such pieces of $\bar{J}_{\mathcal{C}/T}$, Esteves used the "continuous" *polarizations*, that is, polarizations given by vector bundles on $\mathcal{C}$. Let us be more precise.

Let $C$ be a curve and $E$ a vector bundle over $C$. Let $\mathrm{rk}(E)$ denote its rank and $\deg(E)$ its degree. We call the ratio $\mu(E) := \deg(E)/r$ the *slope* of $E$.

We say that $E$ is a *polarization* on $C$ if $\mu(E) \in \mathbb{Z}$. Let $\mathscr{L}$ be a torsion-free rank-1 simple sheaf on $C$ and assume that $E$ is a polarization on $C$.

**Definition 16.** We say that $\mathscr{L}$ is *semistable* (resp. *stable*) with respect to $E$ if $\chi(\mathscr{L}) = -\mu(E)$ and

$$\chi(\mathscr{L}_Y) \geq -\deg_Y(E)/r \text{ (resp. } \chi(\mathscr{L}_Y) > -\deg_Y(E)/r)$$

for each subcurve $Y$ of $C$.

Let $P \in C$ be a smooth point. We say that $\mathscr{L}$ is *$P$-quasistable* with respect to $E$ if $\mathscr{L}$ is semistable with respect to $E$ and

$$\chi(\mathscr{L}_Y) > -\deg_Y(E)/\mathrm{rk}(E)$$

for each subcurve $Y$ of $C$ containing $P$.

A polarization on a family of curves $f : \mathcal{C} \to T$ is a vector bundle $\mathcal{E}$ on $f : \mathcal{C} \to T$ which is a relative polarization. In other words, for each geometric point $t \in T$, $\mathcal{E}_t$ is a vector bundle over $\mathcal{C}_t$ such that its slope $\mu(\mathcal{E}_t) \in \mathbb{Z}$, where $\mathcal{E}_t$ is the restriction of $\mathcal{E}$ to the fiber $\mathcal{C}_t$ of $f : \mathcal{C} \to T$ over $t$.

Let $\mathcal{L}$ be a torsion-free rank-1 simple sheaf on $f : \mathcal{C} \to T$ and $\mathcal{E}$ a polarization on $f : \mathcal{C} \to T$.

**Definition 17.** We say that $\mathcal{L}$ is *semistable* (resp. *stable*) with respect to $\mathcal{E}$ if for each geometric point $t \in T$, $\mathcal{L}_t$ is semistable (resp. stable) with respect to $\mathcal{E}_t$.

Let $\sigma : T \to \mathcal{C}$ be a section through the smooth locus of $f : \mathcal{C} \to T$. We say that $\mathcal{L}$ is *$\sigma$-quasistable* with respect to $\mathcal{E}$ if for each geometric point $t \in T$, $\mathcal{L}_t$ is semistable with respect to $\mathcal{E}_t$ and $\chi(\mathcal{L}_{tY}) > -\deg_Y(\mathcal{E}_t)/\mathrm{rk}(\mathcal{E}_t)$ for each proper subcurve $Y$ of $\mathcal{C}_t$ containing $\sigma(t)$.

In [E01] Prop. 34, p. 3071, Esteves shows that the subspace

$$\bar{J}_{\mathcal{E}}^{ss} \text{ (resp. } \bar{J}_{\mathcal{E}}^{s}, \text{ resp. } \bar{J}_{\mathcal{E}}^{\sigma}) \subseteq \bar{J}_{\mathcal{C}/T}$$

parameterizing semistable (resp. stable, resp. $\sigma$-quasistable) sheaves on $f : \mathcal{C} \to T$ with respect to $\mathcal{E}$ is open. Furthermore, $\bar{J}_{\mathcal{E}}^{ss}$ is of finite type and universally closed, $\bar{J}_{\mathcal{E}}^{s}$ is separated and $\bar{J}_{\mathcal{E}}^{\sigma}$ is proper over $T$; see [E01], Thm. A, p. 3047.

## 1.3 Caporaso's compactification

In this section we talk about the solution given by Caporaso to the problem of compactifying the relative Jacobian over a family of curves. Caporaso solved this problem for families of *stable* curves. More precisely, let $f : \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 3$ and let $J^0_{\mathcal{C}/T}$ denote the corresponding family of *generalized* Jacobians. In her Ph.D. thesis [Ca94], Caporaso shows that there is a "compactification" of $J^0_{\mathcal{C}/T}$. Before going into details, let us fix some notations and definitions.

Let $C$ be a curve and $g := 1 - \chi(\mathscr{O}_C)$ its (arithmetic) genus. We say that a subcurve $Y$ of $C$ is *proper* if $C - Y \neq \varnothing$.

For each proper subcurve $Y$ of $C$, let $Y^c := \overline{C - Y}$, $\delta_Y := \# Y \cap Y^c$ and assume throughout this section $g \geq 3$.

**Definition 18.** We say that $C$ is a *nodal* curve if all its singularities are nodes.

We say that $C$ is a *stable* curve in the sense of Deligne–Mumford if $C$ is nodal, reduced, connected and each rational component of $C$ meets the remaining components in at least three points.

**Definition 19.** Let $T$ be a scheme and let $\mathfrak{Sch}_T$ denote the category of schemes over $T$. Let $\mathcal{F} : \mathfrak{Sch}_T \to \text{Sets}$ be a contravariant functor. We say that a $T$-scheme $F$ *coarsely represents* $\mathcal{F}$ if there exists a functor transformation

$$\Phi : \mathcal{F}(-) \longrightarrow \text{Mor}(-, F)$$

such that if $N$ is an $T$-scheme and $\Psi : \mathcal{F}(-) \to \text{Mor}_T(-, N)$ is a functor transformation, then there is a unique morphism $\pi : F \to N$ over $T$ such that the corresponding functor transformation $\Pi : \text{Mor}_T(-, F) \to \text{Mor}_T(-, N)$ satisfies $\Psi = \Pi \circ \Phi$.

Let $M_g$ be the moduli space of smooth curves of genus $g$ and $\overline{M}_g$ its compactification by stable curves. More precisely, $\overline{M}_g$ coarsely represents the contravariant functor $\overline{\mathscr{M}}_g$, which associates to each scheme $T$ the set

$$\overline{\mathscr{M}}_g(T) = \{\mathcal{C} \to T \text{ family of stable curves of genus } g\}/\sim_T,$$

where $[\mathcal{C} \to T] \sim_T [\mathcal{C}' \to T]$ if there is a $T$-isomorphism between $\mathcal{C}$ and $\mathcal{C}'$; see [Ge82].

For each stable curve $C$ of genus $g$, let $[C]$ denote the point of $\overline{M}_g$ corresponding to its isomorphism class.

Let $M_g^0$ be the locus of $M_g$ parameterizing smooth curves of genus $g$ without nontrivial automorphisms. For each integer $d$, there are a scheme $P_{d,g}$ over $M_g^0$ and a flat morphism $f_{d,g} : P_{d,g} \to M_g^0$ such that for each point $[C] \in M_g^0$, the fiber over $[C]$ is identified with the variety of isomorphism classes of invertible sheaves of degree $d$ on the curve $C$, that is, the fiber is identified with $J_C^d$; see [HM98], p. 41. The scheme $P_{g,d}$ is called the "universal Picard variety" or the "universal Jacobian" of degree $d$.

**Definition 20.** The *generalized* Jacobian of $C$, $J_C^0$, is a smooth, commutative, algebraic group, whose points are identified with isomorphism classes of line bundles having degree 0 on each component of $C$.

In order to solve the compactification problem for generalized Jacobians, Caporaso used Geometric Invariant Theory, GIT, to construct a projective scheme $\overline{P_{d,g}}$ over $\overline{M_g}$ with a surjection to $\overline{M_g}$

$$\phi_d : \overline{P_{d,g}} \to \overline{M_g}$$

such that the preimage of $M_g^0$ is isomorphic to $P_{d,g}$.

Caporaso describes the fiber of $\phi_d$ over any curve $[C] \in \overline{M_g}$ (denoted by $\overline{P_C^d} := \phi_d^{-1}([C])$) as a compactification of the generalized Jacobian of $C$. For the family of curves $f : \mathcal{C} \to T$, Caporaso describes

$$\bar{J}^0_{\mathcal{C}/T} := \overline{P_{d,g}} \times_{\overline{M}_g} T \to T$$

as one compactification of the family of generalized Jacobians of $f : \mathcal{C} \to T$.

To be honest, the term "compactification" is misleading for Caporaso's and Simpson's compactifications. For instance, in Caporaso's compactification, one can not expect that the compactified Jacobian $\overline{P_C^d}$ contains an open dense subset isomorphic to $J_C^0$. This in general is not true, unless $C$ is irreducible. What actually occurs is that $\overline{P_C^d}$ has finitely many irreducible components, each one containing a dense subset isomorphic to $J_C^0$. A similar phenomenon occurs in Simpson's compactification.

However, as the reader can already see, Caporaso's compactification is a projective scheme, while Altman–Kleiman's one may not be a scheme. Furthermore, $\overline{P_{d,g}}$ is a scheme very well behaved, as we may gather from the following theorem.

**Theorem 21.** *Let $d \geq 20(g-1)$ and $g \geq 3$. Then:*

*(1) The projective scheme $\overline{P_{d,g}}$ is reduced, irreducible and Cohen–Macaulay.*

*(2) The proper and surjective morphism*

$$\phi_d : \overline{P_{d,g}} \to \overline{M_g}$$

*is also flat over the locus of stable curves with trivial automorphism group. The preimage of $M_g^0$ under $\phi_d$ is isomorphic to $P_{d,g}$.*

See [Ca94], p. 592. It is important to say that if $d$ and $d'$ are integers for which there is a integer $n$ such that $d \pm d' = n(2g-2)$, then $\overline{P_{d,g}} \cong \overline{P_{d',g}}$, see [Ca94], Lemma 8.1, p. 655, and thus, the same properties hold for $\overline{P_{d,g}}$ for any integer $d$.

Although Caporaso's compactification is a projective scheme with good properties, due to the way it is constructed, it suffers from a disadvantage in relation to Altman–Kleiman's one as it is only a coarse moduli space, that is, it coarsely represents the functor $\overline{\mathscr{P}_{d,g}}$, which we present below, while Altman–Kleiman's one is a fine moduli space.

Now we introduce some preliminaries to define the functor $\overline{\mathscr{P}_{d,g}}$.

**Definition 22.** We say that a curve $C$ is a *quasistable* curve if:

1. $C$ is connected and nodal;
2. each rational component of $C$ meets the remaining components in at least two points;
3. two *exceptional* components never intersect.

An exceptional component is a rational component $E$ such that $\#E \cap \overline{C-E} = 2$. We denote by $C_{\text{exc}}$ the union of the exceptional components of $C$.

**Definition 23.** Let $C$ be a quasistable curve. Let $\omega$ be the dualizing sheaf of $C$ and $d$ an integer. We say that an invertible sheaf $\mathscr{L}$ of degree $d$ on $C$ is *balanced* if the following two properties hold:

1. For each exceptional component $E \subseteq C$, we have $\deg_E(\mathscr{L}) = 1$.

2. For each proper subcurve $Y \subseteq C$, we have

$$\left|\deg_Y(\mathscr{L}) - d\frac{\deg_Y(\omega)}{2g-2}\right| \leq \frac{\delta_Y}{2}. \tag{1.2}$$

**Definition 24.** Let $f : \mathcal{C} \to T$ be a family of quasistable curves over $T$. We say that an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ is relatively very ample (resp. relatively balanced) of degree $d$, if for each geometric point $t \in T$, the fiber $\mathcal{L}_t$ is a very ample (resp. balanced) invertible sheaf of degree $d$ on $\mathcal{C}_t$.

**Definition 25.** Let $\overline{\mathscr{P}_{d,g}}$ be the contravariant functor from schemes to sets, defined on a scheme $T$ by

$$\overline{\mathscr{P}_{d,g}}(T) := \{(f : \mathcal{C} \to T, \mathcal{L})\}/\sim,$$

where $f : \mathcal{C} \to T$ is a family of quasistable curves of genus $g$ over $T$, and $\mathcal{L}$ is a relatively very ample and relatively balanced invertible sheaf of degree $d$ on $f : \mathcal{C} \to T$.

We write $(f : \mathcal{C} \to T, \mathcal{L}_1) \sim_T (f' : \mathcal{C}' \to T, \mathcal{L}_2)$ if there are a $T$-isomorphism

$$\alpha : \mathcal{C} \to \mathcal{C}'$$

and an invertible sheaf $\mathcal{M}$ on $T$ such that

$$\alpha^* \mathcal{L}_2 \cong \mathcal{L}_1 \otimes f^* \mathcal{M}.$$

If $h : T \to T'$ is a morphism of schemes, then

$$\overline{\mathscr{P}_{d,g}}(h) : \overline{\mathscr{P}_{d,g}}(T') \to \overline{\mathscr{P}_{d,g}}(T)$$

is given by using $h$ to pull back to $T'$ the families over $T$.

**Remark 26.** Notice that if $\mathscr{L}$ is balanced, so is $\mathscr{L} \otimes \omega$. Also, if $\mathscr{L}$ is balanced and $d >> 0$, then $\mathscr{L}$ is very ample.

From this remark, it follows that there is a natural isomorphism of functors

$$\overline{\mathscr{P}_{d,g}} \to \overline{\mathscr{P}_{d+m(2g-2),g}},$$

given by tensoring with the $m$th power of the relative dualizing sheaf. Composing with such an isomorphisms of functors, we may assume that $d$ is high enough so that each balanced sheaf on a quasistable curve of genus $g$ is very ample. Thus, with respect to coarse representability of $\overline{\mathscr{P}_{d,g}}$, the condition "very ample" in the definition of $\overline{\mathscr{P}_{d,g}}$ may be ignored.

In [Ca94], Prop. 8.1, (1), p. 653 and [Ca94], Rmk. p. 654, Caporaso shows that $\overline{P_{d,g}}$ coarsely represents $\overline{\mathscr{P}_{d,g}}$.

### 1.3.1 The compactified Jacobian $\overline{P^d_C}$

For any integer $d$ and for any stable curve $C$ of genus $g \geq 3$, Caporaso constructed the compactified Jacobian $\overline{P^d_C}$ as a GIT-quotient

$$V_C \to V_C/G =: \overline{P^d_C}$$

parameterizing closed (and GIT-semistable) orbits of $G$ on $V_C$; see [Ca94], Cor. 5.1, p. 638, where $V_C$ is a subscheme of a certain Hilbert scheme endowed with an action of an algebraic group $G$. Furthermore, she showed that $\overline{P^d_C}$ coarsely represents the *strictly balanced Picard functor* associated to the stable curve $C$, which we define below.

**Definition 27.** Let $C$ be a nodal curve. Let $C_{\text{sing}}$ be the set of singularities of $C$. Let $S \subset C_{\text{sing}}$. We denote by $\nu_S : C^\nu_S \to C$ the normalization of $C$ along $S$, and by $\widehat{C}_S$ the quasistable curve obtained by "blowing up" all the nodes in $S$. More precisely, assume $S = \{P_1, ..., P_n\}$ and for each $i = 1, ..., n$, let $E_i$ be a rational component connecting the points of $\nu_S^{-1}(P_i)$. Then,

$$\widehat{C}_S := \bigcup_{i=1}^{n} E_i \cup C^\nu_S$$

and there is a natural surjective map $\widehat{\nu}_S : \widehat{C}_S \to C$ restricting to $\nu_S$ on $C^\nu_S$ and contracting all the exceptional components $E_i$ of $\widehat{C}_S$, called the *stabilization* of $\widehat{C}_S$. The curve $\widehat{C}_S$ is called a *quasistable model* of $C$.

**Definition 28.** Let $C$ be a quasistable curve and $\omega$ its dualizing sheaf. Let $d$ be an integer and $\mathscr{L}$ an invertible sheaf of degree $d$ on $C$. We say that $\mathscr{L}$ is *strictly balanced* if it is balanced and the inequality

$$\Big|\deg_Y(\mathscr{L}) - d\frac{\deg_Y(\omega)}{2g-2}\Big| \leq \frac{\delta_Y}{2} \tag{1.3}$$

is strict for each proper subcurve $Y \subsetneq C$ such that $Y \cap Y^c \nsubseteq C_{\text{exc}}$.

**Definition 29.** Let $C$ be a quasistable curve and $C_1, ..., C_n$ its irreducible components. We say that $\underline{d} := (d_1, ..., d_n) \in \mathbb{Z}^n$, with $|\underline{d}| := \sum_{i=1}^n = d$, is *balanced* (resp. *strictly balanced*) if there is an invertible sheaf $\mathscr{L}$ balanced (resp. strictly balanced) of degree $d$ on $C$ such that $\underline{d} = (\deg_{C_1}(\mathscr{L}), ..., \deg_{C_n}(\mathscr{L}))$. We let

$$B_d(C) := \{\underline{d} \in \mathbb{Z}^n : |\underline{d}| = d \text{ and } \underline{d} \text{ is balanced on } C\}$$

and

$$SB_d(C) := \{\underline{d} \in \mathbb{Z}^n : |\underline{d}| = d \text{ and } \underline{d} \text{ is strictly balanced on } C\}.$$

**Definition 30.** The strictly balanced Picard functor associated to a stable curve $C$ is the contravariant functor $\overline{\mathcal{P}^d_C}$ from the category of schemes to the category of sets, defined on a scheme $T$ by

$$\overline{\mathcal{P}^d_C(T)} = \{(f : \mathcal{C} \to T, \mathcal{L})\}/\sim$$

where $f : \mathcal{C} \to T$ is a family of quasistable models of $C$ and $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ is a strictly balanced invertible sheaf of relative degree $d$ on $f : \mathcal{C} \to T$.

Furthermore, $(f : \mathcal{C} \to T, \mathcal{L}_1) \sim (f' : \mathcal{C}' \to T, \mathcal{L}_2)$ if the following two facts hold:

1. The stabilizations of $f$ and $f'$ coincide.
2. Let $\mathcal{X} \to T$ be the stabilization of $f$ and $f$'. Let $\sigma : \mathcal{C} \to \mathcal{X}$ and $\sigma' : \mathcal{C}' \to \mathcal{X}$ be the stabilization maps of $f$ and $f$'. There are an isomorphism $\alpha : \mathcal{C} \to \mathcal{C}'$ commuting with $\sigma$ and $\sigma'$ (i.e. $\alpha$ is an $\mathcal{X}$-isomorphism) and a invertible sheaf $\mathscr{M}$ on $T$ such that $\alpha^*\mathcal{L}_2 \cong \mathcal{L}_1 \otimes f^*\mathscr{M}$.

To conclude this section, we present the following theorem on the compactified Jacobian $\overline{P^d_C}$.

**Theorem 31.** *Let $C$ be a stable curve of genus $g \geq 3$ and $d$ an integer. Then:*

1. *The compactified Jacobian $\overline{P^d_C}$ of $C$ is a reduced, connected, projective scheme of pure dimension $g$.*
2. *There is a canonical stratification,*

$$\overline{P^d_C} = \coprod_{(S,\underline{d}) \in I^d_S} J^{\underline{d}^\nu}_{C^\nu_S},$$

*where $I^d_S := \{(S, \underline{d}) : S \subseteq C_{sing}$ and $\underline{d} \in SB_d(\widehat{C_S})\}$ and $\underline{d}^\nu$ is the restriction of $\underline{d} \in SB_d(\widehat{C_S})$ to $C^\nu_S$.*

*Proof.* See [Ca94], Thm. 6.1, p. 641 for (1) and [Ca10], Cor. 2.3, p. 6 for (2). □

## 1.4 Simpson's compactification

In this section we present the solution "given" by Simpson to the problem of compactifying the relative Jacobian over families of curves. More precisely, this compactification is a consequence of a result much more general proven by Simpson; see [Sim], Thm. 1.21, p. 71. We avoid to state this result in its complete generality as we would need to introduce some preliminaries which are unnecessary for our purpose.

In this section all schemes will be defined, by convention, over $\mathrm{Spec}(\mathbb{C})$.

Let $C$ be a nodal curve. We say that a coherent sheaf $\mathscr{L}$ on $C$ is *pure* if for every nonzero subsheaf $\mathscr{M}$ of $\mathscr{L}$, we have

$$\dim(\mathscr{L}) = \dim(\mathscr{M}),$$

where for each coherent sheaf $\mathscr{F}$ on $C$, $\dim(\mathscr{F})$ denotes the dimension of the support of $\mathscr{F}$.

Let $L$ be a very ample invertible sheaf on $C$.

**Definition 32.** For each coherent sheaf $\mathscr{L}$ on $C$, we call the ratio

$$\mu_L(\mathscr{L}) = \frac{a}{r}$$

the *slope* of $\mathscr{L}$ with respect to $L$, where $a$ and $r$ are the coefficients of the Hilbert polynomial $P_L(\mathscr{L}, z) = r \cdot z + a$ of $\mathscr{L}$ with respect to $L$.

**Definition 33.** Let $\mathscr{L}$ be a coherent sheaf on $C$. We say that $\mathscr{L}$ is *slope-semistable* (resp. *slope-stable*) if it is pure and satisfies

$$\mu_L(\mathscr{L}) \leq \mu_L(\mathscr{M}) \text{ (resp. } \mu_L(\mathscr{L}) < \mu_L(\mathscr{M}))$$

for each pure quotient $\mathscr{L} \twoheadrightarrow \mathscr{M}$.

With this definition of slope-semistability (resp. slope-stability) it may seem hard to say when a torsion-free rank-1 sheaf is slope-semistable (resp. slope-stable). However, the following lemma shows that, if $C$ is a nodal curve, this can be done easily.

**Lemma 34.** *Let $\mathscr{L}$ be a torsion-free rank-1 sheaf on a nodal curve $C$. If $q : \mathscr{L} \twoheadrightarrow \mathscr{M}$ is a quotient map such that the quotient $\mathscr{M}$ is a pure sheaf with 1-dimensional support $Y := \mathrm{Supp}(\mathscr{M})$, then $q$ factors as*

$$\mathscr{L} \to i_*(\mathscr{L}_Y) \xrightarrow{\bar{q}} \mathscr{M},$$

*where $i : Y \hookrightarrow C$ is the inclusion and $\bar{q} : i_*(\mathscr{L}_Y) \to \mathscr{M}$ is an isomorphism.*

*Proof.* See [MKV], Lemma 2.2, p. 10. □

Let $\mathscr{L}$ be a torsion-free rank-1 sheaf on $C$. By Lemma 34, p. 26, $\mathscr{L}$ is slope-semistable (resp. slope-stable) with respect to $L$ if and only if

$$\mu_L(\mathscr{L}) \leq (<)\mu_L(i_*(\mathscr{L}_Y))$$

for every proper subcurve $Y$ of $C$, where $i : Y \hookrightarrow C$ is the inclusion map. However, both slopes can be computed explicitly. Indeed, let $g$ be the genus of $C$ and $\omega$ its dualizing sheaf. On one hand, the Hilbert polynomial of $\mathscr{L}$ is

$$\begin{aligned} P_L(\mathscr{L}, z) &= \deg(L) \cdot z + \chi(\mathscr{L}) \\ &= \deg(L) \cdot z + \deg(\mathscr{L}) + 1 - g \\ &= \deg(L) \cdot z + \deg(\mathscr{L}) - \frac{\deg(\omega)}{2}. \end{aligned}$$

Here we used $\deg(\mathscr{L}) = \chi(\mathscr{L}) - \chi(\mathscr{O}_C) = \chi(\mathscr{L}) + \deg(\omega)/2$.

On the other hand, let $Y$ be a proper subcurve of $C$ and $\omega_Y$ its dualizing sheaf. A similar reasoning shows that the Hilbert polynomial of $i_*(\mathscr{L}_Y)$ is

$$\begin{aligned} P_L(i_*\mathscr{L}_Y, z) &= P_{L_Y}(\mathscr{L}_Y, z) \\ &= \deg_Y(L) \cdot z + \deg_Y(\mathscr{L}) - \frac{\deg(\omega_Y)}{2} \\ &= \deg_Y(L) \cdot z + \deg_Y(\mathscr{L}) - \frac{\deg_Y(\omega)}{2} + \frac{\delta_Y}{2}, \end{aligned}$$

where we use the equality $\deg_Y(\omega) = \deg(\omega_Y) + \delta_Y$; see Lemma 60, p. 46.

Thus, we have

$$\mu_L(\mathscr{L}) = \frac{\deg(\mathscr{L}) - 1/2 \cdot \deg(\omega)}{\deg(L)}$$

and

$$\mu_L(i_*(\mathscr{L}_Y)) = \frac{\deg_Y(\mathscr{L}) - 1/2 \cdot \deg_Y(\omega) + 1/2 \cdot \delta_Y}{\deg_Y(L)}.$$

So, $\mu_L(\mathscr{L}) \leq (<)\mu_L(i_*(\mathscr{L}_Y))$ if and only if

$$\frac{\deg(\mathscr{L}) - 1/2 \cdot \deg(\omega)}{\deg(L)} \leq (<)\frac{\deg_Y(\mathscr{L}) - 1/2 \cdot \deg_Y(\omega) + 1/2 \cdot \delta_Y}{\deg_Y(L)},$$

or equivalently

$$\deg_Y(\mathscr{L}) \geq (>)\frac{\deg_Y(L)}{\deg(L)}\Big(\deg(\mathscr{L}) - \frac{\deg(\omega)}{2}\Big) + \frac{\deg_Y(\omega)}{2} - \frac{\delta_Y}{2}. \quad (1.4)$$

Thus, $\mathscr{L}$ is slope-semistable (resp. slope-stable) with respect to $L$ if and only if (1.4) holds for each proper subcurve $Y$ of $C$.

Let $f : \mathcal{C} \to T$ be a family of nodal curves of genus $g$ and $\mathcal{L}$ an invertible relatively ample sheaf on $f : \mathcal{C} \to T$.

**Definition 35.** A relative torsion-free rank-1 sheaf $\mathcal{F}$ on $f : \mathcal{C} \to T$ is called slope-semistable (resp. slope-stable) with respect to $\mathcal{L}$ if so is its restriction with respect to $\mathcal{L}_t$ to the fiber $\mathcal{C}_t$, for each geometric point $t$ of $T$.

Assume that $\mathcal{L}$ has constant relative degree, that is, $t \mapsto \deg(\mathcal{L}_t)$ is constant. Let $d$ be an integer and $t_0 \in T$ a geometric point. Now, let

$$P_d(z) := \deg(\mathcal{L}_{t_0}) \cdot z + d + 1 - g \in \mathbb{Q}[z].$$

Consider the $T$-functor $\bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}/T)$, which associates to each $T$-scheme $S$ the set of slope-semistable sheaves on $p_2 : \mathcal{C}' = \mathcal{C} \times_T S \to S$ with respect to $\mathcal{L} \otimes \mathscr{O}_{\mathcal{C}'}$, such that for each geometric point $s \in S$, their restrictions to the fiber of $p_2 : \mathcal{C}' \to S$ over $s$ have fixed Hilbert polynomial $P_d(z)$.

Casalaina-Martin, Kass and Viviani [MKV], Lemma 2.3, p. 11, used [Sim94], Thm. 1.21, p. 71, in order to show that there is a projective scheme $\bar{J}_{\mathcal{L},d}(\mathcal{C}/T) \to T$, which is called the *relative Simpson's compactified Jacobian* of degree $d$, which coarsely represents $\bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}/T)$. Let us be more precise about the construction of $\bar{J}_{\mathcal{L},d}(\mathcal{C}/T)$.

**The relative quot scheme**. Let $f : X \to T$ be a projective morphism of algebraic schemes, and let $\mathscr{O}_X(1)$ be a relative very ample invertible sheaf on $X$ with respect to $f$. Fix a coherent sheaf $\mathscr{E}$ on $X$ and a numerical polynomial $p(z) \in \mathbb{Q}[z]$.

**Definition 36.** For each $T$-scheme $S$, a *family of quotients of $\mathscr{E}$ parametrized by $S$* is a pair $(\mathscr{F}, q)$ consisting of a $S$-flat coherent quotient $q : \mathscr{E}_S \to \mathscr{F}$, where $\mathscr{E}_S$ is the pullback of $\mathscr{E}$ under the projection $X_S \to X$, such that on each fiber the Hilbert polynomial of $\mathscr{F}$ with respect to $\mathscr{O}_X(1)$ is $p(z)$.

We say that two such families $(\mathscr{F}, q)$ and $(\mathscr{F}', q')$ parametrized by $S$ are equivalent if $\ker(q) = \ker(q')$, and we let $\langle \mathscr{F}, q \rangle$ denote the equivalence class of $(\mathscr{F}, q)$.

Since properness and flatness are preserved by base-change, and as tensor product is right exact, the pullback of $\langle \mathscr{F}, q \rangle$ under a $T$-morphism $S' \to S$ is well-defined, which gives a set-valued contravariant functor $Quot^{X/T}_{\mathscr{E},p(z)}$ defined on each $T$-scheme $S$ by

$$S \mapsto \{\langle \mathscr{F}, q \rangle \text{ parametrized by } S\}.$$

Grothendieck showed that this functor is represented by a projective $T$-scheme denoted by $\mathrm{Quot}^{X/T}_{\mathscr{E},p(z)}$.

Now, let $\mathscr{E} := \mathscr{O}_X(b)^{\oplus r}$, for some $r$ and $b$. Then the scheme $\mathrm{Quot}^{X/T}_{\mathscr{E},p(z)}$ is endowed with a natural action of $\mathrm{SL}_r$ given by changing coordinates on $\mathscr{O}_X^{\oplus r}$. In addition, one can embed $\mathrm{Quot}^{X/T}_{\mathscr{E},p(z)}$ in a certain Grassmannian, which in turn embeds into a certain projective space via the Plücker coordinates, in such way that the following properties are satisfied:
(i) $\mathrm{SL}_r$ acts linearly with respect to the embedding;
(ii) each point $\langle \mathscr{F}, q\rangle \in \mathrm{Quot}^{X/T}_{\mathscr{E},p(z)}$ is GIT-semistable (resp. GIT-stable) if and only if $\mathscr{F}$ is slope-semistable (resp. slope-stable); see [Sim], Thm. 1.19, p. 69.

In order to construct the relative compactified Jacobian $\bar{J}_{\mathcal{L},d}(\mathcal{C}/T) \to T$, Casalaina-Martin, Kass and Viviani followed this strategy: Set $\mathscr{O}_{\mathcal{C}}(1) := \mathcal{L}$. Given $b >> 0$, let $r := P_d(b)$ and consider $\mathrm{Quot}^{\mathcal{C}/T}_{\mathscr{O}_{\mathcal{C}}(-b)^{\oplus r},P_d(z)}$ endowed with an embedding into a certain projective space satisfying properties (i) and (ii) above. By [Sim], p. 66, there is a closed and open subscheme

$$Z \subseteq \mathrm{Quot}^{\mathcal{C}/T}_{\mathscr{O}_{\mathcal{C}}(-b)^{\oplus r},P_d(z)}$$

parameterizing quotient maps

$$q : \mathscr{O}_{\mathcal{C}}(-b)^{\oplus r} \to \mathcal{F}$$

satisfying the following additional conditions: For each geometric point $t \in T$,
(1) $H^1(\mathcal{C}_t, \mathcal{F}_t(b)) = 0$;
(2) $q_t : H^0(\mathcal{C}_t, \mathscr{O}_{\mathcal{C}_t}^{\oplus r}) \to H^0(\mathcal{C}_t, \mathcal{F}_t(b))$ is an isomorphism;
(3) $\mathcal{F}$ is a slope-semistable sheaf on $f : \mathcal{C} \to T$.

Then the natural action of $\mathrm{SL}_r$ restricts to an action on $Z$ and, by the Fundamental GIT Theorem, there is a quotient

$$\pi : Z \to Z/\mathrm{SL}_r.$$

Now let $R \subseteq Z$ be the locus parameterizing quotients $q : \mathscr{O}_{\mathcal{C}}(-b)^{\oplus r} \to \mathcal{F}$ such that $\mathcal{F}$ is a torsion-free rank-1 sheaf on $\mathcal{C}$. From [Pan], Lemma 8.1.1, $R$ is a $\mathrm{SL}_r$-invariant subset that is closed and open in $Z$, and hence its image must be closed and open in $Z/SL_r$; see [Sim], Lemma 1.10, p. 61. Then they set

$$\bar{J}_{\mathcal{L},d}(\mathcal{C}/T) := \pi(R).$$

Finally, by universality, $\bar{J}_{\mathcal{L},d}(\mathcal{C}/T)$ is a categorical quotient of $R$ by $\mathrm{SL}_r$ and an inspection of the proof of [Sim], Thm. 1.21, p. 71 shows that this scheme coarsely represents the functor $\bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}/T)$.

# Chapter 2

# Autoduality for treelike curves

Let $C$ be a connected projective reduced curve defined over an algebraically closed field $k$.

We recall that for each integer $d$, we denote by $J_C^d$ the scheme parameterizing isomorphism classes of invertible sheaves of degree $d$ on $C$, and for each point $P \in C$, we let $\mathfrak{m}_P$ be the sheaf of ideals of $P$ on $C$. Let $\mathscr{L}$ be a line bundle of degree 1 on $C$. The classical *Autoduality Theorem* says the following:

**Theorem 37** (Autoduality Theorem). *If $C$ is smooth, the Abel map*

$$A : C \to J_C^0,\ P \mapsto \mathscr{L} \otimes \mathfrak{m}_P$$

*is well defined and induces an isomorphism*

$$A^* : \mathrm{Pic}^0(J_C^0) \to J_C^0$$

*which is independent of the choice of $\mathscr{L}$.*

See [Mu65], Prop. 6.9, p. 118. We say that $J_C^0$, the *Jacobian* of $C$, is *autodual.*

Case $C$ has singularities, the Autoduality Theorem was first proved by Esteves, Gagné and Kleiman in [EGK] for irreducible curves with at most double points, with $J_C^0$ being replaced by its natural compactification $\bar{J}_C^0$, the moduli space of degree-0 torsion-free rank-1 sheaves over $C$, constructed by Altman and Kleiman [AK]. More precisely, Esteves, Gagné and Kleiman showed the following:

**Theorem 38** (EGK). *Assume $C$ is irreducible and has at most double points. Then, the Abel map*

$$A_{\mathscr{L}} : C \to \bar{J}_C^0,\ P \mapsto \mathscr{L} \otimes \mathfrak{m}_P$$

*is well defined and induces an isomorphism*

$$A^* : \mathrm{Pic}^0(\bar{J}^0_C) \to J^0_C$$

*which is independent of the choice of* $\mathscr{L}$.

Later, Arinkin [A07] extended the validity of the above theorem, also assuming $C$ irreducible, by allowing *planar singularities*, in other words, asking only that the tangent space to $C$ at any point be at most two-dimensional. In this chapter we show autoduality for *treelike* curves whose singularities are all *planar*.

As we have already said in the introduction of this thesis, a generalization of the result obtained in this note has been recently made available at [MRV] by Melo, Rapagnetta and Viviani, with an appendix by López-Martín. They state autoduality for any curve with planar singularities.

For each subcurve $Y$ of $C$ recall that $Y^c = \overline{C - Y}$ and $\delta_Y = \#Y \cap Y^c$. We say that a subcurve $Y$ of $C$ is a *tail* of $C$ if $\#Y \cap Y^c = 1$.

**Definition 39.** We say that a point $P \in C$ is a *separating node* if $P$ is an ordinary node of $C$ and $C - P$ is not connected.

We say that the singularities of $C$ are planar if the tangent space to $C$ at any point is at most two-dimensional.

A point $P \in C$ is called a *crossing* if it lies on two irreducible components of $C$.

**Definition 40.** We say that $C$ is *treelike* if all its crossing points are separating nodes.

**Definition 41.** A *subcurve* of $C$ is a reduced union of irreducible components of $C$. A subcurve is not necessarily connected. We say that a connected subcurve $Y$ of $C$ is a *spine* if each point in $Y \cap Y^c$ is a separating node. Notice that, if $Y$ is spine, each connected component $Z$ of $Y^c$ is a tail intersecting $Y$ transversally at a unique point on the smooth locus of $Y$ and $Z$.

A $n$-tuple $(Z_1, ..., Z_n)$ of spines $Z_i$ covering $C$ with finite pairwise intersection is called a *spine decomposition* of $C$.

**Example 42.** Let $C$ be a treelike curve with irreducible components $C_1, ..., C_n$. Then $(C_1, ..., C_n)$ is a spine decomposition of $C$.

Let $\mathscr{L}$ be a coherent sheaf on $C$. Let $C_1, ..., C_n$ be its irreducible components and $\eta_1, ..., \eta_n$ their generic points. We recall that $\mathscr{L}$ is *torsion-free* if the map

$$\mathscr{L} \longrightarrow \prod_i \mathscr{L}_{\eta_i},$$

where $\mathscr{L}_{\eta_i}$ is the skyscraper sheaf of $\mathscr{L}$ at $\eta_i$, has trivial kernel. It is *rank*-1 if it has generic rank 1 on each irreducible component of $C$. It is *simple* if $\mathrm{Hom}(\mathscr{L}, \mathscr{L}) = k$.

The *degree* of a torsion-free, rank-1 sheaf $\mathscr{L}$ is $\deg(\mathscr{L}) := \chi(\mathscr{L}) - \chi(\mathscr{O}_C)$. It follows from [E01], Prop. 1 p. 3049 that a torsion-free rank-1 sheaf on $C$ is simple only if it is invertible at separating nodes. The converse is true if $C$ is treelike, in which case the restriction of a simple torsion-free rank-1 sheaf to any connected subcurve of $C$ is also simple torsion-free rank-1.

Now, for each connected subcurve $Y$ of $C$, let $\bar{J}_Y$ be the compactification of the Picard scheme of $Y$, that is, the scheme parameterizing torsion-free, rank-1 and simple sheaves on $Y$.

**Proposition 43.** *Let $C$ be a curve and $(Z_1, ..., Z_n)$ a spine decomposition of $C$. Then there is an isomorphism*

$$u : \bar{J}_C \to \bar{J}_{Z_1} \times \cdots \times \bar{J}_{Z_n},\ \mathscr{L} \mapsto ([\mathscr{L}_{Z_1}], ..., [\mathscr{L}_{Z_n}]).$$

*Proof.* See [E09], Prop. 3.2. □

Assume $C$ is treelike and let $C_1, C_2, ..., C_n$ be the irreducible components of $C$. For each $\underline{d} := (d_1, d_2, ..., d_n) \in \mathbb{Z}^n$, let $\bar{J}_C^{\underline{d}}$ be the connected component of $\bar{J}_C$, parameterizing torsion-free, rank-1 simple sheaves $\mathscr{L}$ on $C$ such that $\deg_{C_i}(\mathscr{L}) = d_i$. For each $i = 1, ..., n$, and each $e \in \mathbb{Z}$, let $\bar{J}_{C_i}^e$ be the corresponding scheme for $C_i$.

**Corollary 44.** *Let $C$ be a treelike curve whose irreducible components are $C_1, ..., C_n$. Then for each $\underline{d} = (d_1, ..., d_n) \in \mathbb{Z}^n$ we have an isomorphism*

$$\bar{\pi}^{\underline{d}} : \bar{J}_C^{\underline{d}} \to \bar{J}_{C_1}^{d_1} \times \cdots \times \bar{J}_{C_n}^{d_n},\ \mathscr{I} \mapsto (\mathscr{I}_{C_1}, ..., \mathscr{I}_{C_n}).$$

For each $\underline{d} = (d_1, ..., d_n) \in \mathbb{Z}^n$, let $J_C^{\underline{d}} \subseteq \bar{J}_C^{\underline{d}}$ be the open subscheme parameterizing invertible sheaves. Likewise, for each $i = 1, \ldots, n$ and each integer $e$, let $J_{C_i}^e \subseteq \bar{J}_{C_i}^e$ be the open subscheme parameterizing invertible sheaves. Then $\bar{\pi}^{\underline{d}}$, as defined in the Corollary 43, restricts to an isomorphism

$$\pi^{\underline{d}} = (\pi_1^{\underline{d}}, \ldots, \pi_n^{\underline{d}}) : J_C^{\underline{d}} \longrightarrow J_{C_1}^{d_1} \times \cdots \times J_{C_n}^{d_n}.$$

**Remark 45.** Suppose $C$ is treelike and all its singularities are planar. Then, for each $i = 1, ..., n$, the singularities of $C_i$ are also planar. So, it follows from [AIK], (9), p. 8, that the scheme $\bar{J}_{C_i}^e$ is integral for each $e$. Thus, by [G2], Thm. 3.1, p. 232-06, there is a scheme parameterizing line bundles on $\bar{J}_{C_i}^e$, named $\mathrm{Pic}(\bar{J}_{C_i}^e)$, whose connected component of the identity we denote by $\mathrm{Pic}^0(\bar{J}_{C_i}^e)$. Let $\underline{d} := (d_1, ..., d_n) \in \mathbb{Z}^n$ and $\bar{\pi}^{\underline{d}} : \bar{J}_C^{\underline{d}} \to \bar{J}_{C_1}^{d_1} \times \cdots \times \bar{J}_{C_n}^{d_n}$ be the isomorphism given by Corollary 43. Then, since $\bar{\pi}^{\underline{d}}$ is an isomorphism, we have that $\bar{J}_C^d$ is integral, and there is a corresponding Picard scheme for $\bar{J}_C^d$, whose connected component of the identity we denote by $\mathrm{Pic}^0(\bar{J}_C^d)$.

**Definition 46.** Let $C$ be an integral curve and $d \in \mathbb{Z}$ an integer. Let $A : C \to \bar{J}_C^d$ be a map. We say that $A$ is an *Abel map* if there is an invertible sheaf $\mathscr{L}$ on $C$ such that $A$ sends $P$ to $\mathscr{L} \otimes \mathfrak{m}_P$ for each $P \in C$.

In order to prove our next lemma, we need to use a special type of invertible sheaves called *determinants of cohomology.* For more details about the theory of determinants, see [KM].

Let $f : \mathcal{C} \to T$ be a flat, projective morphism whose geometric fibers are curves. Let $\mathcal{F}$ be a $T$-flat coherent sheaf on $\mathcal{C}$. The determinant of cohomology of $\mathcal{F}$ with respect to $f$ is defined to be the invertible sheaf $\mathcal{D}_f(\mathcal{F})$ on $T$ constructed as follows: Locally on $T$ there is a complex

$$0 \to \mathscr{G}^0 \xrightarrow{\lambda} \mathscr{G}^1 \to 0$$

of free sheaves of finite rank such that, for every coherent sheaf $\mathscr{M}$ on $T$, the cohomology groups of $\mathscr{G}^\bullet \otimes \mathscr{M}$ are equal to the higher direct images of $\mathcal{F} \otimes f^*\mathscr{M}$ under $f$. The complex $\mathscr{G}^\bullet$ is unique up to unique quasi-isomorphism. Hence, its determinant,

$$\det(\mathscr{G}^\bullet) := (\bigwedge^{\mathrm{rank}\,\mathscr{G}^1} \mathscr{G}^1) \otimes (\bigwedge^{\mathrm{rank}\,\mathscr{G}^0} \mathscr{G}^0)^{-1},$$

is unique up to canonical isomorphism. The uniqueness allows us to glue together the local determinants to obtain the invertible sheaf $\mathcal{D}_f(\mathcal{F})$ on $T$.

The determinant of cohomology has the following properties:

1. *Functorial property* : We can functorially associate to each isomorphism, $\phi : \mathcal{F}_1 \cong \mathcal{F}_2$, of $T$-flat coherent sheaves on $\mathcal{C}$ an isomorphism:

$$\mathcal{D}_f(\phi) : \mathcal{D}_f(\mathcal{F}_1) \cong \mathcal{D}_f(\mathcal{F}_2).$$

2. *Additive property* : We can functorially associate to each short exact sequence,

$$\alpha : 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

of $T$-flat coherent sheaves on $\mathcal{C}$ an isomorphism:

$$\mathcal{D}_f(\alpha) : \mathcal{D}_f(\mathcal{F}_2) \cong \mathcal{D}_f(\mathcal{F}_1) \otimes \mathcal{D}_f(\mathcal{F}_3).$$

3. *Projection property*: We can functorially associate to each $T$-flat coherent sheaf $\mathcal{F}$ on $\mathcal{C}$ of relative Euler characteristic $d$ over $T$, and each invertible sheaf $\mathscr{M}$ on $T$, an isomorphism:

$$\mathcal{D}_f(\mathcal{F} \otimes \mathscr{M}) \cong \mathcal{D}_f(\mathcal{F}) \otimes f^* M^{\otimes d}.$$

4. *Base-change property*: We can functorially associate to each $T$-flat coherent sheaf $\mathcal{F}$ on $\mathcal{C}$, and each Cartesian diagram of the form:

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{h_1} & \mathcal{C} \\ {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ T_1 & \xrightarrow{h} & T, \end{array}$$

a base-change isomorphism:

$$h^* \mathcal{D}_f(\mathcal{F}) \cong \mathcal{D}_{f_1}(h_1^* \mathcal{F}).$$

**Lemma 47.** *Let $C$ be an integral curve whose singularities are planar and $\mathscr{L}$ an invertible sheaf of degree 1 on $C$. Let $A_{\mathscr{L}} : C \to \bar{J}_C^0$, $P \mapsto \mathscr{L} \otimes \mathfrak{m}_P$ be the corresponding Abel map. Then there is a natural map*

$$\beta : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\bar{J}_C^0),$$

*such that $A_{\mathscr{L}}^* \circ \beta = 1_{\mathrm{Pic}^0(C)}$, where $A_{\mathscr{L}}^* : \mathrm{Pic}^0(\bar{J}_C^0) \to \mathrm{Pic}^0(C)$ is the pullback of $A_{\mathscr{L}} : C \to \bar{J}_C^0$.*

*Proof.* The proof is in [EGK], Prop. 2.2, p. 595. We reproduce it here for sake of completeness. Let $\mathscr{I}$ be a universal sheaf on $C \times \bar{J}_C^0$ and $\mathscr{M}$ be one on $C \times J_C^0$. Form $C \times \bar{J}_C^0 \times J_C^0$, and let $p_{ij}$ be the projection of $C \times \bar{J}_C^0 \times J_C^0$, onto the product of the indicated factors. Set

$$\mathscr{M}^\diamond := (\mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I} \otimes p_{13}^* \mathscr{M}))^{-1} \otimes \mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I}) \text{ on } \bar{J}_C^0 \times J_C^0 \qquad (2.1)$$

where $\mathcal{D}_{p_{23}}$ denotes the determinant of cohomology. Then $\mathscr{M}^\diamond$ is an invertible sheaf and we claim that it defines the desired map $\beta$. Indeed, the sheaf $\mathscr{I}$ is determined up to tensor product with the pullback of an invertible sheaf $\mathscr{N}$ on $\bar{J}_C^0$. The projection formula for the determinant of cohomology gives us

$$\mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I} \otimes p_2^* \mathscr{N} \otimes p_1^* \mathscr{M}) = \mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I} \otimes p_{13}^* \mathscr{M}) \otimes p_1^* \mathscr{N}^{\otimes m}$$

and

$$\mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I} \otimes p_2^* \mathscr{N}) = \mathcal{D}_{p_{23}}(p_{12}^* \mathscr{I}) \otimes p_1^* \mathscr{N}^{\otimes n},$$

where the $p_{ij}$ are the indicated projections and where $m$ and $n$ are the Euler characteristics of $p_{12}^*\mathscr{I} \otimes p_{13}^*\mathscr{M}$ and $p_{12}^*\mathscr{I}$ on the fibers of $p_{23}$ (hence $m$ and $n$ are locally constant functions on $\bar{J}_C^0 \times J_C^0$). However, since the fibers of $p_{13}^*\mathscr{M}$ have degree 0, we have $m = n$. Therefore, from its definition, (2.1), $\mathscr{M}^\diamond$ does not depend on the choice of $\mathscr{I}$.

Similarly, the sheaf $\mathscr{M}$ is determined up to tensor product with the pullback of an invertible sheaf $\mathscr{P}$ on $J_C^0$. As before, if $\mathscr{M}$ is replaced by its tensor product with the pullback of $\mathscr{P}$, then $\mathscr{M}^\diamond$ is replaced by its tensor product with the pullback of $\mathscr{P}^{\otimes m}$.

Therefore $\mathscr{M}^\diamond$ defines a map $\beta : J_C^0 \to \operatorname{Pic}(\bar{J}_C^0)$. Now we claim that the image of $\beta$ lies in $\operatorname{Pic}^0(\bar{J}_C^0)$. Indeed, since the fiber $\mathscr{M}_{\mathscr{O}_C}$ is $\mathscr{O}_C$ and forming the determinant commutes with passing to the fiber, it follows that $\mathscr{M}^\diamond(0) = \mathscr{O}_{\bar{J}_C^0}$. Thus $\beta(\mathscr{O}_C) = \mathscr{O}_{\bar{J}_C^0}$ implying that $\beta(J_C^0) \subset \operatorname{Pic}^0(\bar{J}_C^0)$.

Finally we show $A_{\mathscr{L}} \circ \beta = 1_{J_C^0}$. Indeed, consider the map

$$1_C \times A_{\mathscr{L}} : C \times C \to C \times \bar{J}_C^0$$

and notice that $A_{\mathscr{L}}$ is defined by $(1_C \times A_{\mathscr{L}})^*\mathscr{I}$, as well as $\mathscr{I}_\Delta \otimes q_1^*\mathscr{L}$, where $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset C \times C$ and $q_{ij}$ indicate the respective projection of $C \times C \times J_C^0$. Hence these two sheaves differ by tensor product with the pullback, along the projection $q_2$, of an invertible sheaf on $C$. But on the other hand, it follows from the base-change property of the determinant of cohomology applied to

$$\begin{array}{ccc} C \times C \times J_C^0 & \xrightarrow{(1_C \times A_{\mathscr{L}} \times 1_{J_C^0})} & C \times \bar{J}_C^0 \times J_C^0 \\ \Big\downarrow{\scriptstyle q_{23}} & & \Big\downarrow{\scriptstyle p_{23}} \\ C \times J_C^0 & \xrightarrow{(A_{\mathscr{L}} \times 1_{J_C^0})} & \bar{J}_C^0 \times J_C^0 \end{array}$$

that

$$(A_{\mathscr{L}} \times 1_{J_C^0})^*\mathscr{M}^\diamond = (\mathcal{D}_{q_{23}}(q_{12}^*\mathscr{I}_\Delta \otimes q_1^*\mathscr{L} \otimes q_{13}^*\mathscr{M}))^{-1} \otimes \mathcal{D}_{q_{23}}(q_{12}^*\mathscr{I}_\Delta \otimes q_1^*\mathscr{L}) \quad (2.2)$$

on $C \times J_C^0$. Hence both sides of this equation define the same map $J_C^0 \to \coprod_n J_C^n$.

In order to evaluate the right-hand side of (2.2), consider the natural sequence

$$0 \to \mathscr{I}_\Delta \to \mathscr{O}_{C \times C} \to \mathscr{O}_\Delta \to 0.$$

Pull it back to $C \times C \times J_C^0$; then tensor with $q_1^*\mathscr{L} \otimes q_{13}^*\mathscr{M}$ and $q_1^*\mathscr{L}$. The property of the additivity of the determinant of cohomology now yields

$$\mathcal{D}_{q_{23}}(q_{12}^*\mathscr{I}_\Delta \otimes q_1^*\mathscr{L} \otimes q_{13}^*\mathscr{M}) = \mathcal{D}_{q_{23}}(q_1^*\mathscr{L} \otimes q_{13}^*\mathscr{M}) \otimes (q_1^*\mathscr{L} \otimes \mathscr{M})^{-1}$$

and

$$\mathcal{D}_{q_{23}}(q_{12}^*\mathscr{I}_\Delta \otimes q_1^*\mathscr{L}) = \mathcal{D}_{q_{23}}(q_1^*\mathscr{L}) \otimes (q_1^*\mathscr{L})^{-1}.$$

Now consider the Cartesian square below

$$\begin{array}{ccc} C \times C \times J_C^0 & \xrightarrow{q_{13}} & C \times J_C^0 \\ \downarrow{\scriptstyle q_{23}} & & \downarrow{\scriptstyle q_2} \\ C \times J_C^0 & \xrightarrow{q_2} & J_C^0 \end{array}$$

Since forming the determinant of cohomology commutes with changing the base, we have

$$\mathcal{D}_{q_{23}}(q_1^*\mathscr{L} \otimes q_{13}^*\mathscr{M}) = q_2^*\mathcal{D}_{q_2}(q_1^*\mathscr{L} \otimes \mathscr{M})$$

and

$$\mathcal{D}_{q_{23}}(q_1^*\mathscr{L}) = q_2^*\mathcal{D}_{q_2}(q_1^*\mathscr{L})$$

on $C \times J_C^0$. Hence the right-hand side of (2.2) differs from $\mathscr{M}$ by tensor product with the pullback of an invertible sheaf on $J_C^0$. Therefore $A_{\mathscr{L}}^* \circ \beta = 1_{J_C^0}$, and the proof is complete. □

**Definition 48.** Let $C$ be a curve. Let $r$ and $s$ be arbitrary integers. For each invertible sheaf $\mathscr{M}$ of degree $s$ on $C$, we define the *translation* by: $\mathscr{M}$

$$\tau_{\mathscr{M}} : \bar{J}_C^r \to \bar{J}_C^{r+s},\ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{M}.$$

**Lemma 49.** *Let $C$ be an integral curve whose singularities are all planar. Let $\mathscr{M}$ be an invertible sheaf on $C$, and $r$ and $s$ integers with $s = \deg(\mathscr{M})$. Then the translation $\tau_{\mathscr{M}}$ induces an isomorphism*

$$\tau_{\mathscr{M}}^* : \mathrm{Pic}^0(\bar{J}_C^{r+s}) \xrightarrow{\sim} \mathrm{Pic}^0(\bar{J}_C^r)$$

*which is independent of $\mathscr{M}$. In particular, if $s = 0$, then $\tau_{\mathscr{M}}^*$ is equal to the identity.*

*Proof.* Since for each invertible sheaf $\mathscr{N}$ on $C$, we have

$$\tau_{\mathscr{M}} \circ \tau_{\mathscr{N}} = \tau_{\mathscr{M} \otimes \mathscr{N}},$$

then $\tau_{\mathscr{M}}$ is an isomorphism whose inverse is $\tau_{\mathscr{M}^{-1}}$. So, $\tau_{\mathscr{M}}^*$ is an isomorphism.

Now, in order to prove that the isomorphism $\tau_{\mathscr{M}}^*$ is independent of $\mathscr{M}$, we may assume $s = 0$, as if also $\mathscr{N}$ is an invertible sheaf of degree $s$ on $C$, then $\mathscr{M} \otimes \mathscr{N}^{-1}$ has degree 0 and it is enough to prove that $\tau^*_{\mathscr{M}\otimes\mathscr{N}^{-1}} = 1$. We may assume $r = 0$ too, as for each invertible sheaf $\mathscr{L}$ of degree 1 on $C$,

$$\tau_{\mathscr{M}} = \tau_{\mathscr{L}^{\otimes r}} \circ \tau_{\mathscr{M}} \circ \tau_{\mathscr{L}^{\otimes -r}}.$$

So, fix an invertible sheaf $\mathscr{L}$ of degree 1 on $C$ and consider the corresponding Abel maps $A_1, A_2 : C \to \bar{J}^0_C$, the first defined by $P \mapsto \mathscr{L} \otimes \mathfrak{m}_P$, the second by $P \mapsto \mathscr{M} \otimes \mathscr{L} \otimes \mathfrak{m}_P$. Plainly we have $A_2 = \tau_{\mathscr{M}} A_1$. By [EGK], Prop. 3.7, p. 605, the pullback maps $A_1^*, A_2^* : \mathrm{Pic}^0(\bar{J}^0_C) \to J^0_C$ are equal. Thus $A_1^* = A_1^* \tau^*_{\mathscr{M}}$.

Finally, it follows from [A07], Thm. C, that a certain map $\rho : J^0_C \to \mathrm{Pic}^0(\bar{J}^0_C)$ is an isomorphism. This map is the one called $\beta$ in Lemma 47, where it is proved that $A_1^*\beta = 1$. So $A_1^*$ is an isomorphism as well. Since $A_1^* = A_1^* \tau^*_{\mathscr{M}}$, it follows that $\tau^*_{\mathscr{M}}$ is the identity.

□

**Theorem 50** (Theorem of the Cube ). *Let $X_1$ and $X_2$ be complete varieties, $X_3$ a connected scheme, and $\mathscr{L}$ an invertible sheaf on $X_1 \times X_2 \times X_3$ whose restrictions to*

$$\{P_1\} \times X_2 \times X_3,\ \ X_1 \times \{P_2\} \times X_3 \text{ and } X_1 \times X_2 \times \{P_3\}$$

*are trivial for some $P_1 \in X_1$, $P_2 \in X_2$ and $P_3 \in X_3$. Then $\mathscr{L}$ is trivial.*

*Proof.* See [Mu74], p. 55. □

**Proposition 51.** *Let $C$ be a treelike curve whose irreducible components are $C_1, ..., C_n$. Let $\underline{d} \in \mathbb{Z}^n$ and suppose $C$ has only planar singularities. For each $j = 1, \ldots, n$, let $\mathscr{I}_j$ be a degree-$d_j$ torsion-free rank-1 sheaf on $C_j$, and let*

$$\iota_j : \bar{J}^{d_j}_{C_j} \longrightarrow \bar{J}^{\underline{d}}_C$$

*be the composition of the inverse of $\overline{\pi}^{\underline{d}}$ with the map given by*

$$\mathscr{I} \mapsto (\mathscr{I}_1, \ldots, \mathscr{I}_{j-1}, \mathscr{I}, \mathscr{I}_{j+1}, \ldots, \mathscr{I}_n).$$

*Then the induced map*

$$\iota^* := (\iota_1^*, \ldots, \iota_n^*) : \mathrm{Pic}^0(\bar{J}^{\underline{d}}_C) \longrightarrow \mathrm{Pic}^0(\bar{J}^{d_1}_{C_1}) \times \cdots \times \mathrm{Pic}^0(\bar{J}^{d_n}_{C_n})$$

*is an isomorphism. Furthermore, if $\mathscr{L}_j$ is a degree-0 invertible sheaf on $C_j$ for $j = 1, \ldots, n$, then replacing each $\mathscr{I}_j$ by $\mathscr{I}_j \otimes \mathscr{L}_j$ does not change $\iota^*$.*

*Proof.* Since $\bar{\pi}^{\underline{d}}$ is an isomorphism, and since the $\bar{J}^{d_j}_{C_j}$ are complete varieties, the proof of the first statement is a simple application of the *Theorem of the Cube*, in an extended version: *Let $X_1, \dots, X_n$ be complete varieties and $P_1, \dots, P_n$ points on each of them. Set $X := X_1 \times \cdots \times X_n$. For each $j = 1, \dots, n$, let $\phi_j : X_j \to X$ be the map taking $P$ to $(P_1, \dots, P_{j-1}, P, P_{j+1}, \dots, P_n)$. Then*

$$\phi^* := (\phi_1^*, \dots, \phi_n^*) : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X_1) \times \cdots \times \mathrm{Pic}^0(X_n)$$

*is an isomorphism.*

Indeed, consider the map

$$\beta : \mathrm{Pic}^0(X_1) \times \cdots \times \mathrm{Pic}^0(X_n) \to \mathrm{Pic}^0(X),\ (\mathscr{I}_1, ..., \mathscr{I}_n) \mapsto \beta_1^*\mathscr{I}_1 \otimes \cdots \otimes \beta_n^*\mathscr{I}_n,$$

where for each $i = 1, ..., n$,

$$\beta_i : X \to X_i$$

is the projection map. Notice that $\phi^*\beta = 1$ trivially, so we need to proof $\beta\phi^* = 1$ as well.

Let $\mathscr{L}$ be the universal invertible sheaf on $X \times \mathrm{Pic}^0(X)$ rigidified along $P \times \mathrm{Pic}^0(X)$, where $P := (P_1, ..., P_n)$. Thus, $\mathscr{L}|_{P \times \mathrm{Pic}^0(X)}$ and $\mathscr{L}|_{X \times \mathscr{O}_X}$ are trivial. For each $i = 1, ..., n$, let

$$\lambda_i := (\phi_i, 1_{\mathscr{O}_X}) : X_i \times \mathrm{Pic}^0(X) \to X \times \mathrm{Pic}^0(X)$$

and

$$\rho_i : X \times \mathrm{Pic}^0(X) \to X_i \times \mathrm{Pic}^0(X)$$

the projection map. Then, $\beta\phi^*$ is induced by

$$\rho_1^*\lambda_1^*\mathscr{L} \otimes \cdots \otimes \rho_n^*\lambda_n^*\mathscr{L}.$$

Let

$$\mathscr{M} := \rho_1^*\lambda_1^*\mathscr{L} \otimes \cdots \otimes \rho_n^*\lambda_n^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

Then $\mathscr{M}$ is an invertible sheaf on $X \times \mathrm{Pic}^0(X)$ such that

$$\begin{aligned}
\lambda_i^*\mathscr{M} &= \mathscr{M}|_{\{P_1\} \times \cdots \times \{P_{i-1}\} \times X_i \times \{P_{i+1}\} \times \cdots \times \{P_n\} \times \mathrm{Pic}^0(X)} \\
&= \lambda_i^*(\rho_1^*\lambda_1^*\mathscr{L} \otimes \cdots \otimes \rho_n^*\lambda_n^*\mathscr{L} \otimes \mathscr{L}^{-1}) \\
&= \lambda_i^*(\rho_i^*\lambda_i^*\mathscr{L} \otimes \mathscr{L}^{-1}) \\
&= \lambda_i^*(\rho_i^*\lambda_i^*\mathscr{L}) \otimes \lambda_i^*\mathscr{L}^{-1} \\
&= \lambda_i^*\mathscr{L} \otimes \lambda_i^*\mathscr{L}^{-1} \\
&= \mathscr{O}_{X_i \times \mathrm{Pic}^0(X)}
\end{aligned}$$

is trivial for each $i = 1, ..., n$, and $\mathscr{M}|_{X\times\{\mathscr{O}_X\}}$ is trivial. In order to prove $\beta\phi^* = 1$, it is enough to prove that $\mathscr{M}$ is trivial.

In case $n = 1$, there is nothing to prove. In case $n = 2$, it follows from the Theorem of the Cube that $\mathscr{M}$ is trivial. For the general case, we may assume by induction that the restriction of $\mathscr{M}$ to $X_1 \times \cdots \times X_{n-1} \times \{P_n\} \times \mathrm{Pic}^0(X)$ is trivial. Then, applying the Theorem of the Cube to the varieties $X_1 \times \cdots \times X_{n-1}$, $X_n$ and $\mathrm{Pic}^0(X)$ we finish the proof of our statement.

As for the second statement, for each $j = 1, \ldots, n$ consider the translation $\tau_{\mathscr{L}_j} : \bar{J}_j^{d_j} \to \bar{J}_j^{d_j}$, sending $\mathscr{I}$ to $\mathscr{I} \otimes \mathscr{L}_j$. It is enough to prove that the induced map $\tau^*_{\mathscr{L}_j}$ on $\mathrm{Pic}^0(\bar{J}_j^{d_j})$ is the identity. But this follows from Lemma 49. □

**Definition 52.** Let $C$ be a treelike curve whose irreducible components are $C_1, ..., C_n$. Let $\underline{d} := (d_1, ..., d_n) \in \mathbb{Z}$ and

$$\bar{\pi}^{\underline{d}} = (\pi_1^{\underline{d}}, ..., \pi_n^{\underline{d}}) : \bar{J}_C^{\underline{d}} \to \bar{J}_{C_1}^{d_1} \times \cdots \times \bar{J}_{C_n}^{d_n}$$

be the isomorphism given by Corollary 43. Let $A : C \to \bar{J}_C^{\underline{d}}$ be a map.

We say that $A$ is a *decomposable Abel map* if $\bar{\pi}_j^{\underline{d}} A|_{C_i}$ is constant for $i \neq j$ and $A_i := \bar{\pi}_i^{\underline{d}} A|_{C_i}$ is an Abel map for each integer $i = 1, \ldots, n$, that is, there is an invertible sheaf $\mathscr{L}_i$ over $C_i$ such that $A_i$ sends $P$ to $\mathscr{L}_i \otimes \mathfrak{m}_P$ for each $P \in C_i$. We say that $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$ *defines* $A$.

Indeed, for $i \neq j$ let $Y_{i,j}$ be the connected component of $\overline{C - C_j}$ containing $C_i$. Since $C$ is treelike, $Y_{i,j}$ meets $C_j$ at a unique point $N_{i,j}$. Then $\bar{\pi}_j^{\underline{d}} A|_{C_i}$ has constant image $\mathscr{L}_j \otimes \mathscr{O}_{C_j}(-N_{i,j})$.

Conversely, given invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_n$ on $C_1, \ldots, C_n$ of degrees $d_1 + 1, \ldots, d_n + 1$, there is a decomposable Abel map $A : C \to \bar{J}^{\underline{d}}$ defined by $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$: For each $P \in C$ and each $j = 1, \ldots, n$, the map $\bar{\pi}_j^{\underline{d}} A$ sends $P$ to $\mathscr{L}_j \otimes \mathscr{O}_{C_j}(-P)$ if $P \in C_j$, and to $\mathscr{L}_j \otimes \mathscr{O}_{C_j}(-N)$ if $P \notin C_j$, where $N$ is the point of intersection with $C_j$ of the connected component of $\overline{C - C_j}$ containing $P$.

The Abel maps constructed in [CCE] are decomposable, as it follows from [CCE], Lemma 3, p. 46.

**Theorem 53.** *Let $C$ be a treelike curve with irreducible components $C_1, ..., C_n$. Suppose all singularities of $C$ are planar and let $\underline{d} := (d_1, ..., d_n) \in \mathbb{Z}^n$. Let $A : C \to \bar{J}_C^{\underline{d}}$ be the decomposable Abel map defined by $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$, where the $\mathscr{L}_i$ are invertible sheaves of degree $d_i + 1$ on $C_i$. Then the induced map*

$$A^* : \mathrm{Pic}^0(\bar{J}_C^{\underline{d}}) \longrightarrow \mathrm{Pic}^0(C)$$

*is an isomorphism which is independent of the choice of $(\mathscr{L}_1, \ldots, \mathscr{L}_n)$.*

*Proof.* For $i \neq j$, let $N_{i,j}$ be the point of intersection of $C_j$ with the connected component of $\overline{C - C_j}$ containing $C_i$. For each $i = 1, \ldots, n$, let $A_i := \pi_i^{\underline{d}} A|_{C_i}$ and define $\iota_i : \bar{J}^{d_i}_{C_i} \to \bar{J}^{\underline{d}}_{C}$ as the composition of the inverse of $\bar{\pi}^{\underline{d}}$ with the map given by

$$\mathscr{I} \mapsto (\mathscr{L}_1(-N_{i,1}), \ldots, \mathscr{L}_{i-1}(-N_{i,i-1}), \mathscr{I}, \mathscr{L}_{i+1}(-N_{i,i+1}), \ldots, \mathscr{L}_n(-N_{i,n})),$$

where $\mathscr{L}_j(-N_{i,j}) := \mathscr{L}_j \otimes \mathscr{O}_{C_j}(-N_{i,j})$ for $j \neq i$. Then, for each $i = 1, ..., n$,

$$\iota_i A_i = A|_{C_i},$$

that is, we have the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{A} & \bar{J}^{\underline{d}}_{C} \\ {\scriptstyle l_i}\uparrow & & \uparrow{\scriptstyle \iota_i} \\ C_i & \xrightarrow{A_i} & \bar{J}^{d_i}_{C_i} \end{array},$$

where $l_i : C_i \to C$ is the inclusion map. Now, by taking the pullback of each map in this diagram, for each $i = 1, ..., n$ we have the diagram below

$$\begin{array}{ccc} \mathrm{Pic}^0(\bar{J}^{\underline{d}}_{C}) & \xrightarrow{A^*} & \mathrm{Pic}^0(C) \\ {\scriptstyle \iota_i^*}\downarrow & & \downarrow{\scriptstyle l_i^*} \\ \mathrm{Pic}^0(\bar{J}^{d_i}_{C_i}) & \xrightarrow{A_i^*} & \mathrm{Pic}^0(C_i) \end{array}$$

where $A_i^* : \mathrm{Pic}^0(\bar{J}^{d_i}_{C_i}) \to \mathrm{Pic}^0(C_i)$ is an isomorphism, by [A07], Thm. C, which is independent of the choice of $\mathscr{L}_i$, by [EGK], Prop. 3.7, p. 605. By combining these diagrams, we obtain the following commutative diagram of maps

$$\begin{array}{ccc} \mathrm{Pic}^0(\bar{J}^{\underline{d}}_{C}) & \xrightarrow{A^*} & \mathrm{Pic}^0(C) \\ {\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle \pi^{\underline{0}}} \\ \mathrm{Pic}^0(\bar{J}^{d_1}_{C_1}) \times \cdots \times \mathrm{Pic}^0(\bar{J}^{d_n}_{C_n}) & \xrightarrow{(A_1^*, \ldots, A_n^*)} & \mathrm{Pic}^0(C_1) \times \cdots \times \mathrm{Pic}^0(C_n), \end{array}$$

where $\iota^* := (\iota_1^*, \ldots, \iota_n^*)$ and $\pi^{\underline{0}} = (l_1^*, ..., l_n^*)$.

It follows from Corollary 43 that $\pi^{\underline{0}}$ is an isomorphism because $C$ is tree-like. Furthermore, also $(A_1^*, ..., A_n^*)$ is an isomorphism, which is independent of the choice of $(\mathscr{L}_1, ..., \mathscr{L}_n)$, because each $A_i^*$ is an isomorphism which is independent of the choice of $\mathscr{L}_i$. Finally, that $\iota^*$ is an isomorphism which is independent of the choice of $(\mathscr{L}_1, ..., \mathscr{L}_n)$, it follows from Proposition 51.

Then the commutativity of the diagram above yields that $A^*$ is an isomorphism which is independent of the choice of $(\mathscr{L}_1, ..., \mathscr{L}_n)$.

□

# Chapter 3

# Abel maps

As we have already said in the introduction to this thesis, one of our goals is to construct two Abel maps for stable curves: a degree-1 Abel map and a degree-0 Abel map with smooth base point; both maps having geometric meaning and having Simpson's compactification as target. Moreover, we show that our Abel maps vary continually along families of stable curves.

The construction of the degree-1 Abel map is the positive answer to a question posed by Viviani to Esteves. More precisely, Viviani asked Esteves whether it would be possible to define degree-1 Abel maps on stable curves, in a natural way, by removing the twisters of the twisted degree-1 Abel map constructed by Caporaso and Esteves; see [CE].

It is not difficult to give a geometric meaning to the maps, we construct here. The hard part is to ensure that our Abel maps vary continuously in families of stable curves. To do this we use continuous polarizations, that is, vector bundles on families of stable curves with constant slope. This chapter is dedicated to construct our Abel maps, at first, only for single stable curves, not for families of stable curves. We consider extending the construction of the Abel maps to families in the next chapter.

Before we begin the construction of our Abel maps, we define the Caporaso–Esteves twisted degree-1 Abel map and we would like to explain why, in our view, one can not define, in a natural way, degree-1 Abel map without twisters having as target Esteves' $\bar{J}^1_C$ or Caporaso's $\overline{P}^1_C$.

Let $C$ be a nodal curve of genus $g$ and $\omega$ its dualizing sheaf. For each point $P \in C$, let $\mathfrak{m}_P$ be the ideal sheaf of $P$ on $C$.

For each proper subcurve $Y$ of $C$, we let

$$g_Y := 1 - \chi(\mathscr{O}_Y)$$

be the (arithmetic) genus of $Y$.

Recall that for each proper subcurve $Y$ of $C$, $\delta_Y = \#Y \cap Y^c$, where $Y^c = \overline{C - Y}$, and that a subcurve $Y \subseteq C$ is called a tail of $C$ if $\delta_Y = 1$.

Let $X$ be a tail of $C$. As $X$ is a tail of $C$, of course so is $X^c$, and if $\{N\} = X \cap X^c$, then $N$ is a separating node connecting $X$ and $X^c$. In this case we say that $X$ and $X^c$ are the tails attached to $N$, or $N$ generates $X$ and $X^c$.

By Proposition 43, p. 38, for each tail $X$ of $C$ there is a unique invertible sheaf on $C$, up to isomorphism, such that its restrictions to $X$ and $X^c$ are $\mathscr{O}_X(-N)$ and $\mathscr{O}_{X^c}(N)$, where $\{N\} = X \cap X^c$. We call this invertible sheaf a *twister*, and we let $\mathscr{O}_C(X)$ denote it. Moreover, for each sum $\sum a_X X$ with integers coefficients $a_X$ and tails $X$, we let

$$\mathscr{O}_C(\sum a_X X) := \bigotimes \mathscr{O}_C(X)^{\otimes a_X}.$$

Now we define the set of *small* tails of $C$. Let $N \in C$ be a separating node and $X$ and $X^c$ the tails generated by $N$. Thus, we have $g = g_{_X} + g_{_{X^c}}$: Indeed, let

$$\nu_N : C_N \to C$$

be the normalization map of $C$ at $N$. Let

$$\mathscr{O}_C \hookrightarrow \mathscr{O}_{C_N}$$

be the associated map of structure sheaves and

$$0 \to H^0(C, \mathscr{O}_C) \to H^0(C_N, \mathscr{O}_{C_N}) \to k \to H^1(C, \mathscr{O}_C) \to H^1(C_N, \mathscr{O}_{C_N}) \to 0$$

the cohomology sequence associated to it. Let $X' \subset C_N$ (resp. $X^{c\prime} \subset C_N$) be the subcurve which is mapped onto $X$ (resp. $X^c$) by $\nu_N$. From this sequence, we have

$$\begin{aligned} g &= h^1(C, \mathscr{O}_C) \\ &= h^1(C_N, \mathscr{O}_{C_N}) \\ &= h^1(X', \mathscr{O}_{X'}) + h^1(X^{c\prime}, \mathscr{O}_{X^{c\prime}}) \\ &= g_{_{X'}} + g_{_{X^{c\prime}}} \\ &= g_{_X} + g_{_{X^c}}. \end{aligned}$$

If $g_{_X} < g/2$ (resp. $g_{_{X^c}} < g/2$) we say that $X$ (resp. $X^c$ ) is the *small tail* generated by $N$ and denote it by $X_N$. On the oder hand, if $g_{_X} = g_{_{X^c}} = g/2$, we choose any one between $X$ and $X^c$ and denote it by $X_N$. Let $\mathscr{S}\mathscr{T}(C)$ denote the set of small tails of $C$.

For each $Q \in C$, let $\mathscr{N}_Q$ be a sheaf on $C$ defined as follows: If $Q$ is not a separating node, let $\mathscr{N}_Q$ be the ideal sheaf of $Q$ on $C$. If $Q$ is a separating node, let $X$ and $X^c$ be the tails generated by $Q$. Then, it follows from Proposition 43, p. 38, that there is a unique, up to isomorphism, invertible sheaf $\mathscr{I}$ on $C$ such that

$$\mathscr{I}|_X \cong \mathscr{O}_X(-Q) \text{ and } \mathscr{I}|_{X^c} \cong \mathscr{O}_{X^c}.$$

So, let $\mathscr{N}_Q := \mathscr{I}$.

Finally we define the Caporaso–Esteves twisted degree-1 Abel map of $C$:

**Definition 54.** Let $C$ be a stable curve of genus $g \geq 3$. We call the map

$$A : C \to \overline{P^1_C},\ Q \mapsto \mathscr{N}_Q^* \otimes \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C) : X \ni Q} X), \tag{3.1}$$

the *twisted degree*-1 *Abel map of* $C$.

Later, Caporaso, Coelho and Esteves extended the definition of the twisted degree-1 Abel map to Gorenstein curves [CCE], p. 50, obtaining as well the degree-0 Abel map for such curves, [CCE], p. 46.

Now we return to the issues which we would like to clarify: Why, in our view, one can not define, in a natural way, a degree-1 Abel map without twisters having as target Esteves' compactified Jacobian $\bar{J}^1_C$ or Caporaso's $\overline{P}^1_C$? Regarding Esteves' compactification, the answer is easier, it follows from the following three lemmas.

Recall that we say a point $P \in C$ is a *separating* node if $P$ is an ordinary node of $C$ and $C - \{P\}$ is not connected, or equivalently, there is subcurve $C_1 \subseteq C$ such that $\{P\} = C_1 \cap C_1^c$.

**Definition 55.** Let $C$ be a curve and $\mathscr{L}$ a torsion-free rank-1 sheaf on $C$. We say that $\mathscr{L}$ is *decomposable* if there are proper subcurves $C_1, C_2 \subseteq C$ such that $\mathscr{L} \cong \mathscr{L}_{C_1} \oplus \mathscr{L}_{C_2}$.

**Lemma 56.** *Let $C$ be a curve and $\mathscr{L}$ a torsion-free rank-1 sheaf on $C$. Then $\mathscr{L}$ is simple if and only if $\mathscr{L}$ is not decomposable.*

*Proof.* See [E01] Prop. 1, p. 3049. □

**Lemma 57** ([JC], p. 22)**.** *Let $C$ be a curve. For each point $P \in C$, $\mathfrak{m}_P$ is a torsion-free rank-1 simple sheaf on $C$ if and only if $P$ is not a separating node.*

*Proof.* Indeed, let $P \in C$ be a point. Suppose $P$ is a separating node. In this case, let $C_1$ and $C_2$ be subcurves of $C$ such that $\{P\} = C_1 \cap C_2$. Now, consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & \mathscr{O}_C & \longrightarrow & \mathscr{O}_P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & (\mathfrak{m}_P)_{C_1} \oplus (\mathfrak{m}_P)_{C_2} & \longrightarrow & \mathscr{O}_{C_1} \oplus \mathscr{O}_{C_2} & \longrightarrow & \mathscr{O}_P \oplus \mathscr{O}_P & \longrightarrow & 0,
\end{array}
$$

where $\alpha, \beta, \gamma$ are restriction maps modulo torsion. Its horizontal sequences are exact. Since $\{P\} = C_1 \cap C_2$, we have $\mathrm{Coker}(\beta) \cong \mathscr{O}_P \cong \mathrm{Coker}(\gamma)$. Then, by the Snake Lemma, $\mathrm{Coker}(\alpha)$ must be 0, and therefore $\mathfrak{m}_P \cong (\mathfrak{m}_P)_{C_1} \oplus (\mathfrak{m}_P)_{C_2}$, that is, $\mathfrak{m}_P$ is not simple.

On the other hand, suppose $\mathfrak{m}_P$ is not simple. From Lemma 56, p. 43, there are subcurves $C_1'$ and $C_2'$ of $C$ such that $C_1' \cup C_2' = C$ and $\mathfrak{m}_P \cong (\mathfrak{m}_P)_{C_1'} \oplus (\mathfrak{m}_P)_{C_2'}$. Therefore we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & \mathscr{O}_C & \longrightarrow & \mathscr{O}_P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma'} & & \\
0 & \longrightarrow & (\mathfrak{m}_P)_{C_1'} \oplus (\mathfrak{m}_P)_{C_2'} & \longrightarrow & \mathscr{O}_{C_1'} \oplus \mathscr{O}_{C_2'} & \longrightarrow & \mathscr{O}_{P \cap C_1'} \oplus \mathscr{O}_{P \cap C_2'} & \longrightarrow & 0,
\end{array}
$$

where $\alpha'$ is an isomorphism, and $\mathscr{O}_{P \cap C_1'}$ and $\mathscr{O}_{P \cap C_2'}$ are either $\mathscr{O}_P$ or 0, depending on whether $P$ is contained in the subcurve in question or not. We claim $P = C_1' \cap C_2'$ scheme-theoretically. In fact, from the diagram above, we have

$$
\begin{aligned}
\chi(\mathscr{O}_{C_1' \cap C_2'}) &= \chi(\mathrm{Coker}(\beta')) \\
&= \chi(\mathrm{Coker}(\gamma')) \\
&= \chi(\mathscr{O}_{P \cap C_1'} \oplus \mathscr{O}_{P \cap C_2'}) - \chi(\mathscr{O}_P) \\
&= \chi(\mathscr{O}_{P \cap C_1'}) + \chi(\mathscr{O}_{P \cap C_2'}) - \chi(\mathscr{O}_P).
\end{aligned}
$$

Now, $\chi(\mathscr{O}_P) = 1$, and $\chi(\mathscr{O}_{P \cap C_1'})$ and $\chi(\mathscr{O}_{P \cap C_2'})$ are either 0 or 1. However, since $C$ is connected, we have $\chi(\mathscr{O}_{C_1' \cap C_2'}) \geq 1$, forcing $\chi(\mathscr{O}_{P \cap C_1'}) = \chi(\mathscr{O}_{P \cap C_2'}) = 1$. But this implies that $P \in C_1' \cap C_2'$ and $\chi(\mathscr{O}_{C_1' \cap C_2'}) = 1$, hence $P = C_1' \cap C_2'$. □

**Lemma 58.** *Let $C$ be a curve and $P$ a point of $C$. Then $\mathfrak{m}_P$ is a torsion-free rank-1 sheaf of degree* -1 *on $C$. Moreover, $\mathfrak{m}_P$ is invertible if and only if $P$ is a smooth point.*

*Proof.* In fact, since $\mathfrak{m}_P \subset \mathscr{O}_C$ and $\mathfrak{m}_P|_{C-\{P\}} \cong \mathscr{O}_C|_{C-\{P\}}$, we have that $\mathfrak{m}_P$ is torsion-free and rank-1.

On the other hand, from the natural exact sequence

$$0 \to \mathfrak{m}_P \to \mathscr{O}_C \to \mathscr{O}_P \to 0,$$

we have

$$\chi(\mathfrak{m}_P) - \chi(\mathscr{O}_C) = -\chi(\mathscr{O}_P) = -1.$$

So $\deg_C(\mathfrak{m}_P) = \chi(\mathfrak{m}_P) - \chi(\mathscr{O}_C) = -1$.

For the second statement, we recall that $\mathfrak{m}_P/\mathfrak{m}_P^2$ is isomorphic to the cotangent space of $C$ at $P$, so it has dimension 1 if and only if $P$ is a smooth point of $C$. □

Let $\lambda : \bar{J}_C^{-1} \to \bar{J}_C^1$, $[\mathscr{I}] \mapsto [\mathscr{I}^*]$, be the isomorphism given by duality. By this isomorphism and by Lemma 57, p. 43, $\mathfrak{m}_P^*$ is a torsion-free rank-1 simple sheaf on $C$ if and only if $P$ is not a separating node of $C$. Thus, since $\bar{J}_C^1$ parametrizes torsion-free rank-1 simple sheaves on $C$, the map

$$A_C : C \to \bar{J}_C^1,\ P \mapsto \mathfrak{m}_P^*$$

is well defined if and only if $C$ has no separating nodes. This explains because one can not define, in a natural way, the degree-1 Abel map having as target Esteves' compactified Jacobian $\bar{J}_C^1$.

As for Caporaso's compactified Jacobian, we have the following explanation. Suppose that $C$ is a stable curve of genus $g \geq 3$. As we have seen in Chapter 1, Caporaso's compactification parametrizes pairs of objects $[Y, \mathscr{M}]$ where $Y$ is a quasistable model of $C$ and $\mathscr{M}$ is a strictly balanced sheaf on $Y$. Now, consider the following setup.

Let $f : \mathcal{C} \to B$ be a *regular smoothing* of $C$, that is, a flat projective morphism between connected and regular schemes such that: $B = \mathrm{Spec}(R)$, where $R$ is a discrete valuation ring, each geometric fiber of $f$ is a curve, $f$ is smooth over the generic point of $B$, and $C$ is the closed fiber of $f$.

Let $p : \mathcal{C} \times_B \mathcal{C} \to \mathcal{C}$ be one of the two projections. Let $\pi : \mathcal{Y} \to \mathcal{C} \times_B \mathcal{C}$ be the blowup along the diagonal $\Delta \subset \mathcal{C} \times_B \mathcal{C}$. Let $\widehat{\Delta} \subset \mathcal{Y}$ be the proper transform of $\Delta$, which is a Cartier divisor on $\mathcal{Y}$. Then, we have a a family of quasistable curves

$$\rho : \mathcal{Y} \xrightarrow{\pi} \mathcal{C} \times_B \mathcal{C} \xrightarrow{p} \mathcal{C},$$

having stabilization $p : \mathcal{C} \times_B \mathcal{C} \to \mathcal{C}$, by [CE], p. 22, and an invertible sheaf $\mathscr{O}_{\mathcal{Y}}(\widehat{\Delta})$ on $\mathcal{Y}$, by [CE], p. 25, satisfying the following properties.

1. $\pi_* \mathscr{O}_{\mathcal{Y}}(\widehat{\Delta}) = \mathcal{I}_\Delta^*$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta$.

2. if $P \in C$ is a smooth point, then $\mathcal{Y}_P = C$ and $\mathscr{O}_C(P) \cong \mathscr{O}_{\mathcal{Y}}(\widehat{\Delta})|_{\mathcal{Y}_P}$, where $\mathcal{Y}_P$ is the fiber of $\rho$ over $P$.

3. if $P \in C$ is a singular point, then $\mathcal{Y}_P = \widehat{C}_P$ and $\mathscr{O}_{\widehat{C}_P}(r) \cong \mathscr{O}_{\mathcal{Y}}(\widehat{\Delta})|_{\mathcal{Y}_P}$, where $r$ is a smooth point of $\widehat{C}_P$ on the exceptional component passing through the two points of $\nu_P^{-1}(P)$, and where $\nu_P : C_P \to C$ is the normalization of $C$ at $P$.

This would be the degree-1 Abel map of $C$, defined in a natural way, having by target Caporaso's compactification

$$A : C \to \overline{P^1_C},\ P \mapsto [\mathcal{Y}_P, \mathscr{O}_{\mathcal{Y}}(\widehat{\Delta})|_{\mathcal{Y}_P}]. \tag{3.2}$$

If $C$ has no separating nodes, it is not hard to prove that it is well defined. However, if $C$ has separating nodes, it follows from [CE], Lemma 4.9 ($i$), p. 19, that if $P \in C$ is a smooth point, the invertible sheaf $\mathscr{O}_C(P)$ is strictly balanced on $C$ if and only if $P$ does not belong to any small tail of $C$. Therefore, the Abel map (3.2) is well defined if and only if $C$ has no separating nodes.

## 3.1 Degree-1 Abel map

In this section we construct degree-1 Abel map for stable curves whose target is Simpson's coarse compactified Jacobian.

Let $C$ be a nodal curve of genus $g$ and $\omega$ its dualizing sheaf. Let $E$ be a vector bundle over $C$. We recall from Chapter 1 that $\mu(E) = \deg(E)/\mathrm{rk}(E)$ is the slope of $E$, where $\mathrm{rk}(E)$ denotes the rank of $E$ and $\deg(E)$ its degree. We say that *$E$ is a polarization of degree $d$* on $C$ if

$$\mu(E) = g - d - 1.$$

For each subcurve $Y$ of $C$ let

$$e_Y := \frac{\deg_Y(\omega)}{2} - \frac{\deg_Y(E)}{\mathrm{rk}(E)}.$$

Let $\mathscr{L}$ be a torsion-free rank-1 sheaf of degree $d$ on $C$.

**Definition 59.** We say that $\mathscr{L}$ is *semistable* with respect to $E$ if $\chi(\mathscr{L}) = -\mu(E)$ and for each proper connected subcurve $Y$ of $C$,

$$\deg_Y(\mathscr{L}) \geq e_Y - \frac{\delta_Y}{2}.$$

**Lemma 60** ([Ca82], Lemma 1.12, p. 61)**.** *Let $C$ be a nodal curve. Let $Y$ be a connected proper subcurve of genus $g_{_Y}$ of $C$. Then*

$$\deg_Y(\omega) = 2g_{_Y} - 2 + \delta_Y.$$

*Proof.* Consider the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \omega \longrightarrow \omega|_{Y^c} \longrightarrow 0\,. \tag{3.3}$$

Claim: $\mathcal{K}$ is the dualizing sheaf of $Y$. In fact, since $\omega$ is the dualizing sheaf of $C$, there is a trace morphism $t : \mathrm{H}^1(C, \omega) \to k$. Composing $t$ with the induced morphism $\mathrm{H}^1(Y, \mathcal{K}) \to \mathrm{H}^1(C, \omega)$, we have a trace morphism $t_Y$ : $\mathrm{H}^1(Y, \mathcal{K}) \to k$ for $\mathcal{K}$.

Let $\mathcal{F}$ be a coherent sheaf on $Y$. In order to prove our Claim, we must show that the composition of the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \times \mathrm{H}^1(Y, \mathcal{F}) \to \mathrm{H}^1(Y, \mathcal{K})$$

with $t_{_Y}$ gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \to \mathrm{H}^1(Y, \mathcal{F})^*.$$

Indeed, let $i : Y \to C$ be the inclusion map and consider the coherent sheaf $i_*\mathcal{F}$ on $C$. Since $\omega$ is a dualizing sheaf for $C$, we have an isomorphism

$$\mathrm{Hom}(i_*\mathcal{F}, \omega) \to \mathrm{H}^1(C, i_*\mathcal{F})^* = \mathrm{H}^1(Y, \mathcal{F})^*$$

induced by $t$.

From Exact Sequence (3.3), we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{K}) \longrightarrow \mathrm{Hom}(i_*\mathcal{F}, \omega) \longrightarrow \mathrm{Hom}(i_*\mathcal{F}, \omega|_{Y^c})\,.$$

We claim $\mathrm{Hom}(i_*\mathcal{F}, \omega|_{Y^c}) = 0$. Indeed, let $f : i_*\mathcal{F} \to \omega|_{Y^c}$ be a morphism. We notice that $f$ has support on $Y \cap Y^c$ because $i_*\mathcal{F}$ has support on $Y$ and $\omega|_{Y^c}$ has support on $Y^c$. Thus, the image of $f$ is a torsion subsheaf of $\omega|_{Y^c}$. But since $\omega$ is an invertible sheaf, $\omega|_{Y^c}$ is a torsion-free sheaf. Then $\mathrm{Hom}(i_*\mathcal{F}, \omega|_{Y^c}) = 0$. Hence we have an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \to \mathrm{H}^1(Y, \mathcal{F})^*$$

induced by $t_Y$, showing our claim.

Thus, since $\mathcal{K}$ is the dualizing sheaf of $Y$, we have $\deg(\mathcal{K}) = 2g_{_Y} - 2$. Now, consider the following commutative natural diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \omega & \longrightarrow & \omega|_{Y^c} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & \omega|_Y & \longrightarrow & \omega|_Y \oplus \omega|_{Y^c} & \longrightarrow & \omega|_{Y^c} & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are exact. By the Snake Lemma, the cokernel of $\alpha$ is equal that of $\beta$. However, the cokernel of $\beta$ is $\omega|_{Y\cap Y^c}$, so

$$\chi(\mathrm{Coker}(\alpha)) = \chi(\omega|_{Y\cap Y^c}).$$

Since $\alpha$ is an injection,

$$\chi(\mathrm{Coker}(\alpha)) = \chi(\omega|_Y) - \chi(\mathcal{K}) = \deg_Y(\omega) - \deg(\mathcal{K}).$$

On the other hand, since $\omega$ is invertible and $\#Y \cap Y^c$ is finite, we have

$$\chi(\omega|_{Y\cap Y^c}) = \chi(\mathscr{O}_{Y\cap Y^c}) = \delta_Y.$$

But, $\deg(\mathcal{K}) = 2g_{_Y} - 2$, so

$$\deg_Y(\omega) - 2g_{_Y} + 2 = \chi(\mathrm{Coker}(\alpha)) = \delta_Y.$$

□

**Lemma 61.** *Let $C$ be a nodal curve of genus $g \geq 2$ and $\omega$ its dualizing sheaf. Then $C$ is semistable if and only if $\omega$ has nonnegative multidegree, that is,* $\deg_Y(\omega) \geq 0$ *for each subcurve $Y$ of $C$.*

*Proof.* It follows from Lemma 60. □

**Remark 62.** Since $C$ is a nodal curve, it follows from Lemma 60, p. 47, that Esteves' semistability notion is equal to the one defined above. In fact, let $Y$ be a connected proper subcurve of $C$. Then

$$\begin{aligned}
\deg_Y(\mathscr{L}) + \chi(\mathscr{O}_Y) &= \chi(\mathscr{L}_Y) \\
&\geq -\frac{\deg_Y(E)}{\mathrm{rk}(E)} \\
&\Leftrightarrow \deg_Y(\mathscr{L}) \\
&\geq -\chi(\mathscr{O}_Y) - \frac{\deg_Y(E)}{\mathrm{rk}(E)} \\
&= g_{_Y} - 1 - \frac{\deg_Y(E)}{\mathrm{rk}(E)} \\
&= \frac{\deg_Y(\omega)}{2} - \frac{\deg_Y(E)}{\mathrm{rk}(E)} - \frac{\delta_Y}{2},
\end{aligned}$$

where we use $\deg_Y(\omega) = 2g_{_Y} - 2 + \delta_Y$.

**Lemma 63.** *Let $C$ be a nodal curve of genus $g \geq 2$ and $\omega$ its dualizing sheaf. Then there is a vector bundle $E$ over $C$ of the form*

$$\mathscr{O}_C^{\oplus a} \oplus \omega^{\otimes b} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)^{\otimes c}$$

*such that $E$ is a polarization of degree 1 on $C$ and $e_Z = 1/2$ for each tail $Z$ of $C$.*

*Proof.* We separate the proof in two cases:
Case 1) $C$ has no tails;
Case 2) $C$ has tails.
Proof in Case (1). Suppose $C$ has no tails. In this case, let

$$E := \mathscr{O}_C^{\oplus 2g-3} \oplus \omega^{\otimes g-2}$$

and observe that

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = (g-2)\frac{2g-2}{2g-2} = g-2,$$

that is, $E$ is a polarization of degree 1 on $C$.

Proof in Case (2). Say that $C$ has tails. Let

$$E = \mathscr{O}_C^{\oplus a} \oplus \omega^{\otimes b} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)^{\otimes c}$$

be a bundle on $C$ where we will choose the integers $a, b, c$ and $n_X$.

So that $E$ be a polarization of degree 1 on $C$, we need

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = b\frac{2g-2}{a+2} = g-2 \Leftrightarrow b = (a+2)\frac{g-2}{2g-2}.$$

So we let $a = 2g-4$ and $b = g-2$.

Now we choose the $n_X$ for each small tail $X$ of $C$, and $c$ such that

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)^{\otimes c}$$

satisfies $e_Z = 1/2$ for each tail $Z$ of $C$. Indeed, let $Z$ be a tail of $C$. Suppose first that $Z$ is small. Then,

$$\begin{aligned} e_Z &= \frac{\deg_Z(\omega)}{2} + c\frac{n_z}{2g-2} - (g-2)\frac{\deg_Z(\omega)}{2g-2} \\ &= c\frac{n_z}{2g-2} + \frac{\deg_Z(\omega)}{2g-2} \\ &= \frac{1}{2} \\ &\Leftrightarrow\ cn_z + \deg_Z(\omega) = g-1 \\ &\Leftrightarrow\ cn_z + 2g_Z - 1 = g-1 \\ &\Leftrightarrow\ cn_z = g - 2g_z, \end{aligned}$$

where we use Lemma 60, p. 47, for the equality $\deg_Z(\omega) = 2g_z - 1$. Then, we may take $c = 1$ and $n_z = g - 2g_z$. Notice that if $Z$ is not small, then $Z^c$ is small and furthermore, $e_Z + e_{Z^c} = 1$. So $e_Z = 1/2 \Leftrightarrow e_{Z^c} = 1/2$. Thus $e_Z = 1/2$ for every tail $Z$ of $C$.

To summarize, we let

$$E := \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_x)X).$$

□

**Theorem 64.** *Let $C$ be a semistable curve of genus $g \geq 2$, $\omega$ its dualizing sheaf and*

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_x)X)$$

*a vector bundle over $C$. Then for each $Q \in C$, the sheaf $\mathfrak{m}_Q^*$ is semistable with respect to $E$.*

*Proof.* Let $Q$ be a point of $C$. Since $\deg_Y(\mathfrak{m}_Q^*) \geq 0$ for each subcurve $Y$ of $C$, in order to show that $\mathfrak{m}_Q^*$ is semistable with respect to $E$, it is enough to show that

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_x)X))}{2(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0 \qquad (3.4)$$

for each connected proper subcurve $Y$ of $C$. So, let $Y$ be a connected subcurve of $C$. The proof that Inequality (3.4) holds for $Y$ is divided in two cases:

1) $C$ contains no tails.
2) $C$ contains tails.

Proof in Case 1) Assume that $C$ contains no tails. In this case, we have

$$\begin{aligned} E &= \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C \\ &= \mathscr{O}_C^{\oplus 2g-3} \oplus \omega^{\otimes g-2}, \end{aligned}$$

that by Lemma 63, p.49, is a polarization of degree 1 on $C$. So, we need to prove

$$e_Y - \delta_Y/2 = \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0.$$

Indeed, on one hand, since $C$ contains no tails, we have $\delta_Y \geq 2$. But on the other hand, since $C$ is semistable, we have from Lemma 61, p. 48, that $\deg_Y(\omega) \leq 2(g-1)$. Hence, $\deg_Y(\omega) \leq 2(g-1) \leq \delta_Y(g-1)$, which implies

$$\frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0.$$

Thus, we have the proof in Case 1.

Proof in Case 2) We divide the proof of this case in the following subcases:

1. There is no tail among the connected components of $Y^c$.
2. $Y$ is a tail.
3. There are at least 2 tails among the connected components of $Y^c$.
4. There is a unique small tail among the connected components of $Y^c$ and $\delta_Y \geq 2$.
5. There is a unique large tail among the connected components of $Y^c$ and $\delta_Y \geq 2$.

Proof in Subcase 1) Say there is no tail among the connected components of $Y^c$. So, $\delta_Y \geq 2$ and

$$\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} = 0.$$

Since $C$ is semistable, we have $\deg_Y(\omega) \leq 2(g-1)$, and therefore, $\deg_Y(\omega) \leq \delta_Y(g-1)$, that is,

$$\frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0.$$

Proof in Subcase 2) Say that $Y$ is a tail. Then $\delta_Y = 1$ and by Lemma 63, p. 49,

$$e_Y = -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} = 1/2.$$

Therefore,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2r(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} = 0.$$

Proof in Subcase 3) Let $X_1, X_2, ..., X_m$ be the connected components of $Y^c$ that are tails. Then $m \geq 2$. Suppose first that they are small. Then,

$$\begin{aligned}
-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} &= \frac{2g_{X_1} - g + \cdots + 2g_{X_m} - g}{2(g-1)} \\
&= \frac{2(g_{X_1} + g_{X_2} \cdots + g_{X_m}) - mg}{2(g-1)} \\
&\leq 0.
\end{aligned}$$

On the other hand, since $\delta_Y \geq 2$ and $C$ is semistable, $\deg_Y(\omega)/2(g-1) - \delta_Y/2 \leq 0$. Therefore,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0.$$

Now, if one of the tails $X_i$ is not small, the reasoning is similar. For example, suppose that $X_1$ is large. Then $X_1^c$ is small. Furthermore,

$$\begin{aligned}
-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} &= \frac{\overbrace{g - 2g_{X_1^c}}^{2g_{X_1}-g} + 2g_{X_2} - g + \cdots 2g_{X_m} - g}{2(g-1)} \\
&= \frac{2(g_{X_1} + g_{X_2} \cdots + g_{X_m}) - mg}{2(g-1)} \\
&\leq 0.
\end{aligned}$$

Proof in Subcase 4) We have $\delta_Y \geq 2$. Let $Z$ be the connected component of

$Y^c$ that is a tail. Then $Z$ is small. It follows that

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} = \frac{2g_Z-g}{2(g-1)} \leq 0 \text{ and } \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0.$$

Proof in Subcase 5) We have $\delta_Y \geq 2$. Let $Z$ be the connected component of $Y^c$ that is a tail. Then it is large. Notice that in this case,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} = \frac{g-2g_{Z^c}}{2(g-1)} = \frac{2g_Z-g}{2(g-1)} \geq 0,$$

what prevents us from using the same reasoning as before. So, in order to prove that

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \leq 0,$$

we proceed in the following way:

$$\begin{aligned}
e_Y - \delta_Y/2 &= -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)}(g-2g_X)X))}{2(g-1)} + \frac{\deg_Y(\omega)}{2(g-1)} - \frac{\delta_Y}{2} \\
&= \frac{2g_Z-g}{2(g-1)} + \frac{2g_Y-2+\delta_Y}{2(g-1)} - \frac{\delta_Y}{2} \\
&= \frac{(\overbrace{2(g_Z+g_Y)}^{\leq 2g}-g-1)+\delta_Y-1}{2(g-1)} - \frac{\delta_Y}{2} \\
&\leq \frac{g-1}{2(g-1)} + \frac{\delta_Y-1}{2(g-1)} - \frac{\delta_Y}{2} \\
&= \frac{1}{2} + \frac{\delta_Y-1-\delta_Y(g-1)}{2(g-1)} \\
&\leq 0 \\
&\Leftrightarrow \frac{\delta_Y-1-\delta_Y(g-1)}{2(g-1)} \leq -\frac{1}{2} \\
&\Leftrightarrow \delta_Y-1-\delta_Y(g-1) \leq -(g-1) \\
&\Leftrightarrow \delta_Y-1 \leq (\delta_Y-1)(g-1) \\
&\Leftrightarrow 1 \leq g-1 \\
&\Leftrightarrow 2 \leq g
\end{aligned}$$

where we used $\delta_Y \geq 2$.

□

Let $L$ be an ample invertible sheaf on $C$. Recall that a torsion-free rank-1 sheaf $\mathscr{L}$ on $C$ is slope-semistable with respect to $L$ if and only

$$\deg_Y(\mathscr{L}) \geq \frac{\deg_Y(L)}{\deg(L)}\Big(\deg(\mathscr{L}) - \frac{\deg(\omega)}{2}\Big) + \frac{\deg(\omega|_Y)}{2} - \frac{\delta_Y}{2}$$

for each connected proper subcurve $Y$ of $C$.

Now, let $f : \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 2$, $\omega_{\mathcal{C}/T}$ its relative dualizing sheaf and $\mathcal{E}$ a relative polarization of degree $d$ on $f : \mathcal{C} \to T$. Let

$$\bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}/T) : (T\text{-schemes})^o \to \text{Sets}$$

be the contravariant $T$-functor which associates to each $T$-scheme $S$ the set of torsion-free rank-1 sheaves on $p_2 : \mathcal{C} \times_T S \to S$ such that for each geometric point $s \in S$, their fibers over $s$ are semistable with respect to $\mathcal{E}_s$. Let $\det(\mathcal{E})$ be the determinant of $\mathcal{E}$.

Let $\mathcal{L}$ be a relative very ample invertible sheaf on $f : \mathcal{C} \to T$ and assume that $t \mapsto \deg(\mathcal{L}_t)$ is constant. Recall the relative Simpson compactified Jacobian functor of $f : \mathcal{C} \to T$ is the $T$-functor $\bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}/T)$ which associates to each $T$-scheme $S$ the set of relative slope-semistable sheaves on $p_2 : \mathcal{C} \times_T S \to S$ such that for each geometric point $s \in S$, their fibers over $s$ have relative Hilbert polynomial $P_d(z) := \deg(\mathcal{L}_s) \cdot z + d + 1 - g$.

In case $T = \operatorname{Spec}(\mathbb{C})$, we set $\bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}) := \bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}/T)$ and $\bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}) := \bar{\mathbf{J}}_{\mathcal{L},d}(\mathcal{C}/T)$.

**Lemma 65.** *Let $f : \mathcal{C} \to T$ be a family if stable curves. For each integer $m$ sufficiently large, the map*

$$\Psi_m : \bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}/T) \to \bar{\mathbf{J}}_{\det(\mathcal{E}\otimes\omega^{\otimes m}_{\mathcal{C}/T}),d-m\mathrm{rk}(\mathcal{E})}(\mathcal{C}/T)$$

*defined by*

$$(p_2 : \mathcal{C}' \to S, \mathcal{N}) \mapsto (p_2 : \mathcal{C}' \to S, \mathcal{N} \otimes \omega^{\otimes -m}_{\mathcal{C}'/S}),$$

*where $p_2 : \mathcal{C}' = \mathcal{C} \times_T S \to S$ is the second projection, is an isomorphism of functors.*

*Proof.* Let $(p_2 : \mathcal{C} \times_T S \to S, \mathcal{N})$ be an element of $\bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}/T)(S)$. Since $f : \mathcal{C} \to T$ is a family of stable curves, $\omega^{\otimes m}_{\mathcal{C}/T}$ is relatively very ample for $m \geq 3$; see [DM69], Thm. 1.2, p. 77. Thus by taking $m >> 0$, we have also that the invertible sheaf $\det(\mathcal{E} \otimes \omega^{\otimes m}_{\mathcal{C}/T}) = \det(\mathcal{E}) \otimes \omega^{\otimes m\mathrm{rk}(\mathcal{E})}_{\mathcal{C}/T}$ is relatively very ample. Now let $s \in S$ be a geometric point, and put $N := \mathcal{N}_s$, $E := \mathcal{E}_s$ and $\omega := \omega_{\mathcal{C}_s}$ the restrictions to the fiber over $s$. Then $N$ is semistable with respect to $E$ if and only if

$$\deg_Y(N) \geq \frac{\deg_Y(\omega)}{2} - \frac{\deg_Y(E)}{\mathrm{rk}(E)} - \frac{\delta_Y}{2}$$

for each subcurve proper $Y$ of $\mathcal{C}_s$, or equivalently,

$$\begin{aligned}
\deg_Y(N\otimes\omega^{\otimes -m}) &\geq -\frac{\deg_Y(E\otimes\omega^{\otimes m})}{\mathrm{rk}(E)}+\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}\\
&= -\frac{\deg_Y(E\otimes\omega^{\otimes m})}{\mathrm{rk}(E)}\cdot\frac{\deg(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\\
&\quad +\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}\\
&= -\frac{\deg_Y(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\cdot\frac{\deg(E\otimes\omega^{\otimes m})}{\mathrm{rk}(E)}+\frac{\deg_Y(\omega)}{2}\\
&\quad -\frac{\delta_Y}{2}\\
&= \frac{\deg_Y(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\Big(-\frac{\deg(E)}{\mathrm{rk}(E)}-m(2g-2)\Big)\\
&\quad +\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}\\
&= \frac{\deg_Y(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\Big(\chi(N)-m(2g-2)\Big)\\
&\quad +\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}\\
&= \frac{\deg_Y(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\Big(\deg(N)-m(2g-2)+1-g\Big)\\
&\quad +\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}\\
&= \frac{\deg_Y(E\otimes\omega^{\otimes m})}{\deg(E\otimes\omega^{\otimes m})}\Big(\deg(N\otimes\omega^{\otimes -m})+\frac{\deg(\omega)}{2}\Big)\\
&\quad +\frac{\deg_Y(\omega)}{2}-\frac{\delta_Y}{2}
\end{aligned}$$

for each connected proper subcurve $Y$ of $\mathcal{C}_s$. Since

$$\deg_Y(E\otimes\omega^{\otimes m})=\deg_Y(\det(E\otimes\omega^{\otimes m}))$$

for each subcurve $Y$ of $\mathcal{C}_s$, including $Y=\mathcal{C}_s$, we have that $N$ is semistable with respect to $E$ if and only if $N\otimes\omega^{\otimes -m}$ is slope-semistable with respect to $\det(E\otimes\omega^{\otimes m})$. Therefore, we have the result.

□

**Remark 66.** Up to taking $m>>0$ we may assume that $\det(\mathcal{E})$ is a relatively very ample invertible sheaf on $\mathcal{C}/T$ and, consequently, that $\Psi_0:\bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C})\to$

$\bar{\mathbf{J}}_{\det(\mathcal{E}),d}(\mathcal{C}/T)$ is an isomorphism. This implies $\bar{\mathbf{J}}^d_{\mathcal{E}}(\mathcal{C}/T)$ to be coarsely represented by the relative Simpson's compactified Jacobian $\bar{J}_{\det(\mathcal{E}),d}(\mathcal{C}/T)$, if the scheme $T$ is defined over Spec($\mathbb{C}$); see Chapter 1, Subsection 1.4.

**Definition 67.** Let $C$ be a stable curve of genus $g \geq 2$ over Spec($\mathbb{C}$). View the second projection $p_2 : C \times C \to C$ as a family of stable curves. Let $\mathcal{I} := \mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset C \times C$ and $\mathcal{I}^*$ its dual sheaf on $C \times C$.

For each point $P \in C$, the fiber $\mathcal{I}_P = \mathfrak{m}_P^*$ is semistable with respect to

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_X)X)$$

by Theorem 64, p. 50. Therefore by Remark 66, the pair $(C \times C/C,\ \mathcal{I}^*)$ defines a map

$$A : C \to \bar{J}_{\det(E),1}(C),\ P \mapsto \mathfrak{m}_P^*.$$

We call $A$ the *degree*-1 *Abel map* of $C$.

This degree-1 Abel map "improves" the one by Caporaso and Esteves because we do not need to use twisters to define it.

## 3.2 Degree-0 Abel map

Let $C$ be a stable curve of genus $g \geq 2$ over Spec($\mathbb{C}$). In the last section, given the polarization of degree 1 on $C$

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_X)X),$$

we constructed degree-1 Abel map of $C$

$$A : C \to \bar{J}_{\det(E),1}(C),\ P \mapsto \mathfrak{m}_P^*.$$

Now, let $P$ be a smooth point of $C$. In this section we construct a polarization $F_P$ of degree 0 on $C$ and degree-0 Abel map of $C$ with base point $P$

$$A_0 : C \to \bar{J}_{\deg(F_P),0}(C),\ Q \mapsto \mathfrak{m}_Q \otimes \mathfrak{m}_P^*.$$

**Lemma 68.** *Let $C$ be a nodal curve of genus $g \geq 2$. Let $P \in C$ be a smooth point. Then there is a bundle $F$ of the form*

$$\mathscr{O}_C^{\oplus a} \oplus \omega^{\otimes b} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)^{\otimes c},$$

*where*

$$n_X = \begin{cases} 1 & if\ P \in X \\ -1 & if\ P \notin X \end{cases}$$

*for each small tail $X$ of $C$, such that $F$ is a polarization of degree $0$ on $C$ with*

$$f_Z = \begin{cases} \frac{1}{2} & if\ P \in Z \\ -\frac{1}{2} & if\ P \notin Z \end{cases}$$

*for each tail $Z$ of $C$.*

*Proof.* We separate the proof in two cases:
Case 1) $C$ contains no tails.
Case 2) $C$ contains tails.

Proof in Case 1). Suppose that $C$ contains no tails. In this case, let

$$F := \mathscr{O}_C \oplus \omega$$

and notice that $\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} = \frac{2g-2}{2} = g-1$, that is, $F$ is a polarization of degree 0 on $C$.

Proof in Case 2). Say that $C$ contains tails. Let

$$F := \mathscr{O}_C^{\oplus a} \oplus \omega^{\otimes b} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)^{\otimes c},$$

where $a, b, c$, and $n_X$ will be chosen integers.

In order that $F$ be a polarization of degree 0 on $C$, we need $\mu(F) = g-1$. Now

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk(F)}} = b\frac{2g-2}{a+2} = g-1 \Leftrightarrow b = \frac{a+2}{2}.$$

So, we let $a = 0$ and $b = 1$. Thus, with these choices for $a$ and $b$, for each subcurve $Y$ of $C$, we have

$$\begin{aligned} f_Y &= \frac{\deg_Y(\omega)}{2} - \frac{\deg_Y(F)}{\mathrm{rk}(F)} \\ &= \frac{\deg_Y(\omega)}{2} - c\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X))}{a+2} - b\frac{\deg_Y(\omega)}{a+2} \\ &= -c\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X))}{a+2} + (a+2-2b)\frac{\deg_Y(\omega)}{2(a+2)} \\ &= -c\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X))}{2}. \end{aligned}$$

Now, let $Z$ be a tail of $C$. Suppose first that $Z$ is small. Then

$$f_Z = -c\frac{\deg_Z(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} = c\frac{n_Z}{2}.$$

So, in order that

$$f_Z = \begin{cases} \frac{1}{2} & \text{if } P \in Z \\ -\frac{1}{2} & \text{if } P \notin Z \end{cases},$$

we let $c = 1$ and

$$n_Z = \begin{cases} 1 & \text{if } \ P \in Z \\ -1 & \text{if } \ P \notin Z \end{cases}.$$

Now if $Z$ is not a small tail, $Z^c$ is a small one. Therefore, since $f_Z = -f_{Z^c}$, we have that

$$f_Z = \begin{cases} \frac{1}{2} & \text{if } \ P \in Z \\ -\frac{1}{2} & \text{if } \ P \notin Z \end{cases} \text{ and } n_Z = \begin{cases} 1 & \text{if } \ P \in Z \\ -1 & \text{if } \ P \notin Z \end{cases}.$$

So in any case we let

$$F = \omega \oplus (\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))$$

where, for each small tail $X$ of $C$, $n_X$=1 if $P \in X$ and $n_X = -1$ otherwise. □

**Theorem 69.** *Let $C$ be a stable curve of genus $g \geq 2$ and $P \in C$ a fixed smooth point. Let*

$$F := \omega \oplus \mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X)$$

*be a polarization of degree 0 on $C$, where for each small tail $X$ of $C$, $n_X = 1$ if $P \in X$ and $n_X = -1$ if $P \notin X$. Then the sheaf $\mathfrak{m}_Q \otimes \mathscr{O}_C(P)$ is semistable with respect to $F$ for each point $Q \in C$.*

*Proof.* Let $Q$ be a point of $C$. In order to show that $\mathfrak{m}_Q \otimes \mathscr{O}_C(P)$ is semistable with respect to $F$, we need to show

$$\deg_Y(\mathfrak{m}_Q \otimes \mathscr{O}_C(P)) \geq -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} \tag{3.5}$$

for each connected proper subcurve $Y \subseteq C$.

However, since

$$\deg_Y(\mathfrak{m}_Q \otimes \mathscr{O}_C(P)) \geq \begin{cases} 0 & \text{if } P \in Y \\ -1 & \text{if } P \notin Y \end{cases}$$

for each proper subcurve $Y$ of $C$, in order to show (3.5) it is enough to show that

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} \leq \begin{cases} 0 & \text{if } P \in Y \\ -1 & \text{if } P \notin Y \end{cases} \tag{3.6}$$

for each connected proper subcurve $Y$ of $C$. So, let $Y$ be a connected proper subcurve of $C$. The proof that Inequality (3.6) holds for $Y$ will be divided in two cases:
1) $C$ contains no tails;
2) $C$ contains tails.

Proof in Case 1) Assume that $C$ contains no tails. In this case, we have

$$\begin{aligned} F &= \omega \oplus \mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X) \\ &= \omega \oplus \mathscr{O}_C, \end{aligned}$$

which by Lemma 68, p. 56, is a polarization of degree 0 on $C$. Thus it is enough to prove that

$$f_Y - \delta_Y/2 = -\frac{\delta_Y}{2} \leq -1.$$

Indeed, since $C$ contains no tails, we have $\delta_Y \geq 2$.

Proof in Case 2) Assume that $C$ contains tails. We divide the proof of this case in two Subcases:

1. There are no tails among the connected components of $Y^c$.
2. There are tails among the connected components of $Y^c$.

Proof in Subcase 1) In this situation, we have

$$\frac{\deg_Y((\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X)))}{2g-2} = \frac{\deg_Y(\mathscr{O}_C)}{2} = 0 \text{ and } \delta_Y \geq 2.$$

Then,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} \leq -1.$$

Proof in Subcase 2). Let $X_1, ..., X_m$ be the tails among the connected components of $Y^c$. Suppose first that all of them are small. Then,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} = \frac{-n_{X_1} - n_{X_2} - \cdots - n_{X_m}}{2}.$$

Now, in order to finish the proof, we need to analyze some cases with respect to the position of the point $P$.

i) $P \in Y$: In this case, it follows from the hypothesis that $n_{X_i}{=}{-1}$ for each $i = 1, ..., n$, and consequently,

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{-n_{X_1} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m - \delta_Y}{2} \\ &\leq 0 \end{aligned}$$

because $\delta_Y \geq m$.

ii) $P \in X_i$ for some $i$: Without loss of generality we may suppose $i = 1$. Then, $n_{X_1} = 1$ and $n_{X_i} = -1$ for each $i = 2, ..., m$, and thus,

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{-n_{X_1} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m-2}{2} - \frac{\delta_Y}{2} \\ &= -1 + \frac{m - \delta_Y}{2} \\ &\leq -1 \end{aligned}$$

iii) $P \notin Y$ and $P \notin X_i$ for $i = 1, ..., m$: In this case, we notice that $\delta_Y \geq m+2$. Indeed, if $\delta_Y = m$, $P \notin C$, a contradiction. And if $\delta_Y = m + 1$, there would be an extra tail $X_{m+1}$ among the connected components of $Y^c$, again a contradiction. Hence $\delta_Y \geq m + 2$. Since $n_{X_i}{=}{-1}$ for each $i = 1, ..., m$, it follows:

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{-n_{X_1} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m - \delta_Y}{2} \\ &\leq \frac{m - m - 2}{2} \\ &\leq -1. \end{aligned}$$

Now, suppose there are large tails among $X_1, ..., X_m$. Since the curve $C$ is stable, there can be at most one large tail among them. Supposing this tail to be $X_1$, without loss of generality, we have that $X_1^c$ is a small tail. Then,

$$-\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} = \frac{n_{X_1^c} - n_{X_2} - \cdots - n_{X_m}}{2}.$$

Again we need to analyze different cases with respect to the position of the point $P$:

i) $P \in Y$: In this situation, $P \in X_1^c$ and $P \notin X_i$ for each $i = 2, ..., m$. Hence, from the hypothesis, $n_{X_1^c} = 1$ and $n_{X_i} = -1$ for each $i = 2, ..., m$. Then,

$$\frac{n_{X_1^c} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} = \frac{m - \delta_Y}{2} \leq 0.$$

ii) $P \in X_i$ for certain $i = 2, ..., m$: Say $P \in X_2$. In this case, we have $n_{X_1^c} = 1$, $n_{X_2} = 1$ and $n_{X_i} = -1$ for each $i = 2, ..., n$. Hence,

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{n_{X_1^c} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m - 2 - \delta_Y}{2} \\ &= -1 + \frac{m - \delta_Y}{2} \\ &\leq -1. \end{aligned}$$

iii) $P \in X_1$. In this case, $n_{X_1^c} = -1$ and $n_{X_i} = -1$ for each $i = 2, ..., m$. Therefore,

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X\in\mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{n_{X_1^c} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m - 2 - \delta_Y}{2} \\ &= -1 + \frac{m - \delta_Y}{2} \\ &\leq -1. \end{aligned}$$

iv) $P \notin Y$ and $P \notin X_i$ for any $i = 1, ..., m$: In this case, $n_{X_1^c} = 1$ and $n_{X_i} = -1$ for each $i = 2, ..., m$. Furthermore, as in Case (iii), we have $\delta_Y \geq m + 2$. Thus,

$$\begin{aligned} -\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2} &= \frac{n_{X_1^c} - n_{X_2} - \cdots - n_{X_m}}{2} - \frac{\delta_Y}{2} \\ &= \frac{m - \delta_Y}{2} \\ &\leq \frac{m - m - 2}{2} \\ &= -1. \end{aligned}$$

To conclude, we have

$$\deg_Y(\mathfrak{m}_Q \otimes \mathscr{O}_C(P)) \geq -\frac{\deg_Y(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X))}{2} - \frac{\delta_Y}{2}.$$

So, as the point $Q \in C$ and the proper subcurve $Y \subseteq C$ are arbitrary, the result follows.

□

**Definition 70.** Let $C$ be a stable curve of genus $g \geq 2$ over Spec($\mathbb{C}$). Let $\omega$ be its dualizing sheaf and $P$ a fixed smooth point of $C$. Let

$$F_P := \omega \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} n_X X)$$

be a polarization of degree 0 on $C$, where for each small tail $X$ of $C$, $n_X = 1$ if $P \in X$ and $n_X = -1$ if $P \notin X$.

Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset C \times C$, and put

$$\mathcal{I} := \mathcal{I}_\Delta \otimes p_1^* \mathscr{O}_C(P),$$

where $p_1 : C \times C \to C$ is the first projection. Since for each point $Q \in C$, the fiber

$$\mathcal{I}_Q = \mathfrak{m}_Q \otimes \mathscr{O}_C(P)$$

is semistable with respect to the polarization $F_P$ by Theorem 69, it follows from Remark 66, p. 55, that the pair $(p_1 : C \times C \to C,\ \mathcal{I})$ defines a map

$$A_P : C \to \bar{J}_{\det(F_P),0}(C),\ Q \mapsto \mathfrak{m}_Q \otimes \mathscr{O}_C(P).$$

We call $A_P$ the *degree-0 Abel map* of $C$ with base point $P$.

# Chapter 4

# Abel maps and the theta divisor

From now until second order, assume $C$ is smooth. Let $d > 0$ and $r \geq 0$ be integers. The set

$$W_d^r(C) := \{\mathscr{L} \in J_C^d : h^0(C, \mathscr{L}) \geq r + 1\}$$

has an algebraic structure and is called a *Brill–Noether variety*. This variety is closely related to the Abel map in degree $d$ of $C$, that is, to the map

$$A_d : C^d \to J_C^d,\ (P_1, ..., P_d) \mapsto \mathfrak{m}_{P_1}^* \otimes \cdots \otimes \mathfrak{m}_{P_d}^*.$$

This is due to the fact that $W_d^0$ is precisely the image of $A_d$ and, if $r > 0$, $W_d^r(C)$ is exactly the locus in $W_d^0(C)$ where the fiber dimension of the Abel map $A_d$ is at least $r$, by [ACGH], Chap. IV, p. 153.

In this section, we focus on a special Brill–Noether variety. Indeed, assume $g \geq 2$. We call the Brill–Noether variety $W_{g-1}^0(C)$ the *theta divisor* of the curve $C$, and denote it by $\Theta(C)$.

In fact, $\Theta(C)$ is a divisor as $\dim W_{g-1}^0(C) = g - 1$, by [Ca08], Rmk. 1.2.3, p. 1389, and $J_C^{g-1}$ is smooth with dimension $g$.

Many properties of the curve $C$ are encoded in the geometry of $\Theta(C)$, by example, if $g \geq 4$, $C$ is hyperelliptic if and only if

$$\dim\Theta(C)_{\text{sing}} = g - 3,$$

where $\Theta(C)_{\text{sing}}$ is the singular locus of $\Theta(C)$, and nonhyperelliptic if and only if $\dim\Theta(C)_{\text{sing}} = g - 4$; see [ACGH], p. 250.

On the other hand, $\dim\Theta(C)_{\text{sing}}$ is precisely described in terms of special invertible sheaves on $C$ (an invertible sheaf is called "special" if its space of global sections has dimension higher that expected). The Riemann Singularity Theorem, [ACGH], p. 226, states that for each $\mathscr{L} \in \Theta(C)$, the multiplicity of $\Theta(C)$ at $\mathscr{L}$ is equal to $h^0(C, \mathscr{L})$. In particular, we have

$$\Theta(C)_{\text{sing}} = W^1_{g-1}(C) = \{\mathscr{L} \in J^{g-1}_C : h^0(C, \mathscr{L}) \geq 2\}.$$

Another important property of the theta divisor, perhaps the most important, is that the isomorphism class of the polarized Abelian variety

$$(J^{g-1}_C, \Theta(C))$$

uniquely determines the isomorphism class of $C$. It is described by the following theorem.

**Theorem 71** (Torelli Theorem, [ACGH], p. 245)**.** *Let $C$ and $C'$ be two smooth connected curves of genus $g \geq 1$. Then $(J^{g-1}_C, \Theta(C)) \cong (J^{g-1}_{C'}, \Theta(C'))$ if and only if $C \cong C'$.*

So, due to the importance of the theta divisor for smooth curves, the notion was extended to nodal curves as we will see in Definition 80, for which we need some preliminaries.

Let $C$ be a nodal curve, not necessarily connected, of genus $g := 1 - \chi(\mathscr{O}_C)$, and $C_1, ..., C_n$ its irreducible components. Let $\delta$ denote the number of nodes of $C$. Let

$$\nu : C^\nu \to C$$

be the normalization map of $C$. Let

$$\mathscr{O}_C \hookrightarrow \mathscr{O}_{C^\nu}$$

be the associated map of structural sheaves and

$$0 \to H^0(C, \mathscr{O}_C) \to H^0(C^\nu, \mathscr{O}_{C^\nu}) \to k^\delta \to H^1(C, \mathscr{O}_C) \to H^1(C^\nu, \mathscr{O}_{C^\nu}) \to 0$$

the cohomology sequence associated to it. From this sequence, we obtain a formula for the genus $g = 1 + h^1(\mathscr{O}_C) - h^0(\mathscr{O}_C)$ of $C$, to know,

$$g = \sum_{i=1}^{n} \widetilde{g}_{C_i} + \delta - n + 1,$$

where $\widetilde{g}_{C_i} = h^1(C_i^\nu, \mathscr{O}_{C_i^\nu})$ is the genus of $C_i^\nu$, the normalization of $C_i$, for $i = 1, ..., n$.

For each subcurve $Y$ of $C$, let $n_Y$ be the number of components of $Y$ and $\delta_Y^{\text{int}}$ the number of nodes of $Y$. Similarly, we have the following formula for the genus of $Y$:

$$g_Y = \sum_{C_i \subseteq Y} \widetilde{g}_{C_i} + \delta_Y^{\text{int}} - n_Y + 1.$$

**Proposition 72.** *Let $C$ be a connected nodal curve over* $\operatorname{Spec}(\mathbb{C})$ *and* $\mathscr{L}$ *a very ample invertible sheaf on $C$. Then the compactified Jacobian $\bar{J}_{\mathscr{L},g-1}(C)$ does not depend on the choice of $\mathscr{L}$.*

*Proof.* Indeed, a torsion-free rank-1 sheaf $\mathscr{N} \in \bar{J}_{\mathscr{L},g-1}$ if and only if $\deg(\mathscr{N}) = g-1$ and

$$\begin{aligned} \deg_Y(\mathscr{N}) &\geq \frac{\deg_Y(\mathscr{L})}{\deg(\mathscr{L})}(\deg(\mathscr{N}) - g + 1) + \frac{\deg_Y(\omega)}{2} - \frac{\delta_Y}{2} \\ &= \frac{\deg_Y(\omega)}{2} - \frac{\delta_Y}{2} \\ &= g_Y - 1 \end{aligned}$$

for each proper subcurve $Y$ of $C$. □

**Corollary 73.** *Let $T$ be a scheme over* $\operatorname{Spec}(\mathbb{C})$ *and $f : \mathcal{C} \to T$ a family of connected nodal curves of genus $g$. Let $\mathcal{L}$ be a relatively very ample invertible sheaf on $f : \mathcal{C} \to T$. Then the relative compactified Jacobian $\bar{J}_{\mathcal{L},g-1}(\mathcal{C}/T)$ does not depend on the choice of $\mathcal{L}$.*

Due to Corollary 73, given a family $f : \mathcal{C} \to T$ of nodal curves of genus $g$ and any relatively very ample invertible sheaf $\mathcal{L}$, we let $\bar{J}_{g-1}(\mathcal{C}/T)$ denote the relative compactified Jacobian parameterizing torsion-free rank-1 sheaves of degree $g-1$ which are relatively slope-semistable on $f : \mathcal{C} \to T$ with respect to $\mathcal{L}$. If $T = \operatorname{Spec}(\mathbb{C})$, we set $\bar{J}_{g-1}(\mathcal{C}) := \bar{J}_{g-1}(\mathcal{C}/T)$.

**Proposition 74.** *Let $C$ be a nodal curve. A coherent sheaf $\mathscr{L}$ of rank-1 on $C$ is torsion-free if and only if it has the form $\mathscr{L} = \nu_*\mathscr{L}'$, where $\mathscr{L}' = \nu^*\mathscr{L}/\text{torsion}$ is an invertible sheaf on a partial normalization $\nu : C' \to C$. The sheaf $\mathscr{L}$ is not invertible precisely at the nodes $P \in C$ over which $\nu$ is not an isomorphism, and for them $\mathscr{L}_P \cong \mathfrak{m}_P$, the ideal sheaf of $P$.*

*Proof.* See [Ses82]. □

**Example 75.** let $C$ be a nodal curve and $P \in C$ a node. Let $\nu_P : C_P \to C$ be the partial normalization at $P$ and $\nu^{-1}(P) =: \{P_1, P_2\}$. Then, we have

$$\nu_{P*}(\mathscr{O}_{C_P}) = \mathfrak{m}_P^* \quad \text{and} \quad \nu_P^*\mathfrak{m}_P^* = \mathscr{O}_{C_P} \oplus k_{P_1} \oplus k_{P_2}.$$

Furthermore,

$$\nu_{P*}(\mathfrak{m}_{P_1} \otimes \mathfrak{m}_{P_2}) = \mathfrak{m}_P \quad \text{and} \quad \nu_P^*\mathfrak{m}_P = (\mathfrak{m}_{P_1} \otimes \mathfrak{m}_{P_2}) \oplus k_{P_1} \oplus k_{P_2},$$

where $k_{P_1}$ and $k_{P_2}$ are the skyscraper sheaves of $P_1$ and $P_2$. In addition, it follows from the projection formula that $\mathfrak{m}_P^* \cong \mathfrak{m}_P \otimes \mathscr{M}$, when $\mathscr{M}$ is any invertible sheaf on $C$ such that $\nu_P^*\mathscr{M} = \mathfrak{m}_{P_1} \otimes \mathfrak{m}_{P_2}$.

**Proposition 76.** *Let $C$ be a nodal curve and $\mathscr{L}$ a torsion-free rank-1 sheaf on $C$. Let $P_1, ..., P_m$ be the points where $\mathscr{L}$ is not invertible. Then $\mathscr{L}$ can be written in the form*

$$\mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_m} \otimes \mathscr{M}_1 \text{ or } \mathfrak{m}^*_{P_1} \otimes \cdots \otimes \mathfrak{m}^*_{P_m} \otimes \mathscr{M}_2,$$

*for invertible sheaves $\mathscr{M}_1$ and $\mathscr{M}_2$ on $C$.*

*Proof.* Let us reason by induction on $m$. Assume that $m = 1$ and let $\nu : C_{P_1} \to C$ be the normalization at $P_1$. By Proposition 74, p. 65, $\mathscr{L} = \nu_*\mathscr{L}'$, where $\mathscr{L}' = \nu^*\mathscr{L}/\text{torsion}$ is an invertible sheaf on $C_{P_1}$. Let $\nu^{-1}(P_1) =: \{P_1^+, P_1^-\}$. Let $\mathscr{N}_{P_1}$ be an invertible sheaf on $C$ such that $\mathfrak{m}_{P_1^+} \otimes \mathfrak{m}_{P_1^-} \otimes \nu^*(\mathscr{N}_{P_1}) \cong \mathscr{L}'$. Then,

$$\mathscr{L} = \nu_*\mathscr{L}' \cong \nu_*(\mathfrak{m}_{P_1^+} \otimes \mathfrak{m}_{P_1^-} \otimes \nu^*(\mathscr{N}_{P_1})) = \mathfrak{m}_{P_1} \otimes \mathscr{N}_{P_1}.$$

Now assume that the result holds for all $l < m$ and let $\gamma : C_\gamma \to C$ be the normalization at $P_m$. Then $\gamma^*\mathscr{L}/\text{torsion}$ is a torsion-free rank-1 sheaf on $C_\gamma$ which is not invertible at $P_1, ..., P_{m-1}$. Hence, by induction, $\gamma^*\mathscr{L}/\text{torsion}$ can be written in the form

$$\mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_{m-1}} \otimes \mathscr{N}$$

for some invertible sheaf $\mathscr{N}$ on $C_\gamma$. Now, let $\gamma^{-1}(P_m) =: \{P_m^+, P_m^-\}$ and let $\mathscr{M}$ be an invertible sheaf on $C$ such that $\mathfrak{m}_{P_m^+} \otimes \mathfrak{m}_{P_m^-} \otimes \gamma^*\mathscr{M} \cong \mathscr{N}$. Then,

$$\begin{aligned}
\mathscr{L} &= \gamma_*(\gamma^*\mathscr{L}/\text{torsion}) \\
&= \gamma_*(\mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_{m-1}} \otimes \mathscr{N}) \\
&\cong \gamma_*(\mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_{m-1}} \otimes \mathfrak{m}_{P_m^+} \otimes \mathfrak{m}_{P_m^-} \otimes \gamma^*\mathscr{M}) \\
&\cong \mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_{m-1}} \otimes \mathfrak{m}_{P_m} \otimes \mathscr{M}.
\end{aligned}$$

Finally, to prove that $\mathscr{L}$ can be written as well in the form $\mathfrak{m}^*_{P_1} \otimes \cdots \otimes \mathfrak{m}^*_{P_m} \otimes \mathscr{M}$, it is enough to write $\mathscr{L}$ in the form $\mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_m} \otimes \mathscr{M}$, and to notice that for each $i = 1, ..., m$, there is an invertible sheaf $\mathscr{M}_i$ on $C$ such that $\mathfrak{m}_{P_i} \cong \mathfrak{m}^*_{P_i} \otimes \mathscr{M}_i$. □

By Proposition 74, p. 65, to each torsion-free rank-1 sheaf $\mathscr{L}$ on $C$, we can associate a unique partial normalization $\nu : C' \to C$ of $C$ and an unique invertible sheaf $\mathscr{L}'$ on $C'$ such that $\nu_*\mathscr{L}' = \mathscr{L}$. Due to this uniqueness, we define the *generalized multidegree* of $\mathscr{L}$ to be the multidegree $\underline{\deg}(\mathscr{L}')$ of $\mathscr{L}'$ on $C'$. We let $\underline{\deg}(\mathscr{L})$ denote the generalized multidegree of $\mathscr{L}$ on $C$. Notice that it follows from Proposition 76 that

$$\deg(\mathscr{L}) = \deg(\mathscr{L}') + \#D_{\mathscr{L}},$$

where $D_{\mathscr{L}} := \{P \in C : \nu \text{ is not an isomorphism over } P\}$. Indeed, if $\mathscr{L}$ is invertible, there is nothing to prove. On the other hand, if $\mathscr{L}$ is not invertible, by Proposition 76, p. 66, there are an invertible sheaf $\mathscr{M}$ on $C$ and nodes $P_1, ..., P_n$ such that

$$\mathscr{L} = \mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_n} \otimes \mathscr{M}.$$

Let $\nu : C_\nu \to C$ be the normalization at $P_1, ..., P_n$, and for each $i = 1, ..., n$, let $\nu^{-1}(P_i) =: \{P_i^+, P_i^-\}$. Then

$$\mathscr{L}' = \mathfrak{m}_{P_1^+} \otimes \mathfrak{m}_{P_1^-} \otimes \cdots \otimes \mathfrak{m}_{P_n^+} \otimes \mathfrak{m}_{P_n^-} \otimes \nu^* \mathscr{M},$$

from which follows that

$$\begin{aligned} \deg(\mathscr{L}') &= \sum_{i=1}^{n} \deg(\mathfrak{m}_{P_i^+}) + \sum_{i=1}^{n} \deg(\mathfrak{m}_{P_i^-}) + \deg(\mathscr{M}) \\ &= -2n + \deg(\mathscr{M}) \\ &= -n + \deg(\mathscr{L}). \end{aligned}$$

**Lemma 77.** *Let $\mathscr{L}$ be a torsion-free rank-1 sheaf on $C$. Let $\nu : C' \to C$ be the normalization along $D_{\mathscr{L}}$. Let $\mathscr{L}'=\nu^*\mathscr{L}/torsion$. If $\deg(\mathscr{L}) = g - 1$, then $\deg(\mathscr{L}') = g' - 1$, where $g'$ is the genus of $C'$.*

*Proof.* Indeed,

$$\begin{aligned} \deg(\mathscr{L}') &= \deg(\mathscr{L}) - \#D_{\mathscr{L}} \\ &= g - 1 - \#D_{\mathscr{L}} \\ &= \underbrace{(\sum_{i=1}^{n} \widetilde{g}_{C_i} + (\delta - \#D_{\mathscr{L}}) - n + 1)}_{g'} - 1 \\ &= g' - 1. \end{aligned}$$

□

Let $\underline{d} = (d_1, ..., d_n) \in \mathbb{Z}^n$ such that $|\underline{d}| := \sum_{i=1}^{n} d_i = g - 1$. For each subcurve $Y$ of $C$, set $d_Y := \sum_{C_i \subset Y} d_i$.

**Definition 78.** We say that $\underline{d}$ is semistable if for each proper subcurve $Y$ of $C$, we have

$$d_Y \geq g_Y - 1.$$

**Lemma 79.** *Let $C$ be a connected nodal curve and $\mathscr{L}$ a torsion-free rank-1 sheaf on $C$. Then $\mathscr{L}$ is slope-semistable if and only if $\underline{\deg}(\mathscr{L}')$ is semistable on $C'$.*

*Proof.* Let $Y$ be a subcurve of $C$ and $Y' \subset C'$ the subcurve such that $\nu(Y') = Y$. Let $W := (C - Y^c) \cap D_{\mathscr{L}}$. From Proposition 72, p. 65, we have:

$$\begin{aligned}
\deg_Y(\mathscr{L}) \geq g_Y - 1 \;&\Leftrightarrow\; \deg_Y(\mathscr{L}) - \#W \geq g_Y - \#W - 1 \\
&\Leftrightarrow\; \deg_Y(\mathscr{L}) - \#W \geq \sum_{C_i \subset Y} \widetilde{g}_{C_i} + \underbrace{\delta_Y^{\text{int}} - \#W}_{\delta_{Y'}^{\text{int}}} - n_Y \\
&\Leftrightarrow\; \deg_Y(\mathscr{L}) - \#W \geq \Big( \sum_{C_i \subset Y} \widetilde{g}_{C_i} + \delta_{Y'}^{\text{int}} - n_{Y'} + 1 \Big) - 1 \\
&\Leftrightarrow\; \deg_{Y'}(\mathscr{L}') \geq g_{Y'} - 1.
\end{aligned}$$

□

**Definition 80.** Let $C$ be a connected nodal curve over $\mathrm{Spec}(\mathbb{C})$ of genus $g \geq 2$. The theta divisor of $C$ is defined to be

$$\Theta(C) := \{\mathscr{L} \in \bar{J}_{g-1}(C) : h^0(C, \mathscr{L}) > 0\}.$$

This definition is motived, mainly, by the following theorem by Beauville.

**Theorem 81.** *Let $\underline{d} \in \mathbb{Z}^n$ be a $n$-tuple such that $\sum_{i=1}^n d_i = g - 1$. Let $J_C^{\underline{d}}$ be the Jacobian parameterizing invertible sheaves on $C$ with multidegree $\underline{d}$. Then the subset*

$$\{\mathscr{L} \in J_C^{\underline{d}} : h^0(C, \mathscr{L}) > 0\} \subset J_C^{\underline{d}}$$

*is a divisor if and only if $\underline{d}$ is semistable.*

*Proof.* See [Bea77], Thm. 2.1. □

Recall that $C_{\text{sing}}$ denotes the set of singularities of $C$. For each $S \subset C_{\text{sing}}$ let $\nu_S : C_S \to C$ be the partial normalization of $C$ along $S$. Let $\Sigma^{\text{ss}}(C_S)$ be the set of semistable $n$-tuples on $C_S$ and $J_{C_S}^{\underline{d}}$ the Jacobian parameterizing invertible sheaves on $C_S$ with multidegree $\underline{d}$. Then, by Proposition 74, p. 65, and Lemma 79, p. 67, $\Theta(C)$ has the following description

$$\Theta(C) = \bigcup_{\substack{\varnothing \subseteq S \subseteq C_{\text{sing}} \\ \underline{d} \in \Sigma^{\text{ss}}(C_S)}} \{\mathscr{L} \in J_{C_S}^{\underline{d}} : h^0(C_S, \mathscr{L}) > 0\}.$$

Given a family of connected nodal curves $f : \mathcal{C} \to T$, let $\Theta(\mathcal{C}/T)$ be the relative theta divisor of the relative compactified Jacobian $\bar{J}_{g-1}(\mathcal{C}/T)$.

**Theorem 82.** *Let $T$ be a scheme over* $\mathrm{Spec}(\mathbb{C})$ *and $\mathcal{C} \to T$ a family of connected nodal curves of genus $g \geq 2$. Then the relative Cartier divisor*

$$\Theta(\mathcal{C}/T) \subset \bar{J}_{g-1}(\mathcal{C}/T)$$

*is relatively ample.*

*Proof.* See [Ale], Thm. 5.3, p. 13. □

From now until the end of this chapter we show how our degree-1 Abel map relates to the theta divisor.

**Definition 83.** Let $C$ be a nodal curve. We say that $C$ is a curve of *compact type* if all its nodes are separating nodes.

**Proposition 84.** *Let $C$ be a stable curve of compact type over* Spec($\mathbb{C}$) *of genus $g \geq 2$ and $\omega$ its dualizing sheaf. Let $C_1, ..., C_n$ be the irreducible components of $C$. Let $\mathscr{L}$ be a line bundle on $C$ with multidegree*

$$(g_{C_1} - 2 + \delta_{C_1}, ..., g_{C_n} - 2 + \delta_{C_n}).$$

*Then the map*

$$B : C \to \bar{J}_{g-1}(C),\ Q \mapsto \mathfrak{m}_Q^* \otimes \mathscr{L}$$

*is well defined. Furthermore if $\mathscr{L}$ is effective, $B$ factors through $\Theta(C)$.*

*Proof.* Let $Q \in C$ be a point. Before we prove that $\mathfrak{m}_Q^* \otimes \mathscr{L}$ is slope-semistable, of course we need to prove that $\deg(\mathfrak{m}_Q^* \otimes \mathscr{L}) = g - 1$, that is, $\deg(\mathscr{L}) = g - 2$ . Indeed,

$$\begin{aligned}\deg(\mathscr{L}) &= \sum_i (g_{C_i} - 2 + \delta_{C_i}) \\ &= \sum_i g_{C_i} \underbrace{-2n + 2\delta + 2}_{\substack{= 0 \text{ because } C \text{ is} \\ \text{of compact type}}} -2 \\ &= g - 2.\end{aligned}$$

Now, we prove that $\mathfrak{m}_Q \otimes \mathscr{L}$ is slope-semistable, that is,

$$\deg_Y(\mathfrak{m}_Q^* \otimes \mathscr{L}) \geq g_Y - 1$$

for each connected subcurve $Y$ of $C$. Indeed, let $Y \subseteq C$ be a connected proper subcurve of $C$. From Theorem 64, p. 50, we have that $\mathfrak{m}_Q^*$ is semistable with respect to the polarization

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega_C^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_X)X),$$

that is,

$$\begin{aligned}\deg_Y(\mathfrak{m}_Q^*) &\geq (2g_{Y_1} - g + \cdots \\ &\quad \cdots + 2g_{Y_n} - g)/(2g-2) + \deg(\omega|_Y)/(2g-2) - \delta_Y/2 \\ &= (2g_{Y_1} - g + \cdots + 2g_{Y_m} - g + 2g_Y - 2 + \delta_Y)/(2g-2) - \delta_Y/2,\end{aligned}$$

where $Y_1, ..., Y_n$ are the tails among the connected components of $Y^c$. However, since $Y$ is of compact type, we have $m = \delta_Y$ and $g_{Y_1} + \cdots + g_{Y_m} + g_Y = g$. So,

$$\begin{aligned} \deg_Y(\mathfrak{m}_Q^*) &\geq (2g_{Y_1} - g + \cdots + 2g_{Y_{\delta_Y}} - g + 2g_Y - 2 + \delta_Y)/(2g-2) - \delta_Y/2 \\ &= (2g - 2 - \delta_Y g + \delta_Y)/(2g-2) - \delta_Y/2 \\ &= 1 - \delta_Y. \end{aligned}$$

Now, since $\underline{\deg}(\mathscr{L}) = (g_{C_1} - 2 + \delta_{C_1}, ..., g_{C_n} - 2 + \delta_{C_n})$ and $Y$ is of compact type, we have

$$\begin{aligned} \deg_Y(\mathscr{L}) &= \sum_{C_i \subseteq Y} (g_{C_i} - 2 + \delta_{C_i}) \\ &= \sum_{C_i \subseteq Y} (g_{C_i} - 2) + \sum_{C_i \subseteq Y} \delta_{C_i} \\ &= \sum_{C_i \subseteq Y} g_{C_i} - 2n_Y + 2\delta_Y^{\text{int}} + \delta_Y \\ &= \sum_{C_i \subseteq Y} g_{C_i} \underbrace{+2\delta_Y^{\text{int}} - 2n_Y + 2}_{\substack{= 0 \text{ because } Y \text{ is} \\ \text{of compact type}}} -2 + \delta_Y \\ &= g_Y - 2 + \delta_Y. \end{aligned}$$

Thus, we have

$$\deg_Y(\mathfrak{m}_Q^*) \geq 1 - \delta_Y \Leftrightarrow \deg_Y(\mathfrak{m}_Q^* \otimes \mathscr{L}) \geq 1 - \delta_Y + g_Y - 2 + \delta_Y = g_Y - 1,$$

that is, $\mathfrak{m}_Q^* \otimes \mathscr{L}$ is slope-semistable.

Finally, we show that the map

$$B : C \to \bar{J}_{g-1}(C), \; Q \mapsto \mathfrak{m}_Q^* \otimes \mathscr{L}$$

is well defined. Indeed, consider the first projection $p_1 : C \times C \to C$ as a family of curves. Let $\mathcal{I}$ be the ideal sheaf of the diagonal $\Delta \subset C \times C$ and $\mathcal{I}^*$ its dual sheaf. Consider the sheaf $\mathcal{I}^* \otimes p_1^*\mathscr{L}$ on $C \times C$ and notice that the fiber $\mathcal{I} \otimes p_1^*\mathscr{L}|_P = \mathfrak{m}_P^* \otimes \mathscr{L}$ is slope-semistable over each point $P \in C$. Therefore, the pair $(p_2 : C \times C \to C, \; \mathcal{I}^* \otimes p_1^*\mathscr{L})$ defines $B : C \to \bar{J}_{g-1}(C)$.

Now, if $\mathscr{L}$ is effective, in order to see that $B$ factors through $\Theta(C)$, it is enough to notice that for each point $Q \in C$, $\mathfrak{m}_Q^*$ has sections, what implies $\mathfrak{m}_Q^* \otimes \mathscr{L}$ has sections too. It is important to notice that since $C$ is a stable curve, the multidegree of $\mathscr{L}$

$$\underline{\deg}(\mathscr{L}) = (g_{C_1} - 2 + \delta_{C_1}, ..., g_{C_n} - 2 + \delta_{C_n})$$

supports an effective invertible sheaf, as $g_{C_i} - 1 + \delta_{C_i} \geq 0$ for $i = 1, ..., n$.

□

**Definition 85.** Let $C$ be a curve and $E$ be a polarization on $C$. We say that $E$ is a *degenerate* polarization if $e_Y - \delta_Y/2 \in \mathbb{Z}$ for each irreducible component $Y$ of $C$, where $e_Y = \deg_Y(\omega)/2 - \deg_Y(E)/\mathrm{rk}(E)$.

The next example is one of a degenerate polarization of degree 1 for curves of compact type.

**Example 86.** Let $C$ be a curve of compact type whose irreducible components are $C_1, ..., C_n$. Let

$$E = \mathscr{O}_C^{\oplus 2g-4} \oplus \omega^{\otimes g-2} \oplus \mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)} (g - 2g_{_X})X)$$

be a polarization of degree 1 on $C$. Then $E$ is degenerate. In fact, for each $i = 1, ..., n$, we have

$$\begin{aligned} e_{C_i} &= -\frac{\deg(\mathscr{O}_C(\sum_{X \in \mathscr{S}\mathscr{T}(C)}(g-2g_{_X})X)|_{C_i})}{2(g-1)} + \frac{\deg_{C_i}(\omega)}{2(g-1)} \\ &= (2g_{_{Y_1}} - g + \cdots + 2g_{_{Y_n}} - g + 2g_{_{C_i}} - 2 + \delta_{C_i})/(2g-2) \end{aligned}$$

where $Y_1, ..., Y_m$ are the tails among the connected components of $C_i^c$. But since $C$ is a curve of compact type, we have $m = \delta_{C_i}$ and $g_{_{Y_1}} + \cdots + g_{_{Y_m}} + g_{_{C_i}} = g$. Thus,

$$\begin{aligned} e_{C_i} &= (2g_{_{Y_1}} - g + \cdots + 2g_{_{Y_{\delta_{C_i}}}} - g + 2g_{_{C_i}} - 2 + \delta_{C_i})/(2g-2) \\ &= [2(g_{_{Y_1}} + \cdots + g_{_{Y_{\delta_{C_i}}}} + g_{_{C_i}}) - \delta_{C_i} g - 2 + \delta_{C_i}]/(2g-2) \\ &= (2g - 2 + \delta_{C_i} - \delta_{C_i} g)/(2g-2) \\ &= 1 - \delta_{C_i}/2. \end{aligned}$$

Hence,

$$e_{C_i} - \delta_{C_i}/2 = 1 - \delta_{C_i}/2 - \delta_{C_i}/2 = 1 - \delta_{C_i} \in \mathbb{Z}$$

for each $i = 1, ..., n$, that is, $E$ is degenerate.

**Proposition 87.** *Let $C$ be a nodal curve. Then there is at least one degenerate polarization of degree 1 on $C$, such that $\mathfrak{m}_Q^*$ is semistable with respect to it for each point $Q \in C$.*

*Proof.* Let $C_1, ..., C_n$ be the irreducible components of $C$. For each $i = 1, ..., n$, let

$$q_i := 1 - \delta_{C_i}/2 + f_i,$$

where the $f_i$ will be chosen integers.

For each subcurve $Y$ of $C$, let $f_{_Y} := \sum_{C_i\subseteq Y} f_i$ and

$$\begin{aligned} q_{_Y} &:= \sum_{C_i\subseteq Y} q_i \\ &= \sum_{C_i\subseteq Y} (1-\delta_{C_i}/2+f_i) \\ &= n_{_Y} - (\delta_Y + 2\delta_Y^{\text{int}})/2 + f_{_Y} \\ &= n_{_Y} - \delta_Y/2 - \delta_Y^{\text{int}} + f_{_Y} \end{aligned}$$

So, suppose the $f_i$ were chosen. Suppose that we constructed a bundle $E$ over $C$ satisfying the following conditions:

1. $\deg(E)/\mathrm{rk}(E) = g-2$.
2. $e_Y = q_{_Y}$ for each subcurve $Y$ of $C$.
3. $e_Y - \delta_Y/2 \leq 0$, or equivalently $1-\delta_Y/2 \leq e_Y$ for each connected subcurve $Y$ of $C$.

Then $E$ would be a degenerate polarization of degree 1 on $C$, as $e_{C_i} - \delta_{C_i}/2 \in \mathbb{Z}$ for each $i = 1, ..., n$. Moreover, for each point $Q \in C$, $\mathfrak{m}_Q^*$ would be semistable with respect to $E$, because by (iii),

$$\deg_Y(\mathfrak{m}_Q^*) \geq 0 \geq e_Y - \delta_Y/2$$

for each connected proper subcurve $Y$ of $C$.

In order to construct such a degenerate polarization $E$, first we obtain the $f_i$. Indeed, let $F$ be a vector bundle over $C$ such that $\deg_{C_i}(F)/\mathrm{rk}(F) = \widetilde{g}_{C_i} - 1/n$ for each $i = 1, ..., n$, and notice that

$$\begin{aligned} \mu(F) &:= \deg(F)/\mathrm{rk}(F) = \sum_{i=1}^{n}(\widetilde{g}_{C_i} - 1/n) \\ &= \sum_{i=1}^{n}\widetilde{g}_{C_i} - 1 \\ &= \sum_{i=1}^{n}\widetilde{g}_{C_i} + \delta - n + 1 - (\delta - n + 1) - 1 \\ &= g - 1 - (\delta - n + 1), \end{aligned}$$

that is, $F$ is a polarization of degree $\delta - n + 1$ on $C$. Let $\mathscr{N}$ be an invertible sheaf of degree $\delta - n + 1$ on $C$ such that $\mathscr{N}$ is semistable with respect to $F$, that is,

$$\deg_Y(\mathscr{N}) \geq g_{_Y} - 1 - \deg_Y(F)/\mathrm{rk}(F)$$

for each connected subcurve $Y$ of $C$. Let $Y$ be a connected proper subcurve of $C$. Then

$$\begin{aligned}
\deg_Y(\mathscr{N}) &\geq g_Y - 1 - \deg_Y(F)/\mathrm{rk}(F) \\
&= g_Y - 1 - \sum_{C_i \subseteq Y} \widetilde{g}_{C_i} + n_Y/n \\
&= g_Y - \underbrace{(\sum_{C_i \subseteq Y} \widetilde{g}_{C_i} + \delta_Y^{\mathrm{int}} - n_Y + 1)}_{g_Y} + \delta_Y^{\mathrm{int}} - n_Y + n_Y/n \\
&= \delta_Y^{\mathrm{int}} - n_Y + n_Y/n.
\end{aligned}$$

However, since $\deg_Y(\mathscr{N}) \in \mathbb{Z}$ and $0 < n_Y/n < 1$, in fact we have

$$\deg_Y(\mathscr{N}) \geq \delta_Y^{\mathrm{int}} - n_Y + 1.$$

Then, for each $i = 1, ..., n$, let $f_i := \deg_{C_i}(\mathscr{N})$, and let $E$ be a vector bundle on $C$ such that $e_{C_i} = q_{C_i}$ for all $i$. Thus, we have

$$f_Y \geq \delta_Y^{\mathrm{int}} - n_Y + 1, \text{ or equivalently, } 1 - \delta_Y/2 \leq e_Y$$

for each connected proper subcurve $Y$ of $C$, that is, $E$ is a degenerate polarization of degree 1 on $C$ such that for each point $Q \in C$, $\mathfrak{m}_Q^*$ is semistable with respect to $E$. □

Proposition 87 has the following importance: Let $E$ be a degenerate polarization of degree 1 on a nodal curve $C$ of genus $g \geq 2$ such that for each point $Q \in C$, $\mathfrak{m}_Q^*$ is semistable with respect to $E$. Let $\mathscr{L}$ be an invertible sheaf on $C$ such that

$$\underline{\deg}(\mathscr{L}) = (\widetilde{g}_{C_1} - 1 - e_{C_1} + \delta_{C_1}/2, ..., \widetilde{g}_{C_n} - 1 - e_{C_n} + \delta_{C_n}/2)$$

and notice that

$$\begin{aligned}
\deg(\mathscr{L}) &= \sum_{i=1}^{n} (\widetilde{g}_{C_i} - 1 - e_{C_i} + \delta_{C_i}/2) \\
&= \sum_{i=1}^{n} (\widetilde{g}_{C_i} + \delta_{C_i}/2 - 1) - \sum_{i=1}^{n} e_{C_i} \\
&= \sum_{i=1}^{n} \widetilde{g}_{C_i} + \delta - n + \deg(E)/\mathrm{rk}(E) - \deg(\omega)/2 \\
&= g - 1 + g - 2 - (g - 1) = g - 2.
\end{aligned}$$

Let $Q$ be a point of $C$ and $Y$ a connected proper subcurve of $C$. By hypothesis, $\mathfrak{m}_Q^*$ is semistable with respect to $E$. Then, $\deg_Y(\mathfrak{m}_Q^*) \geq e_Y - \delta_Y/2$ or equivalently,

$$\begin{aligned}
\deg_Y(\mathfrak{m}_Q^* \otimes \mathscr{L}) &\geq e_Y - \delta_Y/2 + \sum_{C_i \subseteq Y} (\widetilde{g}_{C_i} - 1 - e_{C_i} + \delta_{C_i}/2) \\
&= e_Y - \delta_Y/2 + \sum_{C_i \subseteq Y} \widetilde{g}_{C_i} - n_{_Y} - e_Y + \delta_Y^{\text{int}} + \delta_Y/2 \\
&= \sum_{C_i \subseteq Y} \widetilde{g}_{C_i} + \delta_Y^{\text{int}} - n_{_Y} + 1 - 1 = g_{_Y} - 1.
\end{aligned}$$

Therefore if $C$ is defined over $\operatorname{Spec}(\mathbb{C})$, the map

$$B : C \to \bar{J}_{g-1}(C),\ Q \mapsto \mathfrak{m}_Q^* \otimes \mathscr{L},$$

is well defined, and factors through the theta divisor of $C$ if $\mathscr{L}$ is effective.

**Corollary 88.** *Let $C$ be a stable curve over* $\operatorname{Spec}(\mathbb{C})$ *of genus $g \geq 2$. Let $E$ and $F$ be degenerate polarizations of degree* 1 *on $C$ such that for each point $P \in C$, $\mathfrak{m}_P^*$ is semistable with respect to each one of them. Then, $\bar{J}_{\det(E),1}(C) \cong \bar{J}_{\det(F),1}(C)$.*

*Proof.* Let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on $C$ such that

$$\underline{\deg}(\mathscr{L}) = (\widetilde{g}_{C_1} - 1 - e_{_{C_1}} + \delta_{C_1}/2, ..., \widetilde{g}_{C_n} - 1 - e_{_{C_n}} + \delta_{C_n}/2)$$

and

$$\underline{\deg}(\mathscr{L}) = (\widetilde{g}_{C_1} - 1 - f_{_{C_1}} + \delta_{C_1}/2, ..., \widetilde{g}_{C_n} - 1 - f_{_{C_n}} + \delta_{C_n}/2).$$

In order to prove that $\bar{J}_{\det(E),1}(C) \cong \bar{J}_{\det(F),1}(C)$, it is enough to notice that the translations

$$T_{\mathscr{L}} : \bar{J}_{\det(E),1}(C) \to \bar{J}_{g-1}(C),\ \mathscr{N} \mapsto \mathscr{N} \otimes \mathscr{L}$$

and

$$T_{\mathscr{M}} : \bar{J}_{\det(F),1}(C) \to \bar{J}_{g-1}(C),\ \mathscr{N} \mapsto \mathscr{N} \otimes \mathscr{M}$$

are isomorphisms. ◻

It follows from Corollary 88, that it matters little the choice of degenerate polarization $E$ with respect which $\mathfrak{m}_P^*$ is semistable for every $Q \in C$.

# Chapter 5

# Abel maps for families of curves

The first purpose of this chapter is to describe a construction of a family of stable curves $v : \mathcal{V} \to V$ called a *versal family of stable curves.* It is versal in the following sense: Given any family of stable curves $f : \mathcal{C} \to T$, there is a cover $\cup_{i \in I} T_i$ by open sets of $T$ such that for each $i \in I$, we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & V. \end{array} \tag{5.1}$$

In [HM98], p. 102 there is a construction of such a family, which will be sketched here.

Our second purpose is to extend our Abel maps to this versal family, and our last is to try to extend our Abel maps to any family of stable curves.

Let $X$ be a projective scheme, with very ample invertible sheaf $\mathscr{O}_X(1)$, and $\mathscr{F}$ a coherent sheaf on $X$. The *Hilbert polynomial* of $\mathscr{F}$ with respect to $\mathscr{O}_X(1)$ is defined to be the function

$$\chi(\mathscr{F}(\cdot)) : \mathbb{Z} \to \mathbb{Z},\ n \mapsto \chi(\mathscr{F}(n)),$$

where $\chi(\mathscr{F}(n))$ is the Euler characteristic of $\mathscr{F}(n)$. When $\mathscr{F} = \mathscr{O}_X$, it is also called the Hilbert polynomial of $X$. This terminology is justified by the following proposition:

**Proposition 89.** *Let $X$ be a projective scheme, with very ample invertible sheaf $\mathscr{O}_X(1)$, and $\mathscr{F}$ a coherent sheaf on $X$. Then the Hilbert polynomial of $\mathscr{F}$ with respect to $\mathscr{O}_X(1)$ is a polynomial on $n$, of degree equal to* $\dim \operatorname{Supp} \mathscr{F}$. *It can be expressed as an integer combination of binomial polynomials $\binom{n}{r}$. Finally, for $n >> 0$, we have $\chi(\mathscr{F}(n)) = h^0(\mathscr{F}(n))$.*

*Proof.* The first and second statements follow from [Har], p. 230, Exer. 5.2. The last equality follows from Serre's theorem on vanishing of higher cohomology. Finally, the expression in terms of binomials is a general fact about polynomials taking integer values; see [Har], Prop. 7.3, p. 49. □

Let $C$ be a stable curve of genus $g \geq 2$, $\omega$ its dualizing sheaf and $n \geq 3$ an integer. Under these conditions the sheaf $\omega^{\otimes n}$ is very ample; see [DM69], Thm. 1.2, p. 77. Hence we may use $\omega^{\otimes n}$ to embed $C$ in $\mathbb{P}^r$ for some $r$. Indeed, since

$$\deg_{C_i}(\omega^{\otimes 1-n}) = (1-n)(2g_{C_i} - 2 + \delta_{C_i}) < 0,$$

then $\omega^{\otimes 1-n}$ has no sections, that is, $H^0(C, \omega^{\otimes 1-n}) = 0$. Then by the Riemann–Roch Theorem, $h^0(C, \omega^{\otimes n}) = (2n-1)(g-1)$. So, let

$$r := (2n-1)(g-1) - 1 \text{ and } d := 2n(g-1),$$

and choose a basis for $H^0(\mathrm{C}, \omega^{\otimes n})$. Let $\varphi_{\omega^{\otimes n}} : C \to \mathbb{P}^r$ be the map given by this basis. We call $\varphi_{\omega^{\otimes n}}$ a *n-canonical embedding* of $C$.

Let $X$ denote the image of $\varphi := \varphi_{\omega^{\otimes n}}$ and $\mathcal{I}_{X|\mathbb{P}^r}$ its sheaf of ideals in $\mathbb{P}^r$. Then we have a natural exact sequence:

$$0 \to \mathcal{I}_{X|\mathbb{P}^r} \to \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_X \to 0,$$

and therefore the Hilbert polynomial $P(z)$ of $X$ satisfies

$$P(m) = h^0(\mathscr{O}_X(m)) = h^0(\omega^{\otimes mn}) = 2mn(g-1) + 1 - g$$

for $m >> 0$. That is, $P(z) = dz + 1 - g$, depending only on $n$ and $g$. So, $X$ can be represented by a point in the Hilbert scheme parameterizing subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(z) = dz + 1 - g$.

As we have already said, our aim is to construct a versal family of stable curves. But we do more than this, we construct such a versal family where each fiber is a stable curve of genus $g$, $n$-canonically embedded in $\mathbb{P}^r$ and with Hilbert polynomial $P(z) = dz + 1 - g$.

Before this, we introduce the last preliminary. Let $H$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(z) = dz + 1 - g$.

Let $Y \subset \mathbb{P}^r \times H$ be the universal subscheme. For each point $b \in H$, let $Y_b$ be the fiber of $p_2 : Y \to H$ over $b$, where $p_2 : \mathbb{P}^r \times H \to H$ is the second projection.

For each curve $Z \subset \mathbb{P}^r$, let $\mathcal{I}_{Z|\mathbb{P}^r}$ be the ideal sheaf of $Z$ in $\mathbb{P}^r$ and $\mathcal{N}_{Z|\mathbb{P}^r} := Hom(\mathcal{I}_{Z|\mathbb{P}^r}/\mathcal{I}^2_{Z|\mathbb{P}^r}, \mathscr{O}_Z)$ its normal sheaf.

**Lemma 90.** *Let $H'' \subset H$ be an open subset. Assume that for each point $b \in H''$ the fiber $Y_b$ of $p_2 : Y \to H''$ is a stable curve. Then $H''$ is smooth.*

*Proof.* See [Ca94], Lemma 2.2, p. 609. □

Finally we are ready to describe the construction of the versal family of stable curves given in [HM98]. Indeed, by [HM98], Lemma 3.4, p. 102, there is an open subscheme $H' \subseteq H$ parameterizing the points $b \in H$ whose fiber $Y_b$ of $p_2 : Y \to H$ is a nodal curve.

Let $Y' := Y \cap \mathbb{P}^r \times H'$ and let $\omega_{Y'/H'}$ be the relative dualizing sheaf of $p_2 : Y' \to H'$. Since ampleness is an open property, it follows from semicontinuity that there is an open subscheme $H'' \subseteq H'$ parameterizing the points $b \in H''$ such that $\omega_{Y'/H''}$ restricts to an ample sheaf on the fiber $Y_b$.

Let $Y'' := Y \cap \mathbb{P}^r \times H''$. Then, since the relative dualizing sheaf of $p_2 : Y'' \to H''$ is ample, we have that $p_2 : Y'' \to H''$ is a family of stable curves. Furthermore by Lemma 90, $H''$ is smooth.

Now, notice that $p_2 : Y'' \to H''$ is a family embedded in $\mathbb{P}^r \times H''$, but not all its fibers are $n$-canonically embedded. Due to this, we pass to a subscheme of $H''$. Indeed, let $\omega_{Y''/H''}$ be the relative dualizing sheaf of $p_2 : Y'' \to H''$. Let $\mathscr{O}_{Y''}(1)$ be the very ample sheaf given by the embedding in $\mathbb{P}^r \times H''$. The invertible sheaves $\omega^{\otimes n}_{Y''/H''}$ and $\mathscr{O}_{Y''}(1)$ induce a map

$$H'' \to \mathrm{Pic}^d_{Y''/H''} \times_{H''} \mathrm{Pic}^d_{Y''/H''},$$

where $\mathrm{Pic}^d_{Y''/H''}$ is the algebraic space parameterizing invertible sheaves of degree $d$ on the fibers of $p_2 : Y'' \to H''$. Let $V \subseteq H''$ be the inverse image of the diagonal under the above mapping. Notice that $V$ is not necessarily a closed subscheme of $H''$ because $\mathrm{Pic}^d_{Y''/H''}$ may not be separated over $H''$. Let $\mathcal{V} := Y \cap \mathbb{P}^r \times V$, and denote the projection $p_2$ by $v : \mathcal{V} \to V$.

Thus, each $n$-canonically embedded stable curve of genus $g$ is a fiber of $v$. Reciprocally, for each $b \in V$, we have $\mathscr{O}_{Y_b}(1) \cong \omega^{\otimes n}_{Y_b}$, where $Y_b := v^{-1}(b)$ and $\omega_{Y_b}$ is the dualizing sheaf of $Y_b$. So, the inclusion $Y_b \hookrightarrow \mathbb{P}^r$ may be seen as given by $r+1$ global sections of $\omega^{\otimes n}_{Y_b}$. These $r+1$ sections are linearly independent, because $h^1(\mathbb{P}^r, \mathcal{I}_{Y_b|\mathbb{P}^r}(1)) = 0$, and they span $H^0(Y_b, \omega^{\otimes n}_{Y_b})$ since this vector space has dimension $r+1$. So $Y_b$ is a $n$-canonically embedded stable curve of genus $g$.

Now we state that $V$ is nonsingular. Indeed, to show this we apply the infinitesimal criterion for smoothness, [GD] IV4-17.5.4, p. 69. Let $A$ be a local Artinian $k$-algebra with residue field $k$, and $I \subseteq A$ an ideal isomorphic to $k$. Let $B := A/I$ and $S_A := \mathrm{Spec}(A)$. Denote by $S_B \subseteq S_A$ the subscheme given by $I$. Fix a map $S_B \to V$, and let $b \in V$ denote the point in the image.

According to the infinitesimal criterion for smoothness, to show $V$ is smooth, it is enough to show that $S_B \to V$ extends to a map $S_A \to V$.

Indeed, since $V \subseteq H''$ and $H''$ are nonsingular, there is an extension to a map $S_A \to H''$. So, there is a closed subscheme

$$Y_A \subseteq \mathbb{P}^r \times S_A \tag{5.2}$$

whose intersection with $\mathbb{P}^r \times S_B$ is the subscheme $Y_B$ given by the map $S_B \to V$. Let $\omega_A$ be the relative dualizing sheaf of $p_2 : Y_A \to S_A$ and $\omega_B$ its restriction to $Y_B$.

Let $\sigma_0, ..., \sigma_r$ be the pullbacks to $Y_B$ of a basis of the space of global sections of $\mathscr{O}_{\mathbb{P}^r}(1)$. Since $Y_B$ is given by $S_B \to V$, we have that $\omega_B^{\otimes n} \cong \mathscr{O}_{Y_B}(1)$. So we may regard $\sigma_0, ..., \sigma_r$ as global sections of $\omega_B^{\otimes n}$ giving the embedding of $Y_B$ in $\mathbb{P}^r \times S_B$. Now, since the formation of $H^0(Y_A, \omega_A^{\otimes n})$ commutes with base change, the sections $\sigma_i$ lift to global sections $\sigma_i$ of $\omega_A^{\otimes n}$. And since the $\sigma_i$'s give an embedding of $Y_B$ in $\mathbb{P}^r \times S_B$, the $\sigma_i$'s give as well an embedding of $Y_A$ in $\mathbb{P}^r \times S_A$.

The embedding given by the $\sigma_i$'s may not be that given by the inclusion (5.2), so that it may correspond to a different map $S_A \to H''$. This does not matter because this map lifts the given $S_B \to V$, and it is plain from the construction that it factors through $V$. Then, $V$ is nonsingular, as we stated.

Finally, $v : \mathcal{V} \to V$ is a versal family of stable curves. Indeed, let $f : \mathcal{C} \to S$ be a family of stable curves, and denote by $\omega_{\mathcal{C}/S}$ its relative dualizing sheaf. Then $\omega_{\mathcal{C}/S}^{\otimes n}$ gives a closed embedding of $\mathcal{C}$ into

$$\mathbb{P}_S(f_*(\omega_{\mathcal{C}/S}^{\otimes n})) := \text{Proj}(Sym(f_*(\omega_{\mathcal{C}/S}^{\otimes n}))),$$

where

$$Sym(f_*(\omega_{\mathcal{C}/S}^{\otimes n})) = \bigoplus_{m \geq 0} Sym^m(f_*(\omega_{\mathcal{C}/S}^{\otimes n})),$$

with $Sym^m(f_*(\omega_{\mathcal{C}/S}^{\otimes n}))$ denoting the $m$-th symmetric product of $f_*(\omega_{\mathcal{C}/S}^{\otimes n})$, for each $m \geq 0$.

Now, picking an open covering $S_i$ of $S$ such that for each $i$, $f_*(\omega_{\mathcal{C}/S}^{\otimes n})|_{S_i}$ is free, after choosing a basis, we may embed each $\mathcal{C}_i := f^{-1}(S_i)$ in $\mathbb{P}^r \times S_i$, and hence we get an induced map $S_i \to H$. By construction, this map factors through $V$. Then, $p_2 : \mathcal{C}_i \to S_i$ is the base extension of $\mathcal{C} \to V$ under a map $S_i \to V$.

Moreover, it is important to notice that the family $v : \mathcal{V} \to V$ is a family of stable curves of genus $g \geq 2$.

## 5.1 Degree-1 Abel map for the versal family

Let $v : \mathcal{V} \to V$ be the versal family of stable curves constructed above and consider the following setup due to Esteves. Let $\Sigma \subset \mathcal{V}$ be the scheme of singularities of $v$, given by the Fitting ideal of $\Omega^1_{\mathcal{V}/V}$. Since $v : \mathcal{V} \to V$ is a family of nodal curves, $\Sigma$ intersects each fiber in a reduced scheme. Let $\Sigma^1, ..., \Sigma^l$ be the irreducible components of $\Sigma$. If two of them intersected , say over $b \in V$, then $\Sigma$ would not intersect transversely $\mathcal{V}_b$. Actually, from [DM69], p. 82, one can see that $\Sigma$ is nonsingular and of codimension 2 in $\mathcal{V}$. Thus the $\Sigma^i$ are also the connected components of $\Sigma$.

For each $i = 1, ..., l$, let $B^i = v(\Sigma^i)$. Since $v$ is proper, $B^i$ is an irreducible closed subscheme of $V$. Consider the restriction

$$v^i := v|_{\Sigma^i} : \Sigma^i \to B^i.$$

Then $v^i$ is a proper surjection with finite fibers, whence a finite map. Also it follows from the analysis on [DM69], p. 82, that $v^i$ is an immersion. Also, $B^i$ is a Cartier divisor of $V$, because $V$ is smooth.

Now, reorder the $\Sigma^i$ in such way that $\Sigma^1, ..., \Sigma^m$ are the ones that intersect each fiber in a separating node. For each $i = 1, ..., m$, let $Z^i$ and $W^i$ denote the irreducible components of $v^{-1}(B^i)$. They intersect transversely at $\Sigma^i$. Indeed, for each $b \in B^i$, we have that $B^i_b$ and $W^i_b$ are the two tails of $\mathcal{V}_b$ attached to the node in $\Sigma^i_b$. For each $i = 1, ...m,$, we choose $Z^i$ and $W^i$ such that $Z^i$ has relative genus smaller than $g/2$ over $B^i$. If both $Z^i$ and $W^i$ have relative genus $g/2$, just remove $\Sigma^i$ from the list.

For each $i = 1, ..., m$, let $\mathscr{O}_{\mathcal{V}}(Z^i)$ denote the divisor associated to the $Z^i$. Let $\mathscr{O}_{\mathcal{V}}$ and $\omega_{\mathcal{V}}$ be the structure sheaf and the canonical sheaf of $v : \mathcal{V} \to V$. Now consider the vector bundle

$$\mathcal{E} := \mathscr{O}_{\mathcal{V}}^{\oplus 2g-4} \oplus \omega_{\mathcal{V}}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{V}}(Z^1)^{\otimes g-2g_1} \otimes \cdots \otimes \mathscr{O}_{\mathcal{V}}(Z^m)^{\otimes g-2g_m}$$

on $\mathcal{V}$.

Regard the second projection $p_2 : \mathcal{V} \times_V \mathcal{V} \to \mathcal{V}$ as a family of curves. Let $\mathcal{I} := \mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset \mathcal{V} \times_V \mathcal{V}$ and $\mathcal{I}^*$ its dual sheaf on $\mathcal{V} \times_V \mathcal{V}$. Let $y$ be a geometric point of $V$. Then, we have

$$\mathcal{E}_y = \mathscr{O}_{\mathcal{V}_y}^{\oplus 2g-4} \oplus \omega_{\mathcal{V}_y}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{V}_y}\Big(\sum_{Z \in \mathscr{S}\mathscr{T}(\mathcal{V}_y)} (g - 2g_Z)Z\Big),$$

and furthermore, for each point $P \in \mathcal{V}_y$, $\mathcal{I}^*_P = \mathfrak{m}^*_P$ is semistable with respect to $\mathcal{E}_y$ by Theorem 64, p. 50. So, $\mathcal{I}^*$ is a (relative) torsion-free rank-1 sheaf of degree 1 on $p_2 : \mathcal{V} \times_V \mathcal{V} \to \mathcal{V}$ which is semistable with respect to $\mathcal{E}$, or equivalently, slope-semistable with respect to $\det(\mathcal{E})$; see Remark 66, p. 55.

Hence if $V$ defined over Spec($\mathbb{C}$), the pair $(p_2 : \mathcal{V} \times_V \mathcal{V} \to \mathcal{V},\ \mathcal{I}^*)$ defines a map

$$A : \mathcal{V} \to \bar{J}_{\det(\mathcal{E}),1}(\mathcal{V}/V), \tag{5.3}$$

where $\bar{J}_{\det(\mathcal{E}),1}(\mathcal{V}/V)$ is the Simpson's relative compactified Jacobian, parameterizing torsion free rank-1 sheaves of degree 1 on $v : \mathcal{V} \to V$ which are relatively slope-semistable with respect to $\det(\mathcal{E})$. We call $A$ the *degree*-1 *Abel map* of $v : \mathcal{V} \to V$.

This construction of the degree-1 Abel map of $v : \mathcal{V} \to V$ is a particular one. At first we do not see how to use it to construct degree-1 and degree-0 Abel maps for any family of stable curves of genus $g \geq 2$. In the next section we consider to construct them for any family of stable curves of genus $g \geq 2$ using other techniques.

## 5.2 Degree-1 and degree-0 Abel maps for families of stable curves

Let $f : \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 2$. By the versal property of $v : \mathcal{V} \to V$, there is an open cover $\cup_{j\in J} T_j$ of $T$ and the Cartesian diagrams below

$$\begin{array}{ccc} \mathcal{C}_j & \xrightarrow{g_j} & \mathcal{V} \\ \downarrow & & \downarrow \\ T_j & \longrightarrow & V. \end{array} \tag{5.4}$$

The easiest way to construct, by example, the degree-1 Abel of $f : \mathcal{C} \to T$ is to glue the pullbacks, through the $g_j$, of the invertible sheaves $\mathscr{O}_{\mathcal{V}}(Z^i)$ and to get sheaves $\mathscr{O}_{\mathcal{C}}(Z^i)$ on $\mathcal{C}$ for each $i = 1, 2, ..., n$. Done this, we can construct the relative bundle

$$\mathcal{E} = \mathscr{O}_{\mathcal{C}}^{\oplus 2g-4} \oplus \omega_{\mathcal{C}}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{C}}(Z^1)^{\otimes g-2g_1} \otimes \cdots \otimes \mathscr{O}_{\mathcal{C}}(Z^m)^{\otimes g-2g_m}$$

on $f : \mathcal{C} \to T$ and consequently, the degree-1 Abel map $A : \mathcal{C} \to \bar{J}_{\det(\mathcal{E}),1}(\mathcal{C}/T)$ of $f : \mathcal{C} \to T$.

However, we do not know how to do this gluing. Thus, in order to overcome this difficulty to get the sheaves $\mathscr{O}_{\mathcal{C}}(Z^i)$ on $\mathcal{C}$, we use divisors on *stacks* which will be introduced in the next subsection.

### 5.2.1 Stacks

In this subsection we present a simple introduction to the stacks. The reader who is more interested on this subject may see details in [ACG].

Let $T$ be a scheme and consider the category Sch$/T$ of schemes over $T$. In what follows we will mostly consider the case $T = \mathrm{Spec}(\mathbb{C})$.

**Definition 91.** A *groupoid over* $T$ is a pair $\mathcal{M} = (\mathscr{C}_{\mathcal{M}}, p_{\mathcal{M}})$, where $\mathscr{C}_{\mathcal{M}}$ is a category, and $p_{\mathcal{M}} : \mathscr{C}_{\mathcal{M}} \to \mathrm{Sch}/T$ is a functor such that the following conditions hold:

1) Let $f : S' \to S$ be a morphism in Sch$/T$, and let $\eta$ be an object in $\mathscr{C}_{\mathcal{M}}$ such that $p_{\mathcal{M}}(\eta) = S$. Then there are an object $\xi \in \mathscr{C}_{\mathcal{M}}$ and a morphism $\varphi : \xi \to \eta$ in $\mathscr{C}_{\mathcal{M}}$ with $p_{\mathcal{M}}(\varphi) = f$.

2) Each morphism $\varphi : \xi \to \eta$ in $\mathscr{C}_{\mathcal{M}}$ is *Cartesian* in the following sense. Given any other morphism $\varphi' : \xi' \to \eta$ and a morphism $h : p_{\mathcal{M}}(\xi) \to p_{\mathcal{M}}(\xi')$ such that $p_{\mathcal{M}}(\varphi')h = p_{\mathcal{M}}(\varphi)$, there is a unique morphism $\psi : \xi \to \xi'$ such that $p_{\mathcal{M}}(\psi) = h$ and $\varphi'\psi = \varphi$.

For notational simplicity, we refer to the objects of $\mathscr{C}_{\mathcal{M}}$ as the objects of the groupoid $\mathcal{M}$.

**Definition 92.** A *$n$-pointed smooth curve of genus $g$* is a smooth curve $C$ of genus $g$ together with an ordered collection $P_1, ..., P_n$ of distinct smooth points of $C$.

A *$n$-pointed stable curve of genus $g$* is a curve $C$ together with an ordered collection $P_1, ..., P_n$ of distinct smooth points of $C$ such that for each rational component $E$ of $C$, $\#E \cap \overline{C - E} + \#\{P_i : P_i \in E\} \geq 3$.

If $C$ is a $n$-pointed smooth (resp. stable) curve of genus $g$ with ordered smooth points $P_1, ..., P_n$, we let $(C, P_1, .., P_n)$ denote it.

**Example 93.** Let $\mathscr{C}$ be the category where the objects are families $\xi : \mathcal{C} \to T$ of smooth (resp. stable, $n$-pointed) curves of genus $g$ so that a morphism $\varphi : \xi' \to \xi$ between a family $\xi' : \mathcal{C}' \to T'$ and a family $\xi : \mathcal{C} \to T$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} \\ T' & \xrightarrow{f} & T \end{array}$$

which induces an isomorphism $\mathcal{C}' \cong T' \times_T \mathcal{C}$. Define, the functor $p$, which associates to each family $\xi : \mathcal{C} \to T$ its parameter space $T$, that is, $p(\xi) = T$, and to each morphism $\varphi$, $p(\varphi) = f$. It is not hard to see that the pair $(\mathscr{C}, p)$ satisfies properties 1) and 2).

The groupoid of $n$-pointed smooth (resp. stable) curves of genus $g$ is denoted by $\mathcal{M}_{g,n}$ (resp. $\overline{\mathcal{M}}_{g,n}$).

A morphism $\alpha : \mathcal{M} \to \mathcal{M}'$ of *groupoids over* Sch$/T$ is a functor, which we also denote by $\alpha$, $\alpha : \mathscr{C}_{\mathcal{M}} \to \mathscr{C}_{\mathcal{M}'}$ such that, $p_{\mathcal{M}'} = \alpha p_{\mathcal{M}}$.

**Lemma 94.** *A morphism $F : \mathcal{M} \to \mathcal{M}'$ of groupoids over* Sch$/T$ *is an isomorphism if, and only if, for each $S$ in* Sch$/T$*, the induced functor on the fibers $F_S : \mathcal{M}(S) \to \mathcal{M}'(S)$ is an equivalence of categories.*

*Proof.* See [ACG], Lemma 5.1, p. 282. □

**Lemma 95.** *Every contravariant functor $F$ :* Sch $\to$ Sets *can be considered as a groupoid.*

*Proof.* See [ACG], p. 285. □

**Example 96.** Every scheme can be considered as a groupoid. Indeed, by Yoneda's lemma and by Lemma 95, given a scheme $X$, we can identify $X$ with $\mathrm{Hom}(-, X)$, and then consider $X$ as a groupoid.

Let $\mathcal{M}$ be a groupoid over Sch. Given an object $\xi$ in $\mathcal{M}(T)$, we think of $T$ as a groupoid, and we define an induced morphism of groupoids

$$m_\xi : T \to \mathcal{M},$$

by associating to each object in $T(S)$, that is, to each morphism $f : S \to T$, a pullback $f^*(\xi)$ in $\mathcal{M}(S)$. In particular, given a family of $n$-pointed stable curves $\tau := (\mathcal{C} \to T)$ of genus $g$, it induces a morphism

$$T \to \overline{\mathcal{M}}_{g,n},$$

so that for each morphism $f : S \to T$, we have $f^*(\tau) = (\mathcal{C} \times_T S \to S) \in \mathcal{M}(S)$.

**Definition 97.** Let $\mathcal{M} = (\mathscr{C}, p)$, with $p : \mathscr{C} \to Sch$ be a groupoid. Let $\xi$ be an object of $\mathcal{M}(U)$, $T$ a scheme and $f : U \to T$ an étale surjective morphism. Consider the respective projections

$$p_i : U \times_T U \to U,\ p_{ij} : U \times_T U \times_T U \to U \times_T U \text{ and } q_i : U \times_T U \times_T U \to U,$$

so that $p_1 p_{12} = q_1 = p_1 p_{13}$, $p_2 p_{12} = q_2 = p_1 p_{23}$, and $p_2 p_{13} = q_3 = p_2 p_{23}$.

A *descent datum for $\xi$ relative to $f : U \to T$* is an isomorphism $\varphi : p_1^*\xi \to p_2^*\xi$ such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi : q_1^*\xi \to q_3^*\xi$.

We say that a descent datum for $\xi$ relative to $f$ is *effective* if there are an object $\eta \in \mathcal{M}(T)$ and an isomorphism $\psi : f^*(\eta) \to \xi$ such that $\varphi = (p_2^*\psi) \circ (p_1^*\psi)^{-1}$.

**Definition 98.** A *stack* is a groupoid $\mathcal{M} = (\mathscr{C}, p)$ having the following properties.
1) Every (étale) descent datum is effective.
2) Given a scheme $T$ and objects $\xi$ and $\eta$ in $\mathcal{M}(T)$, the functor

$$\begin{array}{rccc} \mathrm{Isom}_T(\xi,\eta) : & Sch_T & \longrightarrow & Sets \\ & S \xrightarrow{f} T & \longmapsto & \{f^*\xi \cong f^*\eta\} \end{array}$$

is a sheaf in the étale topology.

**Theorem 99.** *The groupoid $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are stacks.*

*Proof.* See [ACG], Thm. 7.6, p. 296. □

We say that a morphism of stacks $f : \mathcal{M} \to \mathcal{N}$ is *representable*, if for each scheme $S$ and each morphism $S \to \mathscr{N}$, the fiber product $\mathcal{M} \times_{\mathcal{N}} S$ is a scheme. (Notice that we are identifying $S$ with $\mathrm{Hom}(-, S)$.)

**Definition 100.** A stack $\mathcal{M}$ is called a *Deligne–Mumford stack* if it has the following properties.

1. The diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable, quasi-compact, and separated.

2. There are a scheme $X$ and a representable étale surjective morphism $X \to \mathcal{M}$.

**Theorem 101.** *$\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are Deligne–Mumford stacks.*

*Proof.* See [ACG], Thm. 8.3, p. 300. □

For the sake of notational simplicity, if $\xi$ and $\eta$ are objects of a groupoid $\mathcal{M}$, and $H$ is a morphism from $\xi$ to $\eta$, then we designate the corresponding morphism $p(\xi) \to p(\eta)$ by $h$.

**Definition 102.** A quasi-coherent sheaf $\mathscr{F}$ on a stack $\mathcal{M}$ consists of the following data:

1. A quasi-coherent sheaf $\mathscr{F}_\alpha$ on $S$ for any morphism $\alpha : S \to \mathcal{M}$, where $S$ is a scheme,

2. An isomorphism
$$\rho_H : h^*(\mathscr{F}_\beta) \xrightarrow{\cong} \mathscr{F}_\alpha$$
for each morphism $H : \alpha \to \beta$ of schemes over $\mathcal{M}$, satisfying the cocycle condition:

3) For each pair of morphisms $H_1 : \alpha_1 \to \alpha_2$ and $H_2 : \alpha_2 \to \alpha_3$, where $\alpha_i : S_i \to \mathcal{M}$, $i = 1, 2, 3$, is a scheme over $\mathcal{M}$, the diagram

$$\begin{array}{ccc} h_1^*(h_2^*(\mathscr{F}_{\alpha_3})) & = & (h_2 \circ h_1)^*(\mathscr{F}_{\alpha_3}) \\ {\scriptstyle h_1^*(\rho_{H_2})}\downarrow & & \downarrow{\scriptstyle \rho_{H_2 \circ H_1}} \\ h_1^*(\mathscr{F}_{\alpha_2}) & \xrightarrow{\rho_{H_1}} & \mathscr{F}_{\alpha_1} \end{array}$$

of isomorphisms of sheaves over $S_1$ commutes.

We say that a sheaf $\mathscr{F}$ on $\mathcal{M}$ is *locally free of rank-r* if all the $\mathscr{F}_\alpha$ are locally free of rank-r.

We say that $\mathscr{F}$ is an invertible sheaf on $\mathcal{M}$ if it is locally free of rank-1. When $\mathcal{M}$ is a Deligne–Mumford stack, the isomorphism classes of invertible sheaves form a group under the tensor product operation called the *Picard group* of the stack and it is denoted by $\mathrm{Pic}(\mathcal{M})$.

Given a family of $n$-pointed stable curves $f : \mathcal{C} \to T$ and an extra section $\sigma : T \to \mathcal{C}$ of $f$, there is a way to get one family of $n+1$-pointed stable curves from it, called *stabilization*; see [ACG], p. 129. We are interested in a particular case. Indeed, let $f : \mathcal{C} \to T$ be a family of stable curves. Consider a projection $p : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ as a family of 0-pointed stable curves and the diagonal $\Delta : \mathcal{C} \to \mathcal{C} \times_T \mathcal{C}$ as a section of $p$. As $\Delta$ is not a section through the smooth locus of $p$, we do the following. Let $Q$ be a node of a fiber of $f$, say over $t_0 \in T$. Close to the node $Q$, $\mathcal{C}$ can be analytically represented as the locus with equation $xy = g$, where $g$ is a function on an open neighborhood $V$ of $t_0$ which vanishes at $t_0$, and thus $\mathcal{C} \times_T \mathcal{C}$ can be locally realized as the locus

$$W = \{((x, y, x', y'), s) \in U \times V : xy = g = x'y'\},$$

where $U$ is a neighborhood of the origin in $\mathbb{C}^4$, and $\Delta$ the locus with equations $x = x',\ y = y'$.

Now, replace $W$ with

$$W' = \{((x, y, x', y'), s, [\lambda : \mu]) \in U \times V \times \mathbb{P}^1 : xy = g = x'y', \lambda x' = \mu x, \lambda y = \mu y'\}$$

and $\Delta$ with the section $\Delta$' corresponding to the locus $\lambda$=$\mu$. Thus, the net effect on the fiber at $x' = y' = 0$, $s = s_0 = 0$, is to replace the node $P$ with a $\mathbb{P}^1$, meeting once each of the two branches of the former node, and crossed by $\Delta'$ at a smooth point. Notice that we can do this local constructions for each node of each fiber over $T$.

Stabilization process shows that all this local constructions that we can do fit together and that the result is algebraic. For our particular case, the final result of the process is a family $h : \mathcal{Y} \to \mathcal{C}$ of 1-pointed stable curves with a section $\Delta' : \mathcal{C} \to \mathcal{Y}$ such that for each smooth point $Q \in \mathcal{C}$, the fiber $\mathcal{Y}_Q$ is the 1-pointed stable curve

$$(\mathcal{C}_{f(Q)}, Q),$$

whereas if $Q$ is singular, $\mathcal{Y}_Q$ is the one

$$(\widehat{\mathcal{C}}_{f(Q)}, R_Q),$$

with $\widehat{\mathcal{C}}_{f(Q)} = E \cup \mathcal{C}^Q_{f(Q)}$, where $\mathcal{C}^Q_{f(Q)}$ is the normalization of $\mathcal{C}_{f(Q)}$ at $Q$, $E$ is the exceptional component joining the points of the inverse image of $Q$ by this normalization, and $R_Q$ is a smooth point of $\widehat{\mathcal{C}}_{p(Q)}$ on $E$.

**Definition 103.** Let $X := (C, P_1, ..., P_n)$ be a $n$-pointed stable curve of genus $g$. Let $P \in X$ be a separating node. Let $\nu_P : X_P \to X$ be the normalization of $X$ at $P$. Let $C_1$ and $C_2$ be the two connected components of $X_P$ of genus $a$ and $b$, respectively. Let $A$ and $B$ be the disjoint subsets of $\{P_1, ..., P_n\}$ indexing, respectively, marked points on $C_1$ and on $C_2$. We say that $P$ is a *separating node of type* $\mathcal{P}$, where $\mathcal{P} = \{(a, A), (b, B)\}$. We shall sometimes refer to such a $\mathcal{P}$ as a *bipartition* of $(g, \{P_1, ..., P_n\})$.

**Definition 104.** A graph $\Gamma$ consists of the following data

1. a finite nonempty set $V = V(\Gamma)$ ( the set of vertices);
2. a finite set $L = L(\Gamma)$ (the set of half-edges);
3. an involution $\iota$ of $L$;
4. a partition of $L$ indexed by $V$, that is, the assignment to each $v \in V$ of a, possibly empty, subset $L_v$ of $L$ such that $L = \cup_{v \in V} L_v$ and $L_v \cap L_w = \varnothing$ if $v \neq w$.

We call an *edge* of the graph to a pair of distinct elements of $L$ interchanged by the involution $\iota$.

A fixed point of the involution is called a *leg* of the graph. The set of edges of $\Gamma$ is denoted by $E(\Gamma)$.

A *dual graph* is the datum of a graph together with the assignment of a nonnegative integer weight $g_v$ to each vertex $v$.

The genus of a dual graph $\Gamma$ is defined to be

$$g = \sum_{v \in V(\Gamma)} g_v + 1 - \chi(\Gamma).$$

A graph (or a dual graph) endowed with a one-to-one correspondence between a finite set $P$ and the set of its legs will be said to be *$P$-marked* or *$n$-marked*, where $n$ is its number of legs.

**Definition 105.** Given a nodal curve $C$ and $D$ a finite set of smooth points of $C$, we can associate a dual graph $\Gamma_{(C,D)}$ to $(C, D)$, given by a 4-tuple $(V, E, L, h)$, where $V$ is the set of vertices, $E$ the set of edges, $L$ the set of legs and $h$ a function on the set V with non-negative integer values. We define this 4-tuple as follows

1. to each irreducible component $C_r$ corresponds a vertex $v_{C_r}$;
2. to each node intersecting the components $C_r$ and $C_s$ (where $C_r$ and $C_s$ can coincide) corresponds an edge connecting the vertices $v_{C_r}$ and $v_{C_s}$;
3. to each marked point $P$ on a component $C_r$, a leg $L_{P,C_r}$;
4. we define $h : V \to \mathbb{Z}_{\geq 0}$ to be the function that associates to each vertex $v$ the geometric genus of the corresponding component of $C$.

Fix a $n$-pointed stable curve $C$ and let $\Gamma$ be its dual graph. Let $\mathcal{D}_{\Gamma_C} \subset \overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford stack where for each scheme $T$, an object in $\mathcal{D}_\Gamma(T)$ is the datum of a family $f : \mathcal{C} \to T$ of $n$-pointed stable curves whose fibers have dual graph which are specializations of $\Gamma$. The codimension of $\mathcal{D}_\Gamma$ in $\overline{\mathcal{M}}_{g,n}$ is equal to the number of edges of $\Gamma$; see [ACG], p. 312.

For each dual graph $\Gamma$ of a $n$-pointed stable curve, we call $\mathcal{D}_\Gamma(T)$ a *boundary strata* of $\overline{\mathcal{M}}_{g,n}$. The simplest boundary strata are those of codimension 1, which correspond to the dual graph of $n$-pointed stable curves with a single node and a single component, which we denote by $\Gamma_{irr}$, or to the dual graph of $n$-pointed stable curves with two components and a single node. The latter, are graphs denoted by $\Gamma_\mathcal{P}$ attached to stable bipartitions $\mathcal{P} = \{(a, A), (b, B)\}$ of $(g, \{1, ..., n\})$. These have two vertices, one of genus $a$ and $\#A$ legs, the other of genus $b = g - a$ ands $\#B$ legs, where $\#A + \#B = n$. We set $\mathcal{D}_{irr} := \mathcal{D}_{\Gamma_{irr}}$ and $\mathcal{D}_\mathcal{P} := \mathcal{D}_{\Gamma_\mathcal{P}}$.

**Fact 106** ( [ACG], p. 339)**.** Let $\overline{\mathcal{M}}_{g,n}$ be a Deligne–Mumford stack. For each boundary strata $\mathcal{D}_{irr}$ and $\mathcal{D}_\mathcal{P}$, where $\mathcal{P} = \{(a, A), (g - a, B)\}$, we can associate a class of invertible sheaves $\mathscr{O}(\mathcal{D}_{irr})$ and $\mathscr{O}(\mathcal{D}_\mathcal{P}) =: \mathscr{O}(\mathcal{P}_{(a,A)})$ in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$, respectively.

**Theorem 107.** *Let $f : \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 2$. Consider a projection $p : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ as a family of stable curves, let $\Delta$ :*

*$\mathcal{C} \to \mathcal{C} \times_T \mathcal{C}$ be the diagonal map and consider $\Delta$ as a section of $p$. For each $i = 1, ..., \lfloor g/2 \rfloor$, where $\lfloor g/2 \rfloor$ is the largest integer less or equal to $g/2$, let*

$$\mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(i,\varnothing)}) := h^* \mathscr{O}(\mathcal{P}_{(i,\varnothing)}).$$

*Let $\omega_{\mathcal{C}}$ be the dualizing sheaf of $\mathcal{C}$ and consider the vector bundle*

$$\mathcal{E} = \mathscr{O}_{\mathcal{C}}^{2g-4} \oplus \omega_{\mathcal{C}}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(1,\varnothing)})^{\otimes(g-2)} \otimes \cdots \otimes \mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(\lfloor g/2 \rfloor,\varnothing)})^{\otimes(g-2\lfloor g/2 \rfloor)}$$

*over $\mathcal{C}$. Then for each geometric point $t \in T$, the restriction of $\mathcal{E}$ to the fiber $\mathcal{C}_t$ is the vector bundle*

$$\mathcal{E}_t = \mathscr{O}_{\mathcal{C}_t}^{\oplus 2g-4} \oplus \omega_{\mathcal{C}_t}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{C}_t}\Big(\sum_{X \in \mathscr{S}\mathscr{T}(\mathcal{C}_t)} (g - 2g_X)X\Big).$$

*Proof.* By the stabilization process, we get a family $h : \mathcal{Y} \to \mathcal{C}$ of 1-pointed stable of genus $g$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{h^*} & \overline{\mathcal{M}}_{g,1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P_r} \\ T & \longrightarrow & \overline{M}_g \end{array} \tag{5.5}$$

where $P_r$ is the 1th *projection*; see [ACG], p. 125.

Then, by commutativity of the diagram (5.5), given a geometric point $t \in T$, the restriction of $\mathcal{E}$ to the fiber $\mathcal{C}_t$ is the vector bundle

$$\mathcal{E}_t = \mathscr{O}_{\mathcal{C}_t}^{\oplus 2g-4} \oplus \omega_{\mathcal{C}_t}^{\otimes g-2} \oplus \mathscr{O}_{\mathcal{C}_t}\Big(\sum_{X \in \mathscr{S}\mathscr{T}(\mathcal{C}_t)} (g - 2g_X)X\Big).$$

□

**Definition 108.** Let $f : \mathcal{C} \to T$ and $\mathcal{E}$ be as in Theorem 107. Regard the second projection $p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ as a family of curves, let $\mathcal{I} := \mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset \mathcal{C} \times_T \mathcal{C}$ and $\mathcal{I}^*$ its dual sheaf on $\mathcal{C} \times_T \mathcal{C}$.

Then, for each point $P \in \mathcal{C}_t$, the sheaf $\mathcal{I}_P^* = \mathfrak{m}_P^*$ on $\mathcal{C}_t$ is semistable with respect to $\mathcal{E}_t$ by Theorem 64 p. 50. Therefore, $\mathcal{I}^*$ is a (relative) torsion-free rank-1 sheaf of degree 1 on $p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ which is semistable with respect to $\mathcal{E}$, or equivalently, slope-semistable with respect to $\det(\mathcal{E})$ by Remark 66, p. 55. Hence the pair $(p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C},\ \mathcal{I}^*)$ defines a map

$$A : \mathcal{C} \to \bar{J}_{\det(\mathcal{E}),1}(\mathcal{C}/T), \tag{5.6}$$

where $\bar{J}_{\det(\mathcal{E},1}(\mathcal{C}/T)$ is the Simpson's relative compactified Jacobian, parameterizing slope-semistable sheaves of degree 1 on $f : \mathcal{C} \to T$ with respect to $\det(\mathcal{E})$. We call $A$ the *degree*-1 Abel map of $f : \mathcal{C} \to T$.

**Theorem 109.** *Let $f : \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 2$ such that there is a section $\sigma : T \to \mathcal{C}$ through the smooth locus of $\mathcal{C}$ and let $\mathscr{O}_{\mathcal{C}}(\Sigma)$ be the relative invertible sheaf associated to $\Sigma := \sigma(T)$.*

*Let*

$$\mathcal{F}^{\Sigma} := \omega_{\mathcal{C}} \oplus \mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(1,\varnothing)})^{\otimes n_1} \otimes \cdots \otimes \mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(\lfloor g/2 \rfloor,\varnothing)})^{\otimes n_{\lfloor g/2 \rfloor}}$$

*be a relative vector bundle on $f : \mathcal{C} \to T$, where for each $i = 1, ..., \lfloor g/2 \rfloor$, we have $n_i = 1$ if Supp$(\mathscr{O}_{\mathcal{C}}(\Sigma)) \cap$ Supp$(\mathscr{O}_{\mathcal{C}}(\mathcal{P}_{(i,\varnothing)})) \neq \varnothing$ and $n_i = -1$ otherwise.*

*Let $\mathcal{I}_{\Delta}$ be the ideal sheaf of the diagonal in the product $\mathcal{C} \times_T \mathcal{C}$ and put*

$$\mathcal{I} := \mathcal{I}_{\Delta} \otimes \mathscr{O}_{\mathcal{C}}(\Sigma).$$

*Extend $(f : \mathcal{C} \to T, \sigma)$ over $f : \mathcal{C} \to T$ and obtain a family of curves $p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ with base section $\sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times_T \mathcal{C}$. Then the sheaf $\mathcal{I}$ is a torsion-free rank-1 sheaf of degree 0 on $p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ which is slope-semistable with respect to* $\det(\mathcal{F}^{\Sigma})$.

*Proof.* Indeed, let $t$ be a geometric point of $T$ and let $P := \sigma(t)$. Now notice that for each point $Q \in \mathcal{C}_t$, we have that $\mathcal{I}_Q = \mathfrak{m}_Q \otimes \mathscr{O}(P)$ is a torsion-free rank-1 sheaf of degree 0 on $\mathcal{C}_t$ which is semistable with respect to $\mathcal{F}_t^{\Sigma}$, where

$$\mathcal{F}_t^{\Sigma} = \omega_{\mathcal{C}_t} \oplus \mathscr{O}_{\mathcal{C}_t}(\sum_{Z \in \mathscr{S}\mathscr{T}(\mathcal{C}_t)} n_Z Z),$$

with $n_Z = 1$ if $P \in Z$ and $n_Z = -1$ otherwise; see Theorem 68, p. 58.

Therefore $\mathcal{I}$ is semistable with respect to $\mathcal{F}^{\Sigma}$, or equivalently, slope-semistable with respect to $\det(\mathcal{F}^{\Sigma})$; see Remark 66, p. 55.

□

**Definition 110.** Let $f : \mathcal{C} \to T$, $\sigma : T \to \mathcal{C}$, $\mathcal{F}^{\Sigma}$ and $\mathcal{I}$ be as in Theorem 109. Extend $(f : \mathcal{C} \to T, \sigma)$ over $f : \mathcal{C} \to T$ and obtain a family of curves $p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C}$ with base section $\sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times_T \mathcal{C}$.

According Theorem 109, the sheaf $\mathcal{I}$ is slope-semistable with respect to $\det(\mathcal{F}^{\Sigma})$ and therefore, the pair $(p_2 : \mathcal{C} \times_T \mathcal{C} \to \mathcal{C},\ \mathcal{I})$ defines a map

$$A_{\sigma} : \mathcal{C} \to \bar{J}_{\det(\mathcal{F}^{\Sigma}),0}(\mathcal{C}/T), \tag{5.7}$$

where $\bar{J}_{\det(\mathcal{F}^{\Sigma}),0}(\mathcal{V}/V)$ is the relative compactified Simpson Jacobian parameterizing torsion-free rank-1 sheaves of degree 0 on $f : \mathcal{C} \to T$ which are relatively slope-semistable with respect to $\det(\mathcal{F}^{\Sigma})$.

We call $A_{\sigma}$ the *degree-0 Abel map* of $f : \mathcal{C} \to T$ with base $\sigma$.

**Remark 111.** It is important to notice that the existence of the degree-0 Abel map $A_{\sigma}$ of $f : \mathcal{C} \to T$ is conditioned to the existence of a section $\sigma : T \to \mathcal{C}$ through the smooth locus of $\mathcal{C}$.

# Chapter 6

# Translations for the compactified Jacobian

Let $C$ be a curve and $C_1, ..., C_n$ its irreducible components. Let

$$\underline{q} = (q_1, ..., q_n) \in \mathbb{Q}^n.$$

If $|\underline{q}| := \sum_{i=1}^{n} q_i \in \mathbb{Z}$, we say that $\underline{q}$ is a *numerical polarization of degree* $|\underline{q}|$ on $C$. For each subcurve $Y$ of $C$, we denote $q_Y := \sum_{C_i \subseteq Y} q_i$.

**Definition 112.** Let $\underline{q}$ be a polarization on $C$ and $i \in \{1, ..., n\}$. We say that $\mathscr{L} \in \bar{J}_C^{|\underline{q}|}$ is *q-i-quasistable* if

$$\deg_Y(\mathscr{L}) \geq q_Y - \delta_Y/2$$

for each connected proper subcurve $Y$ of $C$, with strict inequality if $Y$ contains $C_i$.

**Lemma 113.** *Let $\underline{q}$ be a polarization on $C$, $i \in \{1, ..., n\}$ and $P \in C_i$ a smooth point of $C$. Then there is a polarization $E$ of degree $|\underline{q}|$ on $C$ such that each torsion-free rank-1 simple sheaf $\mathscr{L}$ on $C$ is $\underline{q}$-i-quasistable if and only if it is $P$-quasistable with respect to $E$.*

*Proof.* For each $j = 1, ..., n$, let $a_j, b_j \in \mathbb{Z}$ such that $q_j = a_j/b_j$. For each $j = 1, ..., n$, let $P_j \in C_j$ be a smooth point of $C$. Put $m := b_1 b_2 ... b_n$.

Let

$$E := \mathscr{O}_C^{\oplus 2m-1} \oplus \mathscr{O}_C\Big(\sum_{j=1}^{n} (m\deg(\omega|_{C_j}) - 2mq_j)P_j\Big),$$

where $\omega$ is the dualizing sheaf of $C$. Notice that

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \sum_{j=1}^{n} \frac{m\deg(\omega|_{C_j}) - 2mq_j}{2m} = g - |\underline{q}| - 1,$$

where $g$ is the genus of $C$. So $E$ is a polarization of degree $|\underline{q}|$ on $C$. Moreover, for each proper subcurve $Y$ of $C$, notice that

$$e_Y = \frac{\deg_Y(\omega)}{2} - \frac{\deg(E|_Y)}{\mathrm{rk}(E)} = q_Y.$$

So, from Definition 17, p. 19, Definition 59, p. 46, and Remark 62, p. 48, it follows that $\mathscr{L}$ is $q$-$i$-quasistable if and only if $\mathscr{L}$ is $P$-quasistable with respect to $E$. □

**Remark 114.** Let $q$, $i$, $P$ and $E$ be as in Lemma 113. Let

$$\bar{J}_C^{\underline{q},i} := \{\mathscr{L} \in \bar{J}_C : \mathscr{L} \text{ is } \underline{q}\text{-}i\text{-quasistable}\}.$$

At the end of Chapter 1, we saw that there is a scheme, denoted by $\bar{J}_E^P$, parameterizing torsion-free rank-1 simple sheaves on $C$ which are $P$-quasistable with respect $E$. Then, by Lemma 113 we have

$$\bar{J}_C^{\underline{q},i} = \bar{J}_E^P,$$

that is, $\bar{J}_C^{\underline{q},i}$ is a scheme. We say that $\bar{J}_C^{\underline{q},i}$ is the scheme parameterizing torsion-free rank-1 simple $\underline{q}$-$i$-quasistable sheaves on $C$ and we denote by $J_C^{\underline{q},i} \subseteq \bar{J}_C^{\underline{q},i}$ its open subscheme parameterizing invertible sheaves.

**Definition 115.** Let $\underline{q}$ be a polarization on $C$. Let $\underline{d} := (d_1, ..., d_n) \in \mathbb{Z}^n$ such that $|\underline{d}| := \sum_{i=1}^n d_i = |\underline{q}|$. We say that $\underline{d}$ is *$\underline{q}$-$i$-quasistable* if there is a $\underline{q}$-$i$-quasistable invertible sheaf $\mathscr{L}$ such that $\underline{d} = \underline{\deg}(\mathscr{L}) := (\deg_{C_1}(\mathscr{L}), ..., \deg_{C_n}(\mathscr{L}))$.

For each $\underline{d} \in \mathbb{Z}^n$, we let $J_C^{\underline{d}}$ be the scheme parameterizing invertible sheaves on $C$ whose multidegree is $\underline{d}$.

**Definition 116.** We call a *regular smoothing* of $C$ a proper and flat morphism $f : \mathcal{C} \to \mathrm{Spec}(B)$, where $B$ is a discrete valuation ring having residue field $k$ and quotient field $K$, such that:

1. $C$ is the closed fiber.

2. The total space $\mathcal{C}$ is regular.

3. The generic fiber of $f$, denoted by $\mathcal{C}_K$, is a smooth projective curve over $K$.

Let $f : \mathcal{C} \to \mathrm{Spec}(B)$ be a regular smoothing of $C$. Let

$$D(C) := \left\{ \sum n_i C_i : n_i \in \mathbb{Z} \right\},$$

the group of formal sums. If $Z$ is a subcurve of $C$, we also denote by $Z$ the sum $\sum_{C_i \subseteq Z} C_i \in D(C)$.

Since $\mathcal{C}$ is regular, $C_i$ is a Cartier divisor for each $i = 1, ..., n$, and therefore we have an associated invertible sheaf $\mathcal{O}_C(D)$ on $\mathcal{C}$ for each $D \in D(C)$.

For each regular smoothing $f : \mathcal{C} \to \mathrm{Spec}(B)$ of $C$, let

$$T_f\mathcal{C} := \{\mathcal{O}_{\mathcal{C}}(D)|_C : D \in D(C)\}$$

and

$$T(C) := \{\mathscr{T} \in T_f\mathcal{C} : f \text{ is a regular smoothing of } C\}.$$

The elements of $T(C)$ are called *twisters*.

For each $i, j \in \{1, 2, ..., n\}$, let

$$k_{i,j} := \#(C_i \cap C_j) \text{ if } \ i \neq j$$

and

$$k_{i,j} := -\#(C_i \cap \overline{C - C_j}) \text{ if } \ i = j.$$

Then, for each $i, j \in \{1, ..., n\}$ and each regular smoothing $f : \mathcal{C} \to \mathrm{Spec}(B)$ of $C$, we have

$$\deg_{C_j}(\mathscr{O}_{\mathcal{C}}(C_i)) = k_{i,j}.$$

Notice that $k_{i,j} = k_{j,i}$ and $\sum_j^n k_{i,j} = 0$ for each $i$. Thus, we have for each twister $\mathscr{T}$ on $C$, $\deg(\mathscr{T}) = 0$. Moreover,

$$\underline{\deg}(\mathscr{T}) \in \mathbb{Z}(k_{1,1}, ..., k_{n,1}) + \mathbb{Z}(k_{1,2}, ..., k_{n,2}) + \cdots + \mathbb{Z}(k_{1,n}, ..., k_{n,n}).$$

For each $i \in \{1, ..., n\}$, let

$$\lambda(\underline{q}, i) := \{\underline{e} = (e_1, ..., e_n) \in \mathbb{Z}^n | \ \underline{e} \text{ is } \underline{q}\text{-}i\text{-quasistable}\}.$$

**Proposition 117.** *Let $i \in \{1, ..., n\}$ and $\underline{q}$ be a polarization on $C$. Then, for each $\underline{e} \in \mathbb{Z}^n$ such that $|\underline{e}| = |\underline{q}|$, there is a unique twister multidegree $t_{\underline{e}}$ such that $\underline{e} + t_{\underline{e}} \in \lambda(\underline{q}, i)$.*

*Proof.* See [CEP], p. 10. □

**Corollary 118.** *Let $q$ and $q'$ be polarizations on $C$ and $i, j \in \{1, ..., n\}$. Let $\mathscr{L}$ be an invertible sheaf on $C$ such that $\deg(\mathscr{L}) + |\underline{q}| = |\underline{q}'|$. Then we have a bijection*

$$\lambda(\underline{q}, i) \to \lambda(\underline{q}', j), \ \underline{e} \mapsto \underline{e} + \underline{\deg}(\mathscr{L}) + t_{\underline{e}}(\mathscr{L}),$$

*where $t_{\underline{e}}(\mathscr{L})$ is the unique twister multidegree such that $\underline{e} + \underline{\deg}(\mathscr{L}) + t_{\underline{e}}(\mathscr{L}) \in \lambda(\underline{q}', j)$.*

Let $\underline{q}$ and $\underline{q}'$ be polarizations on $C$. Let $\mathscr{L} \in J_C$ such that

$$\deg(\mathscr{L}) + |\underline{q}| = |\underline{q}'|.$$

Let $i, j \in \{1, .., n\}$ and let $\underline{e} \in \lambda(\underline{q}, i)$. Let $\mathscr{I} \in J_C^{\underline{e}}$. Since

$$\deg(\mathscr{I} \otimes \mathscr{L}) = \deg(\mathscr{I}) + \deg(\mathscr{L}) = \deg(\mathscr{I}) + |\underline{q}| = |\underline{q}'|,$$

it follows from Proposition 117 that there is a twister $\mathscr{T}_{\underline{e}}$ such that

$$\varphi_{\mathscr{L}}(\underline{e}) := \underline{\deg}(\mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_{\underline{e}}) \in \lambda(\underline{q}', j).$$

By Corollary 118, we have a bijection

$$\varphi_{\mathscr{L}} : \lambda(\underline{q}, i) \to \lambda(\underline{q}', j), \ \underline{e} \mapsto \varphi_{\mathscr{L}}(\underline{e}),$$

and by Proposition 117 a well defined map, in fact an isomorphism,

$$B_{\mathscr{L}}^{\underline{e}} : J_C^{\underline{e}} \to J_C^{\varphi_{\mathscr{L}}(\underline{e})}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_{\underline{e}}.$$

Since

$$J_C^{q,i} = \bigcup_{\underline{e} \in \lambda(\underline{q}, i)} J^{\underline{e}} \ \text{ and } \ J_C^{q',j} = \bigcup_{\underline{e} \in \lambda(\underline{q}, i)} J^{\varphi_{\mathscr{L}}(\underline{e})},$$

where the unions are disjoints, the $B_{\mathscr{L}}^{\underline{e}}$ induce a well-defined isomorphism

$$A_{\mathscr{L}} : J_C^{q,i} \to J_C^{q',j}, \ \mathscr{I} \mapsto B_{\mathscr{L}}^{\underline{e}}(\mathscr{I}) \ \text{ if } \ \mathscr{I} \in J_C^{\underline{e}},$$

called a *$\mathscr{L}$-twister-isomorphism*.

It is thus natural to ask: Does $A_{\mathscr{L}}$ extend to an isomorphism

$$\bar{A}_{\mathscr{L}} : \bar{J}_C^{q,i} \to \bar{J}_C^{q',j}?$$

In general, no! Indeed, Viviani and Melo discovered a curve with four components which admits two distinct polarizations $\underline{q}$ and $\underline{q}'$ such that $\bar{J}_C^{q,i} \ncong \bar{J}_C^{q',j}$ for all $i, j \in \{1, 2, 3, 4\}$.

So, since the question we made has a negative answer, we propose the next more natural question: For which curves $C$ such extensions are possible? Of course if $\mathscr{L} = \mathscr{O}_C$, $\underline{q} = \underline{q}'$ and $i = j$, the answer is plain. So we search for nontrivial cases. This chapter is dedicated to constructing two nontrivial examples of curves for which $A_{\mathscr{L}}$ extend. Before this we need some preliminaries.

**Lemma 119.** *Let $C$ be a nodal curve. Let $\mathscr{L}$ and $\mathscr{M}$ be torsion-free rank-1 sheaves on $C$. Suppose that $\mathscr{M}$ is invertible where $\mathscr{L}$ is not. Then $\mathscr{L} \otimes \mathscr{M}$ is a torsion-free rank-1 sheaf and*

$$\deg(\mathscr{L} \otimes \mathscr{M}) = \deg(\mathscr{L}) + \deg(\mathscr{M}).$$

*Proof.* By Proposition 76, p. 66, there are $P_1, ..., P_m, P_{m+1}, ...P_n$ nodes of $C$ such that

$$\mathscr{L} = \mathfrak{m}_{P_1} \otimes \cdots \otimes \mathfrak{m}_{P_m} \otimes \mathscr{N}_1$$

and

$$\mathscr{M} = \mathfrak{m}_{P_{m+1}} \otimes \cdots \mathfrak{m}_{P_n} \otimes \mathscr{N}_2,$$

where $\mathscr{N}_1$ and $\mathscr{N}_2$ are invertible sheaves on $C$. First assume that $\mathscr{L} = \mathfrak{m}_{P_1}$, and consider the natural sequence

$$0 \to \mathfrak{m}_{P_1} \to \mathscr{O}_C \to \mathscr{O}_{P_1} \to 0.$$

Since $\mathscr{M}$ is invertible at $P_1$, by tensoring this sequence by $\mathscr{M}$, we get another exact sequence

$$0 \to \mathfrak{m}_{P_1} \otimes \mathscr{M} \to \mathscr{M} \to \mathscr{O}_{P_1} \to 0,$$

whence we obtain

$$\chi(\mathscr{M}) = \chi(\mathfrak{m}_{P_1} \otimes \mathscr{M}) + 1.$$

Thus,

$$\begin{aligned} \deg(\mathfrak{m}_{P_1} \otimes \mathscr{M}) &= \chi(\mathfrak{m}_{P_1} \otimes \mathscr{M}) - \chi(\mathscr{O}_C) \\ &= \chi(\mathscr{M}) - \chi(\mathscr{O}_C) - 1 = \\ &= \deg(\mathfrak{m}_{P_1}) + \deg(\mathscr{M}). \end{aligned}$$

Then, reasoning by induction, we have

$$\begin{aligned} \deg(\mathscr{L} \otimes \mathscr{M}) &= \Sigma_{i=1}^{m} \deg(\mathfrak{m}_{P_i}) + \Sigma_{j=m+1}^{n} \deg(\mathfrak{m}_{P_j}) + \deg(\mathscr{N}) + \deg(\mathscr{M}) \\ &= \deg(\mathscr{L}) + \deg(\mathscr{M}). \end{aligned}$$

□

Recall that by Proposition 74 p. 65, to each torsion-free rank-1 sheaf $\mathscr{L}$ on $C$ we can associate an unique partial normalization $\nu : C' \to C$ of $C$ and an unique invertible sheaf $\mathscr{L}'$ on $C'$ such that $\nu_*\mathscr{L}' = \mathscr{L}$. Furthermore, the generalized multidegree of $\mathscr{L}$ is, by definition, the multidegree $\underline{\deg}(\mathscr{L}')$ of $\mathscr{L}'$ on $C'$. We also use $\underline{\deg}(\mathscr{L})$ to denote the generalized multidegree of $\mathscr{L}$ on $C$. Recall also that

$$\deg(\mathscr{L}) = \deg(\mathscr{L}') + \#\{P \in C : \nu \text{ is not an isomorphism over } P\}.$$

**Example 120.** Let $C$ be a nodal curve with only two components $C_1$ and $C_2$ such that $C_1 \cap C_2 = \{P, Q\}$. Let $\nu : C' \to C$ be the partial normalization at $P$ and $Q$, and let $\nu^{-1}(P) =: \{P_1, P_2\}$ and $\nu^{-1}(Q) =: \{Q_1, Q_2\}$. For each

$i=1,2$, let $C'_i$ be the subcurve of $C'$ mapped onto $C_i$ through $\nu$ and assume that $P_i, Q_i \in C'_i$. Let
$$\mathscr{L} = \mathfrak{m}_P \otimes \mathfrak{m}_Q \otimes \mathscr{M},$$
where $\mathscr{M}$ is an invertible sheaf on $C$.

From Proposition 74, p. 65, we have that $\mathscr{L}' = \mathfrak{m}_{P_1} \otimes \mathfrak{m}_{P_2} \otimes \mathfrak{m}_{Q_1} \otimes \mathfrak{m}_{Q_2} \otimes \nu^*\mathscr{M}$, whence

$$\begin{aligned}\underline{\deg}(\mathscr{L}') &= \underline{\deg}(\mathfrak{m}_{P_1}) + \underline{\deg}(\mathfrak{m}_{P_2}) + \underline{\deg}(\mathfrak{m}_{Q_1}) + \underline{\deg}(\mathfrak{m}_{Q_2}) + \underline{\deg}(\nu^*\mathscr{M}) \\ &= (\deg_{C'_1}(\nu^*\mathscr{M}) - 2, \deg_{C'_2}(\nu^*\mathscr{M}) - 2) \\ &= (\deg_{C_1}(\mathscr{M}) - 2, \deg_{C_2}(\mathscr{M}) - 2),\end{aligned}$$

where the equality $\deg_{C'_i}(\nu^*\mathscr{M}) = \deg_{C_i}(\mathscr{M})$ follows from the fact that $\mathscr{M}$ is invertible on $C$.

## 6.1 First example

**Lemma 121.** *Let $C = C_1 \cup C_2$ be a nodal curve such that $\{N_1, N_2\} = C_1 \cap C_2$. Let $\underline{q} = (q_1, q_2)$ be a polarization on $C$. Let $\mathscr{L} \in \bar{J}_C^{\underline{q},1}$ be a non invertible sheaf. Then $\mathscr{L} = \mathfrak{m}_{N_j} \otimes \mathscr{M}$ for some $j = 1,2$ and some invertible sheaf $\mathscr{M}$ on $C$.*

*Proof.* Indeed, since $\mathscr{L}$ is not invertible, it follows from Proposition 76, p. 66, that we need to consider only the following cases

1. $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M}$,
2. $\mathscr{L} = \mathfrak{m}_{N_i} \otimes \mathscr{M}$ for some $i = 1,2$,

where $\mathscr{M}$ is an invertible sheaf on $C$. Then it is enough to discard the possibility (1). Indeed, set $d := |\underline{q}|$ and suppose by contradiction that $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M}$. In this case, since

$$\deg(\mathfrak{m}_{N_1}) = \deg(\mathfrak{m}_{N_2}) = -1 \text{ and } \deg(\mathscr{L}) = d,$$

we have $\deg(\mathscr{M}) = d + 2$.

Since
$$\underline{\deg}(\mathscr{L}) = \underline{\deg}(\mathfrak{m}_{N_1}) + \underline{\deg}(\mathfrak{m}_{N_2}) + \underline{\deg}(\mathscr{M}),$$
where $\underline{\deg}(\mathfrak{m}_{N_1}) = \underline{\deg}(\mathfrak{m}_{N_2}) = (-1,-1)$ and $\underline{\deg}(\mathscr{M}) = (x, d+2-x)$ for some $x \in \mathbb{Z}$, we have
$$\underline{\deg}(\mathscr{L}) = (x-2, d-x).$$
Since $\mathscr{L} \in \bar{J}_C^{\underline{q},1}$, we have

$$\deg_{C_i}(\mathscr{L}) - q_i \geq -\delta_{C_i}/2 = -1$$

for each $i = 1, 2$, and moreover, $\deg_{C_1}(\mathscr{L}) - q_1 > -1$. Now, since $\underline{\deg}\mathscr{L} = (x-2, d-x)$,

$$x - 2 - q_1 = \deg_{C_1}(\mathscr{L}) - q_1 > -1$$

gives us $x > q_1 + 1$, whereas

$$-x + q_1 = \deg_{C_2}(\mathscr{L}) - q_2 \geq -1,$$

give us $x \leq q_1 + 1$, an absurd. Hence, $\mathscr{L}$ can not be of the form $\mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M}$. □

**Definition 122.** For each real number $a$, let $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) denote the largest integer less or equal to $a$ (resp. the smallest integer greater or equal to $a$).

**Lemma 123.** *Let $C$ be a nodal curve with only two irreducible components $C_1$ and $C_2$ such that $\#C_1 \cap C_2 = 2$. Let $\underline{q} = (q_1, q_2)$ be a polarization of degree $d$ on $C$. Then*

$$\lambda(\underline{q}, 1) = \{(\lfloor q_1 \rfloor, d - \lfloor q_1 \rfloor), (\lfloor q_1 \rfloor + 1, d - 1 - \lfloor q_1 \rfloor)\}.$$

*Proof.* Let $(p_1, p_2) \in \lambda(\underline{q}, 1)$. Then,

$$p_1 - q_1 > -1 \text{ and } q_1 - p_1 = p_2 - q_2 \geq -1 \Leftrightarrow 1 \geq p_1 - q_1.$$

Since $p_1 \in \mathbb{Z}$ and $0 \geq \lfloor q_1 \rfloor - q_1 > -1$, inevitably we have that

$$p_1 \in \{\lfloor q_1 \rfloor, \lfloor q_1 \rfloor + 1\}.$$

So, since $p_1 + p_2 = d$, if $p_1 = \lfloor q_1 \rfloor$, we get $p_2 = d - \lfloor q_1 \rfloor$. On the other hand, if $p_1 = \lfloor q_1 \rfloor + 1$, we get $p_2 = d - 1 - \lfloor q_1 \rfloor$. Hence,

$$\lambda(\underline{q}, 1) = \{(\lfloor q_1 \rfloor, d - \lfloor q_1 \rfloor), (\lfloor q_1 \rfloor + 1, d - 1 - \lfloor q_1 \rfloor)\}.$$

□

**Proposition 124.** *Let $C = C_1 \cup C_2$ be a curve in $\mathbb{P}^2$ where $C_1$ is a conic and $C_2$ is a line such that $C_1 \cap C_2 = \{N_1, N_2\}$. Let $\underline{p} = (p_1, p_2)$ and $\underline{q} = (q_1, q_2)$ be polarizations on $C$. Let $\mathscr{L} \in J_C$ such that $|\underline{p}| + \deg(\mathscr{L}) = |\underline{q}|$. Let $i, j \in \{1, 2\}$. Then any $\mathscr{L}$-twister-isomorphism $A_{\mathscr{L}} : J_C^{\underline{p},1} \to J_C^{\underline{q},j}$ extends to an isomorphism*

$$\bar{A}_{\mathscr{L}} : \bar{J}_C^{\underline{p},i} \to \bar{J}_C^{\underline{q},j}.$$

*Proof.* For the sake of notational simplicity we assume that $\underline{p}, \underline{q} \in \mathbb{Z}^2$ and we show our statement only for $i$=$j$=1, as for any other $i, j \in \{1, 2\}$ the proof is similar.

By Lemma 123 we have

$$\lambda(\underline{p}, 1) = \{(p_1, p_2), (p_1+1, p_2-1)\} \text{ and } \lambda(\underline{q}, 1) = \{(q_1, q_2), (q_1+1, q_2-1)\}.$$

By Proposition 117, p. 91, for each $\underline{e} \in \lambda(\underline{p}, 1)$, there is a unique twister multidegree $t_{\underline{e}}$ such that $\underline{e} + \underline{\deg}(\mathscr{L}) + t_{\underline{e}} \in \lambda(\underline{q}, 1)$. Thus we have two cases to consider.

1.
$$(p_1, p_2) + \underline{\deg}(\mathscr{L}) + t_{(p_1,p_2)} = (q_1+1, q_2-1) \tag{6.1}$$
and
$$(p_1+1, p_2-1) + \underline{\deg}(\mathscr{L}) + t_{(p_1+1,p_2-1)} = (q_1, q_2), \tag{6.2}$$

2.
$$(p_1, p_2) + \underline{\deg}(\mathscr{L}) + t_{(p_1,p_2)} = (q_1, q_2)$$
and
$$(p_1+1, p_2-1) + \underline{\deg}(\mathscr{L}) + t_{(p_1+1,p_2-1)} = (q_1+1, q_2-1).$$

We give the proof only in Case 1) as the proof in Case 2) is similar.
Proof in Case 1). Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be twisters such that

$$\underline{\deg}(\mathscr{T}_1) = t_{(p_1,p_2)} \text{ and } \underline{\deg}(\mathscr{T}_2) = t_{(p_1+1,p_2-1)}.$$

Now consider the following maps

$$B^1_{\mathscr{L}} : J_C^{(p_1,p_2)} \to J_C^{(q_1+1,q_2-1)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_1,$$

$$B^2_{\mathscr{L}} : J_C^{(p_1+1,p_2-1)} \to J_C^{(q_1,q_2)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_2,$$

and the $\mathscr{L}$-twister-isomorphism

$$A_{\mathscr{L}} : J_C^{\underline{p},1} \to J_C^{\underline{q},1}$$

induced by $B^1_{\mathscr{L}}$ and $B^2_{\mathscr{L}}$.

We claim that $A_{\mathscr{L}} : J_C^{\underline{p},1} \to J_C^{\underline{q},1}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{\underline{p},1} \to \bar{J}_C^{\underline{q},1}$.

Indeed, let $\mathscr{R} \in \bar{J}_C^{p,1}$. Due to the symmetry of the curve $C$, without loss of generality, we may suppose $\mathscr{R} = \mathfrak{m}_{N_1} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on $C$ with multidegree $(p_1+1, p_2)$. Let

$$\{P_t\}_{t\in\mathbb{N}} \subseteq C_1 - \{N_1\} \text{ and } \{Q_t\}_{t\in\mathbb{N}} \subseteq C_2 - \{N_1\}$$

be sequences of smooth points such that

$$\lim_{t\to\infty} P_t = N_1 = \lim_{t\to\infty} Q_t.$$

Now, notice that for each $t \in \mathbb{N}$,

$$\underline{\deg}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) = (p_1, p_2), \text{ that is, } \mathfrak{m}_{P_t} \otimes \mathscr{M} \in J_C^{(p_1,p_2)},$$

$$\underline{\deg}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}) = (p_1+1, p_2-1), \text{ that is, } \mathfrak{m}_{Q_t} \otimes \mathscr{M} \in J_C^{(p_1+1,p_2-1)},$$

and

$$\lim_{t\to\infty} \mathfrak{m}_{P_t} \otimes \mathscr{M} = \mathfrak{m}_{N_1} \otimes \mathscr{M} = \mathscr{R} = \mathfrak{m}_{N_1} \otimes \mathscr{M} = \lim_{t\to\infty} \mathfrak{m}_{Q_t} \otimes \mathscr{M}.$$

In addition, since $C$ has genus 1, notice that these are the only ways to approach $\mathscr{R}$ through the components $J_C^{(p_1,p_2)}$ and $J_C^{(p_1+1,p_2-1)}$. So, in order to show that $A_{\mathscr{L}} : J_C^{p,1} \to J_C^{q,1}$ extends, we need to prove that the limits

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) = \lim_{t\to\infty} \mathfrak{m}_{P_t} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1$$

and

$$\lim_{t\to 0} B^2_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}) = \lim_{t\to\infty} \mathfrak{m}_{Q_t} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_2$$

are isomorphic.
Claim 1: The natural limits

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) = \mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1$$

and

$$\lim_{t\to\infty} B^2_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}) = \mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_2$$

do not belong to $\bar{J}_C^{q,1}$. Indeed, from (6.1) we have

$$\underline{\deg}(\mathscr{L} \otimes \mathscr{T}_1) = (q_1+1, q_2-1) - (p_1, p_2).$$

So, since

$$\begin{aligned}
\underline{\deg}(\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1) &= \underline{\deg}(\mathfrak{m}_{N_1}) + \underline{\deg}(\mathscr{M}) + \underline{\deg}(\mathscr{L} \otimes \mathscr{T}_1) \\
&= (-1,-1) + (p_1+1, p_2) + (q_1+1, q_2-1) - \\
&\quad -(p_1, p_2) \\
&= (q_1+1, q_2-2),
\end{aligned}$$

we have that $\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1 \notin \bar{J}_C^{\underline{q},1}$, as

$$-2 = \deg_{C_2}(\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1) - q_2 \ngeq -1.$$

Similarly, from (6.2), we have

$$\underline{\deg}(\mathscr{L} \otimes \mathscr{T}_2) = (q_1, q_2) - (p_1 + 1, p_2 - 1).$$

Then

$$\begin{aligned} \underline{\deg}(\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_2) &= \underline{\deg}(\mathfrak{m}_{N_1}) + \underline{\deg}(\mathscr{M}) + \underline{\deg}(\mathscr{L} \otimes \mathscr{T}_2) \\ &= (-1, -1) + (p_1 + 1, p_2) + (q_1, q_2) - \\ &\quad -(p_1 + 1, p_2 - 1) \\ &= (q_1 - 1, q_2). \end{aligned}$$

Hence $\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_2 \notin \bar{J}_C^{\underline{q},1}$ because

$$-1 = \deg_{C_1}(\mathfrak{m}_{N_1} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1) - q_1 \ngtr -1.$$

Therefore we have proved Claim 1.

Claim 2: There are torsion-free rank-1 sheaves $\mathscr{L}'$ and $\mathscr{L}''$ on $C$ such that

$$\lim_{t \to 0} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) \cong \mathscr{L}' \in \bar{J}_C^{\underline{q},1}, \ \lim_{t \to 0} B^2_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}) = \mathscr{L}'' \in \bar{J}_C^{\underline{q},1}$$

and $\mathscr{L}' \cong \mathscr{L}''$.

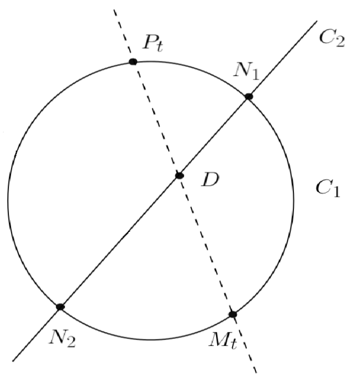


Figure 6.1:

Indeed, first we find the sheaf $\mathscr{L}'$. Fix $D \in C_2$ a smooth point. For each $t \in \mathbb{N}$, let $r_t$ be the line passing through $D$ and $P_t$. Since $C \subseteq \mathbb{P}^2$, it follows from Bezout Theorem that $r_t$ intersects $C_2$ at another smooth, which we denote by $M_t$; see Figure 6.1.

Now, fix another smooth point $B \in C_1$, and let $s_t$ be the line passing through $M_t$ and $B$. Again since $C \subseteq \mathbb{P}^2$, $s_t$ intersect $C_2$ at another smooth point $P_t'$; see Figure 6.2.

Since for each $t \in \mathbb{N}$ the divisor associated to the rational function $r_t/s_t$ on $C$ is $D + P_t + M_t - B - P_t' - M_t$, we have that the divisors $D + P_t$ and $B + P_t'$ are equivalent, that is, $D + P_t \equiv B + P_t'$. Hence, for each $t \in \mathbb{N}$, we have

$$\mathfrak{m}_{P_t} \cong \mathfrak{m}_D^* \otimes \mathfrak{m}_B \otimes \mathfrak{m}_{P_t'}.$$

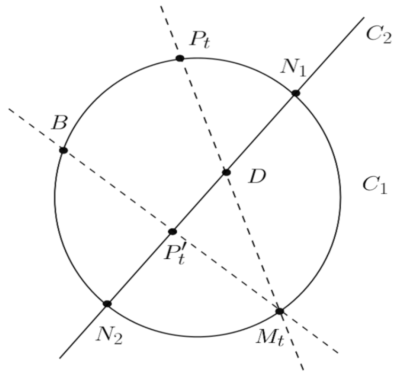


Figure 6.2:

Thus,

$$\begin{aligned}
\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) &= \lim_{t\to\infty} \mathfrak{m}_{P_t} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1 \\
&= \lim_{t\to\infty} \mathfrak{m}_D^* \otimes \mathfrak{m}_B \otimes \mathfrak{m}_{P_t'} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1 \\
&= \mathfrak{m}_D^* \otimes \mathfrak{m}_B \otimes \mathfrak{m}_{N_2} \otimes \mathscr{L} \otimes \mathscr{M} \otimes \mathscr{T}_1.
\end{aligned}$$

Let $\mathscr{L}' := \mathfrak{m}_D^* \otimes \mathfrak{m}_B \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M} \otimes \mathscr{T}_1$. We claim that $\mathscr{L}' \in \bar{J}_C^{\underline{q},1}$. In fact, it is enough to find the multidegree of $\mathscr{L}'$.

$$\begin{aligned}
\underline{\deg}(\mathscr{L}') &= \underline{\deg}(\mathfrak{m}_D^*) + \underline{\deg}(\mathfrak{m}_B) + \underline{\deg}(\mathfrak{m}_{N_2}) + \underline{\deg}(\mathscr{M}) + \underline{\deg}(\mathscr{L} \otimes \mathscr{T}_1) \\
&= (0,1) + (-1,0) + (-1,-1) + (p_1+1, p_2) + (q_1+1, q_2-1) \\
&\quad -(p_1, p_2) \\
&= (q_1, q_2-1).
\end{aligned}$$

Hence, since $\underline{\deg}(\mathscr{L}') = (q_1, q_2-1)$, it is easy to see that $\mathscr{L}' \in \bar{J}_C^{\underline{q},1}$.

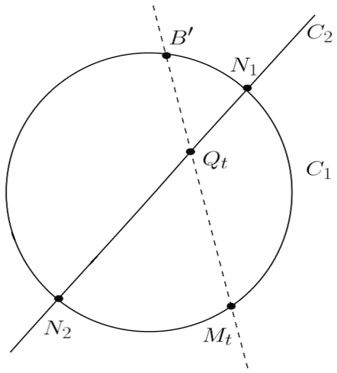


Figure 6.3:

Now we find the sheaf $\mathscr{L}''$ on $C$ such that

$$\lim_{t\to\infty} B^2_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}) = \mathscr{L}'' \in \bar{J}^{q,1}_{\underline{C}}.$$

Fix a smooth point $B' \in C_1$. For each $t \in \mathbb{N}$, let $r'_t$ be the line passing through $B'$ and $Q_t$. Since $C \subseteq \mathbb{P}^2$, $r'_t$ intercept $C_1$ at a point $M'_t$; see Figure 6.3.

Fix a smooth point $D' \in C_2$ and let $s'_t$ be the line passing through $D'$ and $M'_t$. Since $C \subseteq \mathbb{P}^2$, $s'_t$ intercept $C_1$ at a point $Q'_t$; see Figure 6.4. Hence, for each $t \in \mathbb{N}$, the divisors $Q_t + B'$ and $Q'_t + D'$ are congruent because the divisor associated to the rational function $r'_t/s'_t$ is $Q_t + B' - Q'_t - D'$.

Thus, since $Q_t + B' \equiv Q'_t + D'$, we have

$$\mathfrak{m}_{Q_t} \cong \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathfrak{m}_{Q'_t},$$

and therefore

$$\begin{aligned}\lim_{t\to\infty} \mathfrak{m}_{Q_t} \otimes \mathscr{M} \otimes \mathscr{T}_2 &= \lim_{t\to\infty} \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathfrak{m}_{Q'_t} \otimes \mathscr{M} \otimes \mathscr{T}_2 \\ &= \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M} \otimes \mathscr{T}_2.\end{aligned}$$

Let $\mathscr{L}'' := \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M} \otimes \mathscr{T}_2$. Then $\mathscr{L}'' \in \bar{J}^{q,1}_{\underline{C}}$. Indeed, we have

$$\begin{aligned}\underline{\deg}(\mathscr{L}'') &= \underline{\deg}(\mathfrak{m}^*_{B'}) + \underline{\deg}(\mathfrak{m}_{D'}) + \underline{\deg}(\mathfrak{m}_{N_2}) + \underline{\deg}(\mathscr{M}) + \underline{\deg}(\mathscr{L} \otimes \mathscr{T}_2) \\ &= (1,0) + (0,-1) + (-1,-1) + (p_1+1, p_2) + (q_1, q_2) \\ &\quad -(p_1+1, p_2-1) \\ &= (q_1, q_2-1).\end{aligned}$$

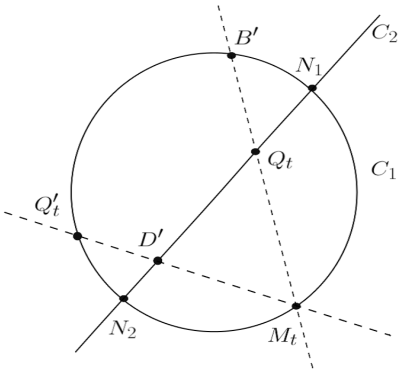


Figure 6.4:

Hence $\mathscr{L}'' \in \bar{J}_C^{q,1}$.

To finish the proof of our claim, we need to prove that $\mathscr{L}' \cong \mathscr{L}''$. Indeed, let $\nu : C' \to C$ be the normalization of $C$ at $N_2$ and let $\{N_{2,1}, N_{2,2}\} := \nu^{-1}(N_2)$. Let

$$\mathscr{I}' := \mathfrak{m}_{N_{2,1}} \otimes \mathfrak{m}_{N_{2,2}} \otimes \nu^*(\mathfrak{m}_D^* \otimes \mathfrak{m}_B \otimes \mathscr{M} \otimes \mathscr{T}_1) = \nu^*(\mathscr{L}')/\text{torsion}$$

and

$$\mathscr{I}'' := \mathfrak{m}_{N_{2,1}} \otimes \mathfrak{m}_{N_{2,2}} \otimes \nu^*(\mathfrak{m}_{B'}^* \otimes \mathfrak{m}_{D'} \otimes \mathscr{M} \otimes \mathscr{T}_2) = \nu^*(\mathscr{L}'')/\text{torsion}.$$

By Proposition 74, p. 65, we have

$$\nu_*\mathscr{I}' = \mathscr{L}' \text{ and } \nu_*\mathscr{I}'' = \mathscr{L}''.$$

Since $C'$ is a curve of compact type, it follows that $\mathscr{I}'$ and $\mathscr{I}''$ are uniquely determined, up to isomorphism, by their restrictions to the irreducible components of $C'$; see Proposition 43, p. 32. However, since

$$\underline{\deg}(\mathscr{I}') = \underline{\deg}(\mathscr{L}') = (q_1, q_2 - 1) = \underline{\deg}(\mathscr{L}'') = \underline{\deg}(\mathscr{I}''),$$

and the irreducible components of $C'$ are $\mathbb{P}^1$, it follows that the restrictions of $\mathscr{I}'$ and $\mathscr{I}'$ are isomorphic, implying $\mathscr{I}' \cong \mathscr{I}''$. Hence

$$\mathscr{L}' = \nu_*\mathscr{I}' \cong \nu_*\mathscr{I}'' = \mathscr{L}'',$$

and we have proved our Claim 2. Therefore, $A_{\mathscr{L}}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{p,1} \to \bar{J}_C^{q,1}$. $\square$

## 6.2 Second example

**Proposition 125.** *Let $C \subseteq \mathbb{P}^2$ be a curve given by the union of three $\mathbb{P}^1$'s. Let $\underline{e} = (e_1, e_2, e_3)$ and $\underline{f} = (f_1, f_2, f_3)$ be polarizations on $C$. Let $\mathscr{L} \in J_C$ such that*

$$|\underline{e}| + \deg(\mathscr{L}) = |\underline{f}|.$$

*Fix $i, j \in \{1,2,3\}$. Then any $\mathscr{L}$-twister-isomorphism $A_{\mathscr{L}} : J_C^{\underline{e},i} \to J_C^{\underline{f},j}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{\underline{e},i} \to \bar{J}_C^{\underline{f},j}$.*

*Proof.* To study this second example, we use the following strategy: Let $\underline{q} := (|\underline{e}|, 0, 0)$ be a polarization on $C$ and $A_{\mathscr{L}} : J_C^{\underline{e},i} \to J_C^{\underline{f},j}$ a $\mathscr{L}$-twister-isomorphism. By Proposition 117, p. 91, we know that for each $\underline{h} \in \lambda(\underline{q}, 1)$ there are unique twisters multidegree $t_1$ and $t_2$ such that $\underline{h} + t_1 \in \lambda(\underline{e}, i)$ and $\underline{h} + \underline{\deg}(\mathscr{L}) + t_2 \in \lambda(\underline{f}, j)$. Thus we are able to construct the following commutative diagram

$$\begin{array}{ccc} J_C^{\underline{q},1} & \xrightarrow{A'_{\mathscr{L}}} & J_C^{\underline{f},j} \\ {\scriptstyle A'_{\mathscr{O}_C}}\downarrow & \nearrow {\scriptstyle A_{\mathscr{L}}} & \\ J_C^{\underline{e},i} & & \end{array}$$

where $A'_{\mathscr{O}_C}$ and $A'_{\mathscr{L}}$ are twister-isomorphisms. Therefore it is enough to show that

$$A'_{\mathscr{O}_C} : J_C^{\underline{q},1} \to J_C^{\underline{e},i} \text{ and } A'_{\mathscr{L}} : J_C^{\underline{q},1} \to J_C^{\underline{f},j}$$

extend to isomorphisms

$$\bar{A}'_{\mathscr{O}_C} : \bar{J}_C^{\underline{q},1} \to \bar{J}_C^{\underline{e},i} \text{ and } \bar{A}'_{\mathscr{L}} : \bar{J}_C^{\underline{q},1} \to \bar{J}_C^{\underline{f},j},$$

respectively, as in this case, as $A_{\mathscr{L}} = A'_{\mathscr{L}} \circ (A'_{\mathscr{O}_C})^{-1}$, this implies that

$$A_{\mathscr{L}} : J_C^{\underline{e},i} \to J_C^{\underline{f},j}$$

extends to the isomorphism

$$\bar{A}_{\mathscr{L}} = \bar{A}'_{\mathscr{L}} \circ (\bar{A}'_{\mathscr{O}_C})^{-1} : \bar{J}_C^{\underline{e},i} \to \bar{J}_C^{\underline{f},j}.$$

The construction that we will give extensions is based on the following four lemmas, whose proof finish that of the proposition.

□

**Lemma 126.** *Let $C$ be a curve and $C_1, C_2, C_3$ its irreducible components. Suppose that $C_i$ is smooth for each $i = 1, 2, 3$, and $\{N_1\} = C_1 \cap C_3$, $\{N_2\} = C_1 \cap C_2$ and $\{N_3\} = C_2 \cap C_3$. Let $d$ be an integer and $\underline{q} = (d, 0, 0)$. Let $\mathscr{L} \in \bar{J}_C^{\underline{q},1} - J_C^{\underline{q},1}$. Then $\mathscr{L} = \mathfrak{m}_{N_i} \otimes \mathscr{M}$ for some $i \in \{1, 2, 3\}$, where $\mathscr{M}$ is an invertible sheaf on $C$.*

*Proof.* Since $\mathscr{L}$ is a torsion-free rank-1 sheaf on $C$, by Proposition 76, p. 66, we need to consider only the following cases:

1. $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathfrak{m}_{N_3} \otimes \mathscr{M}$,
2. $\mathscr{L} = \mathfrak{m}_{N_i} \otimes \mathfrak{m}_{N_j} \otimes \mathscr{M}$,
3. $\mathscr{L} = \mathfrak{m}_{N_i} \otimes \mathscr{M}$,

where $\mathscr{M}$ is an invertible sheaf on $C$.

Thus, in order to prove the lemma, it is enough to exclude Case 1) and Case 2).
Case 1) Suppose by contradiction that $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathfrak{m}_{N_3} \otimes \mathscr{M}$. Since $\deg(\mathfrak{m}_{N_i}) = -1$ for each $i = 1, 2, 3$ and

$$\deg(\mathscr{L}) = \deg(\mathfrak{m}_{N_1}) + \deg(\mathfrak{m}_{N_2}) + \deg(\mathfrak{m}_{N_3}) + \deg(\mathscr{M}),$$

we have $\deg(\mathscr{M}) = d + 3$.

Let $x, y \in \mathbb{Z}$ such that $\underline{\deg}(\mathscr{M}) = (x, y, d + 3 - x - y)$. Since $\underline{\deg}(\mathfrak{m}_{N_1}) = (-1, 0, -1)$, $\underline{\deg}(\mathfrak{m}_{N_2}) = (-1, -1, 0)$, $\underline{\deg}(\mathfrak{m}_{N_3}) = (0, -1 - 1)$ and

$$\underline{\deg}(\mathscr{L}) = \underline{\deg}(\mathfrak{m}_{N_1}) + \underline{\deg}(\mathfrak{m}_{N_2}) + \underline{\deg}(\mathfrak{m}_{N_3}) + \underline{\deg}(\mathscr{M}),$$

we have

$$\underline{\deg}(\mathscr{L}) = (x - 2, y - 2, d + 1 - x - y).$$

Since $\mathscr{L} \in \bar{J}_C^{\underline{q},1}$, we have

$$x - 2 - d = \deg_{C_1}(\mathscr{L}) - d > -1 \text{ and } d - x = \deg_{C_2 \cup C_3}(\mathscr{L}) \geq -1,$$

that is, $x > d + 1$ and $x \leq d + 1$, which is impossible.

Case 2) Suppose by contradiction that $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_3} \otimes \mathscr{M}$. Since $\deg(\mathfrak{m}_{N_i}) = -1$ for $i = 1, 3$, we have $\deg(\mathscr{M}) = d + 2$. Let $x, y \in \mathbb{Z}$ such that

$$\underline{\deg}(\mathscr{M}) = (x, y, d + 2 - x - y).$$

Since $\underline{\deg}(\mathfrak{m}_{N_1}) = (-1, 0, -1)$ and $\underline{\deg}(\mathfrak{m}_{N_3}) = (0, -1 - 1)$, we have

$$\underline{\deg}(\mathscr{L}) = (x - 1, y - 1, d - x - y).$$

Since $\mathscr{L} \in \bar{J}_C^{q,1}$, we have

$$\deg_{C_1}(\mathscr{L}) - d > -1 \text{ and } \deg_{C_2 \cup C_3}(\mathscr{L}) \geq -1,$$

that is, $x = d + 1$. On the other hand, we have

$$y - 1 = \deg_{C_2}(\mathscr{L}) \geq -1 \text{ and } -y = \deg_{C_1 \cup C_3}(\mathscr{L}) - d > -1,$$

that is, $y = 0$. Hence, we have

$$\underline{\deg}(\mathscr{L}) = (d, -1, -1).$$

But we would need as well that $-1 = \deg_{C_1 \cup C_2}(\mathscr{L}) - d > -1$, absurd. Due to symmetry of the curve, $\mathfrak{m}_{N_2} \otimes \mathfrak{m}_{N_3} \otimes \mathscr{M} \notin \bar{J}_C^{q,1}$, for any $\mathscr{M}$ as well.

Suppose by contradiction that $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathfrak{m}_{N_2} \otimes \mathscr{M}$. Then $\deg(\mathscr{M}) = d + 2$. Let $x, y \in \mathbb{Z}$ such that $\underline{\deg}(\mathscr{M}) = (x, y, d + 2 - x - y)$. Since $\underline{\deg}(\mathfrak{m}_{N_1}) = (-1, 0, -1)$ and $\underline{\deg}(\mathfrak{m}_{N_2}) = (-1, -1, 0)$, we have

$$\underline{\deg}(\mathscr{L}) = (x - 2, y - 1, d + 1 - x - y).$$

Since $\mathscr{L} \in \bar{J}_C^{q,1}$, we have

$$x - 2 - d = \deg_{C_1}(\mathscr{L}) - d > -1 \text{ and } d - x = \deg_{C_2 \cup C_3}(\mathscr{L}) \geq -1,$$

that is, $x > d + 1$ and $x \leq d + 1$, absurd.

□

In the next lemma we describe exactly what are the elements of $\bar{J}_C^{q,1} - J_C^{q,1}$.

**Lemma 127.** *Let $C$ and $q$ be as in Lemma 126. Then,*

$$\bar{J}_C^{q,1} - J_C^{q,1} = \{\mathfrak{m}_{N_i} \otimes \mathscr{M} : i = 1, 2, 3 \textit{ and } \mathscr{M} \in J_C^{\underline{a}} \textit{ with } \underline{a} = (d + 1, 0, 0)\}.$$

*Proof.* Let $\mathscr{L} \in \bar{J}_C^{q,1} - J_C^{q,1}$. By Lemma 126, we have $\mathscr{L} = \mathfrak{m}_{N_i} \otimes \mathscr{M}$ for some $i \in \{1, 2, 3\}$, where $\mathscr{M}$ is an invertible sheaf on $C$. We show that all the followings three cases are possible, with $\underline{\deg}(\mathscr{M}) = (d + 1, 0, 0)$:

1. $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathscr{M}$,
2. $\mathscr{L} = \mathfrak{m}_{N_2} \otimes \mathscr{M}$,
3. $\mathscr{L} = \mathfrak{m}_{N_3} \otimes \mathscr{M}$.

Case 1) Suppose $\mathscr{L} = \mathfrak{m}_{N_1} \otimes \mathscr{M}$. Since $\deg(\mathfrak{m}_{N_1}) = -1$ and $\deg(\mathscr{L}) = d$, we have $\deg(\mathscr{M}) = d+1$. Let $x, y \in \mathbb{Z}$ such that $\underline{\deg}(\mathscr{M}) = (x, y, d+1-x-y)$. Since $\underline{\deg}(\mathfrak{m}_{N_1}) = (-1, 0, -1)$ and

$$\underline{\deg}(\mathscr{L}) = \underline{\deg}(\mathfrak{m}_{N_1}) + \underline{\deg}(\mathscr{M}),$$

we have

$$\underline{\deg}(\mathscr{L}) = (x-1, y, d-x-y).$$

In order we have $\mathscr{L} \in \bar{J}_C^{q,1}$, we need

$$x - 1 - d = \deg_{C_1}(\mathscr{L}) - d > -1 \text{ and } d - x = \deg_{C_2 \cup C_3}(\mathscr{L}) \geq -1,$$

that is, $x = d + 1$. But, with $x = d + 1$, we also need

$$-1 - y = \deg_{C_3}(\mathscr{L}) \geq -1 \text{ and } y = \deg_{C_1 \cup C_2}(\mathscr{L}) - d > -1,$$

that is, $y = 0$. Therefore, $\underline{\deg}(\mathscr{M}) = (d + 1, 0, 0)$ and $\underline{\deg}(\mathscr{L}) = (d, 0, -1)$. Finally to see that $\mathscr{L} \in \bar{J}_C^{q,1}$ it is enough to notice that

$$0 = \deg_{C_2}(\mathscr{L}) \geq -1 \text{ and } 0 = \deg_{C_1 \cup C_3}(\mathscr{L}) - d > -1.$$

Case 2) Suppose $\mathscr{L} = \mathfrak{m}_{N_2} \otimes \mathscr{M}$. By analogy with Case 1), it follows from the symmetry of the curve $C$ that $\mathscr{L} \in \bar{J}_C^{q,1}$, with $\underline{\deg}(\mathscr{L}) = (d, -1, 0)$ and $\underline{\deg}(\mathscr{M}) = (d + 1, 0, 0)$.

Case 3) Assume $\mathscr{L} = \mathfrak{m}_{N_3} \otimes \mathscr{M}$. Then we have $\deg(\mathscr{M}) = d + 1$ and $\underline{\deg}(\mathscr{M}) = (x, y, d+1-x-y)$ for some $x, y \in \mathbb{Z}$. Since $\underline{\deg}(\mathfrak{m}_{N_3}) = (0, -1, -1)$ and

$$\underline{\deg}(\mathscr{L}) = \underline{\deg}(\mathfrak{m}_{N_3}) + \underline{\deg}(\mathscr{M}_3),$$

we have $\underline{\deg}(\mathscr{L}) = (x, y-1, d-x-y)$. In order we have $\mathscr{L} \in \bar{J}_C^{q,1}$, we need to have

$$y - 1 = \deg_{C_2}(\mathscr{L}) \geq -1 \text{ and } -y = \deg_{C_1 \cup C_3}(\mathscr{L}) - d > -1,$$

that is, $y = 0$. Then, with $y = 0$, we also need

$$d - x = \deg_{C_3}(\mathscr{L}) \geq -1 \text{ and } x - 1 - d = \deg_{C_1 \cup C_2}(\mathscr{L}) - d > -1,$$

that is, $x = d + 1$. Therefore $\underline{\deg}(\mathscr{M}) = (d + 1, 0, 0)$ and $\underline{\deg}(\mathscr{L}) = (d + 1, -1, -1)$. Finally to see that $\mathscr{L} \in \bar{J}_C^{q,1}$, it is enough to notice that

$$1 = \deg_{C_1}(\mathscr{L}) - d > -1 \text{ and } -1 = \deg_{C_2 \cup C_3}(\mathscr{L}) \geq -1.$$

□

**Lemma 128.** *Let $C$ be a curve and $C_1, C_2, C_3$ its irreducible components. Suppose $\delta_Y = 2$ for each subcurve $Y$ of $C$. Let $\underline{p} = (p_1, p_2, p_3)$ be a polarization of degree $d$ on $C$. Then we have*

$$\lambda(\underline{p}, 1) = \{(\lfloor p_1 \rfloor, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil), (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil),$$
$$(\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1)\}$$

*or*

$$\lambda(\underline{p}, 1) = \{(\lfloor p_1 \rfloor, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil), (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil),$$
$$(\lfloor p_1 \rfloor, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1)\}.$$

*Proof.* We claim that
j) $\underline{q}' := (\lfloor p_1 \rfloor, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil) \in \lambda(\underline{p}, 1)$,

jj) $\underline{q}'' := (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil) \in \lambda(\underline{p}, 1)$.

Proof item (j): We need to prove $q'_{_Y} - p_{_Y} \geq -1$ for each proper subcurve $Y$ of $C$, with strict inequality is strict when $Y$ contains $C_1$. Indeed, on one hand we have $q'_{_{C_1}} - p_{_{C_1}} = \lfloor p_1 \rfloor - p_1 > -1$, $q'_{_{C_2}} - p_{_{C_2}} = \lceil p_2 \rceil - p_2 \geq 0$ and

$$\begin{aligned} q'_{_{C_3}} - p_{_{C_3}} &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_3 \\ &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - d + p_1 + p_2 \\ &= \underbrace{p_1 - \lfloor p_1 \rfloor}_{\geq 0} + \underbrace{p_2 - \lceil p_2 \rceil}_{>-1} > -1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} q'_{_{C_1 \cup C_2}} - p_{_{C_1 \cup C_2}} &= \lfloor p_1 \rfloor + \lceil p_2 \rceil - p_1 - p_2 \\ &= \underbrace{\lfloor p_1 \rfloor - p_1}_{>-1} + \underbrace{\lceil p_2 \rceil - p_2}_{\geq 0} > -1, \end{aligned}$$

$$\begin{aligned} q'_{_{C_1 \cup C_3}} - p_{_{C_1 \cup C_3}} &= \lfloor p_1 \rfloor + d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_1 - p_3 \\ &= d - \lceil p_2 \rceil - d + p_2 = p_2 - \lceil p_2 \rceil > -1 \end{aligned}$$

and

$$\begin{aligned} q'_{_{C_2 \cup C_3}} - p_{_{C_2 \cup C_3}} &= \lceil p_2 \rceil + d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_2 - p_3 \\ &= d - \lfloor p_1 \rfloor - d + p_1 = p_1 - \lfloor p_1 \rfloor \geq 0. \end{aligned}$$

Hence we have item (j).

Proof item (jj): As in item (j), we need to prove $q''_Y - p_Y \geq -1$ for each proper subcurve $Y$ of $C$, with $q'_Y - p_Y > -1$ if $Y$ contains $C_1$. Indeed, first we have $q''_{C_1} - p_{C_1} = \lfloor p_1 \rfloor + 1 - p_1 > 0$, $q''_{C_2} - p_{C_2} = \lceil p_2 \rceil - 1 - p_2 \geq -1$ and

$$\begin{aligned} q''_{C_3} - p_{C_3} &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_3 \\ &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - d + p_1 + p_2 \\ &= \underbrace{p_1 - \lfloor p_1 \rfloor}_{\geq 0} + \underbrace{p_2 - \lceil p_2 \rceil}_{>-1} > -1 \end{aligned}$$

and second we have,

$$\begin{aligned} q''_{C_1 \cup C_2} - p_{C_1 \cup C_2} &= \lfloor p_1 \rfloor + 1 + \lceil p_2 \rceil - 1 - p_1 - p_2 \\ &= \lfloor p_1 \rfloor + \lceil p_2 \rceil - p_1 - p_2 > -1, \end{aligned}$$

$$\begin{aligned} q''_{C_1 \cup C_3} - p_{C_1 \cup C_3} &= \lfloor p_1 \rfloor + 1 + d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_1 - p_3 \\ &= d - \lceil p_2 \rceil - d + p_2 \\ &= p_2 - \lceil p_2 \rceil > 0 \end{aligned}$$

and

$$\begin{aligned} q''_{C_2 \cup C_3} - p_{C_2 \cup C_3} &= \lceil p_2 \rceil - 1 + d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - p_2 - p_3 \\ &= d - \lfloor p_1 \rfloor - d + p_1 - 1 \\ &= p_1 - \lfloor p_1 \rfloor - 1 \geq -1. \end{aligned}$$

Therefore we have item (jj), what finish the proof of our claim.

Now, if $\underline{q} \in \mathbb{Z}^3$ is a polarization on $C$, it is not hard to see that

$$\lambda(\underline{q}, 1) = \{(q_1, q_2, q_2), (q_1 + 1, q_2 - 1, q_3), (q_1 + 1, q_2, q_3 - 1)\},$$

that is, we have $\#\lambda(\underline{q}, 1)$=3. Then, by Corollary 118, p. 91, we get that for any polarization $\underline{q}$ on $C$, $\#\lambda(\underline{q}, 1)$=3. In particular for the polarization $\underline{p}$. So, let $\underline{p}' := (p'_1, p'_2, p'_3) \in \lambda(p, 1) - \{q', q''\}$. We claim that

$$p'_1 \in \{\lfloor p_1 \rfloor, \lfloor p_1 \rfloor + 1\},\ p'_2 \in \{\lceil p_2 \rceil - 1, \lceil p_2 \rceil, \lceil p_2 \rceil + 1\} \text{ and}$$

$$p'_3 \in \{d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1\}.$$

Indeed, since $\underline{p}' = (p'_1, p'_2, p'_3) \in \lambda(\underline{p}, 1)$, we have

$$-1 < p'_1 - p_1 \leq 1,\ -1 \leq p'_2 - p_2 \leq 1 \text{ and } -1 \leq p'_3 - p_3 \leq 1.$$

Since $p'_1, p'_2 \in \mathbb{Z}$, inevitably we have

$$p'_1 \in \{\lfloor p_1 \rfloor, \lfloor p_1 \rfloor + 1\} \text{ and } p'_2 \in \{\lceil p_2 \rceil - 1, \lceil p_2 \rceil, \lceil p_2 \rceil + 1\}.$$

On the other hand, since $p'_3 = d - p'_1 - p'_2$, we have

$$p'_3 \in \{d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1\}.$$

Perhaps the reader may be wondering why

$$p'_3 \in \{d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1\}$$

instead of

$$p'_3 \in \{d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 2\}?$$

Indeed, $p'_3 \neq d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 2$, otherwise,

$$\begin{aligned} p'_3 - p_3 &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 2 - p_3 \\ &= d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 2 - p_3 - d + p_1 + p_2 \\ &= \underbrace{p_1 - \lfloor p_1 \rfloor}_{<1} + \underbrace{p_2 - \lceil p_2 \rceil}_{\leq 0} - 2 < -1, \text{ contradiction.} \end{aligned}$$

Now, since $\underline{p}' \in \lambda(\underline{p}, 1) - \{q', q''\}$ and $\#\lambda(\underline{p}, 1) = 3$, due to the possible choices of $p'_1, p'_2$ and $p'_3$, we have that the only possibilities for $\underline{p}'$ are:
i') $\underline{p}' = (\lfloor p_1 \rfloor, \lceil p_2 \rceil + 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1)$, or

ii') $\underline{p}' = (\lfloor p_1 \rfloor, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil + 1)$, or

iii') $\underline{p}' = (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1)$.

However, we can not have (i'). Indeed, suppose $\underline{p}' = (\lfloor p_1 \rfloor, \lceil p_2 \rceil + 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1) \in \lambda(\underline{p}, 1)$. Since $p_1 + p_2 + p_3 = d$,

$$\begin{aligned} p'_{C_1 \cup C_3} - p_{C_1 \cup C_3} &= \lfloor p_1 \rfloor + d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1 - p_1 - p_3 \\ &= d - \lceil p_2 \rceil - 1 - d + p_2 \\ &= -\lceil p_2 \rceil - 1 + p_2 > -1 \\ &\Leftrightarrow p_2 > \lceil p_2 \rceil, \end{aligned}$$

contradiction. So, $\underline{p}' = (\lfloor p_1 \rfloor, \lceil p_2 \rceil + 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1) \notin \lambda(\underline{p}, 1)$

But on the other hand, (ii') and (iii') may occur. Indeed, suppose that $d$ is even. Suppose $p = (d/2 + 1/10, d/2 + 1/10, -2/10)$. In this case we have

$$\begin{aligned} \underline{p}' = (\lfloor d/2 + 1/10 \rfloor, \lceil d/2 + 1/10 \rceil - 1, d - \lfloor d/2 + 1/10 \rfloor - \lceil d/2 + 1/10 \rceil + 1) \\ = (d/2, d/2, 0) \end{aligned}$$

or

$$p' = (\lfloor d/2+1/10\rfloor +1, \lceil d/2+1/10\rceil, d-\lfloor d/2+1/10\rfloor - \lceil d/2+1/10\rceil -1)$$
$$= (d/2+1, d/2+1, -2).$$

However if $\underline{p}' = (d/2+1, d/2+1, -2)$, then $p'_{C_3} - p_{C_3} = -2+2/10 < -1$; contradiction. So, in this case, (iii') is excluded, and thus $ii'$ holds.

On the other hand, suppose $\underline{p} = (d/2+6/10, d/2-4/10, -2/10)$. Then, we have

$$\underline{p}' = (\lfloor d/2+6/10\rfloor, \lceil d/2-4/10\rceil -1, d-\lfloor d/2+6/10\rfloor - \lceil d/2-4/10\rceil +1)$$
$$= (d/2, d/2-1, 1)$$

or

$$\underline{p}' = (\lfloor d/2+6/10\rfloor +1, \lceil d/2-4/10\rceil, d-\lfloor d/2+6/10\rfloor - \lceil d/2-4/10\rceil -1)$$
$$= (d/2+1, d/2, -1).$$

However, if $\underline{p}' = (d/2, d/2-1, 1)$, then $p'_{C_1\cup C_2} - p_{C_1\cup C_2} = d-1-d-2/10 = -12/10 < -1$; contradiction. In this case, (ii') is excluded, and thus (iii') holds.

To conclude, we have

$$\lambda(p,1) = \{(\lfloor p_1\rfloor, \lceil p_2\rceil, d-\lfloor p_1\rfloor - \lceil p_2\rceil), (\lfloor p_1\rfloor +1, \lceil p_2\rceil -1, d-\lfloor p_1\rfloor - \lceil p_2\rceil),$$
$$(\lfloor p_1\rfloor +1, \lceil p_2\rceil, d-\lfloor p_1\rfloor - \lceil p_2\rceil -1)\}$$

or

$$\lambda(p,1) = \{(\lfloor p_1\rfloor, \lceil p_2\rceil, d-\lfloor p_1\rfloor - \lceil p_2\rceil), (\lfloor p_1\rfloor +1, \lceil p_2\rceil -1, d-\lfloor p_1\rfloor - \lceil p_2\rceil),$$
$$(\lfloor p_1\rfloor, \lceil p_2\rceil -1, d-\lfloor p_1\rfloor - \lceil p_2\rceil +1)\}$$

□

**Lemma 129.** *Let $C$ as in Lemma 119. Assume the components of $C$ are $\mathbb{P}^1$. Let $d$ be an integer. Let $\underline{q} = (d,0,0)$ and $\underline{p} = (p_1,p_2,p_3)$ be polarizations on $C$. Let $\mathscr{L} \in J_C$ such that*

$$d + \deg(\mathscr{L}) = |\underline{p}|.$$

*Then for each $i \in \{1,2,3\}$, any $\mathscr{L}$-twister-isomorphism $A_{\mathscr{L}} : J_C^{\underline{q},1} \to J_C^{\underline{p},i}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{\underline{q},1} \to \bar{J}_C^{\underline{p},i}$.*

*Proof.* We give the proof only for $i$=1 and

$$\lambda(p,1) = \{(\lfloor p_1 \rfloor, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil), (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil), \\ (\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil, d - \lfloor p_1 \rfloor - \lceil p_2 \rceil - 1)\}$$

as for any other situation the proof is similar. For notational simplicity, we assume without loss of generality that $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$. By Lemma 128, p. 106, we have

$$\lambda(\underline{q}, 1) = \{(d, 0, 0), (d + 1, -1, 0), (d + 1, 0, -1)\}$$

and

$$\lambda(\underline{p}, 1) = \{(p_1, p_2, p_3), (p_1 + 1, p_2 - 1, p_3), (p_1 + 1, p_2, p_3 - 1)\}.$$

By Proposition 117, p. 91, for each $\underline{e} \in \lambda(\underline{q}, 1)$ there is a unique twister multidegree $t_{\underline{e}}$ such that $\underline{e} + \underline{\deg}(\mathscr{L}) + t_{\underline{e}} \in \lambda(\underline{p}, 1)$. Then we have six cases to consider:

Case 1)

$$(d, 0, 0) + \underline{\deg}(\mathscr{L}) + t_{(d,0,0)} = (p_1, p_2, p_3), \tag{6.3}$$

$$(d + 1, -1, 0) + \underline{\deg}(\mathscr{L}) + t_{(d+1,-1,0)} = (p_1 + 1, p_2 - 1, p_3)$$

and

$$(d + 1, 0, -1) + \underline{\deg}(\mathscr{L}) + t_{(d+1,0,-1)} = (p_1 + 1, p_2, p_3 - 1). \tag{6.4}$$

Case 2)

$$(d, 0, 0) + \underline{\deg}(\mathscr{L}) + t_{(d,0,0)} = (p_1, p_2, p_3),$$

$$(d + 1, -1, 0) + \underline{\deg}(\mathscr{L}) + t_{(d+1,-1,0)} = (p_1 + 1, p_2, p_3 - 1)$$

and

$$(d + 1, 0, -1) + \underline{\deg}(\mathscr{L}) + t_{(d+1,0,-1)} = (p_1 + 1, p_2 - 1, p_3)$$

Case 3)

$$(d, 0, 0) + \underline{\deg}(\mathscr{L}) + t_{(d,0,0)} = (p_1 + 1, p_2 - 1, p_3),$$

$$(d + 1, -1, 0) + \underline{\deg}(\mathscr{L}) + t_{(d+1,-1,0)} = (p_1, p_2, p_3)$$

and

$$(d + 1, 0, -1) + \underline{\deg}(\mathscr{L}) + t_{(d+1,0,-1)} = (p_1 + 1, p_2, p_3 - 1).$$

Case 4)

$$(d, 0, 0) + \underline{\deg}(\mathscr{L}) + t_{(d,0,0)} = (p_1 + 1, p_2 - 1, p_3),$$

$$(d+1,-1,0)+\underline{\deg}(\mathscr{L})+t_{(d+1,-1,0)}=(p_1+1,p_2,p_3-1)$$

and

$$(d+1,0,-1)+\underline{\deg}(\mathscr{L})+t_{(d+1,0,-1)}=(p_1,p_2,p_3).$$

Case 5)

$$(d,0,0)+\underline{\deg}(\mathscr{L})+t_{(d,0,0)}=(p_1+1,p_2,p_3-1), \tag{6.5}$$

$$(d+1,-1,0)+\underline{\deg}(\mathscr{L})+t_{(d+1,-1,0)}=(p_1,p_2,p_3)$$

and

$$(d+1,0,-1)+\underline{\deg}(\mathscr{L})+t_{(d+1,0,-1)}=(p_1+1,p_2-1,p_3). \tag{6.6}$$

Case 6)

$$(d,0,0)+\underline{\deg}(\mathscr{L})+t_{(d,0,0)}=(p_1+1,p_2,p_3-1),$$

$$(d+1,-1,0)+\underline{\deg}(\mathscr{L})+t_{(d+1,-1,0)}=(p_1+1,p_2-1,p_3)$$

and

$$(d+1,0,-1)+\underline{\deg}(\mathscr{L})+t_{(d+1,0,-1)}=(p_1,p_2,p_3).$$

However, since the set of twister multidegree on $C$ is

$$T:=\mathbb{Z}(-2,1,1)+\mathbb{Z}(1,-2,1)+\mathbb{Z}(1,1,-2),$$

we have that some of these cases can not occur.

Case 1) This case is possible because $t_{(d,0,0)}-t_{(d+1,-1,0)}=t_{(d,0,0)}-t_{(d+1,0,-1)}=(0,0,0)\in T$.

Case 2) In this case we have $t_{(d,0,0)}-t_{(d+1,-1,0)}=(0,-1,1)\notin T$. So this case is not possible.

Case 3) We have $t_{(d,0,0)}-t_{(d+1,-1,0)}=(2,-2,0)\notin T$. This case is not possible.

Case 4) We have $t_{(d,0,0)}-t_{(d+1,-1,0)}=(2,-2,0)\notin T$. This case is not possible.

Case 5) We have $t_{(d,0,0)}-t_{(d+1,-1,0)}=(2,-1,-1)$, $t_{(d,0,0)}-t_{(d+1,0,-1)}=(1,1,-2)$ and $t_{(d+1,-1,0)}-t_{(d+1,0,-1)}=(-1,2,-1)\in T$. This case is possible.

Case 6) We have $t_{(d,0,0)}-t_{(d+1,-1,0)}=(1,0,-1)\notin T$. So this case is not possible.

Therefore, we need to consider only Cases (1) and (5).

Proof in Case 1) Let $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ be twisters on $C$ such that $\underline{\deg}(\mathscr{T}_1) = t_{(d,0,0)}$, $\underline{\deg}(\mathscr{T}_2) = t_{(d+1,-1,0)}$ and $\underline{\deg}(\mathscr{T}_3) = t_{(d+1,0,-1)}$. Let

$$B^1_{\mathscr{L}} : J_C^{(d,0,0)} \to J_C^{(p_1,p_2,p_3)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_1,$$

$$B^2_{\mathscr{L}} : J_C^{(d+1,-1,0)} \to J_C^{(p_1+1,p_2-1,p_3)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_2,$$
$$B^3_{\mathscr{L}} : J_C^{(d+1,0,-1)} \to J_C^{(p_1+1,p_2,p_3-1)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_3,$$

and let $A_{\mathscr{L}} : J_C^{\underline{q},1} \to J_C^{\underline{p},1}$ be the $\mathscr{L}$-twister-isomorphism induced by $B^1_{\mathscr{L}}$, $B^2_{\mathscr{L}}$ and $B^3_{\mathscr{L}}$.
Claim: $A_{\mathscr{L}}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{\underline{q},1} \to \bar{J}_C^{\underline{p},1}$. Indeed, let $\mathscr{L} := \mathfrak{m}_{N_1} \otimes \mathscr{M}_1 \in \bar{J}_C^{\underline{q},1} - J_C^{\underline{q},1}$, where $\mathscr{M}_1$ is an invertible sheaf on $C$ with $\underline{\deg}(\mathscr{M}_1) = (d+1,0,0)$. Let

$$\{P_t\}_{t\in\mathbb{N}} \subseteq C_1 - \{N_1\} \text{ and } \{Q_t\}_{t\in\mathbb{N}} \subseteq C_3 - \{N_1\}$$

be sequences of smooth points of $C$ such that

$$\lim_{t\to\infty} P_t = N_1 = \lim_{t\to\infty} Q_t.$$

Now, notice that for each $t \in \mathbb{N}$, we have

$$\underline{\deg}(\mathfrak{m}_{P_t} \otimes \mathscr{M}_1) = (d,0,0), \text{ that is, } \mathfrak{m}_{P_t} \otimes \mathscr{M}_1 \in J_C^{(d,0,0)}$$

and

$$\underline{\deg}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}_1) = (d+1,0,-1)), \text{ that is, } \mathfrak{m}_{Q_t} \otimes \mathscr{M}_1 \in J_C^{(d+1,0,-1)}.$$

Furthermore,

$$\lim_{t\to\infty} \mathfrak{m}_{P_t} \otimes \mathscr{M}_1 = \mathfrak{m}_{N_1} \otimes \mathscr{M}_1 = \lim_{t\to\infty} \mathfrak{m}_{Q_t} \otimes \mathscr{M}_1.$$

Since the connected components of $J_C^{\underline{q},1}$ are

$$J_C^{(d,0,0)}, \ J_C^{(d+1,-1,0)} \text{ and } J_C^{(d+1,0,-1)},$$

and $C$ has genus 1, we have that $\mathscr{L}$ can be approximated only through the components $J_C^{(d,0,0)}$ and $J_C^{(d+1,0,-1)}$; and only in the way described above.

So, in order to show that $A_{\mathscr{L}}$ extends, we must show

$$\begin{aligned}\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t}\otimes\mathscr{M}_1) &= \mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1\\ &\cong \mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3\\ &= \lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t}\otimes\mathscr{M}_1),\end{aligned}$$

and for this we give a proof only for this $\mathscr{L}$, since the other cases of $\mathscr{L}\in \bar{J}^{\underline{q},1}_C - J^{\underline{q},1}_C$ the proof is similar. Indeed, let

$$\mathscr{L}_1 := \mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1 \text{ and } \mathscr{L}_2 := \mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3.$$

From (6.3) and (6.4), we have

$$\underline{\deg}(\mathscr{L}\otimes\mathscr{T}_1) = (p_1-d, p_2, p_3) = \underline{\deg}(\mathscr{L}\otimes\mathscr{T}_3).$$

Since $\underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1) = (d,0,-1)$, we have

$$\begin{aligned}\underline{\deg}(\mathscr{L}_1) &= \underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1)\\ &= \underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1)+\underline{\deg}(\mathscr{L}\otimes\mathscr{T}_1)\\ &= (p_1,p_2,p_3-1)\\ &= \underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1)+\underline{\deg}(\mathscr{L}\otimes\mathscr{T}_3)\\ &= \underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3)\\ &= \underline{\deg}(\mathscr{L}_2).\end{aligned}$$

Now, we claim that $\mathscr{L}_1,\ \mathscr{L}_2\in \bar{J}^{\underline{p},1}_C$. Indeed, notice that for each $i=1,2$, we have

$$0=\deg_{C_1}(\mathscr{L}_i)-p_1,\ 0=\deg_{C_2}(\mathscr{L}_i)-p_2 \text{ and } -1=\deg(\mathscr{L}_i)-p_3.$$

On the other hand, since

$$0=\deg_{C_1\cup C_2}(\mathscr{L}_i)-p_1-p_2,\ 0=\deg_{C_1\cup C_3}(\mathscr{L}_i)-p_1-p_3$$

and

$$-1=\deg_{C_2\cup C_3}(\mathscr{L}_i)-p_2-p_3,$$

we have that for each $i$, $\mathscr{L}_i$ satisfies the conditions of $\underline{p}$-1-quasistability, which shows our claim.

Finally we prove that $\mathscr{L}_1\cong\mathscr{L}_2$, that is,

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t}\otimes\mathscr{M}_1)\cong \lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t}\otimes\mathscr{M}_1).$$

Indeed, let $\nu_{N_1}: C' \to C$ be the normalization of $C$ at $N_1$. Let $\{N_{1,1}, N_{1,3}\} := \nu_{N_1}^{-1}(N_1)$. Let

$$\mathscr{L}_{1,1} := \mathfrak{m}_{N_{1,1}} \otimes \mathfrak{m}_{N_{1,3}} \otimes \nu_{N_1}^*(\mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1) = \nu_{N_1}^*(\mathscr{L}_1)/\text{torsion}$$

and

$$\mathscr{L}_{1,2} := \mathfrak{m}_{N_{1,1}} \otimes \mathfrak{m}_{N_{1,3}} \otimes \nu_{N_1}^*(\mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3) = \nu_{N_1}^*(\mathscr{L}_2)/\text{torsion}.$$

By Proposition 74, p. 65, we have

$$\nu_{N_1 *}(\mathscr{L}_{1,1}) = \mathscr{L}_1 \text{ and } \nu_{N_1 *}(\mathscr{L}_{1,2}) = \mathscr{L}_2.$$

Since $C'$ is a curve of compact type, by Proposition 43, p. 32, the sheaves $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$ are uniquely determined, up to isomorphism, by their restrictions to the irreducible components of $C'$. However, since

$$\begin{aligned} \underline{\deg}(\mathscr{L}_{1,1}) &= \underline{\deg}(\mathscr{L}_1) \\ &= (p_1, p_2, p_3 - 1) \\ &= \underline{\deg}(\mathscr{L}_2) \\ &= \underline{\deg}(\mathscr{L}_{1,2}), \end{aligned}$$

and since the components of $C'$ are $\mathbb{P}^1$, we have that the restrictions of $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$ are isomorphic, implying $\mathscr{L}_{1,1} \cong \mathscr{L}_{1,2}$.

Hence,

$$\mathscr{L}_1 = \nu_{N_1 *}(\mathscr{L}_{1,1}) \cong \nu_{N_1 *}(\mathscr{L}_{1,2}) = \mathscr{L}_2,$$

that shows our Claim, proving consequently Case 1).

Proof in Case 5) We keep the same notation as in Case 1). Let

$$B^1_{\mathscr{L}} : J_C^{(d,0,0)} \to J_C^{(p_1+1,p_2,p_3-1)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_1,$$

$$B^2_{\mathscr{L}} : J_C^{(d+1,-1,0)} \to J_C^{(p_1,p_2,p_3)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_2,$$

$$B^3_{\mathscr{L}} : J_C^{(d+1,0,-1)} \to J^{(p_1+1,p_2-1,p_3)}, \ \mathscr{I} \mapsto \mathscr{I} \otimes \mathscr{L} \otimes \mathscr{T}_3,$$

and let $A_{\mathscr{L}} : J_C^{q,1} \to J_C^{p,1}$ be the $\mathscr{L}$-twister-isomorphism induced by $B^1_{\mathscr{L}}$, $B^2_{\mathscr{L}}$ and $B^3_{\mathscr{L}}$.

Claim: $A_{\mathscr{L}}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{q,1} \to \bar{J}_C^{p,1}$.

Indeed, let

$$\mathscr{L}, \ \{P_t\}_{t\in\mathbb{N}} \subseteq C_1 - \{N_1\} \text{ and } \{Q_t\}_{t\in\mathbb{N}} \subseteq C_3 - \{N_1\}$$

be exactly as in the proof of Case 1). As before, to prove that $A_{\mathscr{L}}$ extends, it is enough to show

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t}\otimes\mathscr{M}_1)\cong\lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t}\otimes\mathscr{M}_1).$$

However, in contrast with the Case 1), showing this isomorphism is a little trickier, since the natural limits

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t}\otimes\mathscr{M}_1)=\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1$$

and

$$\lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t}\otimes\mathscr{M}_1)=\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3$$

do not belong to $\bar{J}_C^{\underline{p},1}$. Indeed, from (6.5) e (6.6) we have

$$\underline{\deg}(\mathscr{L}\otimes\mathscr{T}_1)=(p_1+1-d,p_2,p_3-1)$$

and

$$\underline{\deg}(\mathscr{L}\otimes\mathscr{T}_3)=(p_1-d,p_2-1,p_3+1).$$

Since $\underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1)=(d,0,-1)$, we have

$$\underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1)=(p_1+1,p_2,p_3-2)\text{ and}$$

$$\underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3)=(p_1,p_2-1,p_3).$$

Then, since $\deg_{C_3}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1)-p_3=-2$, it follows that

$$\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_1\notin\bar{J}_C^{\underline{p},1}.$$

On the other hand, since

$$\underline{\deg}(\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3)=(p_1,p_2-1,p_3),$$

we have that $\deg_{C_1\cup C_2}(\mathscr{L})-p_1-p_2=-1$ which implies that $\mathfrak{m}_{N_1}\otimes\mathscr{M}_1\otimes\mathscr{L}\otimes\mathscr{T}_3\notin\bar{J}_C^{\underline{p},1}$.

So, in order to solve this problem, we look for two sheaves $\mathscr{N}_1$ and $\mathscr{N}_2$ such that

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t}\otimes\mathscr{M})\cong\mathscr{N}_1\cong\mathscr{N}_2\cong\lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t}\otimes\mathscr{M}).$$

First we find the sheaf $\mathscr{N}_1$. Fix a smooth point $B\in C_3$. For each $t\in\mathbb{N}$, let $r_t$ be the line passing trough $B$ and $P_t$. Since $C\subseteq\mathbb{P}^2$, $r_t$ intersects $C_2$ at a smooth point $P'_t$. Fix a smooth point $A \in C_1$ and let $s$ be the line passing through $A$ and $B$. Then $s$ intersects $C_2$ at a smooth point $D$; see Figure 6.5.

Hence, since for each $t \in \mathbb{N}$ the divisor associated to the rational function $r_t/s$ on $C$ is $P_t + P'_t - A - D$, we have

$$\mathfrak{m}_{P_t} \cong \mathfrak{m}^*_{P'_t} \otimes \mathfrak{m}_A \otimes \mathfrak{m}_D.$$

Therefore,

$$\begin{aligned}
\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}_1) &= \lim_{t\to\infty} \mathfrak{m}_{P_t} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1 \\
&\cong \lim_{t\to\infty} \mathfrak{m}^*_{P'_t} \otimes \mathfrak{m}_A \otimes \mathfrak{m}_D \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1 \\
&= \mathfrak{m}^*_{N_3} \otimes \mathfrak{m}_A \otimes \mathfrak{m}_D \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1.
\end{aligned}$$

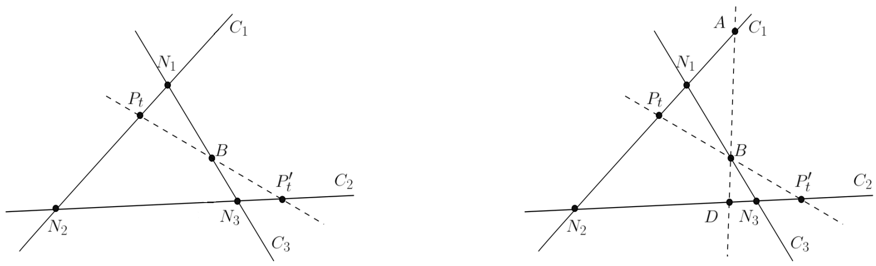


Figure 6.5:

Let $\mathscr{N}_1 := \mathfrak{m}^*_{N_3} \otimes \mathfrak{m}_A \otimes \mathfrak{m}_D \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1$. Then

$$\begin{aligned}
\underline{\deg}(\mathscr{N}_1) &= \underline{\deg}(\mathfrak{m}^*_{N_3} \otimes \mathfrak{m}_A \otimes \mathfrak{m}_D \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1) \\
&= \underline{\deg}(\mathfrak{m}^*_{N_3}) + \underline{\deg}(\mathfrak{m}_A) + \underline{\deg}(\mathfrak{m}_D) + \underline{\deg}(\mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_1) \\
&= (0,0,0) + (-1,0,0) + (0,-1,0) + (p_1+2, p_2, p_3-1) \\
&= (p_1+1, p_2-1, p_3-1)
\end{aligned}$$

which implies that $\mathscr{N}_1 \in \bar{J}^{\underline{p},1}_C$. Indeed, first we have

$$1 = \deg_{C_1}(\mathscr{N}_1) - p_1,\ -1 = \deg_{C_2}(\mathscr{N}_2) - p_2 \text{ and } -1 = \deg_{C_3}(\mathscr{N}_1) - p_3,$$

and second,

$$0 = \deg_{C_1\cup C_2}(\mathscr{N}_1) - p_1 - p_2,\ 0 = \deg_{C_1\cup C_3}(\mathscr{N}_1) - p_1 - p_3$$

and $-1 = \deg_{C_2\cup C_3}(\mathscr{N}_1) - p_2 - p_3$. That is, $\mathscr{N}_1$ satisfies the conditions of $\underline{p}$-1-quasistability.

Now we find the sheaf $\mathscr{N}_2$: Fix a smooth point $B' \in C_2$. For each $t \in \mathbb{N}$, let $r'_t$ be the line passing through $B'$ and $Q_t$. Since $C \subseteq \mathbb{P}^2$, $r'_t$ intersects $C_1$ at a smooth point $C_t$. Fix a smooth point $D' \in C_3$, and for each $t \in \mathbb{N}$, let $s'_t$ be the line passing through $C_t$ and $D'$. So $s'_t$ intersects $C_2$ at a smooth point $Q'_t$; see Figure 6.6.

Then for each $t \in \mathbb{N}$, we have

$$-Q_t \equiv -Q'_t + B' - D'$$

because $Q_t - Q'_t + B' - D'$ is the divisor associated to the rational function $r'_t/s'_t$ on $C$. Thus, for each $t$,

$$\mathfrak{m}_{Q_t} \cong \mathfrak{m}_{Q'_t} \otimes \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'}.$$

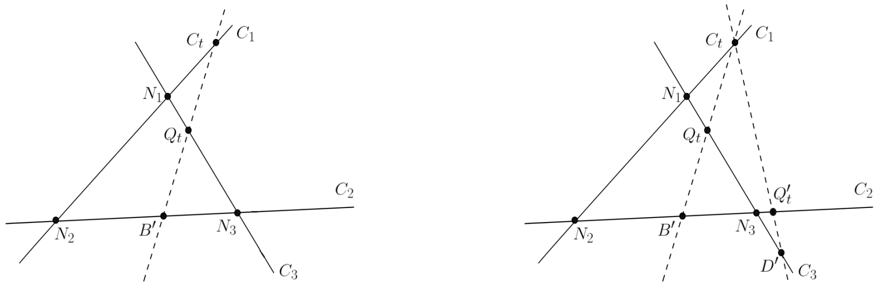


Figure 6.6:

Hence, we have

$$\begin{aligned}
\lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}_1) &= \lim_{t\to\infty} \mathfrak{m}_{Q_t} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3 \\
&\cong \lim_{t\to\infty} \mathfrak{m}_{Q'_t} \otimes \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3 \\
&= \mathfrak{m}_{N_3} \otimes \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3.
\end{aligned}$$

Let $\mathscr{N}_2 := \mathfrak{m}_{N_3} \otimes \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3$. Then

$$\begin{aligned}
\underline{\deg}(\mathscr{N}_2) &= \underline{\deg}(\mathfrak{m}_{N_3} \otimes \mathfrak{m}^*_{B'} \otimes \mathfrak{m}_{D'} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3) \\
&= \underline{\deg}(\mathfrak{m}_{N_3}) + \underline{\deg}(\mathfrak{m}^*_{B'}) + \underline{\deg}(\mathfrak{m}_{D'}) + \underline{\deg}(\mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3) \\
&= (0,-1,-1) + (0,1,0) + (0,0,-1) + (p_1+1, p_2-1, p_3+1) \\
&= (p_1+1, p_2-1, p_3-1).
\end{aligned}$$

Then, since $\underline{\deg}(\mathscr{N}_2) = (p_1+1, p_2-1, p_3-1) = \underline{\deg}(\mathscr{N}_1)$, we also have $\mathscr{N}_2 \in \bar{J}^{\underline{p},1}_C$.

Finally we claim that $\mathscr{N}_1 \cong \mathscr{N}_2$. Indeed, let $\nu_{N_3} : C' \to C$ be the normalization of $C$ at $N_3$. Let $\{N_{3,2}, N_{3,3}\} := (\nu_{N_3})^{-1}(N_3)$. Let

$$\mathscr{L}_1' := \nu_{N_3}^*(\mathfrak{m}_A \otimes \mathfrak{m}_D \otimes \mathscr{M} \otimes \mathscr{L} \otimes \mathscr{T}_1) = \nu_{N_3}^*(\mathscr{N}_1)/\text{torsion and}$$

$$\mathscr{L}_2' := \mathfrak{m}_{N_{3,2}} \otimes \mathfrak{m}_{N_{3,3}} \otimes \nu_{N_3}^*(\mathfrak{m}_{B'}^* \otimes \mathfrak{m}_{C'} \otimes \mathscr{M}_1 \otimes \mathscr{L} \otimes \mathscr{T}_3) = \nu^*{}_{N_3}(\mathscr{N}_2)/\text{torsion.}$$

By Proposition 74, p. 65, we have

$$\nu_{N_3*}(\mathscr{L}_1') = \mathscr{N}_1, \text{ and } \nu_{N_3*}(\mathscr{L}_2') = \mathscr{N}_2.$$

Since $C'$ is a curve of compact type, $\mathscr{L}_1'$ and $\mathscr{L}_2'$ are uniquely determined by their restrictions to the irreducible components of $C'$. Since

$$\begin{aligned} \underline{\deg}(\mathscr{L}_1') &= \underline{\deg}(\mathscr{N}_1) \\ &= (p_1 + 1, p_2 - 1, p_3 - 1) \\ &= \underline{\deg}(\mathscr{N}_2) \\ &= \underline{\deg}(\mathscr{L}_2') \end{aligned}$$

and the irreducible components of $C'$ are $\mathbb{P}^1$, it follows that the restrictions of $\mathscr{L}_1'$ and $\mathscr{L}_2'$ to the irreducible components of $C'$ are isomorphic, implying $\mathscr{L}_1' \cong \mathscr{L}_2'$. Therefore,

$$\mathscr{N}_1 = \nu_{N_3*}(\mathscr{L}_1') \cong \nu_{N_3*}(\mathscr{L}_2') = \mathscr{N}_2,$$

that is,

$$\lim_{t\to\infty} B^1_{\mathscr{L}}(\mathfrak{m}_{P_t} \otimes \mathscr{M}) \cong \mathscr{N}_1 \cong \mathscr{N}_2 \cong \lim_{t\to\infty} B^3_{\mathscr{L}}(\mathfrak{m}_{Q_t} \otimes \mathscr{M}).$$

Therefore $A_{\mathscr{L}}$ extends to an isomorphism $\bar{A}_{\mathscr{L}} : \bar{J}_C^{q,1} \to \bar{J}_C^{p,1}$, finishing the proof of the claim and hence, the proof of the Case (5). □

# Bibliography


[Ale] V. Alexeev, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. 40 (2004), no. 4, 1241-1265. MR 2105707 (2006a:14016).

[AK80] A. Altman, S. Kleiman, *Compactifying the Picard scheme*. Adv. Math. 35 (1980), 50-112.

[AIK] A. Altman, A. Iarrobino, S. Kleiman, *Irreducibility of the compactified Jacobian*, In: Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 1–12. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[AK] A. Altman, S. Kleiman, *Compactifying the Jacobian*, Bull. Amer. Math. Soc. **82** (1976) 947–949.

[A07] D. Arinkin, *Cohomology of line bundles on compactified Jacobians*, At http://arxiv.org/abs/0705.0190.

[A10] D. Arinkin, *Autoduality of compactified Jacobians for curves with plane singularities*, At http://arxiv.org/abs/1001.3868.

[Bea77] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. 41 (1977), no. 2, 149-196.

[BRL] S. Bosch, W. Liitkebohmert and M. Raynaud. *Néron Models*. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 21, Springer Verlag (1990).

[Ca94] L. Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*. Journ. of the Amer. Math. Soc. 7 (1994), 589660.

[Ca08] C. Lucia, *Geometry of the Theta Divisor of a compactified Jacobian*, http:arxiv.org/pdf/0707.4602v4.pdf.

[Ca10] L. Caporaso, *Compactified Jacobians of Néron type*, Rend. Lincei. Mat. Appl. 21 (2010), 1-15.

[CE] L. Caporaso, E. Esteves, *On Abel maps of stable curves*, Michigan Math. J. (2007) 575–607.

[CCE] L. Caporaso, J. Coelho, E. Esteves, *Abel maps of Gorenstein curves*, Rend. Circ. Mat. Palermo **57** (2008) 33–59.

[MKV] S. Casalaina-Martin, J. L. Kass, F. Viniani, *The Local structure of compactified Jacobians*, http://arxiv.org/pdf/1107.4166v2.pdf.

[Ca82] F. Catanese, *Pluricanonical-Gorenstein-Curves*, Progr. Math. 24(1982), 51-95.

[CP] J. Coelho, M. Pacini, *Abel maps for curves of compact type*, J. Pure and Applied Algebra (2010) 1319–1333.

[CEP] J. Coelho, M. Pacini, E. Esteves, *Degree-2 Abel maps for nodal curves*, http://arxiv.org/pdf/1212.1123v1.pdf

[JC] J. Coelho, *Abel maps for reducible curves*, Instituto Nacional de Matemática, IMPA, 2006, http://www.preprint.impa.br/FullText/coelho/main.pdf.

[DM69] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. IHES 36 (1969), 75-109.

[D'S] C. D'Souza, *Compactification of generalized Jacobians*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88** (1979) 419–457.

[Ser] E. Sernesi, *Deformations of Algebraics Schemes*, Grundlehren mathematischen Wissenschaften 334, Springer.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften 267. Springer-Verlag, New York, 1985.

[ACG] E. Arbarello, M. Cornalba, P. A. Griffiths, *Geometry of algebraic curves. Vol. II*, Grundlehren der Mathematischen Wissenschaften 267. Springer-Verlag, New York, 2011.

[E01] E. Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. (2001) 3045–3095.

[E09] E. Esteves, *Compactified Jacobians of curves with spine decomposition*, Geometriae Dedicata **139** (2009) 167–181.

[EGK] E. Esteves, M. Gagné, S. Kleiman, *Autoduality of the compactified Jacobian*, J. London Math. Soc. (2) **65** (2002) 591–610.

[EK] E. Esteves, S. Kleiman, *The compactified Picard scheme of the compactified Jacobian*, Adv. in Math. **198** (2005) 484–503.

[FGAE] B. Fantechi, L. Gottsche, L. lllusie, S. L. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Mathematical Surveys and Monographs, 123, 2005.

[Ge82] D. Gieseker, *Moduli of curves*, Tata Inst. Fund. Res. Lecture Notes, Springer-Verlag, 1982.

[G1] A. Grothendieck, *Fondements de la Géométrie Algébrique.* Extraits du Séminaire Bourbaki, 1957-1962.

[G2] A. Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique V*, Séminaire Bourbaki 232, 1962.

[GD] A. Grothendieck, J. *Dieudonné, Éléments de géométrie algébrique*, Publ. Math. IHES, OIII, Vol. 11, 1961; III2, Vol. 17, 1963; IV3, Vol. 28, 1966; and IV4, Vol. 32, 1967.

[HM98] J. Harris, I. Morrison, *Moduli of Curves.* Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[Har] R. Hartshorne, *Algebraic Geometry.* Graduate texts in Math., Springer- Verlag, 1977.

[H] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987) 91–114.

[Igu56] J. Igusa, *Fiber systems of Jacobian varieties*, Amer. J. Math. 78 (1956), 171- 199.

[LS] S. Lichtenbaum, M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. 128 (1967), 41-70.

[KM] F. Knudsen, D. Mumford, *The projectivity of the moduli space of stable curves I: Preliminaries on "det" and "Div"*, Math. Scand. 39 (1976), 19-55.

[May70] A.L. Mayer, *Compactification of the variety of moduli of curves, lectures 2 and 3*, Seminar on degeneration of algebraic varieties, Institute for Advanced Study, Princeton (1969/70), (mimeographed notes).

[Mu64] D. Mumford, *Further comments on boundary points*, AMS Summer School at Woods Hole (1964), (mimeographed notes).

[MG] D. Mumford, D. Gieseker, *Stability of projective varieties* Enseign. Math. (2) 23 (1977), 39-110.

[MF] D. Mumford, J. Fogarty, *Geometric invariant theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 34. Springer-Verlag, Berlin, 1982.

[MRV] M. Melo, A. Rapagnetta, F. Viviani, *Fourier–Mukai and autoduality for compactified Jacobians. I*, At http://arxiv.org/abs/1207.7233, with an appendix by A. C. López-Martín.

[Mu65] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik 34, Springer, Berlin, 1965.

[Mu74] D. Mumford, *Abelian varieties*, Oxford University Press, 1974.

[SP] *Stacks Project*, http://stacks.math.columbia.edu/, current maintainer: Aise Johan de Jong, Columbia University, 2005.

[Pan] Rahul Pandharipande, *A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bun- dles*, J. Amer. Math. Soc. 9 (1996), no. 2, 425-471. MR 1308406 (96f:14014).

[Ses82] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque 96 (1982).

[OS] T. Oda and C.S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. 253 (1979), 190. MR 82e:14054.

[Sim] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, I, Inst. Hautes Etudes Sci. Publ. Math. (1994), no. 79, 47-129. MR 1307297 (96e:14012).

[Vis] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, In Fundamental algebraic geometry, volume 123 of Math. Surveys Monogr., 1-104. Amer. Math. Soc, Providence, RI, 2005.