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SUFFICIENT CONDITIONS FOR HYPERBOLICITY ON SKEW-PRODUCTS

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Abstract

The goal of this work is to study stable sets of skew-products with dimension 1 on the fibers. By studying the continuation of the periodic points, we prove that assuming absolute stable and infinitesimal stable in the one-parameter family of perturbations associated to the uniform translation is sufficient to imply hyperbolicity. Working with bounded solution we improve the previous result assuming Hölder variation. This means that a set is α-absolute stable by the uniform translation if the distance from the conjugation to the inclusion varies Hölder-continuous according to the distance of the original systems with its perturbation. We prove that if $\alpha > 1/2$, the skew-product is C^2 and preserves orientation on the fibers then the central direction is hyperbolic. After this we study the central topologically hyperbolic sets of Skew-Products. We see that Kupka-Smale condition and topological hyperbolicity property are not enough like it is for diffeomorphisms on surfaces (under the hypothesis of dominated splitting) or endomorphisms in dimension 1 (under the hypothesis of non critical points). Next we find an interesting family of skew-products that we will call the rigid case which has a natural way of perturbing it to obtain hyperbolicity. We finish this thesis by working on the continuation of hyperbolic periodic points proving a dichotomy for hyperbolic sets about the ambient manifold dimension.

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Contents

1 Introduction

One of the objectives in Dynamical Systems is to describe the space of diffeomorphisms of a manifold. One of its classical question is the description of the stable ones. In the study of such a question S. Smale introduced the concept of hyperbolicity in [Sm1]. J. Palis and S. Smale conjectured in [PaSm] that the C^r structural stable diffeomorphisms are the axiom A with strong transversality and the C^r $\Omega-$stable diffeomorphisms are the axiom A with no cycles. Both conjecture were proved in the C^1 topology, the first one the converse was proved by Robinson in [R2] and the direct by R. Mañé in [M3]; for the second one the direct was proved by J. Palis in [Pa] and the converse by S. Smale in [Sm2]. The question in the C^r topology remains open.

One way to approach the stability conjecture is by working with a stronger notion of stability. Given M a C^r Riemannian manifold and $F : M \to M$ a C^r stable diffeomorphism, there exist $\mathcal{U}(F)$ a neighborhood of F in the C^r-topology such that for every $G \in \mathcal{U}(F)$ we have an homeomorphism $\varphi : M \to M$ which verifies $\varphi \circ F = G \circ \varphi$. What can be done now is to ask for a regularity in the variation of φ according to the variation of G. It is said that F is absolute stable if there exist $C > 0$ such that $d(\varphi, Id) \leq Cd(F, G)$. From the analysis via implicit function done by J. Robbin in [R1] we can conclude that Axiom A plus strong transversality implies absolute stability. The converse was proved in the C^1 topology by J. Franks in [F] and J. Guckenheimer in [G]. In the C^r topology context the converse was proved by R. Mañé in [M1]. Since hyperbolicity implies a "C^1" regularity, the open question regarding this approach is what happens when the regularity is lower than Lipschitz, in this context nothing is know.

There is also a related concept to this approach of the stability conjecture which is infinitesimal stability. If $X(M)$ is the space of C^1 vector fields we can define the adjoint map of $F : M \to M$ as $F^* : X(M) \to X(M)$ by

$$F^*(Y)(x) = DF_{F^{-1}(x)}(Y(F^{-1}(x))).$$

We say that F is infinitesimally stable if the map $F^* - Id$ is surjective. From the analysis via implicit function done by J. Robbin in [R1] we can conclude that Axiom A plus strong transversality implies infinitesimal stability. Later R. Mañe in [M1] proved the converse. In Chapter 2 we discuss a point of view to understand the relationship between absolute and infinitesimal stability.

In this work we mainly study the stable Skew-Products systems with dimension 1 on the fibers. This space family is closely related to the partially hyperbolic systems. The techniques studied here work on the C^r context because we do not try to do local perturbations, but to understand our invariant sets as a whole and to find sufficient conditions that will imply hyperbolicity.

Given a C^r diffeomorphism $h : M \to M$, we define the space of Skew-Products related to h as the space of C^r maps $H : M \times \mathbb{R} \to M \times \mathbb{R}$ such that

$$H(x,t) = (h(x), f(x,t)) \text{ where } x \in M,\ t \in \mathbb{R},$$

and $f : M \times \mathbb{R} \to \mathbb{R}$ is a C^r function for which the map $t \mapsto f(x,t)$ is monotone for every $x \in M$.

We will use the notation of $\mathcal{SP}(h)$ or simply $\mathcal{SP}$ to describe this space.

It is a well known fact that the hyperbolic periodic points have a continuation if the system is perturbed. If the perturbation is obtained by a one-parameter curve which is differentiable then the curves that describe this continuation are also differentiable due to implicit function theorem. By studying the first derivative of such curves we proved some results related to absolute and infinitesimal stability. What we did here is to give new proofs of this notions in the more rigid context of skew-products but weakening the hypothesis by asking absolute and infinitesimal stability in the perturbation by the uniform translation.

Given $h : M \to M$, we are going to be interested to work with Skew-Products defined over a locally maximal hyperbolic set of h, if Λ is an hyperbolic set then $\mathcal{SPH}(h, \Lambda)$ or simply $\mathcal{SPH}$ is the space of Skew-Products defined over Λ. We note the Skew-Products that preserve the orientation of the fibers ($\{x\} \times \mathbb{R}$) by $\mathcal{SP}^+$ and $\mathcal{SPH}^+$.

Given $H \in \mathcal{SP}$, the uniform translation will be the one-parameter family $H_s = (h, f_s)$ where $f_s(x,t) = f(x,t) + s$. A compact invariant set $\Lambda_0 \subset M \times \mathbb{R}$ is absolutely stable by the uniform translation if there exist $\epsilon > 0$ and $C > 0$ such that for every $s \in (-\epsilon, \epsilon)$ there exist $\varphi_s : \Lambda_0 \to M$ which verifies $\varphi_s \circ H = H_s \circ \varphi_s$ and $d(\varphi_s, i) \leq C|s|$ where $i : \Lambda_0 \to M \times \mathbb{R}$ is the inclusion. We say that $\Lambda_0 \subset M \times \mathbb{R}$ is infinitesimally stable by the uniform translation if there exist $g : U_0 \to \mathbb{R}$ such that

$$\frac{\partial H}{\partial t}(z) g(z) - g(H(z)) = -1 \ \forall z \in U_0,$$

where U_0 is a neighborhood of Λ_0.

Studying of the continuation of the hyperbolic periodic points we proved:

Theorem 1: *If $H \in \mathcal{SPH}^+$ and Λ_0 is a locally maximal set absolutely stable by the uniform translation then Λ_0 is hyperbolic.*

Theorem 2: *If $H \in \mathcal{SPH}^+$ and Λ_0 is a locally maximal set infinitesimal stable by the uniform translation then Λ_0 is hyperbolic*

Once we assume stability, we could say that a perturbation is bad for the stability if the relationship between the distance of the conjugacy to the inclusion and the distance from the perturbation to the original system has a bad sense of regularity. An interesting conclusion from the previous results and the following theorem is that in the orientation preserving Skew-Products context, the worst perturbation for the stability is the uniform translation and if the regularity for this perturbation can be tamed then the system is hyperbolic.

The key point to prove the previous results is that the first derivative of the continuation of the periodic points are uniformly bounded. What we did next is to weak the hypothesis on the map ϕ_s asking it to vary just Hölder continuous on the parameter s instead of Lipschitz. In this situation the previous technique will not work because we do not have an uniform bound anymore. What we did to overcome this was to adapt the techniques developed in [Ti] for systems with Hölder-Shadowing property. The main idea here is that just Hölder continuity will let us estimate the action of the differential on the center-bundle and also the lack of speed will imply certain slow growth in the perturbations.

Given $\Lambda_0 \subset M \times \mathbb{R}$ a compact transitive invariant set we say that it is α-absolute stable by the uniform translation if there exist $C_0 > 0$ such that $d(\phi_s, i) < C_0 s^\alpha$. We say that Λ_0 is central hyperbolic if it either verifies:

$$\left\| DH^n_{|\{0\}\times T_t\mathbb{R}} \right\| \overset{n\to\infty}{\longrightarrow} 0 \ \forall (x,t) \in \Lambda_0,$$

or

$$\left\| DH^{-n}_{|\{0\}\times T_t\mathbb{R}} \right\| \overset{n\to\infty}{\longrightarrow} 0 \ \forall (x,t) \in \Lambda_0.$$

The theorem then is the following:

Theorem 3: *If $H \in \mathcal{SP}^+$ is C^2 and Λ_0 is a locally maximal set α-absolutely stable by the uniform translation with $\alpha > 1/2$ then Λ_0 is central hyperbolic. If H is just $C^{1+\gamma}$ with $\gamma \in (0,1)$ and $\alpha > 1/(1+\gamma)$ then Λ_0 is central hyperbolic.*

After this, we continued to work in the understanding of the stable Skew-Products by looking for geometric consequences from topological behaviors. This idea was worked first in the complex dynamics context where the first result alike is the Schwartz lemma which finally evolves in proving that if the Julia set is expansive with no critical point inside then it is hyperbolic. After that, in the real dynamics context Singer in [Si] proved that for maps in one dimensional manifolds with negative Schwartzian derivative if all the critical points belong to the basin of some hyperbolic attractor then the system is hyperbolic. This work was generalized by Mañé in [M2] proving that for 1-dimensional manifolds a C^2 Kupka-Smale endomorphism with an expansive invariant set with no critical points has to be hyperbolic. Years later Pujals and Sambarino proved in [PuSa1] that for C^2 Kupka-Smale diffeomorphisms of a surface, an invariant set that has dense periodic points and has dominated splitting must be hyperbolic (They proved more in general a Palis Conjecture). The scheme of the proof is first to see that the manifolds associated to the splitting are in fact stable and unstable manifolds in a topological sense, and using that they prove hyperbolicity.

In dimension 3 in [Pu] there is an example which is a skew-product, Kupka-Smale and topologically hyperbolic but not hyperbolic. This example is not in a generic context for partially hyperbolic systems and is also what we called rigid, yet we managed to perturb it to create one which verifies what we called strong Kupka-Smale. By no means the previous example has some kind of robustness, on the contrary it can be perturbed to become hyperbolic. The key part in this example is the existence of a minimal set which is not hyperbolic.

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically contracting on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| < \epsilon_2, \ \forall k \geq n,$$

where $I_{\epsilon_1}(z) = \{x\} \times [t - \epsilon_1, t + \epsilon_1]$ if $z = (x, t)$.

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically expanding on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| > \epsilon_2, \ \forall k \geq n.$$

We say that H is central topologically hyperbolic on Λ_0 if it is either topologically expanding or topologically contracting on Λ_0.

In our study of central topologically hyperbolic skew-products the first thing to see is the existence of an invariant graph:

Proposition 1.1: *(Invariant Graph) If $H \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic then there exist $b_0 : \Lambda \to \mathbb{R}$ a continuous function such that $\Lambda_0 = graph(b_0)$. In particular $H(x, b_0(x)) = (h(x), b_0(h(x)))$.*

Using this invariant graph, we got the following decomposition lemma:

Lemma 1.2: *(Decomposition) If $H = (h, f) \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic and $b_0 : \Lambda \to \mathbb{R}$ is a continuous function such that $\Lambda_0 = graph(b_0)$. Then there exist U a neighborhood of $\Lambda \times \{0\}$ in $M \times \mathbb{R}$ and $g_0 : U \to M \times \mathbb{R}$ which verifies*

$$f(x,t) = g_0(x, t - b_0(x)) + b_0(h(x)) \text{ and } g_0(x, 0) = 0 \ \forall x \in \Lambda.$$

Moreover if there exist $g_1 : U \to M \times \mathbb{R}$ and $b_1 : \Lambda \to \mathbb{R}$ such that $f(x,t) = g_1(x, t - b_1(x)) + b_1(h(x))$ and $g_1(x, 0) = 0 \ \forall x \in \Lambda$ then $b_0(x) = b_1(x)$ and $g_0(x,t) = g_1(x,t)$ for all $x \in \Lambda$.

At the moment we were working with this, our known examples of skew-products had the map b_0 always C^r. We called them later the rigid case. It is easy to see that if b_0 is C^r we have that g is C^r and therefore we can build and describe all the rigid cases. This family of sets have the property that the dynamics live in a hyper-surface reducing the dimension of the ambient manifold in 1. The problem here is that the central direction, the one which we are intrested in studying, is not tangent to the hyper-surface but transversal. Despite this we proved the following:

Theorem 4: *If $H = (h, f) \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic having b_0 the graph map as differentiable as H then it is approximated by central hyperbolic systems. If $H \in \mathcal{SPH}$ is stable then Λ_0 is hyperbolic.*

The problem with the rigid case is that is not generic in $\mathcal{SPH}$. This is due to the fact that if b_0 is C^r we can see that both the strong stable and strong unstable manifolds belong to the graph of b_0. This implies that for periodic points the strong stable manifold and the strong unstable manifold intersect which is far to be generic. We see that most of skew-products are what we called strong Kupka-Smale. We discuss this with more detail in Chapter 3.

What we did next is to work with the family of locally constant Skew-Products over hyperbolic sets $\mathcal{LCSP}$. This is the set of Skew-Products in $\mathcal{SPH}$ such that if $\Lambda \subset M$ is the hyperbolic set of h, for every $x \in \Lambda$ there exist a neighborhood $U(x)$ for which $f(x,t) = f(y,t)$ $\forall y \in U(x)$ $\forall t \in \mathbb{R}$.

Having in mind the idea of the ambient manifold in which the set Λ_0 lives, we proved the following dichotomy:

Theorem 5: *Given $H \in \mathcal{LCSP}$, and Λ_0 an homoclinic class, if Λ_0 is an hyperbolic set then one of the following two happen:*

- *$H_{|\Lambda_0}$ is normally hyperbolic. If $H_{|\Lambda_0}$ is contracting in the central direction then the tangent bundle of the sub-manifold is $E^{uu} \oplus E^c$ and if it is expanding the tangent bundle is $E^{ss} \oplus E^c$.*
- *H can be approximated on $\mathcal{LCSP}$ by skew-products such that the continuation of Λ_0 contains periodic points with strong connections.*

The theorem says that if we take an hyperbolic homoclinic class of H then in the first case we can reduce the dimension of the ambient manifold. If this do not happen we can perturb it to build strong connections between periodic points. Once we have this if we perturb again we obtain blenders inside Λ_0 due to [BD]. This are known to be dynamical objects with full topological dimension.

One part of the theorem comes from [BC], for the other part we studied the continuation of the periodic points. We proved that if there are two points which belongs to the same strong stable manifold for the contractive case we can perturb our system and create a connection between periodic points. What we would like to do is change the hypothesis of hyperbolicity by the hypothesis of stability. If we could do so, in the normally hyperbolic situation we could apply [PuSa1] for dimension 2 or [PuSa2] for higher dimension, obtaining hyperbolicity for the set and in the blender case create some heterodimensional cycle if we do not have hyperbolicity. This is all deeply related with the Palis conjecture which states in this context that a system can be approximated by either hyperbolic ones or ones that have heterodimensional cycles.

We finished this thesis with a discussion about the techniques worked and some conjectures we formulated about this topic. In Chapter 2 we prove theorems $1, 2$ and 3, in Chapter three we prove theorem 4 and in Chapter 4 we prove theorem 5.

Notations

Let M be a compact orientable riemannian manifold. Given a C^r diffeomorphism $h : M \to M$, we define the space of Skew-Products related to h as the space of C^r maps $H : M \times \mathbb{R} \to M \times \mathbb{R}$ such that

$$H(x,t) = (h(x), f(x,t)) \text{ where } x \in M,\ t \in \mathbb{R},$$

and $f : M \times \mathbb{R} \to \mathbb{R}$ is a C^r function for which the map $t \mapsto f(x,t)$ is monotone for every $x \in M$.

We will use the notation of $\mathcal{SP}(h)$ or simply $\mathcal{SP}$ to describe this space.

We are going to call M the base and for every $x \in M$ the set $\{x\} \times \mathbb{R}$ will be called a fiber.

On our study the dynamics on the base is going to be fix, so we set now the notation h for the diffeomorphism acting on M.

We also use x to represent a point in M, t to represent a point in $\mathbb{R}$ and z to represent a point in $M \times \mathbb{R}$. Let us also define the projections $\pi_M : M \times \mathbb{R} \to M$ and $\pi_c : M \times \mathbb{R} \to \mathbb{R}$.

On $\mathcal{SP}$ we are going to set the C^r topology, in particular the closeness of two skew-products $H_1 = (h, f_1)$ and $H_2 = (h, f_2)$ will be given by the closeness of the maps f_1 and f_2.

If Λ is an invariant set from h, we say that it is hyperbolic if there exists $C > 0, \lambda \in (0,1)$ and $\forall x \in \Lambda$ there exist E^s_x, E^u_x subspaces from T_xM such that $T_xM = E^s_x \oplus E^u_x$ and

$$\left\| Dh^n_{|E^s_x} \right\| \le C\lambda^n \quad \left\| Dh^{-n}_{|E^u_x} \right\| \le C\lambda^n \ \forall x \in \Lambda, \forall n \in \mathbb{N}.$$

We will call $\mathcal{SPH}(h,\Lambda)$ or simply $\mathcal{SPH}$ the space of Skew-Products defined over a hyperbolic set in the base.

If h is hyperbolic, given $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{y \in M : d(h^n(x), h^n(y)) \le \epsilon\},$$

and

$$W^u_\epsilon(x) = \left\{y \in M : d(h^{-n}(x), h^{-n}(y)) \le \epsilon\right\},$$

be the stable and unstable manifolds. It is a well known fact that those sets are C^r manifolds tangent to the spaces E^s_x and E^u_x respectively.

A compact set Λ is locally maximal if there exist $U \subset M$ such that $\Lambda \subset U$ and

$$\Lambda = \bigcap_{n \in \mathbb{Z}} h^n(U).$$

On most situations we are going to be interested in studying the Skew-Product defined over locally maximal set of h which may or may not be M. We therefore set the notation Λ as a locally maximal set of h. If we are working with an hyperbolic set, we require it to be transitive and since it is locally maximal then the periodic points are dense.

We also are going to be interested in studying locally maximal sets of H for which we reserve the notation Λ_0. In particular we want Λ and Λ_0 to be related so we ask from now on that $\pi_M(\Lambda_0) = \Lambda$.

If the map $t \mapsto f(x,t)$ is an increasing monotone map for every $x \in M$ or for every $x \in \Lambda$, then we say that H is an orientation preserving Skew-Product and we are going to note them as $\mathcal{SP}^+$ for the general case and $\mathcal{SPH}^+$ if we are working with an hyperbolic set on the base.

Given $(x,t) \in M \times \mathbb{R}$ and $\epsilon > 0$ we define $I_\epsilon(x,t) = \{(x, t+t_1) : |t_1| < \epsilon\}$.

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically contracting on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| < \epsilon_2, \ \forall k \geq n,$$

where $I_{\epsilon_1}(z) = \{x\} \times [t - \epsilon_1, t + \epsilon_1]$ if $z = (x,t)$.

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically expanding on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| > \epsilon_2, \ \forall k \geq n.$$

We say that H is central topologically hyperbolic on Λ_0 if it is either central topologically expanding or central topologically contracting on Λ_0.

Given $H \in \mathcal{SP}$ and $z \in M \times \mathbb{R}$ we will call the central bundle

$$E^c_z = \{(v,s) \in T_z M \times \mathbb{R} : v = 0\}.$$

Regarding the central bundle E^c, we can construct a continuous vector field e such that $e(z) \in E^c_z$ and $|e(z)| = 1$.

Given $H = (h,f) \in \mathcal{SP}$ we define the function $f' : M \times \mathbb{R} \to \mathbb{R}$ by $f'(x,t) = \frac{\partial f}{\partial t}(x,t)$.

Since h does not depend on $t \in \mathbb{R}$ we have that the differential of the map H acts on tangent bundle leaving the central bundle invariant. In particular due to our notation we have that $DH(e(z)) = f'(z)e(H(z))$. Therefore

$$\left\| DH^n_{|E^c_{(x,t)}} \right\| = \prod_{i=0}^{n-1} |f'(H^i(x,t))|.$$

Given $z \in \Lambda_0$ we define the forward Lyapunov exponents as:

$$\lambda^{+,+}(z) = limsup_n \frac{log(|\prod_{i=0}^{n-1} f'(H^i(z))|)}{n},$$

and

$$\lambda^{+,-}(z) = liminf_n \frac{log(|\prod_{i=0}^{n-1} f'(H^i(z))|)}{n}.$$

We say that H is central hyperbolic on Λ_0 a locally invariant set if there exists $C > 0$ and $\lambda \in (0,1)$ such that either:

$$\left\| DH^n_{|E^c_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_0,$$

or

$$\left\| DH^{-n}_{|E^c_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_0.$$

The last type of Skew-Products remaining to define are the locally constant. We will define them with more detail later, but for now we just say that for every $x \in \Lambda$ there exist a neighborhood $U(x)$ for which $f(x,t) = f(y,t) \; \forall y \in U(x) \; \forall t \in \mathbb{R}$. We call this family $\mathcal{LCSP}$.

The dynamics of a Skew-Product in $\mathcal{LCSP}$ would not be interesting if Λ was not a Cantor set and the dynamics in the fiber were not dominated by the dynamics on the base. The first property comes from a restriction on h. For the second property we require them to be partially hyperbolic.

Let F is a diffeomorphism of a manifold N and Λ_1 a compact invariant set, we say that Λ_1 is partially hyperbolic with the decomposition $T_{\Lambda_1}N = E^s \oplus E^c \oplus E^u$ if there exist $C > 0$ and $\lambda \in (0,1)$ such that :

$$\left\| DF^n_{|E^s_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_1,$$

$$\left\| DF^{-n}_{|E^u_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_1,$$

$$\left\| DF^n_{E^s_z} \right\| \left\| DF^{-n}_{E^c_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_1,$$

and

$$\left\| DF^{-n}_{E^u_x} \right\| \left\| DF^n_{E^c_z} \right\| \leq C\lambda^n \; \forall z \in \Lambda_1.$$

Like in the hyperbolic case, it is know that there exist $W^{cs}(z)$, $W^{cu}(z)$, $W^{ss}(z)$, $W^{uu}(z)$ and $W^c(z)$ C^r manifolds dynamically defined.

We are going to restrict our Skew-Products in $\mathcal{SPH}$ to those which are partially hyperbolic. In particular for those we say that $H \in \mathcal{SPH}$ is strong Kupka-Smale if it is Kupka-Smale and also:

$$W^{ss}(p) \cap W^{uu}(q) = \phi \ \forall p, q \in Per(H).$$

Let us fix now some notation in the context of $\mathcal{LCSP}$. Given $H \in \mathcal{LCSP}$ and a point $z = (x, t) \in \Lambda_0$ we set:

- $E^{ss}(z) = E^{s}(x) \times \{0\}$ and $E^{uu}(z) = E^{u}(x) \times \{0\}$ are invariant under DH, the action of the later its basically the action of Dh and we call this spaces the strong stable and strong unstable respectively.
- Since in this class of Skew-Products the periodic points are dense, we will be interested in working over the homoclinic class of a periodic point which is defined by:
$$\Lambda_0(p) = \overline{W^{s}(\sigma(p)) \pitchfork W^{u}(\sigma(p))},$$
where $\sigma(p)$ is the orbit of p.

If F is partially hyperbolic at Λ_1, we say that it is normally hyperbolic if there exist $S \subset N$ a sub-manifold which contains Λ_1, is locally invariant and its tangent bundle is $E^{s} \oplus E^{c}$ or $E^{u} \oplus E^{c}$.

We will define now the locally constant Skew-Products with more detail. For that we need the notion of Markov partition.

We will start taking h an hyperbolic map on Λ a locally maximal set. A Markov partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a finite covering of Λ such that

- If $x, y \in P_i \cap \Lambda$ then $W^{s}_{\epsilon}(x) \cap W^{u}_{\epsilon}(y)$ contains a unique point which belongs to P_i.
- $int(P_i) \cap int(P_j) = \phi$ if $i \neq j$.
- If $x \in int(P_i) \cap \Lambda$ and $h(x) \in P_j$ then $h(W^{s}_{\epsilon}(x) \cap P_i) \subset W^{s}_{\epsilon}(h(x)) \cap P_j$.
- If $x \in int(P_i) \cap \Lambda$ y $h(x) \in P_j$ then $W^{u}_{\epsilon}(h(x)) \cap P_j \subset h(W^{u}_{\epsilon}(x) \cap P_i)$.

From that definition, we have the next result:

Theorem 1.3: *(Bowen-Sinai) If Λ is an hyperbolic locally invariant set then, given $\beta > 0$ there exists a Markov partition $\mathcal{P}$ such that the rectangles of the partition have diameter smaller than β. Moreover there exists a semi-conjugacy between Λ and a sub-shift defined on $\mathcal{P}^{\mathbb{Z}}$.*

Set $\mathcal{P} = \{P_1, \ldots, P_k\}$ a Markov partition related to Λ and $\pi : \Lambda \to \{1, \ldots, k\}$ defined by $\pi(x) = j$ if $x \in P_j$. We say that a $H = (h, f) \in \mathcal{SP}$ is locally constant if for every $x, y \in \Lambda$ such that $\pi(x) = \pi(y)$ then $f(x, t) = f(y, t)$ $\forall t \in \mathbb{R}$. We will denote the set of locally constant skew-products by $\mathcal{LCSP}(h, \mathcal{P}) = \mathcal{LCSP}$. Pay attention to the fact that our spaces $\mathcal{LCSP}$ have fixed the diffeomorphism on the base h and also the Markov partition.

Let us fix now some notation in the context of $\mathcal{LCSP}$. Given $H \in \mathcal{LCSP}$ and a point $z = (x, t) \in \Lambda_0$ we denote:

- $W^s(x)$ as the stable set which contains the points $y \in M$ which verifies $d(h^n(x), h^n(y)) \to 0$.
- $W^s_{loc}(x)$ as the local stable set which contains the points $y \in W^s(x)$ such that $\pi(h^n(x)) = \pi(h^n(y))\ \forall n \geq 0$.
- $W^{ss}_{loc}(z)$ as the locally strong stable set which is $W^s_{loc}(x) \times \{t\}$.
- $W^{ss}(z)$ as the strong stable set which is the union of $H^{-n}(W^{ss}_{loc}(H^n(z)))\ \forall n \geq 0$.
- $W^u(x)$, $W^u_{loc}(x)$, $W^{uu}_{loc}(z)$ and $W^{uu}(z)$ defined in an analogous way.
- we say that two points z_0 and z_1 belong to the same cylinder if $\pi(\pi_M(z_0)) = \pi(\pi_M(z_1))$.
- $W^{su}(z)$ as the set of points z_1 such that belong to the same cylinder as z and $\pi_c(z_1) = \pi_c(z)$.
- Given two periodic points p_1 and p_2 we say that they have a strong connection if $W^{ss}(p_1) \cap W^{uu}(p_2) \neq \phi$. Observe that if $p_1 \in W^{su}(p_2)$ then p_1 and p_2 have a strong connection.
- Given $H \in \mathcal{LCSP}$, for every $P_i \in \mathcal{P}$ we define the map $f_i : \mathbb{R} \to \mathbb{R}$ by $f_i(t) = f(x, t)$ for a certain $x \in P_i$. These functions will be called central functions.

2 Stability by Translations

Given a stable set Λ of a diffeomorphism $F : M \to M$, we say that it is absolute stable if the variation of the conjugation according to the variation of the perturbation is Lipschitz. This means that there exist $C > 0$ and $\mathcal{U}(F)$ such that for every $G \in \mathcal{U}(F)$ there exist $\varphi_G : \Lambda \to M$ which conjugate the dynamics between Λ and $\varphi_G(\Lambda)$ and verifies:

$$d(i, \varphi_G) \leq Cd(F, G),$$

where $i : \Lambda \to M$ is the inclusion map.

Another way to work this is the following: suppose that Λ is a stable set of F and take a one-parameter family of perturbations F_s with $s \in (-\epsilon, \epsilon)$. We have for every $x \in M$ a curve $\varphi_x(s)$ which is the continuation of the point x. This define us the following map:

$$\varphi : \Lambda \to C((-\epsilon, \epsilon), M),$$

where $C((-\epsilon, \epsilon), M)$ represents the space of maps from $(-\epsilon, \epsilon)$ to M. To us stability means that the continuation is always close to the inclusion. This implies that actually we are working on $C^0((-\epsilon, \epsilon), M)$ and the map φ is continuous if we set the C^0 topology in $C^0((-\epsilon, \epsilon), M)$. The absolute stability condition means that we are working in $C^{Lip}((-\epsilon, \epsilon), M)$ and the map φ is continuous if we set the Lipschitz topology. Observe in particular that if Λ is in fact hyperbolic due to implicit function theorem for Banach spaces we are working with $C^1((-\epsilon, \epsilon), M)$ and φ is continuous if we set the C^1 topology.

Having in mind the stability conjecture, given a one-parameter family of perturbations the worse the regularity we can put in $C((-\epsilon, \epsilon), M)$ the worse is the perturbation to the stability. In our study we realized that for skew-products in $\mathcal{SP}^+$ the uniform translation is the worst perturbation in terms of stability. We prove that if we perturb by the uniform translation and we have some slightly stronger sense of stability than the classical, then we have hyperbolicity.

Another concept associated to work with a stronger sense of stability is the one of infinitesimal stability. If $X(M)$ is the space of C^1 vector fields we can define the adjoint map of F as $F^* : X(M) \to X(M)$ by

$$F^*(Y)(x) = DF_{F^{-1}(x)}(Y(F^{-1}(x))).$$

We say that F is infinitesimally stable if the map $F^* - Id$ is surjective. This is basically equivalent to work with $C^1((-\epsilon, \epsilon), M)$ and having that φ is continuous if we set the C^1 topology.

Let us fix now the main definitions in the context of skew-products for this chapter. Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that is stable if there exist U_0 neighborhood of Λ_0 and $\mathcal{U}(H)$ a neighborhood of H such that for every $H_1 \in \mathcal{U}(H)$ the maximal invariant set of H_1 in U_0 is conjugated to Λ_0.

Given $H = (h, f) \in \mathcal{SP}$ and Λ_0 a locally maximal invariant set of H, we define the uniform translation family by $H_s = (h, f_s) \in \mathcal{SP}$ where $f_s(z) = f(z) + s$ for $s \in (-\epsilon, \epsilon) \subset \mathbb{R}$. We say that Λ_0 is stable by the uniform translation if there exist $\epsilon_0 > 0$ such that the locally maximal set Λ_s of H_s in U_0 is conjugated to Λ_0 and the conjugation ϕ_s is C^0 close to the inclusion $\forall s \in (-\epsilon_0, \epsilon_0)$. We also say that:

- Λ_0 is absolute stable by the uniform translation if there exist $C_0 > 0$ such that $d(Id, \phi_s) < C_0 s$.
- Λ_0 is infinitesimal stable by the uniform translation if there exist $g : U_0 \to \mathbb{R}$ such that
$$\frac{\partial H}{\partial t}(z)g(z) - g(H(z)) = -1 \ \forall z \in U_0.$$
- Λ_0 is α-absolute stable by the uniform translation if there exist $C_0 > 0$ such that $d(Id, \phi_s) < C_0 s^\alpha$.

2.1 Infinitesimal Stability and Absolute Stability

Even though it is well known that general concept of absolute and infinitesimal stability is equivalent to hyperbolicity (c.f. [R1], [F], [G] and [M1]), our hypothesis is weaker since we are only taking the perturbation by the uniform translation and not considering the set of all perturbations.

Studying the continuation of periodic points we prove this two theorems:

Theorem 1: *If $H \in \mathcal{SPH}^+$ and Λ_0 is a locally maximal set absolutely stable by the uniform translation then Λ_0 is hyperbolic*

Proof. If ϕ_s is the conjugacy between H and H_s, given $z \in \Lambda_0$ we call $z(s) = \phi_s(z)$. Since we are not perturbing the base it is clear that $\pi_M(z) = \pi_M(z(s)) \ \forall z \in \Lambda_0$.

Due to the fact that h is hyperbolic on the base and therefore we have a strong shadowing on the base, we have a dense set of periodic points A such that all of them are contractive or all of them are expansive. Let us assume that they are contractive.

Given a periodic point $p \in A$ we have that $p(s)$ is a C^r curve. We will now compute the first derivative of such curve at $s = 0$ obtaining that:

Lemma 2.1:
$$p'(0) = \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} f'(H^{-j}(p)).$$

Proof. We have that $\phi_s(H(p)) = H_s(\phi_s(p))$ which is equivalent to $H(p)(s) = H_s(p(s))$. Now on the central coordinate which is the one that matters to us we have that $\pi_c(H(p)(s)) = f_s(p(s)) = f(p(s)) + s$. If we take the first derivative we conclude:

$$H(p)'(0) = f'(p)p'(0) + 1.$$

By an inductive argument we can see that:

$$p'(0) = \prod_{j=1}^{n} f'(H^{-j}(p)).H^{-n}(p)'(0) + \sum_{i=1}^{n} \prod_{j=1}^{i-1} f'(H^{-j}(p)).$$

Since $\prod_{j=1}^{n} f'(H^{-j}(p))$ converges to 0 by the hyperbolicity of p and $H^{-n}(p)'(0)$ takes values on a finite set, if we take the limit when n goes to infinity we conclude the lemma. $\square$

Observe that since H preserves orientation each term of the sum is a positive number.

The absolute stable condition implies that $p'(0) \leq C_0 \ \forall p \in A$.

If there exist $z \in \Lambda_0$ such that

$$lim_n \prod_{i=0}^{n} f'(H^{-i}(z)) \neq 0,$$

then there exist $\delta > 0$ and $\{n_k\}_{k \in \mathbb{N}}$ an increasing sequence of natural numbers such that

$$\prod_{i=0}^{n_k} f'(H^{-i}(z)) > \delta \ \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$ we can find $p_k \in A$ such that

$$\prod_{i=1}^{n_j} f'(H^{-i}(p_k)) > \frac{\delta}{2} \ \forall j \leq k.$$

In particular $p_k'(0) \geq k\delta/2$ which is a contradiction. Therefore

$$lim_n \prod_{i=0}^{n} f'(H^{-i}(z)) = 0 \ \forall z \in \Lambda_0.$$

This is a known condition to be equivalent to hyperbolicity on a given bundle. $\square$

Theorem 2: *If $H \in \mathcal{SPH}^+$ and Λ_0 is a locally maximal set infinitesimal stable by the uniform translation then Λ_0 is hyperbolic*

Proof. The concept of infinitesimal stable is directly linked with the differentiability of the continuation of the set. Our concept of infinitesimal stable by the uniform translation implies that the continuation when perturbed by the translation will be differentiable.

Using the notation of the previous theorem, let us prove then that $z(s)$ is differentiable.

Suppose that it is differentiable and observe that $z(s)$ must hold the condition

$$H_s(z(s)) = H(z)(s).$$

Since our perturbation only happens on the fiber, the previous equation can be seen as

$$f_s(z(s)) = \pi_c(H(z)(s)).$$

Observe that $f_s(z(s)) = f(z(s)) + s$ and if we take derivatives we conclude that

$$f'(z(s))D\pi_c(z'(s)) + 1 = D\pi_c(H(z)'(s)),$$

which is the equation that verifies $g(z)$ for $s = 0$. Observe that infinitesimal stable by the uniform translation is an open property.

Given s let g_s be the map associated to H_s from the infinitesimal stable property. For a given point z we construct a curve $c_z(s)$ which verifies $D\pi_c(c_z'(s)) = g_s(c_z(s))$, $\pi_M(c_z(s)) = \pi_M(z)$ and $c_z(0) = z$. Since it comes from an ordinary differential equation we know that such a curve exist. Now $c_z(s)$ must verify

$$f'(c_z(s))D\pi_c(c_z'(s)) + 1 = D\pi_c(c_{H(z)}'(s)),$$

and therefore we have that

$$H_s(c_z(s)) = c_{H(z)}(s).$$

Given a family of curves that hold the previous equation, for a s fix, being Λ_s locally maximal one must have that $c_z(s) \in \Lambda_s$. Since $c_z(s)$ is nearby z and Λ_0 is stable we have in fact that $\Lambda_s = \{c_z(s) : z \in \Lambda_0\}$. Therefore $z(s) = c_z(s)$ which implies that $z(s)$ is differentiable. Moreover we have that $z \to z(s)$ is a continuous function taking the C^r topology for the space of curves. In particular $D\pi_c(z'(0)) = g(z)$ which implies that $|D\pi_c(z'(0))| < C_0$. From this point we proceed as in the previous theorem concluding the hyperbolicity. $\square$

2.2 α-Absolute Stability

What we do now is to weak the hypothesis on the map ϕ_s asking it to vary just Hölder continuous on the parameter s instead of Lipschitz. Observe that now the technique using the first derivative of the continuation of the periodic points would not work because just Hölder does not imply that all this derivatives are uniformly bounded which was the key part in the previous theorems.

What we will do is to adapt the techniques developed on [Ti] for systems with Hölder-Shadowing property. The main idea here is that just Hölder continuity will let us estimate the action of the differential on the center-bundle and also the lack of speed will imply certain slow growth in the perturbations.

Theorem 3: *If $H \in \mathcal{SP}^+$ is C^2 and Λ_0 is a locally maximal set α-absolutely stable by the uniform translation with $\alpha > 1/2$ then Λ_0 is central hyperbolic. If H is just $C^{1+\gamma}$ with $\gamma \in (0,1)$ and $\alpha > 1/(1+\gamma)$ then Λ_0 is central hyperbolic.*

Observe first that for this theorem we are not asking to have an hyperbolic set on the base. Also that the inequality on α is strict. It is not clear to us what happens on $\alpha = 1/2$ and we have counterexamples for $\alpha < 1/2$. Nevertheless this examples are weak because can be perturbed to be hyperbolic, this means that the known examples are not generic. We will discuss this later with more detail.

The general framework is the following: Let $\{E_n\}_{n\in\mathbb{Z}}$ be a family of euclidean spaces of dimension m and $\mathcal{A} = \{A_{n\in\mathbb{Z}} : A_n : E_n \to E_{n+1}\}$ a sequence of linear isomorphism such that there exist $R > 0$ with

$$\|A_n\| < R \text{ and } \left\|A_n^{-1}\right\| < R.$$

We say that $\mathcal{A}$ has bounded solution if there exist $L > 0$ such that for all $i \in \mathbb{Z}$, $n > 0$ and $\{w_k \in E_k\}_{k\in[i+1,\ldots,i+n]}$ with $|w_k| \le 1$ there exist $\{v_k \in E_k\}_{k\in[i,i+n]}$ which verifies

$$v_{k+1} = A_k v_k + w_{k+1} \; k \in [i,\ldots,i+n-1],$$

and $|v_k| \le L \; for \; k \in [i,\ldots,i+N]$.

What the previous definition controls is how far you can find an orbit of a perturbation of your system by translations that shadows the 0 orbit in finite steps.

What it is done in [To] and [OPT] is to prove that bounded solution implies hyperbolicity on $\mathcal{A}$.

Given $z \in \Lambda_0$ we define $\mathcal{A}(z) = \{DH_{|E^c} : E^c(z_m) \to E^c(z_{m+1})\}$ where $z_m = H^m(z)$. We say that Λ_0 has uniform bounded solution if there exist Q such that $\mathcal{A}(z)$ has bounded solution and Q is a bound.

Proposition 2.2: *If $H \in \mathcal{SP}^+$ is C^2 and Λ_0 is a locally maximal set α-absolutely stable by the uniform translation with $\alpha > 1/2$ then Λ_0 has uniform bounded solution.*

Proof. The uniformity will come along the proof. It is just needed to see that the constants does not depend on z. We will fix z and prove that $\mathcal{A}(z) = \mathcal{A}$ has bounded solution.

Observe first that $DH_{|E^c}(v) = f'(z)v$ if $v \in E^c(z_m)$. We will identify $E^c(z_m)$ with $\mathbb{R}$. To simplify the notation we will call $a_m = f'(z_m)$.

Given $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, take $w_{m+1}, \ldots, w_{m+n} \in \mathbb{R}$ ($\{w_i\}_{m,n}$) with $|w_i| \leq 1$ let $v_m, \ldots, v_{m+n} \in \mathbb{R}$ ($\{v_i\}_{m,n}$) such that

$$v_{i+1} = a_i v_i + w_{i+1} \textit{ with } m \leq i \leq m+n-1.$$

We define the norm $\|\{v_i\}_{m,n}\| = max\{|v_i| : m \leq i \leq m+n\}$.

To prove the proposition we need to find a number $Q > 0$ which for all $n \in \mathbb{N}$, for all $m \in \mathbb{Z}$, and for all $\{w_i\}_{m,n}$ with $|w_i| \leq 1$ we can find $\{v_i\}_{m,n}$ such that

$$\|\{v_i\}_{m,n}\| \leq Q.$$

It will be clear in the proof that the starting point of the sequences $\{w_i\}_{m,n}$ and $\{v_i\}_{m,n}$ will not be relevant in the computations, therefore we assume $m = 0$ and from now on we note $\{w_i\}_n$ and $\{v_i\}_n$.

In order to find Q, given $\{w_i\}_n$ we define

$$E(\{w_i\}_n) = \{\{v_i\}_n : v_{i+1} = a_i v_i + w_{i+1} \textit{ with } 0 \leq i \leq n-1\},$$

the space of orbits for the perturbation $\{w_i\}_n$. Since we want to find one $\{v_i\}_n \in E(\{w_i\}_n)$ with a small norm we will take the one with the smallest. We define then

$$F(\{w_i\}_n) = min\{\|\{v_i\}_n\| : \{v_i\}_n \in E(\{w_i\}_n)\}.$$

Since $\|\cdot\|$ is a norm the previous definition is good. Now we take the worst perturbation and define

$$Q(n) = \max\{F(\{w_i\}_n) : \{w_i\}_n \textit{ with } |w_i| \leq 1\}.$$

The previous definition is good because F is continuous according to $\{w_i\}_n$ and the space $\{w_i\}_n$ with $|w_i| \leq 1$ is compact.

We therefore have to prove that there exist Q such that $Q(n) \leq Q$.

Let us observe now that from the definition of $Q(n)$ and linearity on the equation $v_{i+1} = a_i v_i + w_{i+1}$ we have the following property: Given $\{w'_i\}_n$ there exist $\{v'_i\}_n \in E(\{w'_i\}_n)$ such that

$$\|\{v'_i\}_n\| \leq Q(n) \|\{w'_i\}_n\|.$$

We now prove that in this algebraic context the algebraic uniform translation is our worst perturbation:

Lemma 2.3: *If $a_i > 0$ $\forall i$ then $Q(n) = F(\{w_i\}_n)$ with $w_i = 1$ $\forall i$.*

Proof. Observe first that if $\{v_i\}_n \in E(\{w_i\}_n)$ then $\{-v_i\}_n \in E(\{-w_i\}_n)$.

Given $\{w_i\}_n$ with $|w_i| \leq 1$ if $\{v_i\}_n \in E(\{w_i\}_n)$ we have that

$$v_i = \prod_{k=0}^{i-1} a_k v_0 + \sum_{k=1}^{i} \prod_{j=k}^{i-1} a_j w_k.$$

If $B_i = \prod_{k=0}^{i-1} a_k$ and $C_i = \sum_{k=1}^{i} \prod_{j=k+1}^{i-1} a_j w_k$ we define $g_i : \mathbb{R} \to \mathbb{R}$ and $G_n : \mathbb{R} \to \mathbb{R}$ such that

$$g_0(v) = v,$$

$$g_i(v) = B_i v + C_i \ with\ 1 \leq i \leq n,$$

and

$$G_n(v) = max\{|g_i(v)| : 0 \leq i \leq n\}.$$

If $D_i = \sum_{k=1}^{i} \prod_{j=k+1}^{i-1} a_j$ in an analogous way we define $f_i : \mathbb{R} \to \mathbb{R}$ and $F_n : \mathbb{R} \to \mathbb{R}$ such that

$$f_0(v) = v,$$

$$f_i(v) = B_i v + D_i \ with\ 1 \leq i \leq n,$$

and

$$F_n(v) = max\{|f_i(v)| : 0 \leq i \leq n\}.$$

We therefore have that $F(\{w_i\}_n) = min\{G_n(v) : v \in \mathbb{R}\}$ and $F(\{1\}_n) = min\{F_n(v) : v \in \mathbb{R}\}$.

Given $i_1, i_2 \leq n$ we define the maps $(g_{i_1}, g_{i_2}) : \mathbb{R} \to \mathbb{R}$ and $(f_{i_1}, f_{i_2}) : \mathbb{R} \to \mathbb{R}$ by

$$(g_{i_1}, g_{i_2})(v) = max\{|g_{i_1}(v)|, |g_{i_2}(v)|\},$$

and

$$(f_{i_1}, f_{i_2})(v) = max\{|f_{i_1}(v)|, |f_{i_2}(v)|\}.$$

We also define the values $min(g_{i_1}, g_{i_2})$ and $min(f_{i_1}, f_{i_2})$ as the minimum value taken by the maps (g_{i_1}, g_{i_2}) and (f_{i_1}, f_{i_2}) respectively. It is easy to verify that the infimum value is in fact a minimum.

Since $|g_i|$ and $|f_i|$ are convex functions G_n and F_n are also convex functions. This implies the following assertion:

$$min(G_n) = max\{min(g_{i_1}, g_{i_2}) : i_1, i_2 \leq n\},$$

and

$$min(F_n) = max\{min(f_{i_1}, f_{i_2}) : i_1, i_2 \leq n\}.$$

The previous assertion tell us that to compute $F(\{w_i\}_n)$ and $F(\{1\}_n)$ we need to compute just $min(g_{i_1}, g_{i_2})$ and $min(f_{i_1}, f_{i_2})$.

We are going to prove now that for i_1 and i_2 fixed we have that

$$min(g_{i_1}, g_{i_2}) \leq min(f_{i_1}, f_{i_2}).$$

This and the previous assertion implies $F(\{w_i\}_n) \leq F(\{1\}_n)$ which concludes the lemma.

For the maps f_i and g_i we have the following property: Given $k, l \in \mathbb{N}$ such that $k + l \leq n$ there exist $B_{k,l}, C_{k,l}$ and $D_{k,l}$ such that:

$$g_{k+l}(v) = B_{k,l} g_k(v) + C_{k,l},$$

and

$$f_{k+l}(v) = B_{k,l} f_k(v) + D_{k,l}.$$

Let us observe now that D_i is always positive. In particular it verifies $D_i \geq |C_i|$ and moreover $D_{k,l} \geq |C_{k,l}|$.

The previous statements are easy computations concluded from the fact that $a_i > 0$.

Fix now i_1 and i_2. Suppose that $i_1 < i_2$ and take $k = i_1$ and $l = i_2 - i_1$.

We have then:

$$g_{i_1}(v) = B_k v + C_k \quad g_{i_2}(v) = B_{k,l} B_k v + B_{k,l} C_k + C_{k,l},$$

$$f_{i_1}(v) = B_k v + D_k \textit{ and } f_{i_2}(v) = B_{k,l} B_k v + B_{k,l} D_k + D_{k,l}.$$

If $min(g_{i_1}, g_{i_2}) = (g_{i_1}, g_{i_2})(v_0)$ then v_0 verifies $|g_{i_1}(v_0)| = |g_{i_2}(v_0)|$. Moreover if $\hat{v_1}$ and $\hat{v_2}$ are such that $g_{i_j}(\hat{v_j}) = 0$ then $v_0 \in [min\{\hat{v_1}, \hat{v_2}\}, max\{\hat{v_1}, \hat{v_2}\}]$.

Suppose that $\hat{v_1} < \hat{v_2}$ then v_0 verifies the equation:

$$g_{i_1}(v_0) = -g_{i_2}(v_0).$$

If we resolve this we conclude that

$$v_0 = \frac{-C_k - C_{k,l} - B_{k,l} C_k}{B_k + B_{k,l}},$$

and therefore

$$min(g_{i_1}, g_{i_2}) = \frac{-C_{k,l} B_k}{B_k + B_{k,l}}.$$

Since B_k and $B_{k,l}$ are positive $C_{k,l}$ must be negative this comes from the condition $\hat{v_1} < \hat{v_2}$. In any case we have

$$min(g_{i_1}, g_{i_2}) = \frac{|C_{k,l}| B_k}{B_k + B_{k,l}}.$$

Computing for f_{i_1} and f_{i_2} we conclude

$$min(f_{i_1}, f_{i_2}) = \frac{|D_{k,l}|B_k}{B_k + B_{k,l}}.$$

Since $D_{k,l} \geq |C_{k,l}|$ we have that $min(f_{i_1}, f_{i_2}) \geq min(g_{i_1}, g_{i_2})$ finishing the proof of the lemma.

□

Observe that in this proof the fact that a_i are positive and the dimension is 1 are key facts. This implies that in a context with a higher dimension in the center bundle or without the hypothesis of orientation preserving it is not clear which is the worst perturbation. In fact this result is powerful because the worst perturbation $\{w_i\}_{m,n}$ does not depend either on n nor m and this is why we can relate the algebraic perturbation to a perturbation of the skew-product. To generalize this result using this technique one should look for a C^r vector field X on U_0 a neighborhood of Λ_0 such that for every n and m, $Q(m,n) = F(\{X(z_i)\}_{m,n})$.

Let us now link the algebraic uniform translation with the uniform translation in the skew-products.

Recall that $H_s = (h, f_s)$ where $f_s = f + s$. Given $z \in \Lambda_0$, we called $z_i = H^i(z)$ and $z_i(s)$ the continuation of z_i by the perturbation H_s. Remember also that $z_i(s)$ only varies on the fiber. The α-absolute hypothesis tell us that

$$|\pi_c(z_i(s)) - \pi_c(z_i)| \leq C_1 s^\alpha.$$

Let us call $u_i(s) = \pi_c(z_i(s)) - \pi_c(z_i)$. We have then that $|u_i(s)| \leq C_1 s^\alpha$.

We now prove that $u_i(s)$ is really close to verify $v_{i+1}(s) = a_i v_i(s) + s$ which is the un-normalized equation of the algebraic translation.

Let us prove then that:

$$|u_{i+1}(s) - a_i u_i(s) - s| \leq C_2 s^{2\alpha}.$$

Observe that

$$u_{i+1}(s) = \pi_c(z_{i+1}(s)) - \pi_c(z_{i+1}) = f_s(z_i(s)) - f(z_i) = f(z_i(s)) - f(z_i) + s.$$

We now apply the Taylor polynomial to f restricted to the center bundle in the point z_i and we have that there exist $\hat{C}_2$ such that

$$|f(z_i(s)) - f(z_i) - f'(z_i)(\pi_c(z_i(s)) - \pi_c(z_i))| \leq \hat{C}_2(\pi_c(z_i(s)) - \pi_c(z_i))^2.$$

Replacing $\pi_c(z_i(s)) - \pi_c(z_i)$ by $u_i(s)$ and combining the previous equation we prove our assertion.

Here is where the condition $\alpha > 1/2$ appears. Since s can be taken small enough, if $2\alpha > 1$ then $C_2 s^{2\alpha} < s$. If we do not have C^2 but $C^{1+\gamma}$ then we can conclude that

$$|u_{i+1}(s) - a_i u_i(s) - s| \leq C_2 s^{(1+\gamma)\alpha},$$

and we will just need that $(1+\gamma)\alpha > 1$.

Anyhow we will continue the proof for the C^2 case.

If we define $r_{i+1}(s) = u_{i+1}(s) - a_i u_i(s) - s$ we just proved that $r_i(s) \leq C_2 s^{2\alpha}$.

Using a previously proved property of $Q(n)$ we have now that there exist $\{e_i(s)\}_n \in E(\{r_i(s)\}_n)$ such that

$$\|\{e_i(s)\}_n\| \leq Q(n) \|\{r_i(s)\}_n\|.$$

This implies that

$$\|\{e_i(s)\}_n\| \leq C_2 Q(n) s^{2\alpha}.$$

It is easy to compute that $\{u_i(s) - e_i(s)\}_n \in E(\{s\}_n)$ and therefore

$$\left\{\frac{u_i(s) - e_i(s)}{s}\right\}_n \in E(\{1\}_n).$$

Since we proved that $Q(n) = F(\{1\}_n)$ by definition of $Q(n)$ we have that

$$\begin{aligned} Q(n) &\leq \left\| \left\{ \tfrac{u_i(s)-e_i(s)}{s} \right\}_n \right\| \leq \tfrac{\|\{u_i(s)\}_n\|}{s} + \tfrac{\|\{e_i(s)\}_n\|}{s} \\ &\leq C_1 s^{\alpha-1} + C_2 Q(n) s^{2\alpha-1} \end{aligned}$$

From this we conclude that

$$Q(n) \leq \frac{C_1 s^{\alpha-1}}{(1 - C_2 s^{2\alpha-1})},$$

if we take s small enough such that $C_2 s^{2\alpha-1} < 1$. We can do this becuase $2\alpha - 1 > 0$ by hypothesis. With this we finish the proof of the proposition. □

Let us prove the theorem.

Proof. Take Q from the previous proposition which does not depend on z. Let us use the same notation $a_i = f'(z_i)$. Given $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define

$$\lambda(m,n) = \prod_{i=m}^{m+n-1} a_i.$$

Let us prove the following lemma:

Lemma 2.4: *There exist n_0 such that for any $m \in \mathbb{Z}$ we have:*

$$\lambda(i, n_0) > 2 \quad or \quad \lambda(i + n_0, n_0) < 1/2.$$

Proof. We will now use once again the algebraic uniform translation. This is: set $\{w_i\}_{m,2n}$ with $w_i = -1$. Since $\mathcal{A}$ has bounded solution there exist $\{v_i\}_{m,2n}$ such that

$$v_{i+1} = a_i v_i - 1,$$

and $\|\{v_i\}_{m,2n}\| < Q$.

Observe that since $a_i > 0$ if $v_i \leq 0$ then $v_j < 0$ for all $j \geq i$.

We have two cases now either: $v_{m+n-1} > 0$ or $v_{m+n-1} \leq 0$. For the first one we have that $v_i > 0$ for $i \in \{m, \ldots, m+n-1\}$.

From the equation $v_{i+1} = a_i v_i - 1$ we have that $a_i = \frac{v_{i+1}+1}{v_i}$. Therefore

$$\lambda(m,n) = \prod_{i=m}^{m+n-1} \frac{v_{i+1}+1}{v_i} = \frac{v_{m+n}+1}{v_0} \prod_{i=m}^{m+n-1} \frac{v_i+1}{v_i} = \frac{v_{m+n}+1}{v_0} \prod_{i=m}^{m+n-1} \left(1 + \frac{1}{v_i}\right).$$

Using the bound Q over v_i we have that

$$\lambda(m,n) \geq \frac{1}{Q}\left(1 + \frac{1}{Q}\right)^n.$$

If $v_{m+n-1} \leq 0$ then $v_i < 0$ for $i \in \{m+n, \ldots, m+2n-1\}$ and then

$$\lambda(m+n,n) \leq (Q+1)\left(1 - \frac{1}{Q}\right)^n.$$

Taking n_0 big enough we conclude our lemma. □

From the previous lemma is easy to see that if $\lambda(m, n_0) > 2$ then $\lambda(m - kn_0, n_0) > 2$ for all $k \in \mathbb{N}$ and if $\lambda(m, n_0) < 1/2$ then $\lambda(m + kn_0, n_0) < 1/2$. Define now $\Lambda^u = \{z \in \Lambda_0 : \lambda(z, m, n_0) > 2\}$ and $\Lambda^s = \{z \in \Lambda_0 : \lambda(z, m, n_0) < 1/2\}$. Due to continuity and the previous assertion we conclude that this two sets are compact, invariant and disjoint and therefore one must be empty. Having that $\Lambda_0 = \Lambda^s$ or $\Lambda_0 = \Lambda^u$ implies that Λ_0 is central hyperbolic. □

3 Topologically Hyperbolic vs Hyperbolicity

Due to Mañé in [M2] it is known that in dimension 1 the expansive property, the hyperbolicity of periodic points and the lack of critical points are enough hypothesis to imply hyperbolicity. In dimension 2 [PuSa1] proved that dominated splitting and hyperbolic periodic points imply the existence of a topological hyperbolic behavior and with that they proved hyperoblicity. In dimension 3 there is an example due to [Pu] which is Kupka-Smale and topologically hyperbolic but it is not hyperbolic. The last example is a Skew-Product which has the lack of a generic property for partially hyperbolic sets which we called the strong Kupka-Smale property.

Let us recall the definition of central topologically hyperbolic

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically contracting on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| < \epsilon_2, \ \forall k \geq n,$$

where $I_{\epsilon_1}(z) = \{x\} \times [t - \epsilon_1, t + \epsilon_1]$ if $z = (x, t)$.

Given $H \in \mathcal{SP}$ and Λ_0 a locally maximal set we say that H is central topologically expanding on Λ_0 if for every $0 < \epsilon_1 < \epsilon_2$ there exist $n(\epsilon_1, \epsilon_2)$ such that for every $z \in \Lambda$

$$|H^k(I_{\epsilon_1}(z))| > \epsilon_2, \ \forall k \geq n.$$

We say that H is central topologically hyperbolic on Λ_0 if it is either central topologically expanding or central topologically contracting on Λ_0.

We are going to begin this chapter by proving some basic results about central topologically hyperbolic sets. The main objects we are studying here are the invariant graphs which are unique. This tell us that the dynamics of these set can be seen C^0 as the dynamics on the base. We will do the proofs mainly for the contractive case, being the expansive case analogous.

Proposition 3.1: *(Invariant Graph) If $H \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic then there exist $b_0 : \Lambda \to \mathbb{R}$ a continuous function such that $\Lambda_0 = graph(b_0)$. In particular $H(x, b_0(x)) = (h(x), b_0(h(x)))$. Moreover there is U_0 a neighborhood of Λ_0 such that if $b_1 : \Lambda \to \mathbb{R}$ verifies $H(x, b_1(x)) = b_1(h(x))$ and $graph(b_1) \subset U_0$ then $b_1 = b_0$.*

Proof. Without loss of generality we can assume that:

- Λ_0 is central topologically contractive.
- $\Lambda_0 = \bigcap_{n \in \mathbb{Z}} H(\Lambda \times [-1, 1])$ since Λ_0 is locally maximal.
- $H \in \mathcal{SP}^+$.
- And from the previous points that $-1 < f(x, -1) < f(x, 1) < 1$.

We will see later the details about the non-preserving orientation case.

Let us see first the existence of b_0. Given $x \in \Lambda$ set $x_n = h^{-n}(x)$. From the previous assumption observe first that we have:

$$\pi_c(H^{n+1}(x_{n+1}, -1)) > \pi_c(H^n(x_n, -1)) \textit{ and } \pi_c(H^{n+1}(x_{n+1}, 1)) < \pi_c(H^n(x_n, 1)).$$

We define then

$$b(x)^- := lim_n \pi_c(H^n(x_n, -1)) \textit{ and } b(x)^+ := lim_n \pi_c(H^n(x_n, 1)).$$

Is clear that $b(x)^- \leq b(x)^+$. By definition we have that $\pi_c(H^{-n}(x, b^{\pm}(x))) \in (-1, 1)$ for every $n \geq 0$ and therefore $\{x\} \times [b(x)^-, b(x)^+] \subset \Lambda_0$.

Moreover, by construction $\{x\} \times [b(x)^-, b(x)^+]$ is maximal among the intervals inside the fiber of x which are contained in Λ_0.

Let us assume that $b(x)^- < b(x)^+$. We have two cases now:

- there exist $\delta_0 > 0$ such that $|H^{-n}(\{x\} \times [b(x)^-, b(x)^+])| > \delta_0 > 0 \ \forall n \geq 0$
- $liminf_n |H^{-n}(\{x\} \times [b(x)^-, b(x)^+])| = 0$.

Suppose there exist $\delta_0 > 0$ such that $|H^{-n}(\{x\} \times [b(x)^-, b(x)^+])| > \delta_0 \ \forall n \geq n_0$, for certain n_0. If we take $y \in \alpha(x)$ we have that $b^+(h^n(y)) - b^-(h^n(y)) \geq \delta_0$. Since H is central topologically contractive, $b^+(h^n(y)) - b^-(h^n(y))$ converge to 0 obtaining a contradiction.

For the second case fix $\epsilon_2 = b(x)^+ - b(x)^-$ and take ϵ_1 arbitrarily small. Let n_0 be from the definition of central topologically contractive. Since $liminf_n |H^{-n}(\{x\} \times [b(x)^-, b(x)^+])| = 0$ there exist $n > n_0$ such that $|H^{-n}(\{x\} \times [b(x)^-, b(x)^+])| < \epsilon_1$. This implies that

$$H^{-n}(\{x\} \times [b(x)^-, b(x)^+]) \subset I_{\epsilon_1}(H^{-n}(z)),$$

and therefore $\{x\} \times [b(x)^-, b(x)^+] \subset H^n(I_{\epsilon_1}(H^{-n}(z)))$. Then $|H^n(I_{\epsilon_1}(H^{-n}(z)))| > \epsilon_2$ which contradicts the definition of central topologically hyperbolic.

In both cases we obtained a contradiction, so we must conclude that $b_0(x) := b^-(x) = b^+(x)$ is well defined and unique.

The uniqueness of $b_0(x)$ and the fact that $(x, b_0(x)) \in \Lambda_0$ implies that $f(x, b_0(x)) = b_0(h(x))$.

If we have a sequence $\{x_n\}_n \subset \Lambda$ which converges to x and $lim_n b_0(x_n)$ does not exist or it is different from $b_0(x)$ then we would have more than one point from Λ_0 on the fiber of x which can not happen. We conclude then that b_0 is continuous.

If we have b_1 which verifies $H(x, b_1(x)) = (h(x), b_1(h(x)))$ and $graph(b_1) \subset U_0$ where U_0 is the one associated to the property of locally maximal from Λ_0, then $grap(b_1)$ is a compact invariant set and therefore $\Lambda_0 = graph(b_1)$. Then by construction, on each fiber of x there is only one point of Λ_0 and therefore $b_0(x) = b_1(x)$.

For the non-preserving orientation case, using the set of points

$$B(x) = \{\pi_c(H^n(x_n, -1))\}_{n\in\mathbb{N}} \cup \{\pi_c(H^n(x_n, 1))\}_{n\in\mathbb{N}},$$

we define $b(x)^- := liminf(B(x))$ and $b(x)^+ := limsup(B(x))$ and all the previous arguments for the rest of the proof are valid. $\square$

Once we have our map b_0 which is continuous we can naturally ask if it has differentiable properties and in that case we will call it the rigid case. Using the uniqueness in the previous proposition and the next lemma we will find a way to describe them all. Also with that construction we will obtain central hyperbolicity for stable maps in the rigid case.

Lemma 3.2: *(Decomposition) If $H = (h, f) \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic and $b_0 : \Lambda \to \mathbb{R}$ is a continuous function such that $\Lambda_0 = graph(b_0)$. Then there exist U a neighborhood of $\Lambda \times \{0\}$ in $M \times \mathbb{R}$ and $g_0 : U \to M \times \mathbb{R}$ which verifies*

$$f(x,t) = g_0(x, t - b_0(x)) + b_0(h(x)) \text{ and } g_0(x, 0) = 0 \ \forall x \in \Lambda.$$

Moreover if there exist $g_1 : U \to M \times \mathbb{R}$ and $b_1 : \Lambda \to \mathbb{R}$ such that $f(x,t) = g_1(x, t - b_1(x)) + b_1(h(x))$ and $g_1(x, 0) = 0$ $\forall x \in \Lambda$ then $b_0(x) = b_1(x)$ and $g_0(x,t) = g_1(x,t)$ for all $x \in \Lambda$.

Proof. For the existence we just define

$$g_0(x, u) = f(x, u + b_0(x)) - b_0(h(x)).$$

For the uniqueness observe that $f(x, b_1(x)) = g_1(x, b_1(x) - b_1(x)) + b_1(h(x)) = b_1(h(x))$ and from the uniqueness of b_0 obtained in the previous proposition we have that $b_0(x) = b_1(x)$ $\forall x \in \Lambda$. From this is immediate that $g_1(x,t) = g_0(x,t)$ $\forall x \in \Lambda$. $\square$

To end this introductory section on central topologically hyperbolic skew-products let us observe which Lyapunov exponent properties they have:

Proposition 3.3: *If $H \in \mathcal{SP}$ and Λ_0 is central topologically contracting then $\lambda^{+,+}(z) \le 0$ $\forall z \in \Lambda_0$. Moreover if there exist $\delta > 0$ such that $\lambda^{+,+}(z) \le -\delta < 0$ then Λ_0 is central hyperbolic. Analogously if Λ_0 is central topologically expanding then $\lambda^{+,-}(z) \ge 0$ $\forall z \in \Lambda_0$ and if $\lambda^{+,-}(z) \ge \delta > 0$ then Λ_0 is central hyperbolic.*

This result is also true for the C^1 Skew-Products yet i would like to do a proof using distortion. We need first the reformulation of a classical result:

Lemma 3.4: *If $H = (h, f) \in \mathcal{SP}$ and f is C^2 then there exist a constant $C_0 > 0$ such that for every z_0 and $z_1 \in M \times \mathbb{R}$ with $\pi_M(z_0) = \pi_M(z_1)$ we have:*

$$\frac{\left|\frac{\partial H^n}{\partial t}(z_0)\right|}{\left|\frac{\partial H^n}{\partial t}(z_1)\right|} \le exp(C_0 \sum_{i=0}^{n-1} |H^i(z_0) - H^i(z_1)|).$$

Proof. Take $C_1 > 0$ and $C_2 > 0$ such that $|f''(z)| < C_1$ and $C_2 < |f'(z)|$ and define $C_0 = C_2/C_1$. Then C_0 is a Lipschitz constant for the map $z \to log|f'(z)|$ on the fiber direction. Then we have

$$log\frac{\left|\frac{\partial H^n}{\partial t}(z_0)\right|}{\left|\frac{\partial H^n}{\partial t}(z_1)\right|} = log\frac{\prod_{i=0}^{n-1}|f'(H^i(z_0))|}{\prod_{i=0}^{n-1}|f'(H^i(z_0))|} =$$

$$\sum_{i=0}^{n-1}\log\left|f'(H^i(z_0))\right| - log\left|f'(H^i(z_0))\right| \leq C_0\sum_{i=0}^{n-1}|H^i(z_0) - H^i(z_1)|.$$

□

Proof. Let us prove the proposition. Suppose that H is central topologically contracting on Λ_0 and that there exist $z \in \Lambda_0$ and $\delta > 0$ such that $\lambda^{+,+}(z) > \delta > 0$. Take $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\delta - C_0\epsilon_2 > 0$ and $|H^n(I_{\epsilon_1}(z))| < \epsilon_2$ $\forall n \geq 0$. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers such that

$$\frac{log(|\prod_{i=0}^{n_k-1} f'(H^i(z))|)}{n_k} \geq \delta.$$

This implies that $|\prod_{i=0}^{n_k-1} f'(H^i(z))| \geq exp(n_k\delta)$.

By the mean value theorem we have for every k a point $z_k \in I_{\epsilon_1}(z)$ such that $|H^{n_k}(I_{\epsilon_1}(z))| = |\frac{\partial H^n}{\partial t}(z_k)|2\epsilon_1$.

Applying the previous lemma to z and z_k we have that

$$\frac{\left|\frac{\partial H^{n_k}}{\partial t}(z)\right|}{\left|\frac{\partial H^{n_k}}{\partial t}(z_k)\right|} \leq exp(C_0\sum_{i=0}^{n_k-1}|H^i(z) - H^i(z_k)|) \leq exp(C_0 n_k\epsilon_2).$$

Then

$$\begin{aligned}|H^{n_k}(I_{\epsilon_1}(z))| &= \left|\tfrac{\partial H^n}{\partial t}(z_k)\right| 2\epsilon_1 \\ &\geq 2\epsilon_1\left|\tfrac{\partial H^{n_k}}{\partial t}(z)\right| exp(-C_0 n_k\epsilon_2) \\ &\geq 2\epsilon_1 exp((\delta - C_0\epsilon_2)n_k)\end{aligned}$$

Since $\delta - C_0\epsilon_2 > 0$, $|H^{n_k}(I_{\epsilon_1}(z))|$ grows exponentially fast which contradicts the central topologically contracting hypothesis.

For the hyperbolic part of the proposition it is clear that $\lambda^{+,+}(z) < -\delta$ implies that

$$lim_n\prod_{i=0}^{n} f'(H^i(z)) = 0.$$

which is a property equivalent to hyperbolicity due to the compactness of Λ_0.

For the central topologically expanding case the proof is analogous. □

3.1 Rigidity

The rigidity case for us will be the case when b_0, the graph map, is C^r if Λ is a manifold or it can be extended in a neighborhood of Λ to a C^r map. We will see that this family of Skew-Products are not generic in $\mathcal{SPH}$ yet they have nice properties.

Observe that in the decomposition lemma 3.2, the map g on the fibers is as differentiable as f on the fibers. If b_0 is differentiable then g_0 is differentiable on M. Therefore given H such that b_0 is differentiable we have g_0 differentiable, and conversely given g_0 and b_0 differentiable such that $g_0(x, 0) = 0$ if we define $f(x, t) = g_0(x, t - b_0(x)) + b_0(h(x))$, then f is C^r and b_0 is going to be the graph map for the skew-product $H = (h, f)$.

Theorem 4: *If $H = (h, f) \in \mathcal{SP}$ and Λ_0 is central topologically hyperbolic having b_0 the graph map as differentiable as H then it is approximated by central hyperbolic systems. If $H \in \mathcal{SPH}$ is stable then Λ_0 is hyperbolic.*

Proof. Let g_0 be such that $f(x, t) = g_0(x, t - b_0(x)) + b_0(h(x))$. Since b_0 is C^r then g is C^r. Take the family of one parameter $g_s : U \to M \times \mathbb{R}$ as $g_s = (1 + s)g_0$ and define

$$f_s(x, t) = g_s(x, t - b_0(x)) + b_0(h(x)) \textit{ and } H_s = (h, f_s).$$

It is clear that if $s < 0$ then the maximal invariant set of H_s in U_0 is contained in the graph of b_0 which is the set Λ_0. In particular if $z \in \Lambda_0$ then its orbit remains in Λ_0 under the action of H_s and therefore Λ_0 is the maximal invariant set of H_s in U_0. It is not hard to compute that if $z \in \Lambda_0$ then $f_s'(z) = (1 + s)f'(z)$. This implies that $\lambda^{+,\pm}(z, H_s) = log(1 + s) + \lambda^{+,\pm}(z, H)$.

Suppose that H is central topologically contractive. Then $\lambda^{+,+}(z, H) \leq 0$. Therefore taking $s < 0$ we have that $\lambda^{+,+}(z, H_s) \leq log(1 + s) < 0$ which implies central hyperbolicity on Λ_0 for H_s.

If the periodic points are dense and we have points $\{z_n\}_{n\in\mathbb{N}}$ such that $\lambda^{+,+}(z_n, H) \to 0$ we can find periodic points $\{p_n\}_{n\in\mathbb{N}}$ such that $\lambda^{+,+}(p_n, H) \to 0$.

Once we have such periodic points, taking $s > 0$ using the stability we conclude once again that Λ_s the maximal invariant set of H_s in U_0 must be Λ_0. For those periodic points we must have that $\lambda^{+,+}(p_n, H_s) > 0$, this implies that we can find an arbitrarily small s which has a periodic point with Lyapunov exponent equal to 0. Such a point can be bifurcated obtaining a contradiction with the stable property. $\square$

Let us see now why the rigid case is not generic. The opposite concept here is the strong Kupka-Smale property.

We are going to restrict our Skew-Products in $\mathcal{SPH}$ to those which are partially hyperbolic. In particular for those we say that $H \in \mathcal{SPH}$ is strong Kupka-Smale if it is Kupka-Smale and also:

$$W^{ss}(p) \cap W^{uu}(q) = \phi \ \forall p, q \in Per(H).$$

Theorem 3.5: *There exist a residual set $\mathcal{SKS} \subset \mathcal{SP}$ such that if $H \in \mathcal{SKS}$ then H is strong Kupka-Smale.*

Proof. It is the same proof of Kupka-Smale theorem but having also the care of making the strong stable and strong unstable manifolds have a transversal intersection which in the context of partially hyperbolic systems means empty intersection. □

Theorem 3.6: *If (H, Λ_0) is rigid then $H \in \mathcal{SKS}^c$.*

Proof. Suppose that H is central topologically contractive. Given a point $z \in \Lambda_0$, we define the cone $C(K, z) = \{v \in T_z(M \times \mathbb{R}) : \ if \ v = v_I + v_M \ then \ \frac{\|v_I\|}{\|v_M\|} < K\}$. If b_0 is differentiable then there exist $K > 0$ and $\epsilon > 0$ such that for every $z_0 \in \Lambda_0$ and $\forall z_1 \in B(z_0, \epsilon) \cap \Lambda_0$ we have that $exp_{z_0}^{-1}(z_1) \in C(K, z_0)$.

If $z_0 \in \Lambda_0$ and $z_1 \in W_\epsilon^{ss}(z_0)$ then we have that $exp_{H^n(z_0)}^{-1}(H^n(z_1)) \in C(K, H^n(z_0))$. Reciprocally if $z_0 \in \Lambda_0$ and $z_1 \in M \times \mathbb{R}$ is such that $\pi_M(z_1) \in W^s(\pi_M(z_0))$, $d(z_0, z_1) \leq \epsilon$ and $exp_{H^n(z_0)}^{-1}(H^n(z_1)) \in C(K, H^n(z_0))$ then $z_1 \in W_\epsilon^{ss}(z_0)$. This implies that if b_0 is C^r, z_0, z_1 in Λ_0 verifying $\pi_M(z_1) \in W^s(\pi_M(z_0))$ then $z_1 \in W_\epsilon^{ss}(z_0)$.

The problem is that the condition $z_1 \in W^{ss}(z_0)$ only depends on its future and the condition $z_1 \in \Lambda_0$ for the central topologically contractive case only depends on its past, therefore if we have $z_0 \in \Lambda_0$, and $x_1 \in W^s(\pi_M(z_0)) \cap \Lambda$ we can perturb our system such that if $t_1 \in \mathbb{R}$ verifies $(x_1, t_1) \in W^{ss}(z_0)$ then $b_0(x_1) \neq t_1$ and therefore b_0 can not be differentiable. □

Observation 1: *In $\mathcal{LCSP}$ the only rigid central topologically hyperbolic systems are the trivial ones. We say that $H \in \mathcal{LCSP}$ is trivial if there exist $t_0 \in \mathbb{R}$ such $f(x, t) = t_0$. This clearly implies that $b_0(x) = t_0$. In $\mathcal{LCSP}$ we have that if $z_1 \in W_{loc}^{ss}(z_0)$ or $z_1 \in W_{loc}^{uu}(z_0)$ then $\pi_I(z_1) = \pi_I(z_0)$. Using this and the previous arguments if b_0 is differentiable then b_0 is locally constant. We leave as an easy exercise for the reader to see that if b_0 is locally constant then H is trivial.*

Let us finish this chapter by perturbing the example in [Pu] to create a $\mathcal{SKS}$ system central topologically hyperbolic which is not Hyperbolic. In particular it is no rigid.

Theorem 3.7: *There exist $H \in \mathcal{SKS}$ central topologically hyperbolic which is not hyperbolic.*

Proof. Take $h : M \to M$ such that it has Λ a locally maximal invariant set hyperbolic and not trivial. We have then that there exist $\Lambda_1 \subset \Lambda$ which is a non trivial minimal set. Take a map $\varphi : M \to \mathbb{R}$ which verifies:

$$\varphi(x) \begin{cases} = 1 \ if \ x \in \Lambda_1 \\ < 1 \ if \ x \notin \Lambda_1 \end{cases}$$

Define now the map $f : M \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,t) = \varphi(x)t - t^3.$$

Let $H_1 \in \mathcal{SP}$ be defined by $H_1 = (h, f)$. Is clear that H is central topologically contractive. Since $f(x,0) = 0$ if $b : M \to \mathbb{R}$ is the graph map associated then $b = 0$. In particular it is rigid.

The lack of hyperbolicity comes from the fact that if $x \in \Lambda_1$ then

$$\frac{\partial H^n}{\partial t}(x,0) = 1 \ \forall n > 0.$$

Observe that for any other point $x \notin \Lambda_1$

$$\frac{\partial H^n}{\partial t}(x,0) < 1 \ \forall n > 0.$$

which implies that H is Kupka-Smale.

Since the strong stable and strong unstable sets belong to $M \times \{0\}$ we conclude that H_1 is not strong Kupka-Smale.

We will perturb now H_1 in a way that it becomes strong Kupka-Smale but the dynamics on the minimal set are not destroyed. For this we define the space

$$\mathcal{B} = \{g : M \times \mathbb{R} \to \mathbb{R} : g(x,t) = g'(x,t) = 0 \ if \ x \in \Lambda_1\}.$$

This is a closed set of a Banach space and therefore is a Banach space. For each $g \in \mathcal{B}$ define $H_g : M \times \mathbb{R} \to \mathbb{R}$ by $H_g = (h, f + g)$. There exist $\mathcal{U}$ an open neighborhood of the map 0 such that $H_g \in \mathcal{SP}$ for every $g \in U$.

To conclude we need to observe the following: Given a periodic point p, since Λ_1 is minimal on h we have that $W^{ss}(p) \cap \Lambda_1 \times \mathbb{R} = \phi$ and $W^{uu}(p) \cap \Lambda_1 \times \mathbb{R} = \phi$. This is because a non trivial minimal set can not intersect the stable manifold or the unstable manifold of a periodic point. With this using the techniques on the $Kupka - Smale$ Theorem we can do our perturbation restricted to the space $\mathcal{B}$ obtaining the strong Kupka-Smale property on a residual set of perturbations. Now since the set of perturbations B does not alter the dynamics on $\Lambda_1 \times \mathbb{R}$ we have that H_g is never hyperbolic. If we take the perturbation small enough the central topologically hyperbolic conditions is not lost and therefore we finish the theorem.

$\square$

4 Analytic continuation on $\mathcal{LCSP}$

In this chapter we are interested in continue our study in the ambient manifold of Λ_0. We do this by working with the continuation of the periodic points.

We recommend to revisit the notations chapter to refresh the symbols we are going to use in this chapter. Let us begin by observing that $\mathcal{LCSP}$ has a Banach manifold structure.

$$\mathcal{B} = \{g : \mathcal{P} \times \mathbb{R} \to \mathbb{R} \text{ of class } C^r\}.$$

Then $\mathcal{B}$ with the C^r topology is a Banach space. Given $H_1, H_2 \in \mathcal{LCSP}$ with $H_i = (h, f_i)$ we can define the map $g : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ by $g(P_i, t) = f_1(x, t) - f_2(x, t)$ where $x \in P_i$. With that construction, it is easy to see that a neighborhood of a given $H \in \mathcal{LCSP}$ is diffeomorphic to a neighborhood of the map 0 in $\mathcal{B}$.

It is also clear that we can define the inclusion of $\mathcal{LCSP} \hookrightarrow \mathcal{B}$ by $i(H)(P_i, t) = f(x, t)$ if $H = (h, f)$ and $x \in P_i$ which is a differentiable map.

Given $g \in \mathcal{B}$ using an abuse of notation when we evaluate it on a point z in $\Lambda \times \mathbb{R}$, $g(z)$ we will referring to $g(\pi(x), t)$ if $z = (x, t)$. It is clear that a map $g \in \mathcal{B}$ is a finite collection of maps of the interval $\mathbb{R}$, we therefore define the maps associated to g, $g_i(t) = g(P_i, t)$.

In this context we have a natural way of taking an arc of perturbations which is: Given $H \in \mathcal{LCSP}$ and $g \in \mathcal{B}$ we define $H_s \in \mathcal{LCSP}$ by

$$H_s(z) = (h(x), f(z) + sg(z)).$$

In particular we set the notation $f_i^s = f_i + sg_i$.

The objective is to study for a given $H \in \mathcal{LCSP}$, Λ_0 an homoclinic class.

4.1 Hyperbolic sets

From Kupka-Smale theorem we now that generically the periodic points are hyperbolic and that the stable and unstable manifolds have transversal intersection. In our context the second property is technically free since by definition the three directions, stable, central and unstable are transversal. We might have conflict with periodic points of different index (which induce heterodimensional cycles), but as we are going to see later the stable and unstable manifolds move in an independent way. To begin taking the flavor of the perturbations we are going to do, let us start by proving the hyperbolicity condition in the Kupka-Smale theorem.

Theorem 4.1: *(Kupka-Smale) There exist a residual subset $KS \subset \mathcal{LCSP}$ such that every skew-product in KS has all the periodic points hyperbolic.*

Proof. It is clear that the hyperbolicity worry us in the central direction. If we fix a word $A = (a_1, \dots, a_m) \in \bigcup_{n=1}^{\infty} \mathcal{P}^n$ of elements of the Markov partition we define the map $f_A = f_{a_m} \circ \dots \circ f_{a_1}$. It is easy to see that we can identify the periodic points of H in Λ_0 with the fixed points of f_A making A vary on $\bigcup_{n=1}^{\infty} \mathcal{P}^n$.

Fixed A and $l \leq m$ set $A_l = (a_1, \dots, a_l)$. Given $t \in \mathbb{R}$, and $g \in \mathcal{B}$ we define $f_A^s = f_{a_m}^s \circ \dots \circ f_{a_1}^s$. We observe that for $s = 0$

$$\frac{\partial f_A^s}{\partial s}(t) = \sum_{i=1}^{m} g(f_{A_i}(t)) \prod_{j=i+1}^{m} f'_{a_j}(f_{A_i}(t)).$$

It is not hard to see that the equation above defines a linear operator over g which goes from $\mathcal{B}$ to $\mathbb{R}$ and which is surjective, therefore using the classic arguments of transversality we obtain a residual set on $\mathcal{LCSP}$ such that the fixed points of f_A are hyperbolic. Since the set of words is countable and the countable intersection of residual set is a residual set we conclude the result. □

The next results show us a way to characterize the hyperbolic set of a locally constant skew-product.

Theorem 5: *Given $H \in \mathcal{LCSP}$, and Λ_0 an homoclinic class, if Λ_0 is an hyperbolic set then one of the following two happen:*

- *$H_{|\Lambda_0}$ is normally hyperbolic. If $H_{|\Lambda_0}$ is contracting in the central direction then the tangent bundle of the sub-manifold is $E^{uu} \oplus E^c$ and if it is expanding the tangent bundle is $E^{ss} \oplus E^c$.*
- *H can be approximated on $\mathcal{LCSP}$ by skew-products such that the continuation of Λ_0 contains periodic points with strong connections.*

Observation 2: *The theorem says that if we take an hyperbolic homoclinic class of H then in the first case we can reduce the dimension of the ambient manifold. If this do not happen we can perturb it to build strong connections between periodic points. Once we have this if we perturb again we obtain blenders inside Λ_0 due to [BD]. This are known to be dynamical objects with full topological dimension.*

What we would like to do is change the hypothesis of hyperbolicity by the hypothesis of stability. If we could do so, in the normally hyperbolic situation we could apply [PuSa1] for dimension 2 or [PuSa2] for higher dimension, obtaining hyperbolicity for the set and in the blender case create some heterodimensional cycle if we do not have hyperbolicity.

To prove the theorem we use the next result due to C. Bonatti and S. Crovisier in [BC].

Theorem 4.2: *(Bonatti-Crovisier) If F is a diffeomorphism of a manifold N, Λ_1 is a partially hyperbolic set with the decomposition $T_{\Lambda_1}N = E^s \oplus E^c \oplus E^u$ and $\forall z \in \Lambda_1$, $W^{ss}(z) \cap \Lambda_1 = \{z\}$ then Λ is normally hyperbolic. In particular the tangent bundle of S is $E^u \oplus E^c$.*

Observation 3: *There exist an analogous version for the case which* $\forall z \in \Lambda_1$, $W^{uu}(z) \cap \Lambda_1 = \{z\}$ *obtaining* S *tangent to* $E^s \oplus E^c$.

Observation 4: *The reciprocal is also true, this is: if* Λ_1 *is partially hyperbolic and normally hyperbolic with* S *tangent to* $E^c \oplus E^u$*, then* $\forall z \in \Lambda_1$, $W^{uu}(z) \cap \Lambda_1 = \{z\}$.

We will split the proof in two cases, one where the center bundle is contracting and the other one were the central bundle is expanding. The proofs are analogous up to certain details which we will reviewed. Let us first see the contractive case.

4.2 Contractive case

Assuming that E^c is contracting and using the theorem of BC we just need to prove the next proposition:

Proposition 4.3: *Let* $H \in \mathcal{LCSP}$*, and* Λ_0 *be an homoclinic class. Suppose that* $H_{|\Lambda_0}$ *is hyperbolic with* E^c *contracting. If there exist* z_0 *and* z_1 *which belong to the same strong stable manifold then there exists* $g \in \mathcal{B}$ *and* $\{s_n\}_{n\in\mathbb{N}}$ *a decreasing sequence to* 0 *such that* H_{s_n} *has periodic points with a strong connection which belongs to the continuation of* Λ_0.

Let us recall that H_s is a Skew-Product defined by:

$$H_s(z) = (h(x), f(z) + sg(z)).$$

Without loss of generality we assume that $z_1 \in W^{ss}_{loc}(z_0)$.

Given a periodic point p of H, it must have an analytic continuation in a neighborhood of H. This means that there exist a curve $p(s)$ of class C^r such that $H_s^{n_p}(p(s)) = p(s)$ where n_p is the period of p. In fact, since we are assuming that Λ_0 is hyperbolic we have that given g there exists an uniform ϵ_0 such that $p(s)$ is defined in $(-\epsilon_0, \epsilon_0)$.

Let us start by calculating the first derivative of the continuation in the parameter s.

Lemma 4.4: *If* $p = (x_0, t_0)$ *is a periodic hyperbolic point of* H *and* n *is the period then we have that* $t_0'(s)$ *verifies*

$$t_0'(s) = \frac{\sum_{i=0}^{n-1} g(H_s^i(p(s))) \prod_{j=i+1}^{n-1}[f' + sg'(H_s^j(p(s)))]}{1 - \prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]},$$

where $f'(x,t) = f'_{\pi(x)}(t)$ *and* $g'(x,t) = g'_{\pi(x)}(t)$.

Proof. The result is obtained by computing the equation $H_s^n(q(s)) = q(s)$ and taking the first derivative in the parameter s. □

Observation 5: *The hyperbolicity of Λ_0 implies that there exist $C > 0$ and $0 < \lambda < 1$ such that*

$$\prod_{j=i+1}^{n-1} [f' + sg'(H_s^j(p(s)))] \leq C\lambda^{n-i-1}.$$

The idea now is to estimate $p(s)$ using its Taylor polynomial of first degree. Since each p has its own continuation, to begin talking about the limit of these continuations we need to have an uniform control over the remainders of such polynomials. For that is the next lemma:

Lemma 4.5: *Given $\epsilon_1 > 0$ exist $\delta_0 > 0$ such that if $p = (x, t)$ is a periodic point then we have*

$$t(s) = t(0) + st'(s) + r(p, s),$$

where $\frac{|r(p,s)|}{|s|} < \epsilon_1$ for every p periodic and for every $s \in (-\delta_0, \delta_0)$.

Proof. Using the later observation it is not hard to see that $t'(s)$ is uniformly bounded which implies that $\{p(s) : p \in \Lambda_0\}$ is an equicontinous family. Moreover if $r \geq 2$ we have that the functions $t''(s)$ are also uniformly bounded which implies that $\{p'(s) : p \in \Lambda_0\}$ es equicontinous and from that point we conclude using Arzelà-Ascoli theorem and Weierstrass theorem. □

If $\{p_m\}_{m\in\mathbb{N}}$ and $\{q_m\}_{m\in\mathbb{N}}$ are two sequences of periodic points of H which converge to z_0 and z_1 respectively, with $z_1 \in W^{ss}_{loc}(z_0)$, we want to understand what happens with $\frac{\partial \pi_c(p_m(s))}{\partial s}_{|s=0}$ (we will note p'_m) and $\frac{\partial \pi_c(q_m(s))}{\partial s}_{|s=0}(q'_m)$ because of the next lemma:

Lemma 4.6: *If lim_m p'_m y lim_m q'_m exist but they are different then for every $s_0 > 0$ arbitrary small there exist $|s| < s_0$ such that H_s has periodic points in $\Lambda_0(s)$ with a strong connection.*

Proof. Let $z'_0 = lim_m\ p'_m$, $z'_1 = lim_m\ q'_m$, $z_0 = (a_0, u_0)$, $z_1 = (a_1, u_1)$, $p_m = (x_m, t_m)$, $q_m = (y_m, r_m)$. Since z_0 and z_1 are on the same local stable manifold they have the same central coordinate, this is $u_0 = u_1$. Since $H_s \in \mathcal{LCSP}$ and it is defined on the same Markov partition of H we have that $x_m(s)$ and $y_m(s)$ are constant and equal to x_m and y_m respectively. What we have to prove then is that there exist $s \in (-s_0, s_0)$ and $m_0, m_1 \in \mathbb{N}$ such that $t_{m_0}(s) = r_{m_1}(s)$. For this we use the Taylor polynomial and we observe that:

$$t_n(s) - r_m(s) = t_n(0) - r_m(0) + s(t'_n - r'_m) + r(s, m, n),$$

where the right part of the equation is equal 0 when $s = \frac{r_m(0) - t_n(0) - r(s,m,n)}{t'_n - r'_m}$. For n and m big enough, we can take the numerator arbitrary small and the denominator we can suppose it far from 0. Therefore we can find s small enough to verify $t_n(s) = r_m(s)$. □

The previous lemma tell us that if we have an uniform control over the remainders of the Taylor polynomials we just need to find a perturbation for which the derivatives of z_0 and z_1 are distinct.

Once we have computed the derivative for the periodic points, we extend to the closure.

Lemma 4.7: *If $\{p_m\}_{m\in\mathbb{N}}$ converges to z_0 then $z_0' = lim_m\ p_m'$ exist and is equal to*

$$\sum_{i=1}^{\infty}[\prod_{j=1}^{i-1} f'(H^{-j}(z_0))]g(H^{-i}(z_0)).$$

Proof. Since we are working on the contractive case, it is more convenient for us to see that we can rewrite the equation for the periodic points as:

$$t_m' = \frac{\sum_{i=1}^{n_m}[\prod_{j=1}^{i-1} f'(H^{-j}(p_m))]g(H^{-i}(p_m))}{1-[\prod_{i=0}^{n_m-1} f'(H^{i}(p_m))]}.$$

The denominator in the previous equation converges to 1 when n is big enough. Therefore we are interested in study the sum. If we fix k we observe that

$$\sum_{i=1}^{k}[\prod_{j=1}^{i-1} f'(H^{-j}(p_m))]g(H^{-i}(p_m)) - \sum_{i=1}^{k}[\prod_{j=1}^{i-1} f'(H^{-j}(z_0))]g(H^{-i}(z_0)),$$

converges to 0 and

$$\left|\sum_{i=k+1}^{n_m}[\prod_{j=1}^{i-1} f'(H^{-j}(p_m))]g(H^{-i}(p_m))\right| < \frac{C\lambda^k}{1-\lambda}.$$

From that we conclude the result making k grow to infinity. $\square$

Observation 6: *The series associated to z_0' is absolutely convergent and dominated by a geometric series.*

Observation 7: *If z_0 is a periodic point then the previous limit coincide with the derivative computed at the beginning.*

Observation 8: *In the equation of t_0' we can appreciate why are we working with these kind of perturbations. Basically what we want is to control through the maps g_i the derivatives of z_0 and z_1. For that, we are going to need to separate points using the g_i functions. Now these functions can only separate points which are on different cylinders or point which belong to the same cylinder but have different central coordinate. The trick is that if we can not separate two points then we are going to construct periodic points which are on the same cylinder with the same central coordinate.*

Let us formalize the previous.

Definition: *Given $A \subset \Lambda_0$, we say that it is* g-Independent *if given two points of A, they belong to different cylinders or if they belong to the same cylinder they have different central coordinate.*

Definition: *Given $A, B \subset \Lambda_0$, we say that they are* g-Independent *if for every $a \in A$ and $b \in B$, the set $\{a, b\}$ is g-independent.*

Definition: *Given a finite set of points $z_0, \ldots, z_n$ en $\Lambda(p)$ we say that they are a su-pseudo-orbit (su-po) if $H(z_i) \in W^{su}(z_{i+1})$ $\forall i < n$. If also $z_0 \in W^{su}(z_n)$ then we say that it is periodic.*

Lemma 4.8: *If $z_0, \ldots, z_n$ is a periodic su-po such that $\{z_0, \ldots, z_{n-1}\}$ is g-independent then there exist q periodic such that its period is n and $H^i(q) \in W^{su}(z_i)$.*

Proof. If $t_0 = \pi_c(z_0)$, by definition of periodic su-po we have that $t_0 = f_{\pi(\pi_M(z_{n-1}))} \circ \cdots \circ f_{\pi(\pi_M(z_0))}(t_0)$. Then we take $x \in \Lambda$ which is the periodic point associated to h induced by the word $\pi(\pi_M(z_0)), \ldots, \pi(\pi_M(z_{n-1}))$, and the point $q = (x, t_0)$ is a periodic point of H which verifies the desired property. $\square$

Observation 9: *It is not immediate that the periodic point obtained in the previous lemma belongs to Λ_0. If it is contracting on the center bundle, then belongs to Λ_0 because since the periodic points are dense there exist one which has an homoclinic connection with q. In the future we will see that q can not be expanding in the center bundle.*

Combining the concept of periodic su-po with g-independent we obtain:

Corollary 4.9: *If $z \in \Lambda_0$ is not g-independent with its future orbit then there exist q periodic such that $q \in W^{su}(z)$.*

Proof. Take n such that $H^n(z) \in W^{su}(z)$. Then $z, H(z), \ldots, H^n(z)$ is a periodic su-po and we apply the previous lemma obtaining q as desired. $\square$

Corollary 4.10: *If q is a periodic point and its orbit is not $g-Independent$ then there exist $q_1 \in \Lambda_0$ a periodic point which has a strong connection with some point in the orbit of q.*

Proof. Let n_q be the period of q. Suppose without losing generality that $H^l(q) \in W^{su}(q)$, where $l < n_q$. Then by the previous corollary we have $q_1 \in W^{su}(q)$ a periodic point of period l which is different from q. If this point were not contracting on the center bundle then using q and q_1 we can construct a family of periodic points q_n all periodic, all contracting on the center bundle, homoclinically related to q which converges to q_1 contradicting the hyperbolicity of Λ_0. $\square$

Lemma 4.11: *The point q obtained on the lemma 4.8 belongs to Λ_0.*

Proof. Let us suppose that q do not belong to Λ_0, If this happens as we have seen before we must have that q is expanding on the center bundle. Using the reasoning of the previous corollary we can suppose that $W^{su}(\theta(q)) \cap per(\Lambda_0) = \phi$. If that is not the case we can contradict the hyperbolicity of Λ_0.

If the orbit of q is not g-Independent then we find another periodic point q_1 for which its orbit belongs to $W^{su}(\theta(q))$. If this point were contracting in the central bundle it would belong to Λ_0, therefore we can assume that expands on the central bundle. Is easy to build $g \in \mathcal{B}$ such that $q' \neq q_1'$. Then for such g there exist $s \in (-\epsilon, \epsilon)$ and $p_1 \in \Lambda_0$ periodic such that $p_1(s)$ has a strong connection with $q_1(s)$ or $q(s)$ obtaining again a contradiction with the fact that Λ_0 is hyperbolic.

Suppose then that the orbit of q is g-Independent. Let $z_0 \in \Lambda_0$, such that $q \in W^{su}(z_0)$. This point exist by hypothesis. We are assuming also that this point is not periodic. If the past orbit of z_0 would not be g-Independent from the orbit of q then we can build q_1 a periodic point which has a strong connection with q and with z_0. By the same reasons as before we obtain a contradiction.

We have now to study the case for which the past orbit of z_0 is g-Independent from the orbit of q.

Let $u_0, \ldots, u_{n_0-1}$ be constants belonging to $[-1, 1]$ such that:

$$d_0 := \frac{\sum_{i=0}^{n_0-1}[\prod_{j=i+1}^{n_0-1} f'(H^j(q))]u_i}{1 - \prod_{i=0}^{n_p-1} f'(H^i(q))} \neq 0.$$

Take N_0 such that

$$|d_0| - \frac{C\lambda^{N_0}}{1-\lambda} > 0.$$

Since we can assume that the past orbit of q is g-Independent, there exist $g_1, \ldots, g_k : I \to [-1, 1]$ such that $g(H^i(q)) = u_i$ y $g(H^{-i}(z_0)) = 0$ if $i < N_0$.

For such g_i we have that

$$|q' - z_0'| = \left| d_0 - \sum_{i=N_0}^{\infty} [\prod_{j=1}^{i-1} f'(H^{-j}(z_0))] g(H^{-i}(z_0)) \right| \geq |d_0| - \frac{C\lambda^{N_0}}{1-\lambda} > 0.$$

This implies that exists $s \in (-\epsilon, \epsilon)$ and $p_1 \in \Lambda_0$ a periodic point such that $p_1(s)$ has a strong connection with $q(s)$ obtaining a contradiction with the hyperbolicity of Λ_0.

□

Let us prove the proposition.

Proof. We are going to see that there exist a perturbation of H, C^r close in $\mathcal{LCSP}$ such that $z_0' - z_1' \neq 0$.

If z_0 and z_1 are not the same point and they belong to the same strong stable manifold then they must have different itineraries in their past orbit. Since $\pi_c(z_0) =$

$\pi_c(z_1)$ while we have that $H^{-i}(z_0)$ and $H^{-i}(z_1)$ belong to the same cylinder we are going to have that

$$[\prod_{j=1}^{i-1} f'(H^{-j}(z_0))]g(H^{-i}(z_0)) = [\prod_{j=1}^{i-1} f'(H^{-j}(z_1))]g(H^{-i}(z_1)).$$

Therefore we can assume without losing generality that $H^{-1}(z_0)$ and $H^{-1}(z_1)$ are not in the same cylinder.

We have two cases now:

- (H1) $\#per(\Lambda) \cap W^{su}(z_0) = 1$
- (H2) $per(\Lambda) \cap W^{su}(z_0) = \phi$

If we had more than one periodic point we would have already a strong connection and we would have finished.

Let us see first the case $(H1)$. In this case, we can assume that z_0 is periodic with period n_0.

Lemma 4.12: *If the past orbit of $H^{-1}(z_1)$ and the orbit of z_0 are not g-Independent then there exist q a periodic point with a strong connection with z_0.*

Proof. Let $i_0 > 1$ and $0 \leq j_0 < n_0$ such that $H^{-i_0}(z_1)$ and $H^{j_0}(z_0)$ are not g-Independent. This means that $H^{j_0}(z_0) \in W^{su}(H^{-i_0}(z_1))$. Then

$$H^{-i_0}(z_1), \ldots, H^{-1}(z_1), z_0, \ldots, H^{j_0}(z_0),$$

is a periodic su-po. Then there exist a periodic point in Λ_0 with a strong connection with z_0. □

Let us assume that the past orbit of $H^{-1}(z_1)$ and the orbit of z_0 are g-Independent.

Take $u_0, \ldots, u_{n_0-1}$ real numbers on $[-1, 1]$ such that

$$d_0 := \frac{\sum_{i=0}^{n_0-1}[\prod_{j=i+1}^{n_0-1} f'(H^j(z_0))]u_i}{1 - \prod_{i=0}^{n_p-1} f'(H^i(z_0))} \neq 0.$$

Take N_0 such that

$$|d_0| - \frac{C\lambda^{N_0}}{1-\lambda} > 0.$$

We can assume that the orbit of z_0 is g-Independent, and therefore there exist $g_1, \ldots, g_k : I \to [-1, 1]$ such that $g(H^i(z_0)) = u_i$ and $g(H^{-i}(z_1)) = 0$ if $i < N_0$.

For such g_i we have that

$$|z_0' - z_1'| = \left| d_0 - \sum_{i=N_0}^{\infty} [\prod_{j=1}^{i-1} f'(H^{-j}(z_1))]g(H^{-i}(z_1)) \right| \geq |d_0| - \frac{C\lambda^{N_0}}{1-\lambda} > 0.$$

Finishing the proof for the case (H1).

Let us see now (H2). We have then that z_0 and z_1 are not periodic and that $H^{-1}(z_0)$ and $H^{-1}(z_1)$ belong to different cylinders.

Lemma 4.13: *If the past orbit of $H^{-1}(z_0)$ is not g-Independent of $H^{-1}(z_0)$ then we have a periodic point in $W^{su}(H^{-1}(z_0))$.*

Proof. Analogue to the proof of the corollary 4.9 □

We now study two cases (H2.a) as the case in which the hypothesis of the previous lemma is true and (H2.b) the opposite. Let us proof first the case (H2.b).

We are assuming then that the past orbit from $H^{-1}(z_0)$ is g-Independent from $H^{-1}(z_0)$.

If there exist $1 < l_0 < l_1$ such that $H^{-l_0}(z_1), H^{-l_1}(z_1) \in W^{su}(H^{-1}(z_0))$, we have that $H^{-l_1}(z_1), \ldots, H^{-l_0}(z_1)$ is a periodic su-po, obtaining a periodic point q which belongs to $W^{su}(H^{-1}(z_0))$. Since the past orbit from $H^{-1}(z_0)$ is g-Independent from $H^{-1}(z_0)$ we have that $H^{-1}(z_0)$ do not belong to $W^{uu}_{loc}(q)$ and then we are again in the case (H1).

Let us assume that there exists only one point in the past orbit from z_1 ($H^{-l_0}(z_1)$) which belongs to $W^{su}(H^{-1}(z_0))$, this point can not be $H^{-1}(z_1)$ by hypothesis.

Lemma 4.14: *If the past orbit from $H^{-1}(z_0)$ is not g-Independent from $H^{-1}(z_1)$ then we have a periodic point in $W^{su}(H^{-1}(z_0))$.*

Proof. Let $H^{-k_0}(z_0) \in W^{su}(H^{-1}(z_1))$ then

$$H^{-k_0}(z_0), \ldots, H^{-2}(z_0), H^{-l_0}(z_1), \ldots, H^{-1}(z_1),$$

is a periodic su-po and therefore we have a periodic point which belongs to $W^{su}(H^{-1}(z_0))$ □

If we are on the hypothesis from the previous lemma we are again in the case (H1). Assuming that we are not, if we had that the past orbit of $H^{-1}(z_1)$ is not g-Independent from $H^{-1}(z_1)$ then we have a periodic point in $W^{su}(H^{-1}(z_1))$. Also, since we are assuming that $H^{-l_0}(z_1)$ belongs to $W^{su}(H^{-1}(z_0))$ and that this point is not in the local strong unstable manifold of any periodic point then we have that $H^{-1}(z_1)$ is not in the local strong unstable manifold from the periodic point obtained and therefore we are again in the case (H1).

From the above reasoning we can assume that $H^{-l_0}(z_1)$ is g-Independent from its past orbit and from the past orbit of $H^{-1}(z_0)$.

Take N_0 such that $1 > \frac{2\lambda_0^N}{1-\lambda}$. We can build $g_1, \ldots, g_k : I \to [-1, 1]$ such that $g(H^{-1}(z_1)) = 1$, $g(H^{-j}(z_0)) = 0$ if $0 < j < N_0$ and $g(H^{-j}(z_1)) = 0$ if $1 < j < N_0$. Then for such functions

$$z_0' = \sum_{i=N_0}^{\infty} [\prod_{j=1}^{i-1} f'(H^{-j}(z_0))] g(H^{-i}(z_0)),$$

$$z_1' = 1 + \sum_{i=N_0}^{\infty} [\prod_{j=1}^{i-1} f'(H^{-j}(z_1))] g(H^{-i}(z_1)),$$

and therefore

$$|z_0' - z_1'| \geq 1 - \frac{2\lambda_0^N}{1-\lambda} > 0,$$

finishing the case (H2.b).

Let us see (H2.a). If q is a periodic point in $W^{su}(H^{-1}(z_0))$ and $H^{-1}(z_0) \notin W^{uu}_{loc}(q)$ then we are in (H1). Assume then that $H^{-1}(z_0) \in W^{uu}_{loc}(q)$, and let us observe that

$$z_0' = f'(H^{-1}(z_0)).H^{-1}(z_0)' + g(H^{-1}(z_0)) = f'(q).q' + g(q) = H(q)',$$

and we conclude that in the previous equation g is only evaluated among $\sigma(q)$ which we may assume g-Independent.

Let us study now what happens with the past orbit from z_1. If this were g-Independent from q then we proceed as in (H1).

Assume therefore that it is not. If the point in the past orbit of z_1 which belongs to $W^{su}(q)$ does not belong to $W^{uu}_{loc}(q)$ we would be again in (H1). We can assume then that it belongs to $W^{uu}_{loc}(q)$. Let j_0 be the smallest natural such that $H^{-j_0}(z_1) \in W^{uu}_{loc}(q)$.

Let $j_1 = max\{0 \leq j < j_0 : H^{-j_0+i}(z_1) \in W^{su}(H^i(q)) \; \forall i \leq j\}$. If $j_1 = j_0 - 1$ then z_0 and z_1 are in $W^{su}(H^{j_0}(q))$ and then we are again on (H1).

If $j_1 < j_0 - 1$ taking the set $A = \{H^{-1}(z_1), \ldots, H^{-j_0+j_1+1}(z_1)\}$ we have that:

Lemma 4.15: *If A is not g-Independent from $\sigma(q)$ then there exist a periodic point which has a strong connection with a point in $\sigma(q)$.*

Proof. Let $j_2 < j_0 - j_1$ and l_0 such that $H^{-j_2}(z_1) \in W^{su}(H^{l_0}(q))$. Then

$$H^{-j_0}(z_1), \ldots, H^{-j_2-1}(z_1), H^{l_0}(q), \ldots, q,$$

is a periodic su-po which give us a periodic point q_2. By construction of A this point is not one of the orbit from q and therefore we conclude. □

Lemma 4.16: *If A is not g-Independent we are in the case (H1).*

Proof. If A is not g-Independent we can find a periodic point which belongs to $W^{su}(z)$ for certain $z \in A$. This periodic point can not be in $W^{uu}_{loc}(z)$ because $H^{-j_0}(z_1)$ is in $W^{uu}_{loc}(q)$. Then z and the periodic point are in (H1). □

Observe now that

$$z_1' = [\prod_{j=1}^{j=j_0-j_1} f'(H^{-j}(z_1))].H^{j_1}(q)' + \sum_{i=1}^{j_0-j_1-1}[\prod_{j=1}^{i-1} f'(H^{-j}(z_1))]g(H^{-i}(z_1)).$$

If A is g-Independent and g-Independent from $\sigma(q)$, there exist $g_1, \dots, g_k : I \to [-1,1]$ such that $g(H^i(q)) = 0$ for every i and

$$\sum_{i=1}^{j_0-j_1-1}[\prod_{j=1}^{i-1} f'(H^{-j}(z_1))]g(H^{-i}(z_1)) \neq 0.$$

Then $z_0' = 0$ and z_1' the previous number, obtaining that $z_0' - z_1' \neq 0$. With this we conclude the proof of (H2.a) and the theorem. $\square$

4.3 Expansive case

Suppose now that E^c is part of the unstable space. By theorem 4.2 we need to prove

Proposition 4.17: *Let $H \in \mathcal{LCSP}$ and Λ_0 an homoclinic class. Suppose that it is hyperbolic and E^c is unstable. Then if there exist z_0 and z_1 which are part of the same strong unstable manifold, there exist $g \in \mathcal{B}$ and s_n a sequence decreasing to 0 such that H_{s_n} has periodic points with strong connections and they belong to the continuation of Λ_0.*

Proof. To prove this we do an analogue construction. Let us observe the important details. Given a periodic point the formula of the first derivative of the continuation of the point it is the same, but it is convenient for us to rewrite it as:

$$\begin{aligned} t'(s) &= \frac{\sum_{i=0}^{n-1} g(H_s^i(p(s))) \prod_{j=i+1}^{n-1}[f' + sg'(H_s^j(p(s)))]}{1 - \prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]} \\ &= \frac{\prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]}{1 - \prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]} \sum_{i=0}^{n-1} g(H_s^i(p(s))) \prod_{j=0}^{i} \frac{1}{f' + sg'(H_s^j(p(s)))}. \end{aligned}$$

Since

$$\frac{\prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]}{1 - \prod_{i=0}^{n-1}[f' + sg'(H_s^i(p(s)))]},$$

converges to -1 when the period of the points grow, it is easy to see that $t'(s)$ is uniformly bounded and taking its derivative once more it is not hard to see that $t''(s)$ is also uniformly bounded and therefore we obtain once again a uniform control over the remainders of the Taylor polynomials for the functions $t(s)$.

If now we take a family of periodic points which converges to z, we have that

$$\frac{\prod_{i=0}^{n-1} f'(H^i(p))}{1-\prod_{i=0}^{n-1} f'(H^i(p))},$$

converges to -1. Fixed k_1, we see that

$$\sum_{i=k_1}^{n-1} g(H^i(p)) \prod_{j=0}^{i} \frac{1}{f'(H^j(p))},$$

is equal to

$$\prod_{j=0}^{k_1} \frac{1}{f'(H^j(p))} \sum_{i=k_1}^{n-1} g(H^i(p)) \prod_{j=k_1+1}^{i} \frac{1}{f'(H^j(p))}.$$

Taking absolute value we see that

$$|\sum_{i=k_1}^{n-1} g(H^i(p)) \prod_{j=0}^{i} \frac{1}{f'(H^j(p))}| \leq \frac{C\lambda^{k_1+1}}{1-\lambda},$$

where λ and C come from the hyperbolicity and therefore it converges to 0. This implies that z' exist and is equal to

$$-\sum_{i=0}^{\infty}[\prod_{j=0}^{i} \frac{1}{f'(H^j(z))}]g(H^i(z)).$$

From this equation we observe that the derivative z' is only dependent of the future orbit of z and therefore if we have z_0 and z_1 which belong to the same local strong unstable manifold but they are not the same point then using the same techniques as in the contractive case we can prove that either already exists periodic points with a strong connection or that there exist $g \in \mathcal{B}$ such that $z_0' \neq z_1'$. $\square$

5 Conclusions

Let us begin this chapter with an immediate corollary of theorem 3. Take $A : M \to SL(2)$ a C^r map and define $F : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$ by

$$F(x, v) = (h(x), A(x)v).$$

Given $\theta \in [0, 2\pi)$ we define $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ as the rotation of angle θ. We say F is stable by rotations if it is conjugated to $R_\theta \circ F$. We say that it is α-absolute stable by rotations if the conjugacy ϕ_θ verifies $d(\phi_\theta, Id) \leq C\theta^\alpha$.

Corollary 5.1: *If F is α-absolute stably by rotations and $\alpha > 1/2$ then it is hyperbolic.*

Proof. Take $\pi : \mathbb{R}^2 - \{0\} \to S^1$ defined by $\pi(v) = \frac{v}{\|v\|}$. We now consider $H_\theta : M \times S^1 \to M \times S^1$ defined by

$$H_\theta(x, v) = (h(x), \pi(R_\theta A(x)v)).$$

In particular H_θ is the perturbation by the uniform translation of H_0 and then we are in the hypothesis of theorem 3. From this we conclude that F has dominated splitting and since we are taking $A(x) \in SL(2)$ we have hyperbolicity. $\square$

We can see now that with a slightly modification of the example in 3.7 the theorem 3 is on the verge of optimal.

Theorem 5.2: *For every $\alpha < 1/2$ there is $H_\alpha \in \mathcal{SKS}$ of class C^2 such that H_α is α-absolute stable and central topologically hyperbolic but not hyperbolic.*

Proof. We define the maps $f_\beta : M \times \mathbb{R} \to \mathbb{R}$ with $\beta > 0$ by

$$f_\beta(x, t) = \varphi(x)t - t^{2+\beta}.$$

We do the same process as in 3.7 using f_β instead of f and we obtain our map H_β which is strong Kupka-Smale. For the points which are not in the minimal set associated to Λ_1 their continuation by the uniform translation is differentiable if the perturbation is taken correctly.

Let $H_{\beta,s}$ be the perturbation family of H_β by the uniform translation. Inside the minimal set associated to Λ_1, the action of the skew-product $H_{\beta,s}$ on the fibers does not depend on the base. This is:

$$(f_\beta + g_\beta + s)(x, t) = t - t^{2+\beta} + s.$$

This implies that to find $b_s(x)$ we have to solve the equation

$$b_s(x) - b_s(x)^{2+\beta} + s = b_s(x),$$

which has solution

$$b_s(x) = s^{\frac{1}{2+\beta}}.$$

If given $\alpha < 1/2$ we take $\beta > 0$ such that $\alpha = \frac{1}{2+\beta}$ then H_β is α-absolute stable but not hyperbolic. $\square$

It is clear that the obstruction to hyperbolicity is on the minimal set of Λ_0 associated to Λ_1. This opened us the following question which we will enunciate it as a conjecture:

Conjecture 5.3: *If Λ_0 is locally maximal, central topologically hyperbolic and verifies: if $\Lambda_1 \subset \Lambda_0$ is a minimal set then Λ_1 is hyperbolic. Then Λ_0 is hyperbolic.*

Observe that the orbit of a periodic point is a minimal set. What we are saying is that if the atoms of our set are all hyperbolic then our set should be hyperbolic. In lower dimensions, 1 and 2, the hyperbolicity of the periodic points was enough to imply hyperbolicity on the whole set and therefore the notion of atoms has always been the periodic points. In dimension 3 there is more space and we can not jump to hyperbolicity from the hyperbolicity of the periodic points. This is why we are proposing to extend the notion of atoms.

There is in fact a more general way to extend this conjecture and is the following:

Conjecture 5.4: *Let $f : M \to M$ be a C^r diffeomorphism. Suppose that there exist Λ a compact locally maximal set which has dominated splitting $T_\Lambda M = E \oplus F$ which verifies the following property: If $\Lambda_1 \subset \Lambda$ is a minimal set then Λ_1 is hyperbolic with $E^s = E$ and $E^u = F$, we should have then that Λ is hyperbolic.*

For this conjecture to be useful we should have a Kupka-Smale's theorem like for minimal sets which should be:

Conjecture 5.5: *There exist $\mathcal{B} \subset Diff^r(M)$ a residual (or dense) set such that if $\Lambda \subset M$ is minimal then Λ is hyperbolic.*

A result in this way is not clear why should be true because a key property in the Kupka-Smale theorem is that generically the periodic points are countable which is not the case for the minimal sets.

Anyhow we have a proposition which points towards our first conjecture:

Proposition 5.6: *Given $H \in \mathcal{SP}$ if Λ_0 is locally maximal, central topologically contracting set which verifies:*

- *If $\Lambda_1 \subset \Lambda_0$ is a minimal set then Λ_1 is hyperbolic*
- *If $z \in \Lambda_0$ then $f'(z) \leq 1$*

Then Λ_0 is hyperbolic.

Proof. Take $z \in \Lambda_0$, in $\omega(z)$ there is a minimal set Λ_1. This can be seen using Zorn's lemma. We take now an open neighborhood U_1 of Λ_1 associated to its hyperbolicity.

When the orbit of z get inside of U_1 the derivative of H in the center bundle decreases exponentially fast. When the orbit of z goes outside of U_1 it can not increase by the second hypothesis therefore we have:

$$lim_n \frac{\partial H^n}{\partial t}(z) = 0 \quad \forall z \in \Lambda_0.$$

This is known to be an equivalent condition to hyperbolicity. $\square$

The previous proposition tell us that to build a counter-example of the first conjecture the example built here is far away to be a starting point.

From the perspective of α-absolute stable by the uniform translation the counter-example shown is also weak. We say that because even though H_α is not hyperbolic, the perturbation by the uniform translation actually makes it hyperbolic. Maybe for $\alpha \leq 1/2$ we can have that even though the system is not hyperbolic there is an open and dense set of parameters for which the perturbation by the uniform translation is hyperbolic.

Another observation from the α-absolute stable theorem proved before that could link all the previous things discussed is that in the proof we never cared for the whereabouts of the orbit in the base.

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