GERMS OF COMPLEX TWO DIMENSIONAL FOLIATIONS

BY A. LINS NETO

ABSTRACT. The purpose of this paper is to show how some results about codimension one foliations in dimension three can be generalized to dimension two foliations in dimension $n \geq 4$.

CONTENTS

0.1. **Notations.** We begin by stablishing some notations that we will use along the text.

1- $\mathcal{O}(U) :=$ set of holomorphic functions defined on a domain $U \subset \mathbb{C}^n$.
$\mathcal{O}^*(U) := \{f \in \mathcal{O}(U) \,|\, f(p) \neq 0\,,\, \forall\, p \in U\}$.
$\mathcal{O}_n :=$ ring of germs at $(\mathbb{C}^n, 0)$ of holomorphic functions, $m_n =$ the maximal ideal of $\mathcal{O}_n$.
$\mathcal{O}_n^* := \{f \in \mathcal{O}_n |\, f(0) \neq 0\}$.
$\widehat{\mathcal{O}}_n$ ring of formal power series.
$\langle f_1, ..., f_k \rangle =$ ideal of $\mathcal{O}_n$ (or $\widehat{\mathcal{O}}_n$) generated by $f_1, ..., f_k$.

2- $\widehat{Diff}(\mathbb{C}^n, 0) :=$ group of formal biholomorphisms at $(\mathbb{C}^n, 0)$ fixing 0.

3- $\Lambda^k(U) :=$ set of holomorphic k-forms defined on a domain $U \subset \mathbb{C}^n$.
$\Lambda_n^k :=$ set of germs at $(\mathbb{C}^n, 0)$ of holomorphic k-forms.
$\widehat{\Lambda}_n^k :=$ set of formal k-forms at $(\mathbb{C}^n, 0)$.

4- $\mathcal{X}(U) :=$ set of holomorphic vector fields defined on a domain $U \subset \mathbb{C}^n$.
$\mathcal{X}_n :=$ set of germs at $(\mathbb{C}^n, 0)$ of holomorphic vector fields.
$\widehat{\mathcal{X}}_n :=$ set of formal vector fields at $(\mathbb{C}^n, 0)$.

5- Given a formal power series $\Phi = \sum_{j \geq 0} \Phi_j$, Φ_j homogeneous of degree j, then $j^k(\Phi) = \sum_{j=0}^k \Phi_j$ denotes the k-jet of Φ, $j \geq 0$.

1991 *Mathematics Subject Classification.* 37F75, 34M15.
Key words and phrases. holomorphic foliation, germ.

6- $i_X\eta :=$ the interior product of the k-form η, $k \geq 1$, by the vector field X.
7- $L_X :=$ the Lie derivative in the direction of the vector field X. When X and Y are vector fields in the same space then $L_X Y := [X, Y]$, the Lie bracket.

1. Basic definitions and statement of the results

A singular holomorphic foliation $\mathcal{F}$ of codimension k, $1 \leq k < n$, on a polydisc $Q \subset \mathbb{C}^n$ can be defined by a holomorphic k-form $\eta \in \Omega^k(Q)$ (see [Me] and [C-C-F]). The form η is *integrable* in the sense that for any $p \in Q$ such that $\eta(p) \neq 0$ then there exists a neighborhood U_p of p such that:

(I). $\eta|_{U_p}$ is locally completely decomposable (briefly l.c.d.). This means that there exist k holomorphic 1-forms $\alpha_1, ..., \alpha_k$ on U_p such that $\eta|_{U_p} = \alpha_1 \wedge ... \wedge \alpha_k$.

(II). For all $1 \leq j \leq k$ we have $d\alpha_j \wedge \eta = 0$.

The singular set of η or $\mathcal{F}$ is defined as

$$sing(\eta) := \{p \in Q \mid \eta(p) = 0\} \ .$$

Conditions (I) and (II) are therefore valid in a neighborhood of any non-singular point of η. The foliation defined by η will be denoted by $\mathcal{F}_\eta$.

Remark 1.1. Condition (I) implies that for any $p \notin sing(\eta)$ the subspace

$$ker(\eta(p)) := \{v \in T_pQ \,|\, i_v\eta(p) = 0\} \subset T_pQ$$

has codimension k. Therefore $ker(\eta)$ defines a holomorphic distribution of codimension k outside $sing(\eta)$. Condition (II) implies that this distribution is integrable and defines a regular foliation $\mathcal{F}_\eta$ outside $sing(\eta)$. In particular, if we take U_p small enough then there exist a coordinate system $w = (w_1, ..., w_n)\colon (U_p, p) \to (\mathbb{C}^n, 0)$ and $f \in \mathcal{O}^*(U_p)$ such that

$$\eta|_{U_p} = f.\, dw_1 \wedge ... \wedge dw_k \ . \tag{1}$$

This means that in these coordinates the leaves of $\mathcal{F}_\eta|_{U_p}$ are the levels $(w_1 = c_1, ..., w_k = c_k)$.

When the foliation has dimension two then η is a $(n-2)$-form and its differential $d\eta$ is a $(n-1)$-form. In particular, if we fix a coordinate system $z = (z_1, ..., z_n)$ of $\mathbb{C}^n$ then we can write

$$d\eta = i_X\,\nu \ , \tag{2}$$

where $\nu = dz_1 \wedge ... \wedge dz_n$ and X is a holomorphic vector field on Q. The vector field X will be called the rotational of η in the coordinate system z. Note that, if $\tilde{X}$ is the rotational of η in another coordinate system $\tilde{z}$ then $\tilde{X} = \phi.\, X$, where $\phi \in \mathcal{O}^*(Q)$. In other words, if $d\eta \not\equiv 0$ then $d\eta$ defines a singular one dimensional foliation on Q. The following basic fact will be proved in § 2:

Proposition 1. *Let η be a holomorphic $(n-2)$-form on the polydisc $Q \subset \mathbb{C}^n$ and X be its rotational. If we assume that η satisfies condition* (I) *then condition* (II) *is equivalent to*

$$i_X\,\eta = 0 \ . \tag{3}$$

Moreover, if $cod_{\mathbb{C}}(sing(X)) \geq 3$ then there exists a holomorphic vector field Y on Q such that

$$\eta = i_Y\, i_X\, \nu = i_Y\, d\eta = L_Y\, \eta \; . \tag{4}$$

In particular, if $p \notin sing(\eta)$ then $X(p) \wedge Y(p) \neq 0$ and $ker(\eta(p)) = \langle X(p), Y(p)\rangle$.

Remark 1.2. The rotational X can be defined for any holomorphic $(n-2)$-form on Q by (2), but in general the form does not define a foliation. When $X \not\equiv 0$ then relation (3) implies also condition (I). When $X \equiv 0$ then η is closed, but does not satisfy condition (I) in general. For instance $\eta = dz_1 \wedge dw_1 + dz_2 \wedge dw_2$ on $\mathbb{C}^4$ is closed but not decomposable.

Remark 1.3. In the above situation, if we assume that $cod_{\mathbb{C}}(sing(X)) \geq 3$ then all irreducible components of $sing(\eta)$ have dimension ≥ 1. In fact, by proposition 1 this implies that $\eta = i_Y\, i_X\, \nu$, and so

$$sing(\eta) = \{p \in Q \,|\, X(p) \wedge Y(p) = 0\} \; .$$

On the other hand, it is known that a set defined as above has no isolated points.

Next, we state the analogous of the Kupka phenomenon for codimension one foliations (see [K] and [Me]). Let η be a germ at $(\mathbb{C}^n, 0)$ of $(n-2)$-form defining a germ of singular two dimensional holomorphic foliation $\mathcal{F}_\eta$ and X be the rotational of η: $d\eta = i_X\, dz_1 \wedge ... \wedge dz_n$.

Proposition 2. *With the above notations assume that $X(0) \neq 0$. Then there exists a coordinate system $w = (w_1, ..., w_n)$ in which the form η does not depends on the variable w_1, that is, it can be written as:*

$$\eta = i_Y\, dw_2 \wedge ... \wedge dw_n = i_Y\, i_{\partial_{w_1}}\, dw_1 \wedge dw_2 \wedge ... \wedge dw_n$$

where in the above formula Y is a holomorphic vector field of the form

$$Y = \sum_{j \geq 2} Y_j\,(w_2, ..., w_n)\, \partial_{w_j} \; .$$

The proof of proposition 2 in a more general situation can be found in [Me].

Remark 1.4. Another way to state proposition 2 is to say that $\mathcal{F}_\eta$ is equivalent to the product of two one dimensional foliations: the singular foliation on $(\mathbb{C}^{n-1}, 0)$ induced by the vector field Y and the fibers of the projection $\Pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ given by $\Pi(w_1, ..., w_n) = (w_2, ..., w_n)$. We can say also that $\mathcal{F}_\eta = \Pi^*(\mathcal{G})$, where $\mathcal{G}$ is the foliation induced by Y. Note also that the curve $\gamma := \Pi^{-1}(0)$ is contained in the singular set of η.

Definition 1. In the situation of proposition 2 and remark 1.4 the curve γ will be called a *singular curve of Kupka type* and the holomorphic class of the vector field Y the *normal type* of γ.

Definition 2. The singularity $0 \in \mathbb{C}^n$ of the $(n-2)$-form η will be called *generalised Kupka* (notation: g.K.) if 0 is an isolated singularity of the rotational X (and so of $d\eta$). A g.K. singularity will be called *non-degenerate* if the linear part $DX(0)$ is non-singular. It will be called *semi-simple* if $DX(0)$ is non-degenerate and has eigenvalues two by two different (notation: s.s.g.K.). It will be called *nilpotent* if the linear part $DX(0)$ is nilpotent (notation: n.g.K.).

We would like to note that the concepts of definition 2 are independent of the n-form used to calculate the rotational X of η. In fact, they depend only of the foliation defined by η, in the sense that:
η is n.g.K. (or s.s.g.K.) $\iff$ $f.\eta$ is n.g.K. (or s.s.g.K.), $\forall\ f \in \mathcal{O}_n^*$.

Next, we will see examples of the above situations.

Example 1. *Semi-simple case.* Consider two linear diagonal vector fields on $\mathbb{C}^n$, $n \geq 3$, $S = \sum_{j=1}^n \lambda_j\, x_j\, \partial_{x_j}$ and $T = \sum_{j=1}^n \mu_j\, x_j\, \partial_{x_j}$. Since $[S,T] = 0$ they generate an action of $\mathbb{C}^2$ on $\mathbb{C}^n$. We will assume that

$$\lambda_i.\, \mu_j - \mu_i.\, \lambda_j \neq 0\ ,\ \forall\ 1 \leq i < j \leq n\ . \tag{5}$$

With condition (5) the generic orbit of the action has dimension two and so S and T generate a singular holomorphic two dimensional foliation on $\mathbb{C}^2$. This foliation is also defined by the $(n-2)$-form $\eta = i_S\, i_T\, \nu$, where $\nu = dx_1 \wedge ... \wedge dx_n$. It can be shown that $d\eta = i_X\, \nu$, where $X = tr(S).\, T - tr(T).\, S$ (tr = trace). Note that condition (5) implies that $X = 0 \iff tr(S) = tr(T) = 0$. In this case, the form η is closed and we say that the foliation can be defined by a holomorphic closed form.

According to our definition, the form η is semi-simple if and only if $tr(S).\, \mu_j - tr(T).\, \lambda_j \neq 0$ for all $j \in \{1, ..., n\}$. Let us remark also that $f(x) = x_1...x_n$ is an *integrating factor* of η, in the sense that $d\left(\frac{1}{f}.\, \eta\right) = 0$. In this case, we say that the foliation can be defined by a meromorphic closed form.

In the next result we will see a situation in which the germ of foliation is equivalent to one generated by a linear action of $\mathbb{C}^2$, as in example 1. Let η be a germ at $0 \in \mathbb{C}^n$ of holomorphic integrable $(n-2)$-form with rotational X. We will assume that 0 is a g.K. non-degenerate singularity of η. In particular, if $S = DX(0)$ then $det(S) \neq 0$. Moreover, there exists a germ of vector field Y such that $\eta = i_Y\, i_X\, \nu$, where $\nu = dz_1 \wedge ... \wedge dz_n$. Let $\lambda_1, ..., \lambda_n$ denote the eigenvalues of S and $\mu_1, ..., \mu_n$ the eigenvalues of $T := DY(0)$. We will asume that there are $1 \leq i < j \leq n$ such that $\lambda_i.\, \mu_j - \lambda_j.\, \mu_i \neq 0$. This is equivalent to $i_S\, i_T\, \nu \neq 0$.

Theorem 1. *In the above situation we have* $tr(S) = 0$, $tr(T) = 1$ *and* $[S,T] = 0$. *In particular, given* $\tau \in \mathbb{C}$ *then the eigenvalues of* $S + \tau.\, T$ *are* $\lambda_j + \tau.\, \mu_j$, $1 \leq j \leq n$. *Moreover:*

(a). *If there exists* $\tau \in \mathbb{C}$ *such that the eigenvalues of* $S + \tau.\, T$ *satisfy Poincaré's non-resonance conditions* (cf. [M]) *and are two by different then* $\mathcal{F}_\eta$ *is formally equivalent to a foliation generated by a linear action of* $\mathbb{C}^2$.

(b). *If there exists* $\tau \in \mathbb{C}$ *such that* $X + \tau.\, Y$ *is linearizable and* $S + \tau.\, T$ *has eigenvalues two by two different then* $\mathcal{F}_\eta$ *is holomorphically equivalent to a foliation generated by a linear action of* $\mathbb{C}^2$. *In particular, if the eigenvalues of* $S + \tau.\, T$ *satisfy Brjuno's condition of small denominators* (see [M]) *then this condition is verified.*

Example 2. *Nilpotent case.* Let $S = \sum_{j=1}^n k_j\, x_j\, \partial_{x_j}$, where $k_j \in \mathbb{N}$, $1 \leq j \leq n$. We say that a germ Z at $0 \in \mathbb{C}^n$, of holomorphic vector field, is *quasi-homogeneous* with respect to S, with weight $\ell \in \mathbb{N} \cup \{0\}$, if $[S, Z] = \ell.\, Z$. In this case, the vector field Z must be polynomial. In fact, if we write $Z = \sum_{j=1}^n Z_j(x).\, \partial_{x_j}$ then $[S, Z] = \ell.\, Z$ is equivalent to

$$S(Z_j) = (\ell + k_j)\, Z_j\ ,\ 1 \leq j \leq n\ , \tag{6}$$

which implies that $Z_1, ..., Z_n$ are polynomials quasi-homogeneous with respect to S:

$$Z_j\,(t^{k_1}.x_1, ..., t^{k_n}.x_n) = t^{\ell+k_j}.\,Z_j\,(x_1, ..., x_n)\ ,\ \forall\, 1 \le j \le n\, ,\ \forall\, t \in \mathbb{C}\, .$$

In this situation, the vector fields S an Z generate an action of the affine group on $\mathbb{C}^n$ and the $(n-2)$-form $\eta = \eta(S, Z) := i_S\, i_Z\, \nu$ is integrable ($\nu = dx_1 \wedge ... \wedge dx_n$). Note that

$$d\eta = d\,(i_S\, i_Z\, \nu) = L_S\,(i_Z\, \nu) - i_S\, d\,(i_Z\, \nu) = i_{[S,Z]}\nu + i_Z\,(L_S\, \nu) - \nabla Z.\, i_S\, \nu\, ,$$

where $\nabla Z = \sum_i \frac{\partial Z_i}{\partial x_i}$. It follows that $d\eta = i_X\, \nu$, where

$$X = (\ell + tr(S))\,.\,Z - \nabla Z.\,S\ .$$

Therefore X is the rotational of η and we can say that η is n.g.K. iff $0 \in \mathbb{C}^n$ is an isolated singularity of X. Note that X satisfies $[S, X] = \ell.\,X$ and $\nabla X = 0$.

Remark 1.5. In this remark we discuss the existence of an example as above. Let $\Sigma(S, \ell) = \{Z\,|\,[S, Z] = \ell.\,Z\}$, $\mathcal{E}(S, \ell) = \{X \in \Sigma(S, \ell)\,|\,\nabla X = 0\}$ and $\mathcal{N}(S, \ell) = \{X \in \mathcal{E}(S, \ell)\,|\,X$ has an isolated singularity at $0 \in \mathbb{C}^n\,\}$. As we have seen before, $\Sigma(S, \ell)$ is a finite dimensional vector space. Since $\mathcal{E}(S, \ell)$ is a linear subspace of $\Sigma(S, \ell)$, it is also a finite dimensional vector space. On the other hand, it is not difficult to see that $\mathcal{N}(S, \ell)$ is a Zariski open subset of $\mathcal{E}(S, \ell)$. In particular, if $\mathcal{N}(S, \ell) \ne \emptyset$ then $\mathcal{N}(S, \ell)$ is a Zariski open and dense subset of $\mathcal{E}(S, \ell)$. It can be verified that, if $\mathcal{N}(S, \ell) \ne \emptyset$ and $X \in \mathcal{N}(S, \ell)$ then the form $\eta = i_S\, i_X\, \nu$ is n.g.K. with rotational $(\ell + tr(S))\, X$.

Let $\mathbb{N}(S) := \{\ell \in \mathbb{N}\,|\,\mathcal{N}(S, \ell) \ne \emptyset\}$. We would like to observe also that for all S the set $\mathbb{N}(S)$ is infinite. We will not prove this assertion in general, but in the next example we will see a situation in which $\mathbb{N}(S) = \mathbb{N}$.

Example 3. Let us assume that the vector field S of example 2 is the radial vector field, $S = \sum_{j=1}^n x_j\, \partial_{x_j}$. In this case it can be proved that $\Sigma(S, \ell) = \{Z\,|$ the coefficients of Z are homogeneous polynomials of degree $\ell + 1\}$. We assert that for all $\ell \ge 1$ then $\mathcal{N}(S, \ell)$ is Zariski open and dense in $\mathcal{E}(S, \ell)$. In order to prove this fact, it is enough to exhibit one example $X \in \mathcal{N}(S, \ell)$. We then consider the vector field

$$J_{\ell+1} := x_n^{\ell+1}\, \partial_{x_1} + x_1^{\ell+1}\, \partial_{x_2} + ... + x_{j-1}^{\ell+1}\, \partial_{x_j} + ... + x_{n-1}^{\ell+1}\, \partial_{x_n}\ .$$

Clearly, $\nabla J_{\ell+1} = 0$ and $0 \in \mathbb{C}^n$ is an isolated singularity of $J_{\ell+1}$. This example is known as the generalized Jouanolou's example of degree $\ell + 1$ (cf. [LN-So]).

In the next result we will see that the situation of example 2 is, in some sense, general.

Theorem 2. *Assume that $0 \in \mathbb{C}^n$ is a n.g.K. singularity of η. Then there exists a holomorphic cordinate system $w = (w_1, ..., w_n)$ around $0 \in \mathbb{C}^n$ where η has polynomial coefficients. More precisely, there exist two polynomial vector fields X and Y in $\mathbb{C}^n$ such that*

(a). *$Y = S + N$, where $S = \sum_{j=1}^n k_j\, w_j\, \partial_{w_j}$ is linear semi-simple with eigenvalues $k_1, ..., k_n \in \mathbb{N}$, $DN(0)$ is linear nilpotent and $[S, N] = 0$.*

(b). *$[N, X] = 0$ and $[S, X] = k.\,X$, where $k \in \mathbb{N}$. In other words, X is quasi-homogeneous with respect to S with weight k.*

(c). *In this coordinate system we have $\eta = i_Y\, i_X\, dw_1 \wedge ... \wedge dw_n$ and $L_Y(\eta) = (k + tr(S))\, \eta$.*

In particular, $\mathcal{F}_\eta$ can be defined by a local action of the affine group.

Definition 3. In the situation of theorem 2, $S = \sum_{j=1}^n k_j\, w_j\, \partial_{w_j}$ and $L_S(X) = k.\, X$, we say that the n.g.K. singularity is of type $(k_1, ..., k_n; k)$.

Remark 1.6. We would like to observe that in many cases it can be proved that vector field N of the statement of theorem 2 vanishes. In order to discuss this assertion it is convenient to introduce some objects. Given two germs of vector fields Z and W set $L_Z(W) := [Z, W]$. Recall that $\Sigma(S, \ell) = \{Z \in \mathcal{X}_n \,|\, L_S(Z) = \ell.\, Z\}$. Let X and $Y = S + N$ be as in theorem 2. Observe that:

- Jacobi's identity implies that if $W \in \Sigma(S, k)$ and $Z \in \Sigma(S, \ell)$ then $[W, Z] \in \Sigma(S, k + \ell)$.
- For all $k \in \mathbb{Z}$ we have $dim_{\mathbb{C}}(\Sigma(S, k)) < \infty$ (because $k_1, ..., k_n \in \mathbb{N}$).
- $N \in \Sigma(S, 0)$, $X \in \Sigma(S, \ell)$ and $L_X(N) = 0$, so that $N \in ker(L_X^0)$, where $L_X^0 := L_X \colon \Sigma(S, 0) \to \Sigma(S, \ell)$. In particular, the vector field $N \in \Sigma(S, 0)$ of theorem 2 necessarily vanishes $\iff ker(L_X^0) = \{0\}$.

In § 3.2 we will see that under a non-resonance condition, which depends only on X, then $ker(L_X^0) = \{0\}$. Let us mention some correlated facts.

(I). If S has no resonances of the type $\langle\sigma, k\rangle - k_j = 0$, where $\langle\sigma, k\rangle = \sum_j \sigma_j.\, k_j$, $k = (k_1, ..., k_n)$ and $\sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{Z}^n_{\geq 0}$, then $ker(L_X) = \{0\}$.

(II). When $n = 3$ and X has an isolated singularity at $0 \in \mathbb{C}^3$ then $ker(L_X) = \{0\}$ (cf. [LN]).

(III). When $N \not\equiv 0$ and $cod_{\mathbb{C}}(sing(N)) = 1$, or $sing(N)$ has an irreducible component of dimension one then it can be proved that X cannot have an isolated singularity at $0 \in \mathbb{C}^n$.

In fact, we think that whenever X has an isolated singularity at $0 \in \mathbb{C}^n$ and $\nabla X = 0$ then $ker(L_X^0) = \{0\}$.

The next result is about the nature of the set $\mathcal{K}(S, \ell) := \{X \in \Sigma(S, \ell) |\, ker(L_X^0) = \{0\}$ and $\nabla X = 0\}$.

Proposition 3. *If $\mathcal{K}(S, \ell) \neq \emptyset$ then $\mathcal{K}(S, \ell)$ is a Zariski open and dense subset of $\mathcal{E}(S, \ell)$. In particular, if there exists $X \in \mathcal{E}(S, \ell)$ satisfying the non-resonance condition mentioned in remark 1.6 then $\mathcal{K}(S, \ell)$ is a Zariski open and dense in $\mathcal{E}(S, \ell)$.*

Proposition 3 is a straightforward consequence of the following facts:

(A). The set of linear maps $\mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$ is finite dimensional vector space. Moreover, the subspace $\mathcal{NI} := \{T \in \mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell)) \,|\, T$ is not injective$\}$ is an algebraic subset of $\mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$.

(B). The map $L\colon\ \mathcal{E}(S, \ell) \to \mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$ defined by $L(X) = L_X^0$ is linear. As a consequence, the set $L^{-1}(\mathcal{NI})$ is an algebraic subset of $\mathcal{E}(S, \ell)$.

(C). $\mathcal{K}(S, \ell) = \mathcal{E}(S, \ell) \setminus L^{-1}(\mathcal{NI})$.

We leave the details to the reader.

Remark 1.7. In the case of the radial vector field, $R := \sum_{j=1}^n z_j\, \partial_{z_j}$, we have $\mathcal{K}(R, \ell) \neq \emptyset$ for all $\ell \geq 1$. In fact, we will prove in § 3.2 that $J_{\ell+1} \in \mathcal{K}(R, \ell)$, where $J_{\ell+1}$ is the generalized Jouanolou's vector field (see example 3).

In the next result we will consider the problem of deformation of two dimensional foliations with a g.K. singularity. Consider a holomorphic family of $(n-2)$-forms,

$(\eta_t)_{t\in U}$, defined on a polydisc Q of $\mathbb{C}^n$, where the space of parameters U is an open set of $\mathbb{C}^k$ with $0 \in U$. Let us assume that:

- For each $t \in U$ the form η_t defines a two dimensional foliation $\mathcal{F}_t$ on Q. Let $(X_t)_{t\in U}$ be the family of holomorphic vector fields on Q such that $d\eta_t = i_{X_t}\,\nu$, $\nu = dz_1 \wedge ... \wedge dz_n$.
- $\mathcal{F}_0$ has a g.K. singularity at $0 \in Q$, either non-degenerate, or nilpotent.

Theorem 3. *In the above situation there exist a neighborhood $0 \in V \subset U$, a polydisk $0 \in P \subset Q$, and a holomorphic map $\mathcal{P}\colon V \to P \subset \mathbb{C}^n$ such that $\mathcal{P}(0) = 0$ and for any $t \in V$ then $\mathcal{P}(t)$ is the nique singularity of $\mathcal{F}_t$ in P. Moreover, $\mathcal{P}(t)$ is of the same type as $\mathcal{P}(0)$, in the sense that:*

(a). *If 0 is a non-degenerate singularity of $\mathcal{F}_0$ then $\mathcal{P}(t)$ is a non-degenerate singularity of $\mathcal{F}_t$, $\forall t \in V$. If 0 is a s.s.g.K. singularity of $\mathcal{F}_0$ then $\mathcal{P}(t)$ is a s.s.g.K. singularity of $\mathcal{F}_t$, $\forall\ t \in V$.*

(b). *If 0 is a n.g.K. singularity of type $(m_1, ..., m_n; \ell)$ of $\mathcal{F}_0$ then $\mathcal{P}(t)$ is a n.g.K. singularity of type $(m_1, ..., m_n; \ell)$ of $\mathcal{F}_t$, $\forall t \in V$.*

As an application of theorem 3 it can be done an easy proof of the fact that there are irreducible components of the space of foliations of dimension two of $\mathbb{P}^n$, $n \geq 3$, which are constituted of linear pull-backs of one dimensional foliations on $\mathbb{P}^{n-1}$ (see the general case in [C-P]). Instead we will prove a generalization of a result of [C-LN] which equally implies this result. Let η be an integrable $(n-2)$-form on $\mathbb{C}^n$, with polynomials coefficients, written as

$$\eta = \eta_0 + ... + \eta_{d+1} = \sum_{j=0}^{d+1} \eta_j \ , \tag{7}$$

where the coefficients of η_j are homogeneous polynomials of degree j, $0 \leq j \leq d+1$, $d \geq 2$.

Theorem 4. *In the above situation, assume that $\eta_{d+1} = i_R\, i_X\, \nu$, where*

(a). *$R = \sum_{j=1}^n x_j\, \partial_{x_j}$ is the radial vector field on $\mathbb{C}^n$ and $\nu = dx_1 \wedge ... \wedge dx_n$.*

(b). *X is a vector field with coefficients homogeneous of degree d such that $\nabla X = 0$ and with an isolated singularity at $0 \in \mathbb{C}^n$.*

Then there exists a translation $\Phi(x) = x + a$, $a \in \mathbb{C}^n$, such that $\Phi^*(\eta) = \eta_{d+1}$.

Remark 1.8. Note that the $(n-2)$-form $\eta_{d+1} = i_R\, i_X\, \nu$ of theorem 4 induces a foliation of dimension one and degree d on $\mathbb{P}^{n-1}$. In particular $\mathcal{F}_{\eta_{d+1}}$, viewed as a two dimensional foliation on $\mathbb{P}^n \supset \mathbb{C}^n$, is the pull-back of a one dimensional foliation of degree d on $\mathbb{P}^{n-1}$ by a linear map $f\colon \mathbb{P}^n - \to \mathbb{P}^{n-1}$ (induced by a linear map $F\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$).

Let $LPB(n,d) := \{\mathcal{F} \mid \mathcal{F} = f^*(\mathcal{G})$, where $\mathcal{G}$ is a one dimensional foliation on $\mathbb{P}^{n-1}$ of degree d and $f\colon \mathbb{P}^n - \to \mathbb{P}^{n-1}$ is a linear map$\}$. As a consequence of theorem 4 we get:

Corollary 1. *For any $d \geq 2$ and $n \geq 3$ the set $LPB(n,d)$ is an irreducible component of the space of two dimensional foliations on $\mathbb{P}^n$.*

2. Proposition 1 and theorem 1

2.1. **Proof of proposition 1.** Let U be a domain of $\mathbb{C}^n$, $n \geq 3$, and $\eta \in \Lambda^{n-2}(U)$, $\eta \not\equiv 0$. We will set $sing(\eta) = \{q \in U \mid \eta(q) = 0\}$ and we will assume that (i). $H^1(U,\mathcal{O})=0$. In particular, if U is a polydisk then this is true.

(ii). η satisfies condition (I) of the integrability condition, that is, for any $q \in U\setminus sing(\eta)$ then there exist a neigborhood V of q, $V\subset U$, and 1-forms $\alpha_1,...,\alpha_{n-2}\in\Lambda^1(V)$ such that

$$\eta|_V=\alpha_1\wedge...\wedge\alpha_{n-2}\ . \tag{8}$$

(iii). η satisfies integrability condition (II) iff for all decomposition as in (ii) then $d\alpha_m\wedge\eta=0$, $\forall\ 1\le m\le n-2$.

We want to prove that, assuming (ii) then, $i_X\,\eta=0\iff$ (iii), where X is the rotational of η: $d\eta=i_X\nu$, $\nu=dz_1\wedge...\wedge dz_n$. First of all observe that, if V and $\alpha_1,...,\alpha_{n-2}$ are as above then

$$d\eta|_V=\sum_{j=1}^{n-2}(-1)^{j-1}\,\alpha_1\wedge...\wedge d\alpha_j\wedge...\wedge\alpha_{n-2}\implies$$

$$d\alpha_m\wedge\eta|_V=\pm\,\alpha_m\wedge d\eta|_V\ ,\ \forall\ m\in\{1,...,n-2\}\ . \tag{9}$$

Proof of $i_X\eta=0\implies$ (iii). We have two possibilities:

Case 1. $X\equiv0$, or equivalently $d\eta\equiv0$. In this case, by (9) we have

$$d\alpha_m\wedge\eta|_V=0\ ,\ \forall\ m\in\{1,...,n-2\}\implies\text{(iii)}.$$

Case 2. $X\not\equiv0$. In this case, $W:=sing(\eta)\cup sing(X)$ is a proper analytic subset of U, so that $U\setminus W$ is open and dense in U.

Let us fix $q\in U\setminus W$ and a neighborhood V of q such that (8) and (9) are true. From $i_X\,\eta=0$ we get

$$i_X\,(\alpha_1\wedge...\wedge\alpha_{n-2})=\sum_{j=1}^{n-2}(-1)^{j-1}i_X(\alpha_j)\,\alpha_1\wedge...\wedge\widehat{\alpha_j}\wedge...\wedge\alpha_{n-2}=0\ ,$$

where $\widehat{\alpha_j}$ means omission of α_j. If we take the wedge product of the above sum by α_m we get

$$0=\alpha_m\wedge\,[(-1)^{m-1}\,i_X(\alpha_m)\,\alpha_1\wedge...\wedge\widehat{\alpha_m}\wedge...\wedge\alpha_{n-2}]=(i_X\,\alpha_m)\,\eta\implies$$

$$i_X\,\alpha_m=0\ ,\ \forall m\in\{1,...,n-2\}\ .$$

Since $i_Xd\eta=0$ we get $i_X(\alpha_m\wedge d\eta)=0$ and this implies that $\alpha_m\wedge d\eta=0$, because $\alpha_m\wedge d\eta$ is a n-form and $X\not\equiv0$. Hence, (9) implies that $d\alpha_m\wedge\eta|_V\equiv0$, $\forall\,m\in\{1,...,n-2\}$, and so (iii) is true.

Proof of (iii) $\implies i_X\,\eta=0$. We can assume $X\not\equiv0$. Remark 1.1 implies that, if we fix $q\in U\setminus sing(\eta)$ then, we can find a coordinate system $w=(w_1,...,w_n)\colon(V,q)\to(\mathbb{C}^n,0)$ and $f\in\mathcal{O}^*(V)$ such that $\eta|_V=f\,dw_3\wedge...\wedge dw_n$. Hence,

$$d\eta|_V=\left[\frac{\partial f}{\partial w_1}\,dw_1+\frac{\partial f}{\partial w_2}\,dw_2\right]\wedge dw_3\wedge...\wedge dw_n=i_{\tilde X}\,dw_1\wedge...\wedge dw_n\ ,$$

where

$$\tilde X=\frac{\partial f}{\partial_{w_2}}\,\partial_{w_1}-\frac{\partial f}{\partial_{w_1}}\,\partial_{w_2}\implies i_{\tilde X}\,\eta=0\ .$$

Since $X|_V=\phi.\tilde X$ for some $\phi\in\mathcal{O}^*(V)$ we get that $i_X\,\eta\,|_V=0$ and this implies that $i_X\,\eta=0$, as wanted.

Let us assume that $cod_{\mathbb{C}}(sing(X)) \geq 3$ and prove that there exists $Y \in \mathcal{X}(U)$ such that $\eta = i_Y\, i_X\, \nu$. Let $W := U \setminus sing(X)$. Since $H^1(U,\mathcal{O}) = 0$ and $cod_{\mathbb{C}}(sing(X)) \geq 3$ it follows from a theorem of H. Cartan (see [H]) that $H^1(W,\mathcal{O}) = 0$.

Now, if we fix $q \in W$ then the relation $i_X\, \eta = 0$ and the division theorem imply that there exist a Stein neighborhood V_q of q and $\zeta_q \in \Lambda^{n-1}(V_q)$ such that $\eta|_{V_q} = i_X\, \zeta_q$. Since $\zeta_q \in \Lambda^{n-1}(V_q)$ there exists $Y_q \in \mathcal{X}(V_q)$ such that $\zeta_q = -i_{Y_q}\, \nu$, or

$$\eta = i_X\, \zeta_q = i_X\, i_{-Y_q}\, \nu = i_{Y_q}\, i_X\, \nu\ .$$

If $V_q \cap V_p \neq \emptyset$ then $i_{(Y_p - Y_q)}\, i_X\, \nu = 0 \implies \exists\ g_{p\,q} \in \mathcal{O}(V_p \cap V_q)$ such that $Y_p - Y_q = g_{p\,q}.\,X$. Note that $(g_{p\,q})_{V_p \cap V_q \neq \emptyset}$ is an additive cocycle. Since $H^1(W,\mathcal{O}) = 0$ the cocycle is trivial and there exists a collection $(h_p)_{q\in W}$, $h_p \in \mathcal{O}(V_p)$ such that $g_{p\,q} = h_p - h_q$ on $V_p \cap V_q \neq \emptyset$. Hence, there exists a holomorphic vector field $Y_1 \in \mathcal{X}(W)$ such that $Y_1|_{V_p} = Y_p - h_p.\,X$. This implies that

$$i_{Y_1}\, d\eta = i_{Y_p}\, d\eta = \eta \text{ on } V_p \implies i_{Y_1}\, d\eta = \eta$$

Since $cod_{\mathbb{C}}(sing(X)) \geq 3$, by Hartog's theorem Y_1 can be extended to a vector field $Y \in \mathcal{X}(U)$ such that $i_Y\, d\eta = \eta$. Finally, since $i_Y\, \eta = 0$ we get

$$L_Y\, \eta = i_Y\, d\eta + d(i_Y\, \eta) = \eta \quad \square$$

2.2. **Proof of theorem 1.** Let $\eta = i_Y\, i_X\, \nu$, where $\nu = dz_1 \wedge ... \wedge dz_n$ and $d\eta = i_X\, \nu$. Set $S := DX(0)$ and $T := DY(0)$. Under the hypothesis that S is non-singular we will prove that $tr(S) = 0$, $tr(T) = 1$ and $[S,T] = 0$.

First of all, let us write $X := \sum_j X_j\, \partial_{z_j}$ and $Y := \sum_j Y_j\, \partial_{z_j}$. Since $d\eta = i_X\, \nu$, we get

$$0 = d(i_X\, \nu) = \nabla X.\, \nu \text{ where } \nabla X = \sum_j \frac{\partial X_j}{\partial z_j} \implies tr(S) = \nabla X(0) = 0\ .$$

Now, note that

$$L_Y\, \eta = \eta \implies L_Y\, d\eta = d\eta \implies i_X\, \nu = L_Y\, i_X\, \nu = i_{[Y,X]}\, \nu + i_X\, L_Y\, \nu =$$

$$= i_{[Y,X]}\, \nu + i_X\, (\nabla Y.\, \nu)\ ,\ \text{where } \nabla Y = \sum_j \frac{\partial Y_j}{\partial z_j} \implies$$

$$[Y,X] = (1 - \nabla Y)\,.\, X = f.\, X\ ,\ \text{where } f = 1 - \nabla Y\ . \tag{10}$$

Taking the 1-jet of both members of the above relation we get $[T,S] = a.\,S$, where $a = f(0) = 1 - tr(T)$. This relation can be written as $S.\,T - T.\,S = a.\,S$ and since S is invertible we obtain

$$S.\,T.\,S^{-1} = T + a.\,I\ ,$$

where I is the identity. Taking the trace in both members we get

$$tr(T) = tr(T) + n.\,a \implies a = 0 \implies tr(T) = 1 \text{ and } [S,T] = 0\ .$$

Let $\lambda_1,...,\lambda_n \neq 0$ and $\mu_1,...,\mu_n$ be the eigenvalues of S and T respectively. Since $[S,T] = 0$, for all $\tau \in \mathbb{C}$ the eigenvalues of $T + \tau.\,S$ are $\mu_j + \tau.\,\lambda_j$, $1 \leq j \leq n$. Let us assume that there is $\tau \in \mathbb{C}$ such that $\rho_j := \mu_j + \tau.\,\lambda_j$, $1 \leq j \leq n$, are two by two different and satisfy Poincaré's non-resonance relations

$$\langle \rho, \sigma \rangle - \rho_j \neq 0\ ,\ \forall\ 1 \leq j \leq n \text{ and } \forall\ \sigma = (\sigma_1,...,\sigma_n) \in \mathbb{Z}_{\geq 0} \text{ with } |\sigma| = \sum_j \sigma_j \geq 2\ .$$

Let $Z := Y + \tau.X$. Note that (10) implies

$$[Z, X] = [Y, X] = f.X$$

On the other hand, by Poincaré's formal linearization theorem, there exists a formal diffeomorphism $\Phi \in \widehat{Diff}(\mathbb{C}^n, 0)$ such that $D\Phi(0) = I$ and $\Phi^*(Z)$ is linear and semi-simple (because $\rho_i \neq \rho_j$, if $i \neq j$). If we set $\widehat{Z} := \Phi^*(Z)$, $\widehat{X} := \Phi^*(X)$, then we have $\widehat{Z} = \sum_j \rho_j.x_j\,\partial_{x_j}$ and $\widehat{X} = \widehat{X}_j.\partial_{x_j}$ and the above relation implies that

$$[\widehat{Z}, \widehat{X}] = \widehat{f}.\widehat{X} \ , \text{ where } \widehat{f} = \Phi^*(f) \ . \tag{11}$$

Note that $\widehat{f}(0) = 0$.

Claim 2.1. *With the above notations we have*

$$\widehat{X}_k(x) = x_k.\psi_k(x) \ , \text{ where } \psi_k(0) = \lambda_k \neq 0 \ , \ 1 \leq k \leq n \ .$$

Proof. Since $D\widehat{X}(0) = \sum_j \lambda_j\,x_j\,\partial_{x_j}$ it is enough to prove that $x_k|X_k$, $1 \leq k \leq n$. Since $\widehat{Z} = \sum_j \rho_j.x_j\,\partial_{x_j}$, relation (11) is equivalent to

$$\widehat{Z}(\widehat{X}_k) = h_k.\widehat{X}_k \ , \text{ where } h_k = \rho_k + \widehat{f} \ , \ 1 \leq k \leq n \ . \tag{12}$$

Let us write the Taylor series of $\widehat{X}_k$ and of h_k as $\widehat{X}_k = \sum_{j \geq 1} G_j(x)$ and $h_k = \sum_{j \geq 0} \phi_j(x)$ where G_j and ϕ_j are homogeneous of degree j, $\forall\, j \geq 1$. The idea is to prove by induction on $j \geq 1$ that $x_k \,|\, G_j$ for all $j \geq 1$.

Step $j = 1$. The linear part of (11) gives $[\widehat{Z}, D\widehat{X}(0)] = 0$. Since $\rho_i \neq \rho_j$ if $i \neq j$ the linear vector field $D\widehat{X}(0)$ is diagonal in the (formal) coordinates $(x_1, ..., x_n)$. Hence, $G_1(x) = \lambda_k.x_k$, and so $x_k \,|\, G_1$.

Step $j - 1 \implies j$, $\forall\, j \geq 2$. Since $\widehat{Z}$ is a linear vector field the homogeneous term of degree j of the left hand of relation (12) is $\widehat{Z}(G_j)$. On the other hand, the homogeneous term of degree j of the right hand of (12) is $\sum_{r+s=j} \phi_r.G_s$ which implies that

$$\widehat{Z}(G_j) = \sum_{r+s=j} \phi_r.G_s = \rho_k.G_j + \sum_{r+s=j, s<j} \phi_r.G_s \implies$$

$$\widehat{Z}(G_j) - \rho_k.G_j = \sum_{r+s=j, s<j} \phi_r.G_s := H_j \ .$$

By the induction hypothesis $x_k \,|\, H_j \implies H_j|_{(x_k=0)} = 0$. If we write $G_j(x) = \sum_\sigma a_\sigma.x^\sigma$ then $\widehat{Z}(G_j) = \sum_\sigma \langle \rho, \sigma \rangle\; a_\sigma\, x^\sigma$ and so

$$\sum_\sigma (\langle \sigma, \rho \rangle - \rho_k)\, a_\sigma\, x^\sigma \Big|_{(x_k=0)} = 0 \implies$$

$a_\sigma = 0$ if $\sigma_k = 0$ (because $\langle \sigma, \rho \rangle - \rho_k \neq 0$) $\implies x_k \,|\, G_j$. Therefore, $x_k|X_k$, $1 \leq k \leq n$ and the claim is proved. □

Now, let us prove assertion (a) of theorem 1. The idea is to prove that there is a linear combination $W = g.\widehat{X} + h.\widehat{Z}$, where $g, h \in \widehat{\mathcal{O}}_n$ and $(g(0), h(0)) \neq (0, 0)$, such that $[\widehat{Z}, W] = 0$.

Recall that we have assumed that there are $i < j$ such that $\lambda_i.\mu_j - \lambda_j.\mu_i \neq 0$. Without lost of generality we will suppose that $i = 1$ and $j = 2$. We assert that there exist $g, h \in \widehat{\mathcal{O}}_n$ such that $(g(0), h(0)) \neq (0,0)$ and $W = g.\widehat{X} + h.\widehat{Z}$ satisfies $W(x_1) = 0$ and $W(x_2) = x_2$.

In fact, by claim 2.1 $\widehat{X}(x_j) = x_j.\psi_j(x)$, $1 \leq j \leq n$. Hence, if W is as above then $W(x_j) = g.x_j.\psi_j(x) + h.\rho_j.x_j$, $1 \leq j \leq n$. In particular, the assertion is equivalent to the fact that the system of linear equations below in $g, h \in \widehat{\mathcal{O}}_n$ has a solution $g, h \in \widehat{\mathcal{O}}_n$ with $(g(0), h(0)) \neq (0,0)$:

$$\begin{cases} \psi_1(x).g + \rho_1.h = 0 \\ \psi_2(x).g + \rho_2.h = 1 \end{cases}$$

This is true because the determinant of the system is $\Delta(x) = \rho_2.\psi_1(x) - \rho_1.\psi_2(x)$ and $\Delta(0) = \rho_2.\lambda_1 - \rho_1.\lambda_2 = \mu_2.\lambda_1 - \mu_1.\lambda_2 \neq 0$. It remains to prove that $[\widehat{Z}, W] = 0$.

First of all, from $[\widehat{Z}, \widehat{X}] = \widehat{f}.\widehat{X}$ and $W = g.\widehat{X} + h.\widehat{Z}$ we get $[\widehat{Z}, W] = g_1.\widehat{X} + h_1.\widehat{Z}$, where $g_1 = \widehat{Z}(g) + g.\widehat{f}$ and $h_1 = \widehat{Z}(h)$. On the other hand, if we set $W(x_j) := W_j$ then

$$[\widehat{Z}, W](x_j) = (\widehat{Z}.W - W.\widehat{Z})(x_j) = \widehat{Z}(W_j) - \rho_j.W_j \ , \ 1 \leq j \leq n \implies$$

$$[\widehat{Z}, W](x_j) = 0 \text{ if } j = 1, 2 \ .$$

This implies that:

$$\begin{matrix} g_1.\widehat{X}(x_1) + h_1.\widehat{Z}(x_1) = 0 \\ g_1.\widehat{X}(x_2) + h_1.\widehat{Z}(x_2) = 0 \end{matrix} \implies \begin{matrix} g_1.\psi_1 + h_1.\rho_1 = 0 \\ g_1.\psi_2 + h_1.\rho_2 = 0 \end{matrix} \implies g_1 = h_1 = 0 \ ,$$

because $\Delta(0) \neq 0$. Therefore, $[\widehat{Z}, W] = 0$ as asserted. Since $\widehat{Z}$ is linear diagonal without resonances the vector field W must be also linear and diagonal, which proves item (a) of theorem 1.

When $Z = Y + \tau.X$ is holomorphically linearizable then we can assume that the diffeomorphism Φ and the vector fields $\widehat{Z}$, $\widehat{X}$ and W are convergent. This proves item (b) of theorem 1. □

3. Theorem 2

In this section we will assume that 0 is a n.g.K. singularity of η: $DX(0)$ is nilpotent, where X is the rotational of η. In this case, by proposition 1 there exists a germ $Y \in \mathcal{X}_n$ such that $\eta = i_Y\, d\eta$, $L_Y\, \eta = \eta$ and $L_Y\, d\eta = d\eta$.

3.1. **Proof of theorem 2.** We will use Poincaré-Dulac normalization theorem for germs of vector fields (see [Me]). Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $DY(0)$. Recall that $\lambda_1, ..., \lambda_n \in \mathbb{C}$ are in the Poicaré domain if $0 \in \mathbb{C}$ is not in the convex hull of the set $\{\lambda_1, ..., \lambda_n\}$.

Theorem 3.1. *There exists a formal diffeomorphism $\Phi \in \widehat{Diff}(\mathbb{C}^n, 0)$ such that $\Phi^*(Y) := \widehat{Y} \in \widehat{\mathcal{X}}_n$ can be written as*

$$\widehat{Y} = S + N \ ,$$

where $S = \sum_{j=1}^n \lambda_j\, w_j\, \partial_{w_j}$ is linear diagonal, N is nilpotent (in a sense that we will precise in remark 3.1) *and $[S, N] = 0$. When $\lambda_1, ..., \lambda_n$ are in the Poicaré domain then we can assume that Φ is convergent.*

Remark 3.1. If we consider $\widehat{Y}$ as a derivation in $\widehat{\mathcal{O}}_n$ then $\widehat{Y}$ induces a linear operator on the finite dimensional vector space of k-jets, $j^k(\widehat{\mathcal{O}}_n) := J^k_n$, say $Y^k\colon J^k_n \to J^k_n$, in such a way that the diagram below commutes:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_n & \stackrel{\widehat{Y}}{\longrightarrow} & \widehat{\mathcal{O}}_n \\ j^k \downarrow & & \downarrow j^k \\ J^k_n & \stackrel{Y^k}{\longrightarrow} & J^k_n \end{array}$$

Similarly, if we denote by $\Gamma^{p\,k} := j^k(\widehat{\Lambda}^p_n)$ the finite dimensional vector space of k-jets of p-forms, then the Lie derivative $L_{\widehat{Y}}\colon \widehat{\Lambda}^p_n \to \widehat{\Lambda}^p_n$ induces a linear operator $L^k_{\widehat{Y}}\colon \Gamma^{p\,k} \to \Gamma^{p\,k}$ in such a way that the diagram below commutes:

$$\begin{array}{ccc} \widehat{\Lambda}^p_n & \stackrel{L_{\widehat{Y}}}{\longrightarrow} & \widehat{\Lambda}^p_n \\ j^k \downarrow & & \downarrow j^k \\ \Gamma^{p\,k} & \stackrel{L^k_{\widehat{Y}}}{\longrightarrow} & \Gamma^{p\,k} \end{array}$$

The vector field N is nilpotent in the sense that it induces the nilpotent parts of the operators Y^k and $L^k_{\widehat{Y}}$. Similarly S induces the semi-simple part of the operators Y^k and $L^k_{\widehat{Y}}$, respectively.

Note also that, if the coordinates are choosen in such a way that $S = \sum_j \lambda_j\, z_j\, \partial_{z_j}$ then the monomial $z^\sigma = z_1^{\sigma(1)}...z_n^{\sigma(n)}$ is an eigenvector of S with $S(z^\sigma) = \langle\lambda,\sigma\rangle\,.\,z^\sigma$, where $\langle\lambda,\sigma\rangle = \sum_j \sigma_j.\lambda_j$. Similarly, a monomial p-form of the type $z^\sigma.\,dz_\mu$, where z^σ is a monomial as above and $dz_\mu = dz_{\mu_1}\wedge ... \wedge dz_{\mu_p}$, $1 \le \mu_1 < ... < \mu_p \le n$, is an eigenvector of of $L_{\widehat{Y}}$ with eigenvalue $\langle\lambda,\sigma\rangle + \sum_{j=1}^p \lambda_{\mu_j}$.

Let $\Phi \in \widehat{Diff}(\mathbb{C}^n, 0)$ be a diffeomorphism that normalizes the vector field Y that satisfies $L_Y\,\eta = i_Y\,d\eta = \eta$. Set $\widehat{\eta} := \Phi^*(\eta)$. Since $L_Y\,\eta = \eta$ we obtain that $L_{\widehat{Y}}\,\widehat{\eta} = \widehat{\eta}$ and $L_{\widehat{Y}}\,d\widehat{\eta} = d\widehat{\eta}$.

Claim 3.1. *We assert that $L_S\,\widehat{\eta} = \widehat{\eta}$ and $L_N\,\widehat{\eta} = 0$. In particular, $L_S\,d\widehat{\eta} = d\widehat{\eta}$ and $L_N\,d\widehat{\eta} = 0$.*

Proof. Set $\widehat{\eta}_k := j^k(\widehat{\eta})$, $k \ge 0$. From remark 3.1 we get $L^k_{\widehat{Y}}\,\widehat{\eta}_k = \widehat{\eta}_k$ for all $k \ge 0$. In particular, $\widehat{\eta}_k$ is an eigenvector of $L^k_{\widehat{Y}}$. Since L^k_S and L^k_N are the semi-simple and nilpotent parts of $L^k_{\widehat{Y}}$, respectively, we get $L^k_S(\widehat{\eta}_k) = \widehat{\eta}_k$ and $L^k_N(\widehat{\eta}_k) = 0$ for all $k \ge 0$. This implies the claim. □

Lemma 3.1. *The eigenvalues $\lambda_1, ..., \lambda_n$ are rational positive and $0 < tr(S) < 1$, where $tr(S) = \sum_j \lambda_j$. In particular, they are in the Poincaré domain and we can assume that Φ converges.*

Proof. First of all we will prove that there are natural numbers $k_1, ..., k_n$ and a function $\ell\colon \{1, ..., n\} \to \{1, ..., n\}$ such that the eigenvalues $\lambda_1, ..., \lambda_n$ satisfy the following system of non-homogeneous linear equations

$$k_j.\,\lambda_j + tr(S) - \lambda_{\ell(j)} = 1\ . \tag{13}$$

In fact, let us write $X = \sum_{j=1}^n X_j(z)\partial_{z_j}$. Since X has an isolated singularity at $0 \in \mathbb{C}^n$ we must have $\langle X_1, ..., X_n\rangle \supset m^p_n$, for some $p \in \mathbb{N}$. Therefore, if we write $\Phi^*(d\eta) = d\widehat{\eta} = i_{\widehat{X}}\nu$, where $\widehat{X} = \sum_{j=1}^n \widehat{X}_j\,\partial_{w_j}$ then $\left\langle \widehat{X}_1, ..., \widehat{X}_n \right\rangle \supset \widehat{m}_n^p$. In particular, the p^{th}-jet of $d\widehat{\eta}$, $j^p(d\widehat{\eta})$ (which has polynomial coefficients) has an isolated singularity at $0 \in \mathbb{C}^n$. If we write

$$j^p(d\widehat{\eta}) = \sum_{j=1} P_j(w)\,dw_1 \wedge ... \wedge \widehat{dw_j} \wedge ... \wedge dw_n \ ,$$

where $P_j \in \mathbb{C}[w_1, ..., w_n]$ has degree $\leq p$, then

$$\{P_1 = ... = P_n = 0\} = \{0\} \ . \tag{14}$$

Note that (14) implies that, for each $j \in \{1, ..., n\}$ there exists $\ell(j) \in \{1, ..., n\}$ such that $P_{\ell(j)}$ contains a monomial of the form $a.\, w_j^{k_j}$, $a \neq 0$, for otherwise we would have $P_r(0, ..., 0, w_j, 0, ..., 0) = 0$, $1 \leq r \leq n$, and (14) would not be true. This is equivalent to say that $j^k(d\widehat{\eta})$ contains a monomial of the form β, where $\beta := a.\, w_j^{k_j}.\, dw_1 \wedge ... \wedge \widehat{dw_{\ell(j)}} \wedge ... \wedge dw_n$, $a \neq 0$. The relation $L_S\, d\widehat{\eta} = d\widehat{\eta}$ implies that $j^k(d\widehat{\eta})$ is an eigenvector of L_S with correspondent eigenvalue 1. Since β is an eigenvector of L_S and

$$L_S(\beta) = \left(k_j.\, \lambda_j + \sum_{j \neq \ell(j)} \lambda_j \right).\, \beta$$

we get

$$k_j.\, \lambda_j + \sum_{j \neq \ell(j)} \lambda_j = 1 \implies (13) \ .$$

In the next arguments we will use the dynamics of the function $\ell\colon I_n \to I_n$, where $I_n = \{1, ..., n\}$. Recall that the orbit of $m \in I_n$ is the set $O(m) = \{\ell^s(m) \,|\, s \geq 0\}$, where $\ell^0(m) = m$ and $\ell^s(m)$, $s \geq 1$, is defined indutively by $\ell^{s+1}(m) = \ell(\ell^s(m))$. We say that $m \in I_n$ is periodic of period $r \geq 1$ if $\ell^r(m) = m$ and $r = min\{s \geq 1 \,|\, \ell^s(m) = m\}$. Since I_n is finite any orbit "finishes" in a periodic orbit. This means that, given $m \in I_n$ then there is $r_o \geq 0$ such that $\ell^{r_o}(m)$ is periodic and

$$O(m) = \{m, \ell(m), ..., \ell^{r_o}(m), ..., \ell^{r_o+r-1}(m) = \ell^{r_o}(m)\} \ ,$$

where $r \geq 1$ is the period of $\ell^{r_o}(m)$. The next step is the following:

Claim 3.2. $tr(S) \neq 1$.

Proof. Let us suppose by contradiction that $tr(S) = 1$. In this case, the system of equations (13) takes the form:

$$k_j.\, \lambda_j - \lambda_{\ell(j)} = 0 \ , \ 1 \leq j \leq n \ . \tag{15}$$

As we will see at the end $tr(S) = 1$ implies also that, after a linear change of variables, we can suppose:

($*$) If $j \in I_n$ is such that $k_j = 1$ then $\ell(j) > j$.

Using this fact, let us prove that (15) implies $\lambda_1 = ... = \lambda_n = 0$, which is a contradiction with $tr(S) = 1$.

Fix $m \in I_n$. If m is a fixed point of ℓ, $\ell(m) = m$, then ($*$) implies $k_m > 1$. On the other hand, (15) implies $(k_m - 1)\,\lambda_m = 0$, and so $\lambda_m = 0$.

From now on we will suppose that m is not a fixed point of ℓ. In this case, since $k_j \geq 1$ for all $j \in I_n$, (15) implies that, if there is $s \geq 1$ such that $\lambda_{\ell^s(m)} = 0$ then

$\lambda_m = 0$. Since any orbit of ℓ contains a periodic point it is sufficient to prove that $\lambda_m = 0$ when m is periodic of period $r \geq 2$.

So, let m be periodic with period $r \geq 2$. Set $m_j := \ell^{j-1}(m)$, $1 \leq j \leq r$, and $m_{r+1} := m_1 = m$. With this notation, we get from (15) that:

$$k_{m_j}.\lambda_{m_j} = \lambda_{m_{j+1}} \ , \ 1 \leq j \leq r \ . \tag{16}$$

Since $r \geq 2$ there is $j_o \in \{1, ..., r\}$ such that $m_{j_o+1} < m_{j_o}$, because m is periodic. In particular, from $(*)$ we get $k_{m_{j_o}} > 1$. On the other hand, (16) implies that

$$(k_{m_1}...k_{m_r} - 1)\,\lambda_{m_1} = 0 \implies \lambda_m = \lambda_{m_1} = 0 \ .$$

It remains to prove that we can suppose $(*)$.

Fix the formal coordinates $z = (z_1, ..., z_n)$ like before, that is where $S = \sum_j \lambda_j\, z_j\, \partial_{z_j}$. Let $\widehat{X}$ be such that $d\widehat{\eta} = i_{\widehat{X}}\nu$, where $\nu = dz_1 \wedge ... \wedge dz_n$. Let us prove first that, if $tr(S) = 1$ then $[S, \widehat{X}_1] = 0$, where $\widehat{X}_1$ denotes $D\widehat{X}(0)$. From $L_S\, d\widehat{\eta} = d\widehat{\eta}$ we obtain

$$d\widehat{\eta} = i_{\widehat{X}}\,\nu = L_S\,(i_{\widehat{X}}\,\nu) = i_{L_S(\widehat{X})}\,\nu + i_{\widehat{X}}(L_S\,\nu) = i_{[S,\widehat{X}]}\,\nu + tr(S).\,i_{\widehat{X}}\,\nu \implies$$

$$[S, \widehat{X}] = (1 - tr(S))\,\widehat{X} = 0 \ .$$

Taking the linear part in the above relation we get $[S, \widehat{X}_1] = 0$. Now, let us note that if $k_j = 1$ then $\widehat{\eta}$ contains a monomial of the form $a\,w_j\,dw_1 \wedge ... \wedge \widehat{dw_{\ell(j)}} \wedge ... \wedge dw_n$, $a \neq 0$, which is equivalent to say that $\widehat{X}_1$ contains a term of the form $\pm a\,w_j\,\partial_{w_{\ell(j)}}$. On the other hand, since $[S, \widehat{X}_1] = 0$ and $\widehat{X}_1$ is nilpotent, after a linear change of variables we can suppose that all the entries of the matrix of $\widehat{X}_1$ in the basis where S is diagonal are below the diagonal. This means exactly that if $k_j = 1$ then $\ell(j) > j$, as the reader can check. This finishes the proof of claim 3.2. $\square$

Let us prove that $\lambda_1, ..., \lambda_n \in \mathbb{Q}_+$ and $0 < tr(S) < 1$. Denote by T be the linear operator of $\mathbb{C}^n$ given by $T(\zeta) = (T_1(\zeta), ..., T_n(\zeta))$, where $T_j(\zeta) = T_j(\zeta_1, ..., \zeta_n) = k_j.\,\zeta_j - \zeta_{\ell(j)}$. If we set $a := 1 - tr(S) \neq 0$, $\lambda = (\lambda_1, ..., \lambda_n)$ and $A = (a, ..., a)$ then system (13) can be written as

$$T_j(\lambda) = a \ , \ \forall\ 1 \leq j \leq r \iff T(\lambda) = A \ . \tag{17}$$

We assert that T is invertible.

In fact, in the proof of claim 3.2 we have seen that the homogeneous system (15), which is equivalent to $T(\zeta) = 0$, has as unique solution $\zeta = 0$ if ℓ satisfies the following property:

$(**)$ For any periodic point $m \in I_n$ of ℓ there exists $s \geq 0$ such that $k_{\ell^s(m)} > 1$.

Since the system (15) is equivalent to $T(\zeta) = 0$, if $(**)$ is true then T is invertible.

On the other hand, if $(**)$ were not true then ℓ would have a periodic orbit $O(m) = \{m, \ell(m), ..., \ell^{(r-1)}(m), \ell^r(m) = m\}$ such that $k_{\ell^s(m)} = 1$, $\forall\ 0 \leq s \leq r - 1$. Since the vector λ satisfies (17) we obtain

$$\lambda_{\ell^{(s-1)}(m)} - \lambda_{\ell^s(m)} = a \ , \ 1 \leq s \leq r \ .$$

This implies $r.\,a = \sum_{s=1}^r \left(\lambda_{\ell^{(s-1)}(m)} - \lambda_{\ell^s(m)}\right) = 0$, which contradicts $a \neq 0$. Therefore $(**)$ is true and T is invertible.

Now, from (17) we get

$$(\lambda_1, ..., \lambda_n) = \lambda = T^{-1}(A) = a.\,T^{-1}(1, ..., 1) \ .$$

Therefore, if set $\rho := (\rho_1, ..., \rho_n) = T^{-1}(1, ..., 1)$ then $\lambda_j = a.\,\rho_j$, $1 \le j \le n$. Note that $\rho \in \mathbb{Q}^n$, because the entries of T are integer numbers. We assert that $\rho_1, ..., \rho_n > 0$.

In fact, $T(\rho) = (1, ..., 1)$ is equivalent to

$$\rho_j = \frac{1}{k_j}\left(1 + \rho_{\ell(j)}\right) \ .$$

An induction argument using the above relation implies the following:

$(***)$ If $m \in I_n$ is such that there exist $s \ge 0$ with $\rho_{\ell^s(m)} \in \mathbb{Q}_+$ then $\rho_m \in \mathbb{Q}_+$.

Since any orbit contains a periodic point it is sufficient to prove that if m is periodic then $\rho_m \in \mathbb{Q}_+$.

Suppose by contradiction that this is not true. In this case, there exists $m \in I_n$ with periodic orbit $O(m) = \{m, \ell(m), ..., \ell^{(r-1)}(m), \ell^r(m)\}$ with $\lambda_{\ell^s(m)} \le 0$, $\forall$ $0 \le s \le r-1$. Since

$$k_{\ell^s(m)}.\,\rho_{\ell^s(m)} - \rho_{\ell^{(s+1)}(m)} = 1 \ , \ \forall\ 0 \le s \le r-1$$

we get

$$0 < r = \sum_{s=0}^{r-1}\left(k_{\ell^s(m)}.\,\rho_{\ell^s(m)} - \rho_{\ell^{(s+1)}(m)}\right) = \sum_{s=0}^{r-1}\left(k_{\ell^s(m)} - 1\right)\,\rho_{\ell^s(m)} \le 0 \ ,$$

because $\rho_{\ell^s(m)} \le 0$ and $k_{\ell^s(m)} - 1 \ge 0$ for all $s = 0, ..., r-1$. This contradiction implies that $(***)$ is true and that $\rho_j \in \mathbb{Q}_+$, $\forall\ 1 \le j \le n$.

Let us prove that $\lambda_j \in \mathbb{Q}_+$, $\forall\ 1 \le j \le n$. Set $\tau := \sum_{j=1}^n \rho_j \in \mathbb{Q}_+$. Since $\lambda_j = a.\,\rho_j = (1 - tr(S)).\,\rho_j$, $1 \le j \le n$, we get

$$tr(S) = \tau.\,(1 - tr(S)) \implies tr(S) = \frac{\tau}{1+\tau} \in \mathbb{Q}_+ \text{ and } 0 < tr(S) < 1 \ .$$

Therefore, $\lambda_j = (1 - tr(S))\,\rho_j \in \mathbb{Q}_+$, $\forall\ 1 \le j \le n$. This finishes the proof of lemma 3.1. □

Let us finish the proof of theorem 2. Observe that $\lambda_1, ..., \lambda_n \in \mathbb{Q}_+$ are in the Poincaré domain and we can assume that Φ converges. In particular, $\widehat{Y} = S + N$, $\widehat{\eta} = \Phi^*(\eta)$ and $d\widehat{\eta}$ are holomorphic. If we write $\Phi(w) = (\Phi_1(w), ..., \Phi_n(w)) = (z_1, ..., z_n)$ then $S = \sum_j \lambda_j\, w_j\, \partial_{w_j}$ is diagonal and semi-simple. Since $\lambda_j \in \mathbb{Q}_+$ and $[S, N] = 0$ then N is also a polynomial vector field. In fact, let us write the Taylor series of N as $\sum_{j\,\sigma} a_{j\,\sigma}\, w^\sigma\, \partial_{w_j}$, where $a_{j\,\sigma} \in \mathbb{C}$. Then the relation $[S, N] = 0$ implies that $(\langle\lambda, \sigma\rangle - \lambda_j)\ a_{j\,\sigma} = 0$. Therefore, if $a_{j\,\sigma} \ne 0$ then we get the ressonance

$$\langle\lambda, \sigma\rangle = \lambda_j \ , \ \forall\,\sigma = (\sigma_1, ..., \sigma_n) \ , \ 1 \le j \le n \ . \tag{18}$$

Since $\lambda_j \in \mathbb{Q}_+$, $\forall j$, the set $\{(j, \sigma) \mid \langle\lambda, \sigma\rangle - \lambda_j = 0\}$ is finite, and so N is a polynomial vector field.

Moreover, if we set $\widehat{\nu} = dw_1 \wedge ... \wedge dw_n$ and $d\widehat{\eta} := i_{\widehat{X}}\,\widehat{\nu}$ then we get $\widehat{\eta} = i_{\widehat{Y}}\, d\widehat{\eta} = i_{\widehat{Y}}\, i_{\widehat{X}}\,\widehat{\nu} = i_S\, i_{\widehat{X}}\,\widehat{\nu}$. On the other hand, from $L_S d\widehat{\eta} = d\widehat{\eta}$ we obtain

$$i_{\widehat{X}}\widehat{\nu} = L_S\, i_{\widehat{X}}\,\widehat{\nu} = i_{[S,\widehat{X}]}\,\widehat{\nu} + i_{\widehat{X}}\, L_S\widehat{\nu} = i_{[S,\widehat{X}]}\,\widehat{\nu} + tr(S)\, i_{\widehat{X}}\,\widehat{\nu} \implies$$

$$[S, \widehat{X}] = (1 - tr(S))\,\widehat{X} \ .$$

This implies that $\widehat{X}$ is also a polynomial vector field. In fact, if $\widehat{X}$ contains non-vanishing monomial of the form $a.\,w^\sigma\,\partial_{w_j}$ then

$$\langle\sigma, \lambda\rangle = 1 - tr(S) > 0 \ .$$

Since $\lambda_1, ..., \lambda_n \in \mathbb{Q}_+$ the set $\{(\sigma, \mu) \mid \langle \sigma, \lambda \rangle = 1 - tr(S)\}$ is finite and so $\widehat{X}$ is a polynomial vector field. Let us prove that $[N, \widehat{X}] = 0$.

Claim 3.3. *After a polynomial change of variables (preserving the form of S) we can assume that $N = \sum_{j=1}^{n} N_j(z)\,\partial_{z_j}$, where $N_1 \equiv 0$ and $N_j = N_j\,(z_1, ..., z_{j-1})$, $\forall j \geq 2$. In other words $\frac{\partial N_j}{\partial z_i} = 0$ if $i \geq j$. In particular, $[N, \widehat{X}] = 0$.*

Proof. First of all, after a permutation of the variables we can assume that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Let $L := DN(0)$ be the linear part of N at $0 \in \mathbb{C}^n$. The relation $[S, N] = 0$ implies that $[S, L] = 0$, because S is linear. Note that L is nilpotent. Therefore, by Jordan's theorem after a linear change of variables that preserves S we can suppose that $L = \sum_{j=2}^{n} \alpha_j\, z_{j-1}\,\partial_{z_j}$, where $\alpha_j \in \{0, 1\}$, $2 \leq j \leq n$. Note that, if $\alpha_j = 1$ then N contains the monomial $z_{j-1}\,\partial_{z_j}$ and by (18) we must have $\lambda_{j-1} = \lambda_j$. On the other hand, if $\lambda_{j-1} < \lambda_j$ for some $j \in \{2, .., n\}$ then for all $i \in \{1, ..., j-1\}$ the component $N_i(z)$ does not depends on $(z_j, ..., z_n)$.

In fact, if $1 \leq i \leq j-1$ and N_i contains a non-vanishing monomial $a.\, z^\sigma$, $\sigma = (\sigma_1, ..., \sigma_n)$, then (18) implies

$$\langle \lambda, \sigma \rangle = \lambda_i \leq \lambda_{j-1} < \lambda_j \leq ... \leq \lambda_n \implies \sigma_r = 0\ ,\ \forall\, r > j-1\ .$$

This proves the first part of the claim. Let us prove that $[N, \widehat{X}] = 0$. From $L_N\, d\widehat{\eta} = 0$ we get

$$0 = L_N\,(i_{\widehat{X}}\,\nu) = i_{[N,\widehat{X}]}\,\nu + i_{\widehat{X}}\,(L_N\,\nu) = i_{[N,\widehat{X}]}\,\nu + \left(\sum_{j=1}^{n} \frac{\partial N_j}{\partial_{z_j}}\right).\, i_{\widehat{X}}\,\nu = i_{[N,\widehat{X}]}\,\nu\ ,$$

because $\frac{\partial N_j}{\partial z_j} = 0$, $1 \leq j \leq n$, by the first part. Therefore, $[N, \widehat{X}] = 0$. □

Now, since $\lambda_1, ..., \lambda_n \in \mathbb{Q}_+$, there exists $k_1 \leq ... \leq k_n \leq r \in \mathbb{N}$ such that $\lambda_j = k_j/r$, $1 \leq j \leq n$, $gcd(k_1, ..., k_n) = 1$ and $\sum_{j=1}^{n} k_j < r$. If we set $S_1 = r.\, S$ then we get $[S_1, N] = 0$ and $[S_1, \widehat{X}] = k\,\widehat{X}$, where $k = r - \sum_j k_j \in \mathbb{N}$. This finishes the proof of theorem 2. □

3.2. **The non-resonance condition.** It remains to specify the non-ressonance condition on the vector field X that implies $ker(L_X^0) = \{0\}$, where $L_X^0 \colon \Sigma_0(S) \to \Sigma_k(S)$.

Let us recall first that the space of orbits of the vector field $S = \sum_{j=1}^{n} k_j\, x_j\, \partial_{x_j}$, $k_1, ..., k_n \in \mathbb{N}$, is an analytic space of dimension $n-1$ known as the weighted projective space with weights $w = (k_1, ..., k_n)$. It will be denoted by $\mathbb{P}_w^{n-1}$. For instance, when $w = (1, ..., 1)$ then $\mathbb{P}_w^{n-1} = \mathbb{P}^{n-1}$, the usual projective space. Let us denote by $\Pi_w \colon \mathbb{C}^n \setminus \{0\} \to \mathbb{P}_w^{n-1}$ the natural projection.

Since $[S, X] = k.\, X$, $k \in \mathbb{N}$, the $(n-2)$-form $\mu = i_S\, i_X\, \nu$ is integrable and induces a two dimensional foliation $\mathcal{F}_\mu$ on $\mathbb{C}^n$. The orbits of S are contained in the leaves of $\mathcal{F}_\mu$, and so there exists a one dimensional foliation on $\mathbb{P}_w^{n-1}$, denoted by $\mathcal{G}_\mu$, such that $\mathcal{F}_\mu = \Pi_w^*(\mathcal{G}_\mu)$. In this way, the orbits of S that are X-invariant can be considered as singularities of $\mathcal{G}_\mu$. These orbits are the analytic separatrices of X through $0 \in \mathbb{C}^n$ and are contained in the singular set of $\mathcal{F}_\mu$. The non-resonance condition will be on one of these orbits.

Let γ be one of these orbits. A straightforward computation gives $d\mu = \ell.\, i_X\, \nu$, where $\ell = k + tr(S)$, and since $0 \in \mathbb{C}^n$ is an isolated singularity of X the curve γ is contained in the Kupka set of $\mathcal{F}_\mu$ and so the normal type of $\mathcal{F}_\mu$ at γ is well defined (see definition 1). Let us denote this normal type by Y_γ. To fix the ideas we will assume that Y_γ is a germ with a singularity at $0 \in \mathbb{C}^{n-1}$.

($\star$) *Non-resonance condition.* There exists a singular orbit γ of $\mathcal{F}_\mu$ such that the linear part $DY_\gamma(0)$ has eigenvalues $\mu_1, ..., \mu_{n-1}$ that satisfy the non-resonance conditions below:

$$\forall\ 1 \le \ell \le n-1\ ,\ \forall\ \sigma \in \mathbb{Z}_{\ge 0}^{n-1}\ ,\ \text{if} \sum_{j=1}^{n-1} \sigma_j . \mu_j = \mu_\ell \text{ then } \sigma_j = 0 \text{ if } j \neq \ell \text{ and } \sigma_\ell = 1\ .$$

Remark 3.2. Let $T = \sum_{j=1}^{n-1} \mu_j\, y_j\, \partial_{y_j}$. We would like to remark that condition ($\star$) implies that:

(a). If Z is a formal vector field in $\widehat{\mathcal{X}}_{n-1}$ such that $[T, Z] = 0$ then Z must be linear and diagonal in the coordinate system y, $Z = \sum_j \alpha_j\, y_j\, \partial_{y_j}$.

(b). $\mu_1, ..., \mu_{n-1}$ satisfy Poincaré's non-resonance conditions. This fact together with (a) implies that the germ of Y_γ is formally equivalent to T.

(c). The derivation $T\colon \widehat{\mathcal{O}}_{n-1} \to \widehat{\mathcal{O}}_{n-1}$ satisfies the following properties:

(c.1). $ker(T) = \mathbb{C}$, that is, if $T(f) = 0$ then f is a constant.

(c.2). The equation $T(\phi) = \psi$, where $\psi(0) = 0$ has an unique solution ϕ with $\phi(0) = 0$.

The proof of these facts is straightforward and is left to the reader.

Example 4. When $S = \sum_j x_j\, \partial_{x_j}$, the radial vector field, then the generalized Jouanolou's example of degree $\ell = k + 1 \ge 2$

$$X = J_\ell(x_1, ..., x_n) = x_n^\ell\, \partial_{x_1} + x_1^\ell\, \partial_{x_2} + ... + x_{n-1}^\ell\, \partial_{x_n}\ .$$

satisfies the non-resonance condition ($\star$).

In fact, note that:

(a). $[S, X] = k. X$. If $\mu = i_S\, i_X\, \nu$, then $d\mu = i_Z\, \nu$, where $Z = (k + n)\, X$.

(b). The orbit $\gamma(t) = (e^t, ..., e^t)$ of S is contained in Kupka set of $\mathcal{F}_\mu$.

The normal type Y_γ of $\mathcal{F}_\mu$ at γ can be computed by taking a normal section Σ to γ at some point, say the point $p = (1, ..., 1)$ and by considering the restriction $\mathcal{F}_\mu|_\Sigma$. We can take for instance $\Sigma = (x_n = 1)$. The restriction $\mathcal{F}_\mu|_\Sigma$ can be computed by projecting X onto the tangent space $T\Sigma$ along S. If $z = (z_1, ..., z_{n-1})$ and $x = (z, 1) \in \Sigma$ then the projection Y_γ at z is given by

$$Y_\gamma(z) = \left(z_n . J_\ell(z) - z_{n-1}^\ell . R(z)\right)|_{(z_n=1)} =$$

$$= \left(1 - z_1 . z_{n-1}^\ell\right) \partial_{z_1} + \sum_{j=2}^{n-2} \left(z_{j-1}^\ell - z_j . z_{n-1}^\ell\right) \partial_{z_j} + \left(z_{n-2}^\ell - z_{n-1}^{\ell+1}\right) \partial_{z_{n-1}}\ .$$

The point $\gamma \cap \Sigma = p = (1, ..., 1)$ is a singularity of Y_γ satisfying codition ($\star$). As the reader can check, the Jacobian matrix of $DY_\gamma(p)$ is of the form $-I + \ell. A$, where A satisfies $A^{n-1} + A^{n-2} + ... + A + I = 0$, I the identity matrix. In particular, the eigenvalues of $DY_\gamma(p)$ are of the form $\mu_1, ..., \mu_{n-1}$, where $\mu_r = -1 + \ell. \delta^r$, $1 \le r \le n-1$ and δ is a primitive n^{th}-root of unity (see also [LN-So]). The proof that $\mu_1, ..., \mu_{n-1}$ satisfy condition ($\star$) is not hard and is left to the reader.

Lemma 3.2. *If X satisfies condition ($\star$) then $ker(L_X^0) = \{0\}$.*

Proof. Let $X = \sum_{j=1}^{n} X_j(z)\,\partial_{z_j}$. We will assume, without lost of generality, that the common orbit γ of X and S that satisfies condition $(\star)$ is contained in $(z_n \neq 0)$ and passes through the point $p = (a,1) = (a_1,...,a_{n-1},1)$. Like in example 4, we compute the normal type Y_γ by projecting the vector field X onto the hyperplane $\Sigma = (z_n = 1)$ through the vector field S. Seting $z = (x,1) = (x_1,...,x_{n-1},1)$ we get:

$$Y_\gamma(x) = \frac{1}{k_n}(S(z_n).\,X - X(z_n).\,S)\Big|_{z=(x,1)} = X - \frac{X_n}{k_n}.\,S\Big|_{z=(x,1)} \tag{19}$$

By assumption, $Y_\gamma(a) = 0$ and $DY_\gamma(a)$ has eigenvalues $\mu_1,...,\mu_{n-1}$ satisfying condition $(\star)$.

In the proof we will use a weighted blow-up at $0 \in \mathbb{C}^n$ with weights $(k_1,...,k_n)$. After ramifications along the hyperplanes $(z_j = 0)$ if necessary, we can write the affine chart of the weighted blow-up associated to the n^{th} coordinate as

$$\Pi(\tau,x) = \Pi(\tau,x_1,...,x_{n-1}) = (\tau^{k_1}.\,x_1,...,\tau^{k_{n-1}}.\,x_{n-1},\tau^{k_n}) = (z_1,...,z_n)\ .$$

Let us prove that $\Pi^*(S) = \tau\,\partial_\tau$ and compute $\Pi^*(X)$. Since $z_n = \tau^{k_n}$ we have

$$S(z_n) = S(\tau^{k_n}) = k_n\,\tau^{k_n-1}\,S(\tau) = k_n\,z_n = k_n\,\tau^{k_n} \implies S(\tau) = \tau\ .$$

On the other hand, if $j < n$ then

$$S(x_j) = S(\tau^{-k_j}.\,z_j) = -k_j\,\tau^{-k_j-1}\,S(\tau)\,z_j + \tau^{-k_j}\,S(z_j) = 0 \implies \Pi^*(S) = \tau\,\partial_\tau\ .$$

Now, using that $[S,X] = k.\,X$ and $X = \sum_j X_j\,\partial_{z_j}$ we obtain

$$X_j \circ \Pi(\tau,x) = X_j(\tau^{k_1}.x_1,...,\tau^{k_{n-1}}.x_{n-1},\tau^{k_n}) = \tau^{k+k_j}.\,X_j(x,1)\ ,\ 1 \le j \le n\ ,$$

and by a straightforward computation

$$\Pi^*(X)(\tau,x) = \tau^k\,(f(x)\,\tau\,\partial_\tau + Y_\gamma(x))\ ,$$

where Y_γ is as in (19) and $f(x) = \frac{1}{k_n}X_n(x,1)$.

Remark 3.3. Set $Y_\gamma(x) = \sum_{j=1}^{n-1} Y_j(x)\,\partial_{x_j}$. From the relation $d(i_X\nu) = 0$, $\nu = dz_1 \wedge ... \wedge dz_n$, we get $d\left(i_{\Pi^*(X)}\,\Pi^*(\nu)\right) = 0$, which is equivalent to

$$\sum_{j=1}^{n-1}\frac{\partial Y_j}{\partial x_j} + (k + tr(S))\,f(x) = 0 \tag{20}$$

In particular, we obtain

$$f(a) = -\frac{\sum_j \mu_j}{k + tr(S)} \neq 0\ .$$

Let us prove that $ker(L_X^0) = \{0\}$. Let $N = \sum_j N_j\,\partial_{z_j} \in \Sigma(S,0)$ be such that $L_X^0(N) = [X,N] = 0$. This relation and $[S,N] = 0$ imply that the orbit γ of X and X is also N-invariant (in fact, $\gamma \subset sing(N)$ because N is nilpotent). Let us compute $\Pi^*(N)$.

Since $[S,N] = 0$, by a similar computation as in the case of X we get $N_j \circ \Pi(\tau,x) = \tau^{k_j}.\,N_j(x,1)$, $1 \le j \le n$, which implies

$$\Pi^*(N)(\tau,x) = g(x)\,\tau\,\partial_\tau + Z(x)\ ,$$

where $g(x) = \frac{1}{k_n} N_n(x,1)$ and $Z(x) = N - \frac{N_n}{k_n} S\Big|_{z=(x,1)}$. Note that the points a and $(0,a)$ are singularities of Z and $\Pi^*(N)$, respectively. Moreover, $g(a) = 0$ by remark 3.3. After a translation we can suppose that $a = 0 \in \mathbb{C}^{n-1}$.

Claim 3.4. *There exists* $\widehat{\Phi} \in \widehat{Diff}(\mathbb{C}^n, 0)$ *of the form* $\widehat{\Phi}(\tau, x) = (\phi(x).\,\tau, \Psi(x)) = (s, y)$, *with* $\phi \in \widehat{\mathcal{O}}^*_{n-1}$ *and* $\Psi \in \widehat{Diff}(\mathbb{C}^{n-1}, 0)$, *such that*

$$\widehat{\Phi}_*(\Pi^*(X)) = u(y).\, s^k.\left(\alpha\, s\, \partial_s + \sum_{j=1}^{n-1} \mu_j\, y_j\, \partial_{y_j}\right) , \tag{21}$$

where $\alpha = -\frac{\sum_j \mu_j}{k + tr(S)}$, $u \in \widehat{\mathcal{O}}_{n-1}$ *and* $u(0) \neq 0$.

Let us assume claim 3.4 and finish the proof of lemma 3.2. Set $T := \sum_{j=1}^{n-1} \mu_j\, y_j\, \partial_{y_j}$ and $L := \alpha\, s\, \partial_s + T$, so that $\widehat{\Phi}_*(\Pi^*(X)) = u(y).\, s^k.\, L$. Note that $\widehat{\Phi}^*(\Pi^*(N))$ is of the form

$$\widehat{\Phi}_*(\Pi^*(N)) = \tilde{g}(y)\, s\, \partial_s + \tilde{Z}(y) := \tilde{N} ,$$

where $\tilde{g}$ and $\tilde{Z}$ are formal series. From $[N, X] = 0$ we get

$$[\widehat{\Phi}^*(\Pi^*(N)), \widehat{\Phi}^*(\Pi^*(X))] = [\tilde{N}, u.\, s^k.\, L] = \tilde{N}\left(u.\, s^k\right) L + u.\, s^k\, [\tilde{N}, L] = 0 \implies$$

$$[L, \tilde{N}] = \frac{\tilde{N}\left(u(y).\, s^k\right)}{u(y).\, s^k}\, L = \phi(y).\, L ,$$

where $\phi(y) = k\, \tilde{g}(y) + \frac{\tilde{Z}(u(y))}{u(y)} \in \widehat{\mathcal{O}}_{n-1}$. Note that $\phi(0) = 0$. Therefore,

$$\phi(y)\, (\alpha\, s\, \partial_s + T) = [L, \tilde{N}] = [\alpha\, s\, \partial_s + T, \tilde{g}(y)\, s\, \partial_s + \tilde{Z}] = T(\tilde{g}(y))\, s\, \partial_s + [T, \tilde{Z}] ,$$

because $[s\, \partial_s, \tilde{g}(y)\, s\, \partial_s] = [s\, \partial_s, \tilde{Z}] = [T, s\, \partial_s] = 0$. This implies

$$\begin{matrix} T(\tilde{g}(y)) = \alpha\, \phi(y) \\ [T, \tilde{Z}] = \phi(y)\, T \end{matrix} .$$

The first relation above implies that $[T, \alpha^{-1}\, \tilde{g}(y)\, T] = \phi(y)\, T$, which together the second relation gives

$$[T, \tilde{Z} - \alpha^{-1}\, \tilde{g}(y)\, T] = 0 .$$

It follows from remark 3.2 that $\tilde{Z} - \alpha^1\, \tilde{g}(y)\, T$ must be linear and diagonal. However, since $D\tilde{Z}(0)$ is nilpotent and $\tilde{g}(0) = 0$ this implies that $\tilde{Z} = \alpha^{-1}\, \tilde{g}(y)\, T \implies$

$$\tilde{N} = \tilde{g}(y)\, s\, \partial_s + \tilde{Z} = \alpha^{-1}\, \tilde{g}(y)\, L \implies \tilde{N} \wedge \widehat{\Phi}_*(\Pi^*(X)) = 0 \implies N \wedge X = 0 \implies$$

$N = h\, X$, where h is holomorphic because X has an isolated singularity at $0 \in \mathbb{C}^n$. However, since $[S, N] = 0$ this implies

$$0 = [S, h\, X] = S(h).\, X + h.\, k.\, X \implies S(h) = -k.\, h \implies h = 0 ,$$

as the reader can check. Hence, $N = 0$ as we wished to prove. □

Proof of claim 3.4. Let $W = \tau^{-k}.\, \Pi^*(X) = f(x)\, \tau\, \partial_\tau + Y_\gamma(x)$. First of all, from remark 3.2 the germ Y_γ is formally linearizable. Therefore, there exists $\Psi \in \widehat{Diff}(\mathbb{C}^{n-1}, 0)$ such that $\Psi_*(Y_\gamma) = \sum_j \mu_j\, y_j\, \partial_{y_j} = T$. In particular, the formal diffeomorphism $\Phi(\tau, x) = (\tau, \Psi(x)) = (\tau, y)$ is such that

$$\Phi_*(W) = \tilde{f}(y)\, \tau\, \partial_\tau + T := \tilde{W} , \quad \tilde{f}(y) = f \circ \phi^{-1}(y) .$$

Note that $\tilde{f}(0) = f(0) = \alpha$. Therefore, by remark 3.2 the equation $T(h) = \alpha - \tilde{f}$ has an unique solution $h \in \widehat{\mathcal{O}}_{n-1}$ such that $h(0) = 0$. Now, set

$$\Phi_1(\tau, y) = (e^{h(y)}.\,\tau, y) = (s, y)\ .$$

We have

$$\tilde{W}(s) = \tilde{W}(e^{h(y)}.\,\tau) = \tilde{W}(e^{h(y)}).\,\tau + e^{h(y)}.\,\tilde{W}(\tau) = T(e^{h(y)}).\,\tau + e^{h(y)}.\,\tilde{f}(y).\,\tau = \alpha.\,s$$

which implies that $\Phi_{1*}(\tilde{W}) = \alpha\, s\, \partial_s + T$ and that

$$(\Phi_1 \circ \Phi)_* \Pi^*(X) = u(y).\, s^k\, (\alpha\, s\, \partial_s + T)\ ,$$

where $u(y) = e^{-k\, h(y)}$. This finishes the proof of claim 3.4 and of lemma 3.2. □

4. Proof of theorem 3

Let $(\eta_t)_{t \in U}$ be a holomorphic family of $(n-2)$-forms on the polydisc $Q \subset \mathbb{C}^n$ as in the hypothesis of theorem 3, $0 \in U \subset \mathbb{C}^k$. Consider the holomorphic family of vector fields $(X_t)_{t \in U}$ given by $d\eta_t = i_{X_t}\nu$, $\nu = dz_1 \wedge ... \wedge dz_n$. We have assumed that $0 \in Q$ is a g.K. singularity of η, so that 0 is an isolated singularity of X_0.

When Y is a holomorphic vector field on an open set of $W \subset \mathbb{C}^n$ and $q \in W$ then the *multiplicity* of Y at q is defined as

$$\mu(Y, q) := dim_{\mathbb{C}} \frac{\mathcal{O}_q}{\mathcal{I}(Y)}\ ,$$

where $\mathcal{I}(Y)$ is the ideal of $\mathcal{O}_q$ generated by the components of Y. Some known facts about the multiplicity are the following:

(i). $\mu(Y, q) < +\infty \iff q$ is an isolated singularity of Y.
(ii). $\mu(Y, q) = 0 \iff Y(q) \neq 0$.
(iii). $\mu(Y, q) = 1 \iff det(DY(q)) \neq 0$, that is the singularity is non-degenerate.

The following result is known for a holomorphic family of vector fields as $(X_t)_{t \in U}$:

Theorem 4.1. *Fix a polydisk $P \subset \overline{P} \subset Q$ such that 0 is the unique singularity of X_0 on $\overline{P}$. Then there exists a polydisk in the parameter space $0 \in V \subset U$ such that for all $t \in V$ then X_t has a finite number of singularities on P and no singularities on the boundary ∂P. Moreover,*

$$\sum_{q \in P} \mu(X_t, q) = \mu(X_0, 0)\ ,\ \forall\, t \in V\ .$$

Let us consider first the case in which η_0 has a non-degenerate singularity at $0 \in Q$. In this case $\mu(X_0, 0) = 1$ by theorem 4.1. Let $P \subset Q$ and V be as in theorem 4.1. Since $\mu(X_0, 0) = 1$ then by theorem 4.1, for every $t \in V$ we have $\sum_{p \in P} \mu(X_t, p) = 1$. Hence, X_t has an unique singularity in P for all $t \in V$. If we call $\mathcal{P}(t)$ this singularity, then the map $t \in V \mapsto \mathcal{P}(t) \in P$ is holomorphic (by the implicit function theorem applyed to the map $(z, t) \mapsto X_t(z)$). If 0 is a s.s.g.K. singularity then the eigenvalues $\lambda_1, ..., \lambda_n$ of $DX_0(0)$ are two by two different, $\lambda_i \neq \lambda_j$ for all $i \neq j$. Hence, by taking a smaller V if necessary, we can assume that the same is true for the eigenvalues of $DX_t(\mathcal{P}(t))$ for all $t \in V$. This proves item (a) of theorem 3.

Let us suppose now that $0 \in \mathbb{C}^n$ is a n.g.K. singularity of η_0 of type $(m_1, ..., m_n; \ell)$. In this case, $det(DX_0(0)) = 0$ because $DX_0(0)$ is nilpotent. Therefore, $\mu(X_0, 0) \geq 2$ by (ii) and (iii). Let P and V be as in theorem 4.1. Since the singularities of X_t on P are isolated, $\forall\, t \in V$, there exists a holomorphic vector field Y_t on P such that $\eta_t = i_{Y_t}\, d\eta_t$ (by proposition 1). Note that the family of vector fields $(Y_t)_{t\in V}$ can be taken holomorphic in the variable $t \in V$ (by the parametric De Rham's division theorem (cf. [DR])). Since Y_0 has a non-degenerate singularity at $0 \in \mathbb{C}^n$, by taking a smaller polydisk $P \subset Q$ and a smaller $V \subset U$ if necessary, then there exists a holomorphic map $\mathcal{P}\colon V \to P$ such that $\mathcal{P}(0) = 0$, $\mathcal{P}(t)$ is a non-degenerate singularity of Y_t and is the unique singularity of Y_t on P, $\forall t \in V$. On the other hand, by theorem 4.1, X_t has a finite number of singularities on P and

$$\sum_{q\in sing(X_t|_P)} \mu(X_t, q) = \mu(X_0, 0) \geq 2 \ , \ \forall t \in V \ .$$

We assert that $sing(X_t|_P) = \{\mathcal{P}(t)\}$, $\forall t \in V$.

In fact, let us fix $t_o \in V$. Denote the local flow of Y_{t_o} by $(s, q) \mapsto \phi_s(q)$. By proposition 1 we have $L_{Y_{t_o}}(d\eta_{t_o}) = d\eta_{t_o}$. In terms of the local flow ϕ_s this means that

$$\frac{d}{ds}\, \phi_s^*(d\eta_{t_o})\bigg|_{s=0} = d\eta_{t_o} \quad \Longrightarrow \quad \phi_s^*(d\eta_{t_o}) = e^s.\, d\eta_{t_o} \ .$$

On the other hand, the second relation above implies that $sing(d\eta_{t_o}) = sing(X_{t_o})$ is invariant by the flow ϕ_s. Hence, if $q \in P$ and $Y_{t_o}(q) \neq 0$ then $X_{t_o}(q) \neq 0$, for otherwise $sing(X_{t_o}|_P)$ would contain a regular orbit of the flow ϕ_s and would not be finite. Since X_{t_o} has at least one singularity in P we must have $sing(X_{t_o}|_P) = sing(Y_{t_o}|_P) = \{\mathcal{P}(t_o)\}$, which proves the assertion. It remains to prove that $\mathcal{P}(t)$ is an n.g.K. singularity of $\mathcal{F}_t$ and has the same type as $\mathcal{P}(0) = 0$.

Let $L_t := DY_t(\mathcal{P}(t))$ and $A_t := DX_t(\mathcal{P}(t))$. Let us prove that A_t is nilpotent for all $t \in V$. We will use the following lemma of linear algebra:

Lemma 4.1. *Let A and L be linear vector fields of $\mathbb{C}^n$ such that $[L, A] = \mu.\, A$, where $\mu \neq 0$. Then A is nilpotent.*

Proof. The idea is to prove by induction on $m \in \mathbb{N}$ that $[L, A^m] = m.\, \mu.\, A^m$. If we admit this fact then we get $tr(A^m) = 0$ because $tr(\,[L, A^m]\,) = 0$, $\forall m \in \mathbb{N}$. This implies that all eigenvalues of A vanish and that A is nilpotent. In fact, if the eigenvalues of A are $\mu_1, ..., \mu_n$ then

$$tr(A^m) = \sum_j \mu_j^m \ , \ \forall m \in \mathbb{N} \ \Longrightarrow \ \sum_j \mu_j^m = 0 \ , \ \forall\, m \in \mathbb{N} \ \Longrightarrow \ \mu_1 = ... = \mu_n = 0 \ .$$

Finally, let us assume by induction that $[L, A^{m-1}] = (m-1).\, \mu.\, A^{m-1}$, $m \geq 2$. Then

$$[L, A^m] = A^m.L - L.A^m = A.(A^{m-1}.L - L.A^{m-1}) + (A.L - L.A).A^{m-1} =$$
$$= A.[L, A^{m-1}] + [L, A].A^{m-1} = m.\, \mu.\, A^m \ ,$$

by the induction hypothesis. □

Let us finish the proof of theorem 3. We have seen in the proof of theorem 2 that $[Y_t, X_t] = (1 - \nabla Y_t)\, X_t$. By taking the linear part of both members we get $[L_t, A_t] = (1 - tr(L_t))\, A_t := \mu(t).\, A_t$. Since $\mu(0) \neq 0$ there exists $\epsilon > 0$ such that $\mu(t) \neq 0$ for $|t| < \epsilon$. Hence, A_t is nilpotent by lemma 4.1, if $|t| < \epsilon$. This can be expressed by $A_t^n = 0$ for all $|t| < \epsilon$. Since the function $t \in V \mapsto A_t^n$ is holomorphic we obtain that $A_t^n = 0$ and that A_t is nilpotent for all $t \in V$. Now, theorem 2 implies that $DY_t(\mathcal{P}(t))$ has positive rational eigenvalues. Hence, the eigenvalues of $DY_t(\mathcal{P}(t))$ do not depend on $t \in V$ and this implies that the type of the singularity is independent of $t \in V$. □

5. Proof of theorem 4

Let η, be an integrable $(n-2)$-form on $\mathbb{C}^n$ such that:

(I). $\eta = \sum_{j=0}^{d+1} \eta_j$, where η_k has coefficients homogeneous of degree k, $0 \le k \le d+1$.

(II). $\eta_{d+1} = i_R\, i_{X_d}\, \nu$, where
- R is the radial vector field on $\mathbb{C}^n$, $\nu = dx_1 \wedge ... \wedge dx_n$,
- X_d is a vector field, homogeneous of degree d, with an isolated singularity at $0 \in \mathbb{C}^n$ and $\nabla X_d = 0$.

We want to prove that there is a translation $\Phi(x) = x + a$ such that $\Phi^*(\eta) = \eta_{d+1}$. The proof will be based in the following lemma:

Lemma 5.1. *Let $\theta = \theta_0 + ... + \theta_\ell + \eta_{d+1}$ be an integrable $(n-2)$-form, where η_{d+1} is as before and the coefficients of θ_j are homogeneous polynomials of degree j, $0 \le j \le \ell$. We assert that:*

(a). *if $\ell < d$ then $\theta_\ell = 0$.*

(b). *if $\ell = d$ then $\theta_d = L_V\, \eta_{d+1}$, where V is a constant vector field on $\mathbb{C}^n$.*

Proof. In the proof we will use the following: if μ_s is a k-form with coefficients homogeneous of degree s then

$$L_R\, \mu_s = i_R\, d\mu_s + d\, i_R\, \mu_s = (k+s)\, \mu_s \ .$$

First of all note that the rotational of η_{d+1} is $(n+d-1)\, X_d$. In fact, we have seen in the proof of theorem 2 that

$$d\eta_{d+1} = d(i_R\, i_{X_d}\, \nu) = i_{Z_d}\, \nu \ ,$$

where

$$Z_d = [R, X_d] + \nabla R.\, X_d - \nabla X_d.\, R = (n+d-1)\, X_d \ ,$$

because $[R, X_d] = (d-1)\, X_d$, $\nabla R = n$ and $\nabla X_d = 0$. In particular, we can write the rotational Z of θ as

$$Z = Z_0 + ... + Z_{\ell-1} + Z_d \ , \text{ where } d\theta_{j+1} = i_{Z_j}\, \nu \ , \ 0 \le j \le \ell - 1 \ .$$

Note that the coefficients of Z_j are homogeneous polynomials of degree j, $0 \le j \le \ell - 1$. Taking the term with homogeneous coefficients of degree $d+\ell$ in the relation $i_Z\, \theta = 0$ (integrability condition), we obtain the relation

$$i_{Z_d}\, \theta_\ell + i_{Z_{\ell-1}}\, \eta_{d+1} = 0 \ .$$

Since

$$i_{Z_{\ell-1}}\, \eta_{d+1} = -i_{X_d}\, i_R\, i_{Z_{\ell-1}}\, \nu = -i_{X_d}\, i_R\, d\theta_\ell \text{ and } Z_d = (n+d-1)\, X_d$$

we get

$$i_{Z_d}\, \theta_\ell + i_{Z_{\ell-1}}\, \eta_{d+1} = i_{X_d}\, [(n+d-1)\, \theta_\ell - i_R\, d\theta_\ell] \implies$$
$$i_{X_d}\, [(n+d-1)\, \theta_\ell - i_R\, d\theta_\ell] = 0 \ .$$

Since X_d has an isolated singularity at $0 \in \mathbb{C}^n$ the above relation and the division theorem imply that $(n+d-1)\, \theta_\ell - i_R\, d\theta_\ell = i_{X_d}\, \zeta$, where by homogeneity of the coefficients we must have

- $\zeta = 0$, if $\ell < d$,
- ζ is a $(n-1)$-form with constant coefficients, if $\ell = d$.

If $\zeta = 0$ then

$$(n+d-1)\,\theta_\ell = i_R\, d\theta_\ell \implies i_R\,\theta_\ell = 0 \implies (n+d-1)\,\theta_\ell = i_R\, d\theta_\ell + d\, i_R\,\theta_\ell = L_R\,\theta_\ell\ .$$

Since θ_ℓ is a $(n-2)$-form with homogeneous coefficients of degree ℓ we must have

$$L_R\,\theta_\ell = (n+\ell-2)\,\theta_\ell \implies \theta_\ell = 0 \text{ if } \ell < d\ .$$

On the other hand, if $\ell = d$ and ζ is a constant form we can write $\zeta = i_U\,\nu$, where U is a constant vector field on $\mathbb{C}^n$. This implies

$$(n+d-1)\,\theta_d - i_R\, d\theta_d = i_{X_d}\,\zeta = -i_U\, i_{X_d}\,\nu = i_V\, d\eta_{\,d+1}\ ,$$

where $V = -\frac{1}{n+d-1}\,U$. From the above relation, we get

$$(n+d-1)\, i_R\,\theta_d = i_R\, i_V\, d\eta_{\,d+1} = -i_V\, i_R\, d\eta_{\,d+1}\ .$$

On the other hand,

$$i_R\, d\eta_{\,d+1} = L_R\,\eta_{\,d+1} - d\, i_R\,\eta_{d+1} = L_R\,\eta_{\,d+1} = (n+d-1)\,\eta_{\,d+1} \implies$$

$$(n+d-1)\, i_R\,\theta_d = -i_V\,\left[(n+d-1)\,\eta_{\,d+1}\right] \implies i_R\,\theta_d = -i_V\,\eta_{\,d+1} \implies$$

$$(n+d-1)\,\theta_d - i_R\, d\theta_d - d\, i_R\,\theta_d = i_V\, d\eta_{\,d+1} + d\, i_V\,\eta_{\,d+1} = L_V\,\eta_{\,d+1}\ .$$

Since $i_R\, d\theta_d + d\, i_R\,\theta_d = L_R\,\theta_d = (n+d-2)\,\theta_d$, from the above relation we obtain $\theta_d = L_V\,\eta_{\,d+1}$ as wished. □

Let us finish the proof of theorem 4. Consider the translation $T_a(x) = x + a$, where $a = (a_1, ..., a_n) \in \mathbb{C}^n$. If $\mu = \sum_I P_I(x)\, dx^I$ is a k-form, where $dx^I = dx_{i_1} \wedge ... \wedge dx_{i_k}$ and $P_I(x)$ is a polynomial, $I = (i_1 < ... < i_k)$, then we can write

$$T_a^*(\mu) = \sum_I P_I(x+a)\, dx^I = \mu + \mu_1(a) + O(|a|^2)$$

where $O(|a|^2)$ denotes a function of a such that $\lim\limits_{a\to 0} \frac{O(|a|^2)}{|a|} = 0$ and

$$\mu_1(a) = \sum_I DP_I(x).a\, dx_I = \sum_I \left(\sum_j a_j.\frac{\partial P_I}{\partial x_j}(x)\right)\, dx_I = L_A\,\mu\ ,$$

where A is the constant vector field $\sum_j a_j\,\partial_{x_j}$.

The above consideration implies that if $\eta = \eta_0 + ... + \eta_{\,d+1}$, a and A are as before, then

$$T_a^*(\eta) = \tilde{\eta}_0 + ... + \tilde{\eta}_d + \eta_{\,d+1}$$

where $\tilde{\eta}_j$ has coefficients homogeneous of degree j, $0 \le j \le d$, and

$$\tilde{\eta}_d = \eta_d + L_A\,\eta_{\,d+1}\ .$$

On the other hand, (b) of lemma 5.1 implies that $\eta_d = L_V\,\eta_{\,d+1}$, for some constant vector field $V = \sum_j v_j\,\partial_{x_j}$, $v_j \in \mathbb{C}$, $1 \le j \le n$. In particular, if $T(x) = x - v$, where $v = (v_1, ..., v_n)$ then the term of order d in $T^*(\eta)$ is

$$\tilde{\eta}_d = \eta_d - L_V\,\eta_{\,d+1} = 0\ .$$

Therefore, $T^*(\eta) = \tilde{\eta}_0 ... + \tilde{\eta}_{\,d-1} + \eta_{\,d+1}$ and an induction argument using (a) of lemma 5.1 implies that $T^*(\eta) = \eta_{\,d+1}$. This finishes the proof of theorem 4. □

References

[Br] M. Brunella : "Birational geometry of foliations"; text book for a course in the First Latin American Congress of Mathematics, IMPA (2000).

[C-LN] C. Camacho and A. Lins Neto: "The Topology of Integrable Differential Forms Near a Singularity"; Publ. Math. I.H.E.S., 55 (1982), 5-35.

[Ce-LN] D. Cerveau, A. Lins Neto: "Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{C}P(n)$, $n \geq 3$"; Ann. of Math. (1996) 577-612.

[C-C-F] Mauricio Corrêa Jr., Omegar Calvo-Andrade, Arturo Fernández-Pérez: "Highter codimension foliations and Kupka singularities"; arxiv:1408.7020

[Ce-Ma] D. Cerveau, J.-F. Mattei: "Formes intégrables holomorphes singulières"; Astérisque, vol.97 (1982).

[C-P] Cukierman, F.; Pereira, J. V.: "Stability of holomorphic foliations with split tangent sheaf"; Amer. J. Math. 130 (2008), no. 2, 413Ű439.

[DR] G. de Rham: "Sur la division des formes et des courants par une forme linéaire"; Comm. Math. Helvetici, 28 (1954), pp. 346-352.

[H] R. Hartshorne: "Algebraic Geometry"; Graduate Texts in Mathematics 52. Springer-Verlag, 1977.

[LN] A. Lins Neto : "Finite determinacy of germs of integrable 1-forms in dimension 3 (a special case)"; Geometric Dynamics, Lect. Notes in Math. n^o 1007 (1981), pp 480-497.

[LN-So] A. Lins Neto, M.G. Soares: "Algebraic solutions of one-dimensional foliations"; J. Diff. Geometry 43 (1996) pg. 652-673.

[M] J. Martinet: "Normalisations des champs de vecteurs holomorphes (d'après A.-D. Brjuno)"; Séminaire Bourbaki, vol. 1980/81, 55-70. Lect. Notes in Math. 901, S.V.

[Me] Medeiros, Airton S.: "Singular foliations and differential p-forms"; Ann. Fac. Sci. Toulouse Math. (6) 9 (2000), no. 3, 451Ű466.

A. Lins Neto
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110
Horto, Rio de Janeiro, Brasil
E-Mail: alcides@impa.br