INTRODUCTION TO THE MINIMAL MODEL PROGRAM IN ALGEBRAIC GEOMETRY

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Abstract. These are lecture notes for the lectures given at the IV CONGRESO LATINOAMERICANO DE MATEMÁTICOS, Córdoba, Argentina, 2012.

Contents

Introduction

The aim of these lectures is to introduce graduate students working on algebraic geometry and related fields to the main ideas of the so called Minimal Model Program, giving an overview of the subject. We have chosen to keep the lectures as elementary as possible, and hope that it is accessible to students who have completed a one-year basic course on algebraic geometry. In particular, we expect the students to be familiar with [7], whose notation we will follow throughout the text.

Given the limitted time, we will not be able to go into the details of the amazing techniques developed in the context of the Minimal Model Program. Great part of these notes are devoted to reviewing the classification of surfaces, and translating it into modern language, while introducing some of the concepts that are key to the Minimal Model Program. We will spend very little time discussing the more recent and very exciting advances in the subject (specially those in [3]). We hope with these

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lectures to motivate the beginners to venture into more advanced material on the subject. We will suggest some further reading at the end of the notes.

Throughout these lectures all varieties are assumed to be irreducible, reduced and defined over the field $\mathbb{C}$ of complex numbers.

Curves and surfaces are always assumed to be irreducible, reduced and projective.

Recall that the Picard group of a projective variety X is the group $\mathrm{Pic}(X)$ of invertible sheaves on X modulo isomorphism. Equivalently, $\mathrm{Pic}(X)$ is the quotient of the group of Cartier divisors on X modulo linear equivalence. By abuse of notaion, we often identify a cartier divisor D on X with its class $[D]$ in $\mathrm{Pic}(X)$.

We denote by Ω^1_X the sheaf of Kähler differentials of X, and by $\omega_X = \wedge^{\dim X}\Omega^1_X$ its canonical sheaf. We denote by $K_X \in \mathrm{Div}(X)$ any divisor on X such that $\omega_X \cong \mathcal{O}_X(K_X)$, and call it a *canonical divisor* or the *canonical class* of X.

1. The classification problem in Algebraic Geometry

We are interested in the following classical problem.

To classify projective varieties up to birational equivalence.

What do we mean by this? Here are some of our goals.

- We want to distinguish vareties by means of *invariants*.
- We want to pick distinguished representatives on each birrational class. In some sense, these representatives should be the "simplest" varieties in their class.
- Given a projective variety, we want to understand the birational transformations needed to bring it to a distinguished representative of it class.

First of all, we recall the Hironaka's famous Resolution of Singularities Theorem: any projective variety X admits a resolution of singularities, i.e., there exists a smooth projective variety $\tilde{X}$ and a birational morphism $f : \tilde{X} \to X$. So we can always assume we start with a smooth variety, even though, as we shall see, it is unavoidable to work with (mildly) singular varieties in order to achieve our classification goals.

1.1 (Classification of curves)**.** A smooth projective curve is nothing but a compact Riemann surface. Two smooth projective curves are birationally equivalent if and only if they are isomorphic. So there is a unique smooth projective model in each birational class of projective curves. We define *genus* $g(X)$ of a smooth projective curve X as

$$g(X) \;=\; h^0(X, \omega_X).$$

This numerical invariant allows us to completely solve the classification problem for curves.

- $g(X) = 0$ if and only if $X \cong \mathbb{P}^1$.
- $g(X) = 1$ if and only if X is an elliptic curve, and there is a 1-dimensional family of those, parametrized by $\mathbb{C}$ (via the j-invariant).
- For each $g \geq 2$, there is an algebraic variety M_g of dimension $3g-3$ parametrizing smooth projective curves of genus g.

For surfaces the situation is not as simple. Given a smooth projective surface S, we can consider the blowup $\tilde{S}$ of S at a point $P \in S$. This is a smooth projective surface birationally equivalent but not isomorphic to S. It is easy to argue that S is "simpler" than $\tilde{S}$. It turns out that any smooth projective surface can be obtained from a distinguished representative of its class by a sequence of blowups. Such distinguished representatives are classically called *minimal surfaces*. We will revise and summarize this theory in Section 2, explaining how the 3 goals pointed out in the beginning of the section are achieved in this case.

While the classification of surfaces was established by the Italian school by the beginning of the 20th century, the first developments on the classification problem in higher dimensions started to take place in the beginning of the 1980's with important ideas from Mori and Reid, among others. With the contributions of many algebraic geometers, such as Kawamata, Kollár, Shokurov, just to mention a few, a powerful theory of classification of projective varieties was then developed. This is called the *Minimal Model Program* (MMP for short). The program was fully established for 3-folds by Mori in [11], yielding him the Fields Medal in 1990. We will give an overview of this program in Section 3.

Only part of the 3-fold theory could be carried out to higher dimensions, and new ideas and techniques were required. A major achievement was obtained recently by Birkar, Cascini, Hacon, and McKernan in [3]. We will address this briefly in Section 4.

2. Classification of projective surfaces

In this section we review the birational classification of complex projective surfaces. We start by recalling the intersection theory on surfaces. Then we state some of the classical results of Castelnuovo and Enriques. We refer to [7, Chapter V] and [2] for details and proofs.

At the end of the section, we rephrase these results from a modern perspective. This reinterpretation suggests generalizations to higher dimensions, which will be explored in the forthcoming sections.

2.1. **Intersection theory on surfaces.** Let S be a smooth surface.

Theorem 2.1 (Intersection form on surfaces)**.** *There exists a unique symmetric bilinear form*

$$\cdot\ : \mathrm{Div}(S) \times \mathrm{Div}(S) \ \to\ \mathbb{Z}$$

satisfying the following conditions.

1. *Given $D, D' \in \mathrm{Div}(S)$, the intersection number $D\cdot D'$ depends only on the linear equilalence classes of D and D'.*
2. *If C and D are curves on S meeting transversely, then $C \cdot D = \sharp(C \cap D)$.*

Definition 2.2. Two divisors $D, D' \in \mathrm{Div}(S)$ are said to be *numerically equivalent* if $D \cdot C = D' \cdot C$ for every curve $C \subset S$. In this case we write $D \equiv D'$. We write $\mathrm{Num}(S)$ for the quotient group $\mathrm{Div}(S)/\equiv$. By the Theorem of the base of Néron-Severi, $\mathrm{Num}(S)$ is a finitely gernerated abelian group. Its rank is called the *Picard number* of S, and is denoted by $\rho(S)$.

Later on it will be important to consider also the $\rho(S)$-dimensional $\mathbb{R}$-vector space $N^1(S) := \mathrm{Num}(S)\otimes_{\mathbb{Z}}\mathbb{R}$. The intersection form on S induces a nondegenerate symmetric bilinear form $\cdot : N^1(S) \times N^1(S) \to \mathbb{R}$.

Example 2.3. $S = \mathbb{P}^2$**.** In this case $\mathrm{Pic}(S) = \mathbb{Z} \cdot [H]$, where $H \subset \mathbb{P}^2$ is a hyperplane section. The intersection form on S is given by $H^2 = 1$.

Example 2.4 (Hirzebruch surfaces)**.** Let $n \in \mathbb{Z}$ be a non-negative integer, and consider the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, with structure morphism $\pi : \mathbb{F}_n \to \mathbb{P}^1$. There is a section $\sigma \subset \mathbb{F}_n$ of π such that $\sigma^2 = -n$. If $n \geq 1$, then such section is unique.

Moreover, if $\sigma' \subset \mathbb{F}_n$ is a section of π different from σ, then $\sigma'^2 \geq n$. We denote by $F \cong \mathbb{P}^1$ a fiber of π. We have

$$\operatorname{Pic}(\mathbb{F}_n) \;=\; \mathbb{Z}\cdot[F] \oplus \mathbb{Z}\cdot[\sigma],$$

and the intersection form on $\mathbb{F}_n$ is given by

- $F^2 = 0$;
- $F \cdot \sigma = 1$; and
- $\sigma^2 = -n$.

For $n = 0$, $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. For $n = 1$, $\mathbb{F}_1$ is isomorphic to the blowup of $\mathbb{P}^2$ at one point, and under this isomorphism σ corresponds to the exceptional divisor. For $n \geq 2$, $\mathbb{F}_n$ admits the following geometric realization. Let $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ be the n^{th} Veronese embedding of $\mathbb{P}^1$, given by $(s:t) \mapsto (s^n : s^{n-1}t : \cdots st^{n-1} : t^n)$. Let $Z \subset \mathbb{P}^{n+1}$ be the cone over $\nu_n(\mathbb{P}^1)$ with vertex P. Then $\mathbb{F}_n$ is isomorphic to the blowup of Z at the point P, and under this isomorphism σ corresponds to the exceptional divisor.

Example 2.5 (Blowups)**.** Let S be a smooth surface and $P \in S$ a point. Let $\pi : \tilde{S} \to S$ be the blowup of S at P. We denote by $E = \pi^{-1}(P) \cong \mathbb{P}^1$ the exceptional divisor of π.
We have

$$\operatorname{Pic}(\tilde{S}) \;=\; \pi^* \operatorname{Pic}(S) \oplus \mathbb{Z}\cdot[E],$$

and the intersection form on $\tilde{S}$ is given by

- $\pi^*D \cdot \pi^*D' = D \cdot D'$ for every $D, D' \in \operatorname{Div}(S)$;
- $\pi^*D \cdot E = 0$ for every $D \in \operatorname{Div}(S)$; and
- $E^2 = -1$.

Similarly, $N^1(\tilde{S}) \;=\; \pi^*N^1(S) \oplus \mathbb{R}\cdot[E]$, and thus $\rho(\tilde{S}) = \rho(S) + 1$. In this sense S is simpler than $\tilde{S}$.

The following concept is very important.

Definition 2.6. Let S be a smooth surface. We say that a divisor $D \in \operatorname{Div}(S)$ is *nef* if $D \cdot C \geq 0$ for every curve $C \subset S$.

Examples 2.7.

(1) Ample divisors are nef. We will see in Theorem 3.4 that nef divisors can be characterized as limits of ample divisors.
(2) Let $\pi : S \to X$ be a morphism into a projective variety X, and let H be an ample Cartier divisor on X. Then π^*H is a nef divisor on S. It is ample if and only if π is finite onto its image.

2.2. **Birational geometry of surfaces.** We now come the problem of determining the "simplest model" in each birrational class of surfaces.

Definition 2.8. A smooth surface S is called a *minimal surface* if the following condition holds. If $\pi : S \to S'$ be a birational morphism onto another smooth surface, then π is an isomorphism.

Given a smooth surface S, how to determine whether it is a minimal surface? We recall that the structure of birational morphisms between smooth projective surfaces is well understood.

Theorem 2.9 (Factorization of birational morphisms of surfaces)**.** *Let $\pi : S \to S'$ be a birational morphism between smooth surfaces. Then π is the composition of a finite number of blowups.*

So, to show that S is a minimal surface, one must show that S is not the blowup of any smooth surface. How do we check this condition?

Definition 2.10. A curve C on a smooth surface S is said to be a -1-*curve* if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

As we saw in Example 2.5, if S is the blowup of a smooth surface, then it contains a -1-curve, namely the exceptional divisor of the blowup. The next theorem says that the converse is true.

Theorem 2.11 (Castelnuovo's contractibility theorem)**.** *Let S be a smooth surface, and $C \subset S$ a -1-curve. Then there exists a smooth projective surface S' and a point $P \in S'$ such that S is isomorphic to the blowup of S' at P, and under this isomorphism C corresponds to the exceptional divisor.*

In the situation of Theorem 2.11, we say that S' is obtained from S by contracting the -1-curve C. Now we can state the classical MMP for surfaces.

2.12 (MMP for surfaces - classical version)**.**

(1) Start with a smooth projective surface S.
(2) Ask: Does S contain a -1-curve? If not, stop! S is a minimal surface. If yes, pick one such curve C and go to (3).
(3) By Castelnuovo's contractibility theorem, there is a blowup $f : S \to S'$ for which C is the exceptional divisor. Go back to (1) with S replaced with S'.

This process must stop after a finite number of steps because the Picard number, which is a positive integer, drops by one every time we contract a -1-curve.

The MMP for surfaces provides a first step in the classification of surfaces: it tells us how to obtain a minimal surface in the birational class of any given surface. Then we ask the following natural questions:

(1) Is the minimal surface in a given birational class unique?
(2) Can we classify minimal surfaces in terms of some numerical invariants?

These questions were also classically answered by the Italian school. We will give the answers by the end of this section. At this point, we antecipate that the answer to question (1) depends on the birational class of the given surface S. More precisely, it depends on the behavour of the canonical class K_S. In fact, as we shall see shortly, the whole MMP for surfaces may be reformulated in terms of numerical properties of the canonical class.

2.3. **The role of the canonical class.** As described in 2.12, it is not at all clear how to generalize the MMP for surfaces to higher dimensions. More precisely, how to generalize the question "*Does S contain a -1-curve?*" to higher dimensions? Our next goal is to rephrase the MMP for surfaces in such a way that it makes sense in arbitrary dimension. The key role will be played by the canonical class K_S of a smooth surface S and its numerical properties.

We start by recalling a very useful result.

Theorem 2.13 (Adjunction formula for surfaces)**.** *Let $C \subset S$ be a curve, and $p_a(C) = h^1(C, \mathcal{O}_C)$ the arithmetic genus of C. Then*

$$2p_a(C) - 2 \ = (K_S + C) \cdot C.$$

Remark 2.14. Let $C \subset S$ be a curve. Then $p_a(C) = 0$ if and only if $C \cong \mathbb{P}^1$.

Examples 2.15.

(1) $S = \mathbb{P}^2$. (Notation as in Example 2.3.) The canonical divisor is $K_{\mathbb{P}^2} = -3H$.
(2) $S = \mathbb{F}_n$. (Notation as in Example 2.4.) The adjuntion formula applied to F and σ yields:
$$K_{\mathbb{F}_n} \;=\; -2E - (2+n)\cdot F.$$
(3) Let $\pi : \tilde{S} \to S$ be the blowup of a smooth surface. (Notation as in Example 2.5.) The canonical divisor of $\tilde{S}$ is given by $K_{\tilde{S}} = \pi^* K_S + E$.

Now we start rephrasing the classical MMP for surfaces in terms of numerical properties of the canonical class. The first step is to give a numerical characterization of -1-curves.

Exercise 2.16. Let S be a smooth surface, and $C \subset S$ a curve. Show that
$$C \text{ is a } -1\text{-curve} \quad \Longleftrightarrow \quad K_S \cdot C < 0 \text{ and } C^2 < 0.$$

It follows from Exercise 2.16, that if K_S is nef (see Definition 2.6), then S is necessarily a minimal surface. The converse is not true, as we shall see in Exercise 2.18.

Definition 2.17. A smooth surface is said to be a *scroll* if there exists a surjective morphism $\pi : S \to B$ onto a smooth curve B whose fibers are all isomorphic to $\mathbb{P}^1$. In this case, it can be shown that there exists a rank 2 vector bundle E on B such that $S \cong \mathbb{P}(E)$. Moreover, if E' is another rank 2 vector bundle on B, then $\mathbb{P}(E) \cong \mathbb{P}(E')$ if and only if there is a line bundle L on B such that $E' \cong E \otimes L$. In particular, since every vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles, rational scrolls are precisely the Hirzebruch surfaces.

A surface birationally equivalent to a scroll is called a *ruled surface*. A surface birationally equivalent to $\mathbb{P}^2$ is called a *rational surface*.

Exercise 2.18.

(1) Show that $\mathbb{P}^2$ and scrolls are minimal surfaces, except for $\mathbb{F}_1$.
(2) Verify that if $S \cong \mathbb{P}^2$ or S is a scroll, then K_S is not nef.
(3) Let $\pi : S \to B$ and $\pi' : S' \to B'$ be scrolls. Show that S and S' are birationally equivalent if and only if $B \cong B'$.

Conversely, it can be shown that the only minimal surfaces whose canonical classes are not nef are $\mathbb{P}^2$ and scrolls. (See Theorem 2.26 for a more precise statement.)

To distinguish between minimal surfaces with K_S nef and not nef, we introduce the following concept, which will generalize to higher dimensions.

Definition 2.19. We say that a surface S is a *minimal model* if K_S is nef.

Exercise 2.20. Let S and S' be birationally equivalent surfaces. Suppose that S and S' are minimal models. Show that $S \cong S'$.

(Hint: use the following structure theorem for birational maps between smooth surfaces: If $\varphi : S \dashrightarrow S'$ is a birational map between smooth surfaces, then there exist compositions of blowups $f : \tilde{S} \to S$ and $g : \tilde{S}' \to S$, and isomorphism $\psi : \tilde{S} \to \tilde{S}'$ such that $\varphi = g \circ \psi \circ f^{-1}$.)

Remark 2.21. At this point one may ask whether a minimal model may be birational equivalent to a scroll. The answer is no. At the end of this section we will introduce birational invariants that can be used to distinguish between these two types of surfaces.

Next we introduce the *Mori cone* of a surface.

Definition 2.22. Let S be a smooth surface. The *Mori cone* of S is the closed convex cone $\overline{NE}(S) \subset N^1(S)$ generated by classes of curves $C \subset S$.

Definition 2.23. An *extremal face* F of a cone $N \subset \mathbb{R}^n$ is a subcone of N satisfying:

$$u, v \in N \text{ and } u + v \in F \quad \Rightarrow \quad u, v \in F.$$

A 1-dimensional extremal face of N is called an *extremal ray.*

Let $D : \mathbb{R}^n \to \mathbb{R}$ be a linear function. We write $N_{D \geq 0}$ for $\{z \in N \mid D(z) \geq 0\}$, and similarly for $N_{D=0}$, $N_{D \leq 0}$, etc. An extremal face $F \subset N$ such that $F \setminus \{0\} \subset N_{D<0}$ is called a *D-negative extremal face.* If $F \subset N_{D=0}$, then we say that F *is supported on* D.

Exercise 2.24.

(1) Let S be a smooth surface and $P \in S$ a point. Let $\pi : \tilde{S} \to S$ be the blowup of S at P. We denote by $E = \pi^{-1}(P) \cong \mathbb{P}^1$ the exceptional divisor of π. Show that $[E] \in N^1(\tilde{S})$ generates an extremal ray of $\overline{NE}(\tilde{S})$.

(2) Let $n \in \mathbb{Z}$ be a non-negative integer, and consider the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, with structure morphism $\pi : \mathbb{F}_n \to \mathbb{P}^1$. Denote by σ a section of π such that $\sigma^2 = -n$, and by F a fiber of π. Recall that $\{[\sigma], [F]\}$ is a basis for $N^1(S)$. Since $\overline{NE}(S)$ is a closed convex cone in a 2-dimensional vector space, it must have exactly 2 extremal rays. Show that these are generated by $[\sigma]$ and $[F]$.
(Hint: for $n \geq 1$, consider the structure morphism $\pi : \mathbb{F}_n \to \mathbb{P}^1$ and the blowup $f : \mathbb{F}_n \to Z$ onto the cone over $\nu_n(\mathbb{P}^1)$ described in Example 2.4.)

In general, the cone $\overline{NE}(S)$ may be "round", and an extremal ray $R \subset \overline{NE}(S)$ may not be generated by the class of a curve. (See [9, Example 1.23].)

We observe that in Exercise 2.24 every extremal ray $R \subset \overline{NE}(S)$ is generated by the class of a curve, and moreover there exists a morphism $\varphi : S \to Y$ with the following property. For any curve $C \subset S$, $\varphi(C)$ is a point if and only if $[C] \in R$. This motivates the following definition.

Definition 2.25. Let S be a smooth surface, and F an extremal face of the Mori cone $\overline{NE}(S)$. A *contraction of F* is a morphism with connected fibers $\varphi_F : S \to Y$ onto a normal projective variety Y satisfying following property. For any curve $C \subset S$, $\varphi_F(C)$ is a point if and only if $[C] \in F$.

If the contraction of an extremal face of $\overline{NE}(S)$ exists, then it is unique by Stein factorization. The next theorem asserts that if R is a K_S-negative extremal ray, then the contraction of R always exists. It also gives a complete description of the contraction in this case.

Theorem 2.26. *Let S be a smooth surface, and R a K_S-negative extremal ray of the cone $\overline{NE}(S)$. Then $R = \mathbb{R}_{\geq 0}[C]$ for some rational curve $C \subset S$ (with $K_S \cdot C < 0$). Moreover, the contraction φ_R of R exists, and is one of the following:*

(1) *If $C^2 < 0$, then $\varphi_R : S \to S'$ is the blowup of a smooth surface S' at one point, and C is the exceptional divisor.*
(2) *If $C^2 = 0$, then $\varphi_R : S \to B$ realizes S as a scroll over a smooth curve B, and C is a fiber of φ_R.*
(3) *If $C^2 > 0$, then $S \cong \mathbb{P}^2$, and $\varphi_R : \mathbb{P}^2 \to pt$.*

Definition 2.27. We call the morphisms of type (2) and (3) above *Mori fiber spaces.*

We are now ready to rephrase the classical MMP described in 2.12 in modern language.

2.28 (MMP for surfaces - modern version)**.**

(1) Start with a smooth projective surface S.
(2) Ask: Is K_S nef? If yes, stop! S is a minimal model. If not, pick a K_S-negative extremal ray R of the cone $\overline{NE}(S)$ and go to (3).
(3) Let $\varphi_R : S \to Y$ be the contraction of R. Ask: Is $\dim Y < 2$? If yes, stop! $\varphi_R : S \to Y$ is a Mori fiber space. If not, φ_R is the blowup of a smooth surface. Go back to (1) with S replaced with Y.

2.4. **Birational invariants.** We introduce some birational invariants for surfaces.

Definition 2.29. Let S be a smooth surface.

(1) The *genus* of S is $p_g(S) := h^0(S, \omega_S)$.
(2) More generally the *plurigenera* of S are $P_n(S) := h^0(S, \omega_S^{\otimes n})$, where n is a positive integer.
(3) The *irregularity* of S is $q(S) := h^0(S, \Omega_S^1) = h^1(S, \mathcal{O}_S)$. (The last equality follows from Hodge duality.)

Theorem 2.30. *The quantities p_g, P_n and q are birational invariants for smooth surfaces.*

Proof. Let S and S' be smooth surfaces, and $\varphi : S \dashrightarrow S'$ a birational map. Then there is a finite subset $\Delta \subset S$ such that $\varphi|_{S\setminus\Delta} : S \setminus \Delta \to S'$ is a morphism. Given a 2-form $\omega \in H^0(S', \omega_{S'})$, we get a form $\varphi^*\omega \in H^0(S \setminus \Delta, \omega_{S\setminus\Delta})$. We may view $\varphi^*\omega$ as a meromorphic form on S with poles along Δ. Since Δ has codimension ≥ 2 in S, $\varphi^*\omega$ extends to a 2-form $\overline{\varphi^*\omega} \in H^0(S, \omega_S)$. This yields an inclusion $H^0(S', \omega_{S'}) \subset H^0(S, \omega_S)$. The same argument gives the reverse inclusion. Hence $p_g(S) = p_g(S')$.

The proof of birational invariance of P_n and q is analogous. $\square$

Exercise 2.31. Compute the birational invariants p_g, P_n and q for rational and ruled surfaces. (Hint: choose a suitable birational model.)

It turns out that the birational invariants p_g, P_n and q may be used to characterize rational and ruled surfaces. This is the content of the next result.

Theorem 2.32 (Numerical characterization of rational and ruled surfaces)**.** *Let S be a smooth surface.*

(1) *(Castelnuovo) S is rational $\iff$ $q(S) = 0$ and $P_n(S) = 0\ \forall n \geq 1$ $\iff$ $q(S) = 0$ and $P_2(S) = 0$.*
(2) *(Enriques) S is ruled $\iff$ $P_n(S) = 0\ \forall n \geq 1$ $\iff$ $P_{12}(S) = 0$.*

Next we define *Kodaira dimension.* We give the definition for arbitrary smooth projective varieties, and then we especialize to the surface case.

2.5. **Kodaira dimension.** Let X be a smooth projective variety of dimension $n \geq 1$. Let $D \in \mathrm{Div}(X)$ be a divisor and suppose that $H^0(X, \mathcal{O}_X(D)) \neq 0$. Pick a basis $\{s_0, \cdots, s_k\}$ for $H^0(X, \mathcal{O}_X(D)) \cong \mathbb{C}^{k+1}$, and consider the rational map $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^k$ that sends a point x at which not all the s_i's vanish to the point $\big(s_0(x) : \cdots : s_k(x)\big) \in \mathbb{P}^k$. We have $0 \leq \dim\big(\varphi_{|D|}(X)\big) \leq n$.

Definition 2.33. Let X be a smooth projective variety of dimension $n \geq 1$, and $D \in \mathrm{Div}(X)$. Define the semigroup of D to be $\mathbb{N}(D) = \big\{ m \geq 0 \;\big|\; H^0(X, \mathcal{O}_X(mD)) \neq 0 \big\}$. The *Iitaka dimension* of D is defined to be

$$\kappa(D) \;=\; \begin{cases} -\infty, & \text{if } \mathbb{N}(D) = \{0\} \\ max\Big\{ \dim\big(\varphi_{|mD|}(X)\big) \;\Big|\; m \in \mathbb{N}(D) \Big\}, & \text{if } \mathbb{N}(D) \neq \{0\} \end{cases}$$

Note that $\kappa(D) \in \{-\infty, 0, 1, \cdots, n\}$. It can be shown that there exist positive constants c_1 and c_2, depending on D, such that, for $m \in \mathbb{N}(D)$ sufficiently large, we have

$$c_1 \cdot m^{\kappa(D)} \;\leq h^0(X, \mathcal{O}_X(mD)) \;\leq c_2 \cdot m^{\kappa(D)}.$$

We say that the divisor D is *big* if $\kappa(D) = n$.

The *Kodaira dimension* of X is defined to be $\kappa(X) := \kappa(K_X)$.

Exercise 2.34. Show that the Kodaira dimension is a birational invariant for smooth projective varieties.

Examples 2.35.

(1) Curves. If $\dim(X) = 1$, then $\kappa(X) \in \{-\infty, 0, 1\}$.
- $g(X) = 0 \iff X \cong \mathbb{P}^1 \iff -K_X$ is ample $\iff \kappa(X) = -\infty$.
- $g(X) = 1 \iff -K_X = 0 \iff \kappa(X) = 0$.
- $g(X) \geq 2 \iff K_X$ is ample $\iff \kappa(X) = 1$.

(2) Hypersurfaces. Let $X = X_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. It follows from the adjunction formula that $K_X = (-n - 2 + d) \cdot H$, where H is the class of a hyperplane in $\mathbb{P}^n$.
- $d < n + 2 \iff -K_X$ is ample $\iff \kappa(X) = -\infty$.
- $d = n + 2 \iff K_X = 0 \iff \kappa(X) = 0$.
- $d > n + 2 \iff K_X$ is ample $\iff \kappa(X) = n$.

Exercise 2.36. Let X and Y be smooth projective varieties, and suppose that $\kappa(X) = 0$. Show that $\kappa(X \times Y) = \kappa(Y)$.

Conclude that, for each positive integer n, and each $\kappa \in \{-\infty, 0, 1, \cdots, n\}$, there exists a smooth projective variety X of dimension n and Kodaira dimension $\kappa(X) = \kappa$.

Definition 2.37. We say that a smooth projective variety X is *of general type* if $\kappa(X) = \dim(X)$.

2.38 (Enriques' classification of minimal surfaces)**.** Let S be a smooth surface. Then $\kappa(S) \in \{-\infty, 0, 1, 2\}$. It follows from Theorem 2.32(2) that $\kappa(S) = -\infty$ if and only if S is a ruled surface.

On the other hand, if $\kappa(S) \geq 0$, then the MMP for S as described in 2.28 ends necessarily with a minimal model S_{min}. Moreover, by Exercise 2.20, S_{min} is unique up to isomorphism.

Minimal models S of surfaces can be divided into the following classes, according to the values of their birational invariants p_g, P_n, q and κ:

(1) $\kappa(S) = 0$. There are 4 classes.
 (a) $p_g(S) = q(S) = 0$. These are called Enriques' surfaces.
 (b) $p_g(S) = 0$ and $q(S) = 1$. These are called bielliptic surfaces.
 (c) $p_g(S) = 1$ and $q(S) = 0$. These are called K3 surfaces.
 (d) $p_g(S) = 1$ and $q(S) = 2$. These are abelian surfaces.

(2) $\kappa(S) = 1$. Such surfaces admit a fibration $f : S \to B$ onto a smooth curve whose generic fiber is an elliptic curve.

(3) $\kappa(S) = 2$. Most surfaces lie in this class. These are called surfaces of general type.

3. The MMP in higher dimensions

Now we want to extend the MMP for surfaces, as described in 2.28, to higher dimensions. Our first task is to introduce the intersection product, spaces of curves and divisors, which are different spaces in dimension bigger than 2, and special cones on them. The reference for most of this section is [9].

3.1. **Intersection product and spaces of curves and divisors.** Throughout this subsection let X be a smooth projective variety.

Definition 3.1. Consider the free abelian group $Z_1(X)$ generated by curves on X. We have an intersection product:

$$\cdot\ : \mathrm{Pic}(X) \times Z_1(X) \ \to\ \mathbb{Z}$$

with the property that, if $D \in \mathrm{Pic}(X)$ and $C \subset X$ is a curve, with normalization $n : \tilde{C} \to C$, then $D \cdot C$ equals the degree of the invertible sheaf $n^*\big(\mathcal{O}_X(D)|_C\big)$.

Two elements $D, D' \in \mathrm{Pic}(X)$ are said to be *numerically equivalent* if $D \cdot \alpha = D' \cdot \alpha$ for every $\alpha \in Z_1(X)$. In this case we write $D \equiv D'$. We write $\mathrm{Num}(X)$ for the quotient group $\mathrm{Pic}(X)/\equiv$. By the Theorem of the base of Néron-Severi, $\mathrm{Num}(X)$ is a finitely gernerated abelian group. Its rank is called the *Picard number* of X, and is denoted by $\rho(X)$. We define the $\rho(X)$-dimensional $\mathbb{R}$-vector space $N^1(X) := \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Similarly, two cycles $\alpha, \alpha' \in Z_1(X)$ are said to be *numerically equivalent* if $D \cdot \alpha = D \cdot \alpha'$ for every $D \in \mathrm{Pic}(X)$. In this case we write $\alpha \equiv \alpha'$. We define the $\rho(X)$-dimensional $\mathbb{R}$-vector space $N_1(X) := \big(Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}\big)/\equiv$.

The intersection product on X induces a perfect pairing $\cdot : N^1(X) \times N_1(X) \to \mathbb{R}$, making $N^1(X)$ and $N_1(X)$ dual vector spaces.

Next we introduce the *Mori cone* and the cone of nef divisors.

Definition 3.2. The *Mori cone* of X is the closed convex cone $\overline{NE}(X) \subset N_1(X)$ generated by classes of curves $C \subset X$.

We say that a divisor $D \in \mathrm{Div}(X)$ is *nef* if $D \cdot C > 0$ for every curve $C \subset X$. This is equivalent to saying that $D \cdot \alpha \geq 0$ for every $\alpha \in \overline{NE}(X)$. So the dual cone of $\overline{NE}(X)$ under the intersection product is the closed convex cone $\mathrm{Nef}(X) \subset N^1(X)$ generated by nef divisors. We call it the *nef cone* of X.

Remark 3.3. Similarly, one can define the *cone of pseudo-effective divisors* of X as the closed convex cone $\mathrm{Pseff}(X) \subset N^1(X)$ generated by classes of effective divisors. It was proved in [4] that the dual cone of $\mathrm{Pseff}(X) \subset N^1(X)$ under the intersection product is the closed convex cone in $N_1(X)$ generated by classes of the so called *strongly movable curves*. Strongly movable curves are images in X of curves obtained as complete intersections of suitable very ample divisors on birational modifications of X.

It is a formidable fact that many geometric properties of divisors depend only on their numerical class. The following are two important manifestations of this phenomenon.

Theorem 3.4. *Let $D \in \mathrm{Div}(X)$ be a divisor.*

(1) *(*Kleiman's ampleness criterion.*) D is ample $\iff$ $D \cdot \ell > 0 \ \ \forall \ell \in \overline{NE}(X) \setminus \{0\}$. This is equivalent to saying that the class of D lies in the interior of $\mathrm{Nef}(X)$.*

(2) *(*Kodaira's lemma.*) D is big $\iff$ the class of D lies in the interior of* $\mathrm{Pseff}(X)$.

3.2. **The first theorems of the MMP.** We want to run the program described in 2.28 starting with a smooth projective variety X of arbitrary dimension. We start by asking whether K_X is nef. If K_X is nef, then we stop and say that X is a minimal model. If K_X is not nef, then we may pick a K_X-negative extremal ray R of the Mori cone $\overline{NE}(X)$. As we shall see in Theorem 3.11 below, $R = \mathbb{R}_{\geq 0}[C]$ for some rational curve $C \subset X$, and the contraction of R (as in Definition 2.25) exists. Let us denote it by $\varphi_R : X \to Y$. If $\dim Y < \dim X$, then, as before, we stop and call $\varphi_R : X \to Y$ a Mori fiber space. If $\dim Y = \dim X$, then φ_R is a birational morphism, and we would like to replace X with Y and go back to the original question. Here we face a problem that did not appear in the surface case: the variety Y may be singular. This situation is illustrated in Example 3.6 below.

Exercise 3.5. Let $Y \subset \mathbb{P}^N$ be a smooth projective variety, and $C(Y) \subset \mathbb{P}^{N+1}$ the cone over Y with vertex P. Let X be the blowup of $C(Y)$ at the point P. Show that X is a smooth projective variety.

Example 3.6. Let $Y \subset \mathbb{P}^5$ be the Veronese embedding of $\mathbb{P}^2$, and $C(Y) \subset \mathbb{P}^6$ the cone over Y with vertex P. One can check that the canonical divisor $K_{C(Y)}$ is not Cartier, while $2K_{C(Y)}$ is Cartier. Let $\pi : X \to C(Y)$ be the blowup of $C(Y)$ at the point P, and denote by $E \cong \mathbb{P}^2$ the exceptional divisor. By Exercise 3.5, X is a smooth projective 3-fold. One can check that $\mathcal{O}_X(E)|_E \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. There is a morphism $p : X \to \mathbb{P}^2$, with fibers ismomorphic to $\mathbb{P}^1$, which resolves the indeterminacy of the projection $C(Y) \dashrightarrow Y \cong \mathbb{P}^2$ from the point P. One can show that $\mathrm{Pic}(X) = \mathbb{Z}\cdot[p^*\mathcal{O}_{\mathbb{P}^2}(1)] \oplus \mathbb{Z}\cdot[E]$, and in $\mathrm{Pic}(X)$ we have

$$2K_X \;=\; \pi^* 2K_Y + E.$$

In $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ we have

$$K_X \;=\; \pi^* K_Y + \frac{1}{2}E.$$

As for the space of 1-cycles, $N_1(X) = \mathbb{R}\cdot f \oplus \mathbb{R}\cdot \ell$, where f denotes the class of a fiber of p, and ℓ denotes the class of a curve on E corresponding to a line in $\mathbb{P}^2$ under the isomorphism $E \cong \mathbb{P}^2$. Note that f and ℓ are classes of curves contained on the fibers of the morphisms p and π, respectively. Hence f and ℓ generate extremal rays of the mori cone $\overline{NE}(X) \subset N_1(X)$. The intersection product of curves and divisors on X gives: $K_X \cdot f = -1$ and $K_X \cdot \ell = -2$. Hence both f and ℓ generate K_X-negative extremal rays of $\overline{NE}(X)$. The contraction of the ray $\mathbb{R}_{\geq 0} f$ is the morphism $p : X \to \mathbb{P}^2$, while the contraction of the ray $\mathbb{R}_{\geq 0}\ell$ is the blowup $\pi : X \to C(Y)$. The latter is a birational morphism onto a *singular* variety.

This simple example brings a point that was understood since the beginning of the development of the MMP for higher dimensional varieties: singularieties are unavoidable, and we must learn how to deal with them. A whole theory of singularieties was developed in the context of the MMP. In these lectures we will only consider a small portion of it. Namely, we will define the smallest class of singularieties $\mathcal{S}$ that unavoidably appear when running the MMP starting with smooth projective varieties, and such that the steps of the MMP are still valid for projective varieties with singularities in the class $\mathcal{S}$.

Recall that we start the MMP by asking if K_X is nef. For this question to make sense, it is necessary that the divisor K_X is at $\mathbb{Q}-$*Cartier*, i.e., some nonzero multiple of it is Cartier. In these lectures we will require something stronger, namely, that X is $\mathbb{Q}$-factorial.

Definition 3.7. Let X be an arbitrary projective variety. A $\mathbb{Q}$-*divisor* on X is a $\mathbb{Q}$-linear combination of prime Weil divisors on X. A $\mathbb{Q}$-divisor D on X is said to be $\mathbb{Q}$-*Cartier* if some nonzero multiple of D is a Cartier divisor. Two $\mathbb{Q}$-divisors D and D' on X are said to be $\mathbb{Q}$-*linearly equivalent* if there exists an integer $m > 0$ such that both mD and mD' are Cartier and $mD \sim mD'$. In this case we write $D \sim_{\mathbb{Q}} D'$.

We say that X is $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

Remark 3.8. The vector spaces $N^1(X)$ and $N_1(X)$, their intersection product, the cones of curves and divisors introduced in the begining of this section, and the Kodaira dimension may all be defined more generally for $\mathbb{Q}$-factorial projective varieties. We leave this easy task to the reader.

If we start with a $\mathbb{Q}$-factorial projective variety X, then we can ask whether K_X is nef. If the answer is no, then we pick a K_X-negative extremal ray R of the Mori cone $\overline{NE}(X)$, and we wish to consider the contraction of R. Now we encounter another problem. The Contraction Theorem that we need here is not valid for arbitrary $\mathbb{Q}$-factorial projective varieties. So we must consider a more restrictive class of possibly singular varieties. The following definition is not intuitive, but it is the right one in our context.

Definition 3.9. Let X be a normal projective variety, and suppose that K_X is $\mathbb{Q}$-Cartier. Let $f : \tilde{X} \to X$ be a log resolution of X. This means that $\tilde{X}$ is a smooth projective variety, f is a birational morphism whose exceptional locus is the union of prime divisors E_i's, and the divisor $\sum E_i$ has simple normal crossing support. There are uniquely defined rational numbers $a(E_i)$'s such that

$$K_{\tilde{X}} \sim_{\mathbb{Q}} f^*K_X + \sum_{E_i} a(E_i)E_i.$$

The $a(E_i)$'s do not depend on the log resolution, but only on the valuations associated to the E_i's.

We say that X is *terminal* if, for some log resolution $f : \tilde{X} \to X$, $a(E_i) > 1$ for every f-exceptional prime divisor E_i. If this condition holds for some log resolution of X, then it holds for every log resolution of X.

Now we can state the first theorems of the MMP, which hold for the class of $\mathbb{Q}$-factorial terminal projective varieties.

Theorem 3.10 (Cone Theorem)**.** *Let X be a $\mathbb{Q}$-factorial terminal projective variety. There is a countable set $\Gamma \subset \overline{NE}_1(X)$ of classes of rational curves $C \subset X$ with $0 < -K_X \cdot C \leq 2\dim(X)$ such that*

(1) *for any ample divisor A on X, there are finitely many classes $[C_1], \ldots, [C_r]$ in Γ such that*

$$\overline{NE}_1(X) = \overline{NE}_1(X)_{(K_X+A)\geq 0} + \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[C_i], \text{ and}$$

(2) $\overline{NE}_1(X) = \overline{NE}_1(X)_{K_X\geq 0} + \sum_{[C]\in\Gamma} \mathbb{R}_{\geq 0}[C]$.

Theorem 3.11 (Contraction Theorem)**.** *Let X be a $\mathbb{Q}$-factorial terminal projective variety. Let F be a K_X-negative extremal face of the Mori cone $\overline{NE}(X)$. Then there* *exists a unique morphism $\varphi_F : X \to Y$ onto a normal projective variety such that $(\varphi_F)_*\mathcal{O}_X = \mathcal{O}_Y$, and, for any curve $C \subset X$, $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

Definition 3.12. Under the assumptions and notation of Theorem 3.11, we say that $\varphi_F : X \to Y$ is *the contraction of* F.

3.13 (Properties of contractions of K_X-negative extremal rays)**.** Let X be a $\mathbb{Q}$-factorial terminal projective variety. Let R be a K_X-negative extremal ray of the cone $\overline{NE}(X)$, and $\varphi_R : X \to Y$ the contraction of R. There is an exact sequence

$$0 \to \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \to \mathbb{Z}, \tag{3.1}$$

where the last map is given by intersection with a curve $C \subset X$ such that $R = \mathbb{R}_{\geq 0}[C]$. In particular, $\rho(X) = \rho(Y)+1$. The *exceptional locus* of φ_R is the locus of X consisting of points at which φ_R fails to be a local isomorphism. One of the following situations occurs.

(1) $\dim(Y) < \dim(X)$. We call such φ_R a *Mori fiber space.*
(2) The morphism φ_R is birational and the exceptional locus of φ_R consists of a prime divisor on X. In this case, Y is a $\mathbb{Q}$-factorial terminal projective variety. We call such φ_R a *divisorial contraction.*
(3) The morphism φ_R is birational and the exceptional locus of φ_R has codimension at least 2 in X. We call such φ_R a *small contraction.*

Definition 3.14. A $\mathbb{Q}$-factorial terminal projective variety X is called a *minimal model* if K_X is nef.

Let us resume our description of the MMP. We start with a smooth (or more, generally $\mathbb{Q}$-factorial terminal) projective variety X, and ask whether K_X is nef. If K_X is nef, then X is a minimal model and we stop. If K_X is not nef, then we pick a K_X-negative extremal ray R of the Mori cone $\overline{NE}(X)$, and consider its contraction $\varphi_R : X \to Y$. According to the description given in 3.13, there are 3 possibilities.

(1) If $\varphi_R : X \to Y$ a Mori fiber space, then we stop.
(2) If $\varphi_R : X \to Y$ is a divisorial contraction, then Y is $\mathbb{Q}$-factorial and terminal, and we go back to the original question with X replaced with Y. In this case $\rho(Y) = \rho(X) - 1$.
(3) If $\varphi_R : X \to Y$ is a small contraction, then we are in trouble for the following reason.

Claim 3.1. *Under the assumptions of 3.13(3), K_Y is not $\mathbb{Q}$-Cartier.*

Proof. Let $C \subset X$ be a rational curve such that $R = \mathbb{R}_{\geq 0}[C]$. Suppose that K_Y is $\mathbb{Q}$-Cartier, and consider the $\mathbb{Q}$-divisor $\varphi_R^* K_Y$ on X. Since $\varphi_R(C)$ is a point, we have $\varphi_R^* K_Y \cdot C = 0$.

On the other hand, $\varphi_R^* K_Y$ coincides with K_X in the open subset of X where φ_R is an isomorphism. Since the exceptional locus of φ_R has codimension at least 2 in X, we must have $\varphi_R^* K_Y = K_X$ on X. However, by assumption, R is a K_X-negative extremal ray, and thus $K_X \cdot C < 0$, yielding a contradiction. □

Since K_Y is not $\mathbb{Q}$-Cartier, in case (3) we cannot hope to continue running the MMP with X replaced with Y. The idea then is to do something different. Instead of contracting the ray R and replacing X with Y, we will perform a *flip* $\psi : X \dashrightarrow X^+$, and go back to the original question with X replaced with X^+. We will explain the notion of flip in the next subsection.

3.3. **Flips.** We now come to a fundamental concept from the MMP.

Definition 3.15 (Flip)**.** Let X be a $\mathbb{Q}$-factorial terminal projective variety, and $f = \varphi_R : X \to Y$ a small contraction associated to a K_X-negative extremal ray $R \subset \overline{NE}(X)$.

A *flip of* f is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "f"'] & & X^+, \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where $\psi : X \dashrightarrow X^+$ is a birational map and $f^+ : X^+ \to Y$ is a birational morphism satisfying the following conditions.

(1) K_{X^+} is $\mathbb{Q}$-Cartier.
(2) The exceptional locus of f^+ has codimension at least 2 in X^+.
(3) $K_{X^+} \cdot C > 0$ for every curve $C \subset X^+$ contracted by f^+.

We refer to [9, Example 2.7] for an example of flip.

It is a difficult task to prove the existence of flips. In dimension 3, it was proved by Mori in [11]. In dimension 4, it was proved by Shokurov in [13]. In [6], Hacon and McKernan proved that flips exist in dimension n provided the existence of minimal models in dimension $n-1$. Using this inductive scheme, existence of flips in any dimension was finally proved in [3]. Given the existence of flips, it is not so difficult to prove that it satisfies the following properties.

3.16 (Properties of flips)**.** Let the notation be as in Definition 3.15.

(1) The flip of f is unique up to isomorphism. In fact, the existence of $f^+ : X^+ \to Y$ is equivalent to the finite generation of the $\mathcal{O}_Y$-algebra $\oplus_{m \in \mathbb{Z}_{\geq 0}} f_*\mathcal{O}_X\big(\lfloor mK_X \rfloor\big)$. Moreover, $f^+ : X^+ \to Y$ is precisely $\mathrm{Proj}_Y\Big(\oplus_{m \in \mathbb{Z}_{\geq 0}} f_*\mathcal{O}_X\big(\lfloor mK_X \rfloor\big)\Big) \to Y$.
(2) X^+ is a $\mathbb{Q}$-factorial terminal projective variety.

Notice moreover that $\psi : X \dashrightarrow X^+$ is an isomorphism in codimension 1. Hence, since both X and X^+ are $\mathbb{Q}$-factorial, we have $\rho(X^+) = \rho(X)$.

Now we can finally describe the MMP in arbitrary dimension.

3.17 (MMP in arbitrary dimension)**.**

(1) Start with a $\mathbb{Q}$-factorial terminal projective variety X.
(2) Ask: Is K_X nef? If yes, stop! X is a minimal model. If not, pick a K_X-negative extremal ray R of the cone $\overline{NE}(X)$ and go to (3).
(3) Let $\varphi_R : S \to Y$ be the contraction of R. There are 3 possibilities.
    (a) If $\varphi_R : X \to Y$ a Mori fiber space, then we stop.
    (b) If $\varphi_R : X \to Y$ is a divisorial contraction, then Y is $\mathbb{Q}$-factorial and terminal. Go back to (1) with X replaced with Y.
    (c) If $\varphi_R : X \to Y$ is a small contraction, then consider the flip $\psi : X \dashrightarrow X^+$ of φ_R. Then X^+ is $\mathbb{Q}$-factorial and terminal. Go back to (1) with X replaced with X^+.

In order to conclude the program, one must show that this process eventually stops. Every time we perform a divisorical contraction $X \to Y$, the Picard number drops by one, $\rho(Y) = \rho(X) - 1$. However, in the case of a flip $X \dashrightarrow X^+$, we have $\rho(X^+) = \rho(X)$.

Therefore this process can only admit a finite number of divisorial contraction, while we have the following question:

Does there exist an infinite sequence of flips?

Termination of flips in dimension 3 was proved in [12]. However, to this date the answer to the question above is not known in arbitrary dimension. So the MMP as described in 3.17 has not been established in higher dimensions. However, in certain cases, a special instance of the MMP, called *MMP with scaling* was proved to terminate in any dimension in [3]. This is the subject of the next section.

4. MMP with scaling

As we mentioned at the end of the previous section, if we start with a $\mathbb{Q}$-factorial terminal projective variety X, and run the MMP as decribed in 3.17, it is not clear that the process terminates. There is however a variation of this program, called the *MMP with scaling*, in which we start with an ample divisor H on X and, instead of choosing an arbitrary extremal ray at each step of the MMP, we use the divisor H to narrow (and sometimes decide) our choice of extremal ray. Here is how it works. At the first step, if K_X is not nef, then, instead of choosing an arbitrary K_X-negative extremal ray of $\overline{NE}(X)$, we proceed as follows. Since H is ample, $\overline{NE}(X)\setminus\{0\}$ is contained in the half-space $\{H>0\}$. We move the hyperplane $\{K_X=0\}$ in $N_1(X)$ toward $\{H=0\}$ until it supports an extremal face F of $\overline{NE}(X)$, and then we choose an extremal ray contained in this face. More precisely, we define

$$\lambda=\inf\Big\{t\geq 0\;\Big|\;\big[K_X+tH\big]\in \mathrm{Nef}(X)\Big\},$$

and choose an extremal ray of $\overline{NE}(X)$ supported on $K_X+\lambda H$. (We invite the reader to draw a picture.) This is necessarily a K_X-negative extremal ray. Then we continue as in the ordinary MMP. If $\psi : X \dashrightarrow Y$ is a birational step in the MMP (i.e., either a divisorical contraction or a flip), then we replace X with Y and H with ψ_*H. The divisor ψ_*H is no longer ample. Nevertheless, the procedure just described can be repeated for Y and ψ_*H.

Definition 4.1. Let X be a $\mathbb{Q}$-factorial terminal projective variety, and H a $\mathbb{Q}$-divisor on X. Suppose that $K_X+\lambda H$ is nef for some $\lambda\geq 0$. (This holds for instance if H is ample.) We define the *nef threshold of* H by

$$\lambda(X,H)=\inf\Big\{\lambda\geq 0\;\Big|\;\big[K_X+\lambda H\big]\in \mathrm{Nef}(X)\Big\}.$$

The *Rationality Theorem* asserts that $\lambda(X,H)\in\mathbb{Q}$.

Now we describe the MMP with scaling in more detail. We start with a $\mathbb{Q}$-factorial terminal projective variety X, and an ample divisor H on X. We will define inductively (possibly finite) sequences of $\mathbb{Q}$-factorial terminal projective varieties X_i's, together with $\mathbb{Q}$-divisors H_i's on them such that $K_{X_i}+\lambda H_i$ is nef for some $\lambda\geq 0$. For each i, $\psi_i : X_i \dashrightarrow X_{i+1}$ will be either a divisorial contraction or a flip from the ordinary MMP, and $H_{i+1}=(\psi_i)_*H_i$.

Step 0. We set $X_0=X$, and $H_0=H$. We move to Step 1 with $n=0$.

Step 1. Suppose we have constructed X_n and H_n. Set $\lambda_n=\inf\Big\{\lambda\geq 0\;\Big|\;\big[K_{X_n}+\lambda H_n\big]\in\mathrm{Nef}(X_n)\Big\}$. We move to Step 2.

Step 2. We ask whether K_{X_n} is nef (or, equivalently, if $\lambda_n=0$).

If K_{X_n} is nef, then we stop and the sequence $\{X_i\}$ ends with the minimal model X_n.

If K_{X_n} is not nef, then there exists at least one K_{X_n}-negative extremal ray $R \subset \overline{NE}(X_n)$ such that $(K_{X_n} + \lambda_n H_n) \cdot R = 0$. We choose one such extremal ray R_n, and let $\varphi_n : X_n \to Y$ be the contraction of R_n. We move to Step 3.

Step 3. We check which of the three possibilities described in 3.13 occurs.

(1) If $\varphi_n : X_n \to Y$ is a Mori fiber space, then we stop and the sequence $\{X_i\}$ ends with X_n.

(2) If $\varphi_n : X_n \to Y$ is a divisorial contraction, then we set $X_{n+1} = Y$ and $H_{n+1} = (\varphi_n)_* H_n$. Since $(K_{X_n} + \lambda_n H_n) \cdot R_n = 0$, by (3.1),

$$K_{X_n} + \lambda_n H_n \sim_{\mathbb{Q}} (\varphi_n)^* \big(K_{X_{n+1}} + \lambda_n H_{n+1}\big).$$

Since $K_{X_n} + \lambda_n H_n$ is nef, this implies that $K_{X_{n+1}} + \lambda_n H_{n+1}$ is also nef. We go back to Step 1 replacing n with $n + 1$.

(3) If $f = \varphi_n : X_n \to Y$ is a small contraction, and $\psi : X_n \dashrightarrow X_n^+$ is the associated flip, then we set $X_{n+1} = X_n^+$, and $H_{n+1} = \psi_* H_n$. Consider the flip diagram:

$$\begin{tikzcd} X_n \arrow[rr, dashed, "\psi"] \arrow[dr, "f"'] & & X_{n+1} \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

Since $(K_{X_n} + \lambda_n H_n) \cdot R_n = 0$, by (3.1), there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D_Y on Y such that $K_{X_n} + \lambda_n H_n \sim_{\mathbb{Q}} f^* D_Y$. Then $K_{X_{n+1}} + \lambda_n H_{n+1} \sim_{\mathbb{Q}} (f^+)^* D_Y$. By hypothesis $K_{X_n} + \lambda_n H_n$ is nef. Thus D_Y is nef and so is $K_{X_{n+1}} + \lambda_n H_{n+1}$. We go back to Step 1 replacing n with $n + 1$.

In [3], the MMP with scaling was proved to terminate in the following two important cases:

(1) X is of general type (this is equivalent to saying that K_X is big, i.e., K_X lies in the interior of $\mathrm{Pseff}(X)$). In this case, the MMP with scaling ends with a minimal model.
(2) X is uniruled (by [4] this is equivalent to saying that $K_X \notin \mathrm{Pseff}(X)$). In this case, the MMP with scaling ends with a Mori fiber space.

Suggested reading

The reader interested in a more detailed and rigorous introduction to the MMP and its techniques is referred to [9]. The texts [8] and [10] also provide a good introduction. All of these cover the "classical" MMP.

There are many notes available in the web discussing the more recent results from [3], including the MMP with scaling. In addition to [3] itself, the reader may consult the expository paper [5]. Those interested in the MMP with scaling for uniruled varieties exclusively may also look at [1].

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